UNIVERSIDADE DE SÃO PAULO
FFCLRP - DEPARTAMENTO DE COMPUTAçÃO E MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Lucas Lisboa Leão

Equações de evolução abstratas com termos não lineares $L^{q,\alpha}$-Hölder
(Abstract evolution equations with $L^{q,\alpha}$-Hölder nonlinear terms)

Fevereiro de 2023
Ribeirão Preto

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Dissertação apresentada à Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, como parte das exigências para obtenção do título de Mestre em Ciências - Matemática.

Aréa de concentração: Matemática.

Orientador: Prof. Dr. Eduardo Alex Hernández Morales.

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1. Equações de evolução abstratas. 2. Semigrupos de operadores lineares limitados. 3 Funções $L^{q,\alpha}$-Hölder.

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Resumo

LEÃO, L.L. **Equações de evolução abstratas com termos não lineares $L^{q,\alpha}$-Hölder**. 2023. 98 p. Dissertação (Mestrado em Ciências Matemática) Faculdade de Filosofia Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, Ribeirão Preto, 2023.

Apresentamos um estudo sobre resultados clássicos da teoria de equações diferencias abstratas, introduzimos as funções $L^{q,\alpha}$-Hölder junto com alguns exemplos e generalizamos alguns resultados já conhecidos a respeito de existência, unicidade e regularidade de soluções para o problema abstrato semilinear de evolução.

Palavras chave: Equações diferenciais abstratas, Semigrupos de operadores lineares limitados, Funções $L^{q,\alpha}$-Hölder.

Abstract

LEÃO, L.L. **Abstract evolution equations with $L^{q,\alpha}$-Hölder nonlinear terms**. 2023. 98 p. Dissertação (Mestrado em Ciências Matemática) Faculdade de Filosofia Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, Ribeirão Preto, 2023.

We present a study of classical results about abstract differential equations, introduce the $L^{q,\alpha}$-Hölder functions including several examples and generalize some well-known existence, uniqueness, and regularity results for the evolution abstract semilinear problem.

Keywords: Abstract differential equations, Semigroups of bounded linear operators, $L^{q,\alpha}$-Hölder function.

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Agradeço, a...

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As diversas amizades que o tempo me trouxe. Correndo o risco de esquecer nomes importantes, cito: Letícia, Kapo, Gelo, Boti, Peppa, Cal, Soli e Vitor. Estes me acompanharam (e acompanham) durante toda a trajetória que percorri para chegar até aqui.

A Comissão Julgadora, Prof. Dr. Eduardo Hernández, Prof. Dra. Marta Gadotti e Prof. Dr. Rodrigo Ponce, pela avaliação e sugestões de melhorias.

E, pelo apoio financeiro, à Capes.

Ao contrário do que acreditamos ser crença generalizada na sociedade, matemáticos não gostam muito de fazer contas. Na verdade, nem sequer são particulamente bons nisso.
(Hilário Alencar & Marcelo Viana)

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Introduction

In this project we are interested in the study of existence, uniqueness and regularity of solutions to the ordinary differential equations on infinite dimensional spaces (in general, a Banach space X) described by

$$\begin{aligned} u'(t) &= Au(t) + f(t) \\ u(0) &= x_0 \in X, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a semigroup $T(t)_{t\geq 0}$ and f is a suitable function.

On the classical literature, see [Pazy, 2012, Engel et al., 2000, Lunardi, 1995] and others, one can find several results concerning the case where f is a Lipschitz (or Hölder) function. The novelty of this work is to consider a more general class of functions, the $L^{q,\alpha}$-Hölder functions.

The concept of $L^{q,\alpha}$-Hölder function is inspired on the paper [Hernandez et al., 2020], where the authors study the controlability of abstract differential equations with state-dependent delay of the form

$$\begin{aligned} u'(t) &= Au(t) + F(t, u(t - \sigma(t, u_t))) + Bv(t),\ t \in [0, a] \\ u(0) &= \varphi \in C([-p, 0]; X) \end{aligned}$$

where σ and v are suitable functions, u_t is the function $u_t : [-p, 0] \to X$ defined by $u_t(\theta) = u(t + \theta)$ and F is a L^q-Lipschitz function, we cite that in this work the authors considered only two simple examples. Now we consider a generalized concept and present several examples.

In order to study the results cited above, this work is organized as follow. In the first chapter, we revisit the basic aspects of semigroups of bounded linear operators theory. In particular, we have studied C_0-semigroups, differentiable semigroups, analytic semigroups and proved the Hille-Yosida theorem, which characterizes when a linear operator is the infinitesimal generator of a C_0-semigroup. This chapter was based on the classic book *Semigroups of linear bounded operators and applications to partial differential equations* by Amnon Pazy [Pazy, 2012] and the master thesis of Andrea Prokopczyk [Prokopczyk, 2005], Denis Fernandes [Silva, 2017] and Michelle Pierri [Pierri, 2006].

In the second chapter, we study the existence and uniqueness of mild, classical, and strong solutions for the inhomogeneous abstract Cauchy's problem

$$\begin{aligned} u'(t) &= Au(t) + f(t) \\ u(0) &= x_0, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the generator of a semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X and $f : [0, a] \to X$ is Lipschitz. We also included some results concerning the regularity of the mild solution of (1)-(1) that can be found on [Pazy, 2012] and [Lunardi, 1995]. We finish this chapter by studying the existence, uniqueness, and regularity of solutions for the semilinear evolution problem

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)) \\ u(0) &= x_0, \end{aligned}$$

where $f : [0, a] \times X \to X$ is continuous and A is the infinitesimal generator of a semigroup of bounded linear operators on a Banach space X.

In the last chapter, we introduce the concepts of L^p_{Lip}-Lipschitz and $L^{p,\alpha}$-Hölder function and build several examples concerning different kinds of functins, we cite, osclilating functions and functions with countable discontinuities points. In addition, we study the existence and uniqueness of local and global solutions to the semilinear problem

$$\begin{aligned} u'(t) &= Au(t)+F(t,u(t)) \\ u(0) &= x_0, \end{aligned}$$

where A is the infinitesimal generator of a semigroup and $F(\cdot)$ is a L^p_{Lip}-Lipschitz or a $L^{p,\alpha}$-Hölder function. We note that the results in this chapter are new and are presented in our pre-print *Abstract differential equation and $L^{q,\alpha}$-Hölder functions* by Hernández, Lisboa and Fernandes that will be submitted for publication in the next weeks.

In the appendices, we note some results concerning Functional Analysis (see [Brézis, 2011] and [Kreyszig, 1978]), Integration theory (see [Bartle, 2014]), and semigroup of bounded linear operators theory.

1. Semigroups of linear operators

In this chapter, we study some aspects of strongly continuous semigroups of bounded linear operators and analytic semigroups. These classes of semigroups are especially interesting in the study of partial differential equations. From now on, $(X, \|\cdot\|_X)$ is a Banach space and $(\mathscr{L}(X), \|\cdot\|)$ denotes the space of linear operators from X into X endowed with the operator norm $\|\cdot\|$.

First of all, we define the concepts of semigroup, uniformly continuous semigroup and the infinitesimal generator of semigroups. Then, we show some generalities of uniformly continuous semigroups.

In Section 1.2, we introduce the concept of strongly continuous semigroups and their basic properties. At this point, we are ready to prove the Hille-Yosida theorem. In Section 1.3, we study differentiable and analytic semigroups.

1.1 Uniformly continuous semigroups of bounded linear operators

We start studying uniformly continuous semigroups of bounded linear operators and their basic properties.

Definition 1.1.1 A one parameter family $(T(t))_{t\geq 0}$ of bounded linear operators from X into X is a **semigroup of bounded linear operators** on X if
i) $T(0) = I$, where I is the identity operator,
ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$.

Definition 1.1.2 A semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X is **uniformly continuous** if $\lim_{t\downarrow 0} \|T(t) - I\| = 0$.

Definition 1.1.3 The linear operator $A : D(A) \subset X \to X$ defined by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)x}{dt}\right|_{t=0}, \text{ for } x \in D(A)$$

where

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$$

is the **infinitesimal generator** of $(T(t))_{t\geq 0}$ and $D(A)$ is the domain of A.

From the definition of a uniformly continuous semigroup of bounded linear operators, the function $t \mapsto T(t)$ from $[0,\infty)$ into $\mathscr{L}(X)$ is continuous at $t = 0$. Moreover, for $t, h \in (0,\infty)$ we see that,

$$\begin{aligned}
\lim_{h\to 0} \|T(t+h) - T(t)\| &= \lim_{h\downarrow 0} \|T(t)T(h) - T(t)\| \\
&= \lim_{h\downarrow 0} \|T(t)(T(h) - I)\| \\
&\leq \lim_{h\downarrow 0} \|T(t)\| \, \|T(h) - I\| .
\end{aligned}$$

Then, $\lim_{h\downarrow 0} \|T(t+h) - T(t)\| = 0$, which proves that the function $t \mapsto T(t)$ is right continuous at $t \geq 0$. Proceeding similarly, we can prove the left continuity at t. From the above, $t \mapsto T(t)$ is continuous on $[0,\infty)$.

We present now a characterization theorem of the infinitesimal generator of a uniformly continuous semigroup.

Theorem 1.1.1 A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Proof: Let A be a bounded linear operator on X and $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$.

We show that $T(t)$ is a bounded linear operator for all $t \geq 0$. For $t \in [0,\infty)$, $x, y \in X$ and $\alpha \in \mathbb{R}$, we note that

$$\begin{aligned}
T(t)(\alpha x + y) &= \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right)(\alpha x + y) \\
&= I(\alpha x + y) + \frac{tA(\alpha x + y)}{1!} + \frac{t^2A^2(\alpha x + y)}{2!} + \ldots \\
&= \alpha \left(\sum_{n=0}^{\infty} \frac{tA^n}{n!} \right)(x) + \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right)(y) \\
&= \alpha T(t)(x) + T(t)(y),
\end{aligned}$$

because A, $A^0 = I$ and $A^n = A \circ A \circ \cdots \circ A$ are linear maps. We also note that

$$\|T(t)\| = \left\| \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{(t\|A\|)^n}{n!} = e^{t\|A\|},$$

which implies that $T(t)$ is a bounded linear operator. Moreover, $T(0) = I + \sum_{n=1}^{\infty} \frac{(0A)^n}{n!} = I$ and

$$\begin{aligned}
T(t+s) &= \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{n!}{(n-i)!i!} t^i s^{n-i} \frac{A^n}{n!} \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \frac{(tA)^i}{i!} \frac{(sA)^{n-i}}{(n-i)!} \\
&= \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} \sum_{k=0}^{\infty} \frac{(sA)^k}{k!} \\
&= T(t)T(s),
\end{aligned}$$

which proves that $(T(t))_{t \geq 0}$ is a semigroup of bounded linear operators. We also note that

$$\begin{aligned}
\|T(t) - I\| &= \left\| \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} - I \right\| \\
&= \left\| I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} - I \right\| \\
&= \left\| tA \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n!} \right\| \\
&= \left\| tA \sum_{n=0}^{\infty} \frac{(tA)^n}{(n+1)!} \right\| \\
&\leq t\|A\| \sum_{n=0}^{\infty} \frac{t^n \|A\|^n}{n!}
\end{aligned}$$

$$\leq \ t\,\|A\|\,e^{t\|A\|},$$

which implies that $\lim_{t\to 0^+}\|T(t)-I\|=0$, because $\|A\|<\infty$ and $t\,\|A\|\,e^{t\|A\|}\to 0$ as $t\to 0$. From the above, we have that $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup.

We prove now that A is the infinitesimal generator of $(T(t))_{t\geq 0}$. For $t\geq 0$, we see that

$$\begin{aligned}
\left\|\frac{T(t)-I}{t}-A\right\| &= \left\|\sum_{n=1}^{\infty}\frac{t^{n-1}A^n}{n!}-A\right\| \\
&= \left\|\sum_{n=2}^{\infty}\frac{t^{n-1}A^n}{n!}\right\| \\
&\leq \sum_{n=2}^{\infty}\frac{t^{n-1}\|A\|^n}{(n-2)!} \\
&= \sum_{n=0}^{\infty} t\|A\|^2\frac{t^{n-2}\|A\|^{n-2}}{(n-2)!} \\
&= t\|A\|^2\sum_{n=0}^{\infty}\frac{(t\|A\|)^n}{n!} = t\|A\|^2 e^{t\|A\|}.
\end{aligned}$$

Using this formula and that $(T(t))_{t\geq 0}$ is uniformly continuous, we have that

$$\lim_{t\downarrow 0}\left\|\frac{T(t)-I}{t}-A\right\|\leq \lim_{t\downarrow 0} t\|A\|^2e^{t\|A\|}=0,$$

which implies that A is the infinitesimal generator of $(T(t))_{t\geq 0}$.

Assume now that $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of bounded linear operators on X. Fixing $\rho>0$ small enough such that $\|I-T(s)\|<1$ if $0<s<\rho$, we have

$$\left\|I-\rho^{-1}\int_0^{\rho}T(s)ds\right\| = \left\|\rho^{-1}\int_0^{\rho}[I-T(s)]\,ds\right\| \leq \rho^{-1}\int_0^{\rho}\|I-T(s)\|ds<1,$$

which implies that $\rho^{-1}\int_0^{\rho}T(s)ds$ and $\int_0^{\rho}T(s)ds$ are invertible (see Proposition A.0.4). Moreover, for $h>0$ we note

$$\begin{aligned}
\left\|\int_0^{\rho+h}T(s)ds-\int_0^{\rho}T(s)ds\right\| &= \left\|\int_0^{\rho}T(s)ds+\int_{\rho}^{\rho+h}T(s)ds-\int_0^{\rho}T(s)ds\right\| \\
&\leq \int_{\rho}^{\rho+h}\|T(s)\|ds \\
&\leq Mh\longrightarrow 0 \text{ as } h\to 0,
\end{aligned}$$

where $M=\sup_{s\in[\rho,\rho+h]}\|T(s)\|$, thus we can conclude that $\int_0^{\rho}T(s)ds$ is a bounded linear operator and from the Bounded Inverse Theorem (see Theorem A.0.5), we conclude that $(\int_0^{\rho}T(s)ds)^{-1}$ is also a bounded linear operator. Moreover, for $0<h<\rho$ we get

$$\begin{aligned}
h^{-1}(T(h)-I)\int_0^{\rho}T(s)ds &= h^{-1}\left[\int_0^{\rho}T(h)T(s)ds-\int_0^{\rho}T(s)ds\right] \\
&= h^{-1}\left[\int_0^{\rho}T(s+h)ds-\int_0^{\rho}T(s)ds\right] \\
&= h^{-1}\left[\int_h^{\rho+h}T(s)ds-\int_0^{\rho}T(s)ds\right]
\end{aligned}$$

$$= h^{-1}\left[\int_{\rho}^{\rho+h} T(s)ds - \int_0^h T(s)ds\right]$$

hence

$$h^{-1}(T(h)-I) = h^{-1}\left[\int_{\rho}^{\rho+h} T(s)ds - \int_0^h T(s)ds\right](\int_0^{\rho} T(s)ds)^{-1}.$$

Using now that $\lim_{h\to 0^+} h^{-1}\int_0^h T(s)ds = I$, we have that

$$\lim_{h\to 0^+}\frac{1}{h}\int_{\rho}^{\rho+h} T(s)ds = \lim_{h\to 0^+}\frac{1}{h}\int_0^h T(s+\rho)ds = T(\rho)\lim_{h\to 0^+}\frac{1}{h}\int_0^h T(s)ds = T(\rho)$$

and

$$\lim_{h\to 0^+}\frac{T(h)-I}{h} = (T(\rho)-I)(\int_0^{\rho} T(s)ds)^{-1} = A,$$

which proves that A is a bounded linear operator on X. This completes the proof. ■

Theorem 1.1.1 shows that a bounded linear operator is the infinitesimal generator of a uniformly continuous semigroup. Now, we guarantee the uniqueness of this semigroup.

Theorem 1.1.2 Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be uniformly continuous semigroups of bounded linear operators. If

$$\lim_{h\to 0^+}\frac{T(h)-I}{h} = A = \lim_{h\to 0^+}\frac{S(h)-I}{h},$$

then $T(t) = S(t)$ for all $t \geq 0$.

Proof: Let $\tau > 0$ be fixed. We show that $T(t) = S(t)$ for every $0 \leq t \leq \tau$. Using that $t \mapsto \|T(t)\|$ and $s \mapsto \|S(s)\|$ are continuous, there exists a constant $C > 0$ such that $\|T(t)\|\|S(s)\| \leq C$ for $s,t \in [0,\tau]$. From the assumption, for $\varepsilon > 0$, we can select $\delta > 0$ such that for $0 \leq h \leq \delta$

$$\begin{aligned}
h^{-1}\|T(h)-S(h)\| &= h^{-1}\|T(h)-I-(S(h)-I)\| \\
&= \left\|\frac{T(h)-I}{h} - A - \left[\frac{S(h)-I}{h} - A\right]\right\| \\
&\leq \left\|\frac{T(h)-I}{h} - A\right\| + \left\|\frac{S(h)-I}{h} - A\right\| \\
&< \frac{\varepsilon}{\tau C}.
\end{aligned}$$

Let $0 \leq t \leq \tau$ and choose $n \geq 1$ such that $\frac{t}{n} < \delta$. From the semigroup property and the last inequality, we see that

$$\begin{aligned}
&\|T(t)-S(t)\| \\
&= \left\|T\left(n\frac{t}{n}\right) - S\left(n\frac{t}{n}\right)\right\| \\
&= \left\|\sum_{k=0}^{n-1} T\left((n-k)\frac{t}{n}\right) S\left(k\frac{t}{n}\right) - T\left((n-k-1)\frac{t}{n}\right) S\left((k+1)\frac{t}{n}\right)\right\| \\
&\leq \sum_{k=0}^{n-1}\left\|T\left((n-k)\frac{t}{n}\right) S\left(k\frac{t}{n}\right) - T\left((n-k-1)\frac{t}{n}\right) S\left((k+1)\frac{t}{n}\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=0}^{n-1}\left\|T\left((n-k-1)\frac{t}{n}\right)T\left(\frac{t}{n}\right)S\left(k\frac{t}{n}\right)-T\left((n-k-1)\frac{t}{n}\right)S\left(\frac{t}{n}\right)S\left(k\frac{t}{n}\right)\right\| \\
&= \sum_{k=0}^{n-1}\left\|T\left((n-k-1)\frac{t}{n}\right)\left[T\left(\frac{t}{n}\right)-S\left(\frac{t}{n}\right)\right]S\left(k\frac{t}{n}\right)\right\| \\
&\le \sum_{k=0}^{n-1}\left\|T\left(\frac{t}{n}\right)-S\left(\frac{t}{n}\right)\right\|\left\|T\left((n-k-1)\frac{t}{n}\right)\right\|\left\|S\left(k\frac{t}{n}\right)\right\| \\
&< \sum_{k=0}^{n-1}\frac{t}{n}\frac{\varepsilon}{\tau C}C \\
&= n\frac{t}{n}\frac{\varepsilon}{\tau}=\frac{t}{\tau}\varepsilon\le\varepsilon,
\end{aligned}
$$

which implies that $T(t)=S(t)$. Noting that $\tau>0$ is arbitrary, we conclude that $T(t)=S(t)$ for all $t\ge 0$. ■

Now, we present some important properties of uniformly continuous semigroups.

Corollary 1.1.3 Let $(T(t))_{t\ge 0}$ be a uniformly continuous semigroup of bounded linear operators on X. Then

- **i)** there exists a constant $\omega\ge 0$ such that $\|T(t)\|\le e^{\omega t}$ for all $t\ge 0$,
- **ii)** there exists a unique bounded linear operator A such that $T(t)=e^{tA}$,
- **iii)** the operator A in (b) is the infinitesimal generator of $(T(t))_{t\ge 0}$,
- **iv)** the function $t\mapsto T(t)$ is differentiable and $\frac{dT(t)}{dt}=AT(t)=T(t)A$.

Proof: All the assertions follow easily from (ii).

(ii) Since $(T(t))_{t\ge 0}$ is an uniformly continuous semigroup, its infinitesimal generator A is a bounded linear operator (see Theorem 1.1.1). Thus, defining $S(t)=e^{tA}$, we have that $(S(t))_{t\ge 0}$ is uniformly continuous and A is its infinitesimal generator. Then, using Theorem 1.1.2, $T(t)=S(t)$ for all $t\ge 0$. Thus, $T(t)=e^{tA}$ and A is the infinitesimal generator of $(T(t))_{t\ge 0}$.

(i) We only note that

$$\|T(t)\|=\left\|e^{tA}\right\|\le\sum_{n=0}^{\infty}\left\|\frac{t^nA^n}{n!}\right\|\le\sum_{n=0}^{\infty}\frac{t^n\|A\|^n}{n!}=e^{\|A\|t}.$$

(iii) Follows from the proof of item (ii).

(iv) For $h\ge 0$, we have that

$$\left\|\frac{T(t+h)-T(t)}{h}-T(t)A\right\|\le\|T(t)\|\left\|\frac{T(h)-I}{h}-A\right\|\le e^{\|A\|t}\left\|\frac{T(h)-I}{h}-A\right\|$$

hence $\frac{d^+T(t)}{dt}=T(t)A$ for all $t\ge 0$.
On the other hand, for $0<h\le t$,

$$
\begin{aligned}
\left\|\frac{T(t)-T(t-h)}{h}-T(t)A\right\| &= \left\|\frac{T(t-h+h)-T(t-h)}{h}-T(t)A\right\| \\
&= \left\|\frac{T(t-h)T(h)-T(t-h)}{h}-T(t)A\right\| \\
&\le \|T(t-h)\|\left\|\frac{T(h)-I}{h}-T(h)A\right\|
\end{aligned}
$$

$$\begin{aligned}
&\leq e^{\|A\|(t-h)}\left\|\frac{T(h)-I}{h}-T(h)A\right\| \\
&\leq e^{\|A\|t}\left\|\frac{T(h)-I}{h}-T(h)A\right\| \\
&\leq e^{\|A\|t}\left\|\frac{T(h)-I}{h}-A+A-T(h)A\right\| \\
&\leq e^{\|A\|t}\left(\left\|\frac{T(h)-I}{h}-A\right\|+\|A-T(h)A\|\right) \\
&\leq e^{\|A\|t}\left\|\frac{T(h)-I}{h}-A\right\|+e^{\|A\|t}\|I-T(h)\|\,\|A\|,
\end{aligned}$$

which implies that $\frac{d^-T(t)}{dt}=T(t)A$ for $t>0$. Thus, $\frac{dT(t)}{dt}=T(t)A$ for all $t\geq 0$. To finish, we can see that $AT(t)=T(t)A$ and switching it in the above limits,

$$T(h+t)-T(t)-hAT(t)=T(h)T(t)-IT(t)-hAT(t)=(T(h)-I-hA)T(t).$$

Then,

$$\lim_{h\to 0}\frac{T(t+h)-T(t)}{h}-AT(t)=\lim_{h\to 0}\left(\frac{T(h)-I}{h}-A\right)T(t)=0,$$

which implies that $\frac{dT(t)}{dt}=AT(t)$ for all $t\geq 0$. ■

1.2 Strongly continuous semigroups of bounded linear operators

In this section, we study the class of strongly continuous semigroups of bounded linear operators and their properties.

Definition 1.2.1 A semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X is a **strongly continuous** semigroup of bounded linear operators if

$$\lim_{t\downarrow 0}T(t)x=x,\quad \forall x\in X. \tag{1.1}$$

A strongly continuous semigroup of bounded linear operators on X will be called a **semigroup of class** C_0 or simply a C_0-semigroup. From Equation (1.1), it follows that the function $t\mapsto T(t)x$ is continuous in $t=0$ for all $x\in X$.

We note that if $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of bounded linear operators, then $\lim_{t\downarrow 0}T(t)=I$. In particular, $\lim_{t\downarrow 0}T(t)x=x$ for all $x\in X$, which shows that $(T(t))_{t\geq 0}$ is a C_0-semigroup.

Next, we present an important property of C_0-semigroups, the exponential boundedness. Using this property, we can obtain the continuity of the map $t\mapsto T(t)x$, as show Corollary 1.2.2.

Theorem 1.2.1 Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. Then, there are constants $\omega\geq 0$ and $M\geq 1$ such that $\|T(t)\|\leq Me^{\omega t},\quad \forall t\in[0,\infty)$.

Proof: We show first that there is $\eta>0$ such that $\|T(t)\|$ is uniformly bounded on $[0,\eta]$. If this is false, there is a sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $t_n\geq 0$, $\lim_{n\to\infty}t_n=0$ and $\|T(t_n)\|\geq n$ for all $n\in\mathbb{N}$. From the Uniform Boundedness Theorem (see Theorem A.0.6), there exists $x\in X$ such that $\{\|T(t_n)x\|:n\in\mathbb{N}\}$ is unbounded. Thus, $\|T(t)\|\leq M$ for $0\leq t\leq\eta$. Moreover, $\|T(0)\|=1$ implies that $M\geq 1$.

Let $\omega = \eta^{-1}\ln(M) \geq 0$. For $t \geq \eta$, we can write $t = m\eta + \delta$, where $m \in \mathbb{N}$ and $0 \leq \delta < \eta$. Indeed, let $\Upsilon = \{n \in \mathbb{N} : \eta n \leq t\}$. Using that Υ is bounded, there exists $m \in \mathbb{N}$ such that $\eta m \leq t$ and $\eta(m+1) > t$, which implies there exists $0 \leq \delta < \eta$ such that $t = \eta m + \delta$. Noting that $\frac{t}{\eta} = m + \frac{\delta}{\eta} > m$, we get

$$\begin{aligned} \|T(t)\| &= \|T(m\eta+\delta)\| \\ &= \|T(\eta)\dots T(\eta)T(\delta)\| \\ &\leq \|T(\eta)\|^m\|T(\delta)\| \\ &\leq M^m M \\ &\leq M^{\frac{t}{\eta}}M \\ &= (e^{\ln M})^{\frac{t}{\eta}}M \\ &= Me^{\omega t}, \end{aligned}$$

which allows us to end the proof. ∎

Corollary 1.2.2 If $(T(t))_{t\geq 0}$ is a C_0-semigroup on X then $t \mapsto T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X, for all $x \in X$.

Proof: Let $x \in X$. For $t \in [0,\infty)$ and $0 \leq t < h$, we get

$$\begin{aligned} \|T(t+h)x - T(t)x\| &= \|T(t)T(h)x - T(t)x\| \\ &= \|T(t)\left[T(h)x - x\right]\| \\ &\leq \|T(t)\|\|T(h)x - x\| \\ &\leq Me^{\omega t}\|T(h)x - x\|. \end{aligned}$$

On the other hand, for $t > 0$ and $0 < h < t$ we see that

$$\begin{aligned} \|T(t)x - T(t-h)x\| &= \|T(t-h)\left[T(h)x - x\right]\| \\ &\leq \|T(t-h)\|\|T(h)x - x\| \\ &\leq Me^{\omega(t-h)}\|\|T(h)x - x\| \\ &\leq Me^{\omega t}\|T(h)x - x\|. \end{aligned}$$

From previous estimatives, $\lim_{h\uparrow 0} T(t-h)x = T(t)x = \lim_{h\downarrow 0} T(t+h)x$, which proves that the map $t \mapsto T(t)x$ is continuous on $\mathbb{R}_0^+$. ∎

In Theorem 1.2.3 below we present some properties of the infinitesimal generator of a C_0-semigroup on X.

Theorem 1.2.3 Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X and let A be its infinitesimal generator. Then,

i) for $x \in X$,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x. \tag{1.2}$$

ii) For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x. \tag{1.3}$$

iii) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{1.4}$$

iv) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau. \tag{1.5}$$

Proof:

i) For $t \geq 0$ and $h > 0$, we see that

$$\begin{aligned} \left\| \frac{1}{h}\int_t^{t+h} T(s)xds - T(t)x \right\| &= \left\| \frac{1}{h}\int_t^{t+h} (T(s)x - T(t)x)ds \right\| \\ &\leq \frac{1}{h}\int_t^{t+h} \|T(s)x - T(t)x\|ds. \end{aligned}$$

Using that $t \mapsto T(t)x$ is continuous, for $\varepsilon > 0$, we can select $h_0 > 0$ such that for $h < h_0$,

$$\frac{1}{h}\int_t^{t+h} \|T(s)x - T(t)x\|ds \leq \frac{1}{h}\int_t^{t+h} \varepsilon ds \leq \frac{\varepsilon}{h}(t+h-t) = \varepsilon$$

hence,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h} T(s)xdx = T(t)x.$$

ii) Let $x \in X$ and $h > 0$. Then,

$$\begin{aligned} \left(\frac{T(h)-I}{h}\right)\left(\int_0^t T(s)xds\right) &= \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds \\ &= \frac{1}{h}\left[\int_0^t T(s+h)xds - \int_0^t T(s)xds\right] \\ &= \frac{1}{h}\left[\int_h^{t+h} T(s)xds - \int_0^t T(s)xds\right] \\ &= \frac{1}{h}\left[\int_t^{t+h} T(s)xds - \int_0^h T(s)xds\right] \\ &= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds. \end{aligned}$$

Noticing that the right-hand side converges to $T(t)x - x$, we infer that $\int_0^t T(s)xds \in D(A)$ and $A(\int_0^t T(s)xds) = T(t)x - x$.

iii) Let $x \in D(A)$ and $h > 0$. Then, $\lim_{h\downarrow 0}\frac{T(h)x-x}{h}$ exists and

$$\lim_{h\downarrow 0}\frac{T(h)-I}{h}T(t)x = \lim_{h\downarrow 0}\frac{T(h)T(t)-T(t)}{h}x = \lim_{h\downarrow 0}T(t)\frac{T(h)-I}{h}x = T(t)Ax.$$

Thus, $T(t)x \in D(A)$ and $\frac{d^+T(t)x}{dt} = AT(t)x = T(t)Ax$. To prove the formula (1.4), for $t > 0$, we show that the left derivative of $T(\cdot)x$ at t exists and is equal to $\frac{d}{dt}T(t)x = T(t)Ax$.
For $0 < h < t$ we set

$$\left\| \frac{T(t)x - T(t-h)x}{h} - T(t)Ax \right\| = \left\| T(t-h)\left((\frac{T(h)x - x}{h}) - T(h)Ax \right) \right\|$$

$$\begin{aligned}
&\leq \|T(t-h)\|\|\frac{T(h)x-x}{h}-Ax+Ax-T(h)Ax\| \\
&\leq Me^{\omega(t-h)}\left(\left\|\frac{T(h)x-x}{h}-Ax\right\|+\|Ax-T(h)Ax\|\right) \\
&\leq Me^{\omega t}\left\|\frac{T(h)x-x}{h}-Ax\right\|+Me^{\omega t}\|Ax-T(h)Ax\|,
\end{aligned}$$

which implies that $\frac{d^-}{dt}(T(t)x)=T(t)Ax$. This concludes the proof of (iii).

iv) Let $x\in D(A)$. From (iii), $T(t)x\in D(A)$ and $\frac{d}{dt}(T(t)x)=T(t)Ax=AT(t)x$. Then, for $t,s\geq 0$,

$$T(t)x-T(s)x=\int_s^t\frac{d}{d\tau}(T(\tau)x)d\tau=\int_s^t AT(\tau)xd\tau=\int_s^t T(\tau)Axd\tau. \blacksquare$$

The Corollary 1.2.4 below proves that if A is an infinitesimal generator, then A is closed (see Definition A.0.1) and $D(A)$ is dense in X.

Corollary 1.2.4 If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, then A is closed and its domain, $D(A)$, is dense on X.

Proof: It is obvious that $\overline{D(A)}\subset X$. Let $x\in X$ and $x_t=\frac{1}{t}\int_0^t T(s)xds$. From item (ii) of Theorem 1.2.3, we know that $\int_0^t T(s)xds\in D(A)$ for $t>0$, hence

$$\lim_{h\downarrow 0}\frac{T(h)-I}{h}\left(\frac{1}{t}\int_0^t T(s)xds\right)=\frac{1}{t}\lim_{h\downarrow 0}\frac{T(h)-I}{h}\left(\int_0^t T(s)xds\right)=\frac{T(t)x-x}{t},$$

which implies that $x_t\in D(A)$ for all $t\geq 0$. Moreover, from Theorem 1.2.3 (i),

$$\lim_{t\downarrow 0}x_t=\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)xds=T(0)x=x.$$

Thus $x_t\to x$ as $t\downarrow 0$, which shows that $x\in\overline{D(A)}$ and $\overline{D(A)}=X$.

The linearity of A is obvious. To prove that A is closed, let $(x_n)_n$ be a sequence in $D(A)$ such that $x_n\to x$ and $Ax_n\to y$ as $n\to\infty$. From Theorem 1.2.3 (iv), for $n\in\mathbb{N}$ we have

$$T(t)x_n-x_n=\int_0^t T(s)Ax_nds. \tag{1.6}$$

Using that $t\mapsto T(t)$ is a continuous operator and

$$\|T(t)Ax_n-T(t)y\|\leq\|T(t)\|\|Ax_n-y\|\leq Me^{\omega t}\|Ax_n-y\|,$$

we conclude that the integrand on the right-hand side of (1.6) converges to $T(s)y$ uniformly on bounded intervals.

Therefore, for $a>0$ and $t\in[0,a]$,

$$\left\|\int_0^t T(s)Ax_nds-\int_0^t T(s)yds\right\|\leq\int_0^t\|T(s)(Ax_n-y)\|ds\leq Me^{\omega a}\|Ax_n-y\|a$$

and

$$\int_0^t T(s)Ax_nds\to\int_0^t T(s)yds.$$

Taking limits on both sides of (1.6), we obtain that

$$T(t)x - x = \int_0^t T(s)y\,ds.$$

Hence,

$$\lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t T(s)y\,ds = T(0)y = y,$$

which implies that $x \in D(A)$ and $Ax = y$. ■

Theorem 1.2.5 Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be C_0-semigroups with infinitesimal generators A and B, respectively. If $A = B$, then $T(t) = S(t)$ for $t \geq 0$.

Proof: Let $x \in D(A) = D(B)$. From Theorem 1.2.3, the map $s \mapsto T(t-s)S(s)x$ is differentiable and

$$\begin{aligned}
&\frac{1}{h}[T(t-(s+h))S(s+h)x - T(t-s)S(s)x] \\
&= \frac{1}{h}[T(t-(s+h))S(s+h)x - T(t-(s+h))S(s)x] \\
&\quad + \frac{1}{h}[T(t-(s+h))S(s)x - T(t-s)S(s)x] \\
&= \frac{1}{h}[T(t-(s+h))(S(s+h)x - S(s)x)] \\
&\quad + \frac{1}{h}[(T(t-(s+h))x - T(t-s)x)S(s)x] \\
&= T(t-(s+h))\underbrace{\frac{S(s+h)-S(s)}{h}}_{\text{converges to } BS(s)x} x - T(t-(s+h))\underbrace{\frac{I-T(h)}{h}}_{\text{converges to } A} S(s)x.
\end{aligned}$$

Using Theorem 1.2.3 and the definition of the infinitesimal generator, we conclude that

$$\begin{aligned}
\frac{dT(t-s)S(s)x}{ds} &= \lim_{h\downarrow 0} \frac{T(t-(s+h))S(s+h)x - T(t-s)S(s)x}{h} \\
&= T(t-s)BS(s)x - T(t-s)AS(s)x = 0.
\end{aligned}$$

Thus, $s \mapsto T(t-s)S(s)x$ is constant. Using $s = 0$ and $s = t$ we have that $T(t)x = S(t)x$ for all $x \in D(A)$. Furthermore, using that $D(A)$ dense in X and $T(t)$ and $S(t)$ are bounded, there is a sequence $(x_n)_n$ in $D(A)$ such that $x_n \to x$ and

$$T(t)x = \lim_{n\to\infty} T(t)x_n = \lim_{n\to\infty} S(t)x_n = S(t)x. \blacksquare$$

Theorem 1.2.6 Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. If $D(A^n)$ is the domain of A^n, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.

Proof: Let

$$\mathscr{D} = \{f : (0,\infty) \to \mathbb{C} : f \text{ is infinitely differentiable with compact support}\}.$$

For $x \in X$ and $\varphi \in \mathscr{D}$, let

$$y = x(\varphi) = \int_0^\infty \varphi(s)T(s)xds. \tag{1.7}$$

Noting that the support $\{s \in (0,\infty) : \varphi(s) \neq 0\}$ is compact, it is easy to see that the integral above is well defined, thus $y \in X$. Moreover, for $h > 0$ we have

$$\begin{aligned}
\frac{T(h)-I}{h}y &= \frac{1}{h}\left[T(h)\int_0^\infty \varphi(s)T(s)xds - \int_0^\infty \varphi(s)T(s)xds\right] \\
&= \frac{1}{h}\left[\int_0^\infty \varphi(s)T(s+h)xds - \int_0^\infty \varphi(s)T(s)xds\right] \\
&= \frac{1}{h}\left[\int_0^\infty \varphi(h-s)T(s)xds - \int_0^\infty \varphi(s)T(s)xds\right] \\
&= \int_0^\infty \frac{\varphi(h-s)-\varphi(s)}{h}T(s)xds.
\end{aligned} \tag{1.8}$$

Noting that the integrand on the right-hand side of (1.8) converges to $-\varphi'(s)T(s)x$ as $h \downarrow 0$ uniformly on $[0,\infty)$ we conclude that $y \in D(A)$ and

$$Ay = \lim_{h\downarrow 0}\frac{T(h)-I}{h}y = -\int_0^\infty \varphi'(s)T(s)xds.$$

In addition,

$$\begin{aligned}
\lim_{h\downarrow 0}\frac{T(h)-I}{h}Ay &= \lim_{h\downarrow 0}\frac{T(h)-I}{h}\left(\int_0^\infty -\varphi'(s)T(s)xds\right) \\
&= \int_0^\infty \lim_{h\downarrow 0} -\frac{\varphi'(h-s)-\varphi'(s)}{h}T(s)xds \\
&= \int_0^\infty \varphi''(s)T(s)xds
\end{aligned}$$

Repeating the previous argument for $n = 1,2,\ldots$, we have that $y \in D(A^n)$ and

$$A^n y = (-1)^n \int_0^\infty \varphi^{(n)}(s)T(s)xds,\ n = 1,2,\ldots$$

which proves that $y \in \bigcap_{n=1}^\infty D(A^n)$.

Let $Y = \{y = x(\varphi) : x \in X \text{ and } \varphi \in \mathscr{D}\}$. From the above $Y \subseteq \bigcap_{n=1}^\infty D(A^n)$.

To conclude the proof we show that Y is dense in X. If this is false, from Hahn-Banach's Theorem (see Theorem A.0.7), there is a non-zero functional $x^* \in X^*$ such that $x^*(y) = 0$ for every $y \in Y$ and therefore

$$0 = x^*\left(\int_0^\infty \varphi(s)T(s)xds\right) = \int_0^\infty \varphi(s)x^*(T(s)x)ds \tag{1.9}$$

for every $x \in X, \varphi \in \mathscr{D}$. Then, for $x \in X$, the continuous function $s \mapsto x^*(T(s)x)$ must vanishes identically on $[0,\infty)$ since otherwise, it would have been possible to choose $\varphi \in \mathscr{D}$ such that the left-hand side of (1.9) does not vanish. Thus, for $s = 0$, $x^*(T(0)x) = x^*(x) = 0$ for all $x \in X$, which implies that $x^* = 0$. This is a contradiction. ■

1.3 The Hille-Yosida Theorem

Now, we study a characterization of infinitesimal generators of C_0-semigroups of contractions. We start defining C_0-semigroup of contractions and the Yosida approximation of an infinitesimal generator together with his resolvent set.

Definition 1.3.1 A C_0-semigroup $(T(t))_{t\geq 0}$ is called **uniformly bounded** if exists $M>0$ such that $\|T(t)\|\leq M$ for all $t\geq 0$. In addiction if $M=1$, it is called a C_0**-semigroup of contractions**.

Definition 1.3.2 Let $A:D(A)\subset X\to X$ be a linear operator.

i) The **resolvent set** of A is the set defined by

$$\rho(A)=\{\lambda\in\mathbb{C}:(\lambda I-A)^{-1}\text{ exists and is continuous on }X\}.$$

ii) The **resolvent operator** is the function $R:\rho(A)\to\mathscr{L}(X)$ given by $R(\lambda:A)=(\lambda I-A)^{-1}$.

iii) The **spectrum** of A is the complementar of the resolvent set, denoted by $\sigma(A)$.

iv) Let $A:D(A)\subset X\to X$ be a linear operator such that $\rho(A)\neq\emptyset$. The **Yosida aproximation** of A is defined by

$$A_\lambda=\lambda AR(\lambda:A)=\lambda^2R(\lambda:A)-\lambda I,\ \lambda\in\rho(A). \tag{1.10}$$

Lemma 1.3.1 Let $A:D(A)\subset X\to X$ be a densely defined closed linear operator and assume $(0,\infty)\subset\rho(A)$ and $\|R(\lambda:A)\|\leq\frac{1}{\lambda}$ for all $\lambda>0$. Then

i) $\lim_{\lambda\to\infty}\lambda R(\lambda:A)x=x$, for all $x\in X$.

ii) $\lim_{\lambda\to\infty}A_\lambda x=Ax$, for all $x\in D(A)$.

iii) For $\lambda\in\rho(A)$, A_λ is a bounded operator in X. Moreover, A_λ is the infinitesimal generator of the uniformly continuous semigroup of contractions $(e^{tA_\lambda})_{t\geq 0}$ and

$$\|e^{tA_\lambda}x-e^{tA_\mu}x\|\leq t\|A_\lambda x-A_\mu x\|,\ \text{for all } x\in X,\lambda>0\text{ and }\mu>0.$$

Proof:

i) For $\lambda\in\rho(A)$ and $x\in D(A)$,

$$\begin{aligned}\|\lambda R(\lambda:A)x-x\| &= \|\lambda(\lambda I-A)^{-1}x-(\lambda I-A)(\lambda I-A)^{-1}x\|\\ &= \|(\lambda I-A)^{-1}[\lambda x-(\lambda I-A)x]\|\\ &\leq \|(\lambda I-A)^{-1}\|\|Ax\|\\ &\leq \frac{1}{\lambda}\|Ax\|.\end{aligned}$$

From the above, $\lim_{\lambda\to\infty}\|\lambda R(\lambda:A)x-x\|\leq\lim_{\lambda\to\infty}\frac{1}{\lambda}\|Ax\|=0$ and

$$\lim_{\lambda\to\infty}\lambda R(\lambda:A)x=x,\forall x\in D(A). \tag{1.11}$$

Let $x\in X$ and $(x_n)_n$ be a sequence in $D(A)$ such that $x_n\to x$. Noticing that $\|\lambda R(\lambda:A)\|\leq 1$, for $\varepsilon>0$, we select $N_\varepsilon\in\mathbb{N}$ such that $\|\lambda R(\lambda:A)(x_n-x)\|\leq\frac{\varepsilon}{3}$ and $\|x_n-x\|\leq\frac{\varepsilon}{3}$ for all $n\geq N_\varepsilon$. Using (1.11), we see that

$$\begin{aligned}\|\lambda R(\lambda:A)x-x\| &= \|\lambda R(\lambda:A)x-\lambda R(\lambda:A)x_{N_\varepsilon}+\lambda R(\lambda:A)x_{N_\varepsilon}-x_{N_\varepsilon}+x_{N_\varepsilon}-x\|\\ &\leq \|\lambda R(\lambda:A)(x_{N_\varepsilon}-x)\|+\|\lambda R(\lambda:A)x_{N_\varepsilon}-x_{N_\varepsilon}\|+\|x_{N_\varepsilon}-x\|\\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\end{aligned}$$

which proves that $\lim_{\lambda\to\infty}\lambda R(\lambda:A)x=x$ for all $x\in X$.

ii) Let $x \in D(A)$. Using the definition of A_λ and item (i),

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda AR(\lambda : A)x = \lim_{\lambda\to\infty} A\lambda R(\lambda : A)x = A \lim_{\lambda\to\infty} \lambda R(\lambda : A)x = Ax.$$

iii) From (1.10), we see that

$$\|A_\lambda\| = \|\lambda^2 R(\lambda : A) - \lambda I\| \leq \lambda^2 \|R(\lambda : A)\| + \|\lambda I\| \leq \lambda^2 \frac{1}{\lambda} + \lambda = 2\lambda$$

and hence A_λ is a bounded linear operator, for each $\lambda \in \rho(A)$. From Theorem 1.1.1, we conclude that A_λ is the infinitesimal generator of $\{e^{tA_\lambda}\}_{t\geq 0}$. Moreover, for $t \geq 0$ we get

$$\|e^{tA_\lambda}\| = \|e^{t(\lambda^2 R(\lambda:A) - \lambda I)}\| \leq e^{t\lambda^2 \|R(\lambda:A)\|} e^{-t\lambda \|I\|} \leq e^{t\lambda^2 \frac{1}{\lambda}} e^{-t\lambda} = \frac{e^{t\lambda}}{e^{t\lambda}} = 1,$$

which proves that $\{e^{tA_\lambda}\}_{t\geq 0}$ is a C_0-semigroup of contractions.
From the above remarks, for $x \in X$ and $\lambda, \mu > 0$,

$$\begin{aligned}
\|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \left\| \int_0^1 \frac{d}{ds}\left(\left(e^{tsA_\lambda} e^{t(1-s)A_\mu}\right) x\right) ds \right\| \\
&= \left\| \int_0^1 (tA_\lambda e^{tsA_\lambda} e^{t(1-s)A_\mu})x + \left(e^{tsA_\lambda}\left(-tA_\mu e^{t(1-s)A_\mu}\right)\right) xds \right\| \\
&\leq \int_0^1 \|e^{tsA_\lambda} e^{t(1-s)A_\mu}\| \|tA_\lambda x - tA_\mu x\| ds \\
&\leq t\|A_\lambda x - A_\mu x\| \int_0^1 \|e^{tsA_\lambda}\| \|e^{t(1-s)A_\mu}\| ds \\
&\leq t\|A_\lambda x - A_\mu x\|. \ \blacksquare
\end{aligned}$$

The last result before the main theorem of this section gives us an important property of closed linear operators that will be used along all the text.

Lemma 1.3.2 Let $A : D(A) \subseteq X \to X$ be a closed linear operator and $a > 0$. If $f \in L^1([0,a];X)$ is such that $Af \in L^1([0,a];X)$, then

$$A\left(\int_0^a f(s)ds\right) = \int_0^a Af(s)ds$$

Proof: Let $\tau = \{\alpha_i : i = 1, 2, \ldots, n\}$ a partition of $[0,a]$ and

$$S(f,\tau) = \sum_{i=1}^{n-1} f(\eta_i)(\alpha_{i+1} - \alpha_i), \eta_i \in [\alpha_i, \alpha_{i+1}].$$

Noting that

$$\begin{aligned}
AS(f,\tau) &= A\left(\sum_{i=1}^{n-1} f(\eta_i)(\alpha_{i+1} - \alpha_i)\right) \\
&= \sum_{i=1}^{n-1} Af(\eta_i)(\alpha_{i+1} - \alpha_i),
\end{aligned}$$

from hypothesis we conclude that $S(f,\tau) \in D(A)$ and

$$S(f,\tau) \longrightarrow \int_0^a f(s)ds \text{ and } AS(f,\tau) \longrightarrow \int_0^a Af(s)ds,$$

as $\Lambda P = \sup_{1\leq k\leq n}|\alpha_{i+1}-\alpha_i| \longrightarrow 0$. Finally, noting that A is a closed linear operator, we infer that

$$A\left(\int_0^a f(s)ds\right) = \int_0^a Af(s)ds. \blacksquare$$

Now we present the main result of this section, which establishes sufficient and necessary conditions on A so that it is the infinitesimal generator of a C_0-semigroup of contractions.

Theorem 1.3.3 — Hille-Yosida. A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ if and only if

i) A is closed and $\overline{D(A)} = X$.

ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}, \ \forall\lambda > 0. \tag{1.12}$$

Proof: Assume that A is the infinitesimal generator of a C_0-semigroup of contractions. From Corollary 1.2.4, A is closed and densely defined on X. To show that $\|R(\lambda : A)\| \leq \frac{1}{\lambda}$, for $\lambda > 0$, we define the operator $R(\lambda) : X \to X$ by

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt. \tag{1.13}$$

Noting that the function $t \mapsto T(t)x$ is continuous and uniformly bounded, the integral on the right-hand side of (1.13) exists and defines a bounded linear operator. Moreover,

$$\|R(\lambda)x\| \leq \int_0^\infty e^{-\lambda t}\|T(t)\|\|x\|dt = \|x\|\int_0^\infty e^{-\lambda t}dt = \frac{\|x\|}{\lambda}.$$

In addition, for $\lambda > 0$, we see that

$$\begin{aligned}
\frac{T(h)-I}{h}R(\lambda)x &= \frac{1}{h}\left[\int_0^\infty e^{-\lambda t}T(t+h)xdt - \int_0^\infty e^{-\lambda t}T(t)xdt\right]\\
&= \frac{1}{h}\left[\int_h^\infty e^{-\lambda(t-h)}T(t)xdt - \int_0^\infty e^{-\lambda t}T(t)xdt\right]\\
&= \frac{1}{h}\int_h^\infty \frac{e^{\lambda h}}{e^{\lambda t}}T(t)xdt + \frac{1}{h}\int_0^h \frac{e^{\lambda h}}{e^{\lambda t}}T(t)xdt\\
&\quad -\frac{1}{h}\int_0^h \frac{e^{\lambda h}}{e^{\lambda t}}T(t)xdt - \frac{1}{h}\int_0^\infty \frac{1}{e^{\lambda t}}T(t)xdt\\
&= \frac{1}{h}\int_0^\infty \frac{e^{\lambda h}}{e^{\lambda t}}T(t)xdt - \frac{1}{h}\int_0^h \frac{e^{\lambda h}}{e^{\lambda t}}T(t) - \frac{1}{h}\int_0^\infty \frac{1}{e^{\lambda t}}T(t)xdt\\
&= \frac{e^{\lambda h}}{h}\int_0^\infty \frac{1}{e^{\lambda t}}T(t)xdt - \frac{e^{\lambda h}}{h}\int_0^h \frac{1}{e^{\lambda t}}T(t) - \int_0^\infty \frac{1}{e^{\lambda t}}T(t)xdt\\
&= \left(\frac{e^{\lambda h}-1}{h}\right)\left(\int_0^\infty \frac{1}{e^{\lambda t}}T(t)xdt\right) - \frac{e^{\lambda h}}{h}\int_0^h \frac{1}{e^{\lambda t}}T(t)xdt\\
&= \left(\frac{e^{\lambda h}-1}{h}\right)R(\lambda)x - \frac{e^{\lambda h}}{h}\int_0^h \frac{1}{e^{\lambda t}}T(t)xdt. \qquad (1.14)
\end{aligned}$$

Noting that $\lim_{h\to 0}\frac{e^{\lambda h}-1}{h}=\lambda$ and that

$$\begin{aligned}\left\|\frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)xdt-x\right\| &\le \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}\|T(t)x-T(0)x\|dt\\ &\le \frac{e^{\lambda h}}{h}\sup_{s\in[0,h]}\|T(s)x-x\|\int_0^h e^{-\lambda t}dt \longrightarrow 0 \text{ as } h\to 0,\end{aligned}$$

we obtain that the right-hand side of (1.14) converges to $\lambda R(\lambda)x-x$ as $h\downarrow 0$. This implies that for every $x\in X$ and $\lambda>0$, $R(\lambda)x\in D(A)$, $AR(\lambda)=\lambda R(\lambda)-I$ and $(\lambda I-A)R(\lambda)=I$.

On the other hand, from Lemma 1.3.2, for $x\in D(A)$, we have that

$$\begin{aligned}R(\lambda)Ax &= \int_0^\infty e^{-\lambda t}T(t)Axdt\\ &= \int_0^\infty e^{-\lambda t}AT(t)xdt\\ &= A\left(\int_0^\infty e^{-\lambda t}T(t)xdt\right)\\ &= AR(\lambda)x,\end{aligned}$$

which implies

$$R(\lambda)(\lambda I-A)x=R(\lambda)\lambda x-R(\lambda)Ax=\lambda R(\lambda)x-AR(\lambda)x=(\lambda I-A)R(\lambda)x=x.$$

From the previous, $R(\lambda)$ is the inverse of $(\lambda I-A)$. Then,

$$\|R(\lambda:A)\|=\|(\lambda I-A)^{-1}\|=\|R(\lambda)\|\le\frac{1}{\lambda},\ \ \lambda>0.$$

Let $x\in D(A)$. From Lemma 1.3.1, we have that

$$\|e^{tA_\lambda}x-e^{tA_\mu}x\|\le t\|A_\lambda x-A_\mu x\|\le t\|A_\lambda x-Ax\|+t\|Ax-A_\mu x\|. \tag{1.15}$$

From the Inequality (1.15) and Lemma 1.3.1, it follows that $(e^{tA_\lambda}x)_{\lambda>0}$ is convergent in X, for all $x\in X$. Moreover, the convergence is uniform on compact intervals of t. We denote this limit by $T(t)x$, which is linear and bounded. Since $D(A)$ is dense on X and $\|T(t)\|\le 1$, there is a unique linear bounded extension of $T(t)x$ on X, which we denote by $T(t)x$. We now note that,

i) $(T(t))_{t\ge 0}$ is a semigroup of bounded linear operators.
For $x\in X$ and $t,s\ge 0$,

$$\begin{aligned}T(0)x &= \lim_{\lambda\to\infty}T_\lambda(0)x=\lim_{\lambda\to\infty}Ix=x,\\ T(t+s)x &= \lim_{\lambda\to\infty}T_\lambda(t+s)x=\lim_{\lambda\to\infty}T_\lambda(t)T_\lambda(s)x=T(t)T(s)x.\end{aligned}$$

ii) $(T(t))_{t\ge 0}$ is a semigroup of contractions.
Let $x\in X$. Using that $(T_\lambda(t))_{t\ge 0}$ is a semigroup of contractions, we have

$$\|T(t)x\|=\left\|\lim_{\lambda\to\infty}T_\lambda(t)x\right\|=\lim_{\lambda\to\infty}\|T_\lambda(t)x\|\le 1,$$

which implies that $\|T(t)\|\le 1$ for all $t\ge 0$.

iii) $(T(t))_{t\geq 0}$ is strongly continuous.
Note that

$$\|T(t+h)x - T(t)x\| \leq \lim_{\lambda\to\infty} \|T_\lambda(t+h)x - T_\lambda(t)x\| \to 0,$$

because $T_\lambda((t)x) \longrightarrow T(t)x$ uniformly on compacts intervals of $[0,\infty)$.

From the above remarks, we conclude that $(T(t))_{t\geq 0}$ is a C_0-semigroup of contractions.

To finish the proof, we show that A is the infinitesimal generator of $(T(t))_{t\geq 0}$. Let $x \in D(A)$. Using the definition of $T(t)$, Theorem 1.2.3 (iv) and that

$$\begin{aligned} \|T_\lambda(s)A_\lambda x - T(s)Ax\| &\leq \|T_\lambda(s)A_\lambda x - T_\lambda(s)Ax\| + \|T_\lambda(s)Ax - T(s)Ax\| \\ &\leq \|T_\lambda(s)Ax\|\|A_\lambda x - Ax\| + \|T_\lambda(s)Ax - T(s)Ax\| \end{aligned}$$

we infer that $\lim_{\lambda\to\infty} T_\lambda(s)A_\lambda x = T(s)Ax$ uniformly on compact sets of s. From the above, for $t > 0$,

$$T(t)x - x = \lim_{\lambda\to\infty} (T_\lambda(t)x - T_\lambda(0)x) = \lim_{\lambda\to\infty} \int_0^t e^{sA_\lambda} A_\lambda x ds = \int_0^t T(s)Axds. \tag{1.16}$$

Let B be the infinitesimal generator of $(T(t))_{t\geq 0}$ and let $x \in D(A)$. From (1.16) and Theorem 1.2.3 (i), we see that

$$Bx = \lim_{h\downarrow 0} \frac{T(h)x - x}{h} = \lim_{h\downarrow 0} \frac{1}{h} \int_0^h T(s)Axds = T(0)Ax = Ax,$$

which implies that $D(A) \subseteq D(B)$ and $A = B$ on $D(A)$.

Noting that B is the infinitesimal generator of $(T(t))_{t\geq 0}$, it follows from the necessary conditions that $1 \in \rho(B)$. On the other hand, by assumption (ii), we have that $1 \in \rho(A)$, then $(I - A)$ and $(I - B)$ are invertible. Remarking that $D(A) \subseteq D(B)$, $(I - B)D(A) = (I - A)D(A) = X$ and

$$(I-B)^{-1}(I-B)D(A) = (I-B)^{-1}X$$

we infer that $D(A) = (I-B)^{-1}X = D(B)$. Therefore, A is the infinitesimal generator of a C_0-semigroup of contractions. ■

Next, we present two corollaries of the Hille-Yosida Theorem. The first is a property that justifies the name "Yosida approximation" and the second is about an estimative for the norm of the resolvent operator.

Corollary 1.3.4 Let A be the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$. If A_λ is the Yosida approximation of A, then

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda} x, \ \ \forall x \in X. \tag{1.17}$$

Proof: From the proof of Theorem 1.3.3 it follows that the right-hand side of (1.17) defines a C_0-semigroup of contractions, $(S(t))_{t\geq 0}$, with infinitesimal generator A. From Theorem 1.2.5, we obtain that $T(t) = S(t)$, for all $t \geq 0$. ■

Corollary 1.3.5 Let A be the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$. Then $\rho(A) \supseteq \{\lambda : \mathrm{Re}\,\lambda > 0\}$ and $\|R(\lambda : A)\| \leq \frac{1}{\mathrm{Re}\,\lambda}$ for all λ with $\mathrm{Re}(\lambda) > 0$.

Proof: Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ and $R(\lambda)$ be the operator

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt.$$

From the proof of Hille-Yosida Theorem (see Theorem 1.3.3), we know that $R(\lambda) = (\lambda I - A)^{-1}$ and, hence, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(A)$. Moreover, for $\lambda \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $x \in X$,

$$\|R(\lambda : A)x\| \leq \int_0^\infty \|e^{-\lambda t}\| \|T(t)x\| dt \leq \int_0^\infty e^{(\operatorname{Re}\lambda)t} \|T(t)\| \|x\| dt \leq \frac{\|x\|}{\operatorname{Re}\lambda},$$

which conclude the proof. ■

The following example shows that the resolvent set of the infinitesimal generator of a C_0-semigroup of contractions need not contain more than the open right half-plane.

■ **Example 1.1** Let $X = BU([0,\infty))$ the space of all bounded uniformly continuous functions on $[0,\infty)$ and $T(t) : X \to X$ be defined by

$$(T(t)f)(x) = f(t+x).$$

We claim that $(T(t))_{t\geq 0}$ is a C_0-semigroup of contractions on X. In fact,

i) $(T(t))_{t\geq 0}$ is a semigroup.
For $f \in X$, $x \in [0,\infty)$ and $t, s \geq 0$, we get

$$\begin{aligned} T(0)(f(x)) &= f(0+x) = f(x), \\ T(t+s)(f(x)) &= f((t+s)+x) = f(t+s+x) = T(t)T(s)(f)(x). \end{aligned}$$

ii) $(T(t))_{t\geq 0}$ is semigroup of contractions.
For $f \in X$,

$$\|T(t)f\| = \sup_{x\geq 0} \|T(t)f(x)\| = \sup_{x\geq 0} \|f(t+x)\| \leq \|f\|.$$

iii) $(T(t))_{t\geq 0}$ is strongly continuous.
For $f \in X$, we have that

$$\lim_{t\to 0} \|T(t)f - f\| = \lim_{t\to 0} \sup_{x\geq 0} \|T(t)f(x) - f(x)\| = \lim_{t\to 0} \sup_{x\geq 0} |f(t+x) - f(x)| = 0$$

because f is uniformly continuous.

Moreover, from the definition of the operator $T(t)$ it is easy to see that the infinitesimal generator of $(T(t))_{t\geq 0}$ is given by

$$(Af)(s) = f'(s), \ f \in D(A), \tag{1.18}$$

where

$$D(A) = \{f : [0,\infty) \to [0,\infty) : f, f' \in X\}. \tag{1.19}$$

From Corollary 1.3.5, we know that $\rho(A) \supseteq \{\lambda : \operatorname{Re}\lambda > 0\}$. In addition, for $\lambda \in \mathbb{C}$, the equation $(\lambda - A)\phi_\lambda = 0$ has nontrivial solution given by $\phi_\lambda(s) = e^{\lambda s}$. Furthermore, if $\operatorname{Re}\lambda \leq 0$ and $\phi_\lambda \in X$ therefore the closed left half-plane is in the spectrum $\sigma(A)$ of A. Then, by definition, the closed left half-plane is not in the resolvent set of A. ■

The Hille-Yosida Theorem can be extend to semigroups such that $\|T(t)\| \leq e^{\omega t}$, for all $t \geq 0$ and some $\omega \geq 0$. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup satisfying the above conditions and define $S(t) = e^{-\omega t} T(t)$. Then,

i) $(S(t))_{t\geq 0}$ is linear for every fixed t.
In fact, for $x,y\in X$ and $\lambda\in\mathbb{C}$,

$$S(t)(x+\lambda y)=e^{-\omega t}T(t)(x+\lambda y)=e^{-\omega t}T(t)x+\lambda e^{-\omega t}T(t)y=S(t)x+\lambda S(t)y.$$

ii) $(S(t))_{t\geq 0}$ is a semigroup.
For $t,s\in[0,\infty)$,

$$\begin{aligned} S(0) &= e^{-\omega(0)}T(0)=e^{0}T(0)=I \\ S(t+s) &= e^{-\omega(t+s)}T(t+s)=e^{-\omega t}T(t)e^{-\omega s}T(s)=S(t)S(s). \end{aligned}$$

iii) $(S(t))_{t\geq 0}$ is strongly continuous.
We just observe that

$$\lim_{t\downarrow 0}S(t)x=\lim_{t\downarrow 0}e^{-\omega t}T(t)x=(\lim_{t\downarrow 0}e^{-\omega t})(\lim_{t\downarrow 0}T(t)x)=\lim_{t\downarrow 0}T(t)x=x.$$

iv) $(S(t))_{t\geq 0}$ is a semigroup of contractions.
For $t\geq 0$,

$$\|S(t)\|=\|e^{-\omega t}T(t)\|\leq\|e^{-\omega t}\|\|T(t)\|\leq\frac{e^{\omega t}}{e^{\omega t}}=1.$$

From above remarks, $(S(t))_{t\geq 0}$ is a C_0-semigroup of contractions. Moreover, if A is the infinitesimal generator of $(T(t))_{t\geq 0}$, then $(A-\omega I)$ is the infinitesimal generator of $(S(t))_{t\geq 0}$ because $\frac{dS(t)}{dt}\Big|_{t=0}=\frac{d}{dt}(e^{-\omega t}T(t))\Big|_{t=0}=A-\omega I$. On the other hand, if A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$, then $(A+\omega I)$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\|\leq e^{\omega t}$. Certainly, $T(t)=e^{\omega t}S(t)$. From the above remarks, we obtain the next characterization of the infinitesimal generators of a C_0-semigroups satisfying $\|T(t)\|\leq e^{\omega t}$.

Corollary 1.3.6 A linear operator A is the infinitesimal generator of a C_0-semigroup satisfying $\|T(t)\|\leq e^{\omega t}$ if and only if

i) A is closed and $\overline{D(A)}=X$.

ii) The resolvent set $\rho(A)$ of A contains $\{\lambda:\operatorname{Im}\lambda=0,\lambda>\omega\}$ and $\|R(\lambda:A)\|\leq\frac{1}{\lambda-\omega}$, for all $\lambda>\omega$.

Proof: Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\|\leq e^{\omega t}$. From the previous remarks, $(A-\omega I)$ is the infinitesimal generator of $S(t)=e^{-\omega t}T(t)$. Thus, from Hille-Yosida Theorem (see Theorem 1.3.3), $(A-\omega I)$ is closed and $\overline{D(A-\omega I)}=X$. Moreover, $\rho(A-\omega I)\supset\mathbb{R}^{+}$ and $\|R(\lambda:A-\omega I)\|\leq\frac{1}{\lambda}$, for all $\lambda>0$.

Using that $D(A)=D(A-\omega I)$, we have $\overline{D(A)}=X$ and noting that $(A-\omega I)$ is closed we conclude that A is closed. Moreover, noting that $R(\lambda:A)=R(\lambda-\omega:A-\omega I)$, we infer that $\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda-0,\lambda>\omega\}\subset\rho(A)$ and

$$\|R(\lambda:A)\|=\|R(\lambda-\omega:A-\omega I)\|\leq\frac{1}{\lambda-\omega},\text{ for }\lambda>\omega.$$

Assume that conditions (i) and (ii) are satisfied. Then, $(A-\omega I)$ is closed, $\overline{D(A-\omega I)}=X$, $\mathbb{R}^{+}\subset\rho(A-\omega I)$ and $\|R(\lambda:A-\omega I)\|=\|R(\lambda+\omega:A)\|\leq\frac{1}{(\lambda+\omega)-\omega}=\frac{1}{\lambda}$. From Hille-Yosida Theorem (see Theorem 1.3.3), $(A-\omega I)$ is the infinitesimal of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Using the above remarks, we infer that A is the infinitesimal generator of $T(t)=e^{\omega t}S(t)$ such that $\|T(t)\|\leq e^{\omega t}$, for all $t\geq 0$. ■

1.4 Examples

In this section, we present some examples of semigroups of bounded linear operators. We start with Cauchy's functional problem and then we study the finite-dimensional matrix semigroup. The main reference to this section is [Engel et al., 2000].

1.4.1 Cauchy's functional equation

In 1821 Cauchy proposed in his *Cours d'Analyse* book the following problem.
Problem 1: Find all the continuous functions $T:\mathbb{R}_+\to\mathbb{C}$ such that

$$\begin{aligned} T(t+s) &= T(t)T(s), t,s\geq 0 && (1.20)\\ T(0) &= 1. && (1.21)\end{aligned}$$

It is obvious that $t\mapsto e^{at}$ satisfies the problem (1.20)-(1.21), for all $a\in\mathbb{C}$. We want to show that all solutions for this problem are given by this family of functions. To this end, we start remembering some properties of the exponential function.

Proposition 1.4.1 Given $a\in\mathbb{C}$, $T(t):=e^{at}$ is differentiable and satisfies the initial value problem (IVP):

$$\begin{aligned} \frac{d}{dt}T(t) &= aT(t), t\geq 0 && (1.22)\\ T(0) &= 1. && (1.23)\end{aligned}$$

Reciprocally, the function $T:\mathbb{R}_+\to\mathbb{C}$ defined by $T(t)=e^{ta}$ for some $a\in\mathbb{C}$ is the only differentiable function satisfying the problem (1.22)-(1.23). Finally, we observe that $a=\frac{d}{dt}T(t)\big|_{t=0}$.
Proof: Noting that

$$\lim_{h\to 0}\frac{e^{(t+h)a}-e^{ta}}{h}=\lim_{h\to 0}\frac{e^{ta}(e^{ha}-1)}{h}=ae^{ta}$$

we conclude that $t\mapsto e^{ta}$ is differentiable and $\frac{d}{dt}e^{ta}=ae^{ta}$. Then, if $T(t)=e^{ta}$ we have $\frac{d}{dt}T(t)=aT(t)$ and $T(0)=1$, that is $t\mapsto e^{at}$ is a solution to IVP.

To show the uniqueness, suppose that $S:\mathbb{R}_+\to\mathbb{C}$ is another differentiable function satisfying (1.22)-(1.23) and define $Q:[0,t]\to\mathbb{C}$ by $Q(s)=T(s)S(t-s)$. Once Q is differentiable, we have

$$\begin{aligned} \frac{d}{ds}Q(s) &= \frac{d}{ds}T(t)S(t-s)-T(t)\frac{d}{ds}S(t-s)\\ &= aT(t)S(t-s)-T(t)aS(t-s)\\ &= 0,\end{aligned}$$

therefore Q is constant. On the other hand, for $s=0$ we have $Q(0)=S(t)$ and for $s=t$, $Q(t)=T(t)$, fromwhere $T(t)=S(t)$ for all $t>0$. ■

Proposition 1.4.2 Let $T:\mathbb{R}_+\to\mathbb{C}$ be a continuous function satisfying (1.20)-(1.21). Then, T is differentiable and there exists a unique $a\in\mathbb{C}$ such that (1.22)-(1.23) holds.
Proof: Once T is continous, we can define

$$V(t):=\int_0^t T(s)ds,\ t\geq 0,$$

which is differentiable and holds $V'(t)=T(t)$. Also, noting that

$$\left|\frac{1}{t}\int_0^t T(s)ds-T(0)\right|\leq\frac{1}{t}\int_0^t|T(s)-T(0)|ds<\frac{1}{t}\varepsilon t=\varepsilon,\ \forall\varepsilon>0,$$

we have $\lim_{t\downarrow 0}\frac{1}{t}V(t) = V'(0) = T(0) = 1$. Therefore, there is $t_0 > 0$ small enough such that $V(t_0) \neq 0$. Hence, V is invertible on t_0, which implies that

$$\begin{aligned} T(t) = V(t_0)^{-1}V(t_0)T(t) &= V(t_0)^{-1}\int_0^{t_0} T(t+s)ds \\ &= V(t_0)^{-1}\int_t^{t+t_0} T(s)ds \\ &= V(t_0)^{-1}(V(t+t_0)-V(t)), \forall t \geq 0. \end{aligned}$$

From previous, we conclude that T is differentiable and

$$\frac{d}{dt}T(t) = \lim_{h\to 0}\frac{T(t+h)-T(t)}{h} = \lim_{h\to 0}\frac{T(h)-T(0)}{h}T(t) = T'(0)T(t).$$

This shows that T satisfies de IVP with $a = T'(0)$. ■

Now, combining the last two results, we get an answer to Cauchy's problem in the following sense.

Theorem 1.4.3 Let $T : \mathbb{R} \to \mathbb{C}$ be a continuous function satisfying (1.20)-(1.21). Then, there exists a unique $a \in \mathbb{C}$ such that

$$T(t) = e^{ta}, \ \forall t \geq 0.$$

R We stress that (1.20)-(1.21) is not just a formal identity, but has meaning in linear dynamical systems description. More precisely, let $x_0 \in \mathbb{C}$ be the state of our system at $t = 0$. Then, $x(t) = T(t)x_0$ describes the state of the system for $t \geq 0$ and

$$x(t+s) = T(t+s)x_0 = T(t)T(s)x_0 = T(t)x(s)$$

describes the system at time $t+s$.

1.4.2 Matrix semigroups

Now, we consider Cauchy's problem stated before on $X = \mathbb{C}^n$. In this case, as we know, $\mathscr{L}(X) = \{T : X \to X : \text{T is linear}\}$ can be identified with $\mathbb{M}_n(\mathbb{C})$, the space of all complex $n\times n$ matrices, and a linear dynamical system on X will be given by

$$T : \mathbb{R}_+ \to \mathbb{M}_n(\mathbb{C})$$

satisfying the functional equation

$$\begin{aligned} T(t+s) &= T(t)T(s), t,s \geq 0 && (1.24) \\ T(0) &= I. && (1.25) \end{aligned}$$

In this new context, we state the next problem as

Problem 2: Find all continuous maps $T : \mathbb{R}_+ \to \mathbb{M}_n(\mathbb{C})$ satisfying the functional equation (1.24)-(1.25).

As before, functions given by $t \mapsto e^{tA}$, where $A \in \mathbb{M}_n(\mathbb{C})$ is any complex matrix, are solutions to the problem. In fact, we have the next result.

Proposition 1.4.4 For any $A \in \mathbb{M}_n(\mathbb{C})$, the map

$$t \mapsto e^{tA} \in \mathbb{M}_n(\mathbb{C})$$

is continuous and satisfies the system

$$\begin{cases} e^{(t+s)A} = e^{tA}e^{sA}, & t,s \geq 0 \\ e^{0A} = I. \end{cases}$$

Proof: Since the series $\sum_{k=0}^{\infty} \frac{t^k\|A\|^k}{k!}$ converges, we have $e^{0A} = \sum_{k=0}^{\infty} \frac{0^kA^k}{k!} = I + \sum_{k=1}^{\infty} \frac{0^kA^k}{k!} = I$ and

$$\begin{aligned} e^{(t+s)A} &= \sum_{n=0}^{\infty} \frac{(t+s)^nA^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{n!}{(n-i)!i!}t^is^{n-i}\frac{A^n}{n!} \\ &= \sum_{i=0}^{\infty}\sum_{n=i}^{\infty} \frac{(tA)^i}{i!}\frac{(sA)^{n-i}}{(n-i)!} \\ &= \sum_{i=0}^{\infty} \frac{(tA)^i}{i!}\sum_{k=0}^{\infty} \frac{(sA)^k}{k!} \\ &= e^{tA}e^{sA}. \end{aligned}$$

In order to show that $t \mapsto e^{tA}$ is continous, we observe that

$$e^{(t+h)A} - e^{tA} = e^{tA}\left(e^{hA} - I\right), \ \forall t,h \geq 0.$$

Thus, it is sufficient to show that $\lim_{h\to 0} e^{hA} = I$, which follows from

$$\|e^{hA} - I\| = \left\| \sum_{k=1}^{\infty} \frac{h^kA^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{|h|^k\|A\|^k}{k!} = e^{|h|\|A\|} - 1. \ \blacksquare$$

Now, we state some properties of the exponential matrix function to conclude that they are the only family of continuous functions satisfying (1.24)-(1.25).

Proposition 1.4.5 Let $T(t) := e^{tA}$ for some $A \in \mathbb{M}_n(\mathbb{C})$. Then, $T : \mathbb{R}_+ \to \mathbb{M}_n(\mathbb{C})$ is differentiable and satisfies the differential equation

$$\frac{d}{dt}T(t) = AT(t), t \geq 0 \quad (1.26)$$

$$T(0) = I. \quad (1.27)$$

Conversely, every differentiable function $T : \mathbb{R}_+ \to \mathbb{M}_n(\mathbb{C})$ satisfying (1.26)-(1.27) is already of the form $T(t) = e^{tA}$ for some $A \in \mathbb{M}_n(\mathbb{C})$. Finally, $A = T'(0)$.

Proof: Note that

$$\frac{T(t+h) - T(t)}{h} = \frac{T(h) - I}{h}T(t), \ \forall t,h \in \mathbb{R}_+.$$

Thus, it is sufficient to show that $\lim_{h\to 0} \frac{T(h)-I}{h} = A$, indeed

$$\left\| \frac{T(h) - I}{h} - A \right\| \leq \sum_{k=0}^{\infty} \frac{|h|^{k-1}\|A\|^k}{k!} = \frac{e^{|h|\|A\|} - 1}{h} - \|A\| \to 0.$$

The remaining is proved as before. $\blacksquare$

Now, remembering that for $t_0 > 0$ small enough such that $\|A(s) - s\| < 1$ for all $|s| < t_0$, we have A is invertible for all $|s| < t_0$. Arguing as before, we have the next theorem.

Theorem 1.4.6 Let $T:\mathbb{R}_+\to\mathbb{C}^n$ be a continuous function satisfying (1.24)-(1.25). Then there exists $A\in\mathbb{M}_n(\mathbb{C})$ such that $T(t)=e^{tA},\ t\geq 0$.

Finally, we present some concrete examples of matrices semigroups.

■ **Example 1.2** **i)** Given a diagonal matrix $A=\operatorname{diag}(a_1,\dots,a_n)$, the semigroup generated by A is

$$(e^{tA})_{t\geq 0}=\begin{pmatrix} e^{ta_1} & 0 & \cdots & 0\\ 0 & e^{ta_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{ta_n}\end{pmatrix}_{t\geq 0}. \tag{1.28}$$

ii) Let A be a $k\times k$ Jordan block

$$A=\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \cdots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & \dots & \dots & 0 & \lambda\end{pmatrix}_{k\times k} \tag{1.29}$$

with eingenvalue $\lambda\in\mathbb{C}$. Decomponding A into $A=D+N$, where $D=\lambda I$ and N is nilpotent, we have

$$(e^{tN})_{t\geq 0}=\begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!}\\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & t\\ 0 & \dots & \dots & 0 & 1\end{pmatrix}_{t\geq 0}. \tag{1.30}$$

Then, $(e^{tA})_{t\geq 0}=(e^{t(D+N)})_{t\geq 0}$.
For arbitrary matrices $A\in\mathbb{M}_n(\mathbb{C})$, compute e^{tA} could be very difficult. But, using the Jordan normal form, there exists an invertible matrix $S\in\mathbb{M}_n(\mathbb{C})$ such that $A=S^{-1}BS$, where B is a Jordan block composition and $(e^{tA})_{t\geq 0}=(S^{-1}e^{tB}S)_{t\geq 0}$.

iii) For some special 2×2 matrices, the semigroup generated is easily computed using the method above. For example, if

$$A=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \tag{1.31}$$

the semigroup generated is

$$(e^{tA})_{t\geq 0}=\begin{pmatrix} \cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}_{t\geq 0}. \tag{1.32}$$

More generally, if

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \tag{1.33}$$

defining $\delta=ad-bc$, $\tau=a+d$ and $\gamma^2=\frac{1}{4}(\tau^2+4\delta)$ we have

$$(e^{tA})_{t\geq 0}=\begin{cases} (e^{t\frac{\tau}{2}}(\frac{1}{\gamma}\sinh(t\gamma)A+(\cosh(t\gamma)-\frac{2\tau}{\gamma})I))_{t\geq 0}, & \gamma\neq 0\\ (e^{t\frac{\tau}{2}}(tA+(1-\frac{t\tau}{2})I))_{t\geq 0}, & \gamma=0\end{cases}. \tag{1.34}$$

■

1.5 Differentiable and analytic semigroups

In this section, the classes of differentiable and analytic semigroups are presented. To begin, we introduce the differentiable semigroups and some results.

1.5.1 Differentiable semigroups

Definition 1.5.1 Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. $(T(t))_{t\geq 0}$ is called **differentiable** for $t > t_0$ if for every $x \in X$, the function $t \mapsto T(t)x$ is differentiable for $t > t_0$. The semigroup $(T(t))_{t\geq 0}$ is called differentiable if it is differentiable for all $t > 0$.

In the next lemma, we present some properties of differentiable semigroups.

Lemma 1.5.1 Let $(T(t))_{t\geq 0}$ be a C_0-semigroup which is differentiable for $t > t_0$ and let A be its infinitesimal generator. Then,

i) for $t > nt_0$, $n = 1,2,\ldots$, $T(t)(X) \subset D(A^n)$, $T^{(n)}(t) = A^n T(t)$ and $T^{(n)}(t)$ is a bounded linear operator.

ii) For $t > nt_0$, $n = 1,2,\ldots$, the function $t \mapsto T^{(n-1)}(t)$ is continuous in the uniform operator topology.

Proof: We start with $n = 1$. Given $t > t_0$ and $x \in X$, by assumption, $s \mapsto T(s)x$ is differentiable for $s > t_0$. Then,

$$\lim_{h\downarrow 0}\left(\frac{T(h)-I}{h}\right)T(t)x = \lim_{h\downarrow 0}\frac{T(t+h)x - T(t)x}{h} = T'(t)x.$$

This implies that $T(t)x \in D(A)$ and $AT(t)x = T'(t)x$, for $t > t_0$ and $x \in X$, which implies that $AT(t)$ is well-defined on X. Therefore, noting that A is closed, $T(t)$ is continuous, and $AT(t)$ is closed, from the Closed Graph Theorem (see Theorem A.0.10), we obtain that $AT(t) = T'(t)$ is a bounded linear operator.

Suppose now that (i) is true for $n > 1$. Next, we show that it is true for $n+1$. Let $t > (n+1)t_0$ and $s > nt_0$ such that $t - s > t_0$, for $y \in X$, we have that $AT(k)y = T(k)Ay$ and

$$T^{(n)}(t)x = A^{(n)}T(t)x = A^{(n)}T(t-s)T(s)x = T(t-s)A^{(n)}T(s)x,\ x \in X.$$

From above, we infer that $T^{(n)}(t)x$ is differentiable, once $T(t-s)A^nT(s)x$ is differentiable for $t - s > t_0$. Moreover,

$$T^{(n+1)}(t)x = AT(t-s)A^nT(s)x = A^{n+1}T(t)x,\ \ \forall t > (n+1)t_0, \forall x \in X$$

and, like the case $n = 1$, using that A is closed and $A^nT(t)$ is continuous, $T^{(n+1)}(t) = A^{n+1}T(t)$ is closed, therefore, bounded for all $t > (n+1)t_0$. This concludes the proof of (i).

To prove (ii), let $\|T(t)\| \leq Me^{\omega t}$, for $T > 0$ and $t \in [0,T]$. Thus, for $n \in \mathbb{N}$ and $nt_0 < t_1 \leq t_2 \leq t_1 + T$, using the part (i), if $s \in [t_1, t_1+T]$ then

$$T^{(n)}(s)x = A^nT(s)x = T(s-t_1)A^nT(t_1)x,\ x \in X.$$

This proves that $s \mapsto T^{(n)}(s)x$ is continuous in $[t_1,t_2]$, once $s \mapsto T(s-t_1)x$ is continuous for all $x \in X$. This implies that

$$\begin{aligned}\left\|T^{(n-1)}(t_2)x - T^{(n-1)}(t_1)x\right\| &= \left\|\int_{t_1}^{t_2}\frac{d}{ds}T^{(n+1)}(s)xds\right\| \\ &= \left\|\int_{t_1}^{t_2}T^{(n)}(s)x\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_1}^{t_2} \|T(s-t_1)A^nT(t_1)x\|ds \\
&\leq \|A^nT(t_1)\|\|x\| \int_{t_1}^{t_2} \|T(s-t_1)\|ds \\
&\leq Me^{\omega T}\|A^nT(t_1)\|x\|(t_2-t_1).
\end{aligned}
$$

In other words,

$$\|T^{(n-1)}(t_2)x - T^{(n-1)}(t_1)x\| \leq Me^{\omega T}\|A^nT(t_1)\|x\|(t_2-t_1).$$

Therefore, $\lim_{t_2\to t_1}\|T^{(n-1)}(t_2)x - T^{(n-1)}(t_1)x\| = 0$, that is, $t \mapsto T^{(n-1)}(t)$ is continuous in $t = t_1$. Consequently, for all $t > nt_0$ and $n \in \mathbb{N}$ the function $t \mapsto T^{(n-1)}(t)$ is continuous in the uniform operator topology. ∎

Corollary 1.5.2 Let $(T(t))_{t\geq 0}$ be a C_0-semigroup differentiable for $t > t_0$. If $t > (n+1)t_0$, then $t \mapsto (T(t))_{t\geq 0}$is n-times differentiable in the uniform operator topology.

Proof: From part (ii) of Lemma 1.5.1, it follows that the function $t \mapsto A^kT(t)$ is continuous in the uniform operator topology, for $t > (n+1)t_0$ and $k \in [1,n]$. Furthermore, for $h > 0, t > (n+1)t_0$ such that $t \pm h > 0$ and $1 \leq k \leq n$,

$$T^{(k-1)}(t) - T^{(k-1)}(t-h) = -\int_t^{t-h} A^kT(s)ds$$

and

$$T^{(k-1)}(t+h) - T^{(k-1)}(t) = \int_t^{t+h} A^kT(s)ds.$$

Therefore,

$$\lim_{h\to 0} \frac{T^{(k-1)}(t+h) - T^{(k-1)}(t)}{h} = A^kT(t),$$

that is, $T^{(k+1)}$ is differentiable in the uniform operator topology for $1 \leq k \leq n$ and $t > (n+1)t_0$. Thus $(T(t))_{t\geq 0}$ is n-times differentiable in the uniform operator topology. ∎

Corollary 1.5.3 If $(T(t))_{t\geq 0}$ is a differentiable C_0-semigroup and A is its infinitesimal generator then the next properties are true

- **i)** For all $t > 0$, $(T(t))_{t\geq 0}$ is differentiable infinitely many times in the uniform operator topology.
- **ii)** For $n \geq 1$, $T^{(n)}(t) = (AT(\frac{t}{n}))^n = (T'(\frac{t}{n}))^n$

Proof: The first item follows directly from Corollary 1.5.2 using that $(T(t))_{t\geq 0}$ is differentiable for $t > 0 = n \cdot 0$, for all $n \in \mathbb{N}$.

From Lemma 1.5.1, $T^{(n)}(t)x = A^nT(t)x$ and $T(t)(X) \subset D(A^n)$, for all $t > 0$ and $n \geq 1$. Thus, for $n \geq 1$ and $x \in X$, once $(T(\frac{t}{n}))x \in D(A)$,

$$T^{(n)}(t)x = (AT(\frac{t}{n})x)^n = (T'(\frac{t}{n}))^n. \ \blacksquare$$

1.5.2 Analytic semigroups

So far, we have just considered semigroups whose domain was the non-negative real line. In this section, we consider semigroups defined on open regions of the complex plane. Next, we restrict ourselves to semigroups defined on regions of the form

$$\Delta(\alpha) = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \alpha, \alpha \in (0, \frac{\pi}{2}]\}.$$

Definition 1.5.2 A family of bounded linear operators $(T(z))_{z\in\Delta(\alpha)\cup\{0\}}$, is an **analytic semigroup** if

i) $T(0) = I$,
ii) $T(z_1+z_2) = T(z_1)T(z_2)$, for all $z_1, z_2 \in \Delta(\alpha)$,
iii) $\lim_{z\to 0} T(z)x = x$ for all $x \in X$ and $z \in \Delta(\alpha)$,
iv) $z \mapsto T(z)$ is analytic on $\Delta(\alpha)$.

Our basic interest is to study conditions under which a semigroup $(T(t))_{t\geq 0}$ is the restriction of an analytic semigroup. For convenience, in the remainder of this section, we assume that $0 \in \rho(A)$. To begin, we consider the next lemma.

Lemma 1.5.4 Let A be a closed and densely defined linear operator in X satisfying the following conditions:

i) For some $0 < \alpha < \frac{\pi}{2}$, $\Sigma_\alpha = \{\lambda : |\arg\lambda| < \frac{\pi}{2} + \alpha\} \cup \{0\} \subset \rho(A)$.
ii) There exists $M > 0$ such that $\|R(\lambda : A)\| \leq \frac{M}{|\lambda|}$ for all non-zero $\lambda \in \Sigma_\alpha$.

Then, A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisying $\|T(t)\| \leq C$ for some constant C. Moreover,

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda : A)d\lambda, \tag{1.35}$$

where Γ is a smooth curve contained in Σ_α, running from $-\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$, for some $\frac{\pi}{2} < \vartheta < \frac{\pi}{2} + \alpha$. In addition, the integral (1.35) converges for $t > 0$ in the uniform operator topology.

Proof: For $t \geq 0$, we define the map

$$U(t) = \frac{1}{2\pi i}\int_\Gamma e^{\mu t}R(\mu : A)d\mu,$$

and consider the path $\tilde{\Gamma} = \underbrace{\{re^{i\vartheta} : r \geq 0\}}_{\Gamma_1} \cup \underbrace{\{-re^{-i\vartheta} : r \geq 0\}}_{\Gamma_2}$, see Figure 1.1. Using that $\mu \mapsto e^{\mu t}$ and $R(\mu : A)$ are analytic functions we can write

$$\int_{\tilde{\Gamma}} e^{\mu t}R(\mu : A)d\mu = \int_{\Gamma_1} e^{\mu t}R(\mu : A)d\mu + \int_{\Gamma_2} e^{\mu t}R(\mu : A)d\mu.$$

Using the conditions in (ii), for $r_1 > 0$ there exist $C > 0$ such that $\|R(\lambda : A)\| < C$ for all $\lambda \in \Gamma_1$ with $|\lambda| \leq r_1$. From the above, for $t > 0$ we note that

$$\begin{aligned}
\left\|\int_{\Gamma_1} e^{\mu t}R(\mu : A)d\mu\right\| &\leq \int_0^{r_1} |e^{\mu t}|\|R(\mu : A)\|d\mu + \int_{r_1}^\infty |e^{\mu t}|\|R(\mu : A)\|d\mu \\
&\leq \int_0^{r_1} |e^{tre^{i\vartheta}}|Ce^{i\vartheta}dr + \int_{r_1}^\infty |e^{tre^{i\vartheta}}|\frac{M}{|re^{i\vartheta}|}e^{i\vartheta}dr \\
&\leq C\int_0^{r_1} e^{r\mathrm{Re}(e^{i\vartheta})t}dr + \frac{M}{r_1}\int_{r_1}^\infty e^{r\mathrm{Re}(e^{i\vartheta})t}dr
\end{aligned}$$

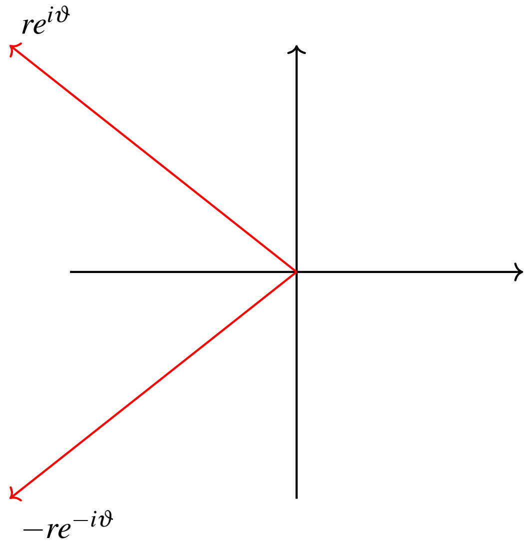

Figure 1.1: The path $\tilde{\Gamma}$ (own figure)

$$= \quad C\left(\frac{e^{r_1\,\mathrm{Re}(e^{i\vartheta})t}-1}{\mathrm{Re}(e^{i\vartheta})t}\right)+\frac{M}{r_1}\left(\frac{e^{r_1\,\mathrm{Re}(e^{i\vartheta})}-1}{\mathrm{Re}(e^{i\vartheta})t}\right).$$

From the above, $\int_{\Gamma_1} e^{\mu t}R(\mu : A)d\mu$ is well-defined, because $\mathrm{Re}(e^{i\vartheta}) < 0$. Similarly, taking $\overline{\Gamma_2}$, the path Γ_2 oriented counterclockwise, we can prove that $\int_{\Gamma_2} e^{\mu t}R(\mu : A)d\mu$ is also well-defined. Thereby, $\int_{\tilde{\Gamma}} e^{\mu t}R(\mu : A)d\mu$ is well-defined for all $t \geq 0$.

Let $\gamma_1(r) = re^{i\vartheta}$ be a parametrization of $\tilde{\Gamma}$ and $\gamma_2(\cdot)$ be a smooth path Γ running from $-\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ in Σ_α, where $\frac{\pi}{2} < \vartheta < \frac{\pi}{2} + \alpha$, such that

$$\lim_{r\to\infty} |\gamma_1(r) - \gamma_2(r)| = 0.$$

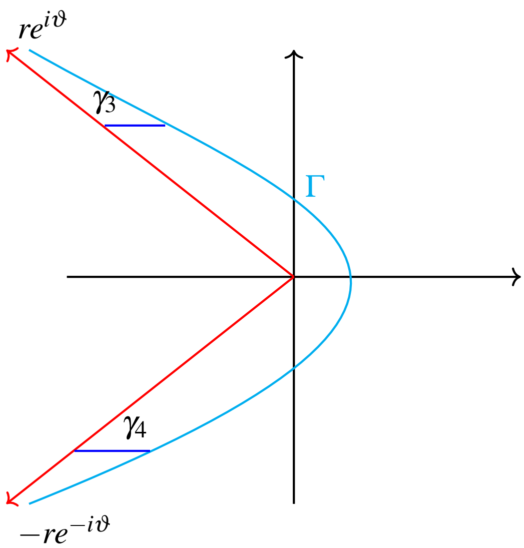

Figure 1.2: The path Υ (own figure)

For $\Upsilon = \tilde{\Gamma} \cup \gamma_3 \cup \Gamma \cup \gamma_4$ as in the Figure 1.2, with γ_3 connecting $\tilde{\Gamma}$ to Γ at $r_0 e^{i\vartheta}$ for r_0 big enough and γ_4 connecting Γ to $\tilde{\Gamma}$ at $r_0 e^{-i\vartheta}$, using the analyticity of the function $\mu \mapsto e^{\mu t}R(\mu : A)$, we have that $\int_\Upsilon e^{\mu t}R(\mu : A)d\mu = 0$.

If $\mu \in \gamma_3$ and $|\mu| > 1$, we note that

$$\left\| \int_{\gamma_3} e^{\mu t} R(\mu : A) d\mu \right\| \leq \int_{\gamma_3} |e^{\mu t}| \frac{M}{|\mu|} d\mu \leq M \int_{\gamma_3} d\mu = M|\gamma_3|,$$

where $|\gamma_3|$ denotes the length of γ_3. Using now that $\lim_{r\to\infty} |\gamma_1(r) - \gamma_2(r)| = 0$, we have that $|\gamma_3| \to 0$ as $r \to \infty$, which implies that $\int_{\gamma_3} e^{\mu t} R(\mu : A) d\mu \to 0$ as $r \to \infty$. Similarly, we can show that $\int_{\gamma_4} e^{\mu t} R(\mu : A) d\mu \to 0$.

From the above, it follows that

$$\int_{\Gamma} e^{\mu t} R(\mu : A) d\mu = \int_{\tilde{\Gamma}} e^{\mu t} R(\mu : A) d\mu.$$

Considering the above, we cand define the path $\Gamma_t = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 = \{-re^{-i\vartheta} : \frac{1}{t} < r < \infty\}$, $\Gamma_2 = \{\frac{1}{t} e^{-i\phi} : -\vartheta < \phi < \vartheta\}$ and $\Gamma_3 = \{re^{i\vartheta} : \frac{1}{t} < r < \infty\}$, see Figure 1.3.

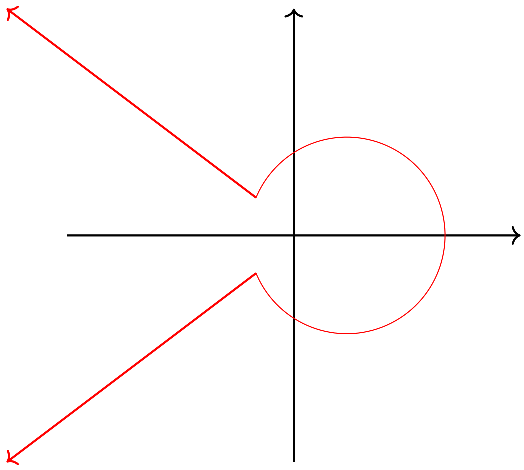

Figure 1.3: The path Γ_t (own figure)

In this case, for $t > 0$ we have that

$$\begin{aligned}
\left\| \frac{1}{2\pi i} \int_{\Gamma_1} e^{\mu t} R(\mu : A) d\mu \right\| &\leq \frac{1}{2\pi} \int_{\frac{1}{t}}^{\infty} \| e^{-re^{-i\vartheta} t} R(-re^{-i\vartheta} : A) e^{-i\vartheta} \| dr \\
&\leq \frac{1}{2\pi} \int_{\frac{1}{t}}^{\infty} |e^{-re^{-i\vartheta} t}| \frac{M}{|re^{-i\vartheta}|} |e^{-i\vartheta}| dr \\
&\leq \frac{1}{2\pi} \int_{\frac{1}{t}}^{\infty} |e^{-re^{-i\vartheta} t}| \frac{M}{r} dr \\
&= \frac{M}{2\pi} \int_{\frac{1}{t}}^{\infty} |e^{-r(\cos\vartheta + i\sin\vartheta)t}| \frac{1}{r} dr \\
&\leq \frac{M}{2\pi} \int_{\frac{1}{t}}^{\infty} |e^{-r(\cos\vartheta + i\sin\vartheta)t}| \frac{1}{\frac{1}{t}} dr \\
&\leq \frac{Mt}{2\pi} \int_{\frac{1}{t}}^{\infty} e^{-r\sin\vartheta t} dr \\
&= -\frac{Mt}{2\pi} \frac{e^{-\sin\vartheta}}{t\sin\vartheta} = C_1.
\end{aligned}$$

We also note that,

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_2} e^{\mu t} R(\mu : A) d\mu \right\| \leq \frac{1}{2\pi} \int_{-\vartheta}^{\vartheta} |e^{\frac{1}{t} e^{-i\phi} t}| \left\| R\left(\frac{1}{t} e^{-i\phi} : A\right) \right\| |\frac{e^{-i\phi}}{t}| |d\phi|$$

$$\begin{aligned}
&\leq \frac{1}{2\pi}\int_{-\vartheta}^{\vartheta}|e^{e^{-i\phi}}|\frac{M}{|\frac{e^{-i\phi}}{t}|}|\frac{e^{-i\phi}}{t}||d\phi| \\
&\leq \frac{M}{2\pi}\int_{-\vartheta}^{\vartheta}e^{\cos\phi}|d\phi| \\
&\leq \frac{M}{2\pi}\sup_{\varphi\in(-\vartheta,\vartheta)}e^{\cos\varphi}2\vartheta = C_2.
\end{aligned}$$

Moreover, proceeding as above, we can show that $\|\int_{\Gamma_3}e^{\mu t}R(\mu:A)d\mu\| \leq C_3$ for some $C_3>0$ which does not depends of t. From the above, there exists $C>0$ such that $\|U(t)\|\leq C$ for all $t>0$.

Next, we show that

$$R(\lambda:A)=\int_0^\infty e^{-\lambda t}U(t)dt, \text{ for all } \lambda>0. \tag{1.36}$$

To this end, for $T>0$, we note that

$$\begin{aligned}
\int_0^T e^{-\lambda t}U(t)dt &= \int_0^T e^{-\lambda t}\left(\frac{1}{2\pi i}\int_\Gamma e^{\mu t}R(\mu:A)d\mu\right)dt \\
&= \frac{1}{2\pi i}\int_0^T\int_\Gamma e^{(\mu-\lambda)t}R(\mu:A)d\mu dt \\
&= \frac{1}{2\pi i}\int_\Gamma\int_0^T e^{(\mu-\lambda)t}R(\mu:A)dt d\mu \\
&= \frac{1}{2\pi i}\int_\Gamma(\frac{e^{t(\mu-\lambda)}}{\mu-\lambda}R(\mu:A))\Big|_0^T d\mu \\
&= \frac{1}{2\pi i}\int_\Gamma\left(\frac{e^{T(\mu-\lambda)}R(\mu:A)}{\mu-\lambda}-\frac{R(\mu:A)}{\mu-\lambda}\right)d\mu \\
&= \frac{1}{2\pi i}\int_\Gamma\frac{e^{T(\mu-\lambda)}R(\mu:A)}{\mu-\lambda}d\mu+\frac{1}{2\pi i}\int_\Gamma\frac{R(\mu:A)}{\lambda-\mu}d\mu.
\end{aligned}$$

Using now that

$$R(\lambda:A)=\frac{1}{2\pi i}\int_\Gamma\frac{R(\mu:A)}{\lambda-\mu}d\mu \text{ (see Theorem A.0.15)},$$

we have that

$$\int_0^T e^{-\lambda t}U(t)dt=\frac{1}{2\pi i}\int_\Gamma\frac{e^{T(\mu-\lambda)}R(\mu:A)}{\mu-\lambda}d\mu+R(\lambda:A).$$

On the other hand,

$$\begin{aligned}
\|\int_\Gamma\frac{e^{T(\mu-\lambda)}R(\mu:A)}{\mu-\lambda}d\mu\| &\leq \int_\Gamma\frac{|e^{T(\mu-\lambda)}|}{|\mu-\lambda|}\|R(\mu:A)\|d\mu \\
&\leq \int_\Gamma\frac{|e^{T(\mu-\lambda)}|}{|\mu-\lambda|}\frac{M}{|\mu|}d\mu \\
&\leq \frac{M}{e^{T\lambda}}\int_\Gamma\frac{|e^{T\mu}|}{|\mu||\lambda-\mu|}d\mu \\
&\leq \frac{M}{e^{T\lambda}}\int_\Gamma\frac{d\mu}{|\mu|(|\mu|-\lambda)}\longrightarrow 0 \text{ as } T\longrightarrow 0.
\end{aligned}$$

From the previous estimates, we obtain the identity (1.36).

Noting that $\|U(t)\| \leq C$, we have that

$$\frac{d}{d\lambda}R(\lambda : A)x = \frac{d}{d\lambda}\left(\int_0^\infty e^{-\lambda t}U(t)dt\right) = \int_0^\infty \frac{d}{d\lambda}(e^{-\lambda t}U(t))dt = \int_0^\infty -te^{-\lambda t}U(t)dt,$$

and

$$\frac{d^2}{d\lambda^2}R(\lambda : A)x = \int_0^\infty \frac{d}{d\lambda}(-te^{-\lambda t}U(t))dt = \int_0^\infty t^2 e^{-\lambda t}U(t)dt.$$

Thereby,

$$\frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda : A)x = (-1)^{n-1}\int_0^\infty t^{n-1}e^{-\lambda t}U(t)dt, \text{ for all } n \in \mathbb{N} \text{ and } \lambda > 0.$$

Moreover, from [Pazy, 2012] (Equation 5.22), we know that $\frac{d^n}{d\lambda^n}R(\lambda : A) = (-1)^n n! R(\lambda : A)^{n+1}$, which allows to infer that

$$\frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda : A) = (-1)^{n-1}(n-1)!R(\lambda : A)^n.$$

Hence,

$$\begin{aligned}
\|R(\lambda : A)^n\| &= \left\|\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}U(t)dt\right\| \\
&\leq \frac{C}{(n-1)!}\int_0^\infty \frac{t^{n-1}}{e^{\lambda t}}dt \\
&= \frac{C}{\lambda^n}.
\end{aligned}$$

From the above and using Theorem A.0.16, we note that A is the infinitesimal generator of a C_0-semigroup such that $\|T(t)\| \leq C$ for all $t > 0$.

To finish, we prove the formula (1.35). Let $x \in D(A^2)$. From Corollary A.0.2, we see that

$$T(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}R(\lambda : A)x d\lambda.$$

Using the contour Γ as before and arguing as above, for $r_0 > 0$ we consider two straight lines connecting Γ to $\gamma + i\theta$ at $r_0 e^{i\vartheta}$ and $\gamma - i\infty$ at $-r_0 e^{i\vartheta}$. Specifically, using the lines $\lambda_1(t) = (t, r_0 \sin\theta)$ and $\lambda_2(t) = (t, -r_0 \sin\theta)$, see Figure 1.4.

From the above, we see that

$$\begin{aligned}
\left|\int_{r_0\cos\theta}^{\gamma} e^{\lambda t}R(\lambda : A)d\lambda\right| &= \left|\int_{r_0\cos\theta}^{\gamma} e^{t(s+ir_0\sin\theta)}R(s + ir_0\sin\theta : A)ds\right| \\
&\leq \int_{r_0\cos\theta}^{\gamma} e^{st}\frac{M}{|s + ir_0\sin\theta|}ds \\
&\leq M\int_{r_0\cos\theta}^{\gamma} \frac{e^{st}}{r_0^2}ds \\
&\leq \left.\frac{e^{st}}{tr_0^2}\right|_{r_0\cos\theta}^{\gamma} \\
&= \frac{e^{r_0\cos\theta} - e^{\gamma\theta}}{tr_0^2}r_0 \longrightarrow \infty \text{ as } r_0 \longrightarrow \infty.
\end{aligned}$$

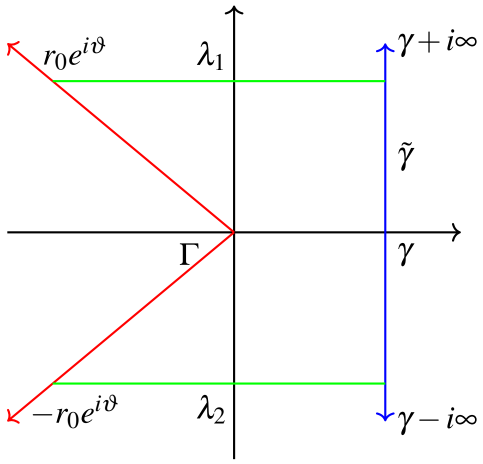

Figure 1.4: The path $\Gamma \cup \lambda_1 \cup \tilde{@g} \cup \lambda_2$ (own figure)

From this estimate, it is easy to infer that

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda : A)xd\lambda = \int_{\Gamma} e^{\lambda t}R(\lambda : A)xd\lambda,$$

which in turn implies that

$$T(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda : A)xd\lambda$$

for every $x \in D(A^2)$. Moreover, using that $D(A^2)$ is dense in X and $\int_\Gamma e^{\lambda t}R(\lambda : A)xd\lambda$ converges uniformly, if $(x_n)_n$ is a sequence in $D(A^2)$ such that $x_n \to x$, we get

$$T(t)x = \lim_n T(t)x_n = \lim_n \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda : A)x_n d\lambda = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda : A)xd\lambda.$$

Which allows us to complete the proof. ■

Next, we prove the main theorem of this section. This result characterizes the C_0-semigroups which are the restriction to $[0,\infty)$ of an analytic semigroup.

Theorem 1.5.5 Let $(T(t))_{t\geq 0}$ be an uniformly bounded C_0-semigroup, A be the infinitesimal generator of $(T(t))_{t\geq 0}$ and assume that $0 \in \rho(A)$. The following conditions are equivalent

i) $(T(t))_{t\geq 0}$ can be extend to an analytic semigroup in $\Delta(\alpha)$ for some α and $\|T(z)\|$ is uniformly bounded in every closed subsector $\overline{\Delta(\alpha')}$, $\alpha' < \alpha$.

ii) There exists a constant $C > 0$ such that

$$\|R(\sigma + i\tau : A)\| \leq \frac{C}{|\tau|}, \text{ for all } \sigma > 0 \text{ and } \tau \neq 0. \tag{1.37}$$

iii) There exists $0 < \alpha < \frac{\pi}{2}$ and $M > 0$ such that

$$\Sigma = \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \alpha\} \cup \{0\} \subset \rho(A)$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|}, \text{ for } \lambda \in \Sigma, \lambda \neq 0.$$

iv) $(T(t))_{t\geq 0}$ is differentiable and there is a constant C such that

$$\|AT(t)\| \leq \frac{C}{t}, \quad \text{for all } t > 0.$$

Proof: $(i \Rightarrow ii)$ Let $0 < \alpha' < \alpha$ such that $\|T(z)\| \leq C_1$ for all $z \in \overline{\Delta(\alpha')}$. From Corollary A.0.16, for $x \in X$ and $\sigma > 0$, we have that

$$R(\sigma + i\tau : A)x = \int_0^\infty e^{-(\sigma+i\tau)t} T(t)x dt.$$

Let $\lambda = \sigma + i\tau$, whith $\tau > 0$. For $r > 0$, we define the C^1 piecewise path $\Lambda_r = \Lambda_r^1 \cup \Lambda_r^2 \cup \Lambda_r^3$, where Λ_r^i are the paths $\Lambda_r^1 = \{\rho e^{i\alpha'} : \rho \in [0,r]\}$, $\Lambda_r^2 = \{re^{-i\vartheta} : \vartheta \in [-\alpha', 0]\}$ and $\Lambda_r^3 = \{t : t \in [0,r]\}$ oriented counterclockwise, see figure 1.5 bellow.

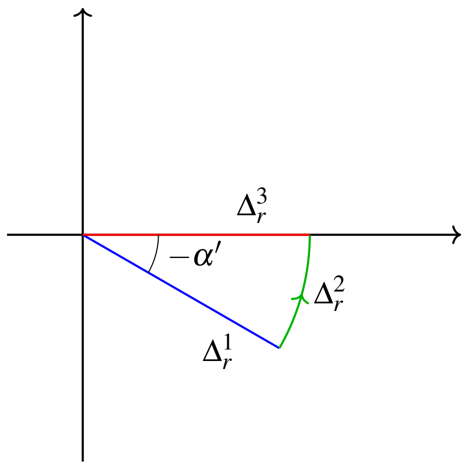

Figure 1.5: Path Λ_r

From the analyticity of the function $z \mapsto T(z)$, we find that $\mu \mapsto e^{-t\mu}T(\mu)$ is analytic and using the Cauchy's Theorem (see Theorem A.0.13), we see that

$$0 = \int_{\Lambda_r^1} e^{-\lambda t} T(t)x dt + \int_{\Lambda_r^2} e^{-\lambda t} T(t)x dt + \int_{\Lambda_r^3} e^{-\lambda t} T(t)x dt.$$

Moreover, noting that

$$\left\| \int_{\Lambda_r^2} e^{-(\sigma+i\tau)t} T(t)x dt \right\| \leq C_1 \|x\| \int_{-\alpha'}^0 |re^{-(\sigma+i\tau)re^{i\vartheta}}| d\vartheta,$$

$re^{-(\sigma+i\tau)re^{i\vartheta}} \xrightarrow{r\to\infty} 0$ and that $|re^{-(\sigma+i\tau)re^{i\vartheta}}|$ is bounded, from the Lebesgue Dominated Convergence Theorem (see Theorem A.0.12) we infer that

$$\left\| \int_{\Lambda_r^2} e^{-(\sigma+i\tau)t} T(t)x dt \right\| \to 0 \text{ as } r \to \infty.$$

From the above, we infer that

$$\lim_{r\to\infty} \int_{\Lambda_r^3} e^{-\lambda t} T(t)x dt = -\lim_{r\to\infty} \int_{\Lambda_r^1} e^{-\lambda t} T(t)x dt.$$

Which in turn implies that

$$R(\sigma+i\tau:A)x=\int_0^\infty e^{-(\sigma+i\tau)t}T(t)x\,dt=\int_{\Pi^+_{\alpha'}} e^{-(\sigma+i\tau)t}T(t)x\,dt, \tag{1.38}$$

for all $\sigma>0$ and $x\in X$, with $\Pi^+_{\alpha'}=\{\rho e^{i\alpha'}:\rho\geq 0\}$ for $0<\alpha'<\alpha$.

From (1.38), for $\sigma,\tau>0$ we have that

$$\begin{aligned}\|R(\sigma+i\tau:A)\| &= \left\|\int_{\Pi^+_{\alpha'}} e^{-(\sigma+i\tau)t}T(t)dt\right\|\\ &\leq \int_0^\infty e^{-\rho(\sigma\cos\alpha'+\tau\sin\alpha')}\|T(\rho)\|d\rho\\ &\leq \int_0^\infty e^{-\rho(\sigma\cos\alpha'+\tau\sin\alpha')}C_1 d\rho\\ &\leq \frac{C_1}{\sigma\cos\alpha'+\tau\sin\alpha'}\\ &\leq \frac{C_1}{\tau\sin\alpha'}\\ &\leq \frac{C}{\tau}.\end{aligned}$$

Similarly, for $\tau<0$, we consider the C^1 piecewise path

$$\Upsilon_r=\{\rho e^{-i\alpha'}:\rho\in[0,r]\}\cup\{re^{i\vartheta}:\vartheta\in[0,\alpha']\}\cup\{t:t\in[0,r]\},$$

illustrated on Figure 1.6.

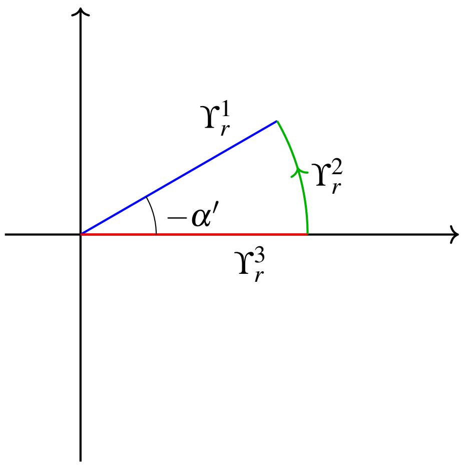

Figure 1.6: The path Υ_r (own figure)

In this case, we have that

$$\|R(\sigma+i\tau:A)\|=\left\|\int_{\Upsilon_1} e^{-(\sigma+i\tau)t}T(t)dt\right\|\leq\int_0^\infty |e^{-(\sigma+i\tau)\rho e^{-i\alpha}}||e^{-i\alpha}|\|T(\rho e^{-i\alpha})\|d\rho\leq\frac{C}{|\tau|}.$$

Then,

$$R(\sigma+i\tau:A)=\int_{\Pi^-_{\alpha'}} e^{-(\sigma+i\tau)t}dt,$$

for all $\sigma > 0$ and $\tau < 0$, where $\Pi^-_{\alpha'} = \{\rho e^{-i\alpha'} : \rho \geq 0\}$. From the previous,

$$\|R(\sigma+i\tau : A)\| \leq \frac{C}{-\tau}, \text{ for all } \sigma > 0 \text{ and } \tau < 0.$$

From the above remarks, we obtain that

$$\|R(\sigma+i\tau : A)\| \leq \frac{C}{|\tau|}, \text{ for all } \sigma > 0 \text{ and } \tau \in \mathbb{R}\setminus\{0\}.$$

$(ii \Rightarrow iii)$ Remarking that A is the infinitesimal generator of a C_0-semigroup, from the Hille-Yosida Theorem (see Theorem A.0.3), we have that $\|R(\lambda : A)\| \leq \frac{1}{\operatorname{Re}\lambda}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. In addition from (1.37), we infer that there exists a constant $C > 0$ such that, for every $\operatorname{Re}\lambda > 0$ and $|\operatorname{Im}\lambda| \neq 0$,

$$\|R(\lambda : A)\| \leq \frac{C}{|\operatorname{Im}\lambda|}.$$

Writing $\lambda = \sigma + i\tau$ we have $|\sigma+i\tau| \leq |\sigma|+|\tau|$. By using the above remarks,

- if $|\sigma| \geq |\tau|$, we have

$$\frac{1}{|\sigma+i\tau|} \geq \frac{1}{2|\sigma|} = \frac{1}{2}\frac{1}{|\sigma|} \geq \frac{1}{2}\|R(\sigma+i\tau : A)\|,$$

that is

$$\|R(\lambda : A)\| \leq \frac{2}{|\lambda|}.$$

- If $|\sigma| < |\tau|$, we get

$$\frac{1}{|\sigma+i\tau|} \geq \frac{1}{2|\tau|} = \frac{1}{2C}\frac{C}{|\tau|} \geq \frac{1}{2C}\|R(\sigma+i\tau : A)\|.$$

Hence,

$$\|R(\lambda : A)\| \leq \frac{2C}{|\lambda|}.$$

From the above remarks, choosing $C_1 = \max\{2, 2C\}$, we have $\|R(\lambda : A)\| \leq \frac{C_1}{|\lambda|}$ for all $\lambda \neq 0$. From the Taylor expansion of $R(\lambda : A)$ for $\sigma + i\tau$, with $\sigma > 0$ and $\tau \neq 0$ we have that

$$\begin{aligned} R(\lambda : A) &= \sum_{n=0}^{\infty} \frac{R(\sigma+i\tau : A)^{(n)}}{n!}(\sigma+i\tau-\lambda)^n \\ &= \sum_{n=0}^{\infty} \frac{|n!R(\sigma+i\tau : A)^{(n+1)}|}{n!}(\sigma+i\tau-\lambda)^n \\ &= \sum_{n=0}^{\infty} |R(\sigma+i\tau : A)^{(n+1)}|(\sigma+i\tau-\lambda)^n. \end{aligned} \tag{1.39}$$

We note that this series converges in $B(X)$ for $\|R(\sigma+i\tau : A)\||\sigma+i\tau-\lambda| \leq k < 1$. Choosing $\lambda = \operatorname{Re}\lambda + i\tau$ in (1.39) and using the hypothesis, we have that

$$\begin{aligned} \|R(\sigma+i\tau : A)\||\sigma+i\tau-\lambda| &= \|R(\sigma+i\tau : A)\||\sigma+\operatorname{Re}\lambda| \\ &\leq \frac{C}{|\tau|}|\sigma+\operatorname{Re}\lambda| \\ &\leq k \end{aligned}$$

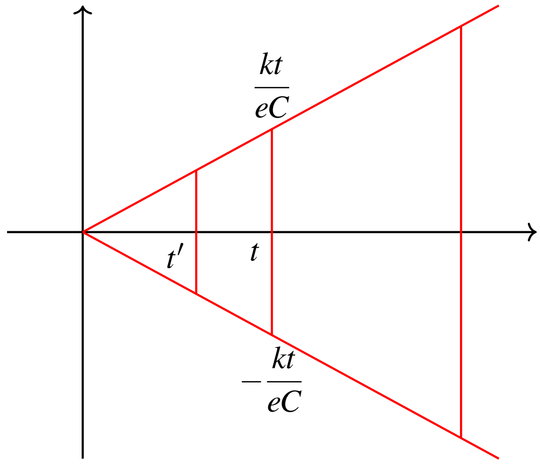

Figure 1.7: Region in which $\|R(\lambda : A)\| \leq \frac{M}{|\lambda|}$ (own figure)

$$\Leftrightarrow \quad |\sigma + \operatorname{Re}\lambda| \leq \frac{k|\tau|}{C}.$$

From the above remarks, we have that the series on (1.39) converges uniformly in $B(X)$ for $|\sigma + \operatorname{Re}\lambda| \leq \frac{k|\tau|}{C}$. Since both $\sigma > 0$ and $k < 1$ are arbitrary numbers, it follows that $\rho(A)$ contains the set of all complex numbers λ such that $\operatorname{Re}\lambda \geq 0$ satisfying $\frac{|\operatorname{Re}\lambda|}{|\operatorname{Im}\lambda|} < \frac{k}{C}$. In particular,

$$\left\{\lambda : |\arg\lambda| \leq \frac{\pi}{2} + \alpha\right\} \subset \rho(A), \text{ where } \alpha = k\arctan\left(\frac{1}{c}\right), \ 0 < k < 1.$$

Moreover, in this region, illustrated on Figure 1.7, we note that

$$\begin{aligned}
\|R(\lambda : A)\| &\leq \sum_{n=0}^{\infty} \|R(\sigma + i\tau : A)^{n+1}\| |(\sigma + i\tau - \lambda)|^n \\
&\leq \|R(\sigma + i\tau)\| \sum_{n=0}^{\infty} (\|R(\sigma + i\tau : A)\| |(\sigma + i\tau - \lambda)|)^n \\
&\leq \frac{C}{|\tau|} \sum_{n=0}^{\infty} k^n \\
&\leq \frac{C}{1-k} \frac{1}{|\tau|} \\
&\leq \frac{\sqrt{C^2+1}}{(1-k)} \frac{1}{|\lambda|} \\
&= \frac{M}{|\lambda|},
\end{aligned}$$

where we use that

$$|\lambda| = \sqrt{(\operatorname{Re}(\lambda))^2 + (\operatorname{Im}(\lambda))^2} \leq \sqrt{\left(\frac{k}{C}\operatorname{Im}(\lambda)\right)^2 + (\operatorname{Im}(\lambda))^2} \leq |\tau| \sqrt{\left(1 + \frac{k^2}{C^2}\right)}.$$

Since, by assumption, $0 \in \rho(A)$, A satisfies (iii).
$(iii \Rightarrow iv)$ Suppose that (iii) is satisfied. Then, from Lemma 1.5.1, A is the infinitesimal generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ such that

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} R(\lambda : A) d\lambda, \tag{1.40}$$

where Γ is the path composed of $re^{i\vartheta}$ and $re^{-i\vartheta}$, $0<r<\infty$ and $\frac{\pi}{2}<\vartheta<\frac{\pi}{2}+\alpha$ oriented so that $\mathrm{Im}\,\lambda$ increases along Γ.

Deriving the expression in (1.40), we get

$$T'(t)=\frac{1}{2\pi i}\int_{\Gamma}\lambda e^{\lambda t}R(\lambda:A)d\lambda,\ \text{for all } t>0.$$

Using now the definition of Γ, we get

$$\begin{aligned}
\|T'(t)\| &= \left\|\frac{1}{2\pi i}\int_{\Gamma}\lambda e^{\lambda t}R(\lambda:A)d\lambda\right\| \\
&\leq \frac{1}{2\pi i}\int_0^{\infty}|re^{i\vartheta}e^{tre^{i\vartheta}}|\|R(re^{i\vartheta}):A)\|dr \\
&\leq \frac{2M}{2\pi}\int_0^{\infty}\frac{re^{tre^{i\vartheta}}}{|\lambda|}dr \\
&\leq \frac{M}{\pi}\int_0^{\infty}\frac{re^{tr(\cos\vartheta+i\sin\vartheta)}}{|\lambda|}dr \\
&\leq \frac{M}{|\lambda|\pi}\int_0^{\infty}re^{rt\cos\vartheta}dr \\
&\leq \frac{M}{\pi}\left(\frac{e^{rt\cos\vartheta}}{t\cos\vartheta}\right)\Bigg|_0^{\infty} \\
&\leq \frac{M}{\pi}\frac{1}{t\cos\vartheta} \\
&\leq \frac{C}{t}. \qquad (1.41)
\end{aligned}$$

Thus, $(T(t))_{t\geq 0}$ is differentiable and $\|AT(t)\|=\|T'(t)\|\leq\frac{C}{t}$, for all $t>0$.
$(iv\Rightarrow i)$ Using that $(T(t))_{t\geq 0}$ is differentiable for $t>0$, from Corollary 1.5.3, we have that

$$\|T^{(n)}(t)\|=\left\|\left(T'\left(\frac{t}{n}\right)\right)^n\right\|\leq\left\|T'\left(\frac{t}{n}\right)\right\|^n.$$

In addition, from (1.41) and noting that $n!e^n\geq n^n$, we have

$$\frac{1}{n!}\|T^{(n)}(t)\|\leq\frac{1}{n!}\left\|T'\left(\frac{t}{n}\right)\right\|^n\leq\frac{1}{n!}\left(\frac{C}{t}\right)^n\leq\left(\frac{Ce}{t}\right)^n.$$

Consider

$$T(z)=T(t)+\sum_{n=1}^{\infty}\frac{T^{(n)}(t)}{n!}(z-t)^n,\ \text{for } t>0 \text{ and } T(0)=I. \qquad (1.42)$$

Using that

$$\begin{aligned}
\left\|\frac{T^{(n)}(t)}{n!}\right\||z-t|^n &\leq \frac{1}{n!}\left\|T'\left(\frac{t}{n}\right)\right\|^n|z-t|^n \\
&\leq \frac{1}{n!}\left(\frac{C}{t}n\right)^n|z-t|^n \\
&\leq \left(\frac{eC}{t}|z-t|\right)^n
\end{aligned}$$

and $(T(t))_{t\geq 0}$ is uniformly bounded, it is easy to see that the series defined in (1.42) uniformly converges in $B(X)$ for $|z-t| \leq k(\frac{t}{eC})$ for every $k<1$. Thus, $T(\cdot)$ is analytic in $\Delta = \{z \in \mathbb{C} : |\arg z| < \arctan(\frac{1}{Ce})\}$. Moreover, observing that $T(z) = T(t)$ for $z \in [0,\infty)$, we have that the family $(T(z))_{z\in\Delta}$ is an extension to Δ of the C_0-semigroup $(T(t))_{t\geq 0}$.

We prove now that $(T(z))_{z\in\Delta}$ is an analytic semigroup and that $\|T(z)\|$ is uniformly bounded in every closed subsector of Δ.

i) From analyticity of the function $z \mapsto T(z)$ it follows that $t \mapsto T(t)$ is analytic for all $t \geq 0$.

ii) Fixed $z_1 \in \Delta$, define $F : \Delta \to \mathscr{L}(X)$ by $F(z) = T(z_1)T(z) - T(z_1+z)$. Noting that $F \equiv 0$ on $[0,\infty)$, from analyticity of F we conclude that $F \equiv 0$ for all $z \in \Delta$. Being z_1 arbitrary, $T(z_1+z_2) = T(z_1)T(z_2)$, for all $z_1, z_2 \in \Delta$.

iii) Reducing the sector Δ to the closed subsector $\overline{\Delta_\varepsilon} = \{z : |\arg z| \leq \arctan(\frac{1}{Ce} - \varepsilon)\}$, we note that there exists $k' < k$ such that $|z-t| \leq k'(\frac{t}{eC})$. Then,

$$
\begin{aligned}
\|T(z)\| &\leq \left\| \sum_{n=1}^{\infty} \frac{T^{(n)}(t)}{n!}(z-t)^n \right\| + \|T(t)\| \\
&\leq \sum_{n=1}^{\infty} \left(\frac{eC}{t}|z-t| \right)^n + M \\
&\leq \sum_{n=1}^{\infty} \left(\frac{eC}{t} k' \left(\frac{t}{eC} \right) \right)^n + M \\
&\leq \sum_{n=1}^{\infty} (k')^n + M \\
&\leq \frac{k'}{1-k'} + M,
\end{aligned}
$$

which proves that $T(\cdot)$ is uniformly bounded on $\overline{\Delta_\varepsilon}$.

To complete the proof, we show that $T(z)x \to x$ as $z \to 0$. For $\varepsilon > 0$, we can select $0 < k' < 1$ such that

$$
\begin{aligned}
\|T(z)x - x\| &\leq \|T(t)x - x\| + \sum_{n=1}^{\infty} \left\| \frac{T^{(n)}(\mathrm{Re}\, z)}{n!}(z - \mathrm{Re}\, z)^n \right\| \|x\| \\
&\leq \|T(t)x - x\| + \sum_{n=1}^{\infty} \left(\frac{Ce}{\mathrm{Re}\, z}|z - \mathrm{Re}\, z| \right)^n \|x\| \\
&\leq \|T(t)x - x\| + \sum_{n=1}^{\infty} \left(\frac{Ce}{\mathrm{Re}\, z} k' \left(\frac{\mathrm{Re}\, z}{Ce} \right) \right)^n \|x\| \\
&\leq \|T(t)x - x\| + \sum_{n=1}^{\infty} (k')^n \|x\| \\
&\leq \varepsilon + \frac{k'}{1+k'}\|x\| < \varepsilon
\end{aligned}
$$

which allows us to end the proof. ■

2. Abstract Differential Equations

2.1 The abstract Cauchy problem

We start this chapter studying existence, uniqueness and regularity of solutions for the inhomogeneous initial value problem

$$\begin{aligned} u'(t) &= Au(t) + f(t),\ t > 0 && (2.1)\\ u(0) &= x \in X, && (2.2)\end{aligned}$$

where $f : [0,a) \to X$ is a suitable function and A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. The main references for this chapter are [Pazy, 2012, Pierri, 2006, Prokopczyk, 2005, Silva, 2017]

2.1.1 The inhomogeneous initial value problem

To begin we define the concepts of classical and mild solutions of (2.1)-(2.2). Then, we study the existence of solutions to this problem. To finish this section, we study the strong solution of (2.1)-(2.2) and make some remarks about it.

Definition 2.1.1 A function $u : [0,a) \to X$ is a **classical solution** of (2.1)-(2.2) on $[0,a)$ if $u(\cdot)$ is continuous on $[0,a)$, continuously differentiable on $(0,a)$, $u(t) \in D(A)$ for all $0 < t < a$ and (2.1)-(2.2) is satisfied on $[0,a)$.

R We may refer to a "classical solution" as a "solution" if there is no ambiguity.

If $u(\cdot)$ is a classical solution of the problem (2.1)-(2.2), then the function $g : [0,a) \to X$ defined by $g(s) = T(t-s)u(s)$ is differentiable on $(0,a)$ and using the Chain Rule, we see that

$$\begin{aligned} g'(s) &= \frac{dT(t-s)}{ds}u(s) + T(t-s)\frac{du(s)}{ds}\\ &= -AT(t-s)u(s) + T(t-s)\frac{du(s)}{ds}\\ &= -AT(t-s)u(s) + T(t-s)(Au(s)+f(s))\\ &= -AT(t-s)u(s) + AT(t-s)u(s) + T(t-s)f(s)\\ &= T(t-s)f(s).\end{aligned}$$

If $f \in L^1([0,a];X)$, the function $s \mapsto T(t-s)f(s)$ is integrable. From the above,

$$\begin{aligned} g(t) - g(0) &= \int_0^t T(t-s)f(s)ds \Leftrightarrow\\ T(0)u(t) - T(t)u(0) &= \int_0^t T(t-s)f(s)ds \Leftrightarrow\\ u(t) - T(t)x &= \int_0^t T(t-s)f(s)ds.\end{aligned}$$

Hence,

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds. \tag{2.3}$$

The above proves the next result.

Corollary 2.1.1 If $f \in L^1([0,a];X)$, $x \in X$ and $u(\cdot)$ is a solution of the problem (2.1)-(2.2) then, this solution is unique and it is given by the formula (2.3).

Motivated by the Corollary 2.1.1, we introduce the next concept of solution.

Definition 2.1.2 Let $x \in X$ and $f \in L^1([0,a];X)$. The function $u \in C([0,a];X)$, given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t < a,$$

is called a **mild solution** of the initial value problem (2.1)-(2.2) on $[0,a]$.

The following example shows that a mild solution may not be a classical solution.

■ **Example 2.1** Assume $x \in X$ such that $T(t)x \notin D(A)$ for all $t \geq 0$. Let the continuous function $f(s) = T(s)x$ and consider the initial value problem

$$\begin{cases} u'(t) &= Au(t) + T(t)x, \ t > 0 \\ u(0) &= 0. \end{cases} \tag{2.4}$$

The mild solution $u(\cdot)$ of the problem (2.4) is given by

$$u(t) = T(t)(0) + \int_0^t T(t-s)T(s)xds = \int_0^t T(t-s+s)xds = \int_0^t T(t)xds = tT(t)x.$$

Noting that the function $t \mapsto tT(t)x$ is not differentiable, we have that $u(\cdot)$ is not be a classical solution of the problem (2.4).

■

At this point a natural question arises: "under which conditions a mild solution is a classical one?", the following studies answer this question.

The next theorem gives us a general criterion to ensure the existence of a classical solution for the initial value problem (2.1)-(2.2).

Theorem 2.1.2 Assume $x \in X$, $f \in L^1([0,a];X) \cap C((0,a])$ and let the function $v : [0,a] \to X$ be defined by

$$v(t) = \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq a. \tag{2.5}$$

The initial value problem (2.1)-(2.2) has a classical solution $u : [0,a) \to X$ if $x \in D(A)$, and one of the following conditions is satisfied:

i) $v(\cdot)$ is continuously differentiable on $(0,a)$,

ii) $v(t) \in D(A)$ for $0 < t < a$ and $Av(\cdot)$ is continuous on $(0,a)$.

Reciprocally, if the initial value problem (2.1)-(2.2) has a solution on $[0,T)$ and $x \in D(A)$, then $v(\cdot)$ satisfies both the conditions (i) and (ii).

Proof: Suppose $x \in D(A)$ and that $u(\cdot)$ is a solution for the initial value problem (2.1)-(2.2). Then

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

Consequently, $v(t) = u(t) - T(t)x$ is continuously differentiable on $(0,a)$, which proves that condition (i) is satisfied. If $x \in D(A)$, we conclude that $T(t)x \in D(A)$ for all $t \geq 0$. Therefore, $v(t) \in D(A)$ and $Av(t) = Au(t) - AT(t)x = u'(t) - f(t) - T(t)Ax$ is continuous on $(0,a)$. Thus, (ii) is also satisfied.

Assume that condition (i) is satisfied. For $t,h\in(0,a)$ such that $t+h\in(0,a)$, we have

$$\begin{aligned}
\frac{T(h)-I}{h}v(t) &= \frac{1}{h}\left[T(h)\int_0^t T(t-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\left[\int_0^t T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\left[\int_0^{t+h} T(t+h-s)f(s)ds-\int_t^{t+h} T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\left[\int_0^{t+h} T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&\quad -\frac{1}{h}\left(\int_t^{t+h} T(t+h-s)f(s)ds\right)\\
&= \frac{v(t+h)-v(t)}{h}-\frac{1}{h}\left(\int_t^{t+h} T(t+h-s)f(s)ds\right). \qquad (2.6)
\end{aligned}$$

Using that $f(\cdot)$ is continuous, from (2.6) we conclude that $v(t)\in D(A)$, $Av(t)=v'(t)-f(t)$ and $v(0)=0$. This implies that the function $u(t)=T(t)x+v(t)$ is a classical solution of the initial value problem (2.1)-(2.2).

To finish, assume that $v(t)\in D(A)$ for $0<t<a$ and that $Av(\cdot)$ is continuous on $(0,a)$. From (2.6), we see that

$$\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}=Av(t)+f(t).$$

Using that $Av(\cdot)$ is continuous on $(0,T)$ and that $v'(t)=Av(t)+f(t)$, we infer that $v(\cdot)$ is continuously differentiable on $(0,a)$ and $v(0)=0$. From the above, we conclude that $u(t)=T(t)x+v(t)$ is a classical solution of the problem (2.1)-(2.2). ■

Next, we present two useful corollaries of Theorem 2.1.2, which establishes conditions on $f(\cdot)$ under which the mild solution of (2.1)-(2.2) is a classical solution. The first one is related to the differentiability of $f(\cdot)$.

Corollary 2.1.3 If $x\in D(A)$ and $f(\cdot)$ is continuously differentiable on $[0,a]$, then the mild solution of the problem (2.1)-(2.2) is a classical one.

Proof: From Theorem 2.1.2, it is sufficient to show that the function $v(\cdot)$ defined on (2.5) is continuously differentiable on $(0,a)$.

For $t,h\in(0,a)$ such that $t+h\in(0,a)$, we see that

$$\begin{aligned}
\frac{v(t+h)-v(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\left[\int_0^{h} T(t+h-s)f(s)ds+\int_h^{t+h} T(t+h-s)f(s)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\left[\int_0^{h} T(t+h-s)f(s)ds+\int_0^{t} T(t-s)f(s+h)ds-\int_0^t T(t-s)f(s)ds\right]\\
&= \frac{1}{h}\int_0^{h} T(t+h-s)f(s)ds+\int_0^{t} T(t-s)\left(\frac{f(s+h)-f(s)}{h}\right)ds.
\end{aligned}$$

Using that $f(\cdot)$ is continuously differentiable on $[0,T]$, we note that

$$\left\|T(t-s)\left(\frac{f(s+h)-f(s)}{h}\right)\right\| \;\leq\; \|T(t-s)\|\int_s^{s+h}\|f'(\tau)\|d\tau$$

$$\leq \|T(t-s)\| \int_0^t \|f'(\tau)\| d\tau$$
$$\leq Me^{\omega a} Ka,$$

where $K = \sup_{0 \leq s \leq T} \|f'(s)\|$ and $\|T(\cdot)\| \leq Me^{\omega a}$. From the above and the Lebesgue Dominated Convergence Theorem (see Theorem A.0.12), we have that

$$v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = T(t)f(0) + \int_0^t T(t-s)f'(s)ds, \;\; 0 < t < T,$$

which proves that $v(\cdot)$ is continuously differentiable on $(0,a)$. ∎

Next, we note conditions under which the condition (ii) on Theorem 2.1.2 is satisfied.

Corollary 2.1.4 If $x \in D(A)$, $f(t) \in D(A)$ for all $t \in [0,a]$, $f \in L^1([0,a];X)$ is continuous on $(0,a)$ and $Af(\cdot) \in L^1([0,a);X)$ then, the mild solution of the problem (2.1)-(2.2) is a classical solution on $[0,a)$.

Proof: Since $f(s) \in D(A)$ for $s > 0$, we have that $T(t-s)f(s) \in D(A)$ and that $AT(t-s)f(s) = T(t-s)Af(s)$ is integrable, because $Af(\cdot) \in L^1([0,a];X)$. Thus, $v(t) = \int_0^t T(t-s)f(s)ds \in D(A)$ and

$$Av(t) = A\int_0^t T(t-s)f(s)ds = \int_0^t AT(t-s)f(s)ds$$

is continuous. Now, using Theorem 2.1.2, it follows that the mild solution of the problem (2.1)-(2.2) is a classical solution on $[0,a)$. ∎

To finish this section, we note the concept of strong solutions to the problem (2.1)-(2.2) and present some results about this class of solutions.

Definition 2.1.3 A function $u(\cdot)$ which is differentiable almost everywhere on $[0,a]$ and $u' \in L^1([0,a];X)$ is called a **strong solution** of the initial value problem (2.1)-(2.2) if $u(0) = x$ and $u'(t) = Au(t) + f(t)$ almost everywhere on $[0,a]$.

▪ **Example 2.2** If $A = 0$ and $f \in L^1([0,T];X)$, the initial value problem (2.1)-(2.2) has usually no solutions unless $f(\cdot)$ is continuous. But it always has a stronger solution $u(t) = u(0) + \int_0^t f(s)ds$. ▪

It is easy to see that a classical solution of the problem (2.1)-(2.2) is a strong solution and that a strong solution is a mild solution of the problem (2.1)-(2.2). It is natural to ask when a mild solution is a strong solution. The next theorem answer this.

Theorem 2.1.5 Assume $x \in D(A)$, $f \in L^1([0,a];X)$ and let $v : [0,a] \to X$ be defined by

$$v(t) = \int_0^t T(t-s)f(s)ds, \;\; 0 \leq t \leq a.$$

The initial value problem (2.1)-(2.2) has a strong solution $u(\cdot)$ on $[0,a]$, if any of the following conditions is satisfied:

i) $v(\cdot)$ is differentiable almost everywhere on $[0,a]$ and $v'(t) \in L^1([0,a];X)$;

ii) $v(t) \in D(A)$ almost everywhere for $0 \leq t \leq a$ and $Av(\cdot) \in L^1([0,a];X)$.

Reciprocally, if (2.1)-(2.2) has a strong solution $u(\cdot)$ on $[0,a]$ and $x \in D(A)$ then $v(\cdot)$ satisfies (i) and (ii).

The proof is similar to the proof of Theorem 2.1.2 and we omit it. We only note that the term

$$\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds \longrightarrow f(t) \; a.e \text{ on } [0,a],$$

because $f \in L^1([0,a];X)$.

As a consequence of Theorem 2.1.5, we have the following corollary, which proof is similar to the proof of Corollary 2.1.3.

Corollary 2.1.6 If $x \in D(A)$, $f(\cdot)$ is differentiable $a.e$ on $[0,a]$ and $f \in L^1([0,a];X)$, then the initial value problem (2.1)-(2.2) has a unique strong solution $u(\cdot)$ on $[0,a]$.

2.1.2 Regularity of mild solutions for analytic semigroups

Consider the initial value problem

$$u'(t) = Au(t) + f(t), \; t > 0 \tag{2.7}$$

$$u(0) = x. \tag{2.8}$$

Next, we study the regularity of the mild solution $u(\cdot)$ of the problem (2.7)-(2.8) in the case in which A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X.

To begin, we establish conditions under which the mild solution of the problem (2.7)-(2.8) is a Hölder continuous (see bellow) classical solution.

Definition 2.1.4 Let I be an interval. A function $f : I \to X$ is **Hölder continuous** with exponent $0 < \alpha < 1$, on I if there is a constant L_f such that

$$\|f(t) - f(s)\| \leq L_f |t-s|^{\alpha}, \quad \text{for all } s,t \in I. \tag{2.9}$$

In a similar way, f is **locally Hölder continuous** if, for each $t \in I$, there is a neighborhood such that is Hölder continuous.

Notation: We denote the space of all Hölder continuous functions with expoent θ by $C^{\theta}(I;X)$.

Theorem 2.1.7 Assume $f \in L^p([0,a];X)$ with $1 < p < \infty$. If $u(\cdot)$ is the mild solution of the problem (2.7)-(2.8) then, $u(\cdot)$ is Hölder continuous with expoent $\frac{p-1}{p}$ on $[\varepsilon, a]$ for every $\varepsilon > 0$. In addition, if $x \in D(A)$ then, $u(\cdot)$ is Hölder continuous with expoent $\frac{p-1}{p}$ on $[0,a]$.

Proof: Assume $\|T(t)\| \leq M$ on $[0,a]$. Using that $(T(t))_{t\geq 0}$ is an analytic semigroup, there is $C_1 \in \mathbb{R}$ such that $\|AT(t)\| \leq \frac{C_1}{t}$ for all $t \in (0,a]$. From the above, for $x \in X$ and $0 < \varepsilon < s < t$ we have

$$\begin{aligned}
\|T(t)x - T(s)x\| &= \left\| \int_s^t AT(\tau)x d\tau \right\| \\
&\leq \int_s^t \|AT(\tau)x\| d\tau \\
&\leq \int_s^t \frac{C_1}{\tau}\|x\| d\tau \\
&\leq \int_s^t \frac{C_1}{s}\|x\| d\tau \\
&= \frac{C_1\|x\|}{s}|t-s| \leq \frac{C_1\|x\|}{\varepsilon}|t-s|.
\end{aligned}$$

This shows that the function $t \mapsto T(t)x$ is Lipschitz continuous on $[\varepsilon, a]$. In addition, if $x \in D(A)$ we have

$$\|T(t)x - T(s)x\| \le \int_s^t \|AT(\tau)x\|d\tau = \int_s^t \|T(\tau)Ax\|d\tau \le \int_s^t \|T(\tau)\|\|Ax\|d\tau = M\|Ax\||t-s|.$$

Hence, the function $t \mapsto T(t)x$ is Lipschitz continuous on $[0,a]$.

From the above, to prove the assertion, it is sufficient to show that the function

$$v(t) = \int_0^t T(t-s)f(s)ds$$

is Hölder continuous with exponent $\frac{p-1}{p}$ on $[0,a]$. For $t,h > 0$ such that $t+h \in [0,a)$, we have

$$\begin{aligned} v(t+h) - v(t) &= \int_0^{t+h} T(t+h-s)f(s)ds - \int_0^t T(t-s)f(s)ds \\ &= \int_t^{t+h} T(t+h-s)f(s)ds + \int_0^t T(t+h-s)f(s)ds - \int_0^t T(t-s)f(s)ds \\ &= \underbrace{\int_t^{t+h} T(t+h-s)f(s)ds}_{I_1} + \underbrace{\int_0^t (T(t+h-s) - T(t-s))f(s)ds}_{I_2} \end{aligned}$$

For I_1, using the Hölder's Inequality (see Theorem A.0.9), we see

$$\begin{aligned} \|I_1\| &\le \int_t^{t+h} \|T(t+h-s)\|\|f(s)\|ds \\ &\le \int_t^{t+h} M\|1 \cdot f(s)\|ds \\ &\le M\left(\int_t^{t+h} \|f(s)\|^p ds\right)^{\frac{1}{p}} \left(\int_t^{t+h} \|1\|^{p'} ds\right)^{\frac{1}{p'}} \\ &\le M\|f\|_p h^{\frac{1}{p'}}. \end{aligned}$$

To estimate I_2, note that $\ln(\rho+1) \le \frac{\rho^\alpha}{\alpha}$ for all $0 < \alpha < 1$, and for $h > 0$,

$$\|T(t+h) - T(t)\| \le \|T(t+h)\| + \|T(t)\| \le 2M.$$

From the above,

$$\begin{aligned} \|I_2\| &\le \int_0^t \|T(t+h-s) - T(t-s)\|\|f(s)\|ds \\ &\le \int_0^{t-h} \|T(t+h-s) - T(t-s)\|\|f(s)\|ds + \int_{t-h}^t \|T(t+h-s) - T(t-s)\|\|f(s)\|ds \\ &\le \int_0^{t-h} \int_{t-s}^{t+h-s} \|AT(\tau)\|\|f(s)\|d\tau ds + \int_{t-h}^t 2M\|f(s)\|ds \\ &\le \int_0^{t-h} C\|f(s)\| \int_{t-s}^{t+h-s} \frac{1}{\tau} d\tau ds + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \\ &\le \int_0^{t-h} C\|f(s)\| \ln(\tau)\Big|_{t-s}^{t+h-s} ds + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{t-h} C\|f(s)\| \ln\left(1+\frac{h}{t-s}\right) ds + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \\
&\le \frac{1}{\alpha}\int_0^{t-h} C\|f(s)\| \left(\frac{h}{t-s}\right)^{\alpha} ds + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \\
&\le \frac{C}{\alpha}\|f\|_{L^p([0,T];X)} \left(\int_0^{t-h} \frac{h^{\alpha p'}}{(t-s)^{\alpha p'}}\right)^{\frac{1}{p'}} ds + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \\
&\le \frac{h^{\alpha p'}}{\alpha} C\|f\|_{L^p([0,T];X)} \left(\frac{(t-h)^{1-\alpha p'}}{1-\alpha p'}\right)^{\frac{1}{p'}} + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}} \\
&\le \frac{h^{\alpha p'}}{\alpha} C\|f\|_{L^p([0,T];X)} \left(\frac{T^{1-\alpha p'}}{1-\alpha p'}\right)^{\frac{1}{p'}} + 2M\|f\|_{L^p([0,T];X)} h^{\frac{1}{p'}}.
\end{aligned}
$$

From the above, we obtain that

$$\|I_2\| \le \|f\|_{L^p([0,T];X)} \left(\frac{CT^{\frac{1}{p'}} + 2M}{\alpha(1-\alpha p')^{\frac{1}{p'}}}\right) h^{\frac{1}{p'}}$$

From the above estimates of I_1 and I_2, we obtain that

$$\|v(t+h)-v(t)\| \le \|I_1\| + \|I_2\| \le \|f\|_p \left(M + \frac{CT^{\frac{1}{p'}} + 2M}{\alpha(1-\alpha p')^{\frac{1}{p'}}}\right) h^{\frac{p-1}{p}}$$

which implies that $v(\cdot)$ is Hölder continuous with expoent $\frac{p-1}{p}$. ■

Next, we present a condition that turns a mild solution into a classical and a strong solution of (2.7)-(2.8). To prove the next result we use the following lemma:

Lemma 2.1.8 For $x \in X$, the homogeneous initial value problem

$$u'(t) = Au(t), \;\; u(0) = x, \tag{2.10}$$

has a unique solution.

This lemma follows from the differentiability of the function $t \mapsto T(t)x$ given by the fact that $(T(t))_{t\ge 0}$ is an analytic semigroup.

Theorem 2.1.9 If $x \in X$, $f \in C([0,a];X)$ and f is Lipschitz on $[0,a]$ then, the mild solution of (2.7)-(2.8) is a classical solution on $[0,a]$.

Proof: Noting that $(T(t))_{t\ge 0}$ is an analytic semigroup, we have that $T(\cdot)x$ is a classical solution of the problem (2.10). From Theorem 2.1.2, it is sufficient to show that $v(t) = \int_0^t T(t-s)f(s)ds \in D(A)$, for all $0 < t < a$ and that $Av(\cdot)$ is continuous on $(0,a)$.

To begin, we note that

$$
\begin{aligned}
v(t) &= \int_0^t T(t-s)f(s)ds \\
&= \int_0^t T(t-s)(f(s)-f(t))ds + \int_0^t T(t-s)f(t)ds
\end{aligned}
$$

$$= \quad v_1(t)+v_2(t).$$

From Theorem 1.2.3 (ii), $v_2(t)\in D(A)$ and $Av_2(t)=(T(t)-I)f(t)$. Moreover, from the estimate

$$\begin{aligned}
&\|Av_2(t+h)-Av_2(t)\| \\
&\le \|T(t+h)f(t+h)-T(t+h)f(t)\|+\|T(t+h)f(t)-T(t)f(t)\|+\|f(t+h)-f(t)\| \\
&\le \|T(t+h)\|\|f(t+h)-f(t)\|+\|T(t+h)f(t)-T(t)f(t)\|+\|f(t+h)-f(t)\| \\
&\le \|T(t+h)\|\|f(t+h)-f(t)\|+\|T(t)\|\|(T(h)-I)f(t)\|+\|f(t+h)-f(t)\| \\
&\le M\|f(t+h)-f(t)\|+M\|(T(h)-I)f(t)\|+\|f(t+h)-f(t)\|,
\end{aligned}$$

we infer that $Av_2(\cdot)$ is continuous on $[0,a]$.

To prove that $v_1(t)\in D(A)$ for $t\in[0,a]$ and that $Av_1(\cdot)$ is continuous, for $\varepsilon>0$ we define the function

$$v_{1,\varepsilon}(t)=\begin{cases}\int_0^{t-\varepsilon}T(t-s)(f(s)-f(t))ds, & \text{for } t>\varepsilon\\ 0, & \text{for } 0<t\le\varepsilon.\end{cases}$$

Let $\varepsilon>0$. If $t>\varepsilon$, we see that

$$\begin{aligned}
\|v_{1,\varepsilon}(t)-v_1(t)\| &\le \int_{t-\varepsilon}^{t}\|T(t-s)\|\|f(s)-f(t)\|ds \\
&\le M\int_{t-\varepsilon}^{t}\|f(s)-f(t)\|ds \\
&\le M\left(\int_{t-\varepsilon}^{t}\|f(s)\|ds+\|f(t)\|\varepsilon\right).
\end{aligned}$$

For $t\in[0,\varepsilon]$, we note that

$$\begin{aligned}
\|v_{1,\varepsilon}(t)-v_1(t)\| &\le \int_0^t\|T(t-s)\|\|f(s)-f(t)\|ds \\
&\le M\left(\int_0^{\varepsilon}\|f(s)\|+\|f(t)\|ds\right) \\
&\le M\|f\|_{L^1([0,\varepsilon])}+M\|f(t)\|\varepsilon.
\end{aligned}$$

From the above, $v_{1,\varepsilon}\to v_1$ for all $t\in[0,T]$. On the other hand, observe that

$$\begin{aligned}
\|Av_{1,\varepsilon}(t)\| &\le \int_0^{t-\varepsilon}\|AT(t-s)\|\|f(s)-f(t)\|ds \\
&\le \int_0^t\frac{C_1}{(t-s)}L_{f,t}(t-s)ds \\
&\le C_1L_{f,t}T, \qquad (2.11)
\end{aligned}$$

where $L_{f,t}$ is the Lipschitz constant of f on $[t-\varepsilon,t]$ and that $\|Av_{1,\varepsilon}(t)\|=0$ for $t\in[0,\varepsilon]$. From the previous, we infer that $v_{1,\varepsilon}(t)\in D(A)$ for all $t\in[0,a]$. Moreover, from the estimative (2.11) we infer that

$$Av_{1,\varepsilon}(t)\longrightarrow\int_0^tAT(t-s)(f(s)-f(t))ds, \text{ as } \varepsilon\to 0.$$

From the above remarks, $v_1(t)\in D(A)$ for all $t\in[0,a]$ and

$$Av_1(t)=\int_0^tAT(t-s)(f(s)-f(t))ds, \text{ for } t\in[0,a].$$

To complete the proof, next we show that $Av_1(\cdot)$ is Hölder continuous on $[0,a]$. For $0 \leq s \leq t \leq a$, we have

$$\begin{aligned}
Av_1(t) - Av_1(s) &= \int_0^t AT(t-\tau)(f(\tau)-f(t))d\tau - \int_0^s AT(s-\tau)(f(\tau)-f(s))d\tau \\
&= \int_0^s AT(t-\tau)(f(\tau)-f(t))d\tau + \int_s^t AT(t-\tau)(f(\tau)-f(t))d\tau \\
&\quad - \int_0^s AT(s-\tau)(f(\tau)-f(s))d\tau \\
&= \int_0^s AT(t-\tau)(f(\tau)-f(s))d\tau + \int_0^s AT(t-\tau)(f(s)-f(t))d\tau \\
&\quad + \int_s^t AT(t-\tau)(f(\tau)-f(t))d\tau - \int_0^s AT(s-\tau)(f(\tau)-f(s))d\tau \\
&= \int_0^s (AT(t-\tau)-AT(s-\tau))(f(\tau)-f(s))d\tau + (T(t)-T(t-s))(f(s)-f(t)) \\
&\quad + \int_s^t AT(t-\tau)(f(\tau)-f(t))d\tau \qquad (2.12) \\
&= \int_0^s \int_{s-\tau}^{t-\tau} A^2T(\theta)(f(\tau)-f(s))d\theta d\tau + (T(t)-T(t-s))(f(s)-f(t)) \\
&\quad + \int_s^t AT(t-\tau)(f(\tau)-f(t))d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
\|Av_1(t)-Av_1(s)\| &\leq \int_0^s \int_{s-\tau}^{t-\tau} \|A^2T(\theta)\|\|f(\tau)-f(s)\|d\theta d\tau + \|T(t)-T(t-s))\|\|f(s)-f(t)\| \\
&\quad + \int_s^t \|AT(t-\tau)\|\|f(\tau)-f(t)\|d\tau \\
&\leq \int_0^s \int_{s-\tau}^{t-\tau} \frac{M_2}{\theta^2} L_f (s-\tau)^\alpha d\theta d\tau + 2ML_f|t-s|^\alpha + \int_s^t \frac{C}{t-\tau} L_f (t-\tau)^\alpha d\tau \\
&\leq M_2 L_f \int_0^s \int_{s-\tau}^{t-\tau} \theta^{\alpha-2} d\theta d\tau + 2ML_f|t-s|^\alpha + CL_f \int_s^t (t-\tau)^{\alpha-1} d\tau \\
&\leq M_2 L_f \int_0^s \left(\frac{(t-\tau)^{\alpha-1}}{\alpha-1} - \frac{(s-\tau)^{\alpha-1}}{\alpha-1} \right) d\tau + 2ML_f|t-s|^\alpha + CL_f \frac{|t-s|^\alpha}{\alpha} \\
&\leq \frac{M_2 L_f}{1-\alpha} \left(\frac{(t-s)^\alpha - t^\alpha - s^\alpha}{\alpha} \right) + 2ML_f|t-s|^\alpha + CL_f \frac{|t-s|^\alpha}{\alpha} \\
&\leq \left(\frac{M_2 L_f}{\alpha(1-\alpha)} + 2ML_f + \frac{CL_f}{\alpha} \right) |t-s|^\alpha. \ \blacksquare
\end{aligned}$$

The main result of this section establishes the Hölder continuity of the solution $u(\cdot)$ of (2.7)-(2.8).

Theorem 2.1.10 Assume $f \in C^\theta([0,a];X)$ and let $u \in C([0,a];X)$ be the mild solution of the initial value problem (2.7)-(2.8). Then, $u(\cdot)$ is a classical solution and

i) for every $\delta > 0$, $Au \in C^\theta([\delta,a];X)$ and $u' \in C^\theta([\delta,a];X)$,

ii) If $x \in D(A)$ then $Au(\cdot)$ and $u'(\cdot)$ are continous on $[0,a]$.

iii) If $x = 0$ and $f(0) = 0$ then $Au(\cdot)$ and $u'(\cdot)$ belongs to $C^\theta([0,a];X)$.

Proof: The fact that $u(\cdot)$ is a classical solution follows from Theorem 2.1.9. Next, considering $v_1(\cdot)$ and $v_2(\cdot)$ the functions defined on the proof of the Theorem 2.1.9, we prove the other assertions.

i) To begin, using that $AT(\cdot)x$ is Lipschitz continous on $[\delta,a]$ it is sufficient to show that $A\nu(t) \in C^\theta([\delta,a];X)$.

From the proof of Theorem 2.1.9, we know that $A\nu_1(t) \in C^\theta([0,a];X)$.

Next, we study the function $A\nu_2(\cdot)$. Using that $A\nu_2(t) = (T(t)-I)f(t)$ and $f \in C^\theta([0,a];X)$, it is enough show that $T(t)f(t) \in C^\theta([\delta,a];X)$ for every $\delta > 0$. For $t \geq \delta$ and $h > 0$, we get

$$\begin{aligned}
&\|T(t+h)f(t+h) - T(t)f(t)\| \\
&= \|T(t+h)f(t+h) - T(t+h)f(t) + T(t+h)f(t) - T(t)f(t)\| \\
&= \|T(t+h)(f(t+h)-f(t)) + f(t)(T(t+h)-T(t))\| \\
&\leq \|T(t+h)\|\|f(t+h)-f(t)\| + \|f(t)\|\|T(t+h)-T(t)\| \\
&\leq ML_f h^\theta + \|f(t)\| \int_t^{t+h} \|AT(\tau)\| d\tau \\
&\leq ML_f h^\theta + \|f(t)\|_{C([0,T];X)} \int_t^{t+h} \frac{C}{\tau} d\tau \\
&\leq ML_f h^\theta + \|f(t)\|_{C([0,T];X)} C\frac{h}{t} \\
&\leq C_1 h^\theta, \qquad (2.13)
\end{aligned}$$

which proves that $A\nu_2(\cdot)$ is Hölder continous on $[\delta,a]$. From the above $Au \in C^\theta([\delta,a];X)$ for all $\delta > 0$, which in turns implies taht $u' \in C^\theta([\delta,a];X)$.

ii) From the proof of the Theorem 2.1.9, we only need to show the continuity of

$$Au(t) = AT(t)x + A\int_0^T T(t-s)(f(s)-f(t))ds + A\int_0^t T(t-s)f(t)ds.$$

For $x \in D(A)$, we have $AT(\cdot)x \in C([0,a];X)$, because $AT(t)x = T(t)Ax$ and the function $t \mapsto T(t)x$ is continuous and, from the proof of the Theorem 2.1.9, we know that $A\nu_1(\cdot) \in C^\theta([0,a];X)$. To finish, we study the continuity of $A\nu_2(t) = (T(t)-I)f(t)$. From the previous, it is sufficient to study the continuity at $t = 0$, we note that

$$\begin{aligned}
\|T(t)f(t) - T(0)f(0)\| &= \|T(t)f(t) - T(0)f(0)\| \\
&= \|T(t)f(t) - T(t)f(0) + T(t)f(0) - T(0)f(0)\| \\
&\leq \|T(t)\|\|f(t)-f(0)\| + \|T(t)f(0) - T(0)f(0)\| \\
&\leq M\|f(t)-f(0)\| + \|T(t)f(0) - T(0)f(0)\|,
\end{aligned}$$

which implies that $T(t)f(t)$ is continous at $t = 0$.

iii) Using a similar argument used above, we have only to show that $T(\cdot)f(\cdot) \in C^\theta([0,a];X)$. For $t,h \in [0,a]$ with $t+h \in [0,a]$, we get

$$\begin{aligned}
&\|T(t+h)f(t+h) - T(t)f(t)\| \\
&= \|T(t+h)f(t+h) - T(t+h)f(t) + T(t+h)f(t) - T(t)f(t)\| \\
&\leq \|T(t+h)\|\|f(t+h)-f(t)\| + \|(T(t+h)-T(t))f(t)\| \\
&\leq ML_f h^\theta + \left\| \int_t^{t+h} AT(\tau)f(t)d\tau \right\| \\
&\leq ML_f h^\theta + \int_t^{t+h} \|AT(\tau)(f(t)-f(0))\| d\tau \\
&\leq ML_f h^\theta + \int_t^{t+h} \frac{C}{\tau} L_f t^\theta d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq ML_f h^\theta + \int_t^{t+h} \frac{C}{\tau} L_f \tau^\theta d\tau \\
&\leq ML_f h^\theta + CL_f \int_t^{t+h} \frac{d\tau}{\tau^{1-\theta}} \\
&\leq L_f \left(M + \frac{C}{\theta} \right) h^\theta. \ \blacksquare
\end{aligned}$$

2.2 Semilinear evolution equations

In this section, we are interested in the semilinear initial value problem

$$u'(t) + Au(t) = f(t, u(t)), \ t > t_0 \tag{2.14}$$

$$u(t_0) = u_0, \tag{2.15}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $f : [t_0, a] \times X \to X$ is continuous in t and satisfies a Lipschitz condition in the second variable.

Before the main result, we present the concept of classical solution of the problem (2.14)-(2.15).

Definition 2.2.1 A function $u : [t_0, a] \to X$ is a **classical solution** of (2.14)-(2.15) if $u(\cdot)$ is continuous on $[t_0, a]$, continuously differentiable on (t_0, a), $u(t) \in D(-A)$ for all $t_0 < t < a$, (2.14) is satisfied on (t_0, a) and $u(t_0) = u_0$.

Remember that if (2.14)-(2.15) has a solution $u(\cdot)$ then, proceeding as in the proof of the Corollary 2.1.1, we have

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^t T(t - s) f(s, u(s)) ds. \tag{2.16}$$

Motivated by the above, we consider the concepts of mild and strong solutions.

Definition 2.2.2 A continuous solution $u(\cdot)$ of the integral equation (2.16) is called a **mild solution** of the initial value problem (2.14)-(2.15).

Definition 2.2.3 A function $u \in C([0, a]; X)$ is called a **strong solution** of the initial value problem (2.14)-(2.15) if $u(\cdot)$ is differentiable almost everywhere on $[t_0, a]$, $u' \in L^1([0, a]; X)$, $u(0) = u_0$ and (2.14) is satisfied almost everywhere on $[t_0, a]$.

The following result concerns the existence and uniqueness of a mild solution to the problem (2.14)-(2.15). In the remainder of this section, we always assume that A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$.

Theorem 2.2.1 Assume that $u_0 \in X$ and that $f : [t_0, a] \times X \to X$ is continuous on the first variable and uniformly Lipschitz continuous (with constant L) on the second variable. Then, the initial value problem (2.14)-(2.15) has a unique mild solution $u \in C([t_0, a]; X)$. Moreover, if $u(\cdot, x(\cdot))$ denotes the mild solution of the problem (2.14)-(2.15) then the map $u_0 \mapsto u(\cdot, u_0)$ is Lipschitz continuous from X to $C([t_0, a]; X)$.

Proof: We start defining the continuous map $\Gamma : C([t_0, a]; X) \to C([t_0, a]; X)$ by

$$(\Gamma u)(t) = T(t - t_0)u_0 + \int_{t_0}^t T(t - s) f(s, u(s)) ds, \ \ t_0 \leq t \leq a.$$

Denoting by $\|\cdot\|_\infty$ the norm of $C([t_0,a];X)$, for $u,v \in C([t_0,a];X)$ and $t \in [t_0,a]$, we have

$$\begin{aligned}
&\|\Gamma u(t)-\Gamma v(t)\| \\
&= \left\| T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds - T(t-t_0)u_0 - \int_{t_0}^{t} T(t-s)f(s,v(s))ds \right\| \\
&\leq \int_{t_0}^{t} \|T(t-s)\|\|f(s,u(s))-f(s,v(s))\|ds \\
&\leq ML\int_{t_0}^{t} \|u(s)-v(s)\|ds \\
&\leq ML\|u-v\|_\infty (t-t_0), \qquad (2.17)
\end{aligned}$$

where we have assumed that $\|T(t)\| \leq M$ for all $t \in [t_0,a]$. Moreover, using (2.17), we see that

$$\begin{aligned}
\|\Gamma^2 u(t)-\Gamma^2 v(t)\| &= \left\| \int_{t_0}^{t} T(t-s)f(s,\Gamma(u(s)))ds - \int_{t_0}^{t} T(t-s)f(s,\Gamma(v(s)))ds \right\| \\
&\leq \int_{t_0}^{t} \|T(t-s)\|\|f(s,\Gamma(u(s)))-f(s,\Gamma(v(s)))\|ds \\
&\leq ML\int_{t_0}^{t} \|\Gamma(u(s))-\Gamma(v(s))\|ds \\
&\leq ML\int_{t_0}^{t} ML\|u-v\|_\infty (s-t_0)ds \\
&\leq \frac{(ML(t-t_0))^2}{2}\|u-v\|_\infty.
\end{aligned}$$

Continuing as above, it is easy to infer that

$$\|\Gamma^n(u(t))-\Gamma^n(v(t))\| \leq \frac{(ML(t-t_0))^n}{n!}\|u-v\|_\infty, \text{ for all } t \in [t_0,a] \text{ and } n \in \mathbb{N}\setminus\{0\},$$

which implies that

$$\|\Gamma^n u-\Gamma^n v\|_\infty \leq \frac{(MLT)^n}{n!}\|u-v\|_\infty.$$

For n large enough such that $\frac{(MLT)^n}{n!} < 1$, we have that Γ^n is a contraction. From the contraction mapping principle, Γ^n has a unique fixed point $u(\cdot)$ in $C([t_0,a];X)$. Noting that $\Gamma u = \Gamma(\Gamma^n u) = \Gamma^n(\Gamma u)$, we conclude that Γu is the fixed point of $\Gamma^n u$. Using the uniqueness of $\Gamma u(\cdot)$ we obtain that $\Gamma u = u$. From the above, $u(\cdot)$ is the unique mild solution of the problem (2.14)-(2.15) on $[t_0,a]$.

Suppose now that $v(\cdot) = v(\cdot,v_0)$ is a mild solution of (2.14) on $[t_0,a]$ with initial condition v_0. Then,

$$\begin{aligned}
\|u(t)-v(t)\| &\leq \|T(t-t_0)u_0 - T(t-t_0)v_0\| \\
&\quad + \int_{t_0}^{t} \|T(t-s)(f(s,u(s))-f(s,v(s)))\|ds \\
&\leq \|T(t-t_0)\|\|u_0-v_0\| + \int_{t_0}^{t} \|T(t-s)\|\|f(s,u(s))-f(s,v(s))\|ds \\
&\leq M\|u_0-v_0\| + ML\int_{t_0}^{t} \|u(s)-v(s)\|ds
\end{aligned}$$

which implies, by Gronwall's Inequality (see A.0.8), that

$$\|u(t)-v(t)\| \leq Me^{ML(T-t_0)}\|u_0-v_0\|.$$

Hence,

$$\|u-v\|_\infty \leq M e^{ML(T-t_0)}\|u_0-v_0\|,$$

which proves that the map $u_0 \mapsto u(\cdot,u_0)$ is Lipschitz. ■

Arguing as in the proof of the Theorem 2.2.1, we obtain the next corollary.

Corollary 2.2.2 Let $g \in C([t_0,a];X)$. Then, the integral equation

$$w(t) = g(t) + \int_{t_0}^{t} T(t-s)f(s,w(s))ds$$

has a unique solution $w \in C([t_0,a];X)$.

Next, we note that the previous results can be generalized. For all $g \in C_{Lip}([t_0,a]\times X;X)$ there exists a unique continuous solution of the integral problem

$$w(t) = g(t,w(t)) + \int_{t_0}^{t} T(t-s)f(s,w(s))ds.$$

The uniform Lipschitz condition on $f(\cdot,u(\cdot))$ implies the existence and uniqueness of a mild solution of (2.14)-(2.15) on the whole interval $[0,a]$. In the following result, we study the case in which f satisfies the next Lipschitz type condition. For $t' \geq 0$ and $c \geq 0$, there exists a constant $L(c,t') > 0$ such that

$$\|f(t,u)-f(t,v)\| \leq L(c,t')\|u-v\|,$$

for all $u,v \in B_c(0,X) = \{x \in X : \|x\| \leq c\}$ and every $t \in [0,t']$.

Theorem 2.2.3 Assume that $f : [0,\infty)\times X \to X$ satisfies the above Lipschitz condition. Then, for $u_0 \in X$, there exists $0 < t_{max} \leq \infty$ such that the initial value problem

$$u'(t) + Au(t) = f(t,u(t)),\ t \geq 0 \qquad (2.18)$$

$$u(0) = u_0 \qquad (2.19)$$

has a unique mild solution $u \in C([0,t_{max});X)$. Moreover, if $t_{max} < \infty$ then $\lim_{t\uparrow t_{max}}\|u(t)\| = \infty$.

Proof: First we show that the initial value problem (2.18)-(2.19) has a unique mild solution on a bounded interval. Let $c > 0$ and $t' > 0$. For $0 < t_1 < t'$ such that $ML(c,t')t_1 < 1$ and $M\|u_0\| + M(L(c,t')c + \|f(\cdot,0)\|)t_1 \leq c$ we define the space

$$\Lambda = \{x \in C([0,t_1];X) : \|x(t)\| \leq c,\ t \in [0,b]\}$$

endowed with the uniform norm denoted by $\|\cdot\|_\infty$ and let $\Gamma : \Lambda \to C([0,t'];X)$ be the function defined by

$$\Gamma(x)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds.$$

For $s \in [0,t']$ and $u,v \in B_c(0,\Lambda)$, we have

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq \int_0^t \|T(t-s)\|\|f(s,u(s)) - f(s,v(s))\|ds \\ &\leq ML(c,t')\|u-v\|t \\ &\leq ML(c,t')t_1\|u-v\|\end{aligned}$$

which implies that $\Gamma(\cdot)$ is Lipschitz. Moreover, for $u \in \Lambda$, we see that

$$\begin{aligned}
\|(\Gamma u)(t)\| &= \left\|T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds\right\| \\
&\leq \|T(t-t_0)\|\|u_0\| + \int_{t_0}^{t} \|T(t-s)\|(\|f(s,u(s))-f(s,0)+f(s,0)\|)ds \\
&\leq M\|u_0\| + \int_{0}^{t} M(L(c,t')\|u\| + \sup_{t\in[0,b]} \|f(t,0)\|)ds \\
&\leq M\|u_0\| + M(L(c,t')c + \|f(\cdot,0)\|)t_1 \\
&\leq c,
\end{aligned}$$

hence $\Gamma(\Lambda) \subset \Lambda$.

From the above, $\Gamma(\cdot)$ is a contraction on Λ, hence there exists a unique mild solution $u \in C([0,t_1];X)$ of (2.18)-(2.19).

We introduce now the set of functions

$$P = \{u : D(u) \subset [0,t_1) \to X : u \text{ is a mild solution of (2.18)-(2.19) on } D(u)\}$$

and the relation $u_1 \leq u_2$ if $D(u_1) \subset D(u_2)$ and $u_1 = u_2$ on $D(u_1)$. It is easy to see that "$\leq$" is a partial order. By defining $D(u) = \cup_{v\in P} D(v)$ and $u : D(u) \to X$ by $u(t) = v(t)$ if $t \in D(v)$, we obtain a mild solution of (2.18)-(2.19) on $D(u)$ such that $v \leq u$ for all $v \in Q \subset P$. From the above and Zorn's Lemma, there exists a maximal solution $u : D(u) \to X$ of (2.18)-(2.19), which we denote $t_{max} = \sup D(u)$.

To finish, we prove that if $t_{max} < \infty$ then, $\lim_{t\to t_{max}} \|u(t)\| = \infty$. Otherwise, there exist $\alpha > 0$ and a sequence $(t_n)_n$ such that $t_n \to t_{max}$ and $\|x(t_n)\| \leq \alpha$, $\forall n \in \mathbb{N}$. Arguing as above, we can conclude that there exists $t_1 > 0$ such that for each $n \in \mathbb{N}$ the initial value problem

$$\begin{aligned}
w'(t) + Aw(t) &= f(t,w(t)),\ t \in (t_n, t_n + t_1), \\
w(t_n) &= w_{t_n},
\end{aligned}$$

has a unique mild solution $w_n \in C([t_n, t_n + t_1];X)$. Let $n \in \mathbb{N}$ large enough such that $t_n + t_1 > t_{max}$. Defining now $\overline{u} : [0, t_n + t_1] \to X$ by $\overline{u} = u(t)$ for $t \in [0,t_n]$ and $\overline{u} = u_n(t)$ for $t \in [t_n, t_1]$ we obtain a mild solution of (2.18)-(2.19) such that $u \leq \overline{u}$, which is contrary to the maximality of $u(\cdot)$. This completes the proof. ■

The next theorem gives us sufficient conditions to guarantee the existence of a classical solution of the initial value problem (2.14)-(2.15).

Theorem 2.2.4 If $f : [t_0, a] \times X \to X$ is continuously differentiable and $u_0 \in D(A)$ then, the mild solution of (2.14)-(2.15) on $[0,a]$ is a classical solution.

Proof: Let $u \in C([t_0,a];X)$ be the mild solution of (2.14)-(2.15). Using that f is a C^1 function, for $s, h \in [t_0, a]$ with $s + h \in [t_0, a]$, we note that

$$\begin{aligned}
& f(s+h,u(s+h)) - f(s,u(s)) \\
&= \int_0^1 \partial_\tau f\left(\tau(s+h,u(s+h)) + (1-\tau)(s,u(s))\right) d\tau \\
&= \int_0^1 \partial_\tau f(\tau(s+h,u(s+h)) + (1-\tau)(s,u(s)))h d\tau \qquad (2.20)
\end{aligned}$$

$$+\int_0^1 \partial_x f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s)))(u(s+h)-u(s))d\tau.$$

Hence

$$\begin{aligned}
&\|f(s+h,u(s+h))-f(s,u(s))\| \\
&\leq \int_0^1 \|\partial_t f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s)))\| h d\tau \\
&\quad + \int_0^1 \|\partial_x f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s)))\| \|u(s+h)-u(s)\| d\tau \\
&\leq \Theta_1 h + \Theta_2 \|u(s+h)-u(s)\|,
\end{aligned}$$

where

$$\Theta_1 = \sup\left\{\partial_t f(\tau(t',x')+(1-\tau)(t,x)) : \tau\in[0,1], t,t'\in[0,T] \text{ and } x,x'\in X\right\},$$

and

$$\Theta_2 = \sup\left\{\partial_x f(\tau(t',x')+(1-\tau)(t,x)) : \tau\in[0,1], t,t'\in[0,T] \text{ and } x,x'\in X\right\}.$$

We also observe that

$$\begin{aligned}
u(t+h)-u(t) &= T(t+h-t_0)u_0 - T(t-t_0)u_0 \\
&\quad + \int_{t_0}^{t+h} T(t+h-s)f(s,u(s))ds - \int_{t_0}^{t} T(t-s)f(s,u(s))ds \\
&= \int_{t-t_0}^{t+h-t_0} AT(s)u_0 ds + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds \\
&\quad + \int_{t_0+h}^{t+h} T(t+h-s)f(s,u(s))ds - \int_{t_0}^{t} T(t-s)f(s,u(s))ds \\
&= \int_{t-t_0}^{t+h-t_0} T(s)Au_0 ds + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds \\
&\quad + \int_{t_0}^{t} T(t-s)f(s+h,u(s+h))ds - \int_{t_0}^{t} T(t-s)f(s,u(s))ds \\
&= \int_{t-t_0}^{t+h-t_0} T(s)Au_0 ds + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds \\
&\quad + \int_{t_0}^{t} T(t-s)(f(s+h,u(s+h))-f(s,u(s)))ds. \qquad (2.21)
\end{aligned}$$

Using the inequalities above, we note that

$$\begin{aligned}
\frac{\|u(t+h)-u(t)\|}{h} &\leq \frac{1}{h}\int_{t-t_0}^{t+h-t_0} \|T(s)\|\|Au_0\| ds + \frac{1}{h}\int_{t_0}^{t_0+h} \|T(t+h-s)\|\|f(s,u(s))\| ds \\
&\quad + \frac{1}{h}\int_{t_0}^{t} \|T(t-s)\|\|f(s+h,u(s+h))-f(s,u(s))\| ds \\
&\leq \frac{1}{h}\int_{t-t_0}^{t+h-t_0} M\|Au_0\| ds + \frac{1}{h}\int_{t_0}^{t_0+h} M\|f(\cdot,u(\cdot))\|_{C([t_0,T];X)} ds \\
&\quad + \int_{t_0}^{t} M(\Theta_1 h + \Theta_2\|u(s+h)-u(s)\|)ds
\end{aligned}$$

$$\leq M(\|Au_0\| + \|f\| + \Theta_1 a) + \int_{t_0}^{t} \Theta_2 \frac{\|u(s+h) - u(s)\|}{h} ds.$$

From Gronwall's Inequality (see Theorem A.0.8), we obtain

$$\frac{\|u(t+h) - u(t)\|}{h} \leq M(\|Au_0\| + \|f\| + \Theta_1 a) e^{\Theta_2 a},$$

which implies that $u(\cdot)$ is Lipschitz continuous.

On the other hand, if $u(\cdot)$ is the classical solution, from (2.20) and (2.21) we have that

$$u'(t) = AT(t-t_0)u_0 + T(t-t_0)f(t_0, u(t_0)) + \int_{t_0}^{t} T(t-s)\partial_s f(s, u(s))ds + \int_{t_0}^{t} \partial_x f(s, u(s))u'(s)ds.$$

Motivated from the above, we define the function $g : [t_0, a] \to X$ by

$$g(t) = T(t-t_0)f(t_0, u(t_0)) + AT(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)\partial_x f(s, u(s))w(s)ds,$$

and study the abstract integral problem

$$w(t) = g(t) + \int_{t_0}^{t} T(t-s)\frac{\partial}{\partial s} f(s, u(s))ds, \ t \in [t_0, a]. \tag{2.22}$$

From Corollary 2.2.2 we know that the problem (2.22) has a unique solution $w \in C([t_0, a]; X)$. Next, we prove that $w = u'$ on $[t_0, a]$.

Let $t, h \in [t_0, a]$ with $t + h \in [t_0, a]$. Defining $w_h(t) = \frac{u(t+h)-u(t)}{h} - w(t)$ and using (2.20) we have that

$$\begin{aligned}
w_h(t) \;=\; & \frac{T(h)-I}{h} T(t-t_0)u_0 - AT(t-t_0)u_0 \\
& + \frac{1}{h}\int_{t_0}^{t_0+h} T(t+h-s)f(s, u(s))ds - T(t-t_0)f(t_0, u(t_0)) \\
& + \frac{1}{h}\int_{t_0}^{t} T(t-s)\int_0^1 \partial_s f(\tau(s+h, u(s+h)) + (1-\tau)(s, u(s)))h d\tau ds \\
& - \int_{t_0}^{t} T(t-s)\partial_s f(s, u(s))ds - \int_{t_0}^{t} T(t-s)\partial_x f(s, u(s))w(s)ds \\
& + \int_{t_0}^{t} T(t-s)\int_0^1 \partial_x f(\tau(s+h, u(s+h)) + (1-\tau)(s, u(s)))\left(\frac{u(s+h)-u(s)}{h}\right) d\tau ds.
\end{aligned}$$

Using that $u_0 \in D(A)$, from the definition of infinitesimal generator,

$$\lim_{h\to 0} \frac{T(t+h-t_0)u_0 - T(t-t_0)u_0}{h} = AT(t-t_0)u_0.$$

From Theorem 1.2.3 (i) and the fact that $f(\cdot, u(\cdot)) \in C([t_0, a]; X)$, we also note that

$$\lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} T(t+h-s)f(s, u(s))ds = T(t-t_0)f(t_0, u(t_0)).$$

Moreover, noting that f is a C^1 function we see that

$$\lim_{h\to 0}\int_{t_0}^{t} T(t-s)\int_0^1 \partial_s f(\tau(s+h, u(s+h)) + (1-\tau)(s, u(s)))d\tau ds = \int_{t_0}^{t} T(t-s)\partial_s f(s, u(s))ds.$$

We also note that,

$$\int_{t_0}^{t} T(t-s)\left(\int_0^1 \partial_x f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s)))(\frac{u(s+h)-u(s)}{h})\right)d\tau - \partial_x f(s,u(s))w(s)ds$$
$$= \int_{t_0}^{t} T(t-s)\int_0^1 (\partial_x f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s))) - \partial_x f(s,u(s)))\left(\frac{u(s+h)-u(s)}{h}\right)d\tau ds$$
$$+\int_{t_0}^{t} T(t-s)\partial_x f(s,u(s))\left(\frac{u(s+h)-u(s)}{h} - w(s)\right)ds.$$

From the previous, the fact that $\theta \mapsto \partial_u f(\theta, u(\theta))$ is continuous and that $\frac{u(s+h)-u(s)}{h}$ is uniformly bounded for $s,h,\theta \in [t_0,a]$ we infer that

$$\lim_{h\to 0}\int_{t_0}^{t} T(t-s)\int_0^1 (\frac{\partial}{\partial u} f(\tau(s+h,u(s+h))+(1-\tau)(s,u(s)))d\tau ds = \int_{t_0}^{t} T(t-s)\frac{\partial}{\partial u} f(s,u(s))).$$

From the above remarks, it follows that

$$\|w_h(t)\| \le \varepsilon(h) + \Theta\int_{t_0}^{t}\|w_h(s)\|ds, \tag{2.23}$$

where $\Theta > 0$ is a constant indenpendent of s and h and $\varepsilon(h)\to 0$ as $h\to 0$. From (2.23) and the Gronwall's Inequality (see Theorem A.0.8), we obtain that

$$\|w_h(t)\| \le \varepsilon(h)e^{(a-t_0)M}$$

which implies that $\|w_h(t)\|\to 0$ as $h\to 0$, that $u' = w$ and that u is a C^1 function on $[t_0,a]$.

Finally, using that $u(\cdot)$ and $f(\cdot)$ are continuously differentiable, we have that $f(\cdot,u(\cdot))$ is continuously differentiable on $[t_0,a]$, and from Corollary 2.1.3, we infer that the mild solution of

$$\frac{dv(t)}{dt} + Av(t) = f(t,u(t)),\ t > t_0 \tag{2.24}$$
$$v(t_0) = u_0. \tag{2.25}$$

is a classical solution.

To finish, remarking that $u(\cdot)$ is the unique solution of (2.24)-(2.25), from the above remarks we infer that $u(\cdot)$ is a classical solution of the problem (2.14)-(2.15). ■

To establish the next result we need to include some remarks. Next, for $x\in D(A)$, we define the graph norm in $D(A)$ by

$$|x|_A := \|x\| + \|Ax\|.$$

Let $Y = D(A)$ endowed with the norm $|\cdot|_A$. We claim that Y is a Banach space. Let $(x_n)_n$ be a Cauchy sequence on Y. For $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m|_A = \|x_n - x_m\| + \|Ax_n - Ax_m\| < \varepsilon,\ \forall n,m > N_\varepsilon.$$

Thus, $(x_n)_n$ and $(Ax_n)_n$ are Cauchy sequences on X and there are $x,y\in X$ such that $x_n\to x$ and $Ax_n\to y$. Using that A is a closed linear operator, we obtain that $Ax = y$, which implies that $|x_n - x|_A \longrightarrow 0$ as $n\to\infty$. Moreover, noting that $Y\subseteq X$ and that $T(t): D(A)\to D(A)$, it follows that $(T(t))_{t\ge 0}$ is a C_0-semigroup on Y. In fact, for $x\in D(A)$, we have that

$$|T(t)x - x|_A = \|T(t)x - x\| + \|AT(t)x - Ax\| = \|T(t)x - x\| + \|T(t)Ax - Ax\| \le \varepsilon,$$

which impliest that $\lim_{t\downarrow 0}|T(t)x - x|_A = 0$.

Next, we prove the existence of a classical solution for the initial value problem (2.14)-(2.15).

Theorem 2.2.5 Assume that $f:[t_0,a]\times Y\to Y$ is uniformly Lipschitz in Y and that for all $y\in Y$ the function $t\mapsto f(t,y)$ is continuous from $[t_0,a]$ into Y. If $u_0\in D(A)$, the initial value problem (2.14)-(2.15) has a unique classical solution on $[t_0,a]$.

Proof: Applying Theorem 2.2.1 in Y, we obtain that there exists a mild solution $u\in C([t_0,T];Y)\subset C([t_0,a];X)$ of the problem, which is given by

$$u(t)=T\big|_Y(t-t_0)u_0+\int_{t_0}^{t}T\big|_Y(t-s)f(s,u(s))ds.$$

Let $g(s)=f(s,u(s))$. From the assumptions, $g(s)\in D(A)$ for all $s\in[t_0,a]$ and the functions $g(\cdot)$ and $Ag(\cdot)$ are continuous in X. Therefore, from Corollary 2.1.4, we infer that the initial value problem

$$\begin{aligned} v'(t)+Av(t) &= g(t),\ \ t>t_0\\ v(t_0) &= u_0, \end{aligned}$$

has a unique classical solution $v\in C([t_0,a];Y)$. Finally, noting that

$$\begin{aligned} v(t) &= T(t-t_0)u_0+\int_{t_0}^{t}T(t-s)g(s)ds\\ &= T(t-t_0)u_0+\int_{t_0}^{t}T(t-s)f(s,u(s))ds=u(t), \end{aligned}$$

we obtain that $u(\cdot)$ is a classical solution of (2.14)-(2.15) on $[t_0,a]$. ■

3. Evolution Abstract Problems with $L^{q,\alpha}$-Hölder nonlinear terms

In this chapter, we introduce the class of $L^{p,\alpha}$- Hölder functions and study the local and global existence and uniqueness of solution for abstract differential equations described in the form

$$\begin{aligned} u'(t) &= Au(t)+F(t,u(t)), \quad t\in[0,a], \\ u(0) &= x_0 \in X, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the generator of an analytic C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$ and $F : [0,a]\times X \to X$ is a $L^{p,\alpha}$-Hölder function.

We note that a $L^{p,\alpha}$-Hölder function is a function satisfying a type of Hölder condition described using a L^p function (see Definition (3.1.1)) and that our current studies are motivated by the concept of L^p_{Lip}-Lipschitz considered initially in [Hernandez et al., 2021]. It is important to remark that each locally Lipschitz function and each locally Hölder function is a $L^{p,\alpha}$-Hölder function. Considering this fact, our main motivation is to extend some classic results about the existence and regularity of mild solution for the case in which $F(\cdot)$ is Lispchtiz or α-Hölder, to the more general case in which $F(\cdot)$ is a $L^{p,\alpha}$-Hölder function.

To conclude this introduction we note some notations used in the remainder of this work. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$, next we use the notation $\mathscr{L}(Z,W)$ for the space of bounded linear operators from Z into W endowed with the operator norm denoted by $\|\cdot\|_{\mathscr{L}(Z,W)}$ and $B_r(z,Z)$ denotes the closed ball $B_r(z,Z) := \{x \in Z : \|x-z\|_Z \leq r\}$. For $p > 1$ we denote by p' the number defined by $\frac{1}{p}+\frac{1}{p'} = 1$.

In the remainder of this chapter, A is the generator of an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on X and, for sake of simplicity, we assume that $0 \in \rho(A)$. For $\eta > 0$, we use the notations $(-A)^\eta$ and $X_\eta = D((-A)^\eta)$ for the η-order fractional power of A and for the domain of $(-A)^\eta$ endowed with the norm $\|\cdot\|_\eta$ defined by $\|x\|_\eta = \|(-A)^\eta x\|$. We also assume that C_i, C_η ($i \in \mathbb{N}$ and $\eta > 0$) are constants such that $\|A^i T(t)\| \leq \frac{C_i}{t^i}$ and $\|(-A)^\eta T(t)\| \leq \frac{C_\eta}{t^\eta}$ for all $t \in [0,a]$.

3.1 $L^{p,\alpha}$-Hölder and L^p_{Lip}-Lipschitz functions

It is well-known the importance of Lipschitz and Hölder functions in the study of the existence and uniqueness of solutions for ordinary differential equations. Considering this fact, the main goal of this section is to present a generalization of these concepts.

For convenience, we note the next concepts. For X and Y Banach spaces, we say that a function $F : [0,a]\times X \to Y$ is Lipschitz if there exists $L_F > 0$ such that

$$\|F(t,x)-F(s,y)\|_Y \leq L_F(|t-s|+\|x-y\|_X), \quad \forall t,s \in [0,a] \text{ and } \forall x,y \in X.$$

Similarly for $\alpha \in (0,1]$ we say that F is α-Hölder continuous if there exists $L_f > 0$ such that

$$\|F(t,x)-F(s,y)\|_Y \leq L_F(|t-s|^\alpha+\|x-y\|_X), \quad \forall t,s \in [0,a] \text{ and } \forall x,y \in X,$$

Next, $(Y_i, \|\cdot\|_{Y_i})$, $i = 1,2$, are Banach spaces and $q \geq 1$.

Inspired by the concept of L^p_{Lip}-Lipschitz function introduced in [Hernandez et al., 2020] we consider the following definition.

Definition 3.1.1 Let $P : [c,d]\times Y_1 \to Y_2$ be a function such that $P(t,\cdot) : Y_1 \to Y_2$ is continuous *almost everywhere* for $t \in [0,a]$ and $P(\cdot,x) : [c,d] \to Y_2$ is strongly measurable for all $x \in Y_1$. If there are $\alpha \in (0,1]$, an integrable function $[P]_{(\cdot,\cdot)} : [c,d]\times[c,d] \to \mathbb{R}^+$ and a non-decreasing function $\mathscr{W}_P : \mathbb{R}^+ \to \mathbb{R}^+$ such that $[P]_{(\cdot,\cdot)}$, $[P]_{(t,\cdot)}$ and $[P]_{(\cdot,0)}$ belongs to $L^q([c,t];\mathbb{R}^+)$ for all

$t \in (c,d]$, and

$$\|P(t,x) - P(s,y)\|_{Y2} \leq \mathscr{W}_P(\max\{\|x\|_{Y_1}, \|y\|_{Y_1}\})[P]_{(t,s)}(|t-s|^{\alpha} + \|x-y\|_{Y_1}),$$

for all $x, y \in Y_1$ and $c \leq s \leq t \leq d$, then we say that $P(\cdot)$ is a $L^{q,\alpha}$-Hölder function if $\alpha \in (0,1)$ and a L^q_{Lip}-function if $\alpha = 1$.

R Next, we use the symbols $L^{q,\alpha}([c,d] \times Y_1; Y_2)$ and $L^q_{Lip}([c,d] \times Y_1; Y_2)$ to denote the sets formed by all the $L^{q,\alpha}$-Hölder functions and all the L^q_{Lip}-Lipschitz functions defined from $[0,a] \times Y_1$ into Y_2.

R The above concepts are defined in connection with the theory of abstract differential equations. We remark that a weaker definition can be considered declining the continuity and the strong measurability.

3.1.1 Examples

As pointed out, the L^q_{Lip}-Lipschitz functions were considered in [Hernandez et al., 2021]. However, only a unique example of this class was presented. Next, we build several examples concerning the functions in the previous definition. In the next examples, we consider $p > 1$ and p' their conjugate.

1) Let $f : [0,a] \to \mathbb{R}$ given by $f(t) = \sqrt[p]{t}$, see Figure 3.1. We claim that $f \in L^q_{Lip}([0,a];\mathbb{R})$ if $q \in (1,p')$.

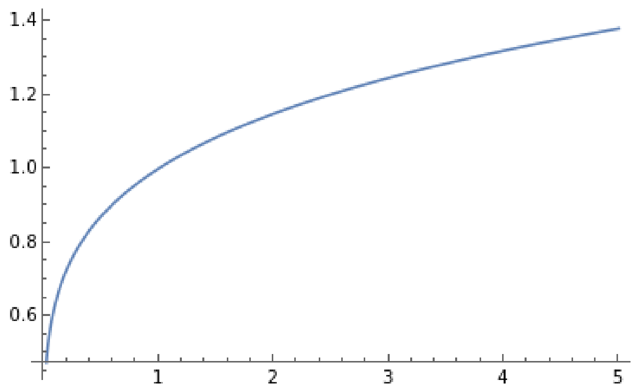

Figure 3.1: The function $f(t) = \sqrt[p]{t}$ (own figure)

In fact, from the Mean Value Theorem, for $0 < s \leq t \leq a$, there is $\xi \in (s,t)$ such that

$$|f(t) - f(s)| = f'(\xi)|t-s| = \frac{1}{p}\xi^{\frac{1}{p}-1}|t-s| \leq \frac{1}{p}s^{\frac{1}{p}-1}|t-s|,$$

we also note that

$$|f(t) - f(0)| = t^{\frac{1}{p}} \leq t^{\frac{1}{p}-1}, \, t > 0.$$

From the above, defining $[f]_{(t,s)} = \frac{1}{p}s^{\frac{1}{p}-1}$, $[f]_{(t,0)} = t^{\frac{1}{p}-1}$ and $[f]_{(0,0)} = 0$, we have

$$\int_0^a [f]^q_{(t,s)} ds = \frac{1}{p}\int_0^a |s^{\frac{1}{p}-1}|^q ds = \int_0^a s^{p'q} ds = \frac{1}{p^q}\frac{a^{1-qp'}}{1-qp'}$$

and noting that $1 - qp' > 0$ for $q \in (1,p')$, we infer that $f(\cdot)$ is a L^q_{Lip}-Lipschitz function.

2) Let $f:[0,2]\to\mathbb{R}$ be given by $f(t)=\sqrt[p]{t}$ for $t\in[0,1]$ and $f(t)=\sqrt[p]{t-1}$ for $t\in(1,2]$, see Figure 3.2.

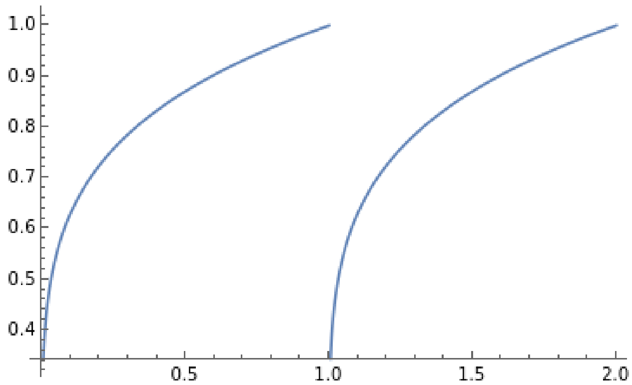

Figure 3.2: The function $f(t)=\sqrt[p]{t}$ for $t\in[0,1]$ and $f(t)=\sqrt[p]{t-1}$ for $t\in(1,2]$ (own figure)

Proceeding as above, we get

1. for $t\in[1,2]$, $s\in[0,1]$ and $t-1>s$,
$$|f(t)-f(s)|\le\sqrt[p]{t-1}-\sqrt[p]{s}\le\frac{|t-s-1|}{ps^{1-\frac{1}{p}}}\le\frac{|t-s|}{ps^{1-\frac{1}{p}}},$$
2. for $t\in[1,2]$ and $s\in[0,1)$ with $t-1<s$,
$$|f(t)-f(s)|\le\sqrt[p]{s}-\sqrt[p]{t-1}|\le\frac{|s-t+1|}{p(t-1)^{1-\frac{1}{p}}}\le\frac{2|t-s|}{p(t-1)^{1-\frac{1}{p}}|t-s|},$$
3. for $t,s\in(0,1]$ with $s<t$,
$$|f(t)-f(s)|\le\sqrt[p]{t}-\sqrt[p]{s}\le\frac{|t-s|}{ps^{1-\frac{1}{p}}},$$
4. for $t,s\in(1,2]$ with $s<t$,
$$|f(t)-f(s)|\le\sqrt[p]{t-1}-\sqrt[p]{s-1}\le\frac{|t-s|}{p(s-1)^{1-\frac{1}{p}}},$$
5. for $t\in[0,1)$, $|f(t)-f(0)|\le\sqrt[p]{t}=\frac{t}{t^{-\frac{1}{p}}}$,
6. for $t\in(1,2]$, $|f(t)-f(0)|\le\sqrt[p]{t-1}\le\frac{(t-1)^{\frac{1}{p}}}{t}t$.

From the above, we define
$$[f]_{(t,s)}=\begin{cases}0, t=s=0,\\ \frac{1}{ps^{1-\frac{1}{p}}}, t-1>s,\\ \frac{2}{p(t-1)^{1-\frac{1}{p}}|t-s|}, \text{ for } t-1<s,\\ \frac{1}{ps^{1-\frac{1}{p}}}, \text{ for } t,s\in[0,1),\\ \frac{1}{p(s-1)^{1-\frac{1}{p}}}, \text{ for } t,s\in(1,2],\\ \frac{1}{t^{-\frac{1}{p}}}, \text{ for } t\in[0,1], s=0,\\ \frac{(t-1)^{\frac{1}{p}}}{t}, \text{ for } t\in(1,2], s=0.\end{cases}$$

Now, observing that for $t \in (1,2]$ we have

$$\begin{aligned}
\int_0^t [f]_{(t,\tau)} d\tau &= \int_0^{t-1} \frac{d\tau}{p\tau^{1-\frac{1}{p}}} + \int_{t-1}^1 \frac{2d\tau}{p(t-1)^{1-\frac{1}{p}}(t-\tau)} + \int_1^t \frac{d\tau}{p(\tau-1)^{1-\frac{1}{p}}} \\
&= \frac{1}{p}\frac{(t-1)^{1-\frac{1}{p}}}{\frac{1}{p}} + \frac{2}{p(t-1)^{1-\frac{1}{p}}}\int_{t-1}^1 \frac{d\tau}{(t-\tau)} + \frac{1}{p}\int_0^t (\tau-1)^{-(1-\frac{1}{p})} d\tau \\
&= (t-1)^{\frac{1}{p}} + \frac{2}{p(t-1)^{1-\frac{1}{p}}}\int_{t-1}^1 \frac{1}{\theta} d\theta + \frac{1}{p}\int_0^{t-1} \theta^{-(1-\frac{1}{p})} d\theta \\
&= (t-1)^{\frac{1}{p}} + \frac{2}{p(t-1)^{1-\frac{1}{p}}} \ln(\frac{1}{t-1}) + (t-1)^{\frac{1}{p}} \\
\int_0^t [f]_{(\tau,0)} d\tau &= \int_0^1 \frac{d\tau}{\tau^{-\frac{1}{p}}} + \int_1^t \frac{(\tau-1)^{\frac{1}{p}}}{\tau} d\tau \\
&\leq \frac{\tau^{\frac{1}{p}}}{\frac{1}{p}}\Big|_0^1 + \int_1^t (\tau-1)^{\frac{1}{p}} d\tau \\
&= p + \frac{(t-1)^{\frac{1}{p}+1}}{\frac{1}{p}+1},
\end{aligned}$$

we conclude that $[f]_{(t,\cdot)}$ and $[f]_{(\cdot,0)}$ are integrable on $[0,t]$ for all $t \in [1,2]$. Similarly, for $t \in (0,1]$, we see that

$$\int_0^t [f]_{(t,\tau)} d\tau = \int_0^t \frac{d\tau}{p\tau^{1-\frac{1}{p}}} = \frac{1}{p}\int_0^t \tau^{\frac{1}{p}-1} d\tau = \frac{1}{p}(p\tau^{\frac{1}{p}})\Big|_0^t = t^{\frac{1}{p}}$$

and

$$\int_0^t [f]_{(t,0)} d\tau = \int_0^t \frac{d\tau}{\tau^{-\frac{1}{p}}} = \frac{\tau^{1+\frac{1}{p}}}{1+\frac{1}{p}}\Big|_0^t = \frac{t^{1+\frac{1}{p}}}{1+\frac{1}{p}},$$

which shows that $[f]_{(t,\cdot)}$ and $[f]_{(\cdot,0)}$ are integrable on $[0,2]$. From the previous remarks, $f \in L^q_{Lip}([0,2];\mathbb{R})$ for $q \in [1,p')$.

3) Let $(t_i)_{i\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $t_i < t_k$ if $k < i$, $(t_i)_{i\in\mathbb{N}} \to 0$ as $i \to \infty$ and the series $\sum_{j\geq 0}(t_j - t_{j+1})^{\frac{1}{p}}$ is convergent. Let $f : [0,1] \to \mathbb{R}$ be given by $f(t) = \sqrt[p]{t-t_i}$ for $t \in (t_i, t_{i-1}]$ and $f(0) = 0$. Arguing as in the previous examples, we note that

1. for $t \in (t_k, t_{k-1}]$ and $s \in (t_i, t_{i-1}]$, with $i > k$ and $t - t_k > s - t_i$, we have that
$$|f(t) - f(s)| \leq \frac{|t-s|}{p(s-t_i)^{1-\frac{1}{p}}},$$
2. for $t \in (t_k, t_{k-1}]$ and $s \in (t_i, t_{i-1}]$, with $i > k$ and $s - t_i > t - t_k$,
$$|f(t) - f(s)| \leq |\sqrt[p]{s-t_i} - \sqrt[p]{t-t_k}| \leq 1 \leq \frac{1}{(t-s)}|t-s|,$$
3. for $t \in [t_k, t_{k+1}]$ and $s \in [t_k, t]$,
$$|f(t) - f(s)| \leq |\sqrt[p]{t-t_k} - \sqrt[p]{s-t_k}| \leq \frac{|t-s|}{(s-t_k)^{1-\frac{1}{p}}}$$

4. for $t \in (t_k, t_{k-1}]$,
$$|f(t) - f(0)| \le (t - t_k)^{\frac{1}{p}} \le t^{\frac{1}{p}}.$$

Next, we show that $f \in L^q_{Lip}([0,1];\mathbb{R})$ for $q \in [1, p')$. Let $t \in (0,1]$ and $k \in \mathbb{N}$ such that $t \in (t_k, t_{k-1}]$. Let $i \in \mathbb{N}$ be the first natural number such that $t - t_k > t_j - t_{j-1}$ for all $j \ge i$. Then

$$\begin{aligned}
\int_0^t [f]^q_{(t,\tau)} d\tau &= \sum_{j=i-1}^{\infty} \int_{t_{j+1}}^{t_j} [f]^q_{(t,\tau)} d\tau + \sum_{j=k}^{i-2} \int_{t_{j+1}}^{t_j} [f]^q_{(t,\tau)} d\tau + \int_{t_k}^{t} [f]^q_{(t,\tau)} d\tau \\
&\le \sum_{j=i-1}^{\infty} \int_{t_{j+1}}^{t_j} \frac{d\tau}{(\tau - t_{j+1})^{(1-\frac{1}{p})q}} + \sum_{j=k}^{i-2} \int_{t_{j+1}}^{t_j} [f]^q_{(t,\tau)} d\tau + \int_{t_k}^{t} \frac{d\tau}{(\tau - t_k)^{(1-\frac{1}{p})q}} \\
&\le \sum_{j=i-1}^{\infty} \frac{(t_j - t_{j+1})^{1-(1-\frac{1}{p})q}}{1-(1-\frac{1}{p})q} + \sum_{j=k}^{i-2} \int_{t_{j+1}}^{t_j} [f]^q_{(t,\tau)} d\tau + \frac{(t-t_k)^{1-(1-\frac{1}{p})q}}{1-(1-\frac{1}{p})q}.
\end{aligned}$$

In addition to the above, defining $s_j := \sup_{s \in [t_j, t_{j-1}]}\{s - t_j \le t - t_k\}$, we also note that

$$\begin{aligned}
\sum_{j=k}^{i-2} \int_{t_{j+1}}^{t_j} [f]^q_{(t,\tau)} d\tau &\le \sum_{j=k}^{i-2} \left(\int_{t_{j+1}}^{s_j} \frac{d\tau}{(\tau - t_{j+1})^{(1-\frac{1}{p})q}} + \int_{s_j}^{t_j} \frac{d\tau}{(t-\tau)^q} \right) \\
&\le \sum_{j=k}^{i-2} \left(\frac{(\tau - t_{j+1})^{1-(1-\frac{1}{p})q}}{1-(1-\frac{1}{p})q} + \int_{s_j}^{t_j} \frac{d\tau}{(t_j-\tau)^q} \right) \\
&\le \sum_{j=k}^{i-2} \left(\frac{(\tau - t_{j+1})^{1-(1-\frac{1}{p})q}}{1-(1-\frac{1}{p})q} + \frac{t_j - s_j}{(t-t_j)^q} \right),
\end{aligned}$$

which allows us to infer that $[f]_{(t,\cdot)}$ is integrable on $[0,t]$. In addition, we also have that

$$\int_0^t [f]^q_{(\tau,0)} d\tau \le \int_0^t \frac{d\tau}{\tau^{(1-\frac{1}{p})q}} \le \frac{t^{1-(1-\frac{1}{p})q}}{1-(1-\frac{1}{p})q},$$

which proves that $[f]_{(\cdot,0)}$ is also integrable on $[0,t]$. From the above remarks we obtain that $f \in L^q_{Lip}([0,1];\mathbb{R})$ and, obviously, $f \in L^{q,\theta}([0,1];\mathbb{R})$ for all $\theta \in (0,1)$.

4) Let $f : [0,1] \to \mathbb{R}$ be the function defined by $f(t) = t\sin(\frac{1}{\sqrt[p]{t}})$ for $t > 0$ and $f(0) = 0$, see Figure 3.3.

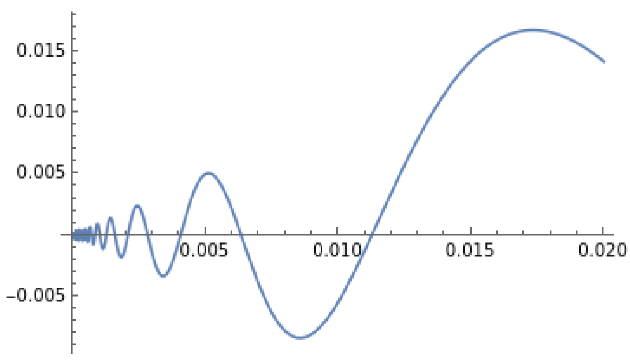

Figure 3.3: The function $f(t) = t\sin(\frac{1}{\sqrt[p]{t}})$ (own figure)

From the Mean Value Theorem, for $< s < t \leq 1$ there exists $\xi(t,s) \in (s,t)$ such that

$$\begin{aligned} |f(t)-f(s)| &= \left|\sin\left(\frac{1}{\sqrt[p]{\xi(t,s)}}\right) - \cos\left(\frac{1}{\sqrt[p]{\xi(t,s)}}\right)\frac{\xi(t,s)}{p\xi^{1+\frac{1}{p}}(t,s)}\right| |t-s| \\ &\leq \left(1+\frac{1}{ps^{\frac{1}{p}}}\right)|t-s|. \end{aligned}$$

Moreover, noting that $|f(t)-f(0)| \leq t$, we infer that $f \in L^q_{Lip}([0,1];\mathbb{R})$ for all $1 < q < p$.

5) Let $f : [0,1] \to \mathbb{R}$ be the function given by $f(t) = t^{-\frac{1}{p}}$ and $f(0) = 0$ with $p > 1$, see Figure 3.4.

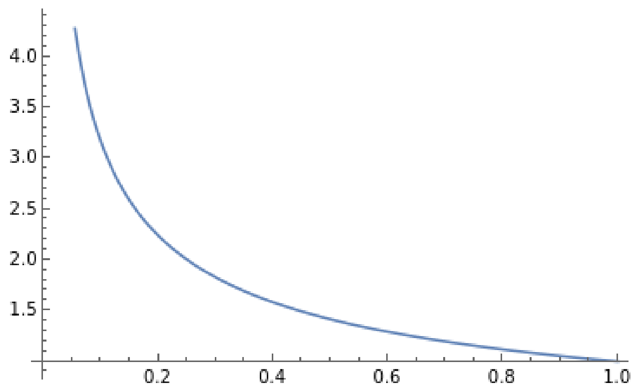

Figure 3.4: The function $f(t) = t^{-\frac{1}{p}}$ on $t \in (0,1]$ (own figure)

For $0 < s \leq t \leq 1$, we get

$$|f(t)-f(s)| = \left|\frac{1}{t^{\frac{1}{p}}} - \frac{1}{s^{\frac{1}{p}}}\right| = \frac{|t^{\frac{1}{p}} - s^{\frac{1}{p}}|}{(ts)^{\frac{1}{p}}} \leq \frac{|t-s|^{\frac{1}{p}}}{(ts)^{\frac{1}{p}}} \leq \frac{|t-s|^{\frac{1}{p}}}{s^{\frac{2}{p}}},$$

and

$$|f(t)-f(0)| \leq t^{-\frac{1}{p}} \leq t^{-\frac{2}{p}} t^{\frac{1}{p}},$$

which allows us to infer that $f \in L^{q,\frac{1}{p}}([0,1])$ for $p > 2q$.

6) Let $p > 2$ and $f : [0,2] \to \mathbb{R}$ be the function defined by $f(t) = t^{-\frac{1}{p}}$ for $t \in (0,1]$, $f(t) = (t-1)^{-\frac{1}{p}}$ for $t \in (1,2]$ and $f(0) = 0$, see Figure 3.5.
Proceeding as usual, we note that

1. for $t > 1$ and $s \in (0,1]$ with $t-1 > s$,

$$|f(t)-f(s)| = \left|\frac{1}{\sqrt[p]{t-1}} - \frac{1}{\sqrt[p]{s}}\right| = \frac{|\sqrt[p]{t-1} - \sqrt[p]{s}|}{\sqrt[p]{s(t-1)}} \leq \frac{|\sqrt[p]{t} - \sqrt[p]{s}|}{\sqrt[p]{s(t-1)}} \leq \frac{|t-s|^{\frac{1}{p}}}{\sqrt[p]{s(t-1)}},$$

2. for $t > 1$ and $s \in (0,1]$ with $t-1 < s$,

$$|f(t)-f(s)| = \frac{|\sqrt[p]{s} - \sqrt[p]{t-1}|}{\sqrt[p]{(t-1)s}} \leq \frac{2}{\sqrt[p]{t-s}\sqrt[p]{(t-1)s}}|t-s|^{\frac{1}{p}},$$

3. for $s,t \in (0,1]$ with $t > s$,

$$|f(t)-f(s)| = \frac{|\sqrt[p]{t} - \sqrt[p]{s}|}{\sqrt[p]{ts}} \leq \frac{|t-s|^{\frac{1}{p}}}{\sqrt[p]{st}},$$

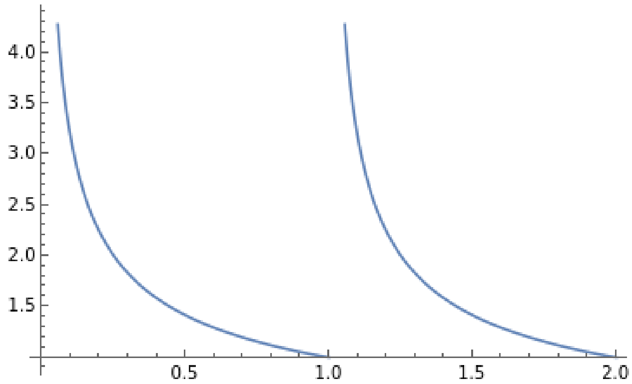

Figure 3.5: The function $f(t) = t^{-\frac{1}{p}}$ for $t \in (0,1]$, $f(t) = (t-1)^{-\frac{1}{p}}$ for $t \in (1,2]$ (own figure)

4. for $s,t \in (1,2]$ with $t > s$,

$$|f(t) - f(s)| \leq \frac{|\sqrt[p]{t-1} - \sqrt[p]{s-1}|}{\sqrt[p]{(t-1)(s-1)}} \leq \frac{|t-s|^{\frac{1}{p}}}{\sqrt[p]{(t-1)(s-1)}},$$

5. for $t \in (0,1]$ and $s \in (1,2]$

$$|f(t) - f(0)| \leq \frac{t^{\frac{1}{p}}}{t^{\frac{2}{p}}} \qquad \text{and} \qquad |f(s) - f(0)| \leq \frac{s^{\frac{1}{p}}}{\sqrt[p]{s(s-1)}}.$$

Considering the above, we define

$$[f]_{(t,s)} = \begin{cases} 0, \quad t = s = 0, \\ \frac{1}{(s(t-1))^{\frac{1}{p}}}, \quad t > 1, s \leq 1 \text{ and } t-1 > s, \\ \frac{2}{((t-s)s(t-1))^{\frac{1}{p}}}, \quad t > 1, s \in (0,1] \text{ and } t-1 < s, \\ \frac{2}{(st)^{\frac{1}{p}}}, \quad s,t \in (0,1], \\ \frac{1}{((t-1)(s-1))^{\frac{1}{p}}}, \quad s,t \in (1,2], \\ \frac{1}{t^{\frac{2}{p}}}, \quad t \in [0,1], s = 0, \\ \frac{1}{(t(t-1))^{\frac{1}{p}}}, \quad t \in [1,2], s = 0. \end{cases}$$

From the above, for $t > 1$ we see that

$$\begin{aligned} \int_0^t [f]_{(t,s)} ds &= \int_0^{t-1} \frac{ds}{((t-1)s)^{\frac{1}{p}}} + \int_{t-1}^t \frac{2}{((t-s)(t-1)s)^{\frac{1}{p}}} ds \\ &= \frac{1}{(t-1)^{\frac{1}{p}}} \int_0^{t-1} \frac{ds}{s^{\frac{1}{p}}} + \frac{2}{(t-1)^{\frac{2}{p}}} \int_{t-1}^t \frac{ds}{(t-s)^{\frac{1}{p}}} \\ &= \frac{1}{(t-1)^{\frac{1}{p}}} \left(\frac{(\tau^{1-\frac{1}{p}})}{1-\frac{1}{p}} \right) \Bigg|_0^{t-1} + \frac{2}{(t-1)^{\frac{2}{p}}} \left(\frac{(t-\tau)^{1-\frac{1}{p}}}{1-\frac{1}{p}} \right) \Bigg|_{t-1}^1 \\ &\leq \frac{2p}{p-1} \left((t-1)^{1-\frac{2}{p}} + (t-1)^{-\frac{2}{p}} ((t-1)^{1-\frac{1}{p}} - 1) \right), \\ \int_0^t [f]_{(s,0)} ds &\leq \int_0^1 \frac{ds}{s^{\frac{2}{p}}} + \int_1^t \frac{ds}{s^{\frac{1}{p}} (s-1)^{\frac{1}{p}}} \leq \frac{p}{p-2} + \frac{p}{p-1} (t-1)^{1-\frac{1}{p}}. \end{aligned}$$

In addition, for $t \in (0,1]$ we note that

$$\begin{aligned}\int_0^t [f]_{(t,s)} ds &= \int_0^t \frac{ds}{(st)^{\frac{1}{p}}} = \frac{1}{t^{\frac{1}{p}}} \frac{s^{1-\frac{1}{p}}}{1-\frac{1}{p}}\Bigg|_0^t = \frac{pt^{1-\frac{2}{p}}}{p-1},\\ \int_0^t [f]_{(s,0)} ds &= \int_0^t \frac{ds}{s^{\frac{2}{p}}} = \frac{p}{p-2} t^{1-\frac{2}{p}}.\end{aligned}$$

From the above estimates we infer that $f \in L^{1,\frac{1}{p}}([0,2])$.

7) Let $p>2$ and $(t_i)_{i\in\mathbb{N}}$ be a sequence on the interval $(0,1)$ such that $t_i < t_k$ if $k<i$, $(t_i)_{i\in\mathbb{N}} \to 0$ as $i\to\infty$ and the series $\sum_{j=1}^{\infty}(t_j - t_{j+1})^{1-\frac{1}{p}}$ is convergent. Let the function $f:[0,1]\to\mathbb{R}$ be given by $f(t) = (t-t_i)^{-\frac{1}{p}}$, for $t\in(t_{i+1},t_i]$ and $f(0)=0$.
Arguing as in the previous examples,

1. for $t\in(t_k,t_{k-1}], s\in(t_i,t_{i-1}]$ with $i<k$ and $t-t_k > s-t_i$

$$|f(t)-f(s)| \le \frac{|s-t|^{\frac{1}{p}}}{(t-t_k)^{\frac{1}{p}}(s-t_i)^{\frac{1}{p}}},$$

2. for $t,s\in(t_k,t_{k-1}]$ with $t>s$

$$|f(t)-f(s)| \le \frac{|s-t|^{\frac{1}{p}}}{(t-t_k)^{\frac{1}{p}}(s-t_k)^{\frac{1}{p}}},$$

3. for $t\in(t_k,t_{k-1}]$ and $s\in(t_i,t_{i-1}]$ with $i<k$ and $t-t_k < s-t_i$

$$|f(t)-f(s)| = \frac{1}{\sqrt[p]{t-t_k}}\left(1-\frac{\sqrt[p]{t-t_k}}{\sqrt[p]{s-t_i}}\right) \le \frac{|t-s|^{\frac{1}{p}}}{|t-s|^{\frac{1}{p}}(t-t_k)^{\frac{1}{p}}},$$

4. for $t\in(t_k,t_{k-1}]$

$$|f(t)-f(0)| \le \frac{t^{\frac{1}{p}}}{t^{\frac{1}{p}}(t-t_k)^{\frac{1}{p}}}.$$

In addition, For $t\in(t_k,t_{k-1}]$ and $i\in\mathbb{N}$ such that $t-t_k > t_j - t_{j+1}$ for all $j\ge i$, we get

$$\begin{aligned}\int_0^t [f]_{(t,\tau)} d\tau &= \sum_{j=i-1}^{\infty}\int_{t_{j+1}}^{t_j}[f]_{(t,\tau)}d\tau + \sum_{j=k}^{i-2}\int_{t_{j+1}}^{t_j}[f]_{(t,\tau)}d\tau + \int_{t_k}^{t}[f]_{(t,\tau)}d\tau\\ &\le \sum_{j=i-1}^{\infty}\frac{1}{(t-t_k)^{\frac{1}{p}}}\int_{t_{j+1}}^{t_j}\frac{d\tau}{(\tau-t_{j+1})^{\frac{1}{p}}} + \sum_{j=k}^{i-2}\int_{t_{j+1}}^{t_j}[f]_{(t,\tau)}d\tau + \frac{1}{(t-t_k)^{\frac{1}{p}}}\int_{t_k}^{t}\frac{d\tau}{(\tau-t_k)^{\frac{1}{p}}}\\ &\le \frac{p}{(t-t_k)^{\frac{1}{p}}(p-1)}\sum_{j=i}^{\infty}(t_j-t_{j+1})^{1-\frac{1}{p}} + \sum_{j=k}^{i-2}\int_{t_{j+1}}^{t_j}[f]_{(t,\tau)}d\tau + \frac{p}{p-1}(t-t_k)^{1-\frac{2}{p}}.\end{aligned}$$

Moreover, if $s_j := \sup_{s\in[t_j,t_{j-1}]}\{s-t_j \le t-t_k\}$, we get

$$\sum_{j=k}^{i-2}\int_{t_{j+1}}^{t_j}[f]_{(t,\tau)}d\tau \le \sum_{j=k}^{i-2}\left(\int_{t_{j+1}}^{s_j}[f]_{(t,\tau)}d\tau + \int_{s_j}^{t_j}[f]_{(t,\tau)}d\tau\right)$$

$$\leq \sum_{j=k}^{i-2}\left(\int_{t_{j+1}}^{s_j}\frac{d\tau}{(t-t_k)^{\frac{1}{p}}(\tau-t_{j+1})^{\frac{1}{p}}}+\int_{s_j}^{t_j}\frac{d\tau}{|t-\tau|^{\frac{1}{p}}(t-t_k)^{\frac{1}{p}}}\right)$$
$$\leq \sum_{j=k}^{i-2}\left(\frac{p(t_j-t_{j+1})^{1-\frac{1}{p}}}{(t-t_k)^{\frac{1}{p}}(p-1)}+\frac{p(t_j-t_{j+1})^{1-\frac{1}{p}}}{(p-1)(t-t_k)^{\frac{2}{p}}}\right),$$

which allows us to infer that $f \in L^{1,\frac{1}{p}}([0,1])$.

8) Let $f:[0,2]\to\mathbb{R}$ be given by $f(t)=\frac{1}{\sqrt[p]{t}}-1$ for $t\in(0,1]$ and $f(t)=\frac{1}{\sqrt[p]{2-t}}-1$ for $t\in[1,2)$ and $f(0)=f(2)=0$, see Figure 3.6.

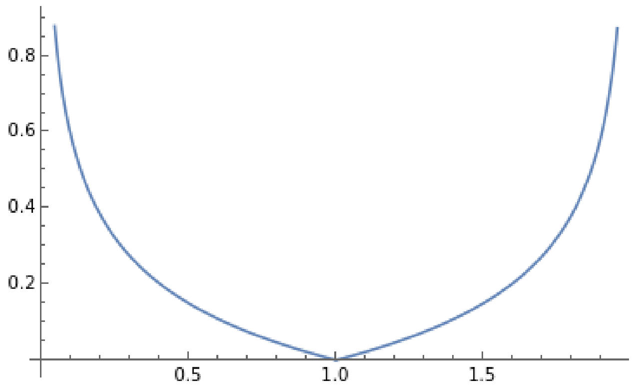

Figure 3.6: The function $f(t)=\frac{1}{\sqrt[p]{t}}-1$ for $t\in(0,1]$ and $f(t)=\frac{1}{\sqrt[p]{2-t}}-1$ for $t\in[1,2)$ (own figure)

Then we have,

1. For $t,s\in(0,1]$,

$$|f(t)-f(s)|=\left|\frac{1}{\sqrt[p]{t}}-1-\frac{1}{\sqrt[p]{s}}+1\right|=\left|\frac{1}{\sqrt[p]{t}}-\frac{1}{\sqrt[p]{s}}\right|\leq\frac{\sqrt[p]{|s-t|}}{\sqrt[p]{st}},$$

2. for $t,s\in[1,2)$,

$$|f(t)-f(s)|\leq\frac{\sqrt[p]{|s-t|}}{\sqrt[p]{(2-s)(2-t)}},$$

3. for $t\in(0,1)$ and $s\in[1,2)$,

$$|f(t)-f(0)|\leq\frac{t^\theta}{t^{\frac{1}{p}+\theta}}\quad\text{and}\quad|f(s)-f(0)|\leq\frac{s^\theta}{s^\theta(2-s)^{\frac{1}{p}}}$$

4. for $t\in[1,2]$ and $s\in(0,1]$ with $2-t<s$,

$$|f(s)-f(t)|\leq\frac{|s^{\frac{1}{p}}-(2-t)^{\frac{1}{p}}|}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}\leq(t-s)^{\frac{1}{p}}\frac{2}{(t-s)^{\frac{1}{p}}s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}.$$

5. for $t\in[1,2]$ and $s\in(0,1]$ with $2-t<s$,

$$|f(t)-f(s)|\leq\frac{(2-t)^{\frac{1}{p}}-s^{\frac{1}{p}}}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}\leq\frac{(2-(t+s))^{\frac{1}{p}}}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}\leq\frac{(2t-2s)^{\frac{1}{p}}}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}\leq\frac{2^{\frac{1}{p}}(t-s)^{\frac{1}{p}}}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}.$$

Considering the above estimates, we define

$$[f]_{(t,s)} = \begin{cases} 0, \ t=s=0, \\ \dfrac{2}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}(t-s)^{\frac{1}{p}}}, \ s\in(0,1], t\in[1,2), 2-t<s \\ \dfrac{2^{\frac{1}{p}}}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}}, \ s\in(0,1], t\in[1,2), 2-t<s \\ \dfrac{1}{(st)^{\frac{1}{p}}}, \ s,t\in(0,1], \\ \dfrac{1}{((2-t)(2-s))^{\frac{1}{p}}}, \ s,t\in[1,2), \\ \dfrac{t^{\theta}}{t^{\frac{1}{p}+\theta}}, \ t\in(0,1], \ s=0, \\ \dfrac{t^{\theta}}{t^{\theta}(2-t)^{\frac{1}{p}}}, \ t\in[1,2), s=0. \end{cases}$$

From the above, for $t\in[0,1]$,

$$\begin{aligned} \int_0^t [f]_{(t,\tau)}d\tau &= \int_0^t \frac{1}{\sqrt[p]{t\tau}}d\tau = -\frac{(t\tau)^{1-\frac{1}{p}}}{1-\frac{1}{p}}, \\ \int_0^t [f]_{(\tau,0)}d\tau &= \int_0^t \frac{\tau^{\theta}}{\tau^{\frac{1}{p}+\theta}}d\tau = \frac{t^{1-\frac{1}{p}}}{1-\frac{1}{p}} \end{aligned}$$

and for $t\in[1,2)$ we have that

$$\begin{aligned} \int_0^t [f]_{(t,s)}ds &\le \int_0^{2-t} \frac{2^{\frac{1}{p}}ds}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}} + \int_{2-t}^1 \frac{ds}{s^{\frac{1}{p}}(2-t)^{\frac{1}{p}}} + \int_1^t \frac{ds}{(2-s)^{\frac{1}{p}}(2-t)^{\frac{1}{p}}} \\ &\le \frac{1}{(2-t)^{\frac{1}{p}}}\left(2^{\frac{1}{p}}\int_0^{2-t} s^{-\frac{1}{p}}ds + \int_{2-t}^1 s^{-\frac{1}{p}}ds + \int_1^t (2-s)^{-\frac{1}{p}}ds\right) \\ &\le 2^{\frac{1}{p}}(2-t)^{1-\frac{2}{p}}, \\ \int_0^t [f]_{(s,0)}ds &\le \int_0^1 \frac{ds}{s^{\frac{1}{p}+\theta}} + \int_1^t \frac{ds}{s^{\theta}(2-s)^{\frac{1}{p}}} \le \frac{1}{1-(\frac{1}{p}+\theta)} + \frac{p}{p-1}(2-t)^{1-\frac{1}{p}}, \end{aligned}$$

which allows us to infer that the functions $[f]_{(t,\cdot)}$ and $[f]_{(\cdot,0)}$ are integrable on $[0,2]$. From the above we have that $f\in L^{1,\min\{\theta,\frac{1}{p}\}}([0,2])$. Moreover, from the above estimates, it is simple to see that $f\in L^{q,\min\{\theta,\frac{1}{p}\}}([0,2])$ for $q>1$ and $\theta\in(0,1)$ such that $q(\frac{1}{p}+\theta)<1$.

9) Assume $f\in L^{q,\alpha}([a,b])$ and that $G\in C(X;X)$ is locally Lipschitz in the following sense: for all $r>0$ there exists $L_G(r)>0$ such that $\|G(x)-G(y)\|\le L_G(r)\|x-y\|$, for all $x,y\in B_r(0,X)$. Then the function $H(t,x)=f(t)G(x)$ belongs to $L^{q,\alpha}([a,b]\times X;X)$. In fact, for $t,s\in[a,b]$ and $x,y\in B_r(0,X)$ we have

$$\begin{aligned} &\|H(t,x)-H(s,y)\| \\ &\quad = \|f(t)G(x)-f(s)G(y)\| \\ &\quad = \|f(t)G(x)-f(s)G(x)+f(s)G(x)-f(s)G(y)\| \end{aligned}$$

$$\begin{aligned}
&\leq |f(t)-f(s)|\|G(x)\|+|f(s)|\|G(x)-G(y)\| \\
&\leq [f]_{(t,s)}|t-s|^{\alpha}\|G(x)\|+|f(s)|L_G(r)\|x-y\| \\
&\leq [f]_{(t,s)}|t-s|^{\alpha}(\|G(x)-G(0)\|+\|G(0)\|)+|f(s)|L_G(r)\|x-y\| \\
&\leq [f]_{(t,s)}|t-s|^{\alpha}(L_G(r)\|x\|+\|G(0)\|)+\|f\|_{C([a,b])}\|L_G(r)\|x-y\| \\
&\leq ([f]_{(t,s)}(L_G(r)r+\|G(0)\|)+\|f\|_{C([a,b])}\|L_G(r))(|t-s|^{\alpha}+\|x-y\|).
\end{aligned}$$

Moreover, if $f\in L^q_{Lip}([a,b])$, then $H=fG\in L^q_{Lip}([a,b]\times X;X)$.

10) Assume $F(t,x)=\zeta(t)G(t,x)$, where $G(\cdot)$ is locally Lipchitz and $\zeta\in C([0,a];\mathbb{R})$. In addition, suppose that $\zeta(\cdot)$ is differentiable almost everywhere on $[0,a]$ and there is a function $\xi:[0,a]\times[0,a]\to\mathbb{R}^+$ such that $|\zeta(t)-\zeta(s)|\leq\zeta'(\xi_{(t,s)})|t-s|$ and $s\leq\xi_{(s,t)}\leq t$ for all $0<s\leq t<a$ and that there exists $q>1$ such that the functions $[\zeta]_{(t,\cdot)}=\zeta'(\xi_{(t,\cdot)})$ and $[\zeta]_{(t,0)}=\zeta'(\xi_{(t,0)})$ belongs to $L^q([0,t])$ for all $t\in[0,a]$.

Under the above conditions, we see that

$$\begin{aligned}
&\|F(t,x)-F(s,y)\| \\
&\quad= \|\zeta(t)G(t,x)-\zeta(t)G(s,y)+\zeta(t)G(s,y)-\zeta(s)G(s,y)\| \\
&\quad\leq |\zeta(t)|\|G(t,x)-G(s,y)\|+|\zeta(t)-\zeta(s)|\|G(s,y)\| \\
&\quad\leq \|\zeta\|_{C([0,a];\mathbb{R})}[G]_{Lip}(|t-s|+\|x-y\|) \\
&\qquad +\zeta'(\xi)|t-s|(\|G(s,y)-G(0,0)\|+\|G(0,0)\|) \\
&\quad\leq ([G]_{Lip}(1+a+r)+\|G(0,0)\|+\|\zeta\|_{C([0,a];\mathbb{R})})[\zeta]_{(t,s)}(|t-s|+\|x-y\|).
\end{aligned}$$

Using that $[\zeta]_{(t,s)}\in L^q([0,a];X)$ and $([G]_{Lip}(1+a+r)+\|G(0,0)\|+\|\zeta\|_{C([0,a];\mathbb{R})})$ is non-decreasing we conclude that $F\in L^q_{Lip}([0,a]\times X;X)$.

11) Let $F(t,x)=\zeta(t)G(t,x)$, with $G\in L^{q,\alpha}([0,a]\times X;X)$ and $\zeta\in L^{p,\alpha}([0,a];\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}\leq 1$. Then,

$$\begin{aligned}
&\|F(t,x)-F(s,y)\| \\
&\quad\leq |\zeta(t)|\|G(t,x)-G(s,y)\|+|\zeta(t)-\zeta(s)|(\|G(s,y)-G(0,0)\|+\|G(0,0)\|) \\
&\quad\leq \|\zeta\|_{C([0,a];\mathbb{R})}\mathscr{W}_G[G]_{(t,s)}(|t-s|^{\alpha}+\|x-y\|) \\
&\qquad +[\zeta]_{(t,s)}|t-s|^{\alpha}(\mathscr{W}_G[G]_{(t,s)}(|s|+\|y\|)+\|G(0,0)\|) \\
&\quad\leq (\mathscr{W}_G(\|\zeta\|+a+r)+\|G(0,0)\|)[G]_{(t,s)}(1+[\zeta]_{(t,s)})(|t-s|^{\alpha}+\|x-y\|)
\end{aligned}$$

hence, $F\in L^{k,\alpha}([0,a]\times X;X)$, with k such that $\frac{1}{k}=\frac{1}{p}+\frac{1}{q}$.

We finish this section proving a important property of the $L^{q,\alpha}$- Hölder functions.

Proposition 3.1.1 $L^{q,\alpha}([c,d]\times X;Y)$ is vectorial space.

Proof: In fact, let $f,g\in L^{q,\alpha}([a,b]\times X;Y)$ and $\beta\in\mathbb{C}$, for $x,y\in B_R(0)$ we have

$$\begin{aligned}
\|(f+\beta g)(t,x)-(f+\beta g)(s,y)\| &\leq \|f(t,x)-f(s,y)\|+|\beta|\|g(t,x)-g(s,y)\| \\
&\leq \mathscr{W}_f(R)[f]_{(t,s)}(|t-s|^{\alpha}+\|x-y\|_X) \\
&\quad +|\beta|\mathscr{W}_g(R)[g]_{(t,s)}(|t-s|^{\alpha}+\|x-y\|_X) \\
&\leq \left(\mathscr{W}_f(R)+|\beta|\mathscr{W}_g(R)\right)\left([f]_{(t,s)}+[g]_{(t,s)}\right)(|t-s|^{\alpha}+\|x-y\|_X),
\end{aligned}$$

hence, $(f+\alpha g)\in L^{q,\alpha}([c,d]\times X;Y)$. This proves our claim. ■

3.2 Existence, uniqueness and regularity of mild solutions

In this section, we study the existence of mild, classical, strong and strict solutions for the problem

$$\begin{aligned} u'(t) &= Au(t)+F(t,u(t)),\ t\in[0,a] && (3.1)\\ u(0) &= x_0\in X && (3.2)\end{aligned}$$

where A is the generator of an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on X and F is a L^q_{Lip}-Lipschitz or a $L^{q,\alpha}$-Hölder function.

To begin we study the inhomogeneous case.

3.2.1 The inhomogeneous Cauchy problem

Consider the non-homogeneous abstract initial value problem (IVP)

$$\begin{aligned} u'(t) &= Au(t)+f(t),\ \ t\in[0,a], && (3.3)\\ u(0) &= x_0\in X. && (3.4)\end{aligned}$$

We remember that for $p>1$ we use the notation p' for the conjugate of p, the number defined by $\frac{1}{p}+\frac{1}{p'}=1$ and remark that the concepts of classical, mild and strong solutions for the problem (3.3)-(3.4) are defined in Definition 2.1.1, Definition 2.1.2 and Definition 2.1.3, respectively. In addition, we remark the concept of a strict solution to the referred problem below.

Definition 3.2.1 Let $u:[0,a]\to X$ be a function. If $u\in C([0,a];X)\cap C^1([0,a];X)$ satisfies the IVP (3.3)-(3.4) on $[0,a]$, we say that $u(\cdot)$ is a **strict solution** to (3.3)-(3.4) on $[0,a]$.

If $f\in L^{q,\alpha}([0,a];Y)$, from the estimative

$$\int_0^a \|f(t)\|dt \ \ \leq\ \ \int_0^a \|f(t)-f(0)\|dt+\int_0^a \|f(0)\|dt \leq \int_0^a [f]_{(t)}t^{\alpha}dt+\|f(0)\|a,$$

we infer that $f(\cdot)$ is integrable, therefore there exists a unique mild solution for the problem (3.3)-(3.4). Arguing as above we conclude that the same holds for $f\in L^q_{Lip}([0,a];X)$.

The next result establishes conditions under which a mild solution of the IVP (3.3)-(3.4) is a classical solution in the case where f is a L^q-Lipschitz function.

Proposition 3.2.1 Suppose $f\in L^q_{Lip}([0,a];X)$, $\Lambda_f:=\sup_{\theta\in[0,a]}\|[f]_{(\theta,\cdot)}\|_{L^q([0,\theta])}<\infty$ and that $f(\cdot)$ is continuous. Let $u\in C([0,a];X)$ be the mild solution of (3.3)-(3.4). If $x_0\in X_\beta$ for some $1<\beta<2$, then

i) $u(\cdot)$ is a classical solution,
ii) $Au(\cdot)\in L^{q,\nu}([0,a];X)$ for all $\nu<\min\{\gamma,\beta-1\}$ and
iii) $u'(\cdot)\in L^{q,\nu}([0,a];X)$ for all $\nu<\min\{\gamma,\beta-1\}$.

Proof: To prove the result, we introduce the decomposition $u(\cdot)=u_1(\cdot)+u_2(\cdot)$ where

$$u_1(t)=T(t)x_0+\int_0^t T(t-\tau)f(t)d\tau \ \text{ and } \ u_2(t)=\int_0^t T(t-\tau)(f(\tau)-f(t))d\tau.$$

From semigroups properties (see Theorem 1.2.3) we note that

$$Au_1(t)=AT(t)x_0+(T(t)-I)f(t),$$

which implies that Au_1 is continuous on $[0,a]$. In addition, noting that

$$\begin{aligned}\|Au_2(t)\| &\leq \int_0^t \|AT(t-\tau)\|\|f(\tau)-f(t)\|d\tau \\ &\leq \int_0^t \frac{C_1}{(t-\tau)}[f]_{(t,\tau)}(t-\tau)d\tau \\ &\leq C_1\|[f]_{(t,\cdot)}\|_{L^1([0,a];X)},\end{aligned}$$

we conclude that $u_2(t) \in D(A)$ for all $t \in [0,a]$. Now, remarking that $\ln(1+\rho) < \frac{\rho^\alpha}{\alpha}$ for all $\alpha \in (0,1)$ and arguing as in (2.12), for $s < t$ in $[0,a]$ and $\gamma < \frac{1}{q'}$, we get

$$\begin{aligned}\|Au_2(t)-Au_2(s)\| &\leq \int_0^s \|(AT(t-\tau)-AT(s-\tau))(f(\tau)-f(s))\|d\tau \\ &\quad +\int_0^s \|AT(s-\tau)((f(\tau)-f(t))-(f(\tau)-f(s)))\|d\tau \\ &\quad +\int_s^t \|AT(t-\tau)(f(\tau)-f(t))\|d\tau \\ &\leq \int_0^s \int_{s-\tau}^{t-\tau} \|A^2T(\xi)\|[f]_{(s,\tau)}(s-\tau)d\xi d\tau + \|(T(s)-I)(f(s)-f(t))\| \\ &\quad +\int_s^t \frac{C}{(t-\tau)}[f]_{(t,\tau)}(t-\tau)d\tau \\ &\leq \int_0^s \int_{s-\tau}^{t-\tau} \frac{C_2}{\xi^2}[f]_{(s,\tau)}\xi d\xi d\tau + (C_0+1)\|f(t)-f(s)\| + \int_s^t C[f]_{(t,\tau)}d\tau \\ &\leq \int_0^s C_2[f]_{(s,\tau)}\ln\left(\frac{t-\tau}{s-\tau}\right)d\tau + (C_0+1)\|f(t)-f(s)\| \\ &\quad +C\left(\int_s^t [f]^q_{(t,\tau)}d\tau\right)^{\frac{1}{q}}\left(\int_s^t 1^{q'}d\tau\right)^{\frac{1}{q'}} \\ &\leq \int_0^s C_2[f]_{(s,t)}\ln\left(1+\frac{t-s}{s-\tau}\right)d\tau + (C_0+1)\|f(t)-f(s)\| \\ &\quad +C\|[f]_{(t,\cdot)}\|_{L^q([0,a];X)}(t-s)^{\frac{1}{q'}} \\ &\leq \frac{C_2}{\gamma}(t-s)^\gamma \int_0^s \frac{[f]_{(s,\tau)}}{(s-\tau)^\gamma}d\tau + (C_0+1)\|f(t)-f(s)\| \\ &\quad +C\|[f]_{(t,\cdot)}\|_{L^q([0,a];X)}(t-s)^{\frac{1}{q'}},\end{aligned}$$

hence, we have

$$\begin{aligned}&\|Au_2(t)-Au_2(s)\| \\ &\leq \frac{C_2\Lambda_f b^{\frac{1}{q'}-\gamma}}{\gamma(1-q'\gamma)^{\frac{1}{q'}}}(t-s)^\gamma + (C_0+1)\|f(t)-f(s)\| + C\|[f]_{(t,\cdot)}\|_{L^q([0,a];X)}(t-s)^{\frac{1}{q'}} \\ &\leq \frac{C_2\Lambda_f b^{\frac{1}{q'}-\gamma}}{\gamma(1-q'\gamma)^{\frac{1}{q'}}}(t-s)^\gamma + (C_0+1)[f]_{(t,s)}(t-s) + C\|[f]_{(t,\cdot)}\|_{L^q([0,a];X)}(t-s)^{\frac{1}{q'}}\end{aligned}$$

from which we conclude that $Au_2(\cdot) \in C([0,a];X) \cap L^{q,\gamma}([0,a];X)$. This implies that $u(\cdot)$ is a classical solution of the problem (3.3)-(3.4) on $[0,a]$ (see Theorem 2.1.2).

On the other hand, for $0 \le s < t \le a$ and $\xi \in (0,1)$, we get

$$\begin{aligned}
&\|Au_1(t) - Au_1(s)\| \\
&\le \|AT(t)x_0 - AT(s)x_0\| + \|(T(t)-I)f(t) - (T(s)-I)f(s)\| \\
&\le \int_s^t \|A^{2-\beta}T(\theta)x_0\| d\theta + \|(T(t)-T(s))f(t)\| + \|(T(s)-I)(f(t)-f(s))\| \\
&\le \frac{C_{2-\beta}\|A^\beta x_0\|}{\beta-1}(t-s)^{\beta-1} + \int_s^t \frac{C_1}{\theta}\|f(t)\| d\theta + (C_0+1)[f]_{(t,s)}(t-s) \\
&\le \frac{C_{2-\beta}\|A^\beta x_0\|}{\beta-1}(t-s)^{\beta-1} + \int_s^t \frac{C}{\theta^\xi \theta^{1-\xi}}\|f\|_{C([0,a])} d\theta + (C_0+1)[f]_{(t,s)}(t-s) \\
&\le \frac{C_{2-\beta}\|A^\beta x_0\|}{\beta-1}(t-s)^{\beta-1} + \frac{C}{\xi s^\xi}\|f\|_{C([0,a];X)}(t-s)^\xi + (C_0+1)[f]_{(t,s)}(t-s),
\end{aligned}$$

which implies that $Au_1 \in L^{q,\beta-1}([0,a];X)$. Therefore, $Au \in L^{q,\nu}([0,a];X)$ for all $\nu < \min\{\gamma, \beta-1\}$. Consequently,

$$\|u'(t) - u'(s)\| \le \|Au(t) - Au(s)\| + \|f(t) - f(s)\| \le \left(\frac{[Au]_{(t,s)}}{s^{\beta-1}} + [f]_{(t,s)}\right)(t-s)^\nu$$

hence $u' \in L^{q,\nu}([0,a];X)$. ∎

Next, we study the regularity of the mild solution of (3.3)-(3.4) with $f \in L^{q,\alpha}([0,a];X)$. In the following, for $t \in [0,a]$, U_t is the set

$$U_t = \{(s,\tau); 0 < \tau < s < t\},$$

$\Theta_{t,\alpha}$ denotes the function $\Theta_{t,\alpha} : U_t \mapsto \mathbb{R}$ defined by

$$\Theta_{t,\alpha}(s,\tau) = \frac{1}{(s-\tau)^{1-\alpha}} - \frac{1}{(t-\tau)^{1-\alpha}},$$

and for a $L^{q,\alpha}$-Hölder function H we write $\Lambda_H = \sup_{s\in[0,a]} \|[H]_{(s,0)}\|_{L^q([0,a];X)}$.

Proposition 3.2.2 Let $f \in L^{q,\alpha}([0,a];X)$ and $u(\cdot)$ be the mild solution of (3.3)-(3.4) on $[0,a]$.

i) If $x_0 \in X$ and $\sup_{t\in[0,a]} \|\frac{[f]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}\|_{L^1([0,t])}$ is finite. Then $u(\cdot)$ is a strong solution and $u'(\cdot) \in L^q([0,a];X)$. In particular, the assertion hold if $\nu = (1-(1-\alpha)q') > 0$ and $\Lambda_f < \infty$.

In addition to the above conditions in (i), if $f \in C([0,a];X)$, $\nu = (1-(1-\alpha)q'') > 0$ and Λ_f is finite. Then,

ii) If $\|[f]_{(s,\cdot)}\Theta_t(s,\cdot)\|_{L^1([0,s];\mathbb{R})} \to 0$ as $s \to t$, then $u(\cdot)$ is a classical solution.

iii) If $\nu = (1-(1-\alpha)q') > 0$, then $u(\cdot)$ is a classical solution. In addition, if $x_0 \in D(A)$, then $u(\cdot)$ is a strict solution on $[0,a]$ and

$$\|u'\|_{C([0,a];X)} \le C_0\|Ax_0\| + 2C_0\|f\|_{C([0,a];X)} + C_1\Lambda_f \frac{a^{\frac{1}{q'}-\alpha}}{(1-\alpha q')^{\frac{1}{q'}}}$$

iv) If $\mu = (1-2(1-\alpha)q') > 0$, $x_0 \in D(A)$ and $AT(\cdot)x_0 \in C^\beta([0,a];X)$ for some $\beta \in (0,1)$, then $u(\cdot)$ is a strict solution and $u' \in L^{p,\min\{\beta,\alpha,1-\alpha,\frac{\nu}{q'}\}}([0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha},\frac{1}{\beta}\}$ and

$$[u']_{(t,s)} \le [AT(\cdot)x_0]_{C^\beta([0,a];X)} a^{\beta-\sigma} + \frac{C_1}{s^{\min\{\frac{1}{\alpha},\frac{1}{\beta}\}}}\|f\|_{C([0,a];X)}$$

$$+(3C_0+1)[f]_{(t,s)}a^{\alpha-\sigma}+\Lambda_f\left(\frac{C_2}{\mu^{\frac{1}{q'}}}a^{1-\alpha-\sigma+\frac{\mu}{q'}}+\frac{C_1}{\nu^{\frac{1}{q'}}}a^{\frac{\nu}{q'}-\sigma}\right),$$

where $\sigma=\min\{\alpha,\beta,1-\alpha,\frac{\nu}{q'}\}$.

v) If $\mu=(1-(1-\alpha)q')>0$, $x_0\in D(A)$ and $(Ax_0+f(0))\in X_\beta$ for some $\beta\in(0,1)$, then $u(\cdot)$ is a strict solution on $[0,a]$ and $u'\in L^{q,\min\{\beta,\alpha,1-\alpha,\frac{\nu}{q'}\}}([0,a];X)$. In particular, $u'\in L^{q,\min\{\alpha,1-\alpha,\frac{\nu}{q'}\}}([0,a];X)$ if $(Ax_0+f(0))\in X_\alpha$ and for $\rho=\min\{\alpha,1-\alpha,\frac{\nu}{q'}\}$,

$$\begin{aligned}[u']_{(t,s)} &\leq \|(-A)^\alpha(Ax_0+f(0))\|C_{1-\alpha}\frac{b^{\alpha-\rho}}{\alpha}+b^{\alpha-\rho}\left(\frac{[f]_{(t,0)}}{\alpha}+(3C_0+1)[f]_{(t,s)}\right)\\ &\quad+\Lambda_f\left(C_2\frac{b^{1-\alpha-\frac{\mu}{q'}-\rho}}{\mu^{\frac{1}{q'}}}+C_1\frac{b^{\frac{\nu}{q'}-\rho}}{\nu^{\frac{1}{q'}}}\right),\ \text{for all } t\in[0,a].\end{aligned}$$

Proof: Consider the decomposition $u(\cdot)=u_1(\cdot)+u_2(\cdot)$ where

$$u_1(t)=T(t)x_0+\int_0^t T(t-\tau)f(t)d\tau \ \text{ and } \ u_2(t)=\int_0^t T(t-\tau)(f(\tau)-f(t))d\tau.$$

If the conditions in (i) hold, by noting that

$$Au_1(t) = AT(t)x_0+(T(t)-I)f(t),\ \text{for all } t\in[0,a]. \tag{3.5}$$

$$\begin{aligned}\|Au_2(t)\| &\leq \int_0^t \|AT(t-\tau)(f(\tau)-f(t))\|d\tau\\ &\leq \int_0^t \frac{C_1}{(t-\tau)}[f]_{(t,\tau)}(t-\tau)^\alpha d\tau\\ &\leq C_1\left\|\frac{[f]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}\right\|_{L^1([0,t])},\end{aligned} \tag{3.6}$$

we infer that $u_i(\cdot)$, $i=1,2$ are $D(A)$-valued functions and that $A(u(\cdot)-T(\cdot)x_0)\in L^q([0,a];X)$, which implies, see Theorem 2.1.5, that $u(\cdot)$ is a strong solution. We can also conclude that $u'\in L^q([0,a];X)$ noting that, by hypothesis, $f(\cdot)\in L^q([0,a];X)$ and from (3.6)

$$\int_0^a \|Au_2(t)\|^q dt\leq \int_0^a\left(C_1\|\frac{[f]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}\|_{L^1([0,t])}\right)^q dt\leq C_1^q\sup\left\|\frac{[f]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}\right\|_{L^1}^q a.$$

In order to study the regularity of $u(\cdot)$, next we study the functions $Au_i(\cdot)$, $i=1,2$ assuming that $f\in C([0,a];X)$ and that Λ_f is finite. To begin, for $0<s\leq t\leq a$ we note that

$$\begin{aligned}&\|Au_2(t)-Au_2(s)\|\\ &\leq \int_0^s \|(AT(t-\tau)-AT(s-\tau))(f(\tau)-f(s))\|d\tau\\ &\quad+\|A\int_0^s T(t-\tau)((f(\tau)-f(t))-(f(\tau)-f(s))d\tau\|\\ &\quad+\int_s^t \|AT(t-\tau)(f(\tau)-f(t))\|d\tau\\ &\leq \int_0^s\int_{s-\tau}^{t-\tau}\|A^2T(\xi)\|[f]_{(s,\tau)}(s-\tau)^\alpha d\xi d\tau\end{aligned}$$

$$
\begin{aligned}
&+\|(T(s)-T(t))(f(s)-f(t))\|+\int_s^t \frac{C_1[f]_{(t,\tau)}}{(t-\tau)^{1-\alpha}}d\tau \\
\leq\ & \int_0^s\int_{s-\tau}^{t-\tau}\|A^2T(\xi)\|[f]_{(s,\tau)}(s-\tau)^{\alpha}d\xi d\tau \\
&+2C_0\|(f(s)-f(t))\|+C_1\left(\int_s^t [f]_{(t,\tau)}^q d\tau\right)^{\frac{1}{q}}\left(\int_s^t (t-\tau)^{-(1-\alpha)q'}d\tau\right)^{\frac{1}{q'}} \\
\leq\ & \int_0^s\int_{s-\tau}^{t-\tau}\frac{C_2}{\xi^{2-\alpha}}[f]_{(s,\tau)}d\xi d\tau+2C_0\|f(s)-f(t)\|+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
\leq\ & \frac{C_2}{(1-\alpha)}\int_0^s [f]_{(s,\tau)}\left[\frac{1}{(s-\tau)^{1-\alpha}}-\frac{1}{(t-\tau)^{1-\alpha}}\right]d\tau+2C_0\|f(s)-f(t)\|+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}}.
\end{aligned}
$$

Using this inequality and the notations $\nu=(1-(1-\alpha)q')$ and $\mu=(1-2(1-\alpha)q')$, we get

$$
\begin{aligned}
\|Au_2(t)-Au_2(s)\| &\leq C_2\|[f]_{(s,\cdot)}\Theta_{t,\alpha}(s,\cdot)\|_{L^1([0,s];\mathbb{R})}+2C_0\|f(t)-f(s)\|+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}}, \\
\|Au_2(t)-Au_2(s)\| &\leq \frac{C_2}{(1-\alpha)}\Lambda_f\left(\int_0^s \Theta_{t,\alpha}(s,\tau)^{q'}d\tau\right)^{\frac{1}{q'}}2C_0\|f(t)-f(s)\|+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}}, \\
\|Au_2(t)-Au_2(s)\| &= \int_0^s C_2\frac{[f]_{(s,\tau)}}{1-\alpha}\left(\frac{(t-\tau)^{1-\alpha}-(s-\tau)^{1-\alpha}}{(s-\tau)^{1-\alpha}(t-\tau)^{1-\alpha}}\right)d\tau \\
&\quad +2C_0\|f(t)-f(s)\|+C_1\Lambda_f\frac{(t-s)^{\frac{\nu}{q'}}}{\nu^{\frac{1}{q'}}} \\
&\leq \frac{C_2(t-s)^{1-\alpha}}{(1-\alpha)}\int_0^s\frac{[f]_{(s,\tau)}}{(s-\tau)^{2(1-\alpha)}}d\tau \\
&\quad +2C_0\|f(t)-f(s)\|+C_1\Lambda_f\frac{(t-s)^{\frac{\nu}{q'}}}{\nu^{\frac{1}{q'}}} \\
&\leq C_2(t-s)^{1-\alpha}\Lambda_f\frac{s^{\frac{\mu}{q'}}}{\mu^{\frac{1}{q'}}}+2C_0\|f(t)-f(s)\|+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \qquad (3.7)
\end{aligned}
$$

If the conditions in (ii), (iii), or (iv) are satisfied we can infer that $Au_2\in C([0,a];X)$. In particular, we note that the continuity of $Au_2(\cdot)$ in (ii) and (iii) follows from the Lebesgue Dominated Convergence Theorem (see Theorem A.0.12).

From the previous remarks we conclude that $Au(\cdot)$ is continuous, therefore $u(\cdot)$ is a classical solution, see Theorem 2.1.2. Moreover, if $x_0\in D(A)$, from Equations (3.5) and (3.7), it is easy to see that $u(\cdot)$ is a strict solution and that

$$
\begin{aligned}
\|u'\|_{C([0,b];X)} &\leq \|AT(t)x_0\|+\|A\int_0^t T(t-\tau)f(t)d\tau\|+\|A\int_0^t T(t-\tau)(f(\tau)-f(t))d\tau\| \\
&\leq C_0\|Ax_0\|+\|(T(t)-I)f(t)\|+\|\int_0^t AT(t-\tau)(f(\tau)-f(t))d\tau\| \\
&\leq C_0\|Ax_0\|+(C_0+1)\|f(t)\|_{C([0,a];X)}+\int_0^t\frac{C_1}{(t-\tau)}[f]_{(t,\tau)}(t-\tau)^{\alpha}d\tau \\
&\leq C_0\|Ax_0\|+(C_0+1)\|f\|_{C([0,a];X)}+C_1\Lambda_f\frac{a^{\frac{1}{q'}-\alpha}}{[1-\alpha q']^{\frac{1}{q'}}}.
\end{aligned}
$$

We prove now the assertions in (iv). Assume $AT(\cdot)x_0 \in C^{\beta}([0,a];X)$. Using (3.5), for $\rho \in (0,1)$ and $0 < s \leq t \leq a$ we see that

$$\begin{aligned}
&\|Au_1(t) - Au_1(s)\| \\
&\leq \|AT(t)x_0 - AT(s)x_0\| + \|(T(t) - T(s))f(t)\| + \|(T(s) - I)(f(t) - f(s))\| \\
&\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta} + \int_s^t \|AT(\tau)f(t)\| d\tau + (C_0+1)\|f(t) - f(s)\| \\
&\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta} + \|f\|_{C([0,a];X)} \int_s^t \frac{C_1}{s^{\rho}\tau^{1-\rho}} d\tau + (C_0+1)\|f(t) - f(s)\| \\
&\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta} + C_1\|f\|_{C([0,a];X)} \frac{(t-s)^{\rho}}{s^{\rho}\rho} + (C_0+1)\|f(t) - f(s)\| \qquad (3.8) \\
&\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta} + C_1\|f\|_{C([0,a];X)} \frac{(t-s)^{\rho}}{s^{\rho}\rho} + (C_0+1)[f]_{(t,s)}(t-s)^{\alpha} \qquad (3.9)
\end{aligned}$$

which implies from (3.8) and (3.9) that $Au_1 \in C([0,a];X) \cap L^{p,\min\{\alpha,\beta\}}([0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$ because $q > \frac{1}{\alpha}$. Moreover, using the notation $\sigma = \min\{\alpha, \beta, 1-\alpha, \frac{\nu}{q'}\}$, from (3.9) and (3.7) we obtain that $u' \in L^{p,\sigma}([0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$ and noting that

$$\begin{aligned}
\|Au(t) - Au(s)\| &\leq \|Au_1(t) - Au_1(s)\| + \|Au_2(t) - Au_2(s)\| \\
&\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta} \\
&\quad + C_1\|f\|_{C([0,a];X)} \frac{(t-s)^{\rho}}{s^{\rho}\rho} + (C_0+1)[f]_{(t,s)}(t-s)^{\alpha} \\
&\quad + C_2(t-s)^{1-\alpha}\Lambda_f \frac{s^{\frac{\mu}{q'}}}{\mu^{\frac{1}{q'}}} + 2C_0[f]_{(t,s)}(t-s)^{\alpha} + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
&\leq (t-s)^{\sigma}\Bigg([AT(\cdot)x_0]_{C^{\beta}([0,a];X)}(t-s)^{\beta-\sigma} + C_1 \frac{\|f\|_{C([0,a];X)}(t-s)^{\rho-\sigma}}{s^{\rho}\rho} \\
&\quad + (C_0+1)[f]_{(t,s)}(t-s)^{\alpha-\sigma} + C_2(t-s)^{1-\alpha-\sigma}\Lambda_f \frac{s^{\frac{\mu}{q'}}}{\mu^{\frac{1}{q'}}} \\
&\quad + 2C_0[f]_{(t,s)}(t-s)^{\alpha-\sigma} + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}-\sigma}\Bigg) \\
&\leq (t-s)^{\sigma}\Bigg([AT(\cdot)x_0]_{C^{\beta}([0,a];X)}a^{\beta-\sigma} + C_1 \frac{\|f\|_{C([0,a];X)}a^{\rho-\sigma}}{s^{\rho}\rho} \\
&\quad + [f]_{(t,s)}(3C_0+1)a^{\alpha-\sigma} + \Lambda_f\Big(C_2 \frac{a^{1-\alpha-\sigma+\frac{\mu}{q'}}}{\mu^{\frac{1}{q'}}} + \frac{C_1}{\nu^{\frac{1}{q'}}}a^{\frac{\nu}{q'}-\sigma}\Big)\Bigg)
\end{aligned}$$

we get

$$\begin{aligned}
[u']_{(t,s)} &\leq [AT(\cdot)x_0]_{C^{\beta}([0,a];X)}a^{\beta-\sigma} + \frac{C_1}{s^{\min\{\frac{1}{\alpha},\frac{1}{\beta}\}}\min\{\frac{1}{\alpha},\frac{1}{\beta}\}}\|f\|_{C([0,a];X)} \\
&\quad + (3C_0+1)[f]_{(t,s)}a^{\alpha-\sigma} + \Lambda_f\left(\frac{C_2}{\mu^{\frac{1}{q'}}}a^{1-\alpha-\sigma+\frac{\mu}{q'}} + \frac{C_1}{\nu^{\frac{1}{q'}}}a^{\frac{\nu}{q'}-\sigma}\right),
\end{aligned}$$

which completes the proof of the assertions in (iv).

To finish, suppose that the conditions in (v) hold, proceeding as above, we get

$$\begin{aligned}
&\|Au_1(t)-Au_1(s)\| \\
&\leq \|(T(t)-T(s))(Ax_0+f(0))\|+\|(T(t)-T(s))(f(s)-f(0))\| \\
&\quad +\|(T(t)-I)(f(t)-f(s))\| \\
&\leq \int_s^t \|AT(\tau)(Ax_0+f(0))\|d\tau+\int_s^t \|AT(\tau)(f(s)-f(0))\|d\tau \\
&\quad +(C_0+1)[f]_{(t,s)}(t-s)^\alpha \\
&\leq \int_s^t \|(-A)^{1-\beta}T(\tau)(-A)^\beta(Ax_0+f(0))\|d\tau+\int_s^t \frac{C_1[f]_{(s,0)}}{\tau}s^\alpha d\tau \\
&\quad +(C_0+1)[f]_{(t,s)}(t-s)^\alpha \\
&\leq \|(-A)^\beta(Ax_0+f(0))\|\int_s^t \frac{C_{1-\beta}}{\tau^{1-\beta}}d\tau+C_1[f]_{(s,0)}\int_s^t \tau^{\alpha-1}d\tau \\
&\quad +(C_0+1)[f]_{(t,s)}(t-s)^\alpha \\
&\leq \|(-A)^\beta(Ax_0+f(0))\|C_{1-\beta}\frac{(t-s)^\beta}{\beta}+C_1[f]_{(s,0)}\frac{(t-s)^\alpha}{\alpha} \\
&\quad +(C_0+1)[f]_{(t,s)}(t-s)^\alpha, \qquad (3.10)
\end{aligned}$$

which implies that $Au_1\in L^{q,\min\{\beta,\alpha\}}([0,a];X)$ and $u'\in L^{q,\min\{\beta,\alpha,1-\alpha,\frac{\nu}{q'}\}}([0,a];X)$, see (3.7).

If $(Ax_0+f(0))\in X_\alpha$, we note that

$$\begin{aligned}
\int_s^t \|AT(\tau)(Ax_0+f(0))\|d\tau &= \int_s^t \|(-A)^{1-\alpha}T(\tau)(-A)^\alpha(Ax_0+f(0))\|d\tau \\
&\leq \|(-A)^\alpha(Ax_0+f(0))\|\int_s^t \frac{C_{1-\alpha}}{\tau^{1-\alpha}}d\tau \\
&\leq \|(-A)^\alpha(Ax_0+f(0))\|C_{1-\alpha}\frac{(t-s)^\alpha}{\alpha}.
\end{aligned}$$

Moreover, arguing as in (3.10), we obtain that $u'\in L^{q,\min\{\alpha,1-\alpha,\frac{\nu}{q'}\}}$ and

$$\begin{aligned}
\|Au(t)-Au(s)\| &\leq \|(-A)^\alpha(Ax_0+f(0))\|C_{1-\alpha}\frac{(t-s)^\alpha}{\alpha}+C_1[f]_{(s,0)}\frac{(t-s)^\alpha}{\alpha} \\
&\quad +(C_0+1)[f]_{(t,s)}(t-s)^\alpha+C_2(t-s)^{1-\alpha}\Lambda_f\frac{s^{\frac{\mu}{q'}}}{\mu^{\frac{1}{q'}}} \\
&\quad +2C_0[f]_{(t,s)}(t-s)^\alpha+\frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
&\leq (t-s)^\rho\Bigg[b^{\alpha-\rho}\left(\|(-A)^\alpha(Ax_0+f(0))\|\frac{C_{1-\alpha}}{\alpha}+\frac{C_1[f]_{(s,0)}}{\alpha}+(3C_0+1)[f]_{(t,s)}\right) \\
&\quad +\Lambda_f\left(\frac{C_2b^{1-\alpha+\frac{\mu}{q'}-\rho}}{\mu^{\frac{1}{q'}}}+\frac{C_1}{\nu^{\frac{1}{q'}}}b^{\frac{\nu}{q'}-\rho}\right)\Bigg].
\end{aligned}$$

Which allows us to end the proof. ■

Arguing as before, next we study the case in which $\max\{\alpha,1-\alpha\}<\frac{1}{q'}$.

Proposition 3.2.3 Assume $f \in L^{q,\alpha}([0,a];X) \cap C([0,a];X)$, Λ_f finite, $\max\{\alpha, 1-\alpha\} < \frac{1}{q'}$ and $\rho \in (\max\{\alpha, 1-\alpha\}, \frac{1}{q'})$. If $u(\cdot)$ is the mild solution of (3.3)-(3.4) on $[0,a]$, then $u(\cdot)$ is a classical solution. In addition to above, if $\nu = 1-(1-\alpha q') > 0$ and

i) $x_0 \in D(A)$ and $AT(\cdot)x_0 \in C^{\beta}([0,a];X)$, then $u(\cdot)$ is a strict solution of (3.3)-(3.4) and $u' \in L^{p,\min\{\beta,\rho-(1-\alpha)\}}([0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$. In particular, the assertion hold if $x_0 \in X_{\beta+1}$,

ii) $x_0 \in D(A)$ and $Ax_0 + f(0) \in X_\beta$ for some $\beta \in (0,1)$, then $u(\cdot)$ is a strict solution on $[0,a]$ and $u' \in L^{q,\min\{\beta,\rho-(1-\alpha)\}}([0,a];X)$. In particular, $u' \in L^{q,\rho-(1-\alpha)}([0,a];X)$ if $Ax_0 + f(0) \in X_\alpha$.

Proof: Let $u_i(\cdot)$, $i = 1,2$, be the functions in the proof of Proposition 3.2.2.

Note that for $0 \le s \le t \le a$ we have

$$
\begin{aligned}
&\|Au_2(t) - Au_2(s)\| \\
&\le \int_0^s \int_{s-\tau}^{t-\tau} \frac{C_2}{\xi^{2-\alpha}} [f]_{(s,\tau)} d\xi d\tau + 2C_0\|f(\tau) - f(t)\| + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
&\le C_2 \int_0^s \frac{[f]_{(s,\tau)}}{(s-\tau)^{\rho}} \int_{s-\tau}^{t-\tau} \frac{d\xi}{\xi^{2-\alpha-\rho}} d\tau + 2C_0\|f(\tau) - f(t)\| + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
&\le C_2 \frac{(t-s)^{\rho-(1-\alpha)}}{\rho-(1-\alpha)} \int_0^s \frac{[f]_{(s,\tau)}}{(s-\tau)^{\rho}} d\tau + 2C_0\|f(\tau) - f(t)\| + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \\
&\le C_2 \frac{(t-s)^{\rho-(1-\alpha)}}{\rho-(1-\alpha)} \Lambda_f \frac{b^{(1-\rho q')\frac{1}{q'}}}{[1-\rho q']^{\frac{1}{q'}}} + 2C_0\|f(\tau) - f(t)\| + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}} \qquad (3.11) \\
&\le C_2 \frac{(t-s)^{\rho-(1-\alpha)}}{\rho-(1-\alpha)} \Lambda_f \frac{b^{(1-\rho q')\frac{1}{q'}}}{[1-\rho q']^{\frac{1}{q'}}} + 2C_0[f]_{(t,s)}(t-s)^{\alpha} + \frac{C_1\Lambda_f}{\nu^{\frac{1}{q'}}}(t-s)^{\frac{\nu}{q'}}, \qquad (3.12)
\end{aligned}
$$

which allows us to infer that $Au_2 \in C([0,a];X) \cap L^{q,\rho-(1-\alpha)}([0,a];X)$ because $\rho - (1-\alpha) < \min\{\alpha, \frac{\nu}{q'}\}$.

Assume that the conditions in (i) hold. Noting that the estimates (3.8) and (3.9) hold and that $q > \frac{1}{\alpha}$, we have that

$$
Au_1 \in C([0,a];X) \cap L^{p,\min\{\alpha,\beta\}}([0,a];X) \; Au_2 \in C([0,a];X) \cap L^{q,\rho-(1-\alpha)}([0,a];X)
$$

for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$, which implies that $u(\cdot)$ is a strict solution of the problem (3.3)-(3.4) on $[0,a]$ and $u' \in L^{p,\min\{\beta,\rho-(1-\alpha)\}}([0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$. Similarly, if the conditions in (ii) hold, then the estimate (3.10) is satisfied, $Au_1 \in L^{q,\min\{\beta,\alpha\}}([0,a];X)$ and $Au_2 \in C([0,a];X) \cap L^{q,\rho-(1-\alpha)}([0,a];X)$, which allows us to infer that $u(\cdot)$ is a strict solution and that $u' \in L^{q,\min\{\beta,\rho-(1-\alpha)\}}([0,a];X)$. This completes the proof. ■

3.2.2 The semilinear case

We consider now the semilinear abstract initial value problem

$$
\begin{aligned}
u'(t) &= Au(t) + F(t,u(t)), \quad t \in [0,a] \qquad (3.13) \\
u(0) &= x_0 \in X, \qquad (3.14)
\end{aligned}
$$

where A is the generator on an analytic C_0-semigroup and F is a suitable function to be especified later.

The concepts of mild, classical and strong solution to (3.13)-(3.14) follows from Definition (2.2.1), Definition (2.2.2) and Definition (2.2.3), respectively. We avoid additional details. In addition, we include the concept of strict solution.

Definition 3.2.2 Let $u : [0,a] \to X$ be a function. If $u \in C([0,a];X) \cap C^1([0,a];X)$ satisfies the IVP, we say that $u(\cdot)$ is a **strict solution** to (3.13)-(3.14) on $[0,a]$.

In the following, for $\alpha \in (0,1)$ and $q > 1$ we use the notations

$$\mu = 1 - (1-\alpha)q' \text{ and } \nu = 1 - 2(1-\alpha)q'.$$

Next, we study the existence and uniqueness of a mild, classical and strict solution for the problem (3.13)-(3.14).

Theorem 3.2.4 Let $\theta, \gamma \in (0,1)$. Assume $F \in L^{q,\theta}([0,a] \times X_\gamma; X)$, $x_0 \in X_\gamma$ and $\delta = 1 - \gamma q' > 0$. Then there exists a unique mild solution $u \in C([0,b];X_\gamma)$ of (3.13)-(3.14) on $[0,b]$ for some $b \in (0,a]$. Moreover, if $T(\cdot)x_0 \in C^\eta([0,a];X_\gamma)$, $\mu > 0$ and $\vartheta = 1 - 2\gamma q' > 0$, then $u(\cdot)$ is a classical solution and $u \in C^{\min\{\eta,\gamma,\frac{\delta}{q'}\}}([0,b];X_\gamma)$.

In addition to the above conditions, assume that $F \in C([0,a] \times X_\gamma; X)$, $T(\cdot)x_0 \in C^\eta([0,a];X_\gamma)$, $x_0 \in D(A)$, $\vartheta > 0$ and that $\Lambda_F = \sup_{s\in[0,a]} \|[F]_{(s,\cdot)}\|_{L^q([0,s])}$ is finite and let $\sigma = \min\{\beta, \alpha, 1-\alpha, \frac{\nu}{q'}\}$, then we have:

- **i)** Let $\alpha = \min\{\eta, \theta, \gamma\}$. If $AT(\cdot)x_0 \in C^\beta([0,a];X)$ and $\mu > 0$, then $u(\cdot)$ is a strict solution and $u' \in L^{p,\sigma}([0,b];X)$ for all $1 < p < \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}$.
- **ii)** Let $\alpha = \min\{\eta, \theta, \gamma, 1-\gamma\}$. If $\mu > 0$ and $Ax_0 + F(0,x_0) \in X_\beta$, then $u(\cdot)$ is a strict solution on $[0,b]$ and $u' \in L^{q,\sigma}([0,b];X)$.

Proof: Let $R > \|x_0\|$ and let $0 < b \le a$ small enough such that

$$\|T(\cdot)x_0 - x_0\|_{C([0,b];X_\gamma)} + \frac{b^{1-\gamma}}{1-\gamma} C_\gamma \|F(0,x_0)\| + C_\gamma \mathscr{W}_F(2R)\Lambda_F \left(\frac{b^{\theta+\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}} + \frac{Rb^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}} \right) \le R, \quad (3.15)$$

$$C_\gamma \mathscr{W}_F(2R) \|[F]_{(\cdot,\cdot)}\|_{L^q([0,b])} \frac{b^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}} < 1, \quad (3.16)$$

where $\Lambda_F = \|[F]_{(\cdot,0)}\|_{L^q([0,b])}$. Let $\mathscr{Z}(b,R)$ be the space

$$\mathscr{Z}(b,R) := \left\{ u \in C([0,b];X_\gamma) : u(0) = x_0,\ \|u - x_0\|_{C([0,b];X_\gamma)} \le R \right\}$$

endowed with the metric $d(u,v) = \|u - v\|_{C([0,b];X_\gamma)}$ and $\Gamma : \mathscr{Z}(b,R) \to C([0,b];X)$ the map given by

$$\Gamma u(t) = T(t)x_0 + \int_0^t T(t-s)F(s,u(s))ds, \quad t \in [0,b]. \quad (3.17)$$

Let $u \in \mathscr{Z}(b,R)$ and $t \in [0,b]$. Noting that

$$\|u\|_{C([0,b];X_\gamma)} \le \|u - x_0\|_{C([0,b];X_\gamma)} + \|x_0\|_{C([0,b];X_\gamma)} \le 2R,$$

we see that

$$\|\Gamma u(t) - x_0\|_\gamma \le \|T(t)x_0 - x_0\|_\gamma + \|\int_0^t T(t-s)F(s,u(s))ds\|_\gamma$$

$$
\begin{aligned}
&\le \|T(\cdot)x_0 - x_0\|_{C([0,b];X_\gamma)} + \int_0^t \|(-A)^\gamma T(t-s)F(s,u(s))(-F(0,x_0)+F(0,x_0))\|ds \\
&\le \|T(\cdot)x_0 - x_0\|_{C([0,b];X_\gamma)} + \int_0^t \|(-A)^\gamma T(t-s)F(0,x_0)\|ds \\
&\quad + \int_0^t \|(-A)^\gamma T(t-s)\|\|F(s,u(s)) - F(0,x_0)\|ds \\
&\le \|T(\cdot)x_0 - x_0\|_{C([0,b];X_\gamma)} + \int_0^t \frac{C_\gamma}{(t-s)^\gamma}\|F(0,x_0)\|ds \\
&\quad + \int_0^t \frac{C_\gamma}{(t-s)^\gamma}\mathscr{W}_F(\max\{\|u(s)\|,\|x_0\|\})[F]_{(s,0)}(s^\theta + \|u(s)-x_0\|_\gamma)ds \\
&\le \|T(\cdot)x_0 - x_0\|_{C([0,b];X_\gamma)} + \frac{b^{1-\gamma}}{1-\gamma}C_\gamma\|F(0,x_0)\| \\
&\quad + C_\gamma\mathscr{W}_F(2R)\|[F]_{(\cdot,0)}\|_{L^q([0,b])}\left(\frac{b^{\theta+\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}} + \frac{Rb^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}\right) \\
&\le R.
\end{aligned}
$$

Which implies that Γu is well defined and that $\Gamma(\mathscr{Z}(b,R)) \subset \mathscr{Z}(b,R)$. Moreover, for $u,v \in \mathscr{Z}(b,R)$ and $t \in [0,b]$ we get

$$
\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|_\gamma &\le \|T(t)x_0 - T(t)x_0\|_\gamma + \int_0^t \|(-A)^\gamma T(t-s)(F(s,u(s)) - F(s,v(s)))\|ds \\
&\le \int_0^t \frac{C_\gamma}{(t-s)^\gamma}\mathscr{W}_F(2R)[F]_{(s,s)}\|u(s)-v(s)\|_\gamma ds \\
&\le C_\gamma\mathscr{W}_F(2R)\|[F]_{(\cdot,\cdot)}\|_{L^q([0,b])}\frac{t^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}\|u-v\|_{C([0,b];X_\gamma)} \\
&\le C_\gamma\mathscr{W}_F(2R)\|[F]_{(\cdot,\cdot)}\|_{L^q([0,b])}\frac{b^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}\|u-v\|_{C([0,b];X_\gamma)},
\end{aligned}
$$

hence, $\Gamma(\cdot)$ is a contraction (see (3.16)). From the above and Banach's Fixed Point Theorem (see A.0.11), there exists a unique mild solution $u \in \mathscr{Z}(b,R)$ of (3.13)-(3.14) on $[0,b]$.

Noting that

$$
\begin{aligned}
g(s) := \|F(s,u(s))\| &\le \|F(s,u(s)) - F(0,x_0)\| + \|F(0,x_0)\| \\
&\le \mathscr{W}_F(2R)[F]_{(s,s)}(s^\theta + R) + \|F(0,x_0)\| \\
&\le \mathscr{W}_F(2R)[F]_{(s,s)}(a^\theta + R) + \|F(0,x_0)\|, \qquad (3.18)
\end{aligned}
$$

we have $g(\cdot) = \|F(\cdot,u(\cdot))\| \in L^q([0,a];X)$. Assuming $T(\cdot)x_0 \in C^\eta([0,a];X_\gamma)$, for $t,h \in (0,b]$ with $t+h \in [0,b]$ we have

$$
\begin{aligned}
&\|u(t+h) - u(t)\|_\gamma \\
&\quad \le \|T(t+h)x_0 - T(t)x_0\|_\gamma + \int_0^t \|(-A)^\gamma(T(t+h-s) - T(t-s))F(s,u(s))\|ds \\
&\qquad + \int_t^{t+h} \|(-A)^\gamma T(t+h-s)\|\|F(s,u(s))\|ds \\
&\quad \le [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta
\end{aligned}
$$

$$
\begin{aligned}
& +\int_0^t\int_{t-s}^{t+h-s}\|(-A)^{1+\gamma}T(\xi)F(s,u(s))\|d\xi ds+\int_t^{t+h}\frac{C_\gamma g(s)}{(t+h-s)^\gamma}ds \qquad (3.19)\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\int_0^t\int_{t-s}^{t+h-s}\frac{C_{1+\gamma}g(s)}{\xi^{1+\gamma}}d\xi ds\\
& +C_\gamma\|g\|_{L^q([t,t+h];X)}\left(\int_t^{t+h}(t+h-s)^{-\gamma q'}ds\right)^{\frac{1}{q'}}\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\frac{C_{1+\gamma}}{\gamma}\int_0^t\frac{(t+h-s)^\gamma-(t-s)^\gamma}{(t+h-s)^\gamma(t-s)^\gamma}g(s)ds+C_\gamma\|g\|_{L^q([t,t+h])}\frac{h^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\frac{C_{1+\gamma}}{\gamma}h^\gamma\int_0^t\frac{g(s)ds}{(t-s)^{2\gamma}}+C_\gamma\|g\|_{L^q([t,t+h])}\frac{h^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}},\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\frac{C_{1+\gamma}}{\gamma}h^\gamma\|g\|_{L^q([0,t];X)}\left(\int_0^t\frac{ds}{(t-s)^{2\gamma q'}}\right)^{\frac{1}{q'}}+C_\gamma\|g\|_{L^q([t,t+h])}\frac{h^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}},\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\frac{C_{1+\gamma}}{\gamma}h^\gamma\|g\|_{L^q([0,t];X)}\frac{t^{\frac{\vartheta}{q'}}}{\vartheta^{\frac{1}{q'}}}+C_\gamma\|g\|_{L^q([t,t+h])}\frac{h^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}},
\end{aligned} \tag{3.20}
$$

that is

$$
\|u(t+h)-u(t)\|_\gamma\leq[T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\|g\|_{L^q([0,b])}\left(\frac{C_{1+\gamma}b^{\frac{\vartheta}{q'}}}{\gamma\vartheta^{\frac{1}{q'}}}h^\gamma+\frac{C_\gamma}{\delta^{\frac{1}{q'}}}h^{\frac{\delta}{q'}}\right), \tag{3.21}
$$

which implies that $u(\cdot)\in C^{\min\{\eta,\gamma,\frac{\delta}{q'}\}}([0,b];X_\gamma)$.

To prove the assertions in (i) and (ii), we note that if F is continuous, then $F(\cdot,u(\cdot))$ is bounded on $[0,a]$, which allows us to simplify the estimates above. To continue, assuming $M>0$ such that $\|F(s,u(s))\|\leq M$, for all $s\in[0,b]$, and $1-2\gamma>0$, from estimate (3.19) we see that

$$
\begin{aligned}
& \|u(t+h)-u(t)\|_\gamma\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\int_0^t\int_{t-s}^{t+h-s}\|(-A)^{1+\gamma}T(\xi)F(s,u(s))\|d\xi ds\\
& +\int_t^{t+h}\frac{C_\gamma\|F(s,u(s))\|}{(t+h-s)^\gamma}ds\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+\int_0^t\int_{t-s}^{t+h-s}\frac{C_{1+\gamma}M}{\xi^{1+\gamma}}d\xi ds+\int_t^{t+h}\frac{C_\gamma M}{(t+h-s)^\gamma}ds\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+M\frac{C_{1+\gamma}}{\gamma}\int_0^t\frac{(t+h-s)^\gamma-(t-s)^\gamma}{(t+h-s)^\gamma(t-s)^\gamma}ds+C_\gamma M\frac{h^{1-\gamma}}{1-\gamma}\\
\leq\ & [T(\cdot)x_0]_{C^\eta([0,b];X_\gamma)}h^\eta+M\frac{C_{1+\gamma}}{\gamma}\frac{b^{1-2\gamma}}{1-2\gamma}h^\gamma+C_\gamma M\frac{h^{1-\gamma}}{1-\gamma},
\end{aligned}
$$

and hence, $u\in C^{\min\{\eta,\gamma,1-\gamma\}}([0,b];X_\gamma)$ which implies

$$
\begin{aligned}
\|F(t,u(t))-F(s,u(s))\| &\leq \mathscr{W}_F(2R)[F]_{(t,s)}((t-s)^\theta+\|u(t)-u(s)\|)\\
&\leq \mathscr{W}_F(2R)[F]_{(t,s)}((t-s)^\theta+(t-s)^{\min\{\eta,\gamma,1-\gamma\}}),
\end{aligned}
$$

that is $F(\cdot,u(\cdot)) \in L^{q,\min\{\theta,\eta,\gamma\}}([0,b];X)$, because $\gamma < 1-\gamma$.

To complete the proof, we only note that the assertions in (i) and (ii) follow from the assertions in (iv) and (v) of Proposition 3.2.2 because $u(\cdot)$ is a mild solution, $F(\cdot,u(\cdot)) \in L^{q,\alpha}([0,b];X) \cap C([0,b];X)$, with $\alpha = \min\{\theta,\eta,\gamma\}$, and by assumptions $x_0 \in D(A)$, $T(\cdot)x_0 \in C^{\eta}([0,a];X_{\gamma})$, $\vartheta > 0$ and $\Lambda_F < \infty$. ■

On the last result, our argument follows from Proposition 3.2.2. By using Proposition 3.2.3 in place of Proposition 3.2.2, we can prove the next result.

Proposition 3.2.5 Suppose, $\theta,\gamma \in (0,1)$, $F \in L^{q,\theta}([0,a]\times X_{\gamma};X) \cap C([0,a]\times X_{\gamma};X)$, $x_0 \in X_{\gamma}$, $\delta = 1-\gamma q' > 0$, $1-2\gamma > 0$ and $T(\cdot)x_0 \in C^{\eta}([0,a];X_{\gamma})$. Then there exists a unique mild solution $u \in C^{\min\{\eta,\gamma\}}([0,b];X_{\gamma})$ of the problem (3.13)-(3.14) on $[0,b]$. Moreover, if $F \in C([0,a]\times X_{\gamma};X)$, let $\alpha = \min\{\eta,\theta,\gamma\}$ and assume $\sup_{s\in[0,a]} \|[F]_{(s,\cdot)}\|_{L^q([0,s])} < \infty$ and $\max\{\alpha,1-\alpha\} < \frac{1}{q'}$. For $\rho \in (\max\{\alpha,1-\alpha\},\frac{1}{q'})$, we get:

- **i)** If $x_0 \in D(A)$ and $AT(\cdot)x_0 \in C^{\beta}([0,a];X)$, then $u(\cdot)$ is a strict solution on $[0,b]$ and $u' \in L^{p,\min\{\beta,\rho-(1-\alpha)\}}((0,a];X)$ for all $1 < p < \min\{\frac{1}{\alpha},\frac{1}{\beta}\}$ In particular, the assertion hold if $x_0 \in X_{\beta+1}$.
- **ii)** If $x_0 \in D(A)$ and $(Ax_0 + f(0)) \in X_{\beta}$ for some $\beta \in (0,1)$, then $u(\cdot)$ is a strict solution on $[0,b]$ and $u' \in L^{q,\min\{\beta,\rho-(1-\alpha)\}}([0,b];X)$. In particular, $u' \in L^{q,\rho-(1-\alpha)}([0,b];X)$ if $(Ax_0 + f(0)) \in X_{\alpha}$.

Using the ideas in the proof of Theorem 3.2.4, it is possible to establish a similar result for the case in which $F \in C([0,a]\times X;X)$ and $x_0 \in X$. In the next result, for $\delta \in (0,1)$ we use the notation $C^{\delta}_{\delta}((0,a];X)$ for the space

$$C^{\delta}_{\delta}((0,a];X) := \left\{ u \in L^{\infty}((0,a];X) : \sup_{0<\varepsilon\le a} \varepsilon^{\delta}[u]_{C^{\delta}([\varepsilon,a];X)} < \infty \right\}.$$

Proposition 3.2.6 Assume $F \in L^{q,\theta}([0,a]\times X;X)$ for some $\theta \in (0,1)$. Then, there exists a unique mild solution $u \in C([0,b];X)$ of (3.13)-(3.14) on $[0,b]$ for some $0 < b \le a$. Moreover, for all $\delta \in (0,\frac{1}{q'})$, $u \in C^{\min\{\eta,\delta\}}_{\delta}((0,b];X)$ and $u \in C^{\min\{\eta,\delta\}}([0,b];X)$ if $T(\cdot)x_0 \in C^{\eta}([0,a];X)$.

Suppose, in addition to the above conditions, $F \in C([0,a]\times X;X)$, $T(\cdot)x_0 \in C^{\eta}([0,a];X)$ and $\sup_{s\in[0,a]} \|[F]_{(s,\cdot)}\|_{L^q([0,s])} < \infty$ and let $\alpha = \min\{\eta,\theta\}$ and $\beta \in (0,1)$.

- **i)** If $\mu = (1-2(1-\alpha)q') > 0$, $x_0 \in D(A)$, $AT(\cdot)x_0 \in C^{\beta}([0,a];X)$, then $u(\cdot)$ is a strict solution and $u' \in L^{p,\min\{\beta,\alpha,1-\alpha,\frac{1}{q'}\}}([0,b];X)$ for all $1 < p < \min\{\frac{1}{\alpha},\frac{1}{\beta}\}$.
- **ii)** If $\mu = (1-2(1-\alpha)q') > 0$, $x_0 \in D(A)$ and $(Ax_0 + F(0,x_0)) \in X_{\beta}$, for some $\beta \in (0,1)$ then $u(\cdot)$ is a strict solution on $[0,b]$ and $u' \in L^{q,\min\{\beta,\alpha,1-\alpha,\frac{1}{q'}\}}([0,b];X)$.

Proof: Let $R > \|x_0\|$. Considering $0 < b \le a$ small enough such that

$$\|T(\cdot)x_0 - x_0\|_{C([0,b])} + C_0\mathscr{W}_F(2R)(b^{\theta}+R)\|[F]_{(0,\cdot)}\|_{L^q([0,b])}b^{\frac{1}{q'}} + C_0\|F(0,x_0)\|b \le R, \quad (3.22)$$

$$C_0\mathscr{W}_F(2R)\|[F]_{(\cdot,\cdot)}\|_{L^q([0,b])}b^{\frac{1}{q'}} < 1. \quad (3.23)$$

Let $\mathscr{X}(b,R)$ be the space

$$\mathscr{X}(b,R) := \left\{ u \in C([0,b];X) : u(0) = x_0,\ \|u - x_0\|_{C([0,b];X)} \le R \right\},$$

endowed with the metric $d(u,v) = \|u-v\|_{C([0,b];X)}$ and $\Gamma : \mathscr{X}(b,R) \to C([0,b];X)$ be the function defined by

$$\Gamma u(t) = T(t)x_0 + \int_0^t T(t-s)F(s,u(s))ds,\ t \in [0,b]. \quad (3.24)$$

Let $u \in \mathscr{Z}(b,R)$ and $t \in [0,b]$.

Noting that $\|u\|_{C([0,b];X)} \leq 2R$, we have that

$$\begin{aligned}
\|\Gamma u(t) - x_0\| &\leq \|T(t)x_0 - x_0\| + \int_0^t \|T(t-s)F(0,x_0)\| ds \\
&\quad + \int_0^t \|T(t-s)\| \|F(s,u(s)) - F(0,x_0)\| ds \\
&\leq \|T(\cdot)x_0 - x_0\| + \int_0^t C_0 \|F(0,x_0)\| ds \\
&\quad + \int_0^t C_0 \mathscr{W}_F(2R)[F]_{(s,0)}(s^\theta + \|u(s) - x_0\|) ds \\
&\leq \|T(\cdot)x_0 - x_0\|_{C([0,b])} + C_0 \mathscr{W}_F(2R)(b^\theta + R)\|[F]_{(0,\cdot)}\|_{L^q([0,b])} b^{\frac{1}{q'}} + C_0\|F(0,x_0)\| b \\
&\leq R,
\end{aligned}$$

which implies that $\Gamma(\mathscr{Z}(b,R)) \subset \mathscr{Z}(b,R)$, once $t \mapsto \int_0^t T(t-s)F(s,u(s))ds$ is continuous. Moreover, for $u, v \in \mathscr{Z}(b,R)$ and $t \in [0,b]$ we see that

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &\leq \int_0^t \|T(t-s)\| \|(F(s,u(s)) - F(s,v(s)))\| ds \\
&\leq \int_0^t C_0 \mathscr{W}_F(2R)[F]_{(s,s)} \|u(s) - v(s)\| ds \\
&\leq C_0 \mathscr{W}_F(2R) \|[F]_{(\cdot,\cdot)}\|_{L^q([0,b])} b^{\frac{1}{q'}} \|u - v\|_{C([0,b];X)},
\end{aligned}$$

and hence, $\Gamma(\cdot)$ is a contraction and, from Banach's Fixed Point Theorem, there exists a unique mild solution $u \in \mathscr{Z}(b,R)$ for the IVP (3.13)-(3.14).

To show the assertions in (i) and (ii), we note that if F is continuous, there is $M > 0$ such that $\|F(s,u(s))\| \leq M$ for all $s \in [0,b]$. Using now that $\ln(1+\rho) \leq \frac{\rho^\vartheta}{\vartheta}$ for all $\rho > 0$ and $\vartheta \in (0,1)$, for $\delta \in (0,1)$ we get

$$\begin{aligned}
&\|u(t+h) - u(t)\| \\
&\leq \|T(t+h)x_0 - T(t)x_0\| + \int_0^t \|(T(t+h-s) - T(t-s))F(s,u(s))\| ds \\
&\quad + \int_t^{t+h} \|T(t+h-s)F(s,u(s))\| ds \\
&\leq [T(\cdot)x_0]_{C^\eta([0,b];X)} h^\eta + \int_0^t \int_{t-s}^{t+h-s} \|AT(\xi)F(s,u(s))\| d\xi ds + \int_t^{t+h} C_0 M ds \\
&\leq [T(\cdot)x_0]_{C^\eta([0,b];X)} h^\eta + \int_0^t \int_{t-s}^{t+h-s} \frac{C_1 M}{\xi} d\xi ds + C_0 M h \\
&\leq [T(\cdot)x_0]_{C^\eta([0,b];X)} h^\eta + C_1 M \int_0^t \ln\left(1 + \frac{h}{t-s}\right) ds + C_0 M h \\
&\leq [T(\cdot)x_0]_{C^\eta([0,b];X)} h^\eta + C_1 M \int_0^t \frac{h^\delta}{\delta (t-s)^\delta} ds + C_0 M h \\
&\leq [T(\cdot)x_0]_{C^\eta([0,b];X)} h^\eta + C_1 \frac{M}{\delta} \frac{b^{1-\delta}}{1-\delta} h^\delta + C_0 M h,
\end{aligned}$$

which implies that $u \in C^\eta([0,b];X)$, and

$$\|F(t,u(t)) - F(s,u(s))\| \leq \mathscr{W}_F(\max\{\|u(t)\|, \|u(s)\|\})[F]_{(t,s)}(|t-s|^\theta + \|u(t) - u(s)\|)$$

$$\begin{aligned} &\leq \mathscr{W}_F(R)[F]_{(t,s)}(|t-s|^{\theta}+[u]_{C^{\eta}([0,b];X)}|t-s|^{\eta}) \\ &\leq \mathscr{W}_F(R)[F]_{(t,s)}(b^{\mu-\theta}+[u]_{C^{\eta}([0,b];X)}b^{\mu-\eta})(t-s)^{\mu} \end{aligned}$$

where $\mu = \min\{\theta,\eta\}$. The above implies that $F(\cdot,u(\cdot)) \in L^{q,\min\{\theta,\eta\}}([0,a];X)\cap C([0,a]\times X;X)$. Now, we can complete the proof using the assertion in (iv) and (v) of Proposition 3.2.2. ■

From the Theorem 3.2.4, we infer the next result on the existence of a solution defined on $[0,a]$.

Corollary 3.2.7 (Solution on $[0,a]$) Assume $\theta,\gamma \in (0,1)$, $F \in L^{q,\theta}([0,a]\times X_{\gamma};X)$, $x_0 \in X_{\gamma}$ and $\delta = 1-\gamma q' > 0$ and let $P_a : [0,a] \mapsto \mathbb{R}$ be the map defined by

$$\begin{aligned} P_a(x) &= \|T(\cdot)x_0 - x_0\|_{C([0,a];X_{\gamma})} + \frac{a^{1-\gamma}}{1-\gamma}C_{0,\gamma}\|F(0,x_0)\| \\ &\quad + C_{0,\gamma}\mathscr{W}_F(2x)\|[F]_{(\cdot,0)}\|_{L^q([0,a])}\left(\frac{a^{\theta+\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}} + \frac{xa^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}\right) - x \\ &\quad + C_{0,\gamma}\mathscr{W}_F(2x)x\|[F]_{(\cdot,\cdot)}\|_{L^q([0,a])}\cdot\frac{a^{\frac{\delta}{q'}}}{\delta^{\frac{1}{q'}}}. \end{aligned} \tag{3.25}$$

If there exists $R>0$ such that $P_a(R)<0$, then there exists a mild solution $u(\cdot)$ of the problem (3.13)-(3.14) defined on $[0,a]$ and $u \in C^{\min\{\eta,\gamma,\frac{\vartheta}{q'}\}}([0,b];X_{\gamma})$ if $T(\cdot)x_0 \in C^{\eta}([0,a];X_{\gamma})$ and $\vartheta = 1-2\gamma q' > 0$. Moreover, the assertions in (i) and (ii) of the Theorem 3.2.4 hold with a in place of b.

Proof: The proof follows from the proof of Theorem 3.2.4 noting that the condition $P(R)<0$, implies that the inequalities (3.15) and (3.16) are satisfied. ■

Existence of solution on $[0,\infty)$.

We complete this section by studying the existence of a solution on the whole semi-axis $[0,\infty)$. To this end, we use the next conditions.

$\mathbf{H_F}$ There are $\theta \in (0,1]$, a measurable function $[F]_{(\cdot,\cdot)} : [0,\infty)\times[0,\infty) \mapsto \mathbb{R}^+$ and a non-decreasing function $\mathscr{W}_F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $[F]_{(\cdot,\cdot)}, [F]_{(t,\cdot)}$ and $[F]_{(\cdot,0)}$ belongs to $L^p([0,a];\mathbb{R}^+)$ for all $t \in [0,a]$ and $a>0$ and

$$\|F(t,x)-F(s,y)\| \leq \mathscr{W}_F(\max\{\|x\|_{\gamma},\|y\|_{\gamma}\})[F]_{(t,s)}(|t-s|^{\theta}+\|x-y\|_{\gamma})$$

for all $t,s \in [0,\infty)$ and $x,y \in X_{\gamma}$.

$\mathbf{H_T}$ There are positive constants ω and $D_{0,\gamma}$ such that

$$\|(-A)^{\gamma}T(t)\| \leq D_{0,\gamma}\frac{e^{-\omega t}}{t^{\gamma}}, \text{ for all } t>0.$$

In the next result, for $x \in X$ we use the notations:

$$\begin{aligned} \Theta_1(x) &= \mathscr{W}_F(2x)\sup_{t>0}\int_0^t \frac{e^{-\omega(t-\tau)}}{(t-\tau)^{\gamma}}[F]_{(\tau,0)}(\tau^{\theta}+x)d\tau, \\ \Theta_2(x) &= \mathscr{W}_F(2x)\sup_{b>0,t\in[0,b]}\int_0^t \frac{e^{-\omega(t-\tau)}}{(t-\tau)^{\gamma}}[F]_{(\tau,\tau)}d\tau. \end{aligned} \tag{3.26}$$

The proof of Corollary 3.2.8, follows combining the ideas in the proof of Theorem 3.2.4 and Corollary 3.2.7.

Corollary 3.2.8 Assume $\theta,\gamma \in (0,1)$, $x_0 \in X_\gamma$, $\delta = 1-\gamma q' > 0$ and that the conditions $\mathbf{H_F}$ and $\mathbf{H_T}$ hold. Let $P:[0,a] \mapsto \mathbb{R}$ be the map defined by

$$P(x) = \|T(\cdot)x_0 - x_0\|_{C([0,\infty);X_\gamma)} + D_{0,\gamma}\|F(0,x_0)\|\left(\frac{1}{\gamma}+\frac{1}{\omega}\right) + D_{0,\gamma}(\Theta_1(x)+\Theta_2(x)) - x.$$

If $P(R) < 0$ for some $R > 0$, then there exists a mild solution $u \in C([0,\infty);X_\gamma)$ of the problem (3.13)-(3.14) on $[0,\infty)$. Moreover, the assertions (i) and (ii) on the Theorem 3.2.4 are satisfied on $[0,a]$ for all $a > 0$.

Proof: Let $a > 0$. Proceeding as in the proof of the Corollary 3.2.7 with $P(\cdot)$ in place of $P_a(\cdot)$, we can prove that there exists a unique mild solution $u^a \in C([0,a];X)$ of the problem (3.13)-(3.14) on $[0,a]$. Defining $u:[0,\infty) \to X$ by $u(t) = u^a(t)$ if $t \in [0,a]$ we obtain that $u(\cdot)$ is a mild solution defined on $[0,\infty)$. Moreover, from the uniqueness of the solution u^a it is easy to see that $u(\cdot)$ is the unique mild solution of the problem (3.13)-(3.14) on $[0,\infty)$.

To finish, we note that the last assertions follow from Theorem 3.2.4, remarking that $u\big|_{[0,a]} = u^a$ is the unique solution of problem (3.13)-(3.14) on $[0,a]$ for all $a > 0$. ■

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A. Appendix

Definition A.0.1 Let X and Y be normed spaces and $T : D(T) \subset X \to Y$ be a linear operator. We say that T is a **closed linear operator** if its graph $G(T) = \{(x,y) : x \in D(T) \subset X, y = Tx\}$ is closed in $X \times Y$.

Lemma A.0.1 Let J be some real interval and $P, Q : J \to \mathscr{L}(X)$ two strongly continuous operator-valued functions on J. Then, the product $(PQ)(\cdot)x : J \to X$, defined by $(PQ)(t)(x) := P(t)Q(t)x$, is strongly continuous as well.

Corollary A.0.2 Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq M$. Let $\gamma > \max\{0, \omega\}$. If $x \in D(A^2)$, then

$$T(t)x = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda : A)x d\lambda$$

and, for every $\delta > 0$, the integral uniformly converges in t for $t \in [\frac{1}{\delta}, \delta]$.

Corollary A.0.3 Let A be the infinitesimal generator of a C_0-semigroup of contractions. The resolvent set of a A contains the open right half-plane, i.e, $\rho(A) \supseteq \{\lambda : \mathrm{Re}(\lambda) > 0\}$ and, for such λ,

$$\|R(\lambda : A)\| \leq \frac{1}{\mathrm{Re}\,\lambda}.$$

Proposition A.0.4 Let B be a bounded linear operator. If $\|I - B\| < 1$ then $B^{-1} = \sum_{j=0}^{\infty}(I-B)^j$.

Theorem A.0.5 — Bounded Inverse. A bounded linear operator T from a Banach space X onto a Banach space Y is an opening mapping. Hence, if T is bijective, T^{-1} is continuous and thus bounded.

Theorem A.0.6 — Uniform boundedness. Let $(T_n)_n$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y such that $(\|T_n x\|)_n$ is bounded for every $x \in X$, say,

$$\|T_n x\| \leq c_x$$

where c_x is a real positive number. Then, the sequence of the norms $(\|T_n\|)_n$ is bounded, that is, there is a positive number c such that

$$\|T_n\| \leq c.$$

Theorem A.0.7 — Hahn-Banach. Let E be a real vector space and $p : E \to \mathbb{R}$ be a function satisfying

$$p(\lambda x) = \lambda p(x), \;\; \forall x \in E \text{ and } \forall \lambda > 0, \tag{A.1}$$

$$p(x+y) \leq p(x) + p(y), \;\; \forall x, y \in E. \tag{A.2}$$

Let $G \subset E$ a linear subspace and $g : G \to \mathbb{R}$ be a linear functional such that

$$g(x) \leq p(x), \;\; \forall x \in G. \tag{A.3}$$

Under theses assumptions, there exists a linear functional f defined on E that extends g, i.e., $g(x) = f(x), \forall x \in G$. Moreover

$$f(x) \leq p(x), \;\; \forall x \in E. \tag{A.4}$$

Theorem A.0.8 — Gronwall's inequality. Assume that $f : [t_0, a] \to \mathbb{R}^+$ is a continuous function such that
$$f(t) \leq C + K \int_{t_0}^{t} f(s)ds, \ \ \forall t \in [t_0, a],$$
where C and K are positive constants. Then, $f(t) \leq C + K \int_{t_0}^{t} e^{Kt}$ for all $t \in [t_0, a]$.

Theorem A.0.9 — Hölder's Inequality. Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $fg \in L^1$ and
$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

Theorem A.0.10 — Closed Graph Theorem. Let E and F two Banach spaces, T be a linear operator from E to F. If the graph of T is closed in $E \times F$. Then, T is continuous.

Theorem A.0.11 — Banach's Fixed Poit Theorem. Let f be a contraction mapping from a closed subset F of a Banach space E into F. Then, there exists a unique $z \in F$ such that $f(z) = z$.

Theorem A.0.12 — Lebesgue Dominated Convergence Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions that converges almost everywhere to a measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then f is integrable and
$$\int f d\mu = \lim \int f_n d\mu. \tag{A.5}$$

Theorem A.0.13 — Cauchy's Integral Theorem. Let $\omega \subset \mathbb{C}$ be a simply connected open set and let $f : \omega \to \mathbb{C}$ be a holomorphic function. If $\gamma : [a, b] \to \omega$ is a smooth closed curve then,
$$\int_\gamma f(z)dz = 0.$$

Theorem A.0.14 Let $f : [0, a] \to \mathbb{R}^+$ be an integrable function. If $h \to 0$ then,
$$\int_0^h f(x)ds \to 0.$$

Theorem A.0.15 — Cauchy's Integral Formula. Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$ we have
$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz.$$

Theorem A.0.16 — Characterization of infinitesimal generator of C_0-semigroups. A linear operator A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq M$ $(M \geq 1)$ if, and only if,

1. A is closed and $D(A)$ is dense in X.
2. The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and

$$\|R(\lambda : A)^n\| \leq \frac{M}{\lambda^n}, \quad \lambda > 0, n = 1, 2, \ldots.$$

Theorem A.0.17 Let (A^α) be the fractional power operator of A, with $\alpha > 0$. Then,

i) A^α is a closed operator with domain $D(A^\alpha) = R(A^{-\alpha})$.
ii) $\alpha \geq \beta > 0$ implies $D(A^\alpha) \subset D(A^\beta)$.
iii) $\overline{D(A^\alpha)} = X$ for all $\alpha \geq 0$.
iv) If α, β are real then

$$A^{\alpha+\beta}x = A^\alpha A^\beta x$$

for every $x \in D(A^\gamma)$ where $\gamma = \min\{\alpha, \beta, \alpha+\beta\}$.
v) For an analytic semigroup $(T(t))_{t\geq 0}$ there exists $C_\alpha \in \mathbb{R}^+$ such that $\|AT(t)\| \leq \frac{C_\alpha}{t^\alpha}$.

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