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**Deflection of mesoscopic atomic superpositions via  
superradiance and superabsorption**

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**Deflection of mesoscopic atomic superpositions via  
superradiance and superabsorption**

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Advisor: Prof. Dr. Miled Hassan Youssef Moussa

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*“Memento Mori”*  
*- Old Roman Saying*



## ABSTRACT

Rocha, L. M. R. **Deflection of mesoscopic atomic superpositions via superradiance and superabsorption.** 2023. 65p. Dissertation (Master in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2023.

In this work we will deal with the effects of a coherent electromagnetic field acting on an atomic dipole gas confined to a cavity. It is known from (5) that in such conditions, if we have a moderately dense sample of  $N$  2-level atoms, we can observe the presence of superradiance, i.e., the enhanced emission of a coherent pulse of intensity proportional to  $N^2$  and, in a smaller scale, the enhanced absorption as well (14). When coupling this sample to a quantum electromagnetic field via Tavis-Cummings interaction we are able to enhance even more the superabsorption as well as the superemission to a lesser degree in such a way as to almost equate the intensities of both processes and give rise to interleaved oscillations between them in both field and sample resulting in the phenomenon of many-body Rabi oscillations described in (1). However, it is to expect that, due to momentum conservation, such a phenomenon would cause a deflection on the atoms of the sample in a small time interval, after which we would have a mesoscopic superposition containing all the information related to the prior state. Our main goal is the study of the deflection suffered by the atoms of such a sample and how it relates to the state before the emission. We solve the time-dependent non-linear Hamiltonian related to the superradiant emission/superabsorption via the Lewis-Riesenfeld invariant method. At the end we will show how the macroscopic states were prepared to obtain our results.

**Keywords:** Superradiance. Superabsorption. Deflection. Tomography.



## RESUMO

Rocha, L. M. R. **Deflexão de superposições atômicas mesoscópicas via superradiância e superabsorção**. 2023. 65p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2023.

Neste trabalho lidaremos com os efeitos de um campo eletromagnético agindo sobre um gás de dipolos atômicos confinado a uma cavidade. Sabe-se por (5) que sob tais condições, se tivermos uma amostra moderadamente densa de  $N$  átomos de dois níveis, podemos observar a presença do fenômeno da super-radiância, i.e., a emissão amplificada de um pulso coerente com intensidade proporcional a  $N^2$  bem como, em menor escala, uma absorção amplificada (14). Ao acoplar esta amostra a um campo eletromagnético através da interação de Tavis-Cummings somos capazes de aumentar ainda mais a intensidade da super-absorção assim como a da super-emissão em menor grau, de tal forma que estas serão praticamente igualadas e darão origem a oscilações que irão intercalar cada processo em ambos campo e amostra resultando no fenômeno de oscilações de Rabi de muitos corpos descritas em (1). No entanto, é esperado que devido a conservação de momento tal fenômeno cause uma deflexão nos átomos da amostra em um pequeno intervalo de tempo, depois do que teríamos uma superposição mesoscópica contendo toda informação do estado anterior. Nosso principal objetivo é o estudo da deflexão sofrida pelos átomos dessa amostra e a relação dessa com o estado antes da emissão. Nós resolveremos o problema da Hamiltoniana não-linear dependente do tempo através do método de invariantes de Lewis-Riesenfeld. E por fim mostraremos como os estados macroscópicos foram preparados para obter os resultados.

**Palavras-chave:** Superradiância. Superabsorção. Deflexão. Tomografia.



## LIST OF FIGURES

- Figure 1 – Atomic sample immersed in cavity with field  $\vec{E}$ . It can be seen that due to the large size of the field’s wavelength in relation to the sample’s size all of its sites are experiencing the same approximate field strength which in turn leads to correlations between them. Source: By the author 20
- Figure 2 – Intensity of superradiant emission vs time for  $N = 10^3, \omega_0 = 10^5, \gamma = 1$ . We can see the rapid ascend near  $t = t_0 - \frac{\tau_c}{2}$  while the function peaks at  $t_0$  (where we have  $\theta = \pi/2$ ) and rapidly decreases as the sample emits and the dipoles assume random disordered positions. Source: By the author . . . . . 34
- Figure 3 – Population of state  $|+, t\rangle$  in time beginning at  $t_0$ . The state is at its maximum population at  $t = t_0$  where the entropy is minimum and quickly decays due to random fluctuations of the EM field into a more disordered state. Source: By the author . . . . . 35
- Figure 4 – Scheme of the trapping apparatus (cavity) with a stationary electric field  $E = E_0 \sin(kx)$  (in red). In the middle, it is possible to see the sample trapped in the node of the field and interacting with it. Since the size of the sample is much smaller than the wavelength of the field, but not zero, we are generally able to approximate  $\sin kx \approx kx$ , as we will see later. Source: By the author . . . . . 36
- Figure 5 – Plotted intensity of Atom (blue) and field (orange) over parameterized time. We can see that the interaction between atom and field gives rise to cycle of superemission-superabsorption by the atom and a reciprocal cycle of superabsorption-superemission by the field. Here we have parameterized time with respect to  $g = 1$  and all the system variables are equal to zero at  $t = 0$  except for  $\langle S_z(0) \rangle = 0.5$  and the values of the parameters used here were  $N = 10^3, \omega_0 = 10^2, \gamma = 10^{-3}, g = 1$ . Source: By the author . . . . . 42
- Figure 6 – Population of the states in time for the same values of the parameters and initial conditions used in 5. It is possible to see the gradual decline in the total energy ( $\langle \sigma_+ \sigma_- \rangle + \langle a^\dagger a \rangle$ ). Source: By the author . . . . . 43

Figure 7 – Intensity plotted for the values of $N = 10^4$ , $\omega_0 = 10^2$ , $\gamma = 10^{-3}$ . It is possible to observe the quick dissipation compared to 5. Here the last term in the Hamiltonian 2.107, which is the dissipative term, starts becoming significant in relation to both $\omega_0$ and $\sqrt{N}g$ and we can notice both the augmented damping effect as well as the raising in the frequency of the Rabi oscillations due to the larger number of atoms. Source: By the author . . . . .	44
Figure 8 – Intensity plotted for the values of $N = 10^3$ , $\omega_0 = 10^2$ , $\gamma = 10^{-2}$ . Very similar to 5 in frequency and amplitude during the first oscillations. Source: By the author . . . . .	44
Figure 9 – Population of the excited state in parameterized time for the same values of the parameters used before. The blue line represents the ordinary superradiant decay ( $g = 0$ ) described in section 2.1.4 while the green line describes the population of the field-coupled sample. It is possible to see that the transition time of the latest is much smaller in comparison (here we use $\tau_c = 2 \times 10^{-3}/\gamma$ ). Source: By the author . . . . .	45
Figure 10 – Population/energy of the excited state for samples with different numbers of atoms and the same value of $\gamma = 10^{-3}$ . As mentioned before and seen in equation 2.123 the larger the number of atoms in the system the more intense is the emission and the faster the energy will dissipate. Source: By the author . . . . .	45
Figure 11 – Illustration of optical Stern-Gerlach experiment found in reference (15). After the superposition state is prepared and the sample is accelerated with the use of a classical field. Once entering the cavity, the field-atom interaction splits the beam symmetrically in two directions on the $x$ axis. The atom impact distributions are represented by the two bumps in the screen. Source: By the author . . . . .	50
Figure 12 – Illustration of the experiment suggested to detect the deflection caused by superradiant emission. First the potential traps the sample positioned in the anti-node, then the potential's concavity is altered creating a moderately dense sample that enters the excited state. In the last part the trapping potential is turned off and the sample is left to fall under gravity or accelerated by a classical EM field leaving the cavity at around $t = t_0$ . As the sample falls the different states acquire different momenta leading to a split trajectory. Source: By the author . . . . .	57



Figure 13 – Plot of the intensities $\mathcal{I}_A$ and $\mathcal{I}_F$ against $\sqrt{N}gt$ , for $N = 10^6, \omega_0 = 10^5g, \theta_0 = 2/N$ and $\phi_0 = 0$ . To the left (a) we have the intensities in the overdamped regime with $\gamma = 10^{-2}g$ with only one SR-SA cycle, whilst to the right (b) we have the intensities in the damped regime with $\gamma = 10^{-3}g$ and two SR-SA cycles before the energy is completely dissipated. Source: By the author . . . . .	59
Figure 14 – Plot of the numerical/approximate solutions for $\theta(t)$ . To the left (a) we have the solutions for the over-damped regime while to the right (b) we have them for the damped regime. Source: By the author . . . . .	60
Figure 15 – Plotting of the momentum transfer for the over-damped (a) and damped (b) regimes. The momentum transfer is proportional to $\kappa$ which tells us that the greatest momentum transfer occurs at $t = t_0$ , after the transfer the sample is split into two beams. Source: By the author . . . . .	60



# CONTENTS

<b>1</b>	<b>INTRODUCTION . . . . .</b>	<b>19</b>
<b>1.1</b>	<b>Defining the phenomenons of superradiance and superabsorption . .</b>	<b>19</b>
<b>1.2</b>	<b>Deflection of the atoms and field . . . . .</b>	<b>20</b>
<b>2</b>	<b>SUPERRADIANCE AND SUPERABSORPTION . . . . .</b>	<b>23</b>
<b>2.1</b>	<b>Coherent pulse emission by an atomic sample . . . . .</b>	<b>23</b>
2.1.1	Liouville equation for an ideal gas interacting with a thermal bath . . . . .	23
2.1.2	The mean field approach and the "dressed" atom . . . . .	27
2.1.3	Solving the non-linear Hamiltonian via Lewis-Riesenfeld . . . . .	30
2.1.4	Energy and intensity . . . . .	34
<b>2.2</b>	<b>Superradiance-superabsorption cycle of the atom-cavity coupling system . . . . .</b>	<b>35</b>
2.2.1	The Tavis-Cummings interaction . . . . .	35
2.2.2	The mean field approximation . . . . .	37
2.2.3	Evolution of the atom-field Liouville equations . . . . .	39
2.2.4	Intensity and many-body Rabi oscillations . . . . .	41
<b>3</b>	<b>SAMPLE DEFLECTION AND STERN-GERLACH EFFECT . . . . .</b>	<b>47</b>
<b>3.1</b>	<b>Deflection of the single-atom state . . . . .</b>	<b>47</b>
<b>3.2</b>	<b>The deflection of the atom-field system . . . . .</b>	<b>50</b>
3.2.1	The solution states for the atom-field interplay . . . . .	50
3.2.2	The approximate solution for the atom and field variables. . . . .	54
3.2.3	Atom-field deflection and the splitting of the sample's trajectory via Stern- Gerlach effect . . . . .	56
<b>4</b>	<b>CONCLUSIONS . . . . .</b>	<b>61</b>
	<b>REFERENCES . . . . .</b>	<b>63</b>



## 1 INTRODUCTION

### 1.1 Defining the phenomenons of superradiance and superabsorption

Superradiance is a phenomenon proposed by Dicke in 1954 (5) in which we have a sample compounded of  $N$  two-level systems interacting with a coherent (monochromatic) radiant field. Consider that our sample is an ensemble of 2-level atoms with a random distribution of dipole moment alignments in a certain configuration (for example, a gas) with an average distancing of  $d$  between them that is interacting with a thermal reservoir. Say the system is prepared in such a way that the atoms are in the excited state, due to fluctuations generated by the collective electromagnetic field in the cavity some atoms will, after a small period of time, start to decay.

Dicke showed that, if the radiation field has a wavelength  $\lambda$  much bigger than the size of the atomic sample considered (or at least is such that  $d \ll \lambda$ ) such that the atoms that compose the sample will effectively be interacting with the same field, case illustrated in fig. 1, correlations will start to be formed between dipoles of different atoms initiating a ripple effect that will gradually revert the system back into its ground state in a cascade of emissions, generating a rapidly decaying coherent pulse of intensity proportional to  $N^2$ , in contrast to the usual incoherent emission of  $N$  uncorrelated atoms that is proportional to  $N$ , and has duration of  $\tau_c \propto 1/N$  that is much smaller than the natural emission time  $T_1$  of the system or the relaxation time  $T_2$  of each independent dipole (atom) emitting in isolation, that by definition are independent of the number of atoms present in the sample. In other words, the presence of other atoms interfere with the decaying time of each atom reducing them on average. The strong pulse generated by the sample is what we call a superradiant pulse.

In addition, due to the time reversal symmetry of the system it is expected that such a phenomenon will have a reciprocal process associated, i.e. an enhanced process of absorption, naturally named "superabsorption", occurring simultaneously with it in the cavity (14). However, in general the probability of an atom absorbing a photon is lower than that of the emitting process (which can easily be seen when the sample itself is found in the vacuum, however it is true even for an intense radiation field), therefore such a process would normally be negligible when compared to the superradiant emission in regular conditions (i.e., coupled with a thermal reservoir), but it can be shown that under certain conditions (namely when coupling the atomic sample with an electromagnetic field in a high  $Q$ -factor cavity) the superabsorption process can be brought to the same level of relevance of the emission process as done in (1, 14).

Still following (1), in order to neglect the effects of the Coulomb interactions be-

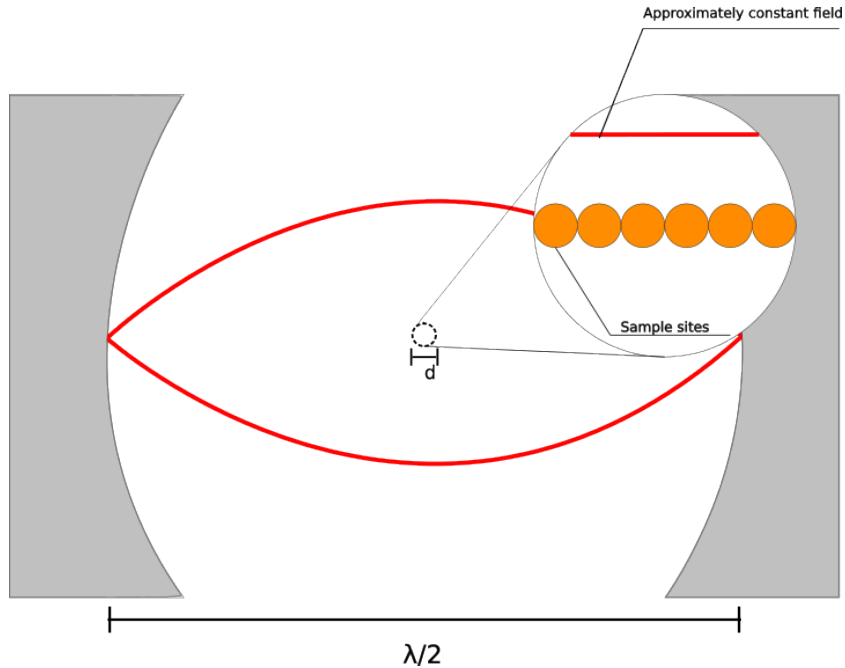


Figure 1 – Atomic sample immersed in cavity with field  $\vec{E}$ . It can be seen that due to the large size of the field's wavelength in relation to the sample's size all of its sites are experiencing the same approximate field strength which in turn leads to correlations between them.

Source: By the author

tween the atoms we will also consider the average distance between them to be sufficiently large so that they are effectively isolated from one another while we still consider them to be relatively close to each other as stated previously, conditions that constitute the so called moderately dense sample regime. Then, as done in (1, 16), we use the mean-field approach in the system's Hamiltonian which leads to a system of differential equations for a "dressed" representative atom (or superatom), and solve the resulting Schrödinger equations via the Lewis-Riesenfeld method as done in (17, 18).

## 1.2 Deflection of the atoms and field

The optical Stern-Gerlach effect is a phenomenon in which a particle beam, composed of two-level on or off-resonant systems in superposition, has its trajectory split in two directions due to the presence of a quantized electromagnetic field that interacts differently with each state and causes a deflection on their paths. As we will see, the photon statistics of the field will directly manifest itself in the momentum distribution of the sample and for this the Stern-Gerlach effect have been applied in studies such as the tomography of the field (15) and atomic lithography (19). With in this mind, in this work we are interested in evaluating the effects that a superradiant emission, together with superabsorption, will have in the momentum states of an atomic mesoscopic sample and how the splitting of an atomic beam might emerge from these phenomena.

In the final sections of this dissertation we will introduce the effects of deflection of a system interacting with a radiation field on the states of both parties in a similar way to what is done in (15), expanding the concepts seen in the aforementioned work to our superradiant/superabsorbing case. However, as we will see, problems emerge from this approach induced by the fact that we are dealing with a position dependent coupling term. To mitigate that fact, we can observe that this phenomenon can be separated in three cases, one where the system oscillates freely in a high  $Q$ -factor cavity leading to several cycles of many-body Rabi oscillations as seen in (1), one where we have a small number of these oscillations due to a large dampening factor and another with an even greater dampening factor that allows for a really precise approximation. This enables us to work with a simplified version of our system in which we then disregard the cyclic nature of the process and neglect the initial effects of superabsorption on the sample.





## 2 SUPERRADIANCE AND SUPERABSORPTION

### 2.1 Coherent pulse emission by an atomic sample

#### 2.1.1 Liouville equation for an ideal gas interacting with a thermal bath

With the purpose of understanding the fundamentals of our work we will introduce first the process of superradiance and then after, we will relate superabsorption and amplification of the superabsorbing process to that. As said before consider the interaction between the a sample of 2-level atomic dipoles and a thermal reservoir, we can describe this by the following the Hamiltonian ( $\hbar = 1$ )

$$H = H_S + H_R + V = H_0 + V, \quad (2.1)$$

where

$$H_S = \omega_0 S_z = \omega_0 \sum_i^N \frac{\sigma_z^{(i)}}{2}, \quad (2.2)$$

and

$$H_R = \sum_k \omega_k b_k^\dagger b_k, \quad (2.3)$$

are the Hamiltonian of the system (represented as an ensemble of  $N$  particles of spin half) and reservoir respectively,  $\omega_0$  and  $\omega_k$  are the Larmor frequency and frequency of the  $k$ -th oscillator in the reservoir,  $\sigma_z^{(i)}$  is the Pauli  $z$ -matrix that represents the spin of the  $i$ -th particle of the system (we will refer to  $S_z$  as the system itself from now on seeing that there are no direct contributions of the spatial part of the particles wave function) and  $V$  is the interaction Hamiltonian between both system and reservoir given by

$$V = \sum v_m (S_- b_m^\dagger + S_+ b_m), \quad (2.4)$$

where  $b_m^\dagger$  and  $b_m$  are the ladder operators for the reservoir's oscillators,  $v_m$  are the coupling terms between the system and each oscillator and  $S_-$  and  $S_+$  are the spin ladder operators of the ensemble of 2-level spin systems (which, for the sake of simplicity, we will call atoms from now on) defined by

$$S_+ = \sum_i^N \sigma_+^{(i)}, \quad (2.5a)$$

$$S_- = \sum_i^N \sigma_-^{(i)}, \quad (2.5b)$$

where  $\sigma_\pm$  are represented in the 2-level basis as

$$\sigma_+^{(i)} = |g_i\rangle \langle e_i|, \quad (2.6a)$$

$$\sigma_-^{(i)} = |e_i\rangle \langle g_i|, \quad (2.6b)$$

such that, together with  $S_z$  these operators obey the SU(2) algebra

$$[S_z, S_{\pm}] = \frac{1}{2} \sum_{i,i'} [\sigma_z, \sigma_{\pm}] = \sum_i \sigma_{\pm} = \pm S_{\pm}, \quad (2.7a)$$

$$[S_+, S_-] = \sum_{i,i'} [\sigma_+, \sigma_-] = 2S_z. \quad (2.7b)$$

To analyze the evolution of the ensemble as it interacts with the reservoir we will need to make use of the quantum Liouville equation given by

$$i\dot{\rho}(t) = [H, \rho(t)], \quad (2.8)$$

which integrated and rearranged gives us

$$\dot{\rho} = -i[H(t), \rho(0)] - \int_0^t dt' [H(t), [H(t'), \rho(t')]], \quad (2.9)$$

where  $\rho(t)$  is the density matrix of the entire system (ensemble and reservoir). Now, in order to develop 2.9, we need to define the density matrix for our problem. Assuming that the reservoir is in thermal equilibrium at temperature  $T$  during the entirety of the interaction we can separate the density matrix in the direct product given below

$$\rho(t) = \rho_S(t) \otimes \rho_R(0), \quad (2.10)$$

where  $\rho_S$  is the density matrix for the spin system and the reservoir's matrix can be written as

$$\rho_R(0) = \frac{e^{-\beta H_R}}{Z_R(\beta)}, \quad (2.11)$$

in which  $Z_R(\beta) = \text{Tr}_R\{e^{-\beta H_R}\}$  is the partition function associated with the Hamiltonian  $H_R$ , where the trace is taken with respect to the reservoir subspace. Analyzing now in the interaction picture, we can write 2.8 as the following

$$i\dot{\rho}_I = [V_I, \rho_I], \quad (2.12)$$

where

$$V_I = U_0^\dagger V U_0 = \sum v_m \left( S_- b_m^\dagger e^{i(\omega_m - \omega_0)t} + S_+ b_m e^{-i(\omega_R - \omega_0)t} \right), \quad (2.13)$$

$\rho_I$  has the usual expression for the density matrix in the interaction picture, namely  $\rho_I = U_0^\dagger \rho U_0$ , and the time evolution operator is given by

$$U_0 = e^{-iH_0 t}. \quad (2.14)$$

Equation 2.9, now can be rewritten as

$$\dot{\rho}_I = -i[V_I(t), \rho(0)] - \int_0^t dt' [V_I(t), [V_I(t'), \rho_I(t')]], \quad (2.15)$$

Since  $\rho_I(0) = \rho(0)$ . Using the following definitions for the operators (16)

$$F_+(t) = \sum_m v_m b_m^\dagger e^{i(\omega_m - \omega_0)t}, \quad (2.16a)$$

$$F_-(t) = \sum_m v_m b_m e^{-i(\omega_m - \omega_0)t} = (F_+)^\dagger, \quad (2.16b)$$

we can write 2.13 as

$$V_I(t) = S_- F_+(t) + S_+ F_-(t). \quad (2.17)$$

Substituting now 2.10 into 2.15 and tracing over the reservoir variables we are able to write the master equation for the spin system as

$$\dot{\rho}_{SI} = - \int_0^t Tr_R \{ [V_I(t), [V_I(t'), \rho_I(t')]] dt' \}, \quad (2.18)$$

or

$$\dot{\rho}_{SI} = -Tr_R \left\{ \int_0^t dt' V_I(t) V_I(t') \rho_I(t') - V_I(t') \rho_I(t') V_I(t) - V_I(t) \rho_I(t') V_I(t') + \right. \\ \left. + \rho_I(t') V_I(t') V_I(t) \right\}, \quad (2.19)$$

where we assume  $Tr_R(V_I \rho_R) = 0$ . Following now the steps taken in (1,16) we can approach such a problem through the use of the Markovian approximation in which, since we have a weak coupling between system and reservoir, the rate with which the density operator of the system changes is much smaller than the rate of change of the reservoir operators\*. We can therefore say that the variation in  $\rho_{SI}$  is negligible in the time required for the correlation functions of the reservoir to become null. We therefore are able to write the approximation

$$\rho_{SI}(t') \approx \rho_{SI}(t). \quad (2.20)$$

With this, we now notice that the following relations, namely

$$Tr_R(F_\pm(t) F_\pm(t') \rho_R) = \sum_{m,m'} v_m v_{m'} \langle b_m^\pm b_{m'}^\pm \rangle f(m, m'; t, t') = 0, \quad (2.21a)$$

and the usual correlation relation between the ladder operators

$$\langle b_m^+ b_{m'}^- \rangle = \delta_{mm'} N, \quad (2.22)$$

(in which we rename the reservoir operators as  $b^\dagger = b_+$  and  $b = b_-$  in order to simplify notation) allow us to substitute 2.17 into 2.19 and simplify the resulting expression with the use of the Lie algebra in 2.7 and the Baker-Campbell-Hausdorff relation as

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\* with the obvious exception of the density operator.

$$\dot{\rho}_{SI}(t) = - \int_0^t dt' e^{-i\omega_0(t-t')} (\xi_{12}(t-t')[S_-, S_+ \rho_{SI}] - \xi_{21}^*(t-t')[S_-, \rho_{SI} S_+] + h.c.), \quad (2.23)$$

where we define the functions

$$\xi_{12}(t-t') = Tr_R(F_+(t)F_-(t')\rho_R), \quad (2.24a)$$

$$\xi_{21}^*(t-t') = Tr_R(F_-(t)F_+(t')\rho_R), \quad (2.24b)$$

which can be rewritten as

$$\tilde{\xi}_{12} = \int_0^t dt' e^{-i\omega_0(t-t')} \xi_{12}(t-t'), \quad (2.25a)$$

$$\tilde{\xi}_{21} = \int_0^t dt' e^{i\omega_0(t-t')} \xi_{21}(t-t'), \quad (2.25b)$$

in order to further simplify our calculations and allowing us to write 2.23 as

$$\dot{\rho}_{SI}(t) = -\tilde{\xi}_{12}[S_-, S_+ \rho_{SI}] - \tilde{\xi}_{21}^*[S_-, \rho_{SI} S_+] - h.c.. \quad (2.26)$$

Now, taking into account the definitions in 2.16 and 2.23 we are able to convert the sums over the frequencies into an integral over a continuous spectrum. On top of this, when considering the Markov approximation we can notice that once we are considering a time scale sufficiently small for our analysis, the timescale in which we will have any change in  $\rho_{SI}$  becomes approximately infinite. We therefore can change the upper limit of the integrals in 2.25 to infinity, namely

$$\tilde{\xi}_{12} = \int_0^t dt' e^{-i\omega_0(t-t')} \xi_{12}(t-t') \approx \int_0^\infty dt' e^{-i\omega_0(t-t')} \xi_{12}(t-t'), \quad (2.27)$$

this allows us to change the integrals in 2.25 to Fourier transforms (2,6). This leaves us with  $\tilde{\xi}$  defined as complex integrals in the frequency space, using Cauchy's Theorem and the definition for the Cauchy Principal Value given by

$$\int_0^\infty dt' e^{\pm i\epsilon t} = \pi \delta(\epsilon) \pm \mathcal{P} \left( \frac{1}{\epsilon} \right), \quad (2.28)$$

we arrive at the expressions

$$\tilde{\xi}_{12} = \frac{\gamma_{12}}{2} - i\Delta\omega_{12}, \quad (2.29a)$$

$$\tilde{\xi}_{21} = \frac{\gamma_{21}}{2} + i\Delta\omega_{21}, \quad (2.29b)$$

where

$$\gamma_{12} = \pi g(\omega_0) |v(\omega_0)|^2 \langle b^+(\omega_0) b^-(\omega_0) \rangle, \quad (2.30a)$$

$$\gamma_{21} = \pi g(\omega_0) |v(\omega_0)|^2 \langle b^-(\omega_0) b^+(\omega_0) \rangle, \quad (2.30b)$$

and

$$\Delta\omega_{12} = \frac{1}{2}\mathcal{P} \left( \int_0^\infty \frac{d\omega}{\omega - \omega} g(\omega) |v(\omega)|^2 \langle b^+(\omega) b^-(\omega) \rangle \right), \quad (2.31a)$$

$$\Delta\omega_{21} = \frac{1}{2}\mathcal{P} \left( \int_0^\infty \frac{d\omega}{\omega - \omega} g(\omega) |v(\omega)|^2 \langle b^-(\omega) b^+(\omega) \rangle \right), \quad (2.31b)$$

with  $g(\omega)$  being defined as the density of frequencies. With this, we are in condition to write the master equation (ME) in the Schrödinger picture given below

$$\dot{\rho}_S = -i[H_S, \rho_S] - \tilde{\xi}_{12}[S_-, S_+ \rho_S] - \tilde{\xi}_{21}^*[S_-, \rho_S S_+] - h.c. \quad (2.32)$$

### 2.1.2 The mean field approach and the "dressed" atom

Now that we have our master equation our next step would be solving it. However since our Hamiltonian is specially complicated with non-linear crossed terms in  $N$  Hilbert spaces (one for each particle) it is convenient to use the mean-field approximation (MFA), as seen in (20), to reduce our space to that of one single representative particle. Such particle will be the key to describe the collective behavior of the system in a approximate manner, its associated time-dependent non-linear spin operators are the result of the averaging process over many atoms and as such it can be referred to as a quasi-particle or a dressed atom.

In order to describe the system in such a way, we need at first to separate the  $N$  particle density matrix in a tensor product of several lower dimensional operators. Considering a number  $p < N$  we begin by defining the  $p$  particle density operator as

$$\rho_p(t) = Tr_{p+1, \dots, N} \{ \rho_N(t) \}, \quad (2.33)$$

in which we take the trace of the total density matrix in relation to the  $N - p$  spaces that do not interest us. This allows us to rewrite equation 2.32 in terms of  $\rho_p$  as

$$\dot{\rho}_p = -i[H_S, \rho_p] - \tilde{\xi}_{12} Tr_{p+1, \dots, N} \{ [S_-, S_+ \rho_S] \} - \tilde{\xi}_{21}^* Tr_{p+1, \dots, N} \{ [S_-, \rho_S S_+] \} - h.c. \quad (2.34)$$

With these definitions, after extensive calculations we are in condition to explicitly write the ME for the  $p$ -particle density matrix as follows (16)

$$\begin{aligned} \dot{\rho}_p = & -i \frac{\omega_0}{2} \left[ \sum_{i=1}^p \sigma_z^i, \rho_p \right] - \tilde{\xi}_{12} \left( \sum_{i,j=1}^p [\sigma_-^i, \sigma_+^j \rho_p] \right. \\ & \left. - (N-p) \left[ \sum_{i=1}^p \sigma_-^i, Tr_{p+1} \{ \sigma_+(p+1) \rho_{p+1} \} \right] \right) - \tilde{\xi}_{21} \left( \sum_{i,j=1}^p [\sigma_+^i, \sigma_-^j \rho_p] \right. \\ & \left. - (N-p) \left[ \sum_{i=1}^p \sigma_+^i, Tr_{p+1} \{ \sigma_-(p+1) \rho_{p+1} \} \right] \right) - h.c.. \end{aligned} \quad (2.35)$$

Since we are interested in the single atom analysis, we must reduce 2.35 to the case  $p = 1$ , which will give us

$$\begin{aligned} \dot{\rho}_1 = -i\frac{\omega_0}{2} [\sigma_z, \rho_1] - \tilde{\xi}_{12} \left( [\sigma_-, \sigma_+ \rho_1] - (N-1) [\sigma_-, Tr_2\{\sigma_+^{(2)} \rho_2\}] \right) - \tilde{\xi}_{21} \left( [\sigma_+, \sigma_- \rho_1] \right. \\ \left. - (N-1) [\sigma_+, Tr_2\{\sigma_-^{(2)} \rho_2\}] \right) - h.c. \end{aligned} \quad (2.36)$$

Now, in order to describe eq. 2.36 in terms of the one atom density matrix we must decompose  $\rho_2$  into a product of the  $\rho_2(1, 2) = \rho_1(1) \otimes \rho_1(2)$  for the first and second atoms. This can be done if we assume the correlations between spins can be ignored in such a way that

$$Tr_2\{\sigma_{\pm}^{(2)} \rho_2\} = \rho_1(1) Tr_2\{\sigma_{\pm}^{(2)} \rho_1(2)\} = \rho_1 \langle \sigma_{\pm} \rangle, \quad (2.37)$$

applying the condition above to equation 2.36 we have

$$\begin{aligned} \dot{\rho} = -i\frac{\omega_0}{2} [\sigma_z, \rho] - \{(N-1)[\tilde{\xi}_{12} \langle \sigma_+ \rangle \sigma_- + \tilde{\xi}_{21} \langle \sigma_- \rangle \sigma_+ - h.c., \rho]\} \\ - \{\tilde{\xi}_{12} [\sigma_-, \sigma_+ \rho] + \tilde{\xi}_{21} [\sigma_+, \sigma_- \rho] + h.c.\}, \end{aligned} \quad (2.38)$$

where we will drop the index of  $\rho_1$  from now on and used the property  $[A, \rho] + h.c. = (A - A^\dagger)\rho - \rho(A - A^\dagger) = [A - h.c., \rho]$ . Considering now the relation between  $\sigma_{\pm}$  and the Pauli  $x$  and  $y$  we can define

$$\sigma_{\pm} = S_x \pm iS_y, \quad (2.39)$$

where

$$S_x = \frac{|e\rangle \langle g| + |g\rangle \langle e|}{2}, \quad (2.40a)$$

$$S_y = \frac{|e\rangle \langle g| - |g\rangle \langle e|}{2i}, \quad (2.40b)$$

that together with the definitions of  $\tilde{\xi}$  in 2.29, allow us to rewrite the first term of the second commutator of eq. 2.38 as

$$\begin{aligned} \tilde{\xi}_{12} \langle \sigma_+ \rangle \sigma_- + \tilde{\xi}_{21} \langle \sigma_- \rangle \sigma_+ - h.c. = i[2\Delta\omega \langle S_x \rangle + \gamma \langle S_y \rangle] S_x \\ + i[2\Delta\omega \langle S_y \rangle - \gamma \langle S_x \rangle] S_y, \end{aligned} \quad (2.41)$$

where we define

$$\Delta\omega = \Delta\omega_{21} - \Delta\omega_{12} = \frac{1}{2} \mathcal{P} \left( \int_0^\infty \frac{g(\omega) |v(\omega)|^2}{\omega_0 - \omega} d\omega \right), \quad (2.42)$$

and

$$\gamma = \gamma_{21} - \gamma_{12} = \pi g(\omega_0) |v(\omega_0)|^2, \quad (2.43)$$

while the last three terms in eq. 2.38 can be rewritten as

$$\begin{aligned} \tilde{\xi}_{12}[\sigma_-, \sigma_+\rho] + \tilde{\xi}_{21}[\sigma_+, \sigma_-\rho] + h.c. &= \left(\frac{\gamma_{12}}{2} - i\Delta\omega_{12}\right) [\sigma_-, \sigma_+\rho] \\ + \left(\frac{\gamma_{21}}{2} + i\Delta\omega_{21}\right) [\sigma_+, \sigma_-\rho] + h.c. &= \left(\frac{\gamma_{12}}{2}[\sigma_-, \sigma_+\rho] + \frac{\gamma_{21}}{2}[\sigma_+, \sigma_-\rho] + h.c.\right) \\ &+ i(\Delta\omega_{21}[\sigma_+, \sigma_-\rho] - \Delta\omega_{12}[\sigma_-, \sigma_+\rho] - h.c.), \end{aligned} \quad (2.44)$$

after developing the imaginary term (not the imaginary part) of the expression 2.38 we have

$$\Delta\omega_{21}[\sigma_+, \sigma_-\rho] - \Delta\omega_{12}[\sigma_-, \sigma_+\rho] - h.c. = \frac{1}{2}(\Delta\omega_{12} + \Delta\omega_{21})[\sigma_z, \rho]. \quad (2.45)$$

Finally, the definitions above allow us to rewrite the ME as the follows

$$\dot{\rho} = -i[H_{SR}, \rho] - \left(\frac{\gamma_{12}}{2}[\sigma_-, \sigma_+\rho] + \frac{\gamma_{21}}{2}[\sigma_+, \sigma_-\rho] + h.c.\right), \quad (2.46)$$

where  $\bar{\omega} = \omega_0 + \Delta\omega_{12} + \Delta\omega_{21}$  is the effective frequency of the representative particle and

$$H_{SR} = \bar{\omega}S_z + (N - 1)[(2\Delta\omega\langle S_x \rangle + \gamma\langle S_y \rangle)S_x + (2\Delta\omega\langle S_y \rangle - \gamma\langle S_x \rangle)S_y], \quad (2.47)$$

is its effective Hamiltonian, where  $S_z$  together with the operators defined in 2.40 are the pseudo-spin operators given by  $S_i = \sigma_i/2$  and obey the algebra

$$[S_i, S_j] = i \sum_{k=1}^3 \epsilon_{ijk} S_k, \quad (2.48)$$

in which  $\epsilon_{ijk}$  is the Levi-Civita symbol.

Equation 2.46 can be further developed using the definitions of  $\gamma$  in 2.43 and  $\gamma_{12}$  and  $\gamma_{21}$  in 2.31 in order to have the master equation rewritten in the Lindblad equation form

$$\begin{aligned} \dot{\rho} = -i[H_{SR}, \rho] - \frac{\gamma}{2}[N(\sigma_-\sigma_+\rho - 2\sigma_+\rho\sigma_- + \rho\sigma_-\sigma_+) + \\ + (N + 1)(\sigma_+\sigma_-\rho - 2\sigma_-\rho\sigma_+ + \rho\sigma_+\sigma_-)], \end{aligned} \quad (2.49)$$

where  $N = \langle b^+b^- \rangle$ .

$H_{SR}$  is an implicitly time-dependent non-linear Hamiltonian that represents the spin of the dressed atom whose emissions are a result of its interactions with an effective mean field of the ensemble, which encompasses the collective magnetic moments and the radiation field. It guides the unitary part of 2.49 and describes the emission of a single coherent superradiant pulse. Now, the second term of 2.49 is responsible for non-unitary evolution of the system and therefore it guides the dissipation process. However, since we are dealing with processes that occur in a much shorter period than the relaxation times of the system that are of order  $\frac{1}{\gamma}$ , we can ignore the dissipation term in our considerations.

### 2.1.3 Solving the non-linear Hamiltonian via Lewis-Riesenfeld

In this section we will take a small deviation from our main theme in order to introduce the necessary mathematical tool that is the Lewis-Riesenfeld invariant method.

In the previous section we have derived the effective Hamiltonian given in eq. 2.47. Considering however that even when discarding the non-unitary terms in 2.47 the Hamiltonian in the Liouville equation

$$\dot{\rho} = -i[H_{SR}, \rho], \quad (2.50)$$

is highly non-linear and time-dependent due to the averages over the  $\sigma_i$ , the job of finding the solution for the states is a herculean task unless we make use of the Lewis-Riesenfeld (LR) method as described in Refs. (17, 18, 21). We know that the Schrödinger equation

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H(t) |\psi(t)\rangle, \quad (2.51)$$

will fully determine the time-dependent state vector  $|\psi(t)\rangle$ . Now, supposing the existence of a time-dependent operator  $I(t)$  such that the given conditions

$$\frac{dI(t)}{dt} \equiv \frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H(t)] = 0, \quad (2.52)$$

and

$$I = I^\dagger, \quad (2.53)$$

are true and assuming it to be part of complete set of commuting observables in order for  $I$  to have complete set of eigenstates associated with it such that

$$I |\lambda, \kappa\rangle = \lambda |\lambda, \kappa\rangle, \quad (2.54a)$$

$$\langle \lambda \kappa | \lambda' \kappa' \rangle = \delta_{\lambda, \lambda'} \delta_{\kappa, \kappa'}, \quad (2.54b)$$

where  $\lambda$  are the eigenvalues of  $I$  and  $\kappa$  are all other quantum numbers associated with the state vector. When deriving both sides of the first equation of 2.54 we have

$$\frac{\partial I}{\partial t} |\lambda, \kappa\rangle + I \frac{\partial}{\partial t} |\lambda, \kappa\rangle = \frac{\partial \lambda}{\partial t} |\lambda, \kappa\rangle + \lambda \frac{\partial}{\partial t} |\lambda, \kappa\rangle. \quad (2.55)$$

Now, applying 2.52 onto the state  $|\lambda, \kappa\rangle$  we have

$$i \frac{\partial I}{\partial t} |\lambda, \kappa\rangle + IH |\lambda, \kappa\rangle - \lambda H |\lambda, \kappa\rangle = 0, \quad (2.56)$$

which combined with 2.55 shows us that

$$\frac{\partial \lambda}{\partial t} = 0, \quad (2.57)$$

this condition shows that in order for  $I$  not to be identically zero  $|\lambda, \kappa\rangle$  must be time dependent. Applying  $\langle \lambda', \kappa' |$  to both 2.55 and 2.56 and comparing them we arrive at the relation

$$i(\lambda - \lambda') \langle \lambda', \kappa' | \frac{\partial}{\partial t} |\lambda, \kappa\rangle = (\lambda - \lambda') \langle \lambda', \kappa' | H |\lambda, \kappa\rangle. \quad (2.58)$$



Though this does not imply that the vectors  $|\lambda, \kappa\rangle$  are solutions of the Schrödinger equation, we are able to redefine the definition of the eigenvectors of  $I$  so that 2.58 is still valid for  $\lambda = \lambda'$  once we have not set a fixed phase. Assuming  $I$  does not contain time-derivative operators, we can define a new set of eigenvectors given by

$$|\lambda, \kappa\rangle_\phi = e^{i\phi_{\lambda\kappa}(t)} |\lambda, \kappa\rangle \quad (2.59)$$

that still obey the orthonormality condition.

Now, in order for us to determine the states that satisfy the necessary condition to be considered a solution of the Schrödinger equation, namely

$$i \langle \lambda, \kappa' | \frac{\partial}{\partial t} | \lambda, \kappa \rangle = \langle \lambda, \kappa' | H | \lambda, \kappa \rangle, \quad (2.60)$$

we must find the phase  $\phi_{\lambda\kappa}(t)$  that satisfy the condition

$$\delta_{\kappa\kappa'} \frac{d\phi_{\lambda\kappa}}{dt} = \langle \lambda, \kappa' | \left( i \frac{\partial}{\partial t} - H \right) | \lambda, \kappa \rangle \quad (2.61)$$

obtained when applying 2.60 to 2.61.

With these considerations, once we have defined an operator  $I(t)$  (also called invariant), a complete set of eigenvectors  $|n, t\rangle$  associated with it and a time-dependent phase that given by

$$\phi_n(t) = \int_0^t d\tau \langle n, \tau | \left( i \frac{\partial}{\partial t} - H(t) \right) | n, \tau \rangle, \quad (2.62)$$

the general solution of the Schrödinger equation can be written as

$$|\psi(t)\rangle = \sum_n c_n e^{i\phi_n(t)} |n, t\rangle. \quad (2.63)$$

with  $c_n = \langle n | \psi(0) \rangle$ .

Now, when applying this method to the case of the Hamiltonian in 2.47 we first notice that it is convenient to write the effective Hamiltonian as

$$H_{SR} = f(t)S_x + g(t)S_y + \omega(t)S_z, \quad (2.64)$$

where we define

$$f(t) = (N - 1)(2\Delta\omega\langle S_x \rangle - \gamma\langle S_y \rangle), \quad (2.65a)$$

$$g(t) = (N - 1)(2\Delta\omega\langle S_y \rangle - \gamma\langle S_x \rangle). \quad (2.65b)$$

Considering this, we can assume that an invariant with the same functional form of the Hamiltonian, namely

$$I(t) = a(t)S_x + b(t)S_y + c(t)S_z \quad (2.66)$$

would satisfy the conditions established in the beginning of this section. Indeed, using 2.64 together with the commutation relations 2.48 to compute the dynamic equations for the  $s_i$  operators and arrive at the system

$$\langle \dot{S}_x \rangle = -i\langle [S_x, H_{SR}] \rangle = -\bar{\omega}(t)\langle S_y \rangle + \langle S_z \rangle(N-1)(2\Delta\omega\langle S_y \rangle + \gamma\langle S_x \rangle), \quad (2.67a)$$

$$\langle \dot{S}_y \rangle = -i\langle [S_y, H_{SR}] \rangle = \bar{\omega}(t)\langle S_x \rangle - \langle S_z \rangle(N-1)(2\Delta\omega\langle S_x \rangle - \gamma\langle S_y \rangle), \quad (2.67b)$$

$$\langle \dot{S}_z \rangle = -i\langle [S_z, H_{SR}] \rangle = -(N-1)\gamma(\langle S_x \rangle^2 + \langle S_y \rangle^2), \quad (2.67c)$$

we can use equations 2.67 to easily verify that the operator

$$I(t) = \langle S_z \rangle S_z + \langle S_x \rangle S_x + \langle S_y \rangle S_y, \quad (2.68)$$

is an invariant and therefore its expected value

$$\langle I(t) \rangle = \langle S_z \rangle^2 + \langle S_x \rangle^2 + \langle S_y \rangle^2 = R^2, \quad (2.69)$$

is a constant of motion. Following Ref. (16) we introduce the Bloch angles  $\theta$  and  $\phi$  such that our pseudo-spin operators can be redefined as

$$\langle S_x \rangle = R \sin \theta \cos \phi, \quad (2.70a)$$

$$\langle S_y \rangle = R \sin \theta \sin \phi \quad (2.70b)$$

$$\langle S_z \rangle = R \cos \theta, \quad (2.70c)$$

in which  $R$  is the radius of the Bloch sphere that for our purposes will be set to  $R = 1/2$ . Using the definitions above we can rewrite the system in 2.65 in terms of the Bloch angles as

$$\dot{\theta}(t) = \frac{1}{2}N\gamma \sin \theta, \quad (2.71a)$$

$$\dot{\phi}(t) = \bar{\omega} - \Delta\omega N \cos \theta \quad (2.71b)$$

where we used the approximation  $N-1 \sim N$ . Solving equations 2.71 as in Ref. (2) we arrive at the expressions for  $\theta(t)$  and  $\phi(t)$  given bellow

$$\phi(t) = \phi_0 + \bar{\omega}t - \frac{2\Delta\omega}{\gamma} \ln \left( \frac{\sin \theta}{\sin \theta_0} \right), \quad (2.72a)$$

$$\sin \theta = \operatorname{sech} \left( \frac{t - t_0}{\tau_c} \right), \quad (2.72b)$$

$$\cos \theta = -\tanh \left( \frac{t - t_0}{\tau_c} \right), \quad (2.72c)$$

where  $\tau_c = 2/\gamma N$  is defined as the characteristic emission time and  $t_0 = \tau_c \ln [\cot g(\theta_0/2)]$  the delay time of the superradiative process. As seen in (20, 22) it can be shown that,  $t_0 \sim \tau_c \ln N$ , which means that, using 2.72 for  $t = 0$ , the relation  $\sin \theta \sim 2/N$  is true and therefore  $\theta_0 \sim 2/N$  is the initial position of the vector in the Bloch sphere. Adding

to this fact, when using 2.71a it is easy to notice that  $\theta = 0$  is an unstable equilibrium point where all atoms are in the excited state, these facts together can be explained when remembering that the state represented by these variables is a collective average. As the Bloch vector stands at  $\theta = 0$  due to the fluctuations of the electromagnetic field of the collective atoms the corresponding state is quickly disturbed and remains in a region around this value.

$$|+, t\rangle = \cos \frac{\theta}{2} |e\rangle + e^{i\phi} \sin \frac{\theta}{2} |g\rangle, \quad (2.73)$$

and

$$|-, t\rangle = \sin \frac{\theta}{2} |e\rangle - e^{i\phi} \cos \frac{\theta}{2} |g\rangle, \quad (2.74)$$

are its eigenvectors. When regarding the LR phases, they can both be written as

$$\begin{aligned} \varphi_{\pm}(t) &= \int_0^t d\tau \langle \pm | (i\partial_t - H_{SR}) | \pm \rangle \\ &= \int_0^t d\tau \langle \pm | i(\partial_t | \pm \rangle) - \int_0^t d\tau \langle \pm | H_{SR} | \pm \rangle = \varphi_{\pm}^g(t) + \varphi_{\pm}^d(t), \end{aligned} \quad (2.75)$$

where we followed the definitions in (2, 23) for the geometrical  $\varphi^g$  and dynamical  $\varphi^d$  phases. Considering from now on only the state  $|+\rangle^\dagger$ , the phases will be given by

$$\varphi_+^g = -\frac{\omega_0}{2} \int_0^t d\tau (1 - \cos \theta(\tau)), \quad (2.76a)$$

$$\varphi_+^d = -\frac{\omega_0}{2} \int_0^t d\tau \cos \theta(\tau), \quad (2.76b)$$

where we are considering the regime where  $\omega_0 \gg \Delta\omega$  and  $\omega_0 \gg N\gamma$ , as done in Refs. (2, 16), since the regime that interest us is of order  $\omega_0 \sim 10^{10} Hz$  and therefore we approximate  $\bar{\omega} = \omega_0$ .

With this, the LR phase becomes

$$\varphi_+(t) = \varphi_+^g(t) + \varphi_+^d(t) = -\frac{\omega_0}{2}t. \quad (2.77)$$

---

<sup>†</sup> Later in this work, when dealing with the both the interplay between superradiance and superabsorption and the deflection we will consider both states.

### 2.1.4 Energy and intensity

The most remarking and characteristic trait of the superradiant emission is the shape of its intensity versus time graph. As mentioned briefly in the introduction, the phenomenon of superradiance consists of pulses that are the result of a collective emission processes in chain whose emitted radiation has an intensity proportional to the number of atoms squared ( $I \propto N^2$ ) and duration of order  $\tau_c \sim \ln N/\gamma N$ . This fact can be shown by calculating the energy of the representative particle.

$$\begin{aligned} \epsilon(t) &\approx \langle \psi(t) | \omega_0 S_z | \psi(t) \rangle = \frac{1}{2} \omega_0 \cos \theta(t) \\ &= -\frac{\omega_0}{2} \tanh \left( \frac{t - t_0}{\tau_c} \right), \end{aligned} \quad (2.78)$$

where we omitted the terms proportional to  $\Delta\omega$  and used eq. 2.72c. As expected the energy is not conserved in the system. The total energy of the system will be given by  $\mathcal{E}(t) = N\epsilon(t)$  and the intensity emitted by all the atoms will then be

$$\mathcal{I}(t) = -N \frac{d\epsilon}{dt} = N \frac{\omega_0}{2\tau_c} \operatorname{sech}^2 \left( \frac{t - t_0}{\tau_c} \right). \quad (2.79)$$

Substituting  $\tau_c = 2/N\gamma$  we have

$$\mathcal{I}(t) = \frac{1}{4} \omega_0 N^2 \gamma \operatorname{sech}^2 \left( \frac{t - t_0}{\tau_c} \right). \quad (2.80)$$

Figure 2 bellow illustrates the behavior of the superradiant pulse in a interval of the order of  $t_0$  while figure 3 shows the population  $\langle \sigma_+ \sigma_- \rangle$  of the of the state in time (time variable re-scaled in units of  $\tau_c$ , and still using  $\hbar = 1$ )

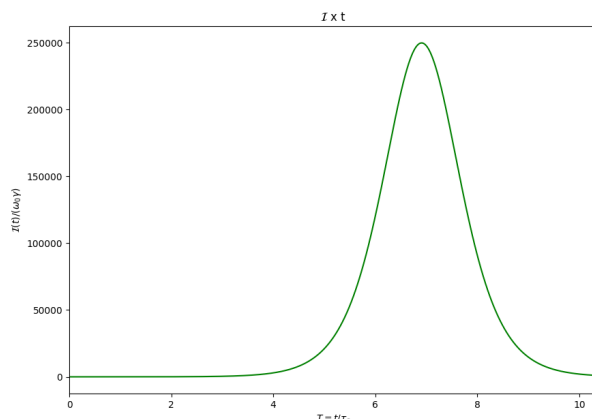


Figure 2 – Intensity of superradiant emission vs time for  $N = 10^3$ ,  $\omega_0 = 10^5$ ,  $\gamma = 1$ . We can see the rapid ascend near  $t = t_0 - \frac{\tau_c}{2}$  while the function peaks at  $t_0$  (where we have  $\theta = \pi/2$ ) and rapidly decreases as the sample emits and the dipoles assume random disordered positions.

Source: By the author

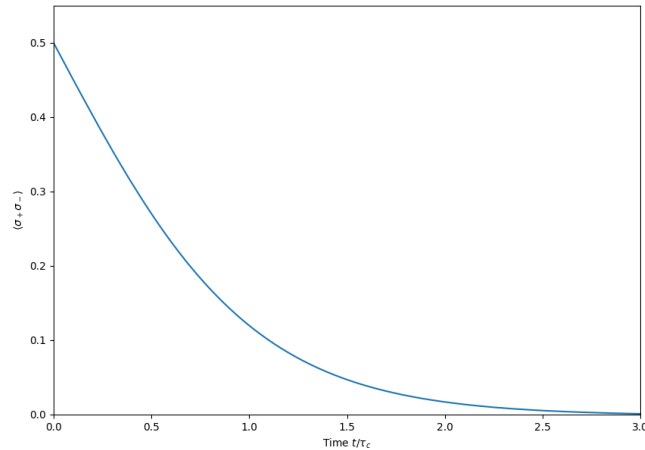


Figure 3 – Population of state  $|+, t\rangle$  in time beginning at  $t_0$ . The state is at its maximum population at  $t = t_0$  where the entropy is minimum and quickly decays due to random fluctuations of the EM field into a more disordered state.

Source: By the author

## 2.2 Superradiance-superabsorption cycle of the atom-cavity coupling system

As mentioned before, due to the time-reversal symmetry of quantum mechanics systems with enhanced emission rates will also present enhanced absorption rates given rise to the process of superabsorption. In a vacuum is natural to expect emission to dominate over superabsorption as there are no photons to absorb, and in fact even under conditions in which both phenomena are balanced one single transition is more likely to emit than absorb.

However as shown in (14) it is possible to engineer a superradiant system in order to enhance the absorption effect in such a way as for its intensity to be of the order of the intensity of the superradiative process.

### 2.2.1 The Tavis-Cummings interaction

When considering now the case in which our sample is interacting with a single mode electromagnetic field resonant with the sample confined in a high Q-factor cavity it can be shown as done in (1) that the interactions between field and atom are such that it will give rise to a cycle of intercalated periods of superabsorption and superemission by both systems considered. The Hamiltonian for this system is similar to that of 2.1 with a change in the interaction potential 2.4 namely

$$H = H_0 + \Delta H, \quad (2.81)$$

with

$$H_0 = H_S + H_f + H_R, \quad (2.82)$$

being  $H_S$  and  $H_R$  the system and reservoir Hamiltonians defined in 2.2 and 2.3,  $H_f$  the free field Hamiltonian

$$H_f = \omega_0 a^\dagger a, \quad (2.83)$$

and  $\Delta H$  defined by

$$\Delta H = \sum_m v_m (S_- b_m^\dagger + S_+ b_m) + H_f^I = V + H_f^I, \quad (2.84)$$

where the first term in the sum represents, just as in the case of superradiance, the interaction between system and reservoir and the new term, is the Hamiltonian of the interaction between the atom and the cavity field which in the dipole approximation gives us

$$H_f^I = -\mathbf{E}\boldsymbol{\mu}, \quad (2.85)$$

with

$$\boldsymbol{\mu} = \mu \sum_i^N (\sigma_+^i + \sigma_-^i) \quad (2.86)$$

being the electric dipole operator of the atoms and  $\mathbf{E}$  the cavity quantum electric field operator. Considering that our cavity is aligned with the  $x$  axis and both our (stationary) field and sample are configured as shown in figure 4

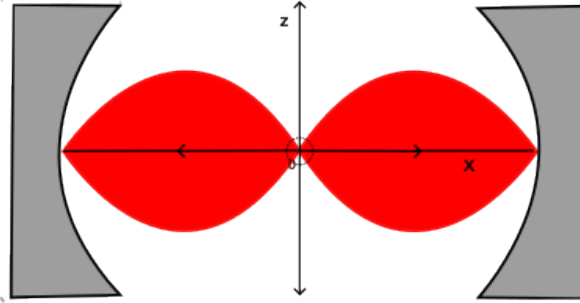


Figure 4 – Scheme of the trapping apparatus (cavity) with a stationary electric field  $E = E_0 \sin(kx)$  (in red). In the middle, it is possible to see the sample trapped in the node of the field and interacting with it. Since the size of the sample is much smaller than the wavelength of the field, but not zero, we are generally able to approximate  $\sin kx \approx kx$ , as we will see later.

Source: By the author

our (monochromatic) quantum field will then be given by

$$\mathbf{E} = (a + a^\dagger)\mathcal{E} \sin(kx), \quad (2.87)$$

where  $\mathcal{E}$  is the effective field per photon and  $k = \omega_0/c$  is the field wave vector. Finally, ap-

plying the rotating wave approximation (RWA) our field-atom interaction can be rewritten as

$$H_f^I = g(aS_+ + a^\dagger S_-) \quad (2.88)$$

known as the Tavis-Cummings Hamiltonian (24, 25), describes the interaction between system and radiation field and  $g = -\mu\mathcal{E} \sin(kx)$  is the time independent radiation-atom coupling term. The total Hamiltonian is given by (26)

$$H = \omega_0 S_z + \omega_0 a^\dagger a + g(aS_+ + a^\dagger S_-) + H_R + V_R, \quad (2.89)$$

With just a few alterations, this is the Hamiltonian we will work with from now on.

## 2.2.2 The mean field approximation

With the Hamiltonian defined in 2.89 in hands we are in condition to describe the dynamics of the system in terms of an effective interaction Hamiltonian from which we will use the LR method to derive the corresponding solutions, as done in the previous sections.

Taking the Hamiltonian  $\Delta H$  to the interaction picture we have

$$\Delta H_I = U_0^\dagger \Delta H U_0 = g(aS_+ + a^\dagger S_-) + V_I, \quad (2.90)$$

with  $V_I$  being defined in 2.15. Using the Liouville equation defined in 2.12 which in here becomes

$$i\dot{\rho}_I = [\Delta H_I, \rho_I], \quad (2.91)$$

which integrating, reorganizing and again tracing over the reservoir variables gives us an equation similar to 2.15

$$\dot{\rho}_{SF} = -i \left[ g(aS_+ + a^\dagger S_-), \rho_{SF} \right] - \int_0^t dt' Tr_R \{ [V_I(t) \cdot [V_I(t'), \rho(0)_R \rho_{SF}(t')]] \}, \quad (2.92)$$

where we have defined

$$\rho_{SF}(t) = Tr_R \{ \rho(t) \} = Tr_R \{ \rho_R(0) \otimes \rho_S(t) \otimes \rho_F(t) \}. \quad (2.93)$$

Taking back to the Schrödinger picture we have an equation very similar to 2.49

$$\dot{\rho}_{SF} = \left[ \omega_0 S_z + \omega_0 a^\dagger a + g(aS_+ + a^\dagger S_-), \rho_{SF} \right] + \mathcal{L}(\rho_{SF}), \quad (2.94)$$

where the term in the commutator represents the unitary Hamiltonian evolution and the Lindblad operator, that gives the non-unitary (dissipative) evolution, is given by

$$\begin{aligned} \mathcal{L}(\rho_{SF}) = & \gamma_{21}(2S_- \rho_{SF} S_+ - S_+ S_- \rho_{SF} - \rho_{SF} S_+ S_-) \\ & - \gamma_{12}(2S_+ \rho_{SF} S_- - S_- S_+ \rho_{SF} - \rho_{SF} S_- S_+), \end{aligned} \quad (2.95)$$

with  $\gamma_{12} = \gamma N/2$  and  $\gamma_{21} = \gamma(N+1)/2$  and  $\gamma$  is defined as in 2.43.

Following now the steps in references (1, 27), we consider  $N$  identical quantum systems contained in certain sites  $i = 1, 2, \dots, N$  associated with the Hilbert spaces  $\mathcal{H}_i$  and corresponding self-Hamiltonians  $h_i$ . Systems on different sites interact through the potential  $V = V_{ij}$  represented by an operator that act on the subspace  $\mathcal{H}_i \otimes \mathcal{H}_j$ . The total Hamiltonian can be written as

$$H_N = \sum_{i=1}^N h_i + \frac{1}{N} \sum_{i \neq j}^N V_{ij}, \quad (2.96)$$

which is an operator that acts in the space

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N. \quad (2.97)$$

Now, being the density matrix at  $t = 0$  a direct product of the  $N$  matrices of each system, we can approximate for any time  $t$

$$\rho_n(t) = \lim_{N \rightarrow \infty} Tr_{n+1} \dots Tr_N \left\{ e^{-iH_N t} \rho_N(0) e^{iH_N t} \right\}, \quad (2.98)$$

where  $\rho_n(t)$  is the direct product of  $n$  one-particle density matrices  $\rho$  at time  $t$  which are the solutions to the Hartree-Fock equation (28), with  $\rho_2(t) \approx \rho(t) \otimes \rho(t)$

$$\dot{\rho}(t) = -i[h, \rho] - iTr_2[V_{12} + V_{21}, \rho_2(t)]. \quad (2.99)$$

In order to transform our Hamiltonian to the form 2.96, so we can apply the mean field theory, we will make use of a mathematical device introduced in (27). If we introduce  $N$  *mathematical* field modes each created by a set of  $a_i^\dagger/a_i$  operators identical to the original creation/annihilation operators, such that they obey the commutation relations  $[a_i, a_j^\dagger] = \delta_{ij}$  and performing the substitutions

$$a \rightarrow \frac{1}{\sqrt{N}} \sum_i^N a_i, \quad (2.100a)$$

$$a^\dagger a \rightarrow \sum_i^N a_i^\dagger a_i, \quad (2.100b)$$

as well as re-scaling the coupling constant to

$$g \rightarrow \frac{\tilde{g}}{\sqrt{N}}, \quad (2.101)$$

we are able to rewrite the Hamiltonian 2.89 (using the definitions for  $S_+$  and  $S_-$  given in 2.5) as

$$H = \sum_i^N \omega_0 a_i^\dagger a_i + \sum_i^N \frac{\omega_0}{2} \sigma_z + \sum_{i \neq j}^N \tilde{g} (a_i \sigma_j^+ + a_i^\dagger \sigma_j^-), \quad (2.102)$$



where we can define

$$h_i = \omega_0 \left( a_i^\dagger a_i + \frac{1}{2} \sigma_z \right), \quad (2.103a)$$

$$V_{ij} = \tilde{g}(a_i \sigma_j^+ + a_i^\dagger \sigma_j^-). \quad (2.103b)$$

Now, in a way similar to what was done in section 2.1.2 we will take the trace of equation 2.94 in order to put it in terms of the single particle density matrix and we apply the above definitions to equation 2.99 resulting in

$$\begin{aligned} \dot{\rho}_1 = & -i\omega_0 \left[ a_i^\dagger a_i + \frac{1}{2} \sigma_z, \rho_1 \right] - i\tilde{g} [Tr_2(\sigma_- \rho_1) a^\dagger + Tr_2(\sigma_+ \rho_1) a, \rho_1] \\ & - i\tilde{g} [Tr_2(a^\dagger \rho_1) \sigma_- + Tr_2(a \rho_1) \sigma_+, \rho_1] + Tr_2 \dots Tr_N(\mathcal{L}(\rho_{SF})), \end{aligned} \quad (2.104)$$

where the terms inside the trace  $Tr_2(\dots)$  are in the space of particle 2 and the terms outside it are in the space of particle 1. We can rewrite equation 2.104 when considering that  $Tr_2(A\rho) = \langle A \rangle$  to

$$\begin{aligned} \dot{\rho}_1 = & -i \left[ \omega_0 \left( a_i^\dagger a_i + \frac{1}{2} \sigma_z \right) + \tilde{g} (\langle \sigma_- \rangle a^\dagger + \langle \sigma_+ \rangle a + \langle a^\dagger \rangle \sigma_- + \langle a \rangle \sigma_+), \rho_1 \right] \\ & + Tr_2 \dots Tr_N(\mathcal{L}(\rho_{SF})). \end{aligned} \quad (2.105)$$

Looking now at the definition of  $\mathcal{L}(\rho_{SF})$  in 2.2.2 and with  $\gamma_{12} = N\gamma/2$  we can write 2.105 in the more familiar form

$$\begin{aligned} \dot{\rho}_1 = & -i[H_{AF}, \rho_1] + \gamma_{21}(2\sigma_- \rho_{SF} \sigma_+ - \sigma_+ \sigma_- \rho_{SF} - \rho_{SF} \sigma_+ \sigma_-) + \\ & + \gamma_{12}(2\sigma_+ \rho_{SF} \sigma_- - \sigma_- \sigma_+ \rho_{SF} - \rho_{SF} \sigma_- \sigma_+), \end{aligned} \quad (2.106)$$

with the effective atom-field Hamiltonian  $H_{AF}$  is defined as

$$\begin{aligned} H_{AF} = & \omega_0 \left( a_i^\dagger a_i + \frac{1}{2} \sigma_z \right) + \sqrt{N}g(\langle \sigma_- \rangle a^\dagger + \langle \sigma_+ \rangle a + \langle a^\dagger \rangle \sigma_- + \langle a \rangle \sigma_+) \\ & - i \frac{N\gamma}{2} (\langle \sigma_- \rangle \sigma_+ - \langle \sigma_+ \rangle \sigma_-). \end{aligned} \quad (2.107)$$

Through this Hamiltonian we will be able to study in detail the behavior of the system. In the following sections we will use it to derive the dynamic equations as well as solving the Schrödinger equation via LR and from this we will be able to observe both the amplification of the phenomenon of superabsorption and the interplay between it and the superradiant emission happening in the sample and in the field.

### 2.2.3 Evolution of the atom-field Liouville equations

With the Hamiltonian given in 2.107 we are able to derive the dynamic equations that will dictate the behavior of the expected values of the operators in time. Rewriting the Hamiltonian 2.107 to the more convenient form

$$H_{AF} = H_A + H_F, \quad (2.108)$$

with

$$H_F = \omega_0 a^\dagger a + \tilde{g}(\langle \sigma_- \rangle a^\dagger + \langle \sigma_+ \rangle a), \quad (2.109a)$$

$$H_A = \frac{\omega_0}{2} \sigma_z + \Lambda \langle \sigma_- \rangle \sigma_+ + \Lambda^* \langle \sigma_+ \rangle \sigma_-, \quad (2.109b)$$

where we define the coefficient

$$\Lambda = \frac{N}{2} \left( \frac{2g\langle a \rangle}{\sqrt{N}\langle \sigma_- \rangle} - i\gamma \right). \quad (2.110)$$

Now, since

$$[H_F, H_A] = 0, \quad (2.111)$$

it is evident that the Hamiltonians in 2.109 act in different Hilbert spaces, one defined by the atom operators  $\sigma$  and one defined by the field  $a$  and  $a^\dagger$  operators. When we derived the mean field Hamiltonian we restricted the interactions to the time-dependent expected values which are mere coefficients in 2.107. We are therefore able to separate the Hamiltonian in a direct sum as we did in 2.108.

With these observations, it is clear that from the Hamiltonian 2.107 we will have two sets of dynamical equations, one that dictates the dynamic of the atom operators and the other dictating the dynamic of the field operators. Rewriting the Hamiltonian in 2.109b in terms of the pseudo-spin operators we have

$$H_A = \omega_0 S_z + 2\Lambda_R (\langle S_x \rangle S_x + \langle S_y \rangle S_y) - 2\Lambda_I (\langle S_x \rangle S_y - \langle S_y \rangle S_x), \quad (2.112)$$

where  $\Lambda_R$  and  $\Lambda_I$  are the real and imaginary parts of  $\Lambda$  given by

$$\Lambda_R = \tilde{g} \frac{\langle S_x \rangle \langle X_1 \rangle - \langle S_y \rangle \langle X_2 \rangle}{\langle S_x \rangle^2 + \langle S_y \rangle^2}, \quad (2.113a)$$

$$\Lambda_I = \tilde{g} \frac{\langle S_x \rangle \langle X_2 \rangle + \langle S_y \rangle \langle X_1 \rangle}{\langle S_x \rangle^2 + \langle S_y \rangle^2} - \frac{\gamma N}{2}, \quad (2.113b)$$

where, from a matter of convenience, we have exchanged the field ladder operators by the ones defined below

$$X_1 = \frac{a + a^\dagger}{2}, \quad (2.114a)$$

$$X_2 = \frac{a - a^\dagger}{2i}, \quad (2.114b)$$

called the field quadrature operators which will be used to represent the field part of the dynamics from now on.

From the commutation relations in 2.48 and the rewritten Hamiltonian above we can derive the following system of equations

$$\dot{S}_z = i[H_A, S_z] = 2\Lambda_I (\langle S_x \rangle S_x + \langle S_y \rangle S_y), \quad (2.115a)$$

$$\dot{S}_x = i[H_A, S_x] = -\omega_0 S_y + 2\Lambda_R \langle S_y \rangle S_z - \Lambda_I \langle S_x \rangle S_z, \quad (2.115b)$$

$$\dot{S}_y = i[H_A, S_y] = \omega_0 S_x - 2\Lambda_R \langle S_x \rangle S_z - \Lambda_I \langle S_y \rangle S_z. \quad (2.115c)$$

When considering the usual commutation relations for the field operators, namely<sup>‡</sup>

$$[a, a^\dagger] = 1, \quad (2.116a)$$

$$[n, a] = -a, \quad (2.116b)$$

$$[n, a^\dagger] = a^\dagger, \quad (2.116c)$$

it is easy to compute the dynamic equations for  $X_1$  and  $X_2$  given by

$$\dot{X}_1 = i[H_F, X_1] = \omega_0 X_2 - \tilde{g} S_y, \quad (2.117a)$$

$$\dot{X}_2 = i[H_F, X_2] = -\omega_0 X_1 - \tilde{g} S_x. \quad (2.117b)$$

Finally, by taking the expected value of the equations 2.115 and 2.117 we arrive at the system of coupled differential equations bellow

$$\langle \dot{S}_z \rangle = 2\Lambda_I (\langle S_x \rangle^2 + \langle S_y \rangle^2), \quad (2.118a)$$

$$\langle \dot{S}_x \rangle = -\omega_0 S_y + 2\langle S_z \rangle (\Lambda_R \langle S_y \rangle - \Lambda_I \langle S_x \rangle), \quad (2.118b)$$

$$\langle \dot{S}_y \rangle = \omega_0 S_x - 2\langle S_z \rangle (\Lambda_R \langle S_x \rangle - \Lambda_I \langle S_y \rangle), \quad (2.118c)$$

$$\langle \dot{X}_1 \rangle = \omega_0 \langle X_2 \rangle - \tilde{g} \langle S_y \rangle, \quad (2.118d)$$

$$\langle \dot{X}_2 \rangle = -\omega_0 \langle X_1 \rangle - \tilde{g} \langle S_x \rangle. \quad (2.118e)$$

We can see that in the system 2.118 the field variables are directly coupled to the atom's while the atoms appear to be in an isolated system of their own. However, due to the definition of  $\Lambda$  in 2.110 the atom system will be coupled to the field as well, albeit in a more indirect way, which will allow us later to show that in a time regime much shorter than one complete oscillation of the atom the superradiative process will dominate for a brief instant. We can also notice that, through the mean field approximation we have eliminated the need for an interaction Hamiltonian once our system variables will be coupled to each other through the time-dependent coefficients  $\langle \sigma_\pm(t) \rangle$  and  $\langle a^\pm(t) \rangle$  and therefore through the time-dependent states  $|\psi_{AF}(t)\rangle$ . We will do further analysis of the Schrödinger equations and its solution state in the following chapters.

#### 2.2.4 Intensity and many-body Rabi oscillations

We will now briefly introduce the concept of the many-body Rabi oscillations that occur from the interplay between superradiation and superabsorption. We know that the energy of the atom is given by  $\epsilon_A = \omega_0 \langle \sigma_z \rangle / 2$ , the intensity of the sample will therefore be given by

$$\mathcal{I}_A = -N \frac{d\epsilon_A}{dt} = -N\omega_0 \Lambda_I (\langle S_x \rangle^2 + \langle S_y \rangle^2), \quad (2.119)$$

<sup>‡</sup> We will use from now on the lower case letter  $n$  to refer to the number of photons or the field number operator.

meanwhile, the field number operator when derived will give us, from the Hamiltonian 2.109

$$\dot{n} = i[H_F, n] = i\tilde{g}(\langle\sigma_+\rangle a - \langle\sigma_-\rangle a^\dagger). \quad (2.120)$$

We know that the energy of the photon is proportional to  $\omega_0$ , therefore the energy of the field per atom is given by  $\epsilon_F = \omega_0\langle n \rangle$ . Using equation 2.114 and the definitions of  $\sigma_+$  and  $\sigma_-$  in terms of the pseudo-spin operators the intensity of the field is given by

$$\mathcal{I}_F = -N\omega_0\langle\dot{n}\rangle = 2N\tilde{g}\omega_0(\langle S_x \rangle \langle X_2 \rangle + \langle S_y \rangle \langle X_1 \rangle). \quad (2.121)$$

With both definitions in hands now we are in condition to describe the interplay between superemission ( $\mathcal{I} > 0$ ) and superabsorption ( $\mathcal{I} < 0$ ) for both the cavity field and the sample (represented here by the "dressed" atom). Figure 5 below, made with the use of a python program written by the author, represents the intermittent oscillations in intensity by the two systems .

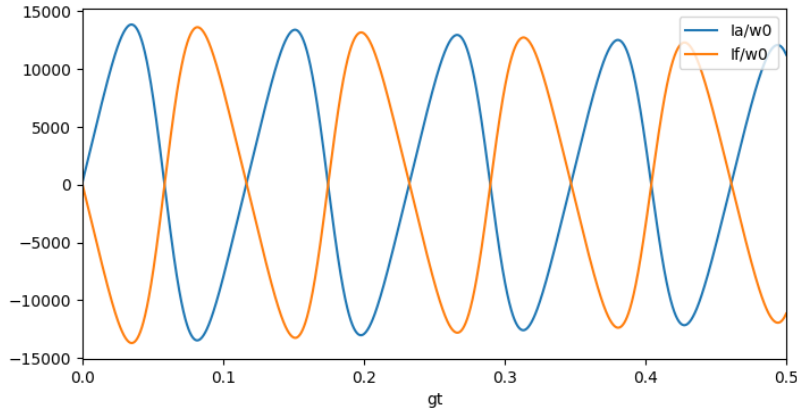


Figure 5 – Plotted intensity of Atom (blue) and field (orange) over parameterized time. We can see that the interaction between atom and field gives rise to cycle of superemission-superabsorption by the atom and a reciprocal cycle of superabsorption-superemission by the field. Here we have parameterized time with respect to  $g = 1$  and all the system variables are equal to zero at  $t = 0$  except for  $\langle S_z(0) \rangle = 0.5$  and the values of the parameters used here were  $N = 10^3$ ,  $\omega_0 = 10^2$ ,  $\gamma = 10^{-3}$ ,  $g = 1$ .

Source: By the author

It is also very useful to analyze the population/energy  $\langle\sigma_+\sigma_-\rangle$  and  $\langle a^\dagger a \rangle$  of the atom-field system in time, which is given in figure 6 below for the same values for the parameters and initial conditions as 5.

In figure 5, as stated before, we can see the reciprocal alternation between the emissions of the field and the atom. This phenomenon happens when the photons emitted by the sample during its superemitting part of the cycle are stored in the cavity and then reabsorbed by the sample once the superemission stops, which again raises the population

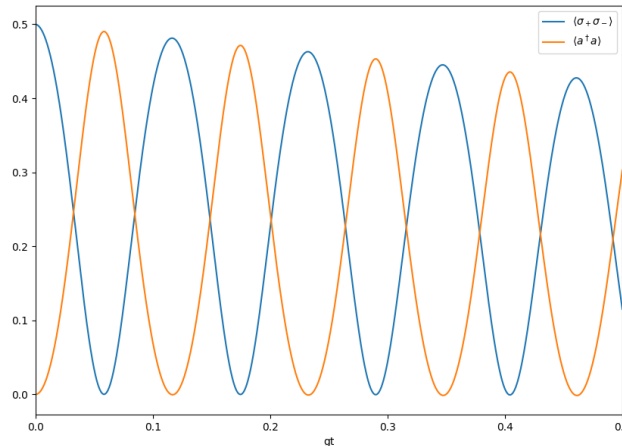


Figure 6 – Population of the states in time for the same values of the parameters and initial conditions used in 5. It is possible to see the gradual decline in the total energy ( $\langle\sigma_+\sigma_-\rangle + \langle a^\dagger a \rangle$ ).

Source: By the author

of the excited state stimulating the atoms to emit again therefore restarting the cycle. When subtracting the intensities, the difference yields

$$\Delta\mathcal{I} = \mathcal{I}_A - \mathcal{I}_F = -\frac{\omega_0}{2}\gamma N^2(\langle S_x \rangle^2 + \langle S_y \rangle^2), \quad (2.122)$$

for all times. Considering the mechanism described before, this difference represents the total radiation emitted by the atom that is not stored in the cavity, and therefore lost to the environment. When put in terms of  $\langle S_z \rangle$  making use of the constraint 2.69 this dissipation in 2.122 becomes

$$\Delta\mathcal{I} = -\frac{\omega_0}{2}\gamma N^2\left(\frac{1}{4} - \langle S_z \rangle^2\right), \quad (2.123)$$

which comes from the dissipative (imaginary) part of 2.107. It can be seen that the absolute value of  $\Delta\mathcal{I}$  is related to  $\langle S_z \rangle^2$  in an inverse manner, and since from 2.69 we have that  $\langle S_z \rangle \leq 1/2$  and  $\langle S_z \rangle = \langle\sigma_+\sigma_-\rangle - 1/2$  it can be inferred that the smaller the population of the state, the greater is the dissipation.

Figures 7 and 8 show the intensity for higher values of  $N$  and  $\gamma$  considering the same initial conditions (time and  $\omega_0$  parameterized in terms of  $g$ ).

From 5, 7 and 8 it is evident that the atom-field coupling greatly increases the effects of the superabsorption that accompany the superradiant emission while decreases the duration of each pulse and therefore raises the frequency of the rotation of the dipoles of the sample's atoms when comparing to the trivial superradiant case  $g = 0$ . In fig 9 we compare the time evolution of the population  $\langle\sigma_+\sigma_-\rangle$  of the excited state for the cases of regular superradiation and for the atom-field coupling case so the difference in the duration of the process can be better seen. As we will see in the next chapter, these

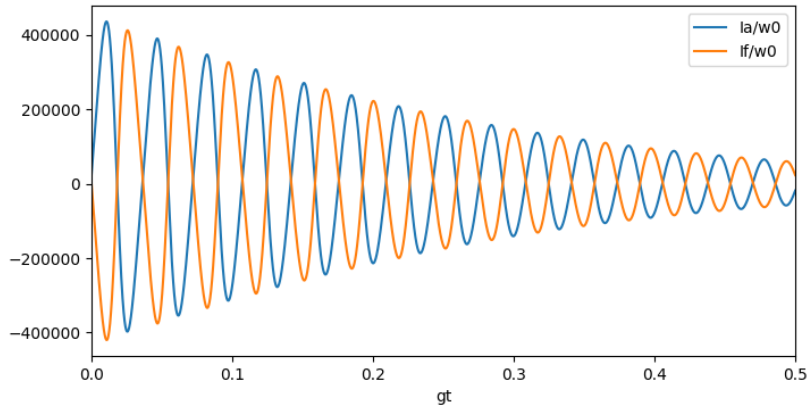


Figure 7 – Intensity plotted for the values of  $N = 10^4$ ,  $\omega_0 = 10^2$ ,  $\gamma = 10^{-3}$ . It is possible to observe the quick dissipation compared to 5. Here the last term in the Hamiltonian 2.107, which is the dissipative term, starts becoming significant in relation to both  $\omega_0$  and  $\sqrt{N}g$  and we can notice both the augmented damping effect as well as the raising in the frequency of the Rabi oscillations due to the larger number of atoms.

Source: By the author

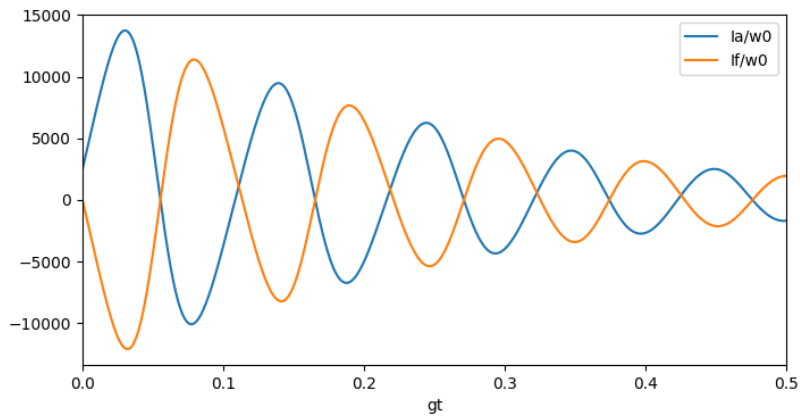


Figure 8 – Intensity plotted for the values of  $N = 10^3$ ,  $\omega_0 = 10^2$ ,  $\gamma = 10^{-2}$ . Very similar to 5 in frequency and amplitude during the first oscillations.

Source: By the author

differences will have an effect in the momentum state of the sample, which during the interaction will suffer a small deflection that will be studied with the use of the methods applied in (15).

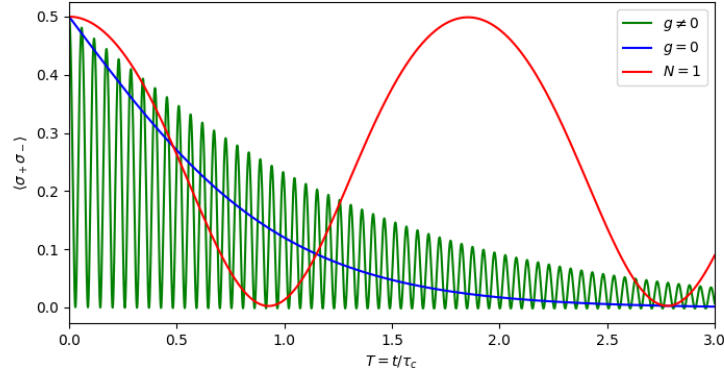


Figure 9 – Population of the excited state in parameterized time for the same values of the parameters used before. The blue line represents the ordinary superradiant decay ( $g = 0$ ) described in section 2.1.4 while the green line describes the population of the field-coupled sample. It is possible to see that the transition time of the latest is much smaller in comparison (here we use  $\tau_c = 2 \times 10^{-3}/\gamma$ ). Source: By the author

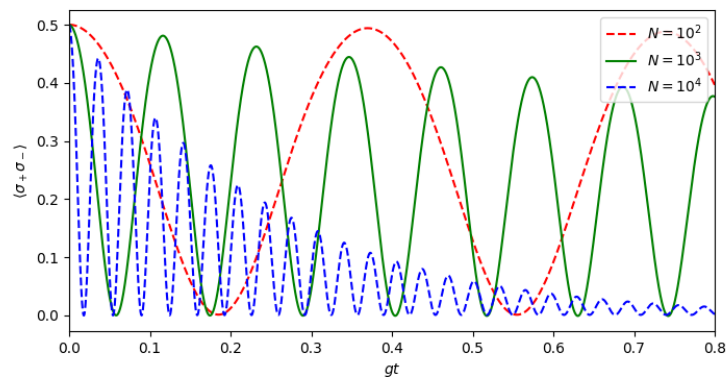


Figure 10 – Population/energy of the excited state for samples with different numbers of atoms and the same value of  $\gamma = 10^{-3}$ . As mentioned before and seen in equation 2.123 the larger the number of atoms in the system the more intense is the emission and the faster the energy will dissipate. Source: By the author





### 3 SAMPLE DEFLECTION AND STERN-GERLACH EFFECT

#### 3.1 Deflection of the single-atom state

With the considerations of the previous chapters and the introduction of the concepts of superradiance, superabsorption, solution with the use of Lewis-Riesenfeld method and Rabi many-body oscillations (1) we are now in condition to analyze some of the effects of these phenomena on an atomic level. In this chapter we will introduce the concept of the deflection, or the alteration of the momentum/position state of an atom subjected to an interaction Hamiltonian such as the Jaynes-Cummings Hamiltonian, and devise an experiment that could properly detect such deflection.

When having an atomic sample of two-level systems tracing a trajectory in an external electromagnetic field it is expected that due to the difference in the interactions of the states with the field a splitting between their paths will occur (29) in what is called the Stern-Gerlach effect (30–32). From the analysis of the deflection suffered by the states it is possible to draw information from both the atomic sample and the field(15). We consider therefore an atomic sample in a superposition of ground and excited states given by

$$|\psi_a\rangle = \frac{1}{\sqrt{2}}(|g\rangle + e^{i\phi}|e\rangle), \quad (3.1)$$

interacting with a standing field in a superposition of Fock states

$$|\psi_f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad (3.2)$$

via the interaction Jaynes-Cummings Hamiltonian defined below

$$H_{JC} = -\mu\mathcal{E} \sin(kx)(a\sigma_+ + a^\dagger\sigma_-), \quad (3.3)$$

which is a single-atom version of 2.88.

Considering the mathematical form of 3.3, if we construct a beam of atoms that passes through a slit with opening  $\Delta x \ll \lambda$  centered at  $x = 0$ , as illustrated in 11, we can rewrite the interaction with the approximation  $\sin(kx) \approx kx$  having then a Hamiltonian linear in  $x$  given by

$$H_{JC} = -\mu\mathcal{E}kx(a\sigma_+ + a^\dagger\sigma_-). \quad (3.4)$$

Now, to analyze the deflection of an atom is necessary to take into consideration its wave function in position space. Considering the atom in the state 3.1, if we associate with it a normalized position distribution function  $f(x)$  we can write its de Broglie positional

state

$$|\psi_{pa}\rangle = \int_{-\infty}^{\infty} dx f(x) |x\rangle |\psi_a\rangle = \int_{-\infty}^{\infty} dx f(x) |x\rangle (|g\rangle + e^{i\phi} |e\rangle), \quad (3.5)$$

where we define  $f(x)$  in such a way as to contain the normalization factors of the whole state. With the definitions given above, we can write the complete system state at  $t = 0$  as a product of the atom and field states and project it in position space giving us

$$\begin{aligned} |\Phi(x', t = 0)\rangle &= \langle x' | \psi_{pa}\rangle |\psi_f\rangle \\ &= \frac{f(x')}{\sqrt{2}} \sum_{n=1}^{\infty} [(c_n + e^{i\phi} c_{n-1}) |+, n\rangle + (c_n - e^{i\phi} c_{n-1}) |-, n\rangle] + f(x') c_0 |g\rangle, \end{aligned} \quad (3.6)$$

where we have used the "dressed" states  $|\pm, n\rangle = (|g\rangle |n\rangle \pm |e\rangle |n-1\rangle)/\sqrt{2}$  which are eigenstates of the operator  $a\sigma_+ + a^\dagger\sigma_-$ .

Now, in order for us to evolve the state 3.1 in time and find the expression for the general state that describes the atom during the experiment we must take into consideration the duration of the interaction. We then divide the experiment in three phases. *i)* The preparation of the atom beam before entering the cavity field. *ii)* The interaction between field and atom during the crossing of the cavity, which we will assume happens in a time  $\tau$ . *iii)* the exiting of the field, happening at a time  $t > \tau$  where the particle is once again governed by the free particle Hamiltonian. If we consider now that the trajectory of the atom is much smaller than the wavelength  $\lambda$  of the field, in such a way that it allow us to make use of the Raman-Nath regime (33) and discard the role of the kinetic energy in the evolution of the atom state during the interaction we are able to evolve 3.1 into

$$\begin{aligned} |\Phi(x, \tau)\rangle &= \frac{f(x)}{\sqrt{2}} \sum_{n=1}^{\infty} [e^{i\sqrt{n}\kappa kx} (c_n + e^{i\phi} c_{n-1}) |+, n\rangle + e^{-i\sqrt{n}\kappa kx} (c_n - e^{i\phi} c_{n-1}) |-, n\rangle] \\ &\quad + f(x) c_0 |g\rangle |0\rangle, \end{aligned} \quad (3.7)$$

where  $\kappa = \mu\mathcal{E}\tau$  is defined as the interaction parameter. In order to better analyze the impact of the interaction on the momentum of the particle we take the Fourier transform of the state 3.1 following a procedure similar to that on Refs. (15, 32). Defining the dimensionless length and momentum variables as  $\theta = kx$  and  $\mathcal{P} = p/k$ , the transform of the distribution function  $f(x)$  can be given by

$$\hat{f}(\mathcal{P} \pm \sqrt{n}\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\mathcal{P} \pm \sqrt{n}\kappa)\theta} \frac{f(\theta/k)}{\sqrt{2}} \frac{d\theta}{k}, \quad (3.8)$$

such that we can write the Fourier transform of the total state as

$$\begin{aligned} |\Phi(\mathcal{P}, \tau)\rangle &= \sum_{n=1}^{\infty} [\hat{f}(\mathcal{P} - \sqrt{n}\kappa) (c_n + e^{i\phi} c_{n-1}) |+, n\rangle + \hat{f}(\mathcal{P} + \sqrt{n}\kappa) (c_n - e^{i\phi} c_{n-1}) |-, n\rangle] \\ &\quad + \sqrt{2} c_0 \hat{f}(\mathcal{P}) |g\rangle |0\rangle. \end{aligned} \quad (3.9)$$

With the system state function defined in the momentum space given above, it is easy to see that the term  $\sqrt{n}\kappa$  originated from the time-evolution operator associated with

the Hamiltonian 3.4 will have a different effect on the particle depending on its state at the start of the interaction. In order to better illustrate this let's consider the probability density function of our state given by

$$W(\mathcal{P}) = k \sum_{n=1}^{\infty} [|\hat{f}(\mathcal{P} - \sqrt{n}\kappa)|^2 |c_n + e^{i\phi} c_{n-1}|^2 + |\hat{f}(\mathcal{P} + \sqrt{n}\kappa)|^2 |c_n - e^{i\phi} c_{n-1}|^2] + 2k |\hat{f}(\mathcal{P})|^2 |c_0|^2, \quad (3.10)$$

where we used the orthogonality of the states  $|\pm, n\rangle$  and  $|g, 0\rangle$  that make the basis of the infinite dimensional  $\mathcal{H}_{(2,\infty)} = \mathcal{H}_2 \otimes \mathcal{H}_\infty$  Hilbert space that describes our system together with the normalization of the state  $|\Phi(\mathcal{P}, \tau)\rangle$  in terms of  $p = \mathcal{P}k$  in order to make  $\int_{-\infty}^{\infty} W(\mathcal{P}) d\mathcal{P} = 1$ . Defining now the distribution function  $f(x)$  as a Gaussian packet defined bellow\*

$$f(x) = \left( \frac{1}{4\pi\Delta x^2} \right)^{\frac{1}{4}} e^{-\frac{x^2}{2\Delta x^2}}, \quad (3.11)$$

in such a way that the function  $\hat{f}$  will be given as

$$\hat{f}(\mathcal{P} \pm \sqrt{n}\kappa) = \left( \frac{\Delta x^2}{64\pi} \right)^{\frac{1}{4}} e^{-\frac{k^2 \Delta x^2}{2} (\mathcal{P} \pm \sqrt{n}\kappa)^2}, \quad (3.12)$$

which is a Gaussian centered at  $\mathcal{P} \pm \sqrt{n}\kappa$  it becomes clear that the different states will lead to different momentum distributions. For high values of  $\kappa$  we can see in eqs. 3.1 and 3.12 that once the terms proportional to the non-vanishing exponentials will be  $e^{-(a\sqrt{n}\kappa)^2} \ll 1$  the probability density becomes

$$W(\mathcal{P} = \pm \sqrt{n}\kappa) \approx \left( \frac{k^2 \Delta x^2}{64\pi} \right)^{\frac{1}{2}} |c_n \pm e^{i\phi} c_{n-1}|^2, \quad (3.13)$$

which shows us two main things: *i*) That the atom beam will receive momentum in the  $x$  direction from the cavity field entirely dependent on the field's state, the intensity of the coupling and, as mentioned before, the state of the atoms before the interaction. *ii*) Due to the superposition of positional states in 3.1 we will observe that the difference in momentum received by each of those basis states will split the beam into two momentum distributions peaking at  $\mathcal{P} = \pm \sqrt{n}\kappa$  that will reflect themselves in the spatial distribution seen in Fig. 11.

From the atomic distribution constructed by the detectors behind the screen it is possible to reconstruct the momentum distribution and moreover the state of the interacting field following the procedures in (15).

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\* Recalling that we defined  $f(x)$  such that it includes the normalization of the state 3.1

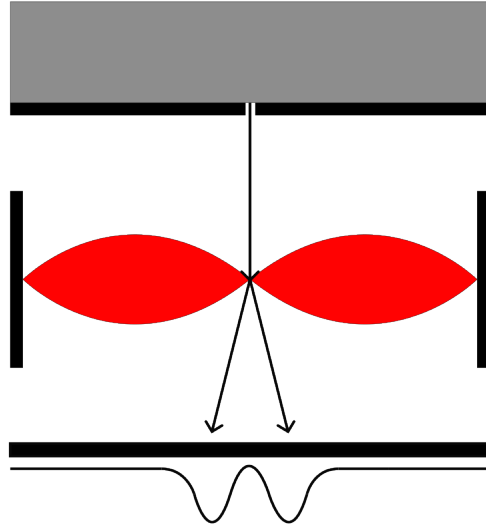


Figure 11 – Illustration of optical Stern-Gerlach experiment found in reference (15). After the superposition state is prepared and the sample is accelerated with the use of a classical field. Once entering the cavity, the field-atom interaction splits the beam symmetrically in two directions on the  $x$  axis. The atom impact distributions are represented by the two bumps in the screen.

Source: By the author

### 3.2 The deflection of the atom-field system

Once we have presented all the necessary fundamentals, we are finally in condition to introduce the core concepts of this dissertation. Furthermore, in this section we combine the techniques already discussed in this chapter as well as those seen in Refs. (31, 33, 34) with the concept of the interplay between superradiance and superabsorption such as to describe the interactions within a cycle of many-body oscillations in terms of the momentum exchange between field and atom, and estimate the deflection suffered by an atomic sample prepared in a superposition state trapped in a cavity.

#### 3.2.1 The solution states for the atom-field interplay

When trying to analyze the interaction between field and atom in the interplay defined in section 2.2 and study it under the lenses of the techniques seen in the previous section, it is necessary to find the solution states of the mean-field Hamiltonian 2.108 in time. Writing down the solution as a product of the ket states for the atom and the field will allow us to define a positional state as done in 3.1 and reproduce all the developments done before.

First, to write down the solution states we may remember that, since 2.111 holds true, the atom-field Hamiltonian 2.108 can be written as a direct sum of Hamiltonian operators in different Hilbert spaces, one being the two-dimensional Hilbert space defined by the  $SU(2)$  algebra in 2.48 associated with the atom operators and the other is the infinite-dimensional Fock space defined by the algebra in 2.116. Even so, when looking at the Schrödinger equation of the system given below

$$i\frac{\partial}{\partial t}|\psi_{AF}(t)\rangle = H_{AF}(t)|\psi_{AF}(t)\rangle = (H_A(t) + H_F(t))|\psi_{AF}(t)\rangle, \quad (3.14)$$

it is not clear that the dynamics of the atom and the field can be separated since the state vector in 3.14 is one for the whole system. In order to solve this problem, we may recall that the Green function of eq. 3.14 is given by the separable form

$$\begin{aligned} U_{AF}(t, t') &= T \left[ \exp \left( -i \int_t^{t'} H_A(\tau) + H_F(\tau) d\tau \right) \right] = \\ &= T \left[ \exp \left( -i \int_t^{t'} H_A(\tau) d\tau \right) \right] T \left[ \exp \left( -i \int_t^{t'} H_F(\tau) d\tau \right) \right] = U_A(t, t') U_F(t, t'), \end{aligned} \quad (3.15)$$

where

$$T[A(t_1)B(t_2)] = \Theta(t_1 - t_2)A(t_1)B(t_2) + \Theta(t_2 - t_1)B(t_2)A(t_1), \quad (3.16)$$

is the time-ordering operator and  $\Theta(\tau)$  is the Heaviside step function. This allow us to write one separate Schrödinger equations for each evolution operator, namely

$$\left( i\frac{\partial}{\partial t} - H_{A \rightarrow F}(t) \right) U_{A \rightarrow F}(t, t') = \delta(t - t'), \quad (3.17)$$

which in turn means we can also separate the state in 3.14 into two solutions  $|\psi_{AF}\rangle = |\psi_A\rangle |\psi_F\rangle$  each with its separate time evolution given by

$$|\psi_{A \rightarrow F}(t)\rangle = U_{A \rightarrow F}(t) |\psi_{A \rightarrow F}(0)\rangle. \quad (3.18)$$

Now with this in mind, we are able to utilize the LR method in a similar manner to what was done in the case of the superradiating atom interacting with a thermal bath solved in section 2.1.3. As such, we are now in need for two invariants to fully describe the system, one whose eigenvectors will give us the atom's solution and one which will give us the field's. For the first, when analyzing the Hamiltonian in 2.115 we can notice that, such as the superradiating Hamiltonian, it is of the form 2.64 and therefore can be also be associated with the invariant given in 2.68. Furthermore, it can be described by its eigenvectors given in 2.73 and 2.74 differing only in the phase which in our new case will be given by the following expression

$$\Phi_{\pm}^A(t) = -\frac{\omega_0 t}{2} \mp \int_0^t \Lambda_R \sin^2 \left( \frac{\theta}{2} \right) dt', \quad (3.19)$$

which for the case  $g = 0$  reduces to the regular superradiant state's phase seen in equation 2.77.

Now that we have characterized both the LR state and its phase we are able to write the general atom state as

$$|\psi_A(t)\rangle = C_+ e^{\Phi_+^A(t)} |+, t\rangle + C_- e^{\Phi_-^A(t)} |-, t\rangle, \quad (3.20)$$

where  $C_+$  and  $C_-$  are complex constants that obey the normalization condition for the state in 3.20.

However, when analyzing the value of  $\langle S_z \rangle$  with respect to the states in 3.20 we arrive at the following expression in terms of the coefficients  $C_+$  and  $C_-$

$$\langle S_z \rangle = \frac{1}{2} \left[ (|C_+|^2 - |C_-|^2) \cos \theta + 2 \operatorname{Re} \left\{ C_+ C_-^* e^{i(\Phi_+ - \Phi_-)} \right\} \sin \theta \right], \quad (3.21)$$

with this definition for  $\langle S_z \rangle$  however we break the definition of the atom invariant in 2.68. In order for the expression in 2.68 to remain invariant we need to equal 3.21 to the definition of  $\langle S_z \rangle$  used in 2.70c, this gives rise to the consistency conditions

$$|C_+|^2 + |C_-|^2 = 1, \quad (3.22a)$$

$$|C_+|^2 - |C_-|^2 = 2R, \quad (3.22b)$$

$$C_+ C_-^* = 0, \quad (3.22c)$$

where the first one is simply the state normalization for  $|\psi_A(t)\rangle$  and the last two indicate that either  $C_- = 0$  and  $|C_+| = \sqrt{2R}$  for  $R > 0$  or  $C_+ = 0$  and  $|C_-| = \sqrt{-2R}$  for  $R < 0$ . For the chosen value of  $R = 1/2$  or atom state will then be

$$|\psi_A(t)\rangle = e^{\Phi_+^A(t)} |+, t\rangle. \quad (3.23)$$

Regarding now the field part of the total state we need first to find an invariant that obeys the LR conditions for the field Hamiltonian  $H_F$ . Looking at Ref. (34), we can at first assume a general form for the invariant that is bi-linear in the field quadrature operators defined in the previous chapters with time dependent coefficients as given by the expression below

$$I_F = a^\dagger a - f(t) X_1 - f^*(t) X_2 + \chi(t), \quad (3.24)$$

where  $f(t)$  is a function to be determined briefly and  $\chi$  is a scalar complex function that generalizes the invariant.

Writing the field Hamiltonian in 2.109a in terms of the quadrature operators we have

$$H_F = \omega_0 a^\dagger a + 2\sqrt{N}g(\langle S_x \rangle X_1 - \langle S_y \rangle X_2), \quad (3.25)$$

and using the LR equation  $\partial_t I_F - i[I_F, H_F] = 0$ , from which we know that we must recover the field equations in the system 2.118, we find that the function  $f$  is such that we can write 3.24 as

$$I_F = a^\dagger a - 2\langle X_1 \rangle X_1 - 2\langle X_2 \rangle X_2 + \chi(t), \quad (3.26)$$

as well as an extra condition for  $\chi(t)$

$$\dot{\chi} = -\sqrt{N}g(\langle S_x \rangle \langle X_2 \rangle + \langle S_y \rangle \langle X_1 \rangle), \quad (3.27)$$

which will soon be relevant.

If we rewrite 3.26 in terms of the creation and annihilation operators, as done in (34) we will have

$$\begin{aligned} I_F &= a^\dagger a - \alpha^*(t)a - \alpha(t)a^\dagger + \chi(t) \\ &= (a^\dagger - \alpha^*)(a - \alpha) - |\alpha|^2 + \chi \\ &= b^\dagger b + \beta, \end{aligned} \quad (3.28)$$

where we define the operators

$$b = a - \alpha(t), \quad b^\dagger = a^\dagger - \alpha^*(t), \quad (3.29)$$

as well as the functions

$$\alpha(t) = \langle X_1(t) \rangle + i\langle X_2(t) \rangle, \quad (3.30a)$$

$$\beta(t) = \chi(t) - |\alpha(t)|^2. \quad (3.30b)$$

Looking now at  $\beta$  we know that in order for our invariant to remain Hermitian it needs to be a real number, and most importantly, taking its time derivative we have

$$\begin{aligned} \frac{d\beta}{dt} &= \frac{d}{dt} (\chi - \langle X_1 \rangle^2 - \langle X_2 \rangle^2) = \\ \dot{\chi} + \sqrt{N}g(\langle S_y \rangle \langle X_1 \rangle + \langle S_x \rangle \langle X_2 \rangle) &= 0, \end{aligned} \quad (3.31)$$

which is easily deduced with the use of 3.27 and the field equations of the system 2.118. From this, together with the definitions 3.29 we can write the field invariant in terms of the operator  $b^\dagger b$  given by

$$b^\dagger b = D[\alpha(t)]a^\dagger a D^\dagger[\alpha(t)], \quad (3.32)$$

where the displacement operator is given by  $D[\alpha(t)] = e^{\alpha a^\dagger - \alpha^* a}$ . The eigenvector basis of the operator  $I_F$  will therefore be equal to the eigenvectors of the displaced number operator  $b^\dagger b$  which are given by the displaced number basis of the original ladder operators

$$|n_b, t\rangle = D[\alpha(t)]|n\rangle. \quad (3.33)$$

From the definition in 2.62, the LR phase for the field wave function will be found to be

$$\Phi^F(t) = \int_0^t \omega_0 |\alpha|^2 - \frac{\sqrt{N}g}{4} (\alpha e^{i\phi} + \alpha^* e^{-i\phi}) \sin \theta dt', \quad (3.34)$$

for all states  $|n\rangle$  and the general field state will be given by

$$|\psi_F(t)\rangle = \sum_n c_n e^{i\Phi^F} D[\alpha(t)]|n\rangle. \quad (3.35)$$

Before defining the system's complete solution however, when looking at the value of  $\langle a \rangle$  in relation to the state defined in 3.35 we have

$$\begin{aligned} \langle \psi_F(t) | a | \psi_F(t) \rangle &= \sum_n \sum_m c_n c_m^* \langle m | D^\dagger[\alpha(t)] a D[\alpha(t)] | n \rangle = \\ &= \sum_n \sum_m c_n c_m^* \langle m | (a + \alpha) | n \rangle = \sum_n \sqrt{n} c_{n-1} c_n^* + \alpha \sum_n |c_n|^2, \end{aligned} \quad (3.36)$$

and yet, from the definitions in 3.30 it is easy to see that  $\alpha = \langle a \rangle$ , which can only be consistent with 3.36 if we have  $c_n = \delta_{n,0}$ . Therefore, applying this condition to 3.35 we are able to write our general field state as

$$|\psi_F(t)\rangle = e^{i\Phi^F(t)} D[\alpha(t)]|0\rangle = e^{i\Phi^F(t)} |\alpha\rangle, \quad (3.37)$$

with  $|\alpha\rangle$  being the coherent state of eigenvalue  $\alpha$ .

Finally, from all the considerations in this chapter, we are able to write the complete wave function of the system

$$|\psi_{AF}(t)\rangle = \eta(t) |+, t\rangle \otimes |\alpha(t)\rangle, \quad (3.38)$$

where we have defined the mutual phase  $\eta(t) = e^{\Phi_+ + \Phi^F}$ .

### 3.2.2 The approximate solution for the atom and field variables.

Now, with the state solution of the Schrödinger equation given by 3.38 in hands we are almost able to approach the coherent deflection of the atomic sample, where we will utilize the techniques used in section 3.1 for the deflection of an atomic beam as well as have some considerations on the experimental implementation of the process. Before proceeding any further however, it is necessary to determine the LR phases of both the atom and field in order to use the Freyberger-Herkommer method to analyse the deflection, and to do so, it is necessary to evaluate the integrals at 3.19 and 3.34. As they stand however, these integrals are not analytically solvable due to the complex time dependences present in the angles  $\theta(t)$  and  $\phi(t)$ , nonetheless it is possible to approximate solutions for these functions through the linearization of the system.

We start by rewriting the atom equations in 2.118 into a system for  $\theta(t)$  and  $\phi(t)$  using 2.70, with this the system's dynamic equations reduce to the four differential



equations bellow

$$\frac{d\theta}{dt} = \frac{N\gamma}{2} \sin \theta - \sqrt{N}g(\langle X_2 \rangle \cos \phi + \langle X_1 \rangle \sin \phi), \quad (3.39a)$$

$$\frac{d\phi}{dt} = \omega_0 - 2\sqrt{N}g(\langle X_1 \rangle \cos \phi - \langle X_2 \rangle \sin \phi) \cot \theta, \quad (3.39b)$$

$$\frac{d\langle X_1 \rangle}{dt} = -\frac{\sqrt{N}g}{2} \sin \theta \sin \phi + \omega_0 \langle X_2 \rangle, \quad (3.39c)$$

$$\frac{d\langle X_2 \rangle}{dt} = -\frac{\sqrt{N}g}{2} \sin \theta \cos \phi - \omega_0 \langle X_1 \rangle. \quad (3.39d)$$

When comparing these to the previous system in 2.118 it is clearer that the interaction parameter  $g$  is the main factor responsible for the phenomenon of many-body Rabi oscillations, such that when setting it to zero the first two equations of 3.39 reduce to the regular superradiating atom's system in 2.71 while the field variables give a trivial solution for an oscillating electromagnetic field of frequency  $\omega_0$ . Given this, it seems reasonable to assume a regime where the atom solutions can be expanded as

$$\theta(t) \approx \theta_h(t) + \epsilon\vartheta(t), \quad (3.40a)$$

$$\phi(t) \approx \phi_h(t) + \epsilon\varphi(t), \quad (3.40b)$$

with  $\theta_h(t)$  and  $\phi_h(t)$  being the solutions of the homogeneous system 2.71 given in 2.72<sup>†</sup>,  $\vartheta$  and  $\varphi$  the new functions in terms of which the atom equations will be rewritten and  $\epsilon$  our perturbative parameter. From the arguments given above, it seems natural to define the parameter  $\epsilon$  in terms of  $g$ , since both need to be zero in the regular superradiative case. However,  $\epsilon$  needs to be regularized in order for our approximations in 3.40 to work. With these considerations and looking at 3.39a, we define  $\epsilon = \sqrt{N}g/N\gamma$  as our expansion parameter.

From the definition of  $\epsilon$ , we observe that we have three regimes for solutions of our superradiance-superabsorption interplay: (i) the under-damped ( $\epsilon \gg 1$ ), (ii) the damped ( $\epsilon \approx 1$ ), (iii) the over-damped ( $\epsilon \ll 1$ ). Here we focus on the over-damped regime where, as said before, an approximated analytic solution in the form 3.40 can be obtained, which also applies, although with less accuracy, to the damped regime.

Now, looking at the solution for the polar angle, namely  $\phi(t) = \phi_0 + \omega_0 t + \epsilon\varphi(t)$ , it is possible to discard the last term once  $\epsilon/\omega_0 \ll \epsilon^2$  which comes from the consideration that  $\omega_0 \sim 10^5 g$ , we can therefore write  $\phi(t) \approx \phi_h(t)$ . With this, and given the fact that we can rewrite the field part of the system of equations in 3.39 as a single differential equation for the function  $\alpha(t)$  defined in 3.30a, we have

$$\frac{d\alpha}{dt} = -i \left[ \omega_0 \alpha + \frac{\sqrt{N}g}{2} \sin \theta e^{-i\phi(t)} \right]. \quad (3.41)$$

<sup>†</sup> Recalling that we have disregarded the last term in 2.71b once  $\omega_0 \gg \Delta\omega$ .

From our initial conditions we have  $\alpha(0) = 0$ , we are able to write the solution to 3.41 as

$$\alpha(t) = -\frac{\sqrt{N}g}{2}(\sin \phi(t) + i \cos \phi(t)) \int_0^t dt' \sin \theta. \quad (3.42)$$

Defining now  $\alpha = |\alpha|e^{i\phi_\alpha}$  we can relate the argument  $\phi_\alpha$  and  $\phi$  using the definition of  $\alpha$  in 3.30a and the equation above which gives us

$$\tan \phi_\alpha = \frac{\langle X_2 \rangle}{\langle X_1 \rangle} = \frac{\cos \phi(t)}{\sin \phi(t)} = \cot \phi(t), \quad (3.43)$$

from which we are able to write  $\phi_\alpha = \pi/2 - \phi = \pi/2 - \omega_0 t$  where we have defined  $\phi_0 = 0$  which can be done without any physical consequences. With these considerations we can rewrite the system 3.39 one more time for the variables  $\vartheta(t)$  and  $|\alpha(t)|$  having then

$$\frac{d\vartheta}{dt} = \frac{N\gamma}{2}(\vartheta \cos \theta_h - 4|\alpha|), \quad (3.44a)$$

$$\frac{d|\alpha|}{dt} = -\frac{\sqrt{N}g}{2}(\sin \theta_h + 4|\alpha|\epsilon). \quad (3.44b)$$

Solving the equation for  $|\alpha|$  above we have

$$|\alpha(t)| = \epsilon e^{-2\sqrt{N}g\epsilon(t-t_0)} |\theta_h(t) - \theta_0|, \quad (3.45)$$

and from the solution of 3.44a we have

$$\theta(t) = \theta_h(t) + 2\sqrt{N}g\epsilon t \operatorname{sech} \left( \frac{t-t_0}{\tau_c} \right), \quad (3.46)$$

where  $t_0$  and  $\tau_c$  are the delay time and characteristic emission time both defined in pg. 30-31.

With the above solutions, we proceeded to analyze the evolution of the state in eq. 3.38 focusing on the sum  $\Phi_+^A(x, t) + \Phi^F(x, t)$  where we are now considering the position dependence of the atom-field coupling. With the original definition of  $g$  based on the placement of the sample in the node of the field, as illustrated in fig. 4, we will have the interaction term given by  $g(x) = \mu\mathcal{E} \sin(kx)$ , which can be approximated by  $g = \mu\mathcal{E}kx$  since the sample is much smaller than the wavelength of the field as discussed before. This however will give rise to complications regarding the definition of the deflection as we will see soon. Furthermore, we now place the sample in the anti-node of the field as done in (35) allowing us to rewrite the interaction as  $g = \mu\mathcal{E} \cos(kx)$  with  $\mu$ ,  $\mathcal{E}$  and  $k$  standing respectively for the atomic dipole moment, the effective electric field per photon, and the wave-vector of the cavity mode.

### 3.2.3 Atom-field deflection and the splitting of the sample's trajectory via Stern-Gerlach effect

Having now both the solution given in 3.38 and an approximation for the angles  $\theta$  and  $\phi$  as well as the function  $\alpha(t)$  we are now able to approach the coherent deflection of

the atomic sample, starting with some considerations on the experimental implementation of the process.

First, we must trap the sample using a convex potential. With the atoms initially in their ground states, the sample is placed near an anti-node of the standing-wave field, as mentioned before. Then, by manipulating the convexity of the trap potential, a moderately dense atomic sample is built. Preparing now the system in such a way as to have all atoms being in the excited state at the time set to be  $t = 0$ , corresponding to the atomic state  $|\psi_A(0)\rangle = \cos(\theta_0/2) |e\rangle + e^{i\phi} \sin(\theta_0/2) |g\rangle$  we then turn off the trapping potential, making the sample exit the cavity under gravity (or accelerated by a classical electric field) and initiating the superemission process. Figure 12 illustrates the scheme of the experiment.

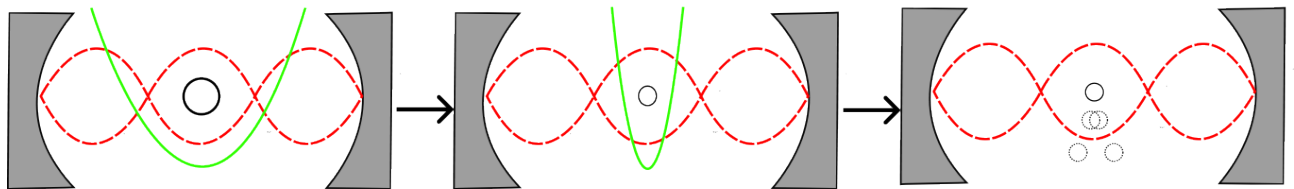


Figure 12 – Illustration of the experiment suggested to detect the deflection caused by superradiant emission. First the potential traps the sample positioned in the anti-node, then the potential's concavity is altered creating a moderately dense sample that enters the excited state. In the last part the trapping potential is turned off and the sample is left to fall under gravity or accelerated by a classical EM field leaving the cavity at around  $t = t_0$ . As the sample falls the different states acquire different momenta leading to a split trajectory.  
Source: By the author

Considering the cavity as being initially in the vacuum, following (15) our state is then given by

$$|\Psi_{AF}(t=0)\rangle = \int_{-\infty}^{+\infty} \Theta(x) |+,0\rangle \otimes |\alpha(0)\rangle \otimes |x\rangle dx, \quad (3.47)$$

where  $\Theta(x)$  is the initial distribution of the sample when sent through the cavity, we thus obtain, after the interaction time  $t$ ,

$$|\Psi_{AF}(t)\rangle = \int_{-\infty}^{+\infty} \eta(t)\Theta(x) |+,t\rangle \otimes |\alpha(t)\rangle \otimes |x\rangle dx. \quad (3.48)$$

Since the sample is released from the trap with zero velocity, we note that the Raman-Nath regime, by which the kinetic energy of the sample is neglected, by assuming that its transverse displacement along the interaction time is small compared to the wavelength of the mode, is here perfectly observed.

Next, by projecting the state  $|\Psi_{AF}\rangle$  onto the position space and expanding the state  $|+, t\rangle$  into (remembering it to be a superposition of the states  $|e\rangle$  and  $|g\rangle$ )

$$|+, t\rangle = \frac{1}{\sqrt{2}}[e^{i\theta(x,t)} |r, t\rangle + e^{-i\theta(x,t)} |l, t\rangle], \quad (3.49)$$

where we define  $|r, t\rangle = (|e\rangle + e^{i\phi(t)} |g\rangle)/\sqrt{2}$  and  $|l, t\rangle = (|e\rangle - e^{i\phi(t)} |g\rangle)/\sqrt{2}$ , we write

$$\begin{aligned} \langle x | \Psi_{AF} \rangle = |\Psi_{AF}(x, t)\rangle &= \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} \Theta(x, t) \left[ e^{i(\theta_h(t) + 2\sqrt{N}gct \operatorname{sech}(\frac{t-t_0}{\tau_c}))} |r, t\rangle + \right. \\ &\quad \left. + e^{-i(\theta_h(t) + 2\sqrt{N}gct \operatorname{sech}(\frac{t-t_0}{\tau_c}))} |l, t\rangle \right] \otimes |\alpha(t)\rangle, \end{aligned} \quad (3.50)$$

noting that, from 3.34 and 3.19, in the over-damped regime we have  $\Phi_+^A(t) = -\omega_0 t$  and  $\Phi^F(t) = 0$ . Defining the effective interaction parameter

$$\kappa = (\mu\mathcal{E})^2 \frac{t}{\gamma} \operatorname{sech}\left(\frac{t-t_0}{\tau_c}\right), \quad (3.51)$$

and Fourier transforming the state vector  $|\Psi_{AF}(x, t)\rangle$  over the momentum representation, it follows that

$$|\Psi_{AF}(p, t)\rangle = \frac{1}{\sqrt{2}} \left[ e^{-i\phi_+} \mathcal{F}_+(p) |r, t\rangle + e^{-i\phi_-} \mathcal{F}_-(p) |l, t\rangle \right], \quad (3.52)$$

with  $\phi_{\pm} = [\omega_0 t \mp \theta_h(t)]/2$ , and

$$\mathcal{F}_{\pm}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i[p x \mp \kappa \cos^2(kx)]} \Theta(x) dx \quad (3.53)$$

Next, we must decide a functional form for  $\Theta(x)$ , following the example in (15) we assume that the spatial distribution of atoms is a narrow Gaussian of width  $\sigma$ , shifted from the origin, such that  $\Theta(x) = \exp[-(x - x_0)^2/\sqrt{2\pi}\sigma^2]$ . In the case where the atomic sample is released from the trap under the action of gravity, this displaced spatial distribution, can be achieved through a proper positioning of the trap in relation to the anti-node. Considering a small enough width of the Gaussian  $k\sigma \ll 1$ , we linearize  $\cos^2 kx$  around  $x = x_0$ . Now, fixing  $kx_0 = \pi/4$ , we obtain  $\cos^2 kx \approx 1/2 - k(x - x_0)$  by expanding the cosine squared in a Taylor series. With these considerations, and defining the scaled atomic momentum  $\mathcal{P} = p/k$  we can perform the integration in eq. 3.53 and rewrite it, apart from a irrelevant phase factor  $e^{\pm i/2}$ , as

$$\mathcal{F}_{\pm}(\mathcal{P}) = \frac{1}{\sqrt{2\pi}} e^{-(k\sigma)^2(\mathcal{P} \pm \kappa)/2} \quad (3.54)$$

Looking at the exponential in eq. 3.53 we can now see the reason to position the particle in the anti-node, as opposed to the usual positioning in the node as seen in fig. 4. Having a sine in the interaction term, and therefore in the aforementioned exponential, would force a linearization involving  $\sin(kx) \approx kx$ , which in turn would lead to a  $(kx)^2$

in the exponential that would not directly translate to an alteration in the momentum states and a splitting in the trajectory of the sample. The cosine allows us to perform a linear expansion that eliminates this factor.

To characterize the superradiant-superabsorption cycle and to estimate the momentum of the deflected atoms in states  $|r, t\rangle$  and  $|l, t\rangle$  we turn to the expressions for the intensities in 2.119 and 2.121. Assuming a mesoscopic sample with  $N = 10^6$  and the following values  $\omega_0 = 10^5 g$ ,  $\theta_0 = 2/N$  and  $\phi_0 = 0$  for the parameters and focusing first in the over-damped regime  $\gamma = 10^{-2} g$  such that  $\epsilon = 0.1$  we can see in fig. 13(a) the existence of a single superradiance-superabsorption (SR-SA) cycle even with the sample-field coupling turned on, which takes place around the delay time  $t_0 \approx 2/\sqrt{N}g$ .

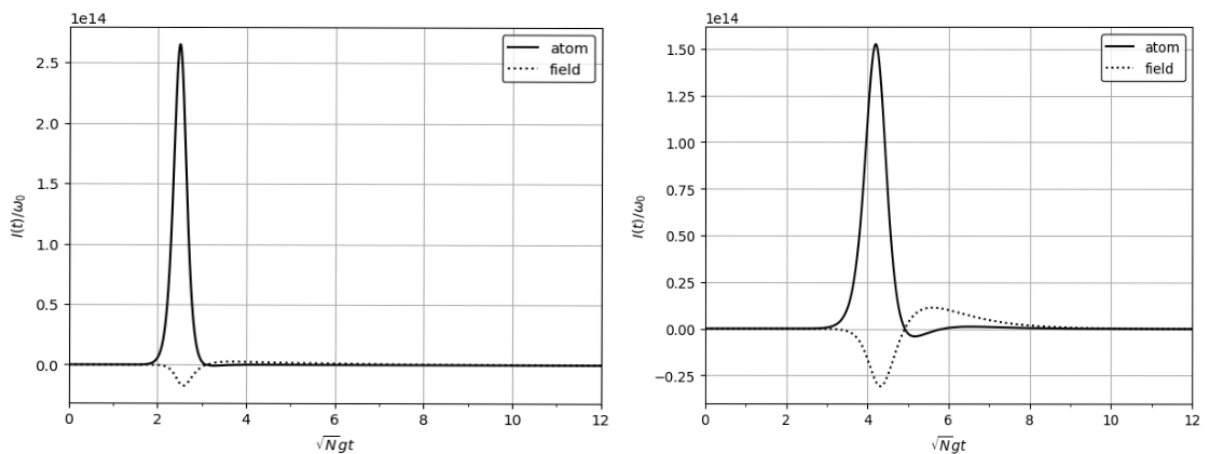


Figure 13 – Plot of the intensities  $\mathcal{I}_A$  and  $\mathcal{I}_F$  against  $\sqrt{N}gt$ , for  $N = 10^6, \omega_0 = 10^5 g$ ,  $\theta_0 = 2/N$  and  $\phi_0 = 0$ . To the left (a) we have the intensities in the overdamped regime with  $\gamma = 10^{-2} g$  with only one SR-SA cycle, whilst to the right (b) we have the intensities in the damped regime with  $\gamma = 10^{-3} g$  and two SR-SA cycles before the energy is completely dissipated.

Source: By the author

In Fig. 13(b), considering the damped regime with  $\gamma = 10^{-3} g$  and  $\epsilon = 0.2$ , we observe two superradiant-superabsorption cycles, the second clearly much less intense than the first. We can see that our approximate solutions also apply to the damped regime through Figs. 14(b) where, considering the same parameters as in Figs. 13(a and b), we plot  $\theta(t)$  against  $\sqrt{N}gt$  as the numerical solution to eqs. 3.39 (solid line) and the approximate solution eq. 3.46 (dotted line). Though not as precise as for the over-damped regime, the approximations still make a reasonable approximation for the damped regime as seen in fig.14(b).

As we see from fig. 13(b), the superradiant pulse initially emitted by the sample and superabsorbed by the field, can return to the sample, depending on the regime of the solution we have considered, thus resulting in as many cycles as allowed by the time of interaction between the atoms and the field. The number of superradiance-superabsorption

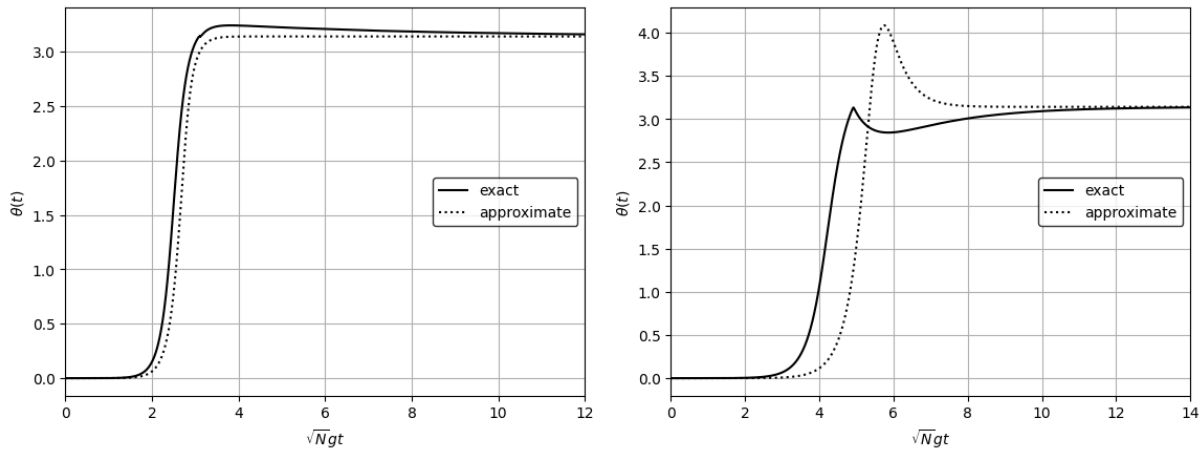


Figure 14 – Plot of the numerical/approximate solutions for  $\theta(t)$ . To the left (a) we have the solutions for the over-damped regime while to the right (b) we have them for the damped regime.

Source: By the author

cycles can be controlled via Stark effect, shifting the sample in and out of resonance with the field. We can see that the duration of the SR-SA cycle is of the order of  $1/\sqrt{N}g$  which is in agreement to what was seen in section 2.2.

Finally, in figs. 15(a and b), we plot  $\kappa$  against  $\sqrt{N}gt$  for the same parameters used in figs. 13(a and b), showing that the maximum momentum exchange between the sample and the field occurs at the delay time, as expected. We then note that with  $t_0 \approx 3/\sqrt{N}g$  for the over-damped regime and  $t_0 \approx 5/\sqrt{N}g$  for the damped as following from figs. 13 and 15, the effective interaction parameter renders a sample-field momentum transfer of maximum absolute value  $k\kappa = k(\mu\mathcal{E})^2 t_0/\gamma$ . From these images, we see that the experiment must be prepared such that the sample leaves the cavity at  $t = t_0$  to maximize transfer and therefore the effects of deflection.

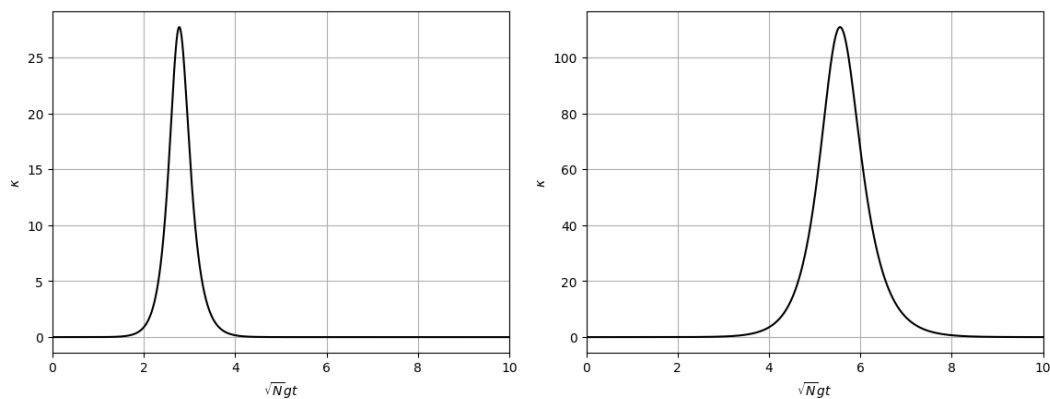


Figure 15 – Plotting of the momentum transfer for the over-damped (a) and damped (b) regimes. The momentum transfer is proportional to  $\kappa$  which tells us that the greatest momentum transfer occurs at  $t = t_0$ , after the transfer the sample is split into two beams.

Source: By the author

## 4 CONCLUSIONS

From the considerations discussed in our work we are able to conclude that the use of the interplay between superradiance and superabsorption, advanced in ref. (1), proves to be a very suitable tool to achieve a coherent deflection of an atomic sample and consequently to achieve a long-sought goal: the preparation of a positional mesoscopic Schrödinger cat-like state. The present proposal poses a challenge to the experimental physics of radiation-matter interaction, seeking to extend the remarkable advances achieved in the last 4 decades (36) to the domain of many-body physics. In particular, we observe that the present development, together with ref. (1), can be used for the proposition of a more efficient quantum lithography protocol based on the deflection of atomic samples instead of individual atoms as in ref. (19). Even though the approach used in this dissertation, i.e. solving the SR-SA equations via a linear approximation of the atom variables, cannot be used for the under-damped regime seen in (1), the present development can still be used in a variety of quantum computing systems such as the implementation of quantum processing with mesoscopic ensembles.





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