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Thermodynamic properties of causal horizons

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ABSTRACT

BARBOSA, M.G. **Thermodynamic properties of causal horizons**. 2020. 86p. Dissertation (Master's Degree in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

Black hole thermodynamics emerged as a surprising connection between thermodynamics and general relativity, showing that one may attribute properties like entropy and temperature to a black hole's event horizon. Inspired by it, studies from the last decades indicate that this analogy extends itself to many models and systems in which one considers the existence of causal horizons. Given certain assumptions, it is even possible to derive Einstein's field equations from the laws of thermodynamics. Nevertheless, this result was not constructed on foundations so solid as those of black hole thermodynamics, leaving room for a formalization of these ideas. Therefore, in order to advance in the generalization of the relation between thermodynamics and gravitation, one may focus on the study of the geometry of causal horizons, which correspond to lightlike hypersurfaces of a spacetime. In this work, we review the laws of black hole thermodynamics, giving special attention to the generalized second law, and show how the mentioned analogy is formulated for general causal horizons. Then, we present the induced geometric objects on lightlike hypersurfaces, providing the basic mathematical tools to analyze the geometry of causal horizons.

Keywords: Black hole thermodynamics. Causal horizons. Lightlike hypersurfaces.

RESUMO

BARBOSA, M.G. **Propriedades termodinâmicas de horizontes causais**. 2020. 86p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

A termodinâmica de buracos negros surgiu como uma surpreendente conexão entre termodinâmica e relatividade geral, mostrando que é possível atribuir propriedades como entropia e temperatura ao horizonte de eventos de um buraco negro. Inspirados por ela, estudos feitos nas últimas décadas indicam que esta analogia se estende para muitos modelos e sistemas nos quais é considerada a existência de horizontes causais. Dadas certas suposições, é possível inclusive derivar as equações de campo de Einstein a partir das leis da termodinâmica. Contudo, tal resultado não foi construído sobre bases tão sólidas quanto aquelas da termodinâmica de buracos negros, abrindo espaço para uma formalização destas ideias. Desta forma, a fim de avançar na generalização da relação entre termodinâmica e gravitação, é possível focar no estudo da geometria de horizontes causais, os quais correspondem a hipersuperfícies tipo-luz de um espaço-tempo. Neste trabalho, serão revisadas as leis da termodinâmica de buracos negros, dando atenção especial para a segunda lei generalizada, e será mostrado como a analogia mencionada é formulada para horizontes causais genéricos. Em seguida, serão apresentados os objetos geométricos induzidos em hipersuperfícies tipo-luz, fornecendo as ferramentas matemáticas básicas para analisar a geometria de horizontes causais.

Palavras-chave: Termodinâmica de buracos negros. Horizontes causais. Hipersuperfícies tipo-luz.

LIST OF FIGURES

LIST OF SYMBOLS

X, Y, ω — Capital letters are used to represent vectors and tensors in general. The letter ω may be used to represent dual vectors when we want to distinguish them from vectors.

$X^a, Y_b, Z^a{}_a$ — Superscript indices are used to represent components of a vector or tensor with respect to a basis of the vector space. Subscript indices are the equivalent for the dual vector space. When stated that X_a is a *vector*, the subscript specifies the vectors in a set. Similarly, for *dual vectors* a superscript denotes an element of some set, like ω^a. Following the Einstein notation, repeated indices imply in a sum over the indice's range, which vary from 0 to 3, unless otherwise stated.

$\mathcal{M}, \mathcal{S}$ — Calligraphic letters represent smooth manifolds or subsets of them, like surfaces, as well as their area.

T_pM, T_p^*M — The tangent space to a manifold $\mathcal{M}$ at a point p and its dual, denoted by a $*$.

TM, T^*M — The tangent bundle of a manifold $\mathcal{M}$, $TM = \cup_{p\in\mathcal{M}} T_pM$, and its dual, denoted by a $*$.

$g(X, Y)$ — The metric tensor field g over a manifold, applied to vector fields X and Y: $g(X, Y) = g_{ab}X^aY^b = X^aY_a$. Unless otherwise stated, it is not degenerate and can be used to raise and lower indices of vectors and one-forms (or tensors in general).

$\gamma(v)$ — A curve γ on a manifold parametrized by v.

$\frac{\mathrm{d}f}{\mathrm{d}v}, \frac{\partial f}{\partial v}$ — Ordinary derivative and partial derivative of a function f with respect to a parameter v.

$\nabla_X T$ — Covariant derivative of a tensor T, of components $T^a{}_b$, with respect to the vector field X. In terms of components, the covariant derivative is denoted by a semicolon: $(T^a{}_b)_{;c}X^c = T^a{}_{b;c}X^c$. Unless otherwise stated, ∇ represents the Levi-Civita (metric and torsion-free) connection, in which $(\nabla_X g)(Y, Z) = 0 = g_{ab;c}$.

$[X, Y](f)$ — Lie bracket of the vectors X and Y acting on a function f over a manifold, equal to $X(Y(f)) - Y(X(f))$.

$L_X Y$	Lie derivative with respect to the vector field X of the tensor Y. In case Y is a vector field, $L_X Y = [X, Y]$. We assume that ∇ is torsion-free, so in terms of components, $(L_X Y)^a = Y^a{}_{;b} X^b - X^a{}_{;b} Y^b$.
$\dfrac{\mathrm{D}T}{\mathrm{d}v}$	The covariant derivative of a tensor T along a curve $\gamma(v)$ with tangent vector X: $\mathrm{D}T/\mathrm{d}v = T^{a\ldots b}{}_{c\ldots d;e} X^e$.
$R(X, Y)Z$	The Riemann curvature tensor R applied to vector fields X, Y and Z: $R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$. In terms of the components: $R^a{}_{bcd} X^c Y^d Z^b = (Z^a{}_{;dc} - Z^a{}_{;cd}) X^c Y^d$.
$T_{(ab)}, T_{[ab]}$	The symmetric and antisymmetric parts of a tensor T_{ab} are denoted by round and square brackets, respectively. So $T_{(ab)} = (1/2!)(T_{ab} + T_{ba})$ and $T_{[ab]} = (1/2!)(T_{ab} - T_{ba})$. The indices which are used in the symmetrization (antisymmetrization) process are isolated from other indices by a $\vert$. So $T_{[a\vert bc\vert d]e}$, for example, equals to $(1/2!)(T_{abcde} - T_{dbcae})$.

CONTENTS

1 INTRODUCTION

"We must confess in all humility that, while number is a product of our mind alone, space has a reality beyond the mind whose rules we cannot completely prescribe."

(Carl Friedrich Gauss)

Geometry is perhaps the branch of mathematics most closely related to physics, laying the basis for and coevolving with classical mechanics as well as with many other modern theories. In particular, general relativity takes this relation to the point of being called geometrodynamics[1] and used and developed the differential geometry of semi-Riemannian manifolds in all of its subareas. One of these is the black hole thermodynamics, which emerged in the 1970s showing an interesting analogy between the laws that describe the evolution of black holes and the laws of thermodynamics, in a completely classical context.[2] This analogy got more attention after Hawking[3] had shown that, including the quantum effects of the vacuum in the surroundings of a black hole, this one irradiates and we can, therefore, assign a temperature and treat it as a thermodynamic object.

As more results were obtained reinforcing the connection between thermodynamics and black holes, it started to seem that there is some more profound meaning in the analogy beyond a simple mathematical coincidence. In fact, Jacobson[4] presented a paper in which he derives the Einstein's field equations from thermodynamic arguments. He assumes that, just like the area of the event horizon of a black hole corresponds to its entropy,[2] the area of an arbitrary causal horizon, like the past light cone of some observer, corresponds to the entropy of the spacetime's region delimited by it. Furthermore, assuming that there is a temperature associated to this horizon given by the Unruh temperature[5] and that the heat flux in this system is given by the flux of energy through the horizon, we may see general relativity as a consequence of the laws of thermodynamics.

Given the importance of causal horizons and their geometric properties in the spacetime thermodynamics, further studies in this area may require a knowledge of the geometry of this class of hypersurfaces, which are generated by null geodesics and called lightlike (or null) hypersurfaces. Thus, after demonstrating how to formulate the generalized second law for black holes and show how it can be extended to general causal horizons, leading to Einstein's field equations, we are going to review[6] some properties of lightlike hypersurfaces, in special the geometric objects induced by the surrounding spacetime.

2 THE CONNECTION BETWEEN GENERAL RELATIVITY AND THERMODYNAMICS

This chapter intends to provide an introduction to the so-called black hole thermodynamics and its proposed extension to general causal horizons, which arise as a consequence of causality and hide regions of spacetime from an observer's view. We focus in the correspondence between the area of spacelike surfaces contained in such horizons and the entropy generated by the flux of energy and momentum through these causal boundaries. To make this relation clearer, we start deriving the Raychaudhuri's equation in the first section, since it is fundamental in comprehending how the second law of black hole dynamics is formulated, what is shown in section 2.2. From this and the generalized second law of thermodynamics, presented in section 2.3, one should perceive the relevance of the area of horizons in this context. In sections 2.4 and 2.5, we present a way to generalize the temperature and entropy, respectively, first established for black holes' event horizons to other causal horizons. The last section also includes the arguments which lead to the interesting perspective that Einstein's field equations can be derived from the laws of thermodynamics, suggesting that gravity may emerge as a statistical phenomenon. In this chapter, it will be considered spacetimes with metric signature $(-,+,+,+)$ and geometrized units $c = G = 1$, unless otherwise stated. Moreover, the main references used to construct each section (or subsection) are cited in its title.

2.1 Raychaudhuri's equation[7]

A first approach to study the gravitational effects described by a 4-dimensional semi-Riemannian manifold $\mathcal{M}$ may consist in the comparison of the Newtonian theory with the acceleration of a test particle in relation to a static frame. This would imply the existence of a timelike Killing vector field[†], which is defined considering a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ that maps the metric tensor g into itself. Namely, the induced map $\phi_* : T_pM \to T_{\phi(p)}M$ applied to the metric $\phi_* g$ equals to g for every point $p \in \mathcal{M}$. Thus, the scalar product is preserved,

$$g(X,Y)|_p = \phi_* g(\phi_* X, \phi_* Y)|_{\phi(p)} = g(\phi_* X, \phi_* Y)|_{\phi(p)}, \tag{2.1}$$

and ϕ is called an isometry. Then W is a *Killing vector field* if it generates a local one-parameter group of diffeomorphisms ϕ_t, which are isometries for every t. Therefore, the Lie derivative

$$L_W g = \lim_{t\to 0} \frac{1}{t}(g - \phi_* g) = 0$$

implies the Killing's equation

$$W_{a;b} + W_{b;a} = 0.$$

† For more details, see Section 3.1 and the derivation of equation (3.6) or [8, Appendix C]. Killing vector fields will not be used in this section, but shall appear in the following ones.

However, such isometries and static frames may not be present in a general spacetime, so we could instead look for the relative acceleration between neighboring particles, or trajectories of photons, and analyze the deformations produced in a family of timelike or null curves. We focus then in this last type and in the expansion of a infinitesimal area with borders defined by these curves, yielding the so-called Raychaudhuris's equation. This equation will be very useful in studying the properties of black holes' event horizons (for they are defined by null geodesics, as we will see) and will be a key element in the proof of the second law of black hole dynamics.

Consider a congruence of null geodesics, i.e. a family of integral curves described by an affine parameter v where each point of the manifold is crossed by one and only one curve, with the tangent vector K obeying

$$g(K,K)=0 \qquad \text{and} \qquad \frac{\mathrm{D}}{\mathrm{d}v}K^a = K^a{}_{;b}K^b = 0. \tag{2.2}$$

As opposed to the case of timelike geodesics, it is not possible to normalize this vector and the last equation still holds if the affine parameter is multiplied by a function which is constant along each curve, being thus K unique up to this factor.

In order to construct a basis for the vectors describing the geodesics' deviation, we first note that the quotient space Q_q of T_qM by K, or the set of equivalence classes of vectors which differ by a multiple of K, is not isomorphic to the subspace of T_qM orthogonal to K, H_q, for K itself belongs to the latter. Therefore, we shall use what is called a quasi-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_qM and its dual $\{e^1, e^2, e^3, e^4\}$ of T_q^*M at a point q on a null geodesic $\gamma(v)$. It can be obtained by setting e_4 to be equal to K, e_3 to be another null vector L, lying in a transversal section of T_qM with relation to K, such that $g(e_3, e_4) = -1$ and $\{e_1, e_2\}$ to be two normalized spacelike vectors orthogonal to each basis vector but themselves. The dual vectors e^3 and e^4 are given then by $-K^a g_{ab}$ and $-L^a g_{ab}$, respectively. In this way we have simultaneously established a basis for H_q, which is e_1, e_2 and e_4, for Q_q, given by the projections into Q_q of e_1, e_2 and e_3, as also for the intersection S_q of Q_q and H_q, namely the projections of e_1 and e_2 (see Figure 1). The parallel transport of these vectors along the geodesics gives a quasi-orthonormal basis at every point q of $\gamma(v)$.

We shall construct now another congruence of curves $\lambda(t,v)$ by moving each point of a curve $\lambda(t)$, with tangent vector $Z = (\frac{\partial}{\partial t})_{\lambda(t)}$, a parameter distance v along the null geodesics with tangent $K = (\frac{\partial}{\partial v})_{\gamma(v)}$, as shown in Figure 2. Defining so Z as $(\frac{\partial}{\partial t})_{\lambda(t,v)}$ and being t and v independent, the vectors Z and K commute, i.e. the Lie derivative $L_K Z = [K, Z]$ vanishes and, being the connection ∇ torsion-free,

$$[K,Z] = \nabla_K Z - \nabla_Z K = 0 \quad \Rightarrow \quad Z^a{}_{;b}K^b - K^a{}_{;b}Z^b = 0,$$

$$\therefore \quad \frac{\mathrm{D}}{\mathrm{d}v}Z^a = Z^a{}_{;b}K^b = K^a{}_{;b}Z^b. \tag{2.3}$$

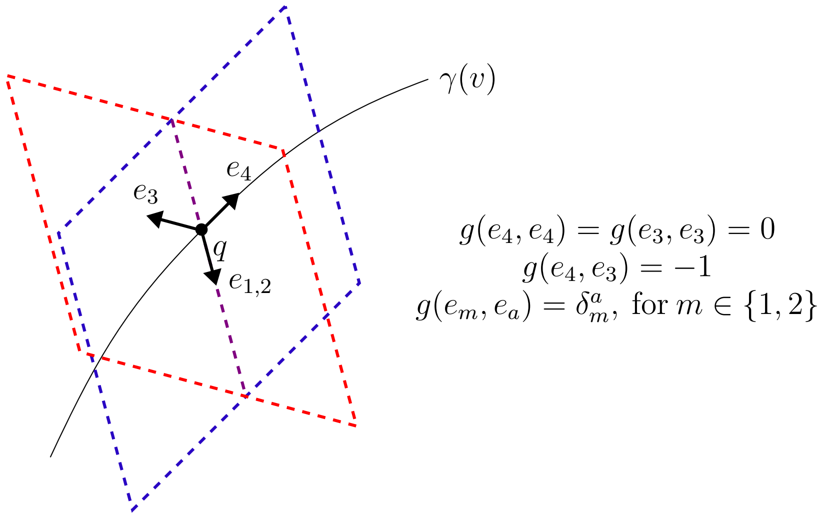


Figure 1 – The quasi-orthonormal basis.
At a point q of one of the null geodesics $\gamma(v)$ we construct a basis with the null vectors $e_4 = K$ and $e_3 = L$ and the spacelike ones, e_1 and e_2, represented in the picture by $e_{1,2}$, which is the plane expanded by e_1 and e_2 in the tangent space T_qM of q. If we choose the quotient space Q_q to be generated by vectors with vanishing component in e_4's direction, then it can be represented by the red hyperplane. The subspace H_q orthogonal to e_4 is given by the blue hyperplane and its intersection with Q_q is the plane S_q represented by the purple line.

Source: By the author.

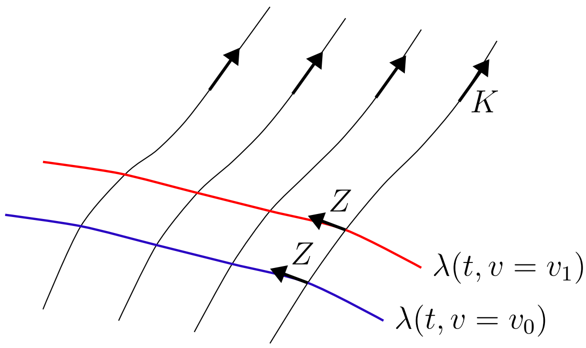


Figure 2 – The congruence which gives the separation between geodesics.
The null congruence of curves $\gamma(v)$ with tangent vectors K is drawn in black. We construct another congruence crossing the former by transporting the points in the blue curve $\lambda(t)$ by the same amount of affine parameter along each $\gamma(v)$. For example, moving each point by $v_1 - v_0$ yields the red curve, or in general, the congruence $\lambda(t, v)$ with tangent vectors Z.

Source: By the author.

Take two neighboring null geodesics and one point at each of them, $p = \lambda(t, v)$ and $q = \lambda(t + \mathrm{d}t, v)$, then their separation $Z\mathrm{d}t$ may be represented by Z as we move both points the same parameter distance v along the geodesics. Adding multiples of K to Z will measure the distance between points starting at different values of v, but still on the same two geodesics. Consequently, the separation of null geodesics, and not of specific points on them, is better represented by Z modulo K, that is the projection of Z into Q_q. In the case of timelike geodesics, this space is equal to H_q and we would be interested in the orthogonal projections of the last equation,

$$_{\perp}\left(\frac{\mathrm{D}}{\mathrm{d}v}{}_{\perp}Z^a\right) = K^a{}_{;b}\,{}_{\perp}Z^b,$$

but since K is a geodesic, $K^a_{;4}$ equals to zero in the quasi-orthonormal basis and, for $\alpha, \beta \in \{1, 2, 3\}$, we may write

$$\frac{\mathrm{D}}{\mathrm{d}v}Z^\alpha = K^\alpha{}_{;\beta}Z^\beta.$$

This equation gives us the desired evolution of Z's projection into Q_q in terms of itself. We also have that

$$0 = (g_{ab}K^aK^b)_{;c} = 2K^a g_{ab}K^b{}_{;c} \quad \Rightarrow \quad K^3{}_{;c} = 0,$$

given the definition of e^3, and, consequently, $Z^3 = -Z^aK_a$ is a constant along the geodesic. Since this component of Z represents a distance in time, it means that light rays emitted from the same source at different times keep this distance constant. So, we can choose Z such that $Z^3 = 0$ and look only for the spacial separation between null geodesics, using the projection into S_q of the former equation,

$$\frac{\mathrm{d}}{\mathrm{d}v}Z^m = K^m{}_{;n}Z^n, \tag{2.4}$$

where $m, n \in \{1, 2\}$ and the covariant derivative becomes an ordinary one, since the components are given with respect to the propagated basis, which is parallel transported, thus

$$\frac{\mathrm{D}}{\mathrm{d}v}(Z^a e_a) = \left(\frac{\mathrm{D}}{\mathrm{d}v}Z^a\right)e_a + Z^a\left(\frac{\mathrm{D}}{\mathrm{d}v}e_a\right) = \left(\frac{\mathrm{d}}{\mathrm{d}v}Z^a\right)e_a.$$

As we have a first order linear ordinary differential equation for the components Z^m, they can be written in terms of their values at a pont q as

$$Z^m(v) = A_{mn}(v)Z^n|_q,$$

and of a matrix $A_{mn}(v)$ that equals to the identity at q and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}v}A_{mn}(v) = K_{m;p}A_{pn}(v). \tag{2.5}$$

This matrix represents the transformation of every Z along each geodesic and we can decomposed it into an orthogonal matrix O_{mp} with positive determinant, responsible for the rotation of neighboring curves with respect to the propagated basis, and a symmetric matrix S_{pn}, which gives the separation between these curves:

$$A_{mn}(v) = O_{mp}(v)S_{pn}(v).$$

Now, given an orthogonal matrix O, its transpose O^T and derivative O' with respect to some parameter v and the identity matrix I, we have that

$$OO^T = I \quad \Rightarrow \quad O'O^T + O(O^T)' = 0 \quad \Rightarrow \quad O' = [-O(O^T)']O.$$

Defining $R = -O(O^T)'$, it is possible to show that O' can be written in terms of an antisymmetrical matrix, for $O'(v) = R(v)O(v)$ and

$$R^T = [-O(O^T)']^T = [-O(O')^T]^T = -O'O^T,$$

$$\therefore \quad R + R^T = -[O(O^T)' + O'O^T] = -(OO^T)' = -(I)' = 0.$$

Thus, at q where $A_{mn} = O_{mp} = S_{pn} = I_{2\times 2}$, $\mathrm{d}O_{mn}/\mathrm{d}v$ is antisymmetric, $\mathrm{d}S_{mn}/\mathrm{d}v$ is symmetric and, from equation (2.5),

$$\left.\frac{\mathrm{d}}{\mathrm{d}v}O_{mn}(v)\right|_q + \left.\frac{\mathrm{d}}{\mathrm{d}v}S_{mn}(v)\right|_q = \left.K_{m;n}\right|_q.$$

This means that, at q, the antisymmetric part of $K_{m;n}$ represents the rate of rotation of neighboring geodesics, while the rate of change of their separation is given by the symmetric part. We shall call them vorticity and expansion tensors, $\omega_{mn} = K_{[m;n]}$ and $\theta_{mn} = K_{(m;n)}$, respectively, being the trace of the latter, or the rate of change of area, called area expansion $\theta = K^m{}_m$. Defining the tensor that projects vectors $X \in T_qM$ into S_q as $h^a{}_b = \delta^a{}_b + L^aK_b + K^aL_b$, it is possible to write

$$\omega_{ab} = h_a{}^c h_b{}^d K_{[c;d]}, \qquad \theta_{ab} = h_a{}^c h_b{}^d K_{(c;d)} \tag{2.6}$$

$$\text{and} \quad \theta = \theta_{ab}h^{ab} = K_{a;b}h^{ab} = K^a{}_{;a},$$

where the last equation is obtained considering that $K_{a;b}K^b = K_{a;b}K^a = 0$. We also define the shear tensor as the trace free part of θ_{ab}:

$$\sigma_{ab} = \theta_{ab} - \frac{1}{2}h_{ab}\theta.$$

Multiplying equation (2.5) by A^{-1}_{np}, changing indices and taking the antisymmetric and symmetric parts, we have that

$$\omega_{mn} = \left(\frac{\mathrm{d}}{\mathrm{d}v}A_{[m|p}\right)A^{-1}_{p|n]} \tag{2.7}$$

$$\text{and} \quad \theta_{mn} = \left(\frac{\mathrm{d}}{\mathrm{d}v}A_{(m|p}\right)A^{-1}_{p|n)}. \tag{2.8}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(A_{mp}A^{-1}_{pn}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}v}A_{mp}\right)A^{-1}_{pn} + A_{mp}\left(\frac{\mathrm{d}}{\mathrm{d}v}A^{-1}_{pn}\right) = 0$$

$$\Rightarrow \quad \left(\frac{\mathrm{d}}{\mathrm{d}v}A_{[m|p}\right)A^{-1}_{p|n]} = -A_{[m|p}\left(\frac{\mathrm{d}}{\mathrm{d}v}A^{-1}_{p|n]}\right)$$

$$\therefore \quad \omega_{mp}\omega_{pn} = -\left(\frac{\mathrm{d}}{\mathrm{d}v}A_{[m|r}\right)A^{-1}_{r|p]}A_{[p|s}\left(\frac{\mathrm{d}}{\mathrm{d}v}A^{-1}_{s|n]}\right).$$

For $\theta_{mp}\theta_{pn}$ we get the same expression with square brackets replaced by round ones; therefore, summing both expressions results in

$$\omega_{mp}\omega_{pn} + \theta_{mp}\theta_{pn} = -\left(\frac{\mathrm{d}}{\mathrm{d}v}A_{mp}\right)\left(\frac{\mathrm{d}}{\mathrm{d}v}A^{-1}_{pn}\right),$$

which is symmetric in m and n, so we can write

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta_{mn} = \left(\frac{\mathrm{d}^2}{\mathrm{d}v^2}A_{(m|p}\right)A^{-1}_{p|n)} + \left(\frac{\mathrm{d}}{\mathrm{d}v}A_{(m|p}\right)\left(\frac{\mathrm{d}}{\mathrm{d}v}A^{-1}_{p|n)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta_{mn} = \left(\frac{\mathrm{d}^2}{\mathrm{d}v^2}A_{(m|p}\right)A^{-1}_{p|n)} - \omega_{mp}\omega_{pn} - \theta_{mp}\theta_{pn}. \tag{2.9}$$

The first term in the last expression is obtained if we take the covariant derivative of equation (2.3),

$$\frac{\mathrm{D}^2}{\mathrm{d}v^2}Z^a = K^a{}_{;bc}Z^bK^c + K^a{}_{;b}Z^b{}_{;c}K^c = K^a{}_{;bc}Z^bK^c + K^a{}_{;b}K^b{}_{;c}Z^c,$$

$$\text{but} \qquad (K^a{}_{;b}K^b)_{;c} = K^a{}_{;bc}K^b + K^a{}_{;b}K^b{}_{;c} = 0,$$

$$\text{so} \qquad \frac{\mathrm{D}^2}{\mathrm{d}v^2}Z^a = K^a{}_{;bc}Z^bK^c - K^a{}_{;bc}K^bZ^c,$$

$$\therefore \quad \frac{\mathrm{D}^2}{\mathrm{d}v^2}Z^a = K^a{}_{;bc}Z^bK^c - K^a{}_{;cb}Z^bK^c = -R^a{}_{bcd}K^bZ^cK^d,$$

since the Riemann curvature tensor components $R^a{}_{bcd}$ are given by

$$R^a{}_{bcd}X^b = X^a{}_{;dc} - X^a{}_{;cd}.$$

Thus, in the quasi-orthonormal basis we have

$$\frac{\mathrm{d}^2}{\mathrm{d}v^2}Z^m = -R^m{}_{4n4}Z^n,$$

$$\Rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}v^2}A_{mp} = -R_{m4n4}A_{np}.$$

Hence, multiplying this equation by A^{-1}_{pn}, taking the symmetric part, considering that $R_{abcd} = R_{cdab}$ and using equation (2.9) yields

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta_{mn} = -R_{m4n4} - \omega_{mp}\omega_{pn} - \theta_{mp}\theta_{pn},$$

which is written with all components as

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta_{ab} = -R_{ecfd}h^e{}_ah^f{}_bK^cK^d - \omega_{ac}\omega^c{}_b - \theta_{ac}\theta^c{}_b. \tag{2.10}$$

Since

$$\begin{aligned} h^{ab}R_{ecfd}h^e{}_a h^f{}_b K^c K^d = R_{ecfd}h^{ef}K^cK^d = R_{ecfd}(g^{ef} + K^eL^f + L^eK^f)K^cK^d \\ = R_{cd}K^cK^d + (K_{e;df} - K_{e;fd})K^dK^eL^f \\ = R_{ab}K^aK^b + (K_{e;d}K^d)_{;f}K^eL^f - K_{e;d}K^d{}_{;f}K^eL^f \\ - (K_{e;f}K^e)_{;d}K^dL^f + K_{e;f}K^e{}_{;d}K^dL^f = R_{ab}K^aK^b, \end{aligned} \tag{2.11}$$

taking the trace of equation (2.10) finally gives us the Raychaudhuri's equation for null congruences:

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta = -R_{ab}K^aK^b + 2\omega^2 - 2\sigma^2 - \frac{1}{2}\theta^2 \tag{2.12}$$

$$\text{where} \quad 2\omega^2 = \omega_{ab}\omega^{ab} \geq 0$$

$$\text{and} \quad 2\sigma^2 = \sigma_{ab}\sigma^{ab} \geq 0,$$

since the rotation and shear are "spacial" tensors.[9]

This equation tells us that rotation causes the expansion of the congruence while shear causes its contraction. The other term can be analyzed considering the energy conditions for the matter distribution, since this one can be very general and the exact form of the energy-momentum tensor unknown in most cases, but it should satisfy some restrictions. The first one is that the energy density must be non-negative for any observer, whose worldline has a tangent timelike vector V^a. Therefore, we have that $T_{ab}V^aV^b \geq 0$ for any $V^a \in T_pM$ and point $p \in \mathcal{M}$, what is called the *weak energy condition* (WEC). It also implies the *null energy condition* (NEC), that is $T_{ab}K^aK^b \geq 0$ for any null vector $K^a \in T_pM$, since K^a can be taken as the limit of a sequence of timelike vectors and the last inequality holds as the sequence converges, namely, it is a consequence of continuity. The other two common restrictions are the dominant and the strong energy conditions (DEC and SEC), where the first requires the WEC and $T^{ab}V_a$ to be non-spacelike, which means that the local energy flow vector is not spacelike and, therefore, mass-energy does not flow with superluminal speeds.

From the Einstein's field equations

$$R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}Tg_{ab}\right) + \Lambda g_{ab} \tag{2.13}$$

and the NEC it follows that $R_{ab}K^aK^b = 8\pi T_{ab}K^aK^b \geq 0$. So, for a null congruence with vanishing vorticity, equation (2.12) implies that

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta \leq \frac{\mathrm{d}}{\mathrm{d}v}\theta + \frac{1}{2}\theta^2 \leq 0, \tag{2.14}$$

which means that θ decreases monotonically and the gravitational effect of mass-energy on null geodesics is nondivergent. In order to extend this reasonable result to a congruence of timelike geodesics, we use the corresponding Raychaudhuri's equation, where we replace null vectors by timelike ones and the factor 1/2 by 1/3, and impose that $R_{ab}V^aV^b \geq 0$. Thus, using the Einstein's field equations with null cosmological constant, this corresponds to

$$T_{ab}V^aV^b \geq \frac{1}{2}TV^aV_a,$$

which is the statement of the SEC.

2.2 The area law of black holes

2.2.1 Causal structures[1,7]

The study of black holes requires, at least for most of the time, to know and to define some global properties of the spacetime in question. In special, the "infinities" or boundaries of the manifold need a proper definition, which can be achieved through the so-called conformal transformations and conformal diagrams. We start then with a sketch of what they are. A conformal transformation is basically a local change of scale, that is, given a nonvanishing differentiable function Ω defined over a manifold $\mathcal{M}$ with metric g, the latter is conformal to $\hat{g}$ if

$$\hat{g} = \Omega^2 g.$$

Consequently, for any vectors $X, Y, V, W \in T_pM$,

$$\frac{g(X,Y)}{g(V,W)} = \frac{\hat{g}(X,Y)}{\hat{g}(V,W)},$$

meaning that angles and ratios of magnitudes are preserved in such transformations. So the causal character of any vector is kept unchanged, i.e. $\hat{g}$ and g have the same sign when applied to the same vector. Particularly, null vectors are null for both metrics and we say that these transformations preserve the causal structure (or the light cone structure) of the manifold.

In order to represent the causal structure a given spacetime in a diagram, something that seems quite complicated at first, we will look for a coordinate transformation that leaves the metric explicitly conformal to another one in which "radial" light cones are drawn with lines at 45° in a plot with a timelike coordinate on one of the axis and a "radial" one on the other. Furthermore, we want a coordinate system that allows us to represent "infinity" with finite coordinate values, so that we can see the whole causal structure of the spacetime.

The simplest example of how this transformations work is seen in Minkowski space, which has the metric written in spherical coordinates as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

and has the coordinate singularities at $r = 0$ and $\theta = 0$. So, formally speaking, we must apply the restrictions $0 < r < \infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$ and use two such coordinate

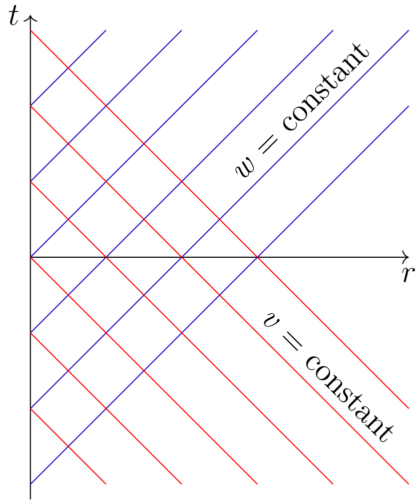


Figure 3 – Null coordinates on Minkowski space.

Source: By the author.

neighborhoods to cover the whole space. Our first transformation aims to highlight the causal structure using the advanced and retarded null coordinates $v = t + r$ and $w = t - r$, which implies that

$$\mathrm{d}s^2 = -\mathrm{d}v\mathrm{d}w + \frac{1}{4}(v - w)^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

for $-\infty < v, w < \infty$. In this coordinate system, light rays travel on the surfaces $v =$ constant and $w =$ constant shown in Figure 3.

Now, to bring infinity to a finite coordinate value, we use another pair of null coordinates $p = \arctan v$ and $q = \arctan w$, which are defined for $-\frac{1}{2}\pi < p, q < \frac{1}{2}\pi$. Then, the metric becomes

$$\mathrm{d}s^2 = \sec^2 p \sec^2 q \left\{-\mathrm{d}p\mathrm{d}q + \frac{1}{4}\sin^2(p - q)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right\},$$

that is conformal to

$$\mathrm{d}\hat{s}^2 = -4\mathrm{d}p\mathrm{d}q + \sin^2(p - q)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

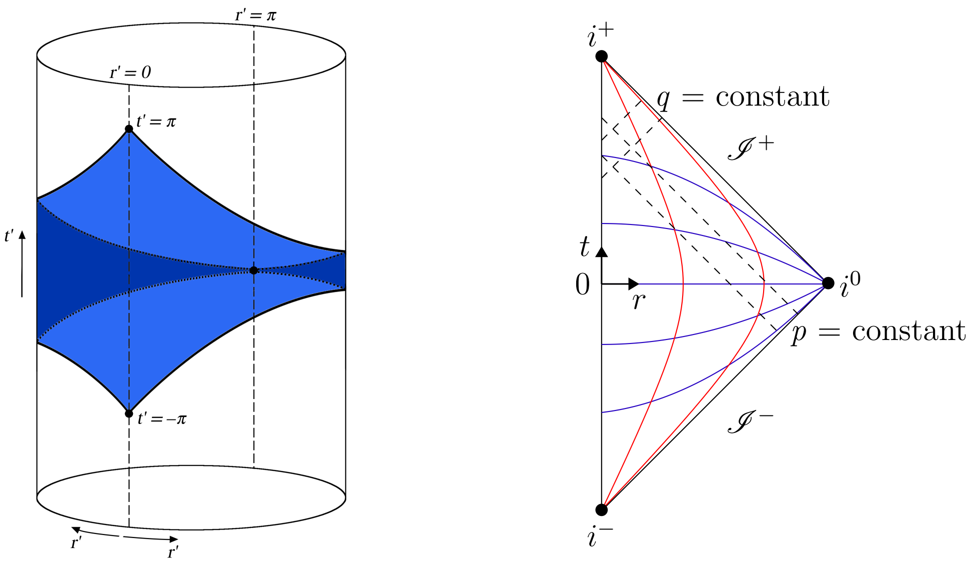
To get back to the more usual form with a timelike coordinate and a radial one, we define $t' = p + q$ and $r' = p - q$ over the range

$$-\pi < t' + r' < \pi, -\pi < t' - r' < \pi, r' \geq 0, \tag{2.15}$$

yielding

$$\mathrm{d}\hat{s}^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + \sin^2(r')(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

This metric coincides locally with the one given by the Einstein static universe, which is a solution to Einstein's equations for a perfect fluid with cosmological constant. However, in that case the coordinates are extended to $-\infty < t' < \infty$, $0 < r' < \pi$ and describe the



(a) (b)

Figure 4 – Representations of Minkowski space.
A representation of the Einstein static universe containing Minkowski space (blue region) is shown in (a). The conformal diagram of Minkowski space is in (b), where the red lines correspond to two spheres of constant radius, while blue lines are surfaces of constant time. Future-directed (past-directed) null geodesics are contained in the surfaces of constant $q(p)$.

Source: By the author.

manifold $\mathbb{R} \times S^3$, being S^3 a spacelike 3-sphere with constant radius. This can be depicted as a cylinder where $t' =$ constant is a circle representing the 3-sphere (in fact, the whole Einstein static universe is given by embedding the cylinder $x^2 + y^2 + z^2 + w^2 = 1$ in the five dimensional semi-Euclidean space with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2$) as in Figure 4(a). Consequently, the causal structure of Minkowski space can be represented in the same way but with the restrictions (2.15), giving the colored area of Figure 4(a).

We can draw the region corresponding to Minkowski space as the interior of one of the triangles from $r' = 0$ to $r' = \pi$, obtaining its coformal diagram (or Penrose diagram), where each point corresponds to a 2-sphere. The boundary of this triangle does not belong to the space, for $r' = 0$ is a coordinate singularity and the rest is called *conformal infinity*, being its union with the interior a *conformal compactification*, that is a manifold with boundary. As shown in Figure 4(b), the conformal infinity can be divided into two lightlike surfaces, $\mathscr{I}^+$ and $\mathscr{I}^-$, and the three points i^+, i^0 and i^-, which are called

$$\mathscr{I}^+ = \text{future null infinity} \quad (p = \frac{1}{2}\pi \quad \text{or} \quad t + r \to \infty,\, t - r = \text{finite}),$$

$$\mathscr{I}^- = \text{past null infinity} \quad (q = -\frac{1}{2}\pi \quad \text{or} \quad t - r \to -\infty,\, t + r = \text{finite}),$$

$$i^+ = \text{future timelike infinity} \quad (p = \frac{1}{2}\pi,\, q = \frac{1}{2}\pi \quad \text{or} \quad t \to \infty,\, r = \text{finite}),$$

$$i^0 = \text{spacial infinity} \quad (p = \frac{1}{2}\pi,\, q = -\frac{1}{2}\pi \quad \text{or} \quad r \to \infty,\, t = \text{finite}) \quad \text{and}$$

$$i^- = \text{past timelike infinity} \quad (p = -\frac{1}{2}\pi,\, q = -\frac{1}{2}\pi \quad \text{or} \quad t \to -\infty,\, r = \text{finite}).$$

As desired, all radial null geodesics are represented by lines at 45°, so we can see that each one of them which is future-directed and starts at some point in the interior of the spacetime approaches $\mathscr{I}^+$ as its affine parameter tends to infinity. Similarly, the past-directed null geodesics approach $\mathscr{I}^-$ if sufficiently extended and future-directed (past-directed) timelike geodesics approach i^+ (i^-). Since the geodesics are "reflected" in the diagram at $r' = 0$, we may say that every null geodesic has its origin on $\mathscr{I}^-$ and ends on $\mathscr{I}^+$, timelike geodesics come from i^- and go to i^+ and the spacial ones have both beginning and end at i^0 for infinite affine parameter values, justifying then the names of each region of conformal infinity.

This kind of representation is very useful in studying more complex spacetimes, like the ones containing black holes. Following this line of thought, one can find a conformal transformation for the maximally extended Schwarzschild spacetime, for example, in which the Schwarzschild metric is transformed in order to cover the whole manifold. The conformal diagram in this case is the one in Figure 5, where regions I and II are exterior to both the black and white holes (III and IV, respectively) and have a geometry which approaches that of Minkowski space far from $r = 2M$, or in other words, close to $\mathscr{I}^+$, $\mathscr{I}^-$ and i^0. We can see this in the diagram as a correspondence between the structure of its conformal infinity and that of Minkowski space. Therefore, we say that a spacetime, or a region of it, is *asymptotically flat* if its future and past null infinities, $\mathscr{I}^+$ and $\mathscr{I}^-$, are drawn in a conformal diagram as lines at 45° joined by i^0, like Minkowski space itself, regions I and II and similar parts in the maximally extended Reissner-Nordström and Kerr solutions, which describe charged and spinning black holes, respectively.

In order to formalize this definition, we first regard the time-orientability of the spacetime $\mathcal{M}$. At any point of it, we may choose which one of the light cones is future-directed and which one is past-directed. If this choice can be made continuously throughout $\mathcal{M}$, then we say that this spacetime is *time-orientable*. Similarly, a spacetime is *space-orientable* if its set of spacelike vector bases can be separated into right handed bases and left handed bases continuously. Then we define:

Definition 2.2.1. *An* asymptotically simple *spacetime is a time- and space-orientable manifold* $(\mathcal{M}, g)$ *which can be embedded** *in a larger manifold* $(\widetilde{\mathcal{M}}, \widetilde{g})$*, such that:*

* A definition of embedding will be given in Section 3.2.

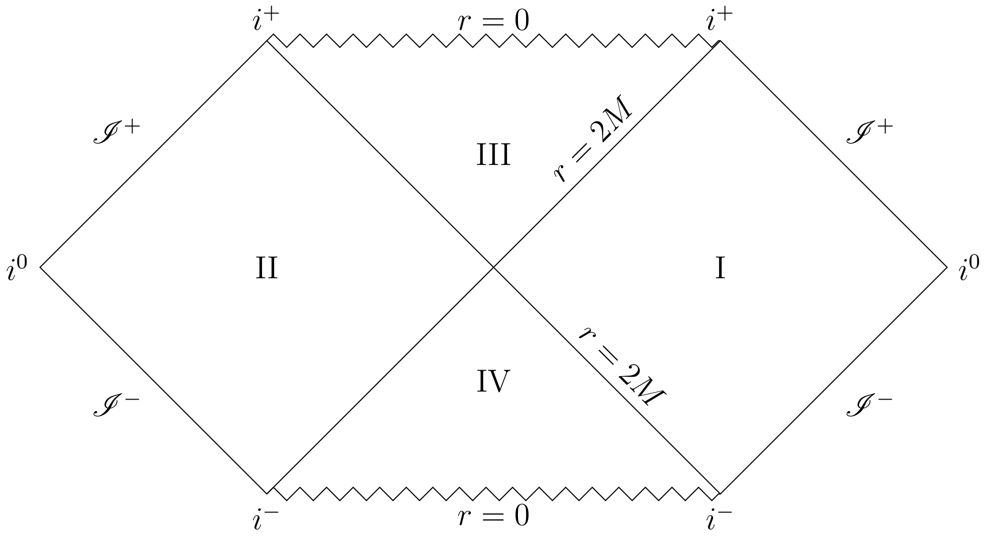


Figure 5 – Conformal diagram of the extended Schwarzschild spacetime.
By analytically extending the Scharwzschild metric one finds that the manifold contains not only the expected asymptotically flat region I and the black hole III, but also mirrored versions of them, which are another asymptotically flat region II and the white hole IV. While every non-spacelike future-directed geodesic inside region III hits the singularity at $r = 0$, inside the white hole they must have sprung from $r = 0$ if sufficiently extended to the past.

Source: By the author.

- *$\theta : \mathcal{M} \to \widetilde{\mathcal{M}}$ results in a manifold with boundary $\partial\mathcal{M} \in \widetilde{\mathcal{M}}$;*
- *$\widetilde{g}$ is conformal to g on $\theta(\mathcal{M})$, so there is a differentiable function Ω for which $\theta_*(\widetilde{g}) = (\Omega \circ \theta)^2 g$;*
- *Ω vanishes on and only on $\partial\mathcal{M}$, where $\mathrm{d}\Omega \neq 0$;*
- *Every null geodesic in $\theta(\mathcal{M})$ has future and past endpoints in $\partial\mathcal{M}$.*

Definition 2.2.2. *Let $\theta(\mathcal{M}) \cup \partial\mathcal{M} \equiv \overline{\mathcal{M}}$. Then an* asymptotically simple and empty *spacetime satisfies the conditions of the previous definition and $R_{ab} = 0$ in an open neighborhood of $\partial\mathcal{M}$ in $\overline{\mathcal{M}}$.*

Definition 2.2.3. *A* weakly asymptotically simple and empty *manifold $\mathcal{M}$ contains an open set $\mathcal{U}$ which is isometric to $\mathcal{U}' \cap \mathcal{M}'$, being $\mathcal{U}'$ a neighborhood of the boundary $\partial\mathcal{M}'$ of some asymptotically simple and empty manifold $\mathcal{M}'$. From now on, we say that a spacetime is* asymptotically flat *if it is weakly asymptotically simple and empty.*

It can be shown[7] that $\partial\mathcal{M}$ is composed of two disconnected lightlike surfaces, $\mathscr{I}^+$ and $\mathscr{I}^-$, which are topologically equal to the future and past null infinities of Minkowski space, matching the intuitive description given before for conformal diagrams. Furthermore, the last definition means that for a spacetime to be asymptotically flat only a part of it must match a neighborhood of the conformal infinity of some asymptotically simple manifold, where *every* null geodesic reaches $\mathscr{I}^+$ and $\mathscr{I}^-$. So the whole extended Schwarzschild spacetime can be said to be asymptotically flat, since it has neighborhoods of $\mathscr{I}^+$ and $\mathscr{I}^-$ meeting the necessary conditions. The same can be said for the maximally extended Reissner-Nordström and Kerr solutions.

Another relevant feature of conformal diagrams is that they allow us to easily see the causal connection between regions of the manifold. Concentrating attention on regions I and III of Figure 5, one can perceive that every null geodesic reaching future null infinity comes from the asymptotically flat region I and none from the black hole in III. That is, there can be no causal relation between "observers at infinity" and events inside the sphere of radius $r = 2M$ which defines the black hole's event horizon at a given time. Past this horizon, every timelike or lightlike curve falls inevitably into the singularity at $r = 0$.

To give a better statement of what an event horizon is, we first consider a time-orientable spacetime and define the following causal relations for events $p, q \in \mathcal{M}$:

Definition 2.2.4. *The event p* precedes *q, $p \ll q$, if there is at least one smooth, future-directed timelike curve γ (i.e. its tangent vectors are timelike and future-directed all along it) going from p to q. This is equivalent to say that q* follows *p, or $q \gg p$.*

Definition 2.2.5. *The event p* causally precedes *q, $p \prec q$, if there is at least one smooth, future-directed causal curve γ (i.e. its tangent vectors are non-spacelike and future-directed all along it) going from p to q. This is equivalent to say that q* causally follows *p, or $q \succ p$.*

Definition 2.2.6. *The* chronological future *$I^+(p)$ of p is the set of all events q that follow p:*

$$I^+(p) = \{q | q \gg p\} .$$

Definition 2.2.7. *The* causal future *$J^+(p)$ of p is the set of all events q that causally follow p:*

$$J^+(p) = \{q | q \succ p\} .$$

Definition 2.2.8. *Being $\mathcal{S}$ a region of spacetime, the chronological and causal futures of $\mathcal{S}$ are, respectively,*

$$I^+(\mathcal{S}) = \bigcup_{p \in \mathcal{S}} I^+(p) \quad \text{and} \quad J^+(\mathcal{S}) = \bigcup_{p \in \mathcal{S}} J^+(p).$$

Definition 2.2.9. *The* boundary of the chronological (causal) future *of $\mathcal{S}$ is denoted by $\partial I^+(\mathcal{S})$ $(\partial J^+(\mathcal{S}))$.*

Similar definitions are made for the *chronological* and *causal pasts* of a point p (or region $\mathcal{S}$), $I^-(p)$ and $J^-(p)$, by changing *follow* for *precede.*

Now, as said before, the *future event horizon* of a black hole divides spacetime into an asymptotically flat region from which light signals can reach the future null infinity and the interior of the black hole. Since there can be many black holes bordering a single asymptotically flat region, we define:

Definition 2.2.10. *The* union of all future event horizons *is given by the boundary of the region of spacetime from which future-directed causal curves can reach future null infinity, that is* $\partial J^-(\mathscr{I}^+)$*.*

Again, the *union of all past event horizons* is defined in a similar manner, as $\partial J^+(\mathscr{I}^-)$.

2.2.2 Penrose's theorem[1]

We shall look now more closely to the structure of a black hole's event horizon and get an important result given by Penrose,[10] which will be a key ingredient in the formulation of the second law of black hole dynamics. In order to do that, we first need to establish a few lemmas about the causal relation between events on $\partial J^-(\mathscr{I}^+)$, as stated below:

Lemma 2.2.1. *If γ_1 and γ_2 are causal, future-directed curves from event p to event q and from q to r, respectively, and $p \not\ll r$, i.e. p is not in the chronological past of r, then γ_1 and γ_2 are null geodesics, being their tangent vectors proportional to each other at q.*

Sketch of proof *: Since γ_2 is a causal curve, q should be on $\partial J^-(r)$ or inside $I^-(r)$ at *q'*, as shown in Figure 6(a). Similarly, p should be on $\partial J^-(q)$ or inside $I^-(q)$. Because $p \not\ll r$, p can not be in $I^-(r)$ and, consequently, neither in $J^-(q')$, for $J^-(q') \subset I^-(r)$. So we must have $q \in \partial J^-(r)$ and $p \in \partial J^-(q)$, for $I^-(q) \subset I^-(r)$. Since this argument holds for any $p \in \gamma_1$ and $q \in \gamma_2$, it implies that $\gamma_1 \in \partial J^-(r)$, $\gamma_2 \in \partial J^-(q)$ and they are null curves.

But the only way to preserve the condition $p \not\ll r$ is to have $p \in \partial J^-(r) \cap \partial J^-(q)$, since any curve like the dashed one lies in $I^-(r)$, that is, $\partial J^-(q) - \partial J^-(r) \cap \partial J^-(q) \in I^-(r)$. As a consequence, γ_1 and γ_2 must be the ones drawn in blue, meaning that their tangent vectors must be parallel transported along themselves, otherwise, a change in any direction other then their own would require the curve to be spacelike at some point, as illustrated in Figure 6(b), contradicting its causal character. Therefore, $\gamma_1 \cup \gamma_2$ form a null geodesic smooth at q. ■

Lemma 2.2.2. *If $p \in \partial J^-(\mathscr{I}^+)$ and $q \in \partial J^-(\mathscr{I}^+)$, then $p \not\ll q$.*

* This proof differs from that of 1. For a rigorous proof see the corollary of Proposition 4.5.1 in 7.

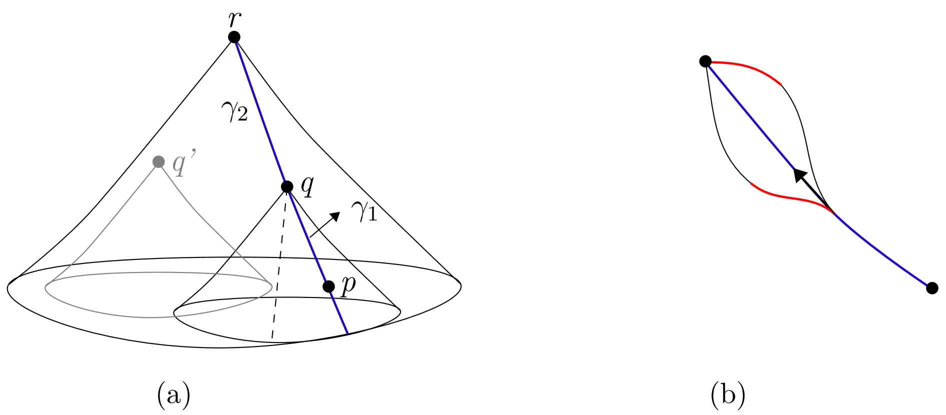


Figure 6 – Schemes for proof of Lemma 2.2.1.
In (a) the cone with vertex at r represents $\partial J^-(r)$, while its interior corresponds to $I^-(r)$. Similar regions are given by the cones with vertex at q' and q. The union of the causal curves γ_1 and γ_2 belongs to the geodesic in blue. The curves in (b) show that deviations from the geodesic curve in blue, with tangent vector represented, forces the resulting curve connecting the two events to be spacelike (red) at some points.

Source: By the author.

Proof: Suppose $p \ll q$. Then there exists a timelike curve from p to q, as shown in Figure 7(a). This curve can be slightly deformed so that it connects the arbitrary points r and s, inside sufficiently small neighborhoods $\mathcal{N}[p]$ and $\mathcal{N}[q]$, but still being timelike . For the points s which belong to $J^-(\mathscr{I}^+)$, it is possible to extend this timelike curve with a causal one from s to $\mathscr{I}^+$. Thus, for any $r \in \mathcal{N}[p]$ there is a causal curve reaching $\mathscr{I}^+$, implying that $\mathcal{N}[p] \subset J^-(\mathscr{I}^+)$ and $p \notin \partial J^-(\mathscr{I}^+)$, which is a contradiction. ∎

Lemma 2.2.3. *Let γ be a causal curve that intersect $\partial J^-(\mathscr{I}^+)$ at event q. Then, any point of that curve causally preceding q belongs to $\partial J^-(\mathscr{I}^+) \cup J^-(\mathscr{I}^+)$.*

Proof: For an arbitrarily small neighborhood $\mathcal{N}[p]$ of some point p on γ, such that $p \prec q$, there will be points like r from which γ can be deformed to a timelike curve λ passing through q, as shown in Figure 7(b). Furthermore, λ can be deformed to another timelike curve σ joining r and $s \in J^-(\mathscr{I}^+)$, because $q \in \partial J^-(\mathscr{I}^+)$ and a part of its neighborhood lies in $J^-(\mathscr{I}^+)$. Thus, σ can be extended as a causal curve until $\mathscr{I}^+$, meaning that $r \in J^-(\mathscr{I}^+)$. Since r belongs to an arbitrarily small neighborhood of p, either $p \in J^-(\mathscr{I}^+)$ or $p \in \partial J^-(\mathscr{I}^+)$. ∎

Using this 3 lemmas, we can prove the following theorem about the structure of future event horizons:

Theorem 2.2.4 (Penrose's theorem)**.** *Let the null geodesics which belong to $\partial J^-(\mathscr{I}^+)$ for at least a finite interval of affine parameter be called the* generators *of $\partial J^-(\mathscr{I}^+)$. When followed to the causal past, a generator may leave $\partial J^-(\mathscr{I}^+)$ at some point, which will be called a* caustic *of $\partial J^-(\mathscr{I}^+)$. If that happens, the generator enters $J^-(\mathscr{I}^+)$. In the*

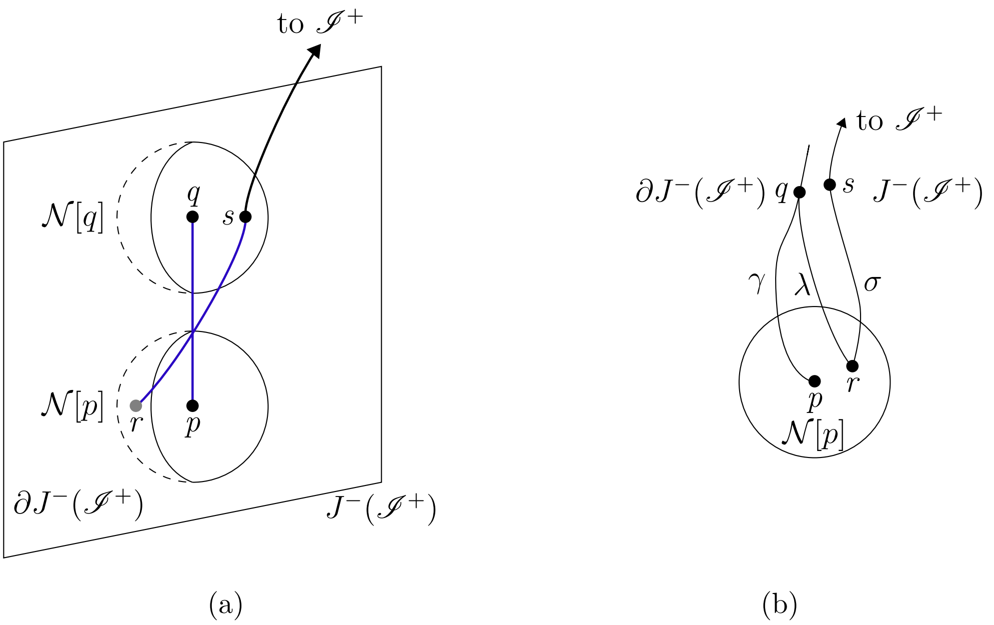


Figure 7 – Schemes for Lemmas 2.2.2 and 2.2.3.
In (a) the blue curves connecting p to q and r to s are timelike, while the black one extending from s to $\mathscr{I}^+$ can be any causal curve. The plane represents $\partial J^-(\mathscr{I}^+)$, where the points p and q are found, and the region to the right corresponds to $J^-(\mathscr{I}^+)$. Note that r may be to the left of the plane. Similarly, in (b) the left part of the figure represents $\partial J^-(\mathscr{I}^+)$ and contains q, while the right one corresponds to $J^-(\mathscr{I}^+)$ and contains r and s.

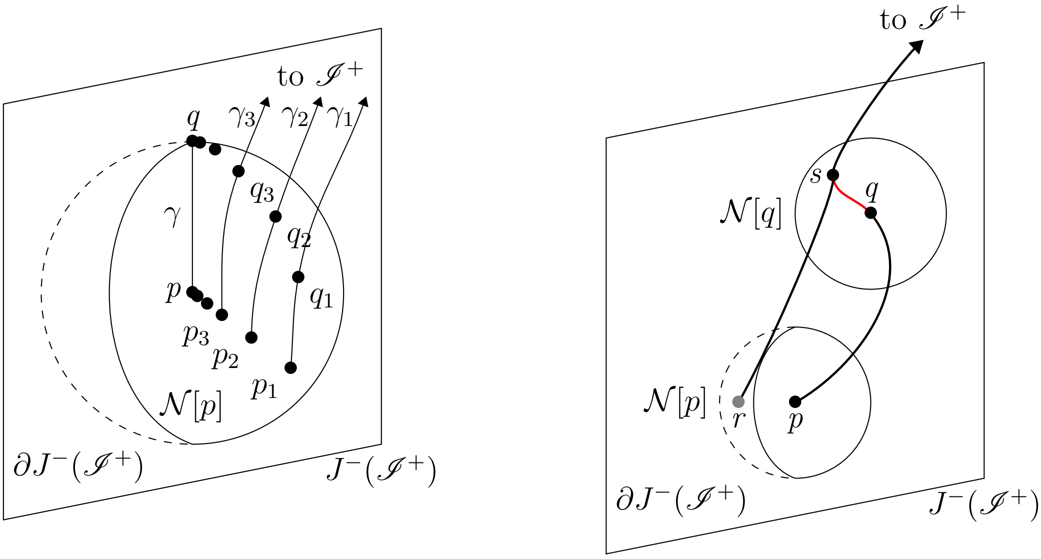
Source: By the author.

future direction, once a generator has entered $\partial J^-(\mathscr{I}^+)$ from $J^-(\mathscr{I}^+)$ through a caustic, it never leaves $\partial J^-(\mathscr{I}^+)$ nor intersect another generator, so they can only intersect at caustics. Moreover, through each noncaustic point of $\partial J^-(\mathscr{I}^+)$ there passes one and only one generator.

Summarizing: $\partial J^-(\mathscr{I}^+)$ is generated by null geodesics which have no endpoints.

Proof: As in Figure 8(a), we can choose an arbitrary point $p \in \partial J^-(\mathscr{I}^+)$ and a sufficiently small neighborhood $\mathcal{N}[p]$ such that it does not contain any singularity. Then we construct a sequence of events $\{p_i\}$ in $\mathcal{N}[p] \cap J^-(\mathscr{I}^+)$ converging to p and, from each $\{p_i\}$, a causal curve γ_i reaching $\mathscr{I}^+$. The intersection of the boundary of the neighborhood with each curve, $\partial\mathcal{N}[p] \cap \gamma_i$, will be denoted by q_i. Being $\partial\mathcal{N}[p]$ compact, the sequence $\{q_i\}$ has a limit point q. Since there is a causal curve between $\{p_i\}$ and $\{q_i\}$, which are arbitrarily close to p and q, *there must be a causal curve γ from p to q.*

For q is the limit of a sequence of points in $J^-(\mathscr{I}^+)$, it either lies in $J^-(\mathscr{I}^+)$ or in $\partial J^-(\mathscr{I}^+)$. To assume that $q \notin \partial J^-(\mathscr{I}^+)$ allows us to construct a neighborhood around it such that $\mathcal{N}[q] \in J^-(\mathscr{I}^+)$, like the one in Figure 8(b). So we can find a causal curve from p to $\mathscr{I}^+$ composed of the causal curve between p and q, a timelike section connecting



(a) (b)

Figure 8 – Schemes for the proof of Penrose's theorem - I.
As before, the plane corresponds to $\partial J^-(\mathscr{I}^+)$ and the region to its right to $J^-(\mathscr{I}^+)$. In (b), the black curves are causal and the segment in red is timelike. The point r may be found to the left of the plane.

Source: By the author.

q to $s \in \mathcal{N}[q]$ and another causal curve from s to $\mathscr{I}^+$. Because of the timelike section, this curve can be deformed into another causal one linking any point r in a sufficiently small neighborhood $\mathcal{N}[p]$ to $\mathscr{I}^+$. Thus, $\mathcal{N}[p] \in J^-(\mathscr{I}^+)$ and, consequently, $p \notin \partial J^-(\mathscr{I}^+)$, which is a contradiction. Therefore, *q must be in $\partial J^-(\mathscr{I}^+)$.*

Lemma 2.2.2 tells us that $p \not\ll q$, for $p, q \in \partial J^-(\mathscr{I}^+)$. But there is a causal curve γ between these points, so Lemma 2.2.1 guarantees that γ is a null geodesic. Moreover, by Lemma 2.2.3 $\gamma \in J^-(\mathscr{I}^+) \cup \partial J^-(\mathscr{I}^+)$. Using the same arguments in the previous paragraph for every point of γ, one finds that $\gamma \in \partial J^-(\mathscr{I}^+)$. Therefore, *through every $p \in \partial J^-(\mathscr{I}^+)$ there passes a null geodesic γ (a generator) which lies in $\partial J^-(\mathscr{I}^+)$ in the direction of the future of p.*

Now, suppose that, by extending γ in the future direction, it leaves $\partial J^-(\mathscr{I}^+)$ at some point $p' \in \partial J^-(\mathscr{I}^+)$ (Figure 9(a)). By repeating the construction shown in Figure 8(a), changing p by p' and q by q', we find that there is a null geodesic $\gamma' \in \partial J^-(\mathscr{I}^+)$ from p' to $q' \in \partial J^-(\mathscr{I}^+)$. So Lemma 2.2.2 implies that $p \not\ll q'$. Therefore, given that γ and γ' are causal curves from p to p' and from p' to q', respectively, their tangent vectors must coincide at p' (if one adjust their parametrizations) by Lemma 2.2.1. Thus, γ' is the extension of γ and it does not leave $\partial J^-(\mathscr{I}^+)$ at p'. So *the generators never leave $\partial J^-(\mathscr{I}^+)$ when followed in the future direction.*

From Lemma 2.2.3, we have that *when a generator enters $\partial J^-(\mathscr{I}^+)$ it comes from*

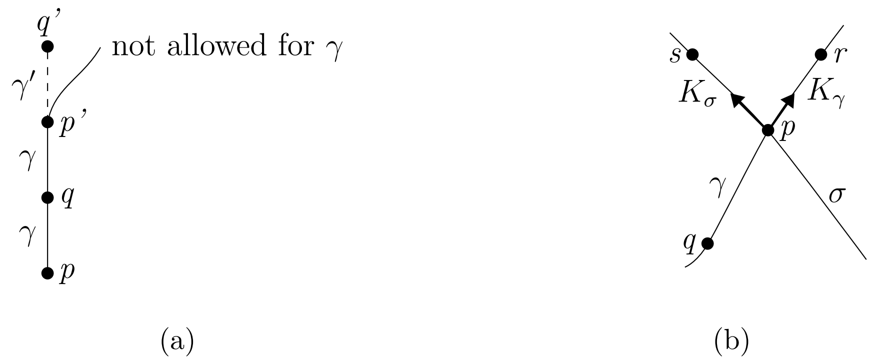


Figure 9 – Schemes for the proof of Penrose's theorem - II.

Source: By the author.

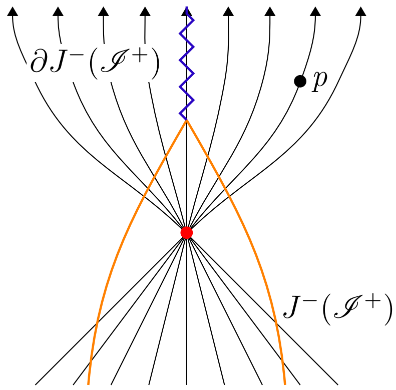


Figure 10 – The structure of $\partial J^-(\mathscr{I}^+)$.
The gravitational collapse of a star, occurring inside the curves in orange, produces a black hole and a singularity (in blue). The generators of the event horizon come from $J^-(\mathscr{I}^+)$ and enter $\partial J^-(\mathscr{I}^+)$ at the caustic in red, following to $\mathscr{I}^+$ without crossing each other again. Furthermore, for each point $p \in \partial J^-(\mathscr{I}^+)$, there is one and only one generator passing through it.

Source: By the author.

$J^-(\mathscr{I}^+)$.

As a final step, consider the possibility of two generators, γ and σ, crossing at an event p causally following the caustic through which γ entered $\partial J^-(\mathscr{I}^+)$, as depicted in Figure 9(b). To the future of p, the generators lie in $\partial J^-(\mathscr{I}^+)$, that is, they have the points $r, s \in \partial J^-(\mathscr{I}^+)$. Also, there is a point q sufficiently close to p and causally preceding it such that $q \in \partial J^-(\mathscr{I}^+) \cap \gamma$. For the Lemma 2.2.2, it means that $q \not\ll s$ and, using Lemma 2.2.1, that the tangent vectors K_γ and K_σ are equal for some parametrization of the curves, i.e. γ and σ are the same null geodesic, contradicting the initial assumption. Given that if p is a caustic there is no $q \in \partial J^-(\mathscr{I}^+) \cap \gamma$ to the past of p, two generators may cross at a caustic. As a result, *once a generator has entered* $\partial J^-(\mathscr{I}^+)$*, it can never thereafter cross any other generator.* This result is illustrated in Figure 10. ■

2.2.3 The second law of black hole dynamics[1]

Given the results obtained so far we can prove and state what has become the so-called second law of black hole dynamics, first presented in 11. As shown in Figure 11, the union of all future event horizons, $\partial J^-(\mathscr{I}^+)$, in an asymptotically flat spacetime can be divided into a collection of infinitesimal bundles of generators which evolve in time. These bundles may be created in caustics at the formation of a black hole after some star's collapse (like bundle 2), by infalling matter through the horizon (bundle 1) or by the collision and merging between two or more black holes (bundle 3). However, Penrose's theorem states that, once they are created, these bundles must follow to future null infinity without merging or being destroyed, for each generator has no future endpoint. Thus, the number of bundles can only increase.

Now, define some spacelike surface as a slice of simultaneity, like $\mathcal{S}_1$ and $\mathcal{S}_2$, and consider its intersection with $\partial J^-(\mathscr{I}^+)$. This would correspond to the event horizons of all black holes as seen by some observer at a given time. Since we expect that these horizons are smooth, spacelike, bidimensional surfaces and they are embedded in the lightlike hypersurface $\partial J^-(\mathscr{I}^+)$[*], they are orthogonal to the vector field generating the geodesics. So Frobenius's theorem[†] tells us that the vorticity of the generators of $\partial J^-(\mathscr{I}^+)$ must vanish. Moreover, we assume that the null energy condition is valid, implying that equation (2.14) holds for each bundle of generators:

$$\frac{\mathrm{d}}{\mathrm{d}v}\theta + \frac{1}{2}\theta^2 \leq 0.$$

By definition, θ represents the rate of change in the infinitesimal transversal area $\mathcal{A}_i$ of each bundle (Figure 11), that is[‡]

$$\theta = \frac{1}{\mathcal{A}_i}\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{A}_i \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}v}{\mathcal{A}_i}^{1/2} = \frac{1}{2}{\mathcal{A}_i}^{1/2}\theta, \tag{2.16}$$

$$\therefore \quad \frac{\mathrm{d}^2}{\mathrm{d}v^2}{\mathcal{A}_i}^{1/2} = \frac{1}{2}{\mathcal{A}_i}^{1/2}\left(\frac{\mathrm{d}}{\mathrm{d}v}\theta + \frac{1}{2}\theta^2\right) \leq 0.$$

If we assume that at some point p along the geodesics $\theta < 0$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}v}{\mathcal{A}_i}^{1/2}\right|_p < 0$$

and, by the previous inequality for the second derivative (which means that the first derivative decreases monotonically),

$$\left.\frac{\mathrm{d}}{\mathrm{d}v}{\mathcal{A}_i}^{1/2}\right|_p \geq \frac{{\mathcal{A}'_i}^{1/2} - {\mathcal{A}_i}^{1/2}}{\Delta v},$$

[*] A lightlike hypersurface is generated by null geodesics, being their tangent vectors orthogonal to the hypersurface. More details will be given in the next chapter.

[†] See Theorem 3.2.2 and equation (3.14).

[‡] The change from the inequality in terms of θ to the one in terms of ${\mathcal{A}_i}^{1/2}$ was based on 12.

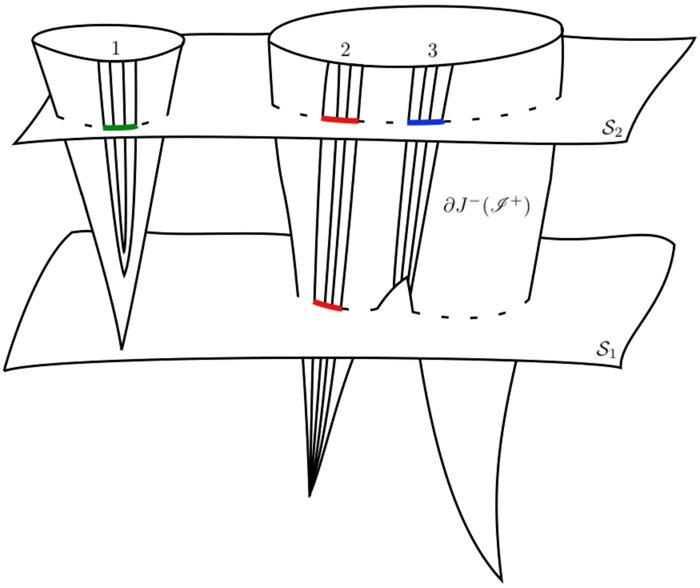


Figure 11 – The dynamics of $\partial J^-(\mathscr{I}^+)$.
The union of the conelike structures represent $\partial J^-(\mathscr{I}^+)$ and their interior are black holes created at the vertex by gravitational collapse of matter. As we follow them in the future direction (upwards), new caustics, and therefore new bundles of generators, may occur if matter falls into a black hole or if they merge, as bundles 1 and 3, respectively. We may "cut" $\partial J^-(\mathscr{I}^+)$ into slices of simultaneity, like the spacelike surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, and each bidimensional intersection between these surfaces and some bundle will have an observer independent area $\mathcal{A}_i$. For example, the intersections in green, red and blue represent the areas $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ of their respective bundles of generators.

Source: By the author.

where ${\mathcal{A}_i}^{1/2}$ and ${\mathcal{A}'_i}^{1/2}$ are, respectively, the initial (at p) and final values for the area at the extremes of the geodesic with affine parameter in the interval Δv. Consequently, for a finite interval

$$\Delta v \leq \left. \frac{{\mathcal{A}_i}^{1/2}}{-\frac{\mathrm{d}}{\mathrm{d}v}{\mathcal{A}_i}^{1/2}} \right|_p$$

the area of the bundle will reach zero. This would mean that the generators of the bundle intersect at this point of null area, contradicting Penrose's theorem. Therefore, either $\theta < 0$ cannot occur along any generator or the bundle for which it happens encounters a singularity before its area becomes zero. We assume that it never happens, what its equivalent to say that observers at future null infinity cannot see any singularity, there are no "naked singularities", and we impose the so-called *cosmic censorship*.

We conclude that $\theta \geq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}v}{\mathcal{A}_i}^{1/2} \geq 0, \tag{2.17}$$

so the area of each bundle never decreases in the future direction, as well as the number of bundles. As a result, *the total cross-sectional area of* $\partial J^-(\mathscr{I}^+)$ *cannot decrease as one* *follows towards the future.* This is the statement of the second law of black hole dynamics and is represented in Figure 11. The sum of the colored areas corresponding to intersections between $\partial J^{-}(\mathscr{I}^{+})$ and the spacelike surfaces increases from $\mathcal{S}_1$ to $\mathcal{S}_2$, if the latter is to the future of $\mathcal{S}_1$ at every point.

In fact, in case two black holes merge, the total cross-sectional area must increase,[13] for each of the separated event horizons is composed by a different closed set of null geodesics before the collision, like bundle 2 in Figure 11. After merging, the resulting black hole will have an event horizon which contains this two disjoint closed sets, but since one single horizon is a connected set of generators, there must be a new open set "gluing" the two previous sets, like bundle 3. Therefore, the number of bundles necessarily increases, and so does the cross-sectional area.

2.2.4 The analogy between black hole dynamics and thermodynamics[2]

The previous result takes the name *second* law of black hole dynamics because of the analogy between it and the second law of thermodynamics. Actually, this analogy appeared together with the other three laws of black hole dynamics and their respective correspondence with the laws of thermodynamics.[2] They were originally formulated considering a vacuum solution of Einstein's field equations containing a stationary axisymmetric black hole, meaning that there is a Killing vector field K associated to both the time translational and the rotational symmetry of spacetime. On the black hole's event horizon, K becomes a null vector field and we define there the so-called *surface gravity* κ by

$$K^{a}{}_{;b}K^{b} = \kappa K^{a}, \tag{2.18}$$

which measures how much the Killing vector field's parametrization differs from an affine one. Physically, it represents the limit to which the acceleration of a test particle corotating with the black hole tends as it approaches the event horizon, measured by observers at infinity. From this definition, we may state the four laws of black hole dynamics (BHD) and their thermodynamic analogues (TA) as

- The zeroth law:

  BHD - For a stationary black hole, the surface gravity κ is constant over its event horizon.

  TA - The temperature T of a system in thermal equilibrium is constant over the system.

- The first law:

  BHD - The change in mass-energy δM between two neighboring stationary solutions containing a black hole is given by

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta\mathcal{A}}{4} + \Omega\delta J + \text{other work terms}, \tag{2.19}$$

where A is the cross-sectional area of the intersection between the event horizon and some spacelike hypersurface, Ω is the black hole's angular velocity and J its angular momentum. Other terms related to work may be present if we consider a charged black hole or some matter-energy around it. Here we already write the terms which correspond to temperature and entropy separately, as $\kappa/2\pi$ and $\delta\mathcal{A}/4$, but it is worth to note that they appeared mixed initially, as $\kappa\delta\mathcal{A}/8\pi$, being the exact coefficients of each one unknown until Hawking calculated the horizon's temperature.

TA - For a closed system, the resulting variation in energy δE after some process is given by

$$\delta E = T\delta S + \text{ work terms},$$

being S the system's total entropy.

- The second law:

  BHD - As shown before, the cross-sectional area of each black hole's event horizon never decreases, no matter what (classical) process takes place:

  $$\delta\mathcal{A} \geq 0.$$

  TA - The total entropy of a closed system never decreases:

  $$\delta S \geq 0.$$

- The third law:

  BHD - There is no physical process by which the surface gravity of a black hole reduces to zero in a finite sequence of operations.

  TA - There is no physical process by which the temperature of a system reduces to zero in a finite sequence of operations.

From these, only the third law has no rigorous mathematical proof, although there are compelling evidences that it should hold. Up to now, all idealized processes by which one could in principle get $\kappa = 0$ require unreasonable assumptions, like infinite divisibility of matter and infinite time. Furthermore, if there exists some finite number of operations by which the third law fails to hold, it could lead to the creation of a naked singularity, so asymptotic predictability* would no longer be a valid argument for many results based on it, like the second law we derived.

Whilst the laws governing the evolution of black holes bear a striking resemblance to the laws of thermodynamics and their terms of work and energy even share the same

* In case there exist naked singularities, they can send signals to, and therefore influence, the external universe. Thus, observers near future null infinity could only predict the evolution of the universe if they knew how singularities behave,[1] what is not possible currently.

physical meaning in both cases, κ and $\mathcal{A}$ could not be interpreted as the temperature and entropy of a black hole, at least classically. Since in this context these objects are "perfect absorbers", they cannot be in thermal equilibrium with any system at a temperature different from zero, being this the only possible value for their temperature. This would mean that entropy could be added to a black hole without changing any aspect of it, so an observer far away would see the total entropy of the universe decrease without consequences, violating the second law of thermodynamics. Even the increase in the black hole's area* could be arbitrarily small, if one could lower a box (containing some definite amount of entropy) on a rope along the axis of rotation at an arbitrarily small velocity. This and other arguments seemed to show that the laws of thermodynamics were not valid when one consider black holes and the analogy described above should not have any deeper physical meaning. However, as we shall see in the next section, subsequently results showed that the quantum nature of matter and vacuum can fix these violations of the laws of thermodynamics and generalize them to cover the physics of black holes.

2.3 The generalized second law[14-15]

A black hole has the remarkable property of hiding information, specially about its pasts. For outside observers, it is impossible to determine what kind of matter-energy was responsible for its formation (despite our guesses) and what fell into it throughout its history. This is formalized in the no-hair theorems,[9] which state that after a black hole has settled to a stationary solution it will be completely described by only a few parameters, like the total mass-energy, angular momentum and electric charge that fell in it, from which we can calculate geometric quantities as the area of its event horizon. Therefore, the second law of thermodynamics could in principle be transcended when someone throws an object (and its entropy) into a black hole†. In order to prevent the casting of doubt on the validity of this law, we should attribute an entropy to the black hole based on its observable parameters. From these, only the area has the essential property of entropy, that is it never decreases. The total energy, for example, can be diminished through the Penrose process,[9] by which one can extract work from the rotational energy. In the case of a Kerr black hole, the irreducible mass

$$M_{ir} = \left(\frac{\mathcal{A}}{16\pi}\right)^{1/2}$$

* From now on, the event horizon's area (or black hole's area) should be understood as the cross-sectional area of the intersection between some spacelike hypersurface and the event horizon.

† As posed by John Wheeler to Jacob Bekenstein (his PhD student at the time) about the irreparable act of allowing the exchange of heat between a hot cup of tea and a cold one:[16] "The consequences of my crime, Jacob, echo down to the end of time," following with a seeming solution, "but if a black hole swims by, and I drop the teacups into it, I conceal from all the world the evidence of my crime. How remarkable!"

corresponds to the energy which cannot be transformed into work in a Penrose process, and we see that again the event horizon's area $\mathcal{A}$ shares a feature with entropy, namely its increase means that less work is available to be done by the system. Given these properties and the analogy with entropy presented before, we may assume that the entropy of a black hole is some monotonically increasing function of its area.

If we use $\mathcal{A}^{1/2}$ as a measure of entropy, then the irreducible mass after the merging of two black holes should be greater than or equal to the total initial irreducible mass. As discussed in the previous section, the merging requires new bundles to be formed, like bundle 3 in Figure 11. By following the opposite time direction, we find that in order to split a black hole into two a bundle should be destroyed, what is not allowed by Penrose's theorem. We conclude that the merging of black holes is an irreversible process, so the choice of $\mathcal{A}^{1/2}$ for the entropy requires the final irreducible mass to be strictly greater than the sum of the initial irreducible mass of each black hole. But if they were initially given by Schwarzschild solutions, then $M = M_{ir}$ and the final total mass-energy of the system should be greater than its initial value. Since the black holes can only lose energy due to gravitational radiation, we see that our choice is not a suitable one.

So the most natural guess is to take the entropy of a black hole to be proportional to its area, being this choice already validated by the analogy between the laws of black hole dynamics and the laws of thermodynamics, specially the second one. The dimensional analysis tells us that this entropy should have the form*

$$S_{BH} = \eta k_B \frac{\mathcal{A}}{L_P^2}, \quad \text{where} \quad L_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} \tag{2.20}$$

is the Planck length, k_B is Boltzmann's constant and η is some dimensionless constant. We use L_P for this is the only universal constant with dimension of length. The appearance of $\hbar$ is to some extent expected, since it also appears in the entropy of classical systems as a remainder of the underlying quantum dynamics of matter giving rise to the macroscopic, classical behavior of thermodynamics. Thus, to get the exact value of η must require a treatment of the quantum and statistical processes occurring in a neighborhood of the event horizon, but a semiclassical analysis should give us a hint of its magnitude.

Before we start such analysis, we remind that, for any system, the entropy may be interpreted as the lack of information about the system's internal configuration given the macroscopic parameters that can be measured, as temperature, energy or pressure. Formally, Shannon's dimensionless entropy

$$S = -\sum_n p_n \ln p_n$$

is given in terms of the probability p_n associated with each of the n possible states that a system can be found in. As we get more information about the system, imposing restrictions

* We briefly recover the use of G and c here, also writing k_B explicitly. In the following equations we set this three constants to 1.

to the values of p_n, like setting some of them to zero, the entropy decreases, so we may relate the corresponding gain in information ΔI as

$$\Delta I = -\Delta S.$$

Hence, the second law can be thought of as a consequence of the information loss about a system's internal configuration that takes place while it evolves to an equilibrium state. In other words, the uncertainties associated to the initial configuration of a system out of equilibrium tend to increase until the equilibrium is reached. We may recover some information by measuring once more the system, but in doing so the total entropy of the universe still increases (solving the problem caused by Maxwell's demon).

It is useful to measure information in multiples of the bit, which is the information contained in the answer of a yes-or-no question. From the last equation, a bit is also minus the entropy associated to such a question when there is no information available. That is, it is given by the value which extremizes $S(p_1, p_2)$ such that $p_1 + p_2 = x(p_1, p_2) = 1$. Therefore, for some Lagrange multiplier λ,

$$\nabla S(p_1, p_2) = \lambda \nabla x(p_1, p_2) \quad \Rightarrow \quad -\ln p_1 - 1 = \lambda = -\ln p_2 - 1,$$

$$\therefore \quad p_1 = p_2 = \frac{1}{2}, \quad \text{and} \quad S(p_1, p_2) = \ln 2,$$

implying that 1 *bit* $=$ $\ln 2$ *nat*, where *nat* is the *natural unit of information* used in Shannon's entropy, which was previously (and will be) omitted since it is dimensionless.

Now we turn to the problem mentioned before of a box being lowered very slowly by an observer far from a black hole of mass M, which we consider initially to be Schwarzschild, for the sake of simplicity. The box contains an object of rest mass m that may lose energy through radiation, represented by the red particles in Figure 12. As the box is being lowered, some mechanism (like a dynamo or another box's lifting) uses the work done by the system to store energy, until the box approaches the horizon and stops. There, the energy of the system measured by the observer, E, may reach arbitrarily small values due to the increasing redshift for radii arbitrarily closer to that of the horizon. Namely, they are given by

$$E = m\left(1 - \frac{2M}{r}\right)^{1/2},$$

so the work done by the system may get very close to m. The box can be programmed to open at this position for a period, allowing the system to radiate an amount of Δm of its proper energy into the black hole, but for the observer this value will be (almost) zero and the black hole will appear unchanged. After that, the mechanism used by the observer will bring the box back to him/her by doing $m - \Delta m$ of work. Consequently, at the end of the experiment, the quantity of heat Δm has been completely converted into work. If we consider the black hole plus the mechanism as a thermodynamic system and

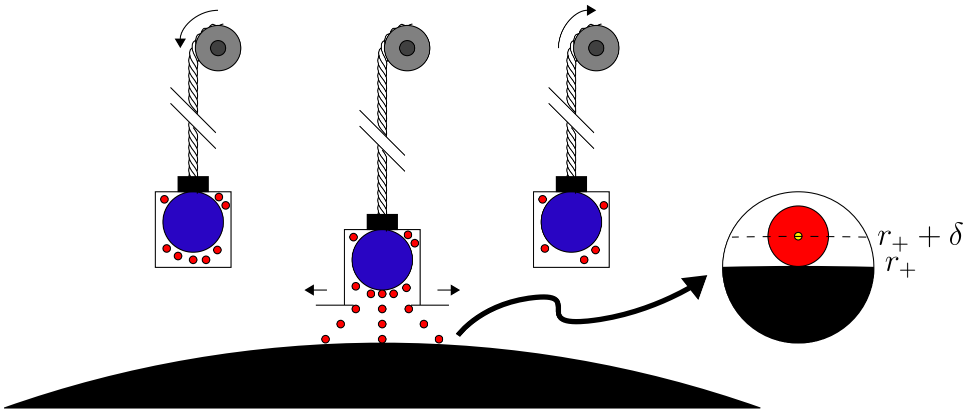


Figure 12 – Attempt to violate the second law.
An observer far away from the black hole lowers (by means of some mechanism, at left) a box containing a system (in blue) at some temperature in a quasi-static process (very slow). At some point near the horizon (at the middle), the box stops and opens, so that radiation (red particles) from the system can enter the black hole, reducing the system's total proper energy. Using the energy stored when the box was being lowered, the mechanism pulls the system back to the observer (at right). But in the end of this process some energy remains, seeming that the heat transferred to the black hole was converted into work without other consequences, violating the second law of thermodynamics. From arguments described in the text, we consider that a particle has crossed the horizon (at r_+) once the red sphere of minimal radius b touches it, as detailed on the right. We see that the coordinate distance δ between the particle's center (in yellow) and the horizon corresponds to the proper length b.

Source: By the author.

the box (which can be brought to thermal equilibrium at the beginning and at the end of the experiment with the atmospheric air, for example) as a heat reservoir, then we see that the Kelvin-Planck statement of the second law is violated:[17]

"No process is possible whose sole result is the absorption of heat from a reservoir and the conversion of this heat into work."

One may generalize this apparent violation of the second law of thermodynamics with other experiments, in which the particles that enter the horizon fall freely all along the way or they are generated by the splitting of some other particle that manages to escape to infinity, for example. But in all cases, their final path, until they are absorbed by the black hole, will be described by free fall, like the distance the red particles in Figure 12 travel between the box and the horizon. Moreover, the constants of motion measured by external observers, as the particle's energy and momentum, will be defined at the beginning of the free fall. Thus, we shall analyze the geodesic motion of particles that cross the horizon in the general context of a charged, rotating black hole, described by the

Kerr-Newman metric, which in Boyer-Lindquist coordinates takes the form[9]

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{2Mr-Q^2}{\rho^2}\right)\mathrm{d}t^2 - \frac{(2Mr-Q^2)a\sin^2\theta}{\rho^2}(\mathrm{d}t\mathrm{d}\phi+\mathrm{d}\phi\mathrm{d}t) \\ &+\frac{\rho^2}{\Delta}\mathrm{d}r^2+\rho^2\mathrm{d}\theta^2+\frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2-a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2, \end{aligned} \tag{2.21}$$

$$\text{being}\quad \Delta(r)=r^2+a^2-2Mr+Q^2,\quad \rho^2(r,\theta)=r^2+a^2\cos^2\theta\quad\text{and}\quad a=\frac{J}{M}.$$

The constants M, J and Q are, respectively, the Komar energy (or mass), Komar angular momentum and charge of this solution.

The event horizon is found at r_+ where $g_{rr}=\rho^2\Delta^{-1}$ diverges, i.e. $\Delta(r_+)=0$, so $r_\pm = M\pm\sqrt{M^2-Q^2-a^2}$ and $\Delta=(r-r_-)(r-r_+)$. In this background, a neutral particle of rest mass μ has as first integral for one of the geodesic equations the following relation:

$$\begin{aligned} &E^2[r^4+a^2(r^2+2Mr-Q^2)]-(\mu^2r^2+q)\Delta = \\ &\qquad 2E(2Mr-Q^2)ap_\phi+(r^2-2Mr+Q^2)p_\phi^2+(p_r\Delta)^2, \end{aligned} \tag{2.22}$$

where q, the particle's energy $E=-p_t$ and the angular momentum in the direction of the axis of symmetry p_ϕ are constants of motion, while p_r is its covariant radial momentum. Solving for E one may find that

$$E = Bap_\phi + \left\{[B^2a^2+A^{-1}(r^2-2Mr+Q^2)]p_\phi^2 + A^{-1}[(\mu^2r^2+q)\Delta+(p_r\Delta)^2]\right\}^{1/2}, \tag{2.23}$$

$$\text{where}\quad A=r^4+a^2(r^2+2Mr-Q^2)\quad\text{and}\quad B=(2Mr-Q^2)A^{-1}.$$

Since $\Delta=0$ at the horizon, we have that $r_+^2+a^2=2Mr_+-Q^2$ and there

$$A=A_+=(r_+^2+a^2)^2\quad\text{and}\quad B=B_+=(r_+^2+a^2)^{-1}.$$

We look now for an expression for the black hole's angular momentum, by considering a photon's trajectory on the equatorial plane ($\theta=\pi/2$) and assuming that the components of its momentum in the r and θ directions are null. Because this trajectory is a null geodesic, we have that[9]

$$\mathrm{d}s^2 = 0 = g_{tt}\mathrm{d}t^2 + g_{t\phi}(\mathrm{d}t\mathrm{d}\phi+\mathrm{d}\phi\mathrm{d}t)+g_{\phi\phi}\mathrm{d}\phi^2,$$

$$\Rightarrow\quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{g_{t\phi}}{g_{\phi\phi}}\pm\sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\frac{g_{tt}}{g_{\phi\phi}}}.$$

Therefore, the event horizon's angular velocity is given by the above equation with $r=r_+$:

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}(r_+) = \frac{a}{r_+^2+a^2} = B_+a. \tag{2.24}$$

Moreover, both $(\mu^2r^2+q)\Delta$ and the term multiplying p_ϕ^2 in equation (2.23) go to zero at the horizon, but

$$p_r\Delta = \Delta g_{rr}p^r = \rho^2p^r = (r^2+a^2\cos^2\theta)p^r$$

may not vanish. As a result, for a particle which crosses the horizon, equation (2.23) becomes

$$E = \Omega p_\phi + A_+^{-1/2} |p_r \Delta|_+. \tag{2.25}$$

After the particle has entered the black hole and transferred its energy and momentum, the first law yields

$$\delta M = \Theta \delta \alpha + \Omega p_\phi, \tag{2.26}$$

where $\alpha = \mathcal{A}/4\pi$ and $\Theta = (r_+ - r_-)/4\alpha$ is the analogue of temperature (given by the surface gravity). Identifying equation (2.25) with δM results

$$\delta\alpha = \Theta^{-1} A_+^{-1/2} |p_r \Delta|_+,$$

which vanishes only if $p_r = 0$ at the horizon, meaning that the particle crosses it at a turning point, where the effective potential equals the particle's total energy. If that happens, then

$$E = \Omega p_\phi$$

and there will be no increase in the event horizon's area, showing that, for a *point* particle, the entropy increase associated with its capture by the black hole can be reduced to zero.

Nevertheless, we shall not assign to a particle a radius b smaller than double the minimum uncertainty in its position δx, which satisfies the uncertainty principle

$$\delta x \delta p \geq \frac{\hbar}{2}.$$

If we consider second quantization, as the momentum increases to values much greater than the particle's rest energy the probability that another particle will be created also increases. When this split occurs, the total momentum decreases considerably (for the kinetic energy is transformed into rest mass) and is divided between the two particles. Thus, a particle's momentum and its uncertainty are expected to assume values below the rest energy, so

$$\mu \geq \delta p \geq \frac{\hbar}{2\delta x} \quad \Rightarrow \quad \delta x \geq \frac{\hbar}{2\mu}.$$

This means that, when the "center of mass" of a particle finds itself at a distance $2\delta x$ from the event horizon, we can no longer tell if the particle has crossed the horizon or not (but it certainly will cross, since it is free falling). We can picture this as the magnified detail in Figure 12, considering that the particle has a radius $b = 2\delta x$ and that it already belongs to the black hole when it touches the horizon. As a result, the radius b should not be smaller than the particle's (reduced) Compton wavelength $\hbar/\mu$. Moreover, we do not consider that the particle is itself a tiny black hole, since we saw that the merging of two of them always increases the total area. So b must also be greater than the particle's Schwarzschild radius 2μ. Given these two restrictions, we have that

$$b \geq \frac{\hbar}{\mu}, \quad \text{if} \quad \mu \leq \left(\frac{\hbar}{2}\right)^{1/2}, \quad \text{or} \quad b \geq 2\mu = \frac{2\mu^2}{\mu} > \frac{\hbar}{\mu} \quad \text{otherwise},$$

$$\therefore \quad b \geq \frac{\hbar}{\mu} \quad \text{in general.} \tag{2.27}$$

For this reason, we shall generalize the previous analysis for the case of a particle with a non-zero radius b, stressing that it starts its free fall while it is completely outside the black hole. Now we consider that the particle is captured by the black hole when its center of mass is at a proper distance b from the horizon, what corresponds to a position with coordinate $r = r_+ + \delta$ such that

$$b = \int_{r_+}^{r_+ + \delta} (g_{rr})^{1/2} \mathrm{d}r.$$

From equation (2.21), and assuming $r_+ - r_- \gg \delta$, we get

$$b \approx 2\delta^{1/2}(r_+^2 + a^2 \cos^2\theta)^{1/2}(r_+ - r_-)^{-1/2}.$$

We evaluate equation (2.23) at this point by expanding the square root term in powers of δ and using the last equation to get

$$E = \Omega p_\phi + [(r_+^2 - a^2)(r_+^2 + a^2)^{-1} p_\phi^2 + \mu^2 r_+^2 + q]^{1/2} \times \\ \times \frac{b}{2}(r_+ - r_-)(r_+^2 + a^2)^{-1}(r_+^2 + a^2\cos^2\theta)^{-1/2} + \mathcal{O}(b^2), \tag{2.28}$$

where it was assumed that $r_+ + \delta$ is a turning point, so that $\delta\alpha$ is minimized.

By requiring that the momentum p_θ be real, it can be shown that

$$q \geq \cos^2\theta \left[a^2(\mu^2 - E^2) + \frac{p_\phi^2}{\sin^2\theta} \right]$$

and equality occurs for $p_\theta = 0$. Thus, to zeroth order in b, $E = \Omega p_\phi$ and

$$q \geq \cos^2\theta \left[a^2\mu^2 + p_\phi^2 \left(\frac{1}{\sin^2\theta} - a^2\Omega^2 \right) \right].$$

From equation (2.24) we see that $a^2\Omega^2 \leq 1$ and so

$$\frac{1}{\sin^2\theta} - a^2\Omega^2 \geq 0 \quad \Rightarrow \quad q \geq a^2\mu^2\cos^2\theta.$$

Then, to $\mathcal{O}(b)$ we have

$$E \geq \Omega p_\phi + \frac{\mu b}{2}(r_+ - r_-)(r_+^2 + a^2)^{-1},$$

with the minimum in E happening when $p_\phi = p_\theta = p^r = 0$ at $r_+ + \delta$. Using equation (2.26) and the value of Θ below it together with the last inequality, we find that the minimal increase in the black hole's area is given by

$$\delta\alpha \geq 2\mu b$$

and depends only on the particle's parameters.

The restriction (2.27) implies that $\delta\alpha \geq 2\hbar$. Consequently, we associate to this universal minimal increase in a black hole's area to the minimal loss of information about a particle's state, that is the bit necessary to answer the question "does the particle exist or not?". Once the particle passes the horizon, we can no longer answer this basic question, for it might have been destroyed and no signal from this event would get out the black hole. Remembering that the entropy associated to 1 bit equals to $\ln 2$ and $S_{BH} = f(\alpha)$, for a monotonically increasing function f, we have that the minimum increase in S_{BH} is such that

$$(\delta S_{BH})_{\min} = 2\hbar \frac{\mathrm{d}f}{\mathrm{d}\alpha} = \ln 2,$$

$$\therefore \quad S_{BH} = \frac{\ln 2}{2} \frac{\alpha}{\hbar},$$

recovering the formula obtained before (equation (2.20)) using dimensional analysis, with the constant η now given explicitly ($\eta \approx 0.03$). If calculated for a black hole of one solar mass, this entropy would be of the order $10^{60}\,\mathrm{erg}\,K^{-1}$, while the entropy of the sun stays close to $10^{42}\,\mathrm{erg}\,K^{-1}$. This shows us how irreversible a black hole's formation is, concentrating an enormous amount of entropy per unit energy.

The above analysis shows that, for a quite general process, the minimal increase in a horizon's area taking place when some mass-energy enters a black hole is different from zero. In addition to reduce $\delta\alpha$ as much as possible, we also ignored some other factors that should contribute to increase the area even more, like the gravitational waves radiated from the system being lowered to the black hole. Besides, explicit calculations for an infalling harmonic oscillator (constituted of two particles connected by a massless spring) or a beam of light, as well as a more careful analysis of the box being lowered (considering that it must have a non-zero volume), demonstrate that, even if we maximize the lost entropy and minimize the increase in area, the latter always compensates the entropy loss. We conclude that a generalization of the second law of thermodynamics for systems including black holes can be given by defining its entropy as the sum of ordinary entropy S_{ord} and all event horizons' areas:

$$\delta S_{\mathrm{system}} = \delta S_{\mathrm{ord}} + \sum_{\text{all BHs}} \delta S_{BH} = \delta S_{\mathrm{ord}} + \eta k_B \sum_i \frac{\delta \mathcal{A}_i}{L_P^2} \geq 0.$$

2.4 Thermal character of horizons[18]

As suggested by Bekenstein, the horizon's area should not be considered as a mere analogue of entropy but a real thermodynamic property. Consistency of this approach requires then a physical temperature to be attributed to the black hole, conflicting with the classical null value for it. This motivated Hawking to try to disprove the claim that black holes indeed have entropy and temperature by studying the behavior of quantum fields in a spacetime containing a black hole.[16] However, he found[3] that the observer dependence of the particle number allows an initial vacuum state (defined in a remote past, before the black hole's formation) to evolve to a thermal state at late times. We interpret this as the result of creation of virtual pairs particle-antiparticle in the vacuum surrounding the event horizon. The negative energy antiparticles can tunnel into the black hole, while the positive energy particles may escape to regions near null infinity, so observers in there measure a black body radiation originating from the black hole and attribute to it a non-null temperature. In this dynamical process, the black hole eventually "evaporates", since its energy is being undermined by the addition of negative energy antiparticles, and the classical second law of black hole dynamics clearly does not hold anymore (in deriving it we used the null energy condition). Still, the generalized second law is satisfied, for the loss in black hole area is exceeded by the entropy in the emitted *Hawking radiation*.

For our purposes, we shall consider a different derivation which applies to static spacetimes, i.e. the observer describes a situation in thermal equilibrium. The Killing vector field related to the time symmetry is timelike for some observers, like the static ones far away from a Schwarzschild black hole, and can be normalized for them as $W = (1, 0, 0, 0)$. It also generates the horizon, so its norm $W^a W_a$ vanishes on this surface and we call it a Killing horizon with surface gravity κ given by equation (2.18). In the extended Schwarzschild spacetime, we saw that there are two horizons enclosing both the black and the white holes, being represented by the X with endpoints at the i^+'s and i^-'s in the conformal diagram 5. These horizons happen to be generated by the same Killing field, which completely vanishes (not only its norm) at the intersection of the horizons, the spacelike 2-surface at the X structure's center. The latter is called a *bifurcation surface* and the union of the two horizons results in a *bifurcate Killing horizon*, which specifically divides a spacelike hypersurface into two regions. In Figure 5, this spacelike hypersurface is the line connecting both i^0's points, which contains the bifurcation surface and is divided in regions belonging to each of the mirrored exterior regions I and II. The important feature here is that observers in one of these regions do not see the other, for the horizons are covering it.

Another important issue is that the usual Schwarzschild coordinates $(t, x, x_\perp)$ (where t stands for the timelike coordinate, x for the radial one and $x_\perp$ for the angular coordinates) are suitable only for one of the four regions in the conformal diagram 5, since these coordinates are singular at the horizon. Thus, to cover the whole manifold one needs a global system of coordinates $(T, X, X_\perp)$, given by Kruskal coordinates for the Schwarzschild solution. All of these characteristics occur in other cases like the de Sitter spacetime or Minkowski space, when one consider the Rindler horizon seen by constantly accelerated observers, as pictured in Figure 13. Therefore, we shall calculate the temperature associated to a bifurcate Killing horizon present in a static spacetime, using the Rindler horizon as a model.

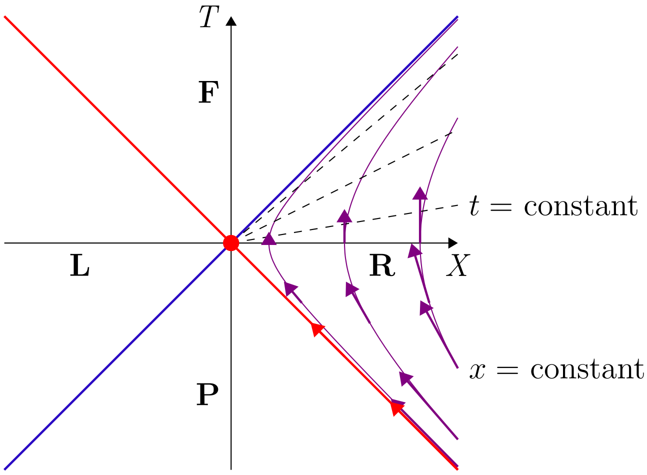


Figure 13 – Bifurcate Killing horizon in Minkowski space.
The vector field which generates Lorentz boosts in the X direction corresponds to a Killing field and, in the usual Cartesian coordinate system (T, X) (suppressing the transverse coordinates $X_\perp$), it is given by $W = (X, T)$, being represented by the drawn arrows. This vector field has as integral curves the violet hyperboles as well as the blue and red planes, which are called Rindler horizons and divide the spacetime into four wedges. The Rindler observers are those following one of the hyperboles with a constant proper acceleration κ in the X direction and can never see events past the blue (red) horizon if they are in the right (left) *Rindler wedge*, indicated by **R** (**L**). We also indicate the past and future wedges, by **P** and **F**. The Rindler metric is constructed from the coordinates (t, x) in which a Rindler observer stays at rest, so the dashed lines correspond to $t =$ constant and the hyperboles to $x =$ constant. At the horizons, $W^a W_a = x = 0$ and $t \to \infty$, so we have a bifurcate Killing horizon with bifurcation surface represented by the red disc, where $W = (0, 0)$.

Source: By the author.

Our approach will make use of the path integral's formalism to study quantum fields in the spacetime in question. Then, we start by defining the probability amplitude for a particle to go from an event $\mathcal{P}_1$ to another event $\mathcal{P}_2$, also known as the kernel*:

$$K(\mathcal{P}_2; \mathcal{P}_1) \equiv K(t_2, q_2; t_1, q_1) = \sum_{\text{paths}} \exp[iA(\text{path})],$$

where q_i represents the set $\{q_a, q_b, ...\}$ of all degrees of freedom of a particle's wave function $\psi(t_i, q_i)$ at a time t_i and $A(\text{path})$ is the action associated to a given path between $\mathcal{P}_1$ and $\mathcal{P}_2$. The probability amplitude provides all the time evolution of a particle if one knows the form of its initial wave function at t_1:

$$\psi(t, q) = \int \mathrm{d}q_1 \; K(t, q; t_1, q_1)\psi(t_1, q_1).$$

In order to describe a quantum field, the discrete degrees of freedom contained in q_i must become a continuous function of the position in space, so for each time coordinate t we have a field configuration $q(x)$ and the last integral turn into a functional integral over the

* In this section, we set $\hbar = 1$.

field configuration at $t = t_1$,

$$\psi(t, q(x)) = \int \mathcal{D}q_1 \; K(t, q(x); t_1, q_1(x))\psi(t_1, q_1(x)).$$

For simplicity, we will keep working with point quantum mechanics, but we shall write the integrals which can be generalized to quantum field theory as functionals like the one above.

In Schrödinger's picture, taking $\hat{q}$ to be the position operator and $|q\rangle$ its eigenstate with eigenvalue q, the kernel may be given in terms of the time-evolution operator $\exp(-iHt)$ as

$$K(t_2, q_2; t_1, q_1) = \langle q_2| \exp[-iH(t_2 - t_1)] \, |q_1\rangle \, ,$$

being H the time-independent Hamiltonian describing the system from some observer's point of view. Thus, given the energy eigenstates $|E_n\rangle$ and the associated energy eigenfunctions $\psi_n(q) = \langle q|E_n\rangle$, we have

$$\begin{aligned} K(T, q_2; 0, q_1) &= \langle q_2| \exp(-iHT) \, |q_1\rangle \\ &= \sum_{n,m} \langle q_2|E_n\rangle \, \langle E_n| \exp(-iHT) \, |E_m\rangle \, \langle E_m|q_1\rangle \\ &= \sum_n \psi_n(q_2)\psi_n^*(q_1) \exp(-iE_nT), \end{aligned}$$

and it is assumed that the ground state (n=0) has energy set to zero and its wave function is chosen to be real. We shall now use the analytic continuation of T to imaginary times, defining $T_E = iT$ as the Euclidean time (the name refers to the change of the Minkowski metric to an Euclidean one) and the Euclidean kernel as

$$K_E(T_E, q_2; 0, q_1) = \sum_n \psi_n(q_2)\psi_n^*(q_1) \exp(-E_nT_E). \tag{2.29}$$

Evaluation of the former expression with $q_1 = q$, $q_2 = 0$ and $T_E \to \infty$ suppresses all the terms in the sum but the one for which $E_n = 0$, so

$$\lim_{T_E \to \infty} K_E(T_E, 0; 0, q) = \psi_0(0)\psi_0(q) \propto \psi_0(q),$$

and, since the proportionality constant can be found by normalization of the wave function, we can write

$$\psi_0(q) = K_E(\infty, 0; 0, q) = K_E(0, q; -\infty, 0).$$

The kernel may be expressed as a path integral in terms of the Euclidean action A_E, what yields

$$\psi_0(q) = \int_{T_E=0, q'=q}^{T_E=\infty, q'=0} \mathcal{D}q' \; e^{-A_E}. \tag{2.30}$$

The Euclidean kernel is also useful in describing a system in a thermal bath with temperature β^{-1}. The mean of some observable $\mathcal{O}(q)$ for a system in an energy eigenstate with wave function $\psi_n(q)$ is given by integrating $\mathcal{O}(q)|\psi_n(q)|^2$ over q. But the canonical ensemble representing the thermal bath requires an averaging over all energy eigenstates weighted by $\exp\{-\beta E_n\}$, so the mean value is

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\sum_n \int \mathrm{d}q\ \psi_n(q)\mathcal{O}(q)\psi_n^*(q)e^{-\beta E_n} \equiv \frac{1}{Z}\int \mathrm{d}q\ \rho(q,q)\mathcal{O}(q),$$

being Z the partition function and

$$\rho(q,q') \equiv \sum_n \psi_n(q)\psi_n^*(q')e^{-\beta E_n}$$

the *density matrix*. If we define the trace as taking $q = q'$ and integrating over q, then

$$\langle\mathcal{O}\rangle = \frac{\mathrm{Tr}(\rho\mathcal{O})}{\mathrm{Tr}(\rho)}$$

and we see that ρ contains both the thermal and quantum information about the system's state. From equation (2.29), one finds that

$$\rho(q,q') = K_E(\beta, q; 0, q'), \tag{2.31}$$

if we substitute the Euclidean time by β.

To proceed with our derivation, we point out an important feature of the systems of coordinates $(T, X, X_\perp)$ and $(t, x, x_\perp)$. The first covers the whole manifold and in Minkowski space is given by the Cartesian coordinates, in which the vacuum state for inertial observers will be defined. The latter divides spacetime into four wedges, as those shown in Figure 13, and are suitable to describe the quantum system as seen by constantly accelerated observers. For an acceleration κ, the Rindler metric can be written as

$$\mathrm{d}s^2 = -\kappa^2 x^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}x_\perp^2,$$

with the coordinates following the transformations

$$T = x\sinh(\kappa t), \quad X = \pm x\cosh(\kappa t) \quad \text{and} \quad X_\perp = x_\perp, \tag{2.32}$$

where the plus sign is used in the **R** wedge and the minus in **L**.

When one does the analytic continuations $T_E = iT$ and $t_E = it$, the previous transformations become

$$T_E = x\sin(\kappa t_E) \quad \text{and} \quad X = \pm x\cos(\kappa t_E). \tag{2.33}$$

Moreover, the horizons collapse into the origin in the Euclidean sector, for they are given by $X^2 - T^2 = 0$, which is equivalent to $X^2 + T_E^2 = 0$. Also, changing the $=$ signs by $<$ implies that both the **F** and **P** quadrants are excluded in the coordinates with imaginary time. So we see that the relations (2.33) transform the Cartesian coordinates to polar

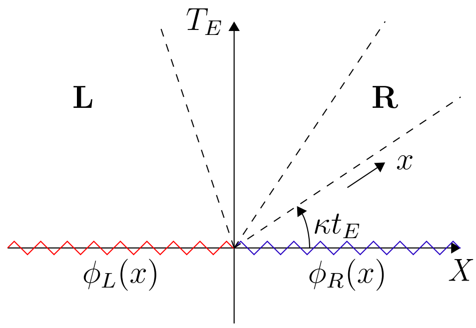


Figure 14 – The Euclidean sector.
The field configuration $q(x)$ defined in the hyperplane $T_E = 0$ is divided in $\phi_L(x) = q(x)$ for $X < 0$ (in red) and $\phi_R(x) = q(x)$ for $X > 0$ (in blue). Since the Cartesian coordinates (T_E, X) are mapped into the polar coordinates (t_E, x), the evolution of the field from $T_E = 0$ to $T_E \to \infty$ corresponds to a evolution from $t_E = 0$ to $t_E = \pi/\kappa$ in the Rindler frame.

Source: By the author.

coordinates $(\theta = \kappa t_E, x)$ and, contrary to (2.32), we may chose only one sign (the positive) to cover both the **R** and **L** regions, varying t_E from 0 to $2\pi/\kappa$. As a result, the time evolution generated by the inertial Hamiltonian H_I carries the field configuration from $T_E = 0$ to $T_E \to \infty$, while the Rindler Hamiltonian H_R does the same evolution on the system, as seen by the accelerated observers, from $t_E = 0$ to $t_E = \pi/\kappa$ (see Figure 14).

If we specify the field configuration as $q = (\phi_L, \phi_R)$, then the ground state wave functional $\langle vac|q\rangle$ can be calculated from equation (2.30):

$$\langle vac|\phi_L, \phi_R\rangle = \int_{T_E=0, q=(\phi_L,\phi_R)}^{T_E=\infty, q=(0,0)} \mathcal{D}q \; e^{-A_E}.$$

As explained above, this integration may be given in terms of the polar coordinate $\theta = \kappa t_E$ from 0 to π, where $q = \phi_R$ and $q = \phi_L$, respectively. Therefore,

$$\langle vac|\phi_L, \phi_R\rangle = \int_{\kappa t_E=0, q=\phi_R}^{\kappa t_E=\pi, q=\phi_L} \mathcal{D}q \; e^{-A_E},$$

which is the Euclidean kernel in the Rindler frame and may be given from the Rindler Hamiltonian as

$$\langle vac|\phi_L, \phi_R\rangle = \langle \phi_L| \exp(-\pi H_R/\kappa) |\phi_R\rangle \, .$$

Remembering that this equality was originally a proportionality, we seek to normalize the wave functional once more, dividing it by C:

$$\begin{aligned} C^2 &= \int \mathcal{D}\phi_L \mathcal{D}\phi_R \; \langle vac|\phi_L, \phi_R\rangle \langle \phi_L, \phi_R|vac\rangle \\ &= \int \mathcal{D}\phi_L \mathcal{D}\phi_R \; \langle \phi_L| \exp(-\pi H_R/\kappa) |\phi_R\rangle \langle \phi_R| \exp(-\pi H_R/\kappa) |\phi_L\rangle \\ &= \mathrm{Tr}[\exp(-2\pi H_R/\kappa)], \end{aligned}$$

$$\therefore \quad \langle vac|\phi_L, \phi_R\rangle = \frac{\langle \phi_L| \exp(-\pi H_R/\kappa) |\phi_R\rangle}{\{\mathrm{Tr}[\exp(-2\pi H_R/\kappa)]\}^{1/2}}.$$

From the relation between the Euclidean kernel and the density matrix (2.31), we can obtain ρ corresponding to observations from the Rindler frame in **R** if we trace the field configuration ϕ_L, since the left wedge is covered by the horizon, so

$$\begin{aligned}
\rho(\phi_R, \phi'_R) &= \int \mathcal{D}\phi_L \; \langle \phi_L, \phi_R | vac\rangle \langle vac | \phi_L, \phi'_R\rangle \\
&= \int \mathcal{D}\phi_L \; \frac{\langle \phi_R | \exp(-\pi H_R/\kappa) | \phi_L\rangle \langle \phi_L | \exp(-\pi H_R/\kappa) | \phi'_R\rangle}{\mathrm{Tr}[\exp(-2\pi H_R/\kappa)]} \\
&= \frac{\langle \phi_R | \exp[-(2\pi/\kappa) H_R] | \phi'_R\rangle}{\mathrm{Tr}(\exp[-(2\pi/\kappa) H_R])} \propto K_E(\beta, \phi_R; 0, \phi'_R),
\end{aligned}$$

with $\beta = 2\pi/\kappa$.

Therefore, the vacuum on $T = 0$ seen by inertial observers appears as a thermal state with temperature $\beta^{-1} = \kappa/2\pi$ to Rindler observers in **R**. This is known as the *Unruh effect*,[5] in contrast to the Hawking radiation. In the latter, our derivation is interpreted as an static observer far away from the horizon, whose Schwarzschild coordinates cover only one quarter of the manifold, measuring a thermal radiation from a state that looks like a vacuum for a global observer using Kruskal coordinates. The bifurcate Killing horizon structure hides certain regions of the manifold from a class of observers, producing a periodic Euclidean time coordinate, which in turn gives rise to a thermal state from an originally vacuum state. In the case of black holes, the temperature

$$T = \frac{\kappa}{2\pi} \tag{2.34}$$

agrees with the laws of black hole dynamics and its analogy with thermodynamics, in addition to provide the numerical factors of the first law (2.19). Further than a definition of temperature, we find that the black hole entropy is given by

$$S_{BH} \equiv \frac{\mathcal{A}}{4},$$

if one sets the physical constants to 1.

2.5 Gravitation from thermodynamics[4,19]

Once established that black holes exhibit a temperature and the laws governing their evolution are not mere analogues to those for thermodynamic systems, the questions of "How general relativity could have this thermodynamic behavior encoded in itself?" and "Is this a particularity of black holes?" started to be investigated. In fact, the last one were answered a few years after the paper describing the Hawking radiation, being shown[20] that these thermodynamic properties also appear in cosmological horizons, as in de Sitter spacetime. A new feature of these cases is the explicit observer dependence of the horizon and, therefore, of its properties. While the black hole's event horizon is defined from null infinity as a whole, in the de Sitter spacetime a horizon is defined for a specific point of null infinity, resulting in many different horizons. Although something similar occurs for a Rindler horizon, the same importance within this thermodynamic approach had not been given to it for long, since only its temperature was well established, but not the first and second laws.

Then, in 1995, Jacobson proposed[4] the inverse relation: assuming the correspondence between area and entropy as well as between temperature and acceleration/surface gravity, he derived the Einstein's field equations from the laws of thermodynamics. So these laws should apply not only for specific kinds of horizons, but for all *causal horizons*, which we define as the boundary of the past of some observer (or set of events). Noting that Bekenstein's argument for attributing an entropy to an event horizon arise from its capability of hiding information, one sees that it should be possible to extend this concept of entropy to every causal horizon. Thus, we will first justify in a more precise way the conjecture that entropy corresponds to horizon area in a very general context, specially the concept of horizon entropy density. Moreover, it will be considered processes occurring in a sufficiently small neighborhood, so that the approximation of spacetime by Minkowski space does not suffer a great deviation and we can say that the metric is nearly-stationary, with a background described by a Killing vector field. Therefore, we shall study the flow of energy-momentum across a bifurcate Killing horizon and proceed as in the previous section, focusing in the Rindler horizon's example.

We derive now a local version of the first law, which differs from that for black holes in considering the increase in horizon area due to a small flow of energy across the horizon, rather than comparing two neighboring solutions of the spacetime as a whole. For this, we consider a stationary background spacetime with Killing vector field K, a Killing horizon and its null generators L with affine parameter v. From the definition of the expansion θ of the congruence generating the horizon (equation 2.16), the change in area of a congruence's bundle is

$$\Delta\mathcal{A} = \int_B \mathrm{d}^2\mathcal{A}\,\mathrm{d}v\;\theta, \tag{2.35}$$

where we integrate over the region B defined by the bundle with finite interval of $v \in [v_1, v_2]$. Given

$$\theta = \frac{\mathrm{d}(v\theta)}{\mathrm{d}v} - v\frac{\mathrm{d}\theta}{\mathrm{d}v}$$

and the Raychaudhuri's equation (2.12) without vorticity*, the previous equation can be written as

$$\Delta\mathcal{A} = \int_{\mathcal{A}_1+\mathcal{A}_2} \mathrm{d}^2\mathcal{A}\;\left(v\theta\Big|_{v_1}^{v_2}\right) + \int_B \mathrm{d}^2\mathcal{A}\,\mathrm{d}v\;2v\sigma^2 + \frac{1}{2}v\theta^2 + R_{ab}L^a vL^b.$$

* Again, Frobenius's theorem 3.2.2 is being used.

Rewriting the first integral in a more concise way and using Einstein's field equations (2.13), we get

$$\Delta\mathcal{A} = v\theta\mathcal{A}\Big|_{v_1}^{v_2} + \int_B \mathrm{d}^2\mathcal{A}\,\mathrm{d}v\; 2v\sigma^2 + \frac{1}{2}v\theta^2 + \int_B \mathrm{d}\Sigma^a\; 8\pi T_{ab} v L^b, \tag{2.36}$$

being the last integral the total orthogonal flux of its integrand across the horizon section B, with $\mathrm{d}\Sigma^a = \mathrm{d}^2\mathcal{A}\,\mathrm{d}v\; L^a$.

This equation holds for every null congruence if Einstein's equations are satisfied, but we now concentrate in the case of small perturbations of the stationary background, induced by an energy-momentum tensor of order ϵ. Einstein's equations imply that the variation in the metric will be of the same order, so $\theta = g^{ab}L_{a;b}$ and σ will also be, since their values in the stationary background are zero ($K_{(a;b)} = 0$ means that $\theta_{ab} = 0$ and switching to an affine parametrization does not change it). Considering that the generators do not form caustics in B, where θ would diverge, then the second and third terms in (2.36) can be dropped, for they are of second order in ϵ. Furthermore, if at v_2 the horizon becomes again stationary, then $\Delta\mathcal{A}$ will be finite and θ must decrease faster than $1/v$, so, for sufficiently large values of v_2, $v\theta\mathcal{A}|_{v_2}$ will be negligible.

To examine $v\theta\mathcal{A}|_{v_1}$, we first look for the relation between the affine parameter v and the Killing parameter u in the background spacetime. Since $K = \partial/\partial u$ and $L = \partial/\partial v$, we have that $K^a = (\partial v/\partial u)L^a$ and

$$K^a{}_{;b}K^b = \left(\frac{\partial v}{\partial u}\right)_{;b} L^b \frac{\partial v}{\partial u} L^a + \left(\frac{\partial v}{\partial u}\right)^2 L^a{}_{;b}L^b = \left(\frac{\partial v}{\partial u}\right)_{;b} L^b K^a = \kappa K^a,$$

given the definition of κ (2.18) and $L^a{}_{;b}L^b = 0$. Thus,

$$\frac{\partial^2 v}{\partial v \partial u} = \kappa \quad \Rightarrow \quad \frac{\partial v}{\partial u} = \kappa v + c \quad \Rightarrow \quad v = a e^{\kappa u} + b, \tag{2.37}$$

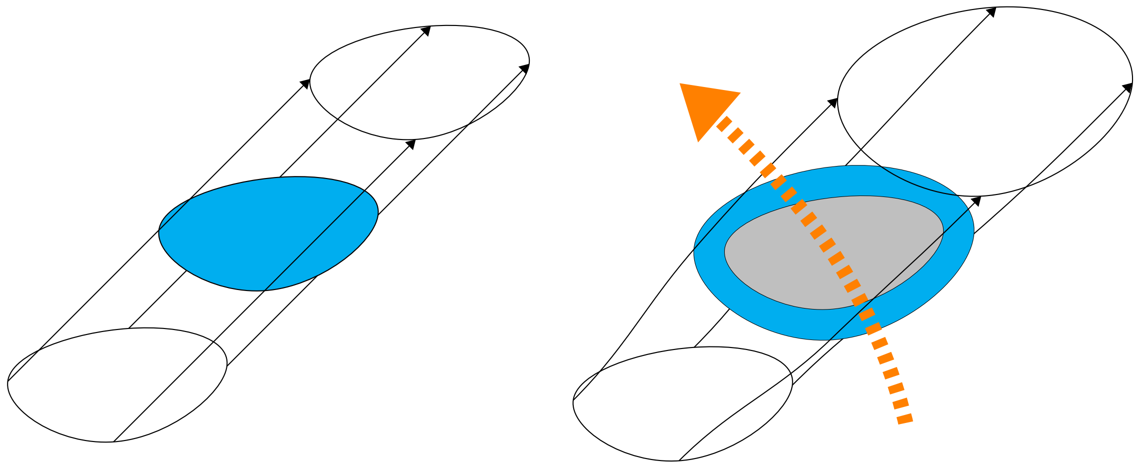
where a, b and c are constants. We see that for $u_1 = u(v_1) < 0$, $|u_1| \gg \kappa^{-1}$ and choosing $b = 0$, the term $v\theta\mathcal{A}|_{v_1}$, which is of order ϵ from θ, goes to zero exponentially fast. This means that, if the Killing time interval is large enough for the horizon to be perturbed and set to "thermal equilibrium" at its "initial and final states", where v equals to v_1 and v_2, respectively, then we may suppress $v\theta\mathcal{A}|_{v_1}$.

Moreover, as the energy-momentum tensor is already of order ϵ, we can use the relation (2.37) with $b = 0$ for the background spacetime to obtain

$$vL^a = v\left(\frac{\partial v}{\partial u}\right)^{-1} K^a = \frac{K^a}{\kappa}. \tag{2.38}$$

Using this result and the above approximations, we find that (2.36) becomes

$$\Delta\mathcal{A} = 4\frac{2\pi}{\kappa}\int_B \mathrm{d}\Sigma^a\; T_{ab}K^b. \tag{2.39}$$



(a) (b)

Figure 15 – Distortion of a causal horizon by an energy flux.
The background stationary spacetime in (a) has a cross-sectional area which does not vary along the interval $[v_1, v_2]$, being represented by the blue region in the middle. We can see this by noting that $\Delta\mathcal{A} = 0$ when $T_{ab} = 0$. On the right (b), we assume that a flux of energy-momentum (in orange) crosses the horizon, so its generators are focused, distorting the background structure. If in addition we impose that both the initial and final sections of the horizon are in nearly-stationary regions, then the resulting cross-sectional areas will be greater than the corresponding ones in the background (in gray) by an amount $\Delta\mathcal{A}$ (in blue).

Source: By the author.

This equation gives the difference in total cross-sectional area between a background Killing horizon H and the resulting perturbed horizon, when a small flux of Killing energy

$$\delta E = \int_H \mathrm{d}\Sigma^a \, T_{ab} K^b$$

passes through H into the region of spacetime hidden by it (see Figure 15). For we can attribute a temperature $T_H = \kappa/2\pi$ to a bifurcate Killing horizon, with its related surface gravity/acceleration κ, and our previous discussion suggests that we should attribute also an entropy to it, we feel compelled to define the entropy S of a causal horizon as one quarter of its total cross-sectional area. In this view, equation (2.39) becomes indeed *the first law of thermodynamics applied to a causal boundary*,

$$\delta S = \frac{\delta E}{T_H}, \tag{2.40}$$

when one assumes Einstein's field equations and an "equilibration time" given by κ, or the temperature T_H.

In order to drop the restriction of a equilibration time, we can assume, instead of a small perturbation in the stationary background, that the process occurs adiabatically. This means that the energy flux crosses the horizon at a rate which is slow when compared with κ. In this case, it is more useful to write Raychaudhuri's equation in terms of the

Killing parameter u,

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}u} = \kappa\hat{\theta} - \frac{1}{2}\hat{\theta}^2 - \frac{1}{2}\hat{\sigma}^2 - R_{ab}K^aK^b,$$

where $\hat{\theta} = (\mathrm{d}v/\mathrm{d}u)\theta$, $\hat{\sigma} = (\mathrm{d}v/\mathrm{d}u)\sigma$ and $\kappa = (\mathrm{d}^2v/\mathrm{d}u^2)/(\mathrm{d}v/\mathrm{d}u)$ (from (2.37)). Since the generators change slowly, $\hat{\theta} \ll \kappa$, $\hat{\sigma} \ll \kappa$, $(\mathrm{d}\hat{\theta}\big/\mathrm{d}u) \ll \kappa\hat{\theta}$ and we may approximate

$$\hat{\theta} \approx \kappa^{-1}R_{ab}K^aK^b.$$

Using Einstein's equations again and equation (2.35), we obtain an even more local first law (2.40), valid for any time interval and finite horizon's subset undergoing an adiabatic process.

With the first law and the correspondence between entropy and area well established in case Einstein's field equations hold, we take now a different point of view, where we assume the thermodynamic relations to derive Einstein's equations. For this, we define* a *Local Rindler Horizon (LRH)* as the boundary of the past of a spacelike 2-surface element $\mathcal{S}$ containing a point p, such that the past directed causal curves are orthogonal to and originate on one side of $\mathcal{S}$, as shown in Figure 16. Additionally, we require the LRH generators' expansion and shear to vanish at p, meaning that the LRH reaches equilibrium at this point.

For a sufficiently small neighborhood of p, a possibly non-null energy-momentum tensor will cause a small perturbation of the local Minkowski space and the previous analysis holds. So a Rindler horizon can be taken as a background stationary Killing horizon for some Rindler observer with an acceleration κ, corresponding to a temperature T_H given by equation (2.34). A LRH may be seen then as a bundle of the resulting perturbed background horizon and the related change in area can be found following the same steps used to get (2.39), but keeping it in terms of the curvature tensor:

$$\Delta\mathcal{A} = \frac{1}{\kappa}\int_{LRH} \mathrm{d}\Sigma^a\ R_{ab}K^b.$$

Imposing the first law (2.40) results in

$$\int_{LRH} \mathrm{d}\Sigma^a\ R_{ab}\frac{K^b}{\kappa} = \int_{LRH} \mathrm{d}\Sigma^a\ 8\pi T_{ab}\frac{K^b}{\kappa}.$$

For K^b/κ is independent of κ (see (2.38)), we can take the limit $\kappa \to \infty$ where the Rindler observer's trajectory approaches the horizon and, consequently, the point p. Furthermore, the condition $|u_1| \gg \kappa^{-1}$, necessary to consider the horizon's initial state as being in equilibrium, is automatically satisfied.

* Curiously, the LRH does not fit in the definition of a causal horizon presented in 19, since in there it is required that the boundary of the past relates to a timelike curve of *infinite* proper length in the future direction.

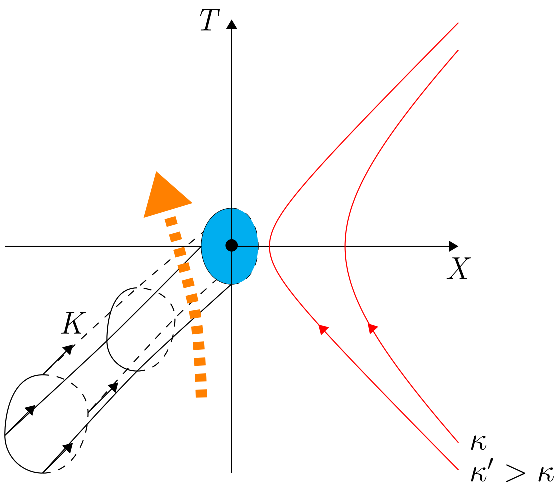


Figure 16 – Local Rindler Horizon.
The LRH is defined from a spacelike 2-surface element $\mathcal{S}$ (in blue, represented in the plane X-T but orthogonal to it) passing through a point p (at the center). The restriction of $\partial J^{-}(\mathcal{S})$ to null geodesics orthogonal to the surface with zero expansion and shear on it results in the curves with tangent vectors K. For a sufficiently small neighborhood of p, these vectors compose a bundle of the Killing vector field generating a Rindler horizon seen by observers following the hyperbolic trajectories in red. When some energy flux (in orange) crosses this bundle, the background Killing horizon is distorted, yielding an increase in the total cross-sectional area. The first law in this case requires a Rindler observer with acceleration κ, which can be increased to bigger values κ', so that the trajectory gets closer to the horizon and p, providing a local thermodynamic analysis.

Source: By the author.

Since the last equation is valid for any LRH, the integrands must be equal,

$$R_{ab}L^a v L^b = 8\pi T_{ab}L^a v L^b,$$

and the arbitrariness of the null vector L and its parametrization implies that

$$R_{ab} = 8\pi T_{ab} + f g_{ab},$$

for some function f. Using the contracted Bianchi identities[7] $R_a{}^b{}_{;b} = (1/2)R_{;a}$ and local conservation of energy and momentum $T_a{}^b{}_{;b} = 0$, one finds that $f_{;a} = -4\pi T_{;a}$, with $T = T^a{}_a$. So

$$f = -4\pi T + \Lambda,$$

where Λ is a constant, and we recover the Einstein's field equations (2.13). Since they were derived from the first law of thermodynamics, we may interpret them as equations of state, describing the dynamics of spacetime that originates from an energy flux.

Returning to the first question of this section, "How general relativity could have this thermodynamic behavior encoded in itself?", we may say that there is no conclusive answer, but we gained some insight into it. From our derivation, it seems that one can take the laws of thermodynamics as more fundamental and gravity appears as a macroscopic consequence of it, so it would be very natural to reproduce the thermodynamic behavior from general relativity and its solutions to spacetimes. It is worth to note that there is a relation between boundary terms of the gravitational action and the area of causal horizons,[18,21] what also suggests that some geometric quantity, most probably the area, plays a central hole in variational principles applied to gravitation. Since this area may diverge and impose difficulties when one considers sections of infinite area or infinite values of generators' affine parameter, we conjecture that the important variable is the horizon's entropy density, what is supported by our previous calculations (we used the area of bundles, not the total cross-sectional area of a horizon). The real significance of these thermodynamic relations remains unclear, but they strongly hint that some microscopic statistical (possibly quantum) mechanics produces in a macroscopic, classical level the effects which we label as gravitational. Possible candidates for this underlying dynamics are given by string theory, loop quantum gravity and the vacuum fluctuations' entanglement occurring across the horizon. In any case, progress in the thermodynamic approach may lead to a better understanding and possibly discard of some candidate theories, contributing to formulate a successful quantum gravity theory (or whatever may substitute it) and overcome some of the present problems in fundamental physics.

3 GEOMETRY OF CAUSAL HORIZONS

Since causal horizons are lightlike hypersurfaces and play a central role in the thermodynamic approach to general relativity, as well as their geometric properties, like cross-sectional areas and expansion of generators, we shall direct our attention now to a purely geometric study of this specific type of surface. In the first section, we define basic concepts to the subsequent text, including a review of the Lie and exterior derivatives. Then, we formalize in the next section the idea of surfaces in a spacetime as embedded submanifolds and give a proof for the Frobenius's theorem, which was used in the previous chapter and can be a useful tool in the study of submanifolds. The third section shows some particularities present in lightlike manifolds, like the restriction for the existence of a Levi-Civita connection to symmetric cases involving Killing vector fields. The last section explains the geometric objects induced by a manifold and its metric on a lightlike hypersurface in a very general manner. The whole chapter is based mainly in 6, so references are made only for specific statements or other sources. Further than explaining a greater number of intermediate steps in the proofs of theorems and propositions, this text differs in some points from 6 by avoiding definitions in terms of vector bundles. Instead, we prefer the use of distributions, since it does not affect the comprehension of the text and the reader may be unfamiliar with the concept of vector bundles.

3.1 Definitions

The main difference between lightlike (or null) surfaces and the ones which are spacelike or timelike is that their generators are given by a null vector field K, so, at each point x of the surface $\mathcal{M}$, the normal vector is orthogonal to itself. Thus, the normal and tangent spaces of $\mathcal{M}$ have an intersection which is not empty, implying that the induced geometric objects must be defined in an unusual manner. We have that

$$K \neq 0, \quad g(K,K) = g(K,X) = 0, \quad \forall X \in T_xM,$$

where g represents the induced metric on the surface from a metric $\bar{g}$ of the manifold $\bar{\mathcal{M}}$ in which $\mathcal{M}$ is embedded. In this case, we say that g is *degenerate* on T_xM, with *radical (or null) space* defined by

$$\operatorname{Rad} T_xM = \{K \in T_xM \mid g(K,X) = 0, X \in T_xM\}.$$

The *nullity degree* of any metric g, $\operatorname{null} T_xM = r$, is given by the dimension of $\operatorname{Rad} T_xM$. Also, the *index* of g ($\operatorname{ind} T_xM$) is the dimension n of the largest subspace $W \subset T_xM$ for which the restriction of g to it is negative definite. From the metric signature of a manifold (or submanifold) of dimension m, we see that r is number of zeros in it and n and p are the number of minus and plus signs, respectively, if $r+n+p=m$. So we say that g is of *type* (p,n,r), being the spacetimes of general relativity of signature $(-,+,+,+)$ and type (3,1,0), with lightlike hypersurfaces of induced metric signature (0,+,+) and type *(2,0,1)*. A *Lorentz (Minkowski) space* with *Lorentz (Minkowski) metric* is a vector space with index $n=1$, $p \geq 1$ and $r=0$.

The complementary subspace to $\mathrm{Rad}\, T_xM$ in T_xM is called a screen subspace of T_xM:

$$T_xM = \mathrm{Rad}\, T_xM \perp S(T_xM),$$

being $\perp$ an orthogonal direct sum. To generalize this concept to the whole tangent bundle

$$TM = \bigcup_{x \in \mathcal{M}} T_xM$$

we first give the following definition:

Definition 3.1.1. *A* distribution *D on $\mathcal{M}$ maps each point $x \in \mathcal{M}$ to a linear subspace of T_xM, D_x, with dimension r. Then we say that D is of rank r and a vector field $X \in TM$ belongs to D if $X(x) \in D_x$ for each point x, so we write $X \in D$. In case there are r smooth linearly independent vector fields belonging to D in a coordinate neighborhood of each x, then D is a smooth distribution and the set of these vector fields forms a* local basis *of D. In the following, every distribution will be taken to be a smooth one.*

We consider that g is of the same type at every $x \in \mathcal{M}$. If g is positive definite, i.e. $p=m$, then we say that $\mathcal{M}$ is a *Riemannian manifold* with a *Riemannian metric*. In general, $\mathcal{M}$ is called a *semi-Riemannian manifold* for $r=0$, with the special case of index n equal to 1 and $m \geq 2$, when we have a *Lorentz manifold*. If $r>0$, $\mathcal{M}$ becomes a *r-lightlike (r-degenerate) manifold* and we suppose that $\mathrm{Rad}\, TM$, the collection of all $\mathrm{Rad}\, T_xM$ at each point of the manifold, defines a smooth distribution of rank r called the *radical distribution*, for which

$$g(K,X)=0, \quad \forall K \in \mathrm{Rad}\, TM \quad \text{and} \quad X \in TM. \tag{3.1}$$

In a similar manner, we define the *screen distribution* $S(TM)$ as the complementary distribution to $\mathrm{Rad}\, TM$ in TM.

The differential operators necessary to study the geometry of semi-Riemannian manifolds and their surfaces are, further than the covariant derivative given by a linear connection ∇, the Lie derivative and the exterior derivative (differential). The Lie derivative refers to the change in a tensor field T along the flow generated by a vector field X. That is, given the integral curves for which X is tangent at every point of a coordinate neighborhood, the mapping ϕ_t takes each point x in each curve to another point $\phi_t(x)$ of the same curve, which is found a parameter distance t from x. Therefore, it induces a mapping ϕ_{t*} (the *pushforward*) between the tangent spaces T_xM and $T_{\phi_t(x)}M$ and, consequently, between tensor fields defined in T_xM and $T_{\phi_t(x)}M$. The *Lie derivative* is defined then as

$$(L_X T)(x) = \lim_{t\to 0} \frac{1}{t}(T(x) - (\phi_{t*}T)(x)).$$

For a function f over the manifold $\mathcal{M}$, vector fields $X, Y, X_i \in TM$ and dual vector fields (1-forms) $\omega, \omega^i \in T^*M$ we have that

$$L_X f = X(f), \quad L_X Y = [X,Y], \quad (L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])$$

and, if $T \in T^n_m M$,

$$\begin{aligned} L_X T(\omega^1,\dots,\omega^n, X_1,\dots,X_m) = X(T(\omega^1,\dots,\omega^n, X_1,\dots,X_m)) \ - \\ \sum_{i=1}^{n} T(\omega^1,\dots,L_X\omega^i,\dots,\omega^n, X_1,\dots,X_m) \ - \\ \sum_{j=1}^{m} T(\omega^1,\dots,\omega^n, X_1,\dots,[X,X_j],\dots,X_m). \end{aligned} \tag{3.2}$$

If the flow of integral curves of a vector field X represents an infinitesimal isometry of the manifold $\mathcal{M}$, that is, the corresponding tangent mapping ϕ_* preserves the scalar product (as in equation (2.1)), then X is a *Killing vector field* and $L_X g = 0$. From equation (3.2), it follows that

$$(L_X g)(Y,Z) = X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]). \tag{3.3}$$

Being ∇ a *torsion-free connection*, $[X,Y] = \nabla_X Y - \nabla_Y X$, and we say that ∇ is a *metric connection* if

$$(\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0. \tag{3.4}$$

Considering that ∇ is the *Levi-Civita connection*, which is the torsion-free metric connection with respect to g, we find that

$$(L_X g)(Y,Z) = 0 \quad \Rightarrow \quad g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0, \quad \forall Y,Z \in TM. \tag{3.5}$$

In terms of coordinates, it yields

$$X_{a;b}Y^bZ^a + X_{b;a}Z^aY^b = 0,$$

so the independence of Y, Z implies the *Killing's equation*:

$$X_{a;b} + X_{b;a} = 0 \quad \Longleftrightarrow \quad L_X g = 0. \tag{3.6}$$

The exterior derivative acts over the so-called *differential q-forms*, which are covariant tensor fields of type $(0,q)$, $W \in T^0_q M$, antisymmetric in all q positions. Namely, given the natural frames field $\partial_i = \partial/\partial x^i$, or coordinate basis, and its dual natural coframes field $\mathrm{d}x^i$, a q-form has coordinates $W_{a_1,\ldots,a_q} = W_{[a_1,\ldots,a_q]}$, with[8]

$$W_{[a_1,\ldots,a_q]} = \frac{1}{q!}\sum_{\pi} \delta_\pi W_{a_{\pi(1)},\ldots,a_{\pi(q)}},$$

where we sum over the permutations π of the indices a_i, being δ_π equal to $+1$ for even permutations and -1 for odd ones. Another way to write it is

$$W = \frac{1}{q!} W_{a_1,\ldots,a_q} \mathrm{d}x^{a_1} \wedge \ldots \wedge \mathrm{d}x^{a_q},$$

with $\wedge$ denoting the *exterior (wedge) product*, a tensor product $\otimes$ antisymmetrized in all its terms. Then, the *exterior derivative* d maps each differential q-form W, which is a function of q vector fields $X_i \in TM$, to a $(q+1)$-form dW given by

$$dW(X_1,\ldots,X_{q+1}) = \frac{1}{q+1}\left\{\sum_{i=1}^{q+1}(-1)^{i+1}X_i(W(X_1,\ldots,\hat{X}_i,\ldots,X_{q+1})) \quad + \right.$$
$$\left. \sum_{1\leq i<j\leq q+1}(-1)^{i+j}W([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{q+1})\right\},$$

where ˆ indicates a term that is omitted. As an example, the exterior derivative of a 1-form ω equals to

$$d\omega(X,Y) = \frac{1}{2}\left\{X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\right\}, \tag{3.7}$$

or in terms of its coordinates $\omega = \omega_a \mathrm{d}x^a$,

$$d\omega = \frac{1}{2!}(d\omega)_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b, \quad \text{with} \quad (d\omega)_{ab} = \frac{1}{2}\left(\frac{\partial\omega_b}{\partial x^a} - \frac{\partial\omega_a}{\partial x^b}\right). \tag{3.8}$$

An important property of the exterior derivative is that $d^2 = 0$ when applied to any differential q-form.

If two vector fields $X, Y \in TM$ commute, i.e. $[X,Y] = 0$, their Lie derivatives vanish, $L_X Y = L_Y X = 0$, meaning that the flows of integral curves of each one of them have parameters which are independent of each other, for

$$[X,Y](f) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) = 0, \quad \text{if} \quad X = \frac{\partial}{\partial t} \quad \text{and} \quad Y = \frac{\partial}{\partial s}.$$

Therefore, these parameters may be taken as coordinates and the corresponding maps $\phi_{X,t}(x)$ and $\phi_{Y,s}(x)$ also commute, as shown in Figure 17. This independence of parameters and commutativity of vector fields is the main characteristic of a natural frames field, which is also called a *holonomic frames field*. However, the study of lightlike surfaces and null curves often requires a *non-holonomic frames field*, for which the basis vectors e_i do not commute ($[e_i, e_j] \neq 0$). An example is the quasi-orthonormal basis used in Section 2.1 and shown in Figure 1. The requirements of quasi-orthonormality and parallel transport of the basis vectors along the curve generated by one of them breaks the independence between the integral curves' parameters in general, implying that there is no coordinate system compatible with such a basis.

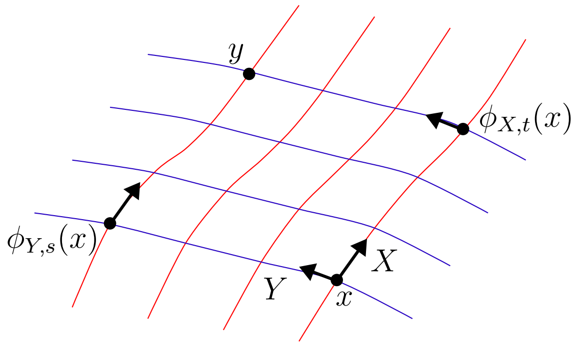


Figure 17 – Vector fields commuting.
If we start at a point x of the manifold and apply consecutively the maps $\phi_{X,t}(x)$ and $\phi_{Y,s}(\phi_{X,t}(x))$, then we find the same final point y obtained using first $\phi_{Y,s}(x)$ and then $\phi_{X,t}(\phi_{Y,s}(x))$, so $\phi_{Y,s}(\phi_{X,t}(x)) = \phi_{X,t}(\phi_{Y,s}(x)) = y$. This means that going from x a parameter distance t along the integral curves of X (in red) and then a parameter distance s along the curves generated by Y (in blue) results in the same final point y if we invert the order $t \leftrightarrow s, X \leftrightarrow Y$. So the points y that can be reached following these integral curves form a 2-dimensional submanifold,[7] where we can use t and s as coordinates and $\{X, Y\}$ as coordinate basis.

Source: By the author.

3.2 Submanifolds and Frobenius's theorem

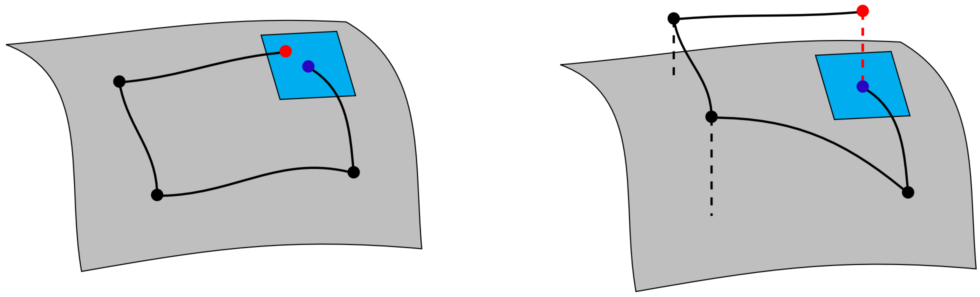
As two commuting vector fields span the tangent bundle of some 2-dimensional submanifold, a set of r linearly independent vector fields with Lie bracket equal to zero for any pair of the set forms a basis for the tangent bundle of a r-dimensional submanifold, for the r parameters of the integral curves generate a system of r coordinates. A trivial case is given by only one vector field, which spans the 1-dimensional tangent bundle of the manifolds corresponding to its integral curves. So a distribution can be seen as the tangent bundle of a submanifold in case the vector fields of its local basis commute. More generally, we will see that a rank r distribution D is the tangent bundle of some submanifold of the m-dimensional manifold $\mathcal{M}$ if the Lie bracket of any two vector fields in the local basis results in a vector belonging to D. Given a local basis of D $\{X_1, \ldots, X_r\}$ and the complementary set $\{X_{r+1}, \ldots, X_m\}$, which together form a local frames field on $\mathcal{M}$, in general

$$[X_\alpha, X_\beta] = C^a_{\alpha\beta} X_a, \tag{3.9}$$

where $\alpha, \beta, \ldots \in \{1, \ldots, r\}$, $a, b, \ldots \in \{1, \ldots, m\}$ and $C^a_{\alpha\beta} = -C^a_{\beta\alpha}$ are smooth functions locally defined on $\mathcal{M}$. So D will locally correspond to a tangent space if it is *involutive*, that is,

$$C^i_{\alpha\beta} = 0, \tag{3.10}$$

with $i, j, \ldots \in \{r+1, \ldots, m\}$. In the rest of this section and in the next one, we shall keep the ranges for $a, b, \alpha, \beta, i, j, \ldots$ restricted as above.



(a) (b)

Figure 18 – Parameters loops and involutivity.
Given two vector fields $X, Y \in D$, where D is a distribution represented by the blue plane, one may try to perform a loop along their integral curves, starting at the blue point and ending at the red one. If $[X, Y] \in D$, the difference between initial and final points is given by a vector $Z \in D$ and D corresponds to the tangent space of the submanifold $\mathcal{S}$ in gray, as in (a). However, $[X, Y] \notin D$ implies that Z points out of the distribution, so neither $\mathcal{S}$ nor any other submanifold has D as its tangent space.

Source: By the author.

We can picture that as in Figure 18: if the one tries to perform a loop, using the integral curves of two vector fields $X = \partial/\partial t$ and $Y = \partial/\partial s$, by going a parameter distance t along X, then s, $-t$ and $-s$ along Y, X and Y, respectively, the final point will differ from the initial if X and Y do not commute, opposed to what is shown in Figure 17. But if the displacement between the initial and final points can be approximated (in a small neighborhood) to a vector belonging to the distribution D containing X and Y, as in Figure 18(a), then D is involutive and will still correspond to the tangent space of a submanifold. On the other hand, if the displacement resulting of the loop in the parameters has a direction given by a vector which does not belong to D, like in Figure 18(b), then one cannot find a smooth submanifold with tangent space given by D. An example of non-involutive distribution that results in a loop similar to the one in Figure 18(b) is the case of the rank 2 distribution on $\mathbb{R}^3$ spanned by[22]

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y}.$$

The Lie bracket $[X, Y] = -\partial/\partial z$ implies that D is not involutive and, therefore, there is no submanifold tangent to it, as represented in Figure 19.

Before we prove the claim about involutive distributions and submanifolds, we give a clearer definition of the latter. Given two smooth manifolds $\mathcal{N}$ and $\mathcal{M}$ of dimension r and m, respectively, one can locally define a map $\phi : \mathcal{N} \to \mathcal{M}$ as

$$x^a = \phi^a(u^1, \ldots, u^r), \quad \text{for} \quad a \in \{1, \ldots, m\},$$

and ϕ^a being smooth functions of the coordinates on $\mathcal{N}$ to coordinates on $\mathcal{M}$. It induces a mapping of functions f on $\mathcal{M}$ to functions $f \circ \phi : \mathcal{N} \to \mathbb{R}$ on $\mathcal{N}$, so we can say that it

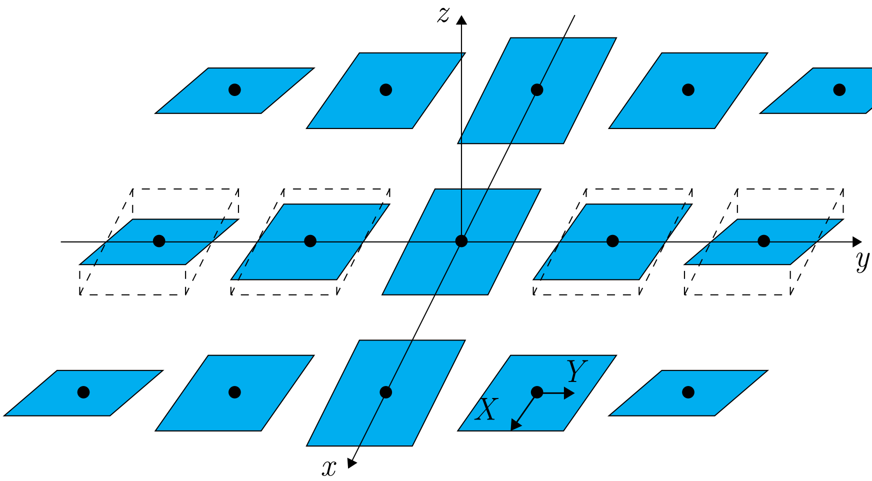


Figure 19 – A non-involutive distribution.
Representation of a non-involutive distribution on $\mathbb{R}^3$, given by the collection of the blue planes, spanned by X and Y. We can see that if we try to put a surface tangent to every plane, this surface would be "torn", for the planes are twisting around axes parallel to the y-axis as one moves in the y direction. We can imagine this tearing as caused by plates glued to some inelastic fabric: if one tries to twist the plates as the distribution, the fabric would be torn.

Source: By the author.

"pulls back" functions from $\mathcal{M}$ to $\mathcal{N}$. We can also construct the tangent mapping at a point $u \in \mathcal{N}$, which is also called the *pushforward* mapping[8]

$$\phi_{*u} : T_u N \to T_{\phi(u)} M,$$

that takes vectors $U \in T_u N$ to $\phi_{*u} U \in T_{\phi(u)} M$ following

$$(\phi_{*u} U)(f) = U(f \circ \phi), \tag{3.11}$$

for all smooth functions f on $\mathcal{M}$. In a similar manner, we can define the *pullback* ϕ_u^* for dual vectors and extend these mappings for purely contravariant or covariant tensors. Mixed tensors (with both lower and upper indices), however, require ϕ to be a diffeomorphism.[8]

The *rank* of ϕ is the dimension of the pushforward's image,[7] $\phi_{*u}(T_u N)$, and is given by the matrix rank

$$\text{rank} \left[\frac{\partial \phi^a}{\partial u^\alpha} \right].$$

So the tangent mapping is *injective* at u if its rank equals to r, case in which we must have $r \leq m$. If ϕ_{*u} is injective for all $u \in \mathcal{N}$, then ϕ is an *immersion* of $\mathcal{N}$ in $\mathcal{M}$ and the image of $\mathcal{N}$ becomes an *immersed manifold* of $\mathcal{M}$. In case ϕ itself is injective, i.e. the image $\phi(\mathcal{N})$ does not intersect itself, it is called an *embedding* and $\mathcal{N}$ is an *embedded submanifold* of $\mathcal{M}$. Since an immersion is locally injective, or a local embedding, we shall consider that a vector field $U \in TN$ is mapped into $\phi_* U \in TM$ if equation (3.11) holds for every point u of a coordinate neighborhood $\mathcal{U} \in \mathcal{N}$. Thus, for local calculations, we shall denote a vector field $\phi_* X$ on a submanifold simply by X and the relation between coordinates in the basis $\partial_\alpha = \partial/\partial u^\alpha$ on $\mathcal{N}$ and in $\partial_a = \partial/\partial x^a$ on $\mathcal{M}$ is given by

$$X = X^\alpha \partial_\alpha \quad \Rightarrow \quad X = X^\alpha B^a_\alpha(u)\partial_a, \quad \text{with} \quad B^a_\alpha(u) = \frac{\partial x^a}{\partial u^\alpha}.$$

Now, if D is a distribution on $\mathcal{M}$ and

$$\phi_{*u}(T_u N) = D_{\phi(u)},$$

for each point u of an immersed manifold $\mathcal{N}$, then $\mathcal{N}$ is called an *integral manifold* of D. Furthermore, $\mathcal{N}$ will be a *maximal integral manifold* or *leaf* of D in case $\mathcal{N}$ is a connected integral manifold of D and there exists no connected integral manifold $\bar{\mathcal{N}}$ for which the immersion $\bar{\phi} : \bar{\mathcal{N}} \to \mathcal{M}$ has image $\bar{\phi}(\bar{\mathcal{N}}) \supset \phi(\mathcal{N})$. We say that the distribution D is *integrable* if every $x \in \mathcal{M}$ is contained in an integral manifold of D. Given these definitions, we state the following propositions and Frobenius's theorem:

Proposition 3.2.1. *Every integrable distribution is involutive.*

Proof:[8] Let D be an integrable distribution on a manifold $\mathcal{M}$. Then, at each point of $\mathcal{M}$, there is an integral manifold $\mathcal{N}$ whose image in $\mathcal{M}$ has a coordinate basis ∂_α, which also serves as a local basis for D. So every vector field $X, Y \in D$ can be written as $X = X^\alpha \partial_\alpha$, $Y = Y^\beta \partial_\beta$ on some open subset $\mathcal{U} \subset \mathcal{M}$. Thus,

$$[X,Y] = X^\alpha \partial_\alpha (Y^\beta \partial_\beta) - Y^\beta \partial_\beta (X^\alpha \partial_\alpha) = \left\{ X^\alpha (\partial_\alpha Y^\beta) - Y^\alpha (\partial_\alpha X^\beta) \right\} \partial_\beta + X^\alpha Y^\beta [\partial_\alpha, \partial_\beta],$$

and since $[\partial_\alpha, \partial_\beta] = 0$,

$$[X,Y] = Z^\alpha \partial_\alpha = Z \in D, \quad \text{with} \quad Z^\alpha = X^\alpha(\partial_\alpha Y^\beta) - Y^\alpha(\partial_\alpha X^\beta),$$

implying that D is involutive. ■

Proposition 3.2.2. *Every involutive distribution is integrable.*

Proof: We saw in the last section and in Figure 17 that, if two vector fields X and Y commute, the parameters of their integral curves form a coordinate system on the subset $\mathcal{U} \subset \mathcal{M}$ corresponding to points which can be reached from an initial point $x \in \mathcal{M}$ through the flows along integral curves of X and Y. So the collection of subsets like $\mathcal{U}$ forms an immersed submanifold with tangent bundle spanned by X and Y themselves. Therefore, if every pair of vector fields in a local basis for a distribution D commutes, then D is integrable. What we have to show is that for every involutive distribution there exists a commuting local basis.[23] So let $\{\partial_1, \ldots, \partial_m\}$ be a coordinate basis on $\mathcal{M}$ and $\{Y_1, \ldots, Y_r\}$ a local basis for D. We can write each Y_α as

$$Y_\alpha = A^a_\alpha \partial_a,$$

being $A = (A^a_\alpha)$ a $r \times m$ matrix of rank r, since it generates the r-dimensional vector spaces of D. Hence, there is some ordering of the vector fields ∂_a for which the $r \times r$ submatrix $A' = (A^\beta_\alpha)$ is invertible, with $A'^{-1} = (\tilde{A}^\beta_\alpha)$. Define another local basis of D,

$$X_\alpha = \tilde{A}^\beta_\alpha Y_\beta = \partial_\alpha + B^i_\alpha \partial_i,$$

for some $r \times (m-r)$ matrix $B = (B^i_\alpha)$. Then, being $[\partial_a, \partial_b] = 0$,

$$\begin{aligned}[X_\alpha, X_\beta] &= [\partial_\alpha + B^i_\alpha \partial_i, \partial_\beta + B^j_\beta \partial_j] \\ &= (\partial_\alpha B^j_\beta)\partial_j - (\partial_\beta B^i_\alpha)\partial_i + B^i_\alpha(\partial_i B^j_\beta)\partial_j - B^j_\beta(\partial_j B^i_\alpha)\partial_i \in \operatorname{span}\{\partial_{r+1}, \ldots, \partial_m\}.\end{aligned}$$

But, for an involutive distribution, $[X_\alpha, X_\beta] \in \operatorname{span}\{X_1, \ldots, X_r\}$, meaning that

$$Z = [X_\alpha, X_\beta] \in \operatorname{span}\{X_1, \ldots, X_r\} \cap \operatorname{span}\{\partial_{r+1}, \ldots, \partial_m\}.$$

So we can express this vector Z in terms of the local basis of D, $Z = Z^\alpha X_\alpha = Z^\alpha \partial_\alpha + Z^\alpha B^i_\alpha \partial_i$, as well as $Z = Z^i \partial_i$. Identifying both expressions implies that $Z^\alpha = 0$, given the linear independence of the basis vectors. Consequently, $[X_\alpha, X_\beta] = 0$ and D is integrable. ■

Theorem 3.2.1 (Frobenius's theorem - vector formulation)**.** *A distribution D on $\mathcal{M}$ is integrable if and only if it is involutive. If so, for each point $x \in \mathcal{M}$ there is a unique leaf of D containing it and any other integral manifold passing through x is an open submanifold of this leaf.*

Proof: The equivalence between integrability and involutivity follows from Propositions 3.2.1 and 3.2.2. Since at each point x there is a commuting local basis of D, as the one obtained in the last proof, it provides a coordinate system for each coordinate neighborhood $\mathcal{U}(x)$ of the local submanifold generated by the flows $\phi_{X_\alpha}(x)$ of basis vector fields X_α. Given a smooth parametrization for the local basis throughout $\mathcal{M}$, the coordinate system for the local submanifolds can be made smooth for a union of local submanifolds. Thus, a collection of neighborhoods $\mathcal{U}(y)$ of points y, which can be reached by the flows $\phi_{X_\alpha}(x)$, in an open subset of $\mathcal{M}$ constitutes an integral submanifold of D. The union of neighborhoods $\mathcal{U}(y)$ of all y which can be reached through successive mappings $\phi_{X_\alpha}(x)$ constitutes a leaf of D. Uniqueness results from the fact that, given a vector field X_α and a point x, there is a unique integral curve tangent to X_α passing through x and, therefore, the flows $\phi_{X_\alpha}(x)$ are unique. Then any other integral manifold at x is a collection of neighborhoods $\mathcal{U}(y)$ contained in the leaf. ■

Another way to specify the distribution D is to use the dual fields of coframes $\{\omega^1, \ldots, \omega^m\}$ of $\{X_1, \ldots, X_m\}$ on $\mathcal{M}$, for which $\{X_1, \ldots, X_r\}$ is a local basis of D. So

$$\omega^i(X) = 0 \quad \Leftrightarrow \quad X \in D,$$

and from equations (3.7) and (3.9),

$$d\omega^i(X_\alpha, X_\beta) = -\frac{1}{2}\omega^i([X_\alpha, X_\beta]) = -\frac{1}{2}C^i_{\alpha\beta}. \tag{3.12}$$

Moreover, equation (3.8) yields

$$d\omega^i = \frac{1}{2} P^i_{ab}\omega^a \wedge \omega^b \quad \Rightarrow \quad P^i_{\alpha\beta} = -C^i_{\alpha\beta},$$

$$\therefore \quad d\omega^i = \frac{1}{2}\left\{-C^i_{\alpha\beta}\omega^\alpha \wedge \omega^\beta + P^i_{\alpha k}\omega^\alpha \wedge \omega^k + P^i_{k\alpha}\omega^k \wedge \omega^\alpha + P^i_{jk}\omega^j \wedge \omega^k\right\} \tag{3.13}$$

for certain smooth functions $P^i_{ab} = -P^i_{ba}$. As a result, we may state Frobenius's theorem as

Theorem 3.2.2 (Frobenius's theorem - dual formulation). *A rank r distribution D on a m-dimensional manifold $\mathcal{M}$ is integrable if and only if its defining differential 1-forms $\{\omega^{r+1}, \dots, \omega^m\}$ satisfy*

$$d\omega^i = \eta^i_k \wedge \omega^k, \quad \textit{with} \quad i, k \in \{r+1, \dots, m\},$$

for some 1-forms $\eta^i_k \in T^*M$.

Proof: If D is integrable, the previous formulation of Frobenius's theorem says that it is also involutive, so equation (3.10) holds and (3.13) becomes

$$d\omega^i = \left\{P^i_{\alpha k}\omega^\alpha + \frac{1}{2}P^i_{jk}\omega^j\right\} \wedge \omega^k,$$

so $\eta^i_k = P^i_{\alpha k}\omega^\alpha + (1/2)P^i_{jk}\omega^j$ fulfills the theorem. In turn,

$$d\omega^i = \eta^i_k \wedge \omega^k \quad \Rightarrow \quad d\omega^i(X_\alpha, X_\beta) = 0,$$

which together with equation (3.12) implies that $C^i_{\alpha\beta} = 0$ and D is involutive. Therefore, the vector formulation of the theorem determines that D is integrable. ■

A vector field $K = K^a\partial_a$ will be orthogonal to a hypersurface if and only if its dual $\kappa = \kappa_a \mathrm{d}x^a = g_{ab}K^b\mathrm{d}x^a$ defines a distribution D tangent to the hypersurface, i.e. $\kappa_a X^a = 0$ for every $X \in D$. Furthermore, equation (3.8) and

$$\kappa_{a;b} = \frac{\partial \kappa_a}{\partial x^b} - \Gamma^c_{ab}\kappa_c \quad \Rightarrow \quad \kappa_{[a;b]} = \frac{1}{2}\left(\frac{\partial \kappa_a}{\partial x^b} - \frac{\partial \kappa_b}{\partial x^a}\right) = -(d\kappa)_{ab},$$

if $\Gamma^a_{bc} = \Gamma^a_{cb}$ are the Christoffel symbols of a torsion-free connection in terms of a coordinate basis [6, p.31]. Then, Frobenius's theorem tells us that, for some $\eta \in T^*M$,

$$d\kappa = \eta \wedge \kappa \quad \Rightarrow \quad (d\kappa)_{ab} = \eta_{[a}\kappa_{b]},$$

and the projection of $\kappa_{[a;b]}$ into D vanishes:

$$\kappa_{[a;b]}X^aY^b = -\eta_{[a}\kappa_{b]}X^aY^b = 0, \quad \forall X, Y \in D. \tag{3.14}$$

Therefore, K is a hypersurface orthogonal vector field if and only if its congruence has a null vorticity (see equations (2.6)), justifying the use of Raychaudhuri's equation without the vorticity term in the sections 2.2.3 and 2.5.

Another consequence of Theorem 3.2.1 is that an integrable distribution D on $\mathcal{M}$ allows us to construct a coordinate system x^a in which each leaf $\mathcal{L}$ is determined by $m-r$ equations $x^i = c^i$, being c^i real constants, and the other r parameters x^α are local coordinates on L. In this case, the collection of all disjoint leaves is called a *foliation* on $\mathcal{M}$, which provides a "slicing" of the manifold that is useful in many calculations. If we consider another coordinate system $\bar{x}^a$ adapted to the foliation, such that $\bar{x}^i = \bar{c}^i$, then $\mathrm{d}\bar{x}^i(\partial_\beta) = \mathrm{d}x^i(\partial_\beta) = 0$, for any basis vector $\partial_\beta \in D$, and

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j}\mathrm{d}x^j + \frac{\partial \bar{x}^i}{\partial x^\alpha}\mathrm{d}x^\alpha \quad \Rightarrow \quad \frac{\partial \bar{x}^i}{\partial x^\alpha}\mathrm{d}x^\alpha(\partial_\beta) = 0, \quad \forall \partial_\beta \in D.$$

Since $\mathrm{d}x^\alpha(\partial_\beta)$ cannot be zero for all ∂_β, for $\mathrm{d}x^\alpha$ is one of the linearly independent 1-forms in the dual basis on $\mathcal{L}$, we have that

$$\frac{\partial \bar{x}^i}{\partial x^\alpha} = 0, \quad \forall i \in \{r+1, \ldots, m\} \quad \text{and} \quad \alpha \in \{1, \ldots, r\}.$$

So a general transformation between coordinates on the same foliation is given by

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, \ldots, x^m) \quad \text{and} \quad \bar{x}^i = \bar{x}^i(x^{r+1}, \ldots, x^m), \tag{3.15}$$

with the transformation of natural frames fields

$$\frac{\partial}{\partial x^\alpha} = B_\alpha^\beta(x)\frac{\partial}{\partial \bar{x}^\beta}, \quad \frac{\partial}{\partial x^i} = B_i^j(x)\frac{\partial}{\partial \bar{x}^j} + B_i^\alpha(x)\frac{\partial}{\partial \bar{x}^\alpha}, \quad \text{for} \quad B_b^a = \frac{\partial \bar{x}^a}{\partial x^b}, \quad x \in \mathcal{M}. \tag{3.16}$$

3.3 Lightlike manifolds

Now, let $\mathcal{M}$ be an r-lightlike manifold of dimension m and g be an r-degenerate metric. Suppose $\mathrm{Rad}\,TM$ is an integrable distribution (as in the case of lightlike hypersurfaces in general relativity). So there is a coordinate system, like the one above, adapted to the foliation on $\mathcal{M}$ given by $\mathrm{Rad}\,TM$. From equation (3.1), it follows that the components of g associated to coordinates x^α on the leaves of $\mathrm{Rad}\,TM$ are null,

$$g_{\alpha\beta} = g_{\alpha i} = g_{i\alpha} = 0, \quad \forall \alpha, \beta \in \{1, \ldots, r\} \quad \text{and} \quad i \in \{r+1, \ldots, m\},$$

and the only non-null components are the g_{ij}. If

$$\frac{\partial g_{ij}}{\partial x^\alpha} = 0, \quad \forall \alpha \in \{1, \ldots, r\} \quad \text{and} \quad i, j \in \{r+1, \ldots, m\}, \tag{3.17}$$

the first equation in (3.16) guarantees that equation (3.17) holds for every coordinate system fitting the foliation of $\mathrm{Rad}\,TM$. Therefore, in case $\mathrm{Rad}\,TM$ is integrable and equation (3.17) is satisfied for some coordinate system, we call $\mathcal{M}$ a *Reinhart lightlike manifold*. Also, we define a *Killing distribution* as a distribution D in which every vector field $X \in D$ is a Killing vector field and a tensor field T on $\mathcal{M}$ is *parallel* with respect to a connection ∇ if $\nabla_X T = 0$, $\forall X \in TM$. Given these definitions, we state the following theorem:

Theorem 3.3.1. *Let* $(\mathcal{M}, g)$ *be a lightlike manifold. Then the following assertions are equivalent:*

(i) $(\mathcal{M}, g)$ *is a Reinhart lightlike manifold.*

(ii) $\operatorname{Rad} TM$ *is a Killing distribution.*

(iii) *There exists a torsion-free linear connection* ∇ *on* $\mathcal{M}$ *such that* g *is a parallel tensor field with respect to* ∇.

Proof: (i)⇒(ii). If $\mathcal{M}$ is a Reinhart lightlike manifold, $\operatorname{Rad} TM$ is integrable and there exists a coordinate system for which $X = X^\alpha \partial_\alpha$, $\forall X \in \operatorname{Rad} TM$. Thus, equations (3.3) and (3.1) yield

$$(L_X g)(Y, Z) = X^\alpha \left\{\partial_\alpha g(Y, Z) - g([\partial_\alpha, Y], Z) - g(Y, [\partial_\alpha, Z])\right\}, \quad \forall Y, Z \in TM.$$

In case at least one the vector fields Y and Z belongs to $\operatorname{Rad} TM$, equation (3.1) and the involutivity of $\operatorname{Rad} TM$ (Proposition 3.2.1) imply that the above equation vanishes. But if $Y, Z \in S(TM)$, then $Y = Y^i \partial_i$, $Z = Z^j \partial_j$ and

$$(L_X g)(Y, Z) = X^\alpha \left\{\partial_\alpha (g_{ij} Y^i Z^j) - g((\partial_\alpha Y^i)\partial_i, Z^j \partial_j) - g(Y^i \partial_i, (\partial_\alpha Z^j)\partial_j)\right\}.$$

Using equation (3.17) for the first term leads to

$$(L_X g)(Y, Z) = X^\alpha \left\{g_{ij} \partial_\alpha (Y^i Z^j) - g_{ij} (\partial_\alpha Y^i) Z^j - g_{ij} Y^i (\partial_\alpha Z^j)\right\} = 0.$$

Consequently, $(L_X g)(Y, Z) = 0 \ \forall Y, Z \in TM$, $X \in \operatorname{Rad} TM$ and $\operatorname{Rad} TM$ is a Killing distribution.

(ii)⇒(i). On the other hand, if $\operatorname{Rad} TM$ is a Killing distribution, equation (3.3) equals to zero for any $X \in \operatorname{Rad} TM$ and $Y, Z \in TM$. So $Y \in \operatorname{Rad} TM$ implies that $g([X, Y], Z) = 0$ and $[X, Y] \in \operatorname{Rad} TM$, meaning that $\operatorname{Rad} TM$ is involutive and, by Frobenius's theorem 3.2.1, integrable. Now, for $X = \partial_\alpha$, $Y = \partial_i$ and $Z = \partial_j$, equation (3.3) becomes $\partial_\alpha g_{ij} = 0$ and equation (3.17) is satisfied, that is, $(\mathcal{M}, g)$ is a Reinhart lightlike manifold.

(iii)⇒(ii). Since there exists a torsion-free linear connection ∇ on $\mathcal{M}$ for which g is parallel, if we do not impose X to be a Killing vector field in equations (3.5) we get

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \quad \forall X, Y, Z \in TM.$$

Thus, equation (3.4) leads to

$$(L_X g)(Y, Z) = Y(g(X, Z)) + Z(g(X, Y)) - g(X, \nabla_Y Z) - g(X, \nabla_Z Y).$$

For any $X \in \operatorname{Rad} TM$, the above equation becomes zero, so $\operatorname{Rad} TM$ is a Killing distribution.

(ii)⇒(iii). As shown in the proof (ii)⇒(i), if $\operatorname{Rad} TM$ is a Killing distribution it is also integrable. So we can consider it as an $(m+r)$-dimensional manifold $\mathcal{R}$ with local coordinates $(x^\alpha, x^i, y^\alpha)$, being (x^α, x^i) coordinates on $\mathcal{M}$ adapted to the foliation given by $\operatorname{Rad} TM$ and (y^α) are coordinates on each vector space constituting $\operatorname{Rad} TM$. The transformation of coordinates are those in equations (3.15) and

$$\bar{y}^\alpha = B^\alpha_\beta(x) y^\beta \quad \Rightarrow \quad \frac{\partial}{\partial y^\alpha} = B^\beta_\alpha(x)\frac{\partial}{\partial \bar{y}^\beta}, \quad \text{with} \quad B^\alpha_\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta}, \quad x \in \mathcal{M}.$$

Therefore, we can define a rank r distribution D over $\mathcal{R}$ spanned by $\{\partial/\partial y^1, \ldots, \partial/\partial y^r\}$, in which case

$$TR = TM \oplus D.$$

If we consider that $\mathcal{M}$ is paracompact, then there exists a Riemannian metric g^* on it* and we can define a screen distribution $S(TM)$ as the complementary orthogonal distribution to $\operatorname{Rad} TM$ in TM with respect to g^*. Hence,

$$TR = S(TM) \oplus \operatorname{Rad} TM \oplus D.$$

Since both $\operatorname{Rad} TM$ and D are distributions of rank r and the matrices for transformations of their basis vector fields ($\partial/\partial x^\alpha$ and $\partial/\partial y^\alpha$, respectively) are the same, any vector field $X = X^\alpha \partial/\partial y^\alpha \in D$ corresponds to another vector field $X^* = X^\alpha \partial/\partial x^\alpha \in \operatorname{Rad} TM$. Given this correspondence and the projection mappings σ, ρ and δ of TR into $S(TM)$, $\operatorname{Rad} TM$ and D, respectively, it is possible to define the tensor $\bar{g} : TR \times TR \to \mathcal{F}(\mathcal{R})$, such that

$$\bar{g}(\bar{X}, \bar{Y}) = g(\sigma\bar{X}, \sigma\bar{Y}) + g^*(\rho\bar{X}, (\delta\bar{Y})^*) + g^*(\rho\bar{Y}, (\delta\bar{X})^*), \tag{3.18}$$

where $\mathcal{F}(\mathcal{R})$ is the space of smooth scalar functions on $\mathcal{R}$ and $\bar{X}, \bar{Y} \in TR$. This tensor corresponds to a semi-Riemannian metric on $\mathcal{R}$, because there are no vector field $\bar{X} \in TR$ orthogonal to every $\bar{Y} \in TR$ with respect to $\bar{g}$, although its restriction to TM equals to the degenerate metric g. So there is a torsion-free metric connection $\bar{\nabla}$ (see [6, p. 39] or [8, p. 35]) on $(\mathcal{R}, \bar{g})$ and we can define a connection ∇ on $\mathcal{M}$ by

$$\bar{\nabla}_X Y = \nabla_X Y + B^\alpha(X, Y)\frac{\partial}{\partial y^\alpha}, \quad \forall X, Y \in TM, \tag{3.19}$$

such that $\nabla_X Y \in TM$ and $B^\alpha(X,Y) \in \mathcal{F}(\mathcal{M})$. The *torsion tensor field* of a connection ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

thus, for $X = \partial/\partial x^a$ and $Y = \partial/\partial x^b$

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X = 0 \quad \Rightarrow \quad \bar{\nabla}_X Y = \bar{\nabla}_Y X. \tag{3.20}$$

* A proof for this claim is in 6. The definition of paracompact manifolds can be found in 22 and 7.

Since $\bar{g}(\nabla_X Y, \partial/\partial x^\beta) = 0$, we have that

$$\bar{g}\left(\bar{\nabla}_X Y, \frac{\partial}{\partial x^\beta}\right) = B^\alpha(X,Y)\bar{g}\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

$$\text{and} \quad \bar{g}\left(\bar{\nabla}_Y X, \frac{\partial}{\partial x^\beta}\right) = B^\alpha(Y,X)\bar{g}\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial x^\beta}\right),$$

then the second equation in (3.20) implies that $B^\alpha(X,Y)$ is symmetric for all $X, Y \in TM$ and the first shows that ∇ is torsion-free. Once again, we equal equation (3.3) to zero and, using equations (3.18), (3.19) and (3.4) for $\bar{g}$, we get

$$0 = (L_X g)(Y,Z) = -\bar{g}(X, \bar{\nabla}_Y Z + \bar{\nabla}_Z Y) = -2B^\alpha(Y,Z)g^*\left(X, \frac{\partial}{\partial x^\alpha}\right),$$

for any $X \in \operatorname{Rad} TM$ and $Y, Z \in TM$. For g^* is a Riemannian metric for vector fields in $\operatorname{Rad} TM$ and X is an arbitrary vector field of $\operatorname{Rad} TM$, it follows that the functions $B^\alpha(Y,Z)$ vanish and $\bar{\nabla}_Y Z = \nabla_Y Z$, meaning that g is parallel with respect to ∇. ■

3.4 Induced geometry on lightlike hypersurfaces

3.4.1 Decomposition of the tangent bundle

Now, we consider that $\mathcal{M}$ is a hypersurface of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$ of dimension $m+2$, with $m > 0$, and $\bar{g}$ has index $q \in \{1, \ldots, m+1\}$. Since $T_u M \subset T_u \bar{M}$ at every $u \in M$, we define the subspace of $T_u \bar{M}$ orthogonal to $T_u M$ as

$$T_u M^\perp = \left\{V_u \in T_u \bar{M} \mid \bar{g}(V_u, W_u) = 0, \quad \forall W_u \in T_u M\right\},$$

$$\text{and} \quad \operatorname{Rad} T_u M = T_u M \cap T_u M^\perp.$$

So we call $\mathcal{M}$ a *lightlike (null, degenerate) hypersurface* of $\bar{\mathcal{M}}$ if it is an immersed manifold of $\bar{\mathcal{M}}$ with $\operatorname{Rad} T_u M \neq \{0\}$, $\forall u \in M$. It follows that $\bar{g}$ induces a 1-degenerate metric g on $\mathcal{M}$ and the rank 1 radical distribution $\operatorname{Rad} TM$ equals to $TM^\perp \subset TM$, being $TM^\perp = \cup_{u \in M} T_u M^\perp$ a distribution on $\mathcal{M}$ [6, p. 78].

As before, we define the screen distribution $S(TM)$ as a (nonunique) complementary distribution of $TM^\perp$ in TM, such that

$$TM = S(TM) \perp TM^\perp \tag{3.21}$$

and the restriction of g to $S(TM)$ is non-degenerate. Given the restriction of $T\bar{M}$ to points in $\mathcal{M}$, $T\bar{M}_{|M}$, we can decompose

$$T\bar{M}_{|M} = S(TM) \perp S(TM)^\perp. \tag{3.22}$$

From the last two equations, one sees that $TM^\perp \subset S(TM)^\perp$ and, since $S(TM)^\perp - TM^\perp \notin TM$ implies that $S(TM)^\perp - TM^\perp$ is not orthogonal to $TM^\perp$, $S(TM)^\perp$ is a non-degenerate distribution of rank 2 on the restriction $\bar{\mathcal{M}}_{|\mathcal{M}}$ (this means that $S(TM)^\perp \subset T\bar{M}$, but it is defined only on points of $\bar{\mathcal{M}} \cap \mathcal{M}$).

Theorem 3.4.1. *Let $(\mathcal{M}, g)$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$. Then, for each screen distribution $S(TM)$, there exists a unique rank 1 distribution $tr(TM)$ on $\bar{\mathcal{M}}_{|\mathcal{M}}$, such that for each vector field $K \in TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset \mathcal{M}$ there exists a unique vector field $L \in tr(TM)$ on $\mathcal{U}$ for which*

$$\bar{g}(L, K) = 1, \quad and \quad \bar{g}(L, L) = \bar{g}(L, X) = 0 \quad \forall X \in S(TM)_{|\mathcal{U}}. \tag{3.23}$$

Proof: Let D be a rank 1 complementary distribution of $TM^{\perp}$ in $S(TM)^{\perp}$ and $0 \neq V \in D_{|\mathcal{U}}$. Given that $S(TM)^{\perp}$ is non-degenerate, it follows that $\bar{g}(K, V) \neq 0$ on $\mathcal{U}$ and, using (3.22), any L satisfying the last equation in (3.23) must be of the form $L = L^V V + L^K K$. Since $K \in TM^{\perp} \Rightarrow \bar{g}(K, K) = 0$, the first equation in (3.23) leads to $L^V = 1/\bar{g}(V, K)$. From $\bar{g}(L, L) = 0$ one gets L^K and

$$L = \frac{1}{\bar{g}(V, K)} \left\{ V - \frac{\bar{g}(V, V)}{2\bar{g}(V, K)} K \right\}, \tag{3.24}$$

which by construction satisfies equations (3.23) on $\mathcal{U}$.

If we consider another coordinate neighborhood $\mathcal{U}^* \subset \mathcal{M}$ such that $\mathcal{U}^* \cap \mathcal{U} \neq \varnothing$, then $K^* = \alpha K$ and $V^* = \beta V$ for some non-zero smooth functions α and β on $\mathcal{U}^* \cap \mathcal{U}$, given that $TM^{\perp}$ and D are of rank 1. As a consequence, equation (3.24) implies that $L^* = (1/\alpha) L$, meaning that D induces a rank 1 distribution $tr(TM)$ spanned by L.

Moreover, if we take another complementary distribution D' of $TM^{\perp}$ in $S(TM)^{\perp}$, we may find another vector field L' by substituting V by $W \in D'_{|\mathcal{U}}$ in the right-hand side of equation (3.24). But the more general form for this vector field is $W = W^V V + W^K K$, so one finds that $L' = L$ and, therefore, the distribution $tr(TM)$ for which $L \in tr(TM)$ satisfies equations (3.23) is unique, as well as L, given by (3.24). ■

This theorem shows us that $tr(TM)$ is a null vector field and $tr(TM)_u \cap T_u M = \{0\}, \forall u \in M$. Thus, from equations (3.22) and (3.21) we get

$$T\bar{M}_{|M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{3.25}$$

Given Theorem 3.4.1 and the above decomposition, we call $tr(TM)$ the *lightlike transversal vector bundle* of $\mathcal{M}$ with respect to $S(TM)$.

In certain cases, it is also possible to first define $tr(TM)$ and then get a screen distribution $S(TM)$. An example is given by the quasi-orthonormal basis of Figure 1: the projector operator used in equations (2.6) is given in terms of K and L and determines the screen distribution spanned by the spacelike basis vectors e_1 and e_2.

3.4.2 The induced differentiation

Let $(\bar{\mathcal{M}}, \bar{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold and $\bar{\nabla}$ its corresponding Levi-Civita connection. If $(\mathcal{M}, g)$ is a lightlike hypersurface of $(\bar{\mathcal{M}}, \bar{g})$ with screen distribution $S(TM)$ and lightlike transversal vector bundle $tr(TM)$, then the last decomposition in (3.25) allows us to write

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \bar{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{3.26}$$

for any $X, Y \in TM$ and $V \in tr(TM)$, where we define the bilinear objects $\nabla_X Y, A_V X \in TM$ and $h(X,Y), \nabla^t_X V \in tr(TM)$. Following steps similar to those in the proof (ii)⇒(iii) of Theorem 3.3.1, changing $B^\alpha(X,Y)\,\partial/\partial y^\alpha$ in equation (3.19) by $h(X,Y)$, one finds that the latter is symmetric in X and Y, implying that ∇ is a torsion-free connection. We say that ∇ and ∇^t are the *induced connections* on $\mathcal{M}$ and $tr(TM)$, respectively. Moreover, h is called the *second fundamental form* and A_V is the *shape operator* of the lightlike immersed submanifold $\mathcal{M}$. So the first and second equations in (3.26) are known as *Gauss* and *Weingarten formulae*, respectively.

Using the vector fields K and L on $\mathcal{U} \subset \mathcal{M}$ of Theorem 3.4.1, we can define a symmetric tensor field B, called the *local second fundamental form*, and a 1-form τ on $\mathcal{U}$ as

$$B(X,Y) = \bar{g}(h(X,Y), K), \quad \forall X, Y \in TM_{|\mathcal{U}} \tag{3.27}$$

$$\text{and} \quad \tau(X) = \bar{g}(\nabla^t_X L, K), \quad \forall X \in TM_{|\mathcal{U}}, \tag{3.28}$$

$$\therefore \quad h(X,Y) = B(X,Y)L \quad \text{and} \quad \nabla^t_X L = \tau(X)L. \tag{3.29}$$

As a result, we have on $\mathcal{U}$ that

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)L \tag{3.30}$$

$$\text{and} \quad \bar{\nabla}_X L = -A_L X + \tau(X)L. \tag{3.31}$$

Considering the possible different screen distributions one may define, we state:

Proposition 3.4.1. *Given two screen distributions $S(TM)$ and $S(TM)'$ on $\mathcal{M}$ and their respective $tr(TM)$ and $tr(TM)'$, one may define B and B' from h and h'. It follows that $B = B'$ on $\mathcal{U}$, so the local second fundamental form does not depend on the choice of a screen distribution.*

Proof: From Gauss formula (3.26) and (3.27) we have that

$$\bar{g}(\bar{\nabla}_X Y, K) = B(X,Y). \tag{3.32}$$

As the left-hand side is independent of $S(TM)$, we find that $B(X,Y) = B(X,Y)'$, $\forall X, Y \in TM_{|\mathcal{U}}$. ■

Taking $Y = K$ in (3.32) and using equation (3.4) leads to

$$B(X, K) = 0, \quad \forall X \in TM_{|\mathcal{U}}, \tag{3.33}$$

As a consequence, we get

Corollary 3.4.1. *The second fundamental form of a lightlike hypersurface is degenerate.*

Even though B is independent of $S(TM)$, it depends on K, for if we choose $\bar{K} = \alpha K$, we have from (3.24) that $\bar{L} = (1/\alpha)L$ and invariance of (3.30) requires that $\bar{B} = \alpha B$. Similarly, applying (3.31) to $L = \alpha\bar{L}$ implies that

$$\tau(X) = \bar{\tau}(X) + X(\ln\alpha), \quad \forall X \in TM_{|\mathcal{U}}. \tag{3.34}$$

Using equation (3.7), one deduces the following:

Proposition 3.4.2. *If τ and $\bar{\tau}$ are 1-forms defined by equation (3.28) using the pairs $\{K, L\}$ and $\left\{\bar{K}, \bar{L}\right\}$, respectively, then $d\tau = d\bar{\tau}$ on $\mathcal{U}$.*

Once defined the connection ∇ on $\mathcal{M}$, we can use the decomposition (3.21) and the projection morphism P of TM on $S(TM)$ to obtain

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \quad \forall X, Y \in TM \tag{3.35}$$

$$\text{and} \quad \nabla_X U = -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t_X U, \quad \forall X \in TM \quad \text{and} \quad U \in TM^\perp, \tag{3.36}$$

such that $\overset{*}{\nabla}_X PY, \overset{*}{A}_U X \in S(TM)$ and $\overset{*}{h}(X, PY), \overset{*}{\nabla}{}^t_X U \in TM^\perp$. As before, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are induced linear connections on $S(TM)$ and $TM^\perp$, respectively, $\overset{*}{h}$ is the second fundamental form and $\overset{*}{A}_U$ is called the shape operator, both associated to $S(TM)$. Then, we call equations (3.35) and (3.36) the Gauss and Weingarten equations for the screen distribution $S(TM)$.

Since g is the restriction of $\bar{g}$ to $\mathcal{M}$, we have from the Weingarten formula (3.26) that

$$g(A_V Y, PW) = \bar{g}(-\bar{\nabla}_Y V + \nabla^t_Y V, PW) = -\bar{g}(\bar{\nabla}_Y V, PW).$$

Thus, using (3.4) and (3.35) results in

$$g(A_V Y, PW) = \bar{g}(V, \overset{*}{h}(Y, PW)). \tag{3.37}$$

We also find that

$$\bar{g}(A_V Y, V) = 0, \quad g(\overset{*}{A}_U X, PY) = \bar{g}(U, h(X, PY)) \quad \text{and} \quad \bar{g}(\overset{*}{A}_U X, V) = 0, \tag{3.38}$$

for any $X, Y, W \in TM$, $U \in TM^\perp$ and $V \in tr(TM)$, if (3.36) is used in the last two equations.

Defining on $\mathcal{U}$

$$C(X, PY) = \bar{g}(\overset{*}{h}(X, PY), L) \quad \text{and} \quad \epsilon(X) = \bar{g}(\overset{*}{\nabla}{}^t_X K, L) \tag{3.39}$$

allow us to write

$$\overset{*}{h}(X, PY) = C(X, PY)K \quad \text{and} \quad \overset{*}{\nabla}{}^t_X K = \epsilon(X)K. \tag{3.40}$$

But, using in sequence equations (3.39), (3.36), (3.30), (3.4), (3.23) and (3.31), one finds that

$$\epsilon(X) = \bar{g}(\nabla_X K, L) = \bar{g}(\bar{\nabla}_X K, L) = -\bar{g}(K, \bar{\nabla}_X L) = -\tau(X),$$

so equations (3.35) and (3.36) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X, PY)K \tag{3.41}$$

$$\text{and} \quad \nabla_X K = -\overset{*}{A}_K X - \tau(X)K. \tag{3.42}$$

Furthermore, (3.37) and (3.38) can be written as

$$g(A_L Y, PW) = C(Y, PW), \quad \bar{g}(A_L Y, L) = 0, \tag{3.43}$$

$$g(\overset{*}{A}_K X, PY) = B(X, PY) \quad \text{and} \quad \bar{g}(\overset{*}{A}_K X, L) = 0. \tag{3.44}$$

It follows from (3.44) and (3.33) that

$$\overset{*}{A}_K K = 0,$$

thus, using (3.30) and (3.42), one finds

$$\bar{\nabla}_K K = \nabla_K K = -\tau(K)K,$$

so the next proposition holds.

Proposition 3.4.3. *Let $(\mathcal{M}, g, S(TM))$ be a lightlike hypersurface of $(\bar{\mathcal{M}}, \bar{g})$. Then, the integral curve of $K \in TM^{\perp}_{|\mathcal{U}}$ is a null geodesic of $\bar{\mathcal{M}}$ with respect to $\bar{\nabla}$, i.e. $\bar{\nabla}_K K = 0$, if and only if it is also a null geodesic of $\mathcal{M}$ with respect to ∇.*

Since a screen distribution $S(TM)$ is non-degenerate (thus, semi-Riemannian), we can define on it an orthonormal basis $\{W_1, \ldots, W_m\}$ of signature $\{\epsilon_1, \ldots, \epsilon_m\}$, such that $\bar{g}(W_i, W_i) = \epsilon_i$, $\forall i \in \{1, \ldots, m\}$. Then, any vector field of TM is spanned by the vector fields W_i and K. If we choose another screen distribution $S(TM)'$ with local orthonormal basis $\{W_i'\}$, there will be a corresponding $tr(TM)'$ and L'. Therefore, the general form for L' and each W_i' can be written as

$$L' = \alpha L + \beta K + \sum_{i=1}^{m} c_i W_i \quad \text{and} \quad W_i' = \sum_{j=1}^{m} A_i^j W_j + \gamma K,$$

being $\alpha, \beta, \gamma, c_i$ and A_i^j smooth functions on a coordinate neighborhood. Imposing equations (3.23) for the sets of induced vector fields $\{W_i, L\}$ and $\{W_i', L'\}$ and considering the same vector field K for both $S(TM)$ and $S(TM)'$ leads to

$$L' = L - \frac{1}{2}\left\{\sum_{i=1}^{m} \epsilon_i (c_i)^2\right\} K + \sum_{i=1}^{m} c_i W_i \quad \text{and} \quad W_i' = \sum_{j=1}^{m} A_i^j (W_j - \epsilon_j c_j K).$$

For the screen distribution $S(TM)'$, we have

$$\bar{\nabla}_X Y = \nabla'_X Y + B'(X,Y)L',$$

which must be equal to equation (3.30), so it follows from this and Proposition 3.4.1 that

$$\nabla'_X Y = \nabla_X Y + B(X,Y)\left\{\frac{1}{2}\left(\sum_{i=1}^{m} \epsilon_i (c_i)^2\right) K - \sum_{i=1}^{m} c_i W_i\right\}, \quad \forall X, Y \in TM,$$

what justifies the following theorem.

Theorem 3.4.2. *Let $(\mathcal{M}, g, S(TM))$ be a lightlike hypersurface of $(\bar{\mathcal{M}}, \bar{g})$. Then, the induced connection ∇ is independent of $S(TM)$, and therefore unique, if and only if $\mathcal{M}$ has a null second fundamental form h.*

Although ∇ is induced by the Levi-Civita connection $\bar{\nabla}$, it is not always a metric connection, as we can see below.

Proposition 3.4.4. (i) *The connection $\overset{*}{\nabla}$ defined in (3.35) is a metric connection on $S(TM)$.* (ii) *The connection ∇ of (3.26) is such that*

$$(\nabla_X g)(Y,Z) = B(X,Y)\bar{g}(Z,L) + B(X,Z)\bar{g}(Y,L), \quad \forall X, Y, Z \in TM. \tag{3.45}$$

Proof: We have that

$$(\overset{*}{\nabla}_{PX}\bar{g})(PY,PZ) = PX(\bar{g}(PY,PZ)) - \bar{g}(\overset{*}{\nabla}_{PX}PY,PZ) - \bar{g}(PY,\overset{*}{\nabla}_{PX}PZ),$$

then, using (3.41), (3.30) and the fact that $\bar{\nabla}$ is a metric connection, (i) is proved. Similarly,

$$\begin{aligned} 0 = (\bar{\nabla}_X \bar{g})(Y,Z) &= X(\bar{g}(Y,Z)) - \bar{g}(\bar{\nabla}_X Y, Z) - \bar{g}(Y, \bar{\nabla}_X Z) \\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - B(X,Y)\bar{g}(Z,L) - B(X,Z)\bar{g}(Y,L) \\ &= (\nabla_X g)(Y,Z) - B(X,Y)\bar{g}(Z,L) - B(X,Z)\bar{g}(Y,L), \end{aligned}$$

and we see that (ii) is satisfied. ∎

Before we state the next theorem, we define a *totally geodesic lightlike hypersurface* as a lightlike hypersurface $\mathcal{M}$ in which every geodesic with respect to the induced connection ∇ is a geodesic of the surrounding manifold $\bar{\mathcal{M}}$ with respect to its connection $\bar{\nabla}$. Also, a distribution D is said to be *parallel* with respect to a connection ∇, with ∇_X defined for $X \in D'$, if $\nabla_X Y \in D, \forall Y \in D, X \in D'$.

Theorem 3.4.3. *Let $(\mathcal{M}, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$. Then, the following assertions are equivalent:*

(i) *$\mathcal{M}$ is totally geodesic.*

(ii) *h vanishes identically on $\mathcal{M}$.*

(iii) *$\overset{*}{A}_U$ vanishes identically on $\mathcal{M}$, for any $U \in TM^{\perp}$.*

(iv) *There exists a unique torsion-free metric connection ∇ induced by $\bar{\nabla}$ on $\mathcal{M}$.*

(v) *$TM^{\perp}$ is a parallel distribution with respect to ∇.*

(vi) *$TM^{\perp}$ is a Killing distribution on $\mathcal{M}$.*

Proof: From (3.26) one gets the equivalence between (i) and (ii). Then, (3.38), (3.29) and (3.33) imply in the correspondence of (ii) and (iii). Equation (3.36) leads to (iii)⇔(v). Using (3.26) and Theorem 3.4.2 follows that (ii)⇔(iv). Moreover, from (3.45), (3.33) and (3.23) we have

$$(\nabla_X g)(K, Z) = B(X, Z) \quad \Rightarrow \quad g(\nabla_X K, Z) = -B(X, Z).$$

So (3.3) and the fact that ∇ torsion-free imply in

$$(L_K g)(X, Z) = K(g(X, Z)) - g(\nabla_K X, Z) - g(X, \nabla_K Z) + g(\nabla_X K, Z) + g(X, \nabla_Z K).$$

From $(\bar{\nabla}_K \bar{g})(X, Z) = 0$, (3.30) and (3.33) we find that the first three terms in the last equation vanish, then

$$(L_K g)(X, Z) = g(\nabla_X K, Z) + g(X, \nabla_Z K) = -2B(X, Z)$$

and the equivalence of (ii) and (vi) is obtained. ■

3.4.3 The Gauss-Codazzi equations

A geometric object that is of great interest for applications in general relativity is the curvature tensor, so now we focus on the induced curvature of a lightlike hypersurface $(\mathcal{M}, g, S(TM))$ immersed in a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$. Let R and $\bar{R}$ be the curvature tensors with respect to ∇ and the Levi-Civita connection $\bar{\nabla}$, respectively. We have from the definition of $\bar{R}$ that

$$\bar{R}(X, Y)Z = \bar{\nabla}_X(\bar{\nabla}_Y Z) - \bar{\nabla}_Y(\bar{\nabla}_X Z) - \bar{\nabla}_{[X,Y]}Z, \quad \forall X, Y, Z \in TM,$$

so equations (3.26) yield

$$\bar{R}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ + \nabla^t_X h(Y,Z) - h(Y,\nabla_X Z) - \nabla^t_Y h(X,Z) + h(X,\nabla_Y Z) - h([X,Y],Z).$$

Since ∇ is a torsion-free connection and h is a bilinear form on TM, it follows that $h([X,Y],Z) = h(\nabla_X Y, Z) - h(\nabla_Y X, Z)$ and

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \quad (3.46)$$

$\forall X,Y,Z \in TM$, where we define

$$(\nabla_X h)(Y,Z) = \nabla^t_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

By using equations (3.46) and (3.37), we deduce the so-called *Gauss-Codazzi equations* of the lightlike hypersurface $(\mathcal{M}, g, S(TM))$:

$$\bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) + \bar{g}(h(X,Z), \overset{*}{h}(Y,PW)) - \bar{g}(h(Y,Z), \overset{*}{h}(X,PW)),$$

$$\bar{g}(\bar{R}(X,Y)Z, U) = \bar{g}((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), U) \qquad \text{and}$$

$$\bar{g}(\bar{R}(X,Y)Z, V) = \bar{g}(R(X,Y)Z, V), \quad \forall X,Y,Z,W \in TM, \;\; U \in TM^{\perp} \text{ and } V \in tr(TM).$$

From these equations, one can calculate the components of the curvature tensor in terms of the induced objects like B, τ and C, defined in (3.27), (3.28) and (3.39), respectively. In order to do that, it is useful to use a local basis, which is not holonomic in general (like the quasi-orthonormal basis of Figure 1), on the $(m+2)$-dimensional manifold $\bar{\mathcal{M}}$, being denoted by

$$\left\{ \frac{\partial}{\partial u^0}, \frac{\delta}{\delta u^i}, L \right\}, \quad \text{with} \quad i \in \{1, \ldots, m\}.$$

Here, $\partial/\partial u^0 = K$ and L are the vector fields defined in Theorem 3.4.1 and the set $\{\delta/\delta u^i\}$ forms a local basis for $S(TM)$.

Then, it is possible to express the induced *Ricci tensor* as

$$Ric(X,Y) = g^{ij} g\left(R\left(X, \frac{\delta}{\delta u^i}\right)Y, \frac{\delta}{\delta u^j}\right) + g\left(R\left(X, \frac{\partial}{\partial u^0}\right)Y, L\right),$$

where j has the same range as i. After some lengthy computation, one finds that [6, p. 99]

$$R_{ij} - R_{ji} = 2d\tau\left(\frac{\delta}{\delta u^i}, \frac{\delta}{\delta u^j}\right), \quad \text{for} \quad R_{ij} = Ric\left(\frac{\delta}{\delta u^j}, \frac{\delta}{\delta u^i}\right), \quad \text{and}$$

$$R_{0i} - R_{i0} = 2d\tau\left(\frac{\partial}{\partial u^0}, \frac{\delta}{\delta u^i}\right), \quad \text{where} \quad R_{0i} = Ric\left(\frac{\delta}{\delta u^i}, \frac{\partial}{\partial u^0}\right).$$

From the above equations and Proposition 3.4.2 one gets the following result.

Theorem 3.4.4. *Let $(\mathcal{M}, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$. Then, the Ricci tensor of the induced connection ∇ is symmetric if and only if each 1-form τ induced by $S(TM)$ is closed, i.e. $d\tau = 0$ on every $\mathcal{U} \subset \mathcal{M}$.*

If $d\tau = 0$, Poincaré lemma[22,24] tells us that $\tau = df$, for a smooth function f on some $\mathcal{U}' \subset \mathcal{U}$ (remember that $d^2 = 0$). So $\tau(X) = X(f)$ and we can take $\alpha = \exp(f)$ in (3.34), leading to $\bar{\tau}(X) = 0$ for any $X \in TM_{|\mathcal{U}'}$ and we may state

Proposition 3.4.5. *Let $(\mathcal{M}, g, S(TM))$ be a lightlike hypersurface of $(\bar{\mathcal{M}}, \bar{g})$. Then, the Ricci tensor of the induced connection ∇ is symmetric if and only if there exists a pair $\{K, L\}$ on $\mathcal{U}$ such that the corresponding 1-form τ from (3.31) vanishes.*

4 CONCLUSIONS

We saw that there are many evidences indicating to a strong relation between general relativity and thermodynamics, with even the possibility to understand the latter as the fundamental source of gravitational effects. Even though this relation may seem quite obscure at first, when one takes into account that causal horizons are major agents in the control of information available to some observer and associates an increase in entropy to information loss, it becomes mandatory to relate some property of causal horizons to entropy. Furthermore, if temperature can be measured from radiation and quantum field theory in curved spacetimes results in an observer dependent particle number, then temperature also becomes an observer dependent concept. Since different observers have different causal horizons, with their corresponding entropy, one may attribute a temperature to these hypersurfaces. This is supported by the fact that the density matrix of a quantum field defined on a spacelike hypersurface must be integrated over the region of spacetime hidden by the horizon. Therefore, the field beyond the horizon provides a statistical ensemble from which one obtains its thermodynamic properties, all related to the causal boundary.

However, the results connecting thermodynamics to general causal horizons are not so well established as those of black hole thermodynamics and even these are not completely understood, lacking, for example, a direct derivation and satisfactory explanation for the entropy formula, what has become an important subject of research in quantum gravity.[25] Given that the connection between area and entropy emerged originally in a classical context, it may be possible to advance in, or at least formalize, the study of spacetime thermodynamics by investigating the geometric properties of causal horizons. With this purpose, we reviewed the geometry induced on lightlike hypersurfaces, finding some characteristics that demand special care when dealing with covariant derivatives. In particular, the existence of a unique Levi-Civita connection and of a symmetric Ricci tensor on the hypersurface depends on exceptional symmetries of the spacetime in question, namely the existence of Killing vector fields and a null transversal induced connection, respectively (as shown by Theorem 3.4.3 and Proposition 3.4.5). Considering these results and the derivations on Sections 2.4 and 2.5, we presume that any progress in the subject for general spacetimes would be based on either local approximations involving Killing vector fields or calculations using only intrinsic properties of the lightlike hypersurface, which do not require the choice of a connection.

We remind that some results point to a relation between boundary terms in the Einstein-Hilbert action and the cross-sectional area of causal horizons.[18,21,26] Moreover, in static spacetimes with bifurcate Killing horizons, the gravitational action may be interpreted as the free energy of the spacetime,[18] so the variational principle leads to an extremization of both the entropy and the area. Since the Einstein-Hilbert action is given by the integral of the scalar curvature, a possible application for the induced geometry on lightlike hypersurfaces may be found in the relation between the action related to a region of spacetime bounded by a causal horizon and the scalar curvature on the boundary. Thus, one could, hopefully, find which geometric object is maximized by the variational principle in a more general context. Such an object would have the fundamental property necessary to correspond to the horizon's entropy, which is monotonic increase over time.

Another approach, similar to the one presented in Section 2.5, is to investigate the relation between the geometry of a causal horizon and the energy flux through it, but described as a flux of information. Using quantum information theory, one may identify the configuration of a causal horizon that leads to the maximum information loss. Given that a temperature needs to be attributed to the horizon and this may relate to an observer of finite lifetime, unlike Rindler observers, whose trajectories are curves with parameters extending infinitely in the future direction, one could use a (perhaps) more appropriate definition of temperature, like the one presented in 27.

In general, studies in this subject contribute to establish this observer dependent thermodynamics and to better understand which degrees of freedom give rise to the thermodynamic behavior of causal horizons. Insofar as this behavior is not restricted to Einstein's theory and occur in many theories of gravity,[18] the observed laws in certain models, like the equipartition of energy, may provide clues to how the theory's microscopic dynamics works, collaborating with the development of theories of quantum gravity. In addition, the thermodynamics of causal horizons not only provides a new approach to gravitation but also may lead to solutions of problems present in other approaches, such as the gravitational coupling of the cosmological constant.[18]

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