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Tese de doutorado

ON PRE-COMPLETE SYSTEMS OF MODAL FUNCTIONS

Tese de doutorado, apresentada ao Departamento de Filosofia da Faculdade de Filosofia, Letras e Ciências Humanas da Universidade de São Paulo, sob a orientação do Prof. Dr. Rodrigo Bacellar.

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Abstract

We present here some highlights of the Theory of Modal Functions, and in particular an important result in the Theory of Systems of Modal Functions: the determination of the pre-complete systems of modal functions. This result is the modal (S5) correlate of Post's criterion of (truth-)functional completeness, and was originally shown by the Moldavian logician M. F. Ratsa (who published it in a paper written in Russian). We present Ratsa's theorems in a framework slightly different from his, and we provide corrections of a few small errors of the original version.

Resumo

Apresentamos alguns fatos relevantes da Teoria das Funções Modais, e em particular um resultado importante na Teoria dos Sistemas de Funções Modais: a determinação dos sistemas pré-completos de funções modais. Esse resultado é o correlato modal (em S5) do critério de completude (vero-)funcional de Post, e é originalmente devido ao lógico moldávio M. F. Ratsa (que o publicou em um artigo em russo). Nós apresentamos os teoremas de Ratsa em um contexto ligeiramente modificado, e fornecemos correções de alguns pequenos erros do artigo original.

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Introduction

'Denying the negation is affirming, but affirming the affirmation is not denying'. This is a sort of slogan that I made up in order to try to explain to non-logicians what my research is about. It may sound like a silly slogan, but it somehow captures the spirit of the research: the study of the relations of definability (or un-definability) between logical connectives. Of course the simplicity of the example is not always reflected in the relations of definability (or un-definability) among logical connectives, and it has also the problem of dealing only with truth-functions, while the research is majorly concerned with modal functions. But, as a slogan, it works. Those who are interested in a deeper consideration about the research are readily introduced to the main concepts of the theory, viz. the concepts of truth-function, modal function, systems of functions, and so on. To those who are *really* interested in the research I recommend reading this thesis.

The most important content in this thesis is a quite complex and very recondite result in the theory of systems of modal functions, due to the Moldavian logician M. F. Ratsa: the determination of the pre-complete systems of modal functions. This result was pursued by me since my Master's Thesis, and I've spent a long time trying to achieve it with my own resources, since at that time I was unaware that the question of the determination of the pre-complete systems of modal functions was already solved. In fact I was unaware that it was ever posed by someone other than me and my supervisor, but it turns out that while the question wasn't (as far as I know) ever posed or solved in print in the English language (or in Portuguese), both things were done in Russian.

The presentation of the result here follows very closely its Russian counterpart, but it isn't exactly a translation, since I have very little knowledge of Russian. What I could do was use some automatic translation tools, and the help of friends and colleagues, in order to make sense of the whole bunch of formulas and matrices that abound in the text. While this was done I carefully checked every lemma and theorem, and, although my version is not a proper translation, I'm pretty sure that the 'logical' content of the original material is present here.

Since the result is deeply embedded in the Theory of Systems of Modal Functions, its significance can't be properly appreciated without a good grasp of this theory, and for that reason I will state some of its basic definitions and a few other results.

I would like to give thanks to some people who helped me while I was writing this thesis. To my supervisor, Roderick Batchelor, for having introduced me to the Theory of Logical Functions and for being such a careful supervisor; to Irina Starikova and Ana Livia Plurabelle Esteves for helping me with the text in Russian; to Vitoria Barbosa for the working environment that her house provided during good part of this research; and to all colleagues in the logic seminars at the Philosophy Department of the University of São Paulo, for the companionship during a research that could be lonely if I wouldn't share it with them. To CAPES, for the scholarship. To Melina Bertholdo, for the love and support during these last years.

Some essential definitions and some basic results

The *language* of modal propositional logic S5 may be taken as consisting of the connectives $\neg$, $\wedge$, $\Box$ (standing for negation, conjunction, and necessity) and propositional variables $p_1, p_2, p_3, \ldots$

The *formulas* of S5 are defined in the usual way: propositional variables are formulas and, if φ and ψ are formulas, so are $\neg\varphi$, $(\varphi \wedge \psi)$, and $\Box\varphi$.

A model for *classical* propositional logic attributes to each propositional variable a truth-value. Now, given a sequence of specific propositions, it might be the case that not every attribution of truth-values for the propositions is really *possible* (e.g.: if p is of the form $r \wedge s$, and q is of the form $r \wedge \neg s$, it is *impossible* to attribute truth to both p and q). But whichever are the possible attributions of truth-values for the sequence, they should be of course a subset of the set of *all* attributions, and among these one must correspond to the actual truth-value of the sequence of propositions. This inspires the following notion of model for S5:

We will say that a *model* M for *classical* propositional logic is a function from the propositional variables into {T, F}. A *model* for S5 is a non-empty set of models for classical propositional logic with a designated element, i.e. a pair $\langle W, w_0\rangle$ where $w_0 \in W \subseteq \{M\colon M \text{ is a model for classical propositional logic}\}$.

(This notion of model is basically the restriction, for propositional S5, of the semantics presented in Kripke 1959. In this paper Kripke shows the completeness and soundness of this semantics for S5.)

A model $\langle W, w_0\rangle$ induces a *valuation*, i.e. a function $\langle W, w_0\rangle^+$ from the formulas of S5 into {T, F}, by the following clauses:

If φ is a propositional variable p, $\langle W, w_0\rangle^+(\varphi) = T$ iff $w_0(p) = T$.

$\langle W, w_0\rangle^+(\neg\varphi) = T$ iff $\langle W, w_0\rangle^+(\varphi) = F$.

$\langle W, w_0\rangle^+(\varphi \wedge \psi) = T$ iff $\langle W, w_0\rangle^+(\varphi) = T$ and $\langle W, w_0\rangle^+(\psi) = T$.

$\langle W, w_0\rangle^+(\Box\varphi) = T$ iff $\langle W, w\rangle^+(\varphi) = T$ for every $w \in W$.

The *valuation of a sequence of formulas* $\langle W, w_0\rangle^+(\langle\varphi_1, \dots, \varphi_n\rangle)$ is understood as $\langle\langle W, w_0\rangle^+(\varphi_1), \dots, \langle W, w_0\rangle^+(\varphi_n)\rangle$.

It is clear from these clauses that, in order to evaluate a formula, all we need to know is the 'behavior' of the model w.r.t. the variables that occur in the formula. If $\varphi(p_1 \dots p_n)$ is a formula with n propositional variables we can evaluate it considering:

(i) The n-sequence of truth-values that the model attributes to the sequence $\langle p_1, \dots, p_n\rangle$.

(ii) The *set* of n-sequences of truth-values that the models $\langle W, w\rangle$ ($w \in W$) attribute to the sequence $\langle p_1, \dots, p_n\rangle$.

If $\langle W, w_0\rangle^+(\varphi) = T$, we say that $\langle W, w_0\rangle$ *satisfies* φ.

If $\langle W, w_0\rangle^+(\varphi) = T$ for every $\langle W, w_0\rangle$, we say that φ is *valid*, and we write $\models \varphi$.

We call an n-sequence of truth-values an *n-ary truth-value*, and we call a non-empty set of n-ary truth-values an *n-ary purely modal value*.

A *n-ary modal value* is a non-empty subset of $\{T, F\}^n$ with a designated element, i.e. a pair $\langle W, w_0\rangle$ where $w_0 \in W \subseteq \{T, F\}^n$. (Intuitively, w_0 stands for the n-ary truth-value that the variables *actually* assume, while W stands for the set of n-ary truth-values that the variables *could possibly* assume.)

A *modal value* is an n-ary modal value, for some n.

An *n-ary modal function* f is a function from n-ary modal values to truth values.

An n-ary modal function is *purely modal* if its value 'depends only on W', i.e. if for every non-empty $W \subseteq \{T, F\}^n$, $f\langle W, w\rangle = f\langle W, v\rangle$ for all $w \neq v \in W$.

The set of all *n-ary* modal functions will be called μ^n.

A modal function is an n-ary modal function, for some n.

The set of *all* modal functions will be called μ.

(This and much of the other notation and terminology in this section is taken from Batchelor 2017.)

We can define in a natural way the notion of a formula *expressing* a modal function, so that every formula of S5 expresses *some* modal function: a formula φ with n variables will express the n-ary modal function that gives T precisely to the n-ary modal values that satisfy φ. We will see that the converse is also true, i.e. that every modal function is expressed by some formula of S5.

Modal values and modal functions find a perspicuous representation in *modal tables*. A modal table is like a truth-table, except that it is constituted of several sub-tables. Each sub-table is also like a truth-table, except that some (but not all) of the rows may be missing. The rows in a modal table for n variables are constituted of n-ary truth-values. The rows *present* in a sub-table indicate which attributions of values to the variables are *possible*, and the rows *absent* from a sub-table indicate which attributions are *impossible*. A sub-table represents a purely modal value, and its rows represent modal values. The rules for evaluating formulas of the form $\neg\varphi$ and $\varphi \wedge \psi$ are the familiar ones, and a formula of the form $\Box\varphi$ has T in a row of a sub-table if and only if φ has T in *all* rows of that sub-table.

There are four unary modal values: $\langle\{T, F\}, T\rangle$ (contingent and true), $\langle\{T, F\}, F\rangle$ (contingent and false), $\langle\{T\}, T\rangle$ (necessary), and $\langle\{F\}, F\rangle$ (impossible). They are represented in this order in the four rows of the unary modal table below, where we can also find some examples of formulas with their valuations.

p	$\neg$p	$\Box$p	$\Box\neg$p	$\neg\Box\neg$p
T	F	F	F	T
F	T	F	F	T
T	F	T	F	T
F	T	F	T	F

Notice that the formula $\neg\Box\neg p$ expresses the unary modal function that only gives F to impossible arguments, and so may well serve as a definition of the symbol of possibility $\Diamond$ ($\Diamond p$ reads 'it is possible that p').

Since modal functions are functions from modal values to truth-values, and there are four unary modal values, it follows that there are sixteen unary modal functions. They are all represented in the following table:

p	$\top$	$\Diamond$	$\neg\Box$	∇	$\neg\nabla^-$	id	$\neg\Diamond\vee\nabla^+$	∇^+	$\neg\nabla^+$	$\Box\vee\nabla^-$	$\neg$	∇^-	Δ	$\Box$	$\neg\Diamond$	$\bot$
T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

Some of these functions may appear a bit recondite, but it is interesting to understand what each of these modal functions 'says', and to find formulas in the language of S5 that express these functions. A formula that expresses a function f may be called a *definition* of f. Once we have defined a function we may use it in the definition of other functions.

id(p) reads 'id p' or simply 'p', and is the identity function (in the present context it requires no definition);

$\top$(p) reads 'verum p', and is the modal function expressed by any tautology with p as the only variable. It is part of a family of functions $\top^n$, the tautologies with n variables;

$\bot$(p) reads 'falsum p', and is the modal function expressed by any contradiction with p as the only variable. It is part of the family $\bot^n$;

$\neg\Diamond p$ reads 'it is impossible that p', and may be expressed as $\Box\neg p$;

$\neg\Box p$ reads 'it is not necessary that p';

∇p reads 'it is contingent that p', and is expressed, for instance, by $\neg\Box p \wedge \neg\Box\neg p$,

Δp reads 'it is rigid that p', and it is expressed by $\Box p \vee \Box\neg p$;

$\nabla^+ p$ reads 'it is contingently true that p', and it is expressed by $p \wedge \neg\Box p$,

$\neg\nabla^+ p$ is the negation of $\nabla^+ p$;

$\nabla^- p$ reads 'it is contingently false that p' and it is expressed by $\neg p \wedge \Diamond p$,

$\neg\nabla^+ p$ is the negation of $\nabla^+ p$;

$\neg\Diamond\vee\nabla^+ p$ reads 'p is either impossible or contingently true'. It can be expressed as $p \leftrightarrow \nabla p$.

$\Box\vee\nabla^- p$ reads 'p is either necessary or contingently false'. It can be expressed as $p \leftrightarrow \Delta p$.

The binary modal functions are far too many (2^{32}) to be introduced one by one. Each of them corresponds to a distribution of T's and F's in the rows of the binary modal table in the next page, where we again find some examples of formulas and their valuations. $\Box(p \rightarrow q)$ figures among the examples because the strict implication is perhaps the most famous of the binary (non-truth-functional) modal functions. It assumes T whenever the binary truth-value $\langle T, F \rangle$ is absent in a sub-table, i.e. whenever it is impossible to attribute the value $\langle T, F \rangle$ to p, q. The relevance of the other examples will soon be clear.

It is important here to notice that we can express formulas which are satisfied by exactly one row of one sub-table. To make this clear we need to establish some definitions.

A *literal* is either a variable or its negation.

The *possibilization* of a formula φ is the formula $\Diamond\varphi$.

The *impossibilization* of a formula φ is the formula $\neg\Diamond\varphi$.

The *classical characteristic formula of a row* is the conjunction of literals where the i-th variable ($1 \leq i \leq n$) appears negated if and only if the i-th term of the n-ary truth-value in the row is F.

The *characteristic formula of a sub-table* is the conjunction whose terms are all the *possibilizations* of the classical characteristic formulas of rows *present* in the sub-table and all the *impossibilizations* of the classical characteristic formulas of rows *absent* from the sub-table.

The *modal characteristic formula* of a row of a sub-table is the conjunction of the classical characteristic formula of the row with the characteristic formula of the sub-table.

It is easy to see that the modal characteristic formula of a row of a sub-table is satisfied only by that row of that sub-table.

For instance, the modal characteristic formula of the first row of the second sub-table

p q	$\Box(p \to q)$	Ind(p, q)	$(p \to q) \to \neg\Diamond p$	$(\neg p \wedge \neg q) \vee (\Box p \wedge \nabla^{+} q)$
T T	F	T	F	F
T F	F	T	T	F
F T	F	T	F	F
F F	F	T	F	T
T T	F	F	F	F
T F	F	F	T	F
F T	F	F	F	F
T T	F	F	F	F
T F	F	F	T	F
F F	F	F	F	T
T T	T	F	F	F
F T	T	F	F	F
F F	T	F	F	T
T F	F	F	T	F
F T	F	F	F	F
F F	F	F	F	T
T T	F	F	F	T
T F	F	F	T	F
T T	T	F	F	F
F T	T	F	F	F
T T	T	F	F	F
F F	T	F	F	T
T F	F	F	T	F
F T	F	F	F	F
T F	F	F	T	F
F F	F	F	F	T
F T	T	F	T	F
F F	T	F	T	T
T T	T	F	F	F
T F	F	F	T	F
F T	T	F	T	F
F F	T	F	T	T

of the binary modal table in last page is

$$p \wedge q \wedge \Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \neg\Diamond(\neg p \wedge \neg q).$$

Theorem (Functional completeness of S5 (Massey 1966)). Every modal function is expressed by some formula of S5.

Proof. It is sufficient to notice that an arbitrary modal function f can be expressed as the disjunction of the modal characteristic formulas of rows where the function has T (if f has T in no rows, it can be expressed by some $\perp^n$). □

The theorem above might well be considered the starting point of the theory of systems of modal functions. Once we know that we can express all modal functions in terms of the usual connectives $\neg$, $\wedge$, $\Box$, i.e. that $\{\neg, \wedge, \Box\}$ is functionally complete, the question presents itself whether there is *a general criterion for establishing, for any given set of modal functions, whether the set is functionally complete or not.* It is reasonably well known (although perhaps not as much as it deserves) that Post has established such a criterion w.r.t. *truth-functions*. In fact, the criterion is a simple corollary of Post's exhaustive classification of the *systems of truth-functions*, or as he calls it 'iteratively closed two-valued systems of functions'. In the next section we will revisit Post's criterion. In what follows in this section we will state some definitions and some simple theorems of the theory of *systems of modal functions*.

Let C be a set of modal functions.

By $\mathcal{L}(C)$ we mean the propositional language whose primitive connectives express the respective functions in C. (We so to speak ignore merely 'orthographic' differences in the connectives, so that this language is always unique for each given C.)

Thus the *formulas of* $\mathcal{L}(C)$ are defined by:

(i) propositional variables are formulas (also called atomic formulas), and

(ii) If f is an n-ary primitive connective in $\mathcal{L}$(C) and $\varphi_1 \dots \varphi_n$ are formulas, then $f(\varphi_1, \dots \varphi_n)$ is a formula.

We say that a set of modal functions C *defines* a modal function f if there is a non-atomic formula φ in $\mathcal{L}(C)$ such that $\models \varphi(p_1, \ldots, p_n) \leftrightarrow f(p_1 \ldots p_n)$.

By [C] we mean set of all modal functions definable by functions in C. We call this set the *system generated by C.*

We may write $[f_1, f_2, \ldots]$ instead of $[\{f_1, f_2 \ldots\}]$.

If $[C] = \mu$ we say that C is *functionally complete.*

If $[\{f\}] = \mu$ we say that f is a *Sheffer-function* for μ.

The third and fourth formulas figuring in the binary modal table above are Sheffer functions for μ. (These Sheffer-functions are due, respectively, to Ratsa and Batchelor.) The following is a very simple, but worth proving

Proposition. $[\{(p \rightarrow q) \rightarrow \neg\Diamond p\}] = \mu$ and $[\{(\neg p \wedge \neg q) \vee (\Box p \wedge \nabla^{+}q)\}] = \mu$.

Proof. We saw that every modal function is expressible in terms of $\{\neg, \wedge, \Box\}$, so it will be enough to define these functions in $\mathcal{L}(\{(p \rightarrow q) \rightarrow \neg\Diamond p\})$. This language has a single primitive symbol, f(p, q), which is equivalent to $(p \rightarrow q) \rightarrow \neg\Diamond p$.

f(p, p) is equivalent to $(p \rightarrow p) \rightarrow \neg\Diamond p$. This formula is an implication with valid antecedent, and so is equivalent to its consequent which is $\neg\Diamond p$. So f(p, p) expresses $\neg\Diamond p$.

Since we have defined $\neg\Diamond p$ we can use it to define $\Diamond p$: $\neg\Diamond\neg\Diamond p$ expresses $\Diamond p$ (we could express it in primitive notation as f(f(p, p), f(p, p))).

Using f and $\Diamond$ we can define $\neg p$: f($\Diamond$p, p) is equivalent to $(\Diamond p \rightarrow p) \rightarrow \neg\Diamond\Diamond p$ which is equivalent to $\Diamond p \rightarrow (\Diamond p \wedge \neg p)$ which is equivalent to $\neg\Diamond p \vee (\Diamond p \wedge \neg p)$, which is equivalent to $\neg p$.

Using f and $\Diamond$ we can also define $p \rightarrow q$: f($\Diamond$p, f(p, q)) is equivalent to

$(\Diamond p \rightarrow ((p \rightarrow q) \rightarrow \neg\Diamond p)) \rightarrow \neg\Diamond\Diamond p =$

$\Diamond p \rightarrow (\Diamond p \wedge \neg((p \rightarrow q) \rightarrow \neg\Diamond p) =$

$\Diamond p \to (\Diamond p \wedge ((p \to q) \wedge \Diamond p) =$

$\Diamond p \to (p \to q) =$

$(p \to q)$.

Using $\neg$, $\Diamond$, and $\to$ it is easy to define the functions in $\{\neg, \wedge, \Box\}$.

So far we argued for the correctness of these definitions appealing to equivalential transformations. One can also use modal tables, and evaluate the formulas considering the modal values of its sub-formulas. It can then be checked that:

If $f = (\neg p \wedge \neg q) \vee (\Box p \wedge \nabla^{+}q)$:

$\neg p = f(p, p)$

$\top p = \neg f(p, \neg p)$.

$p \vee q = \neg f(p, \neg f(q, p))$.

$\nabla^{+}p = f(\top p, p)$.

Again, with these resources it is easy to define the functions in $\{\neg, \wedge, \Box\}$. $\Box$

Let us consider which is the system generated by Ind(p, q) – the second function in the modal table above. This function reads 'p and q are independent', and it can be expressed by $\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q)$. It is true in a sub-table only if all binary truth-values figure in the sub-table. This function is part of a family: for each n there is a function Ind($p_1, \ldots, p_n$), which is true only if all n-ary truth-values figure in a sub-table.

The formulas of $\mathcal{L}$(Ind(p, q)) are either propositional variables or of the form Ind(φ, ψ), where $\varphi, \psi \in \mathcal{L}$(Ind(p, q)).

It happens that, while Ind(p, q) is true if and only if it 'finds' four different sequences of truth values in a sub-table, it is also a purely modal function, i.e., a function that never has different values for rows within a same sub-table. So if at least one of φ, ψ is non-atomic then, since at least one of φ, ψ has only one truth-value in each sub-table, there is no sub-table where we can find all four binary truth-values attributed to φ, ψ. This is enough to see that a non-atomic formula of $\mathcal{L}$(Ind(p, q)) will express either Ind(p, q) itself, or some $\perp^n$. From this it follows that

$$[\mathrm{Ind}(p, q)] = \{\mathrm{Ind}(p, q), \perp^1, \perp^2, \ldots\}.$$

The very same argument shows that

$$[\mathrm{Ind}(p_1 \ldots p_n)] = \{\mathrm{Ind}(p_1 \ldots p_n), \perp^1, \perp^2, \ldots\}.$$

In fact, using this argument we can conclude that, for any $N \subseteq \{1, 2, 3, \ldots\}$:

$$[\{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\}] = \{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\} \cup \{\perp^1, \perp^2, \ldots\}.$$

Proposition (Batchelor 2017). There are $2^{\aleph_0}$ systems of modal functions.

Proof. We first show that there aren't *more* than $2^{\aleph_0}$ systems of modal functions. It is clear that the number of modal functions is $\aleph_0$: for each n there are finitely many n-ary modal functions. It follows that the cardinality of the set of all sets of modal functions is $2^{\aleph_0}$. The set of systems of modal functions is a subset of the set of all sets of modal functions, and so its cardinality $\leq 2^{\aleph_0}$.

Now we show that there aren't *less* than $2^{\aleph_0}$ systems of modal functions.

We just saw that for every $N \subseteq \{1, 2, 3, \ldots\}$,

$$[\{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\}] = \{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\} \cup \{\perp^1, \perp^2, \ldots\}.$$

It is obvious that if $N \neq M \subseteq \{1, 2, 3, \ldots\}$,

$$\{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\} \neq \{\mathrm{Ind}(p_1 \ldots p_m) : m \in M\}$$

and so

$$[\{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\}] = \{\mathrm{Ind}(p_1 \ldots p_n) : n \in N\} \cup \{\perp^1, \perp^2, \ldots\} \neq$$

$$[\{\mathrm{Ind}(p_1 \ldots p_m) : m \in M\}] = \{\mathrm{Ind}(p_1 \ldots p_m) : m \in M\} \cup \{\perp^1, \perp^2, \ldots\}.$$

Since of course $\{N : N \subseteq \{1, 2, 3, \ldots\}\}$ has cardinality $2^{\aleph_0}$, there are then *at least* $2^{\aleph_0}$ systems of modal functions. □

Among the systems of modal functions, some are of special interest, because they can lead us to the *criterion of functional completeness for modal functions*. We say that a system of modal functions C is *pre-complete* if $C \neq \mu$ and, for any $f \notin C$, $[C \cup \{f\}] = \mu$. The determination of the pre-complete systems of modal functions will occupy us for a good part of what follows.

Ratsa appeals to the fact that there is a connection between 'Topo-Boolean Algebras' and modal logics in order to develop his criterion of functional completeness. We preferred to deal directly with S5 and its models, but the adaptation we have made is not in terms of the models presented in the last section, (Kripke 1959)-style, but in terms of models in a (Kripke 1963)-style. The reason to use the former in the definition of modal functions is that, in the framework of (Kripke 1959), there are no two equivalent models w.r.t. a finite set of variables (i.e. models that verify exactly the same formulas), and so the definition of a modal function can be established, as we have seen, in a very direct (and I would say elegant) way. The reason to use Kripke (1963) here is that it is straightforward to adapt Ratsa's theorems to this framework. The *cost* of doing so is that, in order to use Ratsa's result to determine the pre-complete systems of modal functions, as defined in the last section, we will have to establish a correspondence among our *modal functions* and what we will call *W-operations* and to prove that there is also a correspondence between the *systems* of modal functions and *certain* systems of W-operations. Fortunately, that is not very hard to be done. (An adaptation of Ratsa's results directly to an extension of the framework presented in the last section can be found in Batchelor 2017.)

Let we make a slight reformulation of our definition of *model* for S5.

Let W be an arbitrary non-empty set and w_0 an element of this set (intuitively, we think of W as the set of all possible worlds and w_0 as the actual world). A *model* M is an attribution of subsets of W to the propositional variables. A model is extended to a valuation M^+ by the following clauses:

If φ is a propositional variable p, $M^+(\varphi) = M(p)$.

$M^+(\neg\varphi)$ = the complement of $M^+(\varphi)$.

$M^+(\varphi \wedge \psi) = M^+(\varphi) \cap M^+(\psi)$.

$M^+(\Box\varphi) = W$ if $M^+(\varphi) = W$, and $= \varnothing$ if $W^+(\varphi) \neq W$.

The *valuation of a sequence of formulas* $M^+(\langle\varphi_1, \ldots, \varphi_n\rangle)$ is understood as $\langle M^+(\varphi_1), \ldots, M^+(\varphi_n)\rangle$.

We say that a formula φ is *true* in a model M if $w_0 \in M^+(\varphi)$; otherwise we say φ is *false*.

If a formula φ is true in a model M we say that M *satisfies* φ.

If a formula φ is true in every model we say that φ is *valid*, and we write $\vDash \varphi$.

It is clear that, in this new formulation of model, a formula of S5 will express an operation on $\wp(W)$. A formula with n variables will express a function from $(\wp(W))^n$ to $\wp(W)$.

An *n-ary W-relation* is a subset of $(\wp(W))^n$

An *n-ary W-operation* is a function from $(\wp(W))^n$ to $\wp(W)$.

The *n-ary W-relation expressed by a formula* $\varphi(p_1, \ldots, p_n)$ is the set of sequences $\{M^+\langle p_1, \ldots, p_n\rangle : M \text{ satisfies } \varphi(p_1 \ldots p_n)\}$.

We say that f is the *n-ary W-operation expressed by a formula* $\varphi(p_1, \ldots, p_n)$ if, for every attribution M of subsets of W to the propositional variables, $f(M(p_1), \ldots, M(p_n)) = M^+\varphi(p_1, \ldots, p_n)$.

We say that a W-operation *corresponds* to a modal function if they are expressed by a same formula. Not every W-operation corresponds to a modal function (and that is why it is simpler to define a modal function in a (Kripke 1959)-style model).

In the context of W-operations, where C is a set of modal functions, by $\mathcal{L}(C)$ we mean the language whose primitive connectives express W-operations corresponding to functions in C.

We say that f is a *modal W-operation* if f corresponds to some modal function, i.e. if f is the W-operation expressed by a formula of $\mathcal{L}(\neg, \wedge, \Box)$.

Let $W_n = \{w_0, w_1, \ldots, w_{n-1}\}$.

W-relations and W-operations are defined for arbitrary W, so it makes perfectly good sense to talk about W_n-operations and W_n-relations expressed by a formula φ.

If $W' \subseteq W$ we can talk about the W'-reduct of a W-operation expressed by a formula φ, and that is simply the W'-operation expressed by φ. Similarly, we can talk about the W'-reduct of a W-relation.

The following is a really crucial definition:

We say that an n-ary W-operation f *preserves* a W-relation R if

$\forall p, p', \ldots, q, q', \ldots \subseteq W$: $R(p_1, p_1', p_1'', \ldots)$ & $R(p_2, p_2', p_2'', \ldots) \ldots$ & $R(p_n, p_n', p_n'', \ldots) \Rightarrow R(f(p_1, \ldots p_n), f(p_1', \ldots p_n'), f(p_1'', \ldots p_n''), \ldots)$.

It is often useful, when dealing with relations, to think about the matrices whose columns are the tuples of elements in the relation. The definition above of an operation preserving a relation can be formulated in terms of matrices: Let $\mathcal{M}$ be the matrix which has all (and only) the elements of a relation R as columns. We define an $\mathcal{M}$-matrix as a matrix whose columns are columns of $\mathcal{M}$. To say that an n-ary operation preserves $\mathcal{M}$ is to say that for every $\mathcal{M}$-matrix with n columns, the result of applying the operation on the *rows* of the $\mathcal{M}$-matrix is a column of $\mathcal{M}$.

$W_2 = \{w_0, w_1\}$. We will denote W_2 by 1, $\{w_0\}$ by w, $\{w_1\}$ by v and $\{\ \}$ by 0.

It is easy to see that $\neg$, $\wedge$, $\Box$ preserve the following matrix, which is the W_2-relation expressed by the formula $\Delta p \wedge \Delta q \wedge (p \leftrightarrow q)$ .v. $\nabla p \wedge \nabla q \wedge (p \veebar q)$.

$$\begin{vmatrix} 1 & v & w & 0 \\ 1 & w & v & 0 \end{vmatrix}$$

If a W-operation f preserves a W-relation R we say that f is *polymorphism* of R, and that R is an *invariant* for f.

The set of all polymorphisms of a relation R will be denoted by *Pol(R)*.

A set of W-operations C is a *system of W-operations* if, whenever f is a n-ary W-operation $\in$ C and $f_1 \dots f_n$ are W-operations of arbitrary arities $\in$ C, $f(f_1 \dots f_n) \in C$.

Proposition. For any W, and for any W-relation R, the set Pol(R) is a system of W-relations.

Proof. Let W be an arbitrary set, and let R be a W-relation, and suppose for contradiction that Pol(R) is not a system, i.e. there are W-operations g, $g_1 \dots g_n \in$ Pol(R) and a W-operation f $\notin$ Pol(R) such that $f = g(g_1, \dots g_n)$.

Let $\mathcal{M}$ be a matrix whose columns are all and only the tuples in R, and let k be the number of distinct variables occurring in $g_1 \dots g_n$. It is clear that f is a k-ary operation. All we need to see to establish our contradiction is that: for every $\mathcal{M}$-matrix with k columns, the result of applying f in its rows is an $\mathcal{M}$-column (i.e. f $\in$ Pol(R)). Let $\mathcal{M}'$ be an $\mathcal{M}$-matrix with k columns. Let us call $\mathcal{M}'_i$ the matrix constituted by the columns of $\mathcal{M}'$ corresponding to arguments of g_i, for $(1 \leq i \leq n)$. It is clear that each $\mathcal{M}'_i$ is an $\mathcal{M}$-matrix, and so, since $g_i \in$ Pol(R), the result of applying g_i in its columns will be an $\mathcal{M}$-column. This means that the matrix with n columns which we can represent as $g_1(\mathcal{M}'_1), \dots g_n(\mathcal{M}'_n)$ is an $\mathcal{M}$-matrix with n columns, and so, since g $\in$ Pol(R), $g(g_1(\mathcal{M}'_1), \dots g_n(\mathcal{M}'_n)) = f(\mathcal{M}')$ is an $\mathcal{M}$-column, and so f $\in$ Pol(R), with contradiction.

□

We say that a set of functions C is *separated* from a function f by a matrix $\mathcal{M}$ if every function in C preserves $\mathcal{M}$ but f doesn't. Since preserving matrices is the same as preserving relations, and since the set of all functions preserving a relation is always a system, if f is separated from C by some matrix $\mathcal{M}$, f $\notin$ [C]. E.g., the inexpressibility of $\Box$ in terms of $\neg$, $\wedge$ follows from the fact that they are separated by the matrix

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & w & v & 0 \end{vmatrix}$$

We say that a subset S of $\wp(W)$ is a *modal substructure of W* if S is closed under the modal operations, i.e. if s, s′ $\in$ S, then s $\wedge$ s′ $\in$ S, $\neg$s $\in$ S and $\Box$s $\in$ S.

If W′ $\subseteq$ W and S is a modal substructure of W, we say that σ is an *embedding* of S into W′ if σ is a isomorphism of S and W′, i.e. an injective function from S onto W′ such that $\forall$ p, q $\in$ S : $\sigma\neg(p) = \neg\sigma(p)$, $\Box\sigma(p) = \sigma(\Box p)$, and $\sigma(p \wedge q) = \sigma(p) \wedge \sigma(q)$.

Proposition. If W′ $\subseteq$ W, there is a substructure of W which is embeddable in W′.

Proof. It is sufficient to see that if W′ = W – {w}, there is an embedding from a substructure of W into W′, and notice that 'being embeddable' is a transitive relation. □

The structures, substructures and embeddings that will be relevant to our development will be given explicitly in the next sections.

Proposition. For any W′ $\subseteq$ W, the set of W-operations whose W′-reducts preserve a W′-relation is a system of W-operations.

Proof. Since Pol(R) is a system for any R, it is sufficient to notice that, for every W′-relation R′, and for any embedding σ of a substructure of W into W′, σ^{-1}R′ is a W-relation. □

Proposition. For any two systems C, C′, the intersection C $\cap$ C′ is a system.

Proof. Suppose, for contradiction, that both C and C′ are systems and that C $\cap$ C′ is not a system, i.e. there are functions g, $g_1, \ldots, g_n \in$ C $\cap$ C′ and a function f $\notin$ C $\cap$ C′ that are such that f = $g(g_1, \ldots, g_n)$. Since g, $g_1, \ldots, g_n$ are both in C and in C′, and since C and C′ are systems, it follows that f $\in$ C $\cap$ C′, with contradiction. □

Proposition. For any W and for any W-relation R, the set of *modal functions* corresponding to W-operations in Pol(R) is a *system of modal functions.*

Proof. The set of W-operations corresponding to the modal functions is obviously a system, and the set of W-operations in Pol(R) is also a system; and the intersection of two systems is always a system. □

This allows us to determine systems of modal functions *via* W-relations and W-operations. The next proposition shows that the pre-complete systems can be determined via W-operations for *certain* W's.

The *size* of a model is the cardinality of W. If a formula φ is valid in models with size n we write $\vDash^n \varphi$.

It is well known that

$$\{\varphi : \vDash^1 \varphi\} \supseteq \{\varphi : \vDash^2 \varphi\} \supseteq \{\varphi : \vDash^3 \varphi\} \dots$$

and that

$$\{\varphi : \vDash \varphi\} = \{\varphi : \vDash^1 \varphi\} \cap \{\varphi : \vDash^2 \varphi\} \cap \{\varphi : \vDash^3 \varphi\} \dots .$$

We say that a set of modal functions C *n-defines* a modal function $f(p_1 \dots p_m)$ if there is a formula $\varphi \in \mathcal{L}(C)$ such that $\vDash^n \varphi(p_1 \dots p_m) \leftrightarrow f(p_1 \dots p_m)$.

Proposition (Ratsa's Theorem 1). For every set of modal functions C: C is complete iff C is 4-complete.

Proof. ($\Rightarrow$) is quite straightforward, since $\vDash \varphi$ implies $\vDash^4 \varphi$. ($\Leftarrow$) Suppose C is 4-complete. So there are formulas φ, ψ and θ satisfying $\vDash^4 \neg p \leftrightarrow \varphi$, $\vDash^4 \Box p \leftrightarrow \psi$, $\vDash^4 (p \wedge q) \leftrightarrow \theta$; in each case, the variables on the right side are the same as the variables on the left side.

It is known that a formula with n propositional variables is valid iff it is valid w.r.t. models with 2^n possible worlds. It follows that the equivalences above, having no more than two variables and being 4-valid, are S5-valid. So our formulas φ, ψ and θ are such that $\models \neg p \leftrightarrow \varphi$, $\models \Box p \leftrightarrow \psi$, $\models (p \wedge q) \leftrightarrow \theta$. □

Given the proposition above, the main theorem can be established by proving a functional completeness criterion for modal W_4-operations. That will be done after establishing the criterion for modal W_1-, W_2-, and W_3-operations.

We start by dealing with W_1-relations and W_1-operations. In this section we will for convenience write 1 for $\{w_0\}$ and 0 for $\{\ \}$. It is clear that $\models^1 \Box p \leftrightarrow p$, and so the functional completeness criterion w.r.t. modal W_1-operations is

Theorem 1 (essentially Post's functional completeness criterion). A set of modal functions C is 1-complete iff for each of the W_1-relations corresponding to $\neg p$, p, $p \veebar q$, $p \rightarrow q$, $E^4(p, q, r, s)$, there is a function in C that does not preserve this relation (where $E^4(p, q, r, s)$ stands for 'among p, q, r, s there is an even number of truths'). □

The W_1-relation corresponding to $\neg p$ (resp., p) is just $\{0\}$ ($\{1\}$), and the corresponding matrix is (0) ((1)). The class of all operations preserving this relation will be called Π_0 (Π_1).

The W_1-relation corresponding to $p \veebar q$ is $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ and the corresponding matrix is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

The operations preserving this relation are the functions whose W_1-reduct is a self-dual function; the class of all such functions will be called Π_2.

The W_1-relation corresponding to $p \rightarrow q$ is $\{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\}$ and the corresponding matrix is

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

The operations preserving this relation are the monotonic operations; the class of all monotonic operations will be called Π_3.

The W_1-relation corresponding to $E^4(p, q, r, s)$ corresponds to the matrix

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$

The operations preserving this relation are the linear operations; the class of all linear functions will be called Π_4.

In this section we will establish a criterion for functional completeness w.r.t. modal W_2-operations. This is the first and more laborious step in the path from Post's classical criterion to the criterion of functional completeness w.r.t. S5.

The modal W_2-operations are operations on $\{w_0, w_1\}$ which are expressed by some formula of S5. In this section we will for convenience write 1 for $\{w_0, w_1\}$, w for $\{w_0\}$, v for $\{w_1\}$ and 0 for $\{\ \}$. (For the benefit of those who may be consulting Ratsa's paper, we mention that there one finds the symbols 1, σ, ρ, 0 instead of 1, w, v, 0.)

Here are some examples of modal W_2-operations:

p	$\neg$p	$\Box$p	$\Diamond$p	Δp
0	1	0	0	1
v	w	0	1	0
w	v	0	1	0
1	0	1	1	1

p $\wedge$ q

p\q	0	v	w	1
0	0	0	0	0
v	0	v	0	v
w	0	0	w	w
1	0	v	w	1

p $\leftrightarrow$ q

p\q	0	v	w	1
0	1	w	v	0
v	w	1	0	v
w	v	0	1	w
1	0	v	w	1

Note that not every operation on {0, v, w, 1} is expressed by a modal formula. For instance, there is no modal function corresponding to the unary operation defined by the scheme

$g_0(p) = 0$, if $p \in \{0, v\}$;

1, if $p \in \{w, 1\}$.

Although this is not a modal operation, it is used by Ratsa in the definition of one of the maximal classes. We give here a different definition for that class, in terms of a modal operation.

The criterion is based on twenty classes of modal functions, denoted by $\Pi_0, \ldots, \Pi_{19}$; each class will be the class of functions with operational correlate preserving a certain relation. Π_0–Π_4 are as above, i.e. modal functions whose W_1-correlate preserves $\mathcal{M}_1$–$\mathcal{M}_4$; Π_5–Π_{19} are the modal functions whose correlated modal W_2-operations preserve, respectively, the following relations (or their corresponding matrices):

$\mathcal{M}_5$: $\nabla^- p$; $\mathcal{M}_6$: $\neg\nabla^- p$; $\mathcal{M}_7$: ∇p;

$\mathcal{M}_8$: $\Box(p \leftrightarrow \Box q)$; $\mathcal{M}_9$: $\Box(p \leftrightarrow \Diamond q)$; $\mathcal{M}_{10}$: $(\Box p \wedge q) \vee (\neg\Diamond p \wedge \neg q)$;

$\mathcal{M}_{11}$: $\Box(p \leftrightarrow \Box q) \vee \Box(p \leftrightarrow \Diamond q)$; $\mathcal{M}_{12}$: $\Box(p \leftrightarrow \Box q) \vee \Box(\neg p \leftrightarrow \Diamond q)$;

$\mathcal{M}_{13}$: $\Box(\neg p \leftrightarrow \Box q) \vee \Box(p \leftrightarrow \Diamond q)$; $\mathcal{M}_{14}$: $\Box(\nabla^+ p \leftrightarrow \nabla^+ q)$;

$\mathcal{M}_{15}$: $\Delta p \leftrightarrow \Delta q$; $\mathcal{M}_{16}$: $(p \leftrightarrow q) \vee (\nabla p \wedge \nabla q)$; $\mathcal{M}_{17}$: $\Delta p \vee \Delta q$;

$\mathcal{M}_{18}$: $\Delta p \wedge \Delta r \wedge ((p \leftrightarrow r) \vee \Delta q)$;

$\mathcal{M}_{19}$: $\Delta p \wedge \Delta r \wedge ((p \veebar r) \rightarrow \nabla q)$.

In Ratsa's paper the matrix $\mathcal{M}_{10}$ is defined using a formula involving the above-mentioned non-modal operation g_0 (viz. the formula $\Box(p \leftrightarrow g_0(q))$). But we prefer, when possible, to avoid this use of non-modal operations in the characterization of maximal classes; whether this is *always* possible is a question that will come up later.

In the table below we list the classes, together with the matrices corresponding to each of the relations, and we give examples of functions preserving and not preserving each of the matrices. In fact, for each pair of matrices, there is an example of a function preserving one and not preserving the other. In the examples, to save space, we write e.g. $\wedge$ for $p \wedge q$, and similarly for other well-known connectives. 2^3 will stand for 'at least two among the following three propositions are true' (or: $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$). Some less well-known connectives will be referred to using Greek letters:

$$\psi =_{df} (p \rightarrow (\Box(p \vee q) \vee \Box(p \rightarrow q) \vee \Box(q \rightarrow p))) \rightarrow q.$$

$$\zeta =_{df} (\neg\nabla^{+}p \wedge (\Diamond p \vee \neg\nabla^{+}q)) \rightarrow (p \wedge q).$$

$$\theta =_{df} (p \wedge \neg\Diamond q) \vee (\neg\Diamond p \wedge q).$$

$$\xi =_{df} ((\Box(p \vee q) \vee \Box(p \rightarrow q) \vee \Box(q \rightarrow p)) \wedge (\Box(\neg p \vee \neg q) \vee \Box(\neg p \rightarrow \neg q) \vee \Box(\neg q \rightarrow \neg p))) \rightarrow \Box(p \leftrightarrow q).$$

Class	Matrix	$f \in \Pi$	$f \notin \Pi$
Π_0	0	$\bot, \Box, \Diamond, \wedge, \vee, \veebar, \psi$	$\top, \neg, \neg\nabla^-$
Π_1	1	$\top, \Box, \Diamond, \wedge, \vee, \leftrightarrow, \psi$	$\bot, \neg, \nabla^+$
Π_2	0 1 1 0	$\Box, \neg, 2^3, \psi$	$\top, \bot, \neg\nabla^-$
Π_3	0 0 1 0 1 1	$\top, \bot, \Box, \neg\nabla^+, \wedge, \vee, \psi$	$\neg, \neg\Box, \neg\Diamond, \neg\Diamond\vee\nabla+$
Π_4	0 0 0 0 1 1 1 1 0 0 1 1 0 0 1 1 0 1 0 1 0 1 0 1 0 1 1 0 1 0 0 1	$\top, \bot, \neg, \Box, \leftrightarrow,$ $\Box(p \wedge q) \to (p \wedge q), \psi$	$\wedge, \vee, \zeta, \Box(p \vee q), 2^3,$ $p \to \Box q$
Π_5	v	$\neg\nabla^-, \nabla^+, \wedge, \vee, \neg\Diamond\vee\nabla^+, \psi$	$\bot(p), \Box p, \Diamond p, \neg p$
Π_6	0 w 1	$\top, \bot, \neg\Box, \Diamond, \wedge, \vee, \psi$	$\neg, \neg\nabla^+, \nabla^-, \leftrightarrow, \veebar,$ $p \leftrightarrow \Diamond q$
Π_7	v w	$\neg, \neg\nabla^+, \neg\nabla^-, 2^3, \psi$	$\top, \bot, \Box, \Diamond, \wedge$
Π_8	0 0 0 1 0 v w 1	$\top, \bot, \neg\Box, \nabla^-, \wedge, p \vee \Box q, \psi$	$\Diamond, \neg, \neg\Diamond, \neg\nabla^-$
Π_9	0 1 1 1 0 v w 1	$\top, \bot, \neg\Diamond, \neg\nabla^+, \vee, p \wedge \Diamond q, \psi$	$\Box, \neg, \neg\Diamond, \nabla^-$
Π_{10}	0 0 1 1 0 v w 1	$\top, \bot, \neg, \wedge, \vee, \psi$	$\Box, \neg\Box, \neg\Diamond, \neg\nabla^+, \neg\nabla^-$
Π_{11}	0 0 0 1 1 1 0 v w v w 1	$\top, \bot, \neg, \neg\nabla^+, \psi$ $\Diamond(p \wedge q) \to (p \wedge q)$	$\Box, \Delta, \neg\Box, \neg\Diamond, \wedge,$ $p \to \Box q$
Π_{12}	0 0 0 0 1 1 0 v w 1 0 1	$\top, \bot, \neg\Diamond, \nabla^-, \wedge, \psi$	$\neg, \neg\nabla^+, \vee, p \vee \Box q$
Π_{13}	0 0 1 1 1 1 0 1 0 v w 1	$\top, \bot, \neg\Box, \vee, p \to \Box p, \psi$	$\neg, \nabla^-, \wedge, p \wedge \Diamond q$
Π_{14}	0 0 v w 1 1 0 1 v w 0 1	$\top, \bot, \Box, \neg, \neg\nabla^- p \wedge \neg\nabla^- q,$ $\Box(p \leftrightarrow q), \zeta, \psi$	$\wedge, \vee, \leftrightarrow, 2^3, p \to \Box q,$ $\Diamond(p \wedge q) \to (p \wedge q)$

Π_{15}	0 0 v v w w 1 1 0 1 v w v w 0 1	$\top, \bot, \neg, \leftrightarrow, \Box(p \leftrightarrow q)$ $\Box(p \vee q), \psi$	$\wedge, \vee, p \rightarrow \Box q, 2^3, \Diamond(p \wedge q) \rightarrow$ $(p \wedge q), \neg\nabla^- p \wedge \neg\nabla^- q$
Π_{16}	0 v v w w 1 0 v w v w 1	$\top, \bot, \Box, \neg, p \rightarrow \Box q, \zeta, \psi$	$\wedge, \vee, \Diamond(p \wedge q) \rightarrow (p \wedge q),$ $\Box(p \leftrightarrow q), 2^3, \xi$
Π_{17}	0 0 0 0 v v w w 1 1 1 1 0 v w 1 0 1 0 1 0 v w 1	$\top, \bot, \Box, \neg, p \rightarrow \Box q,$ $\Box(p \leftrightarrow q), \psi$	$\wedge, \vee, \leftrightarrow, \zeta, \theta,$ $\Diamond(p \wedge q) \rightarrow (p \wedge q), 2^3$
Π_{18}	0 0 0 0 0 0 1 1 1 1 1 1 0 0 v w 1 1 0 0 v w 1 1 0 1 0 0 0 1 0 1 1 1 0 1	$\top, \bot, \Box, \neg, \Box(p \vee q),$ $\Diamond(p \wedge q) \rightarrow (p \wedge q), \psi$	$\wedge, \vee, \leftrightarrow, \zeta,$ $p \rightarrow \Box q, 2^3$
Π_{19}	0 0 0 0 0 0 1 1 1 1 1 1 0 v v w w 1 0 v v w w 1 0 0 1 0 1 0 1 0 1 0 1 1	$\top, \bot, \neg\Diamond, \neg\nabla^+, \zeta,$ $\Diamond(p \wedge q) \rightarrow (p \wedge q), \psi$	$\Box, \neg\Box, \wedge, \vee, p \rightarrow \Box q,$ $p \wedge (\Box p \rightarrow q)$

Our final goal in this section is to prove that a set of modal functions C is 2-complete iff it is 1-complete and, for each Π_i, $5 \leq i \leq 19$, there is $f \in C$ such that $f \notin \Pi_i$. This is Ratsa's Theorem 2, and it will follow easily from a (rather long) series of lemmas.

In the formulation and in the proofs of these lemmas we use some conventions on certain indefinite descriptions.

The symbol f_i stands for *some* n-ary modal W_2-operation corresponding to modal function $\notin \Pi_i$.

${}^1f$ will stand for *some* modal W_2-operation whose W_1-reduct is f.

(Since the set $\{\{\ \}, \{w_0, w_1\}\}$ (which we are writing here $\{0, 1\}$) is a substructure of W_2 which is embeddable in W_1, the value of a ${}^1f$ will be determined when its arguments assume only the values 0, 1.)

The symbol ${}^{\nabla}\neg$ will stand for *some* modal W_2-operation corresponding to element of the set $\{\neg, \neg\nabla^+, \nabla^-, \Box\vee\nabla^-\}$. (The motivation [or mnemonic] for this last notation is that these are the unary functions which work as negation when applied to contingent arguments.)

We will make extensive use of the fact, mentioned above, that every modal W_2-operation preserves the matrix

$$\begin{vmatrix} 1 & v & w & 0 \\ 1 & w & v & 0 \end{vmatrix}$$

This matrix will be called $\mathcal{M}_{\forall 2}$ (the idea behind this notation is that this is a universal invariant w.r.t. W_2-modal operations).

Lemma 1 (Ratsa's Lemma 4). $\top, \bot \in [f_5, f_7, {}^1\neg, {}^1\bot]$.

Proof. $f_5 \notin \Pi_5$, so f_5 doesn't preserve (v). So $f_5(v \ldots v) \in \{0, w, 1\}$. If $f_5(v \ldots v) \in \{0, 1\}$, we define $\bot(p)$ as ${}^1\bot f_5(p \ldots p)$, and $\top$ as ${}^1\neg{}^1\bot f_5(p \ldots p)$.

If $f_5(v \ldots v) = w$, we use f_7. Since f_7 doesn't preserve (w, v), there is a sequence $\langle\alpha_1 \ldots \alpha_n\rangle$ where $\alpha_i \in \{w, v\}$ and such that $f_7(\alpha_1 \ldots \alpha_n) \in \{0, 1\}$. Let B be the formula $f_7(B_1 \ldots B_n)$ where for $1 \leq i \leq n$:

$B_i =$ p, if $\alpha_i = v$;

$f_5(p \ldots p)$, if $\alpha_i = w$.

$B \in \mathcal{L}\{f_5, f_7\}$. Note that $B_i(v) = \alpha_i$. So $B(v) = f_7(B_1(v) \ldots B_n(v)) = f_7(\alpha_1 \ldots \alpha_n)$. It follows that $B(v) \in \{0, 1\}$ and, since B preserves $\mathcal{M}_{\forall 2}$, we know also that $B(w) \in \{0, 1\}$. We also know that there is no modal W_2-operation mapping $\{1, 0\}$ into $\{w, v\}$. So, for all values of p, $B(p) \in \{1, 0\}$.

We now define $\bot$ as ${}^1\bot B(p)$, and $\top$ as ${}^1\neg{}^1\bot B(p)$. □

Lemma 2 (Ratsa's lemma 5). At least one of $\neg$, $\neg\nabla^+$, ∇^-, $\Box\vee\nabla^- \in [\top, \bot, f_6]$.

Proof. f_6 does not preserve (1, w, 0). So there is a sequence $\langle\underline{\alpha}\rangle$, $\alpha_i \in \{1, w, 0\}$, such that $f_6(\underline{\alpha}) = v$. The letter B will now stand for the formula $f_6(B_1 \ldots B_n)$, where

$B_i = \bot(p)$, if $\alpha_i = 0$;

p, if $\alpha_i = w$;

$\top(p)$, if $\alpha_i = 1$.

$B \in \mathcal{L}\{\neg, \bot, f_6\}$. Note that $B_i(w) = \alpha_i$; so $B(w) = f_6(\underline{\alpha}) = v$. Since $B(w) = v$ and B preserves $\mathcal{M}_{\nabla 2}$ it follows that $B(v) = w$. The table below will help us to see that this proves the lemma.

p	B(p)	$\neg p$	$\neg\nabla^+(p)$	$\nabla^-(p)$	$\Box\vee\nabla^-(p)$
1	?	0	1	0	1
w	v	v	v	v	v
v	w	w	w	w	w
0	?	1	1	0	0

(As mentioned above, the symbol ${}^{\nabla}\neg$ will be an indefinite description for one of these functions.) □

We will state and prove lemma 3 after establishing some auxiliary lemmas.

Lemma 3.1 (Ratsa's lemma A). At least one of $\Box, \Diamond, \Delta \in [\top, \bot, f_{10}, f_{11}, {}^{\nabla}\neg, {}^{1}\neg]$.

Proof. ${}^{1}\neg$ is one of $\neg, \neg\Box, \neg\Diamond, \neg\Diamond\vee\nabla^+$. If ${}^{1}\neg$ is $\neg\Box$ or $\neg\Diamond$ we are done, since ${}^{1}\neg{}^{1}\neg p$ will be then, respectively, $\Box p$ or $\Diamond p$.

There are two cases left.

Case (i): ${}^{1}\neg$ is $\neg\Diamond\vee\nabla^+$. In this case we define formula C by the scheme:

$C =$ ${}^{\nabla}\neg(\neg\Diamond\vee\nabla^+)(p)$, if ${}^{\nabla}\neg = \neg$;

$(\neg\Diamond\vee\nabla^+){}^{\nabla}\neg p$, if ${}^{\nabla}\neg = \neg\nabla^+$;

${}^{\nabla}\neg p$, if ${}^{\nabla}\neg = \nabla^-$ or ${}^{\nabla}\neg = \Box\vee\nabla^-$.

$C \in \mathcal{L}\{{}^{\nabla}\neg, \neg\Diamond\vee\nabla^+\}$, and C satisfies: $C(0) = 0$ and $C(v) = w$.

Since f_{11} doesn't preserve $\mathcal{M}_{11}$, there is a pair of n-sequences $\langle\underline{\alpha}\rangle$ and $\langle\underline{\beta}\rangle$ such that each pair $\langle\alpha_i, \beta_i\rangle$ is an $\mathcal{M}_{11}$-column and $\langle f_{11}(\underline{\alpha}), f_{11}(\underline{\beta})\rangle$ is *not* an $\mathcal{M}_{11}$-column. So

$$\left|\begin{array}{c} f_{11}(\underline{\alpha}) \\ \\ f_{11}(\underline{\beta}) \end{array}\right| \subseteq \left|\begin{array}{cc} 0 & 1 \\ & \\ 1 & 0 \end{array}\right|$$

(Note that, since every modal operation preserves rigidity, we can't 'go' from $\mathcal{M}_{11}$ to any column with w or v on top.)

Let D be the formula $f_{11}(D_1 \ldots D_n)$, where, for $1 \leq i \leq n$:

$D_i =$	$\bot(p)$,	if $\alpha_i = \beta_i = 0$;
	p,	if $\alpha_i = 0$ and $\beta_i = v$;
	C(p),	if $\alpha_i = 0$ and $\beta_i = w$;
	$\neg\Diamond\vee\nabla^+(p)$,	if $\alpha_i = 1$ and $\beta_i = v$;
	$\neg\Diamond\vee\nabla^+(C(p))$,	if $\alpha_i = 1$ and $\beta_i = w$;
	$\top(p)$,	if $\alpha_i = \beta_i = 1$.

The values considered for α_i, β_i exhaust $\mathcal{M}_{11}$. $D \in \mathcal{L}(\top, \bot, C, {}^1\neg, f_{11})$, Var(D) (i.e. the variables of D) = {p}. Since C(0) = 0 and C(v) = w, we can see that $D_i(0) = \alpha_i$ and $D_i(v) = \beta_i$. So

$$D(0) = f_{11}(D_1(0) \ldots D_n(0)) = f_{11}(\underline{\alpha})$$

$$D(v) = f_{11}(D_1(v) \ldots D_n(v)) = f_{11}(\underline{\beta})$$

From the inclusion above we know that

$$D(0) = 0 \text{ and } D(v) = 1$$

or

$$D(0) = 1 \text{ and } D(v) = 0.$$

From D and $\neg\Diamond\vee\nabla^{+}$ we define

$$E = \begin{cases} D(p), & \text{if } D(0) = 1; \\ \neg\Diamond\vee\nabla^{+}(D(p)), & \text{if } D(0) = 0. \end{cases}$$

$E(0) = 1$, $E(v) = 0$. Since E preserves $\mathcal{M}_{\nabla 2}$, $E(w) = 0$. We don't know whether $E(1) = 1$ or $E(1) = 0$, but in either case we can define one of the functions desired in this lemma. The tables below help us see that. The first table deals with case $E(1) = 1$, the second with case $E(1) = 0$.

p	E(p)	Δp
1	1	1
w	0	0
v	0	0
0	1	1

p	E(p)	$\neg\Diamond\vee\nabla^{+}$(E(p))	◊p
1	0	1	1
w	0	1	1
v	0	1	1
0	1	0	0

It remains to consider

Case (ii): $^{1}\neg$ is $\neg$. In this case we use f_{10}. f_{10} does not preserve $\mathcal{M}_{10}$, so there are n-tuples $\langle\underline{\gamma}\rangle$, $\langle\underline{\delta}\rangle$ where each pair $\langle\gamma_i, \delta_i\rangle$ is an $\mathcal{M}_{10}$-column and such that

$$\left|\begin{matrix} f_{10}(\underline{\gamma}) \\ \\ f_{10}(\underline{\delta}) \end{matrix}\right| \subseteq \left|\begin{matrix} 0 & 0 & 1 & 1 \\ \\ w & 1 & 0 & v \end{matrix}\right|$$

The letter F will stand for the formula $f_{10}(F_1 \ldots F_n)$, where, for $1 \leq i \leq n$:

$$F_i = \begin{cases} \bot(p), & \text{if } \gamma_i = \delta_i = 0; \\ p, & \text{if } \gamma_i = 0 \text{ and } \delta_i = v; \\ \neg p, & \text{if } \gamma_i = 1 \text{ and } \delta_i = w; \\ \top(p), & \text{if } \alpha_i = \beta_i = 1. \end{cases}$$

This exhausts $\mathcal{M}_{10}$. $F \in \mathcal{L}\{\top, \bot, \neg, F_{10}\}$, $Var(F) = \{p\}$. It is easy to see that $F_i(0) = \gamma_i$ and $F_i(v) = \delta_i$, from which it follows that $F(0) = f_{11}(\underline{\gamma})$ and $F(v) = f_{11}(\underline{\delta})$. So

$$\left|\begin{matrix} F(0) \\ \\ F(v) \end{matrix}\right| \subseteq \left|\begin{matrix} 0 & 0 & 1 & 1 \\ \\ w & 1 & 0 & v \end{matrix}\right|$$

We reduce the four cases (corresponding to the possible values of F(0), F(v)) to two using formula G, defined by the scheme:

$$G = \begin{cases} F, & \text{if } F(0) = 1; \\ \neg F, & \text{if } F(0) = 0. \end{cases}$$

We know that $G(0) = 1$. $G(v)$ can be either 0 or v; if $G(v) = 0$, since G preserves $\mathcal{M}_{\forall 2}$, $G(w) = 0$ and we are done, as the table below helps us see:

p	G(p)	Δp	¬◊p
1	?	1	0
w	0	0	0
v	0	0	0
0	1	1	1

If G is ¬◊p, we define ◊p as ¬¬◊p.

If G(v) = v we need to use again formula D, but now defined in terms of ⊤, ⊥, G and ¬. D = $f_{11}(D_1 \ldots D_n)$ where, for $1 \leq i \leq n$:

$$D_i = \begin{cases} \bot, & \text{if } \alpha_i = \beta_i = 0; \\ p, & \text{if } \alpha_i = 0 \text{ and } \beta_i = v; \\ \neg G, & \text{if } \alpha_i = 0 \text{ and } \beta_i = w; \\ G, & \text{if } \alpha_i = 1 \text{ and } \beta_i = v; \\ \neg p, & \text{if } \alpha_i = 1 \text{ and } \beta_i = w; \\ \top, & \text{if } \alpha_i = \beta_i = 1. \end{cases}$$

$D_i(0) = \alpha_i$, $D_i(v) = \beta_i$. Recall that D(0) = 0 and D(v) = 1 or D(0) = 1 and D(v) = 0. We reduce these two cases to one using formula E:

$$E = \begin{cases} D, & \text{if } D(0) = 1; \\ \neg D, & \text{if } D(0) = 0. \end{cases}$$

E(0) = 1, E(v) = 0. By preservation of $\mathcal{M}_{\nabla 2}$, E(w) = 0. So we are in the same situation as above with G when G(v) = 0. So E is either Δ or ¬◊ and we are done. □

Lemma 3.2 (Ratsa's lemma Б). At least one of ◊, Δ ∈ [⊤, ⊥, □, f_8, $^{\nabla}\neg$, $^{1}\neg$].

Proof. If ${}^{\nabla}\neg$ is $\neg$, we define $\Diamond p$ as $\neg\Box\neg p$. If ${}^{\nabla}\neg$ is $\neg\nabla^{+}$ we define Δp as $\Box\neg\nabla^{+}p$. If ${}^{\nabla}\neg$ is ∇^{-} or $\Box\vee\nabla^{-}$ we use f_8. Since f_8 doesn't preserve $\mathcal{M}_8$ there are n-tuples $\langle\varepsilon\rangle$, $\langle\zeta\rangle$ such that each pair $\langle\varepsilon_i, \zeta_i\rangle$ is an $\mathcal{M}_8$-column and such that

$$\left|\begin{array}{c} f_8(\varepsilon) \\ \\ f_8(\zeta) \end{array}\right| \subseteq \left|\begin{array}{cccc} 0 & 1 & 1 & 1 \\ \\ 1 & 0 & v & w \end{array}\right|.$$

Let H stand for the formula $F_8(H_1 \ldots H_n)$ where for $1 \leq i \leq n$:

$H_i =$ $\bot(p)$, if $\varepsilon_i = \zeta_i = 0$;

p, if $\varepsilon_i = 0$ and $\zeta_i = v$;

${}^{\nabla}\neg p$, if $\varepsilon_i = 0$ and $\zeta_i = w$;

$\top(p)$, if $\varepsilon_i = \zeta_i = 1$.

This cases exhausts $\mathcal{M}_8$. $H \in \{\top, \bot, f_8, {}^{\nabla}\neg\}$, $Var(H) = \{p\}$. Note that $H_i(0) = \varepsilon_i$ and $H(v) = \zeta_i$. So $H(0) = f_8(\varepsilon)$ and $H(v) = f_8(\zeta)$, from which we get

$$\left|\begin{array}{c} H(0) \\ \\ H(v) \end{array}\right| \subseteq \left|\begin{array}{cccc} 0 & 1 & 1 & 1 \\ \\ 1 & 0 & v & w \end{array}\right|.$$

It follows that

$$\left|\begin{array}{c} \Box H(0) \\ \\ \Box H(v) \end{array}\right| \subseteq \left|\begin{array}{cc} 0 & 1 \\ \\ 1 & 0 \end{array}\right|.$$

As usual, we reduce these two cases to one using a scheme

$P = {}^1\neg\Box H, \quad$ if $H(0) = 0$;

$\qquad \Box H, \quad$ if $H(0) = 1$.

$P \in \mathcal{L}\{\Box, H, {}^1\neg\}$, Var(P) = p. P(0) = 1, P(v) = 0; by $\mathcal{M}_{\forall 2}$, P(w) = 0. If P(1) = 1, P is Δ; if P(1) = 0, P is $\neg\Diamond$. In either case we can define one of Δ, $\Diamond$. This proves lemma 3.2. □

Lemma 3.3 (Ratsa's lemma B). At least one of $\Box, \Delta \in [\top, \bot, \Diamond, f_9, {}^{\nabla}\neg, {}^1\neg]$.

Proof. By dualization of the proof of lemma 3.2. □

Lemma 3 (Ratsa's lemma 6). $\Delta \in \{\top, \bot, f_8, f_9, f_{10}, f_{11}, {}^1\neg, {}^{\nabla}\neg, {}^1\wedge\}$.

Proof. This follows immediately from the auxiliary lemmas and Lemma 4 below. □

Lemma 4 (Ratsa's lemma Γ). $\Delta \in [\Box, \Diamond, {}^1\neg, {}^1\wedge]$.

Proof. Δp can be defined as ${}^1\neg(\Diamond p \; {}^1\wedge \; {}^1\neg\Box p)$. □

Lemma 5 (Ratsa's lemma 7). □ and $\Diamond \in [\top, \bot, \Delta, f_{19}, {}^{\nabla}\neg, {}^1\neg, {}^1\wedge]$.

In fact □ and $\Diamond$ are interdefinable in presence of $\Delta, {}^1\neg, {}^1\wedge$: $\Box p = (\Diamond p \; {}^1\wedge \; \Delta p)$ and $\Diamond p = {}^1\neg({}^1\neg\Box p \; {}^1\wedge \; \Delta p)$. So it will be enough to prove that one of □, $\Diamond$ is expressible in the conditions of the lemma.

As we know, ${}^1\neg$ is one of $\neg, \neg\Box, \neg\Diamond, \neg\Diamond\vee\nabla^+$.

Case (i). ${}^1\neg$ is $\neg\Box$ or $\neg\Diamond$. This is immediate since $\Box p = \neg\Box\neg\Box p$ and $\Diamond p = \neg\Diamond\neg\Diamond p$.

The other cases are much more laborious.

Case (ii). ${}^1\neg$ is $\neg$. Now we need to deal with ${}^1\wedge$. This is *some* modal W_2-operation whose W_1-correlate is $\wedge$. But we don't know how it behaves w.r.t. W_2-arguments other than 0 and 1. But we know that

$(v\ {}^1\!\wedge\ 0) \in \{0, v, w, 1\}$.

Case (*ii-a*). $(v\ {}^1\!\wedge\ 0) \in \{0, 1\}$. In this case we use formula $p\ {}^1\!\wedge\ \Delta p$ which is equivalent either to $\Box p$ (if $(v\ {}^1\!\wedge\ 0) = 0$) or to $\Diamond p$ (if $(v\ {}^1\!\wedge\ 0) = 1$). So in these sub-cases we are done.

Case (iib). $(v\ {}^1\!\wedge\ 0) \in \{v, w\}$. We now define formula C using the scheme:

$$C = \begin{cases} \neg(p\ {}^1\!\wedge \bot), & \text{if } (v\ {}^1\!\wedge\ 0) = v; \\ \neg(\neg p\ {}^1\!\wedge \bot), & \text{if } (v\ {}^1\!\wedge\ 0) = w. \end{cases}$$

$C \in \mathcal{L}\{\bot, \neg, {}^1\!\wedge\}$, $Var(C) = \{p\}$. Since C preserves $\mathcal{M}_{\forall 2}$ and $C(v\ {}^1\!\wedge\ 0) = w$, it follows that $C(w\ {}^1\!\wedge\ 0) = v$. So C is $\neg\nabla^+$:

p	C(p)	$\neg\nabla^+ p$
1	1	1
w	v	v
v	w	w
0	1	1

Now, $(v\ {}^1\!\wedge\ w) \in \{0, w, v, 1\}$. If $(v\ {}^1\!\wedge\ w) \in \{0, 1\}$, we can define one of $\Box$, $\Diamond$ using the formula $p\ {}^1\!\wedge \neg\nabla^+ p$, which will be equivalent to $\Box p$, if $(v\ {}^1\!\wedge\ w) = 0$, or to $\Diamond p$, if $(v\ {}^1\!\wedge\ w) = 1$.

If $(v\ {}^1\!\wedge\ w) \in \{w, v\}$ we first take formula D, defined by the scheme:

$$D(p) = \begin{cases} (\neg\nabla^+ p\ {}^1\!\wedge\ p), & \text{if } (v\ {}^1\!\wedge\ w) = v; \\ (p\ {}^1\!\wedge \neg\nabla^+ p), & \text{if } (v\ {}^1\!\wedge\ w) = w. \end{cases}$$

Since ${}^1\!\wedge$ preserves $\mathcal{M}_{\forall 2}$, $(v\ {}^1\!\wedge\ w) = v$ implies $(w\ {}^1\!\wedge\ v) = w$. From this it is easy to see that D is W_2-equivalent to $\Box\vee\nabla^-$.

Now we need to use f_{19}. Since f_{19} does not preserve $\mathcal{M}_{19}$ there are n-tuples $\langle\underline{\alpha}\rangle$, $\langle\underline{\beta}\rangle$, $\langle\underline{\gamma}\rangle$ such that each $\langle\alpha_i, \beta_i, \gamma_i\rangle$ is an $\mathcal{M}_{19}$-column and such that

$$\begin{array}{|c|c|cccc|}
f_{19}(\underline{\alpha}) & & 0 & 0 & 1 & 1 \\
f_{19}(\underline{\beta}) & \subseteq & 0 & 1 & 0 & 1 \\
f_{19}(\underline{\gamma}) & & 1 & 1 & 0 & 0
\end{array}$$

Let E be formula $f_{19}(E_1 \ldots E_n)$ where for $1 \leq i \leq n$:

$$E_i = \begin{cases}
\bot, & \text{if } \alpha_i = \beta_i = \gamma_i = 0; \\
\neg(\neg\nabla^+ p), & \text{if } \alpha_i = 0,\ \beta_i = v,\ \gamma_i = 0; \\
p, & \text{if } \alpha_i = 0,\ \beta_i = v,\ \gamma_i = 1; \\
\neg(\neg\nabla^+(\neg\nabla^+ p)), & \text{if } \alpha_i = 0,\ \beta_i = w,\ \gamma_i = 0; \\
\Box\vee\nabla^- p, & \text{if } \alpha_i = 0,\ \beta_i = w,\ \gamma_i = 1; \\
\neg(\Delta p), & \text{if } \alpha_i = 0,\ \beta_i = 1,\ \gamma_i = 0; \\
\Delta p, & \text{if } \alpha_i = 1,\ \beta_i = 0,\ \gamma_i = 1; \\
\neg(\Box\vee\nabla^- p), & \text{if } \alpha_i = 1,\ \beta_i = v,\ \gamma_i = 0; \\
\neg\nabla^+(\neg\nabla^+ p), & \text{if } \alpha_i = 1,\ \beta_i = v,\ \gamma_i = 1; \\
\neg p, & \text{if } \alpha_i = 1,\ \beta_i = w,\ \gamma_i = 0; \\
\neg\nabla^+ p, & \text{if } \alpha_i = 1,\ \beta_i = w,\ \gamma_i = 1; \\
\top, & \text{if } \alpha_i = \beta_i = \gamma_i = 1.
\end{cases}$$

This exhausts $\mathcal{M}_{19}$. Var(E) = {p}, $E \in \mathcal{L}\{\top, \bot, \Delta, \neg\nabla^+, \Box\vee\nabla^-, \neg, f_{19}\}$. Note that $E_i(0) = \alpha_i$, $E_i(v) = \beta_i$, and $E_i(1) = \gamma_i$. So

$$\begin{vmatrix} E(0) \\ \\ E(v) \\ \\ E(1) \end{vmatrix} \subseteq \begin{vmatrix} 0 & 0 & 1 & 1 \\ \\ 0 & 1 & 0 & 1 \\ \\ 1 & 1 & 0 & 0 \end{vmatrix}.$$

Since E preserves $\mathcal{M}_{\forall 2}$, E(v) = 0 implies E(w) = 0 (similarly for 1). So E is one of $\Box$, $\Diamond$, $\neg\Diamond$, $\neg\Box$.

<u>Case (iii), ${}^1\neg$ is $\neg\Diamond\vee\nabla^+$,</u> can be reduced to the cases already considered. To do so note first that

$$\neg\Diamond\vee\nabla^+({}^\nabla\neg p) = \begin{cases} \neg\nabla^+ p, & \text{if } {}^\nabla\neg \text{ is } \nabla^-; \\ \neg p, & \text{if } {}^\nabla\neg \text{ is } \Box\vee\nabla^-. \end{cases}$$

This is enough to assure us that at least one of $\neg$, $\neg\nabla^+ \in [{}^\nabla\neg, {}^1\neg]$. The case $\neg$ was our case (ii); in the case $\neg\nabla^+$ we take formula

$$F = \begin{cases} \neg\Diamond\vee\nabla^+ p \ {}^1\!\wedge \neg\nabla^+ p, & \text{if } (v \ {}^1\!\wedge w) = w; \\ \neg\nabla^+ p \ {}^1\!\wedge \neg\Diamond\vee\nabla^+ p, & \text{if } (v \ {}^1\!\wedge w) \neq w. \end{cases}$$

Var(F) = {p}, $F \in \{\neg\nabla^+, {}^1\neg, {}^1\wedge\}$. It is easy to check that F is W_2-equivalent to one of $\neg$, $\neg\Box$, $\neg\Diamond$; in all cases we fall into a case already considered. $\Box$

Lemma 6 (Ratsa's lemma 8). As least one of $\{\neg\}$, $\{\neg\nabla^+, \nabla^-\} \subseteq [\top, \bot, \Box, \Diamond, f_{12}, f_{13}, {}^\nabla\neg, {}^1\neg]$.

Proof. The case where ${}^\nabla\neg$ is $\neg$ is trivial.

<u>Case (i): ${}^\nabla\neg$ is ∇^-.</u> In this case we use f_{12}. f_{12} doesn't preserve $\mathcal{M}_{12}$ and so there are n-tuples $\langle\underline{\alpha}\rangle$ and $\langle\underline{\beta}\rangle$ such that each $\langle\alpha_i, \beta_i\rangle$ is an $\mathcal{M}_{12}$-column and such that

$$\begin{vmatrix} f_{12}(\underline{\alpha}) \\ \\ f_{12}(\underline{\beta}) \end{vmatrix} \subseteq \begin{vmatrix} 1 & 1 \\ \\ v & w \end{vmatrix}.$$

Let C be formula $f_{12}(C_1 \ldots C_n)$ where for $1 \le i \le n$:

$C_i =$ $\bot p$, if $\alpha_i = \beta_i = 0$;

p, if $\alpha_i = 0$ and $\beta_i = v$;

$\nabla^{-} p$, if $\alpha_i = 0$ and $\beta_i = w$;

$\Diamond p$, if $\alpha_i = 0$ and $\beta_i = 1$;

${}^{1}\neg\Diamond p$, if $\alpha_i = 1$ and $\beta_i = 0$;

$\top p$, if $\alpha_i = \beta_i = 1$.

This exhausts $\mathcal{M}_{12}$. Var(C) = {p}, $\mathcal{L}(C) = \{\top, \bot, \Diamond, {}^{\nabla}\neg, {}^{1}\neg, f_{12}\}$. Note that $C_i(0) = \alpha_i$ and $C_i(v) = \beta_i$. So $C(0) = 1$ and $C(v) \in \{v, w\}$. So D, defined by scheme

$D =$ $C(\nabla^{-} p)$, if $C(v) = v$;

$C(p)$, if $C(v) = w$,

is such that $D \in \mathcal{L}\{\nabla^{-}, C\}$ and D satisfies $D(0) = 1$, $D(v) = w$ (also, by $\mathcal{M}_{\nabla 2}$, $D(w) = v$). So D is either $\neg$ or $\neg\nabla^{+}$; in any case expressing what the lemma asks.

It remains to consider

<u>*Case (ii)*: ${}^{\nabla}\neg = \neg\nabla^{+}$ or ${}^{\nabla}\neg = \Box\vee\nabla^{-}$.</u>

In this case we use f_{13}. f_{13} doesn't preserve $\mathcal{M}_{13}$, so there are n-tuples $\langle\underline{\gamma}\rangle$ and $\langle\underline{\delta}\rangle$ etc. such that

$$\left|\begin{array}{c} f_{13}(\underline{\gamma}) \\ \\ f_{13}(\underline{\delta}) \end{array}\right| \subseteq \left|\begin{array}{cc} 0 & 0 \\ \\ v & w \end{array}\right| .$$

Take formula $E = f_{13}(E_1 \ldots E_n)$ where for $1 \le i \le n$:

$$E_i = \begin{cases} \bot p, & \text{if } \gamma_i = \delta_i = 0; \\ \neg\Box p, & \text{if } \gamma_i = 0 \text{ and } \delta_i = 1; \\ \Box p, & \text{if } \gamma_i = 1 \text{ and } \delta_i = 0; \\ p, & \text{if } \gamma_i = 1 \text{ and } \delta_i = v; \\ {}^{\nabla}\neg p, & \text{if } \gamma_i = 1 \text{ and } \delta_i = w; \\ \top p, & \text{if } \gamma_1 = \delta_i = 1. \end{cases}$$

This exhausts $\mathcal{M}_{13}$. $Var(E) = \{p\}$, $E \in \mathcal{L}\{\top, \bot, {}^{\nabla}\neg, \neg, f_{13}\}$. Note that $E_i(1) = \gamma_i$ and $E_i(v) = \delta_i$. So

$$\left|\begin{array}{c} E(1) \\ \\ E(v) \end{array}\right| \subseteq \left|\begin{array}{cc} 0 & 0 \\ \\ v & w \end{array}\right| .$$

We now define formula F by scheme

$$F = \begin{cases} E({}^{\nabla}\neg p), & \text{if } E(v) = v; \\ E(p), & \text{if } E(v) = w. \end{cases}$$

$F(1) = 1$, $F(v) = w$. By $\mathcal{M}_{\forall 2}$, $F(w) = v$ and F is one of $\neg$, $\nabla^{\neg}$; these cases were already considered. □

Lemma 7 (Ratsa's lemma 9). Let Σ be one of the sets $\{\neg\}$, $\{\neg\nabla^+, \nabla^-\}$. Then at least one of $\neg\nabla^+$, $\Box\vee\nabla^- \in [\top, \bot, \Box, \Diamond, f_{17}, \Sigma] = [\{\top, \bot, \Box, \Diamond, f_{17}\} \cup \Sigma]$ *and* this system defines also some formula φ satisfying

$$\varphi(v, 1) = \varphi(1, v) = v.$$

Proof. Suppose

$C =$	$\neg p$,	if $\Sigma = \{\neg\}$;
	$\neg\nabla^+ p$,	if $\Sigma = \{\neg\nabla^+, \nabla^-\}$.

$D =$	$\neg p$,	if $\Sigma = \{\neg\}$;
	$\nabla^- p$,	if $\Sigma = \{\neg\nabla^+, \nabla^-\}$.

$C, D \in \mathcal{L}(\Sigma)$ and satisfy

$$C(0) = 1, C(v) = D(v) = w, \text{ and } D(1) = 0.$$

Since f_{17} does not preserve $\mathcal{M}_{17}$, there are n-tuples $\langle\underline{\alpha}\rangle$, $\langle\underline{\beta}\rangle$ such that each $\langle\alpha_i, \beta_i\rangle$ is an $\mathcal{M}_{17}$-column and such that

$$\left|\begin{matrix} f_{17}(\underline{\alpha}) \\ \\ f_{17}(\underline{\beta}) \end{matrix}\right| \subseteq \left|\begin{matrix} v & v & w & w \\ \\ v & w & v & w \end{matrix}\right|.$$

Let E be formula $f_{17}(E_1 \ldots E_n)$, where for $1 \leq i \leq n$:

$E_i =$	$\bot$,	if $\alpha_i = \beta_i = 0$;
	$D(s)$,	if $\alpha_i = 0$ and $\beta_i = v$;
	$D(r)$,	if $\alpha_i = 0$ and $\beta_i = w$;
	$\Box p$,	if $\alpha_i = 0$ and $\beta_i = 1$;
	$D(q)$,	if $\alpha_i = v$ and $\beta_i = 0$;
	p,	if $\alpha_i = v$ and $\beta_i = 1$;
	$D(p)$,	if $\alpha_i = w$ and $\beta_i = 0$;

q,	if $\alpha_i = w$ and $\beta_i = 1$;
$\Box$r,	if $\alpha_i = 1$ and $\beta_i = 0$;
r,	if $\alpha_i = 1$ and $\beta_i = v$;
s,	if $\alpha_i = 1$ and $\beta_i = w$;
$\top$,	if $\alpha_i = \beta_i = 1$.

This exhausts $\mathcal{M}_{17}$. Var(E) = {p, q, r, s}, E $\in \mathcal{L}\{\top, \bot, \Box, D, f_{17}\}$. Note that, given the conditions on D, $E_i(v, w, 1, 1) = \alpha_i$ and $E_i(1, 1, v, w) = \beta_i$. So we have

$$\begin{vmatrix} E(v, w, 1, 1) \\ \\ E(1, 1, v, w) \end{vmatrix} \subseteq \begin{vmatrix} v & v & w & w \\ \\ v & w & v & w \end{vmatrix}.$$

We define formula F by scheme:

$$F = \begin{cases} E, & \text{if } E(v, w, 1, 1) = E(1, 1, v, w) = v; \\ E(r/s, s/r), & \text{if } E(v, w, 1, 1) = v \text{ and } E(1, 1, v, w) = w; \\ E(p/q, q/p), & \text{if } E(v, w, 1, 1) = w \text{ and } E(1, 1, v, w) = v; \\ D(E), & \text{if } E(v, w, 1, 1) = E(1, 1, v, w) = w. \end{cases}$$

F $\in \mathcal{L}\{D, E\}$. Since F preserves $\mathcal{M}_{\forall 2}$, F satisfies

$$F(v, w, 1, 1) = F(1, 1, v, w) = v.$$

Take unary formula G, defined by the scheme:

$$G = \begin{cases} C(F(\Diamond D, \top, p, D)), & \text{if } F(0, 1, 1, 0) = 0; \\ F(D, p, \top, \Diamond D), & \text{if } F(0, 1, 1, 0) = 1. \end{cases}$$

(Recall that C is either $\neg$ or $\neg\nabla^{+}$.) So G(1) = 1, G(v) = w, G(w) = v. We don't know whether G(0) = 1 or G(0) = 0. If G(0) = 1, G is $\neg\nabla^{+}$; if G(0) = 0, G is $\Box\vee\nabla^{-}$.

Finally, notice that $F(p, G(p), q, G(q))$ is a suitable φ, i.e. a formula φ in two variables such that $\varphi(v, 1) = \varphi(1, v) = v$. □

Lemma 8 (Ratsa's lemma 10). Let E be one of $\neg\nabla^+$, $\Box\vee\nabla^-$, and Σ be one of $\{\neg\}$, $\{\neg\nabla^+, \nabla^-\}$. Then $\neg, \neg\nabla^- \in [\top, \bot, \Box, f_{18}, E, {}^1\neg, {}^1\wedge, \Sigma]$ *and* this system contains also some formula ψ satisfying

$$\psi(1, 0) = 0,\ \psi(v, 1) = v,\ \psi(1, 1) = 1.$$

Proof. Since f_{18} does not preserve $\mathcal{M}_{18}$, there are n-tuples $\langle\underline{\alpha}\rangle$, $\langle\underline{\beta}\rangle$, and $\langle\underline{\gamma}\rangle$ such that each $\langle\alpha_i, \beta_i, \gamma_i\rangle$ is an $\mathcal{M}_{18}$-column and such that $\langle f_{18}(\underline{\alpha}), f_{18}(\underline{\beta}), f_{18}(\underline{\gamma})\rangle$ is *not* an $\mathcal{M}_{18}$-column. Since the α's and γ's are all either 0 or 1, it follows that

$$\left|\begin{array}{c} f_{18}(\underline{\alpha}) \\[2ex] f_{18}(\underline{\beta}) \\[2ex] f_{18}(\underline{\gamma}) \end{array}\right| \subseteq \left|\begin{array}{cccc} 0 & 0 & 1 & 1 \\[2ex] v & w & v & w \\[2ex] 1 & 1 & 0 & 0 \end{array}\right|.$$

Let D be as defined in the proof of lemma 7. $D \in \Sigma$ and D satisfies $D(v) = w$ and $D(1) = 0$. We take then formula $F = f_{18}(F_1 \ldots F_n)$ where for $1 \le i \le n$:

$F_i =$	$\bot$,	if $\alpha_i = \beta_i = \gamma_i = 0$;
	$\Box p\ {}^1\wedge\ q$,	if $\alpha_i = \beta_i = 0$ and $\gamma_i = 1$;
	$D(E(p))$,	if $\alpha_i = 0$ and $\beta_i = v$ and $\gamma_i = 0$;
	$D(p)$,	if $\alpha_i = 0$ and $\beta_i = w$ and $\gamma_i = 0$;
	${}^1\neg\Box p$,	if $\alpha_i = 0$ and $\beta_i = 1$ and $\gamma_i = 0$;
	q,	if $\alpha_i = 0$ and $\beta_i = 1$ and $\gamma_i = 1$;

${}^1\neg q$,	if $\alpha_i = 1$ and $\beta_i = 0$ and $\gamma_i = 0$;
$\Box p$,	if $\alpha_i = 1$ and $\beta_i = 0$ and $\gamma_i = 1$;
p,	if $\alpha_i = 1$ and $\beta_i = v$ and $\gamma_i = 1$;
E(p),	if $\alpha_i = 1$ and $\beta_i = w$ and $\gamma_i = 1$;
${}^1\neg(\Box p\ {}^1\wedge q)$,	if $\alpha_i = 1$ and $\beta_i = 1$ and $\gamma_i = 0$;
$\top$,	if $\alpha_i = \beta_i = \gamma_i = 1$.

This exhausts $\mathcal{M}_{18}$. Var(F) = {p, q}, $F \in \mathcal{L}\{\top, \bot, {}^1\wedge, \Box, D, E\}$. Taking into account the conditions on D ($D(v) = w$, $D(1) = 0$), we can see that $F_i(1, 0) = \alpha_i$, $F_i(v, 1) = \beta_i$, and $F_i(1, 1) = \gamma_i$. So

$$\left|\begin{array}{c} F(1,0) \\ F(v,1) \\ F(1,1) \end{array}\right| \subseteq \left|\begin{array}{cccc} 0 & 0 & 1 & 1 \\ v & w & v & w \\ 1 & 1 & 0 & 0 \end{array}\right|.$$

The formula C satisfying the conditions of the lemma will be defined as:

C =	F,	if F(1, 0) = 0 and F(v, 1) = v and F(1, 1) = 1;
	F(E(p), q),	if F(1, 0) = 0 and F(v, 1) = w and F(1, 1) = 1;
	$F(p, {}^1\neg(\Box p\ {}^1\wedge q))$,	if F(1, 0) = 1 and F(v, 1) = v and F(1, 1) = 0;
	$F(E(p), {}^1\neg(\Box p\ {}^1\wedge q))$,	if F(1, 0) = 1 and F(v, 1) = w and F(1, 1) = 0.

(Recall that E is either $\neg\nabla^+$ or $\Box\vee\nabla^-$.) $C \in \mathcal{L}\{\Box, E, F, {}^1\neg, {}^1\wedge\}$. Since C preserves $\mathcal{M}_{\forall 2}$ it follows that C satisfies the conditions of the lemma.

We now build formulas for $\neg$ and $\neg\nabla^-$. Let Σ be $\{\neg\}$. It is enough then to see that $\neg\nabla^-$ is equivalent to

$\neg C(\neg p, \neg\Box\neg p)$, if $C(0, 1) = 0$;

$C(p, \top(p))$, if $C(0, 1) = 1$.

Finally, let Σ be $\{\neg\nabla^{+}, \nabla^{-}\}$. Note that $\models^{2} C(\neg\nabla^{+}p, \neg\Box p) \leftrightarrow \neg p$. Now $\neg\nabla^{-}p$ is simply $\neg(\nabla^{-}p)$. $\Box$

Lemma 9 (Ratsa's lemma 11). $(\neg\Diamond p \vee \neg\nabla^{-}q) \in [\top, \bot, \neg, \Diamond, \neg\nabla^{-}, f_{14}, f_{15}]$.

Proof. f_{14} does not preserve $\mathcal{M}_{14}$, and so there are n-tuples $\langle\underline{\alpha}\rangle$ and $\langle\underline{\beta}\rangle$ such that each $\langle\alpha_i, \beta_i\rangle$ is an $\mathcal{M}_{14}$-column and such that

$$\left|\begin{matrix} f_{14}(\underline{\alpha}) \\ \\ f_{14}(\underline{\beta}) \end{matrix}\right| \subseteq \left|\begin{matrix} 0 & 0 & v & v & w & w & 1 & 1 & v & w \\ \\ v & w & 0 & 1 & 0 & 1 & v & w & w & v \end{matrix}\right|.$$

Let D be formula $f_{14}(D_1 \ldots D_n)$ where for $1 \leq i \leq n$:

$D_i =$ $\bot$, if $\alpha_i = \beta_i = 0$;

p, if $\alpha_i = 0$ and $\beta_i = 1$;

q, if $\alpha_i = \beta_i = v$;

$\neg q$, if $\alpha_i = \beta_i = w$;

$\neg p$, if $\alpha_i = 1$ and $\beta_i = 0$;

$\top$, if $\alpha_i = \beta_i = 1$.

This exhausts $\mathcal{M}_{14}$. $Var(D) = \{p, q\}$, $D \in \mathcal{L}\{\top, \bot, \neg, f_{14}\}$. Note that $D(0, v) = \alpha_i$ and $D(1, v) = \beta_i$. It follows that

$$\left|\begin{array}{c} D(0, v) \\ \\ D(1, v) \end{array}\right| \subseteq \left|\begin{array}{cccccccccc} 0 & 0 & v & v & w & w & 1 & 1 & v & w \\ \\ v & w & 0 & 1 & 0 & 1 & v & w & w & v \end{array}\right|.$$

We begin by considering the cases corresponding to the first eight columns. To do so take formula E, defined by the scheme:

$$E = \begin{cases} \neg\nabla D(p, q), & \text{if } D(0, v) = 0 \text{ and } D(1, v) = v; \\ \neg D(p, q), & \text{if } D(0, v) = 0 \text{ and } D(1, v) = w; \\ \neg\nabla D(\neg p, q), & \text{if } D(0, v) = v \text{ and } D(1, v) = 0; \\ D(\neg p, q), & \text{if } D(0, v) = v \text{ and } D(1, v) = 1; \\ \neg D(\neg p, q), & \text{if } D(0, v) = w \text{ and } D(1, v) = 0; \\ D(\neg p, \neg q), & \text{if } D(0, v) = w \text{ and } D(1, v) = 1; \\ D(p, q), & \text{if } D(0, v) = 1 \text{ and } D(1, v) = v; \\ D(p, \neg q), & \text{if } D(0, v) = 1 \text{ and } D(1, v) = w. \end{cases}$$

$E \in \mathcal{L}\{\neg, \neg\nabla, D\}$; E satisfies $E(0, v) = 1$, $E(1, v) = v$. It is not hard to see that

$$\models^2 \neg\nabla E(\Diamond p, q) \leftrightarrow (\neg\Diamond p \vee \neg\nabla q).$$

The last two cases, corresponding to the last two columns of the matrix above, will be treated using formula F, defined by scheme:

$$F = \begin{cases} D, & \text{if } D(0, v) = v \text{ and } D(1, v) = w; \\ \neg D, & \text{if } D(0, v) = w \text{ and } D(1, v) = v. \end{cases}$$

$F \in \mathcal{L}\{\neg, D\}$ and F satisfies $F(0, v) = v$, $F(1, v) = w$. Now we use f_{15}. Since f_{15} does not preserve $\mathcal{M}_{15}$, there are n-tuples $\langle \underline{\gamma} \rangle$ and $\langle \underline{\delta} \rangle$ where each $\langle \gamma_i, \delta_i \rangle$ is an $\mathcal{M}_{15}$-column and such that

$$\left|\begin{array}{c|c|cccccccc}f_{15}(\gamma) & \subseteq & 0 & 0 & v & v & w & w & 1 & 1 \\ f_{15}(\delta) & & v & w & 0 & 1 & 0 & 1 & v & w\end{array}\right|.$$

We denote by G the formula $f_{15}(G_1 \ldots G_n)$ where for $1 \leq i \leq n$:

$$G_i = \begin{cases} \bot, & \text{if } \gamma_i = \delta_i = 0; \\ p, & \text{if } \gamma_i = 0 \text{ and } \delta_i = 1; \\ q, & \text{if } \gamma_i = \delta_i = v; \\ F, & \text{if } \gamma_i = v \text{ and } \delta_i = w; \\ \neg F, & \text{if } \gamma_i = w \text{ and } \delta_i = v; \\ \neg q, & \text{if } \gamma_i = \delta_i = w; \\ \neg p, & \text{if } \gamma_i = 1 \text{ and } \delta_i = 0; \\ \top, & \text{if } \gamma_i = \delta_i = 1. \end{cases}$$

This exhausts $\mathcal{M}_{15}$. Var(G) = {p, q}, $G \in \mathcal{L}\{\top, \bot, \neg, F, f_{15}\}$. Since F(0, v) = v and F(1, v) = w we can see that $G_i(0, v) = \gamma_i$ and $G_i(1, v) = \delta_i$. So

$$\left|\begin{array}{c|c|cccccccc}G(0, v) & \subseteq & 0 & 0 & v & v & w & w & 1 & 1 \\ G(1, v) & & v & w & 0 & 1 & 0 & 1 & v & w\end{array}\right|.$$

Note that this matrix = the first eight columns of the matrix above, and so we fall into a case already considered. □

Lemma 10 (Ratsa's lemma 12). $\Diamond(p \wedge q) \in [\top, \bot, \Diamond, f_{16}, {}^1\wedge]$.

Proof. f_{16} does not preserve $\mathcal{M}_{16}$ and so there are n-tuples $\langle \underline{\alpha} \rangle$ and $\langle \underline{\beta} \rangle$ such that each pair $\langle \alpha_i, \beta_i \rangle$ is an $\mathcal{M}_{16}$-column and such that

$$\left| \begin{matrix} f_{16}(\underline{\alpha}) \\ \\ f_{16}(\underline{\beta}) \end{matrix} \right| \subseteq \left| \begin{matrix} 0 & 0 & 0 & v & v & w & w & 1 & 1 & 1 \\ \\ v & w & 1 & 0 & 1 & 0 & 1 & 0 & v & w \end{matrix} \right| .$$

Let D be formula $f_{16}(D_1 \ldots D_n)$ where for $1 \le i \le n$:

$$D_i = \begin{cases} \bot, & \text{if } \alpha_i = \beta_i = 0; \\ p, & \text{if } \alpha_i = \beta_i = v; \\ q, & \text{if } \alpha_i = v \text{ and } \beta_i = w; \\ \neg q, & \text{if } \alpha_i = w \text{ and } \beta_i = v; \\ \neg p, & \text{if } \alpha_i = \beta_i = w; \\ \top, & \text{if } \alpha_i = \beta_i = 0. \end{cases}$$

This exhausts $\mathcal{M}_{16}$. Var(D) = {p, q}, $D \in \mathcal{L}\{\top, \bot, \neg, f_{16}\}$. It is easy to see that $D_i(v, v) = \alpha_i$ and $D_i(v, w) = \beta_i$. So

$$\left| \begin{matrix} D(v, v) \\ \\ D(v, w) \end{matrix} \right| \subseteq \left| \begin{matrix} 0 & 0 & 0 & v & v & w & w & 1 & 1 & 1 \\ \\ v & w & 1 & 0 & 1 & 0 & 1 & 0 & v & w \end{matrix} \right| .$$

We now define formula E as follows:

$$E = \begin{cases} \Diamond D, & \text{if } D(v, v) = 0 \text{ and } D(v, w) \neq 0; \\ \neg\Diamond D, & \text{if } D(v, v) \in \{w, v\} \text{ and } D(v, w) = 0; \\ \Diamond\neg D, & \text{if } D(v, v) = \{1\} \text{ and } D(v, w) \neq 1; \end{cases}$$

$\neg\Diamond\neg D$,   if $D(v, v) \in \{w, v\}$ and $D(v, w) = 1$.

E is a purely modal function on two arguments, and is expressible in terms of $\neg$, $\Diamond$, D. Since $E(v, v) = 0$ and $E(v, w) = 1$, we can conclude that

$$\models^2 (\Diamond p \,{}^1\!\wedge (\Diamond q \,{}^1\!\wedge \neg(E \,{}^1\!\wedge (\neg\Box p \,{}^1\!\wedge \neg\Box q)) \leftrightarrow \Diamond(p \wedge q). \ \Box$$

We are now ready to prove

Theorem 2 (Functional completeness criterion w.r.t. modal W_2-operations – Ratsa's theorem 2). A set of modal functions C is 2-complete iff it is 1-complete and, for each Π_i, $5 \leq i \leq 19$, there is $f \in C$ such that $f \notin \Pi_i$.

Proof. The ($\Rightarrow$) part of this theorem follows from $\{\varphi : \models^2 \varphi\} \subseteq \{\varphi : \models^1 \varphi\}$ and the fact that the classes $\Pi_5 \ldots \Pi_{19}$ are compositionally closed and not 2-complete.

($\Leftarrow$). In view of Lemmas 1–10 we conclude that using ${}^1\neg$, ${}^1\bot$, ${}^1\wedge$ and formulas f_5–f_{19} we can define $\Box$, $\neg$, $\neg\nabla^-$, $\neg\Diamond p \vee \neg\nabla^- q$, $\Diamond(p \wedge q)$, as well as formulas B and C, in two variables, satisfying

$$B(v, 1) = B(1, v) = v$$

$$C(1, 0) = 0,\ C(v, 1) = v,\ C(1, 1) = 1.$$

With these resources we can define $\Diamond(p \wedge q) \rightarrow (p \wedge q)$:

$$\models^2 \neg\nabla^- B[(\neg\Box p \vee \neg\nabla^- q), (\neg\Diamond(p \wedge q) \vee \neg\nabla^- p)] \leftrightarrow (\Diamond(p \wedge q) \rightarrow p \wedge q)).$$

Finally, with this formula and C we can define $p \wedge q$:

$$\models^2 C[\Diamond(p \wedge q) \rightarrow p \wedge q, \Diamond(p \wedge q)] \leftrightarrow p \wedge q. \ \Box$$

Our next step towards the main theorem is to establish the functional completeness criterion w.r.t. modal W_3-operations.

$W_3 = \{w_0, w_1, w_2\}$. In this section, as usual, we will write 1 for $\{w_0, w_1, w_2\}$ and 0 for $\{\ \}$; also we write w for $\{w_0\}$, u for $\{w_1\}$, and v for $\{w_2\}$; moreover we write wu for $\{w_0, w_1\}$, and similarly for the other cases. (Again, in Ratsa's paper one will find different names for the elements of $\wp(W_3)$; the subsets of W_3 which we are representing as 1, wu, wv, uv, w, u, v, 0 are represented in his paper as 1, ω, σ, ν, μ, ρ, ε, 0.)

The convention for indefinite descriptions will also be used, and so e.g. ${}^2\wedge$ will stand for *some* W_3-operation which is W_2-equivalent to $\wedge$.

The sub-structures {0, u, wv, 1}, {0, w, uv, 1}, and {0, v, wu, 1} are isomorphic and embeddable in W_2, from which it follows that the values of unary operations on W_3 are determined by their values w.r.t. W_2. This, of course, doesn't hold for binary operations.

The following will be extensively used below:

Proposition. The matrix below, $\mathcal{M}_{\forall 3}$, is invariant w.r.t. W_3-modal operations.

$$\left|\begin{array}{cccccccc}
0 & u & w & v & wu & uv & wv & 1 \\
0 & u & v & w & uv & wu & wv & 1 \\
0 & w & u & v & wu & wv & uv & 1 \\
0 & w & v & u & wv & wu & uv & 1 \\
0 & v & u & w & uv & wv & wu & 1 \\
0 & v & w & u & wv & uv & wu & 1
\end{array}\right|$$

Proof. It is easy to check that $\Box$, $\neg$, $\wedge$ do preserve this matrix; preserving relations (or their corresponding matrices) is a hereditary property. $\Box$

Thanks to this result it won't be necessary to deal with the full tables for W_3, which would have 64 entries; instead we will use the following pair of partial tables:

$p\backslash q$	0	u	1
0			
u			
1			

$p\backslash q$	u	w	uv
u			
wv			

Using the fact that every modal W_3-operation preserves $\mathcal{M}_{\forall 3}$, we can find the missing values.

We will use below the following W_3-operations, which are *not* modal W_3-operations:

p	g_1	g_2	g_3	g_4
0	0	0	0	0
u	0	wv	u	wv
w	0	wv	uv	wu
v	1	wv	wu	uv
wu	1	u	v	w
uv	0	u	w	v
wv	0	u	wv	u
1	0	1	1	1

The symbols Π_{20}–Π_{23} will denote the classes of modal W_3-operations preserving, respectively, the following predicates: $g_1(p) = 0$, $g_2(p) = q$, $g_3(p) = q$, and $g_4(p) = q$. These predicates correspond to the following matrices:

$\mathcal{M}_{20} = (0, u, w, uv, wv, 1)$;

$$\mathcal{M}_{21} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & wv & wv & wv & u & u & u & 1 \end{vmatrix};$$

$$\mathcal{M}_{22} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & u & uv & wu & v & w & wv & 1 \end{vmatrix};$$

$$\mathcal{M}_{23} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & wv & wu & uv & w & v & u & 1 \end{vmatrix}.$$

In the previous section we were able to avoid using non-modal operations in the definition of the matrix $\mathcal{M}_{10}$. So far we have not been able to do the same w.r.t. the matrices $\mathcal{M}_{20}$–$\mathcal{M}_{23}$. This leads to the above-mentioned

Question. Is it possible to define the matrices $\mathcal{M}_{20}$–$\mathcal{M}_{23}$ using only *modal* W_3-operations? If it is, which formulas could do the job?

We will show in the end of this section that a set of modal functions is 3-complete iff it is 2-complete and not included in any of the classes Π_{20}–Π_{23}; but first we need to show some lemmas, starting with

Lemma 11 (Ratsa's lemma 13). There is a modal W_3-operation Q satisfying $Q(wv, uv) = v$ such that $Q \in [\top, \bot, \neg, f_{20}]$.

Proof. Since $f_{20} \notin \Pi_{20}$, there is an n-tuple $\langle \underline{\alpha} \rangle$, $\alpha_i \in \{0, u, w, uv, wv, 1\}$, such that $f_{20}(\underline{\alpha}) \in \{v, wu\}$.

Let B be formula $f_{20}(B_1 \ldots B_n)$ where for $1 \leq i \leq n$:

$$B_i = \begin{cases} \bot, & \text{if } \alpha_i = 0; \\ \neg p, & \text{if } \alpha_i = u; \\ \neg q, & \text{if } \alpha_i = w; \\ q, & \text{if } \alpha_i = uv; \\ p, & \text{if } \alpha_i = wv; \\ \top, & \text{if } \alpha_i = 1. \end{cases}$$

This exhaust $\mathcal{M}_{20}$, $Var(B) = \{p, q\}$, $B \in \mathcal{L}\{\top, \bot, \neg, f_{20}\}$. Note that $B_i(wv, uv) = \alpha_i$. So $B(wu, uv) = f_{20}(\underline{\alpha})$, and consequently $B(wv, uv) \in \{v, wu\}$. The formula satisfying the condition of the lemma will be

$$Q = \begin{cases} B, & \text{if } B(wv, uv) = v; \\ \neg B, & \text{if } B(wv, uv) = wu. \ \square \end{cases}$$

Lemma 12 (Ratsa's lemma 14). For any formula Q as above, we have $\Box(p \leftrightarrow q) \in [\top, \bot, \Box, \neg, f_{21}, {}^2\wedge, Q]$.

Proof. Since we have $\neg$ and ${}^2\wedge$, we can define ${}^2\vee$, ${}^2\leftrightarrow$, and formula

$$E(p, q) = \Box(p \,{}^2\!\leftrightarrow q) \,{}^2\!\wedge \Box(\neg p \,{}^2\!\leftrightarrow \neg q).$$

Clearly, E(p, q) is W_2-equivalent to $\Box(p \leftrightarrow q)$. We can see also that

$$\models^3 E(p, q) \leftrightarrow E(\neg p, \neg q),$$

and so we have

$$E(u, w) = E(wv, uv) \text{ and } E(u, uv) = E(wv, w).$$

There are two cases to consider: (i) $E(u, w) = E(u, uv)$ and (ii) $E(u, w) \neq E(u, uv)$.

In case (i) we have $\models^3(E(p, q) \,{}^2\!\wedge \neg E(p, \neg q)) \leftrightarrow \Box(p \leftrightarrow q)$.

Case (ii) is sub-divided into (ii-a) $E(u, w) = 0$ and $E(u, uv) = 1$ and (ii-b) $E(u, w) = 1$ and $E(u, uv) = 0$.

In case (ii-a), where $E(u, w) = 0$, let R be formula $\neg Q(p, \neg q)$. We have then

$$R(wv, w) = wu$$

$$[\neg Q(wv, \neg w) = \neg Q(wv, uv) = \neg v = wu]$$

Let B be formula

$$\Box(\Box R \,{}^2\!\vee \neg R) \,{}^2\!\vee (E(p, R) \,{}^2\!\vee E(R, \neg q)).$$

B assumes only values 0 and 1. Since in this case $R(wv, w) = wu$, $E(wv, uv) = 0$ (recall that $E(wv, uv) = E(u, w)$), and since every modal W_3-operation preserves $\mathcal{M}_{\forall 3}$, we can conclude that $B(wv, w) = 0$. Since $R(u, u) \in \{0, u, wv, 1\}$ (modal operations applied to arguments in some substructure return values in the same substructure) we can see also that $B(0, 0) = B(u, u) = B(1, 1) = 1$. It is then easy to check that

$$\models^3 (E(p, q) \wedge^2 (B(p, q) \,{}^2\!\wedge B(\neg p, \neg q))) \leftrightarrow \Box(p \leftrightarrow q).$$

In case (ii-b), where $E(u, w) = 1$, we first show that with Q we can express a formula C on variables p and q, assuming only values 0 and 1, and satisfying

$$C(u, u) = 1, C(wv, uv) = 0$$

$Q(u, u) \in \{0, u, wv, 1\}$. If $Q(u, u) \in \{0, u, 1\}$, our formula C is

$$\Box(\Box Q \,{}^2\!\vee \neg Q) \,{}^2\!\vee E(p, Q).$$

$C \in \mathcal{L}\{\Box, \neg, {}^2\!\vee, E, Q\}$. Given $E(u, uv) = 0$ it follows that $E(wv, w) = 0$; since E preserves $\mathcal{M}_{\forall 3}$ we can conclude that our formula is a suitable C.

If $Q(u, u) = wv$, we use f_{21}. Since $f_{21} \notin \Pi_{21}$ there are n-tuples $\langle\underline{\alpha}\rangle$, $\langle\underline{\beta}\rangle$ etc. such that the pair $\langle f_{21}(\underline{\alpha}), f_{21}(\underline{\beta})\rangle$ is one of the columns of the following matrix:

$$\left|\begin{array}{cccccccccccccccccccccccc} 0 & 0 & 0 & u & u & u & w & w & w & v & v & v & wu & wu & wu & uv & uv & uv & wv & wv & wv & 1 & 1 & 1 \\ u & wv & 1 & 0 & u & 1 & 0 & u & 1 & 0 & u & 1 & 0 & wv & 1 & 0 & wv & 1 & 0 & wv & 1 & 0 & u & wv \end{array}\right|$$

(Note that, although the full W_3-matrix has 64 columns and $\mathcal{M}_{21}$ has only 8 columns, the matrix of columns not in $\mathcal{M}_{21}$ but in the co-domain of some modal W_3-operation applied to an $\mathcal{M}_{21}$-matrix has 'only' 24 columns. That happens because the second row of $\mathcal{M}_{21}$ is constituted only by elements of the substructure $\{0, u, wv, 1\}$.)

We now define formula $F = f_{21}(F_1 \ldots F_n)$ where for $1 \leq i \leq n$:

$F_i =$ $\bot$, if $\alpha_i = \beta_i = 0$;

$\neg p$, if $\alpha_i = u$ and $\beta_i = wv$;

$\neg q$, if $\alpha_i = w$ and $\beta_i = wv$;

Q, if $\alpha_i = v$ and $\beta_i = wv$;

$\neg Q$, if $\alpha_i = wu$ and $\beta_i = u$;

q, if $\alpha_i = uv$ and $\beta_i = u$;

p, if $\alpha_i = wv$ and $\beta_i = u$;

$\top$, if $\alpha_i = \beta_i = 1$.

This exhausts $\mathcal{M}_{21}$. $F \in \mathcal{L}\{\top, \bot, \neg, f_{21}, Q\}$, $\mathrm{Var}(F) = \{p, q\}$. Note that $F_i(wv, uv) = \alpha_i$ and $F_i(u, u) = \beta_i$. So the pair $\langle F(wv, uv), F(u, u)\rangle$ is one of the columns of the big matrix above. In any of the twenty-four relevant cases, our formula can be defined by the scheme:

$$C = \begin{cases} \Box\neg F, & \text{if } F(u, u) = 0; \\ E(p, F) \mathbin{{}^2\wedge} \neg E(\neg p, F), & \text{if } F(u, u) = u; \\ E(\neg p, F) \mathbin{{}^2\wedge} \neg E(p, F), & \text{if } F(u, u) = wv; \\ \Box F, & \text{if } F(u, u) = 1. \end{cases}$$

Using the properties of E (and remembering that $C(u, u) = 1$ and $C(wv, uv) = 0$) it is not hard to check that

$$\models^3 ((E \mathbin{{}^2\wedge} C \mathbin{{}^2\wedge} C(\neg p, \neg q)) \mathbin{{}^2\vee} ((\Box p \wedge \Box q) \mathbin{{}^2\vee} (\Box\neg p \wedge \Box\neg q))) \leftrightarrow \Box(p \leftrightarrow q). \ \Box$$

Lemma 13 (Ratsa's lemma 15). For any such operation Q, the modal W_3-operation $p \wedge q \in [\top, \bot, \Box, \neg, \Box(p \leftrightarrow q), f_{22}, f_{23}, {}^2\wedge, Q]$.

Proof. We first construct a formula B, assuming only values 0 and 1, and satisfying

$$B(wv, w) = 0, B(wv, uv) = 1.$$

Since $Q(wv, uv) = v$, it is easy to see that, when $Q(wv, w) \in \{0, u, w, uv, wv, 1\}$, the formula B will be one of $\Diamond Q$, $\Diamond(p \leftrightarrow Q)$, $\neg\Box(q \leftrightarrow Q)$, $\Diamond(q \leftrightarrow Q)$, $\neg\Box(p \leftrightarrow Q)$, $\neg\Box Q$.

Another simple case is $Q(wv, w) = v$. Since $Q(wv, uv) = v$ and Q preserves $\mathcal{M}_{\forall 3}$, we know that $Q(wv, wu) = w$. In these circumstances, our formula B is

$$\neg\Box(q \leftrightarrow Q(p, \neg Q)).$$

If $Q(wv, w) = wu$, we need to use f_{22}. Since $f_{22} \notin \Pi_{22}$, we have n-tuples $\langle\underline{\alpha}\rangle$ and $\langle\underline{\beta}\rangle$ etc. where the pair $\langle f_{22}(\underline{\alpha}), f_{22}(\underline{\beta})\rangle$ is not a column of the $\mathcal{M}_{22}$-matrix.

We now define formula C as $f_{22}(C_1 \dots C_n)$ where for $1 \leq i \leq n$:

$$C_i = \begin{cases} \bot, & \text{if } \alpha_i = \beta_i = 0; \\ \neg p, & \text{if } \alpha_i = \beta_i = u; \\ q, & \text{if } \alpha_i = w \text{ and } \beta_i = uv; \\ \neg Q, & \text{if } \alpha_i = v \text{ and } \beta_i = wu; \\ Q, & \text{if } \alpha_i = wu \text{ and } \beta_i = v; \\ \neg q, & \text{if } \alpha_i = uv \text{ and } \beta_i = w; \\ p, & \text{if } \alpha_i = \beta_i = wv; \\ \top, & \text{if } \alpha_i = \beta_i = 1. \end{cases}$$

This exhausts $\mathcal{M}_{22}$. Var(C) = {p, q}, $C \in \mathcal{L}\{\top, \bot, \neg, f_{22}, Q\}$. Note that $C_i(wv, w) = \alpha_i$ and $C_i(wv, uv) = \beta_i$. Therefore, the pair $\langle C(wv, w), C(wv, uv)\rangle$ is one of the 56 columns not in $\mathcal{M}_{22}$.

In any of the 56 relevant cases, the required formula is expressible by $\Box$, $\neg$, $\Box(p \leftrightarrow q)$, C and ${}^2v$, as shown by the scheme:

$$B = \begin{cases} \Diamond C, & \text{if } C(wv, w) = 0 \text{ and } C(wv, uv) \neq 0; \\ \Diamond(p \leftrightarrow C), & \text{if } C(wv, w) = u \text{ and } C(wv, uv) \neq u; \\ \neg\Box(q \leftrightarrow C), & \text{if } C(wv, w) = w \text{ and } C(wv, uv) \neq uv; \\ \Diamond(C \leftrightarrow Q), & \text{if } C(wv, w) = v \text{ and } C(wv, uv) \neq wu; \\ \neg\Box(C \leftrightarrow Q), & \text{if } C(wv, w) = wu \text{ and } C(wv, uv) \neq v; \\ \Diamond(q \leftrightarrow C), & \text{if } C(wv, w) = uv \text{ and } C(wv, uv) \neq w; \\ \neg\Box(p \leftrightarrow C), & \text{if } C(wv, w) = wv \text{ and } C(wv, uv) \neq wv; \\ \neg\Box C, & \text{if } C(wv, w) = 1 \text{ and } C(wv, uv) \neq 1. \end{cases}$$

We will also need a formula E, on variables p, q, assuming only values 0 and 1 and such that

$$E(u, w) = 0, E(wv, uv) = 1.$$

If $Q(u, w) \in \{0, wv, w, uv, u, 1\}$ our formula E is one of $\Diamond Q$, $\Diamond(p \leftrightarrow Q)$, $\neg\Box(q \leftrightarrow Q)$, $\Diamond(q \leftrightarrow Q)$, $\neg\Box(p \leftrightarrow Q)$, $\neg\Box Q$. If $Q(u, w) = v$, then E will be $B(\neg Q, q)$; that this is the case can be seen remembering that $Q(wv, uv) = v$, $B(wv, w) = 0$, $B(wv, vu) = 1$ and using $\mathcal{M}_{\forall 3}$.

Let $Q(u, w) = wu$. We now use f_{23}. Since $f_{23} \notin \Pi_{23}$, there are n-tuples $\langle\underline{\gamma}\rangle$, $\langle\underline{\delta}\rangle$ etc. and so $\langle f_{23}(\underline{\gamma}), f_{23}(\underline{\delta})\rangle$ is one of the 56 columns not in $\mathcal{M}_{23}$.

Let G be formula $f_{23}(G_1 \ldots G_n)$ where for $1 \leq i \leq n$:

G =	$\perp$,	if $\gamma_i = \delta_i = 0$;
	p,	if $\gamma_i = u$ and $\delta_i = wv$;
	q,	if $\gamma_i = w$ and $\delta_i = wu$;
	$\neg Q$,	if $\gamma_i = v$ and $\delta_i = uv$;
	Q,	if $\gamma_i = wu$ and $\delta_i = w$:
	$\neg q$,	if $\gamma_i = uv$ and $\delta_i = v$;
	$\neg p$,	if $\gamma_i = wv$ and $\delta_i = u$;
	$\perp$,	if $\gamma_i = \delta_i = 0$.

(This exhausts $\mathcal{M}_{23}$.) Remembering that (by $\mathcal{M}_{\forall 3}$) $Q(wv, uv) = v$ is equivalent to $Q(wv, wu) = w$, it can be seen that $G_i(u, w) = \gamma_i$ and $G_i(wv, wu) = \delta_i$. So $\langle G(u, w), G(wv, wu)\rangle$ is one of the 56 columns not in $\mathcal{M}_{23}$.

Since (by $\mathcal{M}_{\forall 3}$) $E(wv, uv) = 1$ is equivalent to $E(wv, wu) = 1$, it is easy to see that in any of the 56 cases the desired formula E is expressible using $\Box$, $\neg$, $\Box(p \leftrightarrow q)$, G and Q, as the following scheme shows:

$$E = \begin{cases} \Diamond G, & \text{if } G(u, w) = 0 \text{ and } G(wv, wu) \neq 0; \\ \neg\Box(p \leftrightarrow G), & \text{if } G(u, w) = u \text{ and } G(wv, wu) \neq wv; \\ \neg\Box(q \leftrightarrow G), & \text{if } G(u, w) = w \text{ and } G(wv, wu) \neq wu; \\ \Diamond(G \leftrightarrow Q), & \text{if } G(u, w) = v \text{ and } G(wv, wu) \neq uv; \\ \neg\Box(G \leftrightarrow Q), & \text{if } G(u, w) = wu \text{ and } G(wv, wu) \neq w; \\ \Diamond(q \leftrightarrow G), & \text{if } G(u, w) = uv \text{ and } G(wv, wu) \neq v; \\ \Diamond(p \leftrightarrow G), & \text{if } G(u, w) = wv \text{ and } G(wv, wu) \neq u; \\ \neg\Box G, & \text{if } G(u, w) = 1 \text{ and } G(wv, wu) \neq 1. \end{cases}$$

Let R be formula $(B \,{}^2\!\wedge B(q, p)) \,{}^2\!\wedge E$. Since $B(wv, w) = 0$, $B(wv, vu) = 1$ and $E(u, w) = 0$, $E(wv, uv) = 1$ we know that R satisfies conditions

$$R(u, w) = R(u, uv) = R(wv, w) = 0$$

$$R(wv, uv) = 1.$$

Now let F be formula

$$(Q \,{}^2\!\wedge R) \,{}^2\!\vee ((p \,{}^2\!\wedge R(\neg p, q)) \,{}^2\!\vee (q \,{}^2\!\wedge R(p, \neg q))),$$

expressible by $\neg$, ${}^2\!\vee$, ${}^2\!\wedge$, Q and R. Then, from the conditions on Q and R we can find that

$$F(u, w) = 0, F(u, uv) = u, F(wv, w) = w, F(wv, uv) = v.$$

Finally, let H be formula

$$(\Box p \,{}^2\!\vee \Box\neg p) \,{}^2\!\vee (\Box q \,{}^2\!\vee \Box\neg q) \,{}^2\!\vee \Box(p \leftrightarrow q) \,{}^2\!\vee \Box(\neg p \leftrightarrow q).$$

H is expressible via $\Box$, $\neg$, $\Box(p \leftrightarrow q)$, and ${}^2\!\vee$. H has the following table:

p\q	0	u	1
0	1	1	1
u	1	1	1
1	1	1	1

p\q	u	w	uv
u	1	0	0
wv	1	0	0

We can see that

$$\models^3 ((p\ {}^2\!\vee q)\ {}^2\!\vee \neg H)\ {}^2\!\wedge (F\ {}^2\!\vee H)) \leftrightarrow p \wedge q.\ \square$$

Theorem 3 (Ratsa's theorem 3). A set of modal functions is 3-complete iff it is 2-complete and not included in any of the classes Π_{20}–Π_{23}.

Proof. ($\Rightarrow$) follows from $\{\varphi : \models^3 \varphi\} \subseteq \{\varphi : \models^2 \varphi\}$ and the fact that the classes Π_{20}–Π_{23} are closed under definability and not complete w.r.t. modal W_3-operations.

($\Leftarrow$) By hypothesis we have all modal W_2-operations, including ${}^2\!\wedge$ and $\square$. Also by hypothesis, we have functions $f_i \notin \Pi_i$ ($i = 20, 21, 22, 23$), not necessarily distinct and with (so to speak) variables among $p_1 \ldots p_n$. It is enough to show that in these conditions we can define the modal W_3-operation corresponding to $p \wedge q$. This was seen in the previous three lemmas. $\square$

Recall that $W_4 = \{w_0, w_1, w_2, w_3\}$. We will for convenience write w for $\{w_0\}$, u for $\{w_1\}$, v for $\{w_2\}$, and z for $\{w_3\}$; the conventions in the preceding sections are extended to this one in the obvious way. (In Ratsa's notation, the elements of $\wp(W_4)$ here referred to as 1, wuv, wuz, wvz, uvz, wu, wv, uv, wz, uz, vz, w, u, v, z, 0 are denoted by 1, ω, ν, ψ, β, ρ, τ, δ, γ, θ, σ, α, φ, μ, ε, 0.)

The following substructures of W_4 are isomorphic and embeddable in W_3.

{0, wu, v, z, wuv, wuz, vz, 1};

{0, w, v, wv, uz, wuz, uvz, 1};

{0, u, v, uv, wz, wuz, wvz, 1};

{0, w, u, vz, wu, wvz, uvz, 1};

{0, u, z, wv, uz, wuv, wvz, 1};

{0, w, z, uv, wz, wuv, uvz, 1}.

Proposition. The matrix below, $\mathfrak{M}_{\forall 4}$, is invariant w.r.t. modal W_4-operations.

$$\left|\begin{array}{cccccccccccccccc} 0 & w & u & v & z & wu & wv & uv & wz & uz & vz & wuv & wuz & wvz & uvz & 1 \\ 0 & w & u & z & v & wu & wz & uz & wv & uv & vz & wuz & wuv & wvz & uvz & 1 \\ 0 & u & v & z & w & uv & uz & vz & wu & wv & wz & uvz & wuv & wuz & wvz & 1 \end{array}\right|$$

Proof. Again, it is easy to see that ¬, □, and ∧ do preserve this matrix. □

The unit subsets of W_4 will be called *atoms*, their complements *co-atoms*, and the two-element subsets *elements of the middle.* We will call a *middle pair* a pair of non-complementary elements of the middle. So the middle pairs are:

⟨wu, uv⟩,	⟨wz, uz⟩,	⟨wv, uv⟩,	⟨uz, vz⟩,	⟨wv, wz⟩,	⟨wz, wv⟩,
⟨uz, wu⟩,	⟨uv, wu⟩,	⟨vz, uv⟩,	⟨vz, uz⟩,	⟨wz, vz⟩,	⟨wv, vz⟩,
⟨uz, wz⟩,	⟨uv, wv⟩,	⟨wu uv⟩,	⟨wu, uz⟩,	⟨uz, vz⟩,	⟨vz, wz⟩,
⟨vz, wv⟩,	⟨wz, wu⟩,	⟨wv, wu⟩,	⟨uz, uv⟩,	⟨uv, uz⟩,	⟨wv, wz⟩.

Since every modal W_4-operation preserves $\mathcal{M}_{\forall 4}$, given the value of such an operation w.r.t. one of the middle pairs we can find its values w.r.t. any other middle pair.

Given one pair of subsets of W_4, one of two things must happen: (i) they are a middle pair, or (ii) they belong to some of the substructures above. So an adequate table for modal W_4-operations should have cells corresponding to elements of one of the substructures and cells for one of the middle pairs.

Π_{24} is the class of modal functions whose corresponding modal W_4-operations preserve $p \in \{0, wv, wz, uv, vz, 1\}$.

Π_{25} is the class of modal functions whose corresponding modal W_4-operations preserve $p \in \{0, wv, wu, wz, uv, uz, vz, 1\}$.

(Here, again, the question presents itself whether there are modal formulas corresponding to these relations.)

We will show in the end of this section that a system of modal functions is 4-complete iff it is 3-complete and not included in Π_{24} or Π_{25}; that will be done after we establish some (few) lemmas.

Lemma 14. There is a formula B satisfying

$$B(wu, uv) = u$$

and expressible in terms of $\top, \bot, \Box, \neg, f_{24}, f_{25}, \wedge^3$.

Proof. Since $f_{24} \notin \Pi_{24}$ and $\mathcal{M}_{24} = \{0, wv, wz, uv, vz, 1\}$, there is $\langle \underline{\alpha} \rangle$ such that

$f_{24}(\underline{\alpha}) \subseteq \{w, u, v, z, wv, uz, wuv, wuz, wvz, uvz\}$.

Let C be formula $f_{24}(C_1 \dots C_n)$ where for $1 \le i \le n$:

$$C_i = \begin{cases} \bot, & \text{if } \alpha_i = 0; \\ p, & \text{if } \alpha_i = wu; \\ q, & \text{if } \alpha_i = uv; \\ \neg q, & \text{if } \alpha_i = wz; \\ \neg p, & \text{if } \alpha_i = vz; \\ \top, & \text{if } \alpha_i = 1. \end{cases}$$

This exhausts $\mathcal{M}_{24}$, $C \in \mathcal{L}\{\top, \bot, \neg, f_{24}\}$, $C_i(wu, uv) = \alpha_i$. It follows that

$C(wu, uv) \subseteq \{w, u, v, z, wv, uz, wuv, wuz, wvz, uvz\}$.

We now define formula

$$E = \begin{cases} C, & \text{if } C\{wu, uv\} \in \{w, u, v, z, wv\}; \\ \neg C, & \text{if } C(wu, uv\} \in \{uz, wuv, wuz, wvz, uvz\}. \end{cases}$$

$E(wu, uv) \in \{w, u, v, z, wv\}$. When $E(wu, uv) \neq wv$, the desired formula is

$$B = \begin{cases} p \,{}^3\!\wedge \neg E(p, q), & \text{if } E(wu, uv) = w; \\ E(p, q), & \text{if } E(wu, uv) = u; \\ q \,{}^3\!\wedge \neg E(p, q), & \text{if } E(wu, uv) = v; \\ p \wedge^3 \neg(\neg q \,{}^3\!\wedge \neg E(p, q)), & \text{if } E(wu, uv) = z. \end{cases}$$

If $E(wv, uv) = wv$, we use f_{25}. $f_{25} \notin \Pi_{25}$ and so there is an n-tuple $\langle \underline{\beta} \rangle$, $\beta_i \in \{0, wv, wu, wz, uv, uz, vz, 1\}$, such that $f_{25}(\underline{\beta}) \in \{w, u, v, z, wuv, wuz, wvz, uvz\}$.

Let F be formula $f_{25}(F_1 \dots F_n)$ where for $1 \le i \le n$:

$F_i = \bot, \quad \text{if } \beta_i = 0;$

p,	if $\beta_i = wu$;
E(p, q),	if $\beta_i = wv$;
q,	if $\beta_i = uv$;
¬q,	if $\beta_i = wz$;
¬E(p, q),	if $\beta_i = uz$;
¬p,	if $\beta_i = vz$;
⊤,	if $\beta_i = 1$.

(This exhausts $\mathcal{M}_{25}$.) $F \in \mathcal{L}\{\top, \bot, \neg, E, f_{25}\}$. $F_i(wu, uv) = \beta_i$, and so $F(wu, uv) \in \{w, u, v, z, wuv, wuz, wvz, uvz\}$.

We reduce the cases by half using formula

$$G = \begin{cases} F, & \text{if } F(wv, uv) \in \{w, u, v, z\}; \\ \neg F, & \text{if } F(wv, uv) \in \{wuv, wuz, wvz, uvz\}. \end{cases}$$

Comparing G with E above, we can see that this is a case already considered. The lemma is proved. □

We will see now that formula ¬Ind(p, q) is expressible in W_4 by $\Box, \neg, {}^3\!\vee, {}^3\!\wedge$. First we establish the equalities

$$\Box(wu\ {}^3\!\vee\ uv) = \Box(wv\ {}^3\!\vee\ wz) = \Box(vz\ {}^3\!\vee\ uv) = \Box(vz\ {}^3\!\vee\ wz).$$

(This holds since the arguments are middle pairs and the formula can't assume values other than 0 and 1.) We also use formulas

$$(\Box(p\ {}^3\!\vee\ q)\ {}^3\!\wedge\ \Box(\neg p\ {}^3\!\vee\ \neg q))\ {}^3\!\wedge\ (\Box(\neg p\ {}^3\!\vee\ q)\ {}^3\!\wedge\ \Box(p\ {}^3\!\vee\ \neg q))$$

$$(\Box(p\ {}^3\!\vee\ q)\ {}^3\!\vee\ \Box(\neg p\ {}^3\!\vee\ \neg q))\ {}^3\!\vee\ (\Box(\neg p\ {}^3\!\vee\ q)\ {}^3\!\vee\ \Box(p\ {}^3\!\vee\ \neg q))$$

W_3-equivalent, respectively, to 0 and 1. They will be designated by H_1 and H_2.

Given the equalities above we can verify that by $\neg$, H_1 and H_2 we can express $\neg\mathrm{Ind}(p, q)$:

$$\neg\mathrm{Ind}(p, q) = \begin{cases} H_2, & \text{if } \Box(wu \,{}^3\!\vee\, uv) = 0; \\ \neg H_1, & \text{if } \Box(wu \,{}^3\!\vee\, uv) = 1. \end{cases}$$

Finally, we take formula

$$(B(p, q) \,{}^3\!\vee \neg\mathrm{Ind}(p, q)) \,{}^3\!\wedge ((p \,{}^3\!\wedge q) \vee \mathrm{Ind}(p, q))$$

which is W_4-equivalent to $p \wedge q$. □

Now we can officially state and prove

Theorem 4. A system of modal functions is 4-complete iff it is 3-complete and not included in Π_{24} or Π_{25}.

Proof. ($\Rightarrow$) follows from the facts that $\{\varphi : \models^4 \varphi\} \subseteq \{\varphi : \models^3 \varphi\}$ and that Π_{24} and Π_{25} are not complete w.r.t. modal W_4-operations. ($\Leftarrow$) is immediate from the lemmas above. □

A functional completeness criterion for modal functions; determination of the pre-complete systems; results on bases for μ

Proposition 1 and theorems 1–4 yield

Theorem 5. A set of modal functions C is complete iff for each of the classes $\Pi_0, \dots, \Pi_{25}$ there is a function in C that doesn't belong to it. □

For what follows we have to establish the following definitions:

$$S(p, q) =_{df} \Box(p \vee q) \vee \Box(p \to q) \vee \Box(q \to p).$$

$$\varphi(p, q) =_{df} S(p, q) \wedge S(\neg p, q) \wedge S(p, \neg q) \wedge S(\neg p, \neg q).$$

Using the criterion above it is easy to check that the following is an independent basis for μ:

$\Box p$, $\quad \neg\nabla^{-}p$, $\quad \nabla^{+}p$, $\quad p \wedge (\Box p \vee \Diamond q)$, $\quad p \wedge (\Box p \to \Diamond q)$,

$p \leftrightarrow (\Delta q \to \Delta p)$, $\quad p \leftrightarrow (\Delta p \to (\Box q \leftrightarrow \Box r))$,

$(p \to (\Delta q \to \Delta p)) \to (q \wedge (\Delta q \to \Delta p))$, $\quad \varphi(p, q) \to q$,

$(p \to \Delta r) \to (\Diamond p \wedge \Box(p \vee q) \wedge \Delta r)$, $\quad ((p \leftrightarrow q) \to \neg\mathrm{Ind}[p, q]) \leftrightarrow r$,

$((p \vee q \vee r \vee \Box(p \to q) \vee \Box(q \to r) \vee \Box(r \to p)) \to \neg\mathrm{Ind}[p, q]) \to r$,

$(p \vee S[p, q]) \wedge (p \to S(p, \neg q)) \wedge (q \vee (S(\neg p, q) \wedge S(\neg p, \neg q))) \wedge (\varphi \to q)$,

$((p \vee q) \to S) \wedge ((p \to q) \to S[\neg p, q]) \wedge ((q \to p) \vee S[p, \neg q]) \wedge ((p \wedge q) \to S[\neg p, \neg q]) \wedge (\varphi \to q)$.

We show below a table of functions preserving and not preserving matrices $\mathcal{M}_{20}$–$\mathcal{M}_{25}$.

Recall that:

$m_{20} = (0, u, w, uv, wv, 1);$

$$m_{21} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & wv & wv & wv & u & u & u & 1 \end{vmatrix};$$

$$m_{22} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & u & uv & wu & v & w & wv & 1 \end{vmatrix};$$

$$m_{23} = \begin{vmatrix} 0 & u & w & v & wu & uv & wv & 1 \\ 0 & wv & wu & uv & w & v & u & 1 \end{vmatrix};$$

$m_{24} = (0, wv, wz, uv, vz, 1);$

$m_{25} = (0, wv, wu, wz, uv, uz, vz, 1).$

In the following table, θ and ψ are as defined above on page 28.

Class	$f \in \Pi$	$F \notin \Pi$
Π_{20}	$\top, \bot, \Box, \neg, p \to \Box q, \Box(p \leftrightarrow q), \theta,$ S, $(p \to \mathrm{Ind}(p, q)) \to q$	$\leftrightarrow, (p \to (S[\neg p, \neg q] \to S)) \to q,$ $\psi,$ $(p \leftrightarrow S) \wedge (q \to S),$ $(p \vee q \vee S) \wedge ((p \wedge q) \vee S[\neg p, \neg q])$
Π_{21}	$\top, \bot, \Box, \neg, (p \to \Box q), \theta,$ $(p \leftrightarrow S) \wedge (q \to S),$ $(p \to \mathrm{Ind}[p, q]) \to q, (S \wedge S[\neg p, \neg q]) \to \Box(p \leftrightarrow q)$	$\Box(p \leftrightarrow q), \psi$
Π_{22}	$\top, \bot, \Box, \neg, \leftrightarrow, (p \to \Box q), \Box(p \leftrightarrow q), (p \to \mathrm{Ind}[p, q]) \to q, \theta$	$S, (S \wedge S[\neg p, \neg q]) \to \Box(p \leftrightarrow q),$ $(p \vee q \vee S) \wedge ((p \wedge q) \vee S[\neg p, \neg q]), \psi$
Π_{23}	$\top, \bot, \Box, \neg, (p \to \Box q), \Box(p \leftrightarrow q),$ $\theta, (p \to \mathrm{Ind}[p, q]) \to q, (p \vee q \vee S) \wedge ((p \wedge q) \vee S[\neg p, \neg q])$	$\leftrightarrow, S, (p \leftrightarrow S) \wedge (q \to S), \psi$
Π_{24}	$\top, \bot, \Box, \neg, p \to \Box q, \Box(p \leftrightarrow q), \theta,$ S, $(p \to (S[\neg p, \neg q] \to S)) \to q, p \vee q \vee r \vee \Box(p \to q) \vee \Box(q \to r) \vee \Box(r \to p)$	$\leftrightarrow, (p \to \mathrm{Ind}[p, q]) \to q, \psi$
Π_{25}	$\top, \bot, \Box, \neg, \leftrightarrow, p \to \Box q, \Box(p \leftrightarrow q), \theta, S$	$(p \to \mathrm{Ind}[p, q]) \to q, \psi$

Proposition. There are no inclusions among the classes Π_0–Π_{25}.

Proof. Given the list of examples of functions preserving and not preserving each of the matrices $\mathcal{M}_0$–$\mathcal{M}_{25}$, it is mechanical to check that for any two of the classes Π_0–Π_{25} there is a function that belongs to one and doesn't belong to the other. □

Theorem 6. $\Pi_0, \ldots, \Pi_{25}$ are the pre-complete systems of modal functions.

Proof. We start by showing that these systems *are* pre-complete.

Clearly $\Pi_i \neq \mu$ for $i \in \{0, \ldots, 25\}$.

Now suppose for contradiction that Π_n ($n \in \{0, \dots, 25\}$) is properly included in some system $S \neq \mu$. So S contains a function $f \notin \Pi_n$. But by 'no inclusions' proposition above, we know that for every $k \neq n$, $0 \leq k \leq 25$, $\Pi_n \nsubseteq \Pi_k$. This means that, for every $k \neq n$, $0 \leq k \leq 25$, there is some function g such that $g \in \Pi_n$ – whence $g \in S$ – and $g \notin \Pi_k$. But then S satisfies the conditions of Theorem 5, and so $S = \mu$, with contradiction.

We need to show also that no *other* system is pre-complete. Suppose, for contradiction, that there is *another* system S which is also pre-complete. So S is not included in any of the classes Π_0–Π_{25}. But then, by Theorem 5, $S = \mu$, with contradiction. □

The following proposition will be useful in the proof of the next theorem.

Proposition. If a formula φ preserves relations characterized by formulas $P_1, \dots, P_n$ then φ also preserves every relation characterized by formulas obtained from the P_i's via substitution of variables, conjunction and existential quantification.

Proof. Cf. e.g. Bodnarchuk et al. 1969. □

Theorem 7. An independent basis for μ cannot have more than 14 elements.

Proof. We first prove the following inclusions (we use Ratsa's numbering of these):

$\Pi_0 \cap \Pi_{18} \subseteq \Pi_{12}$	(76)	$\Pi_8 \cap \Pi_9 \subseteq \Pi_3$	(83)
$\Pi_1 \cap \Pi_{18} \subseteq \Pi_{13}$	(77)	$\Pi_8 \cap \Pi_{10} \subseteq \Pi_3$	(84)
$\Pi_2 \cap \Pi_3 \subseteq \Pi_0$	(78)	$\Pi_9 \cap \Pi_{10} \subseteq \Pi_3$	(85)
$\Pi_5 \cap \Pi_8 \subseteq \Pi_0$	(79)	$\Pi_{11} \cap \Pi_{15} \subseteq \Pi_{19}$	(86)
$\Pi_5 \cap \Pi_9 \subseteq \Pi_1$	(80)	$\Pi_{12} \cap \Pi_{13} \subseteq \Pi_{17}$	(87)
$\Pi_5 \cap \Pi_{10} \subseteq \Pi_0$	(81)	$\Pi_{14} \cap \Pi_{16} \subseteq \Pi_{15}$	(88)
$\Pi_6 \cap \Pi_7 \subseteq \Pi_5$	(82)	$\Pi_{17} \cap \Pi_{19} \subseteq \Pi_4$	(89)

$\Pi_{17} \cap \Pi_{19} \subseteq \Pi_{18}$ (90)

$\Pi_{22} \cap \Pi_{23} \subseteq \Pi_{20}$ (91)

$\Pi_0 \cap \Pi_4 \cap \Pi_{13} \subseteq \Pi_{17}$ (92)

$\Pi_0 \cap \Pi_{17} \cap \Pi_{19} \subseteq \Pi_{12}$ (93)

$\Pi_1 \cap \Pi_4 \cap \Pi_{12} \subseteq \Pi_{17}$ (94)

$\Pi_1 \cap \Pi_{17} \cap \Pi_{19} \subseteq \Pi_{13}$ (95)

$\Pi_0 \cap \Pi_1 \cap \Pi_4 \cap \Pi_{18} \subseteq \Pi_{17}$ (96)

Given the proposition above, in order to show that e.g. $\Pi_0 \cap \Pi_{18} \subseteq \Pi_{12}$ it will be enough to construct a formula whose W_2-matrix is $\mathcal{M}_{12}$ using predicates R_0 and R_{18} (corresponding to $\mathcal{M}_0$ and $\mathcal{M}_{18}$) and conjunctions, substitution of variables, and existential quantification. The formula in question is $\exists z(R_0(z) \wedge R_{18}(x, y, z))$, and that proves (76).

We will use below formulas: E_1, corresponding to the matrix $\mathcal{M}_{\forall 2}$; E_2, corresponding to the *first two* rows of matrix $\mathcal{M}_{\forall 3}$; and E_3, corresponding to the *last two* rows of matrix $\mathcal{M}_{\forall 3}$.

The formulas proving (77)–(92) and (94) are:

(77) $\exists z(R_1(z) \wedge R_{18}(x, y, z))$; (78) $\exists y(R_2(x, y) \wedge R_3(x, y))$;

(79) $\exists y(R_5(y) \wedge R_8(x, y))$; (80) $\exists y(R_5(y) \wedge R_9(x, y))$;

(81) $\exists y(R_5(y) \wedge R_{10}(x, y))$; (82) $\exists y(R_6(y) \wedge R_7(y) \wedge E_1(x, y))$;

(83) $\exists z(R_8(x, z) \wedge R_9(y, z))$; (84) $\exists z(R_8(x, z) \wedge R_{10}(y, z))$;

(85) $\exists z(R_9(y, z) \wedge R_{10}(x, z))$; (86) $\exists u(R_{11}(x, u) \wedge R_{11}(z, u) \wedge R_{15}(y, u))$;

(87) $\exists z(R_{12}(z, x) \wedge R_{13}(z, y))$; (88) $\exists z(R_{14}(x, z) \wedge R_{16}(y, z))$;

(89) $\exists a\exists b\exists p\exists q\exists r\exists s\exists t\exists v\exists w(R_{17}(a, p) \wedge R_{17}(b, q) \wedge R_{17}(b, r) \wedge R_{17}(p, q) \wedge R_{17}(p, r) \wedge R_{17}(q, r) \wedge R_{19}(x, a, y) \wedge R_{19}(z, a, u) \wedge R_{19}(x, b, z) \wedge R_{19}(y, b, u) \wedge R_{19}(x, p, s) \wedge R_{19}(s, q, t) \wedge R_{19}(t, r, u) \wedge R_{19}(y, p, v) \wedge R_{19}(v, q, w) \wedge R_{19}(w, r, z))$;

(90) $\exists u(R_{17}(y, u) \wedge R_{19}(x, u, z))$;

(91) $\exists y\exists z\exists u(R_{22}(y, z) \wedge R_{23}(y, u) \wedge E_2(x, y) \wedge E_3(z, u))$;

(92) $\exists z\exists u\exists v\exists w(R_0(z) \wedge R_4(z, u, v, w) \wedge R_{13}(u, x) \wedge R_{13}(u, y) \wedge R_{13}(v, x) \wedge R_{13}(w, y))$;

(94) $\exists z\exists u\exists v\exists w(R_1(z) \wedge R_4(z, u, v, w) \wedge R_{12}(u, x) \wedge R_{12}(u, y) \wedge R_{12}(v, x) \wedge R_{12}(w, y))$.

(It is worth noting that in Ratsa's paper the conjunct $R_{17}(b, r)$ is missing from the giant formula above. Without this conjunct the formula can't do its job.)

Inclusion (93) follows from inclusions (76) and (90); inclusion (95) follows from (77) and (90); (96) follows from (77) and (92).

Now take a set of functions $\{A_0 \dots A_{25}\}$ such that, for $0 \leq i \leq 25$, $A_i \notin \Pi_i$. Given these inclusions, we can 'cut' twelve of the functions $A_0 \dots A_{25}$ in such a way that the remaining formulas will also satisfy the conditions of the theorem.

Note that given (82) we know that either $A_5 \notin \Pi_6$ or $A_5 \notin \Pi_7$; that enables us to cut one of A_6, A_7.

Analogously, by (88) we drop one of A_{14}, A_{16}; by (91) we drop one of A_{22}, A_{23}; by (86) one of A_{11}, A_{15}; and by (83) one of A_8, A_9.

Five were done.

Now, we will deal with A_3. Given (83), there are three cases to consider: (i) $A_3 \in \Pi_8 - \Pi_9$, (ii) $A_3 \in \Pi_9 - \Pi_8$, and (iii) $A_3 \notin \Pi_8$ and $A_3 \notin \Pi_9$.

In case (i) we use inclusion (84) to cut A_{10} (since $A_3 \notin \Pi_3$ and $A_3 \in \Pi_8$, from (84) it follows that $A_3 \notin \Pi_{10}$) and inclusion (79) to cut A_5 or A_8 (we are sure that in our list there is some formula corresponding to function not in Π_0; this function cannot be in $\Pi_5 \cap \Pi_8$).

In case (ii) we use inclusion (85) to cut A_{10} and inclusion (80) to cut A_5 or A_9.

In case (iii), given (81) we can cut, besides both A_8 and A_9, one of A_5 and A_{10}.

Seven were done.

Next we will use (78) and (87) to cut the eighth and ninth formulas (one of A_2, A_3 and one of A_{12}, A_{13}).

There are three left.

We deal with A_{17} by cases (cf. (87)). Case (i): If $A_{17} \in \Pi_{12} - \Pi_{13}$. Case (ii): $A_{17} \in \Pi_{13} - \Pi_{12}$. Case (iii) $A_{17} \notin \Pi_{12}$ and $A_{17} \notin \Pi_{13}$.

In case (i) we use (76) to cut A_0 or A_{18} and (94) to cut A_1 or A_4. Also we can use (89) to cut one of A_{17}, A_{19}. [Ratsa says that instead of (89) we could use (93) to cut one of A_0, A_{17}, A_{19}, but that doesn't seem to work. If the function to be cut in this last case is A_0 we are double-counting it.]

In case (ii) we use (77) to cut A_1 or A_{18} and (92) to cut A_0 or A_4. Also we use (89) to cut one of A_{17}, A_{19}. [Same as above, but with (95) instead of (93).]

In case (iii) we cut both A_{12} and A_{13}, and then use (96) to cut one of A_0, A_1, A_4, A_{18}. Finally, we use (89) or (90) to cut one of A_{17}, A_{19}. □

Theorem 8. A function f is a Sheffer-function for μ iff $f \notin \Pi_0, \Pi_1, \Pi_2, \Pi_5, \Pi_6, \Pi_7, \Pi_8, \Pi_9, \Pi_{10}, \Pi_{14}, \Pi_{15}, \Pi_{16}, \Pi_{17}, \Pi_{20}, \Pi_{21}, \Pi_{22}, \Pi_{23}, \Pi_{24}, \Pi_{25}$.

Proof. ($\Rightarrow$) follows from Theorem 5. ($\Leftarrow$) follows from Theorem 5 and the inclusions:

$\Pi_3 \subseteq \Pi_0 \cup \Pi_1$	(100)	$\Pi_{13} \subseteq \Pi_1 \cup \Pi_6$	(104)
$\Pi_4 \subseteq \Pi_0 \cup \Pi_1 \cup \Pi_2$	(101)	$\Pi_{18} \subseteq \Pi_0 \cup \Pi_1 \cup \Pi_{17}$	(105)
$\Pi_{11} \subseteq \Pi_0 \cup \Pi_1 \cup \Pi_7$	(102)	$\Pi_{19} \subseteq \Pi_0 \cup \Pi_1 \cup \Pi_7$	(106)

$\Pi_{12} \subseteq \Pi_0 \cup \Pi_6$ (103)

(100) and (101) are known (cf. Post). To prove (102) suppose $f \in \Pi_{11}$ and $f \notin \Pi_0$ and $f \notin \Pi_7$. So $f(0, \ldots, 0) = 1$ and there is an n-tuple $\langle \underline{\alpha} \rangle$ where, for $1 \leq i \leq n$, $\alpha_i \in \{w, v\}$ and $f(\langle \underline{\alpha} \rangle) \in \{0, 1\}$. Note that the following are $\mathcal{M}_{11}$-matrices.

$$\begin{vmatrix} 0 & \ldots & 0 \\ \alpha_1 & \ldots & \alpha_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & \ldots & 1 \\ \alpha_1 & \ldots & \alpha_n \end{vmatrix}$$

By hypothesis, $f \in \Pi_{11}$. Considering the first matrix, $f(0, \ldots, 0) = 1$ implies $f(\langle \underline{\alpha} \rangle) = 1$. By the second matrix, given $f(\langle \underline{\alpha} \rangle) = 1$ we get $f(1, \ldots, 1) = 1$. So $f \in \Pi_1$.

We now show inclusion (103). Suppose $f \in \Pi_{12}$ and $f \notin \Pi_6$; it follows that there is a tuple $\langle \underline{\alpha} \rangle$ where $\alpha_i \in \{0, w, 1\}$ and such that $f(\langle \underline{\alpha} \rangle) = v$. Take the $\mathcal{M}_{12}$-matrix

$$\begin{vmatrix} 0 & \ldots & 0 \\ \alpha_1 & \ldots & \alpha_n \end{vmatrix}$$

Since $f(\langle \underline{\alpha} \rangle) = v$ and $f \in \Pi_{12}$ it follows that $f(0, \ldots, 0)$, which means that $f \in \Pi_0$. The proof of (104) is similar.

Inclusion (105) can be seen as follows: suppose $f \in \Pi_{18}$ but $f \notin \Pi_0$ and $f \notin \Pi_{17}$. So $f(0, \ldots, 0) = 1$ and there are tuples $\langle \underline{\alpha} \rangle$ and $\langle \underline{\beta} \rangle$, where $\langle \alpha_i, \beta_i \rangle$ are $\mathcal{M}_{17}$-columns, such that $\langle f(\underline{\alpha}), f(\underline{\beta}) \rangle$ is not a $\mathcal{M}_{17}$-column, and so

$$\begin{vmatrix} f(\alpha_1 & \ldots & \alpha_n) \\ f(\beta_1 & \ldots & \beta_n) \end{vmatrix} \subseteq \begin{vmatrix} w & w & v & v \\ w & v & w & v \end{vmatrix}$$

At this point it is worth remembering that $R_{17} = \Delta p \vee \Delta q$, and that

$$R_{18} = \Delta p \wedge \Delta r \wedge ((p \veebar r) \rightarrow \Delta q).$$

We define sequence $\langle \underline{\gamma} \rangle$ using the scheme

$$\gamma_i = 0, \text{ if } \alpha_i \in \{0, 1\}$$

$$1, \text{ if } \alpha_i \in \{w, v\}.$$

(Alternatively, we could say that $\gamma_i = \nabla \alpha_i$.)

We now have

$$\begin{vmatrix} 0 & \cdots & 0 \\ \beta_1 & \cdots & \beta_n \\ \gamma_1 & \cdots & \gamma_n \end{vmatrix} \subseteq \mathcal{m}_{18} \qquad \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \gamma_1 & \cdots & \gamma_n \end{vmatrix} \subseteq \mathcal{m}_{18}$$

Since $f \in \Pi_{18}$ and $f(0, \ldots, 0) = 1$ and $f(\langle \underline{\beta} \rangle) \in \{v, w\}$, it follows that

$$\begin{vmatrix} f(0 & \cdots & 0) \\ f(\beta_1 & \cdots & \beta_n) \end{vmatrix} \subseteq \begin{vmatrix} 1 & 1 \\ w & v \end{vmatrix}$$

This entails

$$\begin{vmatrix} f(0 & \cdots & 0) \\ f(\beta_1 & \cdots & \beta_n) \\ f(\gamma_1 & \cdots & \gamma_n) \end{vmatrix} \subseteq \begin{vmatrix} 1 & 1 \\ w & v \\ 1 & 1 \end{vmatrix}$$

and so we can conclude that $f(\langle \underline{\gamma} \rangle) = 1$; this with $f\langle \underline{\alpha} \rangle \in \{w, v\}$ implies that $f(1, \ldots, 1) = 1$, i.e. $f \in \Pi_1$.

We turn, finally, to (106). Let $f \in \Pi_{19}$ and $f \notin \Pi_0$ and $f \notin \Pi_7$. So $f(0, \ldots, 0) = 1$ and there is $\langle \underline{\alpha} \rangle$, $\alpha_i \in \{w, v\}$, such that $f(\langle \underline{\alpha} \rangle) \in \{0, 1\}$. Remember that $R_{19} = \Delta p \wedge \Delta r \wedge ((p \veebar r) \rightarrow \nabla q)$. So

$$\begin{vmatrix} 0 & \dots & 0 \\ \alpha_1 & \dots & \alpha_n \\ 1 & \dots & 1 \end{vmatrix} \subseteq \mathcal{M}_{19}$$

Since $f \in \Pi_{19}$ and $f(\langle \underline{\alpha} \rangle) \in \{0, 1\}$ it follows that $f(0, \dots, 0) = f(1, \dots, 1) = 1$, i.e. $f \in \Pi_1$.

$\square$

That none of the nineteen classes in Theorem 8 could be dispensed with or replaced by any other pre-complete system is assured by the following list, which gives, for each of the nineteen classes, one function belonging to it but not belonging to any of the other pre-complete classes. Here $S(p, q)$ and $\varphi(p, q)$ are as defined after Theorem 5.

Π_0: $(p \wedge q) \leftrightarrow \neg\Diamond q$; Π_1: $(p \rightarrow q) \leftrightarrow \Delta q$; Π_2: $\neg 2^3(p, q, \square r)$;

Π_5: $\neg(p \vee q) \leftrightarrow (\Delta p \vee \Delta q)$; Π_6: $((\Diamond p \wedge \Diamond q) \rightarrow ((p \wedge q) \vee \square(p \leftrightarrow q))) \wedge \neg\square(p \wedge q)$;

Π_7: $(p \vee \Delta q) \rightarrow (\neg(p \wedge r) \wedge \Delta q)$; Π_8: $\square p \rightarrow ((q \vee (r \leftrightarrow \Delta r)) \wedge \neg\square(q \leftrightarrow r))$;

Π_9: $((p \wedge (q \leftrightarrow \Delta q)) \vee \square(p \leftrightarrow q)) \wedge \neg\Diamond r$; Π_{10}: $\neg(p \wedge q)$;

Π_{14}: $(p \rightarrow \square p) \wedge (\Diamond q \rightarrow q) \wedge \neg\square(p \wedge q)$;

Π_{15}: $((p \vee q \vee (\Diamond p \wedge \Diamond q)) \rightarrow ((p \wedge q) \vee \square(p \leftrightarrow q))) \wedge \neg\square(p \wedge q)$;

Π_{16}: $\neg p \wedge (\Diamond q \rightarrow q) \wedge \neg\square(p \vee q)$; Π_{17}: $((\Diamond p \rightarrow p) \vee (q \leftrightarrow \Delta q)) \wedge (p \rightarrow \neg\square q)$;

Π_{20}: $(\varphi(p, q) \rightarrow ((p \rightarrow q) \wedge \neg\square q)) \wedge (\varphi(p, q) \vee [((p \rightarrow \neg \mathrm{Ind}(p, q)) \rightarrow q) \rightarrow S(p, q)])$;

Π_{21}: $(\varphi(p, q) \rightarrow ((p \rightarrow q) \wedge \neg\square q)) \wedge (\varphi(p, q) \vee [(p \leftrightarrow S(p, q)) \wedge (q \rightarrow S(p, q))]$

Π_{22}: $\neg\square p \wedge (p \leftrightarrow q) \wedge (q \rightarrow \neg \mathrm{Ind}(p, q))$;

Π_{23}: $(\varphi(p, q) \rightarrow ((p \rightarrow q) \wedge \neg\square q)) \wedge (p \vee q \vee S(p, q)) \wedge ((p \wedge q) \vee S(\neg p, \neg q))$;

Π_{24}: $(\neg\mathrm{Ind}(p, q) \to ((p \to q) \wedge \neg\Box q)) \wedge (p \vee q \vee r \vee \Box(p \to q) \vee \Box(q \to r) \vee \Box(r \to p)$ $\vee \neg\mathrm{Ind}(p, q))$; Π_{25}: $(\neg\mathrm{Ind}(p, q) \to ((p \to q) \wedge \neg\Box q)) \wedge ((p \leftrightarrow q) \vee \neg\mathrm{Ind}(p, q))$.

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