UNIVERSIDADE DE SÃO PAULO
FACULDADE DE FILOSOFIA, LETRAS E CIÊNCIAS HUMANAS
DEPARTAMENTO DE FILOSOFIA
PROGRAMA DE PÓS-GRADUAÇÃO EM FILOSOFIA

André Rodrigo Ferreira Coggiola

Deductive Tableaux

São Paulo
2021

André Rodrigo Ferreira Coggiola

Deductive Tableaux

Dissertação apresentada ao Programa de Pós-Graduação em Filosofia do Departamento de Filosofia da Faculdade de Filosofia, Letras e Ciências Humanas da Universidade de São Paulo, para obtenção do título de Mestre em Filosofia sob a orientação do Prof. Dr. Rodrigo Bacellar da Costa e Silva.

São Paulo
2021

Catalogação na Publicação
Serviço de Biblioteca e Documentação
Faculdade de Filosofia, Letras e Ciências Humanas da Universidade de São Paulo

C676d

Coggiola, André Rodrigo Ferreira
Deductive Tableaux / André Rodrigo Ferreira Coggiola; orientador Rodrigo Bacellar da Costa e Silva - São Paulo, 2021.
60 f.

Dissertação (Mestrado)- Faculdade de Filosofia, Letras e Ciências Humanas da Universidade de São Paulo. Departamento de Filosofia. Área de concentração: Filosofia.

1. LÓGICA . 2. LÓGICA MATEMÁTICA. 3. LÓGICA INTUICIONISTA . I. da Costa e Silva, Rodrigo Bacellar, orient. II. Título.

Folha de Aprovação

COGGIOLA, A. R. F. *Deductive Tableaux*. 2021. Dissertação (Mestrado) – Faculdade de Filosofia, Letras e Ciências Humanas. Departamento de Filosofia, Universidade de São Paulo, São Paulo, 2021.

Prof. Dr. Rodrigo de Alvarenga Freire Instituição: Universidade de Brasília

Julgamento:__________________________

Assinatura:___________________________

Prof. Dr. Edelcio Gonçalves de Souza Instituição: Universidade de São Paulo

Julgamento:__________________________

Assinatura:___________________________

Dr. Pedro Alonso Amaral Falcão Instituição: (sem afiliação)

Julgamento:__________________________

Assinatura:___________________________

Para Nina

Agradecimentos

Gostaria de começar agradecendo ao meu orientador Rodrigo Bacellar pelo apoio e incentivo na realização dessa pesquisa. Não é exagero afirmar que a maior parte dessa dissertação decorre de considerações minhas às quais somente pude chegar por meio de suas aulas, notas de curso e artigos.

Agradeço também ao professor Edelcio Gonçalves, cujas aulas e grupos de estudo, que tive o prazer de frequentar, foram essenciais em minha formação lógica. À professora Andrea Loparic, com quem estudei pela primeira vez dedução natural.

A todos meus amigos e colegas de curso, em especial Pedro Falcão e Daniel Nagase.

Aos meus pais e minhas irmãs Bianca, Mariana e Laura.

Sou sobretudo grato a Nina Auras pelo apoio diário e constante nesse período desgraçado do mundo em que essa pesquisa se realizou e por simplesmente tudo.

Agradeço, por fim, à CAPES pelo financiamento desta pesquisa.

RESUMO

COGGIOLA, André Rodrigo Ferreira. *Deductive Tableaux*. 2021. Dissertação (Mestrado) - Faculdade de Filosofia, Letras e Ciências Humanas. Departamento de Filosofia, Universidade de São Paulo, São Paulo, 2021.

O presente estudo tem por objetivo a formalização de um cálculo dedutivo para a lógica clássica baseado em tableaux. São apresentados os cálculos DT, para a lógica proposicional, e QDT, para a lógica de predicados. Uma possível formalização de um cálculo intuicionista nos mesmos moldes é conjecturada. Esses cálculos derivam de uma reformulação do Cálculo de Dados de Roderick Batchelor, que é brevemente apresentado no início.

Palavras-chave: Lógica - Dedução - Tableaux

ABSTRACT

COGGIOLA, André Rodrigo Ferreira. *Deductive Tableaux*. 2021. Dissertation (Master Degree) – Faculdade de Filosofia, Letras e Ciências Humanas. Departamento de Filosofia, Universidade de São Paulo, São Paulo, 2021.

The present study aims at a formalization of a deductive calculus for classical logic based on tableaux. The calculi DT, for propositional logic, and QDT, for predicate logic, are presented. A possible formaliation of an intuitionistic calculus on the same framework is conjectured. These calculi derive from a reformulation of Roderick Batchelor's Data Calculi, which are briefly presented at the beginning.

Key Words: Logic - Deduction - Tableaux

Contents

1 Introduction

In what follows, we will present a new method for logical deductions for classical logic, called the *Deductive Tableaux*. It is based on some ideas from ground theory, having been first devised as a reformulation of the *Data Calculus*, a method of *directional deduction* created by Roderick Batchelor (2019). In these methods, rules have a directionality, either analytic or synthetic, which is analogous and related to the directionality in the relation of *grounds* to their *consequences*.

In section 2 we will give a brief summary of Batchelor's DC and QDC calculi. Then, in sections 3 and 4, we will present our systems for classical propositional and classical predicate logic, respectively.

1.1 Preliminary definitions

Below are some general definitions that will be used in connection with both the data calculi and the deductive tableaux.

We are concerned for now with the propositional logic case only. Necessary definitions for the predicate logic case will be introduced later in due course.

We take $\wedge$, $\vee$ and $\neg$ as our primitive connectives. They are called respectively the *conjunction*, *disjunction* and *negation* connectives.

Besides the connectives, our other symbols will be a denumerable set of *propositional variables* (called simply *variables* when no confusion can arise), together with the left and right parentheses '(' and ')'. We will use the letters $p, q, r, \ldots$, with or without indices (i.e. p_1, p_2 etc.) as the variables.

Formulas are defined by the usual recursive clauses:

1. A variable is a formula (sometimes called an *atomic formula* or an *atom*);
2. If φ and ψ are formulas, then $\neg\varphi$, $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are formulas (called *molecular* formulas).

We will use the Greek letters φ, ψ, χ, with or without indices, as metavariables for *formulas*. Γ, Δ etc. will stand for sets of formulas.

Following the usual custom, we will omit the outer parentheses when considering a whole formula, as well as write $\varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_n$ for $\varphi_1 \circ (\varphi_2 \circ (\ldots \circ (\varphi_n) \ldots))$, where $\circ$ is one of the two binary connectives.

The notion of *immediate subformula* is given by the following clauses:

1. Atomic formulas have no immediate subformula;
2. $\neg\varphi$ has exactly one immediate subformula: φ;
3. $\varphi \wedge \psi$ and $\varphi \vee \psi$ have each exactly two immediate subformulas: φ and ψ.

To each formula of the language we can ascribe a number, called the formula's *complexity*, consisting of the number of occurrences of connectives in the formula. It can also be defined inductively by the following clauses. The complexity number $cp(\varphi)$ of formula φ is:

1. 0, if φ is atomic;
2. $cp(\psi) + 1$, if $\varphi = \neg\psi$;
3. $cp(\psi) + cp(\chi) + 1$, if $\varphi = \psi \circ \chi$.

We use $\varphi \rightarrow \psi$ as abbreviation for $\neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi$ for $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$.

2 The Data Calculi

In this section we will present a summary of Batchelor's DC and QDC calculi, for classical propositional logic and classical predicate logic, respectively. The presentation follows Batchelor (2019) *very* closely, with only a slight difference in the definition of deduction.

2.1 The calculus DC

A *datum* is a finite set of finite sets of formulas. We will use α, β, γ etc. as metavariables for data. The *components* of a datum are its elements.

Instead of using the usual set theoretical notation, data will often be represented with slashes separating the components and commas separating the formulas in each component. So, e.g., instead of $\{\{p, q\}, \{p, \neg q\}, \{\neg p\}, \{r \vee \neg r\}\}$ we write

$$p, q \,/\, p, \neg q \,/\, \neg p \,/\, r \vee \neg r.$$

A datum has the effect of a disjunction of conjunctions, for the inner set-formation is considered as conjunction like and the outer set-formation as 'disjunction-like'. In this way, the slashes correspond to structural disjunctions and the commas to structural conjunctions.

We can say, then, that an interpretation (i.e. assignment of truth-values to the propositional variables) σ *verifies* a datum α if it verifies all the formulas in some component Γ of α. With this, other semantical notions (validity, satisfiability etc.) for data can also be defined.

The definition of a datum implies the existence of $\{\ \}$ and $\{\{\ \}\}$ as data. The first one is clearly unsatisfiable, and will be denoted by $\bot$. The second one is valid and will be denoted by $\top$.

The rules for DC are divided into three categories. First, the *grounding-*

rules (G-rules) concern, roughly speaking, the analysis and synthesis of formulas with each of the connectives in the language as main connective, and negations thereof. They, thus, correspond to the (neutral) grounding conditions for these formulas. Second, the *negation-rules* (N-rules), correspond to principles of classical logic which derive from the general character of negation: the principle of *non-contradiction* (rule NC) and the principle of *excluded middle* (rule EM). Finally, the third category consists of the *structural rules* (S-rules) Del and Exp, which stand, respectively, for deletion and expansion.

The rules are represented below, in tables 1, 2 and 3. Instead of using the traditional horizontal bar for inference, the rules are depicted with an arrow pointing from premiss to conclusion. The rules with a $\downarrow$ are analytic while those with a $\uparrow$ are synthetic. In the G-rules, the $\updownarrow$ indicates, then, a pair of rules, one analytic and one synthetic. Notice that in the synthetic rules the premiss is at the bottom and the conclusion at the top. Finally, Γ, φ stands for $\Gamma \cup \{\varphi\}$ and ρ stands for the 'rest' of a datum, i.e. any other number of components it might have (including the case of none).

$$\wedge\!\updownarrow \quad \begin{array}{c} \Gamma, \varphi \wedge \psi \mathbin{/} \rho \\ \updownarrow \\ \Gamma, \varphi, \psi \mathbin{/} \rho \end{array} \qquad \vee\!\updownarrow \quad \begin{array}{c} \Gamma, \varphi \vee \psi \mathbin{/} \rho \\ \updownarrow \\ \Gamma, \varphi \mathbin{/} \Gamma, \psi \mathbin{/} \rho \end{array}$$

$$\neg\wedge\!\updownarrow \quad \begin{array}{c} \Gamma, \neg(\varphi \wedge \psi) \mathbin{/} \rho \\ \updownarrow \\ \Gamma, \neg\varphi \mathbin{/} \Gamma, \neg\psi \mathbin{/} \rho \end{array} \qquad \neg\vee\!\updownarrow \quad \begin{array}{c} \Gamma, \neg(\varphi \vee \psi) \mathbin{/} \rho \\ \updownarrow \\ \Gamma, \neg\varphi, \neg\psi \mathbin{/} \rho \end{array} \qquad \neg\neg\!\updownarrow \quad \begin{array}{c} \Gamma, \neg\neg\varphi \mathbin{/} \rho \\ \updownarrow \\ \Gamma, \varphi \mathbin{/} \rho \end{array}$$

Table 1: G-rules

These rules are all correct, in the sense that, for any pair of data instantiating the given form, an interpretation that verifies the premiss will also verify the conclusion. The G-rules and N-rules are also equivalential, in the sense that, for each, the premiss is logically equivalent to the conclusion.

	$\Gamma, \varphi, \neg\varphi \;/\; \rho$			$\Gamma, \varphi \;/\; \Gamma, \neg\varphi \;/\; \rho$
NC	$\downarrow$		EM	$\uparrow$
	ρ			$\Gamma \;/\; \rho$

Table 2: N-rules

	$\Gamma, \Delta \;/\; \rho$		$\Gamma \;/\; \rho$
Del	$\downarrow$	Exp	$\uparrow$
	$\Gamma \;/\; \rho$		ρ

Table 3: S-rules

The inverse of each G-rule is itself a G-rule. The inverses of NC and EM are derivable from Exp and Del, respectively. These two S-rules, in turn, are the only rules, then, that allow for weakening of strength in a deduction.

A *development* (in DC) of datum β from datum α is a finite sequence of data, starting with α and ending with β, s.t. each datum in the sequence after the first is immediately inferable from the immediately preceding one by one of the rules above.

A *deduction*, then, of formula φ from the finite set of formulas Γ is defined as any development of $\{\{\varphi\}\}$ from $\{\Gamma\}$. A *proof* of formula φ is any development of $\{\{\varphi\}\}$ from $\top$. A *refutation* of the finite set of formulas Γ is any development of $\bot$ from $\{\Gamma\}$.

A development is *analytic* (or purely analytic, for emphasis) if it uses only analytic rules. Similarly for (purely) *synthetic* developments. A *normal* development is one divisible into two consecutive parts, the first purely analytic and the second purely synthetic.

2.2 The calculus QDC

To extend the methods of directional deduction to classical predicate logic, it is necessary to accommodate in the structure of the datum resources corresponding to the quantifiers. A datum then becomes an expression of the form

$$(\mathrm{E}\underline{f})(\underline{x}) : \Gamma \,/\, \Delta \,/ \ldots$$

where Γ, Δ, ... are finite sets of formulas of first order logic with functional variables.

The rules of QDC include the obvious counterparts of the propositional rules. We list below, then, only the additional rules (tables 4 and 5). These are four pairs of G-rules concerning the analysis and synthesis of the quantifiers (affirmed and denied), and four more S-rules, two weakening rules corresponding to the new structural resources (Inst and Gen), and the two equivalential rules ↓EVE (elimination of vacuous existential variables) and ↑IVU (introduction of vacuous universal variables).

In rules Inst and Gen t is any term.

2.3 Examples of developments

We now show a couple of examples of developments in both DC and QDC.

$\underline{p \to q, (p \wedge q) \to r \vdash_{DC} p \to r}$

$$\begin{array}{lcl} & (\mathrm{E}\underline{f})(\underline{x}) : \forall y\varphi, \Gamma \,/\, \rho & \\ \updownarrow\forall & \updownarrow & \text{provided } y \text{ does not occur free in } \Gamma, \rho. \\ & (\mathrm{E}\underline{f})(\underline{x}, y) : \varphi, \Gamma \,/\, \rho & \end{array}$$

$$\begin{array}{lcl} & (\mathrm{E}\underline{f})(\underline{x}) : \neg\exists y\varphi, \Gamma \,/\, \rho & \\ \updownarrow\neg\exists & \updownarrow & \text{provided } y \text{ does not occur free in } \Gamma, \rho. \\ & (\mathrm{E}\underline{f})(\underline{x}, y) : \neg\varphi, \Gamma \,/\, \rho & \end{array}$$

$$\begin{array}{lcl} & (\mathrm{E}\underline{f})(\underline{x}) : \exists y\varphi(y), \Gamma \,/\, \rho & \\ \updownarrow\exists & \updownarrow & \text{provided } g \text{ does not occur in } \varphi(y),\ \Gamma, \rho. \\ & (\mathrm{E}\underline{f}, g)(\underline{x}) : \varphi(g\underline{x}), \Gamma \,/\, \rho & \end{array}$$

$$\begin{array}{lcl} & (\mathrm{E}\underline{f})(\underline{x}) : \neg\forall y\varphi(y), \Gamma \,/\, \rho & \\ \updownarrow\neg\forall & \updownarrow & \text{provided } g \text{ does not occur in } \varphi(y),\ \Gamma, \rho. \\ & (\mathrm{E}\underline{f}, g)(\underline{x}) : \neg\varphi(g\underline{x}), \Gamma \,/\, \rho & \end{array}$$

Table 4: G-rules for quantifiers

	$(\mathrm{E}\underline{f})(\underline{x},y):\alpha(y)$		$(\mathrm{E}\underline{f},g)(\underline{x}):\alpha(g\underline{x})$
Inst	$\downarrow$	Gen	$\uparrow$
	$(\mathrm{E}\underline{f})(\underline{x}):\alpha(t)$		$(\mathrm{E}\underline{f})(\underline{x}):\alpha(t)$

$$\text{EVE}\quad \begin{array}{c}(\mathrm{E}\underline{f},\underline{g})(\underline{x}):\alpha\\ \downarrow\\ (\mathrm{E}\underline{f})(\underline{x}):\alpha\end{array}\qquad \text{provided } \underline{g} \text{ do not occur in } \alpha.$$

$$\text{IVU}\quad \begin{array}{c}(\mathrm{E}\underline{f})(\underline{x},\underline{y}):\alpha\\ \uparrow\\ (\mathrm{E}\underline{f})(\underline{x}):\alpha\end{array}\qquad \text{provided } \underline{y} \text{ do not occur free in } \alpha.$$

Table 5: S-rules

$\neg p\vee q,\neg(p\wedge q)\vee r$	$\downarrow\vee$
$\neg p\vee q,\neg(p\wedge q)\ /\ \neg p\vee q,r$	$\downarrow\vee(\times 2)$
$\neg p,\neg(p\wedge q)\ /\ q,\neg(p\wedge q)\ /\ \neg p,r\ /\ q,r$	$\downarrow$Del
$\neg p\ /\ q,\neg(p\wedge q)\ /\ \neg p\ /\ r$	$\downarrow\neg\wedge$
$\neg p\ /\ q,\neg p\ /\ q,\neg q\ /\ \neg p\ /\ r$	$\downarrow$NC
$\neg p\ /\ q,\neg p\ /\ \neg p\ /\ r$	$\downarrow$Del
$\neg p\ /\ \neg p\ /\ \neg p\ /\ r\quad (=\neg p\ /\ r)$	$\uparrow\vee$
$\neg p\vee r$	

$\underline{\forall xPx\wedge\forall xQx\vdash_{QDC}\forall x(Px\wedge Qx)}$

$\forall x Px \wedge \forall x Qx$	$\downarrow\wedge$
$\forall x Px, \forall x Qx$	$\downarrow\forall$
$(x) : Px, \forall x Qx$	$\downarrow\forall$
$(x, y) : Px, Qy$	$\downarrow$Inst
$(x) : Px, Qx$	$\uparrow\wedge$
$(x) : Px \wedge Qx$	$\uparrow\forall$
$\forall x(Px \wedge Qx)$	

$\underline{\vdash_{QDC} \forall y(\forall x Px \rightarrow Py)}$

$\{\{\ \}\}$	$\uparrow$EM
$\neg\forall x Px \,/\, \forall y Py$	$\downarrow\forall$
$(y) : \neg\forall x Px \,/\, Py$	$\uparrow\vee$
$(y) : \forall x Px \rightarrow Py$	$\uparrow\forall$
$\forall y(\forall x Px \rightarrow Py)$	

3 Deductive tableaux for classical propositional logic

In this section we will present the calculus DT, a formulation of the deductive tableaux method for classical propositional logic. This calculus is practically the same as the 'deduction trees' presented in Jeffrey 1981 (for some reason present only in the second edition of that book), with a couple of differences. In our formulation we use signed formulas and seek to present the rules in a more symmetrical fashion. Also, Jeffrey's presentation is somewhat brief and superficial, not considering all the definitions and alternate formulations we present here, and his motivations are rather different from ours. Finally, he does not consider any correlate of these methods for predicate logic, which we will do in section 4.

3.1 The calculus DT

By a *signed formula* we mean $T\varphi$ or $F\varphi$ where φ is a formula. $T\varphi$ and $F\varphi$ are thought of as meaning φ is true and φ is false respectively. We will also say that φ is *asserted* or *rejected* as it receives the sign T or F, respectively. As metavariables for signed formulas we will use σ and ξ, with or without indices.

Σ, Ξ will stand for sets of signed formulas (again, with or without indices). If $\Gamma = \{\psi_1, \psi_2, \dots, \psi_n\}$, then by Γ^T we will mean the set of signed formulas $\{T\psi_1, T\psi_2, \dots, T\psi_n\}$, and similarly for Γ^F.

For $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$, by σ_i^*, $1 \leq i \leq n$, we mean

$$\sigma_i^* = \begin{cases} \varphi & \text{if } \sigma_i = T\varphi \\ \neg\varphi & \text{if } \sigma_i = F\varphi \end{cases}$$

and by Σ^* we mean the set of formulas $\{\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*\}$. By a *tableau for* Σ we mean some tableau initiated by $\sigma_1, \sigma_2, \ldots, \sigma_n$ and expanded only according to the rules below. We call this initial segment of a tableau for Σ, consisting of the signed formulas in Σ, the tableau's *origin*.

The primitive rules of the system, for expanding a tableau, are represented on tables 6, 7 and 8.

$$\downarrow\wedge^T \frac{T\varphi \wedge \psi}{T\varphi, T\psi} \qquad \downarrow\vee^T \frac{T\varphi \vee \psi}{T\varphi \mid T\psi} \qquad \downarrow\neg^T \frac{T\neg\varphi}{F\varphi}$$

$$\downarrow\wedge^F \frac{F\varphi \wedge \psi}{F\varphi \mid F\psi} \qquad \downarrow\vee^F \frac{F\varphi \vee \psi}{F\varphi, F\psi} \qquad \downarrow\neg^F \frac{F\neg\varphi}{T\varphi}$$

Table 6: Analytic G-rules

$$\uparrow\wedge^T \frac{T\varphi, T\psi}{T\varphi \wedge \psi} \qquad \uparrow\vee^T \frac{T\varphi \mid T\psi}{T\varphi \vee \psi} \qquad \uparrow\neg^T \frac{F\varphi}{T\neg\varphi}$$

$$\uparrow\wedge^F \frac{F\varphi \mid F\psi}{F\varphi \wedge \psi} \qquad \uparrow\vee^F \frac{F\varphi, F\psi}{F\varphi \vee \psi} \qquad \uparrow\neg^F \frac{T\varphi}{F\neg\varphi}$$

Table 7: Synthetic G-rules

$$\text{ExF} \frac{T\varphi, F\varphi}{\sigma} \qquad \text{PB} \frac{\sigma}{T\varphi \mid F\varphi}$$

Table 8: N-rules

In ExF and PB, σ is any signed formula. ExF stands for *Ex Falso* and PB for *Principle of Bivalence*. It is clear that these rules are neither analytic nor synthetic in the same sense the G-rules are, for they don't concern any connective of the language or relate anyhow formulas with their constituents. But we will treat ExF as an analytic rule itself, or at least as belonging to the analytic part of the system. PB will then belong to the synthetic part. We will discuss the directional character and other aspects of these rules in section 3.5 (page 23).

Formulas separated by a comma belong to the same branch. The expressions of the form $\sigma \mid \xi$ in the conclusion of a rule (so in $\downarrow\vee^T$, $\downarrow\wedge^F$ and PB) indicate branching with σ on the left and ξ on the right. In the premiss of rule (so in $\uparrow\vee^T$ and $\uparrow\wedge^F$) this expression occurs merely to abbreviate a pair of rules, one with σ as single premiss and another with ξ as single premiss. This is done so as to exhibit more fully some symmetries between the rules. We can further justify this use by the following considerations. It can be seen that in all the G-rules we relate single molecular formulas with their immediate subformulas, either as premiss or conclusion. Expressions corresponding to the subformulas can be of two types: conjunctive, where the formulas are separated by a comma (as in e.g. $\uparrow\wedge^T$ or $\downarrow\wedge^T$ rules), or disjunctive, where the formulas are separated by a vertical bar. (For the rules for negation, where there is only one formula in premiss and conclusion, these can be taken as conjunctive.) So, as can be seen, the rules can have multiple formulas as either premiss or conclusion. Multiple premisses of the disjunctive type are eliminable in the sense that these can be seen as indicating a pair of rules, each with just one of the premisses, as described. But multiple premisses of the conjunctive type are non-eliminable. Dually, multiple-conclusions of the conjunctive type are eliminable, in the sense that these can be seen as indicating a pair of rules, each with just one of the conclusions. But multiple-conclusions of the disjunctive type are non-eliminable.

With these considerations we allow that all G-rules, as written down, can be seen as equivalential, as in DC, for we will take formulas in a same branch as conjunctive and formulas in different branches as disjunctive. It also becomes clear that every analytic G-rule is the inverse of some synthetic G-rule and vice versa.

The *dual* of a rule R is the rule obtained from R by inverting premiss and conclusion, dualizing the connectives, and transforming multiple-formulas of conjunctive type into disjunctive type and vice versa (i.e. dualizing also the 'structure'). As can be easily seen, every primitive rule in the system is the dual of another primitive rule. E.g. $\downarrow\wedge^T$ and $\uparrow\vee^T$ are dual to each other, as well as ExF and PB.

We will say that a tableau branch is *closed* if the rule ExF could be applied in it, i.e. if it contains some explicit contradiction. A branch that is not closed is *open*. A tableau as a whole is closed when all its branches are closed and open when at least one branch is open.

We can also use formulas with the material implication $\to$, under the usual definition (already mentioned) $\varphi \to \psi =_{df} \neg\varphi \vee \psi$. The corresponding rules can then be derived from this definition through the rules for $\vee$ and $\neg$. They are represented on table 9.

$$\downarrow\!\to^T \frac{T\varphi \to \psi}{F\varphi \mid T\psi} \qquad \uparrow\!\to^T \frac{F\varphi \mid T\psi}{T\varphi \to \psi}$$

$$\downarrow\!\to^F \frac{F\varphi \to \psi}{T\varphi, F\psi} \qquad \uparrow\!\to^F \frac{T\varphi, F\psi}{F\varphi \to \psi}$$

Table 9: Rules for $\to$

We will also, sometimes, following Smullyan, use a unifying notation, calling molecular signed formulas either α or β, for which we will define signed formulas α_1, α_2 and β_1, β_2, respectively, according to the tables 10 and 11.

α	α_1	α_2
$T\varphi \wedge \psi$	$T\varphi$	$T\psi$
$F\varphi \vee \psi$	$F\varphi$	$F\psi$
$F\varphi \to \psi$	$T\varphi$	$F\psi$
$T\neg\varphi$	$F\varphi$	$F\varphi$
$F\neg\varphi$	$T\varphi$	$T\varphi$

Table 10: Type A: conjunctive formulas

With this notation, all the G-rules can be succinctly summarized in the four rules represented on table 12.

A *development* of a finite set of signed formulas Ξ from a finite set of signed formulas Σ is defined as any tableau for Σ with at least one $\xi \in \Xi$

β	β_1	β_2
$F\varphi \wedge \psi$	$F\varphi$	$F\psi$
$T\varphi \vee \psi$	$T\varphi$	$T\psi$
$T\varphi \to \psi$	$F\varphi$	$T\psi$

Table 11: Type B: disjunctive formulas

$$\downarrow A \quad \frac{\alpha}{\alpha_1, \alpha_2} \qquad \downarrow B \quad \frac{\beta}{\beta_1 \mid \beta_2}$$

$$\uparrow A \quad \frac{\alpha_1, \alpha_2}{\alpha} \qquad \uparrow B \quad \frac{\beta_1 \mid \beta_2}{\beta}$$

Table 12: Summarized G-rules

occurring in each branch.

A *deduction* (in DT) of formula φ from the finite set of formulas Γ is now defined as any development of $\{T\varphi\}$ from Γ^T. Notice this means a tableau in which $T\varphi$ occurs in every branch.

A *proof* (in DT) of a formula φ is defined as any development of $\{T\varphi\}$ from $\{F\varphi\}$.

A *refutation* (in DT) of finite set of formulas Γ is defined as any development of Γ^F from Γ^T.

A development is *normal* when it is constructed in such a way that, in every branch, no synthetic rule is applied before an analytic one. We also say a development is *analytic* (or *purely analytic* for emphasis) when no synthetic rule is applied. Similarly for (*purely*) *synthetic development*. All these definitions naturally apply to deductions, proofs and refutations as well, as these are defined as developments.

Note that deductions, proofs and refutations are all three allowed to proceed in any of the three ways: purely analytic, purely synthetic or combining both analytic and synthetic rules. There are, however, deductions that can only proceed in this third way. In particular, it is obvious that if Γ is consistent and φ has some variable not occurring in Γ, then φ cannot be analytically developed. As for proofs, we can show that there will always be two direct proofs of any valid formula: one purely analytic 'proof by refutation' (as in Smullyan) and one purely synthetic 'proof proper' (theorem 3.4 below). Similarly for refutations, where we can also define one notion of purely synthetic 'refutation by proof' and one purely analytic 'refutation proper'. But note that a 'proof by refutation' is still a proof by our definition, not a refutation, even though it might be considered a refutation in the literal sense. For proofs begin with some formula rejected, while refutations begin with formulas asserted.

The system DT can then be seen as the combination of two systems, each complete for both proofs and refutations, but not deductions: the system DT↓, consisting of only the analytic rules, and the system DT↑, with only the synthetic rules. We can also consider the systems with either no analytic or no synthetic G-rules, but with both N-rules.

3.2 Examples of developments

Here we show some examples of developments in DT. We focus on deductions and synthetic proofs, given that analytic refutations correspond to the familiar tableaux method (as in Smullyan).

Here and in subsequent examples of tableaux, we will use sometimes the symbol $\mathfrak{C}$ to denote the intended conclusion of a development, in order to avoid the repetition of some large formula. Also, when a same rule is applied to more than one branch it will be annotated to the right of the line only once. Otherwise, the rules used in some line are annotated following the order of the branches from left to right.

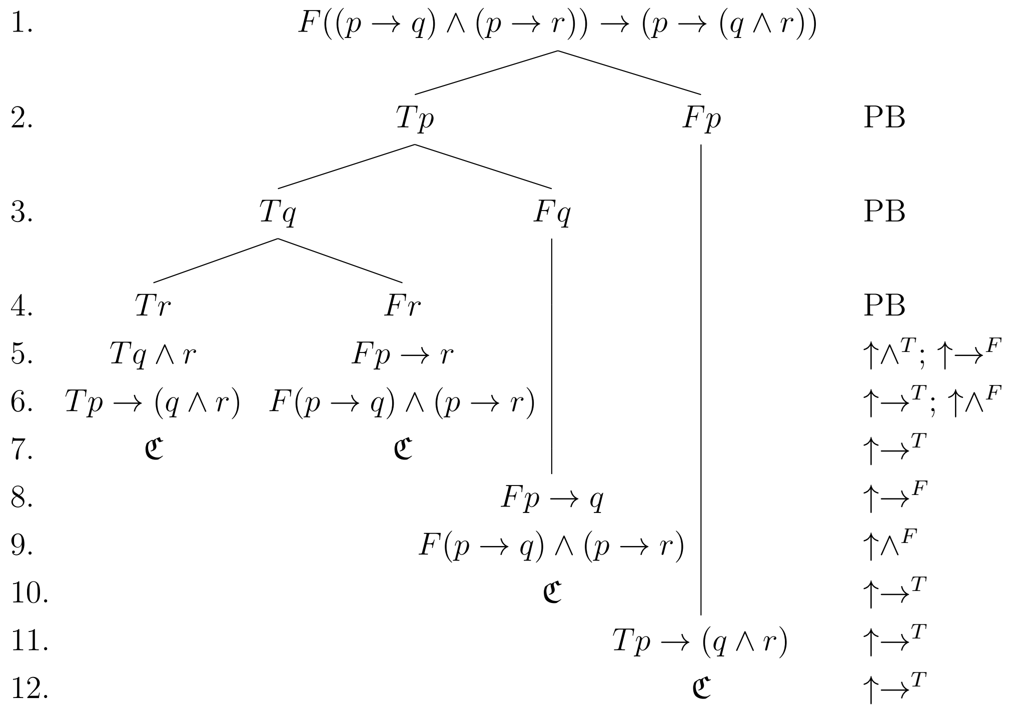
$\underline{p \to q, (p \wedge q) \to r \vdash_{DT} p \to r}$

1.		$Tp \to q$			
2.		$T(p \wedge q) \to r$			
3.	Fp		Tq		$\downarrow\to^T$
4.			$Fp \wedge q$	Tr	$\downarrow\to^T$
5.		Fp	Fq		$\downarrow\wedge^F$
6.			$Tp \to r$		ExF
7.	$Tp \to r$	$Tp \to r$			
8.					$\uparrow\to^T$
				$Tp \to r$	

$p \wedge (\neg q \to \neg p) \vdash_{DT} (p \wedge q) \vee \neg p$

1.	$Tp \wedge (\neg q \to \neg p)$		
2.	Tp		$\downarrow\wedge^T$
3.	$T\neg q \to \neg p$		$\downarrow\wedge^T$
4.	$F\neg q$	$T\neg p$	$\downarrow\to^T$
5.	Tq		$\downarrow\neg^F$
6.	$Tp \wedge q$		$\uparrow\wedge^T$
7.	$T(p \wedge q) \vee \neg p$	$T(p \wedge q) \vee \neg p$	$\uparrow\vee^T$

$\vdash_{DT} ((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$



3.3 Soundness

Assume Γ and Σ are finite sets of formulas and signed formulas, respectively. By $\wedge\theta$ we mean the formula $\sigma_1^* \wedge \sigma_2^* \wedge \ldots \wedge \sigma_m^*$, where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the m signed formulas in some branch θ. Finally, by (θ, σ) we mean $\theta \cup \{\sigma\}$. First we will prove the following lemma:

Lemma 3.1. *Let $\theta_1, \theta_2, \ldots, \theta_n$ be the branches of some tableau for Σ. Then, $\Sigma^* \vDash \wedge\theta_1 \vee \wedge\theta_2 \vee \ldots \vee \wedge\theta_n$.*

Proof. First we note that any tableau for Σ is some extension of the tableau $\mathfrak{T}$ consisting only of the signed formulas in Σ in a single branch θ. Clearly then $\Sigma^* \vDash \wedge\theta$. Next we show that, if the lemma is true for some tableau $\mathfrak{T}_1$, then it is also true for any immediate extension $\mathfrak{T}_2$ of $\mathfrak{T}_1$.

Let $\theta_1', \theta_2', \ldots, \theta_n'$ be the branches in $\mathfrak{T}_1$. Let Ξ be the set of signed formulas at $\mathfrak{T}_1$'s origin. By hypothesis, $\Xi^* \vDash \wedge\theta_1' \vee \wedge\theta_2' \vee \ldots \vee \wedge\theta_n'$. $\mathfrak{T}_2$ is obtained by applying some rule to some of the branches θ'_i, for $1 \leq i \leq n$. We then have three cases:

1. If an A rule is used, then θ'_i contains either an α or both α_1 and α_2 and is extended to $(\theta'_i, \alpha_1, \alpha_2)$ or (θ'_i, α), respectively; either way,

$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, \alpha_1, \alpha_2) \vee \ldots \vee \wedge\theta'_n \quad \text{as well as}$$
$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, \alpha) \vee \ldots \vee \wedge\theta'_n.$$

2. If a B rule is used, then θ'_i contains a β or either β_1 or β_2, and is extended to either both (θ'_i, β_1) and (θ'_i, β_2) or simply to (θ'_i, β); either way,

$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, \beta_1) \vee \wedge(\theta'_i, \beta_2) \vee \ldots \vee \wedge\theta'_n \quad \text{as well as}$$
$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, \beta) \vee \ldots \vee \wedge\theta'_n.$$

3. Finally, one of the N rules might be used. If θ'_i is extended by ExF to (θ'_i, ξ), then θ'^*_i is unsatisfiable, and so

$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, \xi) \vee \ldots \vee \wedge\theta'_n.$$

   If θ'_i is extended by PB to both $(\theta'_i, T\varphi)$ and $(\theta'_i, F\varphi)$, it is clear that

$$\Xi^* \vDash \wedge\theta'_1 \vee \wedge\theta'_2 \vee \ldots \vee \wedge(\theta'_i, T\varphi) \vee \wedge(\theta'_i, F\varphi) \vee \ldots \vee \wedge\theta'_n.$$

□

We now can prove the theorem:

Theorem 3.2 (Soundness of DT)**.** *If* $\Gamma \vdash_{DT} \varphi$*, then* $\Gamma \vDash \varphi$*.*

Proof. Assume $\Gamma \vdash_{DT} \varphi$. This means there exists a tableau for Γ^T with $T\varphi$ occurring in every branch. Let $\theta_1, \theta_2, \ldots, \theta_n$ be its branches. By the previous lemma, and the fact that $(\Gamma^T)^* = \Gamma$, we have that $\Gamma \vDash \wedge\theta_1 \vee \wedge\theta_2 \vee \ldots \vee \wedge\theta_n$. Therefore, since φ is in every θ_i, $1 \leq i \leq n$, we have that $\Gamma \vDash \varphi$. □

3.4 Completeness

In order to prove the completeness of DT, we first prove lemma 3.3. It leads us to theorem 3.4, the proof completeness of DT. From there we have one way of proving the deduction completeness (theorem 3.6), through lemma 3.5. Finally we prove completeness for normal deductions (theorem 3.7), which follows directly from lemma 3.3.

Lemma 3.3. *For any formula φ with variables $p_1, p_2, \ldots, p_n$, and for any valuation v, if $v(\varphi) = T$, then there exists a deduction for $\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \varphi$ using only the synthetic G-rules, where, for $1 \leq i \leq n$,*

$$\pi_i = \begin{cases} p_i & \textit{if } \ v(p_i) = T \\ \neg p_i & \textit{if } \ v(p_i) = F \end{cases}$$

Otherwise, if $v(\varphi) = F$, there exists a deduction for $\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\varphi$ using only the synthetic G-rules.

Proof. We will prove this lemma by induction on the complexity of the formulas. For the base case, we consider φ to be a variable, say $\varphi = p_1$. Then we have the two one line deductions where

$$\text{if } v(p_1) = T \text{ then } \pi_1 = p_1 \vdash_{DT} p_1 = \varphi;$$
$$\text{if } v(p_1) = F \text{ then } \pi_1 = \neg p_1 \vdash_{DT} \neg p_1 = \neg\varphi.$$

Now assume the lemma is true for some formulas ψ_1 and ψ_2. We then have three cases:

1. $\varphi = \neg\psi_1$.

   Note that the variables in ψ_1 and φ are the same. If $v(\varphi) = T$, then

$v(\neg\psi_1) = T$, so $v(\psi_1) = F$. By the induction hypothesis we then have that

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\psi_1 = \varphi.$$

If $v(\varphi) = F$, then $v(\neg\psi_1) = F$, so $v(\psi_1) = T$. So, by the hypothesis,

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \psi_1.$$

Applying now, to every branch, the rule $\uparrow\neg^F$ followed by $\uparrow\neg^T$, we reach a deduction of $\neg\neg\psi_1 = \neg\varphi$.

2. $\varphi = \psi_1 \wedge \psi_2$.
Note that φ contains all the variables in ψ_1 and ψ_2. If $v(\varphi) = T$, then $v(\psi_1 \wedge \psi_2) = T$, so $v(\psi_1) = v(\psi_2) = T$. By the induction hypothesis, we then have both

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \psi_1 \quad \text{and}$$
$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \psi_2.$$

So, applying $\uparrow\wedge^T$ to every branch, we reach a deduction of $\psi_1 \wedge \psi_2 = \varphi$. If $v(\varphi) = F$, then $v(\psi_1 \wedge \psi_2) = F$, so $v(\psi_1) = F$ or $v(\psi_2) = F$. So, by the hypothesis, we will have either

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\psi_1 \quad \text{or}$$
$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\psi_2.$$

Either way, by $\uparrow\wedge^F$ (for, since $T\neg\psi_1$ or $T\neg\psi_2$ was obtained synthetically, $F\psi_1$ or $F\psi_2$ must occur before) followed by $\uparrow\neg^T$, we reach a deduction of $\neg(\psi_1 \wedge \psi_2) = \neg\varphi$.

3. $\varphi = \psi_1 \vee \psi_2$.
Note that φ contains all the variables in ψ_1 and ψ_2. If $v(\varphi) = T$, then $v(\psi_1 \vee \psi_2) = T$, so $v(\psi_1) = T$ or $v(\psi_2) = T$. By the induction

hypothesis, we then have either

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \psi_1 \quad \text{or}$$
$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \psi_2.$$

Either way, applying $\uparrow\vee^T$ to every branch, we reach a deduction of $\psi_1 \vee \psi_2 = \varphi$. If $v(\varphi) = F$, then $v(\psi_1 \vee \psi_2) = F$, so $v(\psi_1) = v(\psi_2) = F$. So, by the hypothesis, we will have both

$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\psi_1 \quad \text{and}$$
$$\pi_1, \pi_2, \ldots, \pi_n \vdash_{DT} \neg\psi_2.$$

So, by $\uparrow\vee^F$ (again, for $F\psi_1$ and $F\psi_2$ must occur before $T\neg\psi_1$ and $T\neg\psi_2$) followed by $\uparrow\neg^T$, we reach a deduction of $\neg(\psi_1 \vee \psi_2) = \neg\varphi$.

□

Theorem 3.4 (Proof completeness of DT)**.** *If $\vDash \varphi$, then there exists a purely synthetic proof of φ in DT.*

Proof. Assume $\vDash \varphi$ and that $p_1, p_2, \ldots, p_n$ are the variables in φ. We begin the proof with $F\varphi$, as by definition, and apply PB (a synthetic rule) successively to every available branch, each time introducing occurrences of one of the variables in φ, asserted and rejected. This procedure will generate branches corresponding to each possible truth-value assignment to the variables. A tautology is precisely a formula that is true under every truth-value assignment to its atoms. So we first apply $\uparrow\neg^T$ to assert the negation of all rejected atoms, and now, by the previous lemma, we have that it is possible to synthetize $T\varphi$ in every branch. □

Note that the proof for φ constructed according to the above procedure makes no essential use of the initial premiss $F\varphi$, for the rule PB is indifferent regarding its premisses. This means, in practice, that any tautology can be correctly synthetically inferred at any point in a tableau (i.e. we can attach this proof minus its first line to the end of any branch and the result will still be a correct tableau). We will make use of this fact in the next proof.

Assume $\Gamma = \{\psi_1, \psi_2, \ldots, \psi_n\}$ for some n.

Lemma 3.5. *If* $\vdash_{DT} \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$*, then* $\Gamma \vdash_{DT} \varphi$*.*

Proof. Assume $\vdash_{DT} \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. Now we begin to construct a tableau for Γ^T. By the hypothesis and the previous theorem, we can synthetically extend this tableau deducing $\neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. Through sereval applications of $\downarrow\vee^T$ and $\downarrow\neg^T$ we reach a tableau with each branch ending with, for $1 \leq i \leq n$, either one of the ψ_i rejected or with φ asserted. So, finally, we apply ExF to each branch with a $F\psi_i$ (since all branches contain also every $T\psi_i$ at the origin) to infer $T\varphi$, obtaining a tableau for $\Gamma \vdash_{DT} \varphi$. □

Theorem 3.6 (Deduction completeness of DT)**.** *If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash_{DT} \varphi$*.*

Proof. Assume $\Gamma \vDash \varphi$. So, $\vDash \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. Then, by theorem 3.4, we have that $\vdash_{DT} \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. Finally, by lemma 3.5, we have that $\Gamma \vdash_{DT} \varphi$. □

Theorem 3.7 (Normal deduction completeness of DT)**.** *If* $\Gamma \vDash \varphi$*, then there exists a normal deduction of* φ *from* Γ*.*

Proof. Assume $\Gamma \vDash \varphi$. We begin the construction of a tableau for Γ^T. First we fully analyze the premisses, i.e. apply every analytic rule possible until we have only atoms, asserted or rejected. To every branch with some formula both asserted and rejected we adjoin $T\varphi$ by ExF. The remaining branches have signed formulas corresponding to some interpretation that verifies Γ. By the hypothesis, every such interpretation also verifies φ. So, either it is possible to synthesize φ already in these branches (or some of them), or first we will have to apply PB as many times as needed to introduce every variable that occurs in φ missing from the branch. It is clear that the branches thus generated will remain open (for we only introduced missing variables). Therefore, they each correspond to an interpretation that verifies Γ and so, by hypothesis, that verifies also φ. Since now they also have every atom occurring in φ, by $\uparrow\neg^T$ and lemma 3.3, we will eventually be able to synthesize $T\varphi$ in every such branch. $\square$

The deductions constructed according to the procedure described in the above proof actually satisfy an even stronger condition for normality than the one defined: no N-rule is applied before an analytic G-rule or after a synthetic G-rule.

3.5 Remarks on PB and ExF

As we said before, we are considering, somewhat arbitrarily, ExF to be an analytic rule and PB a synthetic rule. We will now comment on this choice and discuss some alternative formulations of these rules.

The G-rules, by relating molecular formulas with their immediate subformulas, have a clear analytic or synthetic character. The DC counterparts of ExF and PB (NC and EM, respectively) also have a clear analytic or synthetic character, but this time regarding the constitution of data instead of formulas. But what about ExF and PB? Even if we consider PB synthetic, because it creates a new branch, or even because it allows the introduction of 'new' material to a branch, this hardly says anything about ExF being analytic. There are, though, ways to rectify this.

First, let us say why we chose that particular formulation for these rules. ExF and PB are the only structural rules, in the sense that they only concern the structural resources of the calculus (i.e. the tableau form and the signs). This means also that some aspects of the calculus, like the definitions of deduction, proof and refutation, depend on the particular way we formulate these rules. For example, because every rule (including PB) requires something as premiss, the definition of proof requires some starting point, such as the rejection of the formula to be proved. This might seem artificial, but we believe it is very appropriate for classical logic and has some interesting consequences. In particular, it allows for, as already mentioned, two different forms of normal proof (and similarly for refutations): one analytic and one synthetic. (Actually, there are also analytico-synthetic normal proofs and refutations, that can sometimes be shorter than the other two types.) This initial premiss is somewhat irrelevant in the synthetic proof, but it is necessary for the analytic proof. In this way, if we consider some of the partial systems (i.e. DT↓ or DT↑), we still have proofs and refutations both defined and distinguished from one another. Proofs of either form, in this way, acquire the aspect of the *Consequentia Mirabilis*, or Clavius Law, which states that if something is implied by its own negation, then it must be the case (in symbols: $(\neg A \to A) \to A$). Refutations, in turn, have the form of *reductio ad absurdum* ($(A \to \neg A) \to \neg A$). Finally, this formulation makes all rules relate (signed) formulas in premiss and conclusion, making unnecessary extra symbols, like $\bot$ and $\top$, while also keeping ExF and PB directly dual to each other, in keeping with the general symmetry of the system.

But we could, in fact, use these extra symbols. In particular, it is quite usual to use $\bot$ (sometimes called the 'absurd' or the 'canonical contradiction') in formulations of natural deduction (as in Prawitz 1965). Instead of a single rule ExF (or more properly ExF* from table 15 below, as we consider here for simplicity the unsigned calculus), there are then two rules, one for introducing $\bot$ (from some formula both affirmed and denied) and one for 'eliminating' it (allowing to infer any formula). If negation then is defined (as is also usual) by implication and $\bot$ (i.e. $\neg\varphi =_{df} \varphi \to \bot$), then the first of these rules is actually an instance of modus ponens and is clearly analytic. The second rule, that properly states the *ex falso quodlibet* principle, is then synthetic. So, with $\bot$ (and $\top$) in the language, we could have these rules (call them $\mathrm{NC}^{\bot}$ and $\mathrm{ExF}^{\bot}$) instead of ExF* and redefine refutation of Γ as a development of $\bot$ from Γ^T (i.e. a 'closed' tableau). Maintaining the duality, $\mathrm{EM}^{\bot}$ will have $\top$ as premiss, and now a proof of φ is defined, more generically, as a development of φ from $\top$. Also, dual to $\text{ExF}^{\bot}$ there would be the rule that allows to infer $\top$ from any formula.

This, however, we think, introduces many redundancies and does not correspond well to the idea of ground. For, if $\bot$ and $\top$ are introduced directly to the object language, their rules would have to count (in our approach) as G-rules, which hardly makes sense. If we use signs, there would also be redundant signed formulas $F\bot$, $F\top$, requiring rules of their own. Introducing these resources as special, 'empty', signed formulas make things neater, allowing their rules to be structural. But it is still not quite clear what exactly these objects are, from a conceptual point of view, for they have to be artificially inserted. (As we will see in the next section, generalizing signed formulas as sequents makes these objects appear somewhat more naturally.)

There is still another approach, more in line with the analogy between DC and DT. We could want (synthetic) proofs to start from the empty set of premisses (or from a set of extra-logical axioms in some formalized theory) instead of the rejection of the conclusion or some artificial object. So PB, which, as can be seen by theorems 3.4 and 3.7, is an initial rule for the synthetic part of developments, would have to be formulated as a zero-premiss rule, allowing it to initiate a development. Note that a tableau can also be treated as finite set of finite sets of formulas, just as a datum, the branches of a tableau corresponding to the components of a datum. A (purely logical) proof, then, begins from the tableau with a single empty branch, i.e. from $\{\{\ \}\}$, just as in DC. Dually, there is then a rule analogous do DC's NC, allowing the elimination of a branch in which a formula and its inverse both occur. A refutation of Γ becomes then the empty tableau (i.e. $\{\ \}$), constructed from Γ^T.

In a deduction of (not valid) φ from unsatisfiable Γ, it would be necessary then to apply EM first of all, branching with $T\varphi$ and $F\varphi$, and then eliminate this second branch only. But this deduction would not be normal, because it begins with the use of a synthetic rule. We could then, to rectify this, introduce also a synthetic rule analogous to DC's Exp, allowing the introduction of an arbitrary branch. Since now developments can begin with a branching, i.e. from an empty point (because of the reformulated PB), this rule would introduce an empty point *above* the topmost point in the tableau and a new branch to the right, below this point, the whole tableau before the application of the rule becoming the left branch. (Notice that an empty tableau has no points or branches at all. Applying the rule then leaves us with a tableau with one branch, not two.) In this way, we can synthetically introduce the conclusion *after* analytically destroying the tableau developed from unsatisfiable Γ.

3.6 Signed formulas, implication and intuitionistic logic

The signs T and F in DT actually serve more than one purpose. First, they provide a structural resource for dealing with $\neg$, as the tableaux tree structure provides a resource for dealing with $\wedge$ and $\vee$. In this way we have a structural resource corresponding to each primitive connective. Also, it allows all primitive rules to be *pure*, in the sense of always involving just one occurrence-of-connective. The calculus then is completely generalizable to arbitrary sets of primitive truth-functional connectives, including cases where negation is absent.

The signed formulas also serve to make explicit the relation of the system with the semantical theory. The rules directly correspond to the truth and falsity (or assertion and rejection) conditions for each connective or, in the case of the N-rules, to general logical principles characterizing the semantics. Constructing a development can be seen as a search through the (partial) models that correspond to the valuations given at the tableau's origin. We have exploited this fact, for example, in the completeness proofs above.

Related to this last point, Rumfitt (2000) uses signed formulas in order to argue against a common line of thought in the literature. E.g. Garson (2008) (but also, to some extent, Prawitz, Dummett and Tennant; references to those autors can be found in the two papers mentioned) propounds the idea that intuitionistic rather than classical logic has better claim to being *truly logical*, based on the fact that the rules of natural deduction systems can be shown to express intuitionistic truth-conditions. On the other hand, Rumfitt argues that that line of thought depends on failing to treat assertion and rejection on a par, proposing the use of signed formulas. Intuitionistic logic would then be characterized, in particular, by concentrating exclusively on assertions.

We do not intend to join in this particular discussion here. However, we would only like to point out that the idea that natural deduction favours intuitionistic logic should come as no surprise. For, it is clear, at least in Gentzen's and Prawitz' formulations, that the classical natural deduction calculus comes from the intuitionistic one by the addition of a single rule, that corresponds indirectly to the excluded middle. So it is no wonder the remaining rules, in particular the ones for the connectives, characterize intuitionistic meanings. Von Plato (2014) even suggests that one of the possible sources of Gentzen's natural deduction was Heyting's (1931) characterization of the connectives in terms of proof, which is closely related to intuitionistic logic.

In contrast, DT (as DC) is eminently classical, and not only because of the use of signed formulas. First of all, it has no structural resource for dealing with implication primitively, and that is the main thing we will consider here. Also, DT has more than one rule not valid for intuitionistic logic, in particular, obviously, PB, which is an essential structural rule.

Let us first briefly consider a deductive tableaux system for plain formulas, without signs, which we will call DT*. The rules for this system are represented on tables 13, 14 and 15.

A *development* (in DT*) of finite set of formulas Δ from finite set of

$$\downarrow\wedge \; \frac{\varphi \wedge \psi}{\varphi, \psi} \qquad \downarrow\vee \; \frac{\varphi \vee \psi}{\varphi \mid \psi}$$

$$\downarrow\wedge^{\neg} \; \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \mid \neg\psi} \qquad \downarrow\vee^{\neg} \; \frac{\neg(\varphi \vee \psi)}{\neg\varphi, \neg\psi} \qquad \downarrow\neg^{\neg} \; \frac{\neg\neg\varphi}{\varphi}$$

Table 13: Unsigned analytic G-rules

$$\uparrow\wedge \; \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad \uparrow\vee \; \frac{\varphi \mid \psi}{\varphi \vee \psi}$$

$$\uparrow\wedge^{\neg} \; \frac{\neg\varphi \mid \neg\psi}{\neg(\varphi \wedge \psi)} \qquad \uparrow\vee^{\neg} \; \frac{\neg\varphi, \neg\psi}{\neg(\varphi \vee \psi)} \qquad \uparrow\neg^{\neg} \; \frac{\varphi}{\neg\neg\varphi}$$

Table 14: Unsigned synthetic G-rules

$$\text{ExF}^{*} \; \frac{\varphi, \neg\varphi}{\psi} \qquad \text{PB}^{*} \; \frac{\psi}{\varphi \mid \neg\varphi}$$

Table 15: Unsigned N-rules

formulas Γ is defined as any tableau beginning with formulas in Γ and with at least one $\varphi \in \Delta$ occurring in each branch. A *deduction* of formula φ from Γ is now defined as any development of $\{\varphi\}$ from Γ (i.e. a tableau with φ occurring in every branch). A *proof* of φ is then a development of $\{\varphi\}$ from $\{\neg\varphi\}$, and a *refutation* of Γ, a development of $\Gamma^{\neg}$, a set consisting of formulas in Γ negated, from Γ.

We can translate any deduction in DT into a valid deduction in DT*: we just erase all the T signs, substitute every $F\varphi$ with $\neg\varphi$ and erase formulas obtained with $\uparrow\neg^{T}$ and $\downarrow\neg^{T}$, since these in the unsigned system will be equal to their premisses and thus redundant. It is clear then that DT* is also complete.

In order to accommodate $\rightarrow$ in DT* primitively we could only introduce the familiar resource from natural deduction of taking arbitrary assumptions that can be discharged by conditionalization. A new synthetic rule then, called PM (for *principle of monotonicity*, in keeping with the idea of structural rules corresponding to general logical principles characterizing the semantics), would allow the introduction of any formula inside an 'assumption context'. This can be, for example, a box, as in, e.g. Jáskowski's original natural deduction system (cf. Prawitz 1965, pp. 98-101). The first formula inside the box is the assumption (say φ) and any formula effectively developed inside below it (say ψ) can then be conditionalized to the assumption, discharging it (i.e., $\varphi \to \psi$ can be added outside the box). A whole *conditional box*, then, no matter how many branchings occur inside, occupies a single point in the outer tableau branch. The conditionalization rule (i.e. the introduction of $\to$, call it $\uparrow\!\to$) is obviously synthetic as well.

The analytic rule corresponding to $\to$ could then simply be modus ponens (call it MP). But we could also consider analysing implications into their corresponding structural resource (i.e. a conditional box), as is done for $\wedge$ and $\vee$. A rule $\downarrow\!\to$ would then introduce a box containing φ and ψ, in that order, from single premiss $\varphi \to \psi$. But then we would need as well another analytic structural rule corresponding to modus ponens, i.e. if there is a box containing a development of ψ from assumption φ and φ also occurs above outside the box, then ψ can be added below outside the box. This resembles the cut rule, and so we call it Cut. Clearly, MP is then a derived rule.

Call the calculus formed by DT* plus the above rules for $\to$ (i.e. PM, $\uparrow\!\to$ and either MP or $\downarrow\!\to$ and Cut) DT$^{\to}$. We can question now what restrictions or modifications are necessary in this calculus in order to characterize intuitionistic logic. First of all, rules PB*, $\downarrow\!\neg^{\neg}$ and $\downarrow\!\wedge^{\neg}$ should be dropped. The remaining rules are all sound, but insufficient. The main problem is the introduction of negation. Introducing $\bot$ to the language and its corresponding rules, as well as defining negation in terms of $\to$ and $\bot$ (so dropping $\neg$ as primitive connective), as we considered above, alleviates this problem. Alternatively, if we wish to keep $\neg$ and avoid $\bot$, we could define a rule $\uparrow\!\neg$ in the following way: if $\neg\varphi$ is developed in a conditional box from φ, then $\neg\varphi$ can be added outside the box. (But note that anyhow to $\neg$ and $\to$ correspond now the same structural resource.) In either case, all negative rules could be dropped. Also, proofs now have to be defined as developments from the empty set Γ of initial premisses, i.e. they will always begin with applications of PM.

Conjecture 3.8. *The following sets of rules are adequate for an intuitionistic deductive tableaux calculus:*

- $\downarrow\wedge$, $\downarrow\vee$, $\downarrow\rightarrow$, $\uparrow\wedge$, $\uparrow\vee$, $\uparrow\rightarrow$, *PM, Cut and* $ExF^{\perp}$*. (For* $\neg\varphi =_{df} \varphi \rightarrow \perp$*.)*
- $\downarrow\wedge$, $\downarrow\vee$, $\downarrow\rightarrow$, $\uparrow\wedge$, $\uparrow\vee$, $\uparrow\rightarrow$, $\uparrow\neg$, *PM, Cut and* ExF^{*}*. (For* $\neg$ *primitive.)*

In both, $\downarrow\rightarrow$ *and Cut might be replaced by MP. Also, adding* PB^{*} *to either gives us classical logic.*

But there is also another approach to dealing with $\rightarrow$ related to signed formulas and so closer to DT than to DT*. We can see the signed formulas as special cases of sequents, i.e. sequents in which at most one formula occurs. So $T\varphi$ can be seen as $\vdash \varphi$ and $F\varphi$ as $\varphi \vdash$. Allowing then more than one formula to occur and, in particular, any number of formulas to occur on the antecedent, gives us another way of dealing with $\rightarrow$. I.e. rules for analysing and synthesizing $\rightarrow$, could now be, respectively, something like

$$\frac{\Gamma \vdash \varphi \rightarrow \psi}{\Gamma, \varphi \vdash \psi} \quad \text{and} \quad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}.$$

I.e. we can treat assertion and rejection as limiting cases of implication. We would have, in this way, an analytico-synthetic sequent calculus in tableaux form. There are, however, many subtleties and possibilities in the formulation of such a calculus, and we only intended here to remark how with the use of signed formulas we are only one step away from committing to sequents and, so, to a possible generalization of the calculus to include $\rightarrow$ as primitive. Note that, again, negation becomes a special case of implication, but now even without the introduction of $\perp$ and the definition of negation, for $\neg$ and $\rightarrow$ will also share the same structural resource.

Note that in the first approach we are introducing a resource for implication in the outer structure of the calculus, while on the second approach, we complexify the internal structure of the formal object (i.e. go from formulas to signed formulas and then to sequents). But we could use more than one of these resources at the same time, in particular, we could use e.g. signed formulas and a structural resource of introducing assumptions and conditionalization.

For, notice, and with this we conclude these remarks, despite what was said above, it is possible to maintain the use of signed formulas in a formalization of an intuitionistic calculus. Fitting (1969) does precisely that in an intuitionistic analytic tableaux calculus. But now, instead of $T\varphi$ and $F\varphi$ being read as φ is true and φ is false, respectively, they should be read as φ is (already) proved and φ is (not yet) proved. Or, alternatively, we know that φ is the case and we don't know that φ is the case. On this reading, from the DT calculus, only the rules $\downarrow\neg^F$ and $\uparrow\neg^T$ (i.e. the only rules that go from a rejected premiss to an asserted conclusion) become incorrect for intuitionistic logic and should be dropped. Rule $\downarrow\wedge^F$ and even PB can be maintained. Dropping those two rules for negation makes it impossible to develop the excluded middle or to remove double negations. Obviously, however, other modifications are still necessary in order to get intuitionistic logic.

Remark. We will notice here a curious formulation of a classical propositional calculus that is, in some sense, equivalent to DT, but requires no extra symbols T and F and actually has less rules. For we will need only the affirmative unsigned rules for $\wedge$ and $\vee$ (i.e., the rules $\uparrow\wedge$, $\downarrow\wedge$, $\uparrow\vee$ and $\downarrow\vee$ from the above tables). Then, for negation we have the following peculiar rule: if formula φ occurs in a branch, we can adjoin φ^{flip} to the same branch, where φ^{flip} is φ with either one negation added or one negation removed (if it had any) and *flipped* upside down (i.e. either vertically mirrored or with each symbol rotated 180°), so that conjunctions become disjunctions and vice versa (we then also use – (a n-dash) for negation for symmetry). By avoiding using vertically symmetrical variables, we will always be able to tell if a formula is upright or not. Upright formulas are asserted while upside down formulas are rejected, but then treated as asserted formulas with the flipped (i.e. dualized) connectives. In this way we dispense with the negative or rejective rules. PB and ExF can be formulated in terms of formulas and their flipped versions, but it is probably easier to just use PB* and ExF* instead. This is obviously more the sort of thing a logician does to amuse himself than anything of theoretical importance. But often with amusement comes also a little insight, so that we found it appropriate to include this remark.

4 Deductive tableaux for classical predicate logic

We will now extend the deductive tableaux methods to classical predicate logic. Setting aside the analogies with the data calculi and ground theory, we will seek only to generalize the familiar analytic tableaux method to the analytico-synthetic context of deductions. I.e. we will provide synthetic rules corresponding to the already familiar analytic rules, as well as characterize the necessary new restrictions on developments. In this way, we will have an interesting new method of deduction for classical predicate logic based on tableaux, even though not completely in line with our earlier approaches.

4.1 The calculus QDT

Formulas now are defined in the usual way, concerning a language containing, besides the propositional connectives above, the quantifiers $\forall$ and $\exists$. Also, free and bound variables are going to be typographically distinguished, i.e. we will have a denumerable set of *individual variables* (x, y, z etc.), occurring only bounded, and a denumerable set of *individual parameters* (a, b, c etc.), occurring only free. But we will have no functional variables or identity sign.

The rules of QDT include counterparts of all rules of DT, only now φ, ψ etc. are formulas in the new sense. We then expect to have, as for the other constants, four rules for each quantifier, i.e., the analytic rules $\downarrow\forall^T$, $\downarrow\forall^F$, $\downarrow\exists^T$ and $\downarrow\exists^F$, and the synthetic rules $\uparrow\forall^T$, $\uparrow\forall^F$, $\uparrow\exists^T$ and $\uparrow\exists^F$. We will divide these rules into four categories: the rules of *Universal Generalization* (UG), rules of *Existential Generalization* (EG), rules of *Universal Instantiation* (UI) and rules of *Existential Instantiation* (EI).

We begin by writing down the very straightforward formulation of all these rules (table 16) and then we will describe the restrictions and other necessary adjustments to guarantee the correctness in their applications. Instantiation rules are analytic and generalization rules are synthetic.

UI:	$\downarrow\forall^T$	$\dfrac{T\forall x\varphi}{T\varphi\binom{x}{a}}$		$\downarrow\exists^F$	$\dfrac{F\exists x\varphi}{F\varphi\binom{x}{a}}$	
EI:	$\downarrow\forall^F$	$\dfrac{F\forall x\varphi}{F\varphi\binom{x}{a}}$		$\downarrow\exists^T$	$\dfrac{T\exists x\varphi}{T\varphi\binom{x}{a}}$	
UG:	$\uparrow\forall^T$	$\dfrac{T\varphi}{T\forall x\varphi\binom{a}{x}}$		$\uparrow\exists^F$	$\dfrac{F\varphi}{F\exists x\varphi\binom{a}{x}}$	
EG:	$\uparrow\forall^F$	$\dfrac{F\varphi(\ldots a\ldots a\ldots)}{F\forall x\varphi(\ldots x\ldots a\ldots)}$		$\uparrow\exists^T$	$\dfrac{T\varphi(\ldots a\ldots a\ldots)}{T\exists x\varphi(\ldots x\ldots a\ldots)}$	

Table 16: QDT rules for the quantifiers

The expression $\varphi\binom{x}{a}$ means the formula obtained from φ by replacement of every free occurrence of variable x in φ by parameter a. Likewise, the expression $\varphi\binom{a}{x}$ means the formula obtained from φ by replacement of every occurrence of some parameter a by x. In the EG rules, $\varphi(\ldots a\ldots a\ldots)$ means a formula φ with some (zero or more) occurrences of parameter a, not all of which must be replaced by variable x in the conclusion (indicated by $\varphi(\ldots x\ldots a\ldots)$).

Of all these rules, only the UI and EG rules are really sound as stated. In these rules, a can be any parameter. The remaining rules (i.e. the UG and EI rules) need some restrictions and other considerations to work, which we will describe now.

First of all, we will have to introduce the following procedure in the construction of developments: all the initial signed formulas in a development will have to be marked somehow, in order to distinguish them as *assumptions* from other formulas occurring in the development. We will do this by putting a $\mathfrak{p}$ sign before them. Then, the 'conclusions' of all branching rules (so $\downarrow\vee^T$, $\downarrow\wedge^F$ and PB) and the 'conclusion' of the EI rules have to be marked as assumptions as well.

Next, after the application of any of the rules EI or UG the parameter a has to be *flagged*, i.e. it has to be written somewhere outside the tableau (usually we will write the parameter next to the name of the rule off to the right of the line infered by it). The first restriction common to both EI and UG rules then is: no parameter can be flagged twice (i.e. none of these rules is permissible if a has previously been flagged). The EI rules have the additional restriction that the parameter introduced has to be new to the development. Also, no flagged parameter can occur in the conclusion of a development.

Finally, the UG rules can only be applied provided a does not occur in any assumption (i.e. signed formulas marked with a $\mathfrak{p}$ sign) upon which the signed formulas $T\varphi$ or $F\varphi$ depends. This means these rules can be applied either locally, on a single branch, provided a does not occur in any marked formula on that branch, or *globally*, i.e. to several branches at once, provided the signed formula to be generalized occurs in all branches directly under assumptions with no occurrence of a.

The definition of development for QDT, then, is: a *development* of a finite set of signed formulas Ξ from a finite set of signed formulas Σ is defined as any tableau for Σ with at least one $\xi \in \Xi$ occurring in each branch and with no parameter in Ξ flagged. The definitions of deduction, proof and refutation, then, are all as before.

To make all these things clearer, we will provide now a couple of simple examples of developments and comment on them.

1. $\underline{\forall x Px \vee \forall x Qx \vdash_{QDT} \forall x(Px \vee Qx)}$

1.	$\mathfrak{p}T\forall xPx \vee \forall xQx$		
2.	$\mathfrak{p}T\forall xPx$	$\mathfrak{p}T\forall xQx$	$\downarrow\vee^T$
3.	TPa	TQb	$\downarrow\forall^T$
4.	$TPa \vee Qa$	$TPb \vee Qb$	$\uparrow\vee^T$
5.	$T\forall x(Px \vee Qx)$	$T\forall x(Px \vee Qx)$	$\uparrow\forall^T\ a$; $\uparrow\forall^T\ b$

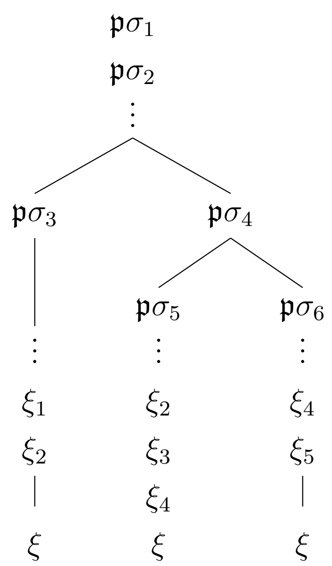
2. $\underline{\forall x(Px \to Qx), \forall x(Qx \to Rx) \vdash_{QDT} \forall x(Px \to Rx)}$

1.	$\mathfrak{p}T\forall x(Px \to Qx)$			
2.	$\mathfrak{p}T\forall x(Qx \to Rx)$			
3.	$TPa \to Qa$			$\downarrow\forall^T$
4.	$TQa \to Ra$			$\downarrow\forall^T$
5.	$\mathfrak{p}FPa$	$\mathfrak{p}TQa$		$\downarrow\to^T$
6.		$\mathfrak{p}FQa$	$\mathfrak{p}TRa$	$\downarrow\to^T$
7.		$TPa \to Ra$	$TPa \to Ra$	ExF; $\uparrow\to^T$
8.	$TPa \to Ra$			$\uparrow\to^T$
9.	$T\forall x(Px \to Rx)$			
10.		$T\forall x(Px \to Rx)$	$T\forall x(Px \to Rx)$	$\uparrow\forall^T\ a$

In the first of these examples we applied the rule $\uparrow\forall^T$ locally, in each branch separately, for the parameters introduced in each don't occur in any of the two marked formulas above $TPa \vee Qa$ or $TPb \vee Qb$. But note we could have instantiated both formulas in line 2 with the same parameter a. Then in line 4 both branches would have $TPa \vee Qa$, which would be a development depending only on the signed formula in line 1. (I.e. when a formula is developed, local assumptions are 'discharged'.) In this case, the rule $\uparrow\forall^T$ could also be applied globally, to both branches at once.

Now, in the second example, the same rule $\uparrow\forall^T$ can only be applied globally, to every branch at once, after developing $TPa \to Ra$. For, above each occurrence of $TPa \to Ra$ there are assumptions in which the parameter a occurs, forbidding the local application of the rule. But the only assumptions common to all these occurrences are the ones in lines 1 and 2, in which a does not occur, so the global application of the rule is allowed.

To make this procedure of 'discharging assumptions' involved in the restriction of the UG rules more clear, consider the following development scheme:



Here, ξ depends only on assumptions σ_1 and σ_2 and we say, in fact, that ξ was *developed* from assumptions σ_1 and σ_2, because it occurs in all branches below these assumptions and no other assumption occurs above common to all these occurrences of ξ. The other premisses occurring above each occurrence of ξ are then considered discharged. Now ξ_1 depends also on the local (i.e. particular to its branch) assumption σ_3, and is a development from σ_1, σ_2 and σ_3, but not from σ_1 and σ_2 alone. ξ_3 then depends also on assumptions σ_4 and σ_5, and ξ_5 on assumptions σ_4 and σ_6. Now, ξ_4 occurs in all branches below σ_4, discharging then assumptions σ_5 and σ_6 (i.e. ξ_4 is a development from σ_1, σ_2 and σ_4). Finally, even though ξ_2 occurs in more than one branch and the only assumptions common to both these branches are σ_1 and σ_2, it is not a development from these assumptions alone, for it does not occur in all branches below these assumptions. Rather, each occurrence of ξ_2 depends on a different set of undischarged assumptions. Also, an assumption cannot be discharged unless all assumptions occurring below are discharged. In this case, then, σ_4 could not be discharged before σ_6.

There is one additional consideration regarding the discharging of assumptions introduced by EI rules. This assumption is discharged for any signed formula developed below it in which the parameter flagged in the application of the rule does not occur.

4.2 Examples of developments

$\forall x Px \wedge \forall x Qx \vdash_{QDT} \forall x(Px \wedge Qx)$

1.	$\mathfrak{p}T\forall x Px \wedge \forall x Qx$	
2.	$T\forall x Px$	$\downarrow\wedge^T$
3.	$T\forall x Qx$	$\downarrow\wedge^T$
4.	TPa	$\downarrow\forall^T$
5.	TQa	$\downarrow\forall^T$
6.	$TPa \wedge Qa$	$\uparrow\wedge^T$
7.	$T\forall x(Px \wedge Qx)$	$\uparrow\forall^T$

$\vdash_{QDT} \forall y(\forall x Px \to Py)$

1.	$\mathfrak{p}F\forall y(\forall x Px \to Py)$		
2.	$\mathfrak{p}T\forall x Px$	$\mathfrak{p}F\forall x Px$	PB
3.	TPa	|	$\downarrow\forall^T$
4.	$T\forall x Px \to Pa$	$T\forall x Px \to Pb$	$\uparrow\to^T$
5.	$T\forall y(\forall x Px \to Py)$	$T\forall y(\forall x Px \to Py)$	$\uparrow\forall^T\ a$; $\uparrow\forall^T\ b$

4.3 Soundness

In the next section, we will produce some arguments which guarantee the completeness of QDT. On the other hand, the soundness of this calculus is a main open question. In particular, we would have to show that the restrictions imposed on the EI and UG rules are the correct ones, given that the remaining rules are in fact sound. In this section we will then try, first, to motivate the given formulation of these rules. Afterwards, we will provide a proof scheme of soundness for QDT, which depends on the still unproved conjecture 4.1.

First, let us show why it is not possible to deduce $\forall x\varphi(x)$ from $\exists x\varphi(x)$. A tableau beginning with $T\exists x\varphi(x)$ would then be as follows:

1. $\mathfrak{p}T\exists x\varphi(x)$
2. $\quad\mathfrak{p}T\varphi(a) \qquad \downarrow\exists^T\ a$

Now, it is not possible to apply $\uparrow\forall^T$ for two different reasons. First, parameter a occurs in an assumption. Second, parameter a is also flagged. To see why we need both restrictions on UG rules, let us see another example of an invalid deduction to which an attempt to construct a development fails. If we tried to deduce $\exists y\forall xPxy$ from $\forall x\exists yPxy$ we would then have:

1. $\mathfrak{p}T\forall x\exists yPxy$
2. $\quad T\exists yPay \qquad \downarrow\forall^T$
3. $\quad\mathfrak{p}TPab \qquad \downarrow\exists^T\ b$

Now, even though parameter a is not flagged, it still occurs in an assumption, so, again, the application of $\uparrow\forall^T$ is blocked. But we have to notice that any assumption introduced by an EI rule is 'discharged' when any formula without the parameter introduced is infered from this assumption. So, if to line 3 we applied $\uparrow\exists^T$, making b disappear, this infered formula would be a development from 1 only, in which a does not occur, allowing the application of $\uparrow\forall^T$. (i.e. we can 'go back' to the premiss).

Conjecture 4.1. *Any interpretation that verifies all the initial assumptions of a development plus every assumption introduced by EI rules will verify also at least one branch.*

Notice that all the propositional rules preserve truth in a tableau (lemma 3.1). This is also the case for UI and EG rules, as they are sound. The EI rules, in turn, introduce assumptions that are considered true under the conjecture's hypothesis. Notice it is possible for an interpretation to verify simultaneously all such assumptions because of the requirement that the parameter introduced by the rule should be new to the development. The only case left to consider in a proof of this conjecture, then, is the application of UG rules. We could argue then in the following way: suppose an interpretation ι that verifies all the initial and EI assumptions. Then ι verifies also $\varphi(a)$ (i.e. the premiss of $\uparrow\forall^T$; another argument analogous to this one works for $\uparrow\exists^F$). By the conditions imposed on the rule, this parameter a is not flagged nor does it occur, in particular, in any initial assumption. This means in fact that a is really arbitrary, i.e. it could have been any other parameter. So ι actually verifies $\varphi(a)$ for any a, which means, in turn, that ι verifies also $\forall x\varphi(x)$. However, since we still lack a more rigorous treatment of this argument, we prefer to leave it here as a motivated conjecture, on which the following theorem depends.

Theorem 4.2. *If $\Gamma \vdash_{QDT} \varphi$, then $\Gamma \vDash \varphi$ (depending on the truth of conjecture 4.1).*

Proof. Assume $\Gamma \vdash_{QDT} \varphi$. This means there exists a development beginning with Γ^T and with $T\varphi$ in every branch. Since it is a development, no flagged parameter occurs in φ or in Γ. So, any assumptions introduced by EI rules have been discharged for $T\varphi$. Any interpretation ι for the language without any of the flagged parameters that verifies Γ can be extended to an interpretation ι' that verifies also all the assumptions introduced by EI rules. Now, if conjecture 4.1 holds, this interpretation will verify at least one branch. Since ι and ι' match in everything unrelated to the flagged parameters, we have that $\Gamma \vDash \varphi$. □

4.4 Completeness

We will prove now that QDT is complete for proofs by showing we can prove the axioms and simulate the rules of a known axiomatic system for classical predicate logic (see e.g. Hilbert & Ackermann 1928). We will call this system HQ and assume its completeness. It has as its axioms every tautology plus every instance of the two schemes $\forall x\varphi(x) \to \varphi(a)$ and $\varphi(a) \to \exists x\varphi(x)$, where a is any parameter free for x in φ. The rules of HQ are *modus ponens* plus the two rules

$$\frac{\psi \to \varphi(a)}{\psi \to \forall x\varphi(x)} \qquad \text{and} \qquad \frac{\varphi(a) \to \psi}{\exists x\varphi(x) \to \psi}$$

where a does not occur in ψ.

We know already that the DT rules (included in QDT) can prove any tautology (theorem 3.4 above). We have to show now we can prove any instance of the axiom schemes (lemma 4.3 below). Next, we will show we can simulate modus ponens (lemma 4.4) as well as the rules above (lemma 4.5).

Lemma 4.3. *For any formula φ, we have (where a is any parameter free for x in φ):*

1. $\vdash_{QDT} \forall x\varphi(x) \to \varphi(a)$
2. $\vdash_{QDT} \varphi(a) \to \exists x\varphi(x)$

Proof. We have the following two proof schemes:

1. $\vdash_{QDT} \forall x\varphi(x) \to \varphi(a)$

1.		$\mathfrak{p}F\forall x\varphi(x) \to \varphi(a)$		
2.	$\mathfrak{p}T\forall x\varphi(x)$		$\mathfrak{p}F\forall x\varphi(x)$	PB
3.	$\mathfrak{p}T\varphi(a)$	$\mathfrak{p}F\varphi(a)$		PB
4.		$F\forall x\varphi(x)$		$\uparrow\forall^F$
5.	$T\forall x\varphi(x) \to \varphi(a)$	$T\forall x\varphi(x) \to \varphi(a)$		$\uparrow\to^T$
6.			$T\forall x\varphi(x) \to \varphi(a)$	$\uparrow\to^T$

2. $\vdash_{QDT} \varphi(a) \to \exists x\varphi(x)$

1.		$\mathfrak{p}F\varphi(t) \to \exists x\varphi(x)$		
2.	$\mathfrak{p}T\exists x\varphi(x)$	$\mathfrak{p}F\exists x\varphi(x)$		PB
3.		$\mathfrak{p}T\varphi(a)$	$\mathfrak{p}F\varphi(a)$	PB
4.		$T\exists\varphi(x)$		$\uparrow\exists^T$
5.		$T\varphi(a) \to \exists x\varphi(x)$	$T\varphi(a) \to \exists x\varphi(x)$	$\uparrow\to^T$
6.	$T\varphi(a) \to \exists x\varphi(x)$			$\uparrow\to^T$

□

Lemma 4.4. *If* $\vdash_{QDT} \varphi$ *and* $\vdash_{QDT} \varphi \to \psi$*, then* $\vdash_{QDT} \psi$*.*

Proof. We begin a tableau with $F\psi$. We apply PB to branch with $T\varphi$ on the left and $F\varphi$ on the right. In the right branch we can then reproduce the proof $\vdash_{QDT} \varphi$, developing $T\varphi$, and then introduce $T\psi$ by ExF. In the left branch, since we have now $T\varphi$ and $F\psi$, we use $\uparrow\to^F$ to develop $F\varphi \to \psi$. We can now reproduce the proof $\vdash_{QDT} \varphi \to \psi$ in this branch, developing $T\varphi \to \psi$. So, by ExF, we can adjoin $T\psi$ to this branch as well, concluding the proof. Below we show also a tableau scheme of this argument.

1.	$\mathfrak{p}F\psi$		
2.	$\mathfrak{p}T\varphi$	$\mathfrak{p}F\varphi$	PB
3.		$\vdots$	
4.		$T\varphi$	By hypothesis
5.	$F\varphi \to \psi$		$\uparrow\!\to^F$
6.	$\vdots$		
7.	$T\varphi \to \psi$		By hypothesis
8.	$T\psi$	$T\psi$	ExF

∎

Lemma 4.5. *For formulas φ and ψ and a not in ψ:*

1. *If $\vdash_{QDT} \psi \to \varphi(a)$, then $\vdash_{QDT} \psi \to \forall x\varphi(x)$.*
2. *If $\vdash_{QDT} \varphi(a) \to \psi$, then $\vdash_{QDT} \exists x\varphi(x) \to \psi$.*

Proof. 1. Assume $\vdash_{QDT} \psi \to \varphi(a)$. Now, from $F\psi \to \forall x\varphi(x)$, we have the following scheme of development of $T\varphi(a)$:

1.	$\mathfrak{p}F\psi \to \forall x\varphi(x)$			
2.	$\mathfrak{p}T\varphi(a)$	$\mathfrak{p}F\varphi(a)$		PB
3.		$\mathfrak{p}T\psi \to \varphi(a)$	$\mathfrak{p}F\psi \to \varphi(a)$	PB
4.	$\mathfrak{p}T\psi$	$\mathfrak{p}F\psi$		PB
5.	$F\psi \to \varphi(a)$	$T\psi \to \forall x\varphi(x)$		$\uparrow\!\to^F$; $\uparrow\!\to^T$
6.	$T\varphi(a)$	$T\varphi(a)$		ExF
7.			$\vdots$	
8.			$T\psi \to \varphi(a)$	From assumption
9.			$T\varphi(a)$	ExF

Now we can apply $\uparrow\forall^T$ to adjoin $T\forall x\varphi(x)$ to every branch, since $T\varphi(a)$ is a development from the formula in line 1, where a does not occur. Then, by $\uparrow\rightarrow^T$ we develop $T\psi \rightarrow \forall x\varphi(x)$, concluding the proof.

2. Assume $\vdash_{QDT} \varphi(a) \rightarrow \psi$. Now, from $F\exists x\varphi(x) \rightarrow \psi$, we have the following scheme of development of $F\varphi(a)$:

1.		$\mathfrak{p}F\exists x\varphi(x) \rightarrow \psi$			
2.		$\mathfrak{p}T\varphi(a)$		$\mathfrak{p}F\varphi(a)$	PB
3.	$\mathfrak{p}T\varphi(a) \rightarrow \psi$		$\mathfrak{p}F\varphi(a) \rightarrow \psi$		PB
4.	$\mathfrak{p}T\psi$	$\mathfrak{p}F\psi$	$\vert$		PB
5.	$T\exists x\varphi(x) \rightarrow \psi$	$F\varphi(a) \rightarrow \psi$	$\vert$		$\uparrow\rightarrow^T$; $\uparrow\rightarrow^F$
6.	$F\varphi(a)$	$F\varphi(a)$	$\vert$		ExF
7.			$\vdots$		
8.			$T\varphi(a) \rightarrow \psi$		From assumption
9.			$F\varphi(a)$		ExF

Now we can apply $\uparrow\exists^F$ to adjoin $F\exists x\varphi(x)$ to every branch, again, since a does not occur in line 1. Then, by $\uparrow\rightarrow^T$ we develop $T\exists x\varphi(x) \rightarrow \psi$, concluding the proof.

□

Theorem 4.6. *If* $\vDash \varphi$*, then* $\vdash_{QDT} \varphi$*.*

Proof. Assume the hypothesis. Then, since HQ is complete, $\vdash_{HQ} \varphi$. By theorem 3.4 of previous section, and the lemmas 4.3, 4.4 and 4.5, we have that QDT can simulate any proof of HQ, and so $\vdash_{QDT} \varphi$. □

By the above proof we have actually that for any valid φ there exists a proof in QDT that is synthetic except for applications of ExF.

Assume now $\Gamma = \{\psi_1, \psi_2, \ldots, \psi_n\}$ for some n.

Theorem 4.7. *If* $\Gamma \vDash \varphi$, *then* $\Gamma \vdash_{QDT} \varphi$.

Proof. Assume the hypothesis. Then $\vDash \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. By the previous theorem we then have that $\vdash_{QDT} \neg\psi_1 \vee \neg\psi_2 \vee \ldots \vee \neg\psi_n \vee \varphi$. Now, by an argument exactly analogous to lemma 3.5 of previous section, we have that $\Gamma \vdash_{QDT} \varphi$. □

Alternatively, to prove the deduction completeness of QDT, instead of assuming the completeness of HQ, we could have only assumed the refutation completeness of QDT↓ (i.e. QDT with only analytic rules, which corresponds to the familiar analytic tableaux). We begin then a tableau with Γ^T and apply PB, branching with $T\varphi$ on the left and $F\varphi$ on the right. In the right branch we then do the usual analytical refutation of $F\varphi$, developing finally by ExF $T\varphi$.

It is still an open question whether QDT is complete for normal deductions, or even at least some other weaker notion.

References

Batchelor, Roderick (2010). 'Grounds and Consequences'. Grazer Philosophische Studien, vol. 80, pp. 65-77.

____________ (2019). Directional Deduction, unpublished manuscript.

Carnap, Rudolf (1943). Formalization of Logic. Cambridge, Mass.: Harvard University Press.

Fitting, Melvin (1969). Intuitionistic Logic, Model Theory and Forcing. Amsterdam: North-Holland Pub. Co..

Gentzen, Gerhard (1969). The Collected Papers of Gerhard Gentzen. Amsterdam: North-Holland Pub. Co..

Heyting, Arend (1931). 'Die intuitionistische Grundlegung der Mathematik'. Erkenntnis 2 (1), pp. 106-115.

Hilbert, David & Ackermann, W. (1928). Grundzüge der Theoretischen Logik. Springer Verlag.

Jeffrey, Richard (1981). Formal logic: Its scope and limits. New York: McGraw Hill.

Kneale, William (1956). 'The province of logic', In: H. D. Lewis (ed.), Contemporary British Philosophy, Third Series, London, Allen & Unwin, pp. 235-61.

Kneale, W. C.; Kneale, M. (1962). The Development of Logic. Oxford University Press.

von Plato, Jan (2014). 'From Axiomatic Logic to Natural Deduction'. Studia Logica 102 (6):1167-1184.

Prawitz, Dag (1965). Natural Deduction: A Proof-Theoretical Study. Dover

Publications.

Rumberg, Antje (2013). ‘Bolzano’s concept of grounding against the background of normal proofs’. Review of Symbolic Logic 6 (3):424-459.

Rumfitt, I. (2000). “‘Yes” and “no”’. Mind 109 (436), pp. 781-823.

Shoesmith, D. J.; Smiley, T. (1978). Multiple-Conclusion Logic. Cambridge University Press.

Smiley, Timothy (1996). ‘Rejection’. Analysis 56 (1):1–9.

Smullyan, Raymond M. (1968). First-Order Logic. Dover (repr. 1995).