# UNIVERSIDADE DE SÃO PAULO INSTITUTO DE FÍSICA DE SÃO CARLOS 

Etevaldo dos Santos Costa Filho

# Construction of new solutions of the electro-vacuum Einstein equation 

## Etevaldo dos Santos Costa Filho

# Construction of new solutions of the electro-vacuum Einstein equation 

Dissertation presented to the Graduate Program in Physics at the Instituto de Física de São Carlos, Universidade de São Paulo, to obtain the degree of Master in Science.

Área de concentração: Basic Physics
Supervisor: Profa. Dra. Betti Hartmann

## Original version

São Carlos

I AUTHORIZE THE REPRODUCTION AND DISSEMINATION OF TOTAL OR PARTIAL COPIES OF THIS DOCUMENT, BY CONVENTIONAL OR ELECTRONIC MEDIA FOR STUDY OR RESEARCH PURPOSE, SINCE IT IS REFERENCED.

Costa Filho, Etevaldo dos Santos
Construction of new solutions of the electro-vacuum Einstein equation / Etevaldo dos Santos Costa Filho; advisor Betti Hartmann -- São Carlos 2020.

149 p.

Dissertation (Master's degree - Graduate Program in Basic Physics) -- Instituto de Física de São Carlos, Universidade de São Paulo - Brasil , 2020.

1. Sibgatullin's method. 2. Black holes. 3. Multipole moments. 4. Ernst's equations. 5. Integrable system. I. Hartmann, Betti, advisor. II. Title.

## FOLHA DE APROVAÇÃO

Etevaldo dos Santos Costa Filho

Dissertação apresentada ao Instituto de Física de São Carlos da Universidade de São Paulo para obtenção do título de Mestre em Ciências. Área de Concentração: Física Básica.

Aprovado(a) em: 17/12/2020

## Comissão Julgadora

Dr(a). Betti Hartmann<br>Instituição: (IFSC/USP)

Dr(a). Yves Brihaye
Instituição: (Université de Mons/Bélgica)

Dr(a). Wojciech Jerzy Maria Zakrzewski
Instituição: (University of Durham/Reino Unido)

Dedico este trabalho à minha avó Lucila: Eu tenho a força.

## ACKNOWLEDGEMENTS

Mais ainda em tempos difíceis, dou-me conta de minha efemeridade e fraqueza. São os que me rodeiam os verdadeiros responsáveis por mim. Nos momentos de indecisão ou decisões difíceis, são os amigos quem me suportam. Quero dizer a cada um que faz parte da minha vida o quão especial você é para mim. Sinto tua falta e garanto-te que o abraço será forte o suficiente para compensar os longos meses de isolamento!!!

Quero agradecer especialmente aqueles que acompanharam de perto essa trajetória.
Agradeço a minha mãe por me apoiar e suportar sempre que preciso e mesmo quando minhas decisões são contra-senso.

Ao meu irmão Matheus, obrigado por todas conversas regadas de conhecimento. Você é uma das pessoas mais inteligentes que já conheci, só não se esqueça: contra toda autoridad... excepto mi mamá.

Muitíssimo obrigado, minhas amigas Julia e Aline. Mesmo distantes, vocês são as pessoas com as quais tenho a honra de conversar todos os dias e me ajudam a aliviar a tensão desses tempos difíceis.

Aos meus amigos Angelo, Felipe, Henrique, Paulão e Ronaldo. Dividimos muito mais do que um teto e histórias. Vocês são como irmãos para mim!! Muito obrigado por me acolher quando eu chorei e cantar comigo quando eu estava feliz. Vocês são uma inspiração para mim, como pessoas e como profissionais. Um agradecimento especial para meu matemático que é quase meu consultor particular. Angelo, muito obrigado por iluminar toda dúvida que tive, por aprender e discutir comigo as inúmeras questões que tive no decorrer deste trabalho.

Aos meus grandes amigos da graduação, Mateus, o raposa, e Doctor Carlos. Vocês alegram meus dias com suas histórias dignas de cinema. Obrigado por serem meus conselheiros de problemas. Amo as tretas que vocês se metem.

A minha grande digníssima, a que emana cremosidade. Sem você, Mariane, metade dessa dissertação não seria possível. Obrigado por me apoiar a seguir meu sonho, me incentivar e me dar coragem. Tem sido uma honra dividir a vida contigo.

Ao meu Professor Gabriel Luchini. Você tem acompanhado praticamente toda minha trajetória na física. Pode-se dizer que você me pegou para criar. Muito obrigado por todas discussões e conselhos.

Professor, Ivan Cabrera. Thank you very much for taking the time and always being available to answer my questions, no matter how simple it is.

Professor Gyula Fodor, you were the first to see me as a researcher and not just as a student. That means a lot to me! It has been an honor to work with you! Thanks!!

Special thanks to my advisor Professor Betti Hartmann who helped, supported and guided my path for the past few years.
"As I raced through the street a phrase repeated itself over and over: Which is the true self? It was this phrase which accompanied me now, racing through the morbid streets of the Bronx. Why was I racing? What was driving me on at this pace?

I slowed down, as if to let the demon overtake me..."

## ABSTRACT

COSTA FILHO, E. S. Construction of new solutions of the electro-vacuum Einstein equation. 2020. 149p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

In this dissertation, we review the Sibgatullin method, discuss its construction and application in the case of N -Solitons interest in time space in the vacuum. Having made this discussion, we introduce the relationship between multipole moments in general relativity and the solutions of the coupled Einstein-Maxwell system of equations. We were able to generalize well known results in the literature.

There was also a brief discussion about the multipolar moments and even more, we corrected a canonical result in the literature.

Keywords: Sibgatullin's method. Black holes. Multipole moments. Ernst's equations. Integrable systems.

## RESUMO

COSTA FILHO, E. S. Construção de novas soluções das equações de Einstein no eletrovácuo. 2020. 149p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

Nesta dissertação, introduzimos o método de Sibgatullin, discutimos sua construção e aplicação no caso de interesse de N-Solitons em espaçostempo no eletrovácuo. Tendo feito essa discussão, introduzimos a relação entre os momentos de multipolo em relatividade geral e as soluções do sistema acoplado de equações de Einstein-Maxwell. Fomos capazes de generalizar resultados já conhecidos na literatura.

Fez-se também uma breve discussão sobre os momentos multipolares em si e mais ainda, corrigimos um resultado já conhecido da literatura. .

Palavras-chave: Método de Sibgatullin. Buracos negros. Momentos multipolares. Equações de Ernst, Sistemas integráveis.

## LIST OF FIGURES

Figure 1 - Representation of the s-plane. ..... 50
Figure 2 - Representation of the s-plane considering the $F$ 's branch cut. ..... 60
Figure 3 - Localization of the compact object with respect to the $z$ axis. a)Considering the roots as real, $\left.\alpha_{ \pm}= \pm \sigma . \mathrm{b}\right)$ Considering the roots as imaginary, $\alpha_{ \pm}= \pm i \sigma$. ..... 67
Figure 4 - Several N-body configurations, representing black holes and hyper- extreme objects ..... 79
Figure 5 - a)This scheme represents the situation between two sub-extreme objects. All roots $\alpha$ are real. b)This scheme represents a interaction between two hyper-extreme. All roots $\alpha$ are complex. c) This scheme represents the interaction between a sub-extreme and a hyper-extreme objects. Two roots are real and two roots are complex. ..... 86
Figure $6-$ a) Overlapping of a sub-extreme configuration. b) Overlapping of one sub-extreme and a hyper-extreme objects. c) Overlapping of a hyper- extreme configuration. ..... 92
Figure 7 - Localization of the extreme objects on the symmetry axis ..... 94

# LIST OF ABBREVIATIONS AND ACRONYMS 

| RHP | Riemann Hilbert Problem |
| :--- | :--- |
| EM | Einstein-Maxwell |
| RHS | Right hand side |
| LHS | Left hand side |

## CONTENTS

1 INTRODUCTION ..... 21
2
FIELD EQUATIONS IN A STATIONARY AXISSYMETRIC SPACE- TIMES ..... 25
2.1 Generalities ..... 25
2.2 Field Equations ..... 27
2.2.1 Ricci Tensor ..... 27
2.2.2 Equations of electromagnetism ..... 28
2.2.3 The Einstein-Maxwell Equations ..... 29
2.3 Weyl-Papapetrou metric and Boundary Conditions ..... 30
3 ERSNT FORMALISM ..... 33
3.1 Derivation of the Field Equations ..... 33
3.2 New Solutions from Symmetry Transformations ..... 36
3.3 An Exact Trivial Solution ..... 37
4 SIBGATULLIN'S METHOD ..... 39
4.1 Lax pair associated with the Einstein-Maxwell Equations ..... 39
4.2 Lie Algebra associated with the Einstein-Maxwell Equations ..... 47
4.3 Correlation between the inner transformation of the group, the ini- tial values and the transformed values ..... 55
4.4 About the Method ..... 62
4.4.1 A Basic Example ..... 64
5 N-SOLITONIC SPACETIME ..... 69
5.1 Derivation of the N -Soliton Solution ..... 69
5.1.1 $\quad \mathrm{N}$-soliton solution for extreme cases ..... 77
5.1.2 On the Equilibrium Equations ..... 78
5.2 2-soliton solution ..... 80
5.2.1 Binary system ..... 85
5.2.2 Tomimatsu-Sato solution with $\delta=2$ ..... 86
5.2.3 Charged, Magnetized Tomimatsu-Sato $\delta=2$ Solution ..... 88
5.2.4 Metric of a rotating charged magnetized object ..... 91
5.3 The Tomimatsu double-Kerr solution ..... 93
THE MULTIPOLE MOMENTS AND AXIS DATA ..... 97
6.1 Relations between the Ernst potentials and multipole moments in vacuum case ..... 98
6.2 Relations between the Ernst potentials and multipole moments in the electrovacuum case ..... 102
6.3 Multipole moments of the $\mathbf{N}$-Soliton solution ..... 106
6.3.1 Metric of a rotating charged magnetized sphere ..... 109
6.3.2 Tomimatsu-Sato solution with $\delta=2$ ..... 111
7 CONCLUSIONS ..... 113
BIBLIOGRAPHY ..... 115
APPENDIX ..... 123
APPENDIX A - THE RIEMANN HILBERT PROBLEM ..... 125
A. 1 The Riemann Hilbert Problem ..... 125
A. 2 Generation of New Solutions ..... 125
APPENDIX B - ERNST POTENTIALS IN THE PRESENCE OF A COSMOLOGICAL CONSTANT ..... 129
APPENDIX C - MULTIPOLE MOMENTS ..... 133
C. 1 Axisymmetric electrovacuum ..... 135
C. 2 Fodor-Hoenselaers-Perjés ..... 138

## 1 INTRODUCTION

The theory of General Relativity (GR) has been successful in describing the gravitational interaction. It is a classical field theory in which the dynamics of the fields generated by a source (a matter distribution) are described by means of highly nonlinear differential equations. ${ }^{1}$ Due to this nonlinearity, these coupled equations are extremely hard to solve analytically in a general case. Therefore, the study of symmetries is essential in GR to simplify and decouple the equations in order to solve them and find exact solutions. ${ }^{2}$

The relevance of the study of exact solutions lies not only in the fact that they describe, exceptionally well, several astrophysical objects but also in the fact that exact solutions bring profound knowledge about the theory's physical structure. Therefore, several authors have put effort into understanding how the symmetries act in GR and have also been trying to construct techniques to find exact solutions possessing the desired symmetries. In particular, the study of stationary axisymmetric spacetimes possesses an enormous physical interest (even more when Einstein's equations are coupled with the electromagnetic fields) because they can describe, in a idealized way, for instance, the exterior region of black holes, neutron stars, and accretion flow . Such spacetimes admit the existence of two commuting Killing: one time-like and the other space-like. By itself, these symmetries simplify a lot the coupled system of Einstein-Maxwell equations. The stationary and axisymmetric spacetimes, which admits $G_{2}$ as isometry group*, possess a set of completely integrable equations. ${ }^{2,4}$ Over the years, different authors developed several solution generating techniques and attempted to give them physical significance.

The majority of these techniques are based on the two Ernst potentials, introduced in 1968 by Ernst (first presented for the vacuum case, ${ }^{5}$ and then also for electrovacuum case $^{6}$ ), that facilitated the study of spacetimes with two commuting Killing vectors because these two equations are highly symmetric, which made it possible to introduce a systematic group theoretical study of the Einstein's equations. Geroch ${ }^{7,8}$ introduced an infinitedimensional group of internal symmetries in which he could generate a two-parameter family of vacuum solutions from a known one. Geroch also conjectured "that any two exact solutions with a pair of commuting Killing fields (and whose two constants vanish) will, at least locally, be related by one of the transformations described here", in particular, that all the asymptotically flat, stationary axisymmetric spacetimes can be generated in this way by starting from the Minkowski spacetime. Then, in a series of papers, Kinnersley ${ }^{9-14}$ extended Geroch's group to consider the coupling with the Maxwell field equations and provided a representation to it in terms of an infinite hierarchy of potentials. They also

[^0]showed that these hierarchies of potentials of a given solution could be associated with a $3 \times 3$ matrix generating function $F$ for the electrovacuum ( $2 \times 2$ matrix for the vacuum case ) that could be linked with the spectral problem and hence, soliton techniques could be used for seeking new solutions.

Ernst and Hauser, ${ }^{15,16}$ exploiting the Kinnersley extended group, constructed a singular linear integral equation for this generating function $F$, which links the desired new function $F$, a seed function and the exponentiated Kinnersley group. After that, they also deduced the same formulae, now not directly making use of the group properties, but basing it on the classical homogeneous Hilbert Riemann problem. ${ }^{17,18}$ Each element of the Geroch Group $\boldsymbol{K}^{\prime}$ ( $\boldsymbol{K}$ for the vacuum case) can be represented by a matrix $u(s)$ (which only depends on the complex parameter $s$ ) and is defined outside of a closed contour $L$ in the complex $s$-plane. They showed explicitly that,for stationary axisymmetric electrovacuum spacetimes, new solutions can be mapped to any other solution with given Ernst potentials on the symmetry axis, through an inner symmetry transformations. And they showed how to construct the solution in the whole spacetime by means of system of linear integral equations, therefore, proving the Geroch conjecture. ${ }^{19}$

Following the previous ideas, Sibgatullin ${ }^{20-22}$ has also introduced a generating technique for the Einstein-Maxwell equations and also introduced the coupling with the neutrino field (massless spin- $1 / 2$ field). Although his derivation was not based on the Ernst and Hauser integral method, the resulting equation can be seen as a simplification of that. Both use a seed function and the unknown function on the symmetry axis and then extended it to the whole spacetime. But for simplification, Sigbatullin decided always to use the Minkowski spacetime as a seed solution and also expressed Geroch's group parameters in terms of the input data. ${ }^{23}$ In particular when the desired solution has rational Ernst functions on the symmetry axis, this implies that the linear integral equation is reduced to an algebraic system (without the need to apply multiple transformations). Compared to the other methods, another advantage of using the Sigbatullin integral method is because it provides a more clear relation between the input parameters in the solution and their physical meaning. ${ }^{24,25}$

Through the Sibgatullin integral method, Ruiz and Manko introduced a new family of asymptotically flat solutions named " $N$-soliton solution", containing $3 N$ complex arbitrary parameters, that can describe a quite variety of compact objects, among them, a set of $N$ aligned Kerr-Newman black-holes in a very compact way. ${ }^{26,27}$ Years later, Manko also presented the equilibrium conditions in the axisymmetric systems for aligned $N$ bodies, in particular, for the $N$-soliton family. ${ }^{28,29}$

It is worth emphasizing that the generating techniques do not carry by themselves the physical meaning of the new solutions. Therefore, the physical study to understand the properties and interpret the solutions correctly is an essential point. One way to
characterize a solution physically is to employ the conservative currents and, hence, use the Komar integrals. ${ }^{30}$ Another procedure is through the multipolar expansion in GR, which links the properties of the fields at the infinity (that is, in an exterior region far away from the source) with the properties of the sources, defined by Geroch ${ }^{31,32}$ (static vacuum case) and Hansen ${ }^{33}$ (stationary vacuum case) and Beig-Simon ${ }^{34,35}$ (stationary electrovacuum case). Although Geroch and Hansen have given a well-posed definition of the multipole moments in GR, it was only after Fodor et al. ${ }^{36}$ introduce an algorithm to evaluate the momenta, that it became practical to use them in order to characterize new solutions.

As one would expect, the multipole moments, as it will be proved, are given in terms of the corresponding data on the axis. A first development that helped us to contour this problem was a previous work provided by Manko, ${ }^{24}$ for the vacuum case, showing how to fully parameterize the $N$-soliton solution in terms of $2 N$ physical parameters introduced by Fodor. In other words, it is possible, through the Sibgatullin method, to write the Ernst potential of a given $N$-soliton solution in terms of the multipole moments.

This work is then organized as follows. In chapter 2 the proper definition of stationary axisymmetric spacetimes as well as the derivation of the equations of motion in terms of the Kinnersley notation is presented.. After that, the Weyl-Papapetrou coordinates are introduced together with a first definition of asymptotically flat spacetimes. In chapter 3 a brief revision of the Ernst potentials and their symmetries is made. Following the Sibgatullin original construction, chapter 4 gives a full revision in the method, starting with the introduction of the Kinnersley-Chitre potentials and some of their properties, explaining the algebra given by Sibgatullin and how he introduced his integral formula, but also giving some physical context. Again, it is a solution method of the EinsteinMaxwell field equations for spacetimes with two Killing vectors, which aims to explore the Einstein-Maxwell equations exact solutions. Instead of solving nonlinear differential equations, all that is needed is to solve linear integral equations. This chapter also presents a homogeneous Hilbert Riemann problem, based on Ernst and Hauser's developments, to clarify some assumptions and choices made by Sibgatullin. In chapter 5, a full revision on the $N$-soliton solution is made, discussing and giving examples how to construct the family solutions, but also how to describe up to $N$ aligned objects, and how the $3 N$ parameters should be interpreted, mathematically, to describe the solution. More importantly, this chapter gives a basis for the new results contained in chapter 6. Following Manko's idea, chapter 6 is an extension of paper 24 to consider electrovacuum spacetimes, that is, to write the Ernst potentials in terms of the multipole moments. Moreover, this chapter also shows that any finite set of multipole moments can be associated with an $N$-soliton solution. Appendix A is brief revision on the Homogeneous Riemann Hilbert problem introducing some necessary concepts used in chapter 4 . Moreover, appendix B provides an extension for the Ernst potentials to consider the presence of a cosmological constant, showing then
that similar approaches could be derived for cosmological models. And finally, appendix C is a revision on the multipole moments introduced by Geroch, on the Fodor algorithm to evaluate them, but also, it presents some corrections which led to wrong electrovacuum multipole moments in literature.

## 2 FIELD EQUATIONS IN A STATIONARY AXISSYMETRIC SPACETIMES

### 2.1 Generalities

The theory of Einstein is a theory to describe the gravitational field geometrically using a 4 -d manifold endowed with a pseudo-RIemannian metric that is symmetric, i.e. $g_{\mu \nu}=g_{\nu \mu}(+---)$. It has then turned out that the equations can describe successfully astrophysical objects and even the evolution of the universe itself. Consider the field equations for Einstein's general theory of relativity in the absence of cosmological constant coupled to the source-free Maxwell equations, then the coupled system of equations are:

$$
\begin{align*}
R_{\mu \nu}-\frac{g_{\mu \nu}}{2} R & =-2 T_{\mu \nu}  \tag{2.1}\\
F^{\mu \nu} ;{ }_{\nu} & =0 \tag{2.2}
\end{align*}
$$

Where $R_{\mu \nu}$ is the Ricci tensor with scalar curvature $R$ and $T_{\mu \nu}$ is the energymomentum tensor of the matter. In principle, a solution of the equation (2.1) fully determines a metric, that is, a spacetime, after the boundary conditions are set. Here, it is set $\frac{8 G}{\pi c^{4}}=2$. The left-hand side of (2.1) is the description of the geometry of the spacetime. On the other hand, the right-hand side gives the information about the distribution and flow of the matter and its associated fields. Therefore, the equation (2.1) represents the coupling between geometry and matter. ${ }^{37}$ This work will only deal with the particular case in which the energy-momentum tensor is due to electromagnetic fields and will only consider the coupled system of Einstein-Maxwell equations exterior of a source (electrovacuum solutions), that is:

$$
\begin{equation*}
T_{\mu \nu}=F_{\mu}{ }^{\sigma} F_{\nu \sigma}-\frac{1}{4} g_{\mu \nu} F_{\sigma}{ }^{\tau} F_{\tau}^{\sigma} \tag{2.3}
\end{equation*}
$$

By its turns, the entity $F^{\mu \nu}$ carries information about the electromagnetic fields; hence, equation (2.2) is the covariant form of the Maxwell's equations which gives the movement equations of the electromagnetic fields.

The physical situation considered in the present work deals with the gravitational and electromagnetic fields, in a region outside the source, such that the generated spacetime is stationary and axisymmetric. In the following, we will explain what we mean with stationary and axisymmetric

A spacetime is said to be stationary if tehre exists a timelike Killing vector $k$, which can be normalized as $k \cdot k=1$. This implies the existence of a one-parameter group $\phi_{t}$,
whose orbits are timelike. Since the curves in the proper time are also timelike, the flows of $\phi_{t}$ can represent time flow in $\mathcal{M}$. The general stationary metric is ${ }^{38}$ :

$$
\begin{equation*}
d s^{2}=g_{1 i}(d t)^{2}+g_{1 i} d t d x^{i}+g_{i j} d x^{i} d x^{j} ; \quad i, j=2,3,4 \tag{2.4}
\end{equation*}
$$

A stationary spacetime is static if $g_{1 i}=0$.
A spacetime is said to be axisymmetric if it has a spacelike Killing vector $\eta$, such that the one-parameter group of isometries associated with it, $\psi_{\varphi}$, possess closed orbits. ${ }^{38}$ $\eta$ is normalized so that the flow $\psi_{\varphi}$ fulfills the condition $\psi_{\varphi}=\psi_{\varphi+2 \pi}$.

The definition of a stationary and axisymmetric spacetime is of a spacetime which admits a timelike Killing vector field and a spacelike Killing vector field with closed integral curves. ${ }^{37}$ It is also a demand that the Killing vectors commute*:

$$
\begin{equation*}
[\eta, k]=0 \tag{2.5}
\end{equation*}
$$

Therefore, it is possible to use theses parameters as time and space coordinates, namely $x^{1}=t$ and $x^{2}=\varphi$. Intuitively, this metric can be associated with an object which has the freedom to rotate on the axis of symmetry. Physically, this also assumes the existence of a reflection symmetry or a motion reversal $(t, \varphi) \rightarrow(-t,-\varphi)$. Mathematically, it means that the metric must be block diagonal ${ }^{9}$

$$
\left[g_{\mu \nu}\right]=\left[\begin{array}{cc}
{\left[f_{a b}\right]} & 0  \tag{2.6}\\
0 & {\left[-h_{m n}\right]}
\end{array}\right]
$$

And the line element may be decomposed as:

$$
\begin{align*}
d s^{2} & =d s_{1}^{2}-d s_{2}^{2}  \tag{2.7}\\
d s_{1}^{2} & =f_{a b} d x^{a} d x^{b}, a, b=1,2  \tag{2.8}\\
d s_{2}^{2} & =h_{m n} d x^{m} d x^{n}, m, n=3,4 \tag{2.9}
\end{align*}
$$

where the metric coefficients are only functions of $x^{3}$ and $x^{4}$. To raise indices in the two-dimensional space with the metric $f_{A B}$, we may use either the inverse metric $\left(f_{A B}\right)^{-1}$ or the alternating symbol $\mathcal{E}^{A B}= \pm 1$, and both choices have certain advantages. Unless otherwise stated ${ }^{\dagger}$, in what follows indices will be raised or lowered using $\mathcal{E}^{A B}$ and $\left(h_{M N}\right)^{-1}$ for the two-dimensional spaces $d s_{1}^{2}$ and $d s_{2}^{2}$, namely $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$, respectively. Otherwise,

[^1]when it becomes necessary to raise an index using $\left(f_{A B}\right)^{-1}$, it will be made explicit. For example:
\[

\mathcal{E}^{a b}=\left($$
\begin{array}{cc}
0 & 1  \tag{2.10}\\
-1 & 0
\end{array}
$$\right) \quad \mathcal{E}_{a b}=\left(\mathcal{E}^{a b}\right)^{-1} \quad f_{a}{ }^{b}=\mathcal{E}^{b c} f_{a c} \quad A^{b}=\mathcal{E}^{b c} A_{c}
\]

And then

$$
\left(f_{a b}\right)^{-1}=-\rho^{-2} f^{a b}=-\rho^{-2} \mathcal{E}^{a c} \mathcal{E}^{b d} f_{c d} \quad f_{a}{ }^{b} f_{b c}=\rho^{2} \mathcal{E}_{a c} \quad V_{b} A^{b}=-V^{b} A_{b}
$$

where $\rho^{2} \equiv-\operatorname{det}\left(f_{A B}\right)$. The covariant derivative associated with $d s_{2}^{2}, \nabla_{2}$, can be written as:

$$
\begin{equation*}
\nabla_{2} \cdot \mathbf{V}=h^{-\frac{1}{2}}\left(h^{\frac{1}{2}} h^{M N} V_{M}\right){,_{N}} \tag{2.11}
\end{equation*}
$$

One should notice that the two-dimensional space $d s_{2}^{2}$ can always be brought to the diagonal form conformal to the $x^{3}-x^{4}$-plane ${ }^{39}$ :

$$
\begin{equation*}
d s_{2}^{2}=e^{\mu}\left[\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2}\right] \tag{2.12}
\end{equation*}
$$

Here $\mu$ is some function of $x^{3}$ and $x^{4}$, which means that the $d s_{2}^{2}$ is conformally invariant. This implies that if $\nabla_{2} \cdot \mathbf{V}$ is zero then $\bar{\nabla} \cdot \mathbf{V}$, where $\bar{\nabla} \equiv\left(\partial_{x^{3}}, \partial_{x^{4}}\right)$, will also be zero. Here, the derivative operator $\bar{\nabla}$ is associated with the two-dimensional space:

$$
\begin{equation*}
d \bar{s}_{2}^{2}=\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2} \tag{2.13}
\end{equation*}
$$

### 2.2 Field Equations

This section aims to obtain the Einstein-Maxwell field equations. It will be shown that this coupled system of equations, written in terms of the given metric, can be put into a total divergence form.

### 2.2.1 Ricci Tensor

It was shown in ${ }^{9}$ that the Ricci tensor can be separated in three blocks: $R_{b c}, R_{a m}$ and $R_{m n}$, although, due to the block diagonal form of the metric, $R_{a m}$ vanishes automatically. As will be shown, only the block $R_{b c}$ of the Ricci tensor brings important information; hence, only it will be considered in the following.

$$
\begin{equation*}
R_{b c}=\Gamma_{b c, \mu}^{\mu}-\Gamma_{\mu b, c}^{\mu}+\Gamma_{\mu \nu}^{\mu} \Gamma_{c b}^{\nu}-\Gamma_{c \nu}^{\mu} \Gamma_{\mu b}^{\nu} \tag{2.14}
\end{equation*}
$$

The non-zero components of the Christoffel symbols are:

$$
\begin{gather*}
\Gamma_{c m}^{a}=\frac{1}{2} g^{a b} g_{b c, m} \quad \Gamma_{a b}^{m}=\frac{-1}{2} g^{m n} g_{a b, n}, \quad a, b, c=1,2 \\
\Gamma_{k l}^{m}=\frac{1}{2} g^{m n}\left(g_{n k, l}+g_{n l, k}-g_{l k, n}\right), \quad k, l, m, n=3,4 \tag{2.15}
\end{gather*}
$$

In terms of the metric, after basic manipulations, the block of the Ricci tensor takes the form:

$$
\begin{equation*}
R_{b c}=\frac{-1}{2}\left[\left(g^{m n} g_{b c, n}\right)_{, m}+\frac{1}{2} g^{m n} g^{d e} g_{d e, m} g_{c b, n}+\frac{1}{2} g^{m l} g^{n k} g_{m l, n} g_{c b, k}-g^{n m} g^{e d} g_{c e, m} g_{d b, n}\right] \tag{2.16}
\end{equation*}
$$

Following Kinnersley, multiplying the equation above by $f^{a b}\left(=-\rho^{2} g^{a b}\right)$ and noting that $f^{a b}{ }_{, m}=f^{a b} g^{e d} g_{e d, m}-f^{a e} g^{b d} g_{e d, m}$, one has:

$$
\begin{array}{r}
f^{a b} R_{b c}=\frac{-1}{2}\left\{f^{a b}\left[\left(g^{m n} g_{b c, n}\right)_{, m}+\frac{1}{2} g^{m n} g^{d e} g_{d e, m} g_{c b, n}+\frac{1}{2} g^{m l} g^{n k} g_{m l, n} g_{c b, k}\right]+\right. \\
\left.+g^{n m}\left[f^{a b} g^{e d} g_{e d, m} g_{b c, n}-g_{b c} f^{a b}{ }_{, m}\right]\right\} \tag{2.17}
\end{array}
$$

Now, using the property that $\partial_{\alpha} g=g g^{\beta \gamma} \partial_{\alpha} g_{\beta \gamma}$, where $g$ is the determinant of the matrix $g_{\alpha \beta}$, it is possible to write the Ricci tensor as divergence:

$$
\begin{equation*}
f^{a b} R_{b c}=\frac{-1}{2} \frac{\rho}{\sqrt{h}}\left(\sqrt{h} \rho^{-1} f^{a b} g_{b c, n}\right)_{, m} \tag{2.18}
\end{equation*}
$$

### 2.2.2 Equations of electromagnetism

Considering a spacetime stationary and axially symmetric, the antisymmetric tensor of the electromagnetic field

$$
F_{\mu \nu}=A_{\nu, \mu}-A_{\mu, \nu}
$$

is independent of $t$ and $\varphi$. Due to the gauge freedom to choose the electromagnetic 4 -potential, $A_{\mu}$ can be made independent of $t$ and $\varphi$ as well. If one also excludes the existence of a possible line current along the symmetry axis, then $A_{m}=0$ and the only non-zero components of the electromagnetic field are:

$$
\begin{equation*}
F_{a m}=-A_{a, m} \tag{2.19}
\end{equation*}
$$

Consequently, the the source-free Maxwell equations to be solved can also be put into the form:

$$
\begin{equation*}
\left(\sqrt{-g} F^{a m}\right)_{, m}=\left(-\rho \sqrt{h} g^{a b} g^{m n} A_{b, n}\right)_{, m}=\left(\rho^{-1} \sqrt{h} f^{a b} g^{m n} A_{b, n}\right)_{, m}=0 \tag{2.20}
\end{equation*}
$$

The energy-momentum tensor for the electromagnetic fields is given by:

$$
\begin{equation*}
T_{\mu \nu}=F_{\mu}{ }^{\sigma} F_{\nu \sigma}-\frac{1}{4} g_{\mu \nu} F_{\sigma}{ }^{\tau} F_{\tau}^{\sigma} \tag{2.21}
\end{equation*}
$$

Notice it is trace free. The components of the block $T_{b c}$ are then given by:

$$
\begin{equation*}
T_{b c}=g^{n m} A_{c, m} A_{b, n}-\frac{1}{2} g_{b c} g^{e d} A_{d, n} A_{e, m} \tag{2.22}
\end{equation*}
$$

Multiplying by $f^{a b}=\left(-\rho^{2} g^{a b}\right)$

$$
\begin{equation*}
f^{a b} T_{b c}=f^{a b} g^{n m} A_{c, m} A_{b, n}-\frac{1}{2} \delta^{a}{ }_{c} f^{e d} g^{n m} A_{d, n} A_{e, m} \tag{2.23}
\end{equation*}
$$

Using the equations of motion, the property $\partial_{\alpha} g=g g^{\beta \gamma} \partial_{\alpha} g_{\beta \gamma}$ and the identity $\varepsilon^{a d} V_{d b}-\varepsilon^{a d} V_{b d}=\delta^{a}{ }_{b} \varepsilon^{c d} V_{d c}$, the energy momentum tensor can be written in a divergent form:

$$
\begin{equation*}
f^{a b} T_{b c}=\frac{1}{2} \frac{\rho}{\sqrt{h}}\left[\rho^{-1} \sqrt{h} g^{m n}\left(f^{a b} A_{c} A_{b, n}+f_{c}{ }^{d} A^{a} A_{d, n}\right)\right]_{, m} \tag{2.24}
\end{equation*}
$$

### 2.2.3 The Einstein-Maxwell Equations

The the Einstein-Maxwell equations (EM equations) is written as: in (2.1). But the electromagnetic energy-momentum tensor is traceless, which implies that the Ricci scalar is zero. Thus, the EM equations multiplied by by $f^{a b}$ can be written as a total divergence form. This reads:

$$
\begin{equation*}
\left[\rho^{-1} \sqrt{h} g^{m n}\left(f^{a b} f_{b c, n}-2 f^{a b} A_{c} A_{b, n}-2 f_{c}^{d} A^{a} A_{d, n}\right)\right]_{, m}=0 \tag{2.25}
\end{equation*}
$$

Remembering the properties discussed in Section 2.1 , if $\nabla_{2} \cdot \mathbf{V}$ is zero then $\bar{\nabla} \cdot \mathbf{V}$ is zero as well. Hence, the equations that describe stationary and axisymmetric gravitational and electromagnetic fields, for the components of the block $a, b=1,2$, become independent of the metric components $g_{m n}$. Thus, in order to solve the EM equations one just needs to find the integrability conditions for the equations:

$$
\begin{align*}
{\left[\rho^{-1}\left(f^{a b} f_{b c, n}-2 f^{a b} A_{c} A_{b, n}-2 f_{c}^{d} A^{a} A_{d, n}\right)\right], n } & =0  \tag{2.26}\\
\left(\rho^{-1} f^{a b} A_{b, n}\right)_{, n} & =0 \tag{2.27}
\end{align*}
$$

### 2.3 Weyl-Papapetrou metric and Boundary Conditions

Papapetrou has shown that the most general line element for a spacetime with the prescribed symmetries, in the absence of cosmological constant, can be written in the following form ${ }^{40}$ :

$$
\begin{equation*}
d s^{2}=f\left(d t^{2}-\omega d \varphi\right)^{2}-f^{-1}\left[e^{2 \gamma}\left(d \rho^{2}+d z^{2}\right)+\rho^{2} d \varphi^{2}\right] \tag{2.28}
\end{equation*}
$$

Here the coordinate system utilized is composed of what is called the Weyl canonical coordinates $(t, \varphi, \rho, z)$. Just as before, the metric functions depend upon the spatial coordinates $x^{3}$ and $x^{4}$, e.g., $z \in(-\infty, \infty)$ and $\rho \in[0, \infty)$. The function $\omega$ is related to the angular momentum of the sources, which can represent rotations around the axis at $\rho=0$. So far, Cartesian derivative operators were considered, but the metric above induces, due to convenience, the use of polar coordinates, that is:

$$
\begin{equation*}
d s_{3}^{2}=d \rho^{2}+d z^{2}+\rho^{2} d \varphi^{2} \tag{2.29}
\end{equation*}
$$

The derivative operators are connected through:

$$
\begin{equation*}
\nabla \cdot \mathbf{V}=\rho^{-1} \bar{\nabla} \cdot(\rho \mathbf{V}) \tag{2.30}
\end{equation*}
$$

Using this system of coordinates, the equations of motion for stationary exterior fields with axial symmetry, one arrives at the following set of coupled equations (see equations (2.26), (2.27)):

$$
\begin{align*}
& \nabla \cdot\left[\rho^{-2} f\left(\nabla A_{2}+\omega \nabla A_{1}\right)\right]=0  \tag{2.31}\\
& \nabla \cdot\left[f^{-1} \nabla A_{1}-\rho^{-2} f \omega\left(\nabla A_{2}+\omega \nabla A_{1}\right)\right]=0  \tag{2.32}\\
& \nabla \cdot\left[\rho^{-2} f^{2} \nabla \omega+4 \rho^{-2} f A_{1}\left(\nabla A_{2}+\omega A_{1}\right)\right]=0  \tag{2.33}\\
& f \nabla^{2} f=(\nabla f)^{2}-\rho^{-2} f^{4}(\nabla \omega)^{2}+2 f\left(\nabla A_{1}\right)^{2}+2 \rho^{-2} f^{3}\left(\nabla A_{2}+\omega \nabla A_{1}\right)^{2}  \tag{2.34}\\
& \gamma, \rho=\frac{1}{4} \rho f^{-2}\left[f_{, \rho}^{2}-f_{, z}^{2}-\rho^{-2} f^{4}\left(\omega,{ }_{\rho}^{2}-\omega,_{z}^{2}\right)\right]+\rho^{-1} f\left[\left(A_{2, \rho}+\omega A_{1, \rho}\right)^{2}-\right. \\
& \left.-\left(A_{2, z}+\omega A_{1, z}\right)^{2}\right]-2 \rho f^{-1}\left(A_{1},{ }_{\rho}^{2}-A_{1},{ }_{z}^{2}\right)  \tag{2.35}\\
& \gamma, z=\frac{1}{2} \rho f^{-2}\left(f,, \rho f_{, z}-\rho^{-2} f^{4} \omega, \rho \omega, z\right)+2 \rho^{-1} f\left(A_{2, \rho}+\omega A_{1, \rho}\right)\left(A_{2, z}+\omega A_{1, z}\right)- \\
& -2 \rho f^{-1} A_{1, \rho} A_{1, z} \tag{2.36}
\end{align*}
$$

One should notice that once the equations for $f, \omega, A_{1}$ and $A_{2}$ are solved, then the remaining function $\gamma$ can be found by integrating equations (2.35) and (2.36). In other words, the integrability conditions for the function $\gamma$ are the system (2.31) - (2.34). Thus, the primary aim is to find the other functions first. The derivative operators $\nabla$ are to be understood as being associated with the "nonphysical" metric (2.29).

The solutions of the EM field equations for stationary spacetimes with axial symmetry, as seen before, are said to be integrable. Then, several techniques to generate new solutions can be applied using a "seed" solution. One of these methods, the Sibgatullin's integral method, will be described in chapter 4.

In order for a given solution to possess some physical meaning, we need to impose appropriate boundary conditions. First of all, the line element (2.28) must be asymptotically flat, that is, at infinity, it must converge to the Minkowski metric, which takes the form in cylindrical coordinates:

$$
\begin{equation*}
d s^{2}=d t^{2}-\rho^{2} d \varphi^{2}-d z^{2}-d \rho^{2} \tag{2.37}
\end{equation*}
$$

This means that the fields tend to zero very far from the sources and that there are no sources at infinity. ${ }^{41}$ :

$$
\begin{equation*}
\lim _{\rho \rightarrow \infty} f(z, \rho)=1, \quad \lim _{\rho \rightarrow \infty} \omega(z, \rho)=0, \quad \lim _{\rho \rightarrow \infty} \gamma(z, \rho)=0 \tag{2.38}
\end{equation*}
$$

By imposing that the azimuthal angular coordinate is periodic, we can impose conditions for the functions $f, \omega$ and $\gamma$ on the symmetry axis $(\rho=0)$.

Being $\eta^{\varphi}$ the spacial Killing vector associated with the axial symmetry, its norm, $\eta^{\varphi} \eta_{\varphi}$ must be equal to zero in the symmetry axis. Thus:

$$
\begin{equation*}
\lim _{\rho \rightarrow 0}|\eta|^{2}=\lim _{\rho \rightarrow 0}\left(f^{-1} \rho^{2}-f \omega\right)=0 \tag{2.39}
\end{equation*}
$$

The limit of $f$ must be different from zero away from singularities in the space-time, i.e. for globally regular spacetimes. Therefore:

$$
\begin{equation*}
\lim _{\rho \rightarrow 0} f(z, \rho) \neq 0, \quad \lim _{\rho \rightarrow 0} \omega(z, \rho)=0 \tag{2.40}
\end{equation*}
$$

On the other hand, the spacetime must be locally conformal to Minkowski spacetime. Consider:

$$
\begin{equation*}
d s^{2}=f^{-1} e^{2 \gamma}\left(f^{2} e^{-2 \gamma}\left(d t^{2}-\omega d \varphi\right)^{2}-d \rho^{2}-d z^{2}-e^{-2 \gamma} \rho^{2} d \varphi^{2}\right) \tag{2.41}
\end{equation*}
$$

for $t, z, \rho$ fixed. This gives:

$$
\begin{equation*}
d s^{2}=f^{-1} e^{2 \gamma} d s_{c}^{2} \tag{2.42}
\end{equation*}
$$

Now, the circumference of such an infinitesimal circle has to be $2 \pi$ times the infinitesimal radius (the ratio between circumference and radius of a small circle on the $z$-axis must approach $2 \pi$ as the radius goes to zero, guaranteeing a locally Minkowski space ${ }^{3}$ ),i.e.:

$$
\begin{equation*}
d s_{c}=e^{-\gamma} \sqrt{f^{2} \omega^{2}-\rho^{2}} d \varphi \tag{2.43}
\end{equation*}
$$

And hence, for the limit where the radius goes to zero ${ }^{42}$ :

$$
\begin{equation*}
\int_{0}^{2 \pi} e^{-\gamma} \sqrt{f^{2} \omega^{2}-\rho^{2}} d \varphi \stackrel{!}{=} 2 \pi \rho \tag{2.44}
\end{equation*}
$$

This condition can only be fulfilled if $\gamma$ is zero in such limit. Then, the conditions that the metric functions must satisfy to ensure elementary flatness on the symmetry axis are ${ }^{41}$ :

$$
\begin{equation*}
\lim _{\rho \rightarrow 0} f(z, \rho) \neq 0, \quad \lim _{\rho \rightarrow 0} \omega(z, \rho)=0, \quad \lim _{\rho \rightarrow 0} \gamma(z, \rho)=0 \tag{2.45}
\end{equation*}
$$

## 3 ERSNT FORMALISM

As seen in chapter 2, the EM equations are written as a system of 4 partial differential equations of second order. In 1968, Ernst introduced a new formalism ${ }^{5,6}$ which simplifies the four EM equations (for four functions, namely $f, \omega, A_{1}$ and $A_{2}$ ) into two differential partial equations (for two complex functions), introducing first the potential $\mathcal{E}$ for vacuum case, ${ }^{5}$ and subsequently for the electrovacuum by adding a second complex potential $\Phi$ describing the electromagnetic field. ${ }^{6}$ These new equations are highly symmetric. Therefore, it is a strong tool when wanting to construct new solutions.

One possibility to deduce the Ernst equations would be by starting from the results of chapter 2. However, to justify and compare with appendix B, the Ernst equations will be deduced in a similar way as Ernst introduced them.

### 3.1 Derivation of the Field Equations

Let $\Psi=\Psi(\rho, z)$ be a function and $\hat{e}_{\varphi}$ be the unitary vector in the azimuthal direction. Then, the following identity is valid:

$$
\begin{equation*}
\nabla \cdot\left(\rho^{-1} \hat{e}_{\varphi} \times \nabla \Psi\right)=0 \tag{3.1}
\end{equation*}
$$

In view of the identity above, the equation (2.31) may be considered as the integrability condition for the existence of a new potential $A_{2}^{\prime}$ :

$$
\begin{equation*}
\rho^{-1} f\left(\nabla A_{2}+\omega \nabla A_{1}\right)=\hat{e}_{\varphi} \times \nabla A_{2}^{\prime} \tag{3.2}
\end{equation*}
$$

Taking the vectorial product with $\hat{e}_{\varphi}$, it is possible to find the following relation:

$$
\begin{equation*}
\rho^{-1} \hat{e}_{\varphi} \times \nabla A_{2}=-\left(f^{-1} \nabla A_{2}^{\prime}+\rho^{-1} \omega \hat{e}_{\varphi} \times \nabla A_{1}\right) \tag{3.3}
\end{equation*}
$$

Taking the divergence of this equation, we have:

$$
\begin{equation*}
\nabla \cdot\left(f^{-1} \nabla A_{2}^{\prime}+\rho^{-1} \omega \hat{e}_{\varphi} \times \nabla A_{1}\right)=0 \tag{3.4}
\end{equation*}
$$

Therefore, the equation (2.32) may be written as:

$$
\begin{equation*}
\nabla \cdot\left(f^{-1} \nabla A_{1}-\rho^{-1} \omega \hat{e}_{\varphi} \times \nabla A_{2}^{\prime}\right)=0 \tag{3.5}
\end{equation*}
$$

Let us introduce the new complex potential:

$$
\begin{equation*}
\Phi=A_{1}+i A_{2}^{\prime} \tag{3.6}
\end{equation*}
$$

which puts the equations (2.31) and (2.32) into the form:

$$
\begin{equation*}
\nabla \cdot\left[f^{-1} \nabla \Phi-i \rho^{-1} \omega \hat{e}_{\varphi} \times \nabla \Phi\right]=0 \tag{3.7}
\end{equation*}
$$

Using the same procedure, the equation (2.33) can be written as:

$$
\begin{equation*}
\nabla \cdot\left[\rho^{-2} f^{2} \nabla \omega-2 \rho^{-1} \hat{e}_{\varphi} \times \operatorname{Im}\left(\Phi^{*} \nabla \Phi\right)\right]=0 \tag{3.8}
\end{equation*}
$$

Here Im stands for taking the imaginary part and the symbol " * " represents the operation of complex conjugation. On the other hand, this equation can be seen as the integrability condition for the existence of a new potential:

$$
\begin{equation*}
\rho^{-1} f^{2} \nabla \omega-2 \hat{e}_{\varphi} \times \operatorname{Im}\left(\Phi^{*} \nabla \Phi\right)=\hat{e}_{\varphi} \times \nabla \Omega \tag{3.9}
\end{equation*}
$$

Thus:

$$
\begin{equation*}
\rho^{-1} \hat{e}_{\varphi} \times \nabla \omega=-f^{-2}\left[\nabla \Omega+2 \operatorname{Im}\left(\Phi^{*} \nabla \Phi\right)\right] \tag{3.10}
\end{equation*}
$$

Thereby:

$$
\begin{equation*}
\nabla \cdot\left\{f^{-2}\left[\nabla \Omega+2 \operatorname{Im}\left(\Phi^{*} \nabla \Phi\right)\right]\right\}=0 \tag{3.11}
\end{equation*}
$$

In terms of the new potentials, the equation (2.34) can be written as:

$$
\begin{equation*}
f \nabla^{2} f=(\nabla f)^{2}-\left[\nabla \Omega+2 \operatorname{Im}\left(\Phi^{*} \nabla \Phi\right)\right]^{2}+2 f \nabla \Phi \cdot \nabla \Phi^{*} \tag{3.12}
\end{equation*}
$$

Introducing the new potential:

$$
\begin{equation*}
\mathcal{E}=f-|\Phi|^{2}+i \Omega \tag{3.13}
\end{equation*}
$$

The equations (2.31)-(2.34) can be written in terms of the following two equations:

$$
\begin{align*}
\left(\operatorname{Re}(\mathcal{E})+|\Phi|^{2}\right) \nabla^{2} \mathcal{E} & =\left(\nabla \mathcal{E}+2 \Phi^{*} \nabla \Phi\right) \cdot \nabla \mathcal{E}  \tag{3.14}\\
\left(\operatorname{Re}(\mathcal{E})+|\Phi|^{2}\right) \nabla^{2} \Phi & =\left(\nabla \mathcal{E}+2 \Phi^{*} \nabla \Phi\right) \cdot \nabla \Phi \tag{3.15}
\end{align*}
$$

In conclusion, the system of equations (2.31)-(2.34) has been reduced into two nonlinear partial differential equations for the functions $\mathcal{E}$ and $\Phi$. That is, the new potentials completely define the stationary axisymmetric gravitational and electromagnetic fields in a region exterior to the sources. Recalling:

$$
\begin{align*}
& f=\operatorname{Re}(\mathcal{E})+|\Phi|^{2}  \tag{3.16}\\
& \omega,_{z}=\rho f^{-2} \operatorname{Im}\left(\mathcal{E},_{\rho}+2 \Phi^{*} \Phi,_{\rho}\right)  \tag{3.17}\\
& \omega,_{\rho}=-\rho f^{-2} \operatorname{Im}\left(\mathcal{E}_{, z}+2 \Phi^{*} \Phi_{z}\right) \tag{3.18}
\end{align*}
$$

Hence, the equations (2.35) and (2.36) for the metric function $\gamma$ written in terms of the Ernst potentials take the form:

$$
\begin{array}{r}
\gamma, \rho=\frac{\rho f^{-2}}{2}\left(\left|\mathcal{E},_{\rho}+2 \Phi^{*} \Phi,\left.\right|^{2}-\left|\mathcal{E},{ }_{z}+2 \Phi^{*} \Phi, z\right|^{2}\right)-\rho f^{-1}\left(|\Phi, \rho|^{2}-|\Phi,|^{2}\right)\right. \\
\gamma, z=\frac{\rho f^{-2}}{2} \operatorname{Re}\left[\left(\mathcal{E},_{\rho}+2 \Phi^{*} \Phi, \rho\right)\left(\mathcal{E},{ }_{z}^{*}+2 \Phi \Phi,_{z}^{*}\right)\right]-2 \rho f^{-1} \operatorname{Re}\left(\Phi,{ }_{\rho}^{*} \Phi, z\right) \tag{3.20}
\end{array}
$$

Moreover, equations (3.14) and (3.15) fall into the class of elliptic differential equations and their properties were studied in great detail, in particular, by Ernst and Hauser. ${ }^{3,19}$

Another useful representation of the Ernst equations can be found by performing the following transformation:

$$
\begin{equation*}
\mathcal{E}=\frac{1-\xi}{1+\xi} \quad \Phi=\frac{q}{1+\xi} \tag{3.21}
\end{equation*}
$$

Performing these transformations, the equations (3.14) and (3.15) can be written into the following way:

$$
\begin{align*}
& \left(|\xi|^{2}-|q|^{2}-1\right) \nabla^{2} \xi=2\left(\xi^{*} \nabla \xi-q^{*} \nabla q\right) \cdot \nabla \xi  \tag{3.22}\\
& \left(|\xi|^{2}-|q|^{2}-1\right) \nabla^{2} q=2\left(\xi^{*} \nabla \xi-q^{*} \nabla q\right) \cdot \nabla q \tag{3.23}
\end{align*}
$$

The formulation of the Ernst equations above allows it to find electromagnetic solutions easily from the corresponding vacuum ones. And admits a phase constant transformation which allows one to set the necessary asymptotically conditions once the problem is specified. In other words, it is easy to verify the following two statements:

- Given any non-zero solution $\xi_{0}$ of the Ernst equation in vacuum, it is always possible to find a solution of the Ernst electrovacuum equations (3.22) and (3.23), by performing
the following transformation:

$$
\begin{align*}
\xi_{0} & =\xi_{0} \longrightarrow \quad \xi=\left(1+|q|^{2}\right)^{1 / 2} \xi_{0}  \tag{3.24}\\
q_{0} & =0 \longrightarrow q \tag{3.25}
\end{align*}
$$

with $q$ constant.

- If the pair $(\xi, q)$ is a solution of equations (3.22) and (3.23), then the pair $\left(e^{i \alpha} \xi, e^{i \alpha} q\right)$ will also be, where $\alpha$ is a real constant.


### 3.2 New Solutions from Symmetry Transformations

This section will present some simple symmetry transformations symmetries associated with the Ernst equations. These symmetries forms a group isomorphic to $S U(2,1)$. In other words, given a solution of (3.14) and (3.15), $\left({ }^{\mathcal{E}}, \circ^{\Phi}\right)$, then any pair $(\mathcal{E}, \Phi)$ generated by the below transformations and its combinations is also a solution.

$$
\begin{align*}
\mathcal{E}=\alpha \alpha^{*} \mathcal{E}_{0} & \Phi & =\alpha \Phi_{0}  \tag{3.26}\\
\mathcal{E}=\mathcal{E}_{0}+i b & \Phi & =\Phi_{0}  \tag{3.27}\\
\mathcal{E}=\frac{\mathcal{E}_{0}}{1+i c \mathcal{E}_{0}} & \Phi & =\frac{\Phi_{0}}{1+i c \mathcal{E}_{0}}  \tag{3.28}\\
\mathcal{E}=\mathcal{E}_{0}-2 \beta \Phi_{0}-\beta \beta^{*} & \Phi & =\Phi_{0}+\beta^{*}  \tag{3.29}\\
\mathcal{E}=\frac{\mathcal{E}_{0}}{1-2 \gamma^{*} \Phi_{0}-\gamma \gamma^{*} \mathcal{E}_{0}} & \Phi & =\frac{\Phi_{0}+\gamma \mathcal{E}_{0}}{1-2 \gamma^{*} \Phi_{0}-\gamma \gamma^{*} \mathcal{E}_{0}} \tag{3.30}
\end{align*}
$$

Here $\alpha, \beta$ and $\gamma$ are complex constants and $a$ and $b$ are real ones, hence the transformations above depend upon 8 arbitrary parameters. Notice that transformations (3.27) and (3.29) are just gauge transformations of the potentials, while (3.26) is an electromagnetic duality combined with rescaling of the potential $\mathcal{E} .^{3,9}$ Only (3.28) and (3.30) generate non-trivial transformations in the Einstein-Maxwell fields. Therefore, they are good tools to construct new solutions. The consideration that the potential $\Phi$ is a function of the potential $\mathcal{E}$ leads to an interesting result:

- Given any non-zero solution $\mathcal{E}_{0}$ of the Ernst vacuum equation, it is always possible to find a new solution of the Ernst electrovacuum equations (3.14) and (3.15), by performing the following transformation:

$$
\begin{align*}
& \mathcal{E}_{0}=\mathcal{E}_{0} \longrightarrow \quad \mathcal{E}=\frac{\mathcal{E}_{0}}{1-|\gamma|^{2} \mathcal{E}_{0}}  \tag{3.31}\\
& \Phi_{0}=0 \longrightarrow \quad \Phi=\frac{\gamma \mathcal{E}_{0}}{1-|\gamma|^{2} \mathcal{E}_{0}} \tag{3.32}
\end{align*}
$$

In conclusion, these symmetries by themselves are already a good justification for the study of the Ernst potentials in General Relativity. And in fact, several non-trivial analytic solutions and methods to construct new solutions are based on the Ernst potentials and their properties. ${ }^{2,4}$

### 3.3 An Exact Trivial Solution

Now consider the Ernst equation for the potential $\xi$ in the vacuum case and rewrite it in terms of prolate spheroidal coordinates instead of cylindrical ones. By performing the transformation below:

$$
\begin{align*}
& \rho^{2}=\left(x^{2}-1\right)\left(1-y^{2}\right)  \tag{3.33}\\
& z=x y \tag{3.34}
\end{align*}
$$

The corresponding operators are:

$$
\begin{array}{r}
\nabla^{2}=\frac{1}{x^{2}-y^{2}}\left\{\partial_{x}\left[\left(x^{2}-1\right) \partial_{x}\right]+\partial_{y}\left[\left(1-y^{2}\right) \partial_{y}\right]\right\} \\
\nabla A \cdot \nabla B=\frac{1}{x^{2}-y^{2}}\left\{\left(x^{2}-1\right) \partial_{x} A \partial_{x} B+\left(1-y^{2}\right) \partial_{y} A \partial_{y} B\right\} \tag{3.36}
\end{array}
$$

One advantage of this coordinate choice is that the operators are left invariant under the interchange of $x$ and $y$. In other words, if $\xi(x, y)$ is a solution, then $\xi(y, x)$ is also a solution. It is straightforward to see that $\xi=x$ is a solution, consequently so is $\xi=y$. Notice that a linear combination of these previous solutions is also a solution:

$$
\begin{equation*}
\xi=x \cos \lambda+i y \sin \lambda \quad \lambda \in \mathbb{R} \tag{3.37}
\end{equation*}
$$

However, this solution is not asymptotically flat, since at infinity $(x \rightarrow \infty) \xi$ diverges. But, the function $\xi(-1)$ is also a solution from (3.22) and possess the right asymptotic behaviour. Now, the goal is to find the correspondent metric functions. Consider then:

$$
\begin{equation*}
\frac{1}{\xi}=e^{i \alpha}(x \cos \lambda+i y \sin \lambda) \tag{3.38}
\end{equation*}
$$

Hence:

$$
\begin{align*}
& f=\frac{1-x^{2} \cos ^{2} \lambda-y^{2} \sin ^{2} \lambda}{(x \cos \lambda+\cos \alpha)^{2}+(y \sin \lambda-\sin \alpha)^{2}}  \tag{3.39}\\
& \Omega=\frac{-2(x \sin \alpha \cos \lambda+y \cos \alpha \sin \lambda)}{(x \cos \lambda+\cos \alpha)^{2}+(y \sin \lambda-\sin \alpha)^{2}} \tag{3.40}
\end{align*}
$$

By solving the equations (3.17) and (3.18):
$\omega=2 \tan \lambda \frac{\left(1-y^{2}\right)(x \cos \lambda \cos \alpha-y \sin \lambda \sin \alpha+1)}{x^{2} \cos ^{2} \lambda+y^{2} \sin ^{2} \lambda-1}+\frac{2 \sin \alpha}{\cos \lambda} y+$ const
It is now clear that $\alpha$ must be set equal to zero in order to impose that the spacetime be asymptotically flat. Then the function $\omega$ can be written as:

$$
\begin{equation*}
\omega=2 \tan \lambda \frac{\left(1-y^{2}\right)(x \cos \lambda+1)}{x^{2} \cos ^{2} \lambda+y^{2} \sin ^{2} \lambda-1} \tag{3.42}
\end{equation*}
$$

In a similar way, the still unknown function, $\gamma$, can be found by integrating equations (3.19) and (3.20), which leads to:

$$
\begin{equation*}
e^{2 \gamma}=C \frac{x^{2} \cos ^{2} \lambda+y^{2} \sin ^{2} \lambda-1}{\cos ^{2} \lambda\left(x^{2}-y^{2}\right)} \tag{3.43}
\end{equation*}
$$

Due to the asymptotic flatness requirement, the integration constant $C$ must be set equal to one. Finally, introducing $\cos \lambda=\sqrt{M^{2}-j^{2}} / M, \sin \lambda=j / M$ and then making the coordinate transformation

$$
\begin{equation*}
r=\sqrt{M^{2}-j^{2}} x+M \quad y=\cos \theta \tag{3.44}
\end{equation*}
$$

The standard Kerr spacetime metric is recovered.

## 4 SIBGATULLIN'S METHOD

Sibgatullin introduced his method in 1984, ${ }^{20-22}$ and since then, several authors have been using it to construct new asymptotically flat metrics and even cosmological models. Although it is a relatively old method, the academic community gave more attention to it after 2015 when gravitational waves were detected directly for the first time. It is because the physical parametrization of solutions constructed with the Sibgatullin method is easier compared to the other methods. Although stationary spacetimes can be constructed with it, which do not generate gravitational waves, time perturbations of those solutions can be used to study the waves theoretically. Therefore, several authors used it to describe a two-body system, which is the expected physical system that generated the gravitational waves detected by LIGO.

Following Geroch,,${ }^{7,8}$ Kinnersley, ${ }^{9-12}$ Hauser and Ernst, ${ }^{15,16}$ and others, the idea behind the construction of the Sibgatullin's integral method is to generate new solutions from known ones. Through a group transformation, the method generates new Ernst potentials directly from the Minkowski spacetime. In order to construct the solutions, only the knowledge of the Ernst potentials' expressions on the symmetry axis need to be known. By choosing Ernst potentials given by a negative power series of $z$ on the symmetry axis, making use of properties of complex singular integrals, the problem of finding new solutions for the Einstein-Maxwell equations is reduced to an algebraic system.

This chapter intends to be a full revision of the method, based on reference 43. But it also discusses the importance of the Riemann Hilbert problem associated with the equations, which enables a better understanding of how the method works.

### 4.1 Lax pair associated with the Einstein-Maxwell Equations

In a series of papers ${ }^{9-12}$ Kinnersley, and collaborators introduced the infinite hierarchy of potentials constructed from a new potential $H_{a b}$ (the Ernst potentials $\mathcal{E}$ and $\Phi$ are contained in $H_{a b}$ in such way that it carries all the physical information of the solution ). Once this hierarchy is constructed, the knowledge of a generating function $F$ is given. ${ }^{44,45}$ This function was used by several authors to construct generating techniques that use an already known solution of the Einstein-Maxwell equations, the seed solution, to construct another solution. ${ }^{2,4}$ The objective of this section is to introduce $H_{a b}$ and $F$, but also to derive a Lax pair for $F$, which will be the basis to introduce an associated Riemann Hilbert problem.

To recap the Weyl-Papapetrou metric and the developments in chapter 2.1, it was shown that the line element could be written in a block form:

$$
\begin{align*}
d s^{2} & =f(d t-\omega d \phi)^{2}-f^{-1}\left[e^{2 \gamma}\left(d \rho^{2}+d z^{2}\right)+\rho^{2} d \phi^{2}\right] \\
& =f_{A B} d x^{A} d x^{B}-f^{-1} e^{2 \gamma} d \xi d \xi^{*}, \quad A, B=1,2 \tag{4.1}
\end{align*}
$$

where $f_{A B}$ is a $2 \times 2$ metric discussed in 2.1. Moreover, $x^{1} \equiv t, x^{2} \equiv \phi, x^{3} \equiv \rho$, $x^{4} \equiv z, \xi \equiv z+i \rho$. Considering this metric, the equation (2.27) is written as:

$$
\begin{equation*}
\nabla \cdot\left(\rho^{-1} f_{A}^{B} \nabla A_{B}\right)=0 \tag{4.2}
\end{equation*}
$$

As before, this equation can be seen as an integrability condition for a new potential. So, it is possible to introduce a potential $B_{A}$, such as:

$$
\begin{equation*}
\rho^{-1} f_{A}{ }^{B} \nabla A_{B}=\tilde{\nabla} B_{A} \tag{4.3}
\end{equation*}
$$

where $\tilde{\nabla} \equiv\left(\partial_{z},-\partial_{\rho}\right)$. That way a new potential $\Phi_{A}$ is introduced in the form $\Phi_{A} \equiv A_{A}+i B_{A}$. Note that the component $\Phi_{1}$ coincides with the definition of the Ernst potential $\Phi$.

However, using the properties from (2.10), we find

$$
\begin{align*}
& \partial_{\rho} A_{B}=f_{B A} g^{A D} \partial_{\rho} A_{D}=-\rho^{2} \mathcal{E}^{D E} \mathcal{E}^{A F} f_{E F} f_{A B} \partial \rho A_{D}=\rho^{-1} f_{B}^{F} \partial_{z} B_{F}  \tag{4.4}\\
& -\partial_{z} A_{B}=-f_{B A} g^{A D} \partial_{z} A_{D}=\rho^{2} \mathcal{E}^{D E} \mathcal{E}^{A F} f_{E F} f_{A B} \partial z A_{D}=\rho^{-1} f_{B}^{F} \partial_{\rho} B_{F} \tag{4.5}
\end{align*}
$$

This means that the relations between $A_{A}$ and $B_{A}$ can be inverted, in other words, $\rho^{-1} f_{A}{ }^{B} \nabla B_{B}=-\tilde{\nabla} A_{A}$, leading to a relation known as Kinnersley condition ${ }^{9}$ :

$$
\begin{equation*}
i \rho \nabla \Phi_{B}=f_{A}{ }^{C} \tilde{\nabla} \Phi_{C} \tag{4.6}
\end{equation*}
$$

In the same way, the idea is to construct a new potential, which obeys the same equation, for the Einstein equation. For this, consider (2.26):

$$
\begin{equation*}
\nabla \cdot\left[\rho^{-1}\left(f_{A}{ }^{C} \nabla f_{C B}-2 A_{B} f_{A}^{C} \nabla A_{C}-2 A_{A} f_{B}{ }^{C} \nabla A_{C}\right)\right]=0 \tag{4.7}
\end{equation*}
$$

This might be interpreted as the integrability condition to the existence of a matrix potential $\Psi_{A B}$ :

$$
\begin{equation*}
\tilde{\nabla} \Psi_{A B}=\rho^{-1}\left(f_{A}^{C} \nabla f_{C B}-2 A_{B} f_{A}^{C} \nabla A_{C}-2 A_{A} f_{B}^{C} \nabla A_{C}\right) \tag{4.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\nabla \Psi_{A B}=-\rho^{-1}\left(f_{A}{ }^{C} \tilde{\nabla} f_{C B}-2 A_{B} f_{A}{ }^{C} \tilde{\nabla} A_{C}-2 A_{A} f_{B}{ }^{C} \tilde{\nabla} A_{C}\right) \tag{4.9}
\end{equation*}
$$

Following Kinnersley, ${ }^{10}$ we now introduce a new potential $E_{A B}=f_{A B}+i\left(\Psi_{A B}+\right.$ $2 A_{A} B_{B}$, which satisfies

$$
\begin{equation*}
\nabla E_{A B}=-i \rho^{-1} f_{A}^{C}\left(\tilde{\nabla} E_{C B}-2 \Phi_{B} \tilde{\nabla} \Phi_{C}^{*}\right) \tag{4.10}
\end{equation*}
$$

Finally, it is possible to define $H_{A B}=E_{A B}-\Phi_{A}^{*} \Phi_{B}-\frac{\mathcal{E}_{A B} K}{2}$, where $\nabla K=2 \Phi^{C_{*}} \nabla \Phi_{C}$. This new potential, $H_{A B}$, satifies the same condintion for $\Phi_{A}{ }^{2}$, namely:

$$
\begin{equation*}
i \rho \nabla H_{A B}=f_{A}{ }^{C} \tilde{\nabla} H_{C B} \tag{4.11}
\end{equation*}
$$

Here, this new potential can be written as ${ }^{10,46}$ :

$$
\begin{equation*}
H_{A B}=f_{A B}-A_{A} A_{B}-B_{A} B_{B}-\frac{\mathcal{E}_{A B} K}{2}+i\left(\Psi_{A B}+A_{A} B_{B}+A_{B} B_{A}\right) \tag{4.12}
\end{equation*}
$$

The condition expressed in (4.6) and (4.11) is known as the Kinnersley condition. ${ }^{10,11}$ Here the component $H_{1}^{2}$ is the definition of the Ernst potential $\mathcal{E}$ multiplied by minus one. Notice then that all physical information is cast into the potentials $H_{A B}$ and $\Phi_{A}$, which satisfy a similar condition. Therefore, it is worth to try to write all physical information into only one equation.

In order to construct a $3 \times 3$ matrix which contains all the information of the system and in such way that the equations of motion can be expressed in one equation, two more potentials ${ }^{10,11}$ are introduced. One may construct a complex matrix:

$$
H_{a}^{b}=\left(\begin{array}{cc}
H_{A}^{B} & \Phi_{A}  \tag{4.13}\\
L^{B} & K
\end{array}\right) \quad a, b=1,2,3
$$

$L^{B}$ and $K$ are defined by the equations*:

$$
\begin{equation*}
\nabla L^{B}=2 \Phi^{C_{*}} \nabla H_{C}^{B} ; \quad \nabla K=2 \Phi^{C_{*}} \nabla \Phi_{C} \tag{4.14}
\end{equation*}
$$

From (4.6) and (4.11), it is possible to write the following simple formula:

$$
\begin{equation*}
2 i \rho \nabla^{2} H_{a}^{b}=\nabla H_{a}^{c} \tilde{\nabla} H_{c}^{b} \tag{4.15}
\end{equation*}
$$

[^2]Since the components $(1,2)$ and $(1,3)$ of $H$ are the Ernst potentials ${ }^{43}(\mathcal{E}$ and $\Phi$, respectively), by solving the equation (4.15), the metric functions can be found using the developments of chapter 3. This equation admits an infinite hierarchy of higher potentials, built in terms of left and right matrix potentials, $Q_{m}$ and $H_{m} .{ }^{9,10}$ In order to show this property it is useful to work with the coordinates $\left(\xi, \xi^{*}\right)$ instead of $(\rho, z)^{\dagger}$. In these coordinates we find:

$$
\begin{equation*}
4 \rho H_{, \xi \xi^{*}}=H_{, \xi} H_{, \xi^{*}}-H_{, \xi^{*}} H_{, \xi} \tag{4.16}
\end{equation*}
$$

which can be put in the divergence free form

$$
\begin{equation*}
\left(2 i \xi H_{, \xi}+H_{, \xi} H\right)_{\xi^{*}}-\left(2 i \xi^{*} H_{\xi^{*}}+H_{\xi^{*}} H\right)_{, \xi}=0 \tag{4.17}
\end{equation*}
$$

That implies the existence of the potential $H_{2}$ :

$$
\begin{equation*}
\left.H_{2, \xi}=2 i \xi H_{, \xi}+H, \xi\right\rangle ; \quad H_{2, \xi^{*}}=2 i \xi^{*} H_{, \xi^{*}}+H_{, \xi^{*}} H \tag{4.18}
\end{equation*}
$$

which satisfies the equation

$$
\begin{equation*}
\left(2 i \xi H_{2, \xi}+H H_{\xi} H_{2}\right)_{\xi^{*}}-\left(2 i \xi^{*} H_{2}, \xi^{*}+H \xi_{\xi^{*}} H_{2}\right)_{, \xi}=0 \tag{4.19}
\end{equation*}
$$

This, as previously shown, can be interpreted as the integrability condition for a new potential $H_{3}$ and so on. The m-th element of the right hierarchy will satisfy:

$$
\begin{equation*}
H_{m, \xi}=2 i \xi H_{m-1}, \xi+H_{, \xi} H_{m-1} ; \quad H_{m, \xi^{*}}=2 i \xi^{*} H_{m-1, \xi}+H_{, \xi^{*}} H_{m-1} \tag{4.20}
\end{equation*}
$$

or in terms of the coordinates $z$ and $\rho$ :

$$
\begin{equation*}
\nabla H_{m}=\nabla H H_{m-1}-\boldsymbol{D} H_{m-1} \quad \boldsymbol{D} \equiv 2 i(\rho \tilde{\nabla}-z \nabla) \tag{4.21}
\end{equation*}
$$

Each of the matrices $H_{n}(n=2,3 \ldots)$ is not unique, since by performing a transformation as defined below, the hierarchy will still be preserved:

[^3]\[

$$
\begin{gather*}
H_{m}^{\prime}=H_{m}+H_{m-1} C_{1}+H_{m-2} C_{2}+\cdots+H_{1} C_{m-1}+C_{m}  \tag{4.22}\\
\vdots  \tag{4.23}\\
H_{3}^{\prime}=H_{3}+H_{2} C_{1}+H_{1} C_{2}+C_{3}  \tag{4.24}\\
H_{2}^{\prime}=H_{2}+H_{1} C_{1}+C_{2}  \tag{4.25}\\
H_{1}^{\prime}=H_{1}+C_{1} \tag{4.26}
\end{gather*}
$$
\]

Here the $C_{m}$ denote constant matrices. Using this freedom and making use of the equation (4.12), one may write:

$$
\begin{equation*}
H_{A B}+H_{B A}^{*}=2\left(f_{A B}-\Phi_{A} \Phi_{B}^{*}+i z \varepsilon_{A B}\right) . \tag{4.27}
\end{equation*}
$$

One can proceed in an analogue way for the left hierarchy elements:

$$
\begin{equation*}
\nabla Q_{m}=Q_{m-1} \nabla H+\boldsymbol{D} Q_{m-1} \tag{4.28}
\end{equation*}
$$

The $H$ matrix also satisfies a similar condition as (4.6), (4.11) and (4.14) that will be the key to find an infinite-dimensional Lie group ${ }^{\ddagger}$, which transforms one solution of the Einstein-Maxwell equations into another solution of these same equations. Using (4.6), (4.11) and the definition of $H$ it is obtained:

$$
\begin{equation*}
(\boldsymbol{\varepsilon} \boldsymbol{D}+\boldsymbol{M} \nabla) H=0 \tag{4.29}
\end{equation*}
$$

Here

$$
\boldsymbol{M}=\boldsymbol{M}^{\dagger}=\boldsymbol{\varepsilon} H-H^{\dagger} \boldsymbol{\varepsilon}-\frac{1}{2} \boldsymbol{\Pi} ; \quad \boldsymbol{\varepsilon}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) ; \quad \boldsymbol{\Pi}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

where " $\dagger$ " represents the Hermitian conjugation. In an equivalent way:

$$
\begin{equation*}
-2 \varepsilon \xi \partial_{\xi} H+\boldsymbol{M} \partial_{\xi} H=0 \quad 2 \varepsilon \xi^{*} \partial_{\xi^{*}} H+\boldsymbol{M} \partial_{\xi^{*}} H=0 \tag{4.30}
\end{equation*}
$$

[^4]Taking the Hermitian conjugation of the right equation and subtracting the left equation, one finds:

$$
\begin{equation*}
2 \xi \partial_{\xi} \boldsymbol{M}+\partial_{\xi} H^{\dagger} \boldsymbol{M}-\boldsymbol{M} \partial_{\xi} H=0 \tag{4.31}
\end{equation*}
$$

It is important to note that the matrices $H_{n}$ also satisfy (4.29). In fact:

$$
\begin{aligned}
(\varepsilon \boldsymbol{D}+\boldsymbol{M} \nabla) H_{n} & =\varepsilon \boldsymbol{D} H_{n}+\boldsymbol{M}\left(\nabla H H_{n-1}-\boldsymbol{D} H_{n-1}\right)=\varepsilon \boldsymbol{D} H_{n}-(\varepsilon \boldsymbol{D} H+\boldsymbol{M} \boldsymbol{D}) H_{n-1} \\
& =2 i \rho\left[\varepsilon \tilde{\nabla} H_{n}-(\varepsilon \tilde{\nabla} H+\boldsymbol{M} \tilde{\nabla}) H_{n-1}\right]-2 i z\left[\varepsilon \nabla H_{n}-(\varepsilon \nabla H+\boldsymbol{M} \nabla) H_{n-1}\right] \\
& =[-2 i \rho(\varepsilon \tilde{D}+\boldsymbol{M} \tilde{\nabla})+2 i z(\varepsilon D+\boldsymbol{M} \nabla)] H_{n-1}
\end{aligned}
$$

Therefore by induction, one sees that:

$$
\begin{equation*}
(\varepsilon \boldsymbol{D}+\boldsymbol{M} \nabla) H_{n}=0 \tag{4.32}
\end{equation*}
$$

In the next section, the existence of a Lie algebra related to these potentials $H_{n}$ will be demonstrated. In order to show that, one needs to consider a small perturbation, $\delta H$ around a given solution $H$, such that $H^{\prime}=H+\delta H$ is also a solution. Such small perturbation obeys the following equation:

$$
\begin{equation*}
(\varepsilon \boldsymbol{D}+\boldsymbol{M} \nabla) \delta H+\delta \boldsymbol{M} \nabla H=0 \tag{4.33}
\end{equation*}
$$

The equation above has solutions of the form:

$$
\begin{equation*}
\delta H=\chi_{n}(\Gamma)=\sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma G_{n-k} \quad G_{k} \equiv H_{k}^{\dagger} \varepsilon+H_{k-1}^{\dagger} \boldsymbol{M} \tag{4.34}
\end{equation*}
$$

Here $\Gamma$ is a constant matrix and satisfies $\Gamma+(-1)^{n} \Gamma^{\dagger}=0$. Note that $G_{k}$ satisfies the recurrence relation (4.28).

In order to show that (4.34) is in fact a solution of (4.33), consider:

$$
\begin{aligned}
& \delta \boldsymbol{M}=\boldsymbol{\varepsilon} \delta H-\delta H^{\dagger} \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon} \sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma G_{n-k}-\sum_{k=0}^{n}(-1)^{k} G_{n-k}^{\dagger} \Gamma^{\dagger} H_{k}^{\dagger} \varepsilon \\
& =\boldsymbol{\varepsilon} \sum_{k=0}^{n}(-1)^{k}\left[H_{k} \Gamma H_{n-k}^{\dagger}+H_{n-k} \Gamma^{\dagger} H_{k}^{\dagger}\right] \varepsilon+\sum_{k=0}^{n}(-1)^{k}\left[\varepsilon H_{k} \Gamma H_{n-k-1}^{\dagger} \boldsymbol{M}-\boldsymbol{M} H_{n-k-1} \Gamma^{\dagger} H_{k}^{\dagger} \varepsilon\right]
\end{aligned}
$$

Since the terms $H_{n}$, with $n$ lower than zero are not defined, set them equal to zero. Now, using that $\Gamma+(-1)^{n} \Gamma^{\dagger}=0$, we find

$$
\begin{aligned}
& \delta \boldsymbol{M}=\varepsilon \sum_{k=0}^{n}(-1)^{k}\left[H_{k} \Gamma H_{n-k}^{\dagger}-(-1)^{n} H_{n-k} \Gamma H_{k}^{\dagger}\right] \boldsymbol{\varepsilon}+ \\
+ & \sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma H_{n-k-1}^{\dagger} \boldsymbol{M}+(-1)^{n} \boldsymbol{M} H_{n-k-1} \Gamma H_{k}^{\dagger} \varepsilon\right]
\end{aligned}
$$

Hence, $\delta \boldsymbol{M}$ can be written in the compact form:

$$
\begin{equation*}
\delta \boldsymbol{M}=\sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma H_{n-k-1}^{\dagger} \boldsymbol{M}-\boldsymbol{M} H_{k} \Gamma H_{n-k-1}^{\dagger} \varepsilon\right] \tag{4.35}
\end{equation*}
$$

Substituting this result into (4.33) gives

$$
\begin{aligned}
& (\varepsilon \boldsymbol{D}+\boldsymbol{M} \nabla) \delta H+\delta \boldsymbol{M} \nabla H= \\
& (\varepsilon \boldsymbol{D}+\boldsymbol{M} \nabla) \sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma G_{n-k}+\sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma H_{n-k-1}^{\dagger} \boldsymbol{M}-\boldsymbol{M} H_{k} \Gamma H_{n-k-1}^{\dagger} \boldsymbol{\varepsilon}\right] \nabla H= \\
& \sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma\left(\boldsymbol{D} G_{n-k}+H_{n-k-1}^{\dagger} \boldsymbol{M} \nabla H\right)+\boldsymbol{M} H_{k} \Gamma\left(\nabla G_{n-k}-H_{n-k-1}^{\dagger} \boldsymbol{\varepsilon} \nabla H\right)\right]= \\
& \sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma\left(\nabla G_{n-k+1}-G_{n-k} \nabla H+H_{n-k-1}^{\dagger} \boldsymbol{M} \nabla H\right)+\right. \\
& \left.+\boldsymbol{M} H_{k} \Gamma\left(\nabla H_{n-k}^{\dagger} \boldsymbol{\varepsilon}+\nabla H_{n-k-1}^{\dagger} \boldsymbol{M}+H_{n-k-1}^{\dagger} \nabla \boldsymbol{M}-H_{n-k-1}^{\dagger} \boldsymbol{\varepsilon} \nabla H\right)\right]= \\
& \sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma\left(\nabla H_{n-k+1}^{\dagger} \boldsymbol{\varepsilon}+\nabla H^{\dagger} \boldsymbol{M}-H_{n-k}^{\dagger} \nabla H^{\dagger} \boldsymbol{\varepsilon}\right)+\right. \\
& \left.+\boldsymbol{M} H_{k} \Gamma\left(\nabla H_{n-k}^{\dagger} \boldsymbol{\varepsilon}+\nabla H_{n-k-1}^{\dagger} \boldsymbol{M}-H_{n-k-1}^{\dagger} \nabla H^{\dagger} \boldsymbol{\varepsilon}\right)\right]= \\
& \sum_{k=0}^{n-1}(-1)^{k}\left[\varepsilon H_{k} \Gamma\left(\boldsymbol{D} H_{n-k}^{\dagger} \boldsymbol{\varepsilon}+\nabla H_{n-k}^{\dagger} \boldsymbol{M}\right)+\boldsymbol{M} H_{k} \Gamma\left(\boldsymbol{D} H_{n-k-1}^{\dagger} \boldsymbol{\varepsilon}+\nabla H_{n-k-1}^{\dagger} \boldsymbol{M}\right)\right]=0 .
\end{aligned}
$$

It is possible to find a recurrence between the solutions of (4.33). In fact:

$$
\nabla \chi_{n+1}=\sum_{k=0}^{n+1}(-1)^{k} \nabla H_{k} \Gamma G_{n+1-k}+\sum_{k=0}^{n+1}(-1)^{k} H_{k} \Gamma \nabla G_{n+1-k}
$$

Although, since $G_{k}$ satisfies the same recurrence relation than $Q_{k}$, then:

$$
\nabla \chi_{n+1}=\sum_{k=0}^{n}(-1)^{k+1}\left(\nabla H H_{k}-\boldsymbol{D} H_{k}\right) \Gamma G_{n-k}+\sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma\left(G_{n-k} \nabla H+\boldsymbol{D} G_{n-k}\right)
$$

which leads to:

$$
\begin{equation*}
\nabla \chi_{n+1}=\chi_{n} \nabla H-\nabla H \chi_{n}+\boldsymbol{D} \chi_{n} \tag{4.36}
\end{equation*}
$$

Using (4.21) and (4.28), it is possible to find by induction the perturbations $\delta H_{m}$ and $\delta G_{m}$ due to the perturbation $\delta H$ in terms of $H_{m}$ and $\chi_{m}$ :

$$
\begin{align*}
& \delta H_{m}=\chi_{n} H_{m-1}-\chi_{n+1} H_{m-2}+\cdots+(-1)^{m-1} \chi_{n+m-1}  \tag{4.37}\\
& \delta G_{m}=G_{m-1} \chi_{n}+G_{m-2} \chi_{n+1}+\cdots+\chi_{n+m-1} \tag{4.38}
\end{align*}
$$

However, it is worth emphasizing that, although all physical properties are contained in $H$, when generating new solutions all potentials $H_{n}$ are necessary. Hence, all information contained in this infinite hierarchy of potentials needs to be put together. Therefore, after the matrices $H_{n}$ have been introduced, one can form a generating function for them, ${ }^{12}$ which will carry by itself all the information about all the potentials in a very simple way. This generating function reads:

$$
\begin{equation*}
F(\rho, z, s)=\sum_{m=0}^{\infty} H_{m}(-i s)^{m} ; \quad H_{0}=1, H_{1}=H \tag{4.39}
\end{equation*}
$$

Here, $s$ is an analytical parameter. Besides that, the expansion coefficients of the $F$ inverse matrix satisfy the recurrence relation (4.28), consequently, the left matrix potentials $Q_{m}$ may be seen as the coefficients of $F^{-1}$, namely:

$$
\begin{equation*}
F^{-1}=\sum_{m=0}^{\infty} Q_{m}(i s)^{m} \tag{4.40}
\end{equation*}
$$

Due to the transformation (4.22), the matrix $F$ is transformed as follows:

$$
\begin{equation*}
F^{\prime}=F C(t) ; \quad C(t)=\sum_{n=0}^{\infty}(-i t)^{n} C_{n} ; \quad C_{0}=1 \tag{4.41}
\end{equation*}
$$

This gauge freedom can be used to limit the number of poles of $F$ in the complex plane, ${ }^{47}$ but it will always be required that $F$ is analytic at and in a neighborhood of the origin ${ }^{\S}$.

As a consequence of (4.20) and from the definition (4.39), $F$ satisfies an overdetermined linear system. In fact:

$$
\begin{align*}
\partial_{\xi} F & =\sum_{m=0}^{\infty}(-i s)^{m} \partial_{\xi} H_{m}=\sum_{m=1}^{\infty}(-i s)^{m}\left(2 i \xi H_{m-1}, \xi+H, \xi H_{m-1}\right) \\
& =2 \xi s \partial_{\xi} F-i s H, \xi \tag{4.42}
\end{align*}
$$

[^5]which leads to:
\[

$$
\begin{equation*}
\partial_{\xi} F=\frac{-i s H_{, \xi}}{1-2 \xi s} F ; \quad \quad \partial_{\xi^{*}} F=\frac{-i s H_{, \xi^{*}}}{1-2 \xi^{*} s} F . \tag{4.43}
\end{equation*}
$$

\]

Thus, the generating matrix $F$ has two branch points, which are at $s_{+}=1 / 2 \xi$ and $s_{-}=1 / 2 \xi^{*}$. Moreover, when $H$ is known, it is straightforward to evaluate $F^{49}$ :

$$
\begin{equation*}
F(\rho, z, s)=F(0,0, s) \exp \left(\int_{c} \frac{-i s H_{, \xi}}{1-2 \xi s} d \xi+\frac{-i s H_{, \xi^{*}}}{1-2 \xi^{*} s} d \xi^{*}\right) \tag{4.44}
\end{equation*}
$$

Thereby, the matrix function $F$ carries not only the information about its associated metric, but also the information to generate new solutions from it. That is, instead of solving the components of the Einstein equation (2.31)(2.34), (2.32), (2.33) or the Ernst potentials (3.14), (3.15), one can simply solve the over determined system above. Notice that the equation (4.43) is a Lax representation for the problem, and the compatibility condition associated to this pair is the equation (4.16) (more will be discussed in appendix A ).

### 4.2 Lie Algebra associated with the Einstein-Maxwell Equations

In this section, it will be demonstrated that a Lie algebra associated with the solutions of (4.29) exists. This algebra can be used to construct a Riemann-Hilbert problem that relates new solutions to an old one. A simplification can be made in the next steps by noticing that

$$
\begin{equation*}
F^{\dagger}(\boldsymbol{\varepsilon}+i s \boldsymbol{M}) F \tag{4.45}
\end{equation*}
$$

does not depend upon the coordinates $\xi$ and $\xi^{*}$. In fact:

$$
\begin{equation*}
\partial_{\xi}\left[F^{\dagger}(\boldsymbol{\varepsilon}+i s \boldsymbol{M}) F\right]=\frac{i s}{1-2 \xi s} F^{\dagger}\left[\partial_{\xi} H^{\dagger}(\boldsymbol{\varepsilon}+i s \boldsymbol{M})+(1+2 i \xi s) \partial_{\xi} \boldsymbol{M}-(\boldsymbol{\varepsilon}+i s \boldsymbol{M}) \partial_{\xi} H\right] F \tag{4.46}
\end{equation*}
$$

which is identically zero due to equation (4.31).
Since $F$ is not unique because of the transformation (4.41), it is possible to choose the $s$ dependence of the equation (4.45). Expanding this equation as series of $s$ we find

$$
\begin{equation*}
F^{\dagger}(\varepsilon+i s \boldsymbol{M}) F=\sum_{k, l=0}^{\infty}(s)^{k+l}(i)^{k-l} G_{k} H_{l}=\boldsymbol{\varepsilon}-\frac{i s}{2} \boldsymbol{\Pi}+\sum_{n=2}^{\infty}(i s)^{n} \sum_{k=0}^{n}(-1)^{k} G_{k} H_{n-k} \tag{4.47}
\end{equation*}
$$

Notice that there exists no matrix $H$ in the first-order expansion (it begins to appear from the second-order forward). Due to this fact, even the arbitrariness of $C(s)$
cannot affect this first-order expansion and, furthermore, it is possible to choose the other coefficients equal to zero. For this choice:

$$
\begin{equation*}
F^{\dagger}(\varepsilon+i s \boldsymbol{M}) F=\Omega \tag{4.48}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega \equiv \varepsilon-\frac{i s}{2} \boldsymbol{\Pi} \tag{4.49}
\end{equation*}
$$

Making use of this gauge choice, one has:

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k} G_{k} H_{n-k}=0 ; \quad \forall n \geq 2 \tag{4.50}
\end{equation*}
$$

With respect to $F^{\dagger}(\varepsilon+i s \boldsymbol{M})=\Omega F^{-1}$, it establishes a relation between $G_{k}$ and $Q_{k}:$

$$
\begin{equation*}
G_{k}=\varepsilon Q_{k}-\frac{1}{2} \boldsymbol{\Pi} Q_{k-1} \tag{4.51}
\end{equation*}
$$

Consider the commutator of two solutions $\chi_{p}(\Gamma)$ and $\chi_{q}(\tilde{\Gamma})$ of (4.33)

$$
\begin{equation*}
\tilde{\delta} \chi_{p}(\Gamma)-\delta \chi_{q}(\tilde{\Gamma}) \tag{4.52}
\end{equation*}
$$

where

$$
\begin{aligned}
& \tilde{\delta} \chi_{p}(\Gamma)=\sum_{k=0}^{p}(-1)^{k}\left[\tilde{\delta} H_{k} \Gamma G_{p-k}+H_{k} \Gamma \tilde{\delta} G_{p-k}\right] \\
& =\sum_{k=0}^{p}(-1)^{k}\left[\sum_{i=0}^{k-1}(-1)^{i} \chi_{q+i}(\tilde{\Gamma}) H_{k-1-i} \Gamma G_{p-k}+H_{k} \Gamma \sum_{i=0}^{p-k-1} G_{p-k-1-i} \chi_{q+i}(\tilde{\Gamma})\right]
\end{aligned}
$$

By a simple expansion, the terms in the sum above can be put in the form:

$$
\tilde{\delta} \chi_{p}(\Gamma)=(-1) \sum_{k=0}^{p-1} \chi_{q+k}(\tilde{\Gamma}) \chi_{p-1-k}+\sum_{k=0}^{p-1} \chi_{p-1-k}(\Gamma) \chi_{q+k}(\tilde{\Gamma})
$$

Consequently, the commutator:

$$
\begin{equation*}
\tilde{\delta} \chi_{p}(\Gamma)-\delta \chi_{q}(\tilde{\Gamma})=\sum_{k=0}^{p+q-1}\left[\chi_{k}(\Gamma) \chi_{p+q-1-k}(\tilde{\Gamma})-\chi_{k}(\tilde{\Gamma}) \chi_{p+q-1-k}(\Gamma)\right] \tag{4.53}
\end{equation*}
$$

Defining $\beta=p+q-1$, the expression above can be put in the form

$$
\begin{equation*}
=(-1)^{\beta} \sum_{k=0}^{\beta} \sum_{l=0}^{\beta-k} \sum_{n=0}^{\beta-l-k}(-1)^{-n-l} H_{k}\left(\Gamma G_{n} H_{\beta-l-k-n} \tilde{\Gamma}-\tilde{\Gamma} G_{n} H_{\beta-l-k-n} \Gamma\right) G_{l} \tag{4.54}
\end{equation*}
$$

Using the relations (4.50) and (4.51) and the expression for $\chi_{k}$, one obtains:

$$
\begin{align*}
& \tilde{\delta} \chi_{p}(\Gamma)-\delta \chi_{q}(\tilde{\Gamma})=\chi_{\beta}(g)-\frac{1}{2} \chi_{\beta-1}(\tilde{g})  \tag{4.55}\\
& g \equiv \Gamma \varepsilon \tilde{\Gamma}-\tilde{\Gamma} \varepsilon \Gamma ; \quad \tilde{g} \equiv \Gamma \Pi \tilde{\Gamma}-\tilde{\Gamma} \Pi \Gamma
\end{align*}
$$

Thus, the commutator of two solutions is also a solution. This implies that the solutions of the form (4.33) produces an infinite-dimensional Lie algebra. In order to find the respective Lie group subtract two solutions of the equation (4.32), $H_{m}$ and $\stackrel{\circ}{H}_{m}$ which differ slightly, multiply the result by $(-i t)^{m}$ and sum over $m$ from 0 to $\infty$, that is:

$$
\begin{gather*}
H_{m}-\stackrel{\circ}{H}_{m}=\delta H_{m}=\sum_{l=0}^{m-1}(-1)^{l} \chi_{n+l} H_{m-1-l}  \tag{4.56}\\
F-\stackrel{\circ}{F}=\sum_{m=0}^{\infty} \sum_{l=0}^{m-1}(-1)^{l}(-i t)^{m} \chi_{n+l} H_{m-1-l}  \tag{4.57}\\
\delta F=-i t \sum_{m=0}^{\infty}(i t)^{m} \chi_{m+n} F \tag{4.58}
\end{gather*}
$$

Consider now:

$$
\begin{align*}
& F \Gamma \Omega F^{-1}=F \Gamma F^{\dagger}(\varepsilon+i s \boldsymbol{M})  \tag{4.59}\\
& =\sum_{k, l=0}^{\infty}(i)^{l-k}(s)^{l+k} H_{k} \Gamma G_{l}=\sum_{n=0}^{\infty}(i s)^{n} \sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma G_{n-k}  \tag{4.60}\\
& F \Gamma \Omega F^{-1}=\sum_{n=0}^{\infty}(i s)^{n} \chi_{n}(\Gamma) \tag{4.61}
\end{align*}
$$

Hence, $F \Gamma \Omega F^{-1}$ might be expanded as a Laurent series in terms of the parameter $s$ with expansion coefficients $\chi_{n} .{ }^{50}$ Consequently, using the Cauchy's integral formula, we find:

$$
\begin{equation*}
\chi_{n}=\frac{1}{2 \pi} \oint_{L} F \Gamma \Omega F^{-1} \frac{d s}{(i s)^{n+1}} \tag{4.62}
\end{equation*}
$$

By assumption, $L$ is a smooth contour that bounds a simply connected region $L_{+}$ in the $s$ complex plane (see figure 1) and includes its origin, such that inside of it the series $\sum_{k=0}^{\infty} H_{k}(-i s)^{k}$ converges.


Figure 1 - Representation of the s-plane.
Source: By the author.

Substituting (4.62) into (4.58) gives:

$$
\begin{equation*}
\delta F F^{-1}=\frac{-1}{2 \pi} \oint_{L} F \Gamma \Omega F^{-1} \sum_{k=0}^{\infty} \frac{t^{k+1} d s}{s^{k+1}(i s)^{n}}=\frac{-1}{2 \pi} \oint_{L} F \Gamma \Omega F^{-1} \frac{t d s}{(s-t)(i s)^{n}} \tag{4.63}
\end{equation*}
$$

So far, what has been considered here is the simple solution (4.34) of the equation (4.33). But a general solution can be given by a linear combination of (4.34) ${ }^{\mathbb{\top}}$ :

$$
\begin{equation*}
\delta H=\sum_{n=1}^{\infty} \sum_{k=0}^{n}(-1)^{k} H_{k} \Gamma_{n-1} G_{n-k} \tag{4.64}
\end{equation*}
$$

It is straightforward to see that the generalization of (4.63) is:

$$
\begin{equation*}
\delta F F^{-1}=\frac{-1}{2 \pi i} \oint_{L} F \Gamma(s) \Omega F^{-1} \frac{t d s}{s(s-t)} \tag{4.65}
\end{equation*}
$$

where $\Gamma(s) \equiv \sum_{k=0}^{\infty}(i s)^{-k} \Gamma_{k}$ is assumed to converge in region $L_{-}=\mathbb{C}-L_{+}$. Despite the written form, this integral may be seen as the integral equation method for effecting Kinnersley-Chitre transformations first derived by Hauser and Ernst. ${ }^{15,16}$

The expression (4.65) is the variation of

$$
\begin{equation*}
\oint_{L} F(s) u(s) \stackrel{\circ}{F}(s)^{-1} \frac{t d s}{s(s-t)}=0, \quad u(s) \equiv \exp (\Gamma(s) \Omega(s)) \tag{4.66}
\end{equation*}
$$

ब Here the index $n-1$ indicates that it is associated with $\chi_{n-1}$.

This is for a small $\Gamma(s)$ and $F(s)$ differing slightly from the initial solution $\stackrel{\circ}{F}$ (known as seed function or seed solution ${ }^{\|}$) along the orbit of the group. ${ }^{43}$ In fact:

$$
\begin{equation*}
\oint_{L}(\stackrel{\circ}{F}+\delta \stackrel{\circ}{F})(\mathbb{1}+\Gamma(s) \Omega) \stackrel{\circ}{F}^{-1} \frac{t d s}{s(s-t)}=0 \tag{4.67}
\end{equation*}
$$

Since the choice of the initial solution is arbitrary, one can omit the symbol superscript and write:

$$
\begin{equation*}
\oint_{L}\left(\mathbb{1}+F \Gamma(s) \Omega F^{-1}+\delta F F^{-1}+\delta F \Gamma(s) \Omega F^{-1}\right) \frac{t d s}{s(s-t)}=0 \tag{4.68}
\end{equation*}
$$

The integration result of the first term is equal to zero and it is possible to neglect the last term since it represents a second order term. Using the expression (4.58), one recovers the equation (4.65). It is important to note that the matrix $u$ obeys the Hauser-Ernst condition ${ }^{17,18 \text { : }}$

$$
u^{\dagger} \Omega u=\Omega, \quad \Omega=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{4.69}\\
-1 & 0 & 0 \\
0 & 0 & -i s / 2
\end{array}\right)
$$

The Sibgatullin method ${ }^{43,51}$ does not use the idea of a group associated with the symmetry transformations of the coordinates and the movement equations. Kinnersley extensively studied such group aspects in a series of papers. ${ }^{9-14}$ However, since its developments are related to each other, discussing its group features qualitatively brings a better physical understanding of the Sibgatullin idea and how its methods work in a physical sense.

Let $\boldsymbol{K}^{\prime}$ ( $\boldsymbol{K}$ for the vacuum case) be the group associated with the symmetry transformations of the coordinates and fields for a stationary spacetime with axial symmetry in the presence of an electromagnetic field, in other words, $\boldsymbol{K}^{\prime}$ is the internal symmetry group. ${ }^{2}$ Based on that, Kinnersley and Chitre developed a method to find new solutions from a transformation of an already known solution that relied on the infinite Lie algebra associated with the $\boldsymbol{K}^{\prime}$ group. Hauser and Ernst ${ }^{17,18}$ introduced an integral equation method for Kinnersley-Chitre (K-C) transformations, by exponentiation of the K-C algebra they have shown that the matrix $u(t)$ can be a representation of an element of the $\boldsymbol{K}^{\prime}$ group. In subsequent papers, ${ }^{15,16}$ they gave a new approach linked with a Hilbert-Riemann problem for constructing the integral equations and, hence, find new solutions from a seed solution. For this latter technique, the idea of K-C transformation was not needed; it just relied on the group element $u(t)$. For such methods, it is required that $u(t)$ is analytic in

[^6]the full region $L_{-} \cup L$ and, if they exist, the poles must be simple and lie on $L_{+}{ }^{47}$ (this is also the scenario for the present study).

The main point of the section, which follows from (4.66), is that there exists a function, holomorphic in the region $L_{-}$and continuous in $L_{-} \cup L,{ }^{52}$ which is connected with $F^{* *}$ :

$$
\begin{equation*}
\chi_{-}(s) \equiv F(s) u(s) \stackrel{\circ}{F}(s)^{-1} \tag{4.70}
\end{equation*}
$$

A linear representation of the group of nonlinear transformations of (4.29) is also given by (4.66). This result can be identified as the Riemann-Hilbert problem (RHP), which might be interpreted, for $s \in L$, as a transformation of the seed solution $\stackrel{\circ}{F}$ under the action of the shift along the orbit group of inner symmetry which produces a new solution $F .{ }^{43}$ The closed contour $L$ is the boundary of regions $L_{-}$and $L_{+}$. Inside of region $L_{+}$, which includes the origin, the matrix function $F$ is holomorphic, and all its singularities (the points $s=1 / 2(z \pm i \rho))$ lie inside of the region $L_{-}$. It is important to note that, despite the branch points in the matrices $F$ and $\stackrel{\circ}{F}^{-1}, \chi_{-}(s)$ is analytic in $L_{-}$. Another comment at hand is related to where the poles of $u(s)$ are in the complex plane: the fact that they are in the region $L_{+}$ensures that (4.70) is not just a gauge transformation.

This is a strong and useful result on which all the development of the Sibgatullin's integral method is based. So, let us discuss its meaning and consequences in a bit more detail.

First of all, let's recall some results from chapter 4. It was shown that a Lax-pair could be associated with the non-linear equation (4.16)

$$
\begin{equation*}
\partial_{\xi} F=\frac{-i s H_{, \xi}}{1-2 \xi s} F ; \quad \quad \partial_{\xi^{*}} F=\frac{-i s H_{, \xi^{*}}}{1-2 \xi^{*} s} F . \tag{4.71}
\end{equation*}
$$

Here, the matrix functions $U$ and $V$ are $3 \times 3$ and have only one simple pole in the $s$-plane:

$$
\begin{equation*}
U \equiv \frac{-i s H, \xi}{1-2 \xi s} ; \quad V \equiv \frac{-i s H_{, \xi^{*}}}{1-2 \xi^{*} s} \tag{4.72}
\end{equation*}
$$

From the compatibility equation, using the $U$ and $V$ above, the equation (4.16) is recovered. In fact:

$$
U_{, \xi^{*}}-V_{, \xi}+[U, V]=\frac{-s^{2}}{(1-2 \xi s)\left(1-2 \xi^{*} s\right)}\left\{-2 i\left(\xi-\xi^{*}\right) H_{, \xi \xi^{*}}-H_{, \xi^{*}} H_{, \xi}+H_{, \xi} H_{, \xi^{*}}\right\} \stackrel{!}{=} 0
$$

[^7]In order for this relation to be valid for all $s$, it is required that the expression in brackets vanishes, and consequently the nonlinear equation (4.16) is recovered.

As mentioned before, the important result of chapter 4 is that there exists a holomorphic function, $\chi_{-}$, in the region $L_{-} \cup L$, given in terms of the function $F$ :

$$
\begin{equation*}
\chi_{-}(s) \equiv F(s) u(s) \stackrel{\circ}{F}(s)^{-1} \tag{4.73}
\end{equation*}
$$

The function above can be interpreted, in terms of the RHP (see appendix A), as the function $\chi_{-}$defined in (A.1). So far, the construction has been made in the complex $s$-plane. Some practicalities due to the introduction of the new complex parameter, $\sigma$, related to the old one as $s=1 / 2(z+i \rho \sigma)$ will be presented now. The matrix functions $\chi_{ \pm}(s)$ are then replaced by $Y_{\mp}(\sigma)$, as follows:

$$
\begin{equation*}
Y_{\mp}(\sigma)=\chi_{ \pm}(s) \quad s=1 / 2(z+i \rho \sigma) . \tag{4.74}
\end{equation*}
$$

The RHP in the $\sigma$-plane is then:

$$
\begin{equation*}
Y_{-} G=Y_{+} \tag{4.75}
\end{equation*}
$$

Due to the map $s \rightarrow \sigma$, the simple closed contour $L$ is mapped into the contour $\Lambda$, in the $\sigma$-plane ${ }^{\dagger \dagger}$, such as, the regions $L_{ \pm}$to $\Lambda_{\mp}$, and $\Lambda$ is a common boundary of the disjoint open regions $\Lambda_{-}$and $\Lambda_{+}$. Consequently, $Y_{+}(\sigma)$ is holomorphic $\forall \sigma \in \Lambda \cup \Lambda_{+}$and $Y_{-}(\sigma)$ is holomorphic $\forall \sigma \in \Lambda \cup \Lambda_{-}$. From now on, the canonical normalization of the RHP will be fixed ${ }^{53}$ i.e., the function $Y_{-}$is normalized to the unit matrix at infinity, $Y_{-}(\infty)=\mathbb{1}$.

The seed function $\stackrel{\circ}{F}$ has its branch points mapped into $\sigma_{ \pm}= \pm 1 \in \Lambda_{+}$. The RHP that will be used in Sibgatullin's method is then defined to be the pair $\Lambda_{\mp}$ of a $3 \times 3$ matrices, functions of $z, \rho$ and $\sigma$, such that:

- $Y_{+}(\sigma)$ is holomorphic $\forall \sigma \in \Lambda \cup \Lambda_{+}$.
- $Y_{-}(\sigma)$ is holomorphic $\forall \sigma \in \Lambda \cup \Lambda_{-}$and $Y_{-}(\infty)=\mathbb{1}$.

Once one solution is known, the output matrix function $F$ is given by:

$$
\begin{equation*}
F=Y_{-} \stackrel{\circ}{F} \tag{4.76}
\end{equation*}
$$

There are some advantages of working in the $\sigma$-plane instead of the old $s$-plane for the RHP formulation in the case of Sibgatullin's method, but the most significant is that $\dagger \dagger$ Note that the contour $\Lambda$ surround the origin of the $\sigma$-plane.
in the $\sigma$-plane the branch points of $F$ are $\sigma_{ \pm}= \pm 1$, whereas in the $s$-plane the branch points are $1 / 2(z \pm i \rho)$. In the former case, the branch points do no longer depend on z and rho, which is an advantage. This implies a procedure how to employ contours. With the branch points depending on $\rho$ and $z$, the choice of $L$ was dependent indirectly on these coordinates. The choice of the simple smooth closed contour is still arbitrary (it just needs to enclose the branch points and the branch cut), but now the branch points are constants, and once one has set the contour $\Lambda$ it is valid for all pairs $(z, \rho) .{ }^{48}$

From the development made in chapter 4, the function $u$ has to be holomorphic in $\Lambda \cup \Lambda_{+}\left(L \cup L_{-}\right)$, which implies $\Lambda \cup \Lambda_{+} \subset \operatorname{dom} u$. Then, it follows that $\sigma_{ \pm} \in \operatorname{dom} u$.

To summarize, let $Y_{ \pm}$be solutions of the RHP corresponding to given $\stackrel{\circ}{F}$ and $u$ as:

$$
\begin{equation*}
Y_{-}=F \stackrel{\circ}{F}^{-1} ; \quad Y_{+}=F u \stackrel{\circ}{F}-1 \tag{4.77}
\end{equation*}
$$

$\stackrel{\circ}{F}$ is defined everywhere except at the branch points $\sigma_{ \pm}$and at the branch cut joining these points (see equation (4.101)). There exists a contour $\Lambda$ so that it encloses $\sigma_{ \pm}$ and the branch cut and, moreover, $\Lambda \cup \Lambda_{+}$is a subset of dom $u$.

Thus, the matrix function $G$ takes the form:

$$
\begin{equation*}
G=\stackrel{\circ}{F} u \stackrel{\circ}{F}^{-1} \tag{4.78}
\end{equation*}
$$

Considering the RHP which has been constructed in this section, the solution $Y_{+}$ is holomorphic at all $\sigma$ at which $u$ is holomorphic given that ${ }^{48}$ :

- $Y_{+}$and $u$ are both holomorphic in $\Lambda \cup \Lambda_{+}$.
- $Y_{+}=Y_{-} G$ where $Y_{-}$is holomorphic in $\Lambda_{-}$, and $G$ is holomorphic at any point in $\Lambda_{-}$at which $u$ is holomorphic.

Analogously, $Y_{-}$is holomorphic at all $\sigma$ at which $\stackrel{\circ}{F}$ is holomorphic since:

- $Y_{-}$and $\stackrel{\circ}{F}$ are both holomorphic in $\Lambda \cup \Lambda_{-}$.
- $Y_{-}=Y_{+} G^{-1}$ where $Y_{+}$is holomorphic in $\Lambda_{+}$, and $G$ is holomorphic at any point in $\Lambda_{+}$at which $\stackrel{\circ}{F}$ is holomorphic.

As a corollary of the equation (4.69) and the discussion above it follows that the module of the determinants of $Y_{+}$and $Y_{-}$are equal. In fact, since $|\operatorname{det} G|=|\operatorname{det} u|=1$, using the equation (4.75):

$$
\begin{equation*}
\left|\operatorname{det} Y_{+}\right|=\left|\operatorname{det} Y_{-}\right| \tag{4.79}
\end{equation*}
$$

Using the same argument of analytic continuation that was used in equation (A.2), it is possible to conclude that $\left|\operatorname{det} Y_{+}\right|=\left|\operatorname{det} Y_{-}\right|$must be constant and, hence, by the normalization condition:

$$
\begin{equation*}
\left|\operatorname{det} Y_{+}\right|=\left|\operatorname{det} Y_{-}\right|=\mathbb{1} \tag{4.80}
\end{equation*}
$$

In conclusion, $F$ is the output potential that is obtained by solving the RHP corresponding to $G=\stackrel{\circ}{F} u \stackrel{\circ}{F}^{-1}$.

### 4.3 Correlation between the inner transformation of the group, the initial values and the transformed values

Now, the holomorphicity of $\chi$ will be exploited. From this, some conditions can be imposed on the matrix $u$. Since the matrix $\chi_{-}(s)$ is analytic everywhere in $L_{-}$including in $F$ branch points, it is necessary to evaluate its behaviour in $s_{ \pm}=1 / 2(z \pm i \rho)$.

Consider then $\chi$ near the symmetry axis, $\rho=0$. It is assumed that the $3 \times 3$ matrix $H$ is an analytic function of $\rho$ near the symmetry axis, and that $H_{1}^{2}=-\mathcal{E}$ and $H_{1}^{3}=\Phi$ are locally holomorphic in this region. ${ }^{19}$ The asymptotic behaviour of the non-zero components of the matrix $H$ at $\rho=0$ is $^{47}$ :

$$
\begin{equation*}
H_{1}^{1}=2 i z \quad H_{1}^{2}=-\mathcal{E}(z, 0) \equiv-e(z) \quad H_{1}^{3}=\Phi(z, 0) \equiv f(z) \tag{4.81}
\end{equation*}
$$

Consequently, the non-zero components of the generating function $F$ are given by:

$$
\begin{array}{cc}
F_{1}^{1}=(1-2 s z)^{-1} & F_{1}^{2}=  \tag{4.82}\\
i s e(z)(1-2 s z)^{-1} \\
F_{1}^{3}=-i s f(z)(1-2 s z)^{-1} & F_{2}^{2}=F_{3}^{3}=1
\end{array}
$$

Or in a matrix notation:

$$
\begin{align*}
& F(z, \rho=0)=\left(\begin{array}{ccc}
(1-2 s z)^{-1} & i s e(z)(1-2 s z)^{-1} & -i s f(z)(1-2 s z)^{-1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)  \tag{4.83}\\
& H=\left(\begin{array}{ccc}
2 i z & -e(z) & f(z) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
\end{align*}
$$

Here, the function $f(z)=\Phi(\rho=0, z)$ should not be confused with the WeylPapapetrou metric function f .

Now, using the equations (4.83), for the asymptotic behaviour of $F$, into the definition of $\chi$, one gets:

$$
\begin{gather*}
\chi_{-}(s)=\left(\begin{array}{ccc}
u_{1}^{1}+i s u_{2}^{1} e(z)-i s u_{3}^{1} f(z) & \frac{A}{1-2 s z} & \frac{B}{1-2 s z} \\
u_{2}^{1}-2 s u_{2}^{1} z & u_{2}^{2}-i s u_{2}^{\circ} e(z) & u_{2}^{3}-i s u_{2}^{1} f(z) \\
u_{3}^{1}-2 s u_{3}^{1} z & u_{3}^{2}-i s u_{3}^{\circ} e(z) & u_{3}^{3}-i s u_{3}^{1} f(z)
\end{array}\right)  \tag{4.84}\\
A=u_{1}^{2}+s e(z)\left(i u_{2}^{2}+s u_{2}^{1} e(z)\right)-i s u_{3}^{2} f(z)-s \odot(z)\left(s u_{3}^{1} f(z)+i u_{1}^{1}\right)  \tag{4.85}\\
B=u_{1}^{3}+i s u_{1}^{1} f(z)+s e(z)\left(i u_{2}^{3}+s u_{2}^{1} f(z)\right)-s f(z)\left(i u_{3}^{3}-s u_{3}^{1} f(z)\right) . \tag{4.86}
\end{gather*}
$$

We obtain that all components of the matrix function $\chi$ are non-singular at the point $s=1 / 2 z$, except for:

$$
\begin{equation*}
\chi_{1}^{2}=\frac{A(s)}{1-2 s z}, \quad \chi_{1}^{3}=\frac{B(s)}{1-2 s z} \tag{4.87}
\end{equation*}
$$

In order to respect the holomorphicity of $\chi$ at $s=1 / 2 z, A$ and $B$ must vanish there.

Considering now the Hermitian of $\left(\chi_{-}(s)\right)^{-1}$ and making use of equations (4.69) and (4.48) one finds:

$$
\begin{equation*}
\left(\chi^{-1}\right)^{\dagger}=(\varepsilon+i s \boldsymbol{M}) \chi(\varepsilon+i s \boldsymbol{\circ})^{-1} \tag{4.88}
\end{equation*}
$$

Now, using the equation (4.43), we get:

$$
\begin{equation*}
\partial_{\xi} \chi=-\frac{i s}{1-2 \xi s}\left(\partial_{\xi} H \chi-\chi \partial_{\xi} \stackrel{\circ}{H}\right) \quad \partial_{\xi^{*}} \chi=-\frac{i s}{1-2 \xi^{*} s}\left(\partial_{\xi^{*}} H \chi-\chi \partial_{\xi^{*}} \stackrel{\circ}{H}\right) \tag{4.89}
\end{equation*}
$$

Due to the analyticity of $\chi$, even in the points $s=i / 2 \xi$ and $s=-i / 2 \xi^{*}$, it must be imposed that:

$$
\begin{equation*}
\left.\left(\partial_{\xi} H \chi-\chi \partial_{\xi} \stackrel{\circ}{H}\right)\right|_{s=1 / 2 \xi}=\left.\left(\partial_{\xi^{*}} H \chi-\chi \partial_{\xi^{*}} \stackrel{\circ}{H}\right)\right|_{s=1 / 2 \xi^{*}}=0 \tag{4.90}
\end{equation*}
$$

As pointed out before, if there are poles in the matrix $\chi_{-}$(remember that $\chi_{-} \rightarrow Y_{+}$), they can only lie in the region $L_{+}$and arise from the poles of $u(s)$ since in this region $F$ and $F^{-1}$ are holomorphic. The relations above are necessary and sufficient to ensure the analyticity of $\chi_{-}$at the branch points.

On the symmetry axis, the boundary condition (4.90) becomes:

$$
\begin{equation*}
\partial_{z} H_{a}^{c} \chi_{c}^{b}-\chi_{a}^{c} \partial_{z} \stackrel{\circ}{H}_{c}^{b}=2 i \delta_{a}^{1} \chi_{1}^{b}-\left(\partial_{z} e\right) \delta_{a}^{1} \chi_{2}^{b}+\left(\partial_{z} f\right) \delta_{a}^{1} \chi_{3}^{b}-\chi_{a}^{1} \partial_{z} \stackrel{\circ}{H_{1}^{b}}, \quad a, b, c=1,2,3 \tag{4.91}
\end{equation*}
$$

Which implies that:

$$
\begin{array}{ll}
\left.\chi_{a}^{1}(s)\right|_{s=1 / 2 z}=0, & a=2,3 . \\
2 i \chi(s)_{1}^{b}=\chi_{1}^{1} \partial_{z} \stackrel{\circ}{H}_{1}^{b}+\left(\partial_{z} e\right) \chi_{2}^{b}-\left.\left(\partial_{z} f\right) \chi_{3}^{b}\right|_{s=1 / 2 z}, & b=2,3 . \tag{4.93}
\end{array}
$$

Sibgatullin chose $u_{2}^{1}=u_{3}^{1}=0$ and $u_{3}^{3}=1$. Then, Sibgatullin found that the general form of $u(s)^{22,43}$ may be written as:

$$
u(s)=\left(\begin{array}{ccc}
a & a s\left(\gamma-i \alpha^{*} \alpha\right) & i s a \alpha  \tag{4.94}\\
0 & 1 / a^{*} & 0 \\
0 & -2 \alpha^{*} & 1
\end{array}\right)
$$

where $a(s)$ and $\alpha(s)$ are arbitrary complex functions of $s$, and $\gamma(s)$ is an arbitrary real function of $s$. These functions are holomorphic in $L_{-}$.

In conclusion, suppose that $\stackrel{\circ}{F}$ and $F$ are any given solutions of (4.43) and $\stackrel{\circ}{\mathcal{E}}, \stackrel{\circ}{\Phi}$ and $\mathcal{E}, \Phi$ their respective Ernst potentials. Let the matrix function $u$ be a member of the group which leads one solution to another, consequently, at $s=1 / 2 z$ :

$$
\begin{align*}
& \mathcal{E}(1 / 2 s, 0)=a(s) a^{*}(s)\left[\stackrel{\circ}{\mathcal{E}}(1 / 2 s, 0)+i \gamma(s)-\alpha(s) \alpha(s)^{*}-2 \alpha(s)^{*} \stackrel{\circ}{\Phi}(1 / 2 s, 0)\right]  \tag{4.95}\\
& \Phi(1 / 2 s, 0)=a(s)[\alpha(s)+\stackrel{\circ}{\Phi}(1 / 2 s, 0)] \tag{4.96}
\end{align*}
$$

Ernst and Hauser ${ }^{19}$ demonstrated that the Ernst potentials fall into the class of elliptic functions, therefore $\mathcal{E}$ and $\Phi$ are holomorphic functions in their domains. As a consequence, they might be analytically continued in some neighborhood of $\rho=0$ and hence, there exists a Taylor series expansion about the symmetry axis. Due to the arbitrariness of the matrix function $u$, Sibgatullin ${ }^{22}$ than argued that for any pair $H \stackrel{\circ}{H}$, solutions of (4.29), there always is a symmetry transformation which leads one solution to the other. By the procedure of analytic continuation, it is possible to write the components of the matrix function $u(s)$, in terms of the arbitrary locally holomorphic functions $e(z)$ and $f(z)$, which corresponds to the Ernst potentials $\mathcal{E}$ and $\Phi$ evaluated on the symmetry axis, respectively, in the whole $s$-complex plane. The above result also shows that the idea of adding soliton solutions to the spacetime is ensured by the poles of the matrix function $u(s)$.

By itself, the precedent result is already a great step in the point of view of constructing new solutions of the Einstein-Maxwell equations. So, it is important to deepen the understanding in order to describe mathematically what is going on.

Let $V^{18}$ be a set of all electrovacuum spacetimes with coordinates $\left(x^{1}, x^{2}, x^{3}, x^{4}\right)$, such that the line element is:

$$
\begin{equation*}
d s^{2}=f_{A B} d x^{A} d x^{B}-h_{M N} d x^{M} d x^{N} \tag{4.97}
\end{equation*}
$$

where the metric components are, at most, function of $x^{3}, x^{4}$.
Given a member of $V^{\prime}$, it is possible to associate it to a potential $F_{0}\left(x^{3}, x^{4}, s\right)$ (in fact, it is associated with a family of potentials $F_{0}^{\prime} s$ which differ only by a gauge transformation, but it was shown by Sibgatullin ${ }^{43}$ that a suitable gauge choice is possible). This potential is holomorphic in a neighborhood of $s=0$. Let the metric be defined in a region of the spacetime, then there is a contour $L$ in the $s$-complex plane, which surrounds $s=0$, such that $F_{0}$ is holomorphic on it and in its interior. Then, let $u(s)$ be a member of $K^{\prime}$ and $V_{0} \in V^{\prime}$ associated to $F_{0}$. The application of $u$ in $F_{0}$ gives to $F$ associated with $V \in V^{\prime}$.

By fixing the seed solution $\stackrel{\circ}{F}$ as the matrix function which generates the Minkowski space, the corresponding Ernst potentials are then $\stackrel{\circ}{\mathcal{E}}=1$ and $\stackrel{\circ}{\Phi}=0$.Given this, it is possible to evaluate the seed solutions $\stackrel{\circ}{H}$ and $\stackrel{\circ}{F}^{20}$ :

$$
\begin{align*}
& \stackrel{\circ}{F}=\left(\begin{array}{ccc}
\lambda^{-1} & i s \lambda^{-1} & 0 \\
i(2 \lambda s)^{-1}(2 z s-1+\lambda) & (2 \lambda)^{-1}(1-2 z s+\lambda) & 0 \\
0 & 0 & 1
\end{array}\right),  \tag{4.98}\\
& \stackrel{\circ}{H}=\left(\begin{array}{ccc}
2 i z & -1 & 0 \\
-\rho^{2} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \lambda=\left[(1-2 s z)^{2}+4 \rho^{2} s^{2}\right]^{1 / 2} .
\end{align*}
$$

It is important to notice that all generation functions $F$ have the same singularities in the $s$-complex plane, which are the zeros of

$$
\begin{equation*}
\lambda=\left[(1-2 s z)^{2}+4 \rho^{2} s^{2}\right]^{1 / 2} \tag{4.99}
\end{equation*}
$$

It can be proven easily by means of the RHP and follows as a consequence of (4.80). In fact, taking the determinant of (4.76), gives:

$$
\begin{equation*}
|\operatorname{det} F|=|\operatorname{det} \stackrel{\circ}{F}| \tag{4.100}
\end{equation*}
$$

By evaluating $|\operatorname{det} \stackrel{\circ}{F}|$, one gets:

$$
\begin{equation*}
|\operatorname{det} F|=\frac{1}{\lambda} \tag{4.101}
\end{equation*}
$$

For all $F$ solutions of the RHP. Also, by considering the Minkowski spacetime, the equations (4.95) and (4.96) are simplified and can be written as:

$$
\begin{align*}
& e(1 / 2 s)=a(s) a^{*}(s)\left[1+i \gamma(s)-\alpha(s) \alpha(s)^{*}\right]  \tag{4.102}\\
& f(1 / 2 s)=a(s) \alpha(s) \tag{4.103}
\end{align*}
$$

Moreover ${ }^{51}$ :

$$
u \stackrel{\circ}{F}^{-1}=\frac{1}{2 a^{*}}\left(\begin{array}{ccc}
\lambda \tilde{e}+(1-2 s z)(e+2 f \tilde{f}) & -2 i s(e+2 f \tilde{f}) & 2 \mathrm{i} s f a^{*}  \tag{4.104}\\
\mathrm{i} s^{-1}(1-2 s z-\lambda) & 2 & 0 \\
-2 i \tilde{f}(1-2 s z-\lambda) & -4 \tilde{f} & 2 a^{*}
\end{array}\right)
$$

where $\tilde{e}(\xi)=e\left(\xi^{*}\right)^{*}$. Consequently, the components of the matrix $\chi(s) \equiv F(s) u(s) \stackrel{\circ}{F}(s)^{-1}$ might be written as ${ }^{20}$ :

$$
\begin{align*}
& \chi_{a}^{3}=F_{a}^{3}+i s f F_{a}^{1} \\
& a^{*} \chi_{a}^{2}=F_{a}^{2}-i s e F_{a}^{1}-2 \tilde{f} \chi_{a}^{3}  \tag{4.105}\\
& 2 a^{*} s \chi_{a}^{1}=-i \lambda\left(i s \tilde{e} F_{a}^{1}+F_{a}^{2}-2 \tilde{f} F_{a}^{3}\right)+(1-2 s z) a^{*} \chi_{a}^{2}
\end{align*}
$$

The great idea behind the Sibgatullin integral method is to perform the analytical continuation of the Riemann Hilbert problem, in order to construct the Ernst Potentials $\mathcal{E}(\rho, z)$ and $\Phi(\rho, z)$, in the whole space, from their behavior on the symmetry axis. Thus, the work hypothesis can be summarized as follows:

It is assumed that $F$ is holomorphic ${ }^{\ddagger \ddagger}$ everywhere except in its unique branch points $s_{ \pm}=1 / 2(z \pm i \rho)$ and in the cut $\mathcal{L}$ which joins these points. It is defined as a simply connected region $L_{+}$that includes the origin, closed by the contour $L$, which does not contain the singularities of $F$. On the other hand, $u$ is holomorphic in the region outside of $L$, namely $L_{-}$. Consequently, all singularities of $u$ lie inside of $L_{+}$and hence, the branch cut of $F$ is in the dom $u$.

Due to the analyticity of $\chi$ in $L_{-}$, its jump in the branch cut $\mathcal{L}$ must be equal to zero, which implies the following conditions on $F$ :

$$
\begin{align*}
& {\left[F_{a}^{3}\right]+i s f\left[F_{a}^{1}\right]=0 . \quad\left[F_{a}^{2}\right]-i s e\left[F_{a}^{1}\right]=0 .} \\
& i s \tilde{e}\left\{F_{a}^{1}\right\}+\left\{F_{a}^{2}\right\}-2 \tilde{f}\left\{F_{a}^{3}\right\}=0 . \quad a=1,2,3 . \tag{4.106}
\end{align*}
$$

$\ddagger \ddagger$ Cosgrove has shown that one can always choose $F$ to be a holomorphic function in $L_{+}$since one can always perform a gauge transformation. The same argument is valid to justify the imposition of $u$ to be holomorphic in $L_{-} .{ }^{47}$

Here $\left.[A] \equiv\left(A_{+}-A_{-}\right)\right|_{\mathcal{L}}$ and $\left.\{A\} \equiv\left(A_{+}+A_{-}\right)\right|_{\mathcal{L}}$, where $A_{ \pm}$are the limiting values of $A$ on the left and right sides of the cut. Note that $\mathcal{L}$ is also a branch cut of the function $\lambda$ defined in (4.98). Let's use the Let exploit the holormophicity of $F$.


Figure 2 - Representation of the s-plane considering the F's branch cut.
Source: By the author.

Consider $F(s)$ to be a holomorphic function everywhere, except in the end points of the curve $\mathcal{L}^{54,55}$. Let $a(t)$ satisfy a Holder condition ${ }^{56}$ for all $t \in \mathcal{L}$, that is:

$$
\begin{equation*}
\left|a\left(t_{2}\right)-a\left(t_{1}\right)\right| \leq c\left|t_{2}-t_{1}\right|^{d} \tag{4.107}
\end{equation*}
$$

where $c$ and $d$ are positive constants. Then, $F(s)$ might be written as:

$$
\begin{equation*}
F(s)=\frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{a(t)}{t-s} d t \tag{4.108}
\end{equation*}
$$

Then, $F(s)$ satisfies the Sokhotski-Plemelj relations:

$$
\begin{align*}
& {[F]=F_{+}\left(t_{0}\right)-F_{-}\left(t_{0}\right)=a\left(t_{0}\right)}  \tag{4.109}\\
& \{F\}=F_{+}\left(t_{0}\right)+F_{-}\left(t_{0}\right)=\frac{1}{\pi i} f_{\mathcal{L}} \frac{a(t)}{t-t_{0}} d t \tag{4.110}
\end{align*}
$$

Here $F_{ \pm}\left(t_{0}\right)=\lim _{t \rightarrow t_{0}^{ \pm}} F(t), t_{0} \in \mathcal{L}$ and the dashed integral $\operatorname{symbol}(f)$ means the evaluation of the principal value. Then, the generating function $F(s)$ can be expressed in terms of a Cauchy-type integral, where its jump is interpreted as the density of the Cauchy integral:

$$
\begin{align*}
F_{a}^{1}(\sigma) & =\frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{d \tau}{s-t}\left[F_{a}^{1}\right] ; & a=1,2,3  \tag{4.111}\\
F_{a}^{b}(t) & =\delta_{a}^{b}+\frac{t}{2 \pi i} \int_{\mathcal{L}} \frac{d s}{s(s-t)}\left[F_{a}^{b}\right] ; & b=2,3 \tag{4.112}
\end{align*}
$$

Such choice is due to the $F(s)$ definition (4.39), which imposes $F(s)=\mathbb{1}$, and is motivated by the holomorphicity of $\chi(s)$ even with $s \rightarrow \infty$ (see equations (4.105)). ${ }^{51,52}$ Therefore, the additional condition must be imposed:

$$
\begin{equation*}
2 \pi i \delta_{a}^{1}=\frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{d s}{s}\left[F_{a}^{1}\right] \tag{4.113}
\end{equation*}
$$

It is immediate from the Sokhotsky theorem ${ }^{57}$ that:

$$
\begin{array}{rlr}
\left\{F_{a}^{1}(t)\right\}=\frac{1}{\pi i} f_{\mathcal{L}} \frac{d s}{s-t}\left[F_{a}^{1}\right] ; & a=1,2,3 \\
\left\{F_{a}^{b}(t)\right\}=\delta_{a}^{b}+\frac{t}{\pi i} f_{\mathcal{L}} \frac{d s}{s(s-t)}\left[F_{a}^{b}\right] ; & b=2,3 \tag{4.115}
\end{array}
$$

The point $t$ should be read as a point lying in the curve $\mathcal{L}$. Using the Sokhotsky equations above with the aid of the equations for $F$ evaluated at the branch cut (4.106), one can find:

$$
\begin{equation*}
\frac{t}{\pi} f_{\mathcal{L}} \frac{\left[F_{a}^{1}\right](\tilde{e}(1 / 2 t)+e(1 / 2 s)+2 \tilde{f}(1 / 2 t) f(1 / 2 s))}{s-t} d s+\delta_{a}^{2}-2 \tilde{f}(1 / 2 t) \delta_{a}^{3}=0 \tag{4.116}
\end{equation*}
$$

Since the poles of $F$ are simplified in the $\sigma$-plane, from now on this plane will be considered instead of the $s$-plane. Introducing the unknown function $\mu_{a} \equiv\left[F_{a}^{1}\right] \lambda$, and noticing that for the integration variable $\sigma$, the integration interval becomes $\sigma \in[-1,1]$, where s $1 / 2 s=z+i \rho \sigma$ (analogously, $1 / 2 t=z+i \rho \tau, \tau \in[-1,1]$ ). The integral above gets the new form:

$$
\begin{equation*}
f_{-1}^{1} \frac{\mu_{a}(\sigma)(\tilde{e}(\eta)+e(\xi)+2 \tilde{f}(\eta) f(\xi))}{(\sigma-\tau) \sqrt{1-\sigma^{2}}} d \sigma=2 \pi \rho\left(\delta_{a}^{2}-2 \tilde{f}(\eta) \delta_{a}^{3}\right) \tag{4.117}
\end{equation*}
$$

where $\xi \equiv z+i \rho \sigma$ and $\eta \equiv z+i \rho \tau$. The components of the required matrix $H$ can be obtained from the equations (4.111) and using $H=\left.i \partial_{t} F\right|_{t=0}$ :

$$
\begin{align*}
& H_{a}^{1}=\frac{2 i}{\pi} \int_{-1}^{1} \frac{\xi \mu_{a}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma  \tag{4.118}\\
& H_{a}^{2}=\frac{-1}{\pi} \int_{-1}^{1} \frac{e(\xi) \mu_{a}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma  \tag{4.119}\\
& H_{a}^{3}=\frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi) \mu_{a}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma \tag{4.120}
\end{align*}
$$

Since the normalization of $F_{a}^{1}$ is still not imposed, the additional condition (4.113), which ensures the uniqueness of the solution, must be satisfied:

$$
\begin{equation*}
\int_{-1}^{1} \frac{\mu_{a}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma=\pi \delta_{a}^{1} \tag{4.121}
\end{equation*}
$$

Therefore, from the given behavior of the Ernst potentials on the symmetry axis, the problem of solving the components of the Einstein equation and finding the metric coefficients reduces to finding the unknown function $\mu_{a}$.

### 4.4 About the Method

As mentioned before, for the metric (2.28), the field equations for vacuum and electrovacuum solutions are nonlinear but are known to be integrable. They can be expressed in terms of the Ernst equations. ${ }^{6}$ Associated with this are various solutiongenerating techniques that can be employed to obtain particular families of solutions from any suitable "seed" solution, the one employed in the present study is known as Sibigatullin's method. ${ }^{43}$ Sibgatullin's method is constructed using solitonic techniques and it is based on the analytic continuation of the Riemann-Hilbert problem which uses the Ernst potentials on the symmetry axis $e(z) \equiv \mathcal{E}(\rho=0, z)$ and $f(z) \equiv \Phi(\rho=0, z)$ in order to construct the potentials $\mathcal{E}(\rho, z)$ and $\Phi(\rho, z)$ in the whole space for $\rho>0$. Using this integral formulation, the Ernst Potentials are given by $\left(H_{a}^{b}=\mathcal{E}^{b c} H_{a c}\right)$ :

$$
\begin{align*}
& \mathcal{E}=H_{11}=\frac{1}{\pi} \int_{-1}^{1} \frac{e(\xi) \mu_{1}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma  \tag{4.122}\\
& \Phi=\Phi_{1}=\frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi) \mu_{1}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma \tag{4.123}
\end{align*}
$$

Other useful potentials are given by:

$$
\begin{align*}
& H_{12}=\frac{2 i}{\pi} \int_{-1}^{1} \frac{\xi \mu_{1}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma  \tag{4.124}\\
& H_{21}=\frac{1}{\pi} \int_{-1}^{1} \frac{e(\xi) \mu_{2}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma  \tag{4.125}\\
& \Phi_{2}=\frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi) \mu_{2}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma \tag{4.126}
\end{align*}
$$

where $\xi$ is a complex variable defined by $\xi=z+i \rho \sigma, \sigma \in[-1,1]$ and $\mu_{a}(\sigma)$ with $a=1,2$ is a unknown function being a solution of the following two integral equations:

$$
\begin{align*}
& f_{-1}^{1} \frac{\mu_{a}(\sigma) h(\xi, \eta)}{(\sigma-\tau) \sqrt{1-\sigma^{2}}} d \sigma=2 \pi \rho\left(\delta_{a}^{2}-2 \tilde{f}(\eta) \delta_{a}^{3}\right)  \tag{4.127}\\
& \int_{-1}^{1} \frac{\mu_{a}(\sigma)}{\sqrt{1-\sigma^{2}}} d \sigma=\pi \delta_{a}^{1} \tag{4.128}
\end{align*}
$$

with $\eta=z+i \rho \tau, \tau \in[-1,1]$ and the function $h(\xi, \eta)$ defined by:

$$
\begin{equation*}
h(\xi, \eta)=e(\xi)+\tilde{e}(\eta)+2 \tilde{f}(\eta) f(\xi) \tag{4.129}
\end{equation*}
$$

Here $\tilde{e}(\xi)=e\left(\xi^{*}\right)^{*}$, in another way, the operation tilde means to write the function in terms of the conjugated variable and then take its conjugated. When $e(\xi)$ and $f(\xi)$ are arbitrary rational functions ${ }^{51} \mu_{a}$ should be of the form:

$$
\begin{equation*}
\mu_{a}(\xi)=-i \xi \delta_{a}^{2}+A_{0}+\sum_{k=1}^{N}\left[\frac{A_{k}^{(1)}}{\xi-\alpha_{k}}+\cdots+\frac{A_{k}^{\left(m_{k}\right)}}{\left(\xi-\alpha_{k}\right)^{m_{k}}}\right] \tag{4.130}
\end{equation*}
$$

Where the $\alpha_{k}$ are roots of the equation $h(\xi, \xi)=e(\xi)+\tilde{e}(\xi)+2 \tilde{f}(\xi) f(\xi)=0$ whose multiplicity is denoted by $m_{k}$ and the coefficients $A$ are only functions of $\rho$ and $z$. By substituting this form into (4.127) and (4.128) one finds a linear system of equations for the determinations of the coefficients $A$ 's

Therefore, the metric coefficients $f$ and $\omega$ can be found through the relations:

$$
\begin{align*}
& f=\operatorname{Re}(\mathcal{E})+|\Phi|^{2}  \tag{4.131}\\
& f \omega=\frac{1}{2}\left(H_{12}+H_{21}^{*}\right)+\Phi \Phi_{2}^{*}-i z \tag{4.132}
\end{align*}
$$

Moreover, the metric function $\gamma$ might be found by solving the system (3.19) and (3.20). These methods have been used extensively by Manko and others in order to construct exact solutions (for example, see references 58-60). Notice that the $\alpha$ 's are the zeros of the metric function $f$ on the symmetry axis, that is, the $\alpha$ represents the event
horizon in the $z$-axis. Due to the ansatz for the Ernst potential in the symmetry axis, $\alpha$ can be real or appear in conjugate pairs. Therefore, a pair of real $\alpha$ 's can represent an event horizon and a pair of complex $\alpha$ 's can represent a naked singularity. In conclusion, this method generates any solution of the Ernst equation whose behaviour in the symmetry axis is a rational function of $z$. A general solution in the case of N poles will be discussed in chapter 5 .

### 4.4.1 A Basic Example

According to the Ernst formalism, ${ }^{5}$ the components of the vaccum Einstein field equations (without electromagnetic field, $\Phi=0$ ) for these particular stationary axisymmetric space-times read:

$$
\begin{equation*}
\operatorname{Re}(\mathcal{E}) \nabla^{2} \mathcal{E}=\overrightarrow{\nabla \mathcal{E}} \cdot \vec{\nabla} \mathcal{E} \tag{4.133}
\end{equation*}
$$

with the Ernst potential being $\mathcal{E}=f+i \Omega$. For any solution of equation (4.133), the metric functions $\omega$ and $\gamma$ of the line element (2.28) can be obtained from the following system of differential equations:

$$
\begin{align*}
\omega,_{\rho} & =\rho f^{-2} \Omega, z & \omega,_{z}=-\rho f^{-2} \Omega,_{\rho}  \tag{4.134}\\
4 \gamma,_{\rho} & =\rho f^{-2}\left(|\mathcal{E}, \rho|^{2}-|\mathcal{E}, z|^{2}\right) & 2 \gamma, z=\rho f^{-2} \operatorname{Re}\left(\mathcal{E}, \mathcal{E}^{*}, z\right) \tag{4.135}
\end{align*}
$$

Consider an Ernst potential whose behaviour on the symmetry axis is given by:

$$
\begin{equation*}
e(z)=\frac{z+a+i c}{z+b+i d}=1+\frac{e}{z-\beta} \tag{4.136}
\end{equation*}
$$

Here $a, b, c$ and $d$ are real parameters. In order to find the shape of the function $\mu_{a}$ it is necessary to evaluate the roots of $h(\xi, \xi)$, where $h(\xi, \eta)$ is given by:

$$
\begin{equation*}
h(\xi, \eta)=e(z)+\tilde{e}(z)=2+\frac{e}{\xi-\beta}+\frac{e^{*}}{\eta-\beta^{*}} \tag{4.137}
\end{equation*}
$$

Thus:

$$
\begin{equation*}
h(\xi, \xi)=\frac{2 \xi^{2}+\left(e+e^{*}-\beta-\beta^{*}\right) \xi-e^{*} \beta-e \beta^{*}}{(\xi-\beta)\left(\xi-\beta^{*}\right)} \tag{4.138}
\end{equation*}
$$

Consequently, the roots of $h(\xi, \xi)=0$ are:

$$
\begin{equation*}
\alpha_{ \pm}=\frac{1}{2}\left(-a-b \pm \sqrt{a^{2}-2 a b+b^{2}-4 c d}\right) \tag{4.139}
\end{equation*}
$$

Then the unknown function $\mu_{1}$ should be of the form:

$$
\begin{equation*}
\mu_{1}=A_{0}+\frac{A_{1}}{\xi-\alpha_{+}}+\frac{A_{2}}{\xi-\alpha_{-}} \tag{4.140}
\end{equation*}
$$

Making use of the relation:

$$
\begin{equation*}
\int_{-1}^{1} \frac{d \sigma}{(z-\alpha+i \rho \sigma) \sqrt{1-\sigma^{2}}}=\frac{\pi}{\sqrt{(z-\alpha)^{2}+\rho^{2}}} \tag{4.141}
\end{equation*}
$$

the formulas (4.128) yields:

$$
\begin{equation*}
A_{0}+\frac{A_{1}}{r_{+}}+\frac{A_{2}}{r_{-}}=1, \quad \quad r_{ \pm}=\sqrt{\left(z-\alpha_{ \pm}\right)^{2}+\rho} \tag{4.142}
\end{equation*}
$$

Now, using equation (4.127) and using the following integral relations:

$$
\begin{gather*}
f_{-1}^{1} \frac{d \sigma}{(\sigma-a) \sqrt{1-\sigma^{2}}}=0, \quad-1 \leq \operatorname{Re}(a) \leq 1 \& \operatorname{Im}(a)=0 .  \tag{4.143}\\
f_{-1}^{1} \frac{d \sigma}{(\sigma-a) \sqrt{1-\sigma^{2}}}=-\frac{\pi}{\sqrt{1+\frac{1}{a^{2}}} a}, \quad \text { otherwise. } \tag{4.144}
\end{gather*}
$$

Consequently, two more equations arise from (5.10) for the three variables ( $A_{0}, A_{1}, A_{2}$ ) when one uses the integral above and hence the the system becomes determined, that is:

$$
\begin{gather*}
A_{0}+\frac{A_{1}}{r_{+}}+\frac{A_{2}}{r_{-}}=1,  \tag{4.145}\\
A_{0}-\frac{A_{1}}{\alpha_{+}-\beta}-\frac{A_{2}}{\alpha_{-}-\beta}=0,  \tag{4.146}\\
\frac{A_{1}}{\left(\alpha_{+}-\beta^{*}\right) r_{+}}+\frac{A_{2}}{\left(\alpha_{-}-\beta^{*}\right) r_{-}}=0 . \tag{4.147}
\end{gather*}
$$

This system is trivially solved and one can find:

$$
\begin{align*}
& A_{0}=\frac{\left(\alpha_{+}-\beta\right)\left(\alpha_{-} \beta^{*}\right) r_{--}\left(\alpha_{-}-\beta\right)\left(\alpha_{+}-\beta^{*}\right) r_{+}}{\left(\alpha_{+}-\beta\right)\left(\alpha_{-} \beta^{*}\right) r_{-}-\left(\alpha_{-}-\beta\right)\left(\alpha_{+}-\beta^{*}\right) r_{+}-\left(\alpha_{+}-\beta^{*}\right)\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta\right)+\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta^{*}\right)\left(\alpha_{-}-\beta\right)}  \tag{4.148}\\
& A_{1}=\frac{-\left(\alpha_{-}-\beta\right)\left(\alpha_{-}-\beta\right)\left(\alpha_{+}-\beta^{*}\right) r_{+}}{\left(\alpha_{+}-\beta\right)\left(\alpha_{-} \beta^{*}\right) r_{-}\left(\alpha_{-}-\beta\right)\left(\alpha_{+}-\beta^{*}\right) r_{+}-\left(\alpha_{+}-\beta^{*}\right)\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta\right)+\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta^{*}\right)\left(\alpha_{-}-\beta\right)}  \tag{4.149}\\
& A_{2}=\frac{\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta^{*}\right)\left(\alpha_{-}-\beta\right) r_{-}}{\left(\alpha_{+}-\beta\right)\left(\alpha_{-} \beta^{*}\right) r_{-}-\left(\alpha_{-}-\beta\right)\left(\alpha_{+}-\beta^{*}\right) r_{+}-\left(\alpha_{+}-\beta^{*}\right)\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta\right)+\left(\alpha_{+}-\beta\right)\left(\alpha_{-}-\beta^{*}\right)\left(\alpha_{-}-\beta\right)} \tag{4.150}
\end{align*}
$$

The Ernst potential $\mathcal{E}$ can be found by performing the integration in equation (4.122), for which the result can be put in the following form:

$$
\begin{gather*}
\mathcal{E}=2 A_{0}-1=\frac{\left|\begin{array}{ccc}
1 & 1 & 1 \\
1 & \frac{r_{+}}{\alpha_{+}-\beta} & \frac{r_{-}}{\alpha_{-}-\beta} \\
0 & \frac{1}{\alpha_{+}-\beta^{*}} & \frac{\alpha_{-}-\beta^{*}}{}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & 1 & 1 \\
-1 & \frac{r_{+}}{\alpha_{+}-\beta} & \frac{r_{-}}{\alpha_{-}-\beta} \\
0 & \frac{1}{\alpha_{+}-\beta^{*}} & \frac{1}{\alpha_{-}-\beta^{*}}
\end{array}\right|}  \tag{4.151}\\
\mathcal{E}=\frac{-(\alpha--\beta)\left(\alpha_{+}^{2}-\alpha_{+}\left(\alpha_{-}+\beta+r_{+}\right)+\alpha-\beta\right)+\beta^{*}\left(\alpha_{+} r_{-}-\alpha_{-} r_{+}+\beta r_{+}-\beta r_{-}\right)+\alpha_{-} r_{-}\left(\beta-\alpha_{+}\right)}{\beta^{*}\left(\alpha_{+} r_{-}-\alpha_{-} r_{+}+\beta r_{+}-\beta r_{-}\right)+\left(\alpha_{-}-\beta\right)\left(\left(\alpha_{+}-\alpha_{-}\right)\left(\alpha_{+}-\beta\right)+\alpha_{+} r_{+}\right)+\alpha_{-} r_{-}\left(\beta-\alpha_{+}\right)} \tag{4.152}
\end{gather*}
$$

Now, we can make use of the multipole expansion in appendix C:

$$
\begin{equation*}
P_{0}=\frac{b-a+i(d-c)}{2}, \quad \quad P_{1}=\frac{a^{2}-b^{2}+d^{2}-c^{2}+i 2(a c-b d)}{4} \tag{4.153}
\end{equation*}
$$

To ensure that the solution is asymptotically flat, the angular monopole must be zero, ${ }^{29,34}$ consequently $c=d$. Thus, $b-a$ should be interpreted as the mass, $2 M$, of the source and $c$ as the density of angular momentum with the opposite sign, $j=J / M$. Another comment is that the origin of the coordinate system does not coincide with the centre of mass. One can see this by the nonzero real component of $P_{1}$, which represents the dipole moment. To set the coordinate system origin to be the centre of mass, one just needs to choose the coordinate $z$ such that $a+b=0$. The Ernst equation becomes:

$$
\begin{equation*}
\mathcal{E}=\frac{\sqrt{M^{2}-j^{2}}\left(-2 M+r_{+}+r_{-}\right)+i j\left(r_{+}-r_{-}\right)}{\sqrt{M^{2}-j^{2}}\left(2 M+r_{+}+r_{-}\right)+i j\left(r_{+}-r_{-}\right)} \tag{4.154}
\end{equation*}
$$

Also notice that $\alpha_{ \pm}$, which represents the location of the compact object, is now symmetric in relation to the origin.

Making use of the prolate spheroidal coordinates:

$$
\begin{gather*}
x=\frac{r_{-}+r_{+}}{2 \sigma}, \quad y=\frac{r_{-}-r_{+}}{2 \sigma} .  \tag{4.155}\\
\sigma=\sqrt{M^{2}-j^{2}} . \tag{4.156}
\end{gather*}
$$

Hence

$$
\begin{equation*}
\mathcal{E}=\frac{x \sqrt{M^{2}-j^{2}}-i j y-M}{x \sqrt{M^{2}-j^{2}}-i j y+M} \tag{4.157}
\end{equation*}
$$



Figure 3 - Localization of the compact object with respect to the $z$ axis. a)Considering the roots as real, $\alpha_{ \pm}= \pm \sigma$. b)Considering the roots as imaginary, $\alpha_{ \pm}= \pm i \sigma$.

Source: By the author.

Rewriting $\cos \lambda=\frac{\sqrt{M^{2}-j^{2}}}{M}$ and $\sin \lambda=\frac{j}{M}$, the solution presented in section 3.3 is recovered when one consider the sub-extreme case.

Now, one should use the relation $f \omega=\frac{1}{2}\left(H_{12}+H_{21}^{*}\right)-i z$ to find the function $\omega$. Using equation (4.124), one finds:

$$
\begin{equation*}
H_{12}=2 i z+2 i \frac{\left(r_{+}+\alpha_{+}-z\right) A_{1}}{r_{+}}+2 i \frac{\left(r_{-}+\alpha_{-}-z\right) A_{2}}{r_{-}} . \tag{4.158}
\end{equation*}
$$

Noticed that, to evaluate $H_{21}$ one should evaluate the function $\mu_{2}$, which is found in a similar procedure than that for $\mu_{1}$. That is, $\mu_{2}$ should be of the form:

$$
\begin{equation*}
\mu_{2}=-i \xi+B_{0}+\frac{B_{1}}{\xi-\alpha_{+}}+\frac{B_{2}}{\xi-\alpha_{-}} \tag{4.159}
\end{equation*}
$$

By means of equations (4.127) and (4.128), one finds the following system of equations for the variables $\left(B_{0}, B_{1}, B_{2}\right)$ :

$$
\begin{gather*}
B_{0}+\frac{B_{1}}{r_{+}}+\frac{B_{2}}{r_{-}}=i z  \tag{4.160}\\
B_{0}-\frac{B_{1}}{\alpha_{1}-\beta}-\frac{B_{2}}{\alpha_{2}-\beta}=i \beta  \tag{4.161}\\
\frac{B_{1}}{\left(\alpha_{+}-\beta^{*}\right) r_{+}}+\frac{B_{2}}{\left(\alpha_{-}-\beta^{*}\right) r_{-}}=i \tag{4.162}
\end{gather*}
$$

Using equation (4.125), one finds:

$$
\begin{equation*}
H_{21}=-2 i(a-b)-2 i z+2 B_{0} \tag{4.163}
\end{equation*}
$$

Consequently:

$$
\begin{equation*}
\omega=2 \frac{j M}{\sqrt{M^{2}-j^{2}}} \frac{\left(1-y^{2}\right)\left(\sqrt{M^{2}-j^{2}} x+M\right)}{\left(M^{2}-j^{2}\right) x^{2}+j^{2} y^{2}-M^{2}} \tag{4.164}
\end{equation*}
$$

The equation above shows that when the parameter associated with the angular momentum, $j$, vanishes, the spacetime is static.

## 5 N-SOLITONIC SPACETIME

In this chapter the construction of N -soliton solutions and their corresponding spacetimes will be discussed, based on the Sibgatullin's method ${ }^{27}$ and the correspondents metric functions. This decribes a system a system of up to $N$-solitons aligned on the symmetry axis*.

The construction of exact axisymmetric solutions of the Einstein-Maxwell equations possessing the required physical properties, implies the existence of solutions in which different parameters would correspond to different relativistic multipole moments, determining the structure of spacetime. Since the starting point is just the behaviour of the Ernst potentials, ${ }^{6} \mathcal{E}$ and $\Phi$, on the symmetry axis, the arbitrariness of the parameters introduced leads to arbitrary multipoles. Thus, it is necessary to find a relation between the free parameters in the solution and their physical meaning. In the following, we will demonstrate that Sibgatullin's method may offer a link between them.

### 5.1 Derivation of the N -Soliton Solution

Consider, now, the general N -soliton electrovacuum solution characterized by the Ernst potentials on the symmetry axis given in terms of a polynomial quotient ${ }^{27}$ :

$$
\begin{align*}
& e(z)=\frac{z^{N}+\sum_{l=1}^{N} a_{l} z^{N-l}}{z^{N}+\sum_{l=1}^{N} b_{l} z^{N-l}}=\frac{P(z)}{R(z)}  \tag{5.1}\\
& f(z)=\frac{\sum_{l=1}^{N} c_{l} z^{N-l}}{z^{N}+\sum_{l=1}^{N} b_{l} z^{N-l}}=\frac{Q(z)}{R(z)} \tag{5.2}
\end{align*}
$$

where $a_{l}, b_{l}, c_{l}, k=1, \cdots N$ are $3 N$ arbitrary complex constants. Notice that the choice of the coefficient of higher order has been made to give an appropriate asymptotic behaviour to the potentials at infinity. The interpretation of these parameters can be revealed by calculating the multipole moments ${ }^{\dagger}$.

Supposing that the previous quotients are irreducible and that $R$ only possesses roots with multiplicity one, then it is possible to write (5.1) and (5.2) in the following form:

[^8]\[

$$
\begin{array}{r}
e(z)=1+\sum_{l=1}^{N} \frac{e_{l}}{z-\beta_{l}} \\
f(z)=\sum_{l=1}^{N} \frac{f_{l}}{z-\beta_{l}} \tag{5.4}
\end{array}
$$
\]

The new parameters $e_{l}, \beta_{l}$ and $f_{l}$ are related with the old ones through the relations.

$$
\begin{gathered}
e_{l}=\frac{P\left(\beta_{l}\right)}{\prod_{k \neq l}^{N}\left(\beta_{l}-\beta_{k}\right)} ; \quad f_{l}=\frac{Q\left(\beta_{l}\right)}{\prod_{k \neq l}^{N}\left(\beta_{l}-\beta_{k}\right)} ; \\
R\left(\beta_{l}\right)=0 .
\end{gathered}
$$

Notice that the numerator of the function $h(\xi, \xi)(4.129)$ is given by a polynomial of order $2 N$ with real coefficients, such that its complex roots $\alpha_{n}$ only appear in conjugate pairs. Supposing that each root $\alpha_{n}$ possesses multiplicity equal to one, we get

$$
\begin{equation*}
h(\xi, \xi)=e(\xi)+\tilde{e}(\xi)+2 f(\xi) \tilde{f}(\xi)=\frac{2 \prod_{n=1}^{2 N}\left(\xi-\alpha_{n}\right)}{\prod_{l=1}^{N}\left(\xi-\beta_{l}\right)\left(\xi-\beta_{l}^{*}\right)} \tag{5.5}
\end{equation*}
$$

Consequently the function $\mu_{1}(\xi)$ must be of the form:

$$
\begin{equation*}
\mu_{1}(\xi)=A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}} \tag{5.6}
\end{equation*}
$$

Before evaluating the potentials $\mathcal{E}$ and $\Phi$, one needs to find the unknown function $\mu$, which in turn, is fully defined by equations (4.127) and (4.128). In order to evaluate the equation (4.127), consider the product ${ }^{61}$ :

$$
\begin{align*}
& \mu_{1}(\xi) h(\xi, \eta)=\left(A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}\right)(e(\xi)+\tilde{e}(\eta)+2 \tilde{f}(\eta) f(\xi))  \tag{5.7}\\
& =A_{0}(\tilde{e}(\eta)+1)+A_{0} \sum_{l=1}^{N} \frac{e_{l}+2 \tilde{f}(\eta) f_{l}}{\xi-\beta_{l}}+(\tilde{e}(\eta+1)) \sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}+ \\
& +\sum_{n=1}^{2 N} \sum_{l=1}^{N} A_{n}\left(e_{l}+2 \tilde{f}(\eta) f_{l}\right)\left(\frac{1}{\left(\alpha_{n}-\beta_{l}\right)\left(\xi-\alpha_{n}\right)}-\frac{1}{\left(\alpha_{n}-\beta_{l}\right)\left(\xi-\beta_{l}\right)}\right)  \tag{5.8}\\
& =A_{0}(\tilde{e}(\eta)+1)+\sum_{l=1}^{N} \mu_{1}\left(\beta_{l}\right) \frac{\left(e_{l}+2 \tilde{f}(\eta) f_{l}\right)}{\xi-\beta_{l}}+\sum_{n=1}^{2 N} h\left(\alpha_{n}, \eta\right) \frac{A_{n}}{\xi-\alpha_{n}} \tag{5.9}
\end{align*}
$$

The result of substituting this compact expression into (4.127) is:

$$
\begin{equation*}
\sum_{l=1}^{N} \mu_{1}\left(\beta_{l}\right) \frac{e_{l}+2 \tilde{f}(\eta) f_{l}}{\left(\eta-\beta_{l}\right) R_{l}}+\sum_{n=1}^{2 N} \frac{h\left(\alpha_{n}, \eta\right) A_{n}}{\left(\eta-\alpha_{n}\right) r_{n}}=0 \tag{5.10}
\end{equation*}
$$

Here, $R_{l}=\sqrt{\rho^{2}+\left(z-\beta_{l}\right)^{2}}$ and $r_{n}=\sqrt{\rho^{2}+\left(z-\alpha_{n}\right)^{2}}$. The equation above must be valid for all $\eta$. Therefore, each of the coefficients, for a different denominator $\eta$, must be zero. In order to achieve a simple form to set the coefficients to zero, consider that:

$$
\begin{equation*}
h\left(\alpha_{n}, \eta\right)=e\left(\alpha_{n}\right)+\tilde{e}(\eta)+2 f\left(\alpha_{n}\right) \tilde{f}(\eta) \tag{5.11}
\end{equation*}
$$

Adding and subtracting $\tilde{e}\left(\alpha_{n}\right)$ and $2 f\left(\alpha_{n}\right) \tilde{f}\left(\alpha_{n}\right)$ in the RHS gives:

$$
\begin{equation*}
=\underline{h\left(\alpha_{n}, \alpha_{n}\right)}+\tilde{e}(\eta)-\tilde{e}\left(\alpha_{n}\right)+2 f\left(\alpha_{n}\right) \tilde{f}(\eta)-2 f\left(\alpha_{n}\right) \tilde{f}\left(\alpha_{n}\right) \tag{5.12}
\end{equation*}
$$

Thus, it is possible to write:

$$
\begin{equation*}
\frac{h\left(\alpha_{n}, \eta\right)}{\eta-\alpha_{n}}=-\sum_{l=1}^{N} \frac{h_{l}\left(\alpha_{n}\right)}{\left(\alpha_{n}-\beta_{l}^{*}\right)\left(\eta-\beta_{l}^{*}\right)} \tag{5.13}
\end{equation*}
$$

where $h_{l}=e_{l}^{*}+2 f\left(\alpha_{n}\right) f_{l}^{*}$.
Substituting this result into (5.10):

$$
\begin{equation*}
\sum_{l=1}^{N} \mu_{1}\left(\beta_{l}\right) \frac{e_{l}+2 \tilde{f}(\eta) f_{l}}{R_{l}\left(\eta-\beta_{l}\right)}-\sum_{l=1}^{N} \sum_{n=1}^{2 N} \frac{h_{l}\left(\alpha_{n}\right) A_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}} \frac{1}{\eta-\beta_{l}^{*}}=0 \tag{5.14}
\end{equation*}
$$

This relation holds if the coefficients of the independents terms $\left(\eta-\beta_{l}\right)^{-1}$ and $\left(\eta-\beta_{l}^{*}\right)^{-1}$ are equal to zero. Hence ${ }^{61}$ :

$$
\begin{align*}
& \mu_{1}\left(\beta_{l}\right)=0 \longrightarrow A_{0}-\sum_{n=1}^{2 N} \frac{A_{n}}{\alpha_{n}-\beta_{l}}=0  \tag{5.15}\\
& \sum_{n=1}^{2 N} \frac{h_{l}\left(\alpha_{n}\right) A_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}}=0 \tag{5.16}
\end{align*}
$$

We then find a set of $2 N$ equations for the coefficients $A_{0}$ and $A_{n}^{\prime} s$. By means of the equation (4.128),

$$
\begin{equation*}
A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{r_{n}}=1 \tag{5.17}
\end{equation*}
$$

The set of equations becomes completely determined. In other words, one needs to solve $2 N+1$ equations for the $2 N+1$ coefficients.

$$
\begin{align*}
& A_{0}-\sum_{n=1}^{2 N} \frac{A_{n}}{\alpha_{n}-\beta_{l}}=0  \tag{5.18}\\
& \sum_{n=1}^{2 N} \frac{h_{l}\left(\alpha_{n}\right) A_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}}=0 \quad l=1, \cdots, N  \tag{5.19}\\
& A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{r_{n}}=1 \tag{5.20}
\end{align*}
$$

Once the function $\mu_{1}(\xi)$ becomes determined, it is possible to evaluate the Ernst potential $\mathcal{E}$ trough the integrals (4.122). Thereby, consider the product:

$$
\begin{align*}
& e(\xi) \mu_{1}(\xi)=\left(1+\sum_{l=1}^{N} \frac{e_{l}}{\xi-\beta_{l}}\right)\left(A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}\right)  \tag{5.21}\\
& =A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}+A_{0} \sum_{l=1}^{N} \frac{e_{l}}{\xi-\beta_{l}}+\sum_{l=1}^{N} \sum_{n=1}^{2 N} e_{l} A_{n}\left(\frac{1}{\left(\beta_{l}-\alpha_{n}\right)\left(\xi-\beta_{l}\right)}+\frac{1}{\left(\alpha_{n}-\beta_{l}\right)\left(\xi-\alpha_{n}\right)}\right) \\
& =A_{0}+\sum_{n=1}^{2 N} \frac{e\left(\alpha_{n}\right) A_{n}}{\left(\xi-\alpha_{n}\right)}+\sum_{l=1}^{N} \frac{e_{l} \mu_{1}\left(\beta_{l}\right)}{\xi-\beta_{l}} \tag{5.22}
\end{align*}
$$

Notice that the third term in (5.22) is zero as a consequence of the equation (5.15). Therefore, the Ernst Potential $\mathcal{E}$ in terms of the coefficients $A$ is written as:

$$
\begin{equation*}
\mathcal{E}=A_{0}+\sum_{n=1}^{2 N} \frac{e\left(\alpha_{n}\right) A_{n}}{r_{n}} \tag{5.23}
\end{equation*}
$$

The equation $h\left(\alpha_{n}, \alpha_{n}\right)=0$ implies that:

$$
\begin{equation*}
e\left(\alpha_{n}\right)=-\tilde{e}\left(\alpha_{n}\right)-2 \tilde{f}\left(\alpha_{n}\right) f\left(\alpha_{n}\right)=-1-\sum_{l=1}^{N} \frac{h_{l}\left(\alpha_{n}\right)}{\alpha_{n}-\beta_{l}^{*}} \tag{5.24}
\end{equation*}
$$

Hence:

$$
\begin{equation*}
\mathcal{E}=A_{0}-\sum_{n=1}^{2 N} \frac{A_{n}}{r_{n}}-\sum_{n=1}^{2 N} \sum_{l=1}^{N} \frac{h_{l}\left(\alpha_{n}\right) A_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}} \tag{5.25}
\end{equation*}
$$

By means of the equations (5.19) and (5.20), it is found that $\mathcal{E}$ can be written in terms of $A_{0}$ only

$$
\begin{equation*}
\mathcal{E}=2 A_{0}-1 \tag{5.26}
\end{equation*}
$$

In a similar way, the potential $\Phi$ is obtained by means of (4.123). Consider, then:

$$
\begin{gather*}
f(\xi) \mu_{1}(\xi)=\left(\sum_{l=1}^{N} \frac{f_{l}}{\xi-\beta_{l}}\right)\left(A_{0}+\sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}\right)  \tag{5.27}\\
=\sum_{l=1}^{N} \frac{f_{l}}{\xi-\beta_{l}} \mu_{1}\left(\beta_{l}\right)+\sum_{n=1}^{2 N} \frac{f\left(\alpha_{n}\right) A_{n}}{\xi-\alpha_{n}} \tag{5.28}
\end{gather*}
$$

Therefore, $\Phi$ can be written as:

$$
\begin{equation*}
\Phi=\sum_{n=1}^{2 N} \frac{f\left(\alpha_{n}\right) A_{n}}{r_{n}} \tag{5.29}
\end{equation*}
$$

Finally, by substituting the values of the $A_{n}$ 's, solutions of (5.18), (5.19) and (5.20), it is possible to write the Ernst potentials in a very compact way as determinants ${ }^{27}$ :

$$
\begin{array}{cc}
\mathcal{E}(\rho, z)=\frac{E_{+}}{E_{-}} ; & \Phi(\rho, z)=\frac{F}{E_{-}} \\
E_{ \pm}=\left|\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\pm 1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \vdots & \ddots & \vdots \\
\pm 1 & \frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right| \quad F=\left|\begin{array}{cccc}
0 & f\left(\alpha_{1}\right) & \cdots & f\left(\alpha_{2 N}\right) \\
-1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \vdots & \ddots & \vdots \\
-1 & \frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right| \tag{5.31}
\end{array}
$$

$E_{ \pm}$and F must be read as $(2 N+1) \times(2 N+1)$ determinants. These solutions of the Ernst equations, which have been found by means of Sibgatullin's integral method, are N -soliton solutions of eletrovacuum stationary spacetimes with axial symmetry. Using the equation (4.131), the metric function can be written as:

$$
\begin{equation*}
f=\frac{E_{+} E_{-}^{*}+E_{+}^{*} E_{-}+2 F F^{*}}{2 E_{-} E_{-}^{*}} \tag{5.32}
\end{equation*}
$$

The metric function $\gamma$ is determined by equations (3.19) and (3.20). Ruiz et al ${ }^{27}$ also gave a compact formula for the metric function $\gamma$ with a proper choice of the integration
constant:

$$
\begin{align*}
e^{2 \gamma} & =\frac{E_{+} E_{-}^{*}+E_{+}^{*} E_{-}+2 F F^{*}}{2 K_{0} K_{0}^{*}} \prod_{n=1}^{2 N} r_{n}  \tag{5.33}\\
K_{0} & =\left|\begin{array}{ccc}
\frac{1}{\alpha_{1}-\beta_{1}} & \cdots & \frac{1}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \ddots & \vdots \\
\frac{1}{\alpha_{1}-\beta_{N}} & \cdots & \frac{1}{\alpha_{2 N}-\beta_{N}} \\
\frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \ddots & \vdots \\
\frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right| \tag{5.34}
\end{align*}
$$

The still unknown function $\omega$ might be found by using equation (4.132), $f \omega=$ $\frac{1}{2}\left(H_{12}+H_{21}^{*}\right)+\Phi \Phi_{2}^{*}-i z$. Recalling the function $H_{12}$ :

$$
\begin{equation*}
H_{12}=\frac{2 i}{\pi} \int_{-1}^{1} \frac{\xi \mu_{1}(\xi) d \sigma}{\sqrt{1-\sigma^{2}}} \tag{5.35}
\end{equation*}
$$

considering the product:

$$
\begin{equation*}
\xi \mu_{1}(\xi)=z A_{0}+i \rho \sigma A_{0}+z \sum_{n=1}^{2 N} \frac{A_{n}}{\xi-\alpha_{n}}+i \rho \sum_{n=1}^{2 N} \frac{A_{n} \sigma}{\xi-\alpha_{n}}, \tag{5.36}
\end{equation*}
$$

using the integral relation:

$$
\begin{align*}
\int_{-1}^{1} \frac{\sigma}{(\sigma+a) \sqrt{1-\sigma^{2}}} d \sigma=\pi- & \frac{\pi}{\sqrt{1-\frac{1}{a^{2}}}} \\
& \operatorname{Im}(a) \neq 0\|\operatorname{Re}(a)=0\| \operatorname{Re}(a) \geq 1 \| \operatorname{Re}(a) \leq-1 \tag{5.37}
\end{align*}
$$

and making use of the condition (5.18), one finds:

$$
\begin{equation*}
H_{12}=2 i z+2 i \sum_{k=1}^{2 N} \frac{\left(r_{n}+\alpha_{n}-z\right) A_{n}}{r_{n}} . \tag{5.38}
\end{equation*}
$$

Now it is necessary to calculate the unknown function $\mu_{2}(\xi)$ which has the form:

$$
\begin{equation*}
\mu_{2}(\xi)=-i \xi+B_{0}+\sum_{n=1}^{2 N} \frac{B_{n}}{\xi-\alpha_{n}} . \tag{5.39}
\end{equation*}
$$

Similar to what was done for $\mu_{1}$, one needs to find the coefficients $B_{k}$ 's. First, analyzing the equation (4.127), consider the product:

$$
\begin{align*}
& \mu_{2}(\xi) h(\xi, \eta)=\left(-i \xi+B_{0}+\sum_{n=1}^{2 N} \frac{B_{n}}{\xi-\alpha_{n}}\right)(e(\xi)+\tilde{e}(\eta)+2 \tilde{f}(\eta) f(\xi))  \tag{5.40}\\
& =\left(-i z+B_{0}\right)(1+\tilde{e}(\eta))+\rho \sigma(1+\tilde{e}(\eta))+\left(-i z+B_{0}\right) \sum_{l=1}^{N} \frac{e_{l}+2 \tilde{f}(\eta) f_{l}}{\xi-\beta_{l}}+ \\
& +\rho \sigma \sum_{l=1}^{N} \frac{e_{l}+2 \tilde{f}(\eta) f_{l}}{\xi-\beta_{l}}+(1+\tilde{e}(\eta)) \sum_{n=1}^{2 N} \frac{B_{n}}{\xi-\alpha_{n}}+  \tag{5.41}\\
& +\sum_{n=1}^{2 N} \sum_{l=1}^{N} B_{n}\left(e_{l}+2 \tilde{f}(\eta) f_{l}\right)\left(\frac{1}{\left(\alpha_{n}-\beta_{l}\right)\left(\xi-\alpha_{n}\right)}-\frac{1}{\left(\alpha_{n}-\beta_{l}\right)\left(\xi-\beta_{l}\right)}\right)
\end{align*}
$$

After integrating, one finds the following two conditions:

$$
\begin{gather*}
B_{0}-\sum_{n=1}^{2 N} \frac{B_{n}}{\alpha_{n}-\beta_{l}}=i \beta_{l}, \\
\sum_{n=1}^{2 N} \frac{h_{l}\left(\alpha_{n}\right) B_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}}=i e_{l}^{*} . \tag{5.42}
\end{gather*}
$$

Arriving then in a set of $2 N$ equations for the coefficients $B_{0}$ and $B_{n}^{\prime} s$. By means of the equation (4.128):

$$
\begin{equation*}
B_{0}+\sum_{n=1}^{2 N} \frac{B_{n}}{r_{n}}=i z \tag{5.43}
\end{equation*}
$$

Hence, the system is complete. Finally, one can find $H_{21}$ and $\Phi_{2}$ by the equations (4.125) and (4.126):

$$
\begin{align*}
& H_{21}=\frac{1}{\pi} \int_{-1}^{1} \frac{e(\xi) \mu_{2}(\sigma) d \sigma}{\sqrt{1-\sigma^{2}}},  \tag{5.44}\\
& \Phi_{2}=\frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi) \mu_{2}(\sigma) d \sigma}{\sqrt{1-\sigma^{2}}} . \tag{5.45}
\end{align*}
$$

Which results in:

$$
\begin{align*}
& H_{21}=-i \sum_{l=1}^{N}\left(e_{l}+e_{l}^{*}\right)-2 i z+2 B_{0}  \tag{5.46}\\
& \Phi_{2}=-i \sum_{l=1}^{N} f_{l}+\sum_{n=1}^{2 N} \frac{f\left(\alpha_{n}\right) B_{n}}{r_{n}} \tag{5.47}
\end{align*}
$$

Using the results above, the metric function $\omega$ is determined as being:

$$
\begin{equation*}
\omega=\frac{2 \operatorname{Im}\left(E_{-} H^{*}-E_{-}^{*} G-F I^{*}\right)}{E_{+} E_{-}^{*}+E_{+}^{*} E_{-}+2 F F^{*}} . \tag{5.48}
\end{equation*}
$$

where

$$
\begin{align*}
& G=\left|\begin{array}{cccc}
0 & r_{1}+\alpha_{1}-z & \ldots & r_{2 N}+\alpha_{2 N}-z \\
-1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \ldots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \vdots & \ddots & \vdots \\
-1 & \frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \ldots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right|,  \tag{5.49}\\
& H=\left|\begin{array}{cccc}
z & 1 & \cdots & 1 \\
-\beta_{1} & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \vdots & \ddots & \vdots \\
-\beta_{N} & \frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
e_{1}^{*} & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \vdots & \ddots & \vdots \\
e_{N}^{*} & \frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right|,  \tag{5.50}\\
& I=\left|\begin{array}{ccccc}
\sum_{l=1}^{N} f_{l} & 0 & f\left(\alpha_{1}\right) & \cdots & f\left(\alpha_{2 N}\right) \\
z & 1 & 1 & \cdots & 1 \\
-\beta_{1} & -1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\beta_{N} & -1 & \frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
e_{1}^{*} & 0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{N}^{*} & 0 & \frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right| . \tag{5.51}
\end{align*}
$$

Here, $H$ and $G$ are determinants of $(2 N+1) \times(2 N+1)$ matrices, while $I$ is the determinant of a $(2 N+2) \times(2 N+2)$ matrix.

Another useful representation of the previous equations was deduced by Ernst ${ }^{62,63}$ in the framework of the Neugebauer family of spacetimes. ${ }^{64}$ Although the $N$-soliton solution given by Manko ${ }^{27}$ and the Ernst's one are constructed from different generating techniques, Manko has shown that both approach are equivalent. ${ }^{24}$ In fact, they only differ in how they are written, that is:

$$
\begin{equation*}
\mathcal{E}(\rho, z)=\frac{U-W}{U+W} ; \quad \quad \Phi(\rho, z)=\frac{F}{U+W} \tag{5.52}
\end{equation*}
$$

where

$$
U=\left|\begin{array}{ccc}
\frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{1}}  \tag{5.53}\\
\vdots & \ddots & \vdots \\
\frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N}}{\alpha_{2 N}-\beta_{N}} \\
\frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{1}^{*}} \\
\vdots & \ddots & \vdots \\
\frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N}\right)}{\alpha_{2 N}-\beta_{N}^{*}}
\end{array}\right| \quad W=\left|\begin{array}{cc}
0 & 1 \\
1 & \\
& (U) \\
0 &
\end{array}\right|
$$

Noticed that all expressions were obtained assuming that the roots $\alpha$ have multiplicity equal to one, that means that the equations above are to be used dealing with sub-extreme and hyper-extreme (naked singularities) objects. But, as Manko et. al. in refrence 26 mentioned, to deal with extreme objects, one can apply L'Hôpital's rule.

### 5.1.1 N -soliton solution for extreme cases

As said before, the $N$-soliton solution was constructed considering only the roots $\alpha_{n}$ with multiplicity one. This means that there roots are real (sub-extreme objects) or they appear in complex conjugate pairs (hyper-extreme objects). Although the formulas presented in the previous section still work when the roots have multiplicity 2 (extreme objects) with the L'Hôpital's rule is applied, sometimes is more practical to already have the right formulas. Hence, in this section we will deduce the $N$-soliton solution considering the case in which $m$ roots $\alpha_{n}$ have multiplicity two. To do so, consider again the function $h(\xi, \eta)$ defined in (4.129), in which $\alpha_{n}$ are the roots of $h(\xi, \xi)$ (compare the equation below with equation (5.5))

$$
\begin{equation*}
h(\xi, \xi)=e(\xi)+\tilde{e}(\xi)+2 f(\xi) \tilde{f}(\xi)=\frac{2 \prod_{j=1}^{m}\left(z-\alpha_{j}\right)^{2} \prod_{n=m+1}^{2 N-m}\left(z-\alpha_{n}\right)}{\prod_{k=1}^{N}\left(z-\beta_{k}\right)\left(z-\bar{\beta}_{k}\right)} \tag{5.54}
\end{equation*}
$$

Notice that now, we have $6 N-m$ independents real parameters. Without loss of generality, we can choose the first $m$ roots to be those with multiplicity two. Also, considering the multiplicity, the function $\mu_{a}(\xi)$ must be of the form:

$$
\begin{equation*}
\mu_{a}(\xi)=-i \xi \delta_{a}^{2}+A_{0}+\sum_{k=1}^{2 N-m} \frac{A_{k}}{\xi-\alpha_{k}}+\sum_{j=1}^{m} \frac{A_{2 N+1-j}}{\left(\xi-\alpha_{j}\right)^{2}} \tag{5.55}
\end{equation*}
$$

Substituting $\mu_{1}$ into the integrals (4.128) and (4.127), and by performing the same calculations as in the previous section, we find the following system of equations ${ }^{65}$ :

$$
\begin{align*}
& A_{0}-\sum_{n=1}^{2 N-m} \frac{A_{n}}{\alpha_{n}-\beta_{l}}+\sum_{j=1}^{m} \frac{r_{j} A_{2 N+1-j}}{\left(\alpha_{j}-\beta_{l}\right)^{2}}=0  \tag{5.56}\\
& \sum_{n=1}^{2 N-m} \frac{h_{l}\left(\alpha_{n}\right) A_{n}}{\left(\alpha_{n}-\beta_{l}^{*}\right) r_{n}}+\sum_{j=1}^{m} \frac{W_{l j} A_{2 N+1-j}}{r_{j}}=0, \quad l=1,2, \ldots N  \tag{5.57}\\
& A_{0}+\sum_{n=1}^{2 N-m} \frac{A_{n}}{r_{n}}+\sum_{j=1}^{m} \frac{P_{j} A_{2 N+1-j}}{r_{j}}=1 \tag{5.58}
\end{align*}
$$

where $W_{l j}=r_{j}^{2} \partial_{a_{j}}\left(\frac{h_{l}\left(\alpha_{j}\right)}{\left(\alpha_{j}-\beta_{l}^{*}\right) r_{j}}\right)$ and $P_{j}=\frac{z-\alpha_{j}}{r_{j}}$. Therefore, the Ernst potentials are ${ }^{61}$ :

$$
\begin{align*}
& \mathcal{E}(\rho, z)=\frac{U^{(m)}-W^{(m)}}{U^{(m)}+W^{(m)}} ; \quad \quad \Phi(\rho, z)=\frac{F^{(m)}}{U^{(m)}+W^{(m)}}  \tag{5.59}\\
& W^{(m)}=\left|\begin{array}{ccc}
0 & 1 & P_{(m)} \\
1 & & \left(U^{(m)}\right) \\
0 &
\end{array}\right|, \quad F^{(m)}=\left|\begin{array}{ccc}
0 & f\left(\alpha_{n}\right) & r_{m}^{2} \frac{\partial}{\partial \alpha_{m}}\left(\frac{f\left(\alpha_{m}\right)}{r_{m}}\right) \\
-1 & \left(U^{(m)}\right) \\
0 &
\end{array}\right|,  \tag{5.60}\\
& U^{(m)}=\left|\begin{array}{cccccc}
\frac{r_{1}}{\alpha_{1}-\beta_{1}} & \cdots & \frac{r_{2 N-m}}{\alpha_{2 N-m}-\beta_{1}} & -\frac{r_{m}^{2}}{\left(\alpha_{m}-\beta_{1}\right)^{2}} & \cdots & -\frac{r_{1}^{2}}{\left(\alpha_{1}-\beta_{1}\right)^{2}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{r_{1}}{\alpha_{1}-\beta_{N}} & \cdots & \frac{r_{2 N-m}}{\alpha_{2 N-m}-\beta_{N}} & -\frac{r_{m}^{2}}{\left(\alpha_{m}-\beta_{N}\right)^{2}} & \cdots & -\frac{r_{1}^{2}}{\left(\alpha_{1}-\beta_{N}\right)^{2}} \\
\frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \cdots & \frac{h_{1}\left(\alpha_{2 N-m}\right)}{\alpha_{2 N-m}-\beta_{1}^{*}} & W_{1 m} & \cdots & W_{11} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{h_{N}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{N}^{*}} & \cdots & \frac{h_{N}\left(\alpha_{2 N-m}\right)}{\alpha_{2 N-m}-\beta_{N}^{*}} & W_{N m} & \cdots & W_{N 1}
\end{array}\right| \tag{5.61}
\end{align*}
$$

### 5.1.2 On the Equilibrium Equations

This section is a brief review on the equilibrium equations of N aligned charged black holes or hyper-extreme objects located in the symmetry axis. ${ }^{28,29}$ Since the labeling
of the roots $\alpha_{n}$ on the axis of symmetry is arbitrary, one can assume, without any lack of generality, that they can be located in such a way that $\operatorname{Re}\left(\alpha_{1}\right) \geq \operatorname{Re}\left(\alpha_{2}\right)>\operatorname{Re}\left(\alpha_{3}\right) \geq$ $\operatorname{Re}\left(\alpha_{4}\right)>\cdots>\operatorname{Re}\left(\alpha_{2 N-1}\right) \geq \operatorname{Re}\left(\alpha_{2 N}\right)$ (see figure 4). For real-valued $\alpha_{2 k-1}, \alpha_{2 k}$, then the $z$-axis interval corresponding to $\alpha_{2 k-1} \geq z \geq \alpha_{2 k}$ will represent the horizon of the $k t h$ black-hole. For complex-valued $\alpha_{2 k-1}, \alpha_{2 k}$, such that $\alpha_{2 k-1}=\alpha_{2 k}^{*}$, the line joining $\alpha_{2 k-1}$ to $\alpha_{2 k}$ will represent a hyper extreme object. ${ }^{61}$


Figure 4 - Several N-body configurations, representing black holes and hyper-extreme objects.

> Source: By the author.

By definition, the metric function $\gamma$ must be zero on the axis in absence of matter (see section 2.3). Through the N -soliton solution, this condition is automatically satisfied in the upper part of the axis, for $z>\operatorname{Re}\left(\alpha_{1}\right)$, and in the lower part, for $z<\operatorname{Re}\left(\alpha_{2 N}\right)$ (see equation (5.33)). Therefore, to ensure the equilibrium equations, $\gamma$ must satisfy:

$$
\begin{equation*}
\gamma\left(\rho=0, z \in \operatorname{Region}_{k}\right)=0, \quad k=2,3, \ldots, N . \tag{5.62}
\end{equation*}
$$

On the other hand, the construction of N -soliton solutions only ensures that the metric function $\omega$ is zero in the upper part of the axis. Therefore, it is necessary to impose the conditions:

$$
\begin{equation*}
\omega\left(\rho=0, z \in \operatorname{Region}_{k}\right)=0, \quad k=2,3, \ldots, N \tag{5.63}
\end{equation*}
$$

Notice that these conditions are related to conditions over the Killing vectors. That is, the vanishing of the space-like Killing vector $\psi_{\varphi}$ corresponds to the fact that this region
should be part of the symmetry axis. The vanishing of the space-like Killing vector $k$ corresponds to elementary flatness (see section 2.3 and also references 41,66). If $k$ does not vanish, this corresponds to a solution with a conical singularity, a line source called "strut", that holds the objects apart. Although this lack of elementary flatness do not seems to be physical, several authors have been some interest in such solutions. ${ }^{67}$

But also the imaginary part of the gravitational monopole momentum (NUT charge) must equal to zero ${ }^{29}$ :

$$
\begin{equation*}
\operatorname{Im}\left[\sum_{k=1}^{N} e_{k}\right]=0 \tag{5.64}
\end{equation*}
$$

Thus, the equilibrium condition gives a set of $2 N-1$ equations for the $3 N$ parameters.

### 5.2 2-soliton solution

In the following, we would like to present examples how the concepts above can be used in concrete cases. In particular, we will show how known solutions fit into the picture. Here, some important discussion will be made together with examples of well-known solutions which fall into the family of ht e2-soliton solutions.

According to the Ernst formalism, ${ }^{6}$ the electrovacuum Einstein field equations, for these particular stationary axisymmetric space-times, read:

$$
\begin{align*}
\left(\operatorname{Re}(\mathcal{E})+|\Phi|^{2}\right) \nabla^{2} \mathcal{E} & =\left(\nabla \mathcal{E}+2 \Phi^{*} \nabla \Phi\right) \cdot \nabla \mathcal{E}  \tag{5.65}\\
\left(\operatorname{Re}(\mathcal{E})+|\Phi|^{2}\right) \nabla^{2} \Phi & =\left(\nabla \mathcal{E}+2 \Phi^{*} \nabla \Phi\right) \cdot \nabla \Phi \tag{5.66}
\end{align*}
$$

with the Ernst potential being $\mathcal{E}=f-|\Phi|^{2}+i \Omega$. For any solution of the above equations, the metric functions $\omega$ and $\gamma$ of the line element (2.28) can be obtained from the following system of differential equations:

$$
\begin{align*}
& \omega,_{z}=-\rho f^{-2} \operatorname{Im}\left(\mathcal{E},_{\rho}+2 \Phi^{*} \Phi,_{\rho}\right)  \tag{5.67}\\
& \omega,_{\rho}=\rho f^{-2} \operatorname{Im}\left(\mathcal{E}, z+2 \Phi^{*} \Phi, z\right)  \tag{5.68}\\
& \gamma,{ }_{, \rho}=\frac{\rho f^{-2}}{2}\left(\left|\mathcal{E},_{\rho}+2 \Phi^{*} \Phi, \rho\right|^{2}-\left|\mathcal{E},{ }_{z}+2 \Phi^{*} \Phi, z\right|^{2}\right)-\rho f^{-1}\left(|\Phi, \rho|^{2}-|\Phi, z|^{2}\right)  \tag{5.69}\\
& \gamma, z=\frac{\rho f^{-2}}{2} \operatorname{Re}\left[\left(\mathcal{E},_{\rho}+2 \Phi^{*} \Phi,_{\rho}\right)\left(\mathcal{E},_{z}^{*}+2 \Phi \Phi,_{z}^{*}\right)\right]-2 \rho f^{-1} \operatorname{Re}\left(\Phi,_{\rho}^{*} \Phi, z\right) \tag{5.70}
\end{align*}
$$

To accomplish the goal of describing a binary configuration, following the method which has been discussed above, we use as a starting point the axis data of the extended 2 -soliton solution. Some calculations may appear repeated, but some physical features
appear from them which become easy to understand in a simple case. It means that, we will consider an Ernst potential whose behaviour on the symmetry axis is given by:

$$
\begin{gather*}
e(z)=\frac{z^{2}+a_{1} z+a_{2}}{z^{2}+b_{1} z+b_{2}}=1+\frac{e_{1}}{z-\beta_{1}}+\frac{e_{2}}{z-\beta_{2}}  \tag{5.71}\\
f(z)=\frac{c_{1} z+c_{2}}{z^{2}+b_{1} z+b_{2}}=\frac{f_{1}}{z-\beta_{1}}+\frac{f_{2}}{z-\beta_{2}} \tag{5.72}
\end{gather*}
$$

Here $a_{1}, a_{2}, b_{1}, b_{2}, c_{1}$ and $c_{2}$ are complex parameters. And:

$$
\begin{array}{cc}
a_{1}=e_{1}+e_{2}-\beta_{1}-\beta_{2} & a_{2}=\beta_{1} \beta_{2}-e_{1} \beta_{2}-e_{2} \beta_{1} \\
b_{1}=-\beta_{1}-\beta_{2} & b_{2}=\beta_{1} \beta_{2} \\
c_{1}=f_{1}+f_{2} & c_{2}=-f_{1} \beta_{2}-f_{2} \beta_{1} .
\end{array}
$$

That is, the above axis data is described by six complex parameters $\left\{a_{i}, b_{i}, c_{i}\right\}$ or $\left\{e_{i}, f_{i}, \beta_{i}\right\}, i=1,2$. In order to find the shape of the function $\mu_{a}$ (4.130), we needed to evaluate the roots of $h(\xi, \xi)$, where $h(\xi, \eta)$ is given by equation (4.129). Let $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and $\alpha_{4}$ denote the roots. Then:

$$
\begin{array}{r}
h(\xi, \xi)=2+\sum_{i=1}^{2}\left(\frac{e_{i}}{\xi-\beta_{i}}+\frac{e_{i}^{*}}{\xi-\beta_{i}^{*}}\right)+2 \sum_{i, j=1}^{2} \frac{f_{i} f_{j}^{*}}{\left(\xi-\beta_{i}\right)\left(\xi-\beta_{j}^{*}\right)}=  \tag{5.73}\\
=\frac{2 \prod_{n=1}^{4}\left(\xi-\alpha_{n}\right)}{\prod_{i=1}^{2}\left(\xi-\beta_{i}\right)\left(\xi-\beta_{i}^{*}\right)}
\end{array}
$$

Thus the roots of $h(\xi, \xi)=0$ are either real or appear in conjugate pairs. So, the system also can be described in terms of the six parameters $\left\{\alpha_{n}, \beta_{i}, f_{i}\right\}$. Then the unknown function $\mu_{1}$ should be of the form:

$$
\begin{equation*}
\mu_{1}=A_{0}+\frac{A_{1}}{\xi-\alpha_{1}}+\frac{A_{2}}{\xi-\alpha_{2}}+\frac{A_{3}}{\xi-\alpha_{3}}+\frac{A_{4}}{\xi-\alpha_{4}} \tag{5.74}
\end{equation*}
$$

Making use of the relations

$$
\begin{align*}
\int_{-1}^{1} \frac{\sigma}{(z-\alpha+i \rho \sigma) \sqrt{1-\sigma^{2}}} & =\frac{\pi}{\sqrt{(z-\alpha)^{2}+\rho^{2}}}  \tag{5.75}\\
\int_{-1}^{1} \frac{1}{\sqrt{1-\sigma^{2}}} d \sigma & =\pi \tag{5.76}
\end{align*}
$$

the formulae (4.128) yields:

$$
\begin{equation*}
A_{0}+\frac{A_{1}}{r_{1}}+\frac{A_{2}}{r_{2}}+\frac{A_{2}}{r_{3}}+\frac{A_{4}}{r_{4}}=1, \quad \quad r_{n}=\sqrt{\left(z-\alpha_{n}\right)^{2}+\rho} \tag{5.77}
\end{equation*}
$$

and satisfies (4.127), (4.128).Now, using the following integral relations:

$$
\begin{equation*}
f_{-1}^{1} \frac{1}{(\sigma-a) \sqrt{1-\sigma^{2}}} d \sigma=0, \quad-1 \leq \operatorname{Re}(a) \leq 1 \& \operatorname{Im}(a)=0 . \tag{5.78}
\end{equation*}
$$

$$
\begin{equation*}
f_{-1}^{1} \frac{1}{(\sigma-a) \sqrt{1-\sigma^{2}}} d \sigma=-\frac{\pi}{\sqrt{1+\frac{1}{a^{2}}} a}, \quad \text { otherwise } \tag{5.79}
\end{equation*}
$$

Consequently, two more equations arise from (5.10) for the three variables ( $A_{0}, A_{1}, A_{2}, A_{3}, A_{4}$ ) when using the above integral results and hence the the system becomes determined:

$$
\begin{gather*}
A_{0}+\frac{A_{1}}{r_{1}}+\frac{A_{2}}{r_{2}}+\frac{A_{2}}{r_{3}}+\frac{A_{4}}{r_{4}}=1,  \tag{5.80}\\
A_{0}-\frac{A_{1}}{\alpha_{1}-\beta_{1}}-\frac{A_{2}}{\alpha_{2}-\beta_{1}}-\frac{A_{3}}{\alpha_{3}-\beta_{1}}-\frac{A_{4}}{\alpha_{4}-\beta_{1}}=0,  \tag{5.81}\\
A_{0}-\frac{A_{1}}{\alpha_{1}-\beta_{2}}-\frac{A_{2}}{\alpha_{2}-\beta_{2}}-\frac{A_{3}}{\alpha_{3}-\beta_{2}}-\frac{A_{4}}{\alpha_{4}-\beta_{2}}=0,  \tag{5.82}\\
\frac{h_{1}\left(\alpha_{1}\right) A_{1}}{\left(\alpha_{1}-\beta_{1}^{*}\right) r_{1}}+\frac{h_{1}\left(\alpha_{2}\right) A_{2}}{\left(\alpha_{2}-\beta_{1}^{*}\right) r_{2}}+\frac{h_{1}\left(\alpha_{3}\right) A_{3}}{\left(\alpha_{3}-\beta_{1}^{*}\right) r_{3}}+\frac{h_{1}\left(\alpha_{4}\right) A_{4}}{\left(\alpha_{4}-\beta_{1}^{*}\right) r_{4}}=0 .  \tag{5.83}\\
\frac{h_{2}\left(\alpha_{1}\right) A_{1}}{\left(\alpha_{1}-\beta_{2}^{*}\right) r_{1}}+\frac{h_{2}\left(\alpha_{2}\right) A_{2}}{\left(\alpha_{2}-\beta_{2}^{*}\right) r_{2}}+\frac{h_{2}\left(\alpha_{3}\right) A_{3}}{\left(\alpha_{3}-\beta_{2}^{*}\right) r_{3}}+\frac{h_{2}\left(\alpha_{4}\right) A_{4}}{\left(\alpha_{4}-\beta_{2}^{*}\right) r_{4}}=0 . \tag{5.84}
\end{gather*}
$$

Here $h_{l}\left(\alpha_{n}\right)=e_{l}^{*}+2 f_{l}^{*} f\left(\alpha_{n}\right), l=1,2, n=1,2,3,4$. Notice that in the vacuum case $e_{l}^{*}$ appears as a common factor in the last two equations above, so it can be canceled. This system is trivially solved using the software Mathematica. ${ }^{68}$ The Ernst potential $\mathcal{E}$ is given in the form ${ }^{26}$ :

$$
\mathcal{E}=\frac{E_{+}}{E_{-}}=\left|\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \frac{r_{2}}{\alpha_{2}-\beta_{1}} & \frac{r_{3}-\beta_{1}}{r_{2}} & \frac{r_{4}-\beta_{1}}{r_{4}} \\
1 & \frac{r_{1}-\beta_{2}}{r_{1}-\beta_{2}} & \frac{\alpha_{2}-\beta_{2}}{\alpha_{3}-\beta_{2}} & \frac{\alpha_{4}-\beta_{2}}{\alpha_{3}}  \tag{5.87}\\
0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{1}^{*}} \\
0 & \frac{h_{2}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{2}^{*}}
\end{array}\right|
$$

In order to evaluate the metric function $\omega$, one should make use of (4.132). To accomplish that, $\mu_{2}$ is necessary. This function should be of the form:

$$
\begin{equation*}
\mu_{2}=-i \xi+B_{0}+\frac{B_{1}}{\xi-\alpha_{1}}+\frac{B_{2}}{\xi-\alpha_{2}}+\frac{B_{3}}{\xi-\alpha_{3}}+\frac{B_{4}}{\xi-\alpha_{4}} \tag{5.88}
\end{equation*}
$$

When one substitutes $\mu_{2}$ into equations (4.128), (4.127), one finds the system for the variables $\left\{B_{0}, B_{1}, B_{2}, B_{3}, B_{4}\right\}$ :

$$
\begin{gather*}
B_{0}+\frac{B_{1}}{r_{1}}+\frac{B_{2}}{r_{2}}+\frac{B_{2}}{r_{3}}+\frac{B_{4}}{r_{4}}=i z,  \tag{5.89}\\
B_{0}-\frac{B_{1}}{\alpha_{1}-\beta_{1}}-\frac{B_{2}}{\alpha_{2}-\beta_{1}}-\frac{B_{3}}{\alpha_{3}-\beta_{1}}-\frac{B_{4}}{\alpha_{4}-\beta_{1}}=i \beta_{1},  \tag{5.90}\\
B_{0}-\frac{B_{1}}{\alpha_{1}-\beta_{2}}-\frac{B_{2}}{\alpha_{2}-\beta_{2}}-\frac{B_{3}}{\alpha_{3}-\beta_{2}}-\frac{B_{4}}{\alpha_{4}-\beta_{2}}=i \beta_{2},  \tag{5.91}\\
\frac{h_{1}\left(\alpha_{1}\right) B_{1}}{\left(\alpha_{1}-\beta_{1}^{*}\right) r_{1}}+\frac{h_{1}\left(\alpha_{2}\right) B_{2}}{\left(\alpha_{2}-\beta_{1}^{*}\right) r_{2}}+\frac{h_{1}\left(\alpha_{3}\right) B_{3}}{\left(\alpha_{3}-\beta_{1}^{*}\right) r_{3}}+\frac{h_{1}\left(\alpha_{4}\right) B_{4}}{\left(\alpha_{4}-\beta_{1}^{*}\right) r_{4}}=i e_{1}^{*} .  \tag{5.92}\\
\frac{h_{2}\left(\alpha_{1}\right) B_{1}}{\left(\alpha_{1}-\beta_{2}^{*}\right) r_{1}}+\frac{h_{2}\left(\alpha_{2}\right) B_{2}}{\left(\alpha_{2}-\beta_{2}^{*}\right) r_{2}}+\frac{h_{2}\left(\alpha_{3}\right) B_{3}}{\left(\alpha_{3}-\beta_{2}^{*}\right) r_{3}}+\frac{h_{2}\left(\alpha_{4}\right) B_{4}}{\left(\alpha_{4}-\beta_{2}^{*}\right) r_{4}}=i e_{2}^{*} . \tag{5.93}
\end{gather*}
$$

Hence, using $f \omega=\frac{1}{2}\left(H_{12}+H_{21}^{*}\right)+\Phi \Phi_{2}^{*}-i z$, one finds:

$$
\begin{equation*}
\omega=\frac{2 \operatorname{Im}\left(E_{-} H^{*}-E_{-}^{*} G-F I^{*}\right)}{E_{+} E_{-}^{*}+E_{+}^{*} E_{-}+2 F F^{*}} . \tag{5.94}
\end{equation*}
$$

where

$$
\begin{align*}
& G=\left|\begin{array}{ccccc}
0 & r_{1}+\alpha_{1}-z & r_{2}+\alpha_{2}-z & r_{3}+\alpha_{3}-z & r_{4}+\alpha_{4}-z \\
-1 & \overline{r_{1}} \\
-1 & \frac{r_{1}-\beta_{1}}{r_{1}} & \overline{\alpha_{2}-\beta_{1}} & \overline{r_{3}} & \frac{r_{4}-\beta_{1}}{\alpha_{1}-\beta_{2}} \\
0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}} & \frac{h_{1}\left(\alpha_{2}\right)}{\alpha_{4}-\beta_{1}} \\
0 & \frac{h_{2}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \frac{h_{2}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{2}} & \frac{h_{1}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{1}^{*}} & \frac{h_{2}}{\alpha_{4}-\beta_{2}} \\
\frac{h_{1}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{1}^{*}} & \frac{h_{2}\left(\beta_{4}\right)}{\alpha_{4}-\beta_{2}^{*}}
\end{array}\right|,  \tag{5.95}\\
& H=\left|\begin{array}{ccccc}
z & 1 & 1 & 1 & 1 \\
-\beta_{1} & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \frac{r_{2}}{\alpha_{2}-\beta_{1}} & \frac{r_{3}}{\alpha_{3}-\beta_{1}} & \frac{r_{4}}{\alpha_{4}-\beta_{1}} \\
-\beta_{2} & \frac{r_{1}-\beta_{2}}{\alpha_{1}} & \frac{\alpha_{2}-\beta_{2}}{\alpha_{2}} & \frac{\alpha_{3}-\beta_{2}}{\alpha_{4}-\beta_{2}} & \frac{\alpha_{4}}{e_{1}^{*}} \frac{\frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}}}{\frac{h_{1}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{1}^{*}}} \\
\frac{h_{1}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{1}^{*}} \\
e_{2}^{*} & \frac{h_{2}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{2}^{*}}
\end{array}\right| \tag{5.96}
\end{align*}
$$

$$
I=\left\lvert\, \begin{array}{cccccc}
\sum_{l=1}^{2} f_{l} & 0 & f\left(\alpha_{1}\right) & f\left(\alpha_{2}\right) & f\left(\alpha_{3}\right) & f\left(\alpha_{4}\right)  \tag{5.97}\\
z & 1 & 1 & 1 & 1 & 1 \\
-\beta_{1} & -1 & \frac{r_{1}}{\alpha_{1}-\beta_{1}} & \frac{r_{2}}{\alpha_{2}-\beta_{1}} & \frac{r_{3}}{\alpha_{3}-\beta_{1}} & \frac{r_{4}}{\alpha_{4}-\beta_{1}} \\
-\beta_{2} & -1 & \frac{r_{1}-\beta_{2}}{\alpha_{1}} & \frac{\alpha_{2}-\beta_{2}}{\alpha_{3}} & \frac{\alpha_{3}-\beta_{2}}{\alpha_{4}-\beta_{2}} & \frac{\alpha_{4}}{\alpha_{1}} \\
e_{1}^{*} & 0 & \frac{h_{1}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{1}^{*}} & \frac{h_{1}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{1}^{*}} \\
e_{2}^{*} & 0 & \frac{h_{2}\left(\alpha_{1}\right)}{\alpha_{1}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{2}\right)}{\alpha_{2}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{3}\right)}{\alpha_{3}-\beta_{2}^{*}} & \frac{h_{2}\left(\alpha_{4}\right)}{\alpha_{4}-\beta_{2}^{*}}
\end{array} .\right.
$$

Notice that due to the arbitrariness of the problem, it takes a very complicated form. But it is still solvable analytically. It is straightforward to see that the type of compact object depends on the nature of the alpha roots. They also give the position of the object on the z axis. Moreover, the coefficients of equations (5.71) and (5.72) are related to the multipole moments ${ }^{36,69}$ defined in appendix C. Therefore, it is possible to put some constrains on those coefficients depending on the problem which one wants to describe. Consider the first moments:

$$
\begin{array}{lr}
P_{0}=-\frac{e_{1}+e_{2}}{2}, & \quad P_{1}=\frac{\left(e_{1}+e_{2}\right)^{2}-2 e_{1} \beta_{1}-2 e_{2} \beta_{2}}{4}  \tag{5.98}\\
Q_{0}=f_{1}+f_{2}, & Q_{1}=\frac{-1}{2}\left(e_{1}+e_{2}\right)\left(f_{1}+f_{2}\right)+\left(f_{1} \beta_{1}+f_{2} \beta_{2}\right) .
\end{array}
$$

To ensure asymptotically flat solutions, it is necessary to set the angular momentum monopole equal zero ${ }^{29,34,58,70}$ (if the angular monopole moment is not zero, NUT charge, then the metric must be asymptotically Taub-NUT, implying a singularity all along half of the symmetry axisc ${ }^{71}$ ), hence, $e_{1}+e_{2}=-2 M$, where $M$ is the mass of the system. Another simplification that can be made is to set the origin of the coordinate system to be the center of mass. Thus, the real part of $P_{1}$ must be zero, which implies $\operatorname{Re}\left(e_{1} \beta_{1}+e_{2} \beta_{2}\right)=2 M^{2}$. One can also identify $\operatorname{Im}\left(e_{1} \beta_{1}+e_{2} \beta_{2}\right)=-2 J$, where $J$ is the total angular momentum of the system. Simon ${ }^{34}$ also has shown that the magnetic monopole must vanish to ensure asymptotically flat solutions, consequently, $f_{1}+f_{2}=Q$ where $Q$ is the total electric charge of the system. ${ }^{29}$

### 5.2.1 Binary system

When one is dealing with a binary system, the roots $\alpha$ might be written in terms of the distance of their center of mass, ${ }^{72}$ that is:

$$
\begin{array}{cl}
\alpha_{1}=\frac{R}{2}+\sigma_{1} & \alpha_{2}=\frac{R}{2}-\sigma_{1} \\
\alpha_{3}=-\frac{R}{2}+\sigma_{2} & \alpha_{4}=-\frac{R}{2}-\sigma_{2} \tag{5.100}
\end{array}
$$

Where $\sigma$ can take either real or imaginary values. If we are dealing with the non-degenerate case, the possible configurations are represented below, that is, the location of the objects on the $z$-axis considering the roots $\alpha$ 's:


Figure $5-\mathrm{a}$ )This scheme represents the situation between two sub-extreme objects. All roots $\alpha$ are real. b)This scheme represents a interaction between two hyperextreme. All roots $\alpha$ are complex. c) This scheme represents the interaction between a sub-extreme and a hyper-extreme objects. Two roots are real and two roots are complex.

Source: By the author.

To ensure that the system is in fact describing a binary system, the equilibrium conditions (5.62) and (5.63) must be satisfied.

### 5.2.2 Tomimatsu-Sato solution with $\delta=2$

In 1972, Tomimatsu and Sato ${ }^{73}$ found a new family if solutions for a rotating mass in vacuum. The potential function $\xi$, on the symmetry axis, has the form ${ }^{\ddagger}$ :

$$
\begin{equation*}
\frac{1}{\xi}=p \frac{(x+1)^{\delta}+(x-1)^{\delta}}{(x+1)^{\delta}-(x-1)^{\delta}}-i q \tag{5.101}
\end{equation*}
$$

Here, the solution is given in terms of prolate coordinates, $(x, y)$, and $q=\frac{a}{M}$ and $p=\sqrt{1-q^{2}}$. For $\delta=1$, the global solution is the Kerr solution. For $\delta=2$, the solution is also an asymptotically flat spacetime, where the Ernst potential $\xi$, which describes the exterior region of a deformed mass, is given in given globally by:

$$
\begin{equation*}
\frac{1}{\xi}=\frac{p^{2} x^{4}-2 \operatorname{ipqxy}\left(x^{2}-y^{2}\right)+q^{2} y^{4}-1}{2 p x\left(x^{2}-1\right)-2 i q y\left(1-y^{2}\right)} \tag{5.102}
\end{equation*}
$$

[^9]and the metric functions are:
\[

$$
\begin{equation*}
f=\frac{A}{B} \quad \omega=2 M q\left(1-y^{2}\right) \frac{C}{A} \quad \gamma=\frac{1}{2} \ln \left(\frac{A}{p^{4}\left(x^{2}-y^{2}\right)^{4}}\right) \tag{5.103}
\end{equation*}
$$

\]

where

$$
\begin{align*}
& A=\left[p^{2}\left(x^{2}-1\right)^{2}+q^{2}\left(1-y^{2}\right)^{2}\right]^{2}-4 p^{2} q^{2}\left(x^{2}-1\right)\left(1-y^{2}\right)\left(x^{2}-y^{2}\right)^{2}  \tag{5.104}\\
& B=\left(p^{2} x^{4}+q^{2} y^{4}-1+2 p x^{3}-2 p x\right)^{2}+4 q^{2} y^{2}\left(p x^{3}-p x y^{2}+1-y^{2}\right)^{2}  \tag{5.105}\\
& C=p^{2}\left(x^{2}-1\right)\left[\left(x^{2}-1\right)\left(1-y^{2}\right)-4 x^{2}\left(x^{2}-y^{2}\right)\right]-p^{3} x\left(x^{2}-1\right) \times  \tag{5.106}\\
& {\left[2\left(x^{4}-1\right)+\left(x^{2}+3\right)\left(1-y^{2}\right)\right]+q^{2}(1+p x)\left(1-y^{2}\right)^{3}}
\end{align*}
$$

The prolate coordinates are related to the Weyl one through:

$$
\begin{equation*}
\rho=\sigma \sqrt{\left(x^{2}-1\right)\left(1-y^{2}\right)}, \quad z=\sigma x y, \quad \sigma \equiv \frac{M p}{\delta} . \tag{5.107}
\end{equation*}
$$

or solving for x and y :

$$
\begin{align*}
& x=\frac{\sqrt{\rho^{2}+(z+\sigma)^{2}}+\sqrt{\rho^{2}+(z-\sigma)^{2}}}{2 \sigma},  \tag{5.108}\\
& y=\frac{\sqrt{\rho^{2}+(z+\sigma)^{2}}-\sqrt{\rho^{2}+(z-\sigma)^{2}}}{2 \sigma} . \tag{5.109}
\end{align*}
$$

It is direct to see that the metric functions are regular in the symmetry axis, and that the functions $\omega$ and $\gamma$ vanish in the region $z>\sigma$, and $z<-\sigma(y=1, y=-1)$ for $\rho=0$. However, in the region $|z|<\sigma, \omega$ and $\gamma$ assume constants values:

$$
\begin{equation*}
\omega=2 M \frac{(1+p)}{q} \quad \gamma=2 \ln \left(\frac{q}{p}\right) \tag{5.110}
\end{equation*}
$$

Notice that there is no choice of $p$ and $q$ which vanish $\omega$ and $\gamma$, simultaneously. Therefore, the Tomimatsu-Sato solution with $\delta=2$ might be interpreted as one single body or as two sources which are not disconnected, and there is a massive region among the them.

Also, on the symmetry axis, $\rho=0$, the Ernst potential $\mathcal{E}$ takes the form:

$$
\begin{equation*}
\mathcal{E}=\frac{1-\xi}{1+\xi}=\frac{z^{2}-(M+i a) z+\frac{M^{2}-a^{2}}{4}}{z^{2}+(M-i a) z+\frac{M^{2}-a^{2}}{4}} \tag{5.111}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathcal{E}=1+\frac{e_{1}}{z-\beta_{1}}+\frac{e_{2}}{z-\beta_{2}} \tag{5.112}
\end{equation*}
$$

where

$$
\beta_{1}=\frac{i}{2}(-(1+i) \sqrt{a} \sqrt{M}+a+i M), \quad \beta_{2}=\frac{i}{2}((1+i) \sqrt{a} \sqrt{M}+a+i M) .
$$

and
$e_{1}=\frac{-2 \sqrt{a} M+(1-i) a \sqrt{M}+(1+i) M^{3 / 2}}{2 \sqrt{a}}, \quad e_{2}=-\frac{2 \sqrt{a} M+(1-i) a \sqrt{M}+(1+i) M^{3 / 2}}{2 \sqrt{a}}$

However, now, $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$, the roots of $h(\xi, \xi)=0$ appear with multiplicity two, that is, there are only two independents roots. For a physical interpretation, the roots should be seen as overlapping themselves. That is:

$$
\begin{equation*}
\alpha_{1}=\alpha_{2}=-\alpha_{3}=-\alpha_{4}=\frac{1}{2} \sqrt{M^{2}-a^{2}} \tag{5.113}
\end{equation*}
$$

Notice that the general formula (5.85) still works, but the l'Hopital's rule should be applied.

### 5.2.3 Charged, Magnetized Tomimatsu-Sato $\delta=2$ Solution

After Tomimatsu-Sato presented their vacuum solution, several authors tried to generalize it including the electromagnetic field.. Manko introduced a generalization ${ }^{74}$ by introducing two new arbitrary real parameters, $q^{\S}$ and $c$, related to electric charge and magnetic dipole moment. This reads
§ Here, $q$ is to be interpreted as the electric charge, not as the same parameter as in the vacuum solution.

$$
\begin{align*}
\mathcal{E}= & \frac{A-2 M B}{A+2 M B}, \quad \Phi=\frac{2 C}{A+2 M B}  \tag{5.114}\\
A= & \left(\sigma^{2} x^{2}-k y^{2}\right)^{2}-\left(\sigma^{2}-k\right)^{2}-2 i \sigma^{3} a x y\left(x^{2}-1\right)  \tag{5.115}\\
& -\left(1-y^{2}\right)\left[a\left(\sigma^{2}-2 k\right)+2 q c\right]\left[a\left(y^{2}+1\right)+2 i \sigma x y\right] \\
B= & \sigma x\left[\sigma^{2}\left(x^{2}-1\right)+k\left(1-y^{2}\right)\right]-i y\left(1-y^{2}\right)\left[a\left(\sigma^{2}-2 k\right)+2 q c\right]  \tag{5.116}\\
C= & \sigma^{2}\left(x^{2}-1\right)(\sigma q x+i c y)+\left(1-y^{2}\right)\{x \sigma(a c+q k)  \tag{5.117}\\
& \left.-i y\left[a q\left(\sigma^{2}-2 k\right)+c\left(2 q^{2}-k\right)\right]\right\} \\
k \equiv & c^{2} /\left(M^{2}-a^{2}-q^{2}\right), \quad \sigma \equiv \sqrt{M^{2}-a^{2}-q^{2}+k} \tag{5.118}
\end{align*}
$$

and the metric functions are given by ${ }^{75}$ :

$$
\begin{align*}
& f= \frac{E}{D}, \quad e^{2 \gamma}=\frac{E}{\sigma^{8}\left(x^{2}-y^{2}\right)^{4}}, \quad \omega=-\frac{2\left(1-y^{2}\right) F}{E}  \tag{5.119}\\
& E=\{ {\left.\left[\sigma^{2}\left(x^{2}-1\right)+k\left(1-y^{2}\right)\right]^{2}+a[a(d-k)+2 q c]\left(1-y^{2}\right)^{2}\right\}^{2} }  \tag{5.120}\\
&-4 \sigma^{2}\left(x^{2}-1\right)\left(1-y^{2}\right)\left[\sigma^{2} a\left(x^{2}-y^{2}\right)+2(a k-q c) y^{2}\right]^{2} \\
& D=\{ \left(\sigma^{2} x^{2}-k y^{2}\right)^{2}+2 \sigma M x\left[\sigma^{2}\left(x^{2}-1\right)+k\left(1-y^{2}\right)\right]  \tag{5.121}\\
&\left.+a[a(d-k)+2 q c]\left(y^{4}-1\right)-d^{2}\right\}^{2} \\
&+4 y^{2}\left\{\sigma^{3} a x\left(x^{2}-1\right)+[a(d-k)+2 q c](\sigma x+M)\left(1-y^{2}\right)\right\}^{2} \\
& F=4 \sigma^{2}\left(x^{2}-1\right)\left[\sigma^{2} a\left(x^{2}-y^{2}\right)+2(a k-q c) y^{2}\right]\left\{\sigma M x \left[\sigma^{2}\left(x^{2}+1\right)-\right.\right.  \tag{5.122}\\
&\left.\left.-k\left(y^{2}+1\right)\right]+\sigma^{2} x^{2}\left(2 M^{2}-q^{2}\right)-k d y^{2}\right\}-\left\{\left[\sigma^{2}\left(x^{2}-1\right)+k\left(1-y^{2}\right)\right]^{2}+\right. \\
&\left.+a[a(d-k)+2 q c]\left(1-y^{2}\right)^{2}\right\}\left\{2 \sigma^{2} q c\left(x^{2}-y^{2}\right)+\right. \\
&\left.+\left(1-y^{2}\right)\left[a d\left(2 \sigma M x+2 M^{2}-q^{2}\right)-(a k-2 q c)\left(2 \sigma M x+M^{2}+a^{2}\right)\right]\right\} \\
& d \equiv M^{2}-a^{2}-q^{2} \tag{5.123}
\end{align*}
$$

The prolate coordinates are related to the Weyl coordinates by:

$$
\begin{equation*}
\rho=\sigma^{\prime} \sqrt{\left(x^{2}-1\right)\left(1-y^{2}\right)}, \quad z=\sigma^{\prime} x y, \quad \sigma^{\prime} \equiv \frac{\sigma}{\delta} \tag{5.124}
\end{equation*}
$$

This gives, when inverting the relations:

$$
\begin{align*}
& x=\frac{\sqrt{\rho^{2}+\left(z+\sigma^{\prime}\right)^{2}}+\sqrt{\rho^{2}+\left(z-\sigma^{\prime}\right)^{2}}}{2 \sigma^{\prime}}  \tag{5.125}\\
& y=\frac{\sqrt{\rho^{2}+\left(z+\sigma^{\prime}\right)^{2}}-\sqrt{\rho^{2}+\left(z-\sigma^{\prime}\right)^{2}}}{2 \sigma^{\prime}} \tag{5.126}
\end{align*}
$$

Notice here that Manko has written the equations missing the parameter $\delta$ causing a mistake in the multipoles evaluation. On the symmetry axis, $\rho=0$, the Ernst potentials $\mathcal{E}$ and $\Phi$ take the form:

$$
\begin{align*}
& \mathcal{E}=\frac{z^{2}-(M+i a) z+\frac{M^{2}-a^{2}-q^{2}+k}{4}}{z^{2}+(M-i a) z+\frac{M^{2}-a^{2}-q^{2}-k}{4}}  \tag{5.127}\\
& \Phi=\frac{q z+\frac{i c}{2}}{z^{2}+(M-i a) z+\frac{M^{2}-a^{2}-q^{2}-k}{4}} \tag{5.128}
\end{align*}
$$

Also, when $c=q=0$, equation (5.111) is recovered. This means:

$$
\begin{gather*}
\mathcal{E}=1+\frac{e_{1}}{z-\beta_{1}}+\frac{e_{2}}{z-\beta_{2}}  \tag{5.129}\\
\Phi=\frac{f_{1}}{z-\beta_{1}}+\frac{f_{2}}{z-\beta_{2}} \tag{5.130}
\end{gather*}
$$

where

$$
\begin{aligned}
\beta_{1} & =\frac{1}{2}\left(i a-M+\sqrt{-2 i a M+q^{2}+k}\right) \\
\beta_{2} & =\frac{1}{2}\left(i a-M-\sqrt{-2 i a M+q^{2}+k}\right), \\
e_{1} & =-\frac{M \sqrt{-2 i a M+q^{2}+k}+i a M-M^{2}}{\sqrt{-2 i a M+q^{2}+k}} \\
e_{2} & =-\frac{M \sqrt{-2 i a M+q^{2}+k}-i a M+M^{2}}{\sqrt{-2 i a M+q^{2}+k}}
\end{aligned}
$$

$$
\begin{aligned}
& f_{1}=-\frac{-q \sqrt{-2 i a M+Q^{2}+k}-i a q-i c+M q}{2 \sqrt{-2 i a M+q^{2}+k}} \\
& f_{2}=-\frac{-q \sqrt{-2 i a M+q^{2}+k}+i a q+i c-M q}{2 \sqrt{-2 i a M+q^{2}+k}}
\end{aligned}
$$

Again, $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$, the roots of $h(\xi, \xi)=0$ appears with multiplicity two and hence, as before, there are only two independents roots.

$$
\begin{equation*}
\alpha_{1}=\alpha_{2}=-\alpha_{3}=-\alpha_{4}=\frac{1}{2} \sqrt{M^{2}-a^{2}-Q^{2}+k} \tag{5.131}
\end{equation*}
$$

### 5.2.4 Metric of a rotating charged magnetized object

Consider the Ernst potentials whose behaviour on the symmetry axis is given by ${ }^{76}$ :

$$
\begin{align*}
& \mathcal{E}(\rho=0, z)=\frac{z^{2}-M z-M^{2} q-i M^{2} j}{z^{2}+M z-M^{2} q+i M^{2} j}  \tag{5.132}\\
& \Phi(\rho=0, z)=\frac{M e z+i M^{2} \mu}{z^{2}+M z-M^{2} q+i M^{2} j} \tag{5.133}
\end{align*}
$$

Here, $M, q, j$ and $\mu$ are real parameters. Or in a equivalent way:

$$
\begin{gather*}
\mathcal{E}(\rho=0, z)=1+\frac{e_{1}}{z-\beta_{1}}+\frac{e_{2}}{z-\beta_{2}}  \tag{5.134}\\
\Phi(\rho=0, z)=\frac{f_{1}}{z-\beta_{1}}+\frac{f_{2}}{z-\beta_{2}} \tag{5.135}
\end{gather*}
$$

where

$$
\begin{gathered}
\beta_{1}=\frac{1}{2}(-M-M \sqrt{-4 i j+4 q+1}), \quad \beta_{2}=\frac{1}{2}(-M+M \sqrt{-4 i j+4 q+1}) . \\
e_{1}=-\frac{M \sqrt{-4 i j+4 q+1}-2 i j M+M}{\sqrt{-4 i j+4 q+1}}, \quad e_{2}=-\frac{M \sqrt{-4 i j+4 q+1}+2 i j M-M}{\sqrt{-4 i j+4 q+1}} . \\
f_{1}=-\frac{-e M \sqrt{-4 i j+4 q+1}-e M+2 i \mu M}{2 \sqrt{-4 i j+4 q+1}}, \quad f_{2}=-\frac{-e M \sqrt{-4 i j+4 q+1}+e M-2 i \mu M}{2 \sqrt{-4 i j+4 q+1}} .
\end{gathered}
$$

But now, $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$, i.e. roots of $h(\xi, \xi)=0$ appear in pairs, where one is the negative of the other. Another point is that the roots alpha must overlap themselves. That is:

$$
\begin{aligned}
& \alpha_{1}=-\alpha_{3} \equiv M \sigma_{1}=M \frac{\sqrt{1+2 q-e^{2}-d}}{\sqrt{2}}, \\
& \alpha_{2}=-\alpha_{4} \equiv M \sigma_{2}=M \frac{\sqrt{1+2 q-e^{2}+d}}{\sqrt{2}} .
\end{aligned}
$$

with

$$
\begin{equation*}
d=\sqrt{\left(1+2 q-e^{2}\right)^{2}+4\left(j^{2}-q^{2}-\mu^{2}\right)} \tag{5.136}
\end{equation*}
$$



Figure 6 - a) Overlapping of a sub-extreme configuration. b) Overlapping of one subextreme and a hyper-extreme objects. c) Overlapping of a hyper-extreme configuration.

Source: By the author.

As Ernst has shown, equations (5.132) and (5.133) represent a six parameter solution possessing equatorial symmetry. ${ }^{77,78}$ Due to the symmetry in the parameters, a common divisor appears in $(5.30)^{79}$ they can be written as ${ }^{76}$ :

$$
\begin{equation*}
\mathcal{E}=\frac{A}{B} \quad \Phi=\frac{C}{B} \tag{5.137}
\end{equation*}
$$

$$
\begin{aligned}
& A=-\left(r_{3}-r_{1}\right)\left(r_{4}-r_{2}\right)\left(j^{2}\left(6 e^{2}+4 q-6\right)-8\left(e^{2}-1\right) q^{2}-2 \mu^{2}\left(3 e^{2}+2 q-3\right)\right)+ \\
& +8 \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(e^{2}(-q)+j^{2}-\mu^{2}+q\right)\left(r_{1} r_{3}+r_{2} r_{4}\right)-M d\left(i ( e ^ { 2 } j - 2 e \mu + j ) \left(\sigma_{2}\left(r_{3}-r_{1}\right)+\right.\right. \\
& \left.+\left(r_{2}-r_{4}\right) \sigma_{1}\right)+2 \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(d\left(r_{1}+r_{2}+r_{3}+r_{4}\right)-\left(e^{2}-1\right)\left(r_{1}-r_{2}+r_{3}-r_{4}\right)\right)+ \\
& \left.+i j d\left(\sigma_{2}\left(r_{3}-r_{1}\right)-\left(r_{2}+r_{4}\right) \sigma_{1}\right)\right)+4\left(r_{1}+r_{3}\right)\left(r_{2}+r_{4}\right) \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(e^{4}-2 e^{2}(q+1)+j^{2}-\right. \\
& \left.-\mu^{2}+2 q+1\right)+i(j-e \mu) d\left(\sigma_{1}\left(r_{1}+r_{3}\right)\left(r_{2}-r_{4}\right)+\sigma_{2}\left(r_{3}-r_{1}\right)\left(r_{2}+r_{4}\right)\right) .
\end{aligned}
$$

$$
\begin{aligned}
& B=-\left(r_{3}-r_{1}\right)\left(r_{4}-r_{2}\right)\left(j^{2}\left(6 e^{2}+4 q-6\right)-8\left(e^{2}-1\right) q^{2}-2 \mu^{2}\left(3 e^{2}+2 q-3\right)\right)+ \\
& +8 \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(e^{2}(-q)+j^{2}-\mu^{2}+q\right)\left(r_{1} r_{3}+r_{2} r_{4}\right)+M d\left(i ( e ^ { 2 } j - 2 e \mu + j ) \left(\sigma_{2}\left(r_{3}-r_{1}\right)+\right.\right. \\
& \left.+\left(r_{2}-r_{4}\right) \sigma_{1}\right)+2 \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(d\left(r_{1}+r_{2}+r_{3}+r_{4}\right)-\left(e^{2}-1\right)\left(r_{1}-r_{2}+r_{3}-r_{4}\right)\right)+ \\
& \left.+i j d\left(\sigma_{2}\left(r_{3}-r_{1}\right)-\left(r_{2}+r_{4}\right) \sigma_{1}\right)\right)+4\left(r_{1}+r_{3}\right)\left(r_{2}+r_{4}\right) \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(e^{4}-2 e^{2}(q+1)+j^{2}-\right. \\
& \left.-\mu^{2}+2 q+1\right)+i(j-e \mu) d\left(\sigma_{1}\left(r_{1}+r_{3}\right)\left(r_{2}-r_{4}\right)+\sigma_{2}\left(r_{3}-r_{1}\right)\left(r_{2}+r_{4}\right)\right) .
\end{aligned}
$$

$$
C=M d\left(-i\left(e^{2} \mu-2 e j+\mu\right)\left(\sigma_{2}\left(r_{3}-r_{1}\right)+\left(r_{2}-r_{4}\right) \sigma 1\right)+2 e \sqrt{-j^{2}+\mu^{2}+q^{2}}\left(d\left(r_{1}+r_{2}+r_{3}+r_{4}\right)-\right.\right.
$$

$$
\left.\left.-\left(e^{2}-1\right)\left(r_{1}-r_{2}+r_{3}-r_{4}\right)\right)+i \mu d\left(\sigma_{2}\left(r_{3}-r_{1}\right)-\left(r_{2}+r_{4}\right) \sigma_{1}\right)\right)
$$

### 5.3 The Tomimatsu double-Kerr solution

In 1983, Tomimatsu ${ }^{80}$ introduced a double Kerr solution in equilibrium. Hoenselaers ${ }^{81}$ showed later that this solution could not be interpreted as two black holes in equilibrium, but as two hyper-extreme objects.

$$
\begin{equation*}
\mathcal{E}(\rho, z)=\frac{A-B}{A+B} ; \quad \quad \Phi(\rho, z)=\frac{F}{U+W} \tag{5.138}
\end{equation*}
$$

$$
\begin{aligned}
& \left.A=2\left(x^{2}-y^{2}\right)^{2}-l\left(x^{4}+y^{4}-2\right)+2 i\left[(x+y)^{2}(1-x y)+l(x-y)\left(2 x y^{2}-x+y\right)\right] 139\right) \\
& \left.B=2\left\{l\left[x\left(x^{2}-1\right)-y\left(1-y^{2}\right)\right]-(x+y)\left(x^{2}-y^{2}\right)+i l(x-y)\left[1-y^{2}-y(x+y)\right\}\right\} 140\right)
\end{aligned}
$$

Here, the parameter $l$ is a real number. The metric functions are ${ }^{61}$ :

$$
\begin{equation*}
f=\frac{N}{D}, \quad e^{2 \gamma}=\frac{N}{(l-2)^{2}\left(x^{2}-y^{2}\right)^{4}}, \quad \omega=\frac{\alpha\left(1-y^{2}\right)(x-1) F}{N} \tag{5.141}
\end{equation*}
$$

$$
\begin{gathered}
N=G^{2}-4\left(x^{2}-1\right)\left(1-y^{2}\right)\left[(x+y)^{2}-2 l y(x-y)\right]^{2} \\
D=N+G P-2\left(1-y^{2}\right)(x-1)\left[(x+y)^{2}-2 l y(x-y)\right] T \\
F=2(x+1)\left[(x+y)^{2}-2 l y(x-y)\right] P-G T \\
G=(2-l)\left(x^{2}-y^{2}\right)^{2}+2 l\left(x^{2}-1\right)\left(1-y^{2}\right) \\
P=4(l-2)^{-1}\left\{2(x+y)^{2}(x-y-1)-l(x+y)\left[(x-y)^{2}+2(x-1)^{2}\right]+l^{2}\left[x(x-1)^{2}+y(1-y)^{2}\right]\right\} \\
T=4 l(l-2)^{-1}[(l-2)(x-y)(x+2 y-1)-2(x+1)]
\end{gathered}
$$

The prolate coordinates are related to the Weyl's one trough:

$$
\begin{equation*}
\rho=\sigma \sqrt{\left(x^{2}-1\right)\left(1-y^{2}\right)}, \quad z=\sigma x y, \quad \sigma \equiv \frac{M}{l} . \tag{5.142}
\end{equation*}
$$

or by the inversion relation:

$$
\begin{align*}
& x=\frac{\sqrt{\rho^{2}+(z+\sigma)^{2}}+\sqrt{\rho^{2}+(z-\sigma)^{2}}}{2 \sigma}  \tag{5.143}\\
& y=\frac{\sqrt{\rho^{2}+(z+\sigma)^{2}}-\sqrt{\rho^{2}+(z-\sigma)^{2}}}{2 \sigma} \tag{5.144}
\end{align*}
$$

Note that the metric functions are regular on the symmetry axis, and that the functions $\omega$ and $\gamma$ vanish in the region $z>\sigma$, and $z<-\sigma(y=1, y=-1)$ for $\rho=0$, but also in the region $|z|<\sigma(x=1)$. Therefore, this solution is to be interpreted as two separated sources.


Figure 7 - Localization of the extreme objects on the symmetry axis Source: By the author.

On the symmetry axis, the Ernst potential takes the form:

$$
\begin{equation*}
\mathcal{E}(\rho=0, z)=\frac{(l-2) l^{2} z^{2}+2 l(l+(-1+i)) M z+((1-4 i) l+2 i) M^{2}}{(l-2) l^{2} z^{2}-2(l+(-1-i)) l M z+(l+(4+2 i)) M^{2}} \tag{5.145}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathcal{E}=1+\frac{e_{1}}{z-\beta_{1}}+\frac{e_{2}}{z-\beta_{2}} \tag{5.146}
\end{equation*}
$$

Where

$$
\begin{gathered}
\beta_{1}=\frac{(l+(-1-i)) l M-\sqrt{2}|l| M \sqrt{((4+3 i)-(2+2 i) l)}}{(l-2) l^{2}}, \\
\beta_{2}=\frac{l(l+(-1-i)) M+\sqrt{2}|l| M \sqrt{((4+3 i)-(2+2 i) l)}}{(l-2) l^{2}}, \\
e_{1}=\frac{2(l-1) M}{(l-2) l}+\frac{(1-i)(l(l+(-2-i))+(1+2 i)) \sqrt{(8+6 i)-(4+4 i) l} M}{(l-2)((2+2 i) l+(-4-3 i))|l|} \\
e_{2}=\frac{(1+i) M\left(i\left(l^{2}-(2+i) l+(1+2 i)\right) \sqrt{(8+6 i)-(4+4 i) l}|l|+4 l^{3}-(11-i) l^{2}+(7-i) l\right)}{(l-2) l^{2}((2+2 i) l+(-4-3 i))}
\end{gathered}
$$

By means of the multipole moments, we see that the total mass, $M_{T}$, is constrained by the parameter $l$ :

$$
\begin{equation*}
M_{T}=\frac{2(1-l) M}{(l-2) l} \tag{5.147}
\end{equation*}
$$

Hence, in order to have positive mass, the parameter $l$ must assume values in the interval $1<l<2$.

## 6 THE MULTIPOLE MOMENTS AND AXIS DATA

Multipole moments turn out to be really important since they permit the description of physical observable quantities of any type of sources in general relativity. Although Geroch ${ }^{32}$ and Hansen ${ }^{33}$ were the first to define the multipole moments in General Relativity for stationary metrics, it was only after Fodor et al. ${ }^{36}$ introduce an algorithm to evaluate them in spacetimes, which are also axisymmetric, that the multipoles received special attention (see Appendix C). It relies on the fact that such algorithm enables one, through a recurrence formula, to evaluate the multipole moments up to the desired order. Multipole moments turned to be a strong tool because of the physical parametrization of astrophysical objects and the spacetime as a whole since, as shown in previous works, the multipoles moments uniquely determine the local structure of the metric. ${ }^{82}$

Since multipole moments describe the far-field behaviour, the moments are not only an interesting theoretical tool, but can also be used in astrophysical experiments; for instance black holes or neutron stars, and observable phenomena related to them like geodesic motion of massive objects, test whether a compact object might be described by a Kerr solution, to study vibration modes in black holes, and even test the no-hair theorem, ${ }^{83,84}$ General Relativity itself, or other theories of Gravity such as scalar-tensor theories. ${ }^{85,86}$ These can be achieved by measuring data from gravitational waves ${ }^{87-90}$ or observing geodesics. ${ }^{91}$ Therefore, connecting the multipole moments with the mathematical parameters that appear when the Einstein-Maxwell equations are solved, turns out to be an important task.

Consider the general N-soliton electrovacuum solution characterized by the Ernst potentials ${ }^{6}$ on the symmetry axis given in terms of a polynomial quotient ${ }^{27}$ :

$$
\begin{align*}
& \mathcal{E}(\rho=0, z)=e(z)=\frac{z^{N}+\sum_{l=1}^{N} a_{l} z^{N-l}}{z^{N}+\sum_{l=1}^{N} b_{l} z^{N-l}}=\frac{P(z)}{R(z)},  \tag{6.1}\\
& \Phi(\rho=0, z)=f(z)=\frac{\sum_{l=1}^{N} c_{l} z^{N-l}}{z^{N}+\sum_{l=1}^{N} b_{l} z^{N-l}}=\frac{Q(z)}{R(z)}, \tag{6.2}
\end{align*}
$$

Here $a_{l}, b_{l}, c_{l}, k=1, \cdots N$ are $3 N$ arbitrary complex constants. Again, the choice of of the coefficients of higher order has been made in order to give the correct asymptotic behaviour at infinity, and it is supposed that the previous quotients are irreducible and that $R$ only possess roots with multiplicity one. In 1995 E . Ruiz et al. ${ }^{27}$ found, through the Sibgatullin integral method, ${ }^{43}$ a very elegant and compact form to write the general solution of this $N$-soliton solution with $3 N$ complex arbitrary parameters. In other words, the Ernst potentials found by the authors and the corresponding metric functions are
written in a very simple way and, as the authors pointed out, these forms have, in fact, facilitated the study of particular physically interesting metrics.

An aspect of that solution is that, at least a priori, these $3 N$ complex parameters do not have physical meaning, being necessary to link them to the Geroch-Hansen multipole moments. ${ }^{34,36,69,92}$ A development that helped to tackle this problem was reported in a paper by V. Manko et al. ${ }^{24}$ which analyzed vacuum solutions $(\Phi=0)$ and related the $2 N$ parameters $a_{l}$ and $b_{l}$ with $2 N$ multipole moments.

The objective of the present chapter is to extend this development to the case with an electromagnetic field. In other words, we will present a way to establish a relation between the $3 N$ arbitrary constants $a_{l}, b_{l}, c_{l}$ with $3 N$ multipole moments. This development is one of the contributions of the present thesis.

### 6.1 Relations between the Ernst potentials and multipole moments in vacuum case

As shown in reference [34], the multipole moments of a given stationary solution with axial symmetry can be obtained from its corresponding Ernst potentials $\xi$ and $q$ evaluated at infinity. That is, when $z \rightarrow \infty$ :

$$
\begin{align*}
\xi & =\sum_{k=0}^{\infty} m_{k} z^{-k-1},  \tag{6.3}\\
q & =\sum_{k=0}^{\infty} q_{k} z^{-k-1} \tag{6.4}
\end{align*}
$$

$\xi$ and $q$ are related to $\mathcal{E}$ and $\Phi$ as follows (see Chapter 3):

$$
\begin{equation*}
\mathcal{E}=\frac{1-\xi}{1+\xi}, \quad \Phi=\frac{q}{1+\xi} . \tag{6.5}
\end{equation*}
$$

Thus, it should be possible to associate the Geroch-Hansen coefficients, $m_{k}$ and $q_{k}$, to the constants $a_{l}, b_{l}$ and $c_{l}$ and, hence, to characterize physically the Ernst equations solutions. As shown in reference [69] and revised in [92]*, the multipole moments $P_{k}$ and $Q_{k}$ are univoquely written in terms of the coefficients $m_{k}$ and $q_{k}{ }^{\dagger}$. The real part of the multipole $P_{k}$ is related with the mass multipole moments, while the imaginary one is related with the angular multipole moments. In the same way, the real part of the multipole $Q_{k}$ is related with the eletric field multipole momentum, while the imaginary one is related with the magnetic field multipole momentum.

[^10]As pointed out in [76], since it is suficient to know the behavior of the Ernst potentials $\mathcal{E}$ and $\Phi$ over the symmetry axis to peform an analytic continuation of them to the whole space, ${ }^{19}$ the coefficients $m_{k}$ and $q_{k}$ appear to have a significant role in the present development. The arbitrariness of theses coefficients raises the question of which condition they have to satisfy in order for the relation between them and $a_{l}, b_{l}$ and $c_{l}$ to be possible. Combining the equations (6.1) and (6.2) with equations (6.3) and (6.4) one finds:

$$
\begin{align*}
\frac{R(z)-P(z)}{R(z)+P(z)} & =\sum_{k=0}^{\infty} m_{k} z^{-k-1}  \tag{6.6}\\
\frac{2 Q(z)}{R(z)+P(z)} & =\sum_{k=0}^{\infty} q_{k} z^{-k-1} \tag{6.7}
\end{align*}
$$

Focusing, first, on the equation (6.6) and equating the coefficients with the same powers of $z$, we obtain:

$$
\begin{aligned}
& \frac{1}{2}\left(b_{1}-a_{1}\right)=m_{0} \\
& \frac{1}{2}\left(b_{2}-a_{2}\right)=m_{1}+\frac{1}{2}\left(b_{1}+a_{1}\right) m_{0} \\
& \vdots \\
& \frac{1}{2}\left(b_{N}-a_{N}\right)=m_{N-1}+\frac{1}{2}\left(b_{1}+a_{1}\right) m_{N-2}+\cdots+\frac{1}{2}\left(b_{N-1}+a_{N-1}\right) m_{0} \\
& 0=m_{n}+\frac{1}{2}\left(b_{1}+a_{1}\right) m_{n-1}+\cdots+\frac{1}{2}\left(b_{N}+a_{N}\right) m_{n-N}, \text { for } n \geq N
\end{aligned}
$$

Defining: $A_{l}=\frac{1}{2}\left(b_{l}-a_{l}\right) ; B_{l}=\frac{1}{2}\left(b_{l}+a_{l}\right) ; A_{0}=B_{0}=1$, we find the following system of equations:

$$
\begin{align*}
& A_{n+1}=\sum_{l=0}^{n} B_{l} m_{n-l}, \quad n=0,1, \ldots, N-1  \tag{6.8}\\
& 0=\sum_{l=0}^{N} B_{l} m_{n-l}, \quad n \geq N . \tag{6.9}
\end{align*}
$$

The system above constitutes a system with infinite equations for a finite number of variables, with the aim to write $a_{l}$ and $b_{l}$ in terms of $m_{n}$. In order to describe the $N$-soliton problem the authors in [24] used a set of $2 N$ arbitrary parameters, $a_{l}$ e $b_{l}$, thus, in principle, it would be possible to use a set of $2 N$ coefficients $m_{k}$ to describe such problem. In what follows, a generalization of the compatibility condition which ensures a system like this one is constructed.

In order to find the $N$ variables $B_{l}, N$ equations are needed. Then, inside the infinity set of parameters $m_{n}$, take $N$ elements $\left\{m_{n_{1}}, m_{n_{2}}, \ldots, m_{n_{N}}\right\}$ in such way that $n_{i} \geq N$ with $i=1,2, \ldots, N$. Consider, now, the equation (6.9) for this set of $n_{i}$.

$$
\begin{gather*}
\sum_{l=1}^{N} B_{l} m_{n_{i}-l}=-m_{n_{i}}  \tag{6.10}\\
\left(\begin{array}{ccccc}
m_{n_{1}-1} & m_{n_{1}-2} & \cdots & m_{n_{1}-N+1} & m_{n_{1}-N} \\
m_{n_{2}-1} & m_{n_{2}-2} & \cdots & m_{n_{2}-N+1} & m_{n_{2}-N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{n_{N}-1} & m_{n_{N}-2} & \cdots & m_{n_{N}-N+1} & m_{n_{N}-N}
\end{array}\right)\left(\begin{array}{c}
B_{1} \\
B_{2} \\
\vdots \\
B_{N}
\end{array}\right)=-\left(\begin{array}{c}
m_{n_{1}} \\
m_{n_{2}} \\
\vdots \\
m_{n_{N}}
\end{array}\right) \tag{6.11}
\end{gather*}
$$

Defining a new object, $L_{i}$, as a determinant of an $i \times i$ matrix:

$$
L_{i}=\left|\begin{array}{ccccc}
m_{n_{1}-N+i-1} & m_{n_{2}-N+i-1} & \cdots & m_{n_{i-1}-N+i-1} & m_{n_{i}-N+i-1}  \tag{6.12}\\
m_{n_{1}-N+i-2} & m_{n_{2}-N+i-2} & \cdots & m_{n_{i-1}-N+i-2} & m_{n_{i}-N+i-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{n_{1}-N} & m_{n_{2}-N} & \cdots & m_{n_{i-1}-N} & m_{n_{i}-N}
\end{array}\right| .
$$

Thus, it is straightforward to see that the system (6.11) only has solutions when $L_{N} \neq 0$. Using Cramer's rule to find the coefficients $B_{l}$, we obtain:

$$
B_{l}=(-1)^{l}\left(L_{N}\right)^{-1}\left|\begin{array}{ccccccc}
m_{n_{1}} & m_{n_{1}-1} & \cdots & m_{n_{1}-(l-1)} & m_{n_{1}-(l+1)} & \cdots & m_{n_{1}-N}  \tag{6.13}\\
m_{n_{2}} & m_{n_{2}-1} & \cdots & m_{n_{2}-(l-1)} & m_{n_{2}-(l+1)} & \cdots & m_{n_{2}-N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
m_{n_{N}} & m_{n_{N}-1} & \cdots & m_{n_{N}-(l-1)} & m_{n_{N}-(l+1)} & \cdots & m_{n_{N}-N}
\end{array}\right|
$$

Due to this highly symmetric structure, it is possible to rewrite $B_{l}$ as:

$$
B_{l}=\left(L_{N}\right)^{-1}\left|\begin{array}{llllll}
0 & m_{n_{1}} & m_{n_{2}} & \cdots & m_{n_{N-1}} & m_{n_{N}}  \tag{6.14}\\
0 & & & & & \\
\vdots & & & & & \\
1 & & & L_{N} & & \\
\vdots & & & &
\end{array}\right|
$$

where the row corresponding to the " 1 " in the first column is the $(l+1)$-th row. By using this result for $B_{l}$ the coefficients $A_{l}$ can be found through the equation (6.8) and, in
a similar way, can be written as:

$$
A_{l+1}=\left(L_{N}\right)^{-1}\left|\begin{array}{cccccc}
m_{l} & m_{n_{1}} & m_{n_{2}} & \cdots & m_{n_{N-1}} & m_{n_{N}}  \tag{6.15}\\
m_{l-1} & & & & & \\
\vdots & & & & & \\
m_{0} & & & L_{N} & & \\
\vdots & & & & & \\
0 & & & & &
\end{array}\right| \text {. }
$$

Nonetheless, $A_{l}$ and $B_{l}$ are described in terms of, at most, $N^{2}+2 N$ independent coefficients $m_{k}$ and by hypothesis it asserts that they must be written in terms of $2 N$ coefficients $m_{k}$. Therefore, it is necessary to restrict the set $\left\{m_{n_{i}}\right\}$, which is solution of the equation (6.10), and its condition is only respected in case when the set $\left\{m_{n_{i}}\right\}$ is chosen with $n_{1}=N$, $n_{2}=N+1, \ldots, n_{N}=2 N-1$. Then, $L_{i}$ can be written as:

$$
L_{i}=\left|\begin{array}{ccccc}
m_{i-1} & m_{i} & \cdots & m_{2 i-3} & m_{2 i-2}  \tag{6.16}\\
m_{i-2} & m_{i-1} & \cdots & m_{2 i-4} & m_{2 i-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{0} & m_{1} & \cdots & m_{i-2} & m_{i-1}
\end{array}\right|,
$$

And then, $A_{l}$ e $B_{l}$, written in terms of $2 N m_{k}$, takes the form:

$$
\begin{gather*}
\left.A_{l+1}=\left(L_{N}\right)^{-1} \left\lvert\, \begin{array}{cccccc}
m_{l} & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1} \\
m_{l-1} & & & & \\
\vdots & & & & \\
m_{0} & & & L_{N} & \\
\vdots & & & & \\
0 & & \\
B_{l} & =\left(L_{N}\right)^{-1} \mid \\
\left.\begin{array}{llllll}
0 & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1} \\
0 & & & &
\end{array} \right\rvert\,,
\end{array}\right.\right]  \tag{6.17}\\
\vdots  \tag{6.18}\\
\vdots \\
0
\end{gather*}
$$

The condition that the coefficients must satisfy when calculating $L_{i}$ for $i>N$ will
now be deduced. Consider:

$$
L_{N+1}=\left|\begin{array}{ccccc}
m_{N} & m_{N+1} & \cdots & m_{2 N-1} & m_{2 N}  \tag{6.19}\\
m_{N-1} & m_{N} & \cdots & m_{2 N-2} & m_{2 N-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{0} & m_{1} & \cdots & m_{N-1} & m_{N}
\end{array}\right|
$$

By performing a cofactor expansion, we:

$$
\begin{equation*}
L_{N+1}=\sum_{k=0}^{N} m_{2 N-k}(-1)^{k+N} M_{k} \tag{6.20}
\end{equation*}
$$

with $M_{k} \equiv(-1)^{k} L_{N} B_{k}$, consequently:

$$
\begin{equation*}
L_{N+1}=(-1)^{N} L_{N} \sum_{k=0}^{N} B_{k} m_{2 N-k} \tag{6.21}
\end{equation*}
$$

However, the sum $\sum_{k=0}^{N} B_{k} m_{n-k}$ is zero by hypothesis for all $n \geq N$. Thus, $m_{k}$ must be chosen such that $L_{N+1}=0$. By induction, it is straightforward to see that all determinants $L_{n}=0$ for all $n>N$, proving the following lemma, which has first been stated (without proof) in reference [24] and reads:

Lemma 1 Once fixed the set of coefficients $\left\{m_{i}\right\}$ with $i=0,1, \ldots, 2 N-1$, the necessary and sufficient condition for this set to describe the behaviour of the Ernst potentials on the symmetry axis as a polynomial quotient (6.1) is that the determinant $L_{n}$ be nonzero for $n=N$ and zero for all $n>N$.

### 6.2 Relations between the Ernst potentials and multipole moments in the electrovacuum case

As mentioned before, the lemma 1 is not new, however its proof is a contribution of mine during my Master thesis. Nonetheless, the paper [24] does not cover the electromagnetic case. In order to try to generalize these results for the cases where the electromagnetic field is present, a similar analysis for the equation (6.7) will be done, so that in the end the $3 N$ variables $a_{l}, b_{l}$ and $c_{l}$ can be written in terms of $3 N$ coefficients $m_{k}$ and $q_{k}$ related with the multipole moments. Equating the coefficients with the same powers of $z$, we find:

$$
\begin{aligned}
& c_{1}=q_{0} \\
& c_{2}=q_{1}+\frac{1}{2}\left(b_{1}+a_{1}\right) q_{0} \\
& \vdots \\
& c_{N}=q_{N-1}+\frac{1}{2}\left(b_{1}+a_{1}\right) q_{N-2}+\cdots+\frac{1}{2}\left(b_{N-1}+a_{N-1}\right) q_{0} \\
& 0=q_{n}+\frac{1}{2}\left(b_{1}+a_{1}\right) q_{n-1}+\cdots+\frac{1}{2}\left(b_{N}+a_{N}\right) q_{n-N} ;, \text { for } n \geq N
\end{aligned}
$$

Such system can be summarized into:

$$
\begin{align*}
& c_{n+1}=\sum_{l=0}^{n} B_{l} q_{n-l}, \quad n=0,1, \ldots, N-1  \tag{6.22}\\
& 0=\sum_{l=0}^{N} B_{l} q_{n-l}, \quad n \geq N \tag{6.23}
\end{align*}
$$

Notice that the above equations have the same structure as the equations (6.8) and (6.9). In addition, the same function $B_{l}$, which was already evaluated in terms of the coefficients $m_{k}$, will be evaluated now in terms of the coefficients $q_{k}$. Since it is the same function $B_{l}$, when it is written in terms of $m_{k}$ or $q_{k}$ it must be equivalent.

While the equations' structure is the same as above, we will introduce a new index here to the determinant $L_{i}$ in order to differentiate whether it is written in terms of $m_{k}$ or $q_{k}$, that is, $L_{i, m}$ and $L_{i, q}$. That is:

$$
L_{i, q}=\left|\begin{array}{ccccc}
q_{i-1} & q_{i} & \cdots & q_{2 i-3} & q_{2 i-2}  \tag{6.24}\\
q_{i-2} & q_{i-1} & \cdots & q_{2 i-4} & q_{2 i-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{0} & q_{1} & \cdots & q_{i-2} & q_{i-1}
\end{array}\right|
$$

Therefore, the equation for $B_{l}$ in terms of $q_{k}$ is given by:

$$
B_{l}=\left(L_{N, q}\right)^{-1}\left|\begin{array}{llllll}
0 & q_{N} & q_{N+1} & \cdots & q_{2 N-2} & q_{2 N-1}  \tag{6.25}\\
0 & & & & & \\
\vdots & & & & & \\
1 & & & L_{N, q} & & \\
\vdots & & & & & \\
0 & & & & &
\end{array}\right|
$$

Notice that in $B_{l}$ the only elements that do not repeat are $m_{0}\left(q_{0}\right)$ and $m_{2 N-1}$ $\left(q_{2 N-1}\right)$, since in the diagonal the principal direction is constituted by equal elements, with
the exception of the elements in the first column. Because of that, $q_{1}$ appears twice, $q_{2}$ appears three times, until $q_{N-1}$ and $q_{N}$ which appear $N$ times, and then, $q_{N+1}$ appears $N-1$ times and so on.

Since the variable $B_{l}$ must be the same independently of whether it is written in terms of the $m_{k}$ or $q_{k}$, the following relation is obtained:
$B_{l}=$


Given that $B_{l}$ is a constant for $m_{k}$ coefficients, from the equality above, we conclude that for a fixed $l$ the variables $q_{k}$ for each $B_{l}$ loose one degree of freedom. Furthermore, knowing that $l$ ranges from 1 to $N$ and $\left|B_{l}\right|=N$, one notices that $N$ of the $\left|q_{k}\right|$ variables are not free, i.e., $N$ variables from the set $q_{k}$ can be described as a function of $2 N$ variables from $m_{k}$ and $N$ variables of $q_{k}$. Consequently, one can generalize lemma 1 .

Lemma 2 Given a set of coefficients $\left\{m_{i}\right\}$ with $i=0,1, \ldots, 2 N-1$ and a subset of $N$ coefficients $q_{i}$ contained in $\left\{q_{i}\right\}$ with $i=0,1, \ldots, 2 N-1$, the necessary and sufficient conditions for those $3 N$ variables to describe the behaviour of the Ernst potentials on the symmetry axis as polynomial quotient (6.1) and (6.2) is that the determinant $L_{N},{ }_{m}^{q}$ be nonzero for $n=N$ and zero for all $n>N$ and that equation (6.26) is valid.

One representation for the $3 N$ variables $A_{l}, B_{l}$ e $c_{l}$ written in terms of $2 N$ coefficients $m_{k}$ and $N$ coefficients $q_{k}$ is given by:

$$
A_{l+1}=\left(L_{N, m}\right)^{-1}\left|\begin{array}{cccccc}
m_{l} & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1}  \tag{6.27}\\
m_{l-1} & & & & \\
\vdots & & & & \\
m_{0} & & & L_{N, m} & \\
\vdots & & & & \\
0 & & & &
\end{array}\right|
$$

$$
\begin{gather*}
B_{l}=\left(L_{N, m}\right)^{-1} \left\lvert\, \begin{array}{cccccc}
0 & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1} \\
0 & & & & \\
\vdots & & & & & \\
1 & & & L_{N, m} & & \\
\vdots & & & & & \\
0 & & & & \\
c_{l+1} & =\left(L_{N, m}\right)^{-1} \mid \\
\left.\begin{array}{cccccc}
q_{l} & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1} \\
q_{l-1} & & & & \\
\vdots & & & & \\
q_{0} & & & L_{N, m} \\
\vdots & & & & \\
0 & & &
\end{array} \right\rvert\,
\end{array} . .\right. \tag{6.28}
\end{gather*}
$$

where the relation between the variables $A_{l}$ e $B_{l}$ and the variables $a_{l}$ e $b_{l}$ defined in (6.1) and (6.2) is given by:

$$
\begin{equation*}
a_{l}=B_{l}-A_{l} \quad b_{l}=B_{l}+A_{l} . \tag{6.30}
\end{equation*}
$$

Thereby, the electrovacuum $N$-soliton solution on the z-axis can be written in terms of the multipole moments in the following compact way:

$$
\begin{gather*}
P(z)=z^{N}+\sum_{l=1}^{N} a_{l} z^{N-l}=z^{N}+\sum_{l=1}^{N}\left(B_{l}-A_{l}\right) z^{N-l}=  \tag{6.31}\\
=\sum_{l=0}^{N} B_{l} z^{N-l}-\sum_{l=1}^{N} \sum_{k=0}^{l-1} B_{k} m_{l-1-k} z^{N-l}=\sum_{l=0}^{N} B_{l} z^{N-l}-\sum_{l=0}^{N-1} \sum_{k=0}^{l} B_{k} m_{l-k} z^{N-l-1} \\
=\sum_{l=0}^{N} B_{l} z^{N-l}-\sum_{l=0}^{N-1} B_{l} \sum_{k=0}^{N-1-l} m_{k} z^{N-l-k-1},
\end{gather*}
$$

where the first and second terms from the last above equality can be written as

$$
\sum_{l=0}^{N} B_{l} z^{N-l}=\left(L_{N}\right)^{-1}\left|\begin{array}{cccccc}
z^{N} & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1}  \tag{6.32}\\
z^{N-1} & & & & & \\
\vdots & & & L_{N} & & \\
z & & & & & \\
1 & & & & &
\end{array}\right|
$$

$$
\sum_{l=0}^{N-1} B_{l} \sum_{k=0}^{N-1-l} m_{k} z^{N-l-k-1}=\left(L_{N, m}\right)^{-1}\left|\begin{array}{ccccc}
\sum_{k=0}^{N-1} m_{k} z^{N-k-1} & m_{N} & m_{N+1} & \cdots & m_{2 N-2}  \tag{6.33}\\
\sum_{k=0}^{N-2} m_{k} z^{N-k-2} & & & m_{2 N-1} \\
\vdots & & & \\
m_{0} & & & \\
0 & & & \\
0 & & & \\
\hline
\end{array}\right|
$$

Hence, we can write $P(z), R(z)$ and $Q(z)$ in the very simple form

$$
P(z)=\left(L_{N, m}\right)^{-1}\left|\begin{array}{ccccc}
z^{N}-\sum_{k=0}^{N-1} m_{k} z^{N-k-1} & m_{N} & m_{N+1} & \cdots & m_{2 N-2}  \tag{6.34}\\
z^{N-1}-\sum_{k=0}^{N-2} m_{k} z^{N-k-2} & & & \\
\vdots & & L_{N, m} & \\
z-m_{0} & & & \\
1 & & &
\end{array}\right|
$$

$$
R(z)=\left(L_{N, m}\right)^{-1}\left|\begin{array}{ccccc}
z^{N}+\sum_{k=0}^{N-1} m_{k} z^{N-k-1} & m_{N} & m_{N+1} & \cdots & m_{2 N-2}
\end{array} m_{2 N-1}\right| \begin{gathered}
z^{N-1}+\sum_{k=0}^{N-2} m_{k} z^{N-k-2}  \tag{6.35}\\
\vdots \\
z+m_{0} \\
1
\end{gathered}
$$

$$
Q(z)=\left(L_{N, m}\right)^{-1}\left|\begin{array}{cccccc}
\sum_{k=0}^{N-1} q_{k} z^{N-k-1} & m_{N} & m_{N+1} & \cdots & m_{2 N-2} & m_{2 N-1}  \tag{6.36}\\
\sum_{k=0}^{N-2} q_{k} z^{N-k-2} & & & & & \\
\vdots & & & L_{N, m} & & \\
q_{0} & & & & \\
0 & & & &
\end{array}\right| .
$$

### 6.3 Multipole moments of the N -Soliton solution

So far, we have given the relations and conditions for writing the $3 N$ parameters of the N -soliton solution, $a_{l}, b_{l}$ and $c_{l}$, in terms of the multipole coefficients $m_{l}$ and $q_{l}$. Now, a stronger result can be achieved by studying the inverse relation of these coefficients. That
is, $L_{N+1}{ }_{m}^{q}$ is always zero for the N -soliton solution, and the conditions in Lemmas 1 and 2 are always satisfied. For this purpose, consider the series below:

$$
\begin{equation*}
\frac{\sum_{l=1}^{N} e_{l} z^{N-l}}{z^{N}+\sum_{k=1}^{N} d_{k} z^{N-k}} \tag{6.37}
\end{equation*}
$$

This series has the same shape shape as the equations (6.6) and (6.7) (they are the same apart from a factor). Therefore, in order to write $m_{l}$ and $q_{l}$ in terms of $a_{l}, b_{l}$ and $c_{l}$, it is necessary to see how to expand the above series in terms of negative powers of $z$. By canceling the term $z^{N}$ and focusing on the denominator we notice that it is possible to expand it in the following power series

$$
\begin{equation*}
\frac{1}{1+\sum_{k=1}^{N} d_{k} z^{-k}}=\sum_{j=0}^{\infty}(-1)^{j}\left(\sum_{k=1}^{N} d_{k} z^{-k}\right)^{j} \tag{6.38}
\end{equation*}
$$

However

$$
\begin{equation*}
\left(d_{1} z^{-1}+d_{2} z^{-2}+\cdots+d_{N} z^{-N}\right)^{j}=\sum_{k_{1}+k_{2}+\cdots+k_{N}=j} \frac{j!}{k_{1}!k_{2}!\cdots k_{N}!} \prod_{t=1}^{N}\left(d_{t} z^{-t}\right)^{k_{t}} \tag{6.39}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\frac{\sum_{l=1}^{N} e_{l} z^{N-l}}{z^{z}+\sum_{k=1}^{N} d_{k} z^{N-k}}=\sum_{l=1}^{N} e_{l} z^{-l} \sum_{j=0}^{\infty} \sum_{k_{1}+k_{2}+\cdots+k_{N}=j} \frac{j!}{k_{1}!k_{2}!\cdots k_{N}!} \prod_{t=1}^{N}\left(d_{t} z^{-t}\right)^{k_{t}} . \tag{6.40}
\end{equation*}
$$

Now, we need to find the general coefficient for this power series. That is, we must write

$$
\begin{equation*}
\frac{\sum_{l=1}^{N} e_{l} z^{N-l}}{z^{z}+\sum_{k=1}^{N} d_{k} z^{N-k}} \equiv \sum_{\alpha=0}^{\infty} h_{\alpha} z^{-\alpha-1} \tag{6.41}
\end{equation*}
$$

and find the coefficients $h_{\alpha}$. After some simple calculations we find

$$
\begin{equation*}
h_{\alpha}=\sum_{l=1}^{N} e_{l} \theta_{l} \tag{6.42}
\end{equation*}
$$

where

$$
\theta_{l}=\left\{\begin{array}{c}
0, \text { if } \alpha<l,  \tag{6.43}\\
\sum_{k_{1}+2 k_{2}+\cdots+N k_{N}=\alpha-l}(-1)^{k_{1}+k_{2}+\cdots+k_{N}} \frac{\left(k_{1}+k_{2}+\cdots+k_{N}\right)!}{k_{1}!k_{2}!\cdots k_{N}!} \prod_{t=1}^{N}\left(d_{t}\right)^{k_{t}}, \text { if } \alpha \geq l .
\end{array}\right.
$$

From the above equation, it is possible to find a relation between the $h_{\alpha}$

$$
\begin{equation*}
h_{\alpha+N}=-\sum_{l=1}^{N} d_{l} h_{\alpha+N-l} . \tag{6.44}
\end{equation*}
$$

This shows that $h_{\alpha+N}$ is a linear combination of the set $\left\{h_{\alpha}, h_{\alpha+1}, \cdots, h_{\alpha+N-1}\right\}$ with fixed coefficients $d_{l}$, that is, the last column of the matrix whose determinant is $L_{N+1},{ }_{m}^{q}$ is a linear combination of the first $N$ columns. Hence, we can write:

$$
\begin{gather*}
m_{\alpha}=\frac{1}{2} \sum_{l=1}^{N}\left(b_{l}-a_{l}\right) \theta_{l}, \quad q_{\alpha}=\sum_{l=1}^{N} c_{l} \theta_{l},  \tag{6.45}\\
\theta_{l}=\left\{\begin{array}{c}
0, \text { if } \alpha<l, \\
\sum_{k_{1}+2 k_{2}+\cdots+N k_{N}=\alpha-l}(-1)^{k_{1}+k_{2}+\cdots+k_{N}} \frac{\left(k_{1}+k_{2}+\cdots+k_{N}\right)!}{k_{1}!k_{2}!\cdots k_{N}!} \prod_{t=1}^{N}\left(\frac{b_{l}+a_{l}}{2}\right)^{k_{t}}, \text { if } \alpha \geq l .
\end{array}\right. \tag{6.46}
\end{gather*}
$$

For this reason, not only $L_{N+1},{ }_{m}^{q}$ but $L_{N+k},{ }_{m}^{q} k \geq 1$, is zero for all $N$-soliton solutions. Moreover, this implies that all $m_{n}$ are determined for $n \geq 2 N$, and $q_{n}$ are determined for $n \geq N$. Finally, using this, we can improve Lemma 2 and state the best form of our result.

Theorem 1 As in Lemma 2 fix a set of coefficients $\left\{m_{i}\right\}_{i=1}^{2 N-1}$ and a subset $\left\{q_{i_{1}}, q_{i_{2}}, \ldots, q_{i_{N}}\right\} \subset$ $\left\{q_{i}\right\}_{i=1}^{2 N-1}$. Then, these $3 N$ variables describe the behavior of Ernst potentials on the symmetry axis as polynomial quotient (5.1) and (5.2) if and only if the determinant $L_{N},{ }_{m}^{q} \neq 0$.

Another interesting outcomes evolves from these previous results. First, notice that Lemma 2 impose to two conditions in the multipoles in order to describe a $N$-soliton solution, the equation (6.26) be valid and $L_{N},{ }_{m}^{q} \neq 0$. However, we proved that equation (6.26) is always valid in the case of a $N$-soliton solution. Arriving in the Theorem 1, which state that $L_{N},{ }_{m}^{q} \neq 0$ is the only condition to establish the relation between the multipoles and the solution. But notice that, if we have $L_{N},{ }_{m}^{q}=0$ and $L_{N-1},{ }_{m}^{q} \neq 0$, we are describing a system of $N-1$ soliton, and the equations are still valid. However, due to the results in the present section, if a given stationary axisymmetric spacetime, solution of the Einstein-Maxwell equations, such that its multipole moments satisfy (6.26) and $L_{N},{ }_{m} \neq 0$, then this solution can be approximated as a $N$-soliton solution.

### 6.3.1 Metric of a rotating charged magnetized sphere

Based on a recent paper by Manko et al [76], let us find the solution associated to the given multipoles moments. In that article, the authors gave the Ernst potentials on the symmetry axis and then found the respective multipole moments. In order to clarify how the method derived in the preceding sections should be used, consider the multipole moments below and then construct the metric solution associated with them. Due to the complexity of higher orders, we will consider a solutions with only the first 6 multipoles $P_{n}$ and $Q_{n}$ :

$$
\begin{aligned}
& P_{0}=M \\
& P_{1}=i j M^{2} \\
& P_{2}=M^{3} q \\
& P_{3}=\frac{1}{5} i M^{4}(e j+\mu(5 q-1)), \\
& P_{4}=\frac{1}{35} M^{3}\left(5 e M^{2}\left(q\left(e^{2}+7 q\right)+\mu^{2}\right)+3 \mathrm{jM}^{2}(e j-\mu)-5 M^{2}(e q+j \mu)\right), \\
& P_{5}=\frac{1}{21} i M^{6}\left(\mu\left(e^{2}(-(q+1))+j^{2}+3 q(7 q-3)+1\right)+e j\left(e^{2}+10 q-1\right)-\mu^{3}\right) .
\end{aligned}
$$

And

$$
\begin{aligned}
& Q_{0}=e M \\
& Q_{1}=i M^{2} \mu \\
& Q_{2}=e M^{3} q, \\
& Q_{3}=\frac{1}{5} i M^{4}(e j+\mu(5 q-1)), \\
& Q_{4}=\frac{1}{35} M^{5}\left(5 e^{3} q+3 e j^{2}+5 e\left(\mu^{2}+q(7 q-1)\right)-8 j \mu\right), \\
& Q_{5}=\frac{1}{21} i M^{6}\left(\mu\left(e^{2}(-(q+1))+j^{2}+3 q(7 q-3)+1\right)+e j\left(e^{2}+10 q-1\right)-\mu^{3}\right) .
\end{aligned}
$$

Hence, using the corrected versions of the equations (23) and (24) from reference [92] (see appendix C), one can find the coefficients $m_{k}$ and $q_{k}$

$$
\begin{array}{cl}
m_{0}=M, & q_{0}=e M \\
m_{1}=i j M^{2}, & q_{1}=i M^{2} \mu \\
m_{2}=M^{3} q, & q_{2}=e M^{3} q \\
m_{3}=i j M^{4} q, & q_{3}=i M^{4} q \mu \\
m_{4}=M^{5} q^{2}, & q_{4}=e M^{5} q^{2} \\
m_{5}=i j M^{6} q^{2}, & q_{5}=I M^{6} q^{2} \mu
\end{array}
$$

With the coefficients $m_{k}$ and $q_{k}$ in hand, it is easy to show from the discussion above that only 6 parameters are needed to describe the Ernst potentials. First of all, notice that:

$$
\begin{equation*}
L_{n, m}=L_{n, q}=0, \quad \forall n \geq 3 . \tag{6.47}
\end{equation*}
$$

In order to relate the multipole coefficients $m_{k}^{\prime} s$ and $q_{k}^{\prime} s$ with the Ernst coeficients $a_{l}, b_{l}$ and $c_{l}$, the following relations must be true:

$$
\begin{align*}
& B_{1}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{lll}
0 & m_{2} & m_{3} \\
1 & m_{1} & m_{2} \\
0 & m_{0} & m_{1}
\end{array}\right|=\left(L_{2, q}\right)^{-1}\left|\begin{array}{lll}
0 & q_{2} & q_{3} \\
1 & q_{1} & q_{2} \\
0 & q_{0} & q_{1}
\end{array}\right|  \tag{6.48}\\
& B_{2}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{lll}
0 & m_{2} & m_{3} \\
0 & m_{1} & m_{2} \\
1 & m_{0} & m_{1}
\end{array}\right|=\left(L_{2, q}\right)^{-1}\left|\begin{array}{lll}
0 & q_{2} & q_{3} \\
0 & q_{1} & q_{2} \\
1 & q_{0} & q_{1}
\end{array}\right| \tag{6.49}
\end{align*}
$$

and

$$
\begin{align*}
& A_{1}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{ccc}
m_{0} & m_{2} & m_{3} \\
0 & m_{1} & m_{2} \\
0 & m_{0} & m_{1}
\end{array}\right|  \tag{6.50}\\
& A_{2}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
m_{0} & m_{1} & m_{2} \\
0 & m_{0} & m_{1}
\end{array}\right| \tag{6.51}
\end{align*}
$$

which can be verified after a straightforward . Consequently:

$$
\begin{array}{lr}
B_{1}=0, & B_{2}=-M^{2} q \\
A_{1}=M, & A_{2}=i j M^{2}
\end{array}
$$

Using the equation (6.30) yields:

$$
\begin{array}{lr}
a_{1}=-M, & a_{2}=-M^{2} q-i j M^{2} \\
b_{1}=M, & b_{2}=-M^{2} q+i j M^{2}
\end{array}
$$

Finally, the coeficients $c_{l}$ can be found using (6.29) and read:

$$
\begin{equation*}
c_{1}=e M, \quad c_{2}=i M^{2} \mu . \tag{6.52}
\end{equation*}
$$

Thus, the Ernst potentials $\mathcal{E}$ and $\Phi$ can now be evaluated with the relations (6.30), (6.1) and (6.2):

$$
\begin{align*}
& \mathcal{E}(\rho=0, z)=\frac{z^{2}-M z-M^{2} q-i M^{2} j}{z^{2}+M z-M^{2} q+i M^{2} j}  \tag{6.53}\\
& \Phi(\rho=0, z)=\frac{M e z+i M^{2} \mu}{z^{2}+M z-M^{2} q+i M^{2} j} \tag{6.54}
\end{align*}
$$

recovering, then, the Ernst potentials presented in [76] (see section 5.2.4).

### 6.3.2 Tomimatsu-Sato solution with $\delta=2$

Now, lets proceed in the same way for the multipoles associated with the TomimatsuSato solution. We will find that the Ernst potentials on the symmetry axis are recovered from the corresponding multipole moments. This helps to interpret this solution as a 2-soliton solution:

$$
\begin{aligned}
& P_{0}=M, \\
& P_{1}=i a M, \\
& P_{2}=-\frac{1}{4} M\left(3 a^{2}+M^{2}\right), \\
& P_{3}=-\frac{1}{2} i a M\left(a^{2}+M^{2}\right), \\
& P_{4}=\frac{1}{112} M\left(35 J^{4}+66 J^{2} M^{2}+11 M^{4}\right), \\
& P_{5}=\frac{1}{112} M\left(35 J^{4}+66 J^{2} M^{2}+11 M^{4}\right) .
\end{aligned}
$$

Therefore, the coefficients $m_{k}$ are:

$$
\begin{aligned}
& m_{0}=M \\
& m_{1}=i a M \\
& m_{2}=-\frac{1}{4} M\left(3 a^{2}+M^{2}\right) \\
& m_{3}=-\frac{1}{2} i a M\left(a^{2}+M^{2}\right) \\
& m_{4}=\frac{1}{16}\left(5 a^{4} M+10 a^{2} M^{3}+M^{5}\right) \\
& m_{5}=\frac{1}{16} i\left(3 a^{5} M+10 a^{3} M^{3}+3 a M^{5}\right)
\end{aligned}
$$

Again, notice that:

$$
\begin{equation*}
L_{n, m}=0, \quad \forall n \geq 3 \tag{6.55}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{2, m}=\frac{1}{4}\left(M^{4}-a^{2} M^{2}\right) . \tag{6.56}
\end{equation*}
$$

In order to relate the multipole coefficients $m_{k}$ and $q_{k}$ with the Ernst coeficients $a_{l}$, $b_{l}$ and $c_{l}$, the following relations must be true:

$$
\begin{gather*}
B_{1}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{lll}
0 & m_{2} & m_{3} \\
1 & m_{1} & m_{2} \\
0 & m_{0} & m_{1}
\end{array}\right|=-i a  \tag{6.57}\\
B_{2}=\left(L_{2, m}\right)^{-1}\left|\begin{array}{lll}
0 & m_{2} & m_{3} \\
0 & m_{1} & m_{2} \\
1 & m_{0} & m_{1}
\end{array}\right|=\frac{1}{4}\left(M^{2}-a^{2}\right) \tag{6.58}
\end{gather*}
$$

which can be verified after a straightforward calculation. Consequently:

$$
\begin{array}{ll}
B_{1}=-i a, & B_{2}=\frac{1}{4}\left(M^{2}-a^{2}\right), \\
A_{1} & =M, \\
A_{2} & =0 .
\end{array}
$$

Using the equation (6.30) yields:

$$
\begin{aligned}
& a_{1}=-(M+i a), \quad a_{2}=\frac{1}{4}\left(M^{2}-a^{2}\right), \\
& b_{1}=M-i a, \quad b_{2}=\frac{1}{4}\left(M^{2}-a^{2}\right) .
\end{aligned}
$$

Thus, the Ernst potentials $\mathcal{E}$ can now be evaluated with the relations (6.30), (6.1):

$$
\begin{equation*}
\mathcal{E}=\frac{1-\xi}{1+\xi}=\frac{z^{2}-(M+i a) z+\frac{M^{2}-a^{2}}{4}}{z^{2}+(M-i a) z+\frac{M^{2}-a^{2}}{4}} \tag{6.59}
\end{equation*}
$$

recovering, then, the Ernst potential on the symmetry axis (5.2.2) for the Tomimatsu-Sato solution with $\delta=2 .{ }^{73}$

## 7 CONCLUSIONS

In this thesis, solutions to the Einstein-Maxwell equations have been discussed that possess a stationary and axisymmetric space-time away from the sources. The main focus has been the study of asymptotically flat solutions, but solutions including a cosmological constant have also been discussed briefly.The main goal has been to gain a deep understanding of the techniques that several authors use to construct binary systems and even cosmological models, which are of huge current interest in the context of observations of gravitational waves (Nobel Prize 2017) as well as motion around black holes residing in the center of galaxies (Nobel prize 2020 and the Event Horizon Telescope observations with the first ever foto of a black hole in 2019). We have discussed the Ernst formalism, which translates all physical information into two coupled systems of differential equations for two complex potentials. We discussed briefly the symmetries of these potentials and how they can be used to seek new solutions.

Also, as far as we know, this work presents the first full revision of Sibgatullin's integral method. Here we followed his ideas in constructing the method, and also discussed qualitatively the methods of other researchers which influenced Sibgatullin. As outlined in this thesis, the mathematical tools necessary for the construction of the method are quite varied, and interesting by themselves, ranging from complex analysis to classical integrability.

Applying Sibgatullin's method, we revised the family of solution introduced by Manko and Ruiz named "N-soliton solution" and its equilibrium equations. Moreover, we focused the method on the " 2 -soliton" case to give some detailed discussions emphasizing that not necessarily the 2 -soliton case and demonstrated that a " 2 -soliton" is not necessarily equal to the space-time of two bodies. Here, we discussed the parameters presented in the Tomimatsu-Sato $\delta=2$ solution in the vacuum and electrovacuum case, as well as the rotating charged magnetized sphere given by Manko.We also interpreted the parameters geometrically and discussed their physical meaning

We demonstrated how to construct exact solutions from their given multipole moments in the general " $N$-soliton" case. That is, this work extends the previous development of Manko and Ruiz to include electromagnetic fields. In this way, a direct link was made between the coefficients of the multipole expansion in General Relativity and the 3 N parameters of the $N$-soliton solution. This result has been summarized in the Lemmas of Chapter 6, and are extensions of the Lemma already presented in the literature by he above mentioned authors. Furthermore, the theorem presented in the same Chapter shows that any set of multipole moments, satisfying a not so restrict condition, can build a $N$-soliton solution. Thus, we conclude that a generic solution of Einstein's equation
coupled with electromagnetism, whose multipole moments satisfy this condition, can be approximated as a $N$-soliton solution.

Another result is given in Appendix C, in which a revision of the multipole moments existing in the literature is made. In this thesis, we give for the first time the correct formulae to evaluate them by means of the Fodor algorithm.

Recently, several authors ${ }^{58,70,93}$ have tried to fully describe binary systems which are relevant for direct gravitational wave measurements. However, several of these solutions describe only stationary spacetimes. Hence, the study of temporal perturbations around these exact solutions is an interesting topic in order to compare theoretical results with the experimental ones.

While we have discussed the electrovacuum case here, it would be interesting to see whether the inclusion of other fields (e.g. non-abelian gauge fields or fermionic fields) allow to develop a similar analytical machinery to produce solutions. Up to now, (mainly) numerical solutions are known. If a corresponding technique existed, one could check whether new solutions, e.g. of binary black holes carrying extra fields, so-called "hair", do exist and what their properties in comparsion to the electrovacuum solutions are.

## BIBLIOGRAPHY

1 LANDAU L.D.; LIFSHITZ, E. The classical theory of fields. 4th. ed. Amsterdam: Butterworth Heinemann, 1980. (The classical theory of fields, v. 2). ISBN 9780750627689.

2 MARTINI, R. Geometric aspects of the Einstein equations and integrable systems. Berlin: Springer-Verlag, 1985. (Lecture Notes in physics). ISBN 9783540160397.

3 STEPHANI, H.; KRAMER, D.; MACCALLUM, M.; HOENSELAERS, C.; HERLT, E. Exact solutions of Einstein's field equations. 2nd. ed. Cambridge: Cambridge University Press, 2003. (Cambridge Monographs on Mathematical Physics).

4 HOENSELAERS, C.; DIETZ, W. Solutions of Einstein's equations: techniques and results. Berlin: Springer-Verlag, 1984. (Lecture notes in physics, v. 2). ISBN 9783540133667.

5 ERNST, F. J. New formulation of the axially symmetric gravitational field problem. Physical Review, American Physical Society, v. 167, p. 1175-1178, 1968. DOI:10.1103/PhysRev.167.1175.

6 ERNST, F. J. New formulation of the axially symmetric gravitational field problem ii. Physical Review, American Physical Society, v. 168, p. 1415-1417, 1968. DOI:10.1103/PhysRev.168.1415.

7 GEROCH, R. A method for generating solutions of Einstein's equations. Journal of Mathematical Physics, v. 12, n. 6, p. 918-924, 1971. DOI:10.1063/1.1665681.

8 GEROCH, R. A method for generating new solutions of Einstein's equation. II. Journal of Mathematical Physics, v. 13, n. 3, p. 394-404, 1972. DOI:10.1063/1.1665990.

9 KINNERSLEY, W. Symmetries of the stationary Einstein-Maxwell field equations. I. Journal of Mathematical Physics, v. 18, n. 8, p. 1529-1537, 1977. DOI:10.1063/1.523458.

10 KINNERSLEY W.; CHITRE, D. M. Symmetries of the stationary Einstein-Maxwell field equations. II. Journal of Mathematical Physics, v. 18, n. 8, p. 1538-1542, 1977. DOI:10.1063/1.523459.

11 KINNERSLEY W.; CHITRE, D. M. Symmetries of the stationary Einstein-Maxwell field equations. III. Journal of Mathematical Physics, v. 19, n. 9, p. 1926-1931, 1978. DOI:10.1063/1.523912.

12 KINNERSLEY W.; CHITRE, D. M. Symmetries of the stationary Einstein-Maxwell equations. IV. transformations which preserve asymptotic flatness. Journal of Mathematical Physics, v. 19, n. 10, p. 2037-2042, 1978. DOI:10.1063/1.523580.

13 HOENSELAERS C.; KINNERSLEY, W. X. B. C. Symmetries of the stationary Einstein-Maxwell equations. VI. transformations which generate asymptotically flat spacetimes with arbitrary multipole moments. Journal of Mathematical Physics, v. 20, n. 12, p. 2530-2536, 1979. DOI:10.1063/1.524058.

14 KINNERSLEY, W. Symmetries of the stationary Einstein-Maxwell field equations. VII. charging transformations. Journal of Mathematical Physics, v. 21, n. 8, p. 2231-2235, 1980. DOI:10.1063/1.524657.

15 HAUSER I.; ERNST, F. J. Integral equation method for effecting Kinnersley-Chitre transformations. Physical Review D, American Physical Society, v. 20, p. 362-369, 1979. DOI:10.1103/PhysRevD.20.362.

16 HAUSER I.; ERNST, F. J. Integral equation method for effecting Kinnersley-Chitre transformations. II. Physical Review D, American Physical Society, v. 20, p. 1783-1790, 1979. DOI:10.1103/PhysRevD.20.1783.

17 HAUSER I.; ERNST, F. J. A homogeneous hilbert problem for the KinnersleyChitre transformations. Journal of Mathematical Physics, v. 21, n. 5, 1980. DOI:10.1063/1.524536.

18 HAUSER I.; ERNST, F. J. A homogeneous hilbert problem for the Kinnersley-Chitre transformations of electrovac space-times. Journal of Mathematical Physics, v. 21, n. 6,1980 . DOI:10.1063/1.524567.

19 HAUSER I.; ERNST, F. J. Proof of a geroch conjecture. Journal of Mathematical Physics, v. 22, n. 5, p. 1051-1063, 1981. DOI:10.1063/1.525012.

20 SIBGATULLIN, N. R. Construction of the general solution of a system of Einstein-Maxwell equations for the stationary axisymmetric case. Doklady Akademii Nauk SSSR, v. 278, p. 1098-1102, 1984.

21 SIBGATULLIN, N. R. On the theory of neutrino electrovacuum with Abelian motion group $G_{2}$ on $V_{2}$. Moscow University Mechanics Bulletin, Springer, v. 40, n. 2, p. 9-17, 1985. ISSN 0027-1330; 1934-8452/e.

22 SIBGATULLIN, N. R. Proof of the geroch conjecture for electromagnetic and neutrino fields in general relativity. Doklady Akademii Nauk SSSR, v. 271, p. 603-607, 1983.

23 ALEKSEEV, G. A. Thirty years of studies of integrable reductions of Einstein's field equations, 2010. Disponível em: <https://arxiv.org/pdf/1011.3846. pdf $>$. Acesso em: 23 jan. 2020.

24 MANKO V. S.; RUIZ, E. Extended multi-soliton solutions of the Einstein field equations. Classical and Quantum Gravity, IOP Publishing, v. 15, n. 7, p. 2007-2016, 1998. DOI:10.1088/0264-9381/15/7/015.

25 MANKO, V. S. Generating techniques and analytically extended solutions of the Einstein-Maxwell equations. General Relativity and Gravitation, v. 31, p. 673, 1999. DOI:10.1023/A:1026697129066.

26 MANKO, V. S.; MARTÍN, J.; RUIZ, E. Extended family of the electrovac two-soliton solutions for the Einstein-Maxwell equations. Physical Review D, American Physical Society, v. 51, p. 4187-4191, 1995. DOI:10.1103/PhysRevD.51.4187.

27 RUIZ, E.; MANKO, V. S.; MARTÍN, J. Extended n-soliton solution of the Einstein-Maxwell equations. Physical Review D, American Physical Society, v. 51, p. 4192-4197, 1995. DOI:10.1103/PhysRevD.51.4192.

28 BRETÓN, N.; MANKO, V. S.; SÁNCHEZ, J. A. On the equilibrium of charged masses in general relativity: the electrostatic case. Classical and Quantum Gravity, IOP Publishing, v. 15, n. 10, p. 3071-3083, 1998. DOI:10.1088/0264-9381/15/10/013.

29 BRETÓN, N.; MANKO, V. S.; SÁNCHEZ, J. A. On the equilibrium of charged masses in general relativity: II. the stationary electrovacuum case. Classical and Quantum Gravity, IOP Publishing, v. 16, n. 11, p. 3725-3734, 1999. DOI:10.1088/0264-9381/16/11/317.

30 KOMAR, A. Covariant conservation laws in general relativity. Physical Review, American Physical Society, v. 113, p. 934-936, 1959. DOI:10.1103/PhysRev.113.934.

31 GEROCH, R. Multipole moments. I. flat space. Journal of Mathematical Physics, v. 11, n. 6, p. 1955-1961, 1970. DOI:10.1063/1.1665348.

32 GEROCH, R. Multipole moments. II. curved space. Journal of Mathematical Physics, v. 11, n. 8, p. 2580-2588, 1970. DOI:10.1063/1.1665427.

33 HANSEN, R. O. Multipole moments of stationary space-times. Journal of Mathematical Physics, v. 15, n. 1, p. 46-52, 1974. DOI:10.1063/1.1666501.

34 SIMON, W. The multipole expansion of stationary Einstein-Maxwell fields. Journal of Mathematical Physics, v. 25, n. 4, p. 1035-1038, 1984. DOI:10.1063/1.526271.

35 SIMON, W.; BEIG, R. The multipole structure of stationary space-times. Journal of Mathematical Physics, v. 24, n. 5, p. 1163-1171, 1983. DOI:10.1063/1.525846.

36 FODOR, G.; HOENSELAERS, C.; PERJÉS, Z. Multipole moments of axisymmetric systems in relativity. Journal of Mathematical Physics, v. 30, n. 10, p. 2252-2257, 1989. DOI:10.1063/1.528551.

37 WALD, R. General Relativity. Chicago: University of Chicago Press, 1984. ISBN 9780226870328.

38 TOWNSEND, P. Black holes: lecture notes. 1997. Disponível em: <https: //arxiv.org/abs/gr-qc/9707012>. Acesso em: 23 jan. 2020.

39 CHANDRASEKHAR, S. The mathematical theory of black holes. New York: Clarendon Press, 1998. (International series of monographs on physics). ISBN 9780198503705.

40 PAPAPETROU, A. Champs gravitationnels stationnaires à symétrie axiale. Annales de l'I.H.P. Physique théorique, Gauthier-Villars, v. 4, n. 2, p. 83-105, 1966.

41 QUEVEDO, H. Multipole moments in general relativity - static and stationary vacuum solutions-. Fortschritte der Physik/ Protein Science, v. 38, p. 733-840, 1990.

42 SYNGE, J. Relativity: the general theory. Amsterdam: North-Holland Publishing Company, 1960. (North-Holland series in physics, v. 1).

43 SIBGATULLIN, N. R. Oscillations and waves in strong gravitational and electromagnetic fields. Berlin: Springer-Verlag, 1991. (Texts and monographs in physics). ISBN 9783540194613.

44 COSGROVE, C. M. Relationships between the group-theoretic and soliton-theoretic techniques for generating stationary axisymmetric gravitational solutions. Journal of Mathematical Physics, v. 21, n. 9, p. 2417-2447, 1980. DOI:10.1063/1.524680.

45 KORDAS, P. Aspects of solution-generating techniques for space-times with two commuting killing vectors. General Relativity and Gravitation, v. 31, p. 1941, 1999. DOI:10.1023/A:1026799124832.

46 JONES, T. C. Gravitational and electromagnetic potentials of the stationary Einstein-Maxwell field equations. Journal of Mathematical Physics, v. 21, n. 7, p. 1790-1797, 1980. DOI:10.1063/1.524631.

47 COSGROVE, C. M. Bäcklund transformations in the hauser-ernst formalism for stationary axisymmetric spacetimes. Journal of Mathematical Physics, v. 22, n. 11, p. 2624-2639, 1981. DOI:10.1063/1.524841.

48 HAUSER, I. On the homogeneous hilbert problem for effecting Kinnersley-Chitre transformations. In: HOENSELAERS, C.; DIETZ, W. (Ed.). Solutions of Einstein's equations: techniques and results. Berlin, Heidelberg: Springer, 1984. p. 128-175. ISBN 978-3-540-38922-4.

49 BABELON, O.; BERNARD, D.; TALON, M. Introduction to classical integrable systems. Cambridge: Cambridge University Press, 2003. (Cambridge monographs on mathematical physics). ISBN 9781139436793.

50 CAHILL, K. Physical mathematics. Cambridge: Cambridge University Press, 2013. ISBN 9781107310735.

51 MANKO, V. S.; SIBGATULLIN, N. R. Construction of exact solutions of the Einstein-Maxwell equations corresponding to a given behaviour of the ernst potentials on the symmetry axis. Classical and Quantum Gravity, v. 10, n. 7, p. 1383, 1993.

52 MUSKHELISHVILI, N. Singular integral equations: boundary problems of function theory and their application to mathematical physics. Mineola: Dover Publications, 2013. (Dover Books on Mathematics). ISBN 9780486145068.

53 ZAKHAROV, V.; SHABAT, A. Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. Functional Analysis and its Applications, Springer New York, v. 13, n. 3, p. 166-174, 1979. ISSN 0016-2663. DOI:10.1007/BF01077483.

54 ENCYCLOPEDIA of Mathematics. Cauchy integral. Accessible at: 11/05/2019. Disponível em: [https://www.encyclopediaofmath.org/index.php/Cauchy_integral](https://www.encyclopediaofmath.org/index.php/Cauchy_integral).

55 WOLFRAM Research. General Identities. Accessible at: 11/05/2019. Disponível em: [http://functions.wolfram.com/PDF/GeneralIdentities.pdf](http://functions.wolfram.com/PDF/GeneralIdentities.pdf).

56 MARKUSHEVICH, A.; SILVERMAN, R. Theory of Functions of a Complex Variable. Providence: AMS Chelsea Pub, 2013. ISBN 9780821837801.

57 ENCYCLOPEDIA of Mathematics. Sokhotskii formulas. Accessible at: 11 May 2019. Disponível em: <Availablefrom:https://www.encyclopediaofmath.org/index.php/ Sokhotskii_formulas>.

58 MANKO, V.; RUIZ, E. Metric for two arbitrary kerr sources. Physics Letters B, v. 794, p. $36-40$, 2019. DOI:https://doi.org/10.1016/j.physletb.2019.05.027.

59 MANKO, V. S.; RUIZ, E. Metric for two equal kerr black holes. Physical Review D, American Physical Society, v. 96, p. 104016, 2017. DOI:10.1103/PhysRevD.96.104016.

60 CABRERA-MUNGUIA, I.; LÄMMERZAHL, C.; MACÍAS, A. Exact solution for a binary system of unequal counter-rotating black holes. Classical and Quantum Gravity, v. 30, n. 17, p. 175020, 2013. DOI:10.1088/0264-9381/30/17/175020.

61 CABRERA-MUNGUIA, I. Hoyos negros extremos en modelos binarios axialsimétricos relativistas. 2010. Tese (Doutorado) - CINVESTAV, IPN, Mexico, MÉXICO, DISTRITO FEDERAL, 2010.

62 ERNST, F. J. Determining parameters of the Neugebauer family of vacuum spacetimes in terms of data specified on the symmetry axis. Physical Review D, American Physical Society, v. 50, p. 4993-4999, 1994. DOI:10.1103/PhysRevD.50.4993.

63 ERNST, F. J. Fully electrified Neugebauer spacetimes. Physical Review D, American Physical Society, v. 50, p. 6179-6189, 1994. DOI:10.1103/PhysRevD.50.6179.

64 NEUGEBAUER, G. A general integral of the axially symmetric stationary Einstein equations. Journal of Physics A, IOP Publishing, v. 13, n. 2, p. L19-L21, 1980. DOI:10.1088/0305-4470/13/2/003.

65 MANKO, V. S.; SANABRIA-GOMEZ, J. D.; MANKO, O. V. Nine parameter electrovac metric involving rational functions. Physical Review D, v. 62, p. 044048, 2000. DOI:10.1103/PhysRevD.62.044048.

66 HENNIG J.; NEUGEBAUER, G. Non-existence of stationary two-black-hole configurations. In: NOVELLO, M.; BERGLIAFFA, S. P.; RUFFINI, R. (Ed.). The twelfth Marcel Grossmann meeting. Paris: World Scientific, 2009. p. 1756-1758. DOI:10.1142/9789814374552_0317.

67 DIETZ, W.; HOENSELAERS, C. Two mass solutions of Einstein's vacuum equations: the double Kerr solution. Annals of Physics, v. 165, n. 2, p. 319 - 383, 1985. ISSN 0003-4916.

68 WOLFRAM RESEARCH. Mathematica, Version 11.3. Champaign, IL: Wolfram Research, Inc., 2018.

69 HOENSELAERS C.; PERJÉS, Z. Multipole moments of axisymmetric electrovacuum spacetimes. Classical and Quantum Gravity, IOP Publishing, v. 7, n. 10, p. 1819-1825, 1990. DOI:10.1088/0264-9381/7/10/012.

70 CABRERA-MUNGUIA, I. Unequal binary configurations of interacting Kerr black holes. Physics Letters B, v. 786, p. $466-471,2018$. ISSN 0370-2693. DOI:https://doi.org/10.1016/j.physletb.2018.10.037.

71 GRIFFITHS, J.; PODOLSKỲ, J. Exact space-times in Einstein's General Relativity. Cambridge: Cambridge University Press, 2009. (Cambridge monographs on mathematical physics). ISBN 9781139481168.

72 CABRERA-MUNGUIA, I. Binary system of unequal counterrotating kerr-newman sources. Physical Review D, American Physical Society, v. 91, p. 044005, 2015. DOI:10.1103/PhysRevD.91.044005.

73 TOMIMATSU, A.; SATO, H. New exact solution for the gravitational field of a spinning mass. Physical Review Letters, American Physical Society, v. 29, p. 1344-1345, 1972. DOI:10.1103/PhysRevLett.29.1344.

74 MANKO, O. V.; MANKO, V. S.; SANABRIA-GóMEZ, J. D. Charged, magnetized tomimatsu-sato $\delta=2$ solution. Progress of Theoretical Physics, v. 100, n. 3, p. 671-673, 1998. ISSN 0033-068X. DOI:10.1143/PTP.100.671.

75 MANKO, O.; MANKO, V.; SANABRIA-GOMEZ, J. Remarks on the charged, magnetized tomimatsu-sato $\delta=2$ solution. General Relativity and Gravitation, v. 31, p. 1539-1548, 1999. DOI:10.1023/A:1026782404418.

76 MANKO, V.; MEJÍA, I.; RUIZ, E. Metric of a rotating charged magnetized sphere. Physics Letters B, v. 803, n. 13, p. 135286, 2020. ISSN 0370-2693. DOI:https://doi.org/10.1016/j.physletb.2020.135286.

77 ERNST, F. J.; MANKO, V. S.; RUIZ, E. Equatorial symmetry/antisymmetry of stationary axisymmetric electrovac spacetimes. Classical and Quantum Gravity, IOP Publishing, v. 23, n. 15, p. 4945-4952, 2006. DOI:10.1088/0264-9381/23/15/013.

78 ERNST, F. J.; MANKO, V. S.; RUIZ, E. Equatorial symmetry/antisymmetry of stationary axisymmetric electrovac spacetimes: II. Classical and Quantum Gravity, IOP Publishing, v. 24, n. 9, p. 2193-2203, 2007. DOI:10.1088/0264-9381/24/9/003.

79 MANKO, V. S.; MARTíN, J.; RUIZ, E. Six-parameter solution of the Einstein-Maxwell equations possessing equatorial symmetry. Journal of Mathematical Physics, v. 36, n. 6, p. 3063-3073, 1995. DOI:10.1063/1.531012.

80 TOMIMATSU, A. On gravitational mass and angular momentum of two black holes in Equilibrium. Progress of Theoretical Physics, v. 70, n. 2, p. 385-393, 1983. ISSN 0033-068X. DOI:10.1143/PTP.70.385.

81 HOENSELAERS, C. Remarks on the double-Kerr solution. Progress of Theoretical Physics, v. 72, n. 4, p. 761-767, 1984. ISSN 0033-068X. DOI:10.1143/PTP.72.761.

82 KUNDU, P. On the analyticity of stationary gravitational fields at spatial infinity. Journal of Mathematical Physics, v. 22, n. 9, p. 2006-2011, 1981. DOI:10.1063/1.525148.

83 CARDOSO, V.; GUALTIERI, L. Testing the black hole 'no-hair' hypothesis. Classical and Quantum Gravity, IOP Publishing, v. 33, n. 17, p. 174001, 2016. DOI:10.1088/0264-9381/33/17/174001.

84 WILL, C. M. Testing the general relativistic "no-hair" theorems using the galactic center black hole sagittarius a*. The Astrophysical Journal, American Astronomical Society, v. 674, n. 1, p. L25-L28, 2008. DOI:10.1086/528847.

85 PAPPAS, G.; SOTIRIOU, T. P. Geodesic properties in terms of multipole moments in scalar-tensor theories of gravity. Monthly Notices of the Royal Astronomical Society, v. 453, n. 3, p. 2862-2876, 2015. ISSN 0035-8711. DOI:10.1093/mnras/stv1819.

86 PAPPAS, G.; SOTIRIOU, T. P. Multipole moments in scalar-tensor theory of gravity. Physical Review D, American Physical Society, v. 91, p. 044011, 2015. DOI:10.1103/PhysRevD.91.044011.

87 SOTIRIOU, T. P.; APOSTOLATOS, T. A. Multipole moments as a tool to infer from gravitational waves the geometry around an axisymmetric body. AIP Conference Proceedings, v. 861, n. 1, p. 756-761, 2006. DOI:10.1063/1.2399654.

88 RYAN, F. D. Accuracy of estimating the multipole moments of a massive body from the gravitational waves of a binary inspiral. Physical Review D, American Physical Society, v. 56, p. 1845-1855, 1997. DOI:10.1103/PhysRevD.56.1845.

89 BARACK, L.; CUTLER, C. Using lisa extreme-mass-ratio inspiral sources to test off-kerr deviations in the geometry of massive black holes. Physical Review D, American Physical Society, v. 75, p. 042003, 2007. DOI:10.1103/PhysRevD.75.042003.

90 BABAK, S.; GAIR, J.; SESANA, A.; BARAUSSE, E.; SOPUERTA, C. F.; BERRY, C. P. L.; BERTI, E.; AMARO-SEOANE, P.; PETITEAU, A.; KLEIN, A. Science with the space-based interferometer lisa. v. extreme mass-ratio inspirals. Physical Review D, American Physical Society, v. 95, p. 103012, 2017. DOI:10.1103/PhysRevD.95.103012.

91 BERTI, E.; BARAUSSE, E.; CARDOSO, V.; GUALTIERI, L.; PANI, P.;
SPERHAKE, U.; STEIN, L. C.; WEX, N.; YAGI, K.; BAKER, T.; BURGESS, C. P.; COELHO, F. S.; DONEVA, D.; FELICE, A. D.; FERREIRA, P. G.; FREIRE, P. C. C.; HEALY, J.; HERDEIRO, C.; HORBATSCH, M.; KLEIHAUS, B.; KLEIN, A.; KOKKOTAS, K.; KUNZ, J.; LAGUNA, P.; LANG, R. N.; LI, T. G. F.; LITTENBERG, T.; MATAS, A.; MIRSHEKARI, S.; OKAWA, H.; RADU, E.; O'SHAUGHNESSY, R.; SATHYAPRAKASH, B. S.; BROECK, C. V. D.; WINTHER, H. A.; WITEK, H.; AGHILI, M. E.; ALSING, J.; BOLEN, B.; BOMBELLI, L.; CAUDILL, S.; CHEN, L.; DEGOLLADO, J. C.; FUJITA, R.; GAO, C.; GEROSA, D.; KAMALI, S.; SILVA, H. O.; ROSA, J. G.; SADEGHIAN, L.; SAMPAIO, M.; SOTANI, H.; ZILHAO, M. Testing general relativity with present and future astrophysical observations. Classical and Quantum Gravity, IOP Publishing, v. 32, n. 24, p. 243001, 2015. DOI:10.1088/0264-9381/32/24/243001.

92 SOTIRIOU, T. P.; APOSTOLATOS, T. A. Corrections and comments on the multipole moments of axisymmetric electrovacuum spacetimes. Classical and Quantum Gravity, IOP Publishing, v. 21, n. 24, p. 5727-5733, 2004. DOI:10.1088/0264-9381/21/24/003.

93 CABRERA-MUNGUIA, I. Corotating dyonic binary black holes.
Physics Letters B, v. 811, p. 135945, 2020. ISSN 0370-2693.
DOI:https://doi.org/10.1016/j.physletb.2020.135945.
94 DOKTOROV, E.; LEBLE, S. A dressing method in mathematical physics. Netherlands: Springer, 2007. (Mathematical physics studies). ISBN 9781402061400.

95 NOVIKOV, S.; MANAKOV, S.; PITAEVSKII, L.; ZAKHAROV, V. Theory of Solitons: the inverse scattering method. New York: Springer, 1984. (Monographs in contemporary mathematics). ISBN 9780306109775.

96 NOGUCHI, J.; SOCIETY, A. M. Introduction to complex analysis. Tokyo: American Mathematical Society, 1998. (Mathematical surveys and monographs). ISBN 9780821803776.

97 ASTORINO, M. Charging axisymmetric space-times with cosmological constant. Journal of High Energy Physics, v. 06, p. 086, 2012. DOI:10.1007/JHEP06(2012)086.

98 CHARMOUSIS, C.; LANGLOIS, D.; STEER, D.; ZEGERS, R. Rotating spacetimes with a cosmological constant. Journal of High Energy Physics, Springer Science and Business Media LLC, v. 2007, n. 02, p. 064-064, 2007. DOI:10.1088/1126-6708/2007/02/064.

99 SÁNCHEZ, A.; MACÍAS, A.; QUEVEDO, H. Generating gowdy cosmological models. Journal of Mathematical Physics, v. 45, n. 5, p. 1849-1858, 2004. DOI:10.1063/1.1695448.

100 THORNE, K. S. Multipole expansions of gravitational radiation. Reviews of Modern Physics, American Physical Society, v. 52, p. 299-339, 1980.
DOI:10.1103/RevModPhys.52.299.
101 BEIG, R. The multipole expansion in general relativity. Acta Physica Austriaca, v. 53, n. 4, p. 249-270, 1981.

102 FODOR, G. Forgó testek nyomatékai az általános relativitáselméletben, master's thesis, in Hungarian. 1989. Dissertação (Mestrado) - Loránd Eötvös University, Budapest, 1989.

103 PERJÉS, Z. Gravitational multipole moments. In: NOVELLO, M.; BERGLIAFFA, S. P.; RUFFINI, R. (Ed.). The tenth Marcel Grossmann meeting (2003). Rio de Janeiro: World Scientific, 2006. p. 59-69.

104 CARNEIRO, D.; FREIRAS, E.; GONCALVES, B.; LIMA, A. de; SHAPIRO, I. On useful conformal tranformations in general relativity. Gravitation and Cosmology, v. 10, p. 305-312, 2004. Disponível em: [https://arxiv.org/pdf/gr-qc/0412113.pdf](https://arxiv.org/pdf/gr-qc/0412113.pdf).

105 FILTER, R. Multipolmomente axialsymmetrisch stationärer Raumzeiten und die Quadrupol-Vermutung. 2008. Dissertação (Mestrado) - Friedrich-SchillerUniversität Jena, Jena, 2008.

106 TRAUTMAN, A. Introduction to gravitational radiation theory. In: BONDI, H.; TRAUTMAN, A.; PIRANI, F. A. E. (Ed.). Lectures on General Relativity. New Jersey: Prentice-Hall, 1965, (Brandeis summer institute in theoretical physics). p. 249-373.

## Appendix

## APPENDIX A - THE RIEMANN HILBERT PROBLEM

The Riemann Hilbert problem (RHP), a classical problem in the theory of functions of a complex variable, holds a central position in the formalism which was used in chapter 4. More details of the construction are given in this appendix together with justifications for some hypothesis that have been assumed.

## A. 1 The Riemann Hilbert Problem

In the complex plane $\mathbb{C}(\infty$ being included $)$ of variable $s^{*}$, let there be any smooth closed contour $L$ surrounding the origin. Consider an $n \times n$ matrix function $G(s)$ defined on $L$, which satisfies a Hölder condition and $\operatorname{det} G(s) \neq 0$, both on $L .{ }^{94}$ The Riemann-Hilbert problem says that $G$ might be factorized as follows. It is required to construct an $n \times n$ matrix function $\chi_{+}(s)$ homomorphic in the region inside $L$, namely $L_{+}$, and an $n \times n$ matrix function $\chi_{-}(s)$ homomorphic in the region outside $L$, namely $L_{-}$, such that:

$$
\begin{equation*}
G(s)=\chi_{+}(s)^{-1} \chi_{-}(s) \quad s \in L \tag{A.1}
\end{equation*}
$$

Let the functions $\chi_{+}(s)$ and $\chi_{-}(s)$ be nondegenerate in their domains. It is straightforward to see that the solution of the RHP is not unique, since if $\chi_{ \pm}(s)$ is a solution, and $g$ is an arbitrary non-degenerate matrix function which does not depend on $s$, then the pair $g \chi_{ \pm}(s)$ will also be a solution of the same RHP. To provide the uniqueness of the solution, it is needed to normalize the problem giving a value of $\chi_{+}(s)$ or $\chi_{-}(s)$ at some point in its domain. In fact, suppose that $\chi_{ \pm}^{(1)}$ and $\chi_{ \pm}^{(2)}$ are both solutions. Then, the equation (A.1) implies on the contour $\left[\chi_{+}(s)^{(1)}\right]^{-1} \chi_{-}(s)^{(1)}=\left[\chi_{+}(s)^{(2)}\right]^{-1} \chi_{-}(s)^{(2)}$. Consider now the function ${ }^{95}$ :

$$
\begin{equation*}
\psi(s)=\chi_{+}(s)^{(2)}\left[\chi_{+}(s)^{(1)}\right]^{-1}=\chi_{-}(s)^{(2)}\left[\chi_{-}(s)^{(1)}\right]^{-1} \tag{A.2}
\end{equation*}
$$

Since this function can be continued analytically from $L$ to the entire complex plane, then, by the Liouville theorem, ${ }^{96} \psi$ must be a constant. Consequently, after normalization, it can be set $\psi=\mathbb{1}$.

## A. 2 Generation of New Solutions

Consider the overdetermined system of differential equations:

$$
\begin{equation*}
F_{{ }_{x}}=U F \quad F_{, t}=V F \tag{A.3}
\end{equation*}
$$

[^11]which is known as the Lax representation (or the zero-curvature representation). ${ }^{94} \mathrm{~A}$ nonlinear equation is said to be integrable when it is possible to associate it with a pair of Lax equations. Here, the matrices $U(x, t, s)$ and $V(x, t, s)$ depend on a solution of the nonlinear equation (in fact, they determines all features of this given nonlinear equation) and, also, on the spectral parameter $s$. The compatibility condition for (A.3) resulting from the equality of mixed derivatives $F,_{x t}=F,{ }_{t x}$, reads:
\[

$$
\begin{equation*}
U_{, t}-V_{, x}+[U, V]=0 \tag{A.4}
\end{equation*}
$$

\]

From the equations above it is possible to infer a gauge invariance. Consider $\stackrel{\circ}{U}$ and $\stackrel{\circ}{V}$ to be solutions of the system (A.4), and $\stackrel{\circ}{F}$ be the corresponding solution of (A.3). Consider the functions ${ }^{95}$ :

$$
\begin{equation*}
U=g \stackrel{\circ}{U} g^{-1}+g,_{x} g^{-1} \quad V=g \stackrel{\circ}{V} g^{-1}+g, t g^{-1} \tag{A.5}
\end{equation*}
$$

Where $g$ is an arbitrary nondegenerate matrix. From a straightforward calculation, one gets that $U$ and $V$ once again satisfy the equation (A.4) and the corresponding solution of the system (A.3) is $F=g \stackrel{\circ}{F}$. All of the solutions that differ from each other by a certain matrix $g$ are said to be gauge equivalent. This invariance of the equation of zero-curvature is an extremely important property. Using this, it is possible to obtain, based on the Riemann Hilbert problem, a class of new solutions of the nonlinear equation involved. Suppose a seed solution is known from the nonlinear equation, then the functions $\stackrel{\circ}{U}, \stackrel{\circ}{V}$ and $\stackrel{\circ}{F}$, associated with this, have an explicit form.

Therefore, we can use the RHP (A.1) to write the matrix $G(x, t, s)$ on the contour $L$ as:

$$
\begin{equation*}
G(x, t, s)=\chi_{+}^{-1}(x, t, s) \chi_{-}(x, t, s) \quad s \in L \tag{A.6}
\end{equation*}
$$

Then we can factorize the matrix function $G(x, t, s)$ on $L$ in terms of a nondegenerate bounded matrix function $G_{0}(s)$, holomorphic in $L_{-}$, for all $x$ and $t$, as ${ }^{53}$ :

$$
\begin{equation*}
G(x, t, s)=\stackrel{\circ}{F}(x, t, s) G_{0}(s) \stackrel{\circ}{F}^{-1}(x, t, s) \quad s \in L \tag{A.7}
\end{equation*}
$$

As a consequence of the above equation one gets that the derivative of $G$ with respect to $x$ can be expressed in terms of a commutator with a seed function: $G, x=[\stackrel{\circ}{U}, F]$. Combining (A.1) and (A.7), after differentiating with respect to x and using (A.3), we find that:

$$
\begin{equation*}
\chi_{+}^{-1},{ }_{x} \chi_{-}+\chi_{+}^{-1} \chi_{-, x}=\stackrel{\circ}{U} \chi_{+}^{-1} \chi_{-}-\chi_{+}^{-1} \chi_{-} \stackrel{\circ}{U} \tag{A.8}
\end{equation*}
$$

Thus, defining the matrix-valued function $U$ by the formula:

$$
\begin{equation*}
U=\chi_{-} \stackrel{\circ}{U} \chi_{-}^{-1}+\chi_{-, x} \chi_{-}^{-1}=\chi_{+} U_{0} \chi_{+}^{-1}+\chi_{+, x} \chi_{+}^{-1} \tag{A.9}
\end{equation*}
$$

Similarly, it is possible to define V as:

$$
\begin{equation*}
V=\chi_{-} \stackrel{\circ}{V} \chi_{-}^{-1}+\chi_{-, t} \chi_{-}^{-1}=\chi_{+} \stackrel{\circ}{V} \chi_{+}^{-1}+\chi_{+, t} \chi_{+}^{-1} \tag{A.10}
\end{equation*}
$$

Expressions (A.9) and (A.10) show that $U, V$ can be continued from $L$ to the whole complex plane and that their poles are the same as the poles of $\stackrel{\circ}{U}$ and $\stackrel{\circ}{V}$, respectively, if the poles do not lie on the contour. The last step is to define $F_{ \pm}=\chi_{ \pm} \stackrel{\circ}{F}$ which satisfies the equation (A.3), meaning $F_{ \pm, x}=U F_{ \pm}$and $F_{ \pm, t}=V F_{ \pm}$. Thus, in this context, the RHP appears as a gauge transformation generating new solutions. It leads to the conclusion that the matrices $U$ and $V$ depend on the new solution in the same way as $U_{0}$ and $V_{0}$ depend on the seed solution and, above all, it should be noticed that the new solution $F_{ \pm}$ has the same poles in the $s$-plane as the seed function $\stackrel{\circ}{F}$.

Consider now, that $U$ and $V$ are $n \times n$ complex-valued matrices depending rationally on the spectral parameter $s$ with simple poles. If the number of poles of $U$, with their multiplicities, is $N_{1}$, while those of the function $V$ are $N_{2}$, it follows from the compatibility equation (A.4) that $U$ and $V$ have $N_{1}+N_{2}+2$ independent matrix functional parameters:

$$
\begin{equation*}
U=u_{0}+\sum_{k=1}^{N_{1}} \frac{u_{k}}{s-a_{k}} ; \quad V=v_{0}+\sum_{k=1}^{N_{2}} \frac{v_{k}}{s-b_{k}} . \tag{A.11}
\end{equation*}
$$

Hence:

$$
\begin{align*}
& u_{0, t}-v_{0}, x_{x}+\left[u_{0}, v_{0}\right]=0  \tag{A.12}\\
& u_{n, t}+\left[u_{n}, R_{n}\right]=0  \tag{A.13}\\
& v_{n},_{x}+\left[v_{n}, T_{n}\right]=0 \tag{A.14}
\end{align*}
$$

Where:

$$
\begin{equation*}
R_{n}=v_{0}+\sum_{k=1}^{N_{2}} \frac{v_{k}}{a_{n}-b_{k}} ; \quad T_{n}=u_{0}+\sum_{k=1}^{N_{1}} \frac{u_{k}}{b_{n}-a_{k}} . \tag{A.15}
\end{equation*}
$$

In application of the gauge freedom defined in (A.5), transforming $\stackrel{\circ}{U}, \stackrel{\circ}{V}$ to $U, V$, implies that:

$$
\begin{array}{ll}
u_{0}=g \stackrel{\circ}{u}_{0} g^{-1}+g,_{x} g^{-1} & u_{n}=g \circ_{n} g^{-1} \\
v_{0}=g \stackrel{\circ}{v}_{0} g^{-1}+g, t g^{-1} & v_{n}=g \stackrel{\circ}{v}_{n} g^{-1} \tag{A.17}
\end{array}
$$

Consequently, due to the gauge factor, it is possible to take $u_{0}=0$ and $v_{0}=0$.
In a general way, these techniques broadly used in soliton physics are applied to the problem of solving non-linear differential equations in General Relativity (the components of the Einstein equation). The idea of this technique is to solve a pair of linear differential equations (Lax-Pair), that is, it is equivalent to solve $\psi$ for $U$ and $V$. Algebraic operations or singular integrals are examples of techniques that might be useful to solve such system. In the Sibgatullin Integral method, in which this problem will be used, $G(s)$ is holomorphic in a neighbourhood of $L$. Then, equation (A.1) can be analytically continued to some region of the complex plane. ${ }^{17}$ Actually, Sibgatullin's method is so specialized that it can be given without the use of contour, , i.e. in that sense it exploits the holomorphicity of the function $\chi_{-}$in its domains.

## APPENDIX B - ERNST POTENTIALS IN THE PRESENCE OF A COSMOLOGICAL CONSTANT

The Ernst formalism has been shown to be a powerful method to construct several non-trivial solutions of the Einstein-Maxwell equations and a very useful tool to study their mathematical and physical properties. Here, we will explore an equivalent formalism, but introduce a cosmological constant into the Einstein equation. As will be shown later, the system is no longer integrable when the cosmological constant is added.

The focus is now on stationary axisymmetric electromagnetic potentials whose metric can be generically written as [97]:

$$
\begin{equation*}
d s^{2}=\alpha e^{\frac{\lambda}{2}}(d t-\omega d \phi)^{2}-\alpha e^{\frac{-\lambda}{2}} d \phi^{2}-\frac{e^{2 \nu}}{\sqrt{\alpha}}\left(d x_{1}^{2}+d x_{2}^{2}\right) \tag{B.1}
\end{equation*}
$$

Just as before, the electromagnetic potential components $A_{t}$ and $A_{\phi}$, and the metric functions $\lambda, \omega, \alpha$ and $\nu$ depend just on $\left(x_{1}, x_{2}\right)$ coordinates. Note that the metric components (B.1) differ from the original Weyl-Papapetrou ones. In fact, the Weyl potential $f$ is now given by $f=\alpha e^{\frac{\lambda}{2}}$. In rather simple terms, integrability breaks down because equations which were homogeneous for $\Lambda=0$ become inhomogeneous when $\Lambda \neq 0$ (for instance, compare with the equation (2.26)).

Thus, the Einstein and Maxwell equations can be written as:

$$
\begin{align*}
& \vec{\nabla} \cdot\left[e^{-\frac{\lambda}{2}} \vec{\nabla} A_{t}+\omega e^{\frac{\lambda}{2}}\left(\vec{\nabla} A_{\phi}-\omega \vec{\nabla} A_{t}\right)\right]=0  \tag{B.2}\\
& \vec{\nabla} \cdot\left[e^{\frac{\lambda}{2}}\left(\vec{\nabla} A_{\phi}-\omega \vec{\nabla} A_{t}\right)\right]=0  \tag{B.3}\\
& \nabla^{2} \alpha+2 \Lambda \sqrt{\alpha} 2 e^{\nu}=0  \tag{B.4}\\
& 4\left[e^{-\frac{\lambda}{2}}\left(\vec{\nabla} A_{t}\right)^{2}+e^{\frac{\lambda}{2}}\left(\left(\vec{\nabla} A_{\phi}\right)^{2}-\omega^{2}\left(\vec{\nabla} A_{t}\right)^{2}\right)\right]-\vec{\nabla} \cdot(\alpha \vec{\nabla} \lambda)- \\
& -2 e^{\frac{\lambda}{2}} \alpha(\vec{\nabla} \omega)^{2}-2 \omega \vec{\nabla} \cdot\left(e^{\frac{\lambda}{2}} \alpha \vec{\nabla} \omega\right)=0  \tag{B.5}\\
& 4 e^{\frac{\lambda}{2}}\left[\omega\left(\vec{\nabla} A_{t}\right)^{2}-\vec{\nabla} A_{t} \cdot \vec{\nabla} A_{\phi}\right]+\vec{\nabla} \cdot\left(\alpha e^{\lambda} \vec{\nabla} \omega\right)=0 \tag{B.6}
\end{align*}
$$

Where $\vec{\nabla} f=\left(\partial_{x_{1}} f, \partial_{x_{2}} f\right)$. Furthermore, when $\Lambda=0$ the component $\alpha$ is harmonic and then the pair $\left(x_{1}, x_{2}\right)$ becomes $(\rho, z)$. In fact, the function $\alpha$ becomes harmonic, $\nabla^{2} \alpha=\left(\nabla_{x_{1}}^{2}+\nabla_{x_{2}}^{2}\right) \alpha=0$, hence one can set $\alpha=\rho$.

One should notice that the equations above do not depend of the function $\nu$, except for equation (B.4). Following Ernst's construction,, ${ }^{6}$ the equation (B.3) may be regarded
as the integrability condition for a magnetic scalar potential $\tilde{A}$ such that:

$$
\begin{equation*}
\hat{e}_{\phi} \times \vec{\nabla} \tilde{A} \equiv e^{\frac{\lambda}{2}}\left(\vec{\nabla} A_{\phi}-\omega \vec{\nabla} A_{t}\right) \tag{B.7}
\end{equation*}
$$

In view of the definition of $\tilde{A}$ it is possible to rewrite (B.2) and (B.3) as:

$$
\begin{equation*}
\vec{\nabla} \cdot\left[e^{-\frac{\lambda}{2}} \vec{\nabla} \tilde{A}-\omega \hat{e}_{\phi} \times \vec{\nabla} A_{t}\right]=0 ; \quad \vec{\nabla} \cdot\left[e^{-\frac{\lambda}{2}} \vec{\nabla} A_{t}+\omega \hat{e}_{\phi} \times \vec{\nabla} \tilde{A}\right]=0 \tag{B.8}
\end{equation*}
$$

Just as in [6] it is possible to introduce a new potential $\Phi=A_{t}+i \tilde{A}$, that allows all the Maxwell equations to be cast in a unique complex equation, where (B.2) is is the real part and and (B.3) is the imaginary part:

$$
\begin{equation*}
\vec{\nabla} \cdot\left[e^{-\frac{\lambda}{2}} \vec{\nabla} \Phi-i \omega \hat{e}_{\phi} \times \vec{\nabla} \Phi\right]=0 \tag{B.9}
\end{equation*}
$$

Rewriting the equation (B.6) in terms of $\Phi$, it takes the form:

$$
\begin{equation*}
\nabla \cdot\left[e^{\lambda} \alpha \vec{\nabla} \omega-2 \hat{e}_{\phi} \times \operatorname{Im}\left(\Phi^{*} \vec{\nabla} \Phi\right)\right]=0 \tag{B.10}
\end{equation*}
$$

Once again, this equation may be considered as an integrability condition for the existence of a new potential $\Omega$ such that:

$$
\begin{equation*}
\hat{e}_{\phi} \times \vec{\nabla} \Omega \equiv e^{\lambda} \alpha \vec{\nabla} \omega-2 \hat{e}_{\phi} \times \operatorname{Im}\left(\Phi^{*} \vec{\nabla} \Phi\right) \tag{B.11}
\end{equation*}
$$

Hence, in terms of $\Omega$ and $\Phi$, (B.6) can be written as:

$$
\begin{equation*}
\vec{\nabla} \cdot\left[\frac{e^{-\lambda}}{\alpha}\left(\vec{\nabla} \Omega+2 \operatorname{Im}\left(\Phi^{*} \vec{\nabla} \Phi\right)\right)\right]=0 \tag{B.12}
\end{equation*}
$$

If one introduces the complex function $\mathcal{E}=\alpha e^{\frac{\lambda}{2}}-|\Phi|^{2}+i \Omega,{ }^{97}$ it is possible to write a pair of coupled complex equations which summarizes the components of the Einstein equation (B.6) and (B.5) and the Maxwell equations (B.3) and (B.2) as:

$$
\begin{align*}
& \left(\operatorname{Re} \mathcal{E}+|\Phi|^{2}\right) \frac{1}{\alpha} \vec{\nabla} \cdot(\alpha \vec{\nabla} \mathcal{E})=\left(\vec{\nabla} \mathcal{E}+2 \Phi^{*} \vec{\nabla} \Phi\right) \cdot \vec{\nabla} \mathcal{E}+\operatorname{Re}^{2}\left(\mathcal{E}+|\Phi|^{2}\right) \frac{\nabla^{2} \alpha}{\alpha}  \tag{B.13}\\
& \left(\operatorname{Re\mathcal {E}}+|\Phi|^{2}\right) \frac{1}{\alpha} \vec{\nabla} \cdot(\alpha \vec{\nabla} \Phi)=\left(\vec{\nabla} \mathcal{E}+2 \Phi^{*} \vec{\nabla} \Phi\right) \cdot \vec{\nabla} \Phi \tag{B.14}
\end{align*}
$$

The equations (B.13) and (B.14) represent the natural generalisation of Ernst's equations ${ }^{6}$ to the case with cosmological constant, so they reduce to the Ernst equations when $\Lambda \rightarrow 0$. It was shown in [98] that it is possible to map a solution in spacetime with $\Lambda=0$ to another with $\Lambda \neq 0$ in the vacuum case. In fact, some authors ${ }^{99}$ have even
used Sibgatullin's method to accomplish it. So, a deep study using the Ernst potentials in presence of the cosmological constant might give some ideas on how to analytically construct solutions based in integrable models.

## APPENDIX C - MULTIPOLE MOMENTS

In General Relativity, it is assumed that a given solution of the Einstein (-Maxwell) field equations in (electro-) vacuum is produced by a source. In a general way, it is not known how to connect a given solution of the field equations in the exterior of the source with field equations in the interior of the object and hence with the physical features of the object which produces such exterior field. ${ }^{37}$ In such a context, the multipole expansion becomes necessary in order to describe the solutions physically.

In Newtonian gravity, the multipole moments may be defined in two ways ${ }^{100}$ :

- as integrals over the mass distribution.
- as coefficients of a multipole expansion.

Working in asymptotically flat static space-times in vacuum, Geroch introduced a tensorial definition of multipole moments. ${ }^{31,32}$ Such metrics have as a solution a potential that satisfies, as in the Newtonian case, the Laplace equation $\nabla^{2} \Phi=0$. The Newtonian multipoles resulting from this equation are a set of totally symmetric, trace-free tensors. He defined the moments in a coordinate-independent way and such that they tend to the respective Newtonian multipoles in the limit of weak gravitational fields. Hansen ${ }^{33}$ extended Geroch's idea to the stationary case, and subsequently, Simon ${ }^{34}$ extended it to the electrovacuum case.

Firstly, it is necessary to introduce a definition of asymptotic flatness which is coordinate-independent. Hansen worked with the 3-dimensional manifold $S$ generated by the orbits of the time-like Killing vector, whose metric

$$
\begin{equation*}
h_{i j}=-g_{00} g_{i j}+g_{0 i} g_{0 j} \tag{C.1}
\end{equation*}
$$

is positive definite.
An asymptotic-flatness coordinate-invariant alternative introduced by GerochHansen consists in adding a point at infinity, $\Lambda$, into the manifold $S$ by performing a conformal transformation. In other words, the asymptotic properties of $S$ are locally determined at the point $\Lambda$ in the new manifold $\tilde{S} .{ }^{32,33,37}$

Definition: A 3-dimensional manifold $S$ is called asymptotically flat if there exists a manifold $\tilde{S}$ with metric $\tilde{h}_{i j}$ and a conformal factor $\Omega$, such that the diffeomorphic map $\psi: S \rightarrow \tilde{S}$ is given by $\tilde{h}_{i j}=\Omega^{2} \psi h_{i j}$, and satisfies the following conditions:

- $\tilde{S}=S \cup \Lambda$
- $\tilde{h}_{i j}=\Omega^{2} h_{i j}$
- at $\Lambda: \Omega=0, \tilde{D}_{i} \Omega=0, \tilde{D}_{i} \tilde{D}_{j} \Omega=2 h_{i j}$
where $\tilde{D}_{i}$ is the covariant derivative associated with $\tilde{h}_{i j}$. The conditions above ensure that $\Omega$ tends to zero fast enough when the coordinates go to the point $\Lambda$. It also ensures that the conformal metric is also flat at the point $\Lambda$. The conformal factor must be chosen so that the space-like Killing vector in $S$ keeps being in $\tilde{S}$.

By introducing a scalar field $\phi$ defined in $S$, the Geroch-Hansen multipoles are defined as a collection of trace-free and totally symmetric tensors written in terms of the derivatives of $\tilde{\phi}=\Omega^{-1 / 2} \phi$ evaluated at $\Lambda$ in $\tilde{S}$, which are constructed recursively from ${ }^{32-34}$ :

$$
\begin{align*}
& P^{(0)}=\left.\tilde{\phi}\right|_{\Lambda}  \tag{C.2}\\
& P_{i}^{(1)}=\left.\tilde{D}_{i} \tilde{\phi}\right|_{\Lambda}  \tag{C.3}\\
& P_{i_{1} i_{2} \cdots i_{n+1}}^{(n+1)}=\left.\mathcal{T} \mathcal{S}\left(\tilde{D}_{i_{n+1}} P_{i_{1} i_{2} \cdots i_{n}}^{(n)}-\frac{1}{2} n(2 n-1) \tilde{R}_{i_{1} i_{2}} P_{i_{3} i_{4} \cdots i_{n+1}}^{(n-1)}\right)\right|_{\Lambda} \tag{C.4}
\end{align*}
$$

Here, the symbols $\mathcal{T}$ and $\mathcal{S}$ denote the operation of taking the trace-free and symmetric parts, respectively. $\tilde{R}_{i j}$ and $\tilde{D}_{i}$ are the Ricci tensor and the covariant derivative associated with the conformal metric $\tilde{h}_{i j}$, respectively. Thus, Hansen provided a formal definition formal definition of multipole moments of a stationary asymptotically flat spacetime. Moreover, it should be emphasized that the multipoles moments in GR, as defined by Geroch, do not have a direct link with the source since it is defined in a region far away from the body instead of an integral over it.

Such a definition of multipole moments does not make any reference to the Einstein equation, different from what one could expect from the Newtonian case. Any function $\phi$ could be used such that its conformal transformation is smooth in $\Lambda$. For instance, if $\phi$ is a solution of the Laplace equation in a 3-dimensional space, one recovers a collection of multipoles for the classical potential. In other words, since the Newton equation (which leads to the Laplace equation) ensures the holomorphic conditions of $\phi$, one should look for the Einstein equations to choose a suitable function $\phi^{41}$ Hansen ${ }^{33}$ showed that if $\phi$ is given by the Einstein's field equation, and if $\tilde{\phi}$ is continuous at $\Lambda$, then equation C. 4 in fact defines the multipole moments for stationary spacetimes.

Also, the condition for asymptotically-flatness do not uniquely determines the conformal factor $\Omega$. That is, one can choose another conformal factor such as $\tilde{\Omega}=\tilde{\omega} \Omega$, where $\tilde{\omega}$ is any smooth function which satisfies $\left.\tilde{\omega}\right|_{\Lambda}=1$. Under this transformation, the
multipole moments tensors transform according to ${ }^{33,101,102}$ :

$$
\begin{equation*}
\left.\tilde{P}_{i_{1} i_{2} \cdots i_{n}}^{(n)}\right|_{\Lambda}=\left.P_{i_{1} i_{2} \cdots i_{n}}^{(n)}\right|_{\Lambda}+\left.\mathcal{T S} \sum_{k=0}^{n-1}\binom{n}{k} \frac{(2 n-1)!!}{(2 k-1)!!}(-2)^{k-n} P_{i_{1} i_{2} \cdots i_{k}}^{(k)}\left(\partial_{i_{k+1}} \tilde{\omega}\right) \cdots\left(\partial_{i_{n}} \tilde{\omega}\right)\right|_{\Lambda} \tag{C.5}
\end{equation*}
$$

Some interesting features arises from here. First, notice that the change of the multipole tensors depend only on $\left.\left(\partial_{i} \tilde{\omega}\right)\right|_{\Lambda}$, which is a vector evaluated at a single point. Also, the change in the $n$th multipole depends on the multipoles of inferior order. Moreover, Geroch ${ }^{32}$ showed that the Newtonian multipoles change in the exactly same way. Finally, by setting $\left.\tilde{\omega}\right|_{\Lambda}=1$ and $\left.\left(\partial_{i} \tilde{\omega}\right)\right|_{\Lambda}=0$, the multipole moments are keep invariant.

## C. 1 Axisymmetric electrovacuum

Considering axial symmetry, if the symmetry axis passes through $\Lambda(\rho=0)$, then the multipole moments must be invariant under the action of the axial space-like Killing vector (since it defines rotations). In order to remain invariant under the action of the axial Killing vector at $\Lambda$, the multipole tensors must necessarily be multiples of the outer product of the axis vector $e^{a}$ with itself. ${ }^{33}$ That is:

$$
\begin{equation*}
P_{i_{1} \cdots i_{n}}^{(n)}=\left.P_{n} \mathcal{S}\left[e_{i_{1}} \cdots e_{i_{n}}\right]\right|_{\Lambda} \tag{C.6}
\end{equation*}
$$

Thus, it is possible to define the scalar moments as being:

$$
\begin{equation*}
P_{n}=\left.\frac{1}{n!} P_{i_{1} \cdots i_{n}}^{(n)} e^{i_{1}} \cdots e^{i_{n}}\right|_{\Lambda} \tag{C.7}
\end{equation*}
$$

Consequently, the scalar moments are just:

$$
\begin{equation*}
P_{n}=\left.\frac{1}{n!} P_{22 \ldots 2}^{(n)}\right|_{\Lambda} \tag{C.8}
\end{equation*}
$$

Beig ${ }^{35}$ and Simon ${ }^{34}$ have shown that the Ernst potentials $\xi$ and $q$ are good scalar fields to be considered in the study for gravitational and electromagnetic multipoles since they are holomorphic functions at infinity and because knowing their behaviour over the symmetry axis is sufficient to analytically continue them into the whole space. ${ }^{19}$ The choice $\phi=\xi$ gives the gravitational moment tensors $P_{i_{1} \cdots i_{n}}^{(n)}$, while the choice $\phi=q$ yields the electromagnetic moments $Q_{i_{1} \cdots i_{n}}^{(n)}$. Fodor ${ }^{36}$ first introduced a simple algorithm to compute the multipole moments in vacuum which is given in section C.2, and then Hoenselaers ${ }^{69}$ extended it for the electromagnetic case. However, two mistakes appear in reference [69]: one in the derivatives of the Ernst potentials and the other in the recurrence formula for the Taylor expansion of the potentials. The first mistake was corrected by Sotiriou, ${ }^{92}$ but the second has gone unnoticed by him. While the second was corrected by Perjés, ${ }^{103}$ but
unfortunately he unfortunately didn't realize the first mistake. Therefore, hopefully, the correct formulae for the multipole moments are given in the present text.

The metric for the 3 -dimensional manifold $S$ is:

$$
h_{i j}(\rho, z, \varphi)=\left(\begin{array}{ccc}
e^{2 \gamma} & 0 & 0  \tag{C.9}\\
0 & e^{2 \gamma} & 0 \\
0 & 0 & \rho^{2}
\end{array}\right)
$$

The field equations for this coordinate system are:

$$
\begin{align*}
& \left(\xi \xi^{*}-q q^{*}-1\right) \nabla^{2} \xi=2\left(\xi^{*} \nabla \xi-q^{*} \nabla q\right) \cdot \nabla \xi  \tag{C.10}\\
& \left(\xi \xi^{*}-q q^{*}-1\right) \nabla^{2} q=2\left(\xi^{*} \nabla \xi-q^{*} \nabla q\right) \cdot \nabla q \tag{C.11}
\end{align*}
$$

The nonzero components of the Ricci tensor for the metric $h_{i j}$ are written as:

$$
\begin{align*}
& R_{11}=-\gamma, \rho \rho-\gamma, z z+\frac{\gamma, \rho}{\rho}  \tag{C.12}\\
& R_{21}=\frac{\gamma, z}{\rho}  \tag{C.13}\\
& R_{22}=-\gamma, \rho \rho-\gamma, z z-\frac{\gamma, \rho}{\rho} . \tag{C.14}
\end{align*}
$$

By using the equations (3.19), (3.20) and (3.21), the Ricci tensor can be written in the simple formula ${ }^{69}$ :

$$
\begin{equation*}
R_{i k}=\left(\xi \xi^{*}-q q^{*}-1\right)^{-2}\left(\partial_{i} \xi \partial_{k} \xi^{*}+\partial_{i} \xi^{*} \partial_{k} \xi-\partial_{i} q \partial_{k} q^{*}-\partial_{i} q^{*} \partial_{k} q+s_{i} s_{k}^{*}+s_{i}^{*} s_{k}\right) \tag{C.15}
\end{equation*}
$$

where $s_{i}=\xi \partial_{i} q-q \partial_{i} \xi$. The indices, $i, j$, here take values 1 or 2 and the other components of the Ricci tensor are zero.

A useful coordinate transformation to manipulate the point $\Lambda$ is ${ }^{36}$ :

$$
\begin{align*}
\bar{\rho} & =\frac{\rho}{\rho^{2}+z^{2}},  \tag{C.16}\\
\bar{z} & =\frac{z}{\rho^{2}+z^{2}},  \tag{C.17}\\
\bar{\varphi} & =\varphi . \tag{C.18}
\end{align*}
$$

In such coordinate system the metric is written as:

$$
h_{\overline{i j}}(\bar{\rho}, \bar{z}, \bar{\varphi})=\frac{1}{\bar{r}^{4}}\left(\begin{array}{ccc}
e^{2 \gamma} & 0 & 0  \tag{C.19}\\
0 & e^{2 \gamma} & 0 \\
0 & 0 & \bar{\rho}^{2}
\end{array}\right)
$$

Where $\bar{r}^{2}=\bar{\rho}^{2}+\bar{z}^{2}$. With this coordinate transformation, $\Lambda$ is the origin of the new coordinate system. An immediate choice of the conformal factor in order for the metric to be holomorphic in $\Lambda$ is

$$
\begin{equation*}
\Omega=\bar{r}^{2}=\bar{\rho}^{2}+\bar{z}^{2} \tag{C.20}
\end{equation*}
$$

Consequently, the metric for the manifold $\tilde{S}$ can be written as:

$$
\tilde{h}_{i j}(\bar{\rho}, \bar{z}, \bar{\varphi})=\left(\begin{array}{ccc}
e^{2 \gamma} & 0 & 0  \tag{C.21}\\
0 & e^{2 \gamma} & 0 \\
0 & 0 & \bar{\rho}^{2}
\end{array}\right)
$$

It is straightforward to see that this choice of conformal factor does fulfill the conditions for $S$ to be considered asymptotically flat*. By performing the transformations $\{\rho, z, \varphi\} \rightarrow\{\bar{\rho}, \bar{z}, \bar{\varphi}\}$ and $\xi \rightarrow \tilde{\xi}$, the field equations (C.10) and (C.11) become ${ }^{69}$ :

$$
\begin{align*}
& \left(\bar{r}^{2} \tilde{\xi} \tilde{\xi}^{*}-\bar{r}^{2} \tilde{q} \tilde{q}^{*}-1\right) \nabla^{2} \tilde{\xi}=  \tag{C.22}\\
& 2\left(\bar{r}^{2}\left(\tilde{\xi}^{*} \nabla \tilde{\xi} \cdot \nabla \tilde{\xi}-\tilde{q}^{*} \nabla \tilde{\xi} \cdot \nabla \tilde{q}\right)+2 \bar{r} \tilde{\xi}^{*} \tilde{\xi} \nabla \tilde{\xi} \cdot \nabla \bar{r}-\bar{r} \tilde{q}^{*} \tilde{\xi} \nabla \tilde{q} \cdot \nabla \bar{r}-\bar{r} \tilde{q}^{*} \tilde{q} \nabla \tilde{\xi} \cdot \nabla \bar{r}+\tilde{\xi}^{*} \tilde{\xi}^{2}-\tilde{q}^{*} \tilde{\xi} \tilde{q}\right) \\
& \left(\bar{r}^{2} \tilde{\xi} \tilde{\xi}^{*}-\bar{r}^{2} \tilde{q} \tilde{q}^{*}-1\right) \nabla^{2} \tilde{q}=  \tag{C.23}\\
& 2\left(\bar{r}^{2}\left(\tilde{\xi}^{*} \nabla \tilde{q} \cdot \nabla \tilde{\xi}-\tilde{q}^{*} \nabla \tilde{q} \cdot \nabla \tilde{q}\right)+\bar{r} \tilde{\xi}^{*} \tilde{q} \nabla \tilde{\xi} \cdot \nabla \bar{r}-2 \bar{r} \tilde{q}^{*} \tilde{q} \nabla \tilde{q} \cdot \nabla \bar{r}+\bar{r}^{*} \tilde{\xi} \nabla \tilde{q} \cdot \nabla \bar{r}+\tilde{\xi}^{*} \tilde{q} \tilde{\xi}-\tilde{q}^{*} \tilde{q}^{2}\right)
\end{align*}
$$

Here, all derivative operators are to be considered as an Euclidean ones in the cylindrical coordinates $\{\bar{\rho}, \bar{z}, \bar{\varphi}\}$. Due to axial symmetry, the derivatives with respect to $\varphi$ may be neglected.

It is important to remark that the new potentials $\tilde{\xi}$ and $\tilde{q}$ are not solutions of the Ernst equations associated with the metric $\tilde{h}_{i j}$. They are just conformal transformations of the Ernst potentials $\xi$ and $q$, which are in fact solutions of the Ernst equations associated with the metric $h_{i j}$. However, since in equation (C.4) the Ricci tensor $\tilde{R}_{i j}$ related to $\tilde{h}_{i j}$, appears, it becomes necessary to know how to write $\tilde{R}_{i j}$ in terms of $\tilde{\xi}$ and $\tilde{q}$. First, notice

[^12]that $\tilde{h}_{i j}$ and $h_{i \bar{j}}$ differ just by a conformal factor. Consequently, the Ricci tensor $\tilde{R}_{i j}$ in $\tilde{h}_{i j}$ can be linked to the Ricci tensor $R_{\bar{i}}$ in $h_{\overline{i j}}$ by the formula ${ }^{104}$ :
\[

$$
\begin{equation*}
\tilde{R}_{i j}=R_{\overline{i j}}-h_{\bar{i} \bar{j}} D^{\bar{k}} \partial_{\bar{k}} \sigma+\left[\partial_{\bar{i}} \sigma \partial_{\bar{j}} \sigma-D_{\bar{i}} \partial_{\bar{j}} \sigma-h_{\bar{i}} \partial^{\bar{k}} \sigma \partial_{\bar{k}} \sigma\right] \tag{C.24}
\end{equation*}
$$

\]

Where $\tilde{h}_{i j}(\bar{\rho}, \bar{z})=e^{2 \sigma(\bar{\rho}, \bar{z})} h_{i j}(\bar{\rho}, \bar{z})$ or $\sigma=\ln (\Omega)$. With the aid of the formulae (C.12),(C.13) and (C.14) which remain valid in $\tilde{S}$, it is possible to write:

$$
\begin{align*}
& \tilde{R}_{11}=\frac{\bar{z}^{2}}{\bar{r}^{2}} R_{1 \overline{11}}-\frac{2 \bar{\rho} \bar{z}}{\bar{r}^{2}} R_{1 \overline{12}}+\frac{\bar{\rho}^{2}}{\bar{r}^{2}} R_{\overline{22}}  \tag{C.25}\\
& \tilde{R}_{12}=\frac{\bar{\rho} \bar{z}}{\bar{r}^{2}} R_{\overline{11}}+\frac{\bar{z}^{2}-\bar{\rho}^{2}}{\bar{r}^{2}} R_{\overline{12}}-\frac{\bar{\rho} \bar{z}}{\bar{r}^{2}} R_{\overline{22}},  \tag{C.26}\\
& \tilde{R}_{22}=\frac{\bar{\rho}^{2}}{\bar{r}^{2}} R_{\overline{11}}+\frac{2 \bar{\rho} \bar{z}}{\bar{r}^{2}} R_{1 \overline{12}}+\frac{\bar{z}^{2}}{\bar{r}^{2}} R_{\overline{22}} \tag{C.27}
\end{align*}
$$

Finally, using equations (3.19) and (3.20) and after some manipulations one can write ${ }^{69}$ :

$$
\begin{equation*}
\left(\bar{r}^{2} \tilde{\xi}^{*} \tilde{\xi}-\bar{r}^{2} \tilde{q}^{*} \tilde{q}-1\right)^{2} \tilde{R}_{i j}=2 \operatorname{Re}\left(\tilde{\nabla}_{i} \tilde{\xi} \tilde{\nabla}_{j} \xi^{*}-\tilde{\nabla}_{i} \tilde{q} \tilde{\nabla}_{j} \tilde{q}^{*}+\tilde{s}_{i} \tilde{s}_{j}^{*}\right) \tag{C.28}
\end{equation*}
$$

Where ${ }^{92}$

$$
\begin{equation*}
\tilde{\nabla}_{1}=\bar{z} \frac{\partial}{\partial \bar{\rho}}-\bar{\rho} \frac{\partial}{\partial \bar{z}} \quad \tilde{\nabla}_{2}=\bar{\rho} \frac{\partial}{\partial \bar{\rho}}+\bar{z} \frac{\partial}{\partial \bar{z}}+1 \quad \tilde{s}_{i}=\bar{r}\left(\tilde{\xi} \tilde{\nabla}_{i} \tilde{q}-\tilde{q} \tilde{\nabla}_{i} \tilde{\xi}\right) . \tag{C.29}
\end{equation*}
$$

These derivatives are the origin of the mistake in Hoenselaers's paper ${ }^{69}$ in which the index 1,2 were interchanged.

## C. 2 Fodor-Hoenselaers-Perjés

It was shown in chapter 4 that the Ernst potentials on the symmetry axis are sufficient to describe the Einstein-Maxwell fields in the entire space. Thus, it is expected that the multipole moments might be evaluated in terms of the power series coefficients of the Ernst potentials on the symmetry axis, that is:

$$
\begin{equation*}
\tilde{\xi}=\sum_{j=0}^{\infty} m_{j} \bar{z}^{j}, \quad \tilde{q}=\sum_{j=0}^{\infty} q_{j} \bar{z}^{j} \tag{C.30}
\end{equation*}
$$

In other words, the multipole moments $P_{n}$ and $Q_{n}$ might be expressed in terms of $m_{j}$ and $q_{j}$. Notice that the recurrence formula for the multipoles (C.4) do not mix the real and imaginary parts, hence, $P_{n}$ and $Q_{n}$ represent a set of four sets of multipoles. Here, the
real part of $P_{n}$ is associated with the mass moment, while the imaginary one is associated with the angular moment (the justification for naming the latter is given by the fact that in the case of a static space-time the imaginary part of $\xi$ vanishes). Moreover, the real and imaginary parts of the moments $Q_{n}$ are the electric and magnetic moments, respectively.

Considering $\tilde{\xi}$ and $\tilde{q}$ holomorphic functions, it is possible to write their power series around $\Lambda$ :

$$
\begin{equation*}
\tilde{\xi}=\sum_{i, j=0}^{\infty} a_{i j} \bar{\rho}^{i} \bar{z}^{j}, \quad \tilde{q}=\sum_{i, j=0}^{\infty} b_{i j} \bar{\rho}^{i} \bar{z}^{j} \tag{C.31}
\end{equation*}
$$

Where $a_{0 j} \equiv m_{j}$ and $b_{0 j} \equiv q_{j}$. Then, by substituting the expansions above in the tilde version of the Ernst equations (C.22)(C.23), one would expect that it is possible to write all $a_{i j}$ and $b_{i j}$ in terms of $m_{j}$ and $q_{j}$. In fact, substituting these expansions into (C.22), the LHS can be written as ${ }^{\dagger}$ :

$$
\begin{gather*}
{\left[\left(p^{2}+z^{2}\right) \sum_{i, j, k, l}^{\infty}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right) p^{i+k} z^{j+l}-1\right] \sum_{m, n}^{\infty} a_{m n}\left(m \rho^{m-2} z^{n}+m(m-1) \rho^{m-2} z^{n}+\right.} \\
\left.+n(n-1) \rho^{m} z^{n-2}\right) \tag{C.32}
\end{gather*}
$$

$$
\begin{equation*}
=\left[\sum_{i, j, k, l}^{\infty}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right)\left(p^{i+k+2} z^{j+l}+p^{i+k} z^{j+l+2}\right)-1\right] \sum_{m, n}^{\infty} a_{m n}\left(\frac{m^{2}}{\rho^{2}}+\frac{n(n-1)}{z^{2}}\right) \rho^{m} z^{n} \tag{C.33}
\end{equation*}
$$

$$
\begin{align*}
&=\sum_{i, j, k, l, m, n}^{\infty} a_{m n}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right)\left(m^{2}+n(n-1)+\frac{m^{2} z^{2}}{\rho^{2}}+\frac{n(n-1) \rho^{2}}{z^{2}}\right) \rho^{i+k+m} z^{j+l+m}- \\
&-\sum_{m, n}^{\infty} a_{m n}\left(\frac{m^{2}}{\rho^{2}}+\frac{n(n-1)}{z^{2}}\right) \rho^{m} z^{n} \tag{C.34}
\end{align*}
$$

$$
=\sum_{e, d} \rho^{e} z^{d}\left(\sum_{\substack{i+k+m=e \\ j+l+n=d}}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right) \times\right.
$$

$$
\times\left(a_{m n}\left(m^{2}+n(n-1)\right)+a_{m-2, n+2}(n+2)(n+1)+a_{m+2, n-2}(m+2)^{2}\right)-
$$

$$
\begin{equation*}
\left.-a_{e+2, d}(e+2)^{2}-a_{e, d+2}(d+1)(d+2)\right) \tag{C.35}
\end{equation*}
$$

[^13]Now, performing the same calculations for the RHS:

$$
\begin{align*}
& 2\left\{( \rho ^ { 2 } + z ^ { 2 } ) \left[\sum_{k, l}^{\infty} a_{k l}^{*} \rho^{k} z^{l}\left(\partial_{\rho} \sum_{i, j}^{\infty} a_{i j} \rho^{i} z^{j} \partial_{\rho} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}+\partial_{z} \sum_{i, j}^{\infty} a_{i j} \rho^{i} z^{j} \partial_{z} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}\right)-\right.\right. \\
& \left.-\sum_{k, l}^{\infty} b_{k l}^{*} \rho^{k} z^{l}\left(\partial_{\rho} \sum_{i, j}^{\infty} b_{i j} \rho^{i} z^{j} \partial_{\rho} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}+\partial_{z} \sum_{i, j}^{\infty} b_{i j} \rho^{i} z^{j} \partial_{z} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}\right)\right]+ \\
& +2 \sum_{i, j, k, l}^{\infty} a_{i j} a_{k l}^{*} \rho^{i+k} z^{j+l}\left(\rho \partial_{\rho} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}+z \partial_{z} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}\right)- \\
& -\sum_{m, n, k, l}^{\infty} a_{m n} b_{k l}^{*} \rho^{m+k} z^{n+l}\left(\rho \partial_{\rho} \sum_{i, j}^{\infty} b_{i j} \rho^{i} z^{j}+z \partial_{z} \sum_{i, j}^{\infty} a_{i j} \rho^{i} z^{j}\right)-  \tag{C.36}\\
& -\sum_{i, j, k, l}^{\infty} b_{i j} b_{k l}^{*} \rho^{i+k} z^{j+l}\left(\rho \partial_{\rho} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}+z \partial_{z} \sum_{m, n}^{\infty} a_{m n} \rho^{m} z^{n}\right)+ \\
& \left.+\sum_{i, j, k, l, m, n}^{\infty}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right) a_{m, n} \rho^{i+k+m} z^{j+l+n}\right\} \\
& =2\left\{\sum_{k, l}^{\infty} a_{k l}^{*}\left(\rho^{k+2} z^{l}+\rho^{k} z^{l+2}\right) \sum_{i, j, m, n}^{\infty}\left[i m a_{i j} a_{m n} \rho^{i+m-2} z^{j+n}+j n a_{i j} a_{m n} \rho^{i+m-2} z^{j+n-2}\right]-\right. \\
& -\sum_{k, l}^{\infty} b_{k l}^{*}\left(\rho^{k+2} z^{l}+\rho^{k} z^{l+2}\right) \sum_{i, j, m, n}^{\infty}\left[i m b_{i j} a_{m n} \rho^{i+m-2} z^{j+n}+j n b_{i j} a_{m n} \rho^{i+m-2} z^{j+n-2}\right]+ \\
& +2 \sum_{i, j, k, l}^{\infty} a_{i j} a_{k l}^{*} \rho^{i+k} z^{j+l} \sum_{m, n}^{\infty} a_{m n}(m+n) \rho^{m} z^{n}- \\
& -\sum_{m, n, k, l}^{\infty} a_{m n} b_{k l}^{*} \rho^{m+k} z^{n+l} \sum_{i, j}^{\infty} b_{i j}(i+j) \rho^{i} z^{j}-  \tag{C.37}\\
& -\sum_{i, j, k, l}^{\infty} b_{i j} b_{k l}^{*} \rho^{i+k} z^{j+l} \sum_{m, n}^{\infty} a_{m n}(m+n) \rho^{m} z^{n}+ \\
& \left.+\sum_{i, j, k, l, m, n}^{\infty}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right) a_{m, n} \rho^{i+k+m} z^{j+l+n}\right\}
\end{align*}
$$

Nonetheless notice that it could also be written as:

$$
\begin{align*}
& 2 \sum_{i, j, k, l}^{\infty} a_{i j} a_{k l}^{*} \rho^{i+k} z^{j+l} \sum_{m, n}^{\infty} a_{m n}(m+n) \rho^{m} z^{n}=  \tag{C.38}\\
& =\sum_{i, j, k, l}^{\infty} a_{i j} a_{k l}^{*} \rho^{i+k} z^{j+l} \sum_{m, n}^{\infty} a_{m n}(m+n) \rho^{m} z^{n}+\sum_{m, n, k, l}^{\infty} a_{m n} a_{k l}^{*} \rho^{m+k} z^{n+l} \sum_{i, j}^{\infty} a_{i j}(i+j) \rho^{i} z^{j}
\end{align*}
$$

Therefore, one gets:

$$
\begin{align*}
& 2\left\{\sum _ { e , d } \rho ^ { e } z ^ { d } \sum _ { \substack { i + k + m = e \\
j + l + n = d } } ( a _ { i j } a _ { k l } ^ { * } - b _ { i j } b _ { k l } ^ { * } ) \left(a_{m n}(i m+j n+m+n+i+j+1)+\right.\right.  \tag{C.39}\\
& \left.\left.\quad+j(n+2) a_{m-2, n+2}+i(m+2) a_{m+2, n-2}\right)\right\}
\end{align*}
$$

Equating the RHS to the LHS, one get:

$$
\begin{align*}
(e+2)^{2} a_{e+2, d} & =-(d+1)(d+2) a_{e, d+2}+\sum_{\substack{i+k+m=e \\
j+l+n=d}}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right)(  \tag{C.40}\\
& a_{m, n}\left(m^{2}+n^{2}-2 i m-2 j n-2 m-3 n-2 i-2 j-2\right)+ \\
& \left.+a_{m-2, n+2}(n+2)(n-2 j+1)+a_{m+2, n-2}(m+2)(m-2 i+2)\right)
\end{align*}
$$

Performing the exact same calculation for equation (C.23), we find the recursion relation for the potential $\tilde{q}$ :

$$
\begin{align*}
(e+2)^{2} b_{e+2, d} & =-(d+1)(d+2) b_{e, d+2}+\sum_{\substack{i+k+m=e \\
j+l+n=d}}\left(a_{i j} a_{k l}^{*}-b_{i j} b_{k l}^{*}\right)(  \tag{C.41}\\
& b_{m, n}\left(m^{2}+n^{2}-2 i m-2 j n-2 m-3 n-2 i-2 j-2\right)+ \\
& \left.+b_{m-2, n+2}(n+2)(n-2 j+1)+b_{m+2, n-2}(m+2)(m-2 i+2)\right)
\end{align*}
$$

These recursion relations, (C.40) and (C.41), imply that $a_{i j}=b_{i j}=0$ for $i$ odd. Notice that these equations also ensure that in fact it is possible to write all $a_{i j}$ and $b_{i j}$ in terms of $m_{j}$ and $q_{j}$, which corroborates that it is always possible to determinate the Ernst potentials everywhere from their behavior on the symmetry axis. ${ }^{102}$ In papers [69, 92] a small mistake appears that resulted in wrong formulae for the multipole moments. The authors found $-4 m-5 n$ instead of the correct $-2 m-3 n-2 i-2 j$ term. The mistake has been corrected in the proceedings paper [ ${ }^{103}$.

Since the Ricci tensor can be written in terms of $\tilde{\xi}$, one can use the power series of $\tilde{\xi}$ to evaluate moments. Following Fodor, ${ }^{36}$ the next step is to construct a suitable algorithm for the calculations of the multipole tensors $P_{i_{1} \cdots i_{n}}^{(n)}$ and $Q_{i_{1} \cdots i_{n}}^{(n)}$. From now on, all procedures will be applied for $P_{i_{1} \cdots i_{n}}^{(n)}$ since the development for $Q_{i_{1} \cdots i_{n}}^{(n)}$ is exactly the same. By exploiting the fact that $P_{i_{1} \cdots i_{n}}^{(n)}$ is symmetric by construction, the notation can be simplified:

$$
\begin{equation*}
P_{a, b, c}^{(n)}=P_{a}^{P_{i}^{(n)} \underbrace{1 \cdots 1}_{b}} \underbrace{2 \cdots 2}_{b} \underbrace{3 \cdot 3}_{c}, \quad a+b+c=n . \tag{C.42}
\end{equation*}
$$

Consider the symmetric part of the covariant derivative:

$$
\begin{equation*}
\mathcal{S}\left(\tilde{D}_{i_{n}} P_{i_{1} \cdots i_{n-1}}^{(n-1)}\right)=\frac{1}{n!} \sum_{\sigma} \tilde{D}_{i_{\sigma}(n)} P_{i_{\sigma}(1) \cdots i_{\sigma}(n-1)}^{(n-1)} . \tag{C.43}
\end{equation*}
$$

Where $\sigma$ runs over all permutations of $(1,2, \cdots n)^{105}$ :

$$
\begin{equation*}
\mathcal{S}\left(\tilde{D}_{i_{n}} P_{i_{1} \cdots i_{n-1}}^{(n-1)}\right)=\frac{1}{n}\left(a \tilde{D}_{1} P_{a-1, b, c}^{(n-1)}+b \tilde{D}_{2} P_{a, b-1, c}^{(n-1)}+c \tilde{D}_{3} P_{a, b, c-1}^{(n-1)}\right) . \tag{C.44}
\end{equation*}
$$

Then:

$$
\begin{align*}
\tilde{D}_{1} P_{i_{1} \cdots i_{n}}^{(n)}=\partial_{\bar{\rho}} P_{a, b, c}^{(n)}-a \gamma,{ }_{\bar{\rho}} P_{a, b, c}^{(n)}-b \gamma, \bar{z} P_{a+1, b-1, c}^{(n)} & +a \gamma_{, \bar{z}} P_{a-1, b+1, c}^{(n)}-  \tag{C.45}\\
& -b \gamma_{, \bar{\rho}} P_{a, b, c}^{(n)}-\frac{c}{\bar{\rho}} P_{a, b, c}^{(n)} .
\end{align*}
$$

$\tilde{D}_{2} P_{i_{1} \cdots i_{n}}^{(n)}=\partial_{\bar{z}} P_{a, b, c}^{(n)}-a \gamma, \bar{z} P_{a, b, c}^{(n)}+b \gamma \gamma_{, \bar{\rho}} P_{a+1, b-1, c}^{(n)}-a \gamma, \bar{\rho} P_{a-1, b+1, c}^{(n)}-b \gamma, \bar{z} P_{a, b, c}^{(n)}$.

$$
\begin{equation*}
\tilde{D}_{23} P_{i_{1} \cdots i_{n}}^{(n)}=c \bar{\rho} e^{-2 \gamma} P_{a+1, b, c-1}^{(n)}-\frac{a}{\bar{\rho}} P_{a-1, b, c+1}^{(n)} \tag{C.47}
\end{equation*}
$$

An analogous calculation leads to:

$$
\begin{equation*}
\mathcal{S}\left(\tilde{R}_{i_{1} i_{2}} P_{i_{2} \cdots i_{n}}^{(n-2)}\right)=\frac{1}{n!} \sum_{\sigma} \tilde{R}_{i_{\sigma(1)} i_{\sigma(2)}} P_{i_{\sigma(2)} \cdots i_{\sigma(n)}}^{(n-2)} . \tag{C.48}
\end{equation*}
$$

$\mathcal{S}\left(\tilde{R}_{i_{1} i_{2}} P_{i_{2} \cdots i_{n}}^{(n-2)}\right)=\frac{1}{n(n-1)}\left(a(a-1) \tilde{R}_{11} P_{a-2, b, c}^{(n-2)}+2 a b \tilde{R}_{12} P_{a-1, b-1, c}^{(n-2)}+b(b-1) \tilde{R}_{22} P_{a, b-2, c}^{(n-2)}\right)$

Hence, equation (C.4) can be expressed as:

$$
\begin{align*}
P_{a, b, c}^{(n)}= & \frac{1}{n} \mathcal{T}\left\{a \partial_{\bar{\rho}} P_{a-1, b, c}^{(n-1)}+b \partial_{\bar{z}} P_{a, b-1, c}^{(n-1)}-\left[(a(a-1)+2 a b) \gamma, \bar{\rho}+2 \frac{a c}{\bar{\rho}}\right] P_{a-1, b, c}^{(n-1)}\right. \\
& -[2 a b+b(b-1)] \gamma \gamma, \bar{z} P_{a, b-1, c}^{(n-1)}+a(a-1) \gamma, \bar{z} P_{a-2, b+1, c}^{(n-1)}+b(b-1) \gamma,{ }_{,} P_{a+1, b-2, c}^{(n-1)} \\
& +c(c-1) \bar{\rho} e^{-2 \gamma} P_{a+1, b, c-2}^{(n-1)}-\left(n-\frac{3}{2}\right)\left[a(a-1) \tilde{R}_{11} P_{a-2, b, c}^{(n-2)}\right.  \tag{C.50}\\
& \left.\left.+2 a b \tilde{R}_{12} P_{a-1, b-1, c}^{(n-2)}+b(b-1) \tilde{R}_{22} P_{a, b-2, c}^{(n-2)}\right]\right\} .
\end{align*}
$$

Now, we will deduce the expression without trace. By induction, it is possible to show that the construction of the trace-free part of a symmetric tensor can be done using the expression ${ }^{100}$ :

$$
\begin{aligned}
\mathcal{T}\left(T_{i_{1} \cdots i_{n}}\right)_{i_{1} \cdots i_{n}}= & T_{i_{1} \cdots i_{n}}+ \\
& +\sum_{k=1}^{[n / 2]} A_{k}^{(n)} \mathcal{S}\left(h_{i_{1} i_{2}} h_{i_{3} i_{4}} \cdots h_{i_{2 k-1} i_{2 k}} h^{r_{1} r_{2}} h^{r_{3} r_{4}} \cdots h^{r_{2 k-1} r_{2 k}} T_{r_{1} r_{2} \cdots r_{2 k-1} r_{2 k} i_{2 k+1} \cdots i_{n}}\right)
\end{aligned}
$$

Where ${ }^{106} A_{m}^{(n)}=\frac{(-1)^{m} n!(2 n-2 m-1)!!}{2^{m} m!(n-2 m)!(2 n-1)!!}$. As a consequence, consider the totally symmetric tensor $T_{i_{1} i_{2} \cdots i_{n}}$, since $\mathcal{T}\left(\mathcal{T}\left(T_{i_{1} \cdots i_{n}}\right)\right)=\mathcal{T}\left(T_{i_{1} \cdots i_{n}}\right)$. Thereby:

$$
\begin{equation*}
A_{1}^{(n)} \mathcal{T S}\left(h_{i_{1} i_{2}} T_{i_{3} i_{4} \cdots i_{n} a b} h^{a b}\right)+A_{2}^{(n)} \mathcal{T} \mathcal{S}\left(h_{i_{1} i_{2}} h_{i_{3} i_{4}} T_{i_{5} i_{6} \cdots i_{n} a b c d} h^{a b} h^{c d}\right)+\text { higher orders }=0 . \tag{C.52}
\end{equation*}
$$

From the arbitrariness of the tensor $T_{i_{1} \cdots i_{n}}$, which order must be equal to zero. In particular, one has as a consequence ${ }^{102}$ :

$$
\begin{equation*}
\mathcal{T S}\left(h_{i_{1} i_{2}} Q_{i_{3} i_{4} \cdots i_{n}}\right)=0 \tag{C.53}
\end{equation*}
$$

Therefore, it is always possible to add a tensor $T_{i_{1} i_{2} \cdots i_{n}}^{(n)}=\mathcal{S}\left(\tilde{h}_{i_{1} i_{2}} Q_{i_{3} i_{4} \cdots i_{n}}^{(n-2)}\right)$ into the definition of $P_{a, b, c}^{(n)}$ since it vanishes by performing the operation of removing the trace. In other words, adding $T_{i_{1} i_{2} \cdots i_{n}}^{(n)}=\mathcal{S}\left(\tilde{h}_{i_{1} i_{2}} Q_{i_{3} i_{4} \cdots i_{n}}^{(n-2)}\right)$ does not change the values of the multipole moments. Physically, it would be expected that the multipole moments do not depend upon the coordinate $\phi$ or that its components do not contribute due to the axial symmetry. Consider then the new tensors:

$$
\begin{align*}
& S_{0,0,0}^{(0)}=P_{0,0,0}^{0}  \tag{C.54}\\
& S_{a, b, c}^{(1)}=P_{a, b, c}^{1},  \tag{C.55}\\
& S_{a, b, c}^{(n)}=P_{a, b, c}^{n}+T_{a, b, c}^{(n)}, \quad n \geq 2 . \tag{C.56}
\end{align*}
$$

It will be shown that the tensor $Q$ can be chosen so that $S_{a, b, c}^{(n)}=0$ for $c \neq 0$. Notice that this is already valid for $n=0$ and $n=1$. Now, substituting $P_{a, b, c}^{(n)}$ by $S_{a, b, c}^{(n)}$ everywhere in (C.50) not considering the operation of removing the trace and adding $T$ to the right hand side, that is:

$$
\begin{align*}
S_{a, b, c}^{(n)}= & \frac{1}{n} \mathcal{T}\left\{a \partial_{\bar{\rho}} S_{a-1, b, c}^{(n-1)}+b \partial_{\bar{z}} S_{a, b-1, c}^{(n-1)}-\left[(a(a-1)+2 a b) \gamma, \bar{\rho}+2 \frac{a c}{\bar{\rho}}\right] S_{a-1, b, c}^{(n-1)}\right. \\
& -[2 a b+b(b-1)] \gamma, \bar{z} S_{a, b-1, c}^{(n-1)}+a(a-1) \gamma, \bar{z} S_{a-2, b+1, c}^{(n-1)}+b(b-1) \gamma, \bar{\rho} S_{a+1, b-2, c}^{(n-1)} \\
& +c(c-1) \bar{\rho} e^{-2 \gamma} S_{a+1, b, c-2}^{(n-1)}-\left(n-\frac{3}{2}\right)\left[a(a-1) \tilde{R}_{11} S_{a-2, b, c}^{(n-2)}\right.  \tag{C.57}\\
& \left.\left.+2 a b \tilde{R}_{12} S_{a-1, b-1, c}^{(n-2)}+b(b-1) \tilde{R}_{22} S_{a, b-2, c}^{(n-2)}\right]\right\}+T_{a, b, c}^{(n)} .
\end{align*}
$$

For $n=2$ :

$$
\begin{align*}
S_{1,0,1}^{(2)} & =T_{1,0,1}^{(2)}, \\
S_{0,1,1}^{(2)} & =T_{0,1,1}^{(2)},  \tag{C.58}\\
S_{0,0,2}^{(2)}=\rho e^{-2 \gamma} S_{1,0,0}^{(1)} & +T_{0,0,2}^{(2)}
\end{align*}
$$

In order to satisfy the requirement $S_{a, b, c}^{(n)}=0$ for $c \neq 0$, it is necessary that:

$$
\begin{equation*}
T_{1,0,1}^{(2)}=T_{0,1,1}^{(2)}=0, \quad T_{0,0,2}^{(2)}=-\rho e^{-2 \gamma} S_{1,0,0}^{(1)} \tag{C.59}
\end{equation*}
$$

For $n=3$ :

$$
\begin{array}{cc}
S_{2,0,1}^{(3)}=T_{2,0,1}^{(3)}, & S_{0,2,1}^{(3)}=T_{0,2,1}^{(3)} \\
S_{1,1,1}^{(3)}=T_{1,1,1}^{(3)}, & S_{0,0,3}^{(3)}=T_{0,0,3}^{(3)}  \tag{C.60}\\
S_{1,0,2}^{(3)}=\frac{2}{3} \rho e^{-2 \gamma} S_{2,0,0}^{(2)}+T_{1,0,2}^{(3)} & S_{0,1,2}^{(3)}=\frac{2}{3} \rho e^{-2 \gamma} S_{1,1,0}^{(2)}+T_{0,1,2}^{(3)}
\end{array}
$$

Thus,

$$
\begin{align*}
& T_{2,0,1}^{(3)}=T_{0,2,1}^{(3)}=T_{1,1,1}^{(3)}=T_{0,0,3}^{(3)}=0 \\
& T_{1,0,2}^{(3)}=-\frac{2}{3} \rho e^{-2 \gamma} S_{2,0,0}^{(2)}  \tag{C.61}\\
& T_{0,1,2}^{(3)}=-\frac{2}{3} \rho e^{-2 \gamma} S_{1,1,0}^{(2)} \tag{C.62}
\end{align*}
$$

By induction, in order to satisfy $S_{a, b, c}^{(n)}=0$ for $c \neq 0^{\ddagger}$ :

$$
\begin{align*}
S_{a, b, 1}^{(n)} & =T_{a, b, 1}^{(n)}  \tag{C.63}\\
S_{a, b, 2}^{(n)} & =\frac{2}{n} \bar{\rho} e^{-2 \gamma} S_{a+1, b, 0}^{(n-1)}+T_{a, b, 2}^{(n)}  \tag{C.64}\\
S_{a, b, c}^{(n)} & =T_{a, b, c}^{(n)}, \quad c \geq 2 \tag{C.65}
\end{align*}
$$

Hence:

$$
\begin{align*}
& T_{a, b, c}^{(n)}=0, \quad c \neq 0,2 .  \tag{C.66}\\
& T_{a, b, 2}^{(n)}=-\frac{2}{n} \bar{\rho} e^{-2 \gamma} S_{a+1, b, 0}^{(n-1)} \tag{C.67}
\end{align*}
$$

In order to find the general formula for $T_{a, b, c}^{(n)}$ it is necessary to know its shape for $c=0$. Remember that:

$$
\begin{align*}
& T_{a, b, c}^{(n)}=\mathcal{S}\left(\tilde{h}_{i_{1} i_{2}} Q_{i_{3} \cdots i_{n}}^{(n-2)}\right)= \\
& \quad=\frac{1}{n(n-1)}\left(a(a-1) \tilde{h}_{11} Q_{a-2, b, c}^{(n-2)}+b(b-1) \tilde{h}_{22} Q_{a, b-2, c}^{(n-2)}+c(c-1) \tilde{h}_{3} Q_{a, b, c-2}^{(n-2)}\right) . \tag{C.68}
\end{align*}
$$

Thus:

$$
\begin{equation*}
T_{a, b, 0}^{(n)}=\frac{1}{n(n-1)}\left(a(a-1) \tilde{h}_{11} Q_{a-2, b, 0}^{(n-2)}+b(b-1) \tilde{h}_{22} Q_{a, b-2,0}^{(n-2)}\right) . \tag{C.69}
\end{equation*}
$$

It is necessary the to find $Q_{a-2, b, 0}^{(n-2)}$ and $Q_{a, b-2,0}^{(n-2)}$. Consider the system:

$$
\begin{align*}
& a(a-1) \tilde{h}_{11} Q_{a-2, b, 1}^{(n-2)}+b(b-1) \tilde{h}_{22} Q_{a, b-2,1}^{(n-2)}=0,  \tag{C.70}\\
& a(a-1) \tilde{h}_{11} Q_{a-2, b, 2}^{(n-2)}+b(b-1) \tilde{h}_{22} Q_{a, b-2,2}^{(n-2)}+2 \tilde{h}_{33} Q_{a, b, 0}^{(n-2)}=-2(n-1) \rho e^{-2 \gamma} S_{a+1, b, 0}^{(n-1)},  \tag{C.71}\\
& a(a-1) \tilde{h}_{11} Q_{a-2, b, c}^{(n-2)}+b(b-1) \tilde{h}_{22} Q_{a, b-2, c}^{(n-2)}+c(c-1) \tilde{h}_{33} Q_{a, b, c-2}^{(n-2)}=0, \quad c \geq 2 . \tag{C.72}
\end{align*}
$$

Due to the arbitrariness of $Q$, it is possible to choose $Q_{a, b, c}^{n}=0$ for $c \neq 0$. As a result:

$$
\begin{equation*}
Q_{a, b, 0}^{(n-2)}=-\frac{n-1}{\tilde{h}_{33}} \rho e^{-2 \gamma} S_{a+1, b, 0}^{(n-1)} . \tag{C.73}
\end{equation*}
$$

$\ddagger$ Except the term $c(c-1) \bar{\rho} e^{-2 \gamma} S_{a+1, b, c-2}^{(n-1)}$, every term in $S_{a, b, c}^{(n)}$ depends upon a lower order in $n$ and the same $c$

Hence, this proves that to evaluate the multipole moments it is necessary only to evaluate the quantities such that $c=0$. Therefore $a+b=n, S_{a}^{n} \equiv S_{a, n-a, 0}^{(n)}$.

$$
\begin{equation*}
T_{a, n-a, 0}^{(n)}=\frac{1}{n}\left(-\frac{a(a-1)}{\rho} S_{a-1}^{(n)}-\frac{(n-a)(n-a-1)}{\rho} S_{a+1}^{(n-a)}\right) \tag{C.74}
\end{equation*}
$$

Substituting this into (C.57), one gets the following recursion relations:

$$
\begin{align*}
S_{0}^{(0)}= & \tilde{\xi}, \quad S_{0}^{(1)}=\partial_{\bar{z}} \tilde{\xi}, \quad S_{1}^{(1)}=\partial_{\bar{\rho}} \tilde{\xi},  \tag{C.75}\\
S_{a}^{(n)}= & \frac{1}{n}\left\{a \partial_{\bar{\rho}} S_{a-1}^{(n-1)}+(n-a) \partial_{\bar{z}} S_{a}^{(n-1)}+\right. \\
& a\left[(a+1-2 n) \partial_{\bar{\rho}} \gamma-\frac{a-1}{\bar{\rho}}\right] S_{a-1}^{(n-1)}+(a-n)(a+n-1) \partial_{\bar{z}} \gamma S_{a}^{(n-1)}+  \tag{C.76}\\
& a(a-1) \partial_{\bar{z}} \gamma S_{a-2}^{(n-1)}+(n-a)(n-a-1)\left(\partial_{\bar{\rho}} \gamma-\frac{1}{\bar{\rho}}\right) S_{a+1}^{(n-1)}- \\
& \left(n-\frac{3}{2}\right)\left(a(a-1) \tilde{R}_{11} S_{a-2}^{(n-2)}+2 a(n-a) \tilde{R}_{12} S_{a-1}^{(n-2)}+\right. \\
& \left.\left.(n-a)(n-a-1) \tilde{R}_{22} S_{a}^{(n-2)}\right)\right\} .
\end{align*}
$$

Where the multipole tensors $P_{a, b, c}^{n}=\mathcal{T}\left(S_{a}^{n}\right)_{a, b, c}$, but from (C.8) the multipole moments are calculated with $a=0$ (or $b=n$ ). From equation (C.51):

$$
\begin{align*}
\mathcal{T}\left(S_{i_{1} \cdots i_{n}}^{(n)}\right)_{22 \cdots 2}= & S_{2 \cdots 2}+  \tag{C.77}\\
& +\sum_{k=1}^{[n / 2]} A_{k}^{(n)} \mathcal{S}\left(h_{22} h_{22} \cdots h_{22} h^{r_{1} r_{2}} h^{r_{3} r_{4}} \cdots h^{r_{2 k-1} r_{2 k}} S_{r_{1} r_{2} \cdots r_{2 k-1} r_{2 k}} \cdots 2\right)
\end{align*}
$$

Fodor et al ${ }^{36}$ have shown that this expression evaluated at $\Lambda\left(\tilde{h}_{22}=\tilde{h}^{22}=1\right)$ can be written as:

$$
\begin{equation*}
\mathcal{T}\left(S_{i_{1} \cdots i_{n}}^{(n)}\right)_{22 \cdots 2}=\frac{n!}{(2 n-1)!!} S_{0}^{(n)} \tag{C.78}
\end{equation*}
$$

Consequently, the scalar moments associated with mass and angular momentum are:

$$
\begin{equation*}
P_{n}=\left.\frac{1}{(2 n-1)!!} S_{0}^{(n)}\right|_{\Lambda} \tag{C.79}
\end{equation*}
$$

In a similar way, moments associated with the electromagnetic fields are given by the exchange of $\xi$ for $q$ in the above definitions:

$$
\begin{align*}
S_{0}^{(0)}= & \tilde{q}, \quad S_{0}^{(1)}=\partial_{\bar{z}} \tilde{q}, \quad S_{1}^{(1)}=\partial_{\bar{\rho}} \tilde{q}  \tag{C.80}\\
S_{a}^{(n)}= & \frac{1}{n}\left\{a \partial_{\bar{\rho}} S_{a-1}^{(n-1)}+(n-a) \partial_{\bar{z}} S_{a}^{(n-1)}+\right. \\
& a\left[(a+1-2 n) \partial_{\bar{\rho}} \gamma-\frac{a-1}{\bar{\rho}}\right] S_{a-1}^{(n-1)}+(a-n)(a+n-1) \partial_{\bar{z}} \gamma S_{a}^{(n-1)}+  \tag{C.81}\\
& a(a-1) \partial_{\bar{z}} \gamma S_{a-2}^{(n-1)}+(n-a)(n-a-1)\left(\partial_{\bar{\rho}} \gamma-\frac{1}{\bar{\rho}}\right) S_{a+1}^{(n-1)}- \\
& \left(n-\frac{3}{2}\right)\left(a(a-1) \tilde{R}_{11} S_{a-2}^{(n-2)}+2 a(n-a) \tilde{R}_{12} S_{a-1}^{(n-2)}+\right. \\
& \left.\left.(n-a)(n-a-1) \tilde{R}_{22} S_{a}^{(n-2)}\right)\right\} .
\end{align*}
$$

Finally, the scalar moments:

$$
\begin{equation*}
Q_{n}=\left.\frac{1}{(2 n-1)!!} S_{0}^{(n)}\right|_{\Lambda} \tag{C.82}
\end{equation*}
$$

If one wishes to evaluate the multipole moments up to the order $n$, the following algorithm should be used:

- Calculate all $a_{i j}$ and $b_{i j}$ in terms of $m_{j}$ and $q_{j}$ using equations (C.40) and (C.41) (the power series of $\tilde{\xi}$ and $\tilde{q}$ ), of to order $i+j \leq n$.
- Evaluate the Ricci tensor $\tilde{R}_{i j}$ and the derivatives of $\gamma$ in terms of the tilded versions of the Ernst potentials, $\tilde{\xi}$ and $\tilde{q}$, using equation (C.28), and using $\gamma, \tilde{\rho}=\frac{\bar{\rho}}{2}\left(\tilde{R}_{11}-\tilde{R}_{22}\right)$ $\gamma,{ }_{z}=\bar{\rho} \tilde{R}_{21}$.
- Compute the tensor $S_{0}^{(0)}$ using (C.75) or (C.80), and then evaluate the tensors $S_{0}^{(1)}=\partial_{\bar{z}} S_{0}^{(0)}$ and $S_{1}^{(1)}=\partial_{\bar{\rho}} S_{0}^{(0)}$.
- Compute all $S_{a}^{(m)}$, such that $a \leq n-m$, by means of the recursion formula.
- Finally, the scalar multipole moments are given by $\left.\frac{1}{(2 n-1)!!} S_{0}^{(n)}\right|_{\Lambda}$.

The values obtained for the first six moments are as follows:
$P_{0}=m_{0}$.

$$
\begin{equation*}
P_{1}=m_{1} \tag{C.84}
\end{equation*}
$$

$$
\begin{align*}
& P_{2}=m_{2} .  \tag{C.85}\\
& P_{3}=m_{3}+\frac{1}{5} q_{0}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right) .  \tag{C.86}\\
& P_{4}=m_{4}+\frac{3}{35} q_{1}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)+\frac{1}{7} q_{0}^{*}\left(3 m_{2} q_{0}-m_{1} q_{1}-2 m_{0} q_{2}\right)+\frac{1}{7}\left(m_{1}^{2}-m_{0} m_{2}\right) m_{0}^{*} .  \tag{C.87}\\
& \begin{array}{r}
P_{5}=m_{5}+\frac{1}{21}\left(\left|m_{0}\right|^{2}\left(m_{1}\left(-\left|q_{0}\right|^{2}\right)+m_{0} q_{1} q_{0}^{*}-7 m_{3}\right)+m_{1}\left|q_{0}\right|^{4}+14 m_{3}\left|q_{0}\right|^{2}-m_{1}\left|q_{1}\right|^{2}+\right. \\
+m_{1}\left|m_{1}\right|^{2}+m_{1} q_{0} q_{2}^{*}+4 m_{2} q_{0} q_{1}^{*}-m_{0} q_{1} q_{2}^{*}-m_{2} q_{1} q_{0}^{*}-m_{0} q_{0} q_{1}\left(q_{0}^{*}\right)^{2}-3 m_{0} q_{2} q_{1}^{*}-6 m_{1} q_{2} q_{0}^{*} \\
\left.-7 m_{0} q_{3} q_{0}^{*}-m_{0} m_{2} m_{1}^{*}+7 m_{1} m_{2} m_{0}^{*}\right) .
\end{array}
\end{align*}
$$

$P_{6}=m_{6}+\frac{1}{1155}\left(\left|m_{0}\right|^{2}\left(m_{0}\left(37\left|q_{1}\right|^{2}+140 q_{2} q_{0}^{*}\right)-210\left(m_{2}\left|q_{0}\right|^{2}+3 m_{4}\right)-37 m_{1} q_{0} q_{1}^{*}\right)+\right.$ $+10\left|q_{0}\right|^{2}\left(-5 m_{0}\left|q_{1}\right|^{2}+5 m_{1} q_{0} q_{1}^{*}+7 m_{1}^{2} m_{0}^{*}+105 m_{4}\right)-100 m_{0}\left|q_{2}\right|^{2}+175 m_{2}\left|q_{0}\right|^{4}+20 m_{2}\left|q_{1}\right|^{2}+$ $+\left|m_{1}\right|^{2}\left(140 m_{2}-13 m_{0}\left|q_{0}\right|^{2}\right)+35 m_{2}\left|m_{0}\right|^{4}-25 m_{0}\left|m_{2}\right|^{2}+35 m_{0} q_{1}^{2}\left(q_{0}^{*}\right)^{2}+35 m_{1} q_{0} q_{3}^{*}+125 m_{2} q_{0} q_{2}^{*}+$ $+350 m_{3} q_{0} q_{1}^{*}+13 m_{0}^{2} q_{1} m_{1}^{*} q_{0}^{*}-35 m_{0} q_{1} q_{3}^{*}-25 m_{1} q_{1} q_{2}^{*}+70 m_{3} q_{1} q_{0}^{*}-70 m_{1} q_{0} q_{1}\left(q_{0}^{*}\right)^{2}-160 m_{1} q_{2} q_{1}^{*}-$ $-280 m_{2} q_{2} q_{0}^{*}-140 m_{0} q_{0} q_{2}\left(q_{0}^{*}\right)^{2}-210 m_{0} q_{3} q_{1}^{*}-420 m_{1} q_{3} q_{0}^{*}-420 m_{0} q_{4} q_{0}^{*}+25 m_{1}^{2} m_{2}^{*}-35 m_{0} m_{1}^{2}\left(m_{0}^{*}\right)^{2}+$

$$
\begin{equation*}
\left.+280 m_{2}^{2} m_{0}^{*}-140 m_{0} m_{3} m_{1}^{*}+350 m_{1} m_{3} m_{0}^{*}\right) \tag{C.89}
\end{equation*}
$$

$Q_{0}=q_{0}$.

$$
\begin{equation*}
Q_{1}=q_{1} . \tag{C.91}
\end{equation*}
$$

$Q_{2}=q_{2}$.
$Q_{3}=q_{3}+\frac{1}{5} m_{0}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)$.
$Q_{4}=q_{4}+\frac{1}{35}\left(3 m_{1}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)+5 m_{0}^{*}\left(2 m_{2} q_{0}+m_{1} q_{1}-3 m_{0} q_{2}\right)-5\left(q_{1}^{2}-q_{0} q_{2}\right) q_{0}^{*}\right)$.
$Q_{5}=q_{5}+\frac{1}{21}\left(\left|q_{0}\right|^{2}\left(-q_{1}\left|m_{0}\right|^{2}+m_{1} q_{0} m_{0}^{*}+7 q_{3}\right)-14 q_{3}\left|m_{0}\right|^{2}+q_{1}\left(\left|m_{0}\right|^{4}+\left|m_{1}\right|^{2}-\left|q_{1}\right|^{2}\right)-\right.$ $\left.-m_{0} m_{1} q_{0}\left(m_{0}^{*}\right)^{2}+m_{2}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)+m_{0}^{*}\left(7 m_{3} q_{0}+6 m_{2} q_{1}+m_{1} q_{2}\right)+m_{1}^{*}\left(3 m_{2} q_{0}-4 m_{0} q_{2}\right)+q_{2}\left(q_{0} q_{1}^{*}-7 q_{1} q_{0}^{*}\right)\right)$.
$Q_{6}=q_{6}+\frac{1}{1155}\left(-37 m_{0} q_{1}\left|q_{0}\right|^{2} m_{1}^{*}+100 q_{0}\left|m_{2}\right|^{2}+37 q_{0}\left|m_{1} q_{0}\right|^{2}-20 q_{2}\left|m_{1}\right|^{2}+35 q_{2}\left|q_{0}\right|^{4}+630 q_{4}\left|q_{0}\right|^{2}+\right.$ $+25 q_{0}\left|q_{2}\right|^{2}-140 q_{2}\left|q_{1}\right|^{2}+210 m_{3} q_{0} m_{1}^{*}+25 m_{1} q_{1} m_{2}^{*}+160 m_{2} q_{1} m_{1}^{*}+35 m_{3}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)-$ $-125 m_{0} q_{2} m_{2}^{*}+35\left(m_{0}^{*}\right)^{2}\left(m_{1}^{2} q_{0}-2 m_{0} m_{1} q_{1}+m_{0}\left(5 m_{0} q_{2}-4 m_{2} q_{0}\right)\right)-350 m_{0} q_{3} m_{1}^{*}+$ $+m_{0}^{*}\left(50 m_{0} m_{1}^{*}\left(m_{0} q_{1}-m_{1} q_{0}\right)+13 q_{0} q_{1}^{*}\left(m_{1} q_{0}-m_{0} q_{1}\right)+70\left(q_{0}^{*}\left(2 m_{2} q_{0}^{2}+m_{0}\left(q_{1}^{2}-3 q_{0} q_{2}\right)\right)+\right.\right.$ $\left.\left.\left.+6 m_{4} q_{0}+6 m_{3} q_{1}+4 m_{2} q_{2}-m_{1} q_{3}-15 m_{0} q_{4}\right)\right)-25 q_{1}^{2} q_{2}^{*}-35 q_{0} q_{1}^{2}\left(q_{0}^{*}\right)^{2}-280 q_{2}^{2} q_{0}^{*}+140 q_{0} q_{3} q_{1}^{*}-350 q_{1} q_{3} q_{0}^{*}\right)$.

For $n \geq 3$ the expressions for $P_{n}$ and $Q_{n}$ are clearly different from the results published earlier. ${ }^{69,92}$


[^0]:    * The set of all killing vectors form an Lie algebra, in which the dimension of the correspondent group, the isometry group, is equal to the number of linearly independent Killing vectors. ${ }^{2,3}$

[^1]:    * The spacetime can be classified according to the isometry group, that is, by the Killing algebra it possesses. Since the Killing vectors commute, this work deals with the denominated abelian $G_{2}$ group. ${ }^{2}$
    $\dagger$ To represent the inverse metric $\left(f_{A B}\right)^{-1}$ we will use $g^{A B}$.

[^2]:    * Following the Sibgatullin notation, here $L^{B}$ and $K$ differ by the ones introduced by Kinnersley by a factor -2 .

[^3]:    $\dagger$ In what follows, both pair of coordinates will be constantly used.

[^4]:    $\ddagger$ Actually, Kinnersley and Chitre ${ }^{9-14}$ exploited the hierarchy potentials related to all of these new potentials in order to find new solutions. However, Sibgatullin showed a simpler technique, just making use of the hierarchy for $H$.

[^5]:    $\S$ This is a requirement not only to ensure that $F$ might be written as a power series given by (4.39), but also to ensure the uniqueness of an associated Hilbert-Riemann problem introduced by Ernst and Hauser. ${ }^{48}$

[^6]:    \| Variables which have o overwritten are related to the seed solution $\stackrel{\circ}{F}$

[^7]:    ** $\quad$ Equation (4.66) also ensures the existence of the inverse of $\chi_{-}(s)$ at all $s \in L_{-} \cup L^{17,52}$

[^8]:    * As we will show in what follows, not necessarily N solitons can be interpreted as N distinct bodies.
    $\dagger$ See appendix C.

[^9]:    $\ddagger$ Here, it is considered the $\xi$ presented by Tomimatsu-Sato powered by -1 due to a different notation

[^10]:    * Noticed that, using the results of the appendix C, even the revision made by SotiriouApostolatos had a mistake.
    $\dagger$ Once $m_{k}$ and $q_{k}$ are known, the multipole moments $P_{k}$ and $Q_{k}$ can be constructed or vice-versa.

[^11]:    * The complex s variable is known as spectral parameter.

[^12]:    * From the previous considerations, it was imposed that $\left.\gamma\right|_{\Lambda}=0$. Notice that it is, again, a requirement.

[^13]:    $\dagger$ In order for the equations to become legible, the bars over the coordinates will be dropped in this calculation

