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## INSTITUTO DE FÍSICA DE SÃO CARLOS

Aspects of active quantum vacua: analogue models and quantization in dispersive media

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Thesis presented to the Graduate Program in Physics at the Instituto de Física de São Carlos, Universidade de São Paulo to obtain the degree of Doctor of Science.

Concentration area: Basic Physics. Advisor: Prof. Dr. Daniel Augusto Turolla Vanzella.

## Corrected Version

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With great love,
to Luana, the person who
keeps me standing

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About four and a half years ago, when I was obtaining my master's degree, my adviser back then, Vitorio De Lorenci, told me about this physicist that could guide my doctoral training. His words at that time were something like: "your scientific eagerness is very similar to Daniel Vanzella's." Since then, he has continuously guided me with his precise advices, and for that I am very happy and thankful, as I certainly ended up in the best place for me. As for Daniel, he accepted the mission, without knowing me at all, and I really hope that I had fulfilled his expectation as a student. Nowadays, if I consider myself prepared to work in any project, for sure that is mainly because he had encouraged me to learn any interesting topic that appeared during this journey, from cosmology to condensed matter theory, including even (a considerably hard, in my opinion) mathematics, when we decided to attend the commutative algebra course at ICMC-USP. For all that, I am truly thankful.

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"At present, as all methods, according to the general persuasion, have been tried in vain, there reigns nought but weariness and complete indifferentism -the mother of chaos and night in the scientific world, but at the same time the source of, or at least the prelude to, the re-creation and reinstallation of a science, when it has fallen into confusion, obscurity, and disuse from ill directed effort."

Immanuel Kant.


#### Abstract

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This is a work about the quantum vacuum. Particularly, unstable vacuum states. These are modified (with respect to Minkowski) vacuum states in which the vacuum polarization effects are so extreme that they eventually dominate the system evolution - a mechanism called vacuum awakening in the gravitational scenario. In this doctoral thesis, we are interested in studying further this phenomenon. We shall focus on interactions between fields and matter that can trigger this sort of instability in two different contexts: analogue models for the vacuum awakening and quantum aspects of active dispersive media. In the former, we shall see that the electromagnetic field in the presence of anisotropic materials behaves as if it were nonminimally coupled to gravity, and we use this analogy to study a couple of scenarios where instability takes place. As for the latter, we shall establish a Lagrangian microscopic model for the interaction between field and matter that gives rise to an effective tachyonic mass, and motivated by this model, we shall quantize the electromagnetic field in the presence of bilayer systems that can sustain instability. Our findings include the appearance of long-range correlations and a new kind of quantum levitation.


Keywords: Vacuum awakening. Analogue models. Active dispersive media.

## RESUMO

RIBEIRO, C.C.H. Aspectos de vácuos quânticos ativos: modelos análogos e quantização em meios dispersivos. 2020. 134p. Tese (Doutorado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

Este é um trabalho sobre o vácuo quântico. Particularmente, estados de vácuo instáveis. Estes são estados de vácuo modificados (em relação a Minkowski) nos quais os efeitos de polarização são tão extremos que a partir de algum momento passam a ditar a evolução do sistema - mecanismo este chamado de despertar do vácuo no contexto gravitacional. Nesta tese de doutorado, nós estamos interessados em estudar outros aspectos deste fenômeno. Nós vamos nos focar em interações campo-matéria que são capazes de ativar este tipo de instabilidade em dois contextos diferentes: em modelos análogos para o despertar do vácuo e em aspectos quânticos de materiais dispersivos ativos. No primeiro, veremos que o campo eletromagnético na presença de materiais anisotrópicos se comporta como se estivesse não-minimalmente acoplado com a gravidade, e nós usaremos esta analogia para estudar exemplos onde esta instabilidade ocorre. No último tópico, vamos estabelecer um modelo microscópico lagrangiano para interação campo-matéria que contém uma massa taquiônica efetiva, e, usando este modelo como motivação, vamos quantizar o campo eletromagnético na presença de sistemas com dois planos de matéria paralelos que sustentam instabilidade. Nossos resultados incluem o surgimento de correlações de longo alcance e um novo tipo de levitação quântica.

Palavras-chave: Despertar do vácuo. Modelos análogos. Materiais dispersivos ativos.

## Contents

1 Prolegomena ..... 15
1.1 Is it possible to build electromagnetically active systems? ..... 17
1.2 How is this thesis organized? ..... 20
1.3 Scalar vacuum near an active dispersionless boundary ..... 20
I Analogue models for the vacuum awakening ..... 29
2 Field quantization in active analogue models ..... 31
2.1 The language of curved spaces ..... 31
2.2 Covariant formulation: effective metrics ..... 33
2.2.1 QED-inspired nonminimal couplings ..... 37
2.3 Plane-symmetric anisotropic medium at rest ..... 40
2.3.1 Instability analysis ..... 43
2.3.2 Example ..... 48
2.4 Spherically-symmetric, stationary anisotropic medium ..... 51
2.4.1 Instability analysis ..... 56
2.4.2 Example ..... 58
2.5 Stabilization: spontaneous vectorization, photo production, and long-range in- duced correlations ..... 60
2.6 Final Remarks ..... 63
II Field quantization in active dispersive media ..... 67
3 Scalar field in dispersive active backgrounds ..... 69
3.1 Scalar field in the presence of dispersive media ..... 70
3.2 Microscopic model for dispersive media ..... 72
3.3 Fano diagonalization revisited ..... 76
3.4 Quantum Langevin Equation ..... 83
3.5 Final Remarks ..... 88
4 Electromagnetism in active bilayered backgrounds ..... 89
4.1 Field equations in dispersive layered backgrounds ..... 90
4.2 Stationary field modes ..... 92
4.2.1 Freely propagating field modes ..... 92
4.2.2 Langevin noises ..... 97
4.3 Bound solutions ..... 101
4.3.1 $\mathscr{S}$-symmetry ..... 102
4.3.2 Unstable field solutions ..... 103
4.3.3 Canonical commutation relation ..... 106
4.3.4 Vacuum polarization and propagators ..... 109
4.4 Casimir effect revisited: quantum levitation ..... 111
4.5 Vacuum-induced superconductivity? ..... 116
4.6 Final remarks ..... 119
5 Concluding Remarks ..... 121
References ..... 123
Appendix A Normalization of modes in the spherically symmetric case ..... 129
A. 1 TE modes ..... 130
A.1.1 Stable ..... 130
A.1.2 Unstable TE modes ..... 131
A. 2 TM modes ..... 132
A.2.1 Stable ..... 132
A.2.2 Unstable ..... 133

## Chapter 1

## Prolegomena

The Minkowski spacetime is the (idealized) structure over which all the (semiclassical) quantum field folklore is founded. In the absence of a quantum theory of gravity, that would incorporate its quantum degrees of freedom into the analysis, it serves as a basis for the process of renormalization. Specifically, it underpins the definition of a preferred reference state - called the Minkowski vacuum - with respect to which field fluctuations are defined (i.e., renormalized). This apparently simple assumption possesses a remarkable consequence: modifications of the Minkowski vacuum are observable. These changes can occur in various ways and perhaps the most famous examples include the Unruh effect, ${ }^{1}$ the Hawking radiation, ${ }^{2}$ and the vacuum polarization in QED and in boundary physics (like the Casimir effect $\left.{ }^{3}\right)$. Moreover, in some cases, this modification is so extreme that the field vacuum becomes active - its observable effects eventually dominate the system evolution -, in which case we say that the vacuum is unstable and that the background is also active. Studies that explored instability of vacuum states date back to the late 70 's and the 80 's, when renowned physicists like L. Ford were looking for tachyonic fields in curved backgrounds. ${ }^{4-6}$ These are fields coupled to gravity in such way that the field equations possess solutions that grow exponentially with time. Further quantum aspects of this phenomenon were unveiled since then, like the mechanism of "spontaneous scalarization" - the spontaneous appearance of classical scalar field configurations —, ${ }^{7,8}$ as well as various applications to astrophysics, ${ }^{9-12}$ including works from the supervisor of this thesis, D. Vanzella, where the quantum instability is also called vacuum awakening.

Despite being fairly understood on curved backgrounds, systems sustaining unstable fields in condensed matter theory did not receive, to our knowledge, a fair attention in the lit-
erature, specially at the quantum level regarding vacuum fluctuations. Classical aspects were explored in some works, ranging from the study of causal properties in such systems ${ }^{13}$ to recent examples of unstable systems built from graphene sheets. ${ }^{14}$ However, when quantum aspects come into play, these systems are usually regarded as inconsistent, either because the authors did not considered all the possible field modes on the quantization scheme, ${ }^{15}$ or simply because it is believed that the linear regime breaks down in such scenarios. ${ }^{16}$ Thus, there is a gap in the literature, as our knowledge of quantum vacuum instability in the gravitational scenario says it should be possible to study quantum vacuum instability of fields in the presence of matter, and, in the least interesting scenario, one should establish consistent arguments to rule out the existence of quantum instability in material media.

It is noteworthy that this analysis was not performed yet, as this sort of instability in material media is far more appealing than in the gravitational context, due to the involved experimental conditions and technological applications. From a scientific perspective, for instance, it could be used to further explore its curved space counterpart. In fact, it is known for almost one hundred years, with a work published by W. Gordon, that electromagnetism in certain material media behaves as if it were coupled to some gravitational field, ${ }^{17}$ and these systems, termed analogue models, have since then attracted a great amount of attention, specially with the works of W. Unruh ${ }^{18}$ and M. Visser. ${ }^{19}$ Thus, the question of if it is possible to build analogue models for the phenomenon of gravity-induced quantum vacuum instability naturally arises.

This doctoral thesis summarizes some of the main results obtained from the effort of studying these topics: analogue models for gravity-induced vacuum instability and quantization in active material configurations. We shall explore in the next sections some intriguing features that were found, specially the possibility of using anisotropic polarizable and/or magnetizable media to model QED-inspired nonminimal couplings in the context of analogue models. As for general aspects of quantization in active materials, we shall present a microscopic model for such a system and establish from it a postulate that seems to rule how quantization should be performed generally. We start our journey in the next two sections with the discussion of the possibility of experimenting with such systems in laboratory and of the notation adopted, and then we proceed to define what we are calling quantum vacuum instability by studying a toy model that captures the phenomenon's essence, before starting working on the proposed problems. Our main findings include the appearance of
strong long-range correlations, that captures the global features of the instability, and a new sort of quantum levitation as an extreme example of the Casimir effect.

### 1.1 Is it possible to build electromagnetically active systems?

The question of how is the structure of unstable field vacua in active materials is logically preceded by the one that asks if there exist active materials at all, or, if existent, how feasible they are. In order to show that our quantization in active scenarios is not vacuous, in this section we investigate the possibility of experimenting with electromagnetically active materials, naturally occurring in nature or engineered (metamaterials). Clearly, a thorough analysis of this subject deserves a complete research on its own, due to the involved theoretical and practical complexity. For this reason, we shall only present a few known examples that are relevant for our purposes.

First of all, there exist active materials occurring naturally. For instance, classical plasmas under certain simple conditions are known to be electromagnetically active since at least 1959, an effect known as Weibel instability, ${ }^{20}$ where classical electromagnetic perturbations grow exponentially with time. Active systems also appear often in solid state physics and classical/quantum optics. In the latter, for instance, perhaps the most famous example is from laser optics, where a medium (like a Fabry-Perot cavity filled with some dielectric material) acts as an amplifier for a pump signal traveling through it. ${ }^{21}$ In these systems, the limit in which the medium becomes active is usually not considered. To our knowledge, only classical aspects were considered in some particular models. ${ }^{13}$ Notwithstanding, in the realm of solid state physics, the situation is different, and a plethora of active systems is known. This is partly due to the fact that some materials possess plasma-like behavior in some regimes -for instance metals and superconductors in general. In a metal presenting plasma-like features, some mechanisms triggering electromagnetic instability are known. For instance, if the metal is traversed by some current (current-induced plasma instability), or the carriers' velocity distribution has a gap in some preferred direction, then the metal becomes an electromagnetic active medium. ${ }^{22,23}$

Thus, the study of unstable quantum vacua in the presence of matter is justified and it may have interesting applications with potential of actual experimentation. The next natural question is: what does characterize materials possessing unstable field solution? For
instance, electromagnetism in the presence of matter is known to be described by a theory that in some way treats the individual degrees of freedom from the matter constituents as effective fields, e.g., the polarization and magnetization vectors. Therefore, the question to be answered now is: how active materials are effectively described? In the absence of free charges and neglecting gravitational effects, all the electromagnetic phenomena in matter obey (effective) Maxwell's equations, where the matter degrees of freedom enter the theory via the constitutive relations $\mathbf{D}=\mathbf{D}(\mathbf{E}, \mathbf{B}), \mathbf{H}=\mathbf{H}(\mathbf{E}, \mathbf{B}),{ }^{24}$ with $\mathbf{D}$ being the electric displacement and $\mathbf{H}$ the magnetic field. In general, these relations must take into account nonlocal effects and thus they are integral equations. Also, electromagnetism is essentially nonlinear, meaning that the dependence of the induced fields $\mathbf{D}, \mathbf{H}$ upon $\mathbf{E}, \mathbf{B}$ is nonlinear. These equations mean that for whichever observed phenomenon, it must be completely described by $\mathbf{D}=\mathbf{D}(\mathbf{E}, \mathbf{B}), \mathbf{H}=\mathbf{H}(\mathbf{E}, \mathbf{B})$, including also the cases of active materials. Let us analyze the Weibel instability. In his seminal paper, ${ }^{20}$ Erich Weibel showed that the electrons in a plasma can act as an active medium if their velocity distribution is sufficiently anisotropic. That is, if their equilibrium velocity distribution is the bi-Maxwellian (in Cartesian coordinates)

$$
\begin{equation*}
f_{0}(\mathbf{v})=\frac{n_{e}}{u_{\perp}^{2} u_{\|}(2 \pi)^{3 / 2}} \mathrm{e}^{-\mathbf{v}_{\perp}^{2} /\left(2 u_{\perp}^{2}\right)-v_{\|}^{2} /\left(2 u_{\|}^{2}\right)}, \tag{1.1}
\end{equation*}
$$

where $\nu_{\|}$is the velocity in the direction $z, \mathbf{v}_{\perp}$ is the velocity in the plane perpendicular to $z$, and $n_{e}, u_{\perp}, u_{\|}$are positive parameters, then the Boltzmann-Maxwell's equations admit an unstable solution. In fact, it is a simple exercise to show from Weibel's results that the electric field for this solution is (the real part of) $\mathbf{E}(t, z)=\mathbf{E}_{0} \exp (-i \omega t+i k z)$, with $\mathbf{E}_{0}$ being a constant vector lying in the plane perpendicular to $z\left(\mathbf{E}_{0} \cdot \hat{z}=0\right), k>0$, and the frequency satisfies the linear dispersion

$$
\begin{equation*}
\omega=i u_{\perp} k, \tag{1.2}
\end{equation*}
$$

valid as long as the anisotropy is high, $u_{\|} \ll u_{\perp}$, and $k \ll \omega_{p}$, with $\omega_{p}^{2}=n_{e} e^{2} / m_{e}$ being the squared plasma frequency, $n_{e}$ the charge carrier density, and $m_{e}$ their masses. Now, if we take a linear constitutive relation in frequency space as $\mathbf{H}_{\perp}=\mu_{\perp}^{-1} \mathbf{B}_{\perp}, \mathbf{D}_{\perp}=\varepsilon_{\perp} \mathbf{E}_{\perp}$, where the sub-index $\perp$ indicates polarization/magnetization only in the plane perpendicular to $z$, then Maxwell's equations imply

$$
\begin{equation*}
\left(\omega^{2} \mu_{\perp} \varepsilon_{\perp}-k^{2}\right) \mathbf{E}=0 \tag{1.3}
\end{equation*}
$$

That is, as the optical quantities $\varepsilon_{\perp}$ and $\mu_{\perp}$ are in general functions of the field's frequency,
(1.3) implies that the product $\mu_{\perp} \varepsilon_{\perp}$ is negative for the $\omega$ and $k$ corresponding to the unstable field solution. Moreover, we notice that this effective description is independent of the polarization/magnetization in the direction parallel to $z$, corresponding to $\varepsilon_{\|}, \mu_{\|}^{-1}$, respectively. Weibel's unstable modes will always exist independently of the values that $\varepsilon_{\|}, \mu_{\|}^{-1}$ take. They can even be positive, corresponding to an extremely anisotropic case.

Soon after Weibel's work, it was shown that this sort of instability occurs even for small anisotropies for all $k$ by taking a different velocity distribution, ${ }^{25}$ and since then this mechanism has been used to elucidate various effects observed in nature. We may quote its recent application to study solar dynamics, ${ }^{26}$ where the presence of a magnetic field makes the problem richer. For our purposes, the "negativeness" of the constitutive functions at unstable frequencies is of more importance, as the problem of finding the dispersion relations in these plasma-based systems even for the simplest cases is very hard. Nevertheless, it should be stressed that this particular anisotropic polarizability/magnetizability is not a special feature of plasmas. Materials possessing these properties are well studied nowadays, and they are called hyperbolic metamaterials, ${ }^{27,28}$ which can be used for producing superlenses and other exquisite optical devices. We assume that the effective description is valid in what follows.

We shall return to the study of electromagnetism in hyperbolic metamaterials later on in this work. Besides this family of examples, we are also interested in a more pedestrian system, that can also be engineered in particular ways that can sustain unstable electromagnetic field solutions without requiring negative constitutive functions. Generally speaking, these are plane-symmetric configurations that consist of dielectric slabs bounded by twodimensional conducting planes. The reasons for studying such configurations are manifold. From a field theoretical perspective, particular instances of such systems were originally used to discover the Casimir forces, ${ }^{3}$ perhaps the most important known effect of vacuum polarization. Moreover, these systems have also a huge experimental appeal. For instance, graphene sheets can be stacked to form lattices presenting extraordinary effects, like superconductivity. ${ }^{29}$ Nevertheless, our main motivation is the fact that high-temperature superconductors are essentially stacks of conducting planes, that support the superconducting phase. ${ }^{30}$ In the search for an ultimate model for superconductivity in such scenarios, one cannot neglect the fact that parallel conducting planes possessing relative current drift can, in certain regimes, sustain unstable electromagnetic solutions. ${ }^{31}$ Thus, we are also go-
ing to study bilayered systems that can sustain instability, from the quantum field theory perspective. In addition to perform quantization in unstable, supposedly inconsistent, configurations, we shall see that this vacuum instability may have profound implications.

### 1.2 How is this thesis organized?

We work with Lorentz-Heaviside units, defined to make $c=\hbar=\varepsilon_{0}=1$, where $c$ is the vacuum speed of light, $\hbar$ is the reduced Planck constant, and $\varepsilon_{0}$ is the vacuum permittivity. The thesis is divided in two parts. The first one is dedicated to the characterization of the proposed analogue models. It consists of a bona fide adaptation of results that are already published, ${ }^{32}$ and thus it follows closely the structure of submitted work. The second one deals with field quantization in active dispersive media. In reading the document, the reader should keep in mind that both parts are (almost) completely independent, both from the involved notation and subject/scientific goals. Part I is focused on curved space analogues, and thus there is great commitment to general covariance, whereas in Part II, fields are scalar/vectorial functions on the flat space and there is no need to have manifest covariance at each step. Thus, notations are slightly different between the parts. For instance, the symbol " $\nabla$ " is used to denote the covariant derivative in the first part and the (flat) gradient operator in the second one. Another difference is the use of Latin letters to denote abstract tensorial indices in Part I and Cartesian components in Part II, where, in order to avoid confusion, letters from the beginning of the alphabet are used in the first part, and from the mid-end in the second one. Nevertheless, notational aspects are explained in detail in the text when needed.

### 1.3 Scalar vacuum near an active dispersionless boundary

The purpose of this section is to outline some of the principal features of the canonical quantization scheme and what is the precise definition of vacuum instability, for pedagogical reasons and to set up the notation for the upcoming sections. With this goal in mind, we could proceed by invoking a preferred book/scientific paper from the vast literature available on the subject to serve as a basis for the presentation. However, this would render a unappealing, technical section which we believe some readers are already familiarized with. Thus, we choose to adopt an alternative approach, which consists in solving a simple (but relevant) model presenting instability, and at each step of the quantization the methodology is
explained.
The system under study consists of a massless scalar field $\phi(t, \mathbf{x})$ in flat spacetime such that its vacuum state is modified by the presence of an idealized plane boundary. Thus, we work with Cartesian coordinates, with $\mathbf{x}=(x, y, z)$. The boundary is supposed to be fixed at $z=0$ and its effect on the field $\phi(t, \mathbf{x})$ is given by

$$
\begin{equation*}
\lim _{z \rightarrow 0}\left[\phi(t, \mathbf{x})+\alpha \operatorname{sgn}(z) \partial_{z} \phi(t, \mathbf{x})\right]=0 \tag{1.4}
\end{equation*}
$$

where $\alpha$ is a real number and $\operatorname{sgn}(z)$ is the signal function. This sort of boundary is called a Robin boundary, and it has received a fair amount of attention in the literature. For instance, the scalar field was quantized in such scenarios in arbitrary dimensions, ${ }^{33}$ and as an application, vacuum polarization effects and the renormalized stress-energy tensor were studied. However, when dealing with this theory, one must consider the fact that it is unstable depending on the value of $\alpha$, a fact completely neglected in the literature.

Let us quote some of the features presented by this system. For $\alpha=0$, the boundary condition (BC, for short) (1.4) reduces to the Dirichlet BC, and for $|\alpha| \rightarrow \infty$, it recovers the Neumann BC. Thus, the $\alpha$ parameter models a "smooth" transition between such boundaries. Moreover, it may appear that this system is just a toy model, but there exists a strong motivation for studying it. It relies on the fact that the electromagnetic field between nondispersive dielectric interfaces can be reduced to the study of a scalar degree of freedom satisfying precisely (1.4) at the interface. We shall return to this matter later on.

The canonical quantization (in the Heisenberg picture) of the scalar field is achieved by finding an operatorial representation for $\phi$ acting in some Hilbert space that is solution of the massless Klein-Gordon equation $\left(\partial_{\mu} \partial^{\mu} \phi=0\right)$ and the BC (1.4) and that satisfies the equaltime commutation relation $\left[\phi(t, \mathbf{x}), \partial_{t} \phi\left(t, \mathbf{x}^{\prime}\right)\right]=i \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$. Clearly, how one find such representation depends on each particular case, and for the systems considered in this work, we can always expand this solution in terms of normal modes. These are particular kind of solutions, usually pertaining to a larger complex function space that in some sense "diagonalizes" the problem (the meaning of this will be made clear in the following sections). The KG equation implies that the "natural" sesquilinear form on the space of these complex solutions, ${ }^{34}$

$$
\begin{equation*}
\left(\phi, \phi^{\prime}\right)_{K G}=i \int \mathrm{~d}^{3} x\left[\bar{\phi} \partial_{t} \phi^{\prime}-\left(\partial_{t} \bar{\phi}\right) \phi^{\prime}\right], \tag{1.5}
\end{equation*}
$$

between two complex solutions $\phi$ and $\phi^{\prime}$, taken along a hypersurface of constant $t$, is conserved along the field development and thus can be used to normalize the field modes. Here the overline means complex conjugation. As the field equation and the boundary condition do not depend on the coordinates $t, x, y$, we can look for (complex) solutions in the form $\phi(t, \mathbf{x})=\exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega \mathbf{k}_{\perp}}(z), \mathbf{k}_{\perp}=\left(k_{x}, k_{y}\right) \in \mathbb{R}^{2}$ and similarly for $\mathbf{x}_{\perp}$. For this particular ansatz, the KG equation reduces to the one dimensional Schrödinger-like equation for $f_{\omega \mathbf{k}_{\perp}}$

$$
\begin{equation*}
\left(-\frac{\mathrm{d}^{2}}{\mathrm{~d} z^{2}}+\mathbf{k}_{\perp}^{2}\right) f_{\omega \mathbf{k}_{\perp}}=\omega^{2} f_{\omega \mathbf{k}_{\perp}}, \tag{1.6}
\end{equation*}
$$

with eigenvalue $\omega^{2}$ and effective potential $\mathbf{k}_{\perp}^{2}$. The BC (1.4) implies that

$$
\begin{equation*}
\lim _{z \rightarrow 0}\left[f_{\omega \mathbf{k}_{\perp}}(z)+\alpha \operatorname{sgn}(z) \frac{\mathrm{d}}{\mathrm{~d} z} f_{\omega \mathbf{k}_{\perp}}(z)\right]=0 . \tag{1.7}
\end{equation*}
$$

As the canonical procedure demands, all physically meaningful solutions $\omega^{2}$ must enter the field expansion. If we call $\mathscr{O}=-\partial_{z}^{2}+\mathbf{k}_{\perp}^{2}$ the operator under study, then its spectrum $\operatorname{spec}(\mathscr{O})$ is the set of all admissible (BC compatible) solutions $\omega^{2}$. It turns out that the spectrum $\operatorname{spec}(\mathscr{O})$ can be further analyzed in two disjoint parts, the essential spectrum and the discrete spectrum. The essential spectrum is the subset of $\operatorname{spec}(\mathscr{O})$ that does not depend on the boundary conditions at infinity. Physically speaking, these are solutions corresponding to freely propagating plane waves that appear in scattering problems. The discrete spectrum consists of all other eigenvalues, and is the part of the spectrum that is dependent on the boundary conditions at infinity. Thus, we must also look for solutions with $f_{\omega \mathbf{k}_{\perp}} \rightarrow 0$ far from the wall, meaning that we are excluding spatial exponential growth on physical grounds. Elements of this set corresponds to "normalizable" solutions, are discrete in the topological sense, and when this set contains negative eigenvalues $\omega^{2}$ the field's vacuum is called unstable. From a mathematical perspective, the eigenvalue equation (1.6) supplemented with these BCs is a singular Sturm-Liouville problem. ${ }^{35}$ Let us see how our problem fits into this nomenclature.

Let us study the spectrum of $\mathscr{O}$. As $\omega^{2}$ is real, we can start by looking for solutions with $\omega^{2}>\mathbf{k}_{\perp}^{2}>0$, that is, we may look for elements in the essential spectrum of $\mathscr{O}$. Thus, the quantity $k_{z}=\left(\omega^{2}-\mathbf{k}_{\perp}^{2}\right)^{1 / 2}$ is a positive real number. Consider the experiment of sending a plane wave originating at positive infinity towards the wall, with wave vector $k_{z}$. Then, part of this wave will get transmitted and part will be reflected, according to the BCs. Thus, this plane
wave builds up to a solution of the general form

$$
f_{k_{z}}^{>}(z)=\frac{1}{\sqrt{2 \pi}}\left\{\begin{array}{cc}
\mathrm{e}^{-i k_{z} z}+R_{k_{k}} \mathrm{e}^{i k_{z} z} & , z>0,  \tag{1.8}\\
T_{k_{z}} \mathrm{e}^{-i k_{z} z} & , z<0 .
\end{array}\right.
$$

The BC (1.4) then implies that we have a perfectly reflecting wall, $T_{k_{z}}=0$, and the reflection coefficient is

$$
\begin{equation*}
R_{k_{z}}=\frac{i \alpha k_{z}-1}{i \alpha k_{z}+1} . \tag{1.9}
\end{equation*}
$$

Moreover, by the symmetry of the problem, the modes originating at minus infinity $f_{k_{z}}^{<}$are given in terms of $f_{k_{z}}^{>}$by $f_{k_{z}}^{<}(z)=f_{k_{z}}^{>}(-z)$, and these exhaust all the possible eigenmodes whose eigenvalues satisfy $\omega^{2}>\mathbf{k}_{\perp}^{2}$, or $k_{z}>0$. The next part of the spectrum corresponds to the interval $\omega^{2}<\mathbf{k}_{\perp}^{2}$. If we define $\gamma=\left(\mathbf{k}_{\perp}^{2}-\omega^{2}\right)^{1 / 2}>0$, then for $z \neq 0$ the general solution of the problem is $f(z)=A \exp (\gamma z)+B \exp (-\gamma z)$. Let us analyze the solution in the half-space $z>0$. In order to have a good (i.e., normalizable) behavior at infinity, we must set $A=0$. The BC (1.4) then requires that $B(1-\alpha \gamma)=0$. Thus, a nonvanishing solution exists if, and only if, $\gamma=1 / \alpha$, and so $\alpha>0$. In this case, the single eigenvalue is completely determined by the parameter $\mathbf{k}_{\perp}^{2}$ as

$$
\begin{equation*}
\omega^{2}=\mathbf{k}_{\perp}^{2}-\frac{1}{\alpha^{2}} \tag{1.10}
\end{equation*}
$$

and the eigenfunction is

$$
\begin{equation*}
g^{>}(z)=\Theta(z) \sqrt{\frac{2}{\alpha}} \mathrm{e}^{-\frac{z}{\alpha}}, \tag{1.11}
\end{equation*}
$$

with $\Theta(z)$ being the unit step function. Therefore, the eigenvalue $\omega^{2}$ becomes negative when $\alpha^{2} \mathbf{k}_{\perp}^{2}<1$, and the theory is unstable. We then conclude that a necessary and sufficient condition for instability is $\alpha>0$. Let us suppose that this is the case. Again, the corresponding mode in the region $z<0$ is obtained through $g^{<}(z)=g^{>}(-z)$. Now we can verify if these solutions exhaust all the possibilities by calculating the completeness relation. For instance, if $z, z^{\prime}>0$,

$$
\begin{equation*}
\overline{g^{>}}(z) g^{>}\left(z^{\prime}\right)+\int_{0}^{\infty} \mathrm{d} k_{z} \overline{f_{k_{z}}^{>}}(z) f_{k_{z}}^{>}\left(z^{\prime}\right)=\delta\left(z-z^{\prime}\right)+\frac{2}{\alpha} \mathrm{e}^{-\frac{\hat{\lambda} z}{\alpha}}+\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} k_{z} \mathrm{e}^{i k_{z} \hat{\Delta} z} \frac{i \alpha k_{z}-1}{i \alpha k_{z}+1}, \tag{1.12}
\end{equation*}
$$

and $\hat{\Delta} z=z+z^{\prime}$. The integrand in the r.h.s. of Eq. (1.12) has a pole at $k_{z}=i / \alpha$, lying in the upper-half plane, and the integral can be solved by means of the Residue Theorem by enclosing the contour of integration from above to ensure convergence. Thus we see that this
integral produces a term that cancels exactly the exponential, rendering only the delta function. Therefore, this set is complete.

The final step in the quantization is to normalize the calculated field modes according to $(,)_{K G}$. Recall that we are looking for modes in the form $\phi(t, \mathbf{x})=\exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega \mathbf{k}_{\perp}}(z)$. Let us consider first the stable, positive-frequency modes ( $\omega>0$ ). Substituting two such solutions in (1.5) results in

$$
\begin{equation*}
\left(\phi, \phi^{\prime}\right)_{K G}=\left(\omega+\omega^{\prime}\right) \mathrm{e}^{-i t\left(\omega^{\prime}-\omega\right)}(2 \pi)^{2} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \int \mathrm{d} z \overline{f_{\omega \mathbf{k}_{\perp}}}(z) f_{\omega^{\prime} \mathbf{k}_{\perp}}(z) . \tag{1.13}
\end{equation*}
$$

We claim that the sesquilinear form $(,)_{K G}$ becomes positive definite on the set of positivefrequency solutions. In order to show this, let $f_{\omega \mathbf{k}_{\perp}}, f_{\omega^{\prime} \mathbf{k}_{\perp}}$ be two solutions of (1.6). Multiplying the equation satisfied by $\overline{f_{\omega \mathbf{k}_{\perp}}}$ (resp., $f_{\omega^{\prime} \mathbf{k}_{\perp}}$ ) by $f_{\omega^{\prime} \mathbf{k}_{\perp}}$ (resp., $\overline{f_{\omega \mathbf{k}_{\perp}}}$ ), and subtracting the results imply the identity

$$
\begin{equation*}
\left(\omega^{2}-\omega^{\prime 2}\right) \int \mathrm{d} z \overline{f_{\omega \mathbf{k}_{\perp}}} f_{\omega^{\prime} \mathbf{k}_{\perp}}=0 \tag{1.14}
\end{equation*}
$$

for all $\omega^{2}, \omega^{\prime 2}$. This means that the above integral is proportional to $\delta\left(\omega^{2}-\omega^{\prime 2}\right), \delta$ being a Dirac (resp., Kronecker) delta if $\omega$ is in the essential (resp., discrete) spectrum. Thus, our claim is proved. Notice that as a by-product the eigenvalues $\omega^{2}$ are real, otherwise Eq. (1.14), amended as $\omega^{2} \rightarrow \bar{\omega}^{2}$, would fail to vanish for $\omega^{2}=\omega^{\prime 2}$.

Taking $f_{\omega \mathbf{k}_{\perp}}=f_{k_{z}}^{>}$as in (1.8) and using the above equation results in the normalized field mode

$$
\begin{equation*}
f_{\mathbf{k}}^{>}(t, \mathbf{x})=\frac{1}{2 \pi} \frac{1}{\sqrt{2 \omega}} \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} f_{k_{z}}^{>}(z) \tag{1.15}
\end{equation*}
$$

where we have used $\int \mathrm{d} z \overline{f_{k_{z}}^{>}}(z) f_{k_{z}^{\prime}}^{>}(z)=\delta\left(k_{z}-k_{z}^{\prime}\right)$, and $\omega=|\mathbf{k}|, k_{z}>0$. For the stable bound solutions $f_{\omega \mathbf{k}_{\perp}}(z)=g^{>}(z)$, we obtain

$$
\begin{equation*}
g_{\mathbf{k}_{\perp}}^{(s)>}(t, \mathbf{x})=\frac{1}{2 \pi} \frac{1}{\sqrt{2 \Omega_{\mathbf{k}_{\perp}}}} \mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp}} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g^{>}(z), \tag{1.16}
\end{equation*}
$$

valid for $\alpha\left|\mathbf{k}_{\perp}\right|>1$, and we have defined $\Omega_{\mathbf{k}_{\perp}}=\left(\mathbf{k}_{\perp}^{2}-1 / \alpha^{2}\right)^{1 / 2}$.
Now we turn our attention to the normalization of unstable modes ( $\omega^{2}<0$ ). Here we should stress the main difference in the quantization procedure when negative eigenvalues $\omega^{2}$ exist. In these cases, $\omega$ is a pure imaginary number, and modes written as $\phi(t, \mathbf{x})=$ $\exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega \mathbf{k}_{\perp}}(z)$ have vanishing norms, $(\phi, \phi)_{K G}=0$. This behavior is not unnatural as the sesquilinear form $(,)_{K G}$ is not positive definite, and can be circumvented by
finding a linear combination of $\exp ( \pm i \omega t)$ with nonvanishing norm. One such combination (between uncountably many) is

$$
\begin{equation*}
g_{\mathbf{k}_{\perp}}^{(u),>}(t, \mathbf{x})=\frac{1}{2 \pi} \frac{1}{\sqrt{\left|\Omega_{\mathbf{k}_{\perp} \mid}\right|}} \cosh \left(\left|\Omega_{\mathbf{k}_{\perp}}\right| t-i \pi / 4\right) \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g^{>}(z), \tag{1.17}
\end{equation*}
$$

valid for $\alpha\left|\mathbf{k}_{\perp}\right|<1$, for which $\left(g_{\mathbf{k}_{\perp}}^{(u),>}, g_{\mathbf{k}_{\perp}^{\prime}}^{(u),>}\right)_{K G}=\delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right)$. This mathematical phenomenon always occurs in the quantization of unstable theories.

The quantized field is now obtained by associating to each found mode an annihilation operator. We shall adopt the method of quantization in the presence of thin films, ${ }^{36}$ and decompose the annihilation operators as depicted in Fig. 1.1. The operator $a_{R}(\mathbf{k})$ (resp., $b_{L}(\mathbf{k})$ ) corresponds to freely rightward-propagating (leftward-propagating) plane waves originating at minus (resp., plus) infinity. The operator $a_{L}(\mathbf{k})$ (resp., $b_{R}(\mathbf{k})$ ) then corresponds to reflected waves only, as there is no transmission. Finally, at each side of the wall there are evanescent modes with annihilation operators $f_{L}\left(\mathbf{k}_{\perp}\right)$ and $f_{R}\left(\mathbf{k}_{\perp}\right)$. The annihilation operators $a_{R}(\mathbf{k})$,


Figure 1.1-Description of the field annihilation operators. The operators $f_{R}\left(\mathbf{k}_{\perp}\right)=f_{L}\left(\mathbf{k}_{\perp}\right)$ correspond to evanescent modes.
Source: By the author.
$b_{L}(\mathbf{k}), f_{L}\left(\mathbf{k}_{\perp}\right)$ satisfy $\left[a_{R}(\mathbf{k}), a_{R}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\left[b_{L}(\mathbf{k}), b_{L}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)$ and $\left[f_{L}\left(\mathbf{k}_{\perp}\right), f_{L}^{\dagger}\left(\mathbf{k}_{\perp}\right)\right]=\delta\left(\mathbf{k}_{\perp}-\right.$ $\mathbf{k}_{\perp}^{\prime}$ ). The "reflected" operators are found to be $a_{L}(\mathbf{k})=R_{k_{z}} a_{R}(\mathbf{k})$ and $b_{R}(\mathbf{k})=R_{k_{z}} b_{L}(\mathbf{k})$, and thus $\left[a_{L}(\mathbf{k}), a_{L}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\left[b_{R}(\mathbf{k}), b_{R}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)$. In this notation, the connection between incident and reflected waves is

$$
\begin{equation*}
\left[a_{L}(\mathbf{k}), a_{R}^{\dagger}(\mathbf{k})\right]=\left[b_{R}(\mathbf{k}), b_{L}^{\dagger}(\mathbf{k})\right]=R_{k_{z}} \delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \tag{1.18}
\end{equation*}
$$

Therefore, for $z>0$,

$$
\begin{align*}
\phi(t, \mathbf{x})= & \frac{1}{\sqrt{(2 \pi)^{3}}} \int_{k_{z}>0} \mathrm{~d}^{3} k \frac{\mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}}}{\sqrt{2 \omega}}\left[b_{L}(\mathbf{k}) \mathrm{e}^{-i k_{z} z}+b_{R}(\mathbf{k}) \mathrm{e}^{i k_{z} z}\right] \\
& +\int_{\alpha\left|\mathbf{k}_{\perp}\right|>1} \mathrm{~d}^{2} k_{\perp} f_{R}\left(\mathbf{k}_{\perp}\right) g_{\mathbf{k}_{\perp}}^{(s),>}(t, \mathbf{x})+\int_{\alpha\left|\mathbf{k}_{\perp}\right|<1} \mathrm{~d}^{2} k_{\perp} f_{R}\left(\mathbf{k}_{\perp}\right) g_{\mathbf{k}_{\perp}}^{(u),>}(t, \mathbf{x})+\text { H.c., } \tag{1.19}
\end{align*}
$$

where "H.c." stands for Hermitian conjugate. This completes the quantization procedure. The method's consistency can be verified by calculating the canonical commutation relation $\left[\phi(t, \mathbf{x}), \partial_{t} \phi\left(t, \mathbf{x}^{\prime}\right)\right]$. Then it is a simple task to show that the completeness relation (1.12) implies the correct commutation relation $\left[\phi(t, \mathbf{x}), \partial_{t} \phi\left(t, \mathbf{x}^{\prime}\right)\right]=i \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$. The vacuum state of this representation of the theory, $|0\rangle$, is defined to be the state vector contained in the kernel of all annihilation operators, that is, $a_{R}(\mathbf{k})|0\rangle=b_{L}(\mathbf{k})|0\rangle=f_{R}\left(\mathbf{k}_{\perp}\right)|0\rangle=0$, and the Fock space is built by acting on it with the corresponding creation operators.

Once the field is expanded as in Eq. (1.19), physical quantities can be calculated according to the general recipe from quantum theory. Among the various possible applications, for unstable field configurations we are interested mainly in one, the vacuum polarization, where the observable is the field operator itself, and the state vector is the vacuum state modified by the presence of the wall, $|0\rangle$. In this case, we have the vacuum expectation value $\langle\phi\rangle=0$, meaning that the measured values of $\phi$ are distributed around the zero value, and the variance reduces to the vacuum expectation value of the squared field $\left\langle(\phi-\langle\phi\rangle)^{2}\right\rangle=\left\langle\phi^{2}\right\rangle$. This quantity (amongst higher moments) measures how distributed around zero a measurement of field is, and as such, the higher $\left\langle\phi^{2}\right\rangle$, more probable is to find nonvanishing field configurations. This variance is readily found through the (positive) Wightman two-point function, defined as the vacuum expectation value $\left\langle\phi(t, \mathbf{x}) \phi\left(t^{\prime}, \mathbf{x}^{\prime}\right)\right\rangle$. Then, we obtain from Eq. (1.19) that, for $z, z^{\prime}>0$,

$$
\begin{align*}
\left\langle\phi(t, \mathbf{x}) \phi\left(t^{\prime}, \mathbf{x}^{\prime}\right)\right\rangle= & \left\langle\phi(t, \mathbf{x}) \phi\left(t^{\prime}, \mathbf{x}^{\prime}\right)\right\rangle_{0}+\frac{1}{2(2 \pi)^{3}} \int \mathrm{~d}^{3} k \frac{1}{\omega} \mathrm{e}^{-i \omega \Delta t+i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}+i k_{z} \hat{\Delta} z} R_{k_{z}} \\
+\frac{\mathrm{e}^{-\frac{\hat{\Delta} z}{\alpha}}}{\alpha(2 \pi)^{2}} & \left\{\int_{\alpha\left|\mathbf{k}_{\perp}\right|>1} \mathrm{~d}^{2} k_{\perp} \frac{\mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp}} \Delta t+i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}}}{\Omega_{\mathbf{k}_{\perp}}}\right. \\
& \left.+\int_{\alpha\left|\mathbf{k}_{\perp}\right|<1} \mathrm{~d}^{2} k_{\perp} \frac{\mathrm{e}^{i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}}}{\left|\Omega_{\mathbf{k}_{\perp}}\right|}\left[\cosh \left(\left|\Omega_{\mathbf{k}_{\perp}}\right| \hat{\Delta} t\right)-i \sinh \left(\left|\Omega_{\mathbf{k}_{\perp}}\right| \Delta t\right)\right]\right\}, \tag{1.20}
\end{align*}
$$

where the first term is the Minkowskian two-point function (in the absence of wall). Thus,
the renormalized value of the variance with respect to the empty space possesses the asymptotic closed form as $t \gg \alpha$,

$$
\begin{equation*}
\left\langle\phi^{2}\right\rangle \approx \frac{1}{8 \pi \alpha t} \mathrm{e}^{2(t-z) / \alpha} . \tag{1.21}
\end{equation*}
$$

This means that as time passes, the field extracts energy from the wall, leading to wild vacuum fluctuations. Eventually, some stabilization mechanism supposedly drives the system to a final stable state. This process is of major theoretical and experimental importance, as it is connected with a sort of vacuum-induced phase transition. Notwithstanding, stabilization would certainly occur in the vicinity of the wall, source of the field's increasing fluctuations, where, besides the renormalization process, Eq. (1.21) does not hold. This is because the first integral in the r.h.s. of Eq. (1.20) diverges as $\left(t^{\prime}, \mathbf{x}^{\prime}\right) \rightarrow(t, \mathbf{x})$, similarly as occurs for simple Dirichlet or Neumann BCs. As it turns out, the physics behind this idealization is that the wall modifies field solutions of arbitrarily large frequency (energy). For instance, if we consider a plane wave coming from the asymptotic infinity with wave vector possessing small $k_{z}$ and very large $\mathbf{k}_{\perp}$ (and therefore frequency), then this solution is reflected according to the BC in the same way as the solutions of small frequencies with the same $k_{z}$, as can be inferred from the reflection coefficient $R_{k_{z}}$. The problem relies of the fact that the renormalization does not subtract the contribution coming from these energetic reflected solutions, and we are left with residual divergences afterward. Thus, in order to proceed with the analysis, we need to impose the physical requirement that the wall should be invisible for very high frequency solutions, or equivalently, that the field high frequency sector is composed solely by empty space solutions. In this case, we say that the wall must be dispersive, and we shall explore quantization in dispersive active systems in great detail later on.

## Part I

## Analogue models for the vacuum awakening

## Chapter 2

## Field quantization in active analogue models

In this chapter we present a family of optical-based analogue models that includes, but is not restricted to, examples of systems that can model the vacuum awakening. We apply Gordon's method to show how anisotropies of the background enter the effective equations in the form of nonminimal couplings, and in the case of strong anisotropy (just like for the Weibel instability), this coupling results in unstable solutions. We also discuss that for these systems the stabilization process occurs through the nonlinear nature of the background, which may seed spontaneous vectorization - the electromagnetic version of the spontaneous scalarization - in analogy to the Einstein's field equations in the gravitational scenario. Our main focus here is on the analogue model, and as such, energy exchange between field and matter, as well as problems related to renormalization are left to the next part of the work.

### 2.1 The language of curved spaces

In this section we quote the basic mathematical language of curved spacetimes to establish the notation adopted in the present chapter. We follow closely Wald's book, ${ }^{37}$ the only exception being the choice of units already explained, and assume the foundations of general relativity as known (the first five chapters of Wald's book), as well as canonical field quantization in curved spacetimes. Thus, a spacetime is an ordered pair ( $\mathscr{M}, g_{a b}$ ), where $\mathscr{M}$ is a 4-dimensional manifold equipped with a Lorentzian metric $g_{a b}$ of signature (-,+,+,+), whose corresponding curvature depends on the matter distribution $T_{a b}$ through Einstein's
field equations $R_{a b}-R g_{a b} / 2=8 \pi G T_{a b} \cdot{ }^{37}$ Here, $R_{a b}$ is the Ricci tensor, given in terms of the Riemann tensor by $R_{a c}=R_{a b c}{ }^{b}$, and the Ricci scalar $R$ is the trace of $R_{a b}$. For the particular case of the flat Minkowski spacetime, the metric is denoted by $\eta_{a b}$. We make use of the abstract index notation, where the Latin letters $a, b, \ldots$ in tensorial equations are kept as reminders of tensorial slots, and do not represent components in a given basis. If a basis is chosen, we employ Greek letters $\alpha, \beta, \ldots$ to express tensorial components in that basis. For instance, the Latin letters in $g_{a b}$ simply indicate that this tensor has two covariant slots, and so on. In this setting, given a tensor field $A_{a b}$ on the manifold, the symmetric (resp., antisymmetric) part of $A_{a b}$ is $2 A_{(a b)}=A_{a b}+A_{b a}\left(\right.$ resp., $\left.2 A_{[a b]}=A_{a b}-A_{b a}\right)$. The symmetrization or antisymmetrization of any number of covariant/contravariant slots is defined similarly, with the appropriate normalization.

Given a time-like vector field $u^{a}$ normalized as $u_{a} u^{a}=-1$, we shall make use of the natural decomposition of arbitrary tensor fields into their spatial and time-like parts with respect to $u^{a}$, defined as follows. Let $h^{a}{ }_{b}=\delta_{b}^{a}+u^{a} u_{b}$ and notice that $h^{a}{ }_{b} u_{a}=0$ and $h^{a}{ }_{b} u^{b}=0$. This operator is a projection operator, as it is idempotent

$$
\begin{equation*}
h_{c}^{a} h_{b}^{c}=\left(\delta_{c}^{a}+u^{a} u_{c}\right)\left(\delta_{b}^{c}+u^{c} u_{b}\right)=\delta_{b}^{a}+u^{a} u_{b}=h_{b}^{a} . \tag{2.1}
\end{equation*}
$$

It can be used to write an arbitrary vector field $A^{a}$ uniquely as

$$
\begin{equation*}
A^{a}=\delta_{b}^{a} A^{b}=\left(h^{a}{ }_{b}-u^{a} u_{b}\right) A^{b} \equiv A_{\perp}^{a}+A_{\|}^{a}, \tag{2.2}
\end{equation*}
$$

where we have identified the perpendicular ( $A_{\perp}^{a}=h^{a}{ }_{b} A^{b}$ ) and parallel ( $A_{\|}^{a}=-u_{b} A^{b} u^{a}$ ) parts of $A^{a}$ and they are clearly orthogonal $\left(g_{a b} A_{\perp}^{a} A_{\|}^{b}=0\right)$. We say that $A_{\perp}^{a}\left(\right.$ resp. $\left.A_{\|}^{a}\right)$ is the spatial (resp., time-like) part of $A^{a}$ with respect to $u^{a}$. For an arbitrary tensor field, we define the decomposition slot-wise by the same rule.

In what follows, this decomposition will be applied mainly for antisymmetric rank-two tensors $F^{a b}$. In this case, we have

$$
\begin{equation*}
F^{a b}=\left(h_{c}^{a}-u^{a} u_{c}\right)\left(h_{d}^{b}-u^{b} u_{d}\right) F^{c d}=2 u^{[a} F^{b] c} u_{c}+h_{c}^{a} h_{d}^{b} F^{c d}, \tag{2.3}
\end{equation*}
$$

where we have used twice that $F^{a b} u_{a} u_{b}=0$ due to the antisymmetry of $F^{a b}$. The last term in Eq. (2.3) can be written in a more explicit way with the aid of the natural volume form $\epsilon_{a b c d}$
induced by the metric $g_{a b}$ and defined through $\epsilon^{a b c d} \epsilon_{a b c d}=-24$. It satisfies the identity $\epsilon^{a_{1} a_{2} a_{3} a_{4}} \epsilon_{b_{1} b_{2} b_{3} b_{4}}=-24 \delta_{b_{1}}^{\left[a_{1}\right.} \delta_{b_{2}}^{a_{2}} \delta_{b_{3}}^{a_{3}} \delta_{b_{4}}^{\left.a_{4}\right]}$, coming from the fact that the tensorial product of two 1-dimensional vector spaces is itself a 1-dimensional vector space. It follows from this last property that

$$
\begin{equation*}
\epsilon^{a b c d} \epsilon_{a e f g}=-6 \delta_{e}^{[b} \delta_{f}^{c} \delta_{g}^{d]} \tag{2.4}
\end{equation*}
$$

Now let us define the vector field

$$
\begin{equation*}
B^{a}=-\frac{1}{2} \epsilon^{a b c d} F_{b c} u_{d} \tag{2.5}
\end{equation*}
$$

It then follows from Eq. (2.4) that

$$
\begin{equation*}
-\epsilon^{a b c d} B_{c} u_{d}=-\frac{1}{2} \epsilon^{c b a d} \epsilon_{c e f t} F^{e f} u^{t} u_{d}=3 F^{[b a} u^{d]} u_{d}=h_{c}^{a} h_{d}^{b} F^{c d} . \tag{2.6}
\end{equation*}
$$

Therefore, by defining the vector field

$$
\begin{equation*}
E^{a}=F^{a b} u_{b} \tag{2.7}
\end{equation*}
$$

an antisymmetric rank-two tensor $F^{a b}$ is uniquely written in terms of $u^{a}$ as

$$
\begin{equation*}
F^{a b}=2 u^{[a} E^{b]}-\epsilon^{a b c d} B_{c} u_{d} \tag{2.8}
\end{equation*}
$$

with $B^{a}$ and $E^{a}$ defined in Eqs. (2.5), (2.7), respectively. This decomposition is particularly useful when the vector field $u^{a}$ coincides with the velocity field of a family of observers. Then, for instance, if $F^{a b}$ is the Maxwell tensor, $E^{a}$ and $B^{a}$ are the electric and magnetic fields measured by those observers. Let us now proceed to study the analogue models.

### 2.2 Covariant formulation: effective metrics

Electromagnetism in material media, in flat spacetime and in the absence of free charges, is described by two antisymmetric (observer-independent) tensors, $F_{a b}$ and $G^{a b}$, satisfying the macroscopic covariant Maxwell's equations,

$$
\begin{equation*}
\partial_{a} G^{a b}=0, \tag{2.9}
\end{equation*}
$$

$$
\begin{equation*}
\partial_{[a} F_{b c]}=0, \tag{2.10}
\end{equation*}
$$

where $\partial_{a}$ is the derivative operator compatible with the flat metric $\eta_{a b}$ (but in arbitrary coordinates). These equations must be supplemented by medium-dependent constitutive relations between $F_{a b}$ and $G^{a b}$, as well as initial and boundary conditions, in order to provide a well-posed problem. These constitutive relations are usually set at the level of (observerdependent) fields $E_{a}, B^{a}, D^{a}$, and $H_{a}$, related to $F_{a b}$ and $G^{a b}$ through (see Eqs. (2.5) and (2.7))

$$
\begin{align*}
E_{a} & =F_{a b} u^{b},  \tag{2.11}\\
D^{a} & =G^{a b} u_{b},  \tag{2.12}\\
B^{a} & =-\frac{1}{2} \epsilon^{a b c d} F_{b c} u_{d},  \tag{2.13}\\
H_{a} & =-\frac{1}{2} \epsilon_{a b c d} G^{b c} u^{d}, \tag{2.14}
\end{align*}
$$

where $u^{a}$ is the four-velocity of the observer measuring these fields. Moreover, the constitutive relations usually take a simpler form in the reference frame in which the medium is (locally and instantaneously) at rest.

Here, we consider a polarizable and magnetizable medium whose constitutive relations in its instantaneous rest frame take the form

$$
\begin{align*}
& D^{a}=\varepsilon^{a b} E_{b},  \tag{2.15}\\
& H_{a}=\mu_{a b} B^{b}, \tag{2.16}
\end{align*}
$$

where the tensors $\varepsilon^{a b}$ and $\mu_{a b}$ are spatial with respect to $u^{a}$, may depend on spacetime coordinates, and the system is assumed dispersionless. We return to this point later. The fact that Eqs. $(2.15,2.16)$ are valid in the medium's instantaneous rest frame means that the fields $E_{a}, B^{a}, D^{a}$, and $H_{a}$ appearing in them are related to $F_{a b}$ and $G^{a b}$ through Eqs. (2.11-2.14) with $u^{a}=v^{a}$, the medium's four-velocity field. We proceed by splitting the tensors $\varepsilon^{a b}$ and $\mu_{a b}$ into isotropic and traceless anisotropic parts,

$$
\begin{align*}
& \varepsilon^{a b}=\varepsilon h^{a b}+\chi_{(\varepsilon)}^{a b},  \tag{2.17}\\
& \mu_{a b}=\mu^{-1} h_{a b}+\chi_{a b}^{(\mu)} \tag{2.18}
\end{align*}
$$

where $h_{b}^{a}=\delta_{b}^{a}+v^{a} \nu_{b}$ is the projection operator orthogonal to $v^{a}$. It follows from Eq. (2.8) with $u^{a}=v^{a}$ that

$$
\begin{equation*}
G^{a b}=2 v^{[a} D^{b]}-\epsilon^{a b c d} H_{c} v_{d}, \tag{2.19}
\end{equation*}
$$

and substituting Eqs. (2.15-2.18) and (2.11,2.13), we obtain

$$
\begin{equation*}
G^{a b}=\left(g^{a c} g^{b d}+\chi^{a b c d}\right) F_{c d}, \tag{2.20}
\end{equation*}
$$

where we have defined the tensors

$$
\begin{align*}
g^{a b} & \equiv \frac{1}{\sqrt{n}}\left[\eta^{a b}-\left(n^{2}-1\right) v^{a} v^{b}\right]  \tag{2.21}\\
\chi^{a b c d} & \equiv\left(\frac{n}{\mu}-1\right) g^{a[c} g^{d] b}-2 \chi_{(\varepsilon)}^{[a \mid[c} v^{d]} v^{[b]}+\frac{1}{2} \epsilon^{a b e f} \epsilon^{c d g h} \chi_{e g}^{(\mu)} v_{f} v_{h} \tag{2.22}
\end{align*}
$$

and the squared refractive index $n^{2}=\mu \varepsilon$. The idea, then, is to consider the symmetric tensor $g_{a b}$, defined through $g_{a b} g^{b c}=\delta_{a}^{c}$, as an effective metric of a curved background spacetime perceived by the electromagnetic field $F_{a b}$. Note that the components of $g^{a b}$ and $\eta^{a b}$ satisfy

$$
\begin{equation*}
\operatorname{det}\left(g^{\alpha \beta}\right)=\operatorname{det}\left(\eta^{\alpha \beta}\right) \tag{2.23}
\end{equation*}
$$

and, thus, $\sqrt{-\eta}=\sqrt{-g}$, where $g \equiv \operatorname{det}\left(g_{\alpha \beta}\right)$. One can easily check that $g_{a b}$ is explicitly given by

$$
\begin{equation*}
g_{a b}=\sqrt{n}\left[\eta_{a b}+\frac{\left(n^{2}-1\right)}{n^{2}} v_{a} v_{b}\right] . \tag{2.24}
\end{equation*}
$$

Therefore, in an arbitrary coordinate system, Eq. (2.9) reads

$$
\begin{equation*}
0=\frac{1}{\sqrt{-\eta}} \partial_{\alpha}\left(\sqrt{-\eta} G^{\alpha \beta}\right)=\frac{1}{\sqrt{-g}} \partial_{\alpha}\left(\sqrt{-g} G^{\alpha \beta}\right) . \tag{2.25}
\end{equation*}
$$

Up to this point, it was understood that the physical background metric $\eta_{a b}$ and its inverse $\eta^{a b}$ were responsible for lowering and raising tensorial indices. Now, with the introduction of an effective metric $g_{a b}$, we should be careful when performing these isomorphisms. In order to minimize chances of confusion, we shall avoid lowering and raising tensorial indices using the effective metric, making explicit most appearances of $g_{a b}$ and $g^{a b}$ in the
equations below, with few exceptions which will be clearly stated. One obvious exception is the definition of $g_{a b}$ as the inverse of $g^{a b}$. Another such exception is the use of $\nabla_{a}$ to denote covariant derivative compatible with $g_{a b}$. With this in mind, from Eqs. (2.10) and (2.25), the electromagnetic tensor $F_{a b}$ satisfies

$$
\begin{align*}
& 0=\nabla_{[a} F_{b c]},  \tag{2.26}\\
& 0=\nabla_{a}\left[\left(g^{a c} g^{b d}+\chi^{a b c d}\right) F_{c d}\right] . \tag{2.27}
\end{align*}
$$

Notice that Eqs. (2.26) and (2.27) applied to homogeneous ( $\nabla_{a} \varepsilon=\nabla_{a} \mu=0$ ), isotropic ( $\chi_{(\varepsilon)}^{a b}=$ $0=\chi_{a b}^{(\mu)}$ ) materials, with arbitrary 4 -velocity field $v^{a}$, lead to the same equations which rule minimally-coupled vacuum electromagnetism in a curved spacetime with metric $\sqrt{\mu / n} g_{a b}$. Optical analogue models in these configurations with $\mu=1$ were studied in the literature. ${ }^{38,39}$ Here, we shall focus on electromagnetism in anisotropic materials, more specifically, materials with only "shear-like" anisotropies: $\chi_{(\varepsilon)}^{[a b]}=0=\chi_{[a b]}^{(\mu)}$. In this case, the tensor $\chi^{a b c d}$ defined in Eq. (2.22) has the same algebraic symmetries as the Riemann curvature tensor, namely, $\chi^{a b c d}=\chi^{c d a b}$ and $\chi^{a[b c d]}=0$ - in addition to $\chi^{a b c d}=\chi^{[a b][c d]}$, which is always true.

The Eqs. (2.26) and (2.27) can be seen as analogous to some nonminimally-coupled electromagnetic field equations in curved spacetime. Although in general $\chi^{\text {abcd }}$ is independent of the Riemann tensor associated with the effective metric $g_{a b}$, one can construct cases where they are related. This is interesting because some one-loop QED corrections to Maxwell's field equations in curved spacetime ${ }^{40,41}$ can be emulated by such nonminimal coupling, as we shall discuss below, in Subsec. 2.2.1.

Before considering particular applications of the equations above, let us define a sesquilinear form on the space of complexified solutions, which will be relevant when applying the canonical quantization procedure. As usual, let us solve Eq. (2.26) by introducing the 4potential $A_{a}$ such that $F_{a b}=\nabla_{a} A_{b}-\nabla_{b} A_{a}$. Then, let $F_{a b}$ and $F_{a b}^{\prime}$ be two complex solutions of Eq. (2.27), associated to $A_{a}$ and $A_{a}^{\prime}$, respectively. With overbars representing complex conjugation, we contract $\bar{A}_{b}$ (resp., $A_{b}^{\prime}$ ) with Eq. (2.27) applied to $F_{c d}^{\prime}$ (resp., $\bar{F}_{c d}$ ) and subtract one from the other, arriving at

$$
\begin{equation*}
\nabla_{a}\left[\left(g^{a c} g^{b d}+\chi^{a b c d}\right)\left(\bar{A}_{b} F_{c d}^{\prime}-A_{b}^{\prime} \bar{F}_{c d}\right)\right]=0 . \tag{2.28}
\end{equation*}
$$

This continuity-like equation ensures that the quantity

$$
\begin{equation*}
\left(A, A^{\prime}\right) \equiv i \int_{\Sigma} \mathrm{d} \Sigma N_{a}\left(g^{a c} g^{b d}+\chi^{a b c d}\right)\left(\bar{A}_{b} F_{c d}^{\prime}-A_{b}^{\prime} \bar{F}_{c d}\right) \tag{2.29}
\end{equation*}
$$

is independent of the space-like hypersurface $\Sigma$ where the integration is performed - provided we restrict attention to solutions satisfying "appropriate" boundary condition —, where $\mathrm{d} \Sigma$ is the physical volume element on $\Sigma$ and $N_{a}=\eta_{a b} N^{b}$, with $N^{a}$ being a unit, futurepointing vector orthogonal to $\Sigma$ (according to $\eta_{a b}$ ). More specifically, considering that the system of interest is contained in the spacetime region $\mathscr{M} \cong T \times \Sigma$, where $T \subseteq \mathbb{R}$ is a real open interval, then the appropriate boundary condition amounts to imposing that the flux of the (sesquilinear) current appearing in Eq. (2.28) vanishes through $T \times \dot{\Sigma}$ (where $\dot{\mathscr{S}}$ denotes the boundary of the space $\mathscr{S}$ ). In particular, in stationary situations which we shall treat here, this condition translates to

$$
\begin{equation*}
\int_{\dot{\Sigma}} \mathrm{d} S s_{a}\left(g^{a c} g^{b d}+\chi^{a b c d}\right)\left(\bar{A}_{b} F_{c d}^{\prime}-A_{b}^{\prime} \bar{F}_{c d}\right)=0 \tag{2.30}
\end{equation*}
$$

where $\mathrm{d} S$ is the physical area element on $\dot{\Sigma}$ and $s^{a}$ is the unit vector field normal to $T \times \dot{\Sigma}$ (according to $\eta_{a b}$ ). Thus, these conditions being satisfied, Eq. (2.29) provides a legitimate sesquilinear form on the space $\mathscr{S}_{\mathbb{C}}$ of complex-valued solutions of Eqs. (2.26) and (2.27). Notice that for pure-gauge solutions - i.e., $A_{a}=\nabla_{a} \psi$, for some scalar function $\psi-,(A, A)=$ 0 . (The converse, however, is not true.)

The relevance of this sesquilinear form is that it provides a legitimate inner product on a (non-unique choice of) subspace $\mathscr{S}_{\mathbb{C}}^{+} \subsetneq \mathscr{S}_{\mathbb{C}}$ of "positive-norm solutions," which, together with its complex conjugate $\mathscr{S}_{\mathbb{C}}^{-} \subsetneq \mathscr{S}_{\mathbb{C}}$, generates all solutions: $\mathscr{S}_{\mathbb{C}}: \mathscr{S}_{\mathbb{C}}^{+} \oplus \mathscr{S}_{\mathbb{C}}^{-}=\mathscr{S}_{\mathbb{C}}$. Loosely speaking, upon completion, $\mathscr{S}_{\mathbb{C}}^{+}$yields a Hilbert space $\mathscr{H}$ from which the (symmetrized) Fock space $\mathscr{F}_{s}(\mathscr{H})$ is canonically constructed to represent states of the electromagnetic field. In particular, choosing $\mathscr{S}_{\mathbb{C}}^{+}$to be generated by positive-frequency solutions (those proportional to $\exp (-i \omega t)$, with $\omega>0$ ), the vacuum state of this Fock representation corresponds to the usual physical vacuum state of the field.

### 2.2.1 QED-inspired nonminimal couplings

As mentioned earlier, Eqs. (2.26) and (2.27) can be interpreted as ruling electromagnetism in curved spacetimes with some QED-inspired nonminimal coupling $\chi^{a b c d}$ with the back-
ground geometry. In fact, in the one-loop-QED approximation, ${ }^{40,41}$

$$
\begin{equation*}
\chi^{a b c d}=\alpha_{1} R^{a b c d}+\alpha_{2} R^{[a \mid[c} g^{d] \mid b]}+\alpha_{3} R g^{a[c} g^{d] b} \tag{2.31}
\end{equation*}
$$

with $\alpha_{1}=-\alpha_{2} / 13=2 \alpha_{3}=-\alpha /\left(90 \pi m_{e}^{2}\right)$, where $\alpha$ is the fine-structure constant, $m_{e}$ is the electron's mass, and $R^{a b c d}, R^{a b}$, and $R$ are now associated with the (effective) metric $g_{a b}$. By leaving $\alpha_{1}, \alpha_{2}$, $\alpha_{3}$ unconstrained, Eq. (2.31) represents a three-parameter family of couplings of the electromagnetic field with the background effective geometry - with some interesting particular cases explored in the literature. ${ }^{42}$

For a generic medium, $\chi^{a b c d}$ is not related to the geometry associated with $g_{a b}$. However, we can simulate couplings given by Eq. (2.31) by conveniently relating $n$ and $v^{a}$ (which determine $g_{a b}$ ) with $\mu$ and the anisotropic tensors $\chi_{(\varepsilon)}^{a b}$ and $\chi_{a b}^{(\mu)}$ (which appear in $\chi^{a b c d}$ ). From Eqs. (2.22) and (2.31), and their contractions with $g_{a b}$,

$$
\begin{align*}
g_{b d} \chi^{a b c d} & =\frac{3}{2}\left(\frac{n}{\mu}-1\right) g^{a c}+\frac{\chi_{(\varepsilon)}^{a c}}{2 n^{3 / 2}}+\frac{n^{3 / 2}}{2} g^{a b} g^{c d} \chi_{b d}^{(\mu)} \\
& =\left(\alpha_{1}+\alpha_{2} / 2\right) R^{a c}+\left(\alpha_{2} / 4+3 \alpha_{3} / 2\right) R g^{a c},  \tag{2.32}\\
g_{a c} g_{b d} \chi^{a b c d} & =6\left(\frac{n}{\mu}-1\right)=\left(\alpha_{1}+3 \alpha_{2} / 2+6 \alpha_{3}\right) R, \tag{2.33}
\end{align*}
$$

we can solve for $\mu$ and the anisotropic tensors, obtaining

$$
\begin{equation*}
\mu=\frac{n}{1+\left(\alpha_{1} / 6+\alpha_{2} / 4+\alpha_{3}\right) R} \tag{2.34}
\end{equation*}
$$

for the isotropic permeability, and

$$
\begin{align*}
n^{-3 / 2} \chi_{(\varepsilon)}^{a b} & =-2 \alpha_{1}\left(R^{a c b d} V_{c} V_{d}+\frac{R}{12} H^{a b}\right)+\frac{\alpha_{2}}{2}\left(R^{a b}-\frac{R}{4} g^{a b}\right)  \tag{2.35}\\
n^{3 / 2} \chi_{a b}^{(\mu)} & =2 \alpha_{1}\left(R_{a c b d} V^{c} V^{d}+\frac{R}{12} H_{a b}\right)+\frac{\left(4 \alpha_{1}+\alpha_{2}\right)}{2}\left(R^{a b}-\frac{R}{4} g^{a b}\right) \tag{2.36}
\end{align*}
$$

for the anisotropic tensors, where $V^{a}=n^{3 / 4} v^{a}$ is the 4-velocity of the medium normalized according to the effective metric $g_{a b}$ and $H_{a b} \equiv g_{a b}+V_{a} V_{b}$. In Eqs. (2.35) and (2.36) indices are lowered and raised by the effective metric and its inverse. In order to find the anisotropic tensors, we have expanded them in terms of the basis $\left\{R_{a c b d} \nu^{c} \nu^{d}, g_{a b}, R_{a b}\right\}$ and solved for the coefficients. Notice that, unless $\alpha_{1}=\alpha_{2}=0$ - which implies $\chi_{(\varepsilon)}^{a b}=0=\chi_{a b}^{(\mu)}$ —, as a consequence of $\chi_{(\varepsilon)}^{a b} \nu_{b}=0=\chi_{a b}^{(\mu)} v^{b}$, only geometries associated with $g_{a b}$ which can be put in
the form given by Eq. (2.24) and satisfying

$$
\begin{equation*}
R_{b}^{a} v^{b}=\frac{R}{4} v^{a}, \tag{2.37}
\end{equation*}
$$

for some timelike 4 -vector $\nu^{a}$, can be emulated by these anisotropic media - with $v^{a}$ then set as the medium's 4 -velocity. Here we note that the constraint of Eq. (2.37) is connected to the classification of spacetimes via the Ricci tensor. ${ }^{43}$ Using Einstein's equations to map this constraint to the stress-energy-momentum tensor $T^{a b}$ of the corresponding gravitational source, we have that

$$
\begin{equation*}
T_{b}^{a} v^{b}=\frac{T}{4} v^{a}, \tag{2.38}
\end{equation*}
$$

where, again, the effective metric and its inverse are used to lower and raise indices (and $T \equiv T_{a}^{a}$ ). One can easily check that in case of perfect fluids - characterized by a proper energy density $\rho$ and (isotropic) pressure $p-$, Eq. (2.38) is only satisfied for $p=-\rho$; i.e., for a cosmological-constant-type "fluid." However, if one allows for sources with anisotropic pressures ( $p_{1}, p_{2}, p_{3}$ ), described by the stress-energy-momentum tensor

$$
\begin{equation*}
T^{a b}=\rho u^{a} u^{b}+\sum_{j=1}^{3} p_{j} \mathbf{e}_{j}^{a} \mathbf{e}_{j}^{b} \tag{2.39}
\end{equation*}
$$

— with $\left\{u^{a}, \mathbf{e}_{1}^{a}, \mathbf{e}_{2}^{a}, \mathbf{e}_{3}^{a}\right\}$ being a tetrad and $u^{a}$ timelike -, then

$$
\begin{equation*}
\rho+\frac{1}{3} \sum_{j=1}^{3} p_{j}=0 \tag{2.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(V_{a} \mathbf{e}_{j}^{a}\right)\left(\rho+p_{j}\right)=0, \quad j=1,2,3 . \tag{2.41}
\end{equation*}
$$

In particular, if $V^{a}=u^{a}$, then Eq. (2.40) is the only additional constraint to be enforced.
Returning attention to the background effective geometry and recalling that all the geometric tensors are obtained from $g_{a b}$ given in Eq. (2.24), we see that Eq. (2.37) actually comprises a system of four differential equations which $n$ and $v^{a}$ must satisfy. Electromagnetism with nonminimal coupling described by Eq. (2.31) can only be simulated in these anisotropic media if the background spacetime geometry is associated to solutions of this
system [via Eq. (2.24)]. We shall treat a particular solution to these differential equations later.

### 2.3 Plane-symmetric anisotropic medium at rest

In this section, we consider the simplest case of an anisotropic medium: a plane-symmetric medium at rest in the inertial lab frame. The purpose of this section is not yet to establish an analogy with some interesting gravitational system, but to present the analysis in a simple context. In Sec. 2.4 we apply the analysis to a more appealing scenario.

Let us consider a medium at rest in an inertial laboratory, such that in inertial Cartesian coordinates $\{(t, x, y, z)\}=\mathscr{I} \times \mathbb{R}^{3} \subseteq \mathbb{R}^{4}$ we have $v^{\mu}=(1,0,0,0), \mu=\mu(z), \varepsilon=\varepsilon(z)$,

$$
\begin{equation*}
\chi_{(\varepsilon)}^{\alpha \beta}=\frac{\Delta^{(\varepsilon)}}{3}\left(2 \delta_{z}^{\alpha} \delta_{z}^{\beta}-\delta_{x}^{\alpha} \delta_{x}^{\beta}-\delta_{y}^{\alpha} \delta_{y}^{\beta}\right), \tag{2.42}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{\alpha \beta}^{(\mu)}=\frac{\Delta^{(\mu)}}{3}\left(2 \delta_{\alpha}^{z} \delta_{\beta}^{z}-\delta_{\alpha}^{x} \delta_{\beta}^{x}-\delta_{\alpha}^{y} \delta_{\beta}^{y}\right) \tag{2.43}
\end{equation*}
$$

with $\Delta^{(\varepsilon)}=\Delta^{(\varepsilon)}(z), \Delta^{(\mu)}=\Delta^{(\mu)}(z), z \in \mathscr{I}$. ( $\mathscr{I}$ is an open real interval.) This simply means that

$$
\begin{align*}
& D^{j}=\varepsilon_{\perp} E_{j}, j=x, y,  \tag{2.44}\\
& D^{z}=\varepsilon_{\|} E_{z},  \tag{2.45}\\
& H_{j}=\mu_{\perp}^{-1} B^{j}, j=x, y,  \tag{2.46}\\
& H_{z}=\mu_{\|}^{-1} B^{z}, \tag{2.47}
\end{align*}
$$

where $\varepsilon_{\|}-\varepsilon_{\perp} \equiv \Delta^{(\varepsilon)}, 2 \varepsilon_{\perp}+\varepsilon_{\|} \equiv 3 \varepsilon, \mu_{\|}^{-1}-\mu_{\perp}^{-1} \equiv \Delta^{(\mu)}$, and $2 \mu_{\perp}^{-1}+\mu_{\|}^{-1} \equiv 3 \mu^{-1}$.
In these coordinates, $g_{\mu \nu}=\sqrt{n} \operatorname{diag}\left(-n^{-2}, 1,1,1\right)$. For convenience, we shall work in the generalized Coulomb gauge ${ }^{44}$ in which $A_{\mu}=(0, \mathbf{A})$ and $\partial_{\perp} \cdot\left(\varepsilon_{\perp} \mathbf{A}_{\perp}\right)+\partial_{z}\left(\varepsilon_{\|} A_{z}\right)=0$, where we have defined $\mathbf{A}_{\perp} \equiv\left(A_{x}, A_{y}\right), \partial_{\perp} \equiv\left(\partial_{x}, \partial_{y}\right)$. In this gauge, the $t$ component of Eq. (2.27) is automatically satisfied, while the spatial components lead to

$$
\begin{equation*}
\left[-\frac{\varepsilon_{\perp}}{\mu_{\perp}} \partial_{t}^{2}+\frac{1}{\mu_{\perp} \mu_{\|}} \partial_{\perp}^{2}+\left(\frac{1}{\mu_{\perp}} \partial_{z}\right)^{2}\right] \mathbf{A}_{\perp}=\frac{1}{\mu_{\perp} \mu_{\|}}\left[\mu_{\|} \partial_{z}\left(\mu_{\perp}^{-1} \partial_{\perp}\right)-\varepsilon_{\perp}^{-1} \partial_{z}\left(\varepsilon_{\|} \partial_{\perp}\right)\right] A_{z} \tag{2.48}
\end{equation*}
$$

$$
\begin{equation*}
\left[-\frac{\mu_{\perp}}{\varepsilon_{\perp}} \partial_{t}^{2}+\frac{1}{\varepsilon_{\perp} \varepsilon_{\|}} \partial_{\perp}^{2}+\left(\frac{1}{\varepsilon_{\perp}} \partial_{z}\right)^{2}\right]\left(\varepsilon_{\|} A_{z}\right)=0 . \tag{2.49}
\end{equation*}
$$

First, let us consider solutions A such that $A_{z}=0$, which describe electric fields which are perpendicular to the $z$ direction - transverse electric modes, $\mathbf{A}^{(\mathrm{TE})}$, for short. ${ }^{45}$ In this case, our gauge condition ensures that there exists a scalar field $\psi$ such that $A_{x}^{(\mathrm{TE})}=\partial_{y} \psi$ and $A_{y}^{(\mathrm{TE})}=-\partial_{x} \psi$. Moreover, making use of the staticity and planar symmetry of the present scenario, we can write $\psi=\exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega k_{\perp}}^{(\mathrm{TE})}(z)$, where $\mathbf{x}_{\perp}=(x, y, 0), \mathbf{k}_{\perp}=\left(k_{x}, k_{y}, 0\right)$, $k_{\perp}=\left\|\mathbf{k}_{\perp}\right\|$, and $f_{\omega k_{\perp}}^{(\mathrm{TE})}(z)$ satisfies

$$
\begin{equation*}
\left[-\frac{\mathrm{d}^{2}}{\mathrm{~d} \zeta^{2}}+\left(\frac{k_{\perp}^{2}}{\mu_{\perp} \mu_{\|}}-\frac{\varepsilon_{\perp} \omega^{2}}{\mu_{\perp}}\right)\right] f_{\omega k_{\perp}}^{(\mathrm{TE})}=0 \tag{2.50}
\end{equation*}
$$

with $\zeta$ being a spatial coordinate such that $\mathrm{d} \zeta=\mu_{\perp} \mathrm{d} z$. The Eq. (2.50) must be supplemented by boundary conditions for $f_{\omega k_{\perp}}^{(\mathrm{TE})}$. Imposing Eq. (2.30) to these modes leads to

$$
\begin{equation*}
\left.\left[\overline{f_{\omega k_{\perp}}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \zeta} f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TE})}-f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \zeta} \overline{f_{\omega k_{\perp}}^{(\mathrm{TE})}}\right]\right|_{\dot{\mathscr{I}}}=0 \tag{2.51}
\end{equation*}
$$

where []$]_{\dot{\mathscr{}}}$ denotes the flux of the quantity in square brackets through $\dot{\mathscr{I}}$. This condition restricts the possible values of $\omega^{2}$. Let $\mathscr{E}_{k_{\perp}}^{(\mathrm{TE})}$ be the ( $k_{\perp}$-dependent) set of $\omega$ values for which Eqs. (2.50) and (2.51) are satisfied for $f_{\omega k_{\perp}}^{(\mathrm{TE})} \not \equiv 0$. For $\omega, \omega^{\prime} \in \mathscr{E}_{k_{\perp}+}^{(\mathrm{TE})} \equiv \mathscr{E}_{k_{\perp}}^{(\mathrm{TE})} \cap \mathbb{R}_{+}^{*}$, we can orthonormalize these modes according to

$$
\begin{align*}
& \left(A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}, A_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TE})}\right)=-\left(\overline{A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}} \overline{A_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TEE}}}\right)=\delta_{\omega \omega^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right),  \tag{2.52}\\
& \left(A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}, \overline{A_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TE})}}\right)=0, \tag{2.53}
\end{align*}
$$

( $\delta_{\omega \omega^{\prime}}$ being the appropriate Dirac-delta distribution on $\mathscr{E}_{k_{\perp}}^{(\mathrm{TE})}$ ), where the sesquilinear form given in Eq. (2.29), applied to the current scenario, takes the form

$$
\begin{equation*}
\left(A, A^{\prime}\right) \equiv i \int_{\Sigma} \mathrm{d}^{3} x\left[\varepsilon_{\perp}\left(\overline{\mathbf{A}}_{\perp} \cdot \partial_{t} \mathbf{A}_{\perp}^{\prime}-\mathbf{A}_{\perp}^{\prime} \cdot \partial_{t} \overline{\mathbf{A}}_{\perp}\right)+\varepsilon_{\|}\left(\bar{A}_{z} \partial_{t} A_{z}^{\prime}-A_{z}^{\prime} \partial_{t} \bar{A}_{z}\right)\right] . \tag{2.54}
\end{equation*}
$$

We obtain (up to a global phase)

$$
\begin{equation*}
\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}=\frac{\mathbf{k}_{\perp} \times \mathbf{n}_{z}}{2 \pi k_{\perp} \sqrt{2 \omega}} \mathrm{e}^{-i\left(\omega t-\mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)} f_{\omega k_{\perp}}^{(\mathrm{TE})}(z) \tag{2.55}
\end{equation*}
$$

with

$$
\begin{equation*}
\int_{\mathscr{I}} \mathrm{d} z \varepsilon_{\perp}(z) \overline{f_{\omega k_{\perp}}^{(\mathrm{TE})}}(z) f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TE})}(z)=\delta_{\omega \omega^{\prime}}, \tag{2.56}
\end{equation*}
$$

and $\mathbf{n}_{z} \equiv(0,0,1)$.

The second set of solutions of Eqs. (2.48) and (2.49), which describe magnetic fields which are perpendicular to the $z$ direction - transverse magnetic modes, $\mathbf{A}^{(\mathrm{TM})}$, for short ,${ }^{45}$ is obtained by conveniently setting $A_{z}^{(\mathrm{TM})}=\varepsilon_{\|}^{-1} \partial_{\perp}^{2} \phi$, where $\phi$ is an auxiliary function. Our gauge condition then leads to $A_{j}^{(\mathrm{TM})}=-\varepsilon_{\perp}^{-1} \partial_{j} \partial_{z} \phi, j=x, y$. Using, again, staticity and planar symmetry, we find solutions of the form $\phi=\exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega k_{\perp}}^{(\mathrm{TM})}(z)$, where $f_{\omega k_{\perp}}^{(\mathrm{TM})}(z)$ satisfies

$$
\begin{equation*}
\left[-\frac{\mathrm{d}^{2}}{\mathrm{~d} \xi^{2}}+\left(\frac{k_{\perp}^{2}}{\varepsilon_{\perp} \varepsilon_{\|}}-\frac{\mu_{\perp} \omega^{2}}{\varepsilon_{\perp}}\right)\right] f_{\omega k_{\perp}}^{(\mathrm{TM})}=0 \tag{2.57}
\end{equation*}
$$

with $\xi$ being a spatial coordinate such that $\mathrm{d} \xi=\varepsilon_{\perp} \mathrm{d} z$. The boundary condition imposed by Eq. (2.30) now leads to

$$
\begin{equation*}
\left.\left[\omega^{2} \overline{f_{\omega k_{\perp}}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \xi} f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TM})}-\omega^{\prime 2} f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TM})} \frac{\mathrm{d}}{\mathrm{~d} \xi} \overline{f_{\omega k_{\perp}}^{(\mathrm{TM})}}\right]\right|_{\dot{\mathscr{I}}}=0 . \tag{2.58}
\end{equation*}
$$

Let $\mathscr{E}_{k_{\perp}}^{(\mathrm{TM})}$ be the ( $k_{\perp}$-dependent) set of $\omega$ values for which Eqs. (2.57) and (2.58) are satisfied for $f_{\omega k_{\perp}}^{(\mathrm{TM})} \not \equiv 0$. For $\omega, \omega^{\prime} \in \mathscr{E}_{k_{\perp}+}^{(\mathrm{TM})} \equiv \mathscr{E}_{k_{\perp}}^{(\mathrm{TM})} \cap \mathbb{R}_{+}^{*}$, we can normalize these modes according to

$$
\begin{align*}
& \left(A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}, A_{\omega^{\left(\mathbf{k}_{\perp}^{\prime}\right.}}^{(\mathrm{TM})}\right)=-\left(\overline{A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}}, \overline{A_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TM})}}\right)=\delta_{\omega \omega^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right),  \tag{2.5}\\
& \left(A_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}, \overline{A_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TM})}}\right)=0, \tag{2.60}
\end{align*}
$$

( $\delta_{\omega \omega^{\prime}}$ now being the appropriate Dirac-delta distribution on $\mathscr{E}_{k_{\perp}}^{(\mathrm{TM})}$ ), obtaining (up to a global phase)

$$
\begin{equation*}
\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}=\frac{\mathrm{e}^{-i\left(\omega t-\mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)}}{2 \pi k_{\perp} \sqrt{2 \omega^{3}}}\left(\frac{k_{\perp}^{2}}{\varepsilon_{\|}} \mathbf{n}_{z}+i \frac{\mathbf{k}_{\perp}}{\varepsilon_{\perp}} \frac{\mathrm{d}}{\mathrm{~d} z}\right) f_{\omega k_{\perp}}^{(\mathrm{TM})}(z), \tag{2.61}
\end{equation*}
$$

with

$$
\begin{equation*}
\int_{\mathscr{I}} \mathrm{d} z \mu_{\perp}(z) \overline{f_{\omega k_{\perp}}^{(\mathrm{TM})}}(z) f_{\omega^{\prime} k_{\perp}}^{(\mathrm{TM})}(z)=\delta_{\omega \omega^{\prime}} . \tag{2.62}
\end{equation*}
$$

Moreover, modes $\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}$ and $\overline{\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}}$ are orthogonal to modes $\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}$ and $\overline{\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}}$.
The solutions expressed in Eqs. (2.55) and (2.61), dubbed positive-frequency normal modes, play a central role in the construction of the Fock (Hilbert) space of the quantized
theory, as described at the end of the previous section. With these solutions, the quantumfield operator $\mathbf{A}$ is represented by

$$
\begin{equation*}
\mathbf{A}=\sum_{\mathrm{J} \in\{\mathrm{TE}, \mathrm{TM}\}} \int_{\mathbb{R}^{2}} \mathrm{~d}^{2} \mathbf{k}_{\perp} \int_{\mathscr{E}_{k_{\perp}}^{(J)}} \mathrm{d} \omega\left[a_{\omega \mathbf{k}_{\perp}}^{(\mathrm{J})} \mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(\mathrm{J})}+\text { H.c. }\right], \tag{2.63}
\end{equation*}
$$

where "H.c." stands for "Hermitian conjugate" of the preceding term and $a_{\omega \mathbf{k}_{\perp}}^{(\mathrm{J})}$ (respectively, $\left.a_{\omega \mathbf{k}_{\perp}}^{(J) \dagger}\right)$ is the annihilation (resp., creation) operator associated with mode $\mathbf{A}_{\omega \mathbf{k}_{\perp}}^{(J)}$, satisfying the canonical commutation relations:

$$
\begin{align*}
& {\left[a_{\omega \mathbf{k}_{\perp}}^{(\mathrm{J}}, a_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{\left(\mathrm{J}^{\prime} \dagger\right.}\right]=\delta^{\mathrm{JJ}} \delta_{\omega \omega^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right),}  \tag{2.64}\\
& {\left[a_{\omega \mathbf{k}_{\perp}}, a_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{\mathrm{J}^{\prime}}\right]=0} \tag{2.65}
\end{align*}
$$

As an application of our quantization scheme one can use the above formulas to obtain, for instance, the Carniglia-Mandel quantization ${ }^{46}$ in a straightforward way. The system in this case is composed by a dielectric-vacuum interface at $z=0$ and a nonmagnetizable ( $\mu_{\|}=$ $\mu_{\perp}=1$ ) homogeneous isotropic nondispersive dielectric ( $\varepsilon_{\|}=\varepsilon_{\perp}=\varepsilon \equiv n^{2}$ ) filling the halfspace $z<0$. These data enter Eqs. (2.50) and (2.57), thus describing the background in terms of effective potentials of one-dimensional Schrödinger-like problems.

### 2.3.1 Instability analysis

In the analysis presented above, it was implicitly assumed that all constitutive functions $\varepsilon_{\perp}$, $\varepsilon_{\|}, \mu_{\perp}$, and $\mu_{\|}$are positive functions of $z \in \mathscr{I}$, mainly on the field modes normalization. This condition ensures that the solutions presented in Eqs. (2.55) and (2.61), together with their complex conjugates, constitute a complete set of (complexified) solutions of Maxwell equations in $\mathscr{I} \times \mathbb{R}^{3}$; in other words, the boundary-value problems defined by Eqs. $(2.50,2.51)$ and Eqs. $(2.57,2.58)$ admit solutions only for (a subset of) $\omega^{2}>0$. This is easily seen by interpreting them as null-eigenvalue problems for the linear operators defined in the square brackets of Eqs. (2.50) and (2.57). Experience with Schrödinger-like equations teaches us that these equations have solutions provided the associated effective potentials (terms in parentheses) become sufficiently negative in a given region - which implies $\omega^{2}>0$ and, typically, the larger the $k_{\perp}^{2}$, the larger the $\omega^{2}$.

However, as anticipated in the introduction, here we shall consider the more interest-
ing situation of materials with some of their constitutive functions assuming negative values. In this case, the effective potentials appearing in Eqs. (2.50) and (2.57) may become sufficiently negative - granting solutions to these boundary-value problems - without demanding $\omega^{2}>0$. For instance, if $\mu_{\|}<0$ (with $\mu_{\perp}, \varepsilon_{\perp}>0$ ), then the larger the value of $k_{\perp}$, the more negatively it contributes to the effective potential of Eq. (2.50), favoring the appearance of solutions with smaller (possibly negative) values of $\omega^{2}$. The same is true for Eq. (2.57) if $\varepsilon_{\|}<0$ and similar analysis can be done if any other constitutive function becomes negative.

At this point, we must introduce an element of reality concerning the constitutive functions. We have been treating these quantities as given functions of $z$ alone - neglecting dispersion effects, since we are, here, interested in gravity analogues. However, these material properties generally depend on characteristics of the electromagnetic field itself, particularly on its time variation (i.e., on $\omega$ ), in which case Eqs. (2.15) and (2.16) would be valid mode by mode, with the constitutive tensors $\varepsilon^{a b}$ and $\mu_{a b}$ possibly being different for different modes. When translated to spacetime-dependent quantities, Eqs. (2.15) and (2.16) would be substituted by sums over the set of allowed field modes - as explained in the next chapter. Therefore, the precise key assumption about our metamaterial media is that some of their anisotropic constitutive functions $\varepsilon_{\perp}, \varepsilon_{\|}, \mu_{\perp}, \mu_{\|}$can become negative for some $\omega$ on the positive imaginary axis, $\omega^{2}<0$. Notwithstanding, the less restrictive condition $\operatorname{Im}(\omega)>0$ would suffice for our purposes. However, dealing with the case $\operatorname{Im}(\omega) \operatorname{Re}(\omega) \neq 0$ would involve a far more complex quantization procedure, which we shall treat in detail in the second part of this work. Moreover, our focus here is to show that the electromagnetic field itself can exhibit interesting behavior without need to exchange energy with the medium (which occurs in dispersive media). This justifies our focus on $\omega^{2}<0$ in what follows.

Let $\omega^{2}=-\Omega^{2}$ (with $\Omega>0$ ) be such value for which at least one of the constitutive functions is negative for $z \in \mathscr{I}$. Thus, both the effective potentials of Eqs. (2.50) and (2.57) take the general form

$$
\begin{equation*}
V_{e f f}=C_{1} k_{\perp}^{2}+C_{2} \Omega^{2}, \tag{2.66}
\end{equation*}
$$

with $C_{1}$ and $C_{2}$ being functions of $z$. Two interesting possibilities arise:

- (i) $C_{1}<0$ : In this case, the larger the value of $k_{\perp}$, the more negative the effective potential gets. Therefore, it is quite reasonable to expect that, for a given size of the interval $\mathscr{I}$, one
can always find "large enough" values of $k_{\perp}-$ certainly satisfying $k_{\perp}^{2}>C_{2} \Omega^{2} /\left|C_{1}\right|-$ such that the Schrödinger-like equation with effective potential $V_{\text {eff }}$ admits null-eingenvalue solutions. We shall refer to this situation as large- $k_{\perp}$ instability;
- (ii) $C_{1}>0$ and $C_{2}<0$ : Under these conditions, the effective potential $V_{\text {eff, }}$, as a function of $k_{\perp}$, is bounded from below: $V_{\text {eff }} \geq-\left|C_{2}\right| \Omega^{2}$. Therefore, a Schrödinger-like equation with effective potential $V_{\text {eff }}$ only admits null-eigenvalue solutions provided $k_{\perp}$ is "sufficiently small" - certainly satisfying $k_{\perp}^{2}<\left|C_{2}\right| \Omega^{2} / C_{1} —$ and the size of the interval where $V_{\text {eff }}$ is negative is "sufficiently large." We shall refer to this situation as minimum-width instability.

Let us call $g_{\Omega k_{\perp}}^{(J)}$ the null-eigenvalue solutions mentioned in either case above, with $\mathrm{J} \in$ \{TE,TM\} depending on whether it refers to Eq. (2.50) or (2.57) with $\omega^{2}=-\Omega^{2}$ (without loss of generality, $\Omega>0$ ). These solutions are associated with unstable electromagnetic modes whose temporal behavior is proportional to $\exp ( \pm \Omega t)$. Although it might be tempting not to consider these "runaway" solutions, ${ }^{15,47}$ they are essential, if they exist, to expand an arbitrary initial field configuration satisfying the boundary-value problems set by Eqs. (2.50,2.51) and (2.57,2.58); in other words, the stationary modes alone do not constitute a complete set of solutions of Maxwell's equations with the given boundary conditions. And even if, on the classical level, one might want to restrict attention to initial field configurations which have no contribution coming from these unstable modes - which is certainly unnatural, for causality forbids the system to constrain its initial configuration based on its future behavior -, inevitable quantum fluctuations of these modes would grow, making them dominant some time e-foldings ( $t \sim N \Omega^{-1}, N \gg 1$ ) after the proper material conditions having been engineered. Therefore, these modes are as physical as the oscillatory ones. In fact, artificial inconsistencies have been reported in the literature, regarding field quantization in active media, ${ }^{15,47}$ which are completely cured when unstable modes are included in the analysis, as shown in the next part of this thesis.

It is interesting to note that depending on which constitutive function is negative, Eqs. (2.50) and (2.57) may incur in different types of instabilities. For instance, if $\mu_{\perp}<0$ for a given $\omega^{2}=-\Omega^{2}<0$, with all other constitutive functions being positive, then Eq. (2.50) exhibits case-(i) instability, while Eq. (2.57) incur in case-(ii) instability. This means that unstable TE modes — with some $k_{\perp}>\sqrt{\mu_{\|} \varepsilon_{\perp}} \Omega$ - would certainly be present, while unstable

TM modes — with some $k_{\perp}<\sqrt{\left|\mu_{\perp}\right| \varepsilon_{\|}} \Omega$ — would only appear if the width of the material (size of the interval $\mathscr{I}$ ) is larger than some critical value. We shall illustrate these facts in a simple example below. But first, let us analyze some features of these unstable modes. In order not to rely on particular initial field configurations, let us focus on the inevitable quantum fluctuations of these modes.

## Unstable TE modes

Repeating the procedure which led us from Eq. (2.50) to Eq. (2.55) for the stable modes, unstable TE modes, $A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}$, properly orthonormalized according to

$$
\begin{align*}
& \left(A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}, A_{\Omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(u \mathrm{TE})}\right)=-\left(\overline{A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}}, \overline{A_{\Omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(u \mathrm{TE})}}\right)=\delta_{\Omega \Omega^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right),  \tag{2.67}\\
& \left(A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}, \overline{\left.A_{\Omega^{\prime}}^{(u \mathrm{k})} \mathbf{k}_{\perp}^{\prime}\right)}\right)=0, \tag{2.68}
\end{align*}
$$

(and orthogonal to all other modes) read (up to a time translation)

$$
\begin{equation*}
\mathbf{A}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}=\frac{\mathbf{k}_{\perp} \times \mathbf{n}_{z}}{2 \pi k_{\perp} \sqrt{\Omega \sin \vartheta}} e^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{\Omega k_{\perp}}^{(\mathrm{TE})}(z) \cosh \left(\Omega t-i s_{\varepsilon}^{\perp} \vartheta / 2\right), \tag{2.69}
\end{equation*}
$$

with $0<\vartheta<\pi, g_{\Omega k_{\perp}}^{(\mathrm{TE})}$ normalized according to

$$
\begin{equation*}
\left|\int_{\mathscr{I}} \mathrm{d} z \varepsilon_{\perp}(z) \overline{g_{\Omega k_{\perp}}^{(\mathrm{TE)}}}(z) g_{\Omega^{\prime} k_{\perp}}^{(\mathrm{TE})}(z)\right|=\delta_{\Omega \Omega^{\prime}}, \tag{2.70}
\end{equation*}
$$

and $s_{\varepsilon}^{\perp}$ being the sign of the integral above. Calculating the electric $\mathbf{E}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}$ and magnetic $\mathbf{B}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{EE})}$ fields associated to these modes, we have:

$$
\begin{align*}
& \mathbf{E}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}=\frac{\sqrt{\Omega}}{2 \pi k_{\perp} \sqrt{\sin \vartheta}}\left(\mathbf{n}_{z} \times \mathbf{k}_{\perp}\right) \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{\Omega k_{\perp}}^{(\mathrm{TE})}(z) \sinh \left(\Omega t-i s_{\varepsilon}^{\perp} \vartheta / 2\right),  \tag{2.71}\\
& \mathbf{B}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TE})}=\frac{\mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}}}{2 \pi k_{\perp} \sqrt{\Omega \sin \vartheta}}\left(-i k_{\perp}^{2} \mathbf{n}_{z}+\mathbf{k}_{\perp} \frac{\mathrm{d}}{\mathrm{~d} z}\right) g_{\Omega k_{\perp}}^{(\mathrm{TE})}(z) \cosh \left(\Omega t-i s_{\varepsilon}^{\perp} \vartheta / 2\right) . \tag{2.72}
\end{align*}
$$

## Unstable TM modes

Now, turning to the TM modes, we repeat the procedure which led us from Eq. (2.57) to Eq. (2.61) for the stable modes. Unstable TM modes, $A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}$, properly orthonormalized ac-
cording to

$$
\begin{align*}
& \left(A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}, A_{\Omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(u \mathrm{TM})}\right)=-\left(\overline{A_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}}, \overline{A_{\Omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(u \mathrm{TM})}}\right)=\delta_{\Omega \Omega^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right),  \tag{2.73}\\
& \left(A_{\left.\Omega \mathbf{k}_{\perp}\right)}^{(u \mathrm{TM})}, \overline{A_{\Omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(u \mathrm{TM})}}\right)=0, \tag{2.74}
\end{align*}
$$

(and orthogonal to all other modes) read (up to a time translation)

$$
\begin{equation*}
\mathbf{A}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}=\frac{\mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}}}{2 \pi k_{\perp} \sqrt{\Omega^{3} \sin \vartheta}}\left(\frac{k_{\perp}^{2}}{\varepsilon_{\|}} \mathbf{n}_{z}+i \frac{\mathbf{k}_{\perp}}{\varepsilon_{\perp}} \frac{\mathrm{d}}{\mathrm{~d} z}\right) g_{\Omega k_{\perp}}^{(\mathrm{TM})}(z) \cosh \left(\Omega t+i s_{\mu}^{\perp} \vartheta / 2\right), \tag{2.75}
\end{equation*}
$$

where, again, $0<\vartheta<\pi, g_{\Omega k_{\perp}}^{(\mathrm{TM})}$ is normalized according to

$$
\begin{equation*}
\left|\int_{\mathscr{I}} \mathrm{d} z \mu_{\perp}(z) \overline{g_{\Omega k_{\perp}}^{(\mathrm{TM})}}(z) g_{\Omega^{\prime} k_{\perp}}^{(\mathrm{TM})}(z)\right|=\delta_{\Omega \Omega^{\prime}}, \tag{2.76}
\end{equation*}
$$

and $s_{\mu}^{\perp}$ is the sign of the integral above. Calculating the electric $\mathbf{E}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}$ and magnetic $\mathbf{B}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}$ fields associated to these modes, we have:

$$
\begin{align*}
& \mathbf{E}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}=-\frac{\mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}}}{2 \pi k_{\perp} \sqrt{\Omega \sin \vartheta}}\left(\frac{k_{\perp}^{2}}{\varepsilon_{\|}} \mathbf{n}_{z}+i \frac{\mathbf{k}_{\perp}}{\varepsilon_{\perp}} \frac{\mathrm{d}}{\mathrm{~d} z}\right) g_{\Omega k_{\perp}}^{(\mathrm{TM})}(z) \sinh \left(\Omega t+i s_{\mu}^{\perp} \vartheta / 2\right),  \tag{2.77}\\
& \mathbf{B}_{\Omega \mathbf{k}_{\perp}}^{(u \mathrm{TM})}=\frac{i \mu_{\perp} \sqrt{\Omega}}{2 \pi k_{\perp} \sqrt{\sin \vartheta}}\left(\mathbf{n}_{z} \times \mathbf{k}_{\perp}\right) \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{\Omega k_{\perp}}^{(\mathrm{TM})}(z) \cosh \left(\Omega t+i s_{\mu}^{\perp} \vartheta / 2\right) . \tag{2.78}
\end{align*}
$$

The modes given by Eqs. (2.69) and (2.75), if present, must be added to the expansion of the field operator A given in Eq. (2.63), along with their complex conjugates - with corresponding annihilation $a_{\Omega \mathbf{k}_{\perp}}^{(u)}$ and creation $a_{\Omega \mathbf{k}_{\perp}}^{(u) \dagger}$ operators, $\mathrm{J} \in\{\mathrm{TE}, \mathrm{TM}\}$. The resulting operator expansion can then be used to calculate electromagnetic-field fluctuations and correlations. In the presence of unstable modes, it is easy to see that the field's vacuum fluctuations are eventually ( $t \gg \Omega^{-1}$ ) dominated by these exponentially-growing modes. Obviously, this instability cannot persist indefinitely as these wild fluctuations will affect the medium's properties, supposedly leading the whole system to a final stable state. In some gravitational contexts, stabilization occurs by decoherence of these growing vacuum fluctuations, ${ }^{48}$ giving rise to a nonzero classical field configuration - a phenomenon called spontaneous scalarization (for spin-0) ${ }^{7,8,49,50}$ or vectorization (for spin-1 fields). ${ }^{51}$ It is possible that something similar might occur in the analogous system. We shall discuss this point further in Sec. 2.5 and in the next part of this work.

### 2.3.2 Example

Let us consider a very simple system just to illustrate the results above in a concrete scenario: a slab of width $L$ (in the region $-L / 2<z<L / 2$ ), made of a homogeneous material with, say, $\mu_{\perp}<0$ for a given $\omega^{2}=-\Omega^{2}(\Omega>0)$ and all other constitutive functions positive. For concreteness sake, here we assume that this value $\omega^{2}=-\Omega^{2}$ is isolated and that it is the most negative value of $\omega^{2}$ for which $\mu_{\perp}<0$. This latter assumption is merely a matter of choice, while the former only affects the measure on the set of quantum numbers $\mathbf{k}_{\perp}$ : $\int \mathrm{d}^{2} \mathbf{k}_{\perp} \rightarrow \int \mathrm{d} \theta \sum_{k_{\perp}} 2 \pi k_{\perp} / L_{\perp}, \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \rightarrow L_{\perp} \delta_{k_{\perp} k_{\perp}^{\prime}} \delta\left(\theta-\theta^{\prime}\right) /\left(2 \pi k_{\perp}\right)$, where $L_{\perp}$ is the legth scale associated with the area of the "infinite" slab ( $L_{\perp} \gg L$ ).

According to the discussion presented earlier, in this scenario, TE modes incur in case-(i) (large- $k_{\perp}$ ) instability, while TM modes undergo case-(ii) (minimum-width) instability. The solutions $g_{\Omega k_{\perp}}^{(J)}$ of Eqs. (2.50) and (2.57) with $\omega^{2}=-\Omega^{2}$ are given by the normalizable - according to Eqs. (2.70) and (2.76) - solutions of the null-eigenvalue, Schrödinger-like equation

$$
\begin{equation*}
\left(-\frac{\mathrm{d}^{2}}{\mathrm{~d} z^{2}}+V_{e f f}\right) g_{\Omega k_{\perp}}^{(\mathrm{J})}=0 \tag{2.79}
\end{equation*}
$$

with $V_{e f f}$ being the well potential represented in Fig. 2.1. The depth of the potential is given


Figure 2.1 - Effective potential well which represents the homogeneous slab with negative $\mu_{\perp}$ for the unstable electromagnetic modes.
Source: By the author.
by

$$
V_{0}=\left\{\begin{array}{ll}
\left|\mu_{\perp}\right| \varepsilon_{\perp} \Omega^{2}+\frac{\left|\mu_{\perp}\right|}{\mu_{\|}} k_{\perp}^{2} & , \mathrm{~J}=\mathrm{TE}  \tag{2.80}\\
\left|\mu_{\perp}\right| \varepsilon_{\perp} \Omega^{2}-\frac{\varepsilon_{\perp}}{\varepsilon_{\|}} k_{\perp}^{2} & , \mathrm{~J}=\mathrm{TM}
\end{array} .\right.
$$

Although here we focus only on unstable modes, associated with $g_{\Omega k_{\perp}}^{(J)}$, note that in this ex-
ample there would also appear stationary bound solutions associated with $f_{\omega_{0} k_{\perp}}^{(\mathrm{TE})}-$ if $\mu_{\perp}<0$ for some $\omega_{0} \in \mathbb{R}$ - for some $k_{\perp}^{2}>\max \left\{\omega_{0}^{2},\left(n_{\perp}^{(\mathrm{TE})} \omega_{0}\right)^{2}\right\}$, where $n_{\perp}^{(\mathrm{TE})} \equiv \sqrt{\mu_{\|} \varepsilon_{\perp}}$ is the transverse refractive index for the TE modes. For such a hypothetical mode, the slab would act as a waveguide, keeping the mode confined due to total internal reflections at its boundaries. The only peculiar feature here is that $k_{\perp}$ would assume arbitrarily large values (in practice, limited only by the inverse length scale below which the continuous-medium idealization breaks down) for a given $\omega_{0}$.

Back to the unstable modes, a straightforward calculation leads to the familiar even and odd solutions to the square-well potential, with $g_{\Omega k_{\perp}}^{(J)}(z)$ exponentially supressed for $|z|>L / 2$ and

$$
g_{\Omega k_{\perp}}^{(J)}(z)= \begin{cases}\frac{\mathcal{N}_{m}^{(0)}}{\cos a_{m}} \cos \left(2 a_{m} z / L\right) & , 0 \leq m \text { even }  \tag{2.81}\\ \frac{\mathcal{N}_{m}^{(J)}}{\sin a_{m}} \sin \left(2 a_{m} z / L\right) & , 1 \leq m \text { odd }\end{cases}
$$

$(-L / 2 \leq z \leq L / 2)$, where $\mathscr{N}_{m}^{(\mathrm{J})}$ are normalization constants and, for the TE modes, $a_{m} \geq$ $\Omega L\left|n_{\|}\right| / 2$, with $n_{\|}=\sqrt{\mu_{\perp} \varepsilon_{\perp}}$, are solutions of the transcendental equations

$$
\sqrt{\left|\mu_{\perp}\right| \mu_{\|}} \sqrt{1-\frac{\Omega^{2} L^{2}}{4 a_{m}^{2}}\left|n_{\|}^{2}\right|\left[1-\left(n_{\perp}^{(\mathrm{TE})}\right)^{-2}\right]}=\left\{\begin{array}{ll}
-\tan a_{m} & , m \text { even }  \tag{2.82}\\
\cot a_{m} & , m \text { odd }
\end{array},\right.
$$

while for the TM modes, $0 \leq a_{m} \leq \Omega L\left|n_{\|}\right| / 2$ and

$$
\sqrt{\varepsilon_{\perp} \varepsilon_{\|}} \sqrt{\frac{\Omega^{2} L^{2}}{4 a_{m}^{2}}\left|n_{\|}^{2}\right|\left[1+\left|n_{\perp}^{(\mathrm{TM})}\right|^{-2}\right]-1}=\left\{\begin{array}{ll}
\tan a_{m} & , m \text { even }  \tag{2.83}\\
-\cot a_{m} & , m \text { odd }
\end{array} .\right.
$$

The transverse momentum $k_{\perp}$ is given in terms of $a_{m}$ by

$$
k_{\perp}=k_{\perp}^{(m)} \equiv \begin{cases}\frac{2}{L} \sqrt{\frac{\mu_{\|}}{\left|\mu_{\perp}\right|}\left(a_{m}^{2}-\left|n_{\|}^{2}\right| \frac{\Omega^{2} L^{2}}{4}\right)} & , \text { TE modes }  \tag{2.84}\\ \frac{2}{L} \sqrt{\frac{\varepsilon_{\|}}{\varepsilon_{\perp}}\left(\left\lvert\, n_{\|}^{2} \frac{\Omega^{2} L^{2}}{4}-a_{m}^{2}\right.\right)} & , \text { TM modes }\end{cases}
$$

The explicit form of $\mathscr{N}_{m}^{(J)}$ is not particularly important, so we only present its asymptotic behavior for $k_{\perp} \rightarrow \infty$ for the TE modes,

$$
\mathscr{N}_{m}^{(\mathrm{TE})} \approx\left\{\begin{array}{ll}
\sqrt{\frac{2\left(1+\mid \mu_{\perp} \backslash \mu_{\|}\right)}{L \varepsilon_{\perp}}}, & m \text { even }  \tag{2.85}\\
\sqrt{\frac{2\left(1+\mid \mu_{\perp} \backslash \mu_{\|}\right)}{L \varepsilon_{\perp} \mid \mu_{\perp} \backslash \mu_{\|}}}, & m \text { odd }
\end{array}, k_{\perp} \gg \Omega,\right.
$$



Figure 2.2 - Graphic representation of solutions of Eqs. (2.82) and (2.83). The solid black curves in the upper (respectively, lower) half plane represent the left-hand side (l.h.s.) of Eq. (2.82) [resp., minus the l.h.s. of Eq. (2.83)] - with $a_{m}$ replaced by the variable $a$-, for different values of $\Omega L$. The dashed blue lines (resp., dotted red lines) represent the function $-\tan a$ (resp., $\cot a$ ). The values $a_{m}$ appearing in Eq. (2.81) are determined by the crossing of the corresponding solid black curve with the dashed blue lines (for $m$ even) and the dotted red lines (for $m$ odd).
Source: By the author.
and for $k_{\perp} \rightarrow 0$ for both TE and TM modes,

$$
\begin{align*}
& \mathscr{N}_{m}^{(\mathrm{TE})} \approx\left\{\begin{array}{ll}
\sqrt{\frac{2\left(\varepsilon_{\perp}+\left|\mu_{\perp}\right|\right)}{\varepsilon_{\perp}^{2}}}, & m \text { even } \\
\sqrt{\frac{2\left(\varepsilon_{\perp}+\left|\mu_{\perp}\right|\right)}{L \varepsilon_{\perp}\left|\mu_{\perp}\right|}}, & m \text { odd }
\end{array}, k_{\perp} \ll \Omega,\right.  \tag{2.86}\\
& \mathscr{N}_{m}^{(\mathrm{TM})} \approx\left\{\begin{array}{ll}
\sqrt{\frac{2\left(\varepsilon_{\perp}+\left|\mu_{\perp}\right|\right)}{L\left|\mu_{\perp}\right|}}, & m \text { even } \\
\sqrt{\frac{2\left(\varepsilon_{\perp}+\left|\mu_{\perp}\right| \mid\right.}{L \varepsilon_{\perp}\left|\mu_{\perp}\right|}}, & m \text { odd }
\end{array}, k_{\perp} \ll \Omega .\right. \tag{2.87}
\end{align*}
$$

In Fig. 2.2, we plot - for different values of $\Omega L$ and given values of $\mu_{\|}, \mu_{\perp}, \varepsilon_{\|}$, and $\varepsilon_{\perp}$ — the left-hand side of Eq. (2.82) (solid black curves in the upper half plane), minus the left-hand side of Eq. (2.83) (solid black curves in the lower half plane) - substituting, in both, $a_{m}$ by the variable $a$-, and the functions $-\tan a$ and $\cot a$ (blue dashed lines and red dotted lines, respectively). Crossing of the blue dashed lines (respectively, red dotted lines) with a fixed solid black curve determines values $a=a_{m}$ for even (resp., odd) solutions $g_{\Omega k_{\perp}}^{(J)}$, for the corresponding value of $\Omega L$. The figure clearly corroborates our preliminary analysis, showing that unstable TE modes appear with arbitrarily large values of $a_{m}$ (and, therefore, of $k_{\perp}$ ) and that unstable TM modes only appear if $L$ is larger than some minimum width $L_{0}$, given by

$$
\begin{equation*}
L_{0}=\frac{2 \Omega^{-1}}{\left|n_{\|}\right|} \tan ^{-1}\left(\sqrt{\frac{\varepsilon_{\perp}}{\left|\mu_{\perp}\right|}}\right) \tag{2.88}
\end{equation*}
$$

The unstable TE and TM modes inside the slab can then be put in the form

$$
\begin{equation*}
\mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(m)}}^{(u \mathrm{TE})}=\frac{\mathscr{N}_{m}^{(\mathrm{TE})}\left(\mathbf{k}_{\perp}^{(m)} \times \mathbf{n}_{z}\right)}{2 \pi k_{\perp}^{(m)} \sqrt{\Omega \sin \vartheta_{m}}} \frac{\cos \left(2 a_{m} z / L+m \pi / 2\right)}{\cos \left(a_{m}+m \pi / 2\right)} \mathrm{e}^{i \mathbf{k}_{\perp}^{(m)} \cdot \mathbf{x}_{\perp}} \cosh \left(\Omega t-i \vartheta_{m} / 2\right), \tag{2.89}
\end{equation*}
$$

for $m \geq m^{(\mathrm{TE})}$, and

$$
\begin{align*}
& \mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(m)}}^{(u \mathrm{TM})}=\frac{\mathscr{N}_{m}^{(\mathrm{TM})} \mathrm{e}^{i \mathbf{k}_{\perp}^{(m)} \cdot \mathbf{x}_{\perp}}}{2 \pi \sqrt{\Omega^{3} \sin \vartheta_{m}}} \cosh \left(\Omega t-i \vartheta_{m} / 2\right) \\
& \times\left[\frac{k_{\perp}^{(m)} \mathbf{n}_{z}}{\varepsilon_{\|}} \frac{\cos \left(2 a_{m} z / L+m \pi / 2\right)}{\cos \left(a_{m}+m \pi / 2\right)}-i \frac{\mathbf{k}_{\perp}^{(m)}}{\varepsilon_{\perp} k_{\perp}^{(m)}} \frac{2 a_{m}}{L} \frac{\sin \left(2 a_{m} z / L+m \pi / 2\right)}{\cos \left(a_{m}+m \pi / 2\right)}\right], \tag{2.90}
\end{align*}
$$

for $0 \leq m \leq m^{(\mathrm{TM})}$, with

$$
\begin{align*}
& m^{(\mathrm{TE})} \equiv\left\lceil 1+\left(\frac{L}{L_{0}}-1\right) \frac{2}{\pi} \tan ^{-1}\left(\sqrt{\frac{\varepsilon_{\perp}}{\left|\mu_{\perp}\right|}}\right)\right\rceil,  \tag{2.91}\\
& m^{(\mathrm{TM})} \equiv\left\lfloor\left(\frac{L}{L_{0}}-1\right) \frac{2}{\pi} \tan ^{-1}\left(\sqrt{\frac{\varepsilon_{\perp}}{\left|\mu_{\perp}\right|}}\right)\right\rfloor, \tag{2.92}
\end{align*}
$$

( $\lceil x\rceil$ represents the smallest integer larger than, or equal to, $x$, while $\lfloor x\rfloor$ represents the largest integer smaller than, or equal to, $x$ ).

Let us recall that these modes give information about fluctuations and correlations of the electromagnetic field; as long as decoherence does not come into play, the expectation values of the field are null, $\langle\mathbf{A}\rangle=\langle\mathbf{E}\rangle=\langle\mathbf{B}\rangle=\mathbf{0}$. We shall use these modes later, when discussing possible consequences of these analogue instabilities. But first, let us explore more interesting analogies.

### 2.4 Spherically-symmetric, stationary anisotropic medium

In the previous section, we presented with great amount of detail the canonical quantization scheme for the electromagnetic field in flat spacetime in the presence of arbitrary planesymmetric anisotropic polarizable/magnetizable media at linear order. The vacuum of such system was then identified with the vacuum of some nonminimally-coupled spin-1 field in a true curved spacetime described by the effective metric $g_{\alpha \beta}=\sqrt{n} \operatorname{diag}\left(-n^{-2}, 1,1,1\right)$. The analysis had the advantage of generalizing in a unified language the quantization of various interesting models coming from quantum optics in terms of simple equations (e.g., the

Carniglia-Mandel modes). However, the analogue spacetime for these configurations is of mathematical interest only and does not capture the symmetry of physical spacetimes. In order to study more appealing analogues, in this section we turn to spherically symmetric configurations, presenting them in a more concise way - for the nuances of the quantization were already explained previously. In this context, we may obtain interesting analogues by also assuming that the medium is able to flow. If the refractive index in a flowing material is high enough, such that the velocity of light becomes smaller than the medium's velocity, then it is clear that a sort of event horizon will form (restricted only to some frequency band which may contain unstable modes). This kind of phenomenon enable us to study analogues of unstable black holes, for instance.

We start working in standard spherical coordinates $(t, r, \theta, \varphi)$, such that the flat metric takes the form $\eta_{\mu \nu}=\operatorname{diag}\left(-1,1, r^{2}, r^{2} \sin ^{2} \theta\right)$. Let the medium four-velocity field be $\nu^{\mu}=$ $\gamma(1, v, 0,0)$, where $v=\nu(r)$ and $\gamma=\left(1-v^{2}\right)^{-1 / 2}$. The effective-metric components then take the form

$$
g_{\alpha \beta}=\sqrt{n}\left(\begin{array}{cccc}
-\gamma^{2}\left(n^{-2}-v^{2}\right) & -\left(1-n^{-2}\right) \gamma^{2} v & 0 & 0  \tag{2.93}\\
-\left(1-n^{-2}\right) \gamma^{2} v & \gamma^{2}\left(1-n^{-2} v^{2}\right) & 0 & 0 \\
0 & 0 & r^{2} & 0 \\
0 & 0 & 0 & r^{2} \sin ^{2} \theta
\end{array}\right)
$$

where the isotropic parts of the constitutive tensors (in the local, instantaneous rest frame of the medium) are functions of $r-\varepsilon=\varepsilon(r), \mu=\mu(r)$ —and, as usual, $n^{2}=\mu \varepsilon$. As for the traceless anisotropic tensors $\chi_{(\varepsilon)}^{a b}$ and $\chi_{a b}^{(\mu)}$, their components read

$$
\begin{equation*}
\chi_{(\varepsilon)}^{\alpha \beta}=\frac{\Delta^{(\varepsilon)}}{3}\left(2 \gamma^{2} v^{2} \delta_{t}^{\alpha} \delta_{t}^{\beta}+4 \gamma^{2} \nu \delta_{t}^{(\alpha} \delta_{r}^{\beta)}+2 \gamma^{2} \delta_{r}^{\alpha} \delta_{r}^{\beta}-\delta_{\theta}^{\alpha} \delta_{\theta}^{\beta} r^{-2}-\delta_{\varphi}^{\alpha} \delta_{\varphi}^{\beta} r^{-2} \sin ^{-2} \theta\right) \tag{2.94}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{\alpha \beta}^{(\mu)}=\frac{\Delta^{(\mu)}}{3}\left(2 \gamma^{2} \nu^{2} \delta_{\alpha}^{t} \delta_{\beta}^{t}-4 \gamma^{2} \nu \delta_{(\alpha}^{t} \delta_{\beta)}^{r}+2 \gamma^{2} \delta_{\alpha}^{r} \delta_{\beta}^{r}-\delta_{\alpha}^{\theta} \delta_{\beta}^{\theta} r^{2}-\delta_{\alpha}^{\varphi} \delta_{\beta}^{\varphi} r^{2} \sin ^{2} \theta\right) \tag{2.95}
\end{equation*}
$$

Similarly to the plane-symmetric case, these anisotropic tensors simply mean that in the instantaneous local rest frame of the medium, its electric permitivity and magnetic permeability in the radial direction ( $\varepsilon_{\|}$and $\mu_{\|}$) and in the angular directions ( $\varepsilon_{\perp}$ and $\mu_{\perp}$ ) satisfy the same relations given below Eqs. (2.44-2.47): $\varepsilon_{\|}-\varepsilon_{\perp} \equiv \Delta^{(\varepsilon)}, 2 \varepsilon_{\perp}+\varepsilon_{\|} \equiv 3 \varepsilon, \mu_{\|}^{-1}-\mu_{\perp}^{-1} \equiv \Delta^{(\mu)}$, and
$2 \mu_{\perp}^{-1}+\mu_{\|}^{-1} \equiv 3 \mu^{-1}$.

Not surprisingly, the lab coordinates $(t, r, \theta, \varphi)$ are not the most convenient ones to express Eqs. (2.26) and (2.27) in the case of a moving medium. One might initially think that coordinates ( $\tau, r, \theta, \varphi$ ) which diagonalize the components of the effective metric, obtained by defining $\tau \equiv t-p(r)$, with $p(r)$ satisfying

$$
\begin{equation*}
\frac{\mathrm{d} p}{\mathrm{~d} r}=-\frac{\left(n^{2}-1\right) v}{1-n^{2} v^{2}}, \tag{2.96}
\end{equation*}
$$

would lead to the simplest form of the field equations. In these coordinates, the effective line element $\mathrm{d} s_{\text {eff }}^{2}$ becomes

$$
\begin{equation*}
\mathrm{d} s_{e f f}^{2}=\sqrt{n}\left[-n^{-2} F \mathrm{~d} \tau^{2}+F^{-1} \mathrm{~d} r^{2}+r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \varphi^{2}\right)\right] \tag{2.97}
\end{equation*}
$$

where $F=\gamma^{2}\left(1-n^{2} v^{2}\right)$. It is noteworthy that for $n=$ constant $>0$ (such that the factors of $n$ in $d s_{e f f}^{2}$ can be absorbed via $\tau \mapsto n^{3 / 4} \tau$ and $r \mapsto n^{-1 / 4} r$ ), then the line element above can be made to represent Schwarzschild spacetime by tuning $v$ so that $F \equiv\left(1-r_{s} / r\right)$, where $r_{s}$ is some positive constant. This is achieved by a velocity field satisfying $v^{2}=\left[1+\left(n^{2}-1\right) r / r_{s}\right]^{-1}$ ( $n \neq 1$ ).

Despite this apparent simplification, the coordinate $\tau=t-p(r)$ with $p$ satisfying Eq. (2.96) is not convenient to express Maxwell's equations in anisotropic media. This is due to the kinematic polarization (resp., magnetization) caused by the magnetic (resp., electric) field. In the case of small velocities and isotropic materials, this effect is modeled by Minkowski's equations. ${ }^{52}$ The coordinates ( $\tau, r, \theta, \varphi$ ) defined using Eq. (2.96) "diagonalizes" only the isotropic part of the theory and do not take into account the anisotropies. It turns out that a much better choice is obtained by setting $\tau \equiv t-p(r)$ and replacing condition given in Eq. (2.96) by

$$
\begin{equation*}
\frac{\mathrm{d} p}{\mathrm{~d} r}=-\frac{\left(n_{\|}^{2}-1\right) v}{1-n_{\|}^{2} v^{2}} \tag{2.98}
\end{equation*}
$$

where, again, $n_{\|}^{2}=\mu_{\perp} \varepsilon_{\perp}$. This choice fully decouples the electromagnetic field modes in the anisotropic, moving material medium, as we shall see below.

Introducing again the 4-potential $A_{\mu}$ via $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$, in these new coordinates
( $\tau, r, \theta, \varphi$ ), the convenient (generalized Coulomb) gauge conditions read $A_{\tau}=0$ and

$$
\begin{equation*}
\partial_{\varrho}\left(\varepsilon_{\|} r^{2} A_{r}\right)+\partial_{\perp} \cdot \mathbf{A}_{\perp}=0 \tag{2.99}
\end{equation*}
$$

where $\varrho$ is merely an auxiliary variable such that $\mathrm{d} r / \mathrm{d} \varrho \equiv \gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right) / \varepsilon_{\perp}, \mathbf{A}_{\perp}=\left(A_{\theta}, A_{\varphi}\right), \partial_{\perp}$ is the derivative operator on the unit sphere compatible with its metric, and it is understood that $r$ is a function of the auxiliary variable $\varrho$. In this gauge, Maxwell's equations lead to

$$
\begin{align*}
& {\left[-\frac{\mu_{\perp}}{\varepsilon_{\perp}} \partial_{\tau}^{2}+\partial_{\rho}^{2}+\frac{\gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right)}{\varepsilon_{\perp} \varepsilon_{\|} r^{2}} \Delta_{S}^{(0)}\right]\left(\varepsilon_{\|} r^{2} A_{r}\right)=0,}  \tag{2.100}\\
& {\left[-\frac{\varepsilon_{\perp}}{\mu_{\perp}} \partial_{\tau}^{2}+\partial_{\rho}^{2}+\frac{r^{2}\left(1-n_{\|}^{2} \nu^{2}\right)}{\mu_{\perp} \mu_{\|} r^{2}}\left(\Delta_{S}^{(1)}-1\right)\right] \mathbf{A}_{\perp}=\partial_{\perp}\left[\partial_{\rho}\left(\frac{\mathrm{d} r}{\mathrm{~d} \rho} A_{r}\right)-\frac{\mu_{\perp}}{r^{2} \mu_{\|} \varepsilon_{\perp}} \frac{\mathrm{d} r}{\mathrm{~d} \rho} \partial_{\rho}\left(\varepsilon_{\|} r^{2} A_{r}\right)\right],} \tag{2.101}
\end{align*}
$$

where $\rho$ appearing in Eq. (2.101) is another auxiliary variable defined through $\mathrm{d} r / \mathrm{d} \rho \equiv \gamma^{2}(1-$ $\left.n_{\|}^{2} \nu^{2}\right) / \mu_{\perp}$ and $\Delta_{S}^{(0)}$ and $\Delta_{S}^{(1)}$ are the Laplacian operators defined on the unit sphere, acting on scalar and covector fields, respectively.

In order to solve these equations, we proceed in close analogy to the plane-symmetric case. First, let us find solutions with $A_{r}=0$ - the transverse electric modes, $\mathbf{A}^{(\mathrm{TE})}$. The gauge conditions imply that these solutions can be written as $\mathbf{A}^{(\mathrm{TE})}=\left(0, \partial_{\varphi} \psi / \sin \theta,-\sin \theta \partial_{\theta} \psi\right)$, where $\psi$ is an auxiliary function to be determined. Making use of the stationarity and spherical symmetry of the present scenario, we can look for field modes of the form $\psi=\exp (-i \omega \tau) Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TE})}(r)$, where $Y_{\ell m}$ are the scalar spherical harmonics. Substituting this into Eq. (2.101), $f_{\omega \ell}^{(\mathrm{TE})}$ must satisfy

$$
\begin{equation*}
\left[-\frac{\mathrm{d}^{2}}{\mathrm{~d} \rho^{2}}+\left(\frac{\gamma^{2}\left(1-n_{\|}^{2} v^{2}\right) \ell(\ell+1)}{r^{2} \mu_{\perp} \mu_{\|}}-\frac{\varepsilon_{\perp} \omega^{2}}{\mu_{\perp}}\right)\right] f_{\omega \ell}^{(\mathrm{TE})}=0, \tag{2.102}
\end{equation*}
$$

where it is understood that $r$ is a function of the auxiliary variable $\rho$. Notice the similarity between this equation and Eq. (2.50). In fact, the boundary condition given by Eq. (2.30) assumes the same form here as it does in the plane-symmetric case:

$$
\begin{equation*}
\left.\left[\overline{f_{\omega \ell}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} f_{\omega^{\prime} \ell}^{(\mathrm{TE})}-f_{\omega^{\prime} \ell}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{f_{\omega \ell}^{(\mathrm{TE})}}\right]\right|_{\dot{\mathscr{I}}}=0 \tag{2.103}
\end{equation*}
$$

This boundary condition ensures that these modes can be orthonormalized according to the
sesquilinear form given in Eq. (2.29), which in this spherically-symmetric scenario assumes the form

$$
\begin{align*}
\left(A, A^{\prime}\right)= & i \int_{\Sigma_{t}} \mathrm{~d} \Sigma\left\{\varepsilon_{\|} \bar{A}_{r} \partial_{\tau} A_{r}^{\prime}+\frac{\varepsilon_{\perp} \overline{\mathbf{A}}_{\perp} \cdot \partial_{\tau} \mathbf{A}_{\perp}^{\prime}}{r^{2}\left(1-n_{\|}^{2} v^{2}\right)}+\frac{r^{2}\left(n_{\|}^{2}-1\right) v}{\mu_{\perp}}\left[\overline{\mathbf{A}}_{\perp} \cdot \partial_{r} \mathbf{A}_{\perp}^{\prime}-\left(\overline{\mathbf{A}}_{\perp} \cdot \partial_{\perp}\right) A_{r}^{\prime}\right]\right\} \\
& -\left(\overline{\mathbf{A}} \leftrightarrow \mathbf{A}^{\prime}\right), \tag{2.104}
\end{align*}
$$

with $\Sigma_{t}$ being a spacelike surface $t=$ constant. After some tedious but straightforward manipulations (presented in the appendix), we obtain the final form of normalized, positivefrequency TE modes:

$$
\begin{equation*}
\mathbf{A}_{\omega \ell m}^{(\mathrm{TE})}=\frac{\left(0, i m / \sin \theta,-\sin \theta \partial_{\theta}\right)}{\sqrt{2 \omega \ell(\ell+1)}} \mathrm{e}^{-i \omega \tau} Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TE})}(r), \tag{2.105}
\end{equation*}
$$

with $f_{\omega \ell}^{(\mathrm{TE)}}$ satisfying Eqs. (2.102) and (2.103), and normalized according to

$$
\begin{equation*}
\int_{\mathscr{\mathscr { C }}_{\ell}} \mathrm{d} \varrho \overline{f_{\omega \ell}^{(\mathrm{TE})}} f_{\omega^{\prime} \ell}^{(\mathrm{TE})}=\delta_{\omega \omega^{\prime}} . \tag{2.106}
\end{equation*}
$$

Note that the integration variable is $\varrho$ [instead of $\rho$ appearing in Eq. (2.102)] and $\mathscr{I}_{\varrho}$ stands for the domain of integration in this variable corresponding to $\mathscr{I}$ in coordinate $r$.

Now, let us look for solutions with $A_{r} \not \equiv 0$ - the transverse magnetic modes, $\mathbf{A}^{(\mathrm{TM})}$. Let $\phi$ be such that $\Delta_{S}^{(0)} \phi=-r^{2} \varepsilon_{\|} A_{r}$. Thus, the gauge conditions lead to the general solution in the form $\mathbf{A}^{(\mathrm{TM})}=\left(-r^{-2} \varepsilon_{\|}^{-1} \Delta_{S}^{(0)}, \partial_{\theta} \partial_{\varrho}, \partial_{\varphi} \partial_{\varrho}\right) \phi$. Using again stationarity and spherical symmetry, $\phi=\exp (-i \omega \tau) Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TM})}(r)$, we obtain that $f_{\omega \ell}^{(\mathrm{TM})}(r)$ satisfies

$$
\begin{equation*}
\left[-\frac{\mathrm{d}^{2}}{\mathrm{~d} \varrho^{2}}+\left(\frac{\gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right) \ell(\ell+1)}{r^{2} \varepsilon_{\perp} \varepsilon_{\|}}-\frac{\mu_{\perp} \omega^{2}}{\varepsilon_{\perp}}\right)\right] f_{\omega \ell}^{(\mathrm{TM})}=0 \tag{2.107}
\end{equation*}
$$

Notice, again, the similarity between this equation and Eq. (2.57). And, again, the boundary condition imposed by Eq. (2.30) to these modes take the same form as in the planesymmetric case:

$$
\begin{equation*}
\left.\left[\omega^{2} \overline{f_{\omega \ell}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \varrho} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}-\omega^{\prime 2} f_{\omega^{\prime} \ell}^{(\mathrm{TM})} \frac{\mathrm{d}}{\mathrm{~d} \varrho} \overline{f_{\omega \ell}^{(\mathrm{TM})}}\right]\right|_{\dot{\mathscr{}}}=0 . \tag{2.108}
\end{equation*}
$$

Properly orthonormalizing these modes using Eq. (2.104) - see appendix -, leads to the
positive-frequency TM normal modes

$$
\begin{equation*}
\mathbf{A}_{\omega \ell m}^{(\mathrm{TM})}=\frac{\left(r^{-2} \varepsilon_{\|}^{-1} \ell(\ell+1), \partial_{\theta} \partial_{\varrho}, i m \partial_{\varrho}\right)}{\sqrt{2 \omega^{3} \ell(\ell+1)}} \mathrm{e}^{-i \omega \tau} Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TM})}(r), \tag{2.109}
\end{equation*}
$$

with $f_{\omega \ell}^{(\mathrm{TM})}$ satisfying Eqs. (2.107) and (2.108), and normalized according to

$$
\begin{equation*}
\int_{\mathscr{I}_{\rho}} \mathrm{d} \rho \overline{f_{\omega \ell}^{(\mathrm{TM})}} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}=\delta_{\omega \omega^{\prime}} . \tag{2.110}
\end{equation*}
$$

Similarly to the TE case, note that the integration variable is not the same which appears in the differential equation, Eq. (2.107). ( $\mathscr{I}_{\rho}$ stands for the domain of integration in the variable $\rho$ corresponding to $\mathscr{I}$ in coordinate $r$.)

The electromagnetic field operator can be represented in terms of the TE and TM modes (and their complex conjugates) as

$$
\begin{equation*}
\mathbf{A}=\sum_{\mathrm{J} \in\{\mathrm{TE}, \mathrm{TM}\} \ell m} \sum_{\mathscr{E}_{\ell+}^{(J)}} \int \mathrm{d} \omega\left[\hat{a}_{\omega \ell m}^{(\mathrm{J})} \mathbf{A}_{\omega \ell m}^{(\mathrm{J})}+\text { H.c. }\right], \tag{2.111}
\end{equation*}
$$

where $\mathscr{E}_{\ell+}^{(J)} \equiv \mathscr{E}_{\ell}^{(J)} \cap \mathbb{R}_{+}^{*}$, with $\mathscr{E}_{\ell}^{(\mathrm{JI})}$ being the set of $\omega$ values for which Eqs. (2.102) and (2.103), for $\mathrm{J}=\mathrm{TE}$, and Eqs. (2.107) and (2.108), for $\mathrm{J}=\mathrm{TM}$, have nontrivial solutions. The orthonormality of TE and TM modes,

$$
\begin{align*}
& \left(A_{\omega \ell m}^{(\mathrm{J})}, A_{\omega^{\prime} \ell^{\prime} m^{\prime}}^{\left(\mathrm{J}^{\prime}\right.}\right)=-\left(\overline{\left.A_{\omega \ell m^{(J)}}^{(\mathrm{J}}, \overline{A_{\omega^{\prime} \ell^{\prime} m^{\prime}}^{\left(\mathrm{J}^{\prime}\right)}}\right)=\delta_{\mathrm{J} J^{\prime}} \delta_{\omega \omega^{\prime}} \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}},}\right.  \tag{2.112}\\
& \left(A_{\omega \ell m^{\prime}}^{(\mathrm{J})}, \overline{A_{\omega^{\prime} \ell^{\prime} m^{\prime}}^{\left(\mathrm{J}^{\prime}\right)}}\right)=0, \tag{2.113}
\end{align*}
$$

requires that the canonical commutation relations

$$
\begin{align*}
& {\left[a_{\omega \ell m}^{(J)}, a_{\omega^{\prime} \ell^{\prime} m^{\prime}}^{\left(\mathrm{J}^{\prime}\right)}\right]=\delta^{\mathrm{J}{ }^{\prime}} \delta_{\omega \omega^{\prime}} \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}},}  \tag{2.114}\\
& {\left[a_{\omega \ell m^{\prime}}^{(J)}, a_{\omega^{\prime} \ell^{\prime} m^{\prime}}^{\left(\mathrm{J}^{\prime}\right)}\right]=0,} \tag{2.115}
\end{align*}
$$

hold.

### 2.4.1 Instability analysis

The close similarity between Eqs. (2.50) and (2.102) and between Eqs. (2.57) and (2.107) make the instability analysis in this spherically-symmetric scenario essentially identical to the one
performed in the plane-symmetric case, with $\ell(\ell+1)$ playing the role $k_{\perp}^{2}$ did in Eq. (2.66). So, putting the effective potentials of Eqs. (2.102) and (2.107), with $\omega^{2}=-\Omega^{2}$, in the form

$$
\begin{equation*}
V_{e f f}=C_{1} \ell(\ell+1)+C_{2} \Omega^{2}, \tag{2.116}
\end{equation*}
$$

we again have two types of instabilities: (i) large- $\ell$ instability, when $C_{1}<0$ somewhere, and (ii) minimum-thickness instability, when $C_{1}>0$ but $C_{2}<0$ in a sufficiently thick spherical shell - see discussion below Eq. (2.66). The only additional feature is that, by allowing the medium to flow, type-(i) (large- $\ell$ ) instability for both TE and TM modes can arise when the medium's velocity $\nu(r)$ exceeds the radial light velocity $n_{\|}^{-1}$.

Let $g_{\Omega \ell}^{(J)}$ represent the solutions of Eqs. (2.102) (for $\mathrm{J}=\mathrm{TE}$ ) and (2.107) (for $\mathrm{J}=\mathrm{TM}$ ), subject to the boundary conditions given by Eqs. (2.103) and (2.108), respectively, with $\omega^{2}=-\Omega^{2}$ ( $\Omega>0$, without loss of generality). The normalized, unstable modes are presented below see appendix for details.

## Unstable TE modes

Unstable TE modes orthonormalized according to the analogous of Eqs. (2.112) and (2.113) read (up to global phase and time translation)

$$
\begin{equation*}
\mathbf{A}_{\Omega \ell m}^{(u \mathrm{TE})}=\frac{\cosh \left(\Omega \tau-i s_{\varepsilon}^{\perp} \vartheta / 2\right)}{\sqrt{\Omega \ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TE})}(r)\left(0, i m / \sin \theta,-\sin \theta \partial_{\theta}\right) Y_{\ell m}(\theta, \varphi), \tag{2.117}
\end{equation*}
$$

with $\vartheta$ being a constant $(0<\vartheta<\pi), g_{\Omega \ell}^{(\mathrm{TE})}$ normalized according to

$$
\begin{equation*}
\left|\int_{\mathscr{I}} \mathrm{d} r \frac{\varepsilon_{\perp}}{r^{2}\left(1-n_{\|}^{2} v^{2}\right)} \overline{g_{\Omega \ell}^{(\mathrm{TE)}}}(r) g_{\Omega^{\prime} \ell}^{(\mathrm{TE})}(r)\right|=\delta_{\Omega \Omega^{\prime}}, \tag{2.118}
\end{equation*}
$$

and $s_{\varepsilon}^{\perp}$ being the sign of the integral above. Calculating the electric $\mathbf{E}_{\Omega \ell m}^{(u T E)}$ and magnetic $\mathbf{B}_{\Omega \ell m}^{(u \mathrm{TE})}$ vector fields associated to these modes in the lab frame, we have:

$$
\begin{align*}
& \mathbf{E}_{\Omega \ell m}^{(u \mathrm{TE})}=\frac{\sqrt{\Omega}\left(-i m \mathbf{e}_{\theta} / \sin \theta+\mathbf{e}_{\varphi} \partial_{\theta}\right)}{r \sqrt{\ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TE})}(r) Y_{\ell m}(\theta, \varphi) \sinh \left(\Omega \tau-i s_{\varepsilon}^{\perp} \vartheta / 2\right),  \tag{2.119}\\
& \mathbf{B}_{\Omega \ell m}^{(u \mathrm{TE})}=\frac{\left[\ell(\ell+1) \mathbf{e}_{r}+\left(i m \mathbf{e}_{\varphi} / \sin \theta+\mathbf{e}_{\theta} \partial_{\theta}\right) r \partial_{r}\right]}{r^{2} \sqrt{\Omega \ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TE})}(r) Y_{\ell m}(\theta, \varphi) \cosh \left(\Omega \tau-i s_{\varepsilon}^{\perp} \vartheta / 2\right) . \tag{2.120}
\end{align*}
$$

## Unstable TM modes

Finally, the unstable TM modes orthonormalized according to the analogous of Eqs. (2.112) and (2.113) read (up to global phase and time translation)

$$
\begin{equation*}
\mathbf{A}_{\Omega \ell m}^{(u \mathrm{TM})}=\frac{\left(r^{-2} \varepsilon_{\|}^{-1} \ell(\ell+1), \partial_{\theta} \partial_{\varrho}, i m \partial_{\varrho}\right)}{\sqrt{\Omega^{3} \ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TM})}(r) Y_{\ell m}(\theta, \varphi) \cosh \left(\Omega \tau+i s_{\mu}^{\perp} \vartheta / 2\right), \tag{2.121}
\end{equation*}
$$

with, again, $\vartheta$ being a constant $(0<\vartheta<\pi), g_{\Omega \ell}^{(\mathrm{TM})}$ normalized according to

$$
\begin{equation*}
\left|\int_{\mathscr{I}} \mathrm{d} r \frac{\mu_{\perp}}{\gamma^{2}\left(1-n_{\|}^{2} v^{2}\right)} \overline{g_{\Omega \ell}^{(\mathrm{TM})}}(r) g_{\Omega^{\prime} \ell}^{(\mathrm{TM})}(r)\right|=\delta_{\Omega \Omega^{\prime}} \tag{2.122}
\end{equation*}
$$

and $s_{\mu}^{\perp}$ being the sign of the integral above. Calculating the electric $\mathbf{E}_{\Omega \ell m}^{(u \mathrm{TM})}$ and magnetic $\mathbf{B}_{\Omega \ell m}^{(u \mathrm{TM})}$ vector fields associated to these modes in the lab frame, we have:

$$
\mathbf{E}_{\Omega \ell m}^{(u \mathrm{TM})}=-\frac{\left[\ell(\ell+1) \mathbf{e}_{r} / \varepsilon_{\|}+\left(i m \mathbf{e}_{\varphi} / \sin \theta+\mathbf{e}_{\theta} \partial_{\theta}\right) r \partial_{\ell}\right]}{r^{2} \sqrt{\Omega \ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TM})}(r) Y_{\ell m}(\theta, \varphi) \sinh \left(\Omega \tau+i s_{\mu}^{\perp} \vartheta / 2\right),
$$

$$
\begin{equation*}
\mathbf{B}_{\Omega \ell m}^{(u \mathrm{TM})}=\frac{\mu_{\perp} \sqrt{\Omega}\left(-i m \mathbf{e}_{\theta} / \sin \theta+\mathbf{e}_{\varphi} \partial_{\theta}\right)}{r \gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right) \sqrt{\ell(\ell+1) \sin \vartheta}} g_{\Omega \ell}^{(\mathrm{TM})}(r) Y_{\ell m}(\theta, \varphi) \cosh \left(\Omega \tau+i s_{\mu}^{\perp} \vartheta / 2\right) . \tag{2.123}
\end{equation*}
$$

As argued in the previous case, when instability is triggered and modes $\mathbf{A}_{\Omega \ell m}^{(u)}$ appear, they must be included in the field expansion given by Eq. (2.111), along with their complex conjugates. Eventually ( $t \gg \Omega^{-1}$ ), these modes dominate the field fluctuations.

### 2.4.2 Example

Now, let us consider a concrete scenario where electromagnetism in a gravitationally interesting system, nonminimally coupled to the background geometry via $\chi^{a b c d}$ given by Eq. (2.31) (but with arbitrary $\alpha_{1}, \alpha_{2}, \alpha_{3}$ ), can be mimicked by an anisotropic, stationary moving medium. We have already seen that setting $n=$ constant and $v^{2}=\left[1+\left(n^{2}-1\right) r / r_{s}\right]^{-1}$, leads to an effective line element which describes the vacuum Schwarzschild spacetime. In this case, Eq. (2.37) is trivially satisfied and Eqs. (2.34-2.36) give

$$
\begin{equation*}
\mu=n, \tag{2.125}
\end{equation*}
$$

$$
\begin{align*}
\Delta^{(\varepsilon)} & =3 \alpha_{1} n^{1 / 2} \frac{r_{s}}{r^{3}},  \tag{2.126}\\
\Delta^{(\mu)} & =\frac{3 \alpha_{1}}{n^{3 / 2}} \frac{r_{s}}{r^{3}}, \tag{2.127}
\end{align*}
$$

which lead to the material properties

$$
\begin{align*}
& \varepsilon_{\perp}=n\left(1-\frac{\alpha_{1} r_{s}}{n^{1 / 2} r^{3}}\right),  \tag{2.128}\\
& \varepsilon_{\|}=n\left(1+\frac{2 \alpha_{1} r_{s}}{n^{1 / 2} r^{3}}\right),  \tag{2.129}\\
& \mu_{\perp}=n\left(1-\frac{\alpha_{1} r_{s}}{n^{1 / 2} r^{3}}\right)^{-1}  \tag{2.130}\\
& \mu_{\|}=n\left(1+\frac{2 \alpha_{1} r_{s}}{n^{1 / 2} r^{3}}\right)^{-1} . \tag{2.131}
\end{align*}
$$

We promptly see that $n_{\|}=\sqrt{\mu_{\perp} \varepsilon_{\perp}}=n$, which shows that the analogue horizon for these nonminimally-coupled modes, located where $v^{2}=n_{\|}^{-2}$, coincides with the analogue Schwarzschild radius $r_{s}$. [Note, however, that this system is analogous to a physical black hole with Schwarzschild radius $R_{s}=n^{1 / 4} r_{s}$, due to absorption of $\sqrt{n}$ in Eq. (2.97).] As for the other refractive indices, $n_{\perp}^{(\mathrm{TE})} \equiv \sqrt{\mu_{\|} \varepsilon_{\perp}}$ and $n_{\perp}^{(\mathrm{TM})} \equiv \sqrt{\mu_{\perp} \varepsilon_{\|}}\left(=n^{2} / n_{\perp}^{(\mathrm{TE})}\right)$, Fig. 2.3 shows their squared values (in black and red, respectively) for positive (solid lines) and negative (dashed lines) values of $\alpha_{1}$. Note that, depending on the values of $\alpha_{1} /\left(n^{1 / 2} r_{s}^{2}\right)$, some kind of metamaterial (possibly with some negative squared refractive indices) may be needed in order to mimic this nonminimal coupling of the electromagnetic field with the Riemann curvature tensor in the exterior region of a Schwarzschild black hole. Conversely, regardless how difficult it may be to set up such an experimental configuration in the lab, it is interesting in its own that QED-inspired nonminimally-coupled electromagnetism in the background of a black hole behaves as in such an exotic metamaterial in flat spacetime.

Turning to the question of possible instabilities, in Fig. 2.4 we show the behavior of the terms $C_{1}$ and $C_{2}$ appearing in Eq. (2.116) for the TE (in blue) and TM (in red) modes - extracted, respectively, from Eqs. (2.102) and (2.107):

$$
\begin{align*}
& C_{1}=\left\{\begin{array}{l}
n^{-2} r^{-9}\left(r-r_{s}\right)\left(r^{3}-\frac{\alpha_{1} r_{s}}{\sqrt{n}}\right)\left(r^{3}+\frac{2 \alpha_{1} r_{s}}{\sqrt{n}}\right) \\
\frac{n^{-2}\left(r-r_{s} r^{3}\right.}{\left(r^{3}-\alpha_{1} r_{s} / \sqrt{n}\right)\left(r^{3}+2 \alpha_{1} r_{s} / \sqrt{n}\right)}
\end{array},\right.  \tag{2.132}\\
& C_{2}=\left\{\begin{array}{l}
\left(1-\frac{\alpha_{1} r_{s}}{r_{s} \sqrt{n}}\right)^{2} \\
\left(1-\frac{\alpha_{1} r_{s}}{r^{3} \sqrt{n}}\right)^{-2},
\end{array}\right. \tag{2.133}
\end{align*}
$$

2.5. Stabilization: spontaneous vectorization, photo production, and long-range induced


Figure 2.3 - Squared values of the refractive indices $n_{\perp}^{(\mathrm{TE})}$ (in black) and $n_{\perp}^{(\mathrm{TM})}$ (in red) for positive (solid lines) and negative (dashed lines) values of $\alpha_{1}$. The black and red dotted lines mark where $n_{\perp}^{(\mathrm{TE})}$ (for negative $\alpha_{1}$ ) and $n_{\perp}^{(\mathrm{TM})}$ (for positive $\alpha_{1}$ ) are singular, respectively.
Source: By the author.
where the first and second lines in the expressions above refer to the TE and TM modes, respectively. The Fig. 2.4(a) is representative of the behavior of $C_{1}$ for $-r_{s}^{2} \sqrt{n} / 2<\alpha_{1}<r_{s}^{2} \sqrt{n}$, while Fig. 2.4(b) gives the correct qualitative behavior of $C_{1}$ for $\alpha_{1}<-r_{s}^{2} \sqrt{n} / 2$ or $\alpha_{1}>r_{s}^{2} \sqrt{n}$. Figs.2.4(c) and 2.4(d) show the behavior of $C_{2}$ for the same values of $\alpha_{1}$ used in Figs. 2.4(a) and 2.4(b).

It is clear, from the expressions above, that $C_{2}$ is everywhere non-negative, while $C_{1}$ assumes negative values in the region with radial coordinate $r$ between $\left(\alpha_{1} r_{s} / \sqrt{n}\right)^{1 / 3}$ and $r_{s}$ (if $\alpha_{1}>0$ ) or between $\left[\left|\alpha_{1}\right| r_{s} /(2 \sqrt{n})\right]^{1 / 3}$ and $r_{s}\left(\right.$ if $\alpha_{1}<0$ ). Therefore, according to the discussion of Subsec. 2.4.1, this nonminimally-coupled electromagnetic theory in Schwarzschild spacetime exhibits large- $\ell$ instability. In particular, if $\alpha_{1}>r_{s}^{2} \sqrt{n}$ or $\alpha_{1}<-2 r_{s}^{2} \sqrt{n}$, then the unstable modes influence the exterior region of the back hole.

### 2.5 Stabilization: spontaneous vectorization, photo production, and long-range induced correlations

We now turn our attention to discussing what can possibly happen to the analogous system when the vacuum instability is triggered. In the gravitational scenario, it has been shown that in some cases (for instance, depending on the field-background coupling), stabilization occurs due to the appearance of a nonzero value for the field (spontaneous scalarization/vectorization, $)^{7,8,49-51}$ seeded by decoherence of the growing initial-vacuum fluctuations. ${ }^{48}$ In this process, field particles/waves are produced ${ }^{49,53}$ and carry away the


Figure 2.4 - Plot of the coefficients $C_{1}-$ (a) and (b) — and $C_{2}-$ (c) and (d) - appearing in Eq. (2.116) for electromagnetic modes TE (blue curves) and TM (red curves), nonminimally coupled to the background geometry of a Schwarzschild black hole via Eq. (2.31). Figs. (a) and (c) illustrate the general behavior of $C_{1}$ and $C_{2}$ for $-r_{s}^{2} \sqrt{n} / 2<\alpha_{1}<r_{s}^{2} \sqrt{n}$, while (b) and (d) are representative of the behavior of $C_{1}$ and $C_{2}$ for $\alpha_{1}<-r_{s}^{2} \sqrt{n} / 2$ or $\alpha_{1}>r_{s}^{2} \sqrt{n}$. According to the instability discussion, only large- $\ell$ instability can appear in this case, since $C_{2} \geq 0$ everywhere. Moreover, for $\alpha_{1}<-r_{s}^{2} \sqrt{n} / 2$ or $\alpha_{1}>r_{s}^{2} \sqrt{n}$, the unstable modes can be mostly supported outside the analogous event horizon, $r>r_{s}$.
Source: By the author.
energy excess of the initial vacuum state in comparison to the stabilized configuration.
If we transpose these conclusions, mutatis mutandis, to our analogous systems, then an electromagnetic field should spontaneously appear in the material, bringing the whole system to a new equilibrium configuration - through nonlinear effects brought in by fielddependent constitutive tensors $\varepsilon^{a b}$ and $\mu_{a b}$ [see Eqs. $(2.15,2.16)$ ] —, with photons being emitted, carrying away the energy excess. Although the detailed dynamics of the stabilization processes in the gravitational and in the analogous systems are quite different - ruled by Einstein equations in the gravitational case and by the macroscopic Maxwell's equations with field-dependent $\varepsilon^{a b}$ and $\mu_{a b}$ in the analogous systems -, the qualitative features of the
2.5. Stabilization: spontaneous vectorization, photo production, and long-range induced
whole process, described above, seem quite reasonable to occur in generic field stabilization processes.

It is important to mention that the time scale set by the instability, $\Omega^{-1}$, is typically of the order of the time light takes to travel the typical size of the system, $L$. Therefore, in the analogous lab scenarios, the stabilization process would occur almost instantaneously $\left(\sim L /(1 \mathrm{~cm}) \times 10^{-10} \mathrm{~s}\right)$ once the instability conditions are met - which, for a given system, may depend on external parameters such as temperature, external fields, etc., through their influence on the constitutive functions $\varepsilon_{\perp}, \varepsilon_{\|}, \mu_{\perp}, \mu_{\|}$. The whole process would most likely be interpreted as a kind of phase transition, where the "long-range" emergent correlations in the material would come from interaction of its constituents with a common (initiallyunstable vacuum) fluctuating mode and/or the stabilized field configuration.

For concreteness sake, let us consider the explict form of the unstable modes found in the example of Sec. 2.3, where instability occurs due to a negative value of $\mu_{\perp}$ - for some (isolated) $\omega^{2}=-\Omega^{2}<0-$ in a homogeneous slab of width $L$. Although this system is not analogous to vacuum nonminimally-coupled electromagnetism in any realistic spacetime, it serves to illustrate general features of the mechanism itself, in addition to being much simpler to setup in the lab. This is no different than looking for fingerprints of analogue Hawking radiation in systems whose only similarity with realistic black holes is the presence of an effective event horizon - which is the common approach in condensed-matter and optical experimental analogues.

As argued before, once instability sets in, the unstable modes must be added to the expansion of the field operator $\mathbf{A}$, along with their complex conjugates, with corresponding annihilation $a_{\Omega \mathbf{k}_{\perp}}^{(u)}$ and creation $a_{\Omega \mathbf{k}_{\perp}}^{(u) \dagger}$ operators. It is easy to see that the field's vacuum fluctuations and correlations are eventually ( $t, t^{\prime} \gg \Omega^{-1}$ ) dominated by these unstable modes - at least as long as decoherence does not come into play. The dominant contribution to the vacuum correlations in the example of Subsec. 2.3.2 reads (the reader should refer to Subsec. 2.3.2 for the definition of all quantities appearing in these expressions):

$$
\begin{aligned}
\left\langle A_{j}(x) A_{l}\left(x^{\prime}\right)\right\rangle \sim & \frac{2 \pi}{L_{\perp}} \int_{0}^{2 \pi} \mathrm{~d} \varphi\left\{\sum_{m=0}^{m^{\mathrm{TMM}}} k_{\perp}^{(m)}\left[\mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(\mathrm{uT)}}}^{(\mathrm{uTM})}(x)\right]_{j}\left[\overline{\mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(m)}}^{(\mathrm{uTM})}}\left(x^{\prime}\right)\right]_{l}\right. \\
& \left.+\sum_{m=m^{(\mathrm{TE})}}^{\infty} k_{\perp}^{(m)}\left[\mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(\mathrm{uTM})}}^{(\mathrm{uT})}(x)\right]_{j}\left[\overline{\mathbf{A}_{\Omega \mathbf{k}_{\perp}^{(m)}}^{(\mathrm{uTE})}}\left(x^{\prime}\right)\right]_{l}\right\} \\
\sim & \frac{e^{\Omega\left(t+t^{\prime}\right)}}{4 L_{\perp} \Omega}\left\{\sum_{m=0}^{m^{(\mathrm{TM})}} \frac{k_{\perp}^{(m)}}{\sin \vartheta_{m}} \mathscr{D}_{j l(m)}^{(\mathrm{TM})}\left(d_{\perp}\right) g_{\Omega k_{\perp}^{(m)}}^{(\mathrm{TM})}(z) \overline{g_{\Omega k_{\perp}^{(m)}}^{(\mathrm{TM})}}\left(z^{\prime}\right)\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.+\sum_{m=m^{(\mathrm{TE})}}^{\infty} \frac{k_{\perp}^{(m)}}{\sin \vartheta_{m}} \mathscr{D}_{j l(m)}^{(\mathrm{TE})}\left(d_{\perp}\right) g_{\Omega k_{\perp}^{(\mathrm{TE})}}^{(\mathrm{TE})}(z) \overline{g_{\Omega k_{\perp}^{(m)}}^{(\mathrm{TE})}}\left(z^{\prime}\right)\right\}, \tag{2.134}
\end{equation*}
$$

where $\varphi$ is the angle between $\mathbf{k}_{\perp}$ and $\left(\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}\right), d_{\perp} \equiv\left\|\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}\right\|$, and the operators $\mathscr{D}_{j l(m)}^{(\mathrm{J})}\left(d_{\perp}\right)$ acting on $g_{\Omega k_{\perp}^{(m)}}^{(J)}(z) \overline{g_{\Omega k_{\perp}^{(m)}}^{(J)}}\left(z^{\prime}\right)$ are defined by

$$
\begin{align*}
\mathscr{D}_{j l(m)}^{(\mathrm{TM})}\left(d_{\perp}\right)= & \frac{1}{\Omega^{2} \varepsilon_{\perp}^{2}}\left[J_{1}^{\prime}\left(k_{\perp}^{(m)} d_{\perp}\right) \delta_{j}^{\ell} \delta_{l}^{\ell}+\frac{J_{1}\left(k_{\perp}^{(m)} d_{\perp}\right)}{k_{\perp}^{(m)} d_{\perp}} \delta_{j}^{\varphi} \delta_{l}^{\varphi}\right] \frac{\mathrm{d}^{2}}{\mathrm{~d} z \mathrm{~d} z^{\prime}} \\
& +\frac{\left(k_{\perp}^{(m)}\right)^{2} J_{0}\left(k_{\perp}^{(m)} d_{\perp}\right)}{\Omega^{2} \varepsilon_{\|}^{2}} \delta_{j}^{z} \delta_{l}^{z}-\frac{k_{\perp}^{(m)} J_{1}\left(k_{\perp}^{(m)} d_{\perp}\right)}{\Omega^{2} \varepsilon_{\|} \varepsilon_{\perp}}\left(\delta_{j}^{\ell} \delta_{l}^{z} \frac{\mathrm{~d}}{\mathrm{~d} z}-\delta_{j}^{z} \delta_{l}^{\ell} \frac{\mathrm{d}}{\mathrm{~d} z^{\prime}}\right)  \tag{2.135}\\
\mathscr{D}_{j l(m)}^{(\mathrm{TE})}\left(d_{\perp}\right)= & \frac{J_{1}\left(k_{\perp}^{(m)} d_{\perp}\right)}{k_{\perp}^{(m)} d_{\perp}} \delta_{j}^{\ell} \delta_{l}^{\ell}+J_{1}^{\prime}\left(k_{\perp}^{(m)} d_{\perp}\right) \delta_{j}^{\varphi} \delta_{l}^{\varphi}, \tag{2.136}
\end{align*}
$$

with indices $\ell$ and $\varphi$ standing for vector components along ( $\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}$ ) and $\mathbf{n}_{z} \times\left(\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}\right)$, respectively; $J_{n}$ and $J_{n}^{\prime}$ stand for the Bessel functions of first kind and their first derivatives, respectively. Field correlations $\left\langle E_{j}(x) E_{l}\left(x^{\prime}\right)\right\rangle$ and $\left\langle B_{j}(x) B_{l}\left(x^{\prime}\right)\right\rangle$ can be similarly obtained in particular, $\left\langle E_{j}(x) E_{l}\left(x^{\prime}\right)\right\rangle \sim \Omega^{2}\left\langle A_{j}(x) A_{l}\left(x^{\prime}\right)\right\rangle$. As an illustration, in Fig. 2.5 we plot the equaltime $\left(t=t^{\prime} \gg \Omega^{-1}\right)$, longitudinal correlation function $\left\langle A_{\ell}(\mathbf{x}) A_{\ell}\left(\mathbf{x}^{\prime}\right)\right\rangle$ for points $\mathbf{x}, \mathbf{x}^{\prime}$ in the plane $z=0$, for the same values of constitutive functions used in Fig. 2.2 and four different values of $\Omega L$. The vertical-axis scale is arbitrary - but the same in all plots - , since the correlations grow exponentially in time, from their typical (stable-vacuum) values of order $\hbar /\left(c L d_{\perp}\right) \sim\left[1 \mathrm{~cm}^{2} /\left(L d_{\perp}\right)\right] \times 10^{-8} \mathrm{eV} /\left(\mathrm{cm}^{3} \mathrm{GHz}^{2}\right)$, until decoherence and vectorization take over. Notice that once minimum-width (TM) instability sets in, macroscopic ( $\sim L$ ) field correlations are enhanced. It is an interesting question whether any such "long-range" correlation would survive or leave an inprint in the final stable configuration. Although not directly relevant for the analogy with gravity-induced instability itself, such correlations might lead to interesting material behavior.

### 2.6 Final Remarks

We have shown that gravity-induced instabilities, related to the vacuum-awakening effect in the quantum context ${ }^{9-11,53}$ and spontaneous scalarization/vectorization in the classical one, ${ }^{7,8,49-51}$ can be mimicked by electromagnetism in anisotropic metamaterials with ap-


Figure 2.5 - Equal-time $\left(t=t^{\prime} \gg \Omega^{-1}\right.$ ), two-point correlation function $\left\langle A_{\ell}(\mathbf{x}) A_{\ell}\left(\mathbf{x}^{\prime}\right)\right\rangle$ of the component of the quantum field $\hat{\mathbf{A}}$ along the vector-separation $\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}$, for points in the $z=0$ plane, for different values of $\Omega L$ - with same values of constitutive functions given in Fig. 2.2. The dotted (blue) lines represent the contribution coming from the TE modes, while the dashed (red) lines depict the contribution coming from the TM modes. The solid (black) lines give the sum of both contributions. Notice that long-range ( $\left\|\mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}\right\| \gtrsim L$ ) correlations are mainly due to the TM modes, which undergo minimum-width instability.
Source: By the author.
propriate constitutive functions. This follows from the formal analogy between electromagnetism in anisotropic media and nonminimally-coupled electromagnetism in curved spacetimes, presented in Sec. 2.2. We explored two concrete scenarios: (i) a plane-symmetric, static slab — whose main interest is its simplicity regarding experimental setup (see Sec. 2.3) - and (ii) a spherically-symmetric, moving media - whose main feature is its analogy with QED-inspired nonminimally-coupled electromagnetism in Schwarzschild spacetime ${ }^{40,41}$ for given velocity and constitutive-functions profiles (see Subsec. 2.2.1 and Sec. 2.4).

Once instability is triggered in the analogous systems, some stabilization process must take place, leading the system to a new stable configuration. The details of this stabilization
process and of the final configuration will most likely depend on specific nonlinear properties of the metamaterial, but it seems reasonable that they might involve the appearance of nonzero electromagnetic fields in the material (analogous to spontaneous vectorization in curved spacetimes) and photo production which carries away the energy excess with respect to the stable configuration. As discussed earlier, the time scale involved in the stabilization process can be very short ( $\sim 10^{-10} \mathrm{~s}$ ), which would make it very difficult to even identify the unstable phase. This is similar to what might occur with negative conductivity, which has never been directly measured but which is predicted to lead to zero-dc-resistance states ${ }^{54}$ which were observed in laboratory ${ }^{55,56}$ - although an alternative explanation has been proposed. ${ }^{57}$

Clearly, the feasibility of such analogues is bound to the existence of material configurations with the required constitutive functions. As briefly pointed out in the introduction, this can be achieved at least for anisotropic neutral plasmas, and the recent advances in metamaterial science offer a plethora of possible candidates, specially the hyperbolic metamaterials, ${ }^{27,28}$ that possess precisely the form given in Eqs. (2.17,2.18) with the required "negativeness." In particular, we call attention to the increase in the spontaneous light emission in such configurations, which may be related to the process of stabilization in active scenarios.

It is also important to mention that the QED-inspired analogues (Subsec. 2.2.1) are not restricted to the study of vacuum instability. For instance, they can be used to study light ray propagation in the corresponding spacetimes and one possible application is the QEDinduced birefringence in the Schwarzschild spacetime. ${ }^{40}$ For this particular experiment, one can work far from the effective horizon, where the constitutive coefficients (2.128)-(2.131) are positive.

Our main purpose here was to lay down a novel class of analogue models of curvedspacetime phenomena, with main interest on the gravitational side of the analogy. Notwithstanding, the consequences of the analogue gravity-induced instability to the metamaterial side may be interesting on its own. The electromagnetic field instability may mark, lead or mediate some kind of phase transition in the metamaterial, where the spontaneously created field and/or its amplified "long-range" correlations may play some important role (see discussion in Sec. 2.5). Investigation in these lines will be explored in the next part of this work.

## Part II

## Field quantization in active dispersive media

## Chapter 3

## Scalar field in dispersive active backgrounds

This chapter is dedicated to the study of the scalar quantum vacuum in flat spacetime and in the presence of active dispersive backgrounds. Its motivation relies on the fact that dispersion is necessary in order to have "well-behaved" vacuum states, as pointed out in the prolegomena. Clearly, the Maxwell field is the most natural choice to conduct this study, because of its major experimental appeal. Nevertheless, the analysis presented here based on the scalar field generalizes in a straightforward way to the electromagnetic field, and have the advantage of being easier, as the internal structure of the scalar field is simpler. We start with the motivation of a nonlocal Klein-Gordon (KG) equation to model frequency-dependent potentials, and in the same way as it happens for the electromagnetic case, we shall show that causality requires that the quantization cannot be performed canonically. Sec. 3.2 contains a microscopic Lagrangian model that will be used to derive the nonlocal KG equation. In Sec. 3.3, we revisit the Fano Diagonalization procedure and show that, in contrast to previous claims, the method is always consistent. This machinery will be used to diagonalize the Lagrangian of Sec. 3.2. In Sec. 3.4 we present the quantum Langevin equation, and the concept of Langevin noises is introduced. It should be stressed that most of the developments in this chapter that concerns the quantization in stable configurations were already explore in the literature, which are explained in the text. Our contribution is that we explore a field-reservoir coupling that adds unstable modes to the diagonalization, revealing how quantization must be done generally. In this chapter we work with Cartesian coordinates, and when the limits of integration are not shown in the integral signs, it is to be assumed
integration over the whole real line.

### 3.1 Scalar field in the presence of dispersive media

The interaction of the scalar field with matter was already briefly studied in the literature. Vacuum fluctuations of the stress tensor in the vicinity of plane walls led naturally to the assumption that such boundaries must be dispersive, just like we saw in the prolegomena. ${ }^{58}$ Moreover, a scalar degree of freedom of the electromagnetic field in dispersive media was considered as a toy model to study such stress tensor fluctuations a couple of decades ago, ${ }^{59}$ and more recently, a nonlocal form of the Klein-Gordon equation was deduced from a microscopic Lagrangian, ${ }^{60}$ where the authors presented a perturbative expansion of Green functions.

These works usually implement the dispersion as a sort of "scalar dielectric function" to keep the analogy with the electromagnetic case as sharp as possible. Here, however, we shall adopt the philosophy that for the scalar field the dispersion must be incorporated through the scalar potential, as there is no fundamental reason to compare effective description of fields of different spins. In order to accomplish this we start from the free massless KG equation in Minkowski spacetime

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi=\left(-\partial_{t}^{2}+\nabla^{2}\right) \phi=0, \tag{3.1}
\end{equation*}
$$

and as usual let the field be decomposed as

$$
\begin{equation*}
\phi(t, \mathbf{x})=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} \omega \phi_{\omega}(\mathbf{x}) \mathrm{e}^{-i \omega t} \tag{3.2}
\end{equation*}
$$

where clearly we are assuming that there is no instability for the moment. This in turn results in the "mode-wise" Klein-Gordon equation

$$
\begin{equation*}
\left(\omega^{2}+\nabla^{2}\right) \phi_{\omega}=0 \tag{3.3}
\end{equation*}
$$

By assuming that the collective net effect of the matter sector upon the scalar field can be expressed in terms of effective potentials, we amend this equation by adding a frequencydependent potential $\hat{\mu}(\omega, \mathbf{x})$ as

$$
\begin{equation*}
\left[\omega^{2}+\nabla^{2}-\hat{\mu}(\omega, \mathbf{x})\right] \phi_{\omega}=0 . \tag{3.4}
\end{equation*}
$$

In this way the requirement of transparency in the high frequency sector is translated as $\hat{\mu}(\omega, \mathbf{x}) \rightarrow 0$ as $|\omega| \rightarrow \infty$. If we define the inverse Fourier transform of $\hat{\mu}(\omega, \mathbf{x})$ by

$$
\begin{equation*}
\mu(t, \mathbf{x})=\frac{1}{2 \pi} \int \mathrm{~d} \omega \mathrm{e}^{-i \omega t} \hat{\mu}(\omega, \mathbf{x}), \tag{3.5}
\end{equation*}
$$

then taking the inverse transform of (3.4) results in

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi(x)-\int_{-\infty}^{\infty} \mathrm{d} \tau \mu(\tau, \mathbf{x}) \phi(t-\tau, \mathbf{x})=0 . \tag{3.6}
\end{equation*}
$$

Notice that the integral is performed along the whole history of the field, implying that the wave equation at a time $t$ depends also on the field's future history. By requiring that causality be preserved, we must have $\mu(t, \mathbf{x})=0$ for $t<0$, in which case $\phi$ satisfies the nonlocal KG equation

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi(t, \mathbf{x})-\int_{0}^{\infty} \mathrm{d} \tau \mu(\tau, \mathbf{x}) \phi(t-\tau, \mathbf{x})=0 \tag{3.7}
\end{equation*}
$$

A similar kind of nonlocality appears in the quantum Langevin equation for a quantum mechanical particle, ${ }^{61}$ where the function $\mu(t, \mathbf{x})$ is usually called the memory function. The frequency space memory function then is found to be

$$
\begin{equation*}
\hat{\mu}(\omega, \mathbf{x})=\int_{0}^{\infty} \mathrm{d} t \mu(t, \mathbf{x}) \mathrm{e}^{i \omega t} \tag{3.8}
\end{equation*}
$$

Notice that the real (resp., imaginary) part of Eq. (3.8) is an even (resp., odd) function of $\omega$, and its extension to complex $\omega$ makes it an analytic function in the upper half-plane. Moreover, as $\mu(t, \mathbf{x})$ is real, its frequency-space form must satisfy the reflection property $\overline{\hat{\mu}}(\omega, \mathbf{x})=$ $\hat{\mu}(-\bar{\omega}, \mathbf{x})$. This analytical property ensures that the function $\hat{\mu}(\omega, \mathbf{x})$ satisfy the Kramers-Kronig relations, ${ }^{62}$ that is, its real and imaginary parts are not independent. In order to show it, consider the contour integral

$$
\begin{equation*}
\oint_{C} \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega+i \epsilon}, \tag{3.9}
\end{equation*}
$$

with $\epsilon$ a small positive real number, $\omega \in \mathbb{R}$, and $C$ is the semi-circle enclosed in the upper half-plane. As the integrand is analytic in the upper half-plane, the integral vanishes. Now using the representation ${ }^{62}$

$$
\begin{equation*}
\frac{1}{\omega^{\prime}-\omega \pm i \epsilon}=\frac{1}{\omega^{\prime}-\omega} \mp i \pi \delta\left(\omega^{\prime}-\omega\right) \tag{3.10}
\end{equation*}
$$

and assuming that $\hat{\mu}(\omega, \mathbf{x})$ goes to zero as $|\omega| \rightarrow \infty$, we are left with

$$
\begin{equation*}
\hat{\mu}(\omega, \mathbf{x})=\frac{1}{i \pi} f \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega}, \tag{3.11}
\end{equation*}
$$

where the bar on the integral sign means that the Cauchy Principal Value is to be taken. Denoting by $\hat{\mu}_{r}=\operatorname{Re}\{\hat{\mu}\}, \hat{\mu}_{i}=\operatorname{Im}\{\hat{\mu}\}$, equation (3.11) is equivalent to

$$
\begin{align*}
& \hat{\mu}_{r}(\omega, \mathbf{x})=\frac{1}{\pi} f \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}_{i}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega}  \tag{3.12}\\
& \hat{\mu}_{i}(\omega, \mathbf{x})=-\frac{1}{\pi} f \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}_{r}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega} \tag{3.13}
\end{align*}
$$

These equations are known as Kramers-Kronig relations, ${ }^{62}$ and they mean that $\hat{\mu}$ is completely characterized by its real or imaginary part. For instance, we have

$$
\begin{equation*}
\hat{\mu}(\omega, \mathbf{x})=\frac{1}{\pi}\left[f \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}_{i}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega}+i \pi \hat{\mu}_{i}(\omega, \mathbf{x})\right]=\frac{1}{\pi} \int \mathrm{~d} \omega^{\prime} \frac{\hat{\mu}_{i}\left(\omega^{\prime}, \mathbf{x}\right)}{\omega^{\prime}-\omega-i \epsilon} . \tag{3.14}
\end{equation*}
$$

In particular, this equation shows that if $\hat{\mu}_{i}$ vanishes, then causality ensures that $\hat{\mu}$ also vanishes. Thus, the problem of quantizing this theory requires the imaginary part of $\hat{\mu}$ not to vanish. In the following sections, we shall see that $\hat{\mu}_{i}$ is related to energy exchanging, and thus a dispersive medium is necessarily an absorbing/gaining one. This means that the field is coupled to some reservoir, and thus the canonical quantization can not be achieved without introducing the medium variables. Nevertheless, a canonical quantization can be performed if $\hat{\mu}_{i}$ is supposed to be small in the same way as was done for the electromagnetic case, ${ }^{63}$ or the experimental apparatus is sensible only to a part of the field spectrum in which $\hat{\mu}_{i}$ can be neglected. We shall not discuss this possibility here, as our main interest is in the general theory. In the next section, we present a microscopic model for dispersion in an infinite homogeneous medium, and we shall see how a consistent quantization scheme can be done by introducing the concept of quantum noise operators to the general case, where instability may be present. This model is also used to justify the form of Eq. (3.7).

### 3.2 Microscopic model for dispersive media

We now present in detail a microscopic model for a dispersive media interacting with the scalar field. The proposed theory is analogous to the model presented by Huttner and Bar-
nett for a dispersive dielectric where the authors modified the Hopfield model by adding a coupling responsible for the losses. ${ }^{15}$ Here we present a more general coupling, one that is capable of modeling active as well as passive media. The notation adopted in this section follows closely the one used by Huttner. ${ }^{15}$ We start with the Lagrangian density

$$
\begin{equation*}
\mathscr{L}=\mathscr{L}_{\text {sf }}+\mathscr{L}_{\text {matter }}+\mathscr{L}_{\text {int }} \tag{3.15}
\end{equation*}
$$

where the Lagrangian for the scalar field $\mathscr{L}_{\text {sf }}$ is ${ }^{34}$

$$
\begin{equation*}
\mathscr{L}_{\mathrm{sf}}=\frac{1}{2}\left[\left(\partial_{t} \phi\right)^{2}-(\nabla \phi)^{2}\right] . \tag{3.16}
\end{equation*}
$$

The matter field is taken to be described by a single homogeneous harmonic oscillator of resonance frequency $\omega_{0}$ plus an infinite continuum of (homogeneous) oscillators

$$
\begin{equation*}
\mathscr{L}_{\text {matter }}=\frac{1}{2}\left[\left(\partial_{t} \varphi\right)^{2}-\omega_{0}^{2} \varphi^{2}\right]+\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega\left[\left(\partial_{t} \kappa_{\omega}\right)^{2}-\omega^{2} \kappa_{\omega}^{2}\right] . \tag{3.17}
\end{equation*}
$$

Notice that homogeneity ensures the theory covariance under Lorentz transformations. The continuum oscillators $\kappa_{\omega}$ are used to model a reservoir, and the single oscillator $\varphi$ is used to intermediate the interaction between the scalar field and the reservoir. Thus, the interaction Lagrangian is a sum of a term that couples the scalar field $\phi$ to the oscillator $\varphi$ plus a term that couples all the oscillators from the reservoir to the oscillator $\varphi$, which we take to be

$$
\begin{equation*}
\mathscr{L}_{\mathrm{int}}=-\phi\left(\nu_{0} \varphi+\nu_{1} \partial_{t} \varphi\right)-\int_{0}^{\infty} \mathrm{d} \omega \nu(\omega) \varphi \partial_{t} \kappa_{\omega}, \tag{3.18}
\end{equation*}
$$

where $\nu_{0}, \nu_{1}$, and $\nu(\omega)$ are real coupling constants. The only hypothesis required on the function $v(\omega)$ is that $v(\omega) \neq 0$ for all $\omega>0$. We shall see that instability is present in this system whenever $\nu_{0} \neq 0$ for modes with wave vector inside the sphere $|\mathbf{k}|=\sqrt{V_{0}}$, with $V_{0}=$ $\nu_{0}^{2} / \omega_{0}^{2}$, whereas the coupling constant $\nu_{1}$ measures the intensity of passive field-reservoir interactions

Next, we take the spatial Fourier decompositions

$$
\begin{align*}
& \phi(t, \mathbf{x})=\frac{1}{\sqrt{(2 \pi)^{3}}} \int \mathrm{~d}^{3} k \hat{\phi}(t, \mathbf{k}) \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}},  \tag{3.19}\\
& \varphi(t, \mathbf{x})=\frac{1}{\sqrt{(2 \pi)^{3}}} \int \mathrm{~d}^{3} k \hat{\varphi}(t, \mathbf{k}) \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}}, \tag{3.20}
\end{align*}
$$

$$
\begin{equation*}
\kappa_{\omega}(t, \mathbf{x})=\frac{1}{\sqrt{(2 \pi)^{3}}} \int \mathrm{~d}^{3} k \hat{\kappa}_{\omega}(t, \mathbf{k}) \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} \tag{3.21}
\end{equation*}
$$

and the reflection $\hat{\phi}(t, \mathbf{k})=\overline{\hat{\phi}}(t,-\mathbf{k})$ is verified in order for $\phi(t, \mathbf{x})$ to be real, and similarly for $\hat{\varphi}$ and $\hat{\kappa}_{\omega}$. Thus, the total Lagrangian becomes

$$
\begin{align*}
L= & \int \mathrm{d}^{3} x \mathscr{L} \\
= & \int^{\prime} \mathrm{d}^{3} k\left\{\left|\partial_{t} \hat{\phi}(\mathbf{k})\right|^{2}-\mathbf{k}^{2}|\hat{\phi}(\mathbf{k})|^{2}+\left|\partial_{t} \hat{\varphi}(\mathbf{k})\right|^{2}-\omega_{0}^{2}|\hat{\varphi}(\mathbf{k})|^{2}+\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\partial_{t} \hat{\kappa}_{\omega}(\mathbf{k})\right|^{2}-\omega^{2}\left|\hat{\kappa}_{\omega}(\mathbf{k})\right|^{2}\right]\right\} \\
& -\int \mathrm{d}^{3} k\left\{\hat{\phi}(\mathbf{k})\left[v_{0} \overline{\hat{\varphi}}(\mathbf{k})+v_{1} \overline{\partial_{t} \hat{\varphi}}(\mathbf{k})\right]+\int_{0}^{\infty} \mathrm{d} \omega v(\omega) \hat{\varphi}(\mathbf{k}) \overline{\partial_{t} \hat{\kappa}_{\omega}}(\mathbf{k})\right\}, \tag{3.22}
\end{align*}
$$

where the prime in the first integral means integration in only half of the reciprocal space, for instance $k_{z}>0$. Also, we have omitted the time-dependence of the operators for notational convenience. In order to find the corresponding Hamiltonian, we use the canonical conjugated momenta defined through

$$
\begin{align*}
& \hat{\pi}(t, \mathbf{k})=\frac{\delta L}{\delta \overline{\partial_{t} \hat{\phi}}(t, \mathbf{k})}=\partial_{t} \hat{\phi}(t, \mathbf{k})  \tag{3.23}\\
& \hat{p}(t, \mathbf{k})=\frac{\delta L}{\delta \overline{\partial_{t} \hat{\varphi}}(t, \mathbf{k})}=\partial_{t} \hat{\varphi}(t, \mathbf{k})-v_{1} \hat{\phi}(t, \mathbf{k})  \tag{3.24}\\
& \hat{q}_{\omega}(t, \mathbf{k})=\frac{\delta L}{\delta \overline{\partial_{t} \hat{\kappa}_{\omega}}(t, \mathbf{k})}=\partial_{t} \hat{\kappa}_{\omega}(t, \mathbf{k})-v(\omega) \hat{\varphi}(t, \mathbf{k}) \tag{3.25}
\end{align*}
$$

and the Legendre transformation

$$
\begin{align*}
H= & \int \mathrm{d}^{3} k\left[\hat{\pi}(\mathbf{k}) \overline{\partial_{t} \hat{\phi}}(\mathbf{k})+\hat{p}(\mathbf{k}) \overline{\partial_{t} \hat{\varphi}}(\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega \hat{q}_{\omega}(\mathbf{k}) \overline{\partial_{t} \hat{\kappa}_{\omega}}(\mathbf{k})\right]-L \\
= & \int^{\prime} \mathrm{d}^{3} k\left\{|\hat{\pi}(\mathbf{k})|^{2}+\tilde{k}^{2}|\hat{\phi}(\mathbf{k})|^{2}+|\hat{p}(\mathbf{k})|^{2}+\tilde{\omega}_{0}^{2}|\hat{\varphi}(\mathbf{k})|^{2}+\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\hat{q}_{\omega}(\mathbf{k})\right|^{2}+\omega^{2}\left|\hat{\kappa}_{\omega}(\mathbf{k})\right|^{2}\right]\right\} \\
& +\int \mathrm{d}^{3} k\left\{\hat{\phi}(\mathbf{k})\left[v_{0} \overline{\hat{\varphi}}(\mathbf{k})+v_{1} \overline{\hat{p}}(\mathbf{k})\right]+\int_{0}^{\infty} \mathrm{d} \omega v(\omega) \hat{q}_{\omega}(\mathbf{k}) \overline{\hat{\varphi}}(\mathbf{k})\right\}, \tag{3.26}
\end{align*}
$$

where we have defined

$$
\begin{align*}
& \tilde{k}=\sqrt{\mathbf{k}^{2}+v_{1}^{2}}  \tag{3.27}\\
& \tilde{\omega}_{0}^{2}=\omega_{0}^{2}+\int_{0}^{\infty} \mathrm{d} \omega v^{2}(\omega) . \tag{3.28}
\end{align*}
$$

We shall define three sets of creation/annihilation operators as

$$
\begin{align*}
& a(t, \mathbf{k})=\sqrt{\frac{1}{2 \tilde{k}}}[\tilde{k} \hat{\phi}(t, \mathbf{k})+i \hat{\pi}(t, \mathbf{k})]  \tag{3.29}\\
& b(t, \mathbf{k})=\sqrt{\frac{1}{2 \tilde{\omega}_{0}}}\left[\tilde{\omega}_{0} \hat{\varphi}(t, \mathbf{k})+i \hat{p}(t, \mathbf{k})\right]  \tag{3.30}\\
& b_{\omega}(t, \mathbf{k})=\sqrt{\frac{1}{2 \omega}}\left[-i \omega \hat{\kappa}_{\omega}(t, \mathbf{k})+\hat{q}_{\omega}(t, \mathbf{k})\right] . \tag{3.31}
\end{align*}
$$

Notice that the global phase of Eq. (3.31) was chosen differently from the other operators. This is not fundamental, and we have just kept the notation as close to Huttner's as possible. ${ }^{15}$ In terms of these operators, the normal ordered Hamiltonian is $H=H_{\mathrm{sf}}+H_{\text {matter }}+H_{\text {int }}$, with

$$
\begin{align*}
& \begin{array}{l}
H_{\mathrm{sf}}=\int \mathrm{d}^{3} k \tilde{k} a^{\dagger}(\mathbf{k}) a(\mathbf{k}), \\
H_{\text {matter }}=\int \mathrm{d}^{3} k\left\{\begin{array}{c}
\tilde{\omega}_{0} b^{\dagger}(\mathbf{k}) b(\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega \omega b_{\omega}^{\dagger}(\mathbf{k}) b_{\omega}(\mathbf{k}) \\
\\
\\
\left.\quad+\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega V(\omega)\left[b^{\dagger}(\mathbf{k})+b(-\mathbf{k})\right]\left[b_{\omega}^{\dagger}(-\mathbf{k})+b_{\omega}(\mathbf{k})\right]\right\}
\end{array}\right. \\
H_{\mathrm{int}}=\frac{1}{2} \int \mathrm{~d}^{3} k\left[a^{\dagger}(\mathbf{k})+a(-\mathbf{k})\right]\left[\bar{\Lambda}(k) b^{\dagger}(-\mathbf{k})+\Lambda(k) b(\mathbf{k})\right]
\end{array} \tag{3.32a}
\end{align*}
$$

where again we omitted the time-dependence of the operators for notational convenience, and

$$
\begin{align*}
& \Lambda(k)=\frac{v_{0}-i \tilde{\omega}_{0} v_{1}}{\sqrt{\tilde{\omega}_{0} \tilde{k}}}  \tag{3.33}\\
& V(\omega)=\sqrt{\frac{\omega}{\tilde{\omega}_{0}}} v(\omega) \tag{3.34}
\end{align*}
$$

The parameters $v_{0}, \nu_{1}$ are responsible for describing how the scalar field interacts with matter. As anticipated, in the following sections we show that the coupling coming from $v_{0}$ models an active medium, that turns the field unstable, and $\nu_{1}$ models a passive interaction. This Hamiltonian can be diagonalized using a method proposed by Ugo Fano, ${ }^{64}$ as was done for the electromagnetic case. ${ }^{15}$ This technique consists of finding a Bogoliubov transformation relating a diagonalizing set of creation/annihilation operators to the original set. ${ }^{34}$ Once this diagonalizing set is known, the quantization is complete. Next section we show that it is always possible to diagonalize the Hamiltonian $H$ for all choices of $v_{0}, \nu_{1}$. The only fundamental requirement is that $v(\omega) \neq 0$ for all $\omega>0$.

### 3.3 Fano diagonalization revisited

In this section we present, in a model-independent fashion, a generalization of the Fano diagonalization scheme applied to the Hamiltonian possessing the general form

$$
\begin{align*}
H= & \int \mathrm{d}^{3} k\left[f(k) a^{\dagger}(\mathbf{k}) a(\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega \omega a_{\omega}^{\dagger}(\mathbf{k}) a_{\omega}(\mathbf{k})\right] \\
& +\frac{1}{2} \int \mathrm{~d}^{3} k\left[a^{\dagger}(\mathbf{k})+a(-\mathbf{k})\right] \int_{0}^{\infty} \mathrm{d} \omega\left[g(k, \omega) a_{\omega}^{\dagger}(-\mathbf{k})+\bar{g}(k, \omega) a_{\omega}(\mathbf{k})\right], \tag{3.35}
\end{align*}
$$

when the collective spectrum contains instabilities. The functions $f(k)$ and $g(k, \omega)$ depend solely on the absolute value of $\mathbf{k}, f(k)>0$ and $|g(k, \omega)|^{2} \neq 0$ for all $k, \omega>0$. The operators $a(\mathbf{k}), a_{\omega}(\mathbf{k})$ satisfy $\left[a(\mathbf{k}), a^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)$ and $\left[a_{\omega}(\mathbf{k}), a_{\omega^{\prime}}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right)$, respectively, and all other commutators vanish. The relevance of this form to our case is that the Hamiltonian of Eq. (3.32) can be diagonalized in two steps, by applying the procedure first to $H_{\text {matter }}$ (that is already in the form of Eq. (3.35)), and then again to the resulting Hamiltonian.

Notice that (recall that we are working in the Heisenberg picture) these operators do not have a simple time evolution, as verified by the application of the Heisenberg equation to them. Then we may ask whether there exists a new set of creation/annihilation operators $A(\mathbf{k}, \omega)$ satisfying $\mathrm{d} A(\mathbf{k}, \omega) / \mathrm{d} t=-i \omega A(\mathbf{k}, \omega)$ and such that the operators $a(\mathbf{k}), a_{\omega}(\mathbf{k})$ are written as a linear combinations of them and their adjoints. We shall show that this is always true for Hamiltonians of the form (3.35), thus generalizing the Fano diagonalization. From a field-theoretical perspective, this simply means that we are looking for a Bogoliubov transformation between both sets of creation/annihilation operators, that is, a linear canonical transformation. Here we shall fill a gap in the literature concerning some "inconsistencies" in this process, that appear in certain cases. We shall show that these inconsistencies are simply an indication that the diagonalizing set used in the literature is incomplete. Once we add all the solutions, including the unstable ones, we show that the transformation is canonical.

Let us now proceed to find such a diagonalizing set. The operators $A(\mathbf{k}, \omega)$ are defined by their time evolution $\mathrm{d} A(t, \mathbf{k}, \omega) / \mathrm{d} t=-i \omega A(t, \mathbf{k}, \omega)$. Then the Heisenberg equation of motion reduces to the commutator

$$
\begin{equation*}
[A(\mathbf{k}, \omega), H]=\omega A(\mathbf{k}, \omega) \tag{3.36}
\end{equation*}
$$

This is the basic equation we need to solve in order to find the linear expansion relating the
operators. It is noteworthy that the $\omega$ in (3.36) is to be interpreted as the frequency of the system collective spectrum, and has nothing to do with the parameter $\omega$ in (3.35). It can even, in principle, be complex. Let us look for a linear expansion of the form

$$
\begin{equation*}
A(\mathbf{k}, \omega)=\alpha_{0}(k, \omega) a(\mathbf{k})+\beta_{0}(k, \omega) a^{\dagger}(-\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[\alpha\left(k, \omega, \omega^{\prime}\right) a_{\omega^{\prime}}(\mathbf{k})+\beta\left(k, \omega, \omega^{\prime}\right) a_{\omega^{\prime}}^{\dagger}(-\mathbf{k})\right] \tag{3.37}
\end{equation*}
$$

Thus, by calculating the commutator between (3.37) and the Hamiltonian (3.35), it follows that the operatorial identity must hold

$$
\begin{align*}
{[A(\mathbf{k}, \omega), H]=} & \left\{f(k) \alpha_{0}(k, \omega)+\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[g\left(k, \omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)-\bar{g}\left(k, \omega^{\prime}\right) \beta\left(k, \omega, \omega^{\prime}\right)\right]\right\} a(\mathbf{k}) \\
& +\left\{-f(k) \beta_{0}(k, \omega)+\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[g\left(k, \omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)-\bar{g}\left(k, \omega^{\prime}\right) \beta\left(k, \omega, \omega^{\prime}\right)\right]\right\} a^{\dagger}(-\mathbf{k}) \\
& +\int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left\{\alpha\left(k, \omega, \omega^{\prime}\right) \omega^{\prime}+\frac{1}{2} \bar{g}\left(k, \omega^{\prime}\right)\left[\alpha_{0}(k, \omega)-\beta_{0}(k, \omega)\right]\right\} a_{\omega^{\prime}}(\mathbf{k}) \\
& +\int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left\{-\beta\left(k, \omega, \omega^{\prime}\right) \omega^{\prime}+\frac{1}{2} g\left(k, \omega^{\prime}\right)\left[\alpha_{0}(k, \omega)-\beta_{0}(k, \omega)\right]\right\} a_{\omega^{\prime}}^{\dagger}(-\mathbf{k}) . \tag{3.38}
\end{align*}
$$

Therefore, equating (3.37) and (3.38) through Eq. (3.36) results in the system

$$
\begin{align*}
& {[\omega-f(k)] \alpha_{0}(k, \omega)=\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[g\left(k, \omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)-\bar{g}\left(k, \omega^{\prime}\right) \beta\left(k, \omega, \omega^{\prime}\right)\right],}  \tag{3.39a}\\
& {[\omega+f(k)] \beta_{0}(k, \omega)=\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[g\left(k, \omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)-\bar{g}\left(k, \omega^{\prime}\right) \beta\left(k, \omega, \omega^{\prime}\right)\right],}  \tag{3.39b}\\
& \left(\omega-\omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)=\frac{1}{2} \bar{g}\left(k, \omega^{\prime}\right)\left[\alpha_{0}(k, \omega)-\beta_{0}(k, \omega)\right],  \tag{3.39c}\\
& \left(\omega+\omega^{\prime}\right) \beta\left(k, \omega, \omega^{\prime}\right)=\frac{1}{2} g\left(k, \omega^{\prime}\right)\left[\alpha_{0}(k, \omega)-\beta_{0}(k, \omega)\right] . \tag{3.39d}
\end{align*}
$$

We can solve Eqs. (3.39) in terms of $\alpha_{0}(k, \omega)$. By subtracting (3.39a) and (3.39b), we find that $\beta_{0}(k, \omega)$ is given by

$$
\begin{equation*}
\beta_{0}(k, \omega)=\frac{\omega-f(k)}{\omega+f(k)} \alpha_{0}(k, \omega) . \tag{3.40}
\end{equation*}
$$

Up to this point, the discussion follows the same lines as presented by Huttner. ${ }^{15}$ However, in order to proceed it is necessary to consider cases with complex $\omega$ separately. Let us look first for solutions with real and positive $\omega$. These are the only ones needed in stable configurations. In this case, thus, Eqs. (3.39c) and (3.39d) are solved using Eq. (3.40) as

$$
\begin{equation*}
\alpha\left(k, \omega, \omega^{\prime}\right)=\left[\frac{1}{\omega-\omega^{\prime}}+y(\omega) \delta\left(\omega-\omega^{\prime}\right)\right] \frac{f(k)}{\omega+f(k)} \bar{g}\left(k, \omega^{\prime}\right) \alpha_{0}(k, \omega), \tag{3.41}
\end{equation*}
$$

$$
\begin{equation*}
\beta\left(k, \omega, \omega^{\prime}\right)=\frac{1}{\omega+\omega^{\prime}} \frac{f(k)}{\omega+f(k)} g\left(k, \omega^{\prime}\right) \alpha_{0}(k, \omega) . \tag{3.42}
\end{equation*}
$$

The delta function appearing in $\alpha\left(k, \omega, \omega^{\prime}\right)$, as pointed out by Fano, ${ }^{64}$ is due to the fact that the general solution of the equation $\left(\omega-\omega^{\prime}\right) \alpha\left(k, \omega, \omega^{\prime}\right)=\gamma\left(k, \omega, \omega^{\prime}\right)$ is a principal part term plus a term supported on the set $\omega=\omega^{\prime}$. The function $y(\omega)$ is fixed by plugging (3.41), (3.42) in (3.39a) and the result is

$$
\begin{equation*}
\frac{f(k)}{2}|g(k, \omega)|^{2} y(\omega)=\omega^{2}-f^{2}(k)-\frac{f(k)}{2} f_{-\infty}^{\infty} \mathrm{d} \omega^{\prime} \frac{\mathfrak{g}\left(k, \omega^{\prime}\right)}{\omega-\omega^{\prime}}, \tag{3.43}
\end{equation*}
$$

where we have defined the auxiliary function $\mathfrak{g}(k, \omega)$ as the odd extension of $|g(k, \omega)|^{2}$ for negative $\omega$, that is, $\mathfrak{g}(k, \omega)=|g(k, \omega)|^{2}$ if $\omega>0$ and $\mathfrak{g}(k,-\omega)=-\mathfrak{g}(k, \omega)$ for $\omega \in \mathbb{R}$. To fix $\alpha_{0}(k, \omega)$, we calculate the commutator between $A(\mathbf{k}, \omega)$ and $A^{\dagger}\left(\mathbf{k}^{\prime}, \omega^{\prime}\right)$. We have

$$
\begin{equation*}
\left[A(\mathbf{k}, \omega), A^{\dagger}\left(\mathbf{k}^{\prime}, \omega^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right)\left|\alpha_{0}(k, \omega)\right|^{2} \frac{f^{2}(k)|g(k, \omega)|^{2}|y(\omega)-i \pi|^{2}}{[\omega+f(k)]^{2}} \tag{3.44}
\end{equation*}
$$

Care must be taken in calculating this commutator. During the calculation, we have used the partial fraction decomposition ${ }^{64}$

$$
\begin{equation*}
\frac{1}{(\omega-x)\left(\omega^{\prime}-x\right)}=\frac{1}{\omega-\omega^{\prime}}\left(\frac{1}{\omega^{\prime}-x}-\frac{1}{\omega-x}\right)+\pi^{2} \delta\left(\omega-\omega^{\prime}\right) \delta(x-\omega), \tag{3.45}
\end{equation*}
$$

where the delta contribution comes from the double pole $\omega=\omega^{\prime}$. The $-i \pi$ in (3.44) comes from this additional term. Thus, by requiring the validity of the canonical commutation relation $\left[A(\mathbf{k}, \omega), A^{\dagger}\left(\mathbf{k}^{\prime}, \omega^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right), \alpha_{0}(k, \omega)$ is determined up to an arbitrary phase and we shall pick the solution

$$
\begin{equation*}
\alpha_{0}(k, \omega)=\frac{\omega+f(k)}{f(k)} \frac{1}{\bar{g}(k, \omega)[y(\omega)-i \pi]} . \tag{3.46}
\end{equation*}
$$

The resulting theory is independent of the chosen phase, a peculiarity of Bogoliubov transformations. We can rearrange Eq. (3.46) in equivalent form by noticing that

$$
\begin{equation*}
\frac{f(k)}{2}|g(k, \omega)|^{2}[y(\omega)-i \pi]=\omega^{2}-\frac{f(k)}{2} \int_{-\infty}^{\infty} \mathrm{d} \omega^{\prime} \frac{\mathfrak{g}\left(k, \omega^{\prime}\right)}{\omega-\omega^{\prime}-i \epsilon}-f^{2}(k) \tag{3.47}
\end{equation*}
$$

as follows from of Eq. (3.10). Thus, (3.46) becomes

$$
\begin{equation*}
\alpha_{0}(k, \omega)=\frac{\omega+f(k)}{2} \frac{g(k, \omega)}{\omega^{2}-z(k, \omega)-f^{2}(k)}, \tag{3.48}
\end{equation*}
$$

where

$$
\begin{equation*}
z(k, \omega)=\frac{f(k)}{2} \int_{-\infty}^{\infty} \mathrm{d} \omega^{\prime} \frac{\mathfrak{g}\left(k, \omega^{\prime}\right)}{\omega-\omega^{\prime}-i \epsilon} \tag{3.49}
\end{equation*}
$$

The other coefficients are

$$
\begin{align*}
& \beta_{0}(k, \omega)=\frac{\omega-f(k)}{2} \frac{g(k, \omega)}{\omega^{2}-z(k, \omega)-f^{2}(k)},  \tag{3.50}\\
& \alpha\left(k, \omega, \omega^{\prime}\right)=\delta\left(\omega-\omega^{\prime}\right)+\frac{f(k)}{2} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{\omega-\omega^{\prime}-i \epsilon} \frac{g(k, \omega)}{\omega^{2}-z(k, \omega)-f^{2}(k)},  \tag{3.51}\\
& \beta\left(k, \omega, \omega^{\prime}\right)=\frac{f(k)}{2} \frac{g\left(k, \omega^{\prime}\right)}{\omega+\omega^{\prime}} \frac{g(k, \omega)}{\omega^{2}-z(k, \omega)-f^{2}(k)} . \tag{3.52}
\end{align*}
$$

These solutions exhaust all possible operators for positive real $\omega$, and in the case of stable systems, they form a "complete" set of operators. In general situations, though, another two (hence discrete) solutions are necessary to complete the spectrum. In order to show this, we start by considering the function $Z(k, \omega) \equiv \omega^{2}-z(k, \omega)-f^{2}(k)$. Then it follows from Eq. (3.49) that $Z(k, \omega)$ is an analytic function of $\omega$ in the lower half-plane. The additional solutions will be present whenever $Z(k, \omega)$ possesses zeros in the lower half-plane. Let us determine in which cases this will happen. Let $\omega=\omega_{r}-i \omega_{i}$, with $\omega_{i}>0$. Then

$$
\begin{align*}
Z(k, \omega)= & \omega_{r}^{2}-\omega_{i}^{2}-f^{2}(k)+\frac{f(k)}{2} \int_{-\infty}^{\infty} \mathrm{d} \omega^{\prime} \frac{\left(\omega^{\prime}-\omega_{r}\right) \mathfrak{g}\left(k, \omega^{\prime}\right)}{\left(\omega^{\prime}-\omega_{r}\right)^{2}+\omega_{i}^{2}} \\
& -2 i \omega_{r} \omega_{i}\left\{1+f(k) \int_{0}^{\infty} \mathrm{d} \omega^{\prime} \frac{\omega^{\prime} \mid g\left(k,\left.\omega^{\prime}\right|^{2}\right.}{\left[\left(\omega^{\prime}-\omega_{r}\right)^{2}+\omega_{i}^{2}\right]\left[\left(\omega^{\prime}+\omega_{r}\right)^{2}+\omega_{i}^{2}\right]}\right\} . \tag{3.53}
\end{align*}
$$

Notice that the imaginary part of $Z(k, \omega)$ is zero if and only if $\omega_{r}=0$, that is, a zero can only exists on the imaginary axis, where $Z(k, \omega)$ takes the form

$$
\begin{equation*}
Z\left(k,-i \omega_{i}\right)=-\omega_{i}^{2}-f^{2}(k)+f(k) \int_{0}^{\infty} \mathrm{d} \omega^{\prime} \frac{\omega^{\prime}\left|g\left(k, \omega^{\prime}\right)\right|^{2}}{\left(\omega^{\prime}\right)^{2}+\omega_{i}^{2}} . \tag{3.54}
\end{equation*}
$$

Viewed as a function of $\omega_{i}, Z\left(k,-i \omega_{i}\right)$ is strictly decreasing, as $\mathrm{d} Z\left(k,-i \omega_{i}\right) / \mathrm{d} \omega_{i}<0$, and it goes to $-\infty$ as $\omega_{i} \rightarrow \infty$. Thus, its maximum value occurs at $\omega_{i}=0$, and it will be positive if

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \omega^{\prime} \frac{\left|g\left(k, \omega^{\prime}\right)\right|^{2}}{\omega^{\prime}}>f(k) . \tag{3.55}
\end{equation*}
$$

Therefore, a necessary and sufficient condition for the function $Z(k, \omega)=0$ in the lower halfplane is that Eq. (3.55) holds true, and in this case it will have precisely one simple zero on the imaginary axis. Let $-i \Omega_{k}, \Omega_{k}>0$ be this zero. As a by-product of (3.54) (that is a real function on the imaginary axis), we have that $Z^{\prime}\left(k,-i \Omega_{k}\right) \neq 0$ is a pure imaginary number, with the prime denoting the complex derivative with respect to $\omega$. Let us now determine the solutions of the system (3.39a)-(3.39d) corresponding to complex $\omega \equiv \Omega$. Clearly, Eq. (3.40) still holds with $\omega \rightarrow \Omega$. As for Eqs. (3.39c), (3.39d), now $\Omega \mp \omega^{\prime}$ is always a nonzero complex number, and thus the solutions are readily found to be

$$
\begin{align*}
& \alpha\left(k, \Omega, \omega^{\prime}\right)=\frac{1}{\Omega-\omega^{\prime}} \frac{f(k)}{\Omega+f(k)} \bar{g}\left(k, \omega^{\prime}\right) \alpha_{0}(k, \Omega),  \tag{3.56}\\
& \beta\left(k, \Omega, \omega^{\prime}\right)=\frac{1}{\Omega+\omega^{\prime}} \frac{f(k)}{\Omega+f(k)} g\left(k, \omega^{\prime}\right) \alpha_{0}(k, \Omega) . \tag{3.57}
\end{align*}
$$

The substitution of Eqs. (3.56), (3.57) in Eq. (3.39a) results in the constraint

$$
\begin{equation*}
\Omega^{2}-\frac{f(k)}{2} \int \mathrm{~d} \omega^{\prime} \frac{\mathfrak{g}\left(k, \omega^{\prime}\right)}{\Omega-\omega^{\prime}}-f^{2}(k)=0 \tag{3.58}
\end{equation*}
$$

the equation determining the possible solutions for $\Omega$. Notice that if $\operatorname{Im} \Omega<0$, then (3.58) coincides with $Z(k, \Omega)=0$, and thus when such a zero exists, it is on the imaginary axis, and it is the only solution in the lower half-plane, with $\Omega=-i \Omega_{k}$. However, it is not true that $Z\left(k, i \Omega_{k}\right)=0$, because the function $Z(k, \omega)$ is analytic only in the lower half-plane. Nevertheless, Eq. (3.58) does not carry information of where $Z(k, \omega)$ is analytic or not, and the odd character of $\mathfrak{g}(k, \omega)$ implies that if $\Omega$ is a solution, then so is $-\Omega$. Therefore, when $-i \Omega_{k}$ is a zero of $Z(k, \omega)$, Eq. (3.58) admits precisely two solutions, $\pm i \Omega_{k}$. Let the corresponding operators be written as

$$
\begin{align*}
A_{ \pm}(\mathbf{k})=\alpha_{ \pm}(k)\{ & {\left[ \pm i \Omega_{k}+f(k)\right] a(\mathbf{k})+\left[ \pm i \Omega_{k}-f(k)\right] a^{\dagger}(-\mathbf{k}) } \\
& \left.+f(k) \int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[\frac{\bar{g}\left(k, \omega^{\prime}\right)}{ \pm i \Omega_{k}-\omega^{\prime}} a_{\omega^{\prime}}(\mathbf{k})+\frac{g\left(k, \omega^{\prime}\right)}{ \pm i \Omega_{k}+\omega^{\prime}} a_{\omega^{\prime}}^{\dagger}(-\mathbf{k})\right]\right\} . \tag{3.59}
\end{align*}
$$

Direct calculation reveals that $\left[A_{ \pm}(\mathbf{k}), A_{ \pm}\left(\mathbf{k}^{\prime}\right)\right]=\left[A_{ \pm}(\mathbf{k}), A\left(\mathbf{k}^{\prime}, \omega\right)\right]=0$. Also, they clearly satisfy the property $A_{ \pm}^{\dagger}(\mathbf{k})=-\left[\overline{\alpha_{ \pm}}(k) / \alpha_{ \pm}(k)\right] A_{ \pm}(-\mathbf{k})$, from which we obtain the "strange" commutation relation $\left[A_{ \pm}(\mathbf{k}), A_{ \pm}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=0$. Despite this unexpected behavior, the algebras deter-
mined by the operators $A_{ \pm}(\mathbf{k})$ are not independent, as the commutator

$$
\begin{equation*}
\left[A_{+}(\mathbf{k}), A_{-}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}+\mathbf{k}^{\prime}\right) 2 f(k) Z^{\prime}\left(k,-i \Omega_{k}\right) \alpha_{+}(k) \alpha_{-}(k) \tag{3.60}
\end{equation*}
$$

does not vanish. This sort of commutation relation is a fingerprint of unstable quantum systems, as will be made clear in this work. In these cases, we can always find a linear combination of $A_{+}(\mathbf{k})$ and $A_{-}(\mathbf{k})$ satisfying the canonical commutation relations. For instance, let $\sqrt{2} A(\mathbf{k})=A_{+}(\mathbf{k})+i A_{-}(\mathbf{k})$. By setting

$$
\begin{equation*}
\alpha_{ \pm}(k) \equiv \alpha(k)=i \sqrt{\frac{-i}{2 f(k) Z^{\prime}\left(k,-i \Omega_{k}\right)}}, \tag{3.61}
\end{equation*}
$$

it follows that $A_{ \pm}^{\dagger}(\mathbf{k})=A_{ \pm}(-\mathbf{k}),\left[A_{+}(\mathbf{k}), A_{-}\left(\mathbf{k}^{\prime}\right)\right]=i \delta\left(\mathbf{k}+\mathbf{k}^{\prime}\right)$, and thus $\left[A(\mathbf{k}), A^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)$. In what follows, we shall fix the choice (3.61). This completes the solution for the diagonalization problem. As the set $\left\{A(\mathbf{k}, \omega), A_{ \pm}(\mathbf{k})\right\}$ is supposed to be complete, it should be possible to write $a(\mathbf{k})$ and $a_{\omega}(\mathbf{k})$ in terms of $A(\mathbf{k}, \omega), A^{\dagger}(\mathbf{k}, \omega)$ and $A_{ \pm}(\mathbf{k})$. By writing out the general linear expression for the inverse transformations, the commutators between $a(\mathbf{k}), a_{\omega}(\mathbf{k})$ and $A(\mathbf{k}, \omega), A^{\dagger}(\mathbf{k}, \omega), A_{ \pm}(\mathbf{k})$ fix the coefficients, and we find

$$
\begin{align*}
a(\mathbf{k})= & i \alpha(k)\left\{\left[i \Omega_{k}+f(k)\right] A_{+}(\mathbf{k})+\left[i \Omega_{k}-f(k)\right] A_{-}(\mathbf{k})\right\} \\
& +\int \mathrm{d} \omega\left[\overline{\alpha_{0}}(k, \omega) A_{\omega}(\mathbf{k})-\beta_{0}(k, \omega) A_{\omega}^{\dagger}(-\mathbf{k})\right],  \tag{3.62}\\
a_{\omega}(\mathbf{k})= & i \alpha(k) f(k) g(k, \omega)\left[\frac{1}{i \Omega_{k}-\omega} A_{+}(\mathbf{k})+\frac{1}{i \Omega_{k}+\omega} A_{-}(\mathbf{k})\right] \\
& +\int \mathrm{d} \omega^{\prime}\left[\bar{\alpha}\left(k, \omega^{\prime}, \omega\right) A_{\omega^{\prime}}(\mathbf{k})-\beta\left(k, \omega^{\prime}, \omega\right) A_{\omega^{\prime}}^{\dagger}(-\mathbf{k})\right] . \tag{3.63}
\end{align*}
$$

Recall that we must have $\left[a(\mathbf{k}), a^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right),\left[a_{\omega}(k), a_{\omega^{\prime}}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right)$ and all the other commutators vanishing. Note that $\left[a(\mathbf{k}), a\left(\mathbf{k}^{\prime}\right)\right]=0$ holds trivially. For $\left[a(\mathbf{k}), a^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]$ and $\left[a_{\omega}(\mathbf{k}), a_{\omega^{\prime}}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]$ we have

$$
\begin{align*}
{\left[a(\mathbf{k}), a^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=} & \delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)\left\{-\frac{2 i \Omega_{k}}{Z^{\prime}\left(k,-i \Omega_{k}\right)}+\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}\right]\right\},  \tag{3.64}\\
{\left[a_{\omega}(\mathbf{k}), a_{\omega^{\prime}}^{\dagger}\left(\mathbf{k}^{\prime}\right)\right]=} & \delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right)\left\{\frac{f(k)}{2 Z^{\prime}\left(k,-i \Omega_{k}\right)}\left[\frac{g(k, \omega)}{i \Omega_{k}+\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}+\omega^{\prime}}-\frac{g(k, \omega)}{i \Omega_{k}-\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}-\omega^{\prime}}\right]\right. \\
& \left.+\int_{0}^{\infty} \mathrm{d} \omega^{\prime \prime}\left[\bar{\alpha}\left(k, \omega^{\prime \prime}, \omega\right) \alpha\left(k, \omega^{\prime \prime}, \omega^{\prime}\right)-\beta\left(k, \omega^{\prime \prime}, \omega\right) \bar{\beta}\left(k, \omega^{\prime \prime}, \omega^{\prime}\right)\right]\right\} . \tag{3.65}
\end{align*}
$$

Thus, the transformation will be canonical if we verify that

$$
\begin{align*}
I_{1} \equiv & -\frac{2 i \Omega_{k}}{Z^{\prime}\left(k,-i \Omega_{k}\right)}+\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}\right]=1,  \tag{3.66}\\
I_{2} \equiv & \frac{f(k)}{2 Z^{\prime}\left(k,-i \Omega_{k}\right)}\left[\frac{g(k, \omega)}{i \Omega_{k}+\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}+\omega^{\prime}}-\frac{g(k, \omega)}{i \Omega_{k}-\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}-\omega^{\prime}}\right] \\
& +\int_{0}^{\infty} \mathrm{d} \omega^{\prime \prime}\left[\bar{\alpha}\left(k, \omega^{\prime \prime}, \omega\right) \alpha\left(k, \omega^{\prime \prime}, \omega^{\prime}\right)-\beta\left(k, \omega^{\prime \prime}, \omega\right) \bar{\beta}\left(k, \omega^{\prime \prime}, \omega^{\prime}\right)\right]=\delta\left(\omega-\omega^{\prime}\right) . \tag{3.67}
\end{align*}
$$

In order to show $I_{1}=1$, it follows from Eqs. (3.48) and (3.50) that

$$
\begin{equation*}
\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}=f(k) \frac{\omega|g(k, \omega)|^{2}}{\left|\omega^{2}-z(k, \omega)-f^{2}(k)\right|^{2}} . \tag{3.68}
\end{equation*}
$$

Moreover, we see from Eq. (3.49) that $\bar{z}(k, \omega)=z(k,-\bar{\omega})$, and for real and positive $\omega$, $\operatorname{Im} z(k, \omega)=f(k) \pi|g(k, \omega)|^{2} / 2$. Thus, the expansion of (3.68) into partial fractions is

$$
\begin{equation*}
\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}=\frac{\omega}{i \pi}\left[\frac{1}{\omega^{2}-z(k, \omega)-f^{2}(k)}-\frac{1}{\omega^{2}-z(k,-\omega)-f^{2}(k)}\right] . \tag{3.69}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}\right]=\frac{1}{i \pi} \int_{-\infty}^{\infty} \mathrm{d} \omega \frac{\omega}{\omega^{2}-z(k, \omega)-f^{2}(k)} \tag{3.70}
\end{equation*}
$$

Now in Eq. (3.70) we have an integral of a function that is meromorphic on the lower halfplane by construction. Thus it can be solved by means of the Residue Theorem by closing the integration contour in the lower half-plane. It follows that

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \omega\left[\left|\alpha_{0}(k, \omega)\right|^{2}-\left|\beta_{0}(k, \omega)\right|^{2}\right]=1-2 \operatorname{Res}\left[\frac{\omega}{Z(k, \omega)},-i \Omega_{k}\right]=1+\frac{2 i \Omega_{k}}{Z^{\prime}\left(k,-i \Omega_{k}\right)} \tag{3.71}
\end{equation*}
$$

and therefore $I_{1}=1$. As for $I_{2}=\delta\left(\omega-\omega^{\prime}\right)$, Eqs. (3.51), (3.52) imply

$$
\begin{align*}
I_{2}= & \delta\left(\omega-\omega^{\prime}\right)+\frac{f(k)}{2 Z^{\prime}\left(k,-i \Omega_{k}\right)}\left[\frac{g(k, \omega)}{i \Omega_{k}+\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}+\omega^{\prime}}-\frac{g(k, \omega)}{i \Omega_{k}-\omega} \frac{\bar{g}\left(k, \omega^{\prime}\right)}{i \Omega_{k}-\omega^{\prime}}\right] \\
& +\frac{f(k)}{2} \frac{g(k, \omega) \bar{g}\left(k, \omega^{\prime}\right)}{\omega-\omega^{\prime}-i \epsilon}\left[\frac{1}{Z(k, \omega)}-\frac{1}{\bar{Z}\left(k, \omega^{\prime}\right)}\right] \\
& +\frac{f(k)}{4 i \pi} g(k, \omega) \bar{g}\left(k, \omega^{\prime}\right) \int_{-\infty}^{\infty} \mathrm{d} x \frac{1}{Z(k, x)} \frac{1}{(x-\omega+i \epsilon)\left(x-\omega^{\prime}-i \epsilon\right)} \\
& -\frac{f(k)}{4 i \pi} g(k, \omega) \bar{g}\left(k, \omega^{\prime}\right) \int_{-\infty}^{\infty} \mathrm{d} x \frac{1}{\bar{Z}(k, x)} \frac{1}{(x-\omega+i \epsilon)\left(x-\omega^{\prime}-i \epsilon\right)} . \tag{3.72}
\end{align*}
$$

Both integrals can be handled by the same procedure used before. For the first one, by closing the contour of integration in the lower half-plane we shall get contribution from two poles, resulting in

$$
\begin{align*}
\int_{-\infty}^{\infty} \mathrm{d} x \frac{1}{Z(k, x)} \frac{1}{(x-\omega+i \epsilon)\left(x-\omega^{\prime}-i \epsilon\right)}= & \frac{-2 i \pi}{Z^{\prime}\left(k,-i \Omega_{k}\right)\left(i \Omega_{k}+\omega\right)\left(i \Omega_{k}+\omega^{\prime}\right)} \\
& +\frac{-2 i \pi}{\left(\omega-\omega^{\prime}-i \epsilon\right)} \frac{1}{Z(k, \omega)} . \tag{3.73}
\end{align*}
$$

As for the second one, instead of solving it directly, we can solve for its complex conjugate, and take the complex conjugation afterward to obtain

$$
\begin{align*}
\int_{-\infty}^{\infty} \mathrm{d} x \frac{1}{\bar{Z}(k, x)} \frac{1}{(x-\omega+i \epsilon)\left(x-\omega^{\prime}-i \epsilon\right)}= & \frac{-2 i \pi}{Z^{\prime}\left(k,-i \Omega_{k}\right)\left(i \Omega_{k}-\omega\right)\left(i \Omega_{k}-\omega^{\prime}\right)} \\
& +\frac{-2 i \pi}{\left(\omega-\omega^{\prime}-i \epsilon\right)} \frac{1}{\bar{Z}(k, \omega)} . \tag{3.74}
\end{align*}
$$

By gathering all the terms, we obtain $I_{2}=\delta\left(\omega-\omega^{\prime}\right)$. The remaining commutators can be calculated using precisely the same steps, and all of them vanish. Therefore, given a Hamiltonian of the general form (3.35), such that $f(k)>0$ and $|g(k, \omega)|^{2} \neq 0$ for positive $\omega$, we can find a Bogoliubov transformation that diagonalizes the system, and the coefficients are explicitly given. The reported inconsistencies in the literature ${ }^{15,16}$ are simply an indication that more solutions are needed in order to complete the set of operators.

### 3.4 Quantum Langevin Equation

Now we return to the Hamiltonian in Eqs. (3.32) and apply the diagonalization procedure to it. We start with the matter Hamiltonian $H_{\text {matter }}$. In this case, as we shall verify, the system does not have unstable modes. Let $B(\mathbf{k}, \omega)$ denote the new set of annihilation operators. Recall that they satisfy $\left[B(\mathbf{k}, \omega), B^{\dagger}\left(\mathbf{k}^{\prime}, \omega^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right)$. Using the results from the previous section, the operators $B(\mathbf{k}, \omega)$ are writen in terms of the original operators as

$$
\begin{equation*}
B(\mathbf{k}, \omega)=\tilde{\alpha}_{0}(\omega) b(\mathbf{k})+\tilde{\beta}_{0}(\omega) b^{\dagger}(-\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[\tilde{\alpha}\left(\omega, \omega^{\prime}\right) b_{\omega^{\prime}}(\mathbf{k})+\tilde{\beta}\left(\omega, \omega^{\prime}\right) b_{\omega^{\prime}}^{\dagger}(-\mathbf{k})\right] \tag{3.75}
\end{equation*}
$$

and the inverse transform is

$$
\begin{equation*}
b(\mathbf{k})=\int_{0}^{\infty} \mathrm{d} \omega\left[\overline{\tilde{\alpha}}_{0}(\omega) B(\mathbf{k}, \omega)-\tilde{\beta}_{0}(\omega) B^{\dagger}(-\mathbf{k}, \omega)\right], \tag{3.76}
\end{equation*}
$$

$$
\begin{equation*}
b_{\omega}(\mathbf{k})=\int_{0}^{\infty} \mathrm{d} \omega^{\prime}\left[\overline{\tilde{\alpha}}\left(\omega^{\prime}, \omega\right) B\left(\mathbf{k}, \omega^{\prime}\right)-\tilde{\beta}\left(\omega^{\prime}, \omega\right) B^{\dagger}\left(-\mathbf{k}, \omega^{\prime}\right)\right] . \tag{3.77}
\end{equation*}
$$

The Bogoliubov coefficients are explicitly given by

$$
\begin{align*}
& \tilde{\alpha}_{0}(\omega)=\frac{\omega+\tilde{\omega}_{0}}{2} \frac{V(\omega)}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}},  \tag{3.78}\\
& \tilde{\beta}_{0}(\omega)=\frac{\omega-\tilde{\omega}_{0}}{2} \frac{V(\omega)}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}},  \tag{3.79}\\
& \tilde{\alpha}\left(\omega, \omega^{\prime}\right)=\delta\left(\omega-\omega^{\prime}\right)+\frac{\tilde{\omega}_{0}}{2} \frac{\bar{V}\left(\omega^{\prime}\right)}{\omega-\omega^{\prime}-i \epsilon} \frac{V(\omega)}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}},  \tag{3.80}\\
& \tilde{\beta}\left(\omega, \omega^{\prime}\right)=\frac{\tilde{\omega}_{0}}{2} \frac{V\left(\omega^{\prime}\right)}{\omega+\omega^{\prime}} \frac{V(\omega)}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}}, \tag{3.81}
\end{align*}
$$

and

$$
\begin{equation*}
\chi(\omega)=-\frac{\tilde{\omega}_{0}}{2} \int \mathrm{~d} \omega^{\prime} \frac{\mathfrak{V}\left(\omega^{\prime}\right)}{\omega^{\prime}-\omega+i \epsilon}, \tag{3.82}
\end{equation*}
$$

where $\mathfrak{V}(\omega)$ is the odd extension of $|V(\omega)|^{2}$. The condition (3.55) for the existence of unstable solutions with $f(k)=\tilde{\omega}_{0}$ and $g(k, \omega)=V(\omega)=\sqrt{\omega / \tilde{\omega}_{0}} \nu(\omega)$ is found to be

$$
\begin{equation*}
\omega_{0}^{2}<0, \tag{3.83}
\end{equation*}
$$

and thus it is never verified. Therefore, the set $\{B(\mathbf{k}, \omega)\}$ exhausts all possible solutions. Eq. (3.76) can be used to write

$$
\begin{equation*}
\Lambda(k) b(\mathbf{k})+\bar{\Lambda}(k) b^{\dagger}(-\mathbf{k})=\int_{0}^{\infty} \mathrm{d} \omega\left[\bar{\zeta}(k, \omega) B(\mathbf{k}, \omega)+\zeta(k, \omega) B^{\dagger}(-\mathbf{k}, \omega)\right] \tag{3.84}
\end{equation*}
$$

where

$$
\begin{align*}
\zeta(k, \omega) & =\bar{\Lambda}(k) \tilde{\alpha}_{0}(\omega)-\Lambda(k) \tilde{\beta}_{0}(\omega) \\
& =\frac{\nu_{0}+i \omega \nu_{1}}{\sqrt{\tilde{k}}} \frac{\sqrt{\tilde{\omega}_{0}} V(\omega)}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}} \equiv \frac{\xi(\omega)}{\sqrt{\tilde{k}}} . \tag{3.85}
\end{align*}
$$

Eq. (3.84) can be used to cast the total Hamiltonian (after a new normal ordering) as

$$
\begin{align*}
H=\int \mathrm{d}^{3} k & \left\{\tilde{k} a^{\dagger}(\mathbf{k}) a(\mathbf{k})+\int_{0}^{\infty} \mathrm{d} \omega \omega B^{\dagger}(\mathbf{k}, \omega) B(\mathbf{k}, \omega)\right. \\
& \left.+\frac{1}{2} \int_{0}^{\infty} \mathrm{d} \omega\left[\zeta(k, \omega) B^{\dagger}(-\mathbf{k}, \omega)+\bar{\zeta}(k, \omega) B(\mathbf{k}, \omega)\right]\left[a^{\dagger}(\mathbf{k})+a(-\mathbf{k})\right]\right\} \tag{3.86}
\end{align*}
$$

Therefore, the structure of $H$ in terms of $B(\mathbf{k}, \omega)$ is precisely the same as $H_{\text {matter }}$ with the obvious identifications, and we can apply the Fano diagonalization scheme again. Let us first determine whether the coupling $\zeta(k, \omega)$ can trigger instabilities or not. We need to set in Eq. (3.55) $f(k)=\tilde{k}$ and $g(k, \omega)=\zeta(k, \omega)$ to obtain

$$
\begin{equation*}
\tilde{k}^{2}=\mathbf{k}^{2}+v_{1}^{2}<\int_{0}^{\infty} \mathrm{d} \omega \frac{|\xi(\omega)|^{2}}{\omega}=\int_{0}^{\infty} \mathrm{d} \omega \frac{\left(v_{0}^{2}+\omega^{2} v_{1}^{2}\right) \tilde{\omega}_{0}|V(\omega)|^{2}}{\omega\left|\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}\right|^{2}} . \tag{3.87}
\end{equation*}
$$

By noticing that

$$
\begin{equation*}
\frac{\tilde{\omega}_{0}|V(\omega)|^{2}}{\left|\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}\right|^{2}}=\frac{1}{i \pi}\left[\frac{1}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}}-\frac{1}{\omega^{2}-\bar{\chi}(\omega)-\tilde{\omega}_{0}^{2}}\right], \tag{3.88}
\end{equation*}
$$

it follows

$$
\begin{align*}
\mathbf{k}^{2}+v_{1}^{2} & <\frac{v_{0}^{2}}{i \pi} \int_{-\infty}^{\infty} \mathrm{d} \omega \frac{1}{\omega} \frac{1}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}}+\frac{v_{1}^{2}}{i \pi} \int_{-\infty}^{\infty} \mathrm{d} \omega \frac{\omega}{\omega^{2}-\chi(\omega)-\tilde{\omega}_{0}^{2}} \\
& <\frac{v_{0}^{2}}{\chi(0)+\tilde{\omega}_{0}^{2}}+v_{1}^{2}=v_{1}^{2}+\frac{v_{0}^{2}}{\omega_{0}^{2}} \tag{3.89}
\end{align*}
$$

where the integrals were solved using the analytic property of $\chi(\omega)$. Thus, unstable modes exists for $\mathbf{k}^{2}<\left(\nu_{0} / \omega_{0}\right)^{2}=V_{0}$, and if $\nu_{0}=0$, the system is stable, as expected. Let $\left\{C(\mathbf{k}, \omega), C_{ \pm}(\mathbf{k})\right\}$ denote the set of diagonalizing operators. In order to complete the quantization, we only need to know how to express $a(\mathbf{k})$ in terms of the new operators. Thus, from Eq. (3.62) we have

$$
\begin{align*}
a(\mathbf{k})= & i \alpha(k)\left[\left(i \Omega_{k}+\tilde{k}\right) C_{+}(\mathbf{k})+\left(i \Omega_{k}-\tilde{k}\right) C_{-}(\mathbf{k})\right] \\
& +\int \mathrm{d} \omega\left[\overline{\alpha_{0}}(k, \omega) C_{\omega}(\mathbf{k})-\beta_{0}(k, \omega) C_{\omega}^{\dagger}(-\mathbf{k})\right] \tag{3.90}
\end{align*}
$$

where the coefficients are given by

$$
\begin{align*}
& \alpha_{0}(k, \omega)=\frac{\omega+\tilde{k}}{2} \frac{\zeta(k, \omega)}{\omega^{2}-\gamma(\omega)-\tilde{k}^{2}},  \tag{3.91}\\
& \beta_{0}(k, \omega)=\frac{\omega-\tilde{k}}{2} \frac{\zeta(k, \omega)}{\omega^{2}-\gamma(\omega)-\tilde{k}^{2}}  \tag{3.92}\\
& \alpha(k)=i \sqrt{\frac{-i}{2 \tilde{k} Z^{\prime}\left(k,-i \Omega_{k}\right)}},  \tag{3.93}\\
& Z(k, \omega)=\omega^{2}-\gamma(\omega)-\tilde{k}^{2} \tag{3.94}
\end{align*}
$$

$$
\begin{equation*}
\gamma(\omega)=\frac{1}{2} \int \mathrm{~d} \omega^{\prime} \frac{\mathfrak{X}\left(\omega^{\prime}\right)}{\omega-i \epsilon-\omega^{\prime}}, \tag{3.95}
\end{equation*}
$$

with $\mathfrak{X}(\omega)$ being the odd extension of $|\xi(\omega)|^{2}$ and the quantity $\Omega_{k}>0$ is the solution of $Z\left(k,-i \Omega_{k}\right)=0$, with $Z(k, \omega)=\omega^{2}-\gamma(\omega)-\tilde{k}^{2}$. Recall that by construction the properties $\left[C(\mathbf{k}, \omega), C^{\dagger}\left(\mathbf{k}^{\prime}, \omega^{\prime}\right)\right]=\delta\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \delta\left(\omega-\omega^{\prime}\right), C_{ \pm}^{\dagger}(\mathbf{k})=C_{ \pm}(-\mathbf{k})$ and $\left[C_{+}(\mathbf{k}), C_{-}\left(\mathbf{k}^{\prime}\right)\right]=i \delta\left(\mathbf{k}+\mathbf{k}^{\prime}\right)$ are verified. Also, from Eq. (3.89) we have the normalization

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \omega \frac{|\xi(\omega)|^{2}}{\omega}=v_{1}^{2}+\frac{v_{0}^{2}}{\omega_{0}^{2}} \tag{3.96}
\end{equation*}
$$

and we identify the memory function $\hat{\mu}(\omega)$ as

$$
\begin{equation*}
\hat{\mu}(\omega)=\overline{\hat{\gamma}}(\omega) . \tag{3.97}
\end{equation*}
$$

Notice that $\hat{\mu}(\omega)$ is analytic in the upper half-plane and tends to zero as $|\omega| \rightarrow \infty$. Thus, it satisfies the Kramers-Kronig relations. Moreover, in terms of $\hat{\mu}(\omega)$, the derivative $Z^{\prime}\left(k,-i \Omega_{k}\right)$ possesses the nice expression

$$
\begin{equation*}
\frac{-i}{Z^{\prime}\left(k,-i \Omega_{k}\right)}=\operatorname{Res}\left[\frac{i}{\omega^{2}-\hat{\mu}(\omega)-\mathbf{k}^{2}-v_{1}^{2}}, i \Omega_{k}\right] \equiv N_{k}^{2}, \tag{3.98}
\end{equation*}
$$

where $\operatorname{Res}\left[f(\omega), \omega_{0}\right]$ is the residue of $f$ at $\omega_{0}$. With these equations, we can conclude the quantization by recalling that (see Eqs. (3.19) and (3.29))

$$
\begin{align*}
\phi(t, \mathbf{x})= & \frac{1}{\sqrt{(2 \pi)^{3}}} \int \mathrm{~d}^{3} k(2 \tilde{k})^{-1 / 2}\left[a(t, \mathbf{k})+a^{\dagger}(t,-\mathbf{k})\right] \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} \\
= & \frac{1}{\sqrt{2(2 \pi)^{3}}} \int_{0}^{\infty} \mathrm{d} \omega \int \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} \frac{\bar{\xi}(\omega)}{\omega^{2}-\hat{\mu}(\omega)-\mathbf{k}^{2}-v_{1}^{2}} C(t, \mathbf{k}, \omega)+\text { H.c. } \\
& -\frac{1}{\sqrt{(2 \pi)^{3}}} \int_{|\mathbf{k}|^{2}<V_{0}} \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} N_{k}\left[C_{+}(t, \mathbf{k})-C_{-}(t, \mathbf{k})\right], \tag{3.99}
\end{align*}
$$

where we have written the operators time-dependence explicitly, and $V_{0}=\left(\nu_{0} / \omega_{0}\right)^{2}$. This expansion can be expressed in a more "canonical form" by using the fact that the diagonalizing operators are defined through their time-derivatives $\mathrm{d} C(t, \mathbf{k}, \omega) / \mathrm{d} t=-i \omega C(t, \mathbf{k}, \omega)$ and $\mathrm{d} C_{ \pm}(t, \mathbf{k}) / \mathrm{d} t= \pm \Omega_{k} C_{ \pm}(t, \mathbf{k})$. They can be integrated as $C(t, \mathbf{k}, \omega)=\exp (-i \omega t) C(0, \mathbf{k}, \omega)$ and $C_{ \pm}(t, \mathbf{k})=\exp \left( \pm \Omega_{k} t\right) C_{ \pm}(0, \mathbf{k})$, where the initial time operators satisfy the same commutation
relations. Therefore, the final form of the quantized field is

$$
\begin{align*}
\phi(t, \mathbf{x})= & \frac{1}{\sqrt{2(2 \pi)^{3}}} \int_{0}^{\infty} \mathrm{d} \omega \int \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} \frac{\bar{\xi}(\omega)}{\omega^{2}-\hat{\mu}(\omega)-\mathbf{k}^{2}-v_{1}^{2}} C(0, \mathbf{k}, \omega) \mathrm{e}^{-i \omega t}+\text { H.c. } \\
& -\frac{1}{\sqrt{(2 \pi)^{3}}} \int_{|\mathbf{k}|^{2}<V_{0}} \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} N_{k}\left[C_{+}(0, \mathbf{k}) \mathrm{e}^{\Omega_{k} t}-C_{-}(0, \mathbf{k}) \mathrm{e}^{-\Omega_{k} t}\right] . \tag{3.100}
\end{align*}
$$

Let us explore the structure of this expansion in order to establish the underlying effective dynamics. We see that the field is written as a sum of a stable, stationary part, represented by the "oscillatory" terms ( $\phi_{s}$ ), and an active, unstable part, that presents exponential growth as time passes $\left(\phi_{u}\right)$. Notice that when $V_{0} \rightarrow 0$, the instability ceases to exists, as shown by the second integral in (3.100). Moreover, the function $\hat{\mu}(\omega)$ was defined in Eq. (3.97) in such way that the term $v_{1}^{2}$ remained explicitly in the denominators appearing in Eq. (3.100). Clearly, the function $\hat{\mu}(\omega)$ can be defined only up to a constant term, and thus this term could be combined with $\hat{\mu}(\omega)$. Nevertheless, careful readers may have noticed that in order to satisfy the transparency condition of Sec. 3.1, namely, $\hat{\mu} \rightarrow 0$ as $\omega$ goes to infinity, only the choice $\hat{\mu} \equiv \bar{\gamma}$ is possible, what then identifies $v_{1}^{2}$ as the field effective mass squared.

We can show that $\phi_{s}$ is solution of a nonlocal KG equation in the form (3.6) with nonvanishing sources by defining the memory function as

$$
\begin{equation*}
\mu(t)=\frac{1}{2 \pi} \int \mathrm{~d} \omega \mathrm{e}^{-i \omega t} \hat{\mu}(\omega) . \tag{3.101}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi_{s}(t, \mathbf{x})-v_{1}^{2} \phi_{s}(t, \mathbf{x})-\int_{0}^{\infty} \mathrm{d} \tau \mu(\tau) \phi_{s}(t-\tau, \mathbf{x})=\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} \mathrm{d} \omega\left[J_{\omega}(\mathbf{x}) \mathrm{e}^{-i \omega t}+J_{\omega}^{\dagger}(\mathbf{x}) \mathrm{e}^{i \omega t}\right], \tag{3.102}
\end{equation*}
$$

where

$$
\begin{equation*}
J_{\omega}(\mathbf{x})=\frac{\bar{\xi}(\omega)}{\sqrt{2(2 \pi)^{2}}} \int \mathrm{~d}^{3} k C(0, \mathbf{k}, \omega) \mathrm{e}^{i \mathbf{k} \cdot \mathbf{x}} \tag{3.103}
\end{equation*}
$$

Therefore, it is straightforward to show that

$$
\begin{equation*}
\left[J_{\omega}(\mathbf{x}), J_{\omega^{\prime}}^{\dagger}\left(\mathbf{x}^{\prime}\right)\right]=\pi|\xi(\omega)|^{2} \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right)=-2 \operatorname{Im}[\hat{\mu}(\omega)] \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \tag{3.104}
\end{equation*}
$$

This commutation relation shows that these sources are uncorrelated for different frequencies and space position. The $J_{\omega}(\mathbf{x})$ are quantum Langevin operators, responsible to add the
matter degrees of freedom to the quantization. Moreover, the commutation relation (3.104) is an instance of the Fluctuation-Dissipation Theorem, and it ensures that the quantized field satisfies the canonical commutation relations, as pointed out by Milonni. ${ }^{65}$

### 3.5 Final Remarks

We close this chapter by stressing that the most important feature revealed by the quantization just performed is the presence of unstable modes in the field expansion. As stated before, this aspect is well known in nondispersive scenarios, but has never been treated in the presence of dispersion. We have shown that the stationary part of the quantized field (see Eq. (3.100)) is solution of the nonlocal KG equation Eq. (3.102), in perfect analogy to the passive electromagnetic case. Notwithstanding, this cannot be concluded for the unstable part, as the integral in the r.h.s. of Eq. (3.102) is meaningless for such solutions. The understanding of this fact is the main reason why such systems have not yet received the fair treatment, which will be presented rigorously in the next chapter for the electromagnetic field, alongside some interesting consequences. We shall use symmetry arguments to incorporate the unstable solutions to the field expansion, that are going to be associated with the algebra we have presented in Eq. (3.60).

## Chapter 4

## Electromagnetism in active bilayered backgrounds

In this chapter, some of this work's main results are presented. We explore the quantization performed in the previous chapter in its generality, establishing a postulate that, in principle, rules field quantization in arbitrary active materials. We have left this discussion open up to now due to the electromagnetic field experimental appeal. In fact, an intriguing family of examples of systems sustaining this vacuum instability is the one made of plane-symmetric configurations (like the graphene bilayer of Ref. ${ }^{14}$ ). This sort of pattern — dielectric slabs bounded by conducting planes - appears in state-of-the-art solid state devices, whose underlying physics remains not completely understood, e.g. the mechanisms of superconductivity in graphene stacks ${ }^{29}$ and in high-temperature superconductors. ${ }^{30}$ We hope that our results will shed some light onto possible unknown phenomena in such scenarios.

We present the quasi-canonical quantization of the electromagnetic field in unstable bilayer systems. Particularly, we focus on configurations in which the background is composed by two parallel, isotropic, nonmagnetizable and dispersive conducting planes enforcing particular boundary conditions upon the Maxwell field. In addition, we also assume that the systems under study are invariant under the transformation $(t, x, y, z) \rightarrow(-t,-x,-y, z)-$ hereafter denoted by $\mathscr{S}$-symmetry -, where $\{t, x, y, z\}$ are the standard Cartesian coordinates. This symmetry plays a central role in the quantization scheme, as it is essential for completing the quantization and, when the instability is present, it is spontaneously broken after the system stabilization (e.g., due to spontaneous vectorization ${ }^{51}$ ). Field quantization in such configurations cannot be performed with standard methods, as the conducting
sheets exchange energy with the electromagnetic radiation, violating the theory's unitarity. For our purposes, we shall use the method of Langevin noises, presented briefly in the previous chapter (see Refs. ${ }^{21,47,66}$ for further information).

We start the chapter by defining the relevant field equations in the next section, where we adopted the philosophy that in the presence of matter, electromagnetic field equations must be defined "mode-wise" to include unstable solutions as well. In Sec. 4.2 we calculate the field solutions corresponding to the stable part of the field's frequency spectrum, that exhaust all possible solutions in stable configurations. In order to achieve quantization, in Sec. 4.3 we shall add the unstable solutions to the field expansion by invoking symmetry arguments, and we postulate that they will be associated to operators satisfying certain commutation relations. We then verify that the obtained quantum field expansion satisfies the canonical commutation relation, i.e., by recalling that the field canonically conjugated to $\mathbf{A}$ is $\partial_{t} \mathbf{A}$, we shall see that for the built representation, $\left[A_{i}(t, \mathbf{x}), \partial_{t} A_{j}\left(t, \mathbf{x}^{\prime}\right)\right]=i \delta_{i j}^{\perp}\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$ holds, where $\delta_{i j}^{\perp}\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$ is the transverse delta ${ }^{67}$

$$
\begin{equation*}
\delta_{i j}^{\perp}\left(\mathbf{x}-\mathbf{x}^{\prime}\right)=\frac{1}{(2 \pi)^{3}} \int \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \Delta \mathbf{x}}\left(\delta_{i j}-\frac{k_{i} k_{j}}{\mathbf{k}^{2}}\right) . \tag{4.1}
\end{equation*}
$$

In Sec. 4.4 we explore a possible stabilization outcome by revisiting the Casimir effect in such systems, where we present a new sort of quantum levitation, and in Sec. 4.5, initial ideas that relate our results to the emergence of superconducting phenomena are presented. We close the work in Sec. 4.6 with some final discussion.

### 4.1 Field equations in dispersive layered backgrounds

Electromagnetic phenomena in the presence of nonmagnetizable matter at rest is described by Maxwell's equations

$$
\begin{align*}
& \nabla \cdot \mathbf{D}=\rho,  \tag{4.2a}\\
& \nabla \times \mathbf{B}=\frac{\partial \mathbf{D}}{\partial t}+\mathbf{J},  \tag{4.2b}\\
& \nabla \cdot \mathbf{B}=0,  \tag{4.2c}\\
& \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} . \tag{4.2d}
\end{align*}
$$

Here, the vector fields E, B represent the electromagnetic field coming from the Maxwell tensor, and $\mathbf{D}$ encapsulates the interaction field-matter in an effective manner. Thus, if we exclude magnetoelectric materials, ${ }^{24}$ generally this field satisfies the functional dependence $\mathbf{D}=\mathbf{D}(\mathbf{E})$, and it must be given for the system (4.2) to be solvable. In order to find the relation $\mathbf{D}=\mathbf{D}(\mathbf{E})$, first we notice that in general settings, it should be nonlocal, and nonlinear. Nevertheless, if we assume that the involved field intensities are small, and that they vary harmonically as $\mathbf{D}_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x})=\mathbf{D}_{\omega \mathbf{k}_{\perp}}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right), \mathbf{E}_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x})=\mathbf{E}_{\omega \mathbf{k}_{\perp}}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, this dependence is linear, and defined "mode-wise." Notice that we have kept the same symbol to denote the full mode and its $z$-dependence. Hereafter, we shall adopt the convention that if a field solution's spacetime dependence is not shown, then it depends on $(t, \mathbf{x})$. We are interested in the specific case of a material "isotropic and homogeneous by parts" (for simplicity) and composed by two planes at $z=0$ and $z=d$ characterized by longitudinal electrical conductivities $\boldsymbol{\sigma}_{1}\left(\omega, \mathbf{k}_{\perp}\right)$ and $\boldsymbol{\sigma}_{2}\left(\omega, \mathbf{k}_{\perp}\right)$, respectively. This means that electric fields parallel to these planes can excite surface currents on them characterized by the Ohm's law

$$
\begin{equation*}
\mathbf{J}(t, \mathbf{x})=\left[\boldsymbol{\sigma}_{1}\left(\omega, \mathbf{k}_{\perp}\right) \delta(z)+\boldsymbol{\sigma}_{2}\left(\omega, \mathbf{k}_{\perp}\right) \delta(z-d)\right] \cdot \mathbf{E}_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x}) . \tag{4.3}
\end{equation*}
$$

Generally, $\boldsymbol{\sigma}_{i}$ are $3 \times 3$ matrices with vanishing third row and column and can account for transport anisotropies in the conducting planes. In our case, isotropy means that $\boldsymbol{\sigma}_{i}=\sigma_{i}\left(\omega, \mathbf{k}_{\perp}\right) \operatorname{diag}(1,1,0)$. In addition, to further simplify the calculations, let us assume that both conducting planes are surrounded by vacuum. Notice that these excited currents feed the electromagnetic field via the four-current $J^{\mu}=(\rho, \mathbf{J})$ as ruled by Maxwell's equations (4.2), where the charge density is connected with $\mathbf{J}$ through charge conservation. We shall assume that the interaction of the planes with the field is completely described by Eq. (4.3), that is, we are neglecting polarization effects $(\mathbf{D}=\mathbf{E})$. Nevertheless, from a mathematical perspective, as these sources are not free, they can be used to define an effective electrical displacement vector such that the system dynamics is given by source-free Maxwell's equations. In order to show this, suppose we have simply $\mathbf{J}=\sigma \mathbf{E}_{\omega \mathbf{k}_{\perp}}$, and recall that the electromagnetic field possesses a harmonic time-dependence $\mathbf{E}_{\omega \mathbf{k}_{\perp}} \propto \exp (-i \omega t)$. Then charge conservation implies $\rho=\nabla \cdot\left[(-i \sigma / \omega) \mathbf{E}_{\omega \mathbf{k}_{\perp}}\right]$, and the sources can be absorbed into the displacement vector $\mathbf{D}_{\omega \mathbf{k}_{\perp}}=(1+i \sigma / \omega) \mathbf{E}_{\omega \mathbf{k}_{\perp}}$ giving the required "source-free" description. We then have

$$
\begin{equation*}
\mathbf{D}_{\omega \mathbf{k}_{\perp}} \equiv \mathbf{E}_{\omega \mathbf{k}_{\perp}}+\boldsymbol{\chi}\left(\omega, \mathbf{k}_{\perp} ; z\right) \cdot \mathbf{E}_{\omega \mathbf{k}_{\perp}} \equiv \boldsymbol{\varepsilon}\left(\omega, \mathbf{k}_{\perp} ; z\right) \cdot \mathbf{E}_{\omega \mathbf{k}_{\perp}}, \tag{4.4}
\end{equation*}
$$

where the dielectric matrix is

$$
\begin{equation*}
\boldsymbol{\varepsilon}_{i j}=\delta_{i j}+\frac{i \sigma_{1}}{\omega} \delta(z) P_{i j}+\frac{i \sigma_{2}}{\omega} \delta(z-d) P_{i j}, \tag{4.5}
\end{equation*}
$$

and $P=\operatorname{diag}(1,1,0)$ is the projection operator on the $x y$-plane. The $z$-dependence in the matrices $\boldsymbol{\chi}, \boldsymbol{\varepsilon}$ model the associated boundary conditions. It is noteworthy that Eq. (4.4) implies that both temporal and spatial (perpendicularly to the walls) dispersions are being considered. The matrix $\boldsymbol{\chi}\left(\omega, \mathbf{k}_{\perp}\right)$ is such that its extension to complex values of $\omega$ in the upper half-plane is analytic, as required by causality. ${ }^{62}$ Moreover, for real $\omega$, we must have $\bar{\chi}\left(\omega, \mathbf{k}_{\perp}\right)=\chi\left(-\omega,-\mathbf{k}_{\perp}\right)$ for the field equation to be real (recall that a bar over a number means complex conjugation.)

Maxwell's equations (4.2c) and (4.2d) in the absence of free sources are solved in terms of the potential vector - the quantum field - as $\mathbf{B}=\nabla \times \mathbf{A}$ and $\mathbf{E}=-\partial_{t} \mathbf{A}$. For the simple harmonic dependencies $\mathbf{A}=\mathbf{A}_{\omega \mathbf{k}_{\perp}}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, Eq. (4.4) and Eqs. (4.2a), (4.2b) reduce to

$$
\begin{equation*}
-\nabla^{2} \mathbf{A}+\nabla(\nabla \cdot \mathbf{A})=\omega^{2} \boldsymbol{\varepsilon} \cdot \mathbf{A}, \tag{4.6}
\end{equation*}
$$

and the generalized Coulomb gauge condition $\nabla \cdot(\boldsymbol{\varepsilon} \cdot \mathbf{A})=0$. It is also important to notice that Eq. (4.6) implies the generalized Coulomb gauge, and thus it alone faithfully describes the system. Once $\boldsymbol{\varepsilon}$ is given, Eq. (4.6) determines the frequency spectrum of the problem. Let us now find the structure of this spectrum.

### 4.2 Stationary field modes

### 4.2.1 Freely propagating field modes

To quantize a system means that one must know the frequency spectrum of the problem, that fundamentally contains all the information on the photon allowed energies. The first observation we make is that it must contain positive, real frequencies, $\omega>0$, corresponding to free field modes that originates at the asymptotic infinities $z= \pm \infty$. Let $\mathbf{A}$ and $\mathbf{A}^{\prime}$ be two such solutions, corresponding to $\left(\omega, \mathbf{k}_{\perp}\right)$ and $\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime}\right)$, respectively. Performing the trick to find the scalar product from field equations (see Eq. (2.28)), Eq. (4.6) implies

$$
\begin{equation*}
\left(\omega-\omega^{\prime}\right) \int \mathrm{d}^{3} x A_{i}^{\prime} \bar{A}_{j} \frac{\omega^{2} \bar{\varepsilon}_{i j}\left(\omega, \mathbf{k}_{\perp} ; z\right)-\left(\omega^{\prime}\right)^{2} \varepsilon_{i j}\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime} ; z\right)}{\omega-\omega^{\prime}}=0, \tag{4.7}
\end{equation*}
$$

and thus the general solution for this integral defines the scalar product $\left(\mathbf{A}, \mathbf{A}^{\prime}\right)=\alpha\left(\omega, \mathbf{k}_{\perp}\right) \delta(\omega-$ $\left.\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right)$. We use this orthogonality relation to properly normalize the eigenfunctions, and associate to each of then appropriate annihilation operators according to the canonical prescription.

The dielectric matrix, Eq. (4.5), combined with Maxwell's equations, Eqs. (4.6), implies

$$
\begin{align*}
& \left(\mathbf{k}_{\perp}^{2}-\omega^{2}\right) A_{z}+\partial_{z}\left(i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}\right)=0,  \tag{4.8a}\\
& \left(-\partial_{z}^{2}+\mathbf{k}_{\perp}^{2}\right) A_{j}+i k_{j}\left(i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}+\partial_{z} A_{z}\right)=\omega^{2} \varepsilon_{\perp} A_{j}, j=x, y, \tag{4.8b}
\end{align*}
$$

where we have defined the "transverse response function"

$$
\begin{equation*}
\varepsilon_{\perp}=1+\frac{i \sigma_{1}}{\omega} \delta(z)+\frac{i \sigma_{2}}{\omega} \delta(z-d) . \tag{4.9}
\end{equation*}
$$

For this particular case, the gauge condition reduces to

$$
\begin{equation*}
\varepsilon_{\perp} i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}+\partial_{z} A_{z}=0 \tag{4.10}
\end{equation*}
$$

## Transverse electric modes (TE)

One of the two families of solutions to equation (4.8) is such that $A_{z} \equiv 0, A_{x}=i k_{y} \exp (-i \omega t+$ $\left.i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega \mathbf{k}_{\perp}}(z)$, and $A_{y}=-i k_{x} \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) f_{\omega \mathbf{k}_{\perp}}(z)$, for which the gauge condition (4.10) is trivially satisfied, and Eq. (4.8b) implies

$$
\begin{equation*}
\left[-\frac{\mathrm{d}^{2}}{\mathrm{~d} z^{2}}-\kappa^{2}-i \omega \sigma_{1} \delta(z)-i \omega \sigma_{2} \delta(z-d)\right] f_{\omega \mathbf{k}_{\perp}}=0 \tag{4.11}
\end{equation*}
$$

Here, and in order to keep notational consistence throughout the chapter, $\kappa=\kappa\left(\omega, \mathbf{k}_{\perp}\right)$ is defined to be the function of $\omega$ analytic in the upper half-plane given by $\kappa=i \sqrt{\mathbf{k}_{\perp}^{2}-(\omega+i \epsilon)^{2}}$, with $\epsilon$ a small positive number. The important aspect at this point is that $\kappa$ coincides with $\sqrt{\omega^{2}-\mathbf{k}_{\perp}^{2}}$ for real $\omega>k_{\perp}$, and satisfies $\bar{\kappa}\left(\omega, \mathbf{k}_{\perp}\right)=-\kappa\left(-\bar{\omega},-\mathbf{k}_{\perp}\right)$ for complex $\omega$.

We then readily recognize that Equation (4.11) is a Schrödinger-like equation with two deltalike sources at $z=0$ and $z=d$. Integration of Eq. (4.11) with respect to $z$ establishes the known boundary conditions of deltalike potentials, namely, $f_{\omega \mathbf{k}_{\perp}}$ is continuous along the
conducting planes, where its derivative possesses a jump discontinuity of

$$
\begin{equation*}
\left.\partial_{z} f_{\omega \mathbf{k}_{\perp}}\right|_{z=0^{+}}-\left.\partial_{z} f_{\omega \mathbf{k}_{\perp}}\right|_{z=0^{-}}=-\left.i \omega \sigma_{1} f_{\omega \mathbf{k}_{\perp}}\right|_{z=0} \tag{4.12}
\end{equation*}
$$

at $z=0$ and similarly at $z=d$. This problem is solved accordingly to the usual procedure in quantum mechanics. For each $\omega>k_{\perp} \equiv\left|\mathbf{k}_{\perp}\right|$, there exists solutions of Eq. (4.12) originating at $z= \pm \infty$, and freely propagating towards the walls. The first one is given by

$$
\begin{equation*}
\mathbf{A}_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(t, \mathbf{x})=\frac{i \hat{\mathbf{e}}_{1}}{\sqrt{2 \kappa(2 \pi)^{3}}} \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} f_{<, \omega \mathbf{k}_{\perp}}^{\mathrm{TE})}(z), \tag{4.13}
\end{equation*}
$$

where " $<$ " is here used to indicate where the mode originates $(z=-\infty)$, and we have defined the normalized vector field $k_{\perp} \hat{\mathbf{e}}_{1}=\left(k_{y},-k_{x}, 0\right)$, and

$$
f_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(z)=\left\{\begin{array}{ccc}
\mathrm{e}^{i \kappa z}+R_{<}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z} & , & z<0  \tag{4.14}\\
A_{<}^{(\mathrm{TE})} \mathrm{e}^{i \kappa z}+B_{<}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z} & , & 0<z<d, \\
T_{<}^{(\mathrm{TE})} \mathrm{e}^{i \kappa z} & , & d<z
\end{array}\right.
$$

The coefficients in $f_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}$ are better expressed in terms of the single-plate reflexion $\left(\mathscr{R}_{i}\right)$ and transmission coefficients $\left(\mathscr{T}_{i} \equiv 1-\mathscr{R}_{i}\right)$ for the TE modes, with $i$ indexing each wall, and

$$
\begin{equation*}
\mathscr{R}_{i}=\frac{\omega \sigma_{i}}{2 \kappa+\omega \sigma_{i}} . \tag{4.15}
\end{equation*}
$$

Thus,

$$
\begin{align*}
& R_{<}^{(\mathrm{TE})}=-\frac{\mathscr{R}_{1}+\mathscr{R}_{2}\left(\mathscr{T}_{1}-\mathscr{R}_{1}\right) \mathrm{e}^{2 i \kappa d}}{1-\mathscr{R}_{1} \mathscr{R}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.16}\\
& T_{<}^{(\mathrm{TE})}=\frac{\mathscr{T}_{1} \mathscr{T}_{2}}{1-\mathscr{R}_{1} \mathscr{R}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.17}\\
& A_{<}^{(\mathrm{TE})}=-\frac{1}{\mathscr{R}_{2}} B_{<}^{(\mathrm{TE})} \mathrm{e}^{-2 i \kappa d}=\frac{1}{\mathscr{T}_{2}} T_{<}^{(\mathrm{TE})} . \tag{4.18}
\end{align*}
$$

These various coefficients are analytic functions of $\omega$ in the upper half-plane, and satisfies the same reflexion property as $\boldsymbol{\varepsilon}$. For instance, $\overline{R_{<}^{(\mathrm{TE)}}}\left(\omega, \mathbf{k}_{\perp}\right)=R_{<}^{(\mathrm{TE)}}\left(-\bar{\omega},-\mathbf{k}_{\perp}\right)$. The other solutions are the ones originating at $z=\infty$, given by

$$
\begin{equation*}
\mathbf{A}_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(t, \mathbf{x})=\frac{i \hat{\mathbf{e}}_{1}}{\sqrt{2 \kappa(2 \pi)^{3}}} \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} f_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(z) . \tag{4.19}
\end{equation*}
$$

The function $f_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(z)$ reads

$$
f_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}(z)=\left\{\begin{array}{ccc}
T_{>}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z} & , & z<0,  \tag{4.20}\\
A_{>}^{(\mathrm{TE})} \mathrm{e}^{i \kappa z}+B_{>}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z} & , & 0<z<d, \\
\mathrm{e}^{-i \kappa z}+R_{>}^{\mathrm{TE})} \mathrm{e}^{i \kappa z} & , & d<z,
\end{array}\right.
$$

with

$$
\begin{align*}
& R_{>}^{(\mathrm{TE})}=-\frac{\mathscr{R}_{2} \mathrm{e}^{-2 i \kappa d}+\mathscr{R}_{1}\left(\mathscr{T}_{2}-\mathscr{R}_{2}\right)}{1-\mathscr{R}_{1} \mathscr{R}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.21}\\
& T_{>}^{(\mathrm{TE})}=T_{<}^{(\mathrm{TE})},  \tag{4.22}\\
& B_{>}^{(\mathrm{TE})}=-\frac{1}{\mathscr{R}_{1}} A_{>}^{(\mathrm{TE})}=\frac{1}{\mathscr{T}_{1}} T_{>}^{(\mathrm{TE})} \tag{4.23}
\end{align*}
$$

## Transverse magnetic modes (TM)

The second family of solutions is found by using the gauge condition (4.10) in (4.8b) to obtain a decoupled equation for $\mathbf{A}_{\perp}$, that is solved by making $A_{j}=-i k_{j} \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right) g_{\omega \mathbf{k}_{\perp}}(z)$, $j=x, y$, with

$$
\begin{equation*}
\left\{-\frac{\mathrm{d}^{2}}{\mathrm{~d} z^{2}}-\kappa^{2}\left[1+\frac{i \sigma_{1}}{\omega} \delta(z)+\frac{i \sigma_{2}}{\omega} \delta(z-d)\right]\right\} g_{\omega \mathbf{k}_{\perp}}=0 \tag{4.24}
\end{equation*}
$$

The $A_{z}$ component is then fixed by the Equation (4.8a) as $A_{z}=\left(\mathbf{k}_{\perp}^{2} / \kappa^{2}\right) \exp \left(-i \omega t+i \mathbf{k}_{\perp}\right.$. $\left.\mathbf{x}_{\perp}\right) \partial_{z} g_{\omega \mathbf{k}_{\perp}}$. We then recognize, in the same way as for the TE modes, that Eq. (4.24) is a Schrödinger-like equation for $g_{\omega \mathbf{k}_{\perp}}$ with deltalike potentials at $z=0, d$. Thus, $g_{\omega \mathbf{k}_{\perp}}$ is continuous at each plane, such that its derivative possesses a jumplike discontinuity of magnitude

$$
\begin{equation*}
\left.\partial_{z} g_{\omega \mathbf{k}_{\perp}}\right|_{z=0^{+}}-\left.\partial_{z} g_{\omega \mathbf{k}_{\perp}}\right|_{z=0^{-}}=-\left.i \frac{\kappa^{2}}{\omega} \sigma_{1} g_{\omega \mathbf{k}_{\perp}}\right|_{z=0} \tag{4.25}
\end{equation*}
$$

at $z=0$ and similarly at $z=d$. As usual, the frequency spectrum is degenerate, as for each $\omega>$ $k_{\perp}$ there exists two solutions originating at the asymptotic infinities that propagate towards the walls. The first one is found to be

$$
\begin{equation*}
\mathbf{A}_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}(t, \mathbf{x})=\frac{i}{\sqrt{2 \kappa(2 \pi)^{2}}} \hat{\mathbf{e}}_{2}\left(-i \frac{\partial_{z}}{\kappa}\right) \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{<, \omega \mathbf{k}_{\perp}}^{\mathrm{TM})}(z), \tag{4.26}
\end{equation*}
$$

where the vector field

$$
\begin{equation*}
\hat{\mathbf{e}}_{2}(\tau)=\frac{1}{\omega k_{\perp}}\left(-k_{x} \kappa,-k_{y} \kappa, \mathbf{k}_{\perp}^{2} \tau\right), \tag{4.27}
\end{equation*}
$$

was identified, and

$$
g_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM}}(z)=\left\{\begin{array}{ccc}
\mathrm{e}^{i \kappa z}+R_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa z} & , & z<0,  \tag{4.28}\\
A_{<}^{(\mathrm{TM})} \mathrm{e}^{i \kappa z}+B_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa z} & , & 0<z<d, \\
T_{<}^{(\mathrm{TM})} \mathrm{e}^{i \kappa z} & , & d<z
\end{array}\right.
$$

with

$$
\begin{align*}
R_{<}^{(\mathrm{TM})} & =-\frac{\mathscr{F}_{1}+\mathscr{J}_{2}\left(\mathscr{K}_{1}-\mathscr{L}_{1}\right) \mathrm{e}^{2 i \kappa d}}{1-\mathscr{J}_{1} \mathscr{J}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.29}\\
T_{<}^{(\mathrm{TM})} & =\frac{\mathscr{K}_{1} \mathscr{K}_{2}}{1-\mathscr{J}_{1} \mathscr{J}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.30}\\
A_{<}^{(\mathrm{TM})} & =-\frac{1}{\mathscr{J}_{2}} B_{<}^{(\mathrm{TM})} \mathrm{e}^{-2 i \kappa d}=\frac{1}{\mathscr{K}_{2}} T_{<}^{(\mathrm{TM})} . \tag{4.31}
\end{align*}
$$

Here, $\mathscr{J}_{i}$ is the single-plate reflexion coefficient for the TM modes,

$$
\begin{equation*}
\mathscr{F}_{i}=\frac{\kappa \sigma_{i}}{2 \omega+\kappa \sigma_{i}}, \tag{4.32}
\end{equation*}
$$

and $\mathscr{K}_{i}=1-\mathscr{J}_{i}$ the transmission coefficient. As for the solutions originating at $z=\infty$, we find that

$$
\begin{equation*}
\mathbf{A}_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}(t, \mathbf{x})=\frac{i}{\sqrt{2 \kappa(2 \pi)^{3}}} \hat{\mathbf{e}}_{2}\left(-i \frac{\partial_{z}}{\kappa}\right) \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}(z), \tag{4.33}
\end{equation*}
$$

and

$$
g_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM}}(z)=\left\{\begin{array}{ccc}
T_{>}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa z} & , & z<0,  \tag{4.34}\\
A_{>}^{(\mathrm{TM})} \mathrm{e}^{i \kappa z}+B_{>}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa z} & , & 0<z<d, \\
\mathrm{e}^{-i \kappa z}+R_{>}^{(\mathrm{TM})} \mathrm{e}^{i \kappa z} & , & d<z,
\end{array}\right.
$$

with

$$
\begin{align*}
& R_{>}^{(\mathrm{TM})}=-\frac{\mathscr{J}_{2} \mathrm{e}^{-2 i \kappa d}+\mathscr{J}_{1}\left(\mathscr{K}_{2}-\mathscr{J}_{2}\right)}{1-\mathscr{J}_{1} \mathscr{J}_{2} \mathrm{e}^{2 i \kappa d}},  \tag{4.35}\\
& T_{>}^{(\mathrm{TM})}=T_{<}^{(\mathrm{TM})},  \tag{4.36}\\
& B_{>}^{(\mathrm{TM})}=-\frac{1}{\mathscr{J}_{1}} A_{>}^{(\mathrm{TM})}=\frac{1}{\mathscr{K}_{1}} T_{>}^{(\mathrm{TM})} . \tag{4.37}
\end{align*}
$$

These solutions exhaust the positive-frequency part of the spectrum, and are normalized according to

$$
\begin{equation*}
\left(\mathbf{A}_{\alpha, \omega \mathbf{k}_{\perp}}^{(\lambda)}, \mathbf{A}_{\alpha^{\prime}, \omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{\left(\lambda^{\prime}\right)}\right)=\frac{1+\left|R_{\alpha}^{(\lambda)}\right|^{2}+\left|T_{\alpha}^{(\lambda)}\right|^{2}}{2} \delta^{\lambda \lambda^{\prime}} \delta_{\alpha \alpha^{\prime}} \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \tag{4.38}
\end{equation*}
$$

where $\lambda, \lambda^{\prime}=\mathrm{TE}, \mathrm{TM}$, and $\alpha, \alpha^{\prime}=<,>$. In general, $\left|R_{\alpha}^{(\lambda)}\right|^{2}+\left|T_{\alpha}^{(\lambda)}\right|^{2} \neq 1$, which is the mathematical condition expressing the loss of unitarity on the walls. The normalization in Eq. (4.38) takes into account the fact that waves incident on the walls from the asymptotic infinities should coincide with Minkowskian vacuum solutions. These modes enter the quantum field expansion multiplied by annihilation operators $a_{\alpha, \omega \mathbf{k}_{\perp}}^{(\lambda)}$ such that

$$
\begin{equation*}
\left[a_{\alpha, \omega \mathbf{k}_{\perp}}^{(\lambda)}, a_{\alpha^{\prime}, \omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{\left(\lambda^{\prime} \dagger\right.}\right]=\delta^{\lambda \lambda^{\prime}} \delta_{\alpha \alpha^{\prime}} \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) . \tag{4.39}
\end{equation*}
$$

### 4.2.2 Langevin noises

The stationary (freely propagating) field modes must be accompanied by appropriate Langevin operators in the field expansion. The method under consideration consists in completing the space of solutions by addition particular field solutions coming from a Langevin current operator $\mathbf{J}$ that satisfies certain commutation relations (see Eq. (3.102)). Associated to $\mathbf{J}$ is the charge density $\rho$, and they are related through charge conservation, $\nabla \cdot \mathbf{J}+\partial_{t} \rho=0$. Let us present a brief review on how to find the particular solution determined by $\mathbf{J}$ if these sources and solutions vanish at the asymptotic past. In this case, the electric displacement and field are related through

$$
\begin{equation*}
\mathbf{D}(t, \mathbf{x})=\mathbf{E}(t, \mathbf{x})+\int \mathrm{d}^{2} x_{\perp}^{\prime} \mathrm{d} t^{\prime} \tilde{\boldsymbol{\chi}}\left(t^{\prime}, \mathbf{x}_{\perp}^{\prime}\right) \cdot \mathbf{E}\left(t-t^{\prime}, \mathbf{x}_{\perp}-\mathbf{x}_{\perp}^{\prime}, z\right) \tag{4.40}
\end{equation*}
$$

that is just the inverse Fourier transform of Eq. (4.4), and the matrix $\tilde{\boldsymbol{\chi}}\left(t, \mathbf{x}_{\perp}\right)$ vanishes for $t<0$, as demanded by the analytical properties of $\boldsymbol{\chi}\left(\omega, \mathbf{k}_{\perp}\right)$. In order to keep notational simplicity, let us denote the convolution of Eq. (4.40) by a " $*$ ", that is, $\mathbf{D}(t, \mathbf{x})=\mathbf{E}(t, \mathbf{x})+(\tilde{\boldsymbol{\chi}} * \mathbf{E})(t, \mathbf{x})$. In terms of the potential vector, the non-homogeneous Maxwell equations read

$$
\begin{align*}
& \nabla \cdot \partial_{t} \mathbf{A}+\nabla \cdot \partial_{t}(\tilde{\boldsymbol{\chi}} * \mathbf{A})=-\rho  \tag{4.41}\\
& \nabla \times \nabla \times \mathbf{A}+\partial_{t}^{2} \mathbf{A}+\partial_{t}^{2}(\tilde{\boldsymbol{\chi}} * \mathbf{A})=\mathbf{J} \tag{4.42}
\end{align*}
$$

Notice in particular that the divergence of Eq. (4.42) together with charge conservation implies Eq. (4.41). The particular solution of this system can be written in terms of the propagator, defined as the solution of

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{G}\left(x, x^{\prime}\right)+\partial_{t}^{2} \mathbf{G}\left(x, x^{\prime}\right)+\partial_{t}^{2}(\tilde{\boldsymbol{\chi}} * \mathbf{G})\left(x, x^{\prime}\right)=\mathbf{I} \delta\left(x-x^{\prime}\right) \tag{4.43}
\end{equation*}
$$

where $\mathbf{I}$ is the $3 \times 3$ identity matrix. Thus, Eq. (4.42) is solved as

$$
\begin{equation*}
\mathbf{A}(x)=\int \mathrm{d}^{4} x^{\prime} \mathbf{G}\left(x, x^{\prime}\right) \cdot \mathbf{J}\left(x^{\prime}\right) \tag{4.44}
\end{equation*}
$$

Moreover, the divergence of Eq. (4.43) implies

$$
\begin{equation*}
\nabla \cdot \partial_{t} \mathbf{G}\left(x, x^{\prime}\right)+\nabla \cdot \partial_{t}(\tilde{\boldsymbol{\chi}} * \mathbf{G})\left(x, x^{\prime}\right)=\Theta\left(t-t^{\prime}\right) \nabla \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \tag{4.45}
\end{equation*}
$$

$\Theta(t)$ being the Heaviside step function. Eq. (4.45) expresses the fact that A given by Equation (4.44) satisfies (4.41). As usual, the propagator is not uniquely defined by Eq. (4.43) and boundary conditions must be imposed to select a particular solution. ${ }^{34}$ For our purposes, we shall use the causal, retarded propagator. We return to this discussion later on.

Let us now find an explicit form of the propagator. We start with the Fourier decompositions

$$
\begin{align*}
& \mathbf{G}\left(x, x^{\prime}\right)=\frac{1}{(2 \pi)^{3}} \int \mathrm{~d} \omega \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{-i \omega \Delta t+i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}} \hat{\mathbf{G}}\left(\omega, \mathbf{k}_{\perp} ; z, z^{\prime}\right),  \tag{4.46}\\
& \mathbf{J}(x)=\frac{1}{(2 \pi)^{3 / 2}} \int \mathrm{~d} \omega \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \hat{\mathbf{J}}\left(\omega, \mathbf{k}_{\perp} ; z\right), \tag{4.47}
\end{align*}
$$

where the relations $\overline{\hat{\mathbf{G}}}\left(\omega, \mathbf{k}_{\perp} ; z, z^{\prime}\right)=\hat{\mathbf{G}}\left(-\omega,-\mathbf{k}_{\perp} ; z, z^{\prime}\right)$ (and similarly for $\hat{\mathbf{J}}$ ) are observed for the fields to be real. Substitution of Eq. (4.46) into Eq. (4.43) results in the differential equation

$$
\begin{equation*}
\mathscr{D}_{i j} \hat{G}_{j k}-\omega^{2} \chi_{i j} \hat{G}_{j k}=\delta_{i k} \delta\left(z-z^{\prime}\right) \tag{4.48}
\end{equation*}
$$

where the matrix $\mathscr{D}$ is given by

$$
\mathscr{D}=\left(\begin{array}{ccc}
-\partial_{z}^{2}+k_{y}^{2}-\omega^{2} & i k_{x} i k_{y} & i k_{x} \partial_{z}  \tag{4.49}\\
i k_{x} i k_{y} & -\partial_{z}^{2}+k_{x}^{2}-\omega^{2} & i k_{y} \partial_{z} \\
i k_{x} \partial_{z} & i k_{y} \partial_{z} & \mathbf{k}_{\perp}^{2}-\omega^{2}
\end{array}\right)
$$

and $\chi_{i j}=\varepsilon_{i j}-\delta_{i j}$ is given by Eq. (4.5). As we shall verify, this particular functional form of $\chi_{i j}$ is such that Eq. (4.48) can be easily solved in its (equivalent) integral form. In order to obtain such equation, let $\hat{G}_{j k}^{0}\left(\omega, \mathbf{k}_{\perp} ; z, z^{\prime}\right)$ be the "free" solution of Eq. (4.48), that is, the one for $\chi_{i j} \equiv 0$. Thus, one may readily verify that

$$
\begin{equation*}
\hat{G}_{j k}\left(z, z^{\prime}\right)=\hat{G}_{j k}^{0}\left(z, z^{\prime}\right)+\omega^{2} \int \mathrm{~d} z^{\prime \prime} \hat{G}_{j i}^{0}\left(z, z^{\prime \prime}\right) \chi_{i l}\left(z^{\prime \prime}\right) \hat{G}_{l k}\left(z^{\prime \prime}, z^{\prime}\right) \tag{4.50}
\end{equation*}
$$

is a solution of Eq. (4.48) by acting on it with $\mathscr{D}_{i j}$. Notice that Eq. (4.50), with $\chi_{i j}$ of equation (4.6), is an algebraic equation for the propagator in terms of the free solution. The only drawback in solving it is that the matrix basis used to expand $\hat{G}_{i j}$ is function of the points $z, z^{\prime}$. The physical principle behind it is that reflected "transverse magnetic" modes are such that their polarization is slightly different from the incident one, and this mechanism must be included in the propagator. Nevertheless, it can be exactly solved for the present case. We start from the free propagator, that possesses the integral representation

$$
\begin{equation*}
\hat{G}_{j l}^{0}\left(z, z^{\prime}\right)=\frac{1}{2 \pi} \int \mathrm{~d} k_{z} \frac{\mathrm{e}^{i k_{z} \Delta z}}{\mathbf{k}^{2}-(\omega+i \epsilon)^{2}}\left[\delta_{j l}-\frac{k_{j} k_{l}}{(\omega+i \epsilon)^{2}}\right] . \tag{4.51}
\end{equation*}
$$

The small $\epsilon>0$ is added to the integral to select the retarded propagator. It should be stressed that it is also necessary in order for the free propagator to satisfy Eq. (4.45). Eq. (4.51) can be integrated as

$$
\begin{equation*}
\hat{G}_{j k}^{0}\left(z, z^{\prime}\right)=\hat{D}^{0}\left(z, z^{\prime}\right) \hat{e}_{1 j} \hat{e}_{1 k}-\frac{1}{(\omega+i \epsilon)^{2}} \delta(\Delta z) \delta_{j z} \delta_{k z}+\hat{D}^{0}\left(z, z^{\prime}\right) \hat{e}_{2 j}(\operatorname{sgn}(\Delta z)) \hat{e}_{2 k}(\operatorname{sgn}(\Delta z)) \tag{4.52}
\end{equation*}
$$

where $\hat{D}^{0}\left(z, z^{\prime}\right)$ is the "scalar" propagator

$$
\begin{equation*}
\hat{D}^{0}\left(z, z^{\prime}\right)=\frac{1}{2 \pi} \int \mathrm{~d} k_{z} \frac{\mathrm{e}^{i k_{z} \Delta z}}{\mathbf{k}^{2}-(\omega+i \epsilon)^{2}}=-\frac{\mathrm{e}^{i \kappa|\Delta z|}}{2 i \kappa} . \tag{4.53}
\end{equation*}
$$

Here, $\operatorname{sgn}(0)=0$, and $\operatorname{sgn}^{2}(z) \equiv 1$ by convention, and the vectors appearing in Eq. (4.52) coincide with the polarization vectors of the TE and TM modes with $\omega \rightarrow \omega+i \epsilon$. Notice that in all cases, their dependence on $\tau$ will only be used flip the sign of the $z$ component, and only the vector fields

$$
\begin{equation*}
\hat{\mathbf{e}}_{1}=\frac{1}{k_{\perp}}\left(k_{y},-k_{x}, 0\right), \tag{4.54}
\end{equation*}
$$

$$
\begin{align*}
& \hat{\mathbf{e}}_{2}^{( \pm)}=\frac{1}{(\omega+i \epsilon) k_{\perp}}\left(-k_{x} \kappa,-k_{y} \kappa, \pm \mathbf{k}_{\perp}^{2}\right),  \tag{4.55}\\
& \hat{\mathbf{e}}_{3}^{( \pm)}=\frac{1}{(\omega+i \epsilon)}\left(k_{x}, k_{y}, \pm \kappa\right) . \tag{4.56}
\end{align*}
$$

appear in the field expansion. In general, these vectors are orthogonal for each set with fixed "+" or "-" and satisfy the completeness relation

$$
\begin{equation*}
\hat{e}_{1 i} \hat{e}_{1 j}+\hat{e}_{2 i}^{( \pm)} \hat{e}_{2 j}^{( \pm)}+\hat{e}_{3 i}^{( \pm)} \hat{e}_{3 j}^{( \pm)}=\delta_{i j} \tag{4.57}
\end{equation*}
$$

The structure of the free propagator, Eq. (4.52), shows that there exists a solution for $\hat{G}_{i j}$ of the same form

$$
\begin{equation*}
\hat{G}_{i j}\left(z, z^{\prime}\right)=\hat{G}^{(\mathrm{TE})}\left(z, z^{\prime}\right) \hat{e}_{1 i} \hat{e}_{1 j}+\hat{G}_{i j}^{(\mathrm{TM})}\left(z, z^{\prime}\right) \tag{4.58}
\end{equation*}
$$

where $0=\hat{e}_{1 i} \hat{G}_{i j}^{(\mathrm{TM})}, 0=\hat{e}_{1 j} \hat{G}_{i j}^{(\mathrm{TM})}$. Thus, we find that

$$
\begin{equation*}
\hat{G}^{(\mathrm{TE})}\left(z, z^{\prime}\right)=\hat{D}^{0}\left(z, z^{\prime}\right)+i \omega \sigma_{1} \hat{D}^{0}(z, 0) \hat{G}^{(\mathrm{TE})}\left(0, z^{\prime}\right)+i \omega \sigma_{2} \hat{D}^{0}(z, d) \hat{G}^{(\mathrm{TE})}\left(d, z^{\prime}\right) \tag{4.59}
\end{equation*}
$$

and

$$
\begin{align*}
\hat{G}_{j k}^{(\mathrm{TM})}\left(z, z^{\prime}\right)= & \hat{G}_{j k}^{0(\mathrm{TM})}\left(z, z^{\prime}\right)+i \omega \sigma_{1} \hat{G}_{j l}^{0(\mathrm{TM})}(z, 0) P_{l m} \hat{G}_{m k}^{(\mathrm{TM})}\left(0, z^{\prime}\right) \\
& +i \omega \sigma_{2} \hat{G}_{j l}^{0(\mathrm{TM})}(z, d) P_{l m} \hat{G}_{m k}^{(\mathrm{TM})}\left(d, z^{\prime}\right) . \tag{4.60}
\end{align*}
$$

Both equations are solved by making $z=0$ and $z=d$ and after some lengthy manipulations we find that

$$
\begin{align*}
\hat{G}^{(\mathrm{TE})}\left(z, z^{\prime}\right)= & \hat{D}^{0}\left(z, z^{\prime}\right)+\frac{2 i \kappa}{\mathscr{P}}\left\{\mathscr{R}_{1} \hat{D}^{0}(z, 0) \hat{D}^{0}\left(0, z^{\prime}\right)+\mathscr{R}_{2} \hat{D}^{0}(z, d) \hat{D}^{0}\left(d, z^{\prime}\right)\right. \\
& \left.-\mathscr{R}_{1} \mathscr{R}_{2} \mathrm{e}^{i \kappa d}\left[\hat{D}^{0}(z, 0) \hat{D}^{0}\left(d, z^{\prime}\right)+\hat{D}^{0}(z, d) \hat{D}^{0}\left(0, z^{\prime}\right)\right]\right\}, \tag{4.61}
\end{align*}
$$

and

$$
\begin{align*}
\hat{G}_{j k}^{(\mathrm{TM})}\left(z, z^{\prime}\right)=\hat{G}_{j k}^{0(\mathrm{TM})}\left(z, z^{\prime}\right) & +\frac{2 i \kappa}{\mathscr{Q}} \hat{e}_{2 j}\left(\frac{\partial_{z}}{i \kappa}\right) \hat{e}_{2 k}\left(\frac{i \partial_{z^{\prime}}}{\kappa}\right)\left\{\mathscr{J}_{1} \hat{D}^{0}(z, 0) \hat{D}^{0}\left(0, z^{\prime}\right)+\mathscr{J}_{2} \hat{D}^{0}(z, d) \hat{D}^{0}\left(d, z^{\prime}\right)\right. \\
& \left.-\mathscr{F}_{1} \mathscr{J}_{2} \mathrm{e}^{i \kappa d}\left[\hat{D}^{0}(z, 0) \hat{D}^{0}\left(d, z^{\prime}\right)+\hat{D}^{0}(z, d) \hat{D}^{0}\left(0, z^{\prime}\right)\right]\right\} . \tag{4.62}
\end{align*}
$$

Thus, this propagator is used to find the particular field solution of Eq. (4.42). It is instructive
to separate the particular solution in its positive and negative-frequency parts

$$
\begin{equation*}
A_{i}^{\text {noise }}(t, \mathbf{x})=\frac{1}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \int_{0}^{\infty} \mathrm{d} \omega \mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \int \mathrm{d} z^{\prime} \hat{G}_{i j}\left(\omega, \mathbf{k}_{\perp} ; z, z^{\prime}\right) \hat{J}_{j}\left(\omega, \mathbf{k}_{\perp} ; z^{\prime}\right)+\text { H.c. } \tag{4.63}
\end{equation*}
$$

and H.c. stands for Hermitian conjugate. The quantization is achieved by supposing that the operator $\mathbf{J}$ represents noises that are uncorrelated for different $\left(\omega, \mathbf{k}_{\perp}, z\right)$ and $\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime}, z^{\prime}\right)$, and thus we must have (the Fluctuation-Dissipation Theorem)

$$
\begin{equation*}
\left[\hat{J}_{i}\left(\omega, \mathbf{k}_{\perp} ; z\right), \hat{J}_{j}^{\dagger}\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime} ; z^{\prime}\right)\right]=2 \omega^{2} \operatorname{Im}\left\{\varepsilon_{i j}\left(\omega, \mathbf{k}_{\perp}, z\right)\right\} \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \delta\left(z-z^{\prime}\right) \tag{4.64}
\end{equation*}
$$

In general, the constant appearing in the commutator of Eq. (4.64) is fixed by Eq. (4.43), and its determination is straightforward when the propagator is sufficiently symmetric. ${ }^{47,68}$ For instance, if there is no spatial dispersion, the system solutions satisfy the Lorentz reciprocity theorem, which is used to prove Eq. (4.64). In our case, the problem seems to be intricate, and we shall assume it to hold as well without presenting a proof. The validity of this choice will be established once we show that it ensures that the quantum field satisfies the canonical commutation relations. We shall explore this expansion in what follows.

### 4.3 Bound solutions

Let us now study instances where this system's vacuum also contains unstable bound states. As already mentioned in the introduction, this aspect was studied for several similar systems, including the case composed by two parallel graphene sheets with one of them carrying a dc current, where classical solutions were found. In addition, bound solutions for deltalike mirrors, characterized by $\omega<k_{\perp}$ corresponding to modes trapped inside the mirror, are also known. ${ }^{69}$ In the present case, we are interested in how unstable solutions modify the quantum vacuum. Therefore, the case of interest here is when the system admits bound solutions with $\operatorname{Im} \omega>0$, and thus these are solutions grow exponentially with time $[\exp (-i \omega t)]$. In order to accommodate these modes in the field expansion, we proceed as follows, by postulating that the bound states are $\mathscr{S}$-symmetric. And by this we mean that the field equations satisfied by those modes are invariant under this transformation, in the sense that if we have a given solution, its transformation under the $\mathscr{S}$ operation is also a solution.

### 4.3.1 $\mathscr{S}$-symmetry

Let us establish the meaning of this operation on the stable (real $\omega$ ) modes first. The symmetry properties of the electromagnetic field under the transformation $\mathscr{S}$ can be deduced easily from the invariance of the interaction term $J^{\mu} A_{\mu}$ between field and sources under general conditions. As we are interested in electric neutral scenarios, invariance of the term $\mathbf{J} \cdot \mathbf{A}$ under the transformation $\mathscr{S}$ means that the field transforms as $\mathbf{A}_{\perp}\left(t, \mathbf{x}_{\perp}, z\right) \xrightarrow{\mathscr{S}} \mathbf{A}_{\perp}\left(-t,-\mathbf{x}_{\perp}, z\right)$ and $A_{z}\left(t, \mathbf{x}_{\perp}, z\right) \xrightarrow{\mathscr{S}}-A_{z}\left(-t,-\mathbf{x}_{\perp}, z\right)$. Let $\mathbf{A}(t, \mathbf{x})=\operatorname{Re}\left\{\mathbf{A}^{+}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)\right\}$ be a solution of the field equations. Then it transforms under $\mathscr{S}$ to $\mathbf{A}^{-}(t, \mathbf{x})=\operatorname{Re}\left\{\mathbf{A}^{-}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)\right\}$, where

$$
\begin{align*}
& \mathbf{A}_{\perp}^{-}(z)=\overline{\mathbf{A}_{\perp}^{+}}(z),  \tag{4.65a}\\
& A_{z}^{-}(z)=-\overline{A_{z}^{+}}(z) . \tag{4.65b}
\end{align*}
$$

Thus, $\mathscr{S}$-invariance means that $\mathbf{A}^{-}(t, \mathbf{x})$ must also be a solution of the system. The equation satisfied by $\mathrm{A}^{-}$is

$$
\begin{align*}
& \left(\mathbf{k}_{\perp}^{2}-\omega^{2}\right) A_{z}^{-}+\partial_{z}\left(i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}^{-}\right)=0  \tag{4.66a}\\
& \left(-\partial_{z}^{2}+\mathbf{k}_{\perp}^{2}\right) A_{j}^{-}+i k_{j}\left(i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}^{-}+\partial_{z} A_{z}^{-}\right)=\omega^{2} \bar{\varepsilon}_{\perp} A_{j}^{-}, \quad j=x, y \tag{4.66b}
\end{align*}
$$

together with $\bar{\varepsilon}_{\perp} i \mathbf{k}_{\perp} \cdot \mathbf{A}_{\perp}^{-}+\partial_{z} A_{z}^{-}=0$. Notice that this is simply Eq. (4.6) with $\overline{\boldsymbol{\varepsilon}}$ replacing $\boldsymbol{\varepsilon}$. For our purposes, this is the definition of the $\mathscr{S}$-operation acting on stable Fourier modes: (the real part of) $\mathbf{A}^{+}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$ is mapped to (the real part of) $\mathbf{A}^{-}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, with $\mathbf{A}^{-}(z)$ solution of Eqs. (4.66).

Let us return to the case where $\mathbf{A}(t, \mathbf{x})=\operatorname{Re}\left\{\mathbf{A}^{+}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)\right\}$ is a field mode with $\operatorname{Im} \omega>0$. Application of the operation $\mathscr{S}$ on it results in a field mode $\mathbf{A}^{-}(t, \mathbf{x})=\operatorname{Re}\left\{\mathbf{A}^{-}(z) \exp \left(-i \bar{\omega} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)\right\}$ with $\mathbf{A}^{-}(z)$ satisfying the system (4.66) with the corresponding complex $\bar{\omega}$. Thus, for each bound, unstable solution, $\mathscr{S}$-invariance implies that there should correspond a $\mathscr{S}$-symmetric one. Particularly, it is noteworthy that this symmetry operation is not directly related to complex conjugation in the general case, as usually stated in quantum theory textbooks. This analysis shows that when a $\mathscr{S}$-symmetric system possesses bound, growing solutions, the field expansion cannot be composed of such modes alone, for the corresponding vacuum would not be $\mathscr{S}$-invariant, the case under study here.

In this case, it is necessary to add the $\mathscr{S}$-symmetric solutions to the expansion, that is, the solutions $\mathbf{A}^{-}(t, \mathbf{x})=\mathbf{A}^{-}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, with $\mathbf{A}^{-}(z)$ and also $\omega$ being determined by (4.66). We shall use these modes to complete the field spectrum.

For each $\mathbf{k}_{\perp}$, let the "plus" solutions be indexed by $\lambda$ as $\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(z) \exp \left(-i \Omega_{\mathbf{k}_{\perp} \lambda}^{+} t+\right.$ $\left.i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, with $\Omega_{\mathbf{k}_{\perp} \lambda}^{+}$denoting the complex frequency. Then, as the field must be real, the reflection properties of $\boldsymbol{\varepsilon}$ imply that if $\Omega_{\mathbf{k}_{\perp} \lambda}^{+}$is a solution, then so is $-\overline{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}}$. As this solution coincides with one labeled by $-\mathbf{k}_{\perp}$, we shall take the labeling to be compatible with the property $\Omega_{-\mathbf{k}_{\perp} \lambda}^{+}=-\overline{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}}$. For each such solution, we should also add to the expansion the solutions of the system (4.66) with $\operatorname{Im} \omega<0$, denoted by $\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{-}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{-}(z) \exp \left(-i \Omega_{\mathbf{k}_{\perp} \lambda}^{-} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, with $\Omega_{\mathbf{k}_{\perp} \lambda}^{-}$being its frequency, indexed according to $\overline{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}}=\Omega_{\mathbf{k}_{\perp} \lambda}^{-}$.

### 4.3.2 Unstable field solutions

The "active" field modes are calculated following exactly the same lines as was done for the propagating modes, and in order to find the "plus" modes, we only need to solve equations (4.11) and (4.24) imposing Dirichlet boundary conditions at the asymptotic infinities to pick out the bounded solutions. We then find for the TE modes

$$
\begin{align*}
& \mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(z) \mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp} \lambda}^{+} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}},  \tag{4.67a}\\
& \mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(z)=\frac{i \hat{\mathbf{e}}_{1}}{2 \pi} \mathscr{N}_{\mathbf{k}_{\perp} \lambda}^{+} f_{\mathbf{k}_{\perp} \lambda}^{+}(z), \tag{4.67b}
\end{align*}
$$

where

$$
f_{\mathbf{k}_{\perp} \lambda}^{+}(z)=\left\{\begin{array}{ccc}
\mathscr{T}_{1}^{+} \mathrm{e}^{-i \kappa^{+} z} & , & z<0,  \tag{4.68}\\
-\mathscr{R}_{1}^{+} \mathrm{e}^{i \kappa^{+} z}+\mathrm{e}^{-i \kappa^{+} z} & , \quad 0<z<d, \\
-\mathscr{R}_{1}^{+} \mathscr{T}_{2}^{+} \mathrm{e}^{i \kappa^{+} z} & , \quad d<z,
\end{array}\right.
$$

with the " + " sign added to $\mathscr{R}_{i}, \mathscr{T}_{i}$ and $\kappa$ to stress that these coefficients are evaluated at $\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}, \mathbf{k}_{\perp}\right)$, and the complex frequencies are defined implicitly through

$$
\begin{equation*}
\mathscr{P} \equiv 1-\mathscr{R}_{1}^{+} \mathscr{R}_{2}^{+} \mathrm{e}^{2 i \kappa^{+} d}=0 . \tag{4.69}
\end{equation*}
$$

Here, $\mathscr{N}_{\mathbf{k}_{\perp} \lambda}^{+}$is the normalization constant, that will be defined later. For the TM modes, we denote the complex frequencies by $\Sigma_{\mathbf{k}_{\perp} \lambda}^{+}$. We find that

$$
\begin{equation*}
\mathbf{C}_{\mathbf{k}_{\perp} \lambda}^{+}(t, \mathbf{x})=\mathbf{C}_{\mathbf{k}_{\perp} \lambda}^{+}(z) \mathrm{e}^{-i \Sigma_{\mathbf{k}_{\perp} \lambda}^{+} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \tag{4.70a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{C}_{\mathbf{k}_{\perp} \lambda}^{+}(z)=\frac{i}{2 \pi} \hat{\mathbf{e}}_{2}\left(-i \frac{\partial_{z}}{\kappa^{+}}\right) \mathscr{M}_{\mathbf{k}_{\perp} \lambda}^{+} g_{\mathbf{k}_{\perp} \lambda}^{+}(z), \tag{4.70b}
\end{equation*}
$$

where

$$
g_{\mathbf{k}_{\perp} \lambda}^{+}(z)=\left\{\begin{array}{cc}
\mathscr{K}_{1}^{+} \mathrm{e}^{-i \kappa^{+} z} & , \quad z<0,  \tag{4.71}\\
-\mathscr{J}_{1}^{+} \mathrm{e}^{i \kappa^{+} z}+\mathrm{e}^{-i \kappa^{+} z} & , \quad 0<z<d, \\
-\mathscr{J}_{1}^{+} \mathcal{K}_{2}^{+} \mathrm{e}^{i \kappa^{+} z} & , \quad d<z,
\end{array}\right.
$$

$\mathscr{M}_{\mathbf{k}_{\perp} \lambda}^{+}$is a constant, and the frequencies are given by the zeros

$$
\begin{equation*}
\mathscr{Q} \equiv 1-\mathscr{J}_{1}^{+} \mathscr{J}_{2}^{+} \mathrm{e}^{2 i \kappa^{+} d}=0 \tag{4.72}
\end{equation*}
$$

We have kept the same notation $\kappa^{+}$for both modes, and care should be taken when dealing with the full field expansion. The functions $\mathscr{P}$ and $\mathscr{Q}$, for each $\mathbf{k}_{\perp}$ fixed, are analytic functions of $\omega$ in the upper half-plane, and as such, their zeros are isolated (in the topological sense).

The "minus" field modes can be calculated directly from their defining Equations (4.66). However, a particularly important property can be inferred from the reflexivity $\overline{\boldsymbol{\varepsilon}}\left(\omega, \mathbf{k}_{\perp}\right)=$ $\boldsymbol{\varepsilon}\left(-\bar{\omega},-\mathbf{k}_{\perp}\right)$, whose application to Eqs. (4.66) implies that these modes possess a solution in the form

$$
\begin{align*}
& \mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{-}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{-}(z) \mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp} \lambda}^{-} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}},  \tag{4.73a}\\
& \mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{-}(z)=i \mathbf{A}_{-\mathbf{k}_{\perp} \lambda}^{+}(z), \tag{4.73b}
\end{align*}
$$

for the TE modes, and

$$
\begin{align*}
& \mathbf{C}_{\mathbf{k}_{\perp} \lambda}^{-}(t, \mathbf{x})=\mathbf{C}_{\mathbf{k}_{\perp} \lambda}^{-}(z) \mathrm{e}^{-i \Sigma_{\mathbf{k}_{\perp}}^{-} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}},  \tag{4.74a}\\
& C_{\mathbf{k}_{\perp} \lambda, j}^{-}(z)=i C_{-\mathbf{k}_{\perp} \lambda, j}^{+}(z), \quad j=x, y,  \tag{4.74b}\\
& C_{\mathbf{k}_{\perp} \lambda, z}^{-}(z)=-i C_{-\mathbf{k}_{\perp} \lambda, z}^{+}(z), \tag{4.74c}
\end{align*}
$$

for the TM field modes.

Let us discuss how to normalize these modes. Recall that these norms should not depend on time. The problem then consists in finding a scalar product such that this can be achieved. Let $\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}(z) \exp \left(-i \Omega_{\mathbf{k}_{\perp} \lambda}^{+} t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right), \mathbf{A}_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}(t, \mathbf{x})=\mathbf{A}_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}(z) \exp \left(-i \Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-} t+\right.$ $i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}$ ) denote two such solutions, TE or TM. The manipulation of the field equations (4.6)
implies that

$$
\begin{equation*}
\int \mathrm{d}^{3} x A_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}, i}^{-}(t, \mathbf{x}) A_{\mathbf{k}_{\perp} \lambda, j}^{+}(t, \mathbf{x})\left[\left(\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)^{2} \bar{\varepsilon}_{i j}\left(\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{+}, \mathbf{k}_{\perp}^{\prime}\right)-\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}\right)^{2} \varepsilon_{i j}\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}, \mathbf{k}_{\perp}\right)\right]=0, \tag{4.75}
\end{equation*}
$$

holds in general. Notice that in Eq. (4.75) the integral in $\mathbf{x}_{\perp}$ produces the delta $\delta\left(\mathbf{k}_{\perp}+\mathbf{k}_{\perp}^{\prime}\right)$ and the remaining integral will be multiplied by the time-dependent factor $\exp \left[-i t\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}+\right.\right.$ $\left.\left.\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)\right]$. Consider now the multiplication of the integral in Eq. (4.75) by $\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}+\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)^{-1}$

$$
\begin{equation*}
\frac{1}{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}+\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}} \int \mathrm{d}^{3} x A_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}, i}^{-}(t, \mathbf{x}) A_{\mathbf{k}_{\perp} \lambda, j}^{+}(t, \mathbf{x})\left[\left(\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)^{2} \bar{\varepsilon}_{i j}\left(\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{+}, \mathbf{k}_{\perp}^{\prime}\right)-\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}\right)^{2} \varepsilon_{i j}\left(\Omega_{\mathbf{k}_{\perp} \lambda}^{+}, \mathbf{k}_{\perp}\right)\right] . \tag{4.76}
\end{equation*}
$$

Thus, for $\mathbf{k}_{\perp}^{\prime} \neq-\mathbf{k}_{\perp}$, or $\mathbf{k}_{\perp}^{\prime}=-\mathbf{k}_{\perp}$ and $\lambda \neq \lambda^{\prime}$, the integral is zero, independent of time. Therefore, it must be equal to

$$
\begin{equation*}
\mathrm{e}^{-i t\left(\Omega_{\mathbf{k}_{\perp} \lambda^{\prime}}^{+}+\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)} h\left(\mathbf{k}_{\perp}, \mathbf{k}_{\perp}^{\prime}\right) \delta_{\lambda \lambda^{\prime}} \delta\left(\mathbf{k}_{\perp}+\mathbf{k}_{\perp}^{\prime}\right), \tag{4.77}
\end{equation*}
$$

for some function $h\left(\mathbf{k}_{\perp}, \mathbf{k}_{\perp}^{\prime}\right)$. In particular, its support is contained in the diagonal ( $\left.\mathbf{k}_{\perp}^{\prime}, \lambda^{\prime}\right)=$ $\left(-\mathbf{k}_{\perp}, \lambda\right)$, where $\Omega_{\mathbf{k}_{\perp} \lambda}^{+}+\Omega_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}=\Omega_{\mathbf{k}_{\perp} \lambda}^{+}+\Omega_{-\mathbf{k}_{\perp} \lambda}^{-}=0$. Thus, Eq. (4.76) is independent of time and can be taken as a legitimate sesquilinear form. In order to determine the function $h\left(\mathbf{k}_{\perp},-\mathbf{k}_{\perp}\right)$, we shall assume that $\mathbf{k}_{\perp}^{\prime}=-\mathbf{k}_{\perp}-\eta \mathbf{q}$, where $\mathbf{q}$ is some unit vector in the plane $k_{x} k_{y}$ such that $\boldsymbol{\varepsilon}\left(\omega, \mathbf{k}_{\perp}+\eta \mathbf{q}\right)$ does not depend on $\eta$ near $\eta=0$. Then, by using the property $\overline{\boldsymbol{\varepsilon}}\left(\omega, \mathbf{k}_{\perp}\right)=\boldsymbol{\varepsilon}\left(-\bar{\omega},-\mathbf{k}_{\perp}\right)$, and making $\eta \rightarrow 0$ we obtain that

$$
\begin{equation*}
h\left(\mathbf{k}_{\perp},-\mathbf{k}_{\perp}\right)=-\left.(2 \pi)^{2} \int \mathrm{~d} z A_{-\mathbf{k}_{\perp} \lambda, i}^{-} A_{\mathbf{k}_{\perp} \lambda, j}^{+}\left[\partial_{\omega} \omega^{2} \varepsilon_{i j}\left(\omega, \mathbf{k}_{\perp}\right)\right]\right|_{\omega=\Omega_{\mathbf{k}_{\perp} \lambda}^{+}} . \tag{4.78}
\end{equation*}
$$

In the case where such vector $\mathbf{q}$ does not exist, additional terms will be present. In our case, we shall consider only cases where it is possible to find such $\mathbf{q}$. We shall use this sesquilinear form to normalize the modes as

$$
\begin{equation*}
\left(\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{+}, \mathbf{A}_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right)_{\text {bounded }}=i \delta_{\lambda \lambda^{\prime}} \delta\left(\mathbf{k}_{\perp}+\mathbf{k}_{\perp}^{\prime}\right) \tag{4.79}
\end{equation*}
$$

Notice that this still leaves a multiplicative degree of freedom in the normalized solutions, as happens for canonical quantization in general.

The explicit solutions (4.67), (4.70) and the $\mathscr{S}$-symmetric counterparts (4.73) and (4.74) can be inserted in the sesquilinear form (4.79) in order to fix the normalization constants, and we find, after some lengthy simplifications, that

$$
\begin{align*}
& \mathscr{N}_{\mathbf{k}_{\perp} \lambda}^{+}=\frac{1}{\left(2 i \Omega_{\mathbf{k}_{\perp} \lambda}^{+} \lambda^{\frac{1}{2}} \mathscr{T}_{1}^{+}\right.}\left\{\operatorname{Res}\left[\frac{\omega}{\kappa} R_{<}^{(\mathrm{TE})}, \Omega_{\mathbf{k}_{\perp} \lambda}^{+}\right]\right\}^{\frac{1}{2}},  \tag{4.80}\\
& \mathscr{M}_{\mathbf{k}_{\perp} \lambda}^{+}=\frac{1}{\left(2 i \Sigma_{\mathbf{k}_{\perp} \lambda}^{+} \lambda^{\frac{1}{2}} \mathcal{K}_{1}^{+}\right.}\left\{\operatorname{Res}\left[\frac{\omega}{\kappa} R_{<}^{(\mathrm{TM})}, \Sigma_{\mathbf{k}_{\perp} \lambda}^{+}\right]\right\}^{\frac{1}{2}}, \tag{4.81}
\end{align*}
$$

where $\operatorname{Res}\left[f(\omega), \omega_{0}\right]$ denotes the residue of the function $f$ at the point $\omega=\omega_{0}$. In determining Eqs. (4.80) and (4.81) we assumed that the poles of $R_{<}^{(\lambda)}, \lambda \in\{T E, T M\}$, at the complex frequencies are simple. We further restrict the validity of our analysis to such cases, and leave as a conjecture that these exhaust all the physically relevant cases.

We now arrived at the core of this quantization. As we have demonstrated in the previous chapter, these (normalized) unstable modes enter the field expansion multiplied by operators $a_{\mathbf{k}_{\perp} \lambda}^{ \pm}$. As the property $\mathbf{A}_{\mathbf{k}_{\perp} \lambda}^{ \pm}=\mathbf{A}_{-\mathbf{k}_{\perp} \lambda}^{ \pm}$holds classically, these operators should satisfy the "reality condition" $\left(a_{\mathbf{k}_{\perp} \lambda}^{ \pm}\right)^{\dagger}=a_{-\mathbf{k}_{\perp} \lambda}^{ \pm}$. Then we state the fundamental postulate:

$$
\begin{equation*}
\left[a_{\mathbf{k}_{\perp} \lambda}^{+}, a_{\mathbf{k}_{\perp}^{\prime} \lambda^{\prime}}^{-}\right]=i \delta_{\lambda \lambda^{\prime}} \delta\left(\mathbf{k}_{\perp}+\mathbf{k}_{\perp}^{\prime}\right), \tag{4.82}
\end{equation*}
$$

with the commutator being compatible with the sesquilinear form (4.79). Notice that this is simply an alternative way of expressing the canonical commutation relation satisfied by creation/annihilation operators. In fact, if we define $\sqrt{2} a_{\mathbf{k}_{\perp} \lambda}=a_{\mathbf{k}_{\perp} \lambda}^{+}+i a_{\mathbf{k}_{\perp} \lambda}^{-}$, then $\left[a_{\mathbf{k}_{\perp} \lambda}, a_{\mathbf{k}_{\perp} \lambda^{\prime}}^{\dagger}\right]=$ $\delta_{\lambda \lambda^{\prime}} \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right)$. This can be used to write the field expansion in a more conventional form and thus identify the theory's vacuum state in this $\mathscr{S}$-symmetric representation.

### 4.3.3 Canonical commutation relation

Now we synthesize the calculated field modes in the field expansion and show how the canonical commutation relations are satisfied once we add the unstable solutions following the prescription just elaborated. In order to keep things simpler, let us focus on the region $z \leq 0$. We start with the noises. Notice that Eq. (4.64), for $\varepsilon_{i j}$ defined in Eq. (4.5), is solved by defining

$$
\begin{equation*}
\hat{J}_{i}\left(\omega, \mathbf{k}_{\perp} ; z\right)=\hat{f}_{1 i}\left(\omega, \mathbf{k}_{\perp}\right) \delta(z)+\hat{f}_{2 i}\left(\omega, \mathbf{k}_{\perp}\right) \delta(z-d) \tag{4.83}
\end{equation*}
$$

where the operators $f_{i j}\left(\omega, \mathbf{k}_{\perp}\right)$ satisfy the commutation relation

$$
\begin{equation*}
\left[\hat{f}_{i j}\left(\omega, \mathbf{k}_{\perp}\right), \hat{f}_{l k}^{\dagger}\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime}\right)\right]=\omega\left(\sigma_{i}+\bar{\sigma}_{i}\right) \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) P_{i l} P_{j k} . \tag{4.84}
\end{equation*}
$$

Equation (4.83) can be used to find an explicit form for the particular solution,

$$
\begin{align*}
A_{i}^{\mathrm{noise}}(t, \mathbf{x})= & \frac{i}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \int_{0}^{\infty} \mathrm{d} \omega \frac{\mathrm{e}^{-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}-i \kappa z}}{2 \kappa}\left\{\hat{e}_{1 i} \hat{e}_{1 j}\left[\left(1+R_{<}^{(\mathrm{TE})}\right) \hat{f}_{1 j}+\mathrm{e}^{i \kappa d} T_{<}^{(\mathrm{TE})} \hat{f}_{2 j}\right]\right. \\
& \left.-\hat{e}_{2 i}^{(-)} \frac{k_{\perp j}}{k_{\perp}} \frac{\kappa}{\omega}\left[\left(1+R_{<}^{(\mathrm{TM})}\right) \hat{f}_{1 j}+\mathrm{e}^{i \kappa d} T_{<}^{(\mathrm{TM})} \hat{f}_{2 j}\right]\right\}+ \text { H.c.. } \tag{4.85}
\end{align*}
$$

These field modes should be complemented with the freely propagating solutions. The region in which we are analyzing the field expansion, $z \leq 0$, is composed by field modes that propagate towards and away from the wall at $z=0$. In order to avoid extremely lengthy equations, let us treat separately the TE and TM parts of the field expansion. For the TE polarization, the photons propagating towards the wall are represented by the annihilation operator $a_{r, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} \equiv a_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}$ satisfying the commutation relation (4.39). The photons propagating away from the wall are made from a reflected part of the incident photons from the left, by a transmitted portion coming from incident photons on the right of the wall $\left[a_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}\right]$, and also by the noises in the particular solution (4.85). By gathering all these contributions, and also the bound field modes, the transverse electric part of the full quantized field can be cast as

$$
\begin{align*}
A_{i}^{(\mathrm{TE})}(t, \mathbf{x})= & \frac{i}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \hat{e}_{1 i} \int_{k_{\perp}}^{\infty} \mathrm{d} \omega \frac{\mathrm{e}^{-i \omega t}}{\sqrt{2 \kappa}}\left[a_{r, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} \mathrm{e}^{i \kappa z}+a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z}\right] \\
& +\frac{i}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \hat{e}_{1 i} \int_{0}^{k_{\perp}} \mathrm{d} \omega \frac{\mathrm{e}^{-i \omega t}}{2 \kappa} F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa z}+\text { H.c. } \\
& +\int \mathrm{d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \sum_{\lambda}\left[a_{\mathbf{k}_{\perp} \lambda}^{+} A_{\mathbf{k}_{\perp} \lambda, i}^{+}(z) \mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp} \lambda}^{+} t}+a_{\mathbf{k}_{\perp} \lambda}^{-} A_{\mathbf{k}_{\perp} \lambda, i}^{-}(z) \mathrm{e}^{-i \Omega_{\mathbf{k}_{\perp} \lambda}^{-} t}\right], \tag{4.86}
\end{align*}
$$

where

$$
\begin{equation*}
a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}=a_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} R_{<}^{(\mathrm{TE})}+a_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})} T_{>}^{(\mathrm{TE})}+\frac{1}{\sqrt{2 \kappa}} F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}, \tag{4.87}
\end{equation*}
$$

and the operator $F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}$ satisfies

$$
\begin{align*}
{\left[F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}, F_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TE}) \dagger}\right]=} & 2 \bar{\kappa} \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right)\left\{\Theta\left(k_{\perp}-\omega\right)\left[R_{<}^{(\mathrm{TE})}-\overline{R_{<}^{(\mathrm{TE})}}\right]\right. \\
& \left.+\Theta\left(\omega-k_{\perp}\right)\left[1-\left|R_{<}^{(\mathrm{TE})}\right|^{2}-\left|T_{>}^{(\mathrm{TE})}\right|^{2}\right]\right\} . \tag{4.88}
\end{align*}
$$

This commutation relation ensures that

$$
\begin{equation*}
\left[a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}, a_{\ell, \omega^{\prime} \mathbf{k}_{\perp}^{\prime}}^{(\mathrm{TE}) \dagger}\right]=\delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \tag{4.89}
\end{equation*}
$$

as it should occur. Notice also that $a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TE})}$ is correlated with the rightward-propagating photons. This field expansion can be used to calculate the TE component of the canonical commutation relation, and we find that

$$
\begin{align*}
& {\left[A_{i}^{(\mathrm{TE})}(t, \mathbf{x}), \partial_{t} A_{j}^{(\mathrm{TE}) \dagger}\left(t, \mathbf{x}^{\prime}\right)\right]=\frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \Delta \mathbf{x}} \hat{e}_{1 i} \hat{e}_{1 j}} \\
& +\frac{1}{(2 \pi)^{2}} \int \mathrm{~d}^{2} k_{\perp} \cos \left(\mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}\right) \hat{e}_{1 i} \hat{e}_{1 j}\left\{\frac{-1}{2 \pi i} \int_{-\infty}^{\infty} \mathrm{d} \omega \frac{\omega}{\kappa} R_{<}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}\right. \\
& \left.+\sum_{\lambda} \operatorname{Res}\left[\frac{\omega}{\kappa} R_{<}^{(\mathrm{TE})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}, \Omega_{\mathbf{k}_{\perp} \downarrow}^{+}\right]\right\} . \tag{4.90}
\end{align*}
$$

as expected. The same steps can be followed for the TM part, except that now care must be taken with the polarization vectors, that depend on the "direction" of propagation. We shall denote the annihilation operators corresponding to photons propagating towards the wall by $a_{r, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} \equiv a_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}$, and the photons propagating away from the wall add together to the operator $a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}$ given by

$$
\begin{equation*}
a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}=a_{<, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} R_{<}^{(\mathrm{TM})}+a_{>, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} T_{>}^{(\mathrm{TM})}+\frac{1}{\sqrt{2 \kappa}} F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}, \tag{4.91}
\end{equation*}
$$

where

$$
\begin{align*}
& {\left[F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})}, F_{\omega^{\prime} \mathbf{k}_{\perp}^{\prime} \dagger}^{(\mathrm{TM})}\right]=2 \kappa \delta\left(\omega-\omega^{\prime}\right) \delta\left(\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right)\left\{\Theta\left(k_{\perp}-\omega\right)\left[R_{<}^{(\mathrm{TM})}-\overline{R_{<}^{(\mathrm{TM})}}\right]\right.} \\
& \left.\quad+\Theta\left(\omega-k_{\perp}\right)\left[1-\left|R_{<}^{(\mathrm{TM})}\right|^{2}-\left|T_{>}^{(\mathrm{TM})}\right|^{2}\right]\right\} . \tag{4.92}
\end{align*}
$$

The transverse magnetic part of the quantized field then reads

$$
\begin{align*}
A_{i}^{(\mathrm{TM})}(t, \mathbf{x})= & \frac{i}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \int_{k_{\perp}}^{\infty} \mathrm{d} \omega \frac{\mathrm{e}^{-i \omega t}}{\sqrt{2 \kappa}}\left[a_{r, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} \hat{e}_{2 i}^{(+)} \mathrm{e}^{i \kappa z}+a_{\ell, \omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} \hat{e}_{2 i}^{(-)} \mathrm{e}^{-i \kappa z}\right] \\
& +\frac{i}{(2 \pi)^{3 / 2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \int_{0}^{k_{\perp}} \mathrm{d} \omega \frac{\mathrm{e}^{-i \omega t}}{2 \kappa} F_{\omega \mathbf{k}_{\perp}}^{(\mathrm{TM})} \hat{e}_{2 i}^{(-)} \mathrm{e}^{-i \kappa z}+\text { H.c. } \\
& +\int \mathrm{d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \sum_{\lambda}\left[c_{\mathbf{k}_{\perp} \lambda}^{+} C_{\mathbf{k}_{\perp} \lambda, i}^{+}(z) \mathrm{e}^{\left.-i \sum_{\mathbf{k}_{\perp} \lambda}^{+}+c_{\mathbf{k}_{\perp} \lambda}^{-} C_{\mathbf{k}_{\perp} \lambda, i}^{-}(z) \mathrm{e}^{-i \Sigma_{\mathbf{k}_{\perp} \lambda}^{-} t}\right],}\right. \tag{4.93}
\end{align*}
$$

and the canonical commutator can be cast as

$$
\begin{align*}
{\left[A_{i}^{(\mathrm{TM})}(t, \mathbf{x}), \partial_{t} A_{j}^{(\mathrm{TM}) \dagger}\left(t, \mathbf{x}^{\prime}\right)\right]=} & \frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} k \mathrm{e}^{i \mathbf{k} \cdot \Delta \mathbf{x}} \hat{e}_{2 i}^{(+)} \hat{e}_{2 j}^{(+)} \\
+ & \frac{1}{(2 \pi)^{2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}}\left\{\frac{-1}{2 \pi i} \int_{-\infty}^{\infty} \mathrm{d} \omega \hat{e}_{2 i}^{(-)} \hat{e}_{2 j}^{(+)} \frac{\omega}{2 \kappa} R_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}\right. \\
& \left.+\sum_{\lambda} \operatorname{Res}\left[\hat{e}_{2 i}^{(-)} \hat{e}_{2 j}^{(+)} \frac{\omega}{2 \kappa} R_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}, \Sigma_{\mathbf{k}_{\perp} \lambda}^{+}\right]\right\} \\
+ & \frac{1}{(2 \pi)^{2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{-i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}}\left\{\frac{-1}{2 \pi i} \int_{-\infty}^{\infty} \mathrm{d} \omega \hat{e}_{2 i}^{(+)} \hat{e}_{2 j}^{(-)} \frac{\omega}{2 \kappa} R_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}\right. \\
& \left.+\sum_{\lambda} \operatorname{Res}\left[\hat{e}_{2 i}^{(+)} \hat{e}_{2 j}^{(-)} \frac{\omega}{2 \kappa} R_{<}^{(\mathrm{TM})} \mathrm{e}^{-i \kappa\left(z+z^{\prime}\right)}, \Sigma_{\mathbf{k}_{\perp} \lambda}^{+}\right]\right\} . \tag{4.94}
\end{align*}
$$

In the first line, the vector $\hat{\mathbf{e}}_{2}^{(+)}$is given by $\omega \hat{\mathbf{e}}_{2}^{(+)}=\left(-k_{x} k_{z},-k_{y} k_{z}, \mathbf{k}_{\perp}^{2}\right)$, where $\omega=|\mathbf{k}|$, and this integral combines with the first term of Eq. (4.90) to reproduce the transverse delta of Eq. (4.1). All the remaining terms cancel exactly by means of the Residue Theorem by closing the countour of integration in the upper half-plane, with the only observations that all poles are simple, and that the vectors $\hat{e}_{2 i}^{( \pm)} \hat{e}_{2 j}^{(\mp)}$ do not add poles at $\omega=0$ (see Eq. (4.55)). We thus have concluded the field quantization, and the postulate of Sec. 4.3 is justified.

### 4.3.4 Vacuum polarization and propagators

In order to start exploring the physical content of the theory under study, we need to calculate a couple of propagators from the quantized field. Recall that the propagators of the theory can be written in terms of the (positive) two-point Wightman function, defined to be the vacuum expectation value $G_{i j}^{+}\left(x, x^{\prime}\right)=\left\langle A_{i}(x) A_{j}\left(x^{\prime}\right)\right\rangle$. The "natural" vacuum state of the theory, $|0\rangle$, is defined as usual, being the only nonvanishing state lying in the kernel of all the annihilation operators. The effect of the operators $a_{\mathbf{k}_{\perp} \lambda}^{ \pm}, c_{\mathbf{k}_{\perp} \lambda}^{ \pm}$on it is determined once they are expressed as

$$
\begin{align*}
& a_{\mathbf{k}_{\perp} \lambda}^{+}=\frac{1}{\sqrt{2}}\left(a_{\mathbf{k}_{\perp} \lambda}+a_{-\mathbf{k}_{\perp} \lambda}^{\dagger}\right),  \tag{4.95}\\
& a_{\mathbf{k}_{\perp} \lambda}^{-}=\frac{1}{i \sqrt{2}}\left(a_{\mathbf{k}_{\perp} \lambda}-a_{-\mathbf{k}_{\perp} \lambda}^{\dagger}\right), \tag{4.96}
\end{align*}
$$

and similarly for $c_{\mathbf{k}_{\perp} \lambda}^{ \pm}$, where $a_{\mathbf{k}_{\perp} \lambda}|0\rangle, c_{\mathbf{k}_{\perp} \lambda}|0\rangle=0$. The relevant propagators for our purposes are the Hadamard function and the retarded propagator, defined in terms of the Wightman
function as (clearly, for a bosonic, real field) ${ }^{34}$

$$
\begin{align*}
& G_{i j}^{(1)}\left(x, x^{\prime}\right)=\operatorname{Re} G_{i j}^{+}\left(x, x^{\prime}\right),  \tag{4.97}\\
& G_{R, i j}\left(x, x^{\prime}\right)=2 \Theta\left(t-t^{\prime}\right) \operatorname{Im} G_{i j}^{+}\left(x, x^{\prime}\right), \tag{4.98}
\end{align*}
$$

respectively. The Hadamard function appears when one is interested in study vacuum polarization effects, like Casimir forces. Direct calculation of the Hadamard function reveals that it naturally splits as

$$
\begin{equation*}
G_{i j}^{(1)}\left(x, x^{\prime}\right)=G_{0, i j}^{(1)}\left(x, x^{\prime}\right)+G_{\text {Ren }, i j}^{(1)}\left(x, x^{\prime}\right), \tag{4.99}
\end{equation*}
$$

where the sub-index 0 stands for free space, and Ren for renormalized. Suppose that the system configuration do admit unstable solutions. Thus, its vacuum state will fluctuate around its zero value $(\langle\mathbf{A}\rangle=0)$ with squared standard deviation $\left\langle A_{i}^{2}\right\rangle \equiv G_{\text {Ren, }, i i}^{(1)}(x, x) / 2$. This fluctuation behaves as

$$
\begin{equation*}
\left\langle A_{i}^{2}\right\rangle \approx \frac{1}{2} \int \mathrm{~d}^{2} k_{\perp} \sum_{\lambda}\left|A_{\mathbf{k}_{\perp} \lambda, i}^{+}(z)\right|^{2} \mathrm{e}^{2 t \operatorname{Im}\left\{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}\right\}}+\frac{1}{2} \int \mathrm{~d}^{2} k_{\perp} \sum_{\lambda}\left|C_{\mathbf{k}_{\perp} \lambda, i}^{+}(z)\right|^{2} \mathrm{e}^{2 t \operatorname{Im}\left\{\Sigma_{\mathbf{k}_{\perp}{ }^{\prime}}^{+}\right\}}, \tag{4.100}
\end{equation*}
$$

as $t \gg \min _{\mathbf{k}_{\perp}, \lambda}\left\{\left(\operatorname{Im}\left\{\Omega_{\mathbf{k}_{\perp} \lambda}^{+}\right\}\right)^{-1},\left(\operatorname{Im}\left\{\Sigma_{\mathbf{k}_{\perp} \lambda}^{+}\right\}\right)^{-1}\right\}$, and thus it grows exponentially with time. Clearly, similar expressions hold true for the fluctuations of the electric and magnetic fields. We shall return to the study of vacuum polarization in the next sections, where some interesting consequences of this phenomenon are explored.

It is noteworthy that the retarded propagator defined in Eq. (4.98) does not coincide with the causal propagator used to add the noises to the field quantization, solution of Eq. (4.43). The reason for it is that they do not satisfy the same equation. In fact, the field equation, in the general setting, is defined for each harmonic mode as $\nabla \times \nabla \times \mathbf{A}+\partial_{t}^{2} \boldsymbol{\varepsilon}\left(\omega, \mathbf{k}_{\perp}\right) \cdot \mathbf{A}=0$ (see Eq. (4.6)), where $\mathbf{A}(t, \mathbf{x})=\mathbf{A}(z) \exp \left(-i \omega t+i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)$, and this equation can also admit solutions possessing exponential growth/decay, that cannot be represented as Fourier transforms/convolutions. Thus, the quantum field is not solution of the homogeneous part of Eq. (4.41). Moreover, if we consider the field equation (4.6) acting "mode-wise" on (4.98), it follows from the canonical commutation relation that

$$
\begin{equation*}
\left(-\delta_{i j} \nabla^{2}+\partial_{i} \partial_{j}+\partial_{t}^{2} \varepsilon_{i j}\right) G_{R, j k}\left(x, x^{\prime}\right)=\delta\left(t-t^{\prime}\right) \delta_{i k}^{\perp}\left(\mathbf{x}-\mathbf{x}^{\prime}\right), \tag{4.101}
\end{equation*}
$$

as expected.
We close the field quantization with some comments on the origin of the literature's socalled inconsistency in such active theories, when the propagator of Eq. (4.48), as function of $\omega$, possesses poles in the upper half-plane. As was shown, the existence of such poles is a requirement for the theory to be consistently quantized not only for passive configurations, but generally. The alleged inconsistency relies on the fact that not all solutions to the field equation can be found by assuming that the field can be expressed as a Fourier transform.

### 4.4 Casimir effect revisited: quantum levitation

The most natural consequence of the quantization just performed, at least from a philosophical perspective, is related to vacuum polarization near the conducting walls when instability is present. As already mentioned, this configuration (when the walls are perfect conductors) is known to give rise to attractive Casimir forces. When unstable modes are present, we expect that these Casimir forces will be eventually dominated by the additional contribution from the unstable field solutions. In this section we explore this effect.

Our analysis thus far assumes that the walls can sustain instability. We know that this can be achieved at least for graphene sheets with one of them carrying a dc current. ${ }^{14}$ Let us show that this is generic, and can be implemented under "simple" assumptions. Recall that the system modifying the electromagnetic vacuum consists of two plane and parallel conducting walls at $z=0, d$, such that the plane at $z=0$ is supporting a constant current density $\mathbf{J}$ in the $x$ direction, as depicted in Fig. 4.1. In order to find explicit examples, we


Figure 4.1 - Schematics of the system configuration.
Source: By the author.
must provide the in-plane conductivities $\sigma_{i}\left(\omega, \mathbf{k}_{\perp}\right), i=1,2$. For the plane without current,
we shall assume the Drude form

$$
\begin{equation*}
\sigma_{2}\left(\omega, \mathbf{k}_{\perp}\right)=\frac{\sigma_{0}}{1-i \tau \omega} \tag{4.102}
\end{equation*}
$$

where $\sigma_{0}$ is the dc conductivity and $\tau$ is the the sample relaxation time. As for the plane with the current, the electrical conductivity is modified by this charge flux. As we are assuming an electric neutral scenario, this current is characterized by the four vector $J^{\mu}=(0, \mathbf{J})$. Notice that as this vector is spacelike, there do not exist a Lorentz frame in which the spatial current vanishes -where we could indeed write down the conductivity. However, let us assume that the major electrical response is caused by one type of charge carrier, e.g., the electrons. This means that we can find a Lorentz frame in which these carriers are at rest and thus in this frame the spatial current is maintained by positively charged particles moving in the opposite direction. Therefore, in this frame, despite the existence of a charge flux, the electrical response can be taken as in Eq. (4.102). Thus, if we know how conductivities in these different Lorentz frames are related, then the problem is done. However, this is not as simple as one would expect. How dielectric functions transform under Lorentz boosts is known since $1910,{ }^{52}$ but the problem remains not completely understood until now. In fact, these transformations were recently reconsidered ${ }^{70}$ and new aspects were unveiled. For our purposes, we only need to use the fact that this system is covariant under Lorentz boosts parallel to the plane, and that under these transformations, the in-plane dielectric function (4.5) is scalar. This means that for a boost in the $x$-direction with velocity $v$ such that in the new frame we have a rest frame electrical response, we should have

$$
\begin{equation*}
\frac{\sigma_{1}\left(\omega, \mathbf{k}_{\perp}\right)}{\omega}=\frac{\sigma_{1}^{\prime}\left(\omega^{\prime}, \mathbf{k}_{\perp}^{\prime}\right)}{\omega^{\prime}} \approx \frac{\sigma_{1}^{\prime}\left(\omega-k_{x} v, \mathbf{k}_{\perp}\right)}{\omega-k_{x} v} \tag{4.103}
\end{equation*}
$$

to leading order in powers of $v$, where the prime "'" refers to the quantity in the transformed frame. Thus, if we assume that in this new frame the response has the same functional form of Eq. (4.102), we have

$$
\begin{equation*}
\sigma_{1}\left(\omega, \mathbf{k}_{\perp}\right)=\frac{\omega}{\omega-v k_{x}} \sigma_{2}\left(\omega-v k_{x}, \mathbf{k}_{\perp}\right) . \tag{4.104}
\end{equation*}
$$

This same result was demonstrated for graphene sheets ${ }^{14}$ by solving an Hamiltonian, in a rather sophisticated approach. Here we showed through (not so simple) symmetry arguments that it must hold for all materials as well if a preferred Lorentz frame exists.

In this kind of analysis, it is customary to further simplify the equations by looking for
solutions with $|\omega| \ll\left|v k_{x}\right|$. Notice that under this assumption, the conductivity (4.104) is essentially a degree 1 polynomial, and the equations defining the bound solutions - Eqs. (4.69) and (4.72) — are polynomials of degrees 3 and 2, respectively. In the Fig. 4.2 we present the imaginary part of the calculated unstable frequencies for TE modes. The parameters we


Figure 4.2 - Imaginary part of unstable TE frequencies. The solid line is the solution found exactly within the approximation $|\omega| \ll \nu k_{x}$, and the dots correspond to numerically calculated frequencies without approximations. The parameters are $\sigma_{0}=0.01 \mathrm{~S} / \mathrm{m}, v=10 \mathrm{~m} / \mathrm{s}, \tau=10^{-12} \mathrm{~s}$ and $d=5 \mathrm{~nm}$.
Source: By the author.
have chosen correspond to an ordinary semiconductor at ambient temperature. It should be stressed that this instability is robust, and quite generic. In fact, if the parameters were to correspond to a high-mobility material like graphene ( $\nu=c / 600$ ), or to a good conductor $\sigma_{0} \sim 10^{7} \mathrm{~S} / \mathrm{m}$, we would have a stronger instability. This phenomenon was linked a long time ago to the weakened Coulomb interaction characteristic of two-dimensional electronic systems. ${ }^{31}$

The dots in Fig. 4.2 show a good agreement between the approximate solutions and the "exact" ones. A curious aspect of them is that despite the fact that the polynomial solutions are defined for all $\mathbf{k}_{\perp}$, they fail to satisfy Eq. (4.69) for higher values of $k_{\perp}$, because the regime $|\omega| \ll\left|\nu k_{x}\right|$ breaks down. We have implemented the numerical algorithm of Davidenko ${ }^{71}$ to solve the complete equation in order to find the solutions in the limit, and we have found that they cease to exist in a discontinuous manner, probably due to the square root inside the exponential of Eq. (4.69). Moreover, we have implemented an algorithm in order to find the frequency with higher imaginary part, that corresponds to the value of $k_{y} \approx 1.6 \times 10^{8} \mathrm{~m}^{-1}$ appearing in the figure. It is noteworthy to mention that the calculated solutions were found in the first quadrant of the $k_{x} k_{y}$ plane and recall that unstable modes appear in groups of four solutions, as the dependence of Eq. (4.69) on $k_{y}$ comes through $k_{y}^{2}$, and for each solution
numbered by $k_{x}$, there is one with $-k_{x}$. Thus, the solutions in Fig. 4.2 are accompanied by three others.

Now that we have an explicit recipe for constructing an unstable system, we can start to explore the physical consequences. Clearly, this means that we wish to study how the system will come to stabilization. As mentioned before, perhaps the most natural processes in such systems are related to Casimir forces on the walls. In particular, if we suppose that the wall at $z=0$ is fixed, and the one at $z=d$ is allowed to move, then it is reasonable to expect that this fluctuating vacuum will stabilize the system by moving the wall in some direction. Let us explore this possibility now.

The energy content/balance of the theory is given by the stress-energy tensor. As is a well known fact in effective electromagnetic theories, it is not possible in general to define the stress tensor consistently if one does not have the knowledge of the underlying microscopic model. ${ }^{72}$ However, notice that in our case the region between the conducting walls is vacuum, and thus in this region the stress tensor must be simply the Maxwell stress tensor, independently of the microscopic dynamics on the walls. It is given by ${ }^{37}$

$$
\begin{equation*}
T_{\mu v}=F_{\mu \alpha} F_{v}{ }^{\alpha}-\frac{1}{4} \eta_{\mu v} F_{\alpha \beta} F^{\alpha \beta}, \tag{4.105}
\end{equation*}
$$

where $\eta_{\mu \nu}=\operatorname{diag}(-1,1,1,1)$ is the Minkowski metric in Cartesian coordinates, and $F_{\mu \nu}=$ $\partial_{\mu} A_{v}-\partial_{v} A_{\mu}$ is the Maxwell tensor. Thus, the conservation law $\partial_{v} T^{\mu \nu}=0$ holds, from which we obtain the desired effect as follows. Let $u^{\mu}=(1,0,0,0)$ be the four-velocity field of the laboratory frame. Then $p_{\mu}=-T_{\mu \nu} u^{v}=-T_{\mu 0}$ is the electromagnetic momentum density. As we are interested in forces perpendicular to the walls, the conservation law for $\mu=3$ implies

$$
\begin{equation*}
\partial_{t} p_{3}+\nabla \cdot \mathbf{f}=0 \tag{4.106}
\end{equation*}
$$

where we have defined the vector field $\mathbf{f}=\left(T_{31}, T_{32}, T_{33}\right)$. Therefore, by integrating Eq. (4.106) on a small cylindrical volume with faces parallel to the wall at $z=d$ gives us the $z$ component of the pressure

$$
\begin{equation*}
P=\lim _{\epsilon \rightarrow 0^{+}}\left(\left.T_{33}\right|_{z=d-\epsilon}-\left.T_{33}\right|_{z=d+\epsilon}\right), \tag{4.107}
\end{equation*}
$$

which is the required expression to calculate the Casimir forces on the wall. As usual, we assume that this formula holds at the quantum level, where $P$ becomes an operator, and thus the (renormalized) vacuum expectation value $\langle P\rangle$ is interpreted as the average force per
unit area on the wall, which we calculate now. The $T_{33}$ component is found from (4.105) to be

$$
\begin{equation*}
2 T_{33}=\mathbf{E}_{\perp}^{2}-E_{z}^{2}+\mathbf{B}_{\perp}^{2}-B_{z}^{2} . \tag{4.108}
\end{equation*}
$$

Notice that the vacuum expectation value of Eq. (4.107) is essentially the discontinuity of $\left\langle T_{33}\right\rangle$ along the walls. If we suppose that the instability takes place in the TE modes as in our example, then a couple of simplifications occur. For instance, the solutions are such that $A_{z}, E_{z} \equiv 0$, and the perpendicular component of the vector potential $\left(\mathbf{A}_{\perp}\right)$ is continuous (see Eq. (4.67)). This means that electric field fluctuations are continuous along the wall, and only the perpendicular components of the magnetic field $\mathbf{B}=\left(-\partial_{z} A_{y}, \partial_{z} A_{x}, \partial_{x} A_{y}-\partial_{y} A_{x}\right)$ will contribute. The calculation gets further simplified if we restrict the analysis to the asymptotic behavior $t \gg 1$. In this regime, the fluctuations are dominated by the unstable field modes, and the renormalized equal-time two-point function reduces to

$$
\begin{equation*}
\left\langle A_{i}(t, \mathbf{x}) A_{j}\left(t, \mathbf{x}^{\prime}\right)\right\rangle_{\mathrm{ren}} \approx \frac{1}{2} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{2 t \operatorname{Im}\left\{\Omega_{\mathbf{k}_{\perp}}^{+}\right\}+i \mathbf{k}_{\perp} \cdot \Delta \mathbf{x}_{\perp}} A_{\mathbf{k}_{\perp}, i}^{+}(z) A_{-\mathbf{k}_{\perp}, j}^{+}\left(z^{\prime}\right) . \tag{4.109}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left\langle\mathbf{B}_{\perp}^{2}\right\rangle_{\mathrm{ren}}(z) \approx \frac{1}{8 \pi^{2}} \int \mathrm{~d}^{2} k_{\perp} \mathrm{e}^{2 t \mathrm{Im}\left\{\Omega_{\mathbf{k}_{\perp}}^{+}\right\}}\left|\mathscr{N}_{\mathbf{k}_{\perp}}^{+}\right|^{2}\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(z)\right|^{2}, \tag{4.110}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle P\rangle_{\text {ren }} \approx-\frac{1}{16 \pi^{2}} \int \mathrm{~d}^{2} k_{\perp}\left|\mathscr{N}_{\mathbf{k}_{\perp}}^{+}\right|^{2} \mathrm{e}^{2 t \operatorname{Im}\left\{\Omega_{\mathbf{k}_{\perp}}^{+}\right\}} \lim _{\epsilon \rightarrow 0^{+}}\left[\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(d+\epsilon)\right|^{2}-\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(d-\epsilon)\right|^{2}\right] . \tag{4.111}
\end{equation*}
$$

Notice that if the discontinuity presented by $\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(z)\right|$ at $z=d$ is negative, then a vacuuminduced repulsive force will emerge forcing the walls to separate, in sharp distinction with the "canonical" Casimir force. In this case, as the wall at $z=d$ is allowed to move, clearly the system will come to a stabilization as the distance between the walls increases, and a sort of quantum levitation is achieved. Let us now see that this is actually what happens. From Eq. (4.68), it follows that

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0^{+}}\left[\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(d+\epsilon)\right|^{2}-\left|\partial_{z} f_{\mathbf{k}_{\perp}}^{+}(d-\epsilon)\right|^{2}\right]=-4\left|\kappa^{+} \mathrm{e}^{-i \kappa^{+} d}\right|^{2} \operatorname{Re} \frac{1}{\mathscr{R}_{2}^{+}} . \tag{4.112}
\end{equation*}
$$

Moreover, if we denote $\omega=\omega_{r}+i \omega_{i}, \kappa=\kappa_{r}+i \kappa_{i}$ for an arbitrary frequency, we have that

$$
\begin{equation*}
\operatorname{Re} \frac{1}{\mathscr{R}_{2}}=1+\frac{2}{\sigma_{0}\left(\omega_{r}^{2}+\omega_{i}^{2}\right)}\left[\omega_{r} \kappa_{r}+\omega_{r}^{2} \kappa_{i} \tau+\omega_{i} \kappa_{i}\left(1+\omega_{i} \tau\right)\right] \geq 0 \tag{4.113}
\end{equation*}
$$

for $\omega_{i} \geq 0$ and the wall characterized by Eq. (4.102). Therefore, the pressure (4.111) is positive, and the result follows. The Fig. 4.3 depicts the (negative) discontinuity presented by one of the dominant modes of Fig. 4.2 at $z=d$.


Figure 4.3 - Plot of the profile $\left|\partial_{z} f_{\mathbf{q}}^{+}(z)\right|$ as function of $z$. We have used $\mathbf{q}=(7.9,1.6) \times 10^{8} \mathrm{~m}^{-1}$, and $\Omega_{\mathbf{q}}^{+}=(1.29+$ $1.10 i) \times 10^{9} \mathrm{~s}^{-1}$. There is also a (positive) discontinuity at $z=0$.
Source: By the author.

### 4.5 Vacuum-induced superconductivity?

We now call attention to a possibly intriguing application of the quantization procedure just studied, that can in principle occur for such systems with active vacua when certain conditions are met. As noted before, the most important feature of systems possessing active vacua is how they will reach an eventual stabilized state, and it should be clear by now that there is a great number of such possible outcomes. In fact, we argued in Sec. 2.5 that the stabilization can occur through particle production, spontaneous vectorization, and by the action of long-range correlations. Moreover, for the particular system under study in this chapter, we showed in the last section that stabilization through mechanical movement of the system components is quite natural, as it consists of a Casimir-like system. In this section we call attention to another possible outcome for the stabilized state - the spontaneous appearance of a superconducting phase.

Notice that when the electromagnetic vacuum instability lives long enough without be-
ing disrupted by decoherence phenomena, the difficulty in predicting the system fate relies on our lack of knowledge of the favorable (energy) configurations, as it is reasonable to assume that the system will evolve to the "nearest" stable state of (locally) minimum energy. In this direction, it is a curious fact that stabilization by the appearance of a superconducting phase can be predicted to occur. This follows directly from a theorem presented by de Gennes ${ }^{73}$ if the material is isotropic and homogeneous, and it is the core of the phenomenological description of superconductors in the general setting, ${ }^{74}$ where the superconducting phase is the state that minimizes the system free energy. Let us see how "natural" is the emergence of a superconducting state from the stabilization process.

Let us start with a hypothetical homogeneous and isotropic, electrically neutral portion of a material such that the electromagnetic field in its interior becomes unstable (in the sense presented here) when one acts on it with external fields (for instance, the system under study in this chapter becomes unstable when there are relative current drifts between the walls), and suppose that the growing vacuum fluctuations somehow induce the formation of Cooper pairs. Then this system will become a superconductor. In order to prove this, we start by noticing that Cooper pairs possess a much higher mobility than unpaired electrons, and thus the kinetic energy of the resulting current density will be solely due to their conduction. Next, recall that the electromagnetism in the stabilized system is described by the vector potential $\mathbf{A}$ that in the absence of free charges and due to the material homogeneity can be taken to satisfy $\nabla \cdot \mathbf{A}=0$. In this gauge, the electric and magnetic fields are written in terms of the vector potential as $\mathbf{E}=-\partial_{t} \mathbf{A}$ and $\mathbf{B}=\nabla \times \mathbf{A}$, respectively. Thus, as the stabilized system is stationary, the electric field inside the material vanishes, and its energy is

$$
\begin{equation*}
\mathscr{E}=\frac{1}{2} \int_{V} \mathrm{~d}^{3} x\left[\mathbf{B}^{2}+\lambda^{2}(\nabla \times \mathbf{B})^{2}\right], \tag{4.114}
\end{equation*}
$$

where the second term is the kinetic energy of the current charge carriers combined with $\nabla \times \mathbf{B}=\mathbf{J} . \boldsymbol{\lambda}$ is a constant that depends on the transport properties of the material, and it is supposed to be independent of $\mathbf{B}=\nabla \times \mathbf{A}$ for weak fields. Notice that because of the gauge condition $\nabla \cdot \mathbf{A}=0$, perturbations of $\mathbf{B}$ are in one-to-one correspondence with divergenceless perturbations of A via the Helmholtz Theorem. Thus, A is an extremal solution of $\mathscr{E}$ if and only if it is solution of $\mathbf{A}-\lambda^{2} \nabla^{2} \mathbf{A}=0$, the so-called London's equations, that provides a phenomenological description of type-I superconductors.

In the case of a material possessing plane-symmetric properties (for instance a layered
material), the same analysis holds and leads to the Landau-Ginzburg theory. Suppose that in such material, the unstable vacuum fluctuations promote the formation of Copper pairs, described by the (nonrelativistic) Schrödinger field $\Psi$. Due to the anisotropic properties of the material, the energy of the stabilized state can be expanded as ${ }^{74}$

$$
\begin{equation*}
\mathscr{E}=\mathscr{E}_{0}+\int \mathrm{d}^{3} x\left[\frac{\left|\hat{\nabla}_{\perp} \Psi\right|^{2}}{2 m_{\perp}}+\frac{\left|\hat{\nabla}_{\|} \Psi\right|^{2}}{2 m_{\|}}+\gamma_{1}|\Psi|^{2}+\gamma_{2}|\Psi|^{4}+\frac{1}{2} \mathbf{B}^{2}\right], \tag{4.115}
\end{equation*}
$$

where $\mathscr{E}_{0}$ is the energy of the normal state, assumed independent of $\Psi, \hat{\nabla}=\nabla+2 i e \mathbf{A}$ describes the minimal coupling, and $m_{\perp}, m_{\|}, \gamma_{1}, \gamma_{2}$ are real numbers. Thus, if Eq. (4.115) is the energy of the stabilized state, requiring it to be minimal with respect to $\Psi$ leads to the LandauGinzburg equation, and minimizing it with respect to $\mathbf{A}$, leads to the Ampère law. ${ }^{74}$

We shall call this mechanism vacuum-induced superconductivity. In synthesis, it relies on two fundamental hypothesis: if the (modified) electromagnetic vacuum in a material becomes unstable when external fields are applied on it, and if these growing vacuum fluctuations promote the formation of Cooper pairs, this system will behave as a superconductor. Clearly, this is not the whole story, for the energies in Eqs. (4.114) and (4.115) describe quite particular systems. Nevertheless, as long as the involved fields are small, and there is no "exotic" interaction between the Cooper pairs, nor the superconducting phase is formed by other kind of bound states, Eq. (4.115) should provide the correct description for the system. Therefore, if a (probably dynamical) mechanism of Copper pairing is present in materials with this kind of instability, a superconductor will emerge.

Concerning the question of what are the precise mechanisms giving rise to attractive electron-electron interactions, it should be clear that a thorough analysis in this direction deserves a dedicated work, for the requirement of having in the final state only Cooper pairs and magnetic induction still leaves the transient regime completely arbitrary. For instance, an obvious effect that will happen on the conducting walls is the modification of the phonon spectrum, as individual ions of the lattice are subjected to the fluctuating force $q \mathbf{E}$, where $q$ is the ion charge, and $\mathbf{E}$ is the quantized electric field. Our experience with active quantum vacuum states then teach us that if the electromagnetic field is prepared in its natural vacuum state, this force vanishes on average, but as time passes, higher fluctuations become more and more probable, thus modifying the system entirely.

Another interesting phenomenon that may play an important role in the transient regime
is the appearance of long-range correlations, already explored in Sec. 2.5, and that is also the case here. Recall that the starting point of the quantization scheme we have presented is the collective description of the matter degrees of freedom through a few parameters, that are effectively incorporated into the electromagnetic field dynamics via polarization, and thus quantum aspects of the field are sustained by the whole system. These include the longrange correlations, that could reveal novel effects between distant charged particles. In fact, strange correlations in such systems were measured quite recently, as reported recently. ${ }^{75}$ Further developments in this direction will be explored in future works.

### 4.6 Final remarks

Stacked layers of conducting materials become sensible to the relative motion of electrical currents, what leads to the appearance of classical instabilities in some situations. In this chapter, based on the findings of Chapter 3, we have shown how to properly quantize the electromagnetic field in bilayer systems when these classical instabilities are present, by invoking symmetry arguments that revealed how usually missing solutions must be added to the quantum field expansion. These solutions were associated to operators whose algebra spans the same state space as some annihilation/creation operators, thus recovering the usual notion of Fock space. The postulated method was justified a posteriori, by showing that the quantum field expansion satisfy the canonical commutation relation.

The motivation for studying such systems, besides the development of fundamental quantum field theory methods, relies on the fact that they are connected to state-of-the-art solid state devices, e.g. graphene stacks and high-temperature superconductors. As an application of the quantized field expansion, we have predicted a new type of stabilization through mechanical movement - a sort of quantum levitation - that is an instance of the Casimir effect in such scenarios. It is noteworthy that this represents a first step in studying the phenomenology coming from our analysis, that may reveal interesting physics in the near future. One such possibility is related to the spontaneous appearance of long-range correlations when the quantum vacuum is unstable, as reported in the first part of this thesis. This phenomenon is also present in the bilayer system, and the appearance of correlations between distant particles in the material may lead to strongly correlated quasi-particles' behavior, that is observed in related systems. ${ }^{75}$ Possible links with superconducting phenomena will
be explored elsewhere.
Note that some aspects in the quantization procedure we have presented still need development. For instance, how is structured the quantum vacuum when the $\mathscr{S}$-symmetry does not hold, and is it possible to have physical situations such that the zeros of Eqs. (4.69) and (4.72) are not simple, are examples of open questions that require careful thinking. These are aspects that when answered will strengthen the analysis presented here, and will be explored elsewhere. It is also important to notice that our motivation was to study the unstable quantum vacuum in bilayer systems, that naturally led to the use of the "impedance boundary conditions" encapsulated in the dielectric matrix of Eq. (4.5). Clearly, the devised method does not depend on this particular background, and should work for every equilibrium configuration in which the field equations admit symmetries involving time-reversal, that can be used to correctly add the unstable field modes to the quantization procedure.

It should be clear that this quantization is only the tip of the iceberg, and the scientific discoveries that can potentially emerge from our analysis are still to be found, as briefly discussed in the text. We have neglected completely other features of actual systems, like temperature, that certainly will play an important role concerning decoherence and how stabilization occurs. We hope that further development of these ideas will result in interesting physics, improving even more our understanding of Nature.

## Chapter 5

## Concluding Remarks

In this chapter we conclude this work by presenting some features of its development and by-products that were not contemplated in the main text. The original goal of this doctoral training was to study the phenomenon of gravity-induced vacuum instability from the point of view of the field spin, as the effect was (at the project's planning period) fairly studied only for the spin 0 field. This means that we were interested in studying how to couple fields of spin $1 / 2,1$ and 2 to gravity in such way to trigger instabilities. We then immediately realized that the analysis for spin $1 / 2$ fields is trivial, as any reasonable nonminimal coupling between these fields and gravity must result in stable theories. The argument for it is actually simple, and relies on the self-adjoint character of the evolution operators in these theories. In the static case, any spin $1 / 2$ field satisfies an equation of the form $i \partial_{t} \psi=H \psi$, where $H$ is Hermitian under natural boundary conditions. ${ }^{34}$ This means that the operator $i \partial_{t}$ only possesses real eigenvalues, and thus the theory is stable. This lead us to question the nomenclature in recent papers, ${ }^{76}$ as exponentially growing solutions of spin $1 / 2$ fields must come from couplings with another (actually unstable) fields.

In this way, due to the experimental appeal and simplicity in comparison with spin 2 fields, the natural field to be analyzed was the electromagnetic field. We started by studying one loop corrections to the Maxwell field coming from QED in the Schwarzschild spacetime, anticipating the analysis presented in Subsec. 2.4.2, by working at linear order. As a result, we have found that when the black hole radius is comparable to the electron Compton wavelength, the theory predicts the existence of instability. However, in this regime gravity is so intense (in the vicinity of the black hole) that the linear analysis is not reliable, and nonlinear contributions may change the field dynamics considerably. We then decided to leave this
(quite lengthy) analysis outside the final document, as in the context of analogue models, the theory is always valid by assumption.

Another development that did not achieved critical mass to enter the document is an analysis performed in the context of scalar QED, following Ford's idea ${ }^{5}$ as motivation. Basically, it consists in choosing the unitary gauge, in which the theory is described by a single scalar degree of freedom plus a Proca field. Thus, by working at linear order, the dynamics of the Proca field is such that $\nabla_{a} \nabla^{a} A_{b}+R_{b c} A^{c}+e^{2} \rho^{2} A_{b}=0$, where the massive scalar field $\rho$ satisfies the (minimally coupled) KG equation. If we assume that $\rho$ is prepared in its vacuum state and take the vacuum expectation value of the equation for $A_{a}$, we are left with an effective equation modeling a varying mass squared $\left(e^{2}\left\langle\rho^{2}\right\rangle\right)$ for $A_{a}$. Thus, the problem reduces to know promising cases where we can actually calculate $\left\langle\rho^{2}\right\rangle$, and it must become negative in some region (subvacuum effect). This is in fact the main problem with this analysis, because vacuum polarization effects can only be calculated in very special cases.

From the possible scenarios we have analyzed, one in special deserves mentioning: the determination of $\left\langle\rho^{2}\right\rangle$ in the context of linearized gravity. This analysis was performed a few years ago for massless configurations, ${ }^{77}$ and we have extended it for the massive case as well. As a preliminary result,we verified that, in contrast to the massless case, where there is an arbitrary mass scale in the final expression, here the field mass fix this scale naturally. Remarkably, we found that an interesting application of this vacuum polarization is not related to its original purpose, but to the very nature of the field dynamics. Numerical simulations reveal that for some reasonable choices of the spacetime metric, the vacuum polarization $\left\langle\rho^{2}\right\rangle_{\text {Ren }}$ possesses a characteristic oscillatory behavior. As it turns out, this seems to be a systematic fingerprint of the Huygens' principle breakdown caused by the field mass, that can be detected when nonlocal effects are measured, as suggested recently in the literature. ${ }^{78}$ This seems to be the case here, and perhaps it could be measured in the near future for the Maxwell field in the presence of matter.

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## Appendix A

## Normalization of modes in the spherically

## symmetric case

Here, we present in detail the calculations involved in normalizing the electromagnetic modes in the spherically symmetric case. Since we are dealing with analogues to which there is a natural physical notion of time - the lab-frame time $t$-, it is convenient to use $t=$ constant surfaces $\left(\Sigma_{t}\right)$ to normalize the modes. Obviously, this choice bears no physical consequence on our results. The sesquilinear form given in Eq. (2.29), applied to the scenario described in Sec. 2.4, takes the form - notice that the integrand is a scalar and, as such, can be evaluated in any coordinate system:

$$
\begin{align*}
\left(A, A^{\prime}\right)= & i \int_{\Sigma_{t}} \mathrm{~d} \Sigma\left\{\varepsilon_{\|} \bar{A}_{r} \partial_{\tau} A_{r}^{\prime}+\frac{\varepsilon_{\perp} \overline{\mathbf{A}}_{\perp} \cdot \partial_{\tau} \mathbf{A}_{\perp}^{\prime}}{\gamma^{2}\left(1-n_{\|}^{2} v^{2}\right)}+\frac{\gamma^{2}\left(n_{\|}^{2}-1\right) v}{\mu_{\perp}}\left[\overline{\mathbf{A}}_{\perp} \cdot \partial_{r} \mathbf{A}_{\perp}^{\prime}-\left(\overline{\mathbf{A}}_{\perp} \cdot \partial_{\perp}\right) A_{r}^{\prime}\right]\right\} \\
& -\left(\overline{\left.\mathbf{A} \leftrightarrow \mathbf{A}^{\prime}\right) .}\right. \tag{A.1}
\end{align*}
$$

Below, we evaluate this expression for each type of mode.

## A. 1 TE modes

## A.1.1 Stable

Substituting $\quad \mathbf{A}^{(\mathrm{TE})} \quad=\quad\left(0, \partial_{\varphi} \psi / \sin \theta,-\sin \theta \partial_{\theta} \psi\right) \quad$ into $\quad$ Eq. (A.1), with $\psi=\exp (-i \omega \tau) Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TE})}(r)$, one gets:

$$
\begin{align*}
\left(A^{(\mathrm{TE})}, A^{\prime(\mathrm{TE})}\right)= & \int_{S^{2}} \mathrm{~d} S\left[\left(\partial_{\theta} \overline{Y_{\ell m}}\right)\left(\partial_{\theta} Y_{\ell^{\prime} m^{\prime}}\right)+\frac{m m^{\prime}}{\sin ^{2} \theta} \overline{Y_{\ell m}} Y_{\ell^{\prime} m^{\prime}}\right] \\
& \times \int_{\mathscr{I}} \mathrm{d} \rho \mathrm{e}^{i\left(\omega-\omega^{\prime}\right) \tau}\left[\left(\omega+\omega^{\prime}\right) \frac{\varepsilon_{\perp}}{\mu_{\perp}} \overline{f_{\omega \ell}^{(\mathrm{TE})}} f_{\omega^{\prime} \ell^{\prime}}^{(\mathrm{TE})}\right. \\
& \left.+\frac{i \gamma^{2}\left(n_{\|}^{2}-1\right) v}{\mu_{\perp}}\left(\overline{f_{\omega \ell}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} f_{\omega^{\prime} \ell^{\prime}}^{(\mathrm{TE})}-f_{\omega^{\prime} \ell^{\prime}}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{f_{\omega \ell}^{(\mathrm{TE})}}\right)\right], \tag{A.2}
\end{align*}
$$

where $S^{2}$ is the unit sphere, recall that $\mathrm{d} r / \mathrm{d} \rho=\gamma^{2}\left(1-n_{\|}^{2} v^{2}\right) / \mu_{\perp}$, and it is understood that this last integral must be evaluated at $\tau+p(r)=t=$ constant [recall definition of $\tau$ right above Eq. (2.98)]. It is straightforward to show that the first integral evaluates to $\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}$ provided we normalize $Y_{\ell m}$ according to $\int_{S^{2}} \mathrm{~d} S \overline{Y_{\ell m}} Y_{\ell^{\prime} m^{\prime}}=\delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}$. As for the second integral, let us first consider the quantity

$$
\begin{equation*}
W_{\omega \omega^{\prime}}^{(\ell)} \equiv \frac{1}{\left(\omega-\omega^{\prime}\right)}\left(\overline{f_{\omega \ell}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} f_{\omega^{\prime} \ell}^{(\mathrm{TE})}-f_{\omega^{\prime} \ell}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{f_{\omega \ell}^{(\mathrm{TE})}}\right) . \tag{A.3}
\end{equation*}
$$

Making use of Eq. (2.102), $W_{\omega \omega^{\prime}}^{(\ell)}$ clearly satisfies

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} \rho} W_{\omega \omega^{\prime}}^{(\ell)}=\frac{\varepsilon_{\perp}}{\mu_{\perp}}\left(\omega+\omega^{\prime}\right) \overline{f_{\omega \ell}^{(\mathrm{TE})}} f_{\omega^{\prime} \ell}^{\mathrm{(TE})} . \tag{A.4}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\left(A^{(\mathrm{TE})}, A^{\prime(\mathrm{TE})}\right) & =\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \int_{\mathscr{I}} \mathrm{d} \rho \mathrm{e}^{i\left(\omega-\omega^{\prime}\right) \tau}\left[\frac{\mathrm{d}}{\mathrm{~d} \rho} W_{\omega \omega^{\prime}}^{(\ell)}-i\left(\omega-\omega^{\prime}\right) \frac{\mathrm{d} p}{\mathrm{~d} \rho} W_{\omega \omega^{\prime}}^{(\ell)}\right] \\
& =\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \int_{\mathscr{I}} \mathrm{d} \rho \mathrm{e}^{i\left(\omega-\omega^{\prime}\right)[\tau+p(r)]} \frac{\mathrm{d}}{\mathrm{~d} \rho}\left(\mathrm{e}^{-i\left(\omega-\omega^{\prime}\right) p(r)} W_{\omega \omega^{\prime}}^{(\ell)}\right) \\
& =\left.\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \mathrm{e}^{i\left(\omega-\omega^{\prime}\right) t}\left[\mathrm{e}^{-i\left(\omega-\omega^{\prime}\right) p(r)} W_{\omega \omega^{\prime}}^{(\ell)}\right]\right|_{\dot{\mathscr{I}}} \tag{A.5}
\end{align*}
$$

where we made use that $t=\tau+p(r)$ is kept constant along integration in $r$ (or $\rho$ ) and $\left[\|_{\dot{\mathscr{I}}}\right.$ indicates that we must calculate the flux of the quantity in square brackets at the boundaries of $\mathscr{I}$. We see that in order to guarantee orthogonality between modes with different $\omega$, with-
out worring about the specific form of $p(r)$, we must impose boundary conditions at $\dot{\mathscr{I}}$ such that, in Eq. (A.3), $\left.W_{\omega \omega^{\prime}}^{(\ell)}\right|_{\dot{\mathscr{I}}}=0$ for $\omega \neq \omega^{\prime}$. Then, referring back to Eq. (A.4) and writing

$$
\begin{equation*}
W_{\omega \omega^{\prime}}^{(\ell)}(\rho)=\left(\omega+\omega^{\prime}\right) \int_{\rho_{-}}^{\rho} \mathrm{d} \rho^{\prime} \frac{\varepsilon_{\perp}}{\mu_{\perp}} \overline{f_{\omega \ell}^{(\mathrm{TE})}} f_{\omega^{\prime} \ell}^{(\mathrm{TE})} \tag{A.6}
\end{equation*}
$$

we finally obtain

$$
\begin{equation*}
\left(A^{(\mathrm{TE})}, A^{\prime(\mathrm{TE})}\right)=2 \omega \ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \int_{\mathscr{I}} \mathrm{d} \varrho \overline{f_{\omega \ell}^{(\mathrm{TE})}} f_{\omega^{\prime} \ell}^{(\mathrm{TE})} \tag{A.7}
\end{equation*}
$$

which justifies the normalization of the TE modes in Sec. 2.4. (Notice that the integration variable is $\varrho$, defined through $\mathrm{d} r / \mathrm{d} \varrho=\gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right) / \varepsilon_{\perp}$.)

## A.1.2 Unstable TE modes

Generic unstable TE modes are given by $\mathbf{A}^{(u \mathrm{TE})}=\left(0, \partial_{\varphi} \psi / \sin \theta,-\sin \theta \partial_{\theta} \psi\right)$ with

$$
\begin{equation*}
\psi=\left(\alpha_{\Omega \ell} \mathrm{e}^{\Omega \tau}+\beta_{\Omega \ell} \mathrm{e}^{-\Omega \tau}\right) Y_{\ell m}(\theta, \varphi) g_{\Omega \ell}^{(\mathrm{TE})}(r), \tag{A.8}
\end{equation*}
$$

where $\alpha_{\Omega \ell}$ and $\beta_{\Omega \ell}$ are complex constants and $g_{\Omega \ell}^{(\mathrm{TE})}(r)$ is a solution of Eq. (2.102) with $\omega^{2}=$ $-\Omega^{2}(\Omega>0$, without loss of generality) and proper boundary conditions (see below). Sesquilinearity of Eq. (2.104) makes it easy to calculate $\left(A^{(u \mathrm{TE})}, A^{\prime(u \mathrm{TE})}\right)$ from Eq. (A.5) with the appropriate substitution $\omega \mapsto \mp i \Omega$ and $\omega^{\prime} \mapsto \pm i \Omega^{\prime}$ :

$$
\begin{align*}
\left(A^{(u \mathrm{TE})}, A^{\prime(u \mathrm{TE})}\right)= & \ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}\left[\overline{\alpha_{\Omega \ell}} \alpha_{\Omega^{\prime} \ell} \mathrm{e}^{\left(\Omega+\Omega^{\prime}\right) \tau} W_{\Omega \Omega^{\prime}}^{(u \ell)}+\overline{\beta_{\Omega \ell}} \beta_{\Omega^{\prime} \ell} \mathrm{e}^{-\left(\Omega+\Omega^{\prime}\right) \tau} W_{-\Omega-\Omega^{\prime}}^{(u \ell)}\right. \\
& +\overline{\alpha_{\Omega \ell}} \beta_{\Omega^{\prime} \ell} \mathrm{e}^{\left(\Omega-\Omega^{\prime}\right) \tau} W_{\Omega-\Omega^{\prime}}^{(u \ell)}+\left.\overline{\beta_{\Omega \ell}} \alpha_{\Omega^{\prime} \ell} \mathrm{e}^{-\left(\Omega-\Omega^{\prime}\right) \tau} W_{-\Omega \Omega^{\prime}}^{(u \ell)}\right|_{\dot{\mathscr{g}}}, \tag{A.9}
\end{align*}
$$

where

$$
\begin{equation*}
W_{ \pm \Omega \pm \Omega^{\prime}}^{(u \ell)} \equiv \frac{i}{\left( \pm \Omega \pm \Omega^{\prime}\right)}\left(\overline{g_{\Omega \ell}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} g_{\Omega^{\prime} \ell}^{(\mathrm{TE})}-g_{\Omega^{\prime} \ell}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{g_{\Omega \ell}^{(\mathrm{TE})}}\right) . \tag{A.10}
\end{equation*}
$$

As with the stable case, we must impose boundary conditions on $g_{\Omega \ell}^{(\mathrm{TE})}(r)$ such that

$$
\left.\left(\overline{g_{\Omega \ell}^{(\mathrm{TE})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} g_{\Omega^{\prime} \ell}^{(\mathrm{TE})}-g_{\Omega^{\prime} \ell}^{(\mathrm{TE})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{g_{\Omega \ell}^{(\mathrm{TE})}}\right)\right|_{\dot{\mathscr{I}}}=0,
$$

which implies $\left.W_{\Omega \Omega^{\prime}}^{(u \ell)}\right|_{\dot{\mathscr{I}}}=\left.W_{-\Omega-\Omega^{\prime}}^{(u \ell)}\right|_{\dot{\mathscr{I}}}=0$ and, for $\Omega \neq \Omega^{\prime},\left.W_{\Omega-\Omega^{\prime}}^{(u \ell)}\right|_{\dot{\mathscr{I}}}=\left.W_{-\Omega \Omega^{\prime}}^{(u \ell)}\right|_{\dot{\mathscr{I}}}=0$. Therefore, using the analogous of Eq. (A.6),

$$
\begin{equation*}
W_{ \pm \Omega \pm \Omega^{\prime}}^{(u \ell)}(\rho)=-i\left( \pm \Omega \mp \Omega^{\prime}\right) \int_{\rho_{-}}^{\rho} \mathrm{d} \rho^{\prime} \frac{\varepsilon_{\perp}}{\mu_{\perp}} \overline{g_{\Omega \ell}^{(\mathrm{TE})}} g_{\Omega^{\prime} \ell}^{(\mathrm{TE})} \tag{A.11}
\end{equation*}
$$

into Eq. (A.9), we finally obtain

$$
\begin{equation*}
\left(A^{(u \mathrm{TE})}, A^{\prime(u \mathrm{TE})}\right)=4 \Omega \ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \operatorname{Im}\left(\overline{\alpha_{\Omega \ell}} \beta_{\Omega \ell}\right) \int_{\mathscr{I}} \mathrm{d} \varrho \overline{g_{\Omega \ell}^{(\mathrm{TE})}} g_{\Omega^{\prime} \ell}^{(\mathrm{TE})}, \tag{A.12}
\end{equation*}
$$

where $\operatorname{Im}(z)$ stands for the coefficient of the imaginary part of the complex number $z$. Thus, imposing orthonormality of these modes - for orthonomalized $g_{\Omega \ell}^{(\mathrm{TE})}$ (in the $L^{2}(\mathscr{I}, \mathrm{~d} \varrho)$ inner product) —, the general expression for $\alpha_{\Omega \ell}$ and $\beta_{\Omega \ell}$ (up to rephasing, $\alpha_{\Omega \ell} \mapsto \mathrm{e}^{i \delta} \alpha_{\Omega \ell}, \beta_{\Omega \ell} \mapsto$ $\mathrm{e}^{i \delta} \beta_{\Omega \ell}$, and time resetting, $\alpha_{\Omega \ell} \mapsto \mathrm{e}^{\Omega t_{0}} \alpha_{\Omega \ell}, \beta_{\Omega \ell} \mapsto \mathrm{e}^{-\Omega t_{0}} \beta_{\Omega \ell}$ ) read

$$
\begin{align*}
& \alpha_{\Omega \ell}=\frac{\mathrm{e}^{-i \vartheta / 2}}{2 \sqrt{\Omega \ell(\ell+1) \sin \vartheta}},  \tag{A.13}\\
& \beta_{\Omega \ell}=\frac{\mathrm{e}^{i \vartheta / 2}}{2 \sqrt{\Omega \ell(\ell+1) \sin \vartheta}}, \tag{A.14}
\end{align*}
$$

with $0<\vartheta<\pi$ being an arbitrary constant.

## A. 2 TM modes

## A.2.1 Stable

Now, substituting $\mathbf{A}^{(\mathrm{TM})}=\left(r^{-2} \varepsilon_{\|}^{-1} \Delta_{S}^{(0)}, \partial_{\theta} \partial_{\varrho}, \partial_{\varphi} \partial_{\varrho}\right) \phi$ into Eq. (2.104), with $\phi=\exp (-i \omega \tau) Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(\mathrm{TM})}(r)$, and evaluating the angular integrals (similarly to the previous TE case), we obtain:

$$
\begin{align*}
&\left(A^{(\mathrm{TM})}, A^{\prime(\mathrm{TM})}\right)=\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \int_{\mathscr{\mathscr { I }}} \mathrm{d} \rho \mathrm{e}^{i\left(\omega-\omega^{\prime}\right) \tau}\left\{( \omega + \omega ^ { \prime } ) \left[\frac{\mathrm{d}}{\mathrm{~d} \varrho} \overline{f_{\omega \ell}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} \rho_{\omega^{\prime} \ell}^{(\mathrm{TM})}\right.\right. \\
&+\frac{r^{2}\left(1-n_{\|}^{2} v^{2}\right) \ell(\ell+1)}{\varepsilon_{\|} \varepsilon_{\perp} r^{2}} \frac{f_{\omega \ell}^{(\mathrm{TM})}}{\left.f_{\omega^{\prime} \ell}^{(\mathrm{TM})}\right]} \\
&\left.+\frac{i \gamma^{2}\left(n_{\|}^{2}-1\right) v}{\varepsilon_{\perp}}\left(\omega^{2} \overline{f_{\omega \ell}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \rho} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}-\omega^{\prime 2} f_{\omega^{\prime} \ell}^{(\mathrm{TM})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{f_{\omega \ell}^{(\mathrm{TM})}}\right)\right\}, \tag{A.15}
\end{align*}
$$

where recall that $\mathrm{d} r / \mathrm{d} \varrho=\gamma^{2}\left(1-n_{\|}^{2} \nu^{2}\right) / \varepsilon_{\perp}$. The strategy to simplify the expression above is the same applied in the TE case. Define

$$
\begin{equation*}
\mathscr{W}_{\omega \omega^{\prime}}^{(\ell)}=\frac{1}{\left(\omega-\omega^{\prime}\right)}\left(\omega^{2} \overline{f_{\omega \ell}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \varrho} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}-\omega^{\prime 2} f_{\omega^{\prime} \ell}^{(\mathrm{TM})} \frac{\mathrm{d}}{\mathrm{~d} \varrho} \overline{f_{\omega \ell}^{(\mathrm{TM})}}\right) . \tag{A.16}
\end{equation*}
$$

One can easily check, using Eq. (2.107), that

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} \varrho} \mathscr{W}_{\omega \omega^{\prime}}^{(\ell)}=\left(\omega+\omega^{\prime}\right)\left[\frac{\mathrm{d}}{\mathrm{~d} \varrho} \overline{f_{\omega \ell}^{(\mathrm{TM})}} \frac{\mathrm{d}}{\mathrm{~d} \varrho} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}+\frac{r^{2}\left(1-n_{\|}^{2} v^{2}\right) \ell(\ell+1)}{\varepsilon_{\|} \varepsilon_{\perp} r^{2}} \overline{f_{\omega \ell}^{(\mathrm{TM})}} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}\right] . \tag{A.17}
\end{equation*}
$$

Therefore, we can put Eq. (A.15) in the same form as Eq. (A.5), with $W \mapsto \mathscr{W}$ and $\rho \mapsto \varrho$. Now, orthogonality of the modes demand that $f_{\omega \ell}^{(\mathrm{TM})}$ satisfy either Dirichlet or Neumann boundary conditions at $\dot{\mathscr{I}}$, which leads to

$$
\begin{equation*}
\left(A^{(\mathrm{TM})}, A^{\prime(\mathrm{TM})}\right)=\left.\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}\left[\mathbb{W}_{\omega \omega^{\prime}}^{(\ell)}\right]\right|_{\dot{\mathscr{F}}}, \tag{A.18}
\end{equation*}
$$

In order to simplify even further the expression above, note that using again Eq. (2.107) in Eq. (A.17), we can write

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} \varrho} \mathscr{W}_{\omega \omega^{\prime}}^{(\ell)}=\frac{\left(\omega+\omega^{\prime}\right)}{2}\left[\frac{\mathrm{~d}^{2}}{\mathrm{~d} \varrho^{2}}+\frac{\mu_{\perp}}{\varepsilon_{\perp}}\left(\omega^{2}+\omega^{\prime 2}\right)\right]\left(\overline{f_{\omega \ell}^{(\mathrm{TM})}} f_{\omega^{\prime} \ell}^{(\mathrm{TM})}\right), \tag{A.19}
\end{equation*}
$$

whose integration on $\mathscr{I}$ gives us $\left.\left[\mathscr{W}_{\omega \omega^{\prime}}^{(\ell)}\right]\right|_{\mathscr{\mathscr { G }}}$, which substituted into Eq. (A.18) finally leads to

$$
\begin{equation*}
\left(A^{(\mathrm{TM})}, A^{\prime(\mathrm{TM})}\right)=2 \omega^{3} \ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \int_{\mathscr{I}} \mathrm{d} \rho \overline{f_{\omega \ell}^{(\mathrm{TM})}} f_{\omega^{\prime} \ell}^{(\mathrm{TM})} . \tag{A.20}
\end{equation*}
$$

(Notice that the integration variable is $\rho$.)

## A.2.2 Unstable

Generic unstable TM modes are given by $\mathbf{A}^{(u T M)}=\left(r^{-2} \varepsilon_{\|}^{-1} \Delta_{S}^{(0)}, \partial_{\theta} \partial_{\varrho}, \partial_{\varphi} \partial_{\varrho}\right) \phi$ with

$$
\begin{equation*}
\phi=\left(\alpha_{\Omega \ell} \mathrm{e}^{\Omega \tau}+\beta_{\Omega \ell} \mathrm{e}^{-\Omega \tau}\right) Y_{\ell m}(\theta, \varphi) g_{\Omega \ell}^{(\mathrm{TM})}(r), \tag{A.21}
\end{equation*}
$$

where $\alpha_{\Omega \ell}$ and $\beta_{\Omega \ell}$ are complex constants and $g_{\Omega \ell}^{(\mathrm{TM})}(r)$ is a solution of Eq. (2.107) with $\omega^{2}=$ $-\Omega^{2}(\Omega>0$, without loss of generality) satisfying Dirichlet or Neumann boundary conditions. Once more, sesquilinearity of Eq. (2.104) makes it easy to calculate ( $A^{(u \mathrm{TM})}, A^{\prime(u \mathrm{TM})}$ ) from

Eq. (A.18) with the substitution $\omega \mapsto \mp i \Omega$ and $\omega^{\prime} \mapsto \pm i \Omega^{\prime}$ :

$$
\begin{equation*}
\left(A^{(u \mathrm{TM})}, A^{\prime(u \mathrm{TM})}\right)=\left.\ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}\left[\overline{\alpha_{\Omega \ell}} \beta_{\Omega^{\prime} \ell} W_{\Omega-\Omega^{\prime}}^{(u \ell)}+\overline{\beta_{\Omega \ell}} \alpha_{\Omega^{\prime} \ell} W_{-\Omega \Omega^{\prime}}^{(u \ell)}\right]\right|_{\dot{\mathscr{I}}} \tag{A.22}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{W}_{ \pm \Omega \pm \Omega^{\prime}}^{(u \ell)} & =\frac{-i}{\left( \pm \Omega \pm \Omega^{\prime}\right)}\left(\Omega^{2} \overline{g_{\Omega \ell}^{(T M)}} \frac{\mathrm{d}}{\mathrm{~d} \rho} g_{\Omega^{\prime} \ell}^{(\mathrm{TM})}-\Omega^{\prime 2} g_{\Omega^{\prime} \ell}^{(\mathrm{TM})} \frac{\mathrm{d}}{\mathrm{~d} \rho} \overline{g_{\Omega \ell}^{(\mathrm{TM})}}\right) \\
& =\frac{i\left( \pm \Omega \mp \Omega^{\prime}\right)}{2}\left[-\frac{\mathrm{d}}{\mathrm{~d} \varrho}\left(\overline{g_{\Omega \ell}^{(\mathrm{TM})}} g_{\Omega^{\prime} \ell}^{(\mathrm{TM})}\right)+\left(\Omega^{2}+\Omega^{\prime 2}\right) \int_{\varrho_{-}}^{\varrho} \mathrm{d} \varrho^{\prime} \frac{\mu_{\perp}}{\varepsilon_{\perp}} \overline{g_{\Omega \ell}^{(\mathrm{TM})}} g_{\Omega^{\prime} \ell}^{(\mathrm{TM})}\right] . \tag{A.23}
\end{align*}
$$

In the last passage of the expression above we used the analogous of Eq. (A.19). Thus

$$
\begin{equation*}
\left(A^{(u \mathrm{TM})}, A^{\prime(u \mathrm{TM})}\right)=-4 \Omega^{3} \ell(\ell+1) \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \operatorname{Im}\left(\overline{\alpha_{\Omega \ell}} \beta_{\Omega \ell}\right) \int_{\mathscr{I}} \mathrm{d} \rho \overline{g_{\Omega \ell}^{(\mathrm{TM})}} g_{\Omega^{\prime} \ell}^{(\mathrm{TM})} \tag{A.24}
\end{equation*}
$$

Imposing orthonormality of these modes — for orthonomalized $g_{\Omega \ell}^{(\mathrm{TM})}$ (in the $L^{2}(\mathscr{I}, \mathrm{~d} \rho)$ inner product) —, the general expression for $\alpha_{\Omega \ell}$ and $\beta_{\Omega \ell}$ (again, up to rephasing and time resetting) can be expressed as

$$
\begin{align*}
& \alpha_{\Omega \ell}=\frac{\mathrm{e}^{i \vartheta / 2}}{2 \sqrt{\Omega^{3} \ell(\ell+1) \sin \vartheta}},  \tag{A.25}\\
& \beta_{\Omega \ell}=\frac{\mathrm{e}^{-i \vartheta / 2}}{2 \sqrt{\Omega^{3} \ell(\ell+1) \sin \vartheta}}, \tag{A.26}
\end{align*}
$$

with $0<\vartheta<\pi$.

