# UNIVERSIDADE DE SÃO PAULO INSTITUTO DE FÍSICA DE SÃO CARLOS 

MARCUS VINÍCIUS GONZALEZ RODRIGUES

Higher-order behaviour of heavy-quark current correlators in the small-momentum expansion

São Carlos

# Higher-order behaviour of heavy-quark current correlators in the small-momentum expansion 

Dissertation presented to the Graduate Program in Physics at the Instituto de Física de São Carlos, Universidade de São<br>Paulo to obtain the degree of Master of Science.

Concentration area: Theoretical and Experimental Physics

Advisor: Prof. Dr. Diogo Rodrigues Boito

Corrected version
(Original version available on the Program Unit)

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Rodrigues, Marcus Vinícius Gonzalez
Higher-order behaviour of heavy-quark current
correlators in the small-momentum expansion / Marcus
Vinícius Gonzalez Rodrigues; advisor Diogo Rodrigues Boito

- corrected version -- São Carlos 2021.

114 p.

Dissertation (Master's degree - Graduate Program in Theoretical and Experimental Physics) -- Instituto de Física de São Carlos, Universidade de São Paulo - Brasil , 2021.

1. Quark masses. 2. Strong coupling. 3. Perturbative QCD. 4. Renormalons. I. Boito, Diogo Rodrigues, advisor. II. Title.

## ACKNOWLEDGEMENTS

- Gostaria de agradecer à minha Mãe por todo o suporte e incentivo durante os momentos da minha vida, seja pessoal ou profissional.
- I would like to thank my girlfriend Danyellen for all her love, friendship and making me happier day after day.
- I would like to thank Diogo Boito for being the best advisor one can have. All his teachings were essential for my growth as a scientist.
- I would like to thank Diogo Boito, Vicent Mateu, Maarten Golterman, Kim Maltman, Santi Peris and Wilder Schaaf for their collaborations in the projects I worked during my Master's degree. Working with them gave me valuable experiences on how to do science at the highest level.
- I would like to thank Rep Zeppelin for all the laughs. Living there was one the best experiences of my life.
- I would like to thank Abô for all physics discussions. He is helping me since my first semester as an undergrad student and because of him I'm now working with Particle Physics.
- I would like to thank all my friends from IFSC, who made my years as an undergrad student more pleasant.
- I would like to thank IFSC for all their support during my bachelor and Master dissertation.
- I would like to thank FAPESP for the financial support of this dissertation (PROCESS 2019/16957-9).


#### Abstract

RODRIGUES, M. V. G. Higher-order behaviour of heavy-quark current correlators in the small-momentum expansion. 2021. 114 p. Dissertation (Masters in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2021.

In the absence of direct observations of new physics at the LHC, precision physics plays an important role in the search for phenomena beyond the Standard Model (SM). In this situation, the fundamental parameters of the SM, such as quark masses and the strong coupling, $\alpha_{s}$, must be known with high precision. One of the most powerful methods to extract the charm- and bottom-quark masses, $m_{c}$ and $m_{b}$, as well as the strong coupling, is the use of sum-rules for the vector and pseudo-scalar heavy-quark current correlators. At present, the increasing precision of experimental data and lattice simulations may not directly translate into more accurate determinations of $m_{c}$, $m_{b}$ and $\alpha_{s}$, as the perturbative uncertainty due to the residual renormalization-scale dependence dominates the final error of these parameters. In this work, we extended the analysis of Grozin and Sturm for the vector correlator and calculated, for the first time, the small-momentum expansion of the pseudo-scalar, scalar and axial-vector correlators in the large- $\beta_{0}$ limit. We performed a detailed study of the singularities of the Borel transforms and the higher-order behaviour of the perturbative series of these correlators. With the knowledge of the structure of renormalon singularities in the Borel transforms, we can design new observables with tamed perturbative behaviour that can lead to improved determinations of quark masses and $\alpha_{s}$ based on heavy-quark current correlators.


Keywords: Quark masses. Strong coupling. Perturbative QCD. Renormalons.

## RESUMO

RODRIGUES, M. V. G. Correlatores de quark pesado na expansão de baixos momentos em ordens mais altas. 2021. 114 p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2021.

Na ausência de observação direta de nova física no LHC, a física de precisão desempenha um papel fundamental na busca por fenômenos além do Modelo Padrão (SM). Nesta situação, os parâmetros fundamentais do SM, como as massas dos quarks e o acoplamento forte, $\alpha_{s}$, devem ser conhecidos com alta precisão. Um dos métodos mais poderosos para se extrair as massas dos quarks charm e bottom, $m_{c}$ e $m_{b}$, bem como acoplamento forte, é o uso de regras de soma para os correlatores de quark pesado vetorial e pseudo-escalar. No momento, um aumento na precisão dos dados experimentais ou simulações de QCD na rede não devem permitir determinações ainda mais precisas de $m_{c}, m_{b}$ e $\alpha_{s}$, visto que a incerteza perturbativa relacionada à dependência residual na escala de renormalização domina os erros finais desses parâmetros. Neste trabalho, nós estendemos a análise de Grozin e Sturm no correlator vetorial e calculamos, pela primeira vez, a expansão de baixos momentos para os correlatores pseudo-escalar, escalar e vetor-axial no limite large- $\beta_{0}$. Fizemos um estudo detalhado sobre as singularidades nas transformadas de Borel e sobre o comportamento perturbativo em ordens mais altas desses correlatores. Com o conhecimento da estrutura de singularidades to tipo renormalons nas transformadas de Borel, nós podemos construir novos observáveis com melhor comportamento perturbativo que podem permitir uma melhora nas extrações das massas dos quarks e do acoplamento forte através dos correlatores de corrente com quark pesado.

Palavras-chave: Massas dos quarks. Acoplamento forte. QCD perturbativa. Renormalons.

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## 1 INTRODUCTION

The Standard Model (SM) of particle physics ${ }^{1-3}$ is one of the most successful theories in the history of science. Seminal contributions to its construction are due to the outstanding works of Feynman, ${ }^{4}$ Schwinger ${ }^{5}$ and Tomonaga ${ }^{6}$ in the 40 s with the development of Quantum Electrodynamics (QED), the theory that explains the electromagnetic aspects of nature within the Quantum Field Theory framework, and that now, alongside with the Weak Interactions, ${ }^{1,7,8}$ describes the electroweak sector of the SM. Nowadays, the SM successfully predicts with high accuracy the results of many physical observables. The anomalous magnetic moment of the electron, for instance, has an impressive precision of 0.62 parts per billion, ${ }^{9}$ which is considered one the most precise determinations of a physical quantity in the history of physics. However, tensions between experiments and the SM do exist. To give prominent examples, the SM can not satisfactorily describe massive neutrinos ${ }^{10}$ and, more recently, a new measurement of the anomalous magnetic moment of the muon at Fermilab ${ }^{11}$ revealed a larger discrepancy between experiment and the theory prediction for this quantity, reaching about $4.2 \sigma$. This provides an additional evidence that the SM might need to incorporate some (yet unknown) new physics.

Without direct observations of physics beyond the SM at the Large Hadron Collider (LHC), precision physics remains a crucial tool to search for new phenomena. ${ }^{12}$ Due to the enormous progress in multi-loop calculations in the past years, the uncertainties in the theory predictions of many key observables are now mainly dominated by the errors on the fundamental parameters of the SM. With the forthcoming $e^{+} e^{-}$facilities such as the Future Circular Collider ${ }^{13}$ that should aim at Higgs and top-quark physics, the quark masses and the strong coupling, $\alpha_{s}$, must be known with higher precision. ${ }^{14}$ These are the fundamental parameters in the strong interaction sector of the SM, described by Quantum Chromodynamics (QCD). However, due to quark confinement, the extraction of these parameters is a very complex task; quarks are not physical observables in the strict sense and some of their properties depend on conventions related to regularization and renormalization of the theory. ${ }^{15}$

The procedure adopted for extractions of quark masses and the strong coupling follows essentially the same general strategy. First, one calculates in the state of the art of QCD an observable that can be measured with reasonable accuracy. The quark masses and the strong coupling are then treated, within a precise definition, as free parameters of the theoretical prediction; from comparisons with the experimental data it is possible to extract these parameters through a rigorous statistical procedure.

One of the most powerful methods for the extraction of the charm- and bottom-quark masses is the use of QCD sum rules for the cross-section $\sigma\left(e^{+} e^{-} \rightarrow q \bar{q}\right)$ with $q=c, b .^{16,17}$

The observables are the vector moments $M_{q, n}^{V}$, defined as weighted integrals over the normalized cross-section with weights of the type $1 / s^{n+1}$, where $s$ is the invariant mass of the $e^{+} e^{-}$pair and $n$ is a positive integer. With the use of unitarity and analyticity in Quantum Field Theories, these moments can be described in the theoretical framework as the coefficients in the small-momentum expansion of the heavy-quark vector-current correlator, which is essentially the Taylor expansion of the correlator for $s \sim 0$. For not too high values of $n$ these coefficients are largely dominated by the perturbative contribution in the Operator Product Expansion (OPE). As of today, the perturbative expansion in the strong coupling is known up to four-loop accuracy for $n<5$, i.e., $\mathscr{O}\left(\alpha_{s}^{3}\right) .{ }^{18-25}$ Due to the high sensitivity in the heavy-quark mass, these moments have been the basis of precise determinations of the charm and bottom masses. ${ }^{26-32}$

Analogous strategies make use of the pseudo-scalar moments, $M_{q, n}^{P}$, calculated with the coefficients of the small-momentum expansion of the heavy-quark pseudo-scalar correlator. Although these moments can not be determined experimentally - there is no such pseudoscalar photon -, high precision determinations are obtained with lattice simulations in the case of the charm-quark. ${ }^{33-37}$ (Lattice simulations for the vector and axial-vector correlators do also exist in the literature, but are not as competitive as the pseudo-scalar charm moments. ${ }^{34}$ )

Considering dimensionless ratios of roots of the vector and pseudo-scalar moments is also an important tool for extractions of the strong coupling. This strategy has been extensively used in the lattice community in the case of the pseudo-scalar correlator, ${ }^{33-37}$ and it was recently shown that considering these ratios in the charm vector moments also leads to precise determinations of $\alpha_{s}$, thanks to the present status of experimental data of $e^{+} e^{-}$annihilation in the low-energy region. ${ }^{38,39}$

In all studies of this type, it is crucial to estimate the theoretical errors arising from the use of truncated perturbative expansions. These uncertainties give a large contribution to the final error of quark masses and $\alpha_{s}$ extractions from the vector and pseudo-scalar moments, and therefore they should be carefully studied through estimates of higher-order unknown coefficients in perturbative QCD and/or through conservative renormalizationscale variations. These studies benefit from the partial knowledge about the yet unknown coefficients and the higher-order behaviour of the perturbative expansion. In this context, the large- $\beta_{0}$ limit of QCD is an important tool. ${ }^{40,41}$ In this approximation, one first consider the large- $N_{f}$ limit, where the number of active quark flavours, $N_{f}$, is considered a large parameter but the product $N_{f} \alpha_{s}$ is kept constant and $\mathscr{O}(1)$. The leading- $N_{f}$ coefficients in the perturbative expansion are then calculated to all orders in the strong coupling through the computation of QED-like Feynman diagrams. To obtain the large- $\beta_{0}$ limit, the non-abelian nature of QCD is then introduced in this approximation through a procedure known as naive non-abelianization, ${ }^{42,43}$ where the fermionic contribution of the leading-order coefficient of the $\mathrm{QCD} \beta$-function, $\beta_{0, f}$, is replaced by the complete one-loop
coefficient $\beta_{0}$. As a result, one will have a series at $1 / \beta_{0}$ accuracy of the given observable that is known to all orders in perturbation theory and whose Borel transform can be studied exactly. Through a detailed study of the singularities on the real axis, known as renormalons, ${ }^{40}$ of the Borel transform, one can obtain valuable information about the divergent behaviour of the perturbative series. The singularities arising from the infrared region of loop subgraphs are of particular importance due to its one-to-one correspondence with non-perturbative aspects of the theory encoded in the condensates of the OPE. In some situations, the large- $\beta_{0}$ limit provides reasonable predictions of the coefficients in the perturbative series obtained in the full theory. However, even when this is not the case, this approximation gives valuable information about the perturbative expansion, since the positions of the renormalon singularities in large- $\beta_{0}$, related to the weight of their contribution to the series coefficients, are unchanged in full QCD.

The small-momentum expansion of the vector current correlator in the large- $\beta_{0}$ limit was already known in the literature ${ }^{44}$ but, to the best of our knowledge, no phenomenological applications have ever been exploited. In this work we extend the calculation of Grozin and Sturm ${ }^{44}$ to the pseudo-scalar, scalar and axial-vector correlators, and perform a detailed analysis of the renormalon content of the respective Borel transforms to obtain a better understanding of the perturbative series of the vector and pseudo-scalar moments, since these are extensively used in precise determinations of quark masses and the strong coupling. With the results obtained in this work for the higher-order behaviour of the perturbative series of the moments and the closed form of the Borel transforms, one can design new combinations of moments based on renormalon cancellations that can be translated into series with better perturbative behaviour that permit an improvement of the determinations of $m_{c}, m_{b}$ and $\alpha_{s}$ based on heavy-quark current correlators.

Some of main results of this work have been recently published by the Journal of High Energy Physics in collaboration with Diogo Boito and Vicent Mateu. ${ }^{45}$ Here we give a more detailed derivation of these results and perform, for the very first time, an analysis about the higher-order behaviour of the vector and pseudo-scalar moments.

This work is structured as follows. In Chap. 2 we give an overview about QCD, renormalization, loop calculations and asymptotic series. The current correlators of our interest are introduced in Chap. 3, together with the computation of the leading order contribution of the moments, and an overview about the determination of the experimental values of $M_{q, n}^{V}$. In Chap. 4 we give a detailed derivation of the calculation framework in the large- $\beta_{0}$ limit and present our results for the Borel transforms of the moments in this approximation. The analysis of the perturbative series and a framework to design combinations of moments with improved perturbative behaviour is left to Chap. 5. Our conclusions are presented in Chap. 6.

## 2 QUANTUM CHROMODYNAMICS

In this chapter we aim at developing the strong interaction sector of the SM, described by QCD, and introduce notations and conventions that will be used along this work.

The $\mathrm{SM}^{1-3}$ is a gauge theory based on the symmetry group $S U(3)_{C} \otimes S U(2)_{L} \otimes U(1)_{Y}$. The strong sector is described by the $S U(3)_{C}$ group while the electroweak sector is described by the $S U(2)_{L} \otimes U(1)_{Y}$. To generate the masses of the gauge bosons $Z$ and $W^{ \pm}$and all massive fermions, the introduction of a scalar particle able to acquire a non-zero vacuum expectation value and spontaneously break the symmetry is made necessary. This scalar particle is the so-called Higgs Boson ${ }^{46-48}$ discovered in 2012 by the collaborations ATLAS ${ }^{49}$ and $\mathrm{CMS}^{50}$ at the Large Hadron Collider (LHC).

The $S U(3)_{C}$ nature of QCD is based both on theoretical and experimental aspects. ${ }^{51}$ From a theoretical point of view, at first sight hadrons - states composed by quarks would face problems concerning the quantum statistics: the baryon $\Delta^{++}(u u u)$, for instance, is a composite system with total angular momentum $J=3 / 2$ and no orbital angular momentum in the fundamental state. This corresponds to have all three $u$-quarks spins aligned to the same direction $\left(s_{z}=1 / 2\right)$ and zero relative angular momentum, implying a symmetric wavefunction for the $\Delta^{++}$in disagreement with the Fermi-Dirac statistics of particles with half-integer values of angular momentum. This problem is solved if one assumes the existence of a new quantum number for quarks: colour. ${ }^{52}$ If each flavour of quark has $N_{c}=3$ possible colours, then the $\Delta^{++}$can be reinterpreted as an antisymmetric state:

$$
\begin{equation*}
\Delta^{++}=\frac{\varepsilon^{\alpha \beta \gamma}}{\sqrt{6}}\left|u_{\alpha}^{s_{z}=1 / 2}, u_{\beta}^{s_{z}=1 / 2}, u_{\gamma}^{s_{z}=1 / 2}\right\rangle, \tag{2.1}
\end{equation*}
$$

where $\alpha, \beta, \gamma$ are colour indices and $\varepsilon^{\alpha \beta \gamma}$ is the 3 -dimensional fully antisymmetric LeviCività tensor. Because the quantum number colour has never been observed in nature, it is necessary to impose the colour confinement hypothesis, a postulate that all asymptotic states $^{\mathrm{I}}$ are colourless. In terms of group theory, this means that hadronic states must be invariant under rotations in the colour space, i.e, are colour singlets in the $\operatorname{SU}\left(N_{c}\right)$ algebra. The confinement hypothesis is supported by the asymptotic freedom ${ }^{53,54}$ of the strong coupling $\alpha_{s}$ typical of Yang-Mills gauge theories, i.e., the intensity of the strong coupling decreases with the energy scale.

These fundamental properties of QCD are extensively studied in the observable

$$
\begin{equation*}
R_{\mathrm{tot}}(s)=\frac{\sigma\left(e^{+} e^{-} \rightarrow \text { hadrons }\right)}{4 \pi \alpha_{e m}^{2}(s) / 3 s} \tag{2.2}
\end{equation*}
$$

[^0]where $\sigma\left(e^{+} e^{-} \rightarrow\right.$ hadrons) is the total inclusive cross section for $e^{+} e^{-}$annihilation into hadrons and $\alpha_{e m}(s)$ is the effective electromagnetic coupling. By means of the optical theorem, ${ }^{55}$ the $R_{\mathrm{tot}}(s)$ ratio can be expressed as the imaginary part of the vector-current correlator and, for sufficiently high energies and far away from resonances, the perturbative expansion
\[

$$
\begin{equation*}
R_{\mathrm{tot}}(s)=N_{c} \sum_{f} Q_{f}^{2}\left(1+\frac{\alpha_{s}(s)}{\pi}+\cdots\right)+\mathscr{O}\left(\frac{m_{f}^{4}}{s^{2}}\right)+\mathscr{O}\left(\frac{\Lambda_{\mathrm{QCD}}^{4}}{s^{2}}\right) \tag{2.3}
\end{equation*}
$$

\]

is valid. The pre-factor $N_{c} \sum_{f} Q_{f}^{2}$ is the partonic result obtained at lowest order in perturbation theory, where no gluon exchange between quarks is considered. This factor comes only from the postulate that quarks have $N_{c}$ colours and different electric charges $Q_{f}$. The presence of a energy-dependent coupling $\alpha_{s}(s)$ is a consequence of the renormalization program that will be discussed later in this chapter.

In Fig. 1 we display a compilation of the experimental data for $R_{\text {tot }}(s) .{ }^{56}$ The dashed red line is the partonic prediction with $N_{c}=3$ and, although it is not statistically equivalent with the experimental data at low energies, it gives a reasonable prediction for energies above 10 GeV and below the $Z$-boson resonance. This is a strong evidence of the colour quantum number. A better description of the experimental data is achieved when the first order correction in $\alpha_{s}$, depicted by the solid blue line, is taken into account. The asymptotic freedom of the strong coupling can then be visualized since QCD corrections (i.e., from $\alpha_{s}$ ) are at about $10 \%$ around 2.5 GeV , but it turns out to be only about $5 \%$ around 20 GeV .

### 2.1 The QCD Lagrangian

Having presented some general aspects of QCD we now turn to a derivation of the QCD Lagrangian based on the gauge principle with the gauge group $S U(3)$ in the free Dirac Lagrangian that describes relativistic spin- $1 / 2$ particles.

Let us denote $q_{f}^{\alpha}$ a quark field with flavour $f$ and colour $\alpha$. The free Dirac Lagrangian is given by

$$
\begin{equation*}
\mathscr{L}_{0}=\sum_{f} \bar{q}_{f}\left(i \gamma_{\mu} \partial^{\mu}-m_{f}\right) q_{f}, \tag{2.4}
\end{equation*}
$$

where $q_{f} \equiv\left(q_{f}^{1}, q_{f}^{2}, q_{f}^{3}\right)^{T}, m_{f}$ are quark masses and $\gamma_{\mu}$ are the $4 \times 4$ Dirac matrices. The Lagrangian $\mathscr{L}_{0}$ is invariant under global $S U(3)$ transformations represented by the unitary matrices

$$
\begin{equation*}
U=\exp \left(i t^{a} \theta_{a}\right) \tag{2.5}
\end{equation*}
$$

for arbitrary parameters $\theta_{a}$, with $a=1, \ldots, 8$, and $t^{a}$ denoting the eight generators of the $S U(3)$ algebra in the fundamental representation. The gauge principle requires the


Figure 1 - Experimental data for $R_{\text {tot }}(s)$ provided by the Particle Data Group ${ }^{56}$ with the theoretical predictions at leading and next-to-leading order. Each jump in the theoretical prediction correspond to the presence of a new active quark flavour.

Source: By the author.

Lagrangian to be also invariant under local transformations characterized by space-time dependent parameters $\theta_{a}(x)$. In order to satisfy this requirement, it is mandatory to replace the quark derivatives $\partial^{\mu} q_{f}$ by covariant derivatives, $D^{\mu} q_{f}$, as

$$
\begin{equation*}
D^{\mu} q_{f}=\left[\partial^{\mu}+i g_{s} t^{a} G_{a}^{\mu}(x)\right] q_{f} \tag{2.6}
\end{equation*}
$$

For this purpose, we need to introduce eight gauge bosons $G_{a}^{\mu}(x)$ - the gluons - and an arbitrary parameter $g_{s}$ that later will be related to the strong coupling $\alpha_{s}$.

We complete the construction of the QCD Lagrangian adding a gauge-invariant kinect term to allow the gluon to be a real propagating field. The kinect Lagrangian is given by

$$
\begin{equation*}
\mathscr{L}_{\text {kin }}=-\frac{1}{4} G_{a}^{\mu \nu} G_{\mu \nu}^{a} \tag{2.7}
\end{equation*}
$$

where the gluon field-strengh tensor $G_{a}^{\mu \nu}$ is defined as

$$
\begin{equation*}
G_{a}^{\mu \nu}=\partial^{\mu} G_{a}^{\nu}-\partial^{\nu} G_{a}^{\mu}-g_{s} f^{a b c} G_{b}^{\mu} G_{c}^{\nu} \tag{2.8}
\end{equation*}
$$

and $f^{a b c}$ are the $S U(3)$ structure constants that dictate the commutation relations of the generators:

$$
\begin{equation*}
\left[t^{a}, t^{b}\right]=i f^{a b c} t^{c} \tag{2.9}
\end{equation*}
$$

The presence of the structure constants $f^{a b c}$ in the gluon field-strengh tensor is a consequence of the non-abelian nature of the gauge group $S U(3)$ and is responsible for gluon

(a) quark-gluon interaction

(b) cubic gluon self-interaction

(c) quartic gluon self-interaction

Figure 2 - Feynman diagrams of the interaction vertices in the QCD Lagrangian.
Source: By the author.
self-interactions that give rise to the asymptotic freedom in QCD.
The QCD Lagrangian is therefore

$$
\begin{align*}
\mathscr{L}_{\mathrm{QCD}}= & \underbrace{-\frac{1}{4}\left(\partial^{\mu} G_{a}^{\nu}-\partial^{\nu} G_{a}^{\mu}\right)\left(\partial_{\mu} G_{\nu}^{a}-\partial_{\nu} G_{\mu}^{a}\right)}_{\text {gluon propagation }}+\underbrace{\sum_{f} \bar{q}_{f}\left(i \gamma_{\mu} \partial^{\mu}-m_{f}\right) q_{f}}_{\text {quark propagation }} \\
& \underbrace{-g_{s} G_{a}^{\mu} \sum_{f} \bar{q}_{f}^{\alpha} \gamma_{\mu}\left(t^{a}\right)_{\alpha \beta} q_{f}^{\beta}}_{\text {quark-gluon interaction }}  \tag{2.10}\\
& \underbrace{+\frac{g_{s}}{2} f^{a b c}\left(\partial^{\mu} G_{\alpha}^{\mu}-\partial^{\nu} G_{a}^{\mu}\right) G_{\mu}^{b} G_{\nu}^{c}}_{\text {cubic gluon self-interaction }}-\underbrace{\frac{g_{s}^{2}}{4} f^{a b c} f_{a d e} G_{b}^{\mu} G_{c}^{\nu} G_{\mu}^{d} G_{\nu}^{e}}_{\text {quartic gluon self-interaction }} .
\end{align*}
$$

The first line is composed only by kinetic terms that are related to the free quark and gluon propagators. The second line represents the interaction between a gluon and a quark-antiquark pair accompanied by a coupling parameter $g_{s}$. The last line contains only gluon self-interactions. The Feynman diagrams that represent these interactions are depicted in Fig. 2.

Now one can proceed with the quantization of the QCD Lagrangian. ${ }^{57}$ Let us denote $q(x)$ a quark field with mass $m$. Its Fourier decomposition in terms of the creation and annihilation operators $a(\vec{p}, \lambda)$ and $b^{\dagger}(\vec{p}, \lambda)$ is given by

$$
\begin{equation*}
q(x)=\int \frac{\mathrm{d}^{3} p}{(2 \pi)^{3} 2 E(\vec{p})} \sum_{\lambda}\left[u(\vec{p}, \lambda) a(\vec{p}, \lambda) e^{-i p x}+v(\vec{p}, \lambda) b^{\dagger}(\vec{p}, \lambda) e^{i p x}\right], \tag{2.11}
\end{equation*}
$$

where the four-spinors $u(\vec{p}, \lambda)$ and $v(\vec{p}, \lambda)$ are the solutions of the free Dirac Lagrangian in momentum space, $E(\vec{p})$ is the relativistic energy and the summation covers all possible helicity states. The quantization of $q(x)$ follows by imposing anti-commutation relations for the creation and annihilation operators and the quark propagator $S_{F, i j}^{(0)}(x-y)$ in coordinate
space can be obtained as

$$
\begin{equation*}
S_{F, i j}^{(0)}(x-y) \equiv\langle 0| T q_{i}(x) \bar{q}_{j}(y)|0\rangle=\int \frac{\mathrm{d}^{4} p}{(2 \pi)^{4}}\left[\frac{i(\not p+m) \delta_{i j}}{p^{2}-m^{2}+i 0}\right] e^{-i p(x-y)} \tag{2.12}
\end{equation*}
$$

where $i, j$ are colour indices, $|0\rangle$ is the vacuum of the free theory, $T$ denotes the timeordering operator, $\not p \equiv p^{\mu} \gamma_{\mu}$ is the slash notation and the $i 0$-prescription in the denominator is to shift the pole of the propagator and maintain causality in the theory. ${ }^{55}$ The term inside the square brackets in the RHS is the quark propagator in momentum space, $S_{F, i j}^{(0)}(p)$, which differs from a generic fermion propagator $S_{F}^{(0)}(p)$ only in the Kronecker delta $\delta_{i j}$. More precisely,

$$
\begin{equation*}
S_{F, i j}^{(0)}(p) \equiv \int \mathrm{d} x e^{i p x}\langle 0| T q_{i}(x) \bar{q}_{j}(0)|0\rangle=\delta_{i j} \frac{i(\not p+m)}{p^{2}-m^{2}+i 0} \tag{2.13}
\end{equation*}
$$

For gluon fields the procedure is more cumbersome since it is necessary to add a gauge-fixing term in the Lagrangian and introduce a set of non-physical fields that couple only to gluons - the so-called Faddeev-Popov ghosts - due to gluon self-interactions. ${ }^{55}$ After following these steps and imposing the anti-commutation relations of the creation and annihilation operators the gluon propagator $D_{F}^{(0), a b}(x-y)$ in coordinate space can be obtained as

$$
\begin{equation*}
D_{\mu \nu}^{(0), a b}(x-y) \equiv\langle 0| T G_{\mu}^{a}(x) G_{\nu}^{b}(y)|0\rangle=\int \frac{\mathrm{d}^{4} k}{(2 \pi)^{4}}\left[\frac{-i \delta^{a b}}{k^{2}+i 0}\left(g_{\mu \nu}-\xi \frac{k_{\mu} k_{\nu}}{k^{2}+i 0}\right)\right] e^{-i k(x-y)} \tag{2.14}
\end{equation*}
$$

Here $\xi$ is a gauge-fixing parameter that does not change physics, i.e., all physical results are independent of $\xi$. In particular, the suitable choice $\xi=0$ is called Feynman gauge, while $\xi=1$ is called Landau gauge. The term inside the square brackets in the RHS is the gluon propagator in momentum space, $D_{\mu \nu}^{(0), a b}(k)$, which differs from a generic spin-1 massless boson propagator $D_{\mu \nu}^{(0)}(k)$ only in the Kronecker delta $\delta^{a b}$. More precisely,

$$
\begin{equation*}
D_{\mu \nu}^{(0), a b}(k) \equiv \int \mathrm{d} x e^{i k x}\langle 0| T G_{\mu}^{a}(x) G_{\nu}^{b}(0)|0\rangle=\delta^{a b} \frac{-i}{k^{2}+i 0}\left(g_{\mu \nu}-\xi \frac{k_{\mu} k_{\nu}}{k^{2}+i 0}\right) \tag{2.15}
\end{equation*}
$$

### 2.2 The renormalization program in QCD

In the last section the quark and gluon propagators were presented in the free theory. However, as one can see in the QCD Lagrangian of Eq. (2.10), there are interactions between fields accompanied by a coupling parameter $g_{s}$. These interactions, in many calculations, are related to divergent integrals. The procedure to obtain finite physical quantities is known as the renormalization program of a Quantum Field Theory: one should first (i) regularize the expressions by explicitly isolating the divergences of the


Figure 3 - Feynman diagram for the first order correction of the quark propagator.
Source: By the author.
integrals, (ii) recognize that the non-interacting particles (also known as bare particles) on which perturbation theory is based are not the real physical particles that interact (the interactions modify some particle properties like charge and mass), and, finally, (iii) one should renormalize the theory subtracting the infinities under a prescription that relates bare particles to the physical particles. An important consequence of the renormalization program is that some properties like charge and mass acquire an energy dependence when it is imposed that a physical quantity must be independent of the regularization procedure adopted.

In this section we renormalize the quark mass at first order in perturbation theory going through the main steps of the calculation to give an overview of the standard procedure to calculate loop integrals.

The quark propagator in momentum space and in the full interacting theory is given by

$$
\begin{equation*}
S_{F, i j}(p)=\int \mathrm{d}^{4} x e^{i p x}\langle 0| T q_{i}(x) \bar{q}_{j}(0) e^{i \int \mathrm{~d}^{4} z \mathscr{L}_{I}}|0\rangle \tag{2.16}
\end{equation*}
$$

where $\mathscr{L}_{I}$ includes only the interaction terms of the QCD Lagrangian. The exponential is expanded as a power series in $g_{s}$ and dictates the perturbative expansion. Since the expansion starts at 1 , at leading order the quark propagator in the full theory is just $S_{F, i j}^{(0)}(p)$. The next contribution is order $g_{s}^{2}$ and is generated by the quark-gluon interaction term in the QCD Lagrangian. After contracting the fields with Wick's theorem ${ }^{55}$ to calculate the time-ordered product between the free vacuum states and evaluating the coordinate space integrals, the first order correction to the quark propagator is found to be

$$
\begin{equation*}
S_{F, i j}^{(1)}(p)=\int \frac{\mathrm{d}^{4} k}{(2 \pi)^{4}}\left[S_{F}^{(0)}(p)\left(-i g_{s} \gamma^{\mu} t^{a}\right) S_{F}^{(0)}(p-k)\left(-i g_{s} \gamma^{\nu} t^{b}\right) S_{F}^{(0)}(p)\right]_{i j} D_{\mu \nu}^{(0), a b}(k) \tag{2.17}
\end{equation*}
$$

which is represented by the self-energy Feynman diagram in Fig. 3.
Since the gluon propagator is proportional to $\delta^{a b}$ and

$$
\begin{equation*}
\left[t^{a}, t^{a}\right]_{i j}=C_{F} \delta_{i j} \tag{2.18}
\end{equation*}
$$

where $C_{F}=\left(N_{c}^{2}-1\right) / 2 N_{c}$ is the Casimir operator of the gauge group $S U\left(N_{c}\right)$ in the fundamental representation, a Kronecker $\delta_{i j}$ can be factorized. The quark propagator in the full theory can then be written as

$$
\begin{equation*}
S_{F, i j}(p)=\delta_{i j}\left[S_{F}^{(0)}(p)+S_{F}^{(0)}(p)\left[-i \Sigma^{(1)}(p)\right] S_{F}^{(0)}(p)+\ldots\right] \tag{2.19}
\end{equation*}
$$

The structure of $S_{F, i j}(p)$ in terms of $S_{F}^{(0)}(p)$ is a geometric series that can be summed to obtain

$$
\begin{equation*}
S_{F, i j}(p)=\delta_{i j} \frac{i}{[p-m-\Sigma(p)+i 0]} \tag{2.20}
\end{equation*}
$$

in a process known as Dyson resummation. The new function $\Sigma(p)$, called quark self-energy, at first order takes the form

$$
\begin{equation*}
\Sigma^{(1)}(p)=-i g_{s}^{2} C_{F} \int \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}}\left[\gamma^{\mu} S_{F}^{(0)}(p-k) \gamma^{\nu}\right] D_{\mu \nu}^{(0)}(k) \tag{2.21}
\end{equation*}
$$

As it is usual in loop calculations, the integral over the internal momentum $k$ diverges and therefore a process of regularization is needed. The modern procedure to regularize loop integrals in a way that Lorentz symmetry and gauge invariance is preserved is based on the assumption that the number of space-time dimension being 4 is the source of divergence. This procedure is known as dimensional regularization, ${ }^{58}$ and its main idea relies on reformulating the whole theory with an arbitrary number of dimensions $d$. When the physical limit $d \rightarrow 4$ is taken the divergences appear as poles in the resulting expression. This limit is employed by conveniently writing $d=4-2 \epsilon$ and then taking the limit $\epsilon \rightarrow 0^{+}$.

Some of the major changes in the theory after promoting it to $d$ dimensions are:

- The integrations have to be performed in $d$ dimensions, and therefore

$$
\begin{equation*}
\int \frac{\mathrm{d}^{4} k}{(2 \pi)^{4}} \quad \rightarrow \quad \int \frac{\mathrm{~d}^{d} k}{(2 \pi)^{d}} \tag{2.22}
\end{equation*}
$$

- The introduction of an arbitrary energy scale $\mu$, also known as 't Hooft mass, is needed to maintain the dimension of correlation functions;
- Since Lorentz indices $\mu, \nu$ now should run from 0 to $d-1$, the contractions of the Dirac-matrices $\gamma_{\mu}$ and the metric tensor have to be generalized to $d$ dimensions. Some examples are:

$$
\begin{align*}
& g_{\mu \nu} g^{\mu \nu}=4 \rightarrow g_{\mu \nu} g^{\mu \nu}=d  \tag{2.23}\\
& \gamma_{\mu} \gamma^{\mu}=4 \rightarrow \gamma_{\mu} \gamma^{\mu}=d  \tag{2.24}\\
& \gamma_{\mu} \gamma_{\nu} \gamma^{\mu}=-2 \gamma_{\nu} \rightarrow \gamma_{\mu} \gamma_{\nu} \gamma^{\mu}=(2-d) \gamma_{\nu} \tag{2.25}
\end{align*}
$$

The anti-commutation relations of the Dirac matrices, $\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 g_{\mu \nu}$, is preserved, and therefore no changes in their traces are expected.

Under the dimensional regularization prescription we can go back to the calculation of the quark self-energy function and write

$$
\begin{equation*}
\Sigma^{(1)}(p)=-i\left[g_{s}^{2} \mu^{-(4-d)}\right] C_{F} \mu^{4-d} \int \frac{\mathrm{~d}^{d} k}{(2 \pi)^{d}}\left[\gamma^{\mu}\left(\frac{i(p-\not k+m)}{(p-k)^{2}-m^{2}}\right) \gamma^{\nu}\right] \frac{-i g_{\mu \nu}}{k^{2}}, \tag{2.26}
\end{equation*}
$$

where the quark and gluon propagators (in the Feynman gauge) were already replaced accordingly to Eqs. (2.12) and (2.14). The Dirac matrices are then contracted using Eqs. (2.24) and (2.25) such that $\Sigma^{(1)}(p)$ becomes

$$
\begin{equation*}
\Sigma^{(1)}(p)=-i\left[g_{s}^{2} \mu^{-(4-d)}\right] C_{F} \mu^{4-d} \int \frac{\mathrm{~d}^{d} k}{(2 \pi)^{d}} \frac{[(2-d)(\not p-\not p)+d m]}{k^{2}\left[(p-k)^{2}-m^{2}\right]} . \tag{2.27}
\end{equation*}
$$

One of the most convenient strategies to solve the loop integral consists in first combining the denominators into a single one using the Feynman parameter

$$
\begin{equation*}
\frac{1}{A^{n_{1}} B^{n_{2}}}=\frac{\Gamma\left(n_{1}+n_{2}\right)}{\Gamma\left(n_{1}\right) \Gamma\left(n_{2}\right)} \int_{0}^{1} \mathrm{~d} x \frac{x^{n_{1}-1}(1-x)^{n_{2}-1}}{[A x+B(1-x)]^{n_{1}+n_{2}}}, \tag{2.28}
\end{equation*}
$$

and then shifting the internal momentum to $k \rightarrow k^{\prime}=k-p x$. Then

$$
\begin{equation*}
\Sigma^{(1)}(p)=-i\left[g_{s}^{2} \mu^{-(4-d)}\right] C_{F} \mu^{4-d} \int_{0}^{1} \mathrm{~d} x \int \frac{\mathrm{~d}^{d} k^{\prime}}{(2 \pi)^{d}} \frac{p(2-d)(1-x)+d m+\left(\text { linear in } k^{\prime}\right)}{\left[\left(k^{\prime}\right)^{2}-\Delta\right]^{2}} \tag{2.29}
\end{equation*}
$$

where $\Delta \equiv m^{2} x-p^{2} x(1-x)$. Linear terms in $k^{\prime}$ are odd functions that result in zero when integrated over all space by symmetry. The integral over $k^{\prime}$ is solved performing a Wick rotation ${ }^{55}\left(k_{0}^{\prime}, \vec{k}^{\prime}\right) \rightarrow\left(i k_{0}^{\prime}, \vec{k}^{\prime}\right)$ to transform the integration variable $k^{\prime}$ into an Euclidean one and using $d$-dimensional spherical coordinates. In particular,

$$
\begin{equation*}
\int \frac{\mathrm{d}^{d} k^{\prime}}{(2 \pi)^{d}} \frac{1}{\left[\left(k^{\prime}\right)^{2}-\Delta\right]^{n}}=\frac{(-1)^{n} i}{(4 \pi)^{d / 2}} \frac{\Gamma(n-d / 2)}{\Gamma(n)} \Delta^{d / 2-n} \tag{2.30}
\end{equation*}
$$

With the above equation the integral over the internal momentum in $\Sigma^{(1)}(p)$ is solved and we arrive at the expression

$$
\begin{equation*}
\Sigma^{(1)}(p)=\not p \Sigma_{p}^{(1)}(p)+m \Sigma_{m}^{(1)}(p), \tag{2.31}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma_{p}^{(1)}(p)=\frac{C_{F} \alpha_{s}(\mu)}{4 \pi}\left(\frac{4 \pi \mu^{2}}{p^{2}}\right)^{\epsilon} \Gamma(\epsilon)(2 \epsilon-2) \int_{0}^{1}(1-x)\left[\frac{m^{2}}{p^{2}} x-x(1-x)\right]^{-\epsilon} \tag{2.32}
\end{equation*}
$$

and

$$
\begin{equation*}
\Sigma_{m}^{(1)}(p)=\frac{C_{F} \alpha_{s}(\mu)}{4 \pi}\left(\frac{4 \pi \mu^{2}}{p^{2}}\right)^{\epsilon} \Gamma(\epsilon)(4-2 \epsilon) \int_{0}^{1}\left[\frac{m^{2}}{p^{2}} x-x(1-x)\right]^{-\epsilon} \tag{2.33}
\end{equation*}
$$

Here we introduced the strong coupling in terms of $g_{s}$ as $\alpha_{s}(\mu)=g_{s}^{2} \mu^{-2 \epsilon} /(4 \pi)$. Expanding the expressions around $\epsilon=0$ and then integrating over the Feynman parameter $x$ results in

$$
\begin{align*}
& \Sigma_{p}^{(1)}(p)=\frac{C_{F}}{4} \frac{\alpha_{s}(\mu)}{\pi}\left[-\frac{1}{\hat{\epsilon}}+\ln \left(\frac{m^{2}}{\mu^{2}}\right)-1-\frac{m^{2}}{p^{2}}+\left(1-\frac{m^{4}}{p^{4}}\right) \ln \left(1-\frac{p^{2}}{m^{2}}\right)\right],  \tag{2.34}\\
& \Sigma_{m}^{(1)}(p)=\frac{C_{F}}{4} \frac{\alpha_{s}(\mu)}{\pi}\left[4\left(\frac{1}{\hat{\epsilon}}-\ln \left(\frac{m^{2}}{\mu^{2}}\right)\right)+6-4\left(1-\frac{m^{2}}{p^{2}}\right) \ln \left(1-\frac{p^{2}}{m^{2}}\right)\right], \tag{2.35}
\end{align*}
$$

where

$$
\begin{equation*}
\frac{1}{\hat{\epsilon}} \equiv \frac{1}{\epsilon}-\gamma_{E}+\ln 4 \pi, \tag{2.36}
\end{equation*}
$$

with $\gamma_{E} \approx 0.5772$ being the Euler-Mascheroni constant.
As expected, the divergence of the loop integral was made manifest as poles in the variable $\epsilon$ and therefore we say that the expression is regularized. The next step consists in redefining the fields, masses and couplings in order to obtain finite Green functions. At first order in perturbation theory, for the quark propagator one only needs to redefine the quark field and the mass. We then define

$$
\begin{equation*}
q(x)=\sqrt{Z_{2 F}} q^{R}(x) \quad \text { and } \quad m=Z_{m} m^{R} . \tag{2.37}
\end{equation*}
$$

The quantities on the LHS are the ones that appear in the QCD Lagrangian, and are usually called bare quantities. On the RHS, the quantities with the superscript $R$ are the (finite) renormalized quantities. The infinities that lead to divergent Green functions are absorbed order by order in the renormalization constants

$$
\begin{equation*}
Z_{j} \equiv 1+Z_{j}^{(1)} \frac{\alpha_{s}}{\pi}+\mathscr{O}\left(\alpha_{s}^{2}\right) . \tag{2.38}
\end{equation*}
$$

The renormalization constants $Z_{2 F}$ and $Z_{m}$ are now determined by rewriting the quark propagator in terms of renormalized quantities and demanding the expression to be finite. At first order in perturbation theory the inverse of the quark propagator (c.f. Eq. (2.20)) is given by ${ }^{\text {II }}$

$$
\begin{equation*}
S_{F, i j}^{-1}(p)=Z_{2 \mathrm{~F}}\left[\not p-Z_{m} m^{R}-\not p \Sigma_{p}^{(1)}(p)-Z_{m} m^{R} \Sigma_{m}^{(1)}(p)\right], \tag{2.39}
\end{equation*}
$$

which can be expanded in the strong coupling as

[^1]$S_{F, i j}^{-1}(p)=\not p Z_{2 \mathrm{~F}}^{(1)} \frac{\alpha_{s}}{\pi}-m^{R}\left(Z_{2 \mathrm{~F}}^{(1)}+Z_{m}^{(1)}\right) \frac{\alpha_{s}}{\pi}+\not p \frac{C_{F}}{4} \frac{\alpha_{s}}{\pi} \frac{1}{\hat{\epsilon}}-m^{R} C_{F} \frac{\alpha_{s}}{\pi} \frac{1}{\hat{\epsilon}}+($ finite terms $)+\mathscr{O}\left(\alpha_{s}^{2}\right)$.
The $1 / \hat{\epsilon}$ contain, besides the pole $1 / \epsilon$, the finite terms $\gamma_{E}$ and $\ln 4 \pi$. The choice of subtracting not only the pole $1 / \epsilon$, but $1 / \hat{\epsilon}$, is known as the Modified Minimal Subtraction renormalization scheme, $\overline{\mathrm{MS}}$, and this can be systematically performed to all orders in perturbation theory by subtracting only the pole $1 / \epsilon$ assuming the 't Hooft mass is transformed as $\mu^{2} \rightarrow \mu^{2} \frac{e^{\gamma} E}{4 \pi}$. Therefore, the renormalization constants $Z_{2 F}$ and $Z_{m}$ in the $\overline{\mathrm{MS}}$-scheme and in the Feynman gauge are given by
\[

$$
\begin{equation*}
Z_{2 F}=1-\frac{C_{F}}{4} \frac{\alpha_{s}}{\pi} \frac{1}{\hat{\epsilon}}+\mathscr{O}\left(\alpha_{s}^{2}\right) \quad \text { and } \quad Z_{m}=1-\frac{3}{4} C_{F} \frac{\alpha_{s}}{\pi} \frac{1}{\hat{\epsilon}}+\mathscr{O}\left(\alpha_{s}^{2}\right) . \tag{2.41}
\end{equation*}
$$

\]

The determination of the coupling renormalization constant $Z_{\alpha}$ involves the calculation of more Feynman diagrams and is beyond the scope of this work. The result at first order reads ${ }^{57}$ :

$$
\begin{equation*}
Z_{\alpha}=1-\left[\frac{11 N_{c}-2 N_{f}}{12}\right] \frac{\alpha_{s}}{\pi} \frac{1}{\hat{\epsilon}}+\mathscr{O}\left(\alpha_{s}^{2}\right) \tag{2.42}
\end{equation*}
$$

where $N_{f}$ is the number of active quark flavours. The term with $N_{f}$ comes from corrections that encompass quark-gluon interactions, while the term with $N_{c}$ comes from gluon self-interactions and ghost bubbles.

### 2.2.1 Running of the strong coupling and quark masses

In the development of the renormalization program, the introduction of a new parameter is needed, and in the case of a dimensional regularization procedure this parameter is the energy scale $\mu$. Any physical observable, however, must be independent of the regularization method used in the calculations, and the equation that expresses this regularization invariance is known as the Renormalization Group Equation (RGE). Let us denote $R\left(\alpha_{s}, m\right)$ a physical observable with one single quark - for simplicity - with a renormalized mass $m .{ }^{\text {III }}$ The RGE for $R\left(\alpha_{s}, m\right)$ is

$$
\begin{equation*}
\mu \frac{\mathrm{d}}{\mathrm{~d} \mu} R\left(\alpha_{s}, m\right)=\left\{\mu \frac{\partial}{\partial \mu}-2 \beta\left(\alpha_{s}\right) \frac{\partial}{\partial \ln \alpha_{s}}-\gamma\left(\alpha_{s}\right) \frac{\partial}{\partial \ln m}\right\} R\left(\alpha_{s}, m\right)=0 \tag{2.43}
\end{equation*}
$$

where the renormalization group functions are defined as ${ }^{\mathrm{IV}}$

[^2]\[

$$
\begin{align*}
& \beta\left(\alpha_{s}\right) \equiv \frac{\mu}{2} \frac{\mathrm{~d} \ln Z_{\alpha}}{\mathrm{d} \mu}=\beta_{0}\left(\frac{\alpha_{s}}{4 \pi}\right)+\beta_{1}\left(\frac{\alpha_{s}}{4 \pi}\right)^{2}+\ldots \quad \beta \text {-function, }  \tag{2.44}\\
& \gamma_{m}\left(\alpha_{s}\right) \equiv \mu \frac{\mathrm{d} \ln Z_{m}}{\mathrm{~d} \mu}=\gamma_{m, 0}\left(\frac{\alpha_{s}}{4 \pi}\right)+\gamma_{m, 1}\left(\frac{\alpha_{s}}{4 \pi}\right)^{2}+\ldots \quad \text { mass anomalous dimension. } \tag{2.45}
\end{align*}
$$
\]

The $\beta$-function tells us how the coupling varies with the energy, while the energy dependence for the mass is governed by the anomalous dimension $\gamma_{m}$. At present, both renormalization group functions are known up to five-loop accuracy, i.e., up to terms $\beta_{4}$ and $\gamma_{m, 4}{ }^{60-64}$ The choice of a renormalization scale $\mu$ modifies the values of $\alpha_{s}$ and the mass, but in a way that physical observables remain invariant. However, since only truncated perturbative expansions are known, some residual scale dependence remains.

The coefficients of the $\beta$-function are determined by the renormalization constant $Z_{\alpha}$ order by order, whose $\mu$-dependence comes only from $\alpha_{s}$, as the bare strong coupling, $\alpha_{s}^{0}$, is related to $\mu$ by $\alpha_{s}^{0}=g_{s}^{2} \mu^{-2 \epsilon} / 4 \pi$ and $g_{s}$ - the parameter that appear in the QCD Lagrangian - is scale independent. In particular, employing the chain rule gives

$$
\begin{equation*}
\frac{\mu}{2} \frac{\mathrm{~d} \ln Z_{\alpha}}{\mathrm{d} \mu}=\frac{\mu}{2} \frac{\mathrm{~d}}{\mathrm{~d} \alpha_{s}^{0}} \ln \left[1-\frac{11 N_{c}-2 N_{f}}{12} \frac{\alpha_{s}^{0}}{\pi}\right] \frac{\mathrm{d} \alpha_{s}^{0}}{\mathrm{~d} \mu}+\mathscr{O}\left(\alpha_{s}^{2}\right), \tag{2.46}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\frac{\mu}{2} \frac{\mathrm{~d} \ln Z_{\alpha}}{\mathrm{d} \mu}=\frac{11 N_{c}-2 N_{f}}{3} \frac{\alpha_{s}}{4 \pi}+\mathscr{O}\left(\alpha_{s}^{2}\right) . \tag{2.47}
\end{equation*}
$$

The above equation can be compared with the perturbative expansion of the $\beta$-function given in Eq. (2.44), from which we obtain

$$
\begin{equation*}
\beta_{0}=\frac{1}{3}\left(11 N_{c}-2 N_{f}\right) . \tag{2.48}
\end{equation*}
$$

The energy dependence of $\alpha_{s}$ is now obtained by solving the differential equation of the $\beta$-function in the physical limit $\epsilon \rightarrow 0^{+}$, which at first order in perturbation theory becomes

$$
\begin{equation*}
-\frac{\mu}{2} \frac{\mathrm{~d} \ln \alpha_{s}}{\mathrm{~d} \mu}=\beta_{0} \frac{\alpha_{s}}{4 \pi} \tag{2.49}
\end{equation*}
$$

Integrating this equation on both sides one gets

$$
\begin{equation*}
\int_{\alpha_{s}\left(\mu_{1}\right)}^{\alpha_{s}\left(\mu_{2}\right)} \frac{\mathrm{d} \alpha_{s}}{\alpha_{s}^{2}}=-\frac{\beta_{0}}{2 \pi} \int_{\mu_{1}}^{\mu_{2}} \frac{\mathrm{~d} \mu}{\mu} \tag{2.50}
\end{equation*}
$$

and after carrying out the integrations and isolating $\alpha_{s}\left(\mu_{2}\right)$ one obtains


Figure 4 - Summary of extractions of $\alpha_{s}$ as a function of the energy scale.
Source: Adapted from ZYLA et al. ${ }^{65}$

$$
\begin{equation*}
\alpha_{s}\left(\mu_{2}\right)=\frac{\alpha_{s}\left(\mu_{1}\right)}{1-\alpha_{s}\left(\mu_{1}\right) \frac{\beta_{0}}{2 \pi} \ln \frac{\mu_{1}}{\mu_{2}}} . \tag{2.51}
\end{equation*}
$$

This equation dictates the one-loop running of the strong coupling, assuming the knowledge of $\alpha_{s}$ in a reference scale $\mu_{1}$. Since in QCD $\beta_{0}$ is a positive parameter for $N_{f} \leq 16$, $\alpha_{s}\left(\mu_{2}\right)$ decreases logarithmically and goes to zero in the limit $\mu_{2} \rightarrow \infty$, as illustrated in Fig. 4, where a compilation of $\alpha_{s}$ extractions in different energy scales shows an amazing verification of the running coupling predicted by the $\beta$-function. This is the celebrated asymptotic freedom of the strong coupling that yielded the Nobel Prize of Physics in 2004 for David J. Gross, Hugh D. Politzer and Frank Wilczek. ${ }^{53,54}$ Nowadays, the world average value for the strong coupling at the $Z$-mass $\left(m_{Z} \approx 91.19 \mathrm{GeV}\right)$ recommended by the Particle Data Group is ${ }^{65}$

$$
\begin{equation*}
\alpha_{s}\left(m_{Z}\right)=0.1179 \pm 0.0010 \tag{2.52}
\end{equation*}
$$

Another interesting aspect of Eq. (2.51) is that when $\mu_{2}$ approaches the scale

$$
\begin{equation*}
\Lambda_{\mathrm{QCD}} \equiv \mu_{1} e^{-\frac{2 \pi}{\beta_{0} \alpha_{s}\left(\mu_{1}\right)}} \tag{2.53}
\end{equation*}
$$

the strong coupling diverges and perturbation theory ceases to work. The Landau pole in $\Lambda_{\mathrm{QCD}}$ is a renormalization group invariant (i.e., its derivative with respect to $\mu_{1}$ vanishes) and its numerical value is of about 200 MeV in the $\overline{\mathrm{MS}}$-scheme. ${ }^{65}$ The one-loop running coupling written in terms of $\Lambda_{\mathrm{QCD}}$ reads

$$
\begin{equation*}
\alpha_{s}(\mu)=\frac{2 \pi}{\beta_{0} \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}} . \tag{2.54}
\end{equation*}
$$

Analogously to the determination of $\beta_{0}$, the first coefficient of the mass anomalous dimension, $\gamma_{m, 0}$, is obtained with the mass renormalization constant $Z_{m}$ given in Eq. (2.41). The result is

$$
\begin{equation*}
\gamma_{m, 0}=6 C_{F} . \tag{2.55}
\end{equation*}
$$

Taking the ratio between the definitions of $\gamma_{m}$ and the $\beta$-function in the physical limit and integrating on both sides, the running of quark masses is derived:

$$
\begin{equation*}
m\left(\mu_{2}\right)=m\left(\mu_{1}\right) \exp \int_{\alpha_{s}\left(\mu_{1}\right)}^{\alpha_{s}\left(\mu_{2}\right)} \mathrm{d} \ln \alpha_{s} \frac{\gamma_{m}\left(\alpha_{s}\right)}{2 \beta\left(\alpha_{s}\right)} . \tag{2.56}
\end{equation*}
$$

As in the case of $\alpha_{s}$, the masses also decrease with the energy.
In Appendix A we show the perturbative series in logs of both $\alpha_{s}$ and the quark masses. The formulas quoted in this appendix provide a good approximation for the runnings when $\mu_{2} \approx \mu_{1}$ and can also be used to recover logarithms in resumed perturbative series (c.f. Chap. 3).

### 2.3 Borel transform and the large- $\boldsymbol{\beta}_{0}$ limit of QCD

A better understanding about perturbative series in QCD can be obtained from the so-called large- $\beta_{0}$ limit. ${ }^{40,41}$ This limit relies first on the large- $N_{f}$ limit, where the number of quark flavours is considered a large parameter but the product $N_{f} \alpha_{s}$ is kept constant and $\mathscr{O}(1)$. Accordingly, in this power-counting, $\alpha_{s} \sim 1 / N_{f}$. Within this constraint, we consider only Feynman diagrams that generate the leading- $N_{f}$ terms in the perturbative expansion, i.e., the ones responsible for the terms in which $N_{f}$ appears as the highest power in the coefficients of $\alpha_{s}$. Additional powers of $N_{f}$ in the coefficients of $\alpha_{s}$ generates terms that are beyond $1 / N_{f}$ accuracy and thus are dropped as a first approximation. The large- $N_{f}$ limit, therefore, reduces drastically the topologies of Feynman diagrams to be calculated in perturbation theory. However, as we need to consider gluon self-interactions to obtain a reliable representation of QCD, an additional step is needed: the naive nonabelianization. ${ }^{42,43}$ In this procedure we replace the fermionic term $\left(-2 N_{f} / 3\right)$ present in the gluon propagator with fermionic bubble loop corrections by the full $\beta_{0}$ coefficient, which includes an additional colour term $\left(11 N_{c} / 3\right)$ related to gluon self-interactions, to naively take into account a set of non-abelian diagrams responsible for the asymptotic freedom of the strong coupling. The large- $\beta_{0}$ limit is thus obtained from the naive non-abelianization in the large- $N_{f}$ limit. ${ }^{V}$ When the leading- $N_{f}$ terms in the perturbative expansion come only from corrections of the gluon propagator from fermion bubble loops, one can assign

[^3]a summed value to a given observable in large- $\beta_{0}$ through a Borel representation, which is essentially an inverse Laplace transform. This summed value is what we call the "true value" which the perturbative expansion of the observable should approach.

In several processes, the large- $\beta_{0}$ limit reproduces well the results obtained for observables from full QCD. This similarity can be assessed directly from the coefficients in the perturbative expansion or looking into the full amplitude. However, even when this is not the case, the large- $\beta_{0}$ limit still brings valuable information about the higher-order behaviour of the perturbative series through a detailed study of the poles in the Borel transform of the series - the so-called renormalons. Getting this information is important due to the enormous difficulty in obtaining results in the full theory when the number of loops increases. As a concrete example, the recently calculated five-loop coefficient of the QCD $\beta$-function, $\beta_{4}$, required the computation of about one and a half million Feynman diagrams ${ }^{61}$ and had a time-span of almost 19 years from the previous four-loop result. ${ }^{66}$

In the current correlators that we calculate in this work, extra renormalizations are needed beyond the renormalization of the fermionic bubble loops in the gluon propagator, making the mathematical formulation of the large- $\beta_{0}$ limit more intricate. Therefore, here we focus on the more general aspects of divergent series, Borel transform and renormalons, and leave to Chap. 4 the more detailed derivation of the large- $\beta_{0}$ limit.

### 2.3.1 Asymptotic series and the Borel transform

Since the outstanding work by Dyson in $1952,{ }^{67}$ it is well known that perturbative series in realistic Quantum Field Theories, like QED and QCD, are divergent and, at best, asymptotic. The argument used by Dyson in the context of QED is based on a vacuum instability produced by an analytic continuation of the electromagnetic coupling to the negative axis. This instability is the cornerstone that makes the perturbative series have zero radius of convergence. With the enormous progress in the subsequent years to obtain higher-order contributions in simplistic scalar Quantum Field Theories, such as $\lambda \phi^{3}$ and $\lambda \phi^{4}$, Dyson's argument could be verified in practice from explicit computations of higher-order diagrams. ${ }^{68}$

Within the current techniques available in Quantum Field Theories, interactions between particles described by the SM are only known when the elementary coupling parameters are weak, through a perturbative expansion in the renormalized couplings. The knowledge of a given observable $R(\alpha)$ is thus limited to series as ${ }^{40}$

$$
\begin{equation*}
R(\alpha)=\sum_{n=0}^{\infty} r_{n} \alpha^{n+1} \tag{2.57}
\end{equation*}
$$

where $\alpha$ is a renormalized coupling. The divergence of the perturbative series is encoded in the asymptotic limit of the coefficients $r_{n}$ when $n$ goes to infinity,

$$
\begin{equation*}
r_{n} \stackrel{n \rightarrow \infty}{\sim} K a^{n} n^{b} n! \tag{2.58}
\end{equation*}
$$

for constants $K, a, b .{ }^{40}$ Evidently, Eq. (2.57) should not even be understood as an equality in the strict sense, and its real meaning is far from being obvious. Our best hope about the series expansion in the LHS of Eq. (2.57) is that it is asymptotic to the true quantity $R(\alpha)$ before diverging. By asymptotic, we mean that there are numbers $K_{N}$ such that

$$
\begin{equation*}
\left|R(\alpha)-\sum_{n=0}^{N} r_{n} \alpha^{n+1}\right|<K_{N+1} \alpha^{N+2} \tag{2.59}
\end{equation*}
$$

for all $\alpha$ in a region of the complex $\alpha$-plane. From the pattern of the coefficients $r_{n}$ given in Eq. (2.58) we identify $K_{N} \propto a^{N} N^{b} N$ !. The truncation error follows the same pattern, thus the precision of the series expansion to describe $R(\alpha)$ increases until it reaches an order

$$
\begin{equation*}
N_{*} \sim \frac{1}{|a| \alpha} \tag{2.60}
\end{equation*}
$$

beyond which no improvements are seen and the series starts to diverge.
An important question about Eq. (2.57) is how we actually sum the divergent series in the LHS. In principle there are many ways to deal with divergent series, but the most convenient to our case is based on the Borel method inspired in the integral representation of the factorial. The Borel method softens the factorial behaviour of the coefficients $r_{n}$ at large $n$. To demonstrate this, let us first insert a $n$ ! in both the numerator and denominator of Eq. (2.57):

$$
\begin{equation*}
R(\alpha)=\sum_{n=0}^{\infty} n!\frac{r_{n}}{n!} \alpha^{n+1} . \tag{2.61}
\end{equation*}
$$

Now we write the $n$ ! in the numerator in the integral representation

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} t e^{-t} t^{n}=n! \tag{2.62}
\end{equation*}
$$

and invert the order of the summation with the integral to obtain

$$
\begin{equation*}
R(\alpha)=\alpha \int_{0}^{\infty} \mathrm{d} t e^{-t} \sum_{n=0}^{\infty}\left(\frac{r_{n}}{n!} t^{n} \alpha^{n}\right) \tag{2.63}
\end{equation*}
$$

Within the change of variable $t=u / \alpha$ we arrive at the Borel representation

$$
\begin{equation*}
\bar{R}(\alpha)=\int_{0}^{\infty} \mathrm{d} u e^{-u / \alpha} B[R](u), \tag{2.64}
\end{equation*}
$$

where

$$
\begin{equation*}
B[R](u)=\sum_{n=0}^{\infty} r_{n} \frac{u^{n}}{n!} \tag{2.65}
\end{equation*}
$$

is the Borel transform of the quantity $R(\alpha)$ and the integral $\bar{R}(\alpha)$ is a quantity whose perturbative expansion is the same as for $R(\alpha)$. In Eq. (2.65) we see that the coefficients of $B[R](u)$ are factorially suppressed with respect to $r_{n}$. Therefore, we expect the series of the Borel transform to be better behaved than the one of $R(\alpha)$.

The integral representation for $\bar{R}(\alpha)$ is ill-defined if the Borel transform contains singularities in the real positive axis, which is quite generally the case since the divergent behaviour of the expansion of $R(\alpha)$ is translated into singularities (in the negative and positive axis) in the Borel transform. In order to obtain a finite number to the integral in Eq. (2.64) the contour must be distorted to avoid the singularities in the real axis under a certain prescription. This makes the integral acquire an ambiguous imaginary part that is related to ambiguities in the Borel sum. Throughout this work we will use the Principal Value prescription to obtain the central value and the ambiguity of the Borel sum; a detailed implementation of this calculation for an arbitrary number of poles and multiplicities in the Borel transform is described in Appendix B.

The true value of a physical quantity, however, must be unambiguous. In fact, a one-toone correspondence between the so-called renormalons and the Wilsonean Operator Product Expansion (OPE), responsible to encode non-perturbative contributions of observables, is conjectured to ensure a cancellation of the ambiguity in the Borel sum ${ }^{40}$. VI

### 2.3.2 Renormalons

Renormalon is the name given to singularities that may appear in the Borel transform $B[R](u)$ of a given quantity $R(\alpha)$. We separate the renormalon singularities of the Borel transform in two classes: we call infrared (IR) renormalons the divergences due to the lowenergy region in loop subgraphs, and we call ultraviolet (UV) renormalons the divergences due to the high-energy region. In QCD, while usually IR renormalons are located at the positive $u$-axis, UV renormalons are usually located at the negative $u$-axis.

The higher-order behaviour of the perturbative expansion of $R(\alpha)$ is determined by the closest renormalon to the origin - also called leading renormalon -, but the asymptotic behaviour can be postponed in the perturbative expansion depending on the structure of the subleading renormalons. To show this, consider that the Borel transform of $R(\alpha)$ is a function with two singularities of multiplicity $\gamma$ located at $u=p_{1}$ and $u=p_{2}$, with $\left|p_{2}\right|>\left|p_{1}\right|$, such that

$$
\begin{equation*}
B[R](u)=\frac{A_{1}}{\left(u-p_{1}\right)^{\gamma}}+\frac{A_{2}}{\left(u-p_{2}\right)^{\gamma}} \tag{2.66}
\end{equation*}
$$

for arbitrary constants $A_{1}, A_{2}$. Expanding $B[R](u)$ around $u=0$ results in

[^4]

Figure 5 - Higher-order behaviour of the asymptotic series dictated by the coefficients given in Eq. (2.68). On the left-hand panel the residues $A_{i}$ are set to unity and on the right-hand panel the residue $A_{2}$ is twenty times larger than $A_{1}$. The series were normalized by their Borel sum to make the comparison easier.

Source: By the author.

$$
\begin{equation*}
B[R](u)=\sum_{n=0}^{\infty}\left[\frac{(\gamma+n-1)!(-1)^{\gamma}}{(\gamma-1)!}\left(\frac{A_{1}}{\left(p_{1}\right)^{\gamma+n}}+\frac{A_{2}}{\left(p_{2}\right)^{\gamma+n}}\right) \frac{u^{n}}{n!}\right], \tag{2.67}
\end{equation*}
$$

from which we can identify the coefficients $r_{n}$ of $R(\alpha)$ (c.f. Eq.(2.57)) as

$$
\begin{equation*}
r_{n}=(-1)^{\gamma} \frac{(\gamma+n-1)!}{(\gamma-1)!}\left(\frac{A_{1}}{\left(p_{1}\right)^{\gamma+n}}+\frac{A_{2}}{\left(p_{2}\right)^{\gamma+n}}\right) . \tag{2.68}
\end{equation*}
$$

Some important remarks can be extracted from the above expression. Due to the factor $A_{i} /\left(p_{i}\right)^{\gamma+n}$, (i) renormalons located at the negative axis contribute with a sign-alternating perturbative series, while renormalons located at the positive axis contribute with a fixed-sign perturbative series; (ii) the behaviour is dominated by the closest renormalon to the origin; (iii) the higher-order behaviour of the perturbative series is postponed if the numerators $A_{i}$ of the more distant renormalons suppress the contribution from the leading renormalon.

These remarks about the relation between the renormalon structure of the Borel transform and the perturbative behaviour of the true series can be visualized in Fig. 5. We considered the Borel transform in Eq. (2.66) keeping $A_{1}=1$ but changing the values of $A_{2}$. The Borel sum were calculated using the Principal Value prescription detailed in Appendix B - in both situations the ambiguity of the Borel sum was completely negligible - and the coefficients of the perturbative expansion of $R(\alpha)$ were calculated with Eq. (2.68). We used $\alpha=0.15$ and considered only simple-poles (i.e., $\gamma=1$ ) located at


Figure 6 - Run-away behaviour of a perturbative series dominated by the IR renormalon and with a large coupling parameter. The gray band represents the ambiguity of the Borel sum. The series is normalized by the Borel sum.

Source: By the author.
$p_{1}=-1$ and $p_{2}=2$. On the left-hand panel of Fig. 5 the residues of the singularities are equal and the sign-alternating behaviour determined by the leading renormalon at $u=-1$ is immediately achieved at low orders. Making the residue $A_{2}$ twenty times larger than $A_{1}$ suppresses the sign-alternating behaviour, as it is depicted on the right-hand panel of Fig. 5. Increasing the importance of IR renormalons seems to make the perturbative series more "convergent", but this might be dangerous in realistic situations for several reasons.

Perturbative series highly dominated by IR renormalons could need more orders in the expansion to become closer to the expected value. Given the enormous difficulty to do multiloop calculations, realistic perturbative series are only known to a few orders in the coupling parameter, and thus the truncated series might not give a reliable description of the observable.

If the IR residue is much larger than the UV residue, higher values of the coupling parameter might lead to an enhancement of the leading IR renormalon that postpone the sign-alternating behaviour in a way that the perturbative series crosses the true value at a certain order and goes away, without a plateau between the series and the true value. ${ }^{69,70}$ This phenomenon can be visualized in Fig. 6, where we maintained $A_{2}=20 A_{1}$, but doubled the coupling to $\alpha=0.3$.

In QCD, IR renormalons generate ambiguities in the Borel sum to be compensated by non-perturbative operators in the OPE. These operators are accompanied by power corrections of order $\left(\frac{\Lambda_{\mathrm{QCD}}}{z}\right)^{p}$, where $\Lambda_{\mathrm{QCD}}$ is the typical scale of the theory from which the perturbative expansion ceases to work, $z$ can be either the total energy of the process or a mass and $p$ is the dimension of the operator. Higher contributions from IR renormalons
are thus related to higher contributions from non-perturbative effects of the theory; in this situation, a simple expansion in the coupling parameter is not sufficient to have a reliable description of the observable.

In order to get a better understanding about the physical meaning of IR and UV renormalons, let us present the canonical example based on Ref. ${ }^{68}$ Consider the two-point vector correlation function $\Pi^{V}\left(Q^{2}\right)$ in the massless limit given by

$$
\begin{equation*}
\left(q^{2} g_{\mu \nu}-q_{\mu} q_{\nu}\right) \Pi^{V}\left(Q^{2}\right)=-i \int \mathrm{~d} x e^{i q x}\langle\Omega| T j_{\mu}^{V}(x) j_{\nu}^{V, \dagger}(0)|\Omega\rangle \tag{2.69}
\end{equation*}
$$

with Euclidean momenta $Q^{2}=-q^{2}$ and $j_{\mu}^{V}(x)=\bar{q}(x) \gamma_{\mu} q(x)$. The correlator $\Pi^{V}\left(Q^{2}\right)$ by itself is renormalization scheme dependent and therefore is not a physical quantity. This scheme dependence vanishes when one takes derivatives of $\Pi^{V}\left(Q^{2}\right)$ with respect to the Euclidean momenta $Q^{2}$, so we work with the Adler function

$$
\begin{equation*}
D\left(Q^{2}\right)=-4 \pi^{2} Q^{2} \frac{\mathrm{~d} \Pi^{V}\left(Q^{2}\right)}{\mathrm{d} Q^{2}} \tag{2.70}
\end{equation*}
$$

normalized to 1 at leading order. To identify the renormalons we can focus only on the diagrams depicted in Fig. 7. These diagrams, which are basically composed by the insertion of $n$ fermionic bubble loops in the gluon propagator, give rise to the coefficients of order $N_{f}^{n} \alpha_{s}^{n+1}$ in the perturbative expansion of the Adler function. Considering only this class of Feynman diagrams is the basis of the large- $N_{f}$ limit for the Adler function, and after replacing the fermionic contribution that appears in the corrections of the gluon propagator from fermion bubbles by the full $\beta_{0}$ coefficient we obtain the result in the large- $\beta_{0}$ limit. For illustration purposes we will consider only the simplified expression

$$
\begin{equation*}
D\left(Q^{2}\right) \propto Q^{2} \int \mathrm{~d} k^{2} \frac{k^{2} \alpha_{s}\left(k^{2}\right)}{\left(k^{2}+Q^{2}\right)^{3}} \tag{2.71}
\end{equation*}
$$

which coincides with the exact result ${ }^{71}$ in the limits $k^{2} \gg Q^{2}$ and $k^{2} \ll Q^{2}$. In this simplified expression, related to a two-loop Feynman diagram, the integral over one loopmomentum, as well as the angular dependence of the remaining internal momentum, was already carried out. Moreover, the strong coupling $\alpha_{s}$ here is conveniently parametrized in terms of the squared momenta $k^{2}$. The IR and UV regimes of the loop integral, given by $k^{2} \ll Q^{2}$ and $k^{2} \gg Q^{2}$ respectively, capture the divergent behaviour of the perturbation theory. The IR domain is particularly important in QCD as it represents the low-energy region in which the strong coupling blows up. To analyse these regimes we write $\alpha_{s}\left(k^{2}\right)$ in terms of $\alpha_{s}\left(Q^{2}\right)$ using Eq. (2.51) and then expand in $k^{2}$.

In the IR domain the Adler function can be written as


Figure 7 - Diagrams with fermionic bubble loop corrections in the gluon propagator for the computation of the Adler function.

Source: By the author.

$$
\begin{equation*}
D\left(Q^{2}\right) \propto \frac{\alpha_{s}^{Q}}{Q^{4}} \sum_{n=0}^{\infty}\left(\frac{\beta_{0} \alpha_{s}^{Q}}{4 \pi}\right) \int \mathrm{d} k^{2} k^{2}\left(\ln \frac{Q^{2}}{k^{2}}\right)^{n} \tag{2.72}
\end{equation*}
$$

where $\alpha_{s}^{Q} \equiv \alpha_{s}\left(Q^{2}\right)$. With the suitable change of variable $k^{2} \rightarrow Q^{2} e^{-y / 2}$ the remaining integral, up to constants, becomes

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} y y^{n} e^{-y} \tag{2.73}
\end{equation*}
$$

which is the integral representation of the $\Gamma$-function. For positive integer $n$ the result is simply $n!$. The Adler function in the IR regime of the loop variable is thus given by

$$
\begin{equation*}
D\left(Q^{2}\right) \propto \frac{\alpha_{s}^{Q}}{2} \sum_{n=0}^{\infty}\left(\frac{\beta_{0} \alpha_{s}^{Q}}{8 \pi}\right)^{n} n!, \quad k^{2} \ll Q^{2} . \tag{2.74}
\end{equation*}
$$

In an analogous fashion one can derive the Adler function in the UV regime of the loop variable, whose perturbative expansion is given by

$$
\begin{equation*}
D\left(Q^{2}\right) \propto \alpha_{s}^{Q} \sum_{n=0}^{\infty}\left(\frac{\beta_{0} \alpha_{s}^{Q}}{4 \pi}\right)^{n}(-1)^{n} n!, \quad k^{2} \gg Q^{2} . \tag{2.75}
\end{equation*}
$$

In both Eqs. (2.74) and (2.75) the factorial behaviour of the series expansion is manifest. The sum is thus asymptotically divergent and therefore we consider the Borel prescription discussed in the previous section to sum the series,

$$
\begin{equation*}
\bar{D}\left(Q^{2}\right)=\int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} B[D](u), \tag{2.76}
\end{equation*}
$$

where we conveniently parametrized the Borel integral in terms of $\beta \equiv \beta_{0} \alpha_{s}^{Q} / 4 \pi$. The Borel transform $B[D](u)$ is thus calculated from the coefficients in the perturbative expansion of $D\left(Q^{2}\right)$ in $\beta_{0} \alpha_{s} / 4 \pi$. The result for $B[D](u)$ reads

$$
\begin{align*}
& B[D](u) \propto \sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^{n}=\frac{1}{1-\frac{u}{2}}, \quad \text { IR regime },  \tag{2.77}\\
& B[D](u) \propto \sum_{n=0}^{\infty}(-u)^{n}=\frac{1}{1+u}, \quad \text { UV regime. } \tag{2.78}
\end{align*}
$$

The factorial divergence of the perturbative series of the Adler function manifests itself as singularities in the Borel plane. The divergence from the low-energy behaviour was translated into a pole in the positive axis of the Borel plane, while the divergence from the high-energy behaviour is encoded at the pole in the negative axis. One can also notice from the Borel transform that the leading renormalon is located at $u=-1$, so at higher orders the perturbative expansion of the Adler function is sign alternating.

The renormalon structure of the Borel transform of the Adler function is more involved if one use the exact result for the Adler function ${ }^{71}$ instead of the simplified expression given in Eq. (2.71). The complete Borel transform $B[D](u)$ in large- $\beta_{0}$ contain an infinity number of IR renormalons located at positive integers starting at $u=2$ and an infinity number of UV renormalons located at negative integers starting at $u=-1$.

In full QCD the singularities are no longer poles; they become branch cuts starting at branch points at the same location of the poles obtained in large- $\beta_{0}$. This statement follows from the imposition that ambiguities of the Borel transform in the full theory, which we do not know exactly, should also cancel the ambiguities generated by higher dimensional operators in the OPE. ${ }^{40,72}$ This requires a modification on the structure of the Borel transform obtained in large- $\beta_{0}$. In particular, the exponents in the singular expansion of the Borel transform in the full theory become non-integer numbers, which, in turn, transform the poles into branch cuts. ${ }^{72}$ As a concrete example, it is shown in Eq. (2.77) that the exponent of the leading IR renormalon of the Borel transform of the Adler function in large- $\beta_{0}$ equals unity, so we a have a simple-pole located at $u=2$. In full QCD this exponent is no longer 1, it is a non-integer number that depends on the two-loop coefficient of the $\beta$-function and on the anomalous dimension of the 4 -dimensional gluon condensate operator. ${ }^{72}$ Therefore, the singularity becomes a branch cut starting at $u=2$. Although it is not essential for our work, we show below the structure of the singularity at $u=2$ in the Borel transform of the full theory:

$$
\begin{equation*}
B[D](u) \propto \frac{1}{(2-u)^{\xi}}[1+\mathscr{O}[(2-u)]] \tag{2.79}
\end{equation*}
$$

where at next-to-leading order in perturbation theory

$$
\begin{equation*}
\xi=1+2 \frac{\beta_{1}}{\beta_{0}^{2}}-\frac{\gamma_{G, 0}}{2 \beta_{0}} \tag{2.80}
\end{equation*}
$$

and $\gamma_{G, 0}=0$ is the first coefficient of the anomalous dimension of the gluon condensate
operator. ${ }^{72}$
Having studied the basics of QCD, renormalons and the Borel transform, we turn to an introduction of the current correlators that are the main objects of this work.

## 3 CURRENT CORRELATORS AND THE ONE-LOOP MOMENTS

In this chapter we introduce the current correlators that are used to construct the moments $M_{q, n}^{\delta}$ in the theoretical framework. A detailed derivation of the leading order result employing some of the most convenient strategies to perform the small-momentum asymptotic expansion of loop integrals is given.

Current-current correlators can describe a large set of observables. The vector ( $V$ ) correlator is the core of hadronic electroproduction, ${ }^{73}$ the scalar $(S)$ correlator is used in Higgs physics, ${ }^{74}$ some extensions of the Higgs sector beyond the Standard Model make use of the pseudo-scalar $(P)$ correlator, ${ }^{74}$ and the axial-vector $(A)$ correlator, together with the vector correlator, is an important tool in $Z$-decays, ${ }^{73}$ to mention a few applications. With the increase in precision in lattice simulations, the pseudo-scalar correlator has also proven to be an important tool for precise determinations of the charm-quark mass and of the strong coupling. ${ }^{33-38}$

The current correlators $\Pi^{\delta}(s)$ of our interest are defined as

$$
\begin{equation*}
\left(q^{2} g_{\mu \nu}-q_{\mu} q_{\nu}\right) \Pi^{\delta}(s)+q_{\mu} q_{\nu} \Pi_{L}^{\delta}(s)=-i \int \mathrm{~d} x e^{i q x}\langle\Omega| T j_{\mu}^{\delta}(x) j_{\nu}^{\delta, \dagger}(0)|\Omega\rangle \tag{3.1}
\end{equation*}
$$

for $\delta=V, A$, whereas

$$
\begin{equation*}
\Pi^{\delta}(s)=i \int \mathrm{~d} x e^{i q x}\langle\Omega| T j^{\delta}(x) j^{\delta, \dagger}(0)|\Omega\rangle \tag{3.2}
\end{equation*}
$$

for $\delta=S, P$. In the above equations $s=q^{2}$ and the bilinear quark currents are defined as

$$
\begin{align*}
j_{\mu}^{V}(x) & =\bar{q}(x) \gamma_{\mu} q(x), & j_{\mu}^{A}(x) & =\bar{q}(x) \gamma_{\mu} \gamma_{5} q(x) \\
j^{S}(x) & =2 m_{q} \bar{q}(x) q(x), \quad \text { and } & j^{P}(x) & =2 \operatorname{im}_{q} \bar{q}(x) \gamma_{5} q(x) . \tag{3.3}
\end{align*}
$$

The longitudinal contribution $\Pi_{L}^{V}(s)$ of the vector correlator is zero by means of Ward's identity, ${ }^{55}$ while $\Pi_{L}^{A}(s)$ can be obtained by applying the projector $q_{\mu} q_{\nu}$. The mass factor $2 m_{q}$, which in this context is the bare quark mass, in the scalar and pseudo-scalar correlators is needed to ensure renormalization group invariance. ${ }^{73}$

One of the most powerful methods to extract charm- and bottom-quark masses is based on the use of QCD sum-rules ${ }^{16,17}$ with the vector moments $M_{q, n}^{V}$ defined as

$$
\begin{equation*}
M_{q, n}^{V}=\int_{s_{\mathrm{th}}}^{\infty} \mathrm{d} s \frac{R_{q \bar{q}}(s)}{s^{n+1}} \tag{3.4}
\end{equation*}
$$

where $s_{\text {th }}$ is the threshold energy for multi-particle state production and $R_{q \bar{q}}(s)$ (with
$q=c, b)$ is the experimentally accessible normalized cross-section

$$
\begin{equation*}
R_{q \bar{q}}(s)=\frac{\sigma\left(e^{+} e^{-} \rightarrow q \bar{q}\right)}{\sigma\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)_{\mathrm{LO}}}=\frac{\sigma\left(e^{+} e^{-} \rightarrow q \bar{q}\right)}{4 \pi \alpha_{e m}^{2}(s) / 3 s}, \tag{3.5}
\end{equation*}
$$

where $\sigma\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)_{\mathrm{LO}}$ is the leading order cross-section for muon production in $e^{+} e^{-}$ annihilation and $\alpha_{e m}(s)$ is the effective electromagnetic coupling constant. With the use of the theoretical prediction of $R_{q \bar{q}}(s)$ derived from the optical theorem, ${ }^{55}$

$$
\begin{equation*}
R_{q \bar{q}}(s)=12 \pi Q_{q}^{2} \operatorname{Im}\left[\Pi_{q}^{V}(s+i 0)\right], \tag{3.6}
\end{equation*}
$$

with $Q_{q}$ being the quark electric charge, and dispersive relations of complex analysis, the vector moments $M_{q, n}^{V}$ can be written in terms of the Taylor expansion of $\Pi_{q}^{V}(s)$ around $s=0$. As the correlator $\Pi_{q}^{V}(s)$ satisfies the Schwarz reflection principle $\Pi_{q}^{V}(s+i 0)=\left[\Pi_{q}^{V}(s-i 0)\right]^{*}$ for all $s$ in the complex $s$-plane,,${ }^{55,75}$ the relation

$$
\begin{equation*}
2 i \operatorname{Im}\left[\Pi_{q}^{V}(s+i 0)\right]=\Pi_{q}^{V}(s+i 0)-\Pi_{q}^{V}(s-i 0) \tag{3.7}
\end{equation*}
$$

holds. Thus, with Eq. (3.6) in Eq. (3.4) and using Cauchy's theorem one can write the vector moments as

$$
\begin{equation*}
M_{q, n}^{V}=12 \pi^{2} Q_{q}^{2} \frac{1}{2 \pi i} \oint_{\mathscr{C}} \mathrm{d} s \frac{\Pi_{q}^{V}(s)}{s^{n+1}} \tag{3.8}
\end{equation*}
$$

where $\mathscr{C}$ is the contour depicted in Fig. 8. Since $\Pi_{q}^{V}(s)$ is an analytic function inside the contour $\mathscr{C}$, Cauchy's theorem is used again to write

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{\mathscr{C}} \mathrm{d} s \frac{\Pi_{q}^{V}(s)}{s^{n+1}}=\left.\frac{1}{n!} \frac{\mathrm{d}^{n}}{\mathrm{~d} s^{n}} \Pi_{q}^{V}(s)\right|_{s=0} \tag{3.9}
\end{equation*}
$$

Therefore, the vector moments in terms of the vector correlator defined in Eq. (3.1) are given by

$$
\begin{equation*}
M_{q, n}^{V}=\int_{s_{\mathrm{th}}}^{\infty} \mathrm{d} s \frac{R_{q \bar{q}}(s)}{s^{n+1}}=\left.\frac{12 \pi^{2} Q_{q}^{2}}{n!} \frac{\mathrm{d}^{n}}{\mathrm{~d} s^{n}} \Pi_{q}^{V}(s)\right|_{s=0} \tag{3.10}
\end{equation*}
$$

The integral over $s$ in the equation above is understood to be evaluated with experimental data while the RHS is to be calculated in theory. Comparing both prescriptions to represent $M_{q, n}^{V}$ through a statistical analysis one can extract the mass of the quark $q$. It is also important to comment that the derivatives at zero energy in Eq. (3.10) are not in conflict with perturbative QCD, as the typical scale of the Feynman diagrams, in this case, is the heavy-quark mass.

We generalize the definition of the moments beyond the vector current as

$$
\begin{equation*}
M_{q, n}^{\delta}=\left.\frac{12 \pi^{2} Q_{q}^{2}}{n!} \frac{\mathrm{d}^{n}}{\mathrm{~d} s^{n}} \Pi_{q}^{\delta}(s)\right|_{s=0} \tag{3.11}
\end{equation*}
$$



Figure 8 - Contour of integration $\mathscr{C}$ in the complex $s$-plane.
Source: By the author.

The pseudo-scalar moments, although not available from experimental data, are of particular importance due to their high-precision determinations from lattice QCD simulations. ${ }^{33-37}$ The axial-vector and vector moments, at the date of this work, couldn't be determined with reasonable accuracy from the lattice yet. ${ }^{34}$ As will be discussed later in this work, our analysis will be restricted to the physical moments, i.e., those that do not require an additional, scheme-dependent, subtraction beyond coupling and mass renormalization. For the vector and axial-vector correlators this means $n \geq 1$, while for the pseudo-scalar and scalar correlators this means $n \geq 0$. In addition, our phenomenological analysis should be restricted to the first four physical moments. This constraint comes from a break down of the standard perturbative QCD supplemented by non-perturbative corrections encoded in the OPE condensates at large- $n$. For high values of $n$ the typical energy scale of the moments is of order $m_{q} / n^{76}$ and approaches the Landau pole in $\Lambda_{\mathrm{QCD}}$ at $n \gtrsim 4$, which means perturbative QCD ceases to work. Moreover, for large values of $n$, on the experimental side of the sum-rule of Eq. (3.10) the moments become dominated by resonance contributions, and therefore a treatment in the framework of non-relativistic $\mathrm{QCD}^{77}$ becomes mandatory. ${ }^{78}$

### 3.1 Experimental determination of the vector moments

In this section we give an overview, for the interested reader, about the determination of the experimental values of the vector moments $M_{q, n}^{V}$. From the sum-rule of Eq. (3.10) with the knowledge of the experimental values of these moments, as well as the covariance matrix between them, we are able to extract quark masses or $\alpha_{s}$ from the comparison of the experimental results with the theory prediction through a careful statistical analysis.

The experimental values of $M_{q, n}^{V}$, as given in Eq. (3.4), are determined by the sum of three main contributions, ${ }^{31,32}$

$$
\begin{equation*}
M_{q, n}^{V}=M_{q, n}^{V, \text { res }}+M_{q, n}^{V, \text { data }}+M_{q, n}^{V, \text { pert }} . \tag{3.12}
\end{equation*}
$$

The first contribution, $M_{q, n}^{V, \text { res }}$, comes from narrow quarkonium resonances below the open quark-antiquark threshold, which is located at $\sqrt{s} \approx 3.73 \mathrm{GeV}(\sqrt{s} \approx 10.54 \mathrm{GeV})$ for the charm (bottom) moments. The second contribution, $M_{q, n}^{V, \text { data }}$, is determined by the region where we use experimental data to evaluate part of the integral in Eq. (3.4). The last contribution, $M_{q, n}^{V, \text { pert }}$, is related to the high-energy region where no experimental data is available and perturbation theory must be used as an estimate. Since the integrand in Eq. (3.4) is suppressed with weights of the type $1 / s^{n+1}, M_{q, n}^{V, \text { pert }}$ gives only a small contribution to the total value of $M_{q, n}^{V}$ and tends to zero as $n$ grows.

Below we give an overview of how the different contributions to $M_{q, n}^{V}$ are obtained. The main references for this section are Refs. ${ }^{31,32}$

### 3.1.1 Narrow resonances

The contribution from narrow resonances to the moments are obtained with the use of the Breit-Wigner shape for the $R$-ratio in the narrow width approximation $\left(\Gamma_{e e, X} / M_{X} \rightarrow\right.$ $0),{ }^{79}$

$$
\begin{equation*}
R^{\mathrm{BW}}(s)=\frac{9 M_{X}^{2} \Gamma_{e e, X}}{\alpha_{e m}^{2}\left(M_{X}^{2}\right)}\left[\frac{\pi}{M_{X}} \delta\left(s-M_{X}^{2}\right)\right] \tag{3.13}
\end{equation*}
$$

where $\Gamma_{e e, X}$ and $M_{X}$ are the electronic decay width and mass of the resonance $X$, respectively. In the above equation $\delta$ is the Dirac delta distribution. With the use of Eq. (3.13) in Eq. (3.4) we obtain the resonance contribution to the vector moments as

$$
\begin{equation*}
M_{q, n}^{V, \text { res }}=\sum_{X} \frac{9 \pi \Gamma_{e e, X}}{\alpha_{e m}^{2}\left(M_{X}^{2}\right) M_{X}^{2 n+1}}, \tag{3.14}
\end{equation*}
$$

where we sum over all the narrow resonances contributions. In the case of the charm moments we must consider the charmounium states $J / \psi$ and $\psi^{\prime}$, while for the bottom moments we need the contribution from the bottonium resonances $\Upsilon(1 S), \Upsilon(2 S), \Upsilon(3 S)$ and $\Upsilon(4 S)$.

### 3.1.2 Data contribution

For the charm moments the data contribution covers the energy range between 3.73 GeV and 10.538 GeV and makes use of a large set of experimental data for the total hadronic cross section $\sigma\left(e^{+} e^{-} \rightarrow\right.$ hadrons) extensively measured by the collaborations BES, ${ }^{80-85}$ CrystalBall,,${ }^{86,87}$ CLEO, ${ }^{88-90}$ MD1, ${ }^{91}$ PLUTO $^{92}$ and MARK II. ${ }^{93}$ In order to obtain the most precise value for the data contribution using all the experimental information available, one should first combine all datasets into a single-one using a method of data combination to reduce errors. In Ref. ${ }^{31}$ the authors used an algorithm of data combination advocated


Figure 9 - Typical experimental data for the total hadronic $R_{\mathrm{tot}}(s),{ }^{8-93}$ from which we subtract the $u d s$ background to obtain the exclusive contribution $R_{c \bar{c}}(s)$.

Source: By the author.
in Ref. ${ }^{94}$ in the context of the dispersive evaluation of the anomalous magnetic moment of the muon, $(g-2)_{\mu}$. The general strategy for data combination relies on first assigning consecutive data points into clusters. The energy of each cluster is defined as the weighted average of the energies of the data points within that given cluster, and the values of the cross sections at the clusters energies are determined through a minimization procedure that takes into account all the correlations between the experimental data. As we are interested only on the $c \bar{c}$ or $b \bar{b}$ production in $e^{+} e^{-}$annihilation to obtain the experimental value of $M_{q, n}^{V}$, in the case of the charm vector moments the non-charm background must be subtracted from the fully inclusive experimental data for $\sigma\left(e^{+} e^{-} \rightarrow\right.$ hadrons $)$. This amounts to subtract the $u d s$ and the real and virtual secondary $c \bar{c}$ radiation backgrounds, which is often done with perturbation theory. ${ }^{31}$ Typical experimental data for $R_{\mathrm{tot}}(s)$ (c.f. Eq. (2.2)), from which we subtract the $u d s$ background to obtain the exclusive contribution $R_{c \bar{c}}(s)$ to be used in the determination of the experimental values of $M_{c, n}^{V}$, are displayed in Fig. 9.

For the bottom moments the data contribution covers the energy range between 10.62 GeV and 11.2062 GeV and is based only on the high-statistics BaBar data for $\sigma\left(e^{+} e^{-} \rightarrow b \bar{b}\right) .{ }^{95}$ Since we only consider a single dataset to evaluate the experimental value of $M_{b, n}^{V}$, no data combination is required. Moreover, as BaBar already measured the exclusive cross section of $b \bar{b}$ production (thanks to the $b$-tagging procedure ${ }^{95}$ ), there is no need to subtract any background. However, BaBar data still need to be subtracted for the $\Upsilon(4 S)$ radiative tail and corrected for initial-state radiation and vacuum polarization effects. ${ }^{32}$ Typical experimental data to be used in the determination of the experimental values of $M_{b, n}^{V}$ are displayed in Fig. 10.

With the data points $\left(s_{i}, R_{q \bar{q}}\left(s_{i}\right)\right)$ for the exclusive contribution of $q \bar{q}$ production in


Figure 10 - Corrected BaBar experimental data for $R_{b \bar{b}}(s) .{ }^{95}$
Source: By the author.
$e^{+} e^{-}$annihilation in hands, it is then straightforward to evaluate the integral in Eq. (3.4) with experimental data using trapezoidal integration.

### 3.1.3 Continuum region

The integral for the vector moments in Eq. (3.4) must be done all the way to the infinity. Therefore, above the region where no experimental data is available one should use the theory prediction for $R_{q \bar{q}}(s)$. The perturbative QCD series to be used includes the non-singlet massless quark contribution supplemented by quark mass corrections of the type $\left(m_{q}^{2} / s\right)$. The corresponding formulas can be found in Refs. ${ }^{31,32}$

### 3.1.4 Total contribution and correlations

After computing the three main contributions to the experimental vector moments described above, the final values for $M_{q, n}^{V}$ are obtained with Eq. (3.12). The uncertainties and the correlations between these moments are determined by the covariance matrix $C_{q}^{V}\left(n, n^{\prime}\right)$ that is given by

$$
\begin{equation*}
C_{q}^{V}\left(n, n^{\prime}\right)=\sum_{i, j}\left(\frac{\partial M_{q, n}^{V}}{\partial p_{i}}\right)\left(\frac{\partial M_{q, n^{\prime}}^{V}}{\partial p_{j}}\right) V^{p}(i, j), \tag{3.15}
\end{equation*}
$$

where $p$ is a vector with entries $p=\left(\left\{R_{q \bar{q}}\left(s_{i}\right)\right\},\left\{\Gamma_{e e, X}\right\},\left\{M_{X}\right\}\right)$ and $V^{p}(i, j)$ is the covariance matrix that includes the covariances of the experimental data, as well as of the electronic decay widths and masses of the resonances. ${ }^{\text {VII }}$ With the knowledge of $C_{q}^{V}\left(n, n^{\prime}\right)$

[^5]we can obtain the experimental error of any combination of moments (c.f. Sec. 5.4).
It is important to point out that as $n$ grows the final values for $M_{q, n}^{V}$ become dominated by the resonance contribution which, in turn, has much smaller errors than $M_{q, n}^{V, \text { data }}$. However, since for large- $n$ the main contribution comes from bound states, the theory prediction for the vector moments can no longer be described by standard perturbative QCD. In this situation, it is thus imperative to use non-relativistic $\mathrm{QCD}^{77,78}$ on the theory side. In this work, we use $n \leq 4$ and the non-relativistic regime is not explored.

We have successfully reproduced all the results for the charm and bottom vector experimental moments quoted in Refs. ${ }^{31,32}$ We will therefore directly use the results presented in these references throughout this dissertation.

Finally, the method of data combination used in the determination of the charm moments is a very general and important tool for any situation (or problem) where the use of multiple datasets from different measurements is required. For example, in Ref. ${ }^{96}$ we used a method for data combination (advocated in the works of Refs. ${ }^{97,98}$ ) in the context of $\tau$-decays to construct an improved $\tau$ vector isovector spectral function. Using our new spectral function with smaller experimental errors we could determine the strong coupling at the $\tau$ mass scale with high precision. ${ }^{96}$

### 3.2 Theoretical description of the moments

Quite generally, the perturbative expansion of the renormalized moments $M_{q, n}^{\delta}$ in $\alpha_{s}$ should be parametrized in terms of two different renormalization scales: one, $\mu_{\alpha}$, for the strong coupling and one another, $\mu_{m}$, for the mass. Alternative treatments of these scales, such as keeping $\mu_{\alpha}=\mu_{m}$ or leaving them as uncorrelated quantities, lead to significant discrepancies in the final perturbative uncertainties of quark mass extractions. ${ }^{28-32}$ In the literature, results for the coefficients in the perturbative expansion are commonly presented with resumed logarithms within the choice $\mu_{\alpha}=\mu_{m}=m_{q}\left(m_{q}\right) \equiv \bar{m}_{q}$ that should be recovered through renormalization group equations. With the logarithms resumed the perturbative expansion of $M_{q, n}^{\delta}$ reads

$$
\begin{equation*}
M_{q, n}^{\delta}=\frac{1}{\left[2 \bar{m}_{q}\right]^{2 n}} \sum_{i=0}\left[\frac{\alpha_{s}\left(\bar{m}_{q}\right)}{\pi}\right]^{i} c_{i}^{\delta,(n)} \tag{3.16}
\end{equation*}
$$

where $c_{i}^{\delta,(n)}$ are the independent (non-log) coefficients calculated in perturbative QCD. Nowadays, for the four currents the analytical expression of these coefficients are known up to $\mathscr{O}\left(\alpha_{s}^{3}\right)$ for the first three physical moments. ${ }^{18-24}$ For the vector and pseudo-scalar correlators the fourth physical moment is also known exactly, ${ }^{24,25}$ while higher moments at $\mathscr{O}\left(\alpha_{s}^{3}\right)$ have been estimated in Refs. ${ }^{99-102}$ The coefficients $c_{i}^{\delta,(n)}$ have a quark charge $Q_{q}$ and a number of flavours $N_{f}$ dependence that we are omitting for notational simplicity.

To recover the logarithms, we first set both scales equal and write

$$
\begin{equation*}
M_{q, n}^{\delta}=\frac{1}{\left[2 m_{q}\left(\mu_{m}\right)\right]^{2 n}} \sum_{i=0}\left[\frac{\alpha_{s}\left(\mu_{m}\right)}{\pi}\right]^{i} \sum_{j=0}^{i} c_{i, j}^{\delta,(n)} \ln ^{j}\left[\frac{\mu_{m}}{m_{q}\left(\mu_{m}\right)}\right], \tag{3.17}
\end{equation*}
$$

where $c_{i, 0}^{\delta,(n)} \equiv c_{i}^{\delta,(n)}$. The coefficients $c_{i, j>0}^{\delta,(n)}$ are found by imposing that Eq. (3.17) satisfies the renormalization group equation given in Eq. (2.43). Because of the $\beta$-function and the mass anomalous dimension in Eq. (2.43) a composite of summations will be generated and the $\mu_{m}$-independence should be imposed at each order in $\alpha_{s}$. This procedure allows us to find the coefficients $c_{i, j}^{\delta,(n)}$ through the recurrence relation

$$
\begin{align*}
c_{i, j}^{\delta,(n)} & =\frac{2}{j} \sum_{k=1}^{i-j} \frac{1}{4^{k}}\left[\left((i-k) \beta_{k-1}-n \gamma_{k-1}\right) c_{i-k, j-1}^{\delta,(n)}-\frac{j}{2} \gamma_{k-1} c_{i-k, j}^{\delta,(n)}\right]  \tag{3.18}\\
& +\frac{2}{j} \frac{1}{4^{i-j+1}}\left[(j-1) \beta_{i-j}-n \gamma_{i-j}\right] c_{j-1, j-1}^{\delta,(n)} .
\end{align*}
$$

(These coefficients can also be determined using the log expansions of $\alpha_{s}$ and $m_{q}$ given in Appendix A.) The second renormalization scale is now recovered with the introduction with one more logarithm:

$$
\begin{equation*}
M_{q, n}^{\delta}=\frac{1}{\left[2 m_{q}\left(\mu_{m}\right)\right]^{2 n}} \sum_{i=0}\left[\frac{\alpha_{s}\left(\mu_{\alpha}\right)}{\pi}\right]^{i} \sum_{j=0}^{i} \sum_{k=0}^{[i-1]} c_{i, j, k}^{\delta,(n)} \ln ^{j}\left[\frac{\mu_{m}}{m_{q}\left(\mu_{m}\right)}\right] \ln ^{k}\left[\frac{\mu_{\alpha}}{\mu_{m}}\right] \tag{3.19}
\end{equation*}
$$

where $[i-1] \equiv \max (i-1,0)$ and $c_{i, j, 0}^{\delta,(n)} \equiv c_{i, j}^{\delta,(n)}$. In an analogous fashion, the coefficients $c_{i, j, k>0}^{\delta,(n)}$ are obtained by imposing the $\mu_{\alpha}$-independence through the renormalization group equation in Eq. (2.43). The recurrence relation for these coefficients reads

$$
\begin{equation*}
c_{i, j, k}^{\delta,(n)}=\frac{2}{k} \sum_{l=j}^{i-1} \frac{l \beta_{i-l-1}}{4^{i-l}} c_{l, j, k-1}^{\delta,(n)} . \tag{3.20}
\end{equation*}
$$

The high sensitivity of the moments $M_{q, n}^{\delta}$ to the quark mass through the explicit global factor $1 /\left[2 m_{q}\left(\mu_{m}\right)\right]^{2 n}$ and logarithms at $\mathscr{O}\left(\alpha_{s}\right)$ make them the basis for precise determinations of the charm- and bottom-quark masses since some time. ${ }^{26-37}$ When one is interested in extractions of $\alpha_{s}$ it is useful to work with the dimensionless ratios (for $n>0$ )

$$
\begin{equation*}
R_{q, n}^{\delta} \equiv \frac{\left(M_{q, n}^{\delta}\right)^{\frac{1}{n}}}{\left(M_{q, n+1}^{\delta}\right)^{\frac{1}{n+1}}}, \tag{3.21}
\end{equation*}
$$

whose perturbative expansion

$$
\begin{equation*}
R_{q, n}^{\delta}=\sum_{i=0}\left[\frac{\alpha_{s}\left(\mu_{\alpha}\right)}{\pi}\right]^{i} \sum_{j=0}^{[i-1]} \sum_{k=0}^{[i-2]} r_{i, j, k}^{\delta,(n)} \ln ^{j}\left[\frac{\mu_{m}}{m_{q}\left(\mu_{m}\right)}\right] \ln ^{k}\left[\frac{\mu_{\alpha}}{\mu_{m}}\right] \tag{3.22}
\end{equation*}
$$

is obtained with the use of Eq. (3.19) in Eq. (3.21) and consistently re-expanding the expression in the strong coupling. These ratios (with $\delta=V, P$ ) have already proven to be an important tool for $\alpha_{s}$ determinations, ${ }^{33-39}$ since the mass dependence, by construction, is highly suppressed, entering only logarithmically and starting at $\mathscr{O}\left(\alpha_{s}^{2}\right)$.

We remark that the dimensionless combination of the moments presented in Eq. (3.21) is not unique. In fact, with the knowledge of the renormalon structure in large- $\beta_{0}$ obtained in this work, we are able to design improved combinations of moments that display a better perturbative behaviour capable to improve the determinations of $m_{c}, m_{b}$ and $\alpha_{s}$.

### 3.3 Current renormalization and $\gamma_{5}$ in $d$-dimensions

Before going through the calculations of the heavy-quark current-current correlators, it is imperative to understand the current renormalizations that may take place in the correlators $\Pi^{\delta}(s)$ and the $d$-dimensional generalization of the chirality operator $\gamma_{5}$. These topics are covered in this section.

In the present work we are interested in currents with the form

$$
\begin{equation*}
j(x)=\bar{q}(x) \Gamma q(x) \tag{3.23}
\end{equation*}
$$

where $\Gamma$ stands for a combination of $\gamma$-matrices. Since $j(x)$ is a composite operator, it is possible that Green functions with the insertion of one current $j(x)$ remain divergent after wave function renormalization, i.e., redefining the quark field as

$$
\begin{equation*}
q_{0}(x)=\sqrt{Z_{2 F}} q(x), \tag{3.24}
\end{equation*}
$$

where $q_{0}$ is the bare quark field, coupling and masses renormalization. Henceforth, an extra renormalization constant $Z_{j}$ to absord the remaining divergences should be introduced, such that the complete renormalized current is given by

$$
\begin{equation*}
j(x)=Z_{j} \bar{q}(x) \Gamma q(x) \tag{3.25}
\end{equation*}
$$

Through the renormalization constant $Z_{j}$ one can also derive the anomalous dimension of the current $j(x)$,

$$
\begin{equation*}
\gamma_{j}=\mu \frac{\mathrm{d}}{\mathrm{~d} \mu} \ln \left(\frac{Z_{j}}{Z_{2 F}}\right) \tag{3.26}
\end{equation*}
$$

The determination of the vector renormalization constant $Z_{V}$ of $j_{\mu}^{V}(x)$ follows from Ward's identity ${ }^{55}$

$$
\begin{equation*}
i\left(p_{1}-p_{2}\right)^{\mu} G_{V, \mu}\left(p_{1}, p_{2}\right)=S_{F}^{(0)}\left(p_{1}\right)-S_{F}^{(0)}\left(p_{2}\right) \tag{3.27}
\end{equation*}
$$

where $S_{F}^{(0)}(p)$ is the free quark propagator in momentum space given in Eq. (2.13) and $G_{V, \mu}\left(p_{1}, p_{2}\right)$ is the vector vertex function

$$
\begin{equation*}
G_{V, \mu}\left(p_{1}, p_{2}\right)=\int \mathrm{d} x_{1} \mathrm{~d} x_{2} e^{i p_{1} x_{1}-i p_{2} x_{2}}\langle 0| T q\left(x_{1}\right) j_{\mu}^{V}(0) \bar{q}\left(x_{2}\right)|0\rangle . \tag{3.28}
\end{equation*}
$$

(Colour indices were suppressed without loss of generality.) Eq. (3.27) is valid either when written in terms of the bare or renormalized fields, masses and couplings. Comparing the renormalization constants that should be introduced in the LHS and RHS of Eq. (3.27), the vector renormalization constant is found to be

$$
\begin{equation*}
Z_{V}=Z_{2 F}, \tag{3.29}
\end{equation*}
$$

and therefore, as per Eq. (3.26), the anomalous dimension of the vector current vanishes.
This procedure is easily generalized for arbitrary bilinear currents $j(x)$ using the generic vertex function

$$
\begin{equation*}
G_{j}\left(p_{1}, p_{2}\right)=\int \mathrm{d} x_{1} \mathrm{~d} x_{2} e^{i p_{1} x_{1}-i p_{2} x_{2}}\langle 0| T q\left(x_{1}\right) j(0) \bar{q}\left(x_{2}\right)|0\rangle . \tag{3.30}
\end{equation*}
$$

In the case of the scalar current, for instance, the relation between the vertex function and the free quark propagator reads

$$
\begin{equation*}
i G_{S}\left(p_{1}, p_{1}\right)=-\frac{\partial}{\partial m} S_{F}^{(0)}\left(p_{1}\right) \tag{3.31}
\end{equation*}
$$

The scalar renormalization constant is thus found to be

$$
\begin{equation*}
Z_{S}=Z_{m} Z_{2 F}, \tag{3.32}
\end{equation*}
$$

which results in a non-vanishing anomalous dimension for the scalar current. Since the extra renormalization constant is $Z_{m}$, a mass factor should be introdued in the bilinear $\bar{q} q$ to ensure renormalization group invariance.

Similar arguments are used to show that the axial-vector current has zero anomalous dimension and the pseudo-scalar current, as in the scalar one, has an anomalous dimension equals to the mass anomalous dimension. However, these results are obtained assuming a hermitian, anti-commuting $\gamma_{5}$ satisfying the relations

$$
\begin{equation*}
\left(\gamma_{5}\right)^{2}=1 \quad \text { and } \quad\left\{\gamma_{5}, \gamma_{\mu}\right\}=0 \tag{3.33}
\end{equation*}
$$

that may not be valid in an arbitrary number of dimensions $d$. In particular, one possible 4-dimensional definition for $\gamma_{5}$ is ${ }^{103}$

$$
\begin{equation*}
\gamma_{5}=\frac{i}{4!} \varepsilon_{\mu \nu \rho \sigma} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}, \tag{3.34}
\end{equation*}
$$

where the totally anti-symmetric Levi-Cività tensor $\varepsilon_{\mu \nu \rho \sigma}$ is unavoidably a 4 -dimensional
object. With this definition of $\gamma_{5}$ and using trace properties of the $\gamma$-matrices, one can show that in 4 dimensions

$$
\begin{equation*}
\operatorname{Tr}\left[\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma_{5}\right]=4 i \varepsilon_{\mu \nu \rho \sigma} \tag{3.35}
\end{equation*}
$$

Now, in $d$ dimensions and assuming that $\gamma_{5}$ still satisfies the anti-commutation relations, from the computation of $\operatorname{Tr}\left[\gamma_{\omega} \gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma^{\omega} \gamma_{5}\right]$ one can derive the relation

$$
\begin{equation*}
(d-4) \operatorname{Tr}\left[\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma_{5}\right]=0 \tag{3.36}
\end{equation*}
$$

which is in conflict with the 4 -dimensional result for this trace (Eq. (3.35)) unless $d=4$.
A practical way to deal with $\gamma_{5}$ in $d$ dimensions is due to 't Hooft and Veltman, ${ }^{58}$ where the definition from Eq. (3.34) is maintained but the anti-commutation relation $\left\{\gamma_{5}, \gamma_{\mu}\right\}=0$ is dropped. By doing this, axial and Ward identities, that are valid in regularization methods in 4 dimensions, are now violated and should be restored with the introduction of extra, finite, renormalization constants. ${ }^{103}$ In practice, these extra renormalization constants recover the 4 -dimensional axial-vector and pseudo-scalar anomalous dimension and, as we will see later, in the large- $\beta_{0}$ limit of current correlators only the anomalous dimension is necessary. Therefore, in the context of this work, there is no need to calculate these extra renormalization constants, and we can directly employ the 't Hooft-Veltman extension of $\gamma_{5}$.

One final obstacle that should be overcome is related to the axial-vector current. Since the anti-commutativity of $\gamma_{5}$ is now violated by definition, $j_{\mu}^{A}(x)$ should be rewritten in the more symmetric form

$$
\begin{equation*}
j_{\mu}^{A}(x)=\frac{1}{2} \bar{q}(x)\left(\gamma_{\mu} \gamma_{5}-\gamma_{5} \gamma_{\mu}\right) q(x) \tag{3.37}
\end{equation*}
$$

Using the anti-commutation relations of the ordinary $\gamma$-matrices and Eq. (3.34), the axial-vector current becomes

$$
\begin{equation*}
j_{\mu}^{A}(x)=\frac{i}{3!} \varepsilon_{\mu \nu_{1} \nu_{2} \nu_{3}} \bar{q}(x) \gamma^{\nu_{1}} \gamma^{\nu_{2}} \gamma^{\nu_{3}} q(x) \tag{3.38}
\end{equation*}
$$

This is the form that we will use in $d$ dimensions.

### 3.4 The one-loop moments

Having discussed the renormalization of bilinear currents and the definition of $\gamma_{5}$ that should be used along the calculations, we are now at a position to introduce the small-momentum expansion and obtain the one-loop moments of the current correlators $\Pi^{\delta}(s)$.

Calculating the perturbative series of current correlators with arbitrary masses is a challenging task. In practice, some asymptotic expansion must performed lest the loop
integrals become intractable. In the present analysis we are particularly interested in the small-momentum expansion $s \ll 4 m^{2}$ due to its relation with the moments $M_{q, n}^{\delta}$ given in Eq. (3.11). In this section a detailed introduction of the main methods to perform this expansion is given in the context of one-loop integrals. We will rely on the formalisms based on (i) hypergeometric functions, ${ }^{104}$ (ii) the Feynman-Mellin-Barnes representation ${ }^{105}$ and (iii) a simple Taylor series. ${ }^{106}$ As we will see in the remainder of this section, expanding the correlators through a Taylor series is much less powerful than the other two methods, but the former is more convenient in the large- $\beta_{0}$ limit since there, as we will see, the application of Integration-By-Parts (IBP) ${ }^{107}$ to reduce the loop integrals to a small set of master integrals is not possible. Nevertheless, the hypergeometric functions will be of great important to solve the two-loop single-scale integrals arising in the Taylor expansion in the large- $\beta_{0}$ limit.

With respect to the current correlators of Eqs. (3.1) and (3.2), the leading order contribution to the bare correlator $\Pi_{0}^{\delta}(s)$ has the general form

$$
\begin{equation*}
\Pi_{0}^{\delta,(1)}(s) \propto \int \mathrm{d}^{d} k \frac{\operatorname{Tr}\left[\Gamma^{\delta}\left(\not k+q+m_{q}\right) \Gamma^{\delta}\left(\not k+m_{q}\right)\right]}{\left[(k+q)^{2}-m_{q}^{2}\right]\left[k^{2}-m_{q}^{2}\right]}, \tag{3.39}
\end{equation*}
$$

where $\Gamma^{\delta}$ is the combination of $\gamma$-matrices of the current $j^{\delta}(x)$ and the superscript (1) is added to emphasize that we are working at the one-loop level. At this stage $m_{q}$ is the bare quark mass. The related Feynman diagram is displayed in Fig. (11). After computing the trace over the fermion loop and contracting all Lorentz indices (for $\delta=V, A$ ) only scalar products of the moments appear in the numerator. These scalar products can then be written in terms of $q^{2}$ and of the denominators of the integrand in Eq. (3.39), e.g.,

$$
\begin{equation*}
2 k \cdot q=\left[(k+q)^{2}-m_{q}^{2}\right]-\left[k^{2}-m_{q}^{2}\right]-q^{2}, \tag{3.40}
\end{equation*}
$$

which reduces the problem to the study of the scalar one-loop integral

$$
\begin{equation*}
J_{1}\left(n_{1}, n_{2}\right)=\int \frac{\mathrm{d}^{d} k}{\left[(k+q)^{2}-m_{q}^{2}\right]^{n_{1}}\left[k^{2}-m_{q}^{2}\right]^{n_{2}}} . \tag{3.41}
\end{equation*}
$$

For illustration purposes we give the bare vector correlator at leading order written in terms of $J_{1}\left(n_{1}, n_{2}\right)$ :

$$
\begin{equation*}
\Pi_{0}^{V,(1)}(s)=\frac{i N_{c}}{(2 \pi)^{d}(1-d) s}\left[(4-2 d)\left(J_{1}(1,0)+J_{1}(0,1)\right)+\left(8 m_{q}^{2}-(4-2 d) s\right) J_{1}(1,1)\right] . \tag{3.42}
\end{equation*}
$$

### 3.4.1 Expansion using hypergeometric functions

In order to obtain both the expansion by means of hypergeometric functions or Feynman-Mellin-Barnes representation, one should first write the loop integral in terms of Feynman


Figure 11 - One-loop Feynman diagram for the computation of the leading order contribution of $\Pi^{\delta}(s)$. Crosses stand for the insertion of the currents defined in Eq. (3.3) with an external momentum $q$.

Source: By the author.
parameters. Combining the denominators of $J_{1}\left(n_{1}, n_{2}\right)$ using the Feynman parameter of Eq. (2.28) and solving the integral over the loop momentum with Eq. (2.30) the expression for $J_{1}\left(n_{1}, n_{2}\right)$ becomes

$$
\begin{align*}
J_{1}\left(n_{1}, n_{2}\right) & =i \pi^{d / 2}(-1)^{n_{1}+n_{2}} \frac{\Gamma\left(n_{1}+n_{2}-d / 2\right)}{\Gamma\left(n_{1}\right) \Gamma\left(n_{2}\right)}  \tag{3.43}\\
& \int_{0}^{1} \mathrm{~d} x \frac{x^{n_{1}-1}(1-x)^{n_{2}-1}}{\left(m_{q}^{2}\right)^{n_{1}+n_{2}-d / 2}}\left[\frac{1}{1-\frac{q^{2}}{m_{q}^{2}} x(1-x)}\right]^{n_{1}+n_{2}-d / 2} .
\end{align*}
$$

In the special case $n_{1}=n_{2}=1$, with the auxiliary of the identity

$$
\begin{equation*}
\int_{0}^{1} \mathrm{~d} x f(x(1-x))=\frac{1}{2} \int_{0}^{1} \frac{\mathrm{~d} w}{\sqrt{1-w}} f\left(\frac{w}{4}\right) \tag{3.44}
\end{equation*}
$$

the expression for $J_{1}(1,1)$ can be conveniently written as

$$
\begin{equation*}
J_{1}(1,1)=i \pi^{d / 2} \frac{\Gamma(2-d / 2)}{2\left(m_{q}^{2}\right)^{2-d / 2}} \int_{0}^{1} \mathrm{~d} w \frac{w^{0}(1-w)^{-1 / 2}}{\left[1-\frac{q^{2}}{4 m_{q}^{2}} w\right]^{2-d / 2}} \tag{3.45}
\end{equation*}
$$

The asymptotic expansion of $J_{1}(1,1)$ is now easily obtained since the remaining integral over $w$ has the same form of the parametric representation of the hypergeometric function

$$
\begin{align*}
{ }_{2} F_{1}(a, b ; c ; z) & \equiv \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} \mathrm{~d} w \frac{w^{b-1}(1-w)^{c-b-1}}{(1-z w)^{a}}  \tag{3.46}\\
& =\sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n} n!} z^{n}
\end{align*}
$$

where $(x)_{n} \equiv \Gamma(x+n) / \Gamma(x)$ is the Pochhammer symbol. In particular,

$$
\begin{align*}
J_{1}(1,1) & =i \pi^{d / 2} \frac{\Gamma(2-d / 2)}{2\left(m_{q}^{2}\right)^{2-d / 2}}\left[\frac{\Gamma(1) \Gamma(1 / 2)}{\Gamma(3 / 2)}{ }_{2} F_{1}\left(2-d / 2,1 ; 3 / 2 ; \frac{q^{2}}{4 m_{q}^{2}}\right)\right]  \tag{3.47}\\
& =i \pi^{d / 2} \frac{\Gamma(3 / 2)}{\left(m_{q}^{2}\right)^{2-d / 2}} \sum_{n=0}^{\infty} \frac{\Gamma(2-d / 2+n)}{\Gamma(3 / 2+n)}\left(\frac{q^{2}}{4 m_{q}^{2}}\right)^{n} .
\end{align*}
$$

When the number of propagators increases the introduction of more Feynman parameters might be necessary. In this situation the generalized hypergeometric functions

$$
\begin{align*}
& { }_{p+1} F_{q+1}\left(a_{1}, \cdots, a_{p}, c ; b_{1}, \cdots, b_{q}, d ; z\right) \equiv \sum_{n=0}^{\infty} \frac{\left(a_{1}\right)_{n} \cdots\left(a_{p}\right)_{n}(c)_{n}}{\left(b_{1}\right)_{n} \cdots\left(b_{q}\right)_{n}(d)_{n}} \frac{z^{n}}{n!}  \tag{3.48}\\
& =\frac{\Gamma(d)}{\Gamma(c) \Gamma(d-c)} \int_{0}^{1} \mathrm{~d} w w^{c-1}(1-w)^{d-c-1}{ }_{p} F_{q}\left(a_{1}, \cdots, a_{p} ; b_{1}, \cdots, b_{q} ; w z\right)
\end{align*}
$$

are important tools.
The remaining integrals $J_{1}(1,0)$ and $J_{1}(0,1)$ (recall Eq. (3.42)) are equal and can be immediatly solved with Eq. (2.30). The result is

$$
\begin{equation*}
J_{1}(1,0)=J_{1}(0,1)=-i \pi^{d / 2} \frac{\Gamma(1-d / 2)}{\left(m_{q}^{2}\right)^{1-d / 2}} \tag{3.49}
\end{equation*}
$$

With the expressions for $J_{1}(1,1)$ and $J_{1}(0,1)$ given in Eqs. (3.47) and (3.49) it is already possible to obtain the one-loop result in the small-momentum expansion for the four correlators. However, it is interesting to discuss the case where an extra integral $J_{1}\left(n_{1}, n_{2}\right)$ with $n_{1}$ and $n_{2} \neq 1$ needs to be calculated. In this situation one should use the IBP method ${ }^{107}$ to reduce $J_{1}\left(n_{1}, n_{2}\right)$ to a linear combination of $\left\{J_{1}(0,1), J_{1}(1,1)\right\}$, which is a possible set of the so-called master integrals of the topology of $J_{1}\left(n_{1}, n_{2}\right) .{ }^{108}$ The IBP method relies on the idea that surface terms on the integrand vanishes, i.e.,

$$
\begin{equation*}
\int \mathrm{d}^{d} k \frac{\partial}{\partial k_{\mu}}\left[\frac{v_{\mu}}{\left[(k+q)^{2}-m_{q}^{2}\right]^{n_{1}}\left[k^{2}-m_{q}^{2}\right]^{n_{2}}}\right]=0 \tag{3.50}
\end{equation*}
$$

where $v_{\mu}$ is an external or internal momentum of the loop integral. After evaluating the partial derivative, it is possible to find recurrence relations between $J_{1}\left(n_{1}, n_{2}\right)$. For instance, if one set $v_{\mu}=k_{\mu}$ the recurrence relation

$$
\begin{equation*}
J_{1}\left(n_{1}, n_{2}\right)=\frac{2 n_{2} m_{q}^{2} J_{1}\left(n_{1}, n_{2}+1\right)+n_{1} J_{1}\left(n_{1}+1, n_{2}-1\right)-n_{1}\left(q^{2}-2 m_{q}^{2}\right) J_{1}\left(n_{1}+1, n_{2}\right)}{d-n_{1}-2 n_{2}} \tag{3.51}
\end{equation*}
$$

is derived. A second recurrence relation can be obtained with $v_{\mu}=q_{\mu}$. When combined, these recurrence relations can be used to reduce any $J_{1}\left(n_{1}, n_{2}\right)$ to a linear combination of
$\left\{J_{1}(0,1), J_{1}(1,1)\right\}$. The integral $J_{1}(2,1)$, for instance, is given by

$$
\begin{equation*}
J_{1}(2,1)=\frac{d-3}{4 m_{q}^{2}-q^{2}} J_{1}(1,1)-\frac{d-2}{2 m_{q}^{2}\left(4 m_{q}^{2}-q^{2}\right)} J_{1}(0,1) \tag{3.52}
\end{equation*}
$$

Using IBP to reduce the computation of Feynman diagrams to the calculation of a (luckily small) set of master integrals is crucial when the number of loops increases. Some useful computer codes to perform this hard-work are FIRE ${ }^{109}$ and LiteRed, ${ }^{110}$ both based on Laporta's algorithm. ${ }^{111}$

### 3.4.2 Expansion using the Feynman-Mellin-Barnes representation

A second and more powerful approach to find asymptotic expansions of loop integrals is based on the Feynman-Mellin-Barnes (FMB) integral representation and was first proposed in Ref. ${ }^{105}$

Let us consider a generic function $\mathscr{F}(\rho)$ that have a Mellin-Barnes (MB) integral representation

$$
\begin{equation*}
\mathscr{F}(\rho)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \mathrm{~d} z \rho^{-z} \mathscr{M}[\mathscr{F}](z), \tag{3.53}
\end{equation*}
$$

where $\mathscr{M}[\mathscr{F}](z)$ is the MB transform of $\mathscr{F}(\rho)$,

$$
\begin{equation*}
\mathscr{M}[\mathscr{F}](z)=\int_{0}^{\infty} \mathrm{d} \rho \rho^{z-1} \mathscr{F}(\rho), \tag{3.54}
\end{equation*}
$$

and $c$ is a real number in the fundamental strip defined as the largest interval between two poles of $\mathscr{F}(\rho)$. By means of the converse mapping theorem, ${ }^{112}$ the position of poles, multiplicity and residues at the LHS of the fundamental strip encode the aymptotic behaviour of $\mathscr{F}(\rho)$ for $\rho \rightarrow 0$. The behaviour for $\rho \rightarrow \infty$ is encoded by the poles at the RHS of the fundamental strip.

If a singular expansion $\mathscr{M}[\mathscr{F}](z)$ in the LHS of the fundamental strip reads

$$
\begin{equation*}
\mathscr{M}[\mathscr{F}](z) \asymp \sum_{\xi, k} \frac{b_{k, \xi}}{(z+\xi)^{k+1}} \tag{3.55}
\end{equation*}
$$

where $\asymp$ means "singular part of", the asymptotic behaviour of $\mathscr{F}(\rho)$ for $\rho \rightarrow 0$ is

$$
\begin{equation*}
\mathscr{F}(\rho)=\sum_{\xi, k} \frac{(-1)^{k}}{k!} b_{\xi, k} \rho^{\xi} \ln ^{k} \rho . \tag{3.56}
\end{equation*}
$$

This strategy is applied to $J_{1}\left(n_{1}, n_{2}\right)$ by considering the MB transform

$$
\begin{equation*}
\frac{1}{(1+X)^{\nu}}=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \mathrm{~d} z(X)^{-z} \frac{\Gamma(z) \Gamma(\nu-z)}{\Gamma(\nu)} \tag{3.57}
\end{equation*}
$$

in the representation of $J_{1}\left(n_{1}, n_{2}\right)$ using Feynman parameters, as given in Eq. (3.43). $J_{1}\left(n_{1}, n_{2}\right)$ can be written in the FMB representation as

$$
\begin{equation*}
J_{1}\left(n_{1}, n_{2}\right)=\frac{i \pi^{d / 2}(-1)^{n_{1}+n_{2}}}{\left(m_{q}^{2}\right)^{n_{1}+n_{2}-d / 2} \Gamma\left(n_{1}\right) \Gamma\left(n_{2}\right)} \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \mathrm{~d} z\left(\frac{q^{2}}{m_{q}^{2}}\right)^{-z} \mathscr{M}\left[J_{1}\left(n_{1}, n_{2}\right)\right](z), \tag{3.58}
\end{equation*}
$$

where the MB transform is given by

$$
\begin{align*}
\mathscr{M}\left[J_{1}\left(n_{1}, n_{2}\right)\right](z) & =(-1)^{-z} \Gamma(z) \Gamma\left(n_{1}+n_{2}-d / 2-z\right) \int_{0}^{1} \mathrm{~d} x x^{n_{1}-1-z}(1-x)^{n_{2}-1-z}  \tag{3.59}\\
& =(-1)^{-z} \Gamma(z) \Gamma\left(n_{1}+n_{2}-d / 2-z\right)\left[\frac{\Gamma\left(n_{1}-z\right) \Gamma\left(n_{2}-z\right)}{\Gamma\left(n_{1}+n_{2}-2 z\right)}\right],
\end{align*}
$$

where in the last step we used the definition of the Euler-Beta function,

$$
\begin{equation*}
B(a, b) \equiv \int_{0}^{1} \mathrm{~d} x x^{a-1}(1-x)^{b-1}=\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} . \tag{3.60}
\end{equation*}
$$

The singular expansion of $\Gamma(s)$, which has single poles at negative integer values of $s$, is very simple and reads

$$
\begin{equation*}
\Gamma(s) \asymp \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{1}{(s+n)} \tag{3.61}
\end{equation*}
$$

Thus calculating the residues of $\mathscr{M}\left[J_{1}\left(n_{1}, n_{2}\right)\right]$ at the LHS of the fundamental strip is straightforward. The singular expansion reads

$$
\begin{equation*}
\mathscr{M}\left[J_{1}\left(n_{1}, n_{2}\right)\right](z) \asymp \sum_{n=0}^{\infty} \frac{\Gamma\left(n+n_{1}\right) \Gamma\left(n+n_{2}\right) \Gamma\left(n+n_{1}+n_{2}-d / 2\right)}{n!\Gamma\left(2 n+n_{1}+n_{2}\right)} \frac{1}{(z+n)} \tag{3.62}
\end{equation*}
$$

and is already in the desired form. The small-momentum asymptotic expansion of $J_{1}\left(n_{1}, n_{2}\right)$ is, therefore, accordingly to Eq. (3.56),

$$
\begin{equation*}
J_{1}\left(n_{1}, n_{2}\right)=\frac{i \pi^{d / 2}(-1)^{n_{1}+n_{2}}}{\left(m_{q}^{2}\right)^{n_{1}+n_{2}-d / 2} \Gamma\left(n_{1}\right) \Gamma\left(n_{2}\right)} \sum_{n=0}^{\infty} b_{n}\left(n_{1}, n_{2}\right)\left(\frac{q^{2}}{m_{q}^{2}}\right)^{n}, \tag{3.63}
\end{equation*}
$$

where the coefficients $b_{n}\left(n_{1}, n_{2}\right)$ read

$$
\begin{equation*}
b_{n}\left(n_{1}, n_{2}\right)=\frac{\Gamma\left(n+n_{1}\right) \Gamma\left(n+n_{2}\right) \Gamma\left(n+n_{1}+n_{2}-d / 2\right)}{n!\Gamma\left(2 n+n_{1}+n_{2}\right)} . \tag{3.64}
\end{equation*}
$$

The last equation is in agreement with the result for the special case $n_{1}=n_{2}=1$ already obtained with hypergeometric functions.

The fact that we obtained an analytical expression for $J_{1}\left(n_{1}, n_{2}\right)$ for arbitrary values of $n_{1}$ and $n_{2}$ was just a consequence of the simplicity of the one-loop integrals. When the number of loops increases this is no longer feasible, and therefore the IBP method should
be used to reduce the number of integrals to be calculated.

### 3.4.3 Expansion with Taylor series

The last approach considered in this work to perform the asymptotic expansion is based on a Taylor series with the application of the momentum space d'Alembertian operator

$$
\begin{equation*}
\square_{q}=\frac{\partial}{\partial q^{\mu} q_{\mu}} . \tag{3.65}
\end{equation*}
$$

If $J\left(q^{2}\right)$ is a scalar function regular at $q^{2}=0$, then the following expansion is valid ${ }^{106}$ :

$$
\begin{equation*}
J\left(q^{2}\right)=\left.\sum_{j=0}^{\infty} \frac{1}{j!(d / 2)_{j}}\left(\frac{q^{2}}{4}\right)^{j}\left(\square_{q}^{j} J\left(q^{2}\right)\right)\right|_{q=0} \tag{3.66}
\end{equation*}
$$

In the one-loop topology, the application of the d'Alembertian operator in $J_{1}\left(n_{1}, n_{2}\right)$ reads

$$
\begin{equation*}
\square_{q} J_{1}\left(n_{1}, n_{2}\right)=4\left[\left(n_{1}+1-d / 2\right) n_{1} J_{1}\left(n_{1}+1, n_{2}\right)+n_{1}\left(n_{1}+1\right) m_{q}^{2} J_{1}\left(n_{1}+2, n_{2}\right)\right], \tag{3.67}
\end{equation*}
$$

and higher orders are obtained by recursively applying the operator in the RHS of the above equation. The remaining single-scale integrals after setting $q=0$ are easier to solve (see Eq. (2.30)) and read

$$
\begin{equation*}
\left.J_{1}\left(n_{1}, n_{2}\right)\right|_{q=0}=\int \frac{\mathrm{d}^{d} k}{\left[k^{2}-m_{q}^{2}\right]^{n_{1}+n_{2}}}=i(-1)^{n_{1}+n_{2}} \pi^{d / 2} \frac{\Gamma\left(n_{1}+n_{2}-d / 2\right)}{\Gamma\left(n_{1}+n_{2}\right)}\left(m_{q}^{2}\right)^{d / 2-n_{1}-n_{2}} \tag{3.68}
\end{equation*}
$$

The method of applying the d'Alembertian operator to perform the small-momentum expansion is less powerful when compared to other methods. This is because instead of finding expressions to all orders in $q^{2}$, in this method the expansion is calculated order by order, and therefore we are limited by computational power. However, as we will see in the next chapter, in the large- $\beta_{0}$ limit one of the propagators has an analytically regularized exponent that prevents us to use IBP to reduce the loop integrals to a set of master integrals that can be evaluated in the small-momentum expansion. Since in the Taylor method the action of setting $q=0$ reduces drastically the complexity of the integrals, this will be the only feasible method in the large- $\beta_{0}$ limit. The hypergeometric functions are, however, still of great importance to solve the two-loop integrals with $q=0$.

### 3.4.4 Results

We conclude this chapter giving the leading order results of the four correlators. For illustration purposes, let us work out explicitly the vector correlator. Using the results for

Table 1 - Leading order results $N_{n}^{\delta}$.

| $n$ | $N_{n}^{P}$ | $N_{n}^{V}$ | $N_{n}^{S}$ | $N_{n}^{A}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\frac{4}{3}$ | - | $\frac{4}{5}$ | - |
| 1 | $\frac{8}{15}$ | $\frac{16}{15}$ | $\frac{8}{35}$ | $\frac{8}{15}$ |
| 2 | $\frac{32}{105}$ | $\frac{16}{35}$ | $\frac{32}{315}$ | $\frac{16}{105}$ |
| 3 | $\frac{64}{315}$ | $\frac{256}{945}$ | $\frac{64}{1155}$ | $\frac{64}{945}$ |
| 4 | $\frac{512}{3465}$ | $\frac{128}{693}$ | $\frac{512}{15015}$ | $\frac{128}{3465}$ |
| 5 | $\frac{1024}{9009}$ | $\frac{2048}{15015}$ | $\frac{1024}{45045}$ | $\frac{1024}{45045}$ |

Source: By the author.
$J_{1}(1,1), J_{1}(1,0)$ and $J_{1}(0,1)$ given in Eqs. (3.47) and (3.49) in Eq. (3.42) and collecting the coefficients in the expansion of $s / 4 m_{q}^{2}$ we obtain for the bare vector correlator at one-loop order

$$
\begin{align*}
\Pi_{0}^{V,(1)}(s) & =\frac{N_{c}}{16 \pi^{2}}\left\{2\left(\frac{4 m_{q}^{2}}{s}\right) \frac{16^{\epsilon} \pi^{\epsilon} \Gamma(\epsilon)}{\left(4 m_{q}^{2}\right)^{\epsilon}(3-2 \epsilon)}+\frac{4}{3} \frac{16^{\epsilon} \pi^{\epsilon} \Gamma(\epsilon)}{\left(4 m_{q}^{2}\right)^{\epsilon}}\right.  \tag{3.69}\\
& \left.+\frac{16}{15}\left(\frac{s}{4 m_{q}^{2}}\right) \frac{16^{\epsilon} \pi^{\epsilon} \Gamma(1+\epsilon)}{\left(4 m_{q}^{2}\right)^{\epsilon}}+\frac{16}{35}\left(\frac{s}{4 m_{q}^{2}}\right)^{2} \frac{16^{\epsilon} \pi^{\epsilon} \Gamma(2+\epsilon)}{\left(4 m_{q}^{2}\right)^{\epsilon}}+\mathscr{O}\left[\left(\frac{s}{4 m_{q}^{2}}\right)^{3}\right]\right\}
\end{align*}
$$

where we already used $d=4-2 \epsilon$. In the limit $\epsilon \rightarrow 0$ the coefficients of $\left(4 m_{q} / s\right)$ and $\left(s / 4 m_{q}^{2}\right)^{0}$ are UV divergent and require a subtraction that is beyond mass and coupling renormalization, since these have no effect at leading order in perturbation theory. These terms, therefore, are renormalization scheme-dependent and can not describe a physical quantity. Hence, we cast our results for the renormalized correlators in the form of

$$
\begin{equation*}
\widehat{\Pi}^{\delta}(s)=\frac{N_{c}}{16 \pi^{2}} \sum_{n=n_{\delta}}^{\infty}\left(\frac{s}{4 m_{q}^{2}(\mu)}\right)^{n} N_{n}^{\delta} C_{n}^{\delta}(\mu) \tag{3.70}
\end{equation*}
$$

where $N_{n}^{\delta}$ is the one-loop result in $d=4$ dimensions and $C_{n}^{\delta}(\mu)$ is a perturbative series in $\alpha_{s}$ starting at 1. Since we are interested only in the physical moments $M_{q, n}^{\delta}$, i.e., those that do not require an additional, scheme-dependent, subtraction beyond mass and coupling renormalization, we remove from the definition of $\widehat{\Pi}^{\delta}(s)$ the unphysical terms that contain an UV divergence setting $n_{S}=n_{P}=0$ and $n_{V}=n_{A}=1$. The results for $N_{1}^{V}$ and $N_{2}^{V}$ can be read off from the second line of Eq. (3.69).

According to the definition of the moments given in Eq. (3.11), and already using $N_{c}=3$, we have

$$
\begin{equation*}
M_{q, n}^{\delta}=\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[2 m_{q}(\mu)\right]^{2 n}} C_{n}^{\delta}(\mu) \tag{3.71}
\end{equation*}
$$

The first few coefficients $N_{n}^{\delta}$ obtained from the expansion of the correlators at one-loop, as explained above in the text, are displayed in Tab. 1 and are in full agreement with Ref. ${ }^{113}$ Having defined the heavy-quark current-current correlators of our interest and introduced the technology necessary to obtain the small-momentum expansion of loop integrals, now we are at a position to introduce the large- $\beta_{0}$ limit to study the higher-order behaviour of the perturbative series that govern the moments $M_{q, n}^{\delta}$ and the ratios $R_{q, n}^{\delta}$. This is the subject of the next chapter.

## 4 THE LARGE- $\boldsymbol{\beta}_{0}$ LIMIT: TWO-LOOPS AND BEYOND

The large- $\beta_{0}$ limit is a method to obtain semi-quantitative information about the higher-order behaviour of perturbative series in QCD. ${ }^{40,41}$ It is based first on the large- $N_{f}$ limit, where the number of quark flavours $N_{f}$ is considered a large parameter with the constraint $N_{f} \alpha_{s} \sim 1$, followed by a procedure known as naive non-abelianization, ${ }^{42,43}$ which consists in the replacement of the leading fermionic contribution of the QCD $\beta$-function, $\beta_{0, f}$, by the full $\beta_{0}$ coefficient. This transformation includes, in an effective way, the non-abelian character of QCD into the calculation.

The results presented in this and in the next chapter were already published in the Journal of High Energy Physics. ${ }^{45}$

### 4.1 The large- $\boldsymbol{\beta}_{0}$ limit as an expansion in $1 / \beta_{0}$

The aim of this section is to develop the large- $\beta_{0}$ limit as an expansion in $1 / \beta_{0}$, where $\beta_{0}$ is considered a large parameter. This section is based on Ref. ${ }^{41}$, from which most of our formalism derives.

Let us consider a quantity $C$ that contains, at one-loop level, no gluon-propagators. The general form of the perturbative expansion in terms of powers of the bare QCD coupling $g_{s}$ of the one-loop normalized bare quantity, $C_{0}$, is given by

$$
\begin{align*}
C_{0} & =1+\hat{c}_{1,0}\left(\frac{g_{s}^{2}}{(4 \pi)^{d / 2}}\right)+\left(\hat{c}_{2,0}+\hat{c}_{2,1} N_{f}\right)\left(\frac{g_{s}^{2}}{(4 \pi)^{d / 2}}\right)^{2}+\cdots  \tag{4.1}\\
& =1+\sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \hat{c}_{j, k} N_{f}^{k}\left(\frac{g_{s}^{2}}{(4 \pi)^{d / 2}}\right)^{j}
\end{align*}
$$

where $\hat{c}_{j, k}$ are coefficients that should be computed in perturbation theory and $d=4-2 \epsilon$ is the number of space-time dimensions under the dimensional regularization prescription described in Sec. 2.2. The determination of the coefficients $\hat{c}_{j, k}$ in the fixed-order expansion of Eq. (4.1) requires the computation of Feynman diagrams with $L=j+1$ number of loops. With other coefficients $c_{j, k}$ the fixed-order expansion of $C_{0}$ can also be written in terms of $\beta_{0}$, which we recall from Eq. (2.48) that is given by

$$
\begin{equation*}
\beta_{0}=\frac{11}{3} N_{c}-\frac{2}{3} N_{f} \equiv \beta_{0, c}+\beta_{0, f}, \tag{4.2}
\end{equation*}
$$

where $\beta_{0, c}$ is the non-abelian contribution for $\beta_{0}$. In terms of $\beta_{0}$ the perturbative expansion of $C_{0}$ is


Figure 12 - Feynman diagram for the gluon propagator with the insertion of one massless quark bubble.

Source: By the author.

$$
\begin{equation*}
C_{0}=1+\sum_{j=1}^{\infty} \sum_{k=0}^{j-1} c_{j, k} \beta_{0}^{k}\left(\frac{g_{s}^{2}}{(4 \pi)^{d / 2}}\right)^{j} \tag{4.3}
\end{equation*}
$$

which can be expanded in $1 / \beta_{0}$ as

$$
\begin{equation*}
C_{0}=1+\frac{1}{\beta_{0}} f\left(\frac{\beta_{0} g_{s}^{2}}{(4 \pi)^{d / 2}}\right)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right) \tag{4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
f(x)=\sum_{j=1}^{\infty} c_{j, j-1} x^{j} \tag{4.5}
\end{equation*}
$$

If the gluon propagator that appears at two-loop level does not couple to quarks beyond the one-loop bubble, under the naive non-abelianization the coefficients $c_{j, j-1}$ are determined exclusively by the insertion of $j-1$ massless quark bubble loops in the bare gluon propagator. In order to get an analytic expression for the gluon propagator dressed with $j-1$ massless quark bubbles, let us first work in detail the expression for the gluon propagator with one single massless quark bubble loop, $D_{\mu \nu}^{1, a b}(k)$, as depicted by the Feynman diagram in Fig. 12. Using Feynman rules ${ }^{55}$ the expression for $D_{\mu \nu}^{1, a b}(k)$ reads

$$
\begin{equation*}
D_{\mu \nu}^{1, a b}(k)=\left[\frac{-i g_{\mu \rho_{1}} \delta^{a c_{1}}}{k^{2}}\right]\left[i \Pi_{c_{1} c_{2}}^{\rho_{1} \rho_{2}}(k)\right]\left[\frac{-i g_{\rho_{2}} \delta^{c_{2} b}}{k^{2}}\right] \tag{4.6}
\end{equation*}
$$

where the amplitude $\prod_{c_{1} c_{2}}^{\rho_{1} \rho_{2}}(k)$ that represents the massless quark bubble should have its Lorentz structure decomposed as

$$
\begin{equation*}
\Pi_{c_{1} c_{2}}^{\rho_{1} \rho_{2}}(k)=\left(k^{2} g^{\rho_{1} \rho_{2}}-k^{\rho_{1}} k^{\rho_{2}}\right) \Pi_{c_{1} c_{2}}\left(k^{2}\right) \tag{4.7}
\end{equation*}
$$

due to Ward's identity. ${ }^{55}$ Therefore,

$$
\begin{equation*}
D_{\mu \nu}^{1, a b}(k)=\frac{-i \delta^{a c_{1}} \delta^{c_{2} b}}{\left(-k^{2}\right)}\left(g_{\mu \nu}-\frac{k_{\mu} k_{\nu}}{k^{2}}\right) \Pi_{c_{1} c_{2}}\left(k^{2}\right) . \tag{4.8}
\end{equation*}
$$

Using Feynman rules the expression of $\Pi_{c_{1} c_{2}}\left(k^{2}\right)$ is obtained as

$$
\begin{align*}
i\left(k^{2} g^{\rho_{1} \rho_{2}}-k^{\rho_{1}} k^{\rho_{2}}\right) \Pi_{c_{1} c_{2}}\left(k^{2}\right) & =i \Pi_{c_{1} c_{2}}^{\rho_{1} \rho_{2}}(k)  \tag{4.9}\\
& =-N_{f} \int \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} \operatorname{Tr}\left[\left(-i g_{s} \gamma^{\rho_{1}} t^{c_{1}}\right) \frac{i \not p}{p^{2}}\left(-i g_{s} \gamma^{\rho_{2}} t^{c_{2}}\right) \frac{i(\not p+\not p)}{(p+k)^{2}}\right],
\end{align*}
$$

where we are summing over the $N_{f}$ flavours that contribute to the loop. Since the generators $t^{c_{i}}$ of the gauge group $S U\left(N_{c}\right)$ do not live in the same sub-space of the Dirac matrices, their trace can be calculated separately using

$$
\begin{equation*}
\operatorname{Tr}\left[t^{c_{1}} t^{c_{2}}\right]=T_{f} \delta^{c_{1} c_{2}} \tag{4.10}
\end{equation*}
$$

where $T_{f}=1 / 2$. The integral over the loop momentum is then easily solved using the methods already described in Sec. 2.2: one first calculate the trace over the fermion loop, combine the denominators within the introduction of the Feynman parameter in Eq. (2.28) and solve the loop integral with Wick's rotation and $d$-dimensional spherical coordinates. ${ }^{55}$ In particular, we can conveniently write $\Pi_{c_{1} c_{2}}\left(k^{2}\right)$ as

$$
\begin{equation*}
\Pi_{c_{1} c_{2}}\left(k^{2}\right)=\left[\left(-\frac{2}{3} N_{f}\right) \frac{g_{s}^{2}}{(4 \pi)^{d / 2}} \frac{D(\epsilon)}{\epsilon} e^{-\gamma_{E} \epsilon}\right] \frac{1}{\left(-k^{2}\right)^{\epsilon}} \delta^{c_{1} c_{2}}, \tag{4.11}
\end{equation*}
$$

where

$$
\begin{equation*}
D(\epsilon)=6 e^{\gamma_{E} \epsilon} \frac{\Gamma(1+\epsilon) \Gamma^{2}(2-\epsilon)}{\Gamma(4-2 \epsilon)} . \tag{4.12}
\end{equation*}
$$

Combining the results of Eqs. (4.8) and (4.11) we finally obtain the expression for the gluon propagator with the insertion of one single massless quark bubble as

$$
\begin{equation*}
D_{\mu \nu}^{1, a b}(k)=\left[\left(-\frac{2}{3} N_{f}\right) \frac{g_{s}^{2}}{(4 \pi)^{d / 2}} \frac{D(\epsilon)}{\epsilon} e^{-\gamma_{E} \epsilon}\right] \frac{-i \delta^{a b}}{\left(-k^{2}\right)^{1+\epsilon}}\left(g_{\mu \nu}-\frac{k_{\mu} k_{\nu}}{k^{2}}\right) . \tag{4.13}
\end{equation*}
$$

Notice in the previous equation that both the Lorentz and colour structures of $\Pi_{c_{1} c_{2}}^{\rho_{1} \rho_{2}}(k)$ were contracted with the free gluon propagators in Eq. (4.6) and the contribution from $\Pi_{c_{1} c_{2}}\left(k^{2}\right) \equiv \Pi\left(k^{2}\right) \delta^{c_{1} c_{2}}$ was factored out in the term inside the square brackets. It is thus straightforward to obtain the expression for the gluon propagator with the insertion of $j-1$ massless quark bubble loops as

$$
\begin{equation*}
D_{\mu \nu}^{j-1, a b}(k)=\left[\left(-\frac{2}{3} N_{f}\right) \frac{g_{s}^{2}}{(4 \pi)^{d / 2}} \frac{D(\epsilon)}{\epsilon} e^{-\gamma_{E} \epsilon}\right]^{j-1} \frac{-i \delta^{a b}}{\left(-k^{2}\right)^{1+(j-1) \epsilon}}\left(g_{\mu \nu}-\frac{k_{\mu} k_{\nu}}{k^{2}}\right) . \tag{4.14}
\end{equation*}
$$

The dressed propagator $D_{\mu \nu}^{j-1, a b}(k)$ has the same structure of the free gluon propagator in the Landau gauge, but with a constant global factor and a shifted, analytically regular-
ized, exponent in the momentum transfer $k$. The term in parenthesis inside the square brackets, $-2 / 3 N_{f}$, is $\beta_{0, f}$, the fermionic contribution to $\beta_{0}$. Summing the bubble loops $D_{\mu \nu}^{1, a b}(k)+D_{\mu \nu}^{2, a b}(k)+D_{\mu \nu}^{3, a b}(k)+\cdots$ all the way to infinity effectively introduces the one-loop running coupling (c.f. Eq. (2.51)) with $\beta_{0, f}$ in the quark-gluon vertex. It is thus natural to replace this term by the full $\beta_{0}$ coefficient to effectively take into account vacuum polarization corrections from gluon self-interactions in the procedure known as naive nonabelianization. ${ }^{42,43}$ Hence, if we denote $c(h)$ the two-loop contribution to $C_{0}$ in the Landau gauge and with the gluon propagator analytically regularized by $\left(-k^{2}\right)^{1} \rightarrow\left(-k^{2}\right)^{1+h}$, we have that

$$
\begin{equation*}
c_{j, j-1}=\left[\frac{D(\epsilon)}{\epsilon} e^{-\gamma_{E} \epsilon}\right]^{j-1} c(j \epsilon-\epsilon) . \tag{4.15}
\end{equation*}
$$

Having obtained the coefficients $c_{j, j-1}$ needed at $1 / \beta_{0}$ accuracy we proceed with coupling renormalization. Since higher order coefficients of the QCD $\beta$-function scale as $\beta_{i \geq 1} \sim \beta_{0}^{i}$, at $1 / \beta_{0}$ accuracy the $\beta$-function must be truncated at its first term, i.e.,

$$
\begin{equation*}
\beta=\beta_{0} \frac{\alpha_{s}}{4 \pi} . \tag{4.16}
\end{equation*}
$$

Accordingly, the coupling renormalization constant is given by

$$
\begin{equation*}
Z_{\alpha}=\frac{1}{1+\beta / \epsilon}=1-\beta_{0} \frac{\alpha_{s}}{4 \pi} \frac{1}{\epsilon}+\mathscr{O}\left(\alpha_{s}^{2}\right) \tag{4.17}
\end{equation*}
$$

in the $\overline{\mathrm{MS}}$-scheme. The bare coupling $g_{s}$ is then replaced by the renormalized strong coupling $\alpha_{s}$ through the relation

$$
\begin{equation*}
\frac{g_{s}^{2}}{(4 \pi)^{d / 2}}=Z_{\alpha} \frac{\alpha_{s}}{4 \pi}\left(\mu^{2} e^{\gamma_{E}}\right)^{\epsilon}=\frac{1}{\beta_{0}} \frac{\epsilon \beta}{\epsilon+\beta}\left(\mu^{2} e^{\gamma_{E}}\right)^{\epsilon} . \tag{4.18}
\end{equation*}
$$

After coupling renormalization, therefore, the quantity $C_{0}$ takes the form

$$
\begin{equation*}
C_{0}=1+\frac{1}{\beta_{0}} \sum_{j=1}^{\infty} \frac{F(\epsilon, j \epsilon)}{j}\left(\frac{\beta}{\epsilon+\beta}\right)^{j}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right) \tag{4.19}
\end{equation*}
$$

where the auxiliary function $F(\epsilon, u)$ is given by

$$
\begin{equation*}
F(\epsilon, u) \equiv u e^{\gamma_{E} \epsilon} c(u-\epsilon) \mu^{2 u} D(\epsilon)^{u / \epsilon-1} . \tag{4.20}
\end{equation*}
$$

However, even after coupling renormalization, additional subtractions might be required. These remaining divergences are absorbed with the introduction of an extra renormalization constant

$$
\begin{equation*}
Z=1+\sum_{i=1}^{\infty} \frac{Z_{i}}{\epsilon^{i}}, \tag{4.21}
\end{equation*}
$$

such that the completely renormalized quantity $C=Z^{-1} C_{0}$ is determined by the term of order $\epsilon^{0}$ in the $\overline{\mathrm{MS}}$-scheme. The renormalization constant $Z$ is then used to determine the anomalous dimension of $C$,

$$
\begin{equation*}
\gamma=\mu \frac{\mathrm{d} \ln Z}{\mathrm{~d} \mu} \tag{4.22}
\end{equation*}
$$

Employing the chain rule, the derivative with respect to $\mu$ reads

$$
\begin{equation*}
\mu \frac{\mathrm{d}}{\mathrm{~d} \mu}=\mu \frac{\partial}{\partial \mu}+\mu \frac{\mathrm{d} \alpha_{s}}{\mathrm{~d} \mu} \frac{\partial}{\partial \alpha_{s}}=\mu \frac{\partial}{\partial \mu}-2 \alpha_{s}(\epsilon+\beta) \frac{\partial}{\partial \alpha_{s}}, \tag{4.23}
\end{equation*}
$$

where the last equality holds for in the large- $\beta_{0}$ limit. The only contribution from $Z$ that can yield finite results for $\gamma$ is $Z_{1}$, under the action of $-2 \alpha_{s} \epsilon \frac{\partial}{\partial \alpha_{s}}=-2 \beta \epsilon \frac{\partial}{\partial \beta}$. Thus the anomalous dimension of $C$ is given by

$$
\begin{equation*}
\gamma=-2 \beta \frac{\partial Z_{1}}{\partial \beta} \tag{4.24}
\end{equation*}
$$

In order to extract the coefficients of order $\epsilon^{-1}$ and $\epsilon^{0}$ in the pertubative series of $C_{0}$ given in Eq. (4.19) we employ the expansions

$$
\begin{equation*}
\left(\frac{\beta}{\epsilon+\beta}\right)^{j}=\left(\frac{\beta}{\epsilon}\right)^{j} \sum_{r=0}^{\infty} \frac{(-1)^{r}}{r!} \frac{\Gamma(j+r)}{\Gamma(j)}\left(\frac{\beta}{\epsilon}\right)^{r} \tag{4.25}
\end{equation*}
$$

and

$$
\begin{equation*}
F(\epsilon, u)=\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} F_{m, l} \epsilon^{m} u^{l} \tag{4.26}
\end{equation*}
$$

In particular, collecting the coefficients of order $\epsilon^{-1}$ yields the result

$$
\begin{align*}
\beta_{0} Z_{1} & =\beta F_{0,0}-\beta^{2} \frac{F_{1,0}}{2}+\beta^{3} \frac{F_{2,0}}{3}+\ldots  \tag{4.27}\\
& =-\sum_{i=1}^{\infty}(-\beta)^{i} \frac{F_{i-1,0}}{i} \tag{4.28}
\end{align*}
$$

from which we can derive the anomalous dimension of $C$ in terms of $F(\epsilon, u)$ :

$$
\begin{equation*}
\gamma=-2 \frac{\beta}{\beta_{0}} F(-\beta, 0) \tag{4.29}
\end{equation*}
$$

Analogously, extracting the coefficients of order $\epsilon^{0}$ yields the result

$$
\begin{equation*}
C=1+\frac{1}{\beta_{0}} \sum_{i=1}^{\infty} \Gamma(i) \beta^{i} F_{0, i}-\frac{1}{\beta_{0}} \sum_{i=1}^{\infty}(-\beta)^{i} \frac{F_{i, 0}}{i} \tag{4.30}
\end{equation*}
$$

for the renormalized quantity $C$. Using $F(0, u)-F(0,0)=\sum_{i=1}^{\infty} u^{i} F_{0, i}$ the first sum of the equation above can be written in an integral representation as

$$
\begin{equation*}
\frac{1}{\beta_{0}} \sum_{i=1}^{\infty} \Gamma(i) \beta^{i} F_{0, i}=\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} \frac{\sum_{i=1}^{\infty} u^{i} F_{0, i}}{u}=\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} \frac{F(0, u)-F(0,0)}{u} \tag{4.31}
\end{equation*}
$$

and a similar procedure is adopted for the second sum to obtain

$$
\begin{equation*}
-\frac{1}{\beta_{0}} \sum_{i=1}^{\infty}(-\beta)^{i} \frac{F_{i, 0}}{i}=\frac{1}{\beta_{0}} \int_{-\beta}^{0} \mathrm{~d} \epsilon \frac{\sum_{i=1}^{\infty} \epsilon^{i} F_{i, 0}}{\epsilon}=-\frac{1}{\beta_{0}} \int_{0}^{\beta} \mathrm{d} \epsilon \frac{F(-\epsilon, 0)-F(0,0)}{\epsilon} \tag{4.32}
\end{equation*}
$$

One more step can be done in the integral over $\epsilon$ due to Eq. (4.29), from which we get

$$
\begin{equation*}
\gamma-\gamma_{0} \frac{\alpha_{s}}{4 \pi}=-\frac{2 \beta}{\beta_{0}}[F(-\beta, 0)-F(0,0)], \tag{4.33}
\end{equation*}
$$

where $\gamma_{0}$ is the first coefficient of the anomalous dimension in the fixed-order perturbative expansion (c.f. Eq. (2.45)).

Finally, the renormalized quantity $C$ can be cast in the convenient form

$$
\begin{equation*}
C(\mu)=1+\frac{1}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma}{\alpha_{s}}-\frac{\gamma_{0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right), \tag{4.34}
\end{equation*}
$$

where $\beta$ is understood to be calculated at the scale $\mu$, i.e., $\beta=\beta_{0} \frac{\alpha_{s}(\mu)}{4 \pi}$, and

$$
\begin{equation*}
S(u) \equiv \frac{F(0, u)-F(0,0)}{u} . \tag{4.35}
\end{equation*}
$$

The first integral in Eq. (4.34), over $\gamma$, is present only in quantities that require additional subtractions beyond the renormalization of the massless fermionic chain in the gluon propagator. We identify the function $S(u)$ as the Borel transform of $C(\mu)$ since it is related to $C(\mu)$ through a Borel summation structure (c.f. Eq. (2.64)).

Eq. (4.34) represents the large- $\beta_{0}$ limit of the renormalized quantity $C$. As we are working only at leading order in $1 / \beta_{0}$, the perturbative expansion of Eq. (4.34) in the strong coupling must reproduce only the leading- $N_{f}$ coefficients obtained in full QCD, i.e., those of order $N_{f}^{j} Q_{s}^{j+1}$. Details about how to obtain the perturbative expansion of $C$ are given in the remainder of this chapter.

We end this section with a brief overview about the $d$-dimensional generalization of $\gamma_{5}$ discussed in Sec. 3.3. To restore the axial and Ward identities that are violated in dimensional regularization the introduction of extra, finite renormalization constants is needed. ${ }^{103}$ However, as it is shown in Eq. (4.34), all the information about extra renormalizations beyond the dressed gluon propagator is encoded in the anomalous dimension. Since the introduction of the extra, finite renormalization constants, in practice, recovers the 4-dimensional anomalous dimension of the pseudo-scalar and axial-vector currents, they do not need to be calculated; the knowledge of the 4-dimensional anomalous dimension is sufficient for our purposes.

### 4.2 The moments $M_{q, n}^{\delta}$ in the large- $\beta_{0}$ limit

In this section we present the results for the small-momentum expansion of the vector, axial-vector, scalar and pseudo-scalar correlators in the large- $\beta_{0}$ limit formulated in the previous section.

The correlators $\widehat{\Pi}^{\delta}(s)$ defined in Eq. (3.70), as well as the moments $M_{n}^{\delta}$ (Eq. (3.71)), are determined by non-trivial functions $C_{n}^{\delta}(\mu)$ that in large- $\beta_{0}$ take the form

$$
\begin{equation*}
C_{n}^{\delta}(\mu)=1+\frac{1}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right) \tag{4.36}
\end{equation*}
$$

accordingly to Eq. (4.34). Since the moments $M_{q, n}^{\delta}$ are accompanied by an explicit global factor $\left[2 m_{q}\right]^{-2 n}$, an extra renormalization constant $Z_{m}^{2 n}$ should be introduced besides coupling renormalization. This renormalization is included in $C_{n}^{\delta}(\mu)$ through the anomalous dimension $\gamma\left(\alpha_{s}\right)$ which, therefore, is given by

$$
\begin{equation*}
\gamma\left(\alpha_{s}\right)=2 n \gamma_{m}\left(\alpha_{s}\right), \tag{4.37}
\end{equation*}
$$

where $\gamma_{m}\left(\alpha_{s}\right)$ is the mass anomalous dimension calculated at $1 / \beta_{0}$ accuracy, ${ }^{41,114}$

$$
\begin{equation*}
\gamma_{m}\left(\alpha_{s}\right)=2 C_{F} \frac{\beta}{\beta_{0}} \frac{[1+(2 / 3) \beta] \Gamma(4+2 \beta)}{\Gamma^{2}(2+\beta) \Gamma(3+\beta) \Gamma(1-\beta)}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right), \tag{4.38}
\end{equation*}
$$

and $\beta$ is given in Eq. (4.16). As discussed in Sec. 3.3, the vector and axial-vector currents do not require current renormalization, while the current renormalization of the scalar and pseudo-scalar currents was already introduced when we defined $j^{S}$ and $j^{P}$ with an explicit mass factor.

The Borel transforms $S_{n}^{\delta}(u)$ are obtained as

$$
\begin{equation*}
S_{n}^{\delta}(u)=\frac{F_{n}^{\delta}(0, u)-F_{n}^{\delta}(0,0)}{u} \tag{4.39}
\end{equation*}
$$

where the auxiliary functions $F_{n}^{\delta}(\epsilon, u)$ are given by

$$
\begin{equation*}
F_{n}^{\delta}(\epsilon, u) \equiv u e^{\gamma_{E} \epsilon} c_{n}^{\delta}(u-\epsilon) \mu^{2 u} D(\epsilon)^{u / \epsilon-1} \tag{4.40}
\end{equation*}
$$

with $c_{n}^{\delta}(u-\epsilon)$ being the one-loop normalized coefficients in the small-momentum expansion of the two-loop correction of the correlators $\widehat{\Pi}^{\delta}(s)$ with the Landau-gauge gluon propagator analytically regularized,

$$
\begin{equation*}
c^{\delta}(u-\epsilon)=\sum_{n}\left(\frac{s}{4 m_{q}^{2}}\right)^{n} N_{n}^{\delta}(\epsilon) c_{n}^{\delta}(u-\epsilon) \tag{4.41}
\end{equation*}
$$



Figure 13 - Two-loop Feynman diagrams for the computation of heavy-quark current correlators in the large- $\beta_{0}$ limit. The diagram on the RHS must be counted twice. The dashed line represents the gluon propagator with the introduction of massless quark bubbles (Fig. 14). Crosses stand for the insertion of the currents defined in Eq. (3.3) with an external momentum $q$.

Source: By the author.

$$
\cdots \cdots \cdots \cdots \cdots=1000000000000000
$$

Figure 14 - Dressed gluon propagator with the introduction of massless quark bubble loops.
Source: By the author.

The result of the above equation is obtained from the computation of the Feynman diagrams depicted in Fig. 13. These diagrams are a composition of a heavy-quark bubble loop and a dressed gluon propagator with the insertion of massless quark bubble loops, as shown in Fig. 14. Accordingly, the fermionic contribution in the dressed gluon propagator is calculated with $N_{l}=N_{f}-1$ quark flavours. For this reason, in heavy-quark current correlators in large- $\beta_{0}$ the coefficient $\beta_{0}$ must be calculated with $N_{l}$ quark flavours instead of $N_{f}$, i.e.,

$$
\begin{equation*}
\beta_{0}=\frac{11}{3} N_{c}-\frac{2}{3} N_{l} . \tag{4.42}
\end{equation*}
$$

Similar to the one-loop calculation, the two-loop correction of the correlators $\widehat{\Pi}^{\delta}(s)$ has a general form given by

$$
\begin{align*}
& c^{\delta}(u-\epsilon) \propto \int \mathrm{d}^{d} k_{1} \mathrm{~d}^{d} k_{2}  \tag{4.43}\\
& \times\left[\left(\frac{g_{\sigma \rho}-k_{\sigma} k_{\rho} / k^{2}}{\left(-k^{2}\right)^{1+u-\epsilon}}\right) \operatorname{Tr}\left(\Gamma^{\delta} \frac{1}{k_{1}+q-m_{q}} \gamma^{\sigma} \frac{1}{k_{2}+q-m_{q}} \Gamma^{\delta} \frac{1}{k_{2}-m_{q}} \gamma^{\rho} \frac{1}{k_{1}-m_{q}}\right)\right. \\
& \left.+2\left(\frac{g_{\sigma \rho}-k_{\sigma} k_{\rho} / k^{2}}{\left(-k^{2}\right)^{1+u-\epsilon}}\right) \operatorname{Tr}\left(\Gamma^{\delta} \frac{1}{k_{1}+q-m_{q}} \Gamma^{\delta} \frac{1}{k_{1}-m_{q}} \gamma^{\sigma} \frac{1}{k_{2}-m_{q}} \gamma^{\rho} \frac{1}{k_{1}-m_{q}}\right)\right],
\end{align*}
$$

where $k \equiv k_{2}-k_{1}$ and $\Gamma^{\delta}$ is the combination of $\gamma$-matrices in the current $j^{\delta}$. The two-loop
scalar integrals to be studied, however, are more complicated and take the form

$$
\begin{align*}
& J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)= \\
& =\int \frac{\mathrm{d}^{d} k_{1} \mathrm{~d}^{d} k_{2}}{\left[\left(k_{1}+q\right)^{2}-m_{q}^{2}\right]^{n_{1}}\left[\left(k_{2}+q\right)^{2}-m_{q}^{2}\right]^{n_{2}}\left[k_{1}^{2}-m_{q}^{2}\right]^{n_{3}}\left[k_{2}^{2}-m_{q}^{2}\right]^{n_{4}}\left[\left(k_{2}-k_{1}\right)^{2}\right]^{n_{5}}} . \tag{4.44}
\end{align*}
$$

The exponents $n_{i}$ with $1 \leq i \leq 4$ are integer numbers while $n_{5}$, the power of the gluon propagator, is a combination of $u$ and $\epsilon$. Even after symmetry considerations, it remains to be computed about 30 integrals. It is not feasible to calculate many of these integrals using the techniques based on hypergeometric functions or the FMB transform described in Sec. 3.4, and they can not be reduced to simpler integrals using $\mathrm{IBP}^{107}$ due to the analytically regularized $n_{5}$. Thus, to compute the small-momentum expansion we use the Taylor expansion given in Eq. (3.66). In particular, the application of the momentum space d'Alembertian operator on $J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)$ reads

$$
\begin{align*}
\square_{q} J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right) & =4\left\{\left(n_{1}+n_{2}+1-d / 2\right)\left[n_{1} \mathbf{1}^{+} J_{2}+n_{2} \mathbf{2}^{+} J_{2}\right]\right. \\
& +m_{q}^{2}\left[n_{1}\left(n_{1}+1\right) \mathbf{1}^{++} J_{2}+n_{2}\left(n_{2}+1\right) \mathbf{2}^{++} J_{2}\right] \\
& \left.+n_{1} n_{2}\left[2 m_{q}^{2} \mathbf{1}^{+} \mathbf{2}^{+} J_{2}-\mathbf{1}^{+} \mathbf{2}^{+} \mathbf{5}^{-} J_{2}\right]\right\} \tag{4.45}
\end{align*}
$$

where we used the notation $\mathbf{1}^{ \pm} J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)=J_{2}\left(n_{1} \pm 1, n_{2}, n_{3}, n_{4}, n_{5}\right)$ and analogously for $\boldsymbol{2}^{ \pm}$and $\boldsymbol{5}^{ \pm}$. We also used $\boldsymbol{n}^{++} \equiv\left(\boldsymbol{n}^{+}\right)^{2}$. Recursively applying the d'Alembertian operator to obtain higher order terms in the small-momentum expansion requests a large amount of computer efforts. For this purpose, therefore, we used the computer algebra program FORM. ${ }^{115}$

After expanding the integrals $J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)$ up to a desired order in $q^{2}$ one needs to calculate the remaining single-scale tadpole integrals

$$
\begin{equation*}
\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\int \frac{\mathrm{d}^{d} k_{1} \mathrm{~d}^{d} k_{2}}{\left[k_{1}^{2}-m_{q}^{2}\right]^{\lambda_{1}}\left[k_{2}^{2}-m_{q}^{2}\right]^{\lambda_{2}}\left[\left(k_{2}-k_{1}\right)^{2}\right]^{\lambda_{3}}} . \tag{4.46}
\end{equation*}
$$

To solve this integral we first combine the denominators $\left[k_{2}^{2}-m_{q}^{2}\right]^{\lambda_{2}}$ and $\left[\left(k_{2}-k_{1}\right)^{2}\right]^{\lambda_{3}}$ with the introduction of a Feynman parameter given in Eq. (2.28) to arrive at an integral over $k_{2}$ that can be immediatly solved with Eq. (2.30). At this stage, the integral $\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ becomes

$$
\begin{equation*}
\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\frac{(2 \pi)^{d}}{(4 \pi)^{d / 2}} i(-1)^{\lambda_{1}+\lambda_{2}} \frac{\Gamma\left(\lambda_{2}+\lambda_{3}-d / 2\right)}{\Gamma\left(\lambda_{2}\right) \Gamma\left(\lambda_{3}\right)} \int_{0}^{1} \mathrm{~d} x x^{-\lambda_{3}-1+d / 2}(1-x)^{-\lambda_{2}-1+d / 2} \tag{4.47}
\end{equation*}
$$

$$
\times(-1)^{\lambda_{2}+\lambda_{3}-d / 2} \int \frac{\mathrm{~d}^{d} k_{1}}{\left[k_{1}^{2}-m_{q}^{2}\right]^{\lambda_{1}}\left[k_{1}^{2}-m_{q}^{2} /(1-x)\right]^{\lambda_{2}+\lambda_{3}-d / 2}} .
$$

We repeat the procedure to solve the integral over $k_{1}$ with the price of introducing an additional Feynman parameter. The result is

$$
\begin{align*}
\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) & =-\pi^{d}(-1)^{\lambda_{1}+\lambda_{2}+\lambda_{3}}\left(m_{q}^{2}\right)^{d-\lambda_{1}-\lambda_{2}-\lambda_{3}} \frac{\Gamma\left(\lambda_{1}+\lambda_{2}+\lambda_{3}-d\right)}{\Gamma\left(\lambda_{1}\right) \Gamma\left(\lambda_{2}\right) \Gamma\left(\lambda_{3}\right)}  \tag{4.48}\\
& \times \int_{0}^{1} \mathrm{~d} y y^{\lambda_{1}-1}(1-y)^{\lambda_{2}+\lambda_{3}-d / 2-1} \int_{0}^{1} \mathrm{~d} x \frac{x^{-\lambda_{3}-1+d / 2}(1-x)^{-\lambda_{2}-1+d / 2}}{[y+(1-y) /(1-x)]^{\lambda_{1}+\lambda_{2}+\lambda_{3}-d}} .
\end{align*}
$$

Factorizing the term $(1-x)$ in the denominator we arrive at an integral over $x$ that has the same structure of the integral representation of hypergeometric functions ${ }_{2} F_{1}$ given in Eq. (3.46). Thus,

$$
\begin{align*}
\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) & =-\pi^{d}(-1)^{\lambda_{1}+\lambda_{2}+\lambda_{3}}\left(m_{q}^{2}\right)^{d-\lambda_{1}-\lambda_{2}-\lambda_{3}}  \tag{4.49}\\
& \times \frac{\Gamma\left(\lambda_{1}+\lambda_{2}+\lambda_{3}-d\right)}{\Gamma\left(\lambda_{1}\right) \Gamma\left(\lambda_{2}\right) \Gamma\left(\lambda_{3}\right)} \frac{\Gamma\left(d / 2-\lambda_{3}\right) \Gamma\left(\lambda_{1}+\lambda_{3}-d / 2\right)}{\Gamma\left(\lambda_{1}\right)} \\
& \times \int_{0}^{1} \mathrm{~d} y y^{\lambda_{1}-1}(1-y)^{\lambda_{2}+\lambda_{3}-d / 2-1}{ }_{2} F_{1}\left(\lambda_{1}+\lambda_{2}+\lambda_{3}-d, d / 2-\lambda_{3} ; \lambda_{1} ; y\right)
\end{align*}
$$

The final integral over $y$ is now solved using the definition of the generalized hypergeometric functions given in Eq. (3.48). The result is expressed in terms of the hypergeometric function ${ }_{3} F_{2}\left(-d+\lambda_{1}+\lambda_{2}+\lambda_{3}, d / 2-\lambda_{3}, \lambda_{1} ; \lambda_{1}, \lambda_{1}+\lambda_{2}+\lambda+3-d / 2 ; 1\right)$ which in turn have two identical indices and, therefore, can be immediately reduced to ${ }_{2} F_{1}$. We then use the relation

$$
\begin{equation*}
{ }_{2} F_{1}(a, b ; c ; 1)=\frac{\Gamma(c) \Gamma(c-b-a)}{\Gamma(c-b) \Gamma(c-a)} \tag{4.50}
\end{equation*}
$$

to finally obtain an analytical result for $\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ :

$$
\begin{align*}
\widehat{J}_{2}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) & =-\pi^{d}(-1)^{\lambda_{1}+\lambda_{2}+\lambda_{3}}\left(m_{q}^{2}\right)^{d-\lambda_{1}-\lambda_{2}-\lambda_{3}}  \tag{4.51}\\
& \times \frac{\Gamma\left(\lambda_{1}+\lambda_{3}-d / 2\right) \Gamma\left(\lambda_{2}+\lambda_{3}-d / 2\right) \Gamma\left(d / 2-\lambda_{3}\right) \Gamma\left(\lambda_{1}+\lambda_{2}+\lambda_{3}-d\right)}{\Gamma\left(\lambda_{1}\right) \Gamma\left(\lambda_{2}\right) \Gamma\left(\lambda_{1}+\lambda_{2}+2 \lambda_{3}-d\right) \Gamma(d / 2)} .
\end{align*}
$$

This result completes the computation of all elements needed to the evaluation of $J_{2}\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)$ in the small-momentum asymptotic expansion.

### 4.2.1 Results

Following the procedure outlined above, we calculated the correlators $\widehat{\Pi}^{\delta}(s)$, for $\delta=$ $V, A, S, P$, in the large- $\beta_{0}$ limit. The Borel transforms $S_{n}^{\delta}(u)$ can be conveniently cast in terms of non-trivial polynomials of $u, P_{n}^{\delta}(u)$, that must be determined order by order in
the small-momentum expansion for each current. The results are

$$
\begin{align*}
& S_{n}^{V}(u)=\frac{6 C_{F} n}{u}-3 C_{F}\left[\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right]^{u} \frac{4^{n} \Gamma(2-u) \Gamma(u) \Gamma(2+n+u)^{2}}{(n+u) \Gamma(3+2 n+2 u)} P_{n}^{V}(u),  \tag{4.52a}\\
& S_{n}^{A}(u)=\frac{6 C_{F} n}{u}-3 C_{F}\left[\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right]^{u} \frac{4^{n} \Gamma(2-u) \Gamma(u) \Gamma(2+n+u)^{2}}{(n+u)(1+n+u) \Gamma(3+2 n+2 u)} P_{n}^{A}(u),  \tag{4.52b}\\
& S_{n}^{S}(u)=\frac{6 C_{F} n}{u}-3 C_{F}\left[\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right]^{u} \frac{4^{n} \Gamma(2-u) \Gamma(u) \Gamma(1+n+u)^{2}}{(3+2 n+2 u) \Gamma(2+2 n+2 u)} P_{n}^{S}(u),  \tag{4.52c}\\
& S_{n}^{P}(u)=\frac{6 C_{F} n}{u}-3 C_{F}\left[\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right]^{u} \frac{4^{n} \Gamma(2-u) \Gamma(u) \Gamma(2+n+u)^{2}}{(1+n+u) \Gamma(3+2 n+2 u)} P_{n}^{P}(u) . \tag{4.52d}
\end{align*}
$$

The polynomial $P_{1}^{V}(u)$, for instance, is given by

$$
\begin{equation*}
P_{1}^{V}(u)=3+\frac{92 u}{27}+\frac{29 u^{2}}{27}+\frac{u^{3}}{9} . \tag{4.53}
\end{equation*}
$$

The first few polynomials $P_{n}^{\delta}(u)$ are quoted in Appendix C and higher orders in $n$ can be found in Ref. ${ }^{116}$ (We remind the reader that $n$ starts at 1 for the vector and axial-vector currents, while for the scalar and pseudo-scalar currents it starts at $n=0$.) For the vector correlator we computed the Borel transforms $S_{n}^{V}(u)$ up to $n=12$, having full agreement with the results up to $n=2$ quoted in Ref. ${ }^{44}$ The results for the correlatores $\delta=P, S, A$ appeared in the literature for the first time in our recently published paper. ${ }^{45}$ Having the Borel transform of the vector correlator up to higher values of $n$ is of particular importance since it allow for a connection with non-relativistic QCD, the effective field theory used to describe the moments $M_{q, n}^{V}$ with large- $n$. For the remaining correlators there is no interest in moments with large values of $n$ since lattice errors increase significantly as $n$ grows, in contrast with the experimental moments $M_{q, n}^{V}$.

The general structure of the functions $S_{n}^{\delta}(u)$ fulfils the expectation of Borel transforms in the large- $\beta_{0}$ limit. Terms with a global factor $\left[e^{5 / 3} \mu^{2} / m_{q}(\mu)\right]^{u}$ lead to a Borel sum that is scale and scheme independent. ${ }^{40}$ This is easily verified when one combine the kernel weight $e^{-u / \beta}$ in the Borel integral of Eq. (4.36) with the factor $\left[e^{5 / 3} \mu^{2} / m_{q}(\mu)\right]^{u}$ and use the one-loop running coupling in terms of the renormalization group invariant $\Lambda_{\mathrm{QCD}}$ displayed in Eq. (2.54). In particular,

$$
\begin{equation*}
e^{-u / \beta}\left[\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right]^{u}=\left[\frac{e^{5 / 3} \Lambda_{\mathrm{QCD}}^{2}}{m_{q}^{2}(\mu)}\right]^{u}=\left[\frac{e^{5 / 3} \Lambda_{\mathrm{QCD}}^{2}}{\bar{m}_{q}^{2}}\right]^{u}+\mathscr{O}\left(\frac{1}{\beta_{0}}\right) . \tag{4.54}
\end{equation*}
$$

The explicit dependence on the scale through the factor $\mu^{2}$ is cancelled and the apparent scale dependence from the quark mass is dropped in large- $\beta_{0}$. Writing the quark mass at a scale $\mu$ in terms of a reference scale $\bar{m}_{q} \equiv m_{q}\left(m_{q}\right)$ brings corrections of order $1 / \beta_{0}$ and superior. As the Borel integral over $S_{n}^{\delta}(u)$ is already accompanied by an explicit $1 / \beta_{0}$,
these corrections are dropped at $1 / \beta_{0}$ accuracy. The functions $S_{n}^{\delta}(u)$ also present a term $1 / u$ that does not contain the factor $\left[e^{5 / 3} \mu^{2} / m_{q}(\mu)\right]^{u}$, which is a remainder of the quark mass renormalization. In fact, quite generally, the $1 / u$ term can be written as $n \gamma_{m, 0} / u$, where $\gamma_{m, 0}=6 C_{F}$ is the first coefficient of mass anomalous dimension (c.f. Eq. (2.55)). The apparent $1 / u$ pole is exactly cancelled due to the $\Gamma(u)$ in the numerators of $S_{n}^{\delta}(u)$ and the scheme and scale dependence from this term is compensated by the integral over the anomalous dimension in Eq. (4.36) and the global mass factor present in the moments $M_{q, n}^{\delta}$. Accordingly, the full expression of the moments, which in large- $\beta_{0}$ takes the form

$$
\begin{align*}
M_{q, n}^{\delta} & =\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 m_{q}^{2}(\mu)\right]^{n}}  \tag{4.55}\\
& \times\left[1+\frac{2 n}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right],
\end{align*}
$$

accordingly to Eqs. (3.71), (4.36) and (4.37), is $\mu$-independent. To show this one should write $m_{q}(\mu)$ in terms of $\bar{m}_{q}$, a reference value, and re-expand everything consistently at $1 / \beta_{0}$ accuracy. A detailed demonstration of this statement is presented in Appendix D.

The functions $S_{n}^{\delta}(u)$ contain poles on the real positive axis starting at $u=2$ arising from the $\Gamma(2-u)$ in the numerators of Eq. (4.52). As discussed in Sec. 2.3, these singularities are the IR renormalons of the Borel transform and because of their existence the Borel integral of the moments in Eq. (4.55) are ill-defined and should be calculated under a prescription. In this work we use the Principal Value prescription detailed in Appendix B to calculate the real part of the Borel integral and the intrinsic ambiguity arising from the deformation of the integration contour. This ambiguity, however, should compensated by non-perturbative corrections since in QCD IR renormalons have a one-to-one correspondence with higher order operators in the OPE. In particular, the OPE of the correlators $\widehat{\Pi}^{\delta}(s)$ is given by

$$
\begin{equation*}
\widehat{\Pi}^{\delta}(s)=\mathscr{C}_{0}^{\delta}(s)+\sum_{i} \mathscr{C}_{i}^{\delta}(s) \frac{\mathscr{O}_{i}}{\left(2 m_{\text {pole }}\right)^{d_{i}}}, \tag{4.56}
\end{equation*}
$$

where $\mathscr{C}_{0}^{\delta}(s)$ is the standard perturbative contribution calculated in this work, $\mathscr{C}_{i}^{\delta}(s)$ are the Wilson coefficients of the vacuum condensate operators $\mathscr{O}_{i}$, with dimension $d_{i}$, that encode non-perturbative corrections to the correlators and $m_{\text {pole }}$ is the renormalized mass in the on-shell renormalization scheme.

The leading non-perturbative correction comes from the 4-dimensional gluon condensate $\mathscr{O}_{G} \equiv\langle\Omega| \alpha_{s} G^{a \mu \nu} G_{\mu \nu}^{a}|\Omega\rangle$, whose Wilson coefficients in the small-momentum expansion are
given by ${ }^{117}$

$$
\begin{align*}
& \mathscr{C}_{G}^{V}(s)=\frac{1}{\pi} \sum_{n=1}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n}\left[-\frac{2 n+2}{15} \frac{(4)_{n}}{(7 / 2)_{n}}+\mathscr{O}\left(\frac{1}{\beta_{0}}\right)\right] \equiv \sum_{n=1}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n} \mathscr{C}_{G, n}^{V},  \tag{4.57a}\\
& \mathscr{C}_{G}^{P}(s)=\frac{1}{\pi} \sum_{n=0}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n}\left[-\frac{n-2}{12} \frac{(2)_{n+2}}{(3 / 2)_{n+2}}+\mathscr{O}\left(\frac{1}{\beta_{0}}\right)\right] \equiv \sum_{n=0}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n} \mathscr{C}_{G, n}^{P},  \tag{4.57b}\\
& \mathscr{C}_{G}^{A}(s)=\frac{1}{\pi} \sum_{n=1}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n}\left[-\frac{1}{3} \frac{(3)_{n+1}}{(5 / 2)_{n+1}}+\mathscr{O}\left(\frac{1}{\beta_{0}}\right)\right] \equiv \sum_{n=1}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n} \mathscr{C}_{G, n}^{A},  \tag{4.57c}\\
& \mathscr{C}_{G}^{S}(s)=\frac{1}{\pi} \sum_{n=0}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n}\left[-\frac{3 n+10}{12} \frac{(2)_{n+2}}{(5 / 2)_{n+2}}+\mathscr{O}\left(\frac{1}{\beta_{0}}\right)\right] \equiv \sum_{n=0}^{\infty}\left(\frac{s}{4 m_{\text {pole }}^{2}}\right)^{n} \mathscr{C}_{G, n}^{S}, \tag{4.57d}
\end{align*}
$$

with $(x)_{n} \equiv \Gamma(x+n) / \Gamma(x)$ being the Pochhammer symbol. The ambiguity related to $\mathscr{O}_{G}$, $\delta_{\Lambda}\left\{\mathscr{O}_{G}\right\}$, is calculated through the evaluation of the gluon condensate operator in large- $\beta_{0}$ limit, which consists essentially in the computation of a dressed gluon tadpole. ${ }^{44}$ The same result can be obtained using the low-energy theorem ${ }^{118}$ applied to the scalar gluonium correlator in large- $\beta_{0} .{ }^{119}$ The result for $\delta_{\Lambda}\left\{\mathscr{O}_{G}\right\}$ reads ${ }^{44}$

$$
\begin{equation*}
\delta_{\Lambda}\left\{\mathscr{O}_{G}\right\}=\frac{3}{2 \pi} \frac{N_{g}}{\beta_{0}} e^{10 / 3} \Lambda_{\mathrm{QCD}}^{4} \tag{4.58}
\end{equation*}
$$

where $N_{g}=C_{F} N_{c} / T_{F}\left(=8\right.$ in QCD) is the total number of gauge bosons in the $S U\left(N_{c}\right)$ Yang-Mills gauge theory. The ambiguity of the Borel integral arising from IR renormalons at $u=m$ in $S_{n}^{\delta}(u)$, accordingly to Eq. (B.10), is given by the residue of $e^{-u / \beta} S_{n}^{\delta}(u)$ at $u=m$. Since $\widehat{\Pi}^{\delta}(s)$ must be unambiguous, the relation

$$
\begin{equation*}
\frac{N_{c}}{16 \pi^{2}} \frac{1}{\beta_{0}} \operatorname{Res}\left[e^{-u / \beta} S_{n}^{\delta}(u) ; u=2\right]+\mathscr{C}_{G, n}^{\delta}(s) \frac{\delta_{\Lambda}\left\{\mathscr{O}_{G}\right\}}{\left[2 m_{\text {pole }}\right]^{4}}=0 \tag{4.59}
\end{equation*}
$$

must hold for all values of $n$. In the equation above the pole mass $m_{\text {pole }}$ can be replaced by the $\overline{\mathrm{MS}}$ mass, $m_{q}(\mu)$, since corrections will be of order $1 / \beta_{0}^{2}$ due to an already global $1 / \beta_{0}$ in the gluon condensate ambiguity given in Eq. (4.58). We have verified the ambiguity cancellation from the $u=2$ renormalon for all four correlators, enforcing the correctness of our results.

Apart from the already mentioned IR renormalons, the functions $S_{n}^{\delta}(u)$ also contain UV renormalons lying at the real negative axis starting at $u=-1$. As the IR renormalons are generated exclusively by the $\Gamma(2-u)$ in the numerators of Eq. (4.52), they are all simple poles. For the case of UV renormalons, however, the pattern is a little more intricate and its structure depends on the denominator of $S_{n}^{\delta}(u)$. The Borel transforms $S_{n}^{V}(u)$ of the
vector correlator, for instance, have singularities with a double- plus simple-pole structure at $u=-n,-(n+2),-(n+3), \ldots$ while all other UV poles are simple. In the case of the pseudo-scalar Borel transforms $S_{n}^{P}(u)$ the double- plus simple-pole structure starts at $u=-(n+1)$. Some exceptions might take place depending on the polynomials $P_{n}^{\delta}(u)$, tough. Since $u=-7$ is a root of $P_{0}^{P}(u)$, for example, the UV pole at $u=-7$ is simple in $S_{0}^{P}(u)$.

As discussed in Sec. 2.3.2, the position of the leading renormalon in the Borel transform dictates the higher-order behaviour of the perturbative series. In all functions $S_{n}^{\delta}(u)$ the leading renormalon is located at $u=-1$ and therefore a sign-alternating behaviour is expected. However, this behaviour can be postponed depending on the scale $\mu$. The simple-pole contribution of the leading UV renormalon is suppressed by the leading IR renormalon located at $u=2$, and we measure the strength of this suppression by taking the ratio of the residues at $u=2$ and $u=-1$, which yields

$$
\begin{equation*}
\frac{\operatorname{Res}\left[S_{n}^{\delta}, u=2\right]}{\operatorname{Res}\left[S_{n}^{\delta}, u=-1\right]} \propto\left(\frac{\mu}{m_{q}(\mu)}\right)^{6} \tag{4.60}
\end{equation*}
$$

Hence, for higher values of $\mu$ a fixed-sign behaviour should take place up to intermediate orders, postponing the asymptotic regime dictated by the leading UV renormalon.

Before discussing further the results in the $\overline{\mathrm{MS}}$-scheme, let us first make a short digression about the on-shell scheme. Due to the explicit mass factor in $M_{q, n}^{\delta}$, the Borel representation of the moments in large- $\beta_{0}$ can also be written in the on-shell renormalization scheme, which, in turn, changes significantly the renormalon structure of their Borel transforms. The $\overline{\mathrm{MS}}$ quark mass is transformed into the pole mass in large- $\beta_{0}$ using the relation ${ }^{120}$

$$
\begin{align*}
m_{q}(\mu)=m_{\text {pole }}[1 & +\frac{1}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]  \tag{4.61}\\
& \left.+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{m}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right],
\end{align*}
$$

where the Borel transform $S_{m}(u)$ is given by

$$
\begin{equation*}
S_{m}(u)=6 C_{F}\left[\frac{1}{2 u}-\left(\frac{e^{5 / 3} \mu^{2}}{m_{q}^{2}(\mu)}\right)^{u} \frac{\Gamma(u) \Gamma(1-2 u)}{\Gamma(3-u)}(1-u)\right] . \tag{4.62}
\end{equation*}
$$

Using Eq. (4.61) in Eq. (4.55) we obtain the integral representation of $M_{q, n}^{\delta}$ in the on-shell scheme as

$$
\begin{equation*}
M_{q, n}^{\delta}=\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 m_{\mathrm{pole}}^{2}\right]^{n}}\left[1+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{\mathrm{OS}, n}^{\delta}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right], \tag{4.63}
\end{equation*}
$$



Figure 15 - Absolute values of the residues of $S_{\mathrm{OS}, n}^{V}(u)$ and $S_{n}^{V}(u)$ at $u=-1$ and $u=2$.
Source: By the author.
where

$$
\begin{equation*}
S_{\mathrm{OS}, n}^{\delta}(u)=S_{n}^{\delta}(u)-2 n S_{m}(u) \tag{4.64}
\end{equation*}
$$

The integral over the anomalous dimension, as well as the explicit $1 / u$ term in the Borel transform, is exactly cancelled when $M_{q, n}^{\delta}$ is expressed in the on-shell scheme. This is in agreement with the no-running mass when the quark mass is expressed in the on-shell renormalization scheme.

Since the on-shell scheme Borel transform $S_{\mathrm{OS}, n}^{\delta}(u)$ is given by a difference between $S_{n}^{\delta}(u)$ and $S_{m}(u)$, some partial renormalon cancellation is expected. As shown in Fig. 15, in the vector correlator the IR pole at $u=2$ and the UV pole at $u=-1$ have smaller residues in the on-shell scheme, specially for lower values of $n$. However, the Borel transform $S_{m}(u)$ brings a new UV renormalon located at positive $u=1 / 2,{ }^{120}$ as is well known, in addition to an extra pole at $u=1$. The pole at $u=1 / 2$ is now the leading renormalon and it significantly enhances the divergence of the perturbative series of the moments. For that reason, one should not consider the on-shell scheme in reliable phenomenology studies of $M_{q, n}^{\delta}$ within the standard perturbative expansion in $\alpha_{s}$.

Going back to $M_{q, n}^{\delta}$ in the $\overline{\mathrm{MS}}$ renormalization scheme, the perturbative expansion in the strong coupling of the moments $M_{q, n}^{\delta}$ is derived from $C_{n}^{\delta}(\mu)$ in Eq. (4.36) with $\gamma\left(\alpha_{s}\right)=2 n \gamma_{m}\left(\alpha_{s}\right)$. The first contribution comes from the integral over the anomalous dimension, which can be expanded in $\alpha_{s}$ using the fixed-order perturbative expansion of the anomalous dimension

$$
\begin{equation*}
\gamma_{m}\left(\alpha_{s}\right)=\sum_{k=0}^{\infty} \gamma_{m, k}\left(\frac{\alpha_{s}}{4 \pi}\right)^{k+1} \tag{4.65}
\end{equation*}
$$

where the coefficients $\gamma_{m, k}$ are assumed to be calculated at $1 / \beta_{0}$ accuracy using Eq. (4.38). The first few coefficients are given by

$$
\begin{equation*}
\gamma_{m, 0}=6 C_{F}, \quad \gamma_{m, 1}=5 C_{F} \beta_{0} \quad \text { and } \quad \gamma_{m, 2}=-\frac{35}{6} C_{F} \beta_{0}^{2} . \tag{4.66}
\end{equation*}
$$

Within the perturbative expansion of $\gamma_{m}\left(\alpha_{s}\right)$ the first integral of $C_{n}^{\delta}(\mu)$ can be written as a power series in the strong coupling as

$$
\begin{equation*}
\frac{1}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{0}}{2}\right]=\frac{n}{\beta_{0}} \sum_{k=1}^{\infty} \frac{\gamma_{m, k}}{k}\left(\frac{\alpha_{s}(\mu)}{4 \pi}\right)^{k} \tag{4.67}
\end{equation*}
$$

To obtain the perturbative contribution arising from the Borel transform, we replace the function $S_{n}^{\delta}(u)$ by its Taylor expansion

$$
\begin{equation*}
S_{n}^{\delta}(u)=\sum_{k=0}^{\infty} s_{n, k}^{\delta} u^{k} \tag{4.68}
\end{equation*}
$$

in the Borel integral, yielding

$$
\begin{equation*}
\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)=\frac{1}{\beta_{0}} \sum_{k=1}^{\infty} s_{n, k-1}^{\delta}(k-1)!\beta_{0}^{k}\left(\frac{\alpha_{s}(\mu)}{4 \pi}\right)^{k} . \tag{4.69}
\end{equation*}
$$

Combining the results of Eqs. (4.67) and (4.69) we finally obtain the perturbative expansion of $C_{n}^{\delta}(\mu)$ :

$$
\begin{equation*}
C_{n}^{\delta}(\mu)=1+\frac{1}{\beta_{0}} \sum_{k=1}^{\infty}\left(\frac{n \gamma_{m, k}}{4^{k} k}+s_{n, k-1}^{\delta}(k-1)!\frac{\beta_{0}^{k}}{4^{k}}\right)\left(\frac{\alpha_{s}(\mu)}{\pi}\right)^{k}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right) . \tag{4.70}
\end{equation*}
$$

Both the anomalous dimension $\gamma_{m}(\alpha)$ at $1 / \beta_{0}$ accuracy and the Borel transforms $S_{n}^{\delta}(u)$ are a composition of $\Gamma$-functions and polynomials in their respective variables. In order to obtain the coefficients $\gamma_{m, k}$ and $s_{n, k}^{\delta}$ efficiently we use the compact form

$$
\begin{equation*}
\Gamma(n+x)=(n-1)!\exp \left\{x\left(H_{n-1}^{(1)}-\gamma_{E}\right)+\sum_{k=2}^{\infty}\left[\frac{(-x)^{k}}{k}\left(\zeta(k)-H_{n-1}^{(k)}\right)\right]\right\} \tag{4.71}
\end{equation*}
$$

valid for $n \geq 0$, to expand the various $\Gamma$-functions. In the previous equation $\zeta(k)$ is the Riemann zeta-function and $H_{n}^{(k)} \equiv \sum_{i=1}^{n} n^{-k}$ is the harmonic number. All the exponentials arising from the expanded $\Gamma$-functions can be combined into a single one and be expanded using the relation ${ }^{121}$

$$
\begin{equation*}
\exp \left[\sum_{k=1}^{\infty} a_{k} x^{k}\right]=\sum_{k=0}^{\infty} f_{k} x^{k} \tag{4.72}
\end{equation*}
$$

where $f_{0}=1$ and $f_{k \geq 1}$ are recursively obtained as

$$
\begin{equation*}
f_{k+1}=\frac{1}{k+1} \sum_{i=0}^{k}(i+1) f_{k-i} a_{i+1} . \tag{4.73}
\end{equation*}
$$

The expanded exponentials are finally combined with the finite polynomials into a single expansion using

$$
\begin{equation*}
\sum_{i=n}^{\infty} a_{i} x^{i} \sum_{j=m}^{N} b_{j} x^{j}=\sum_{i=n+m}^{\infty} x^{i} \sum_{j=m}^{\min (N, i-n)} a_{i-j} b_{j} . \tag{4.74}
\end{equation*}
$$

Below we give the leading- $N_{l}$ coefficients in the $\alpha_{s}$ expansion of the combination $\tilde{C}_{n}^{\delta} \equiv N_{n}^{\delta} C_{n}^{\delta}$ with $\mu=\bar{m}_{q}$ up to $\mathscr{O}\left(\alpha_{s}^{4}\right)$, which are the first unknown coefficients for the pseudo-scalar, scalar and axial-vector correlators, for the first four physical moments. Additional terms are easily generated with the procedure described above. In terms of $a_{s} \equiv \alpha_{s} / \pi$ we have, for the vector correlator,

$$
\begin{align*}
\tilde{C}_{1}^{V}= & 1.0667+2.5547 a_{s}+\left(\cdots+0.66228 N_{l}\right) a_{s}^{2}  \tag{4.75}\\
& +\left(\cdots+0.096101 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.096093 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{2}^{V}= & 0.45714+1.1096 a_{s}+\left(\cdots+0.45492 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.01595 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.036331 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{3}^{V}= & 0.27090+0.51940 a_{s}+\left(\cdots+0.42886 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.039596 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.033047 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{4}^{V}= & 0.18471+0.20312 a_{s}+\left(\cdots+0.42483 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.052774 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.033935 N_{l}^{3}\right) a_{s}^{4} .
\end{align*}
$$

For the pseudo-scalar correlator we find

$$
\begin{align*}
\tilde{C}_{0}^{P}= & 1.3333+3.1111 a_{s}+\left(\cdots+0.61729 N_{l}\right) a_{s}^{2}  \tag{4.76}\\
& +\left(\cdots+0.37997 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.22899 N_{l}^{3}\right) a_{s}^{4} \\
\tilde{C}_{1}^{P}= & 0.53333+2.0642 a_{s}+\left(\cdots+0.28971 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots+0.070202 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.035807 N_{l}^{3}\right) a_{s}^{4} \\
\tilde{C}_{2}^{P}= & 0.30477+1.2117 a_{s}+\left(\cdots+0.26782 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots+0.015357 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.021840 N_{l}^{3}\right) a_{s}^{4} \\
\tilde{C}_{3}^{P}= & 0.20318+0.71276 a_{s}+\left(\cdots+0.28628 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.0091663 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.021261 N_{l}^{3}\right) a_{s}^{4},
\end{align*}
$$

while for the scalar correlator we have

$$
\begin{align*}
\tilde{C}_{0}^{S}= & 0.8+0.60247 a_{s}+\left(\cdots+0.58765 N_{l}\right) a_{s}^{2}  \tag{4.77}\\
& +\left(\cdots+0.23981 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.20536 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{1}^{S}= & 0.22857+0.42582 a_{s}+\left(\cdots+0.23664 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots+0.0039812 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.030916 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{2}^{S}= & 0.10159+0.15356 a_{s}+\left(\cdots+0.15634 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.018026 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.017163 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{3}^{S}= & 0.055411+0.032800 a_{s}+\left(\cdots+0.12383 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.020909 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.013605 N_{l}^{3}\right) a_{s}^{4} .
\end{align*}
$$

Finally, for the axial-vector correlator we obtain

$$
\begin{align*}
\tilde{C}_{1}^{A}= & 0.53333+0.84609 a_{s}+\left(\cdots+0.41317 N_{l}\right) a_{s}^{2}  \tag{4.78}\\
& +\left(\cdots+0.047848 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.069840 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{2}^{A}= & 0.15238+0.14166 a_{s}+\left(\cdots+0.19218 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.020498 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.017170 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{3}^{A}= & 0.067725-0.012760 a_{s}+\left(\cdots+0.13562 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.022336 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.012418 N_{l}^{3}\right) a_{s}^{4}, \\
\tilde{C}_{4}^{A}= & 0.036941-0.057469 a_{s}+\left(\cdots+0.10678 N_{l}\right) a_{s}^{2} \\
& +\left(\cdots-0.020499 N_{l}^{2}\right) a_{s}^{3}+\left(\cdots+0.010501 N_{l}^{3}\right) a_{s}^{4} .
\end{align*}
$$

Before going to the computation of the ratios of moments $R_{q, n}^{\delta}$ in large- $\beta_{0}$, let us briefly emphasize the points that enforces the correctness of our results:

- The leading- $N_{l}$ coefficients in the perturbative expansion of the moments $M_{q, n}^{\delta}$ in large- $\beta_{0}$ we obtained in this work are in full agreement with those that are already known in full QCD. ${ }^{20-24}$
- The IR renormalons in the Borel transforms $S_{n}^{\delta}(u)$ when written in the $\overline{\mathrm{MS}}$-scheme are all simple and start at $u=2$. No pole at $u=1$ is present, as is expected since there is no dimension- 2 condensate contribution. ${ }^{40}$
- We have verified the exactly ambiguity cancellation related to the $u=2$ IR pole and the gluon condensate operator in the four correlators.


### 4.3 The ratios of moments $R_{q, n}^{\delta}$ in the large- $\beta_{0}$ limit

With the knowledge of the moments $M_{q, n}^{\delta}$ at $1 / \beta_{0}$ accuracy it is straightforward to obtain the dimensionless ratios $R_{q, n}^{\delta}$ in the large- $\beta_{0}$ limit. Using the definition of $R_{q, n}^{\delta}$
given in Eq. (3.21), together with the Borel representation of $C_{n}^{\delta}(\mu)$ in Eq. (4.36) and re-expanding the expression in $1 / \beta_{0}$ one can derive the representation of $R_{q, n}^{\delta}$ in the large- $\beta_{0}$ limit as

$$
\begin{equation*}
R_{q, n}^{\delta}=\left(\frac{9}{4} Q_{q}^{2}\right)^{\frac{1}{n(n+1)}} \frac{\left(N_{n}^{\delta}\right)^{\frac{1}{n}}}{\left(N_{n+1}^{\delta}\right)^{\frac{1}{n+1}}}\left[1+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} B_{n}^{\delta}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right] \tag{4.79}
\end{equation*}
$$

where the new Borel transforms $B_{n}^{\delta}(u)$ are obtained in terms of $S_{n}^{\delta}(u)$ as

$$
\begin{equation*}
B_{n}^{\delta}(u)=\frac{S_{n}^{\delta}(u)}{n}-\frac{S_{n+1}^{\delta}(u)}{n+1} \tag{4.80}
\end{equation*}
$$

Since the ratios $R_{q, n}^{\delta}$ are designed to exactly cancel the explicit mass dependence of the moments $M_{q, n}^{\delta}$, no integral over an anomalous dimension is present in the Borel representation of $R_{q, n}^{\delta}$. Accordingly, the corresponding Borel transforms $B_{n}^{\delta}(u)$ do not contain a scale and scheme dependent term proportional to $\gamma_{m, 0} / u$, which vanishes in Eq. (4.80). The Borel integral by itself is already scale and scheme independent thanks to the now global $\left[e^{5 / 3} \mu^{2} / m_{q}(\mu)\right]^{u}$ factor in $B_{n}^{\delta}(u)$. An important comment is that, in contrast with the moments $M_{q, n}^{\delta}$, where the running mass takes an important role to ensure the scale independence, the ratios $R_{q, n}^{\delta}$ in large- $\beta_{0}$ have no dependence at all on the scale and renormalization scheme of the quark mass. Since $m_{q}(\mu)$ now enters only through the already $1 / \beta_{0}$ suppressed Borel transforms $B_{n}^{\delta}(u)$, any correction due to a running mass or a changing in scheme will be of order $1 / \beta_{0}^{2}$ and, therefore, is dropped in large- $\beta_{0}$.

Analogously to Eq. (4.70), the perturbative expansion of the ratios is given by

$$
\begin{equation*}
R_{q, n}^{\delta}=\left(\frac{9}{4} Q_{q}^{2}\right)^{\frac{1}{n(n+1)}} \frac{\left(N_{n}^{\delta}\right)^{\frac{1}{n}}}{\left(N_{n+1}^{\delta}\right)^{\frac{1}{n+1}}}\left[1+\sum_{k=1}^{\infty} b_{n, k-1}^{\delta}(k-1)!\frac{\beta_{0}^{k}}{4^{k}}\left(\frac{\alpha_{s}(\mu)}{\pi}\right)^{k}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right], \tag{4.81}
\end{equation*}
$$

where $b_{n, k}^{\delta}$ are the coefficients in the Taylor expansion

$$
\begin{equation*}
B_{n}^{\delta}(u)=\sum_{k=0}^{\infty} b_{n, k}^{\delta} u^{k} . \tag{4.82}
\end{equation*}
$$

The fact that the functions $B_{n}^{\delta}(u)$ are given by a difference between two Borel transforms $S_{n}^{\delta}(u)$ suggests that renormalon cancellations might take place. Both the leading IR and UV renormalons are significantly smaller in $B_{n}^{\delta}(u)$ compared to their $S_{n}^{\delta}(u)$ counterparts, as it is depicted in Fig. 16a. The residue of the leading UV renormalon in $B_{3}^{V}$, for instance, is $31(38)$ times smaller than the one in $S_{3}^{V}\left(S_{4}^{V}\right)$. For the leading IR renormalon, at $u=2$, the residue of $B_{3}^{V}$ is only $16.0 \%(8.1 \%)$ than that of $S_{3}^{V}\left(S_{4}^{V}\right)$. This strongly suggests that the perturbative series of the ratios $R_{q, n}^{\delta}$ should be better behaved than those of $M_{q, n}^{\delta}$, due to a smaller contribution of the poles that govern the asymptotic regime of perturbative


Figure 16 - (a) Absolute value of the residues of $B_{n}^{V}$ relative to those of $S_{n}^{V}$ for the leading UV and IR poles at $p=-1$ and $p=2$, respectively. (b) Absolute value of the residues of $B_{n}^{V}$ for the same two poles.

Source: By the author.
series. Furthermore, as shown in Fig. 16b, in absolute terms the residue of $B_{n}^{V}(u)$ at $u=-1$ decreases as $n$ grows, suggesting an exact cancellation at $n \rightarrow \infty$. Since the residue at $u=2$ increases, it is expected that the asymptotic behaviour of $R_{q, n}^{\delta}$ dictated by the leading UV renormalon should be postponed more and more as $n$ grows.

These statements about the renormalon structure of $B_{n}^{\delta}(u)$ are corroborated through the following analysis. For the vector correlator the non-polynomial part of $S_{n}^{V}(u)$, i.e., the one from the $\Gamma$-functions, tends to $6 C_{F} e^{-5 / 3} \sqrt{\pi / n}$ when expanded both for $u=-1$ and $n=\infty$, while it tends to $(3 / 64) e^{10 / 3} \sqrt{\pi / n}$ when expanded both for $u=2$ and $n=\infty$. In both cases the behaviour at large- $n$ is of order $1 / n^{1 / 2}$. The polynomials $P_{n}^{V}(u)$ evaluated at $u=-1$ behave as $P_{n}^{V}(-1) \simeq 0.7 n^{3 / 2}$ and thus the residue of the leading UV renormalon in $S_{n}^{V}(u)$ can be approximated by the linear expression $1.4 C_{F} n$. As the Borel transforms $B_{n}^{\delta}(u)$ are differences between two $S_{n}^{\delta}(u)$ in the combination $S_{n}^{\delta}(u) / n$, the residue of $B_{n}^{V}(u)$ at $u=-1$ goes to zero as $n$ grows. A similar analysis can be performed for the leading IR renormalon, where the polynomials $P_{n}^{V}(u)$ behave as $P_{n}^{V}(2) \simeq 2.1 n^{7 / 2}$, which correspond to a non-linear dependence in $n$ for the residues at $u=2$, approximately given by $4.9 C_{F} n^{3}$. Hence, the residues at $u=2$ in $B_{n}^{V}(u)$ display a behaviour with a quadratic dependence in $n$. Similar conclusions can be drawn for the pseudo-scalar correlator.

Since the singularities in the ratios are softened with respect to $M_{q, n}^{\delta}$, we expect that the perturbative series of $R_{q, n}^{\delta}$ will be significantly improved. In the next chapter we investigate the higher-order behaviour of the perturbative series using the results in large- $\beta_{0}$ obtained
in this work. We shall see importance of the renormalization scale $\mu$ in the interplay between IR and UV renormalons, as expected from Eq. (4.60), as well as the consequences of renormalon cancellations in $B_{n}^{\delta}(u)$.

## 5 ANALYSIS OF PERTURBATIVE SERIES

In this chapter we perform a detailed analysis of the perturbative series of the moments $M_{q, n}^{\delta}$ and the ratios of moments $R_{q, n}^{\delta}$ for the vector and pseudo-scalar correlators, which are the ones that can be reliably obtained with experimental data or through lattice simulations. We also show that our results can guide us in the design of combinations of moments or dimensionless ratios with better perturbative behaviour, which can be used to improve the determinations of quark masses and $\alpha_{s}$ based on heavy-quark current correlators.

Throughout our analysis we will always use the fixed reference values $\bar{m}_{b} \equiv m_{b}\left(m_{b}\right)=$ 4.18 GeV and $\bar{m}_{c} \equiv m_{c}\left(m_{c}\right)=1.28 \mathrm{GeV}$ for the bottom- and charm-quark masses, respectively, in the $\overline{\mathrm{MS}}$-scheme. For the central value of the strong coupling we use the world average value recommend by the Particle Data Group $\alpha_{s}^{\left(N_{f}=5\right)}\left(m_{Z}\right)=0.1179,{ }^{65}$ with $m_{Z}=91.19 \mathrm{GeV}$, which yields the values $\alpha_{s}^{\left(N_{f}=4\right)}\left(\bar{m}_{b}\right)=0.2245$ and $\alpha_{s}^{\left(N_{f}=3\right)}\left(\bar{m}_{c}\right)=$ 0.3865 using the five-loop running coupling ${ }^{61,63,64,122}$ and four-loop matching ${ }^{123,124}$ at the thresholds, both in full QCD, obtained with REvolver. ${ }^{125}$ For consistency, in large- $\beta_{0}$ the running coupling will be performed at one-loop level, with three and four active flavours for the charm and bottom moments, respectively.

The perturbative coefficients in $\alpha_{s}$ for $M_{q, n}^{\delta}$ are obtained combining Eqs. (3.71) and (4.70), while the perturbative expansion for the ratios $R_{q, n}^{\delta}$ are directly obtained with Eq. (4.81). In large- $\beta_{0}$, the "true value" that the perturbative series should approach is determined by the Borel sum of the series given by the integral representation at $1 / \beta_{0}$ accuracy quoted in Eqs. (4.55) and (4.79) for the moments and ratios, respectively. The integral over the anomalous dimension, which is present in $M_{q, n}^{\delta}$, is easily solved numerically, while the integral over the Borel transform needs to be solved under a certain prescription to circumvent the singularities of IR origin located at the positive real axis. Here we adopt the Principal Value (PV) prescription detailed in Appendix B.

### 5.1 The higher-order behaviour of $M_{q, n}^{\delta}$

In this section we focus on the higher-order behaviour of the moments $M_{q, n}^{\delta}$ extensively used in precise determinations of the charm- and bottom-quark masses. Using Eq. (4.70) we could verify up to high orders in the perturbative series that the effective parameter of expansion is indeed $\alpha_{s} \sqrt{n}$, as is well known. ${ }^{76}$ This means that for large- $n$ the expansion parameter is no longer small and we have a breakdown of the perturbative series. Therefore, we restrict our analysis to the first few physical moments only, for which standard perturbative QCD remains valid.

Due to the global mass factor $\left[2 m_{q}(\mu)\right]^{-2 n}$ in the moments, we must take into account
corrections from the running mass in large- $\beta_{0}$. Using the running mass of Eq. (2.56) with $\beta$ and $\gamma_{m}\left(\alpha_{s}\right)$ at $1 / \beta_{0}$ accuracy as given in Eqs. (4.16) and (4.38), we expand the mass factor in $1 / \beta_{0}$ as

$$
\begin{equation*}
\frac{1}{\left[2 m_{q}(\mu)\right]^{2 n}}=\frac{1}{\left[2 \bar{m}_{q}\right]^{2 n}}\left[1-\frac{2 n}{\beta_{0}} \int_{\alpha_{s}\left(\bar{m}_{q}\right)}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s} \frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right] \tag{5.1}
\end{equation*}
$$

to obtain the perturbative expansion of $M_{q, n}^{\delta}$ as

$$
\begin{equation*}
M_{q, n}^{\delta}=\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[2 \bar{m}_{q}\right]^{2 n}}\left[C_{n}^{\delta}(\mu)-\frac{2 n}{\beta_{0}} \int_{\alpha_{s}\left(\bar{m}_{q}\right)}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s} \frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right] \tag{5.2}
\end{equation*}
$$

where the perturbative series of $C_{n}^{\delta}(\mu)$ in the strong coupling is given in Eq. (4.70). The integral in the equation above is related to the running mass, and therefore we do not expand it in $\alpha_{s}$. There is also a mass-dependence in $C_{n}^{\delta}(\mu)$ through $\alpha_{s}$-suppressed logarithms arising from the Taylor expansion of $S_{n}^{\delta}(u)$. Employing the running mass in these terms is equivalent to directly replace $m_{q}(\mu)$ by $\bar{m}_{q}$, since corrections from the integral over the anomalous dimension will be of order $1 / \beta_{0}^{2}$ and therefore are dropped in our large- $\beta_{0}$ analysis.

Before going through our results, one more comment is necessary. In full generality, the perturbative series of $M_{q, n}^{\delta}$ in QCD, as shown in Eq. (3.19), is given in terms of two renormalization scales: one for the strong coupling and one another for the mass. In large- $\beta_{0}$, however, we can not use two scales since in this case the Borel integral will be scheme and scale dependent. Therefore, we should use a single-scale in the analysis of $M_{q, n}^{\delta}$, which is equivalent to set $\mu_{\alpha}=\mu_{m} \equiv \mu$ in Eq. (3.19).

In Fig. 17 we display the perturbative series of the first three vector charm moments $M_{c, n}^{V}$ for three choices of the scale $\mu$. These plots are normalized by the Borel sum such that the true value that all series should approach is 1 . One can easily see in Fig. 17a the role of the scale $\mu$ in the asymptotic behaviour of the perturbative series. Since the leading renormalon of the Borel transforms $S_{n}^{\delta}(u)$ is located at $u=-1$, a sign-alternating behaviour is expected in the asymptotic regime. However, since the ratio between the residues of $S_{n}^{\delta}(u)$ at $u=2$ and $u=-1$ scales as $\mu^{6}$, as given in Eq. (4.60), for large values of $\mu$ the sign-alternating behaviour is postponed to higher orders in the perturbative expansion. In Fig. 17b we can see for $M_{c, 2}^{V}$ that a fixed-sign behaviour takes place up to intermediate orders even for low values of $\mu$, indicating a stronger contribution from IR renormalons as $n$ grows. The perturbative series of Fig. 17b also shows a pattern of crossing the true value, without a plateau of convergence. As we discussed in Sec. 2.3.2, this is a behaviour typically seen in perturbative series highly dominated by the leading IR renormalon and with a large value of the expansion parameter. In Fig. 17c we see that the ambiguity of the Borel integral arising from IR renormalons increases significantly and


Figure 17 - Perturbative series of the moments $M_{c, n}^{V}$ normalized by the Borel sum calculated in the PV prescription. The gray band represents the ambiguity.

Source: By the author.
the perturbative series does not display any sign of a reliable description of the true value. In fact, for $n=5$ the ambiguity of the Borel integral is of about $160 \%$, which is a clear indication that non-perturbative corrections are very large.

Similar plots for the pseudo-scalar charm moments $M_{c, n}^{P}$ with $n=1,2,3$ are shown in Fig. 18. The perturbative series of $M_{c, 1}^{P}$ displayed in Fig. 18a has a somewhat different pattern since it approaches the Borel sum from bellow, while the other moments approach from above. Compared to the vector moments, the pseudo-scalar charm moments have a weaker dependence on the leading IR renormalon. In Fig. 18b we see that $M_{c, 2}^{P}$ does not present a run-away behaviour and has a tiny Borel ambiguity that can not even be visualized. This was expected since there is no renormalon at $u=2$ in $M_{c, 2}^{P}$. The weaker contribution of IR renormalons in the pseudo-scalar moments postpones the value of $n$ at which the typical run-away behaviour of the perturbative series is achieved. The fourth pseudo-scalar moment $M_{c, 3}^{P}$, Fig. 18c, for instance, has the same behaviour of the second vector moment $M_{c, 2}^{V}$ shown in Fig. 17 b .

Finally, in Fig. 19 we show the perturbative series of the first three physical vector bottom moments $M_{b, n}^{V}$. Since at the bottom mass scale the strong coupling is significantly smaller than in the charm mass scale, large contributions from the leading IR renormalon are not translated into perturbative series with a run-away behaviour, but to an almost convergent series, showing signs of reaching the asymptotic regime only at about the 14 th order and with low values of $\mu$, as one can see in Fig. 19a. A general pattern seen in the bottom moments $M_{b, n}^{V}$ is that the perturbative series approach the true value slower as $n$ increases, which is a sign of the dominance of the leading IR renormalon. Compared to $M_{b, 1}^{V}$, at order $\alpha_{s}^{3}$ the third moment $M_{b, 3}^{V}$ shown in Fig. 19c is significantly far from the Borel sum and presents a much larger spread on its value depending on the choice


Figure 18 - Perturbative series of the moments $M_{c, n}^{P}$ normalized by the Borel sum calculated in the PV prescription. The gray band represents the ambiguity.

Source: By the author.


Figure 19 - Perturbative series of the moments $M_{b, n}^{V}$ normalized by the Borel sum calculated in the PV prescription.

Source: By the author.
of the scale $\mu$. In the plots of Fig. 19 the ambiguities of the Borel integral are tiny and can not be visualized. This reinforces that in the bottom moments the non-perturbative corrections from condensates are completely negligible for all practical purposes.

### 5.2 The higher-order behaviour of $\boldsymbol{R}_{q, n}^{\delta}$

We now turn to the analysis of the perturbative series of the dimensionless ratios of moments $R_{q, n}^{\delta}$ with $\delta=V, P$. Using Eq. (4.81) we verified that in $R_{q, n}^{\delta}$ the effective parameter of expansion is also $\alpha_{s} \sqrt{n}$, but with a global $1 / n^{2}$ common to all coefficients. Although this extra factor softens the effective expansion parameter, we should still restrict


Figure 20 - Perturbative series of the dimensionless ratios $R_{c, n}^{V}$ normalized by the Borel sum calculated in the PV prescription. The gray band represents the ambiguity.

Source: By the author.
our analysis to ratios that do not contain moments with $n>4$. In the ratios $R_{q, n}^{\delta}$ the mass dependence enter only through $\alpha_{s}$-suppressed logarithms, and therefore any corrections due to a running mass or changes in the mass renormalization scheme are beyond $1 / \beta_{0}$ accuracy and should be dropped in our large- $\beta_{0}$ analysis. Accordingly, we use only the fixed values $\bar{m}_{c}$ and $\bar{m}_{b}$ everywhere a mass appears in $R_{q, n}^{\delta}$.

Before going through the results of the perturbative series of the ratios of moments, it is imperative to let clear that the upcoming plots for $R_{q, n}^{\delta}$ are with a significant smaller scale in the $y$-axis to facilitate the discussion. A direct comparison between the plots of $R_{q, n}^{\delta}$ and $M_{q, n}^{\delta}$ without taking into account the different scale in the $y$-axis might give a misleading impression that the perturbative series of the moments are better than those of the ratios.

In Fig. 20 we display the perturbative series of the first vector charm ratios $R_{c, n}^{V}$ for three choices of the scale $\mu$. Again, these plots are normalized by the Borel sum such that all series should approach unity. The scale $\mu$ plays also an important role in the interplay between IR and UV renormalons in the ratios of moments. In Fig. 20a we see a clear onset of the sign-alternating behaviour for $\mu=1.5 \mathrm{GeV}$ that is significantly suppressed when we double the scale to $\mu=3 \mathrm{GeV}$. In Figs. 20b and 20c the partial renormalon cancellation in the Borel transform $B_{n}^{\delta}(u)$ takes place and the oscillation of the perturbative series is postponed even for low values of $\mu$. In Fig. 20c we have a perturbative series with almost no sign of UV renormalon, and therefore, due to the high value of $\alpha_{s}$ at the charm mass scale, the series just crosses the true value and runs away without a plateau of convergence. A direct consequence of the partial IR renormalon cancellation is that in the ratios the ambiguities of the Borel integral are significantly smaller than in the moments $M_{q, n}^{\delta}$. We see in Fig. 20b that the ambiguity of $R_{c, 2}^{V}$ is in the order of $0.2 \%$, a much smaller value


Figure 21 - Perturbative series of the dimensionless ratios $R_{c, n}^{P}$ normalized by the Borel sum calculated in the PV prescription. The gray band represents the ambiguity.

Source: By the author.
compared to $0.9 \%$ (2.9\%) in its counterparts $M_{c, 2}^{V}\left(M_{c, 3}^{V}\right)$.
Analogous plots for the pseudo-scalar ratios $R_{c, n}^{P}$ with $n=0,1,2$, where $R_{c, 0}^{P} \equiv M_{c, 0}^{P}$, are shown in Fig. 21. The 0-th pseudo-scalar moment $M_{c, 0}^{P}$ has the same dimensionless structure of $R_{q, n}^{\delta}$ and therefore is an important observable for $\alpha_{s}$-extractions. However, $M_{c, 0}^{P}$ can not benefit from the partial renormalon cancellation present in the ratios $R_{q, n}^{\delta}$, as its Borel transform is given solely by $S_{0}^{P}(u)$. We see in Fig. 21a that $M_{c, 0}^{P}$ has a large contribution from the leading UV renormalon, which translates into a perturbative series with strong oscillatory behaviour, especially for low $\mu$. In Figs. 21b and 21c the partial renormalon cancellation takes place and we have a perturbative series for the ratios comparable to the ones presented in the vector case (Fig. 20).

Finally, in Fig. 22 we display the results for the first three vector bottom ratios $R_{b, n}^{V}$. As in the case of the bottom moments $M_{b, n}^{V}$, the bottom ratios display a much better perturbative behaviour, which simply reflects the dominance of IR renormalons with a relatively small expansion parameter $\alpha_{s}$. The onset of the sign-alternating asymptotic behaviour is postponed to significantly higher orders in the expansion. The asymptotic regime dictated by the leading UV renormalon is visible only in the first ratio $R_{b, 1}^{V}$ and with a relatively small value of $\mu$, as shown in Fig. 22a. In Fig. 22b we do not see any sign of an oscillatory pattern up to 15th order, and the series uniformly approach to the true value. Albeit very well behaved, the perturbative series of the ratios $R_{b, c}^{V}$ approach the true value relatively slowly and small values of $\mu$ are required to have a series that could reliably describe the true value at the first orders in $\alpha_{s}$. In the plots of Fig. 22 the ambiguity of the Borel integral arising from IR renormalons are tiny and not visible. This feature of the bottom ratios $R_{b, n}^{V}$ was already present in the bottom moments $M_{b, n}^{V}$.


Figure 22 - Perturbative series of the dimensionless ratios $R_{b, n}^{V}$ normalized by the Borel sum calculated in the PV prescription.

Source: By the author.

### 5.3 From large- $\boldsymbol{\beta}_{0}$ to QCD

Before we use the large- $\beta_{0}$ limit to draw conclusions that may apply to full QCD, it is important to compare the series up to order $\alpha_{s}^{3}$, the last order exactly known in full QCD. We do not intend to use the large- $\beta_{0}$ limit to estimate higher-order unknown coefficients in QCD. Rather, our goal is to derive more general conclusions that can guide the phenomenological applications of $M_{q, n}^{\delta}$ and $R_{q, n}^{\delta}$, namely quark masses and strong coupling extractions. In full QCD we should use four and five active flavours for the charm and bottom moments, respectively. Accordingly, in QCD perturbative series we use as reference values $\alpha_{s}^{\left(N_{f}=4\right)}\left(\bar{m}_{c}\right)=0.3849$ and $\alpha_{s}^{\left(N_{f}=5\right)}\left(\bar{m}_{b}\right)=0.2243$, and the five-loop running to RG-evolve the coupling and the quark mass. In order to mimic as much as possible the analysis we did in large- $\beta_{0}$, in the QCD series of the moments $M_{q, n}^{\delta}$ we use the same scale for the mass and the coupling, which amounts to use $\mu_{m}=\mu_{\alpha}=\mu$ in Eq. (3.19). For the ratios we use $\mu=\mu_{\alpha}$, but we fix the mass scale to $\mu_{m}=\bar{m}_{q}$, since any information about the running mass had to be dropped in the large- $\beta_{0}$ analysis of $R_{q, n}^{\delta}$.

Let us start with a direct comparison of the series obtained in QCD and in large- $\beta_{0}$ for the moments $M_{q, 1}^{\delta}$ with a somewhat large renormalization scale $\mu \sim 2 \bar{m}_{q}$, as depicted in the upper panels of Fig. 23. We see in Figs. 23a and 23b that for the vector moments the large- $\beta_{0}$ series captures most of the features of the series in QCD. However, this is not entirely true for the pseudo-scalar moment $M_{c, 1}^{P}$ illustrated in Fig. 23c, where the large- $\beta_{0}$ series seems to be mirrored with respect to its QCD counterpart. Looking into the lower panels of Fig. 23 we see that there is a significant discrepancy in the behaviour of both series at order $\alpha_{s}^{2}$ related to the leading UV renormalon. At low values of $\mu$ the perturbative series becomes dominated by the UV renormalon at $u=-1$, but in QCD this dominance is not as salient as in the large- $\beta_{0}$ series. With $\mu \sim \bar{m}_{q}$ the large- $\beta_{0}$ series


Figure 23 - Perturbative series up to $\mathscr{O}\left(\alpha_{s}^{3}\right)$ for $M_{q, 1}^{\delta}$ in large- $\beta_{0}$ and full QCD for $\mu \sim 2 \bar{m}_{q}$ (upper panels) and $\mu \sim \bar{m}_{q}$ (lower panels). The solid horizontal line represents the large- $\beta_{0}$ Borel sum while the dashed lines are the values of the moments obtained from experimental data or lattice simulations $M_{b, 1}^{V}=(4.526 \pm 0.111) 10^{-3} \mathrm{GeV}^{-2}{ }^{32}$, $M_{c, 1}^{V}=(21.21 \pm 0.36) 10^{-2} \mathrm{GeV}^{-2}{ }^{31}$ and $M_{c, 1}^{P}=(1.402 \pm 0.020) 10^{-1} \mathrm{GeV}^{-2} .{ }^{34}$

Source: By the author.
of $M_{q, n}^{\delta}$ flips sign at $\mathscr{O}\left(\alpha_{s}^{2}\right)$, but this is not observed in the QCD series. Hence, in full QCD a competition between UV and IR renormalons persists to intermediate orders even for low values of scale. In particular, the non-log coefficients $c_{2,0,0}^{\delta,(n)}$ in Eq. (3.19) are not well reproduced in large- $\beta_{0}$.

Analogous plots for the dimensionless ratios $R_{q, 1}^{\delta}$ are shown in Fig. 24. We see in Figs. 24a, 24b and 24c that for $\mu \sim 2 \bar{m}_{q}$ the QCD and the large- $\beta_{0}$ series behave almost identically. In particular, the mirrored pattern observed in $M_{c, 1}^{P}$ is no longer seen in the ratio $R_{c, 1}^{P}$. The discrepancy at order $\alpha_{s}^{2}$ observed in the moments $M_{q, n}^{\delta}$ for low values of $\mu$ is still present in the pseudo-scalar ratio, but this discrepancy is postponed to $\mathscr{O}\left(\alpha_{s}^{3}\right)$ in the vector ratios, as one can see in the lower panels of Fig. 24. Fortunately, the QCD series of the ratios appear to approach the experimental- or simulation-based values faster


Figure 24 - Perturbative series up to $\mathscr{O}\left(\alpha_{s}^{3}\right)$ for $R_{q, 1}^{\delta}$ in large- $\beta_{0}$ and full QCD for $\mu \sim 2 \bar{m}_{q}$ (upper panels) and $\mu \sim \bar{m}_{q}$ (lower panels). The solid horizontal line represents the large- $\beta_{0}$ Borel sum while the dashed lines are the values of the ratios obtained from experimental data $R_{b, 1}^{V}=0.8502 \pm 0.0014, R_{c, 1}^{V}=1.770 \pm 0.017^{38,39}$ or lattice simulations $R_{c, 1}^{P}=1.199 \pm 0.004 .{ }^{36}$

Source: By the author.
than the series in large- $\beta_{0}$ approach the Borel sum. This remark does not apply to the moments $M_{q, n}^{\delta}$, since the QCD and the large- $\beta_{0}$ series seem to approach their "true values" equally fast.

### 5.4 Combined moments and dimensionless ratios

With the knowledge of the renormalon structure of the moments $M_{q, n}^{\delta}$ and ratios of moments $R_{q, n}^{\delta}$ in large- $\beta_{0}$ we can construct new combinations of moments designed as to suppress, or even exactly cancel, specific renormalon contributions. Ideally, one should rely on combinations that do not involve moments with $n>4$, since these are not well described within standard perturbative QCD. For dimensional quantities, i.e., the ones that retain a global mass factor, we consider the combinations

$$
\begin{equation*}
\widehat{M}_{q}^{\delta}(a, b, c) \equiv\left[M_{q, 1}^{\delta}\right]^{a}\left[M_{q, 2}^{\delta}\right]^{b}\left[M_{q, 3}^{\delta}\right]^{c}, \tag{5.3}
\end{equation*}
$$

for the vector and pseudo-scalar correlators, whereas for dimensionless quantities, ideal for $\alpha_{s}$-extractions, we consider the combinations

$$
\begin{align*}
& \widehat{R}_{q}^{V}(a, b, c) \equiv\left[R_{q, 1}^{V}\right]^{a}\left[R_{q, 2}^{V}\right]^{b}\left[R_{q, 3}^{\delta}\right]^{c},  \tag{5.4}\\
& \widehat{R}_{q}^{P}(a, b, c) \equiv\left[M_{q, 0}^{P}\right]^{a}\left[R_{q, 1}^{P}\right]^{b}\left[R_{q, 2}^{P}\right]^{c},
\end{align*}
$$

with arbitrary parameters $a, b$ and $c$. The large- $\beta_{0}$ representation of $\widehat{M}_{q}^{\delta}$ is obtained inserting Eq. (4.55) in its definition and re-expanding the expression in $1 / \beta_{0}$. In an analogous fashion the large- $\beta_{0}$ representation of $\widehat{R}_{q}^{\delta}$ is obtained by consistently re-expanding the expression in $1 / \beta_{0}$ using the results of Eq. (4.79). The Borel transforms of $\widehat{M}_{q}^{\delta}$ and $\widehat{R}_{q}^{\delta}$ can then be easily written as linear combinations of $S_{n}^{\delta}(u)$ or $B_{n}^{\delta}(u)$.

The numerators of the leading UV and IR renormalons in the new Borel transforms become a linear combination on the parameters $a, b$ and $c$. Suitable choices of these values can lead to significant reductions of renormalon singularities, which can translate into better behaved perturbative series. Reducing the contribution from the leading IR renormalon located at $u=2$ is of particular importance due to its one-to-one correspondence with the leading non-perturbartive correction parametrized by the gluon condensate. Moreover, large values of the residue of the leading IR renormalon are responsible for the run-away behaviour of perturbative series observed in the charm moments and ratios of moments of Figs. 17, 18, 20 and 21. However, working with combinations aimed at reducing only the IR contribution can yield perturbative series largely dominated by the leading UV renormalon, which translates into badly behaved perturbative series with strong signalternating behaviour already at low orders in $\alpha_{s}$. Therefore, one must achieve some compromise in the interplay between the leading IR and UV renormalons.

On what concerns the combination of moments, one can also make use of the strong positive correlations between the moments $M_{q, n}^{\delta}$ to achieve a new $\widehat{M_{q}^{\delta}}$ with considerably lower experimental (or lattice simulation) error. Whereas the final uncertainties in charm mass and $\alpha_{s}$ determinations from charm moments have a large contribution from the perturbative error (mainly determined by scale variations), bottom mass and $\alpha_{s}$ extractions from bottonium moments are largely dominated by the experimental error. ${ }^{31,32,38}$ Choosing values of $a, b$ and $c$ in $\widehat{M}_{q}^{\delta}$ such that $a$ and $c$ have equal sign, but opposite to $b$, can lead to significant reductions in the final experimental error of the combined moment. It is also desirable to obtain a combined moment with a linearised dependence on the quark mass through the global mass factor to avoid some numerical complications in fit procedures. ${ }^{31}$ This amounts to set an extra constraint given by $2 a+4 b+6 c=1$ in the case of $\widehat{M}_{q}^{\delta}$. In Fig. 25 we show the perturbative series of the combined bottom moment $\widehat{M}_{b}^{V}$ with the choice $a=-2, b=2.41$ and $c=-0.77$, normalized by the Borel sum.


Figure 25 - Perturbative series of the combined bottom moment $\widehat{M}_{b}^{V}(-2,2.41,-0.77)$ normalized by the Borel sum.

Source: By the author.

Compared to $M_{b, 1}^{V}$, Fig. 19a, this choice for the parameters reduces the leading IR (UV) renormalon contribution in about $98 \%$ ( $87 \%$ ) and linearises the mass dependence in the denominator. The perturbative series was slightly improved, being less sensitive to scale variations, but due to the introduction of the strongly correlated moments $M_{b, 2}^{V}$ and $M_{b, 3}^{V}$ the experimental error in $\widehat{M}_{b}^{V}$ was reduced in almost $37 \%$ (c.f. Sec. 3.1 for a discussion about the determination of the experimental values of $M_{q, n}^{V}$ ). Therefore, we end up with a combined moment with a relative experimental error comparable to the ones in $M_{b, 3}^{V}$ and $M_{b, 4}^{V}$, which are at the limit of being well described with perturbative QCD, and improved the perturbative behaviour seen in $M_{b, 1}^{V}$.

In the combined dimensionless ratios $\widehat{R}_{q}^{\delta}$ of Eq. (5.4) one should only focus on values of $a, b$ and $c$ that reduce the leading IR and UV renormalons, since we do not have the global mass factor to linearise and the strong correlations between moments might not give further improvement in the experimental error, as the dimensionless ratios $R_{q, n}^{\delta}$ already benefit from these positive correlations. In Fig. 26 we show the normalized perturbative series of the combined charm ratio $\widehat{R}_{c}^{V}(-1 / 3,1,-1 / 3)$ for three values of $\mu$. Compared to the ratio $R_{c, 2}^{V}$ depicted in Fig. 20b, from which the main results of Refs. ${ }^{38,39}$ are based, the perturbative series is significant improved, since it approaches the true value faster, is less sensitive to scale variations and does not has a run-away behaviour typically seen in series largely dominated by the leading IR renormalon, as it was expected since the residue of both the leading UV and IR renormalons were reduced by about $70 \%$.

Interestingly, for the combined moment $\widehat{M}_{b}^{V}$ the large- $\beta_{0}$ and QCD series are relatively


Figure 26 - Perturbative series of the combined charm ratio $\widehat{R}_{c}^{V}(-1 / 3,1,-1 / 3)$ normalized by the Borel sum.

Source: By the author.
close. As one can see in the equations below,

$$
\begin{align*}
\left.\widehat{M}_{b}^{V}(-2,2.41,-0.77)\right|_{\mathrm{QCD}} & =\frac{1}{\left[2 \bar{m}_{b}\right]}\left[0.6000-0.2502\left(\frac{\alpha_{s}}{\pi}\right)\right.  \tag{5.5}\\
& \left.+3.2267\left(\frac{\alpha_{s}}{\pi}\right)^{2}-8.1871\left(\frac{\alpha_{s}}{\pi}\right)^{3}\right],
\end{align*}
$$

$$
\begin{align*}
\left.\widehat{M}_{b}^{V}(-2,2.41,-0.77)\right|_{\text {large }-\beta_{0}} & =\frac{1}{\left[2 \bar{m}_{b}\right]}\left[0.6000-0.2502\left(\frac{\alpha_{s}}{\pi}\right)\right.  \tag{5.6}\\
& \left.+0.4866\left(\frac{\alpha_{s}}{\pi}\right)^{2}-14.2240\left(\frac{\alpha_{s}}{\pi}\right)^{3}\right]
\end{align*}
$$

the non-log coefficients in QCD and large- $\beta_{0}$ have the same sign and order of magnitude up to $\mathscr{O}\left(\alpha_{s}^{3}\right)$. For the combined ratio $\widehat{R}_{c}^{V}$ the situation is even better and the large- $\beta_{0}$ series predicts with high precision the non-log coefficients obtained in full QCD:

$$
\begin{gather*}
\left.\widehat{R}_{c}^{V}(-1 / 3,1,-1 / 3)\right|_{\mathrm{QCD}}=0.9015-0.05343\left(\frac{\alpha_{s}}{\pi}\right)+0.6315\left(\frac{\alpha_{s}}{\pi}\right)^{2}-2.4339\left(\frac{\alpha_{s}}{\pi}\right)^{3},  \tag{5.7}\\
\left.\widehat{R}_{c}^{V}(-1 / 3,1,-1 / 3)\right|_{\text {large }-\beta_{0}}=0.9015-0.05343\left(\frac{\alpha_{s}}{\pi}\right)+0.6752\left(\frac{\alpha_{s}}{\pi}\right)^{2}-1.9944\left(\frac{\alpha_{s}}{\pi}\right)^{3} . \tag{5.8}
\end{gather*}
$$

This better agreement between the large- $\beta_{0}$ and QCD series even at low values of $\mu$ is probably due to the significant reduction of the leading UV renormalon in our combined moments and ratios, since the main source of disagreement between both series in $M_{q, n}^{\delta}$ and $R_{q, n}^{\delta}$ was due to a flip in sign at order $\alpha_{s}^{2}$ or $\alpha_{s}^{3}$, as already discussed in the previous section.

### 5.5 Discussion

With the observations done along this chapter, we are in a position to draw some conclusions and make some plausible hypothesis about the results in full QCD and in the impact of quark masses and $\alpha_{s}$ extractions based on heavy-quark current correlators. Concerning the moments $M_{q, n}^{\delta}$ and the ratios $R_{q, n}^{\delta}$ with $\delta=V, P$ :

- We demonstrated that the ratios $R_{q, n}^{\delta}$ benefit from a reduction in the leading UV renormalon as $n$ grows. This leads to perturbative series dominated by the leading IR renormalon that approach their true values uniformly, but somewhat slowly. For larger $n$ the perturbative series would require larger values of $\alpha_{s}$ to achieve the true value at order $\alpha_{s}^{3}$. This behaviour was seen in the QCD analysis of Refs. ${ }^{38,39}$ and is compatible with the partial renormalon cancellation found in this work.
- We have shown that using combinations of moments or dimensionless ratios based on renormalon suppressions can produce series with improved perturbative behaviour. Moreover, due to the strong positive correlations, combined moments can also benefit from significant reductions in the experimental error. Using these combinations to reduce the spread of scale variation at $\mathscr{O}\left(\alpha_{s}^{3}\right)$ could lead to significant improvements in determinations of quark masses, as well as $\alpha_{s}$, based on heavy-quark current correlators.
- The perturbative series of the bottom moments $M_{b, n}^{V}$ and the ratios $R_{q, n}^{\delta}$, with $\delta=V, P$, are well behaved for not too low values of $\mu$, but display a rather large spread arising from scale variations. This spread is reduced at $\mathscr{O}\left(\alpha_{s}^{4}\right)$, indicating that the perturbative error in bottom mass extractions from $M_{b, n}^{V}$, as well as $\alpha_{s}$ determinations from $R_{q, n}^{\delta}$, could be significantly reduced when the $\alpha_{s}^{4}$ coefficients in full QCD become available.
- With the perturbative series in large- $\beta_{0}$ for the vector charm moments $M_{c, n}^{V}$ we confirmed the expectations that one should not consider moments with large values of $n$ under standard perturbative QCD for reliable phenomenological applications. The perturbative series of the third vector charm moment $M_{c, 3}^{V}$, for instance, does not show any sign of describing the true value.
- Finally, it is important to point out that in large- $\beta_{0}$ the charm vector and pseudoscalar moments (and ratios) behave rather similar with respect to scale variations. This is not in agreement with what is observed in QCD, where results obtained with the pseudo-scalar correlator tend to have larger perturbative errors. ${ }^{31,38}$ The source of this qualitative difference is probably beyond $1 / \beta_{0}$ accuracy.


## 6 CONCLUSION

In this work we obtained the small-momentum expansion of the vector, axial-vector, scalar and pseudo-scalar correlators in the large- $\beta_{0}$ limit. For the vector correlator we extended the work by Grozin and Sturm ${ }^{44}$ to higher values of $n$, while the results for the other correlators are new, appearing for the first time in the literature in our work of Ref. ${ }^{45}$ The exact Borel transforms of these correlators are quoted in Eq. (4.52).

With our results we were able to reproduce the leading- $N_{l}$ coefficients already known in the full theory up to four-loop accuracy for the first few physical moments and, in the case of the axial-vector, scalar and pseudo-scalar correlators, calculate for the very first time the previously unknown corresponding leading- $N_{l}$ five-loop (and higher) terms, as it is displayed in Eqs. (4.75), (4.76), (4.77) and (4.78).

We used our results for the Borel transforms to gain understanding about the higherorder behaviour of the vector and pseudo-scalar moments, $M_{q, n}^{\delta}$, and their dimensionless ratios, $R_{q, n}^{\delta}$. These moments have been for a long time the basis for precise determinations of the charm- and bottom-quark masses. ${ }^{26-32}$ The pseudo-scalar ratios $R_{c, n}^{P}$ have also been used since some time for precise determinations of the strong coupling, ${ }^{33-37}$ and recently it was shown that one can also obtain precise values of $\alpha_{s}$ using the vector charm ratios $R_{c, n}^{V}$ thanks to the current status of the experimental data of $e^{+} e^{-}$annihilation in the low-energy region. ${ }^{38,39}$ (The bottom vector ratios, however, can not give competitive results yet due to the large experimental errors. ${ }^{38}$ ) We identified partial renormalon cancellations in $R_{q, n}^{\delta}$ that make the perturbative series for the ratios better behaved than their counterparts $M_{q, n}^{\delta}$. The higher-order behaviour of $M_{q, n}^{\delta}$ and $R_{q, n}^{\delta}$ can be seen in Figs. 17, 18, 19, 20, 21 and 22 .

We verified that the large- $\beta_{0}$ series of the heavy-quark current correlators predict rather well the results obtained in the full theory for higher values of the renormalization scale, as is shown in Figs. 23 and 24. Some discrepancies arise for low values of the scale, since the leading UV renormalon seems to be less pronounced in QCD, making the interplay between UV and IR renormalons persist at intermediate orders. Compared to full QCD, the large- $\beta_{0}$ series, therefore, present a stronger oscillatory behaviour.

Moreover, we have checked that in large- $\beta_{0}$ both for the moments and the ratios the perturbative series at $\mathscr{O}\left(\alpha_{s}^{3}\right)$ is somewhat far from the expected value calculated by the Borel sum, and have a large spread from scale variations. The situation is considerably better at $\mathscr{O}\left(\alpha_{s}^{4}\right)$, which means quark masses and $\alpha_{s}$ extractions from heavy-quark current correlators can probably be significantly improved should the five-loop results in QCD be available.

Using the knowledge of the renormalon structure in the Borel transforms one can
also design new combinations of moments or of their dimensionless ratios with improved perturbative behaviour. Suitable combinations of $M_{q, n}^{\delta}$, as in Eq. (5.3), can yield new quantities with less sensitivity to scale variations and smaller experimental errors. Bottom quark mass extractions from $M_{b, n}^{V}$ can be significantly improved from these combined moments as the large errors in the high-energy region of $e^{+} e^{-}$annihilation experimental data give a large contribution in the final uncertainty of $m_{b}$. Constructing a combination of bottom moments with smaller experimental errors and good perturbative behaviour is the key for improving the determinations of $m_{b}$. Combined ratios (c.f. Eq. (5.4)) might not benefit much from smaller experimental errors since the positive correlations between ratios are not as strong as in the case of the moments. However, significant reductions in the spread due to renormalization scale variations can lead to an improvement in the final error on $\alpha_{s}$ from the charm vector and pseudo-scalar ratios.

Finally, the results for higher values of $n$ in the vector correlators obtained in this work should allow for a connection with non-relativistic QCD, the effective field theory needed to properly describe the moments $M_{q, n}^{V}$ at large- $n$. Exploring this connection can be of particular importance since the experimental error of $M_{q, n}^{V}$ decreases as $n$ grows. This is left for future work.

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## Appendix A Expansion of $\alpha_{s}$ and quark masses in logs

A powerful method to recover the logarithms in resumed perturbative series is make use of the expansions of $\alpha_{s}$ and $m_{q}$ in logs.

The expansion of $\alpha_{s}\left(\mu_{2}\right)$ in terms of $\alpha_{s}\left(\mu_{1}\right)$ is obtained with the QCD $\beta$-function given in Eq. (2.44) in the physical limit $\epsilon \rightarrow 0^{+}$. We write $\alpha_{s}\left(\mu_{2}\right)$ as

$$
\begin{equation*}
\alpha_{s}\left(\mu_{2}\right)=\sum_{i=1}^{\infty}\left[\alpha_{s}\left(\mu_{1}\right)\right]^{i} \sum_{j=0}^{i-1} d_{i j}^{\alpha} L^{j}, \tag{A.1}
\end{equation*}
$$

where $L \equiv \ln \left(\frac{\mu_{1}}{\mu_{2}}\right)$, and impose it to satisfy the $\beta$-function

$$
\begin{equation*}
-\frac{\mu_{2}}{2} \frac{\mathrm{~d} \ln \alpha_{s}\left(\mu_{2}\right)}{\mathrm{d} \mu_{2}}=\beta\left(\alpha_{s}\left(\mu_{2}\right)\right)=\sum_{i=0}^{\infty} \beta_{i}\left(\frac{\alpha_{s}\left(\mu_{2}\right)}{4 \pi}\right)^{i+1} \tag{A.2}
\end{equation*}
$$

to determine the coefficients $d_{i j}^{\alpha}$. In particular, the expansion of $a_{s}\left(\mu_{2}\right) \equiv \alpha_{s}\left(\mu_{1}\right) / \pi$ reads

$$
\begin{equation*}
a_{s}\left(\mu_{2}\right)=a_{s}\left(\mu_{1}\right)+a_{s}^{2}\left(\mu_{1}\right)\left[-\frac{1}{2} \beta_{0} L\right]+a_{s}^{3}\left(\mu_{1}\right)\left[\frac{1}{4} \beta_{0}^{2} L^{2}-\frac{1}{8} \beta_{1} L\right]+\mathscr{O}\left(a_{s}^{4}\right) . \tag{A.3}
\end{equation*}
$$

The procedure is completely analogous for the quark mass. The renormalization group equation is a bit more intricate, tough. We first write the quark mass at a scale $\mu_{2}$ as

$$
\begin{equation*}
m_{q}\left(\mu_{2}\right)=m_{q}\left(\mu_{1}\right) \sum_{i=0}^{\infty}\left[\alpha_{s}\left(\mu_{1}\right)\right]^{i} \sum_{j=0}^{i} d_{i j}^{m} L^{j} \tag{A.4}
\end{equation*}
$$

and then impose it to satisfy the renormalization group equation of Eq. (2.43). In particular,

$$
\begin{align*}
& \frac{m_{q}\left(\mu_{2}\right)}{m_{q}\left(\mu_{1}\right)}=1+a_{s}\left(\mu_{1}\right) \frac{\gamma_{0} L}{4}+a_{s}^{2}\left(\mu_{1}\right)\left[2 L^{2}\left(\frac{\beta_{0} \gamma_{0}}{32}+\frac{\gamma_{0}^{2}}{64}\right)+\frac{\gamma_{1} L}{16}\right]  \tag{A.5}\\
& +a_{s}^{3}\left(\mu_{1}\right)\left[\frac{4}{3} L^{3}\left(\frac{\beta_{0}^{2} \gamma_{0}}{64}+\frac{3 \beta_{0} \gamma_{0}^{2}}{256}+\frac{\gamma_{0}^{3}}{512}\right)+2 L^{2}\left(\frac{\beta_{1} \gamma_{0}}{128}+\frac{\beta_{0} \gamma_{1}}{64}+\frac{\gamma_{0} \gamma_{1}}{128}\right)+\frac{\gamma_{2} L}{64}\right]+\mathscr{O}\left(a_{s}^{4}\right) .
\end{align*}
$$

## Appendix B Principal Value prescription

This appendix was based on Ref. ${ }^{121}$
The integral's Principal Value (PV) among the positive real axis of a function $f(u)$ is defined as the average

$$
\begin{equation*}
\operatorname{PV}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\} \equiv \frac{1}{2}\left[\mathrm{PV}_{+}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\}+\mathrm{PV}_{-}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\}\right] \tag{B.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{PV}_{ \pm}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\} \equiv \int_{C^{ \pm}} \mathrm{d} u f(u) \tag{B.2}
\end{equation*}
$$

and $C^{+(-)}$are paths that contour the poles of $f(u)$ with (anti-)clockwise infinitesimal semi-circles. In principle, the shape of the semi-circles could lead to different results to the integrals ${ }^{\text {VIIII }}$, so the PV prescription has an intrinsic ambiguity given by

$$
\begin{equation*}
\delta_{\Lambda}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\} \equiv \frac{1}{2 \pi i}\left[\mathrm{PV}_{+}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\}-\mathrm{PV}_{-}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\}\right] \tag{B.3}
\end{equation*}
$$

which is a path enclosing the poles in a way that the residue theorem can be used.
A better understanding about the computation of the integral's PV is obtained if we first write $f(u)$ in the singular expansion

$$
\begin{equation*}
f(u) \asymp \sum_{n=1}^{n_{p}} \sum_{i \in \mathscr{I}_{n}} \frac{f_{n}^{i}}{\left[u-u_{n}\right]^{i}}, \tag{B.4}
\end{equation*}
$$

with $n_{p}$ denoting the total number of poles in the real positive axis of $f(u)$ and $\mathscr{I}_{n}$ denoting the set of multiplicities of the poles $u_{n}$, and then split the integral over the path $C^{ \pm}$as

$$
\begin{align*}
& \int_{C^{ \pm}} \mathrm{d} u f(u)=\left(\int_{0}^{u_{1}-\delta_{1}}+\sum_{n=1}^{n_{p}-1} \int_{u_{n}+\delta_{n}}^{u_{n+1}-\delta_{n-1}}+\int_{u_{n_{p}}+\delta_{n_{p}}}^{\infty}\right) \mathrm{d} u f(u)  \tag{B.5}\\
& +\sum_{n=1}^{n_{p}} \sum_{i \in \mathscr{I}_{n}}\left[\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \mathrm{~d} u\left(f(u)-\frac{f_{n}^{i}}{\left[u-u_{n}\right]^{i}}\right)+f_{n}^{i} \mathrm{PV}_{ \pm}\left\{\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \frac{\mathrm{~d} u}{\left[u-u_{n}\right]^{i}}\right\}\right]
\end{align*}
$$

for finite numbers $\delta_{n}$ satisfying the condition $u_{n}+\delta_{n} \leq u_{n+1}-\delta_{n+1}$. The integrals in the first line are free of singularities, while in the second line the first integral is finite due to the pole subtraction. The last integral in the second line must be regularized in the PV

[^6]prescription as
\[

$$
\begin{equation*}
\mathrm{PV}_{ \pm}\left\{\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \frac{\mathrm{~d} u}{\left[u-u_{n}\right]^{i}}\right\} \equiv \lim _{\epsilon \rightarrow 0^{+}}\left(\int_{u_{n}-\delta_{n}}^{u_{n}-\epsilon}+\int_{C_{u_{n}, \epsilon}^{ \pm}}+\int_{u_{n}+\epsilon}^{u_{n}+\delta_{n}}\right) \frac{\mathrm{d} u}{\left[u-u_{n}\right]^{]}}, \tag{B.6}
\end{equation*}
$$

\]

where $C_{u_{n}, \epsilon}^{+(-)}$denotes an upper (lower) semi-circle centered at $u_{n}$ and with radius $\epsilon$. For $i$ odd the integral vanishes since the integrand becomes an even function around the $u_{n}$ pole. On the other hand, for $i$ even we have

$$
\begin{equation*}
\left(\int_{u_{n}-\delta_{n}}^{u_{n}-\epsilon}+\int_{u_{n}+\epsilon}^{u_{n}+\delta_{n}}\right) \frac{\mathrm{d} u}{\left[u-u_{n}\right]^{i}}=\frac{2}{i-1}\left(\frac{1}{\epsilon^{i-1}}-\frac{2}{\delta_{n}^{i-1}}\right) \tag{B.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{2}\left[\int_{C_{u_{n}, \epsilon}^{+}}+\int_{C_{u_{n}, \epsilon}^{-}}\right] \frac{\mathrm{d} u}{\left[u-u_{n}\right]^{i}}=-\frac{2}{i-1} \frac{1}{\epsilon^{i-1}} . \tag{B.8}
\end{equation*}
$$

Thus, for all positive integer $i$ the average

$$
\begin{equation*}
\operatorname{PV}\left\{\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \frac{\mathrm{~d} u}{\left[u-u_{n}\right]^{]}}\right\}=\frac{1}{2}\left[\mathrm{PV}_{+}\left\{\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \frac{\mathrm{~d} u}{\left[u-u_{n}\right]^{i}}\right\}+\mathrm{PV}_{-}\left\{\int_{u_{n}-\delta_{n}}^{u_{n}+\delta_{n}} \frac{\mathrm{~d} u}{\left[u-u_{n}\right]^{i}}\right\}\right] \tag{B.9}
\end{equation*}
$$

is free of the $\epsilon$-divergence, and so is Eq. (B.1).
The computation of the ambiguity (c.f. Eq. (B.3)) is much simpler. It is given by the sum of the simple-poles residues:

$$
\begin{equation*}
\delta_{\Lambda}\left\{\int_{0}^{\infty} \mathrm{d} u f(u)\right\}=\sum_{n=1}^{n_{p}} f_{n}^{1} . \tag{B.10}
\end{equation*}
$$

## Appendix C Polynomials

Here we display the first three polynomials $P_{n}^{\delta}(u)$ for each current. Higher orders in $n$ are available in Ref. ${ }^{116}$

Vector current:

$$
\begin{align*}
& P_{1}^{V}(u)=3+\frac{92 u}{27}+\frac{29 u^{2}}{27}+\frac{u^{3}}{9},  \tag{C.1}\\
& P_{2}^{V}(u)=10+\frac{2095 u}{162}+\frac{7393 u^{2}}{1296}+\frac{2887 u^{3}}{2592}+\frac{7 u^{4}}{54}+\frac{u^{5}}{96}, \\
& P_{3}^{V}(u)=\frac{315}{16}+\frac{54791 u}{1920}+\frac{62653 u^{2}}{3840}+\frac{3039 u^{3}}{640}+\frac{1037 u^{4}}{1280}+\frac{u^{5}}{10}+\frac{19 u^{6}}{1920}+\frac{u^{7}}{1920} .
\end{align*}
$$

Pseudo-scalar current:

$$
\begin{align*}
& P_{0}^{P}(u)=-\frac{2 u}{3}(7+u),  \tag{C.2}\\
& P_{1}^{P}(u)=6-\frac{11 u}{18}-\frac{49 u^{2}}{12}-\frac{8 u^{3}}{9}-\frac{u^{4}}{12}, \\
& P_{2}^{P}(u)=\frac{2-u}{2}\left(\frac{u^{5}}{192}+\frac{19 u^{4}}{192}+\frac{467 u^{3}}{576}+\frac{2311 u^{2}}{576}+\frac{2677 u}{288}+\frac{15}{2}\right) .
\end{align*}
$$

Scalar current:

$$
\begin{align*}
P_{0}^{S}(u) & =u\left(-\frac{61}{27}+\frac{235 u}{27}+\frac{260 u^{2}}{27}+\frac{20 u^{3}}{9}+\frac{2 u^{4}}{9}\right),  \tag{C.3}\\
P_{1}^{S}(u) & =15+\frac{703 u}{36}+\frac{2333 u^{2}}{72}+\frac{2539 u^{3}}{72}+\frac{305 u^{4}}{18}+\frac{197 u^{5}}{48}+\frac{7 u^{6}}{12}+\frac{5 u^{7}}{144}, \\
P_{2}^{S}(u) & =\frac{105}{2}+\frac{41357 u}{480}+\frac{15517 u^{2}}{160}+\frac{513613 u^{3}}{5760}+\frac{99889 u^{4}}{1920}+\frac{35993 u^{5}}{1920} \\
& +\frac{1711 u^{6}}{384}+\frac{223 u^{7}}{320}+\frac{u^{8}}{16}+\frac{7 u^{9}}{2880} .
\end{align*}
$$

Axial-vector current:

$$
\begin{align*}
P_{1}^{A}(u) & =6+\frac{661 u}{54}+\frac{1423 u^{2}}{108}+\frac{271 u^{3}}{36}+\frac{205 u^{4}}{108}+\frac{7 u^{5}}{36}  \tag{C.4}\\
P_{2}^{A}(u) & =30+\frac{2161 u}{36}+\frac{8315 u^{2}}{144}+\frac{30793 u^{3}}{864}+\frac{1555 u^{4}}{108}+\frac{98 u^{5}}{27}+\frac{25 u^{6}}{48}+\frac{u^{7}}{32}, \\
P_{3}^{A}(u) & =\frac{315}{4}+\frac{77507 u}{480}+\frac{798 u^{2}}{5}+\frac{59687 u^{3}}{576}+\frac{337453 u^{4}}{6912}+\frac{580397 u^{5}}{34560} \\
& +\frac{69961 u^{6}}{17280}+\frac{10969 u^{7}}{17280}+\frac{1973 u^{8}}{34560}+\frac{77 u^{9}}{34560} .
\end{align*}
$$

## Appendix D $\quad$ Scale independence of $M_{q, n}^{\delta}$

We show that the moments $M_{q, n}^{\delta}$ are $\mu$-independent in the following way: in large- $\beta_{0}$ the moments are given by

$$
\begin{align*}
M_{q, n}^{\delta} & =\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 m_{q}^{2}(\mu)\right]^{n}}  \tag{D.1}\\
& \times\left[1+\frac{2 n}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right],
\end{align*}
$$

accordingly to Eqs. (3.71), (4.36) and (4.37). Using $\bar{m}_{q}$ and $\alpha_{s}\left(\bar{m}_{q}\right)$ as reference values we now write the explicit mass factor of the moments in terms of $\bar{m}_{q}$ using the running mass of Eq. (2.56) with $\gamma_{m}$ and $\beta$ at $1 / \beta_{0}$ accuracy and re-expand the expression in $1 / \beta_{0}$. The result is

$$
\begin{align*}
M_{q, n}^{\delta} & =\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 \bar{m}_{q}^{2}\right]^{n}}  \tag{D.2}\\
& \times\left[1+\frac{2 n}{\beta_{0}} \int_{0}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)\right. \\
& \left.-\frac{2 n}{\beta_{0}} \int_{\alpha_{s}\left(\bar{m}_{q}\right)}^{\alpha_{s}(\mu)} \mathrm{d} \ln \alpha_{s} \frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right] .
\end{align*}
$$

Now we add and subtract a factor $\gamma_{m, 0} / 2$ in the integrand of the last integral to combine with the first integral and arrive with only one integral over the anomalous dimension, from 0 to $\alpha_{s}\left(\bar{m}_{q}\right)$. This gives

$$
\begin{align*}
M_{q, n}^{\delta} & =\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 \bar{m}_{q}^{2}\right]^{n}}  \tag{D.3}\\
& \times\left[1+\frac{2 n}{\beta_{0}} \int_{0}^{\alpha_{s}\left(\bar{m}_{q}\right)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]+\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta} S_{n}^{\delta}(u)\right. \\
& \left.-\frac{n \gamma_{m, 0}}{\beta_{0}} \ln \left(\frac{\alpha_{s}(\mu)}{\alpha_{s}\left(\bar{m}_{q}\right)}\right)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right] .
\end{align*}
$$

To proceed with the proof of the $\mu$-independence of the moments now we turn to the evaluation of the Borel integral. Quite generally, the functions $S_{n}^{\delta}(u)$ have the form ${ }^{\mathrm{IX}}$

[^7]\[

$$
\begin{equation*}
S_{n}^{\delta}(u)=\frac{n \gamma_{m, 0}}{u}\left[1-\left(\frac{e^{5 / 3} \mu^{2}}{\bar{m}_{q}^{2}}\right)^{u}\right]+\left(\frac{e^{5 / 3} \mu^{2}}{\bar{m}_{q}^{2}}\right)^{u}\left[G_{n}^{\delta}(u)+\frac{n \gamma_{m, 0}}{u}\right] \tag{D.4}
\end{equation*}
$$

\]

where $G_{n}^{\delta}(u)$ are functions of $u$ only. Using the relation

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} u \frac{e^{-u / \beta}}{u}\left(1-A^{u}\right)=\ln (1-\ln (A) \beta) \tag{D.5}
\end{equation*}
$$

the Borel integral of the first piece of $S_{n}^{\delta}(u)$ in Eq. (D.4) can be solved and we obtain

$$
\begin{align*}
\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u \frac{n \gamma_{m, 0}}{u} e^{-u / \beta}\left[1-\left(\frac{e^{5 / 3} \mu^{2}}{\bar{m}_{q}^{2}}\right)^{u}\right] & =\frac{n \gamma_{m, 0}}{\beta_{0}} \ln \left(1-\ln \left(\frac{e^{5 / 3} \mu^{2}}{\bar{m}_{q}^{2}}\right) \frac{\beta_{0} \alpha_{s}(\mu)}{4 \pi}\right)  \tag{D.6}\\
& =-\frac{n \gamma_{m, 0}}{\beta_{0}} \ln \left(\frac{\alpha_{s}\left(e^{-5 / 6} \bar{m}_{q}\right)}{\alpha_{s}(\mu)}\right)
\end{align*}
$$

where in the last step we used the one-loop running coupling in Eq. (2.51). Finally, the $\alpha_{s}(\mu)$ inside the logarithm in the equation above is cancelled with the one in Eq. (D.3) and we arrive at the expression

$$
\begin{align*}
M_{q, n}^{\delta} & =\left(\frac{9}{4} Q_{q}^{2}\right) \frac{N_{n}^{\delta}}{\left[4 \bar{m}_{q}^{2}\right]^{n}}  \tag{D.7}\\
& \times\left[1+\frac{2 n}{\beta_{0}} \int_{0}^{\alpha_{s}\left(\bar{m}_{q}\right)} \mathrm{d} \ln \alpha_{s}\left[\frac{2 \pi \gamma_{m}\left(\alpha_{s}\right)}{\alpha_{s}}-\frac{\gamma_{m, 0}}{2}\right]\right. \\
& +\frac{1}{\beta_{0}} \int_{0}^{\infty} \mathrm{d} u e^{-u / \beta}\left(\frac{e^{5 / 3} \mu^{2}}{\bar{m}_{q}^{2}}\right)^{u}\left[G_{n}^{\delta}(u)+\frac{n \gamma_{m, 0}}{u}\right] \\
& \left.-\frac{n \gamma_{m, 0}}{\beta_{0}} \ln \left(\frac{\alpha_{s}\left(e^{-5 / 6} \bar{m}_{q}\right)}{\alpha_{s}\left(\bar{m}_{q}\right)}\right)+\mathscr{O}\left(\frac{1}{\beta_{0}^{2}}\right)\right],
\end{align*}
$$

that ensures the scale (and scheme) independence of the moments $M_{q, n}^{\delta}$, as it depends only on the reference values $\bar{m}_{q}$ and $\alpha_{s}\left(\bar{m}_{q}\right)\left(\alpha_{s}\left(e^{-5 / 6} \bar{m}_{q}\right)\right.$ can be expressed in terms of $\alpha_{s}\left(\bar{m}_{q}\right)$ using the one-loop ruuning coupling), and the Borel integral is over an integrand with a global factor $\left[e^{5 / 3} \mu^{2} / \bar{m}_{q}^{2}\right]^{u}$.


[^0]:    ${ }^{\text {I }}$ By asymptotic we mean states at a time $t \rightarrow+\infty(-\infty)$ after (before) an interaction.

[^1]:    ${ }^{\text {II }} \mathrm{We}$ are omitting a Kronecker $\delta_{i j}$ and a factor of $i$ since we are only interested in removing the divergences.

[^2]:    ${ }^{\text {III }}$ Henceforth the superscript $R$ for renormalized quantities will be omitted.
    ${ }^{I V}$ The definitions of the renormalization group functions vary a lot in the literature. Here we are adopting the same definition of Grozin. ${ }^{59}$

[^3]:    ${ }^{\mathrm{V}}$ The large- $\beta_{0}$ limit should not be understood as a large- $N_{c}$ limit.

[^4]:    $\overline{{ }^{\mathrm{VI}} \mathrm{At} \text { the date of this work, no general proof of this conjecture exists for realistic Quantum Field Theories }}$ as QCD , only for $\sigma$-models. ${ }^{40}$

[^5]:    ${ }^{\text {VII }}$ Refs. ${ }^{31,32}$ also consider an extra parameter to assign an error arising from the use of perturbative QCD. We are omitting it in our discussion for the sake of simplicity.

[^6]:    ${ }^{\text {VIII }}$ For instance, one could make a hole $\left[u_{n}-\epsilon, u_{n}+\epsilon\right]$ with the limit $\epsilon \rightarrow 0$ around the $u_{n}$ pole, or make a hole $\left[u_{n}-\epsilon, u_{n}+2 \epsilon\right]$.

[^7]:    ${ }^{\text {IX }}$ We already replaced $m_{q}(\mu)$ by $\bar{m}_{q}$ in $S_{n}^{\delta}(u)$ since we are working at $1 / \beta_{0}$ accuracy.

