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Contact foliations: closed leaves and generalised Weinstein conjectures

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Folheações de contato: folhas fechadas e conjecturas de Weinstein generalizadas

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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To Camila and Elizete,
my truest beacons.

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"Ich weiß nicht, warum ich das tue
Es ist fast ein innerer Zwang, der mich dazu treibt."
Aufbruch - Der Weg einer Freiheit

RESUMO

FINAMORE, D. **Folheações de contato: folhas fechadas e conjecturas de Weinstein generalizadas**. 2023. 141 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

A conjectura de Weinstein, que diz respeito a existência de órbitas periódicas para fluxos de Reeb, é um dos problemas mais clássicos da Geometria de Contato. Almeida, em sua tese de doutorado (ALMEIDA, 2018), introduziu uma generalização do conceito clássico de estrutura de contato que possibilita a definição de *folheações de contato,* i.e., folheações de dimensão maior que 1 que generalizam as principais propriedades do fluxo de Reeb.

Neste trabalho, inspirados pela conjectura de Weinstein para o caso clássico, buscamos encontrar folhas fechadas para folheações de contato. Generalizando ideias usadas com êxito anteriormente para provar a conjectura em variedades de contato com propriedades adicionais, obtemos a existência de folhas fechadas nos casos particulares em que a folheação é hiperbólica ou C^1-*equicontínua.* Esta última classe engloba folheações de contato quasiconformais, conformais, isométricas e riemannianas. Além disso, usando técnicas da Teoria de Morse, relacionamos as folhas fechadas à cohomologia básica de uma folheação de contato C^1-equicontínua, obtendo uma cota inferior para a quantidade de folhas fechadas diretamente proporcional a codimensão da folheação.

Palavras-chave: ações de grupos, folheações, estruturas de contato generalizadas, dinâmica de contato.

ABSTRACT

FINAMORE, D. **Contact foliations: closed leaves and generalised Weinstein conjectures**. 2023. 141 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

The Weinstein conjecture, which regards the existence of periodic orbits for Reeb flows, is a classic problem in Contact Geometry. In his doctoral dissertation, Almeida (ALMEIDA, 2018) introduced a novel geometric structure, which generalises contact structures and provides a notion of *contact foliation*, i.e., higher dimensional analogues for the Reeb flow.

In this work, inspired by the classical Weinstein conjecture, we seek to find closed leaves for such contact foliations. By generalising ideas already employed successfully in proving the Weinstein conjecture in the past, we obtain the existence of closed leaves in particular cases when the foliation is either hyperbolic or C^1-equicontinuous. This later class encompasses those of quasiconformal, conformal, isometric, and Riemannian contact foliations. Moreover, using techniques from Morse Theory, we were able to relate the closed leaves of a C^1-equicontinuous contact foliation to its basic cohomology, obtaining a lower bound for the number of closed orbits, as a function of the foliation's codimension.

Keywords: group actions, foliations, generalised contact structures, contact dynamics.

LIST OF SYMBOLS

$\mathrm{Gl}_n(\mathbb{R})$ — nth real general linear group

$\mathrm{Sl}_n(\mathbb{R})$ — nth real special linear group

$\mathrm{So}_n(\mathbb{R})$ — nth real special orthogonal group

$\mathrm{Co}_n(\mathbb{R})$ — group of conformal transformations of $\mathbb{R}^n$

$C_n(\mathbb{R})$ — space of conformal metrics of $\mathbb{R}^n$

$\Gamma(E)$ — space of (smooth) sections of the vector bundle E

$\Gamma(M)$ — short for $\Gamma(\mathrm{T}M)$, the space of (smooth) sections of the vector bundle $\mathrm{T}M$

$\Gamma(\mathcal{F}$ — short for $\Gamma(\mathrm{T}\mathcal{F})$, the space of (smooth) sections of $\mathrm{T}M$ tangent to $\mathcal{F}$

$\mathcal{L}_X$ — Lie derivative in the direction of X

$\iota_X\omega$ — interior product of differential form ω with X

$\mathrm{Diff}(M)$ — group of diffeomorphisms of the manifold M

$\mathrm{Diff}_{\mathrm{vol}}(M)$ — group of volume-preserving diffeomorphisms of the manifold M

$\mathrm{Iso}(M,g)$ — group of isometries of the Riemannian manifold (M,g)

$H^*_{dR}(M)$ — de Rham cohomology algebra of the manifold M

$\wedge^k(M)$ — set of differential k-forms on M

$(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ — q-contact manifold

ξ — q-contact distribution

$\mathcal{R}$ — Reeb distribution

$\vec{\lambda}$ — adapted coframe of a q-contact structure

$\mathrm{d}M_i$ — i-th volume form of M

$\mathfrak{C}_i$ — characteristic distribution of λ_i

$\mathcal{C}_i$ — characteristic foliation of λ_i

δ_{ij} — Dirac's delta notation: $\delta_{ij}1$ if $i=j$; $\delta_{ij}=0$ otherwise

R_i — i-th Reeb vector field of the q-contact structure

F — contact action associated to the q-contact distribution

$\mathcal{F}$ — contact foliation associated to the q-contact action

g^{τ} — adapted metric for the q-contact structure

WC — Weinstein Conjecture

WGWC — Weak Generalised Weinstein Conjecture

SGWC — Strong Generalised Weinstein Conjecture

Δ — Laplace-Beltrami operator

E^1_F — C^1-enveloping group of the action F.

$\mathfrak{B}(\mathcal{F})$ — Lie algebra of $\mathcal{F}$-foliate vector fields

$\mathfrak{t}(\mathcal{F})$ — Lie algebra of $\mathcal{F}$-transverse vector fields

$\delta(\vec{\lambda})$ — basic dimension of the adapted coframe $\vec{\lambda}$

$L_{F,V}(x,a)$ — maximum distortion of the unity ball at V_x under $\mathrm{d}F^a_x$

$L_F(x,a)$ — maximum distortion of the unity ball at T_xM under $\mathrm{d}F^a_x$

$l_{F,V}(x,a)$ — minimum distortion of the unity ball at V_x under $\mathrm{d}F^a_x$

$l_F(x,a)$ — minimum distortion of the unity ball at T_xM under $\mathrm{d}F^a_x$

$E_{F,V}(x,a)$ — V-eccentricity of F evaluated at $(x,a) \in M \times \mathbb{R}^q$

$E_F(x,a)$ — $\mathrm{T}M$-eccentricity of F evaluated at $(x,a) \in M \times \mathbb{R}^q$

$\wedge^k(M)^G$ — differential k-forms on M invariant under the action of the group G

$\wedge^k(\mathcal{F})$ — set of $\mathcal{F}$-basic k-forms

$H^*_B(\mathcal{F})$ — basic cohomology algebra of a foliation $\mathcal{F}$

CONTENTS

CHAPTER

1

INTRODUCTION

A contact manifold is an odd-dimensional manifold M together with a totally non-integrable distribution $\xi \subset \mathrm{T}M$ of maximal rank. In other words, M has dimension $2n+1$ and ξ has rank $2n$. The non-integrability condition means ξ supports non-degenerate closed forms and is, therefore, a symplectic bundle, hence the usual assertion that contact geometry is but an odd-dimensional analogue of symplectic geometry. We usually ask for the condition that ξ is coorientable, which implies the existence of a 1-form λ on M such that $\ker\lambda = \xi$. Given a coorientable contact manifold (M,ξ), a choice of defining 1-form λ for the contact structure ξ gives rise to a unique vector field R on M satisfying

$$\lambda(R) \equiv 1,$$
$$\iota_R \mathrm{d}\lambda \equiv 0.$$

The field R is known as the *Reeb field*, and its flow is called the *Reeb flow*, or *contact flow*, of M. Reeb (or Contact) Dynamics is the name given to the study of the Reeb flow defined by a coorientable contact structure.

The most famous question in the field of Contact Dynamics is undoubtedly the Weinstein Conjecture. It states that every Reeb flow defined on a closed contact manifold must have a closed orbit, i.e., an orbit homeomorphic to S^1. Much progress has been made in the quest for an answer since the question was first posed by Alan Weinstein in 1978 (WEINSTEIN, 1979). In particular, the conjecture is known to hold true for any three-dimensional contact manifold (TAUBES, 2007), for a class of contact structures known as *overtwisted* (BORMAN; ELIASHBERG; MURPHY, 2015; NIEDERKRÜGER, 2006), for contact flows supporting an invariant metric (BANYAGA, 1993), and for many other particular cases. In its full generality, however, the Weinstein Conjecture remains an open problem.

There are several ways to generalise contact manifolds. For instance, an *almost contact manifold* is an odd-dimensional manifold together with a distribution $\xi \subset \mathrm{T}M$

and a non-vanishing vector field R such that

(i) R is always transverse to ξ;

(ii) ξ admits an almost complex structure $J : \xi \to \xi$.

In this case, ξ plays the role of non-integrable distribution. However, it still has codimension 1, so the foliation associated with the structure (that is, the flow of R) is still one-dimensional.

In order to obtain generalisations with more than one "Reeb field", one possibility is to look at some particular cases of f-structures. An f-structure on a $(2n+1)$-dimensional manifold M, a concept introduced by Yano in (YANO, 1982), is a tensor field of type $(1,1)$ and rank $2n$ satisfying the equation $f^3 + f = 0$. The cases where $q = 0$ and $q = 1$ are simply almost complex and almost contact structures, respectively. An f-structure always comes with a splitting of the tangent bundle

$$\mathrm{T}M = \mathrm{Im}(f) \oplus \ker(f).$$

When there are on M vector fields $R_1, \cdots R_q$ spanning $\ker(f)$ and 1-forms $\lambda_1, \cdots, \lambda_q$ such that

(i) $\lambda_i(R_j) = \delta_{ij}$;

(ii) $\lambda_i \circ f = 0$;

(iii) $f^2 = -\mathrm{id} + \sum_i R_i \otimes \lambda_i$,

we say the f-structure has a *paralellisable kernel* or *complemented frames*. In an f-structure with complemented frames, there is always an associated metric g such that

$$g(X,Y) = g(f(X), f(Y)) + \sum_{i=1}^{q} \lambda_i(X)\lambda_i(Y),$$

and the triple (M, f, g) is said to be an *metric f-structure*). Using one such metric, we define the *fundamental* (or *Sasakian*) *form* of the f-structure to be

$$\omega(X,Y) = g(X, f(Y)).$$

When it happens that $\mathrm{d}\lambda_1 = \cdots = \mathrm{d}\lambda_q = \omega$, the structure is called an *almost $\mathcal{S}$-structure* (or simply an *$\mathcal{S}$-structure*, depending on whether or not some other integrability conditions are satisfied (BLAIR, 1970; DI TERLIZZI, 2006)). In this case, the pair $(\mathrm{Im}(f), \omega)$ is a symplectic bundle over M, and the fields R_i play the role of Reeb fields. On an $\mathcal{S}$-manifold the fields R_i are pairwise commutative (CABRERIZO; FERNÁNDEZ; FERNÁNDEZ, 1990, Corollary 2.4), and therefore define a foliation on M, which is analogue to the Reeb flow.

A similar but simpler notion is due to Bolle. In his work (BOLLE, 1996) he considers the C_p*-condition*, a property of submanifolds of a symplectic manifold (M,ω) that generalises the idea of a contact hypersurface. More specifically, if $\dim M = 2n$, then for $1 \leq p \leq n$, we say a submanifold S satisfies the C_p-condition if

(i) S is a compact co-isotropic submanifold of codimension p. In other words, $\ker(\omega|_S)$ is a vector bundle of constant rank p over S;

(ii) There are 1-forms $\lambda_1, \cdots, \lambda_p$ on S such that $\mathrm{d}\lambda_1 = \cdots = \mathrm{d}\lambda_q = \omega|_S$;

(iii) the application $X \mapsto (\lambda_1(X), \cdots, \lambda_p(X))$ is a bundle isomorphism between $\ker(\omega|_S)$ and $\mathbb{R}^p$.

A compact almost $\mathcal{S}$-manifold M can also be interpreted as a submanifold satisfying a C_p-condition if one regards it as the zero section in its symplectisation (see Appendix B). The converse is usually not true since submanifolds satisfying the C_p-condition carry much less structure.

In this work, we deal with geometrical structures called q*-contact structures* (cf. Definition 1), first defined by Almeida in (ALMEIDA, 2018). While they bear many similarities to the structures defined by Bolle and to almost $\mathcal{S}$-structures, there's a crucial difference in that the derivatives of the defining forms need not be all the same, but only *share the same kernel.* The notion of q-contact structure seeks to encompass only the absolute minimum requirements necessary to obtain a higher dimensional analogue to the Reeb flow. Basically, a such structure consists of a collection of q linearly independent 1-forms $\{\lambda_1, \cdots, \lambda_q\}$, which we call an *adapted coframe*, with the property that all the derivatives $d\lambda_i$ share the same kernel, and are non-degenerate on the intersection $\xi := \cap_i \ker \lambda_i$. In such a setup, there is a unique global frame $\{R_1, \cdots, R_q\}$ of pairwise commutative fields satisfying $\lambda_i(R_j)$, which we call the Reeb fields for the structure. The underlying foliation of the bundle spanned by the Reeb fields is the *contact foliation*, the higher dimensional analogue to the Reeb flow we sought.

There are many examples of q-contact manifolds besides the already established notions that a q-contact structure seeks to generalise. Of course, every contact structure is a 1-contact structure. By taking products of contact manifolds one can construct 2-contact structures whose adapted coframe $\{\lambda_1, \lambda_2\}$ is such that $\mathrm{d}\lambda_1 \neq \mathrm{d}\lambda_2$ (cf. Example 5), hence such products can not be an almost $\mathcal{S}$-manifold or a C_2-condition submanifold of any larger symplectic manifold. We can construct further examples by looking at mapping tori and flat torus bundles over q-contact manifolds (cf. Examples 7 and 8). Examples of algebraic nature are provided by Almeida in his thesis (ALMEIDA, 2018), where algebraic contact structures displaying hyperbolic properties are the main concern. For instance, he shows the *Weyl chamber action* is of contact nature.

Here, we expand the basic theory of q-contact structures by providing several new examples, properties and constructions. We show that the mapping tori of a q-contact manifold is a $(q+1)$-contact manifold (cf. Example 8) and provide a partial converse to Almeida's "extension" construction (Example 7) in the form of the "reduction" procedure described in Theorem 2. Shortly, if the contact action $F : \mathbb{R}^q \to \mathsf{Diff}(M)$ has non-trivial kernel $\mathbb{Z}^l$ *and a dense leave*, then we can construct a manifold M_0 over which M is a principal $\mathbb{T}^l$-bundle. Moreover, M_0 carries a contact foliation $\mathcal{F}_0$ of which $\mathcal{F}$ is an extension, in the sense of Example 7.

We also investigate a novel object, the *characteristic foliation*, which have no analogue in the contact case, i.e., for $q = 1$. The characteristic foliation $\mathcal{C}_i$ of the 1-form λ_i consists of the integral foliation of the bundle $\mathsf{Span}\{R_1, \cdots, \widehat{R}_i, \cdots, R_q\}$, and it has the remarkable property that whenever N is a transversal to $\mathcal{C}_i$, the pair $(N, \lambda_i|_{\mathrm{T}N})$ is a contact manifold. The holonomy pseudogroup of $\mathcal{C}_i$ consists of contactomorphisms between these manifolds. These foliations play a crucial role in our first major result, Theorem 4, which is a classification result asserting that every closed manifold supporting a q-dimensional uniform contact foliation is a fibration over the torus $\mathbb{T}^{q-1}$. Here, *uniform* means all the 1-forms comprising the adapted coframe $\vec{\lambda}$ have the same exterior derivatives (this is the case for almost $\mathcal{S}$-manifolds and C_p-condition submanifolds, for instance). This theorem is in line with already known results for other contact-like structures (cf. (GOERTSCHES; LOIUDICE, 2020a, Theorem 4.4)).

Since the Weinstein conjecture is such a noteworthy part of the theory of Contact Dynamics, it is only natural that we ask ourselves how the problem of the existence of closed orbits translates to a set of general contact foliations. This is the focus of this work. First, we note that the q-dimensional contact foliation associated with a q-contact structure is the underlying foliation of a smooth, locally free action of the Euclidean plane $\mathbb{R}^q$, whose infinitesimal generators are the Reeb fields. Consequently, the topological type of any leaf is the quotient of $\mathbb{R}^q$ by the isotropy subgroup of the corresponding orbit. This, in turn, implies that each leaf must be homeomorphic to a cylinder

$$\mathbb{R}^{q-l} \times \mathbb{T}^l,$$

for some $0 \leq l \leq q$. In the 1-dimensional (contact) case, this means each orbit must either be a line $(l = 0)$ or a circle $(l = 1)$ and what Weinstein conjectured was that in a closed manifold, one must have $l = 1$ for at least one orbit. In the general case, we have at least two possibilities for a generalised Weinstein conjecture: one can ask whether $l \neq 0$, for at least one leaf, or $l = q$ for at least one leaf. The first case simply says that a contact foliation on a closed manifold is not a foliation by planes. In contrast, in the second situation, we ask for the existence of a closed leaf. In this work, we propose and investigate two novel conjectures, both generalising the classical Weinstein conjecture (**WC**):

The weak generalised Weinstein conjecture (**WGWC**)**.** Every contact foliation on a closed manifold has a leaf homeomorphic to $\mathbb{T}^l \times \mathbb{R}^{q-l}$, for some $l \geq 1$.

The strong generalised Weinstein conjecture (**SGWC**)**.** Every contact foliation on a closed manifold has a leaf homeomorphic to $\mathbb{T}^q$.

The first thing that comes to mind is whether or not the conjectures are really different. For instance, it is clear that the **WC** and the **SGWC** do not allow the existence of non-trivial minimal contact foliations. However, one could, *a priori*, have cylindrical leaves whose closure is the entire ambient manifold. Our first significant result (a consequence of Proposition 2 and Theorem 3) states precisely that this can not happen. In other words, neither of the Weinstein conjectures allows the existence of non-trivial minimal contact foliations.

Notice that, for now, there's no guarantee that the conjectures are not, in reality, the same. In fact, for uniform contact foliations, it will follow from Theorem 4 that the **WGWC** is equivalent to the **WC**.

Our main results, Theorems A, B and C, concern the existence of closed leaves for contact foliations satisfying additional geometric or dynamical properties, as well as the classification of such foliations.

Regarding the validity of our conjectures, we give partial affirmative answers to the **SGWC** in at least two cases: when the contact foliation is partially hyperbolic [1] and when the contact foliation is *quasiconformal*. The latter is a class of contact foliations whose holonomies' distortion of transverse sections is bounded (in a sense to be made precise in Definition 31). Riemannian and conformal contact foliations belong to such class, as well as contact actions supporting an invariant metric. The latter condition is the strongest of them all, and foliations satisfying it are called *isometric*. This is the first class we investigate, before attacking the more general case. The first of our main theorems reads as

Theorem A. Every isometric contact foliation satisfies the **SGWC**.

This is first shown in Theorem 15, where we take advantage of the regularity of leaf closures in Riemannian foliations to restrict the contact foliation to specific submanifolds, in a procedure that can be iterated until it ends in a closed leave. This technique ensures the existence of at least two closed leaves. We then make use of Morse Theory to improve

[1] Partially hyperbolic contact foliations include, for instance, all the contact foliations appearing in Almeida's work. See Section 2.3.1 for a detailed discussion

on this lower bound in Theorem 18, greatly generalising previously known constructions from the literature. More specifically, in (RUKIMBIRA, 1995), Rukumbira studies the class of K-contact manifolds, contact manifolds for which the Reeb field is Killing with respect to a contact metric. A such metric is a Riemannian metric g satisfying a couple of compatibility conditions with a tensor $f : TM \to TM$ of the form $f = J \oplus 0$, where J is an almost complex structure on the contact structure (in particular, there is an f-structure on M, of which g is the associated metric). Using the particular properties of f and g, Rukimbira defines a Morse function on the ambient manifold, which he uses to count closed orbits for the Reeb field.

The class of K-contact manifolds has a natural generalisation in the class of *metric f-K-contact manifolds*, which is a particular kind of f-structure and also an example of a uniform isometric q-contact structure. Goertsches and Loiudice, using properties of the tensor f, adapted the calculations of Rukimbira and developed a generalisation of his Morse-Bott function to the case of metric f-K-contact-structures.

Here, we further adapt the arguments of (RUKIMBIRA, 1995; GOERTSCHES; LOIUDICE, 2020b) by showing that they do not depend on the particular properties of the tensor f or the contact metric g but only on the fact that the infinitesimal generators of the contact action are Killing fields. In fact, if a Reeb field R_i is Killing for some metric tensor g with Levi-Civita connection ∇, then the mapping

$$X \mapsto \nabla_{R_i} X$$

is a $(1,1)$-tensor on M with some properties akin to those of a f-structure. Using such tensor, we can adapt some of the lemmas from Rukimbira and Goertsches-Loiudice to the case of isometric contact foliations, allowing us to relate the basic cohomology of the contact foliation to the existence of closed orbits. In particular, we show that any isometric contact foliation of codimension $2n$ on a closed manifold M has at least $n+1$ closed leaves (cf. Theorem 18). This lower bound depends on the properties of a coframe of infinitesimal generators for the contact action and the first Betti number. It will generally be greater than just $n+1$.

As remarked before, quasiconformal foliations generalise the concepts of Riemannian, isometric and conformal foliations. For general foliations, these four classes are all distinct. A surprising and remarkable discovery we made while working on this project is that, for contact foliations on compact manifolds, all these notions are actually equivalent. More precisely, *up to a choice of Riemannian metric tensor, every quasiconformal contact foliation on a compact manifold can be considered isometric.*

One of the most useful dynamical properties of an isometric action $F : \mathbb{R}^q \to M$ is the fact that its C^1-closure in the group of diffeomorphisms of M, which we call the *C^1-enveloping group* $\mathrm{E}^1_F = \overline{F(\mathbb{R}^q)}$, is compact. This condition implies a strong form of equicontinuity, dubbed here C^1-equicontinuity. With this nomenclature, our result can be stated as

Theorem B. Let $\mathcal{F}$ be a contact foliation on a compact manifold M. The following are equivalent.

(i) $\mathcal{F}$ is C^1-equicontinuous;

(ii) the C^1-enveloping group E^1_F is a torus;

(iii) $\mathcal{F}$ admits a bundle-like metric (i.e., $\mathcal{F}$ is a Riemannian foliation);

(iv) $\mathcal{F}$ admits an invariant metric;

(v) $\mathcal{F}$ is quasiconformal;

(vi) $\mathcal{F}$ admits an invariant conformal structure.

Theorem B, combined with Theorem 18, asserts that every quasiconformal contact foliation on a closed manifold has at least 2 distinct closed leaves (generally many more). This proves our main result in this work.

Theorem C. Every quasiconformal contact foliation on a closed manifold satisfies the SGWC.

In particular, our results show that the classical Weinstein Conjecture is satisfied by every conformal Reeb field, a result previously not known, to the best of the author's knowledge.

On the structure of this work

In Chapter 2, we introduce the notion of q-contact structure, its basic properties and the generalised Weinstein conjectures, as well as some examples. In chapter 3, we develop the fundamentals of contact foliations and discuss the topological properties of manifolds supporting uniform contact foliations.

Chapter 4 is devoted to study metrics and the property of C^1-equicontinuity. We investigate the geometrical and topological properties of manifolds supporting contact foliations with invariant metrics, obtaining results regarding their cohomology and the curvature of the invariant metric. We also prove that C^1-equicontinuous contact manifolds satisfy the **SGWC** and provide lower bounds on the number of closed orbits using the basic cohomology of the contact foliation.

Finally, in Chapter 5, we look at quasiconformal and conformal contact foliations and show how these two properties reduce to C^1-equicontinuity. We discuss some open problems in Chapter 6.

On the nomenclature

In his work, Almeida refers to q-contact structures as *generalised k-contact structures*. We decided to change the index letter from k to q in order to avoid confusion with the already established notion of a K-contact structure. Moreover, Almeida also employs the notion of *k-contact structure*, taken from (BOLLE, 1996; MONTANO, 2008; TOMASSINI; VEZZONI, 2008), which generalises the similar notion of submanifolds satisfying a C_k-condition. Here, we call such objects *uniform q-contact structures* (Definition 9). Our decision to rename these concepts was based, first, on the belief that our nomenclature is less cumbersome and more informative. Moreover, during our research efforts, we stumbled upon the work of Gaset et al. (GASET *et al.*, 2020), in which yet another object called a k-contact structure is defined. In the sense of Gaset et al., a k-contact structure is even more general than the objects present in this work since they do not require the kernels of the derivatives of the defining 1-forms to be all the same. In light of that, we felt it made little sense to call our structures "generalised," so we decided to abandon the adjective for good.

Besides the geometric structures from Gaset et al., uniform q-contact structures (that is, C_q-condition submanifolds) and f-structures as discussed above, there are several other geometric structures similar or related to the structures studied here. We would like to give special mention to the notions of Contact Pair (cf. (BANDE; HADJAR, 2005)); Multicontat Structures (cf. (VITAGLIANO, 2015)); Pluricontact and Polycontact Structures (cf. (APOSTOLOV *et al.*, 2018; VAN ERP, 2011)). The interested reader is referred to (ALMEIDA, 2018, Section 3.2.2) for a more detailed comparison between all these different concepts.

CHAPTER 2

CONTACT FOLIATIONS

2.1 Contact actions of the Euclidean space

Intuitively, a contact action is an action of $\mathbb{R}^q$ whose frame of infinitesimal co-generators consists of "contact forms", i.e. 1-forms whose kernels intersect to define a completely non-integrable bundle on which their derivatives are symplectic.

Definition 1 (*q-contact manifolds*)**.** Let n,q be positive integers and consider a $2n+q$ dimensional manifold M. A **q-contact structure** on M is a collection $\vec{\lambda} = (\lambda_1, ..., \lambda_q)$ of q linearly independent non-vanishing 1-forms λ_i, together with a splitting

$$\mathrm{T}M = \mathcal{R} \oplus \xi$$

of the tangent bundle, satisfying the following conditions:

(i) $\xi := \cap_i \ker \lambda_i$;

(ii) for every i, the restriction $\mathrm{d}\lambda_i|_\xi$ is non-degenerate. In other words, $(\xi, \mathrm{d}\lambda_i)$ is a *symplectic bundle* over M for every i;

(iii) for every i, one has $\ker \mathrm{d}\lambda_i = \mathcal{R}$.

A manifold endowed with such structure is called a **q-contact manifold** and denoted by $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$, or simply by M when the context permits. The bundles $\mathcal{R}$ and ξ are called the **Reeb distribution** and **q-contact distribution,** respectively.

There are several properties implicit in this definition:

- $\ker \lambda_i$ is the bundle whose fibre at p is the kernel of the linear functional $\lambda_i|_p : \mathrm{T}_pM \to \mathbb{R}$. It is a classical result in Functional Analysis that if $L, L_1, \cdots, L_k$ are linear functionals such that $\cap_i \ker L_i \subset \ker L$, then L is a linear combination of the L_i. Hence linear independence of the λ_i means that at every point p, the intersection of these kernels is a $2n$-dimensional subspace of T_pM, that is, ξ is a vector bundle of constant rank $2n$.

- The kernel of a 2-form ω is defined as the set of fields X such that the contraction $\iota_X\omega := \omega(X,\cdot)$ is identically zero. Thus, condition (ii), together with the splitting of $\mathrm{T}M$, already gives $\ker \mathrm{d}\lambda_i \subset \mathcal{R}$, and condition (iii) could be restated as $\mathcal{R} \subset \ker \mathrm{d}\lambda_i$.
- Moreover, the linear independence of the 1-forms λ_i is equivalent to

$$\lambda := \lambda_1 \wedge \cdots \wedge \lambda_q \neq 0$$

at every point of M. In other words, λ is a volume form for the Reeb distribution $\mathcal{R}$.

Definition 2 (*Characteristic form*)**.** The q-form

$$\lambda := \lambda_1 \wedge \cdots \wedge \lambda_q$$

is called the **characteristic form** of the q-contact structure.

- As $\operatorname{rank}\xi \equiv 2n$, condition (ii) is equivalent to $\mathrm{d}\lambda_i^n|_\xi \neq 0$. From this we can conclude that $\mathrm{d}M_i := \lambda \wedge \mathrm{d}\lambda_i^n$ is a volume form on M for every i. In particular, M is orientable. We write

$$\mathrm{vol}_i(M) := \int_M \mathrm{d}M_i$$

for the volume of M concerning the volume form $\mathrm{d}M_i$.

Remark 1. The splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$ is as smooth as the least smooth of forms λ_i comprising the collection $\vec{\lambda}$, or, equivalently, as smooth as its characteristic form λ. *In this work, we will always assume the characteristic form λ to be at least of class C^1.*

Remark 2. In (YANO, 1982), Yano began the study of manifolds M supporting a $(1,1)$-tensor f satisfying the equation $f^3 + f = 0$. Such a structure induces a splitting $\mathrm{T}M = \ker f \oplus \operatorname{Im} f$, which is said to be an *f-structure with complemented frames* if $\ker f$ is parallelisable (GOLDBERG; YANO, 1970). Given an f-manifold with complemented frames, additional hypotheses on the kernel $\ker f$ give rise to different geometric structures such as $\mathcal{K}$-manifolds and $\mathcal{S}$-manifolds. These last two are particular cases of q-contact structures as considered here (cf. (BLAIR, 1970)).

On the other hand, given a q-contact manifold $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$, for every defining 1-form λ_i, the symplectic bundle $(\xi, d\lambda_i)$ admits a compatible almost complex structure J_i. If we define f_i to be equal J_i on ξ and identically zero on $\mathcal{R}$, then it is clear that $f_i^3 + f_i = 0$. So M supports (at least) as many different f-structures with complemented frames as there are different 2-forms $d\lambda_i$. The concept of an f-structure with complemented frames is more general than that of a q-contact structure.

As in the contact case, for q-contact structures the bundle ξ is maximally non-integrable:

Proposition 1. For any vector field X tangent to ξ there's another vector field $Y \in \Gamma(\xi)$ such that $[X,Y]$ is not a section of ξ.

Proof. Assume there is $X \in \Gamma(\xi)$ such that for any other vector field Y tangent to ξ, their Lie bracket $[X,Y]$ is also tangent to ξ. Then $\lambda_i(X) = \lambda_i(Y) = \lambda_i([X,Y]) = 0$ for every i, since ξ is the intersection of their kernels. From this, we obtain $\mathrm{d}\lambda_i(X,Y) = X\lambda_i(Y) - Y\lambda_i(X) - \lambda_i([X,Y]) = 0$ for every i, which can never happen as every $\mathrm{d}\lambda_i$ is non-degenerate on ξ.

□

Each 1-form λ_i has an associated tangent distribution $\mathfrak{C}_i(p) \subset \mathrm{T}_pM$, called the *characteristic distribution*:

Definition 3 (*Characteristic distribution*)**.** The **characteristic distribution** of λ_i is the distribution

$$\begin{aligned}\mathfrak{C}_i(p) &\coloneqq \{X_p \in \mathrm{T}_pM;\ \lambda_i(X_p) = 0 \text{ and } \iota_{X_p}\mathrm{d}\lambda_i = 0\}\\ &= \ker\lambda_i|_p \cap \mathcal{R}_p.\end{aligned}$$

Note that this distribution has a constant rank equal to $q-1$. Indeed, since $\mathcal{R}_p$ has dimension q and $\ker\lambda_i|_p$ is a hyperplane, the vector space $\mathfrak{C}_i(p) \subset \mathrm{T}_pM$ has dimension either q, in case $\mathcal{R}_p \subset \ker\lambda_i|_p$, or $q-1$ otherwise. Now, the fact that $\mathrm{d}M_i$ is a volume form for M, together with $\lambda_i|_\xi = 0$ implies $\lambda_i|_\mathcal{R} \neq 0$, hence $\mathfrak{C}_i(p)$ has dimension $q-1$. More than that, if the splitting $\mathcal{R} \oplus \xi$ is smooth, so are the characteristic distributions $\mathfrak{C}_i$. These are all integrable since $X, Y \in \mathfrak{C}_i$ imply that

$$\lambda_i([X,Y]) = X\lambda_i(Y) - Y\lambda_i(X) - \mathrm{d}\lambda_i(X,Y) = 0,$$

and, using Cartan's formula,

$$\iota_{[X,Y]}\mathrm{d}\lambda_i = \mathcal{L}_{[X,Y]}\lambda_i = \mathcal{L}_X\mathcal{L}_Y\lambda_i - \mathcal{L}_Y\mathcal{L}_X\lambda_i = \mathcal{L}_X 0 - \mathcal{L}_Y 0 = 0,$$

hence $\mathfrak{C}_i$ is involutive.

Definition 4 (*Characteristic Foliation*)**.** The underlying foliation $\mathcal{C}_i$ associated with the characteristic distribution $\mathfrak{C}_i$ is called the **characteristic foliation of** λ_i.

We proceed by defining a suitable global frame for the bundle $\mathcal{R}$

Proposition 2. There is a unique global frame $R_1, \cdots, R_q \in \Gamma(\mathcal{R})$ for $\mathcal{R}$ satisfying the relations $\lambda_i(R_j) = \delta_{ij}$.

Proof. As the 1-forms λ_i are all linearly independent, we conclude that

$$K_i := \bigcap_{j\neq i} \ker \lambda_j$$

is a bundle of rank $2n+1$ containing ξ, for every $i = 1, \cdots, q$. From the equality $\mathcal{R} \oplus \xi = \mathrm{T}M$, it follows that

$$\mathcal{R} \cap K_i \neq 0$$

and that

$$\dim(\mathcal{R} \cap K_i) = 1.$$

Thus, for each i there is only one vector field R_i tangent to $\mathcal{R}$ such that $\lambda_i(R_i) = 1$ and $\lambda_j(R_i) = 0$ for every j other than i.

It is then straightforward to check that the R_i are everywhere linearly independent, for if there was a $p \in M$ such that $R_i|_p = \sum_{j\neq i} t_j R_j|_p$ then we would have

$$1 = \lambda_i|_p(R_i|_p) = \sum_{j\neq i} t_j \lambda_i|_p(R_j|_p) = 0.$$

□

If the splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$ is smooth then so are the vector fields R_i. It follows immediately from their construction that $\mathcal{R} = \mathrm{Span}\{R_1, \cdots, R_q\}$.

Definition 5 (*Reeb vector fields*)**.** The (unique) linearly independent vector fields $\{R_i\} \subset \Gamma(\mathcal{R})$ spanning $\mathcal{R}$ and satisfying

$$\lambda_i(R_j) = \delta_{ij} \quad \text{and} \quad \mathrm{d}\lambda_i(R_j, \cdot) = 0$$

are called the **Reeb vector fields** of the q-contact structure.

We remarked earlier that every q-contact manifold admits volume forms and is therefore orientable. The existence of the Reeb fields immediately implies another topological obstruction.

Proposition 3. If M is a q-contact manifold, then its Euler characteristic is zero.

Proof. Each field R_i is non-vanishing, and therefore its index at every point is zero. Applying the Poincaré-Hopf Theorem to R_i gets us $\chi(M) = 0$.

□

To further justify calling the fields from Proposition 2 Reeb fields, note that they satisfy

$$\mathcal{L}_{R_j}\lambda_i = \mathrm{d}(\iota_{R_j}\lambda_i) + \iota_{R_j}\mathrm{d}\lambda_i = 0,$$

meaning every Reeb flow preserves every 1-form λ_i and hence the bundle ξ. One see as easily that $\mathcal{L}_{R_j} \mathrm{d}\lambda_i = 0$ as well, so that the Reeb flows actually preserve the symplectic bundles $(\xi, \mathrm{d}\lambda_i)$ over M.

Perhaps the most crucial property of the spanning set $\{R_i\}$ is that these fields commute with one another.

Proposition 4. $[R_i, R_j] = 0$ for every i, j.

Proof. Since $\mathrm{d}M_l = \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n$ is a volume form, it's sufficient to show that

$$\iota_{[R_i,R_j]} \mathrm{d}M_l = 0.$$

First, note that $\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j^n = 0$ for any i and j, for this is a $(2n+2)$-form, and any choice of $2n+2$ linearly independent vectors on $\mathrm{T}M$ would have at least one vector belonging to $\mathcal{R}$, on which $\mathrm{d}\lambda_l$ is zero. Thus $\lambda_i \wedge \mathrm{d}\lambda_j^n$ is a closed $(2n+1)$-form and, for any choices $1 \le i_1 < \cdots i_s \le q$, we have $\mathrm{d}(\lambda_{i_1} \wedge \cdots \wedge \lambda_{i_s} \wedge \mathrm{d}\lambda_j^n) = 0$. From this we calculate, using $\lambda_i(R_j) = \delta_{ij}$:

$$\mathrm{d}(\iota_{R_j} \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n) = \mathrm{d}(\lambda_1 \wedge \cdots \wedge \lambda_{j-1} \wedge \lambda_{j+1} \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n) = 0$$

and similarly

$$\mathrm{d}(\iota_{R_j}(\iota_{R_i} \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n)) = 0.$$

From here we use the relation $\iota_{[R_i,R_j]} = [\iota_{R_i}, \mathcal{L}_{R_j}]$ together with Cartan's Formula to get

$$\begin{aligned}\iota_{[R_i,R_j]}(\lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n) &= [\iota_{R_i}, \mathcal{L}_{R_j}](\lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n) \\ &= [\iota_{R_i}, \quad \mathrm{d} \circ (\iota_{R_j}) + (\iota_{R_j}) \circ \mathrm{d}](\lambda_1 \wedge \cdots \wedge \lambda_q \wedge \mathrm{d}\lambda_l^n) = 0.\end{aligned}$$

□

Thus, the bundle $\mathcal{R}$ is parallelisable, hence trivial, while the bundle ξ is symplectic, for each $\mathrm{d}\lambda_i$ is a non-degenerate closed 2-form. This gives us yet another obstruction to the existence of a q-contact structure:

Proposition 5. If a manifold M of dimension admits a q-contact structure, then the structure group of $\mathrm{T}M$ reduces to $\{\mathrm{id}_q\} \times \mathrm{Sp}(2n, \mathbb{R})$, where

$$\mathrm{Sp}(2n,\mathbb{R}) := \left\{ A \in \mathrm{Sl}(2n,\mathbb{R});\ A^t \begin{bmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{bmatrix} A = \begin{bmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{bmatrix} \right\}$$

is the symplectic group.

As all the Reeb vector fields commute, there is a well-defined action of $\mathbb{R}^q$ on M:

Definition 6 (*q-contact action*)**.** Let M be a q-contact manifold with Reeb vector fields $R_1, \cdots, R_q$, and denote by $\exp(tR_i)$ the flow of R_i. We can define an action

$$\begin{aligned} F : \mathbb{R}^q &\longrightarrow \mathrm{Diff}(M) \\ a &\longmapsto F^a : M \to M, \end{aligned}$$

by setting

$$F^a := \exp(t_1 R_1) \circ \cdots \circ \exp(t_q R_q),$$

for $a = (t_1, \cdots, t_q) \in \mathbb{R}^q$. This will be called the **q-contact action on M associated to $\vec{\lambda}$**, or simply the **contact action**, context permitting.

Remark 3.

(i) We will sometimes write the action as a mapping $F : \mathbb{R}^q \times M \to M$ instead of a homomorphism from $(\mathbb{R}^q, +)$ to the group of diffeomorphisms of M. In such case, we write $F(a,x) := F^a(x)$.

(ii) If the splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$ is smooth then so is the Reeb action F. The 1-forms λ_i are non-vanishing, thus the vector fields R_i are non-singular. This means the Reeb action of $\mathbb{R}^q$ is locally free.

(iii) As we pointed out before, for any pair $(i, j) \in \{1, \cdots q\}^2$ we have

$$\mathcal{L}_{R_i} \lambda_j = \mathcal{L}_{R_i} \mathrm{d}\lambda_j = 0,$$

so that the flow of every Reeb field preserves every other flow, as well as every symplectic bundle $(\xi, \mathrm{d}\lambda_j)$. This means the range of the Reeb action homomorphism F is actually the much smaller set

$$\mathrm{Symp}(\xi, \vec{\lambda}) := \left(\bigcap_{i=1}^{q} \mathrm{Symp}(\xi, \mathrm{d}\lambda_i) \right) \bigcap \left(\bigcap_{i=1}^{q} \mathrm{Diff}(M, \varphi_i) \right),$$

where $\mathrm{Symp}(\xi, \mathrm{d}\lambda_i)$ is the group of bundle isomorphisms $\psi : (M, \xi) \to (M, \xi)$ such that $\psi^* \mathrm{d}\lambda_i = \mathrm{d}\lambda_i$, and $\mathrm{Diff}(M, \varphi)$ is the group of diffeomorphisms of M preserving the flowlines of φ.

The commutativity of the R_i also implies that $\mathcal{R}$ is involutive and, therefore, integrable. We denote by $\mathcal{F}$ the underlying foliation of F, that is, the q-dimensional foliation of M whose leaves are the orbits of F.

Definition 7 (*Contact foliation*)**.** The foliation $\mathcal{F}$ underlying a contact action of $\mathbb{R}^q$ is called a **contact foliation.**

Note that $\mathrm{T}\mathcal{F} = \mathcal{R}$, that is, $\mathcal{R}_p$ is exactly the tangent space of $\mathcal{F}(p)$, the leaf (orbit of F) through p. This means the characteristic form λ is a leaf-wise volume form to $\mathcal{F}$, i.e., it restricts to a volume form on each leaf $\mathcal{F}(x)$. Moreover, we can consider a Riemannian metric tensor on the Reeb distribution, making the global frame $\{R_i\}$ into an orthonormal frame. Namely, the metric

$$g^{\tau} = \sum_{i=1}^{q} \lambda_i \otimes \lambda_i. \tag{2.1}$$

With respect to this metric, the form λ satisfies

$$\lambda(X_1, \cdots, X_q) = \det\{g^{\tau}(X_i, R_j)\}_{ij},$$

so that λ is also the characteristic form of the contact foliation $\mathcal{F}$ with respect to the metric g^{τ}, in the classic sense (cf. (TONDEUR, 1997, Chapter 4)).

Definition 8 (*Adapted metric*)**.** The Riemannian metric g^{τ} defined as in Equation (2.1) is called the **adapted metric** for the coframe $\vec{\lambda}$.

Furthermore, since each characteristic distribution $\mathfrak{C}_i$ satisfies $\mathfrak{C}_i(p) \subset \mathcal{R}_p$ for every $p \in M$, it follows that each leaf $\mathcal{C}_i(p)$ is a submanifold of the leaf $\mathcal{F}(p)$, that is, each leaf of $\mathcal{F}$ is itself a foliated manifold, with q codimension 1 foliations $\mathcal{C}_i$.

Each leaf $\mathcal{F}(p)$ has a canonical parametrisation

$$\begin{aligned} F_p : \mathbb{R}^q &\to M \\ (t_1, \cdots, t_q) &\mapsto F(t_1, \cdots, t_q, p). \end{aligned}$$

The Reeb vector fields R_i satisfy

$$\mathrm{d}F_p|_{e_i}(0) = R_i|_p,$$

where $\{e_1, \ldots, e_q\}$ is the canonical basis of $\mathbb{R}^q$, and are therefore called the **frame of infinitesimal generators of F.** Any ordered set of 1-forms $\{\alpha_1, \cdot \cdots, \alpha_q\}$ such that $\alpha_i(R_j) = \delta_{ij}$ is called a **frame of infinitesimal cogenerators of** F or a **coframe adapted to F.** The set $\{\lambda_i\}$ is a natural choice of adapted coframe.

2.2 Examples

Example 1 (Contact manifolds)**.** Every contact manifold (M, ξ) with a transversely orientable contact distribution ξ is a 1-contact manifold once a defining form λ is chosen. Here $\mathcal{R}$ is the span of the Reeb vector field, and the action F is the one induced by the Reeb field's flow. We will refer to this as the **contact case.**

Example 2 (Structures on $\mathbb{R}^{2n+q}$)**.** There's a simple q-contact structure on $\mathbb{R}^{2n+q}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n, z_1, \cdots, z_q)$, given by the 1-forms

$$\lambda_i := dz_i + \sum_{j=1}^{n} x_j \, \mathrm{d}y_j.$$

In this case, using the identification $T_x\mathbb{R}^{2n+q} = \mathbb{R}^{2n+q}$, the Reeb vector fields are $R_i = \partial_{z_i}$; therefore $\mathcal{R} = \mathrm{Span}(\partial_{z_1}, \cdots, \partial_{z_q})$. The action F is simply a translation

$$F(t_1, \cdots, t_q)x = x + (0, \cdots, 0, t_1, \cdots, t_q),$$

and the contact foliation $\mathcal{F}$ consists of all planes parallel to $\{0\} \times \mathbb{R}^q$. The q-contact distribution is

$$\xi = \mathrm{Span}(\partial_{x_1}, Y_1, \cdots, \partial_{x_n}, Y_n),$$

where $Y_i := \partial_{y_i} - x_i \sum_j \partial_{z_j}$, and the volume form $\mathrm{d}M_i$ is the canonical volume form of $\mathbb{R}^{2n+q}$, for every i.

More generally, if $\omega = \mathrm{d}\alpha$ is an exact non-degenerate form on $\mathbb{R}^{2n}$, then we regard α as an 1-form in $\mathbb{R}^{2n+q}$ by means of the identification $\mathbb{R}^{2n} \approx \{(x, y, 0)\} \subset \mathbb{R}^{2n+q}$, setting it to be zero elsewhere. Then all of the above holds for

$$\lambda_i := \mathrm{d}z_i + \alpha.$$

In the same way, given a collection of such non-degenerate forms $\omega_i = d\alpha_i$, then for any choice $\{\alpha_{j_1}, \cdots, \alpha_{j_q}\} \in \{\alpha_1, \cdots, \alpha_l\}^q$, the 1-forms

$$\lambda_i := \mathrm{d}z_i + \alpha_{j_i}$$

comprise an adapted coframe for a q-contact structure on $\mathbb{R}^{2n+q}$. In particular, the $\mathrm{d}\lambda_i = \omega_{j_i}$ need not be equal.

Definition 9 (*Uniform q-contact structures*)**.** A q-contact structure with adapted coframe $\vec{\lambda} = (\lambda_1, \cdots, \lambda_q)$ is called **uniform** if all the exterior derivatives $\mathrm{d}\lambda_i$ are the same, that is,

$$\mathrm{d}\lambda_i = \mathrm{d}\lambda_j \quad \forall i, j.$$

Both $\mathcal{K}$-structures and $\mathcal{S}$-structures (cf. Remark 2) are particular cases of uniform q-contact structures. Uniform q-contact structures were also studied by Bolle in (BOLLE, 1996). He defines them as codimension p submanifolds of a symplectic manifold (W, ω) satisfying what he calls "C_p-condition". The concepts are the same because every compact uniform q-contact manifold is a codimension q submanifold of its *symplectisation* satisfying the C_q-condition. The symplectisation of $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ consists of the product $W := M \times \mathbb{R}^q$.

Given coordinates $(x, t_1, \cdots, t_q)$ on W, if we equip it with the non-degenerate 2-form

$$\omega := \sum_{i=1}^{q} (dt_i \wedge \lambda_i + t_i d\lambda_1),$$

then the pair (W, ω) is symplectic, and $M \times 0$ satisfies the C_q-condition (see Appendix B).

At this point, one might ask if *every* contact foliation supports a uniform adapted coframe. The answer is no. The algebraic contact foliations constructed by Almeida in his doctoral dissertation do not admit any uniform underlying coframe. See (ALMEIDA, 2018, Proposition 3.3.5) for the complete construction.

Example 3 (Weyl Chamber Actions)**.** Suppose G is a semi-simple Lie group and $\mathfrak{a}$ is a Cartan subspace with centraliser $\mathfrak{a} \oplus \mathfrak{k}$. Let K be a compact subgroup of G associated with $\mathfrak{k}$ and Γ a uniform lattice on G acting freely on G/K. The exponentials of elements in $\mathfrak{a}$ act on $M := \Gamma \setminus (G/K)$ by right translation. This is called the **Weyl Chamber Action**. Almeida proved (ALMEIDA, 2018, Theorem 1.0.2) that this is equivalent to a q-contact action of $\mathbb{R}^q$ on M.

The following "product-like" constructions give examples of closed q-contact manifolds other than closed contact manifolds and Weyl Chamber Actions.

Example 4 (Products of contact manifolds)**.** Suppose (M_1, α_1) and (M_2, α_2) are contact manifolds, and let

$$\lambda_1 := \pi_1^* \alpha_1 - \pi_2^* \alpha_2$$
$$\lambda_2 := \pi_1^* \alpha_1 + \pi_2^* \alpha_2$$

be 1-forms defined on the product $M := M_1 \times M_2$ by means of the canonical projections $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$. It is clear that λ_1 and λ_2 are linearly independent. Let X_i be the Reeb vector field of α_i, and $\xi_i = \ker \alpha_i$ the associated contact structure. As

$$\ker \lambda_i = \pi_1^* \xi_1 \cap \pi_2^* \xi_2 \oplus \operatorname{Span}\{\pi_1^* X_1 \pm \pi_2^* X_2\},$$

it is immediate that

$$\xi := \ker \lambda_1 \cap \ker \lambda_2 = \pi_1^* \xi_1 \cap \pi_2^* \xi_2.$$

Moreover, if we let

$$R_1 := \frac{1}{2}(\pi_1^* X_1 - \pi_2^* X_2),$$
$$R_2 := \frac{1}{2}(\pi_1^* X_1 + \pi_2^* X_2)$$

then $\lambda_i(R_j) = \delta_{ij}$ and it is straightforward to check that $\mathcal{R} := \mathrm{Span}\{R_1, R_2\}$ satisfies

$$\mathrm{T}M = \mathrm{T}M_1 \oplus \mathrm{T}M_2 = \mathcal{R} \oplus \xi.$$

It remains to show the derivatives $\mathrm{d}\lambda_i$ are non-degenerate on ξ. Given Y tangent to ξ, let $p = (p_1, p_2)$ be a point in M, and write $Y = Y_1 \oplus Y_2$, with $Y_i \in \Gamma(\xi_i) \subset \Gamma(M_i)$. The equality $\mathrm{d}\lambda_i|_p(Y_p, \cdot) \equiv 0$ implies

$$\mathrm{d}\alpha_1|_{p_1}(Y_1|_{p_1}, \cdot) = \pm\,\mathrm{d}\alpha_2|_{p_2}(Y_2|_{p_2}, \cdot),$$

or, in more explicit terms, that for any choice of $Z_i \in \mathrm{T}_{p_i}M_i$, we have

$$\mathrm{d}\alpha_1|_{p_1}(Y_1|_{p_1}, Z_1) = \pm\,\mathrm{d}\alpha_2|_{p_2}(Y_2|_{p_2}, Z_2).$$

This can only happen when $Y_1|_{p_1} = Y_2|_{p_2} = 0$, for both α_1 and α_2 are non-degenerate. Hence

$$\iota_Y\,\mathrm{d}\lambda_i \equiv 0 \iff Y = 0.$$

Furthermore, the $\mathrm{d}\lambda_i$ are non-degenerate on ξ, as we wished. Note that, because the fields $\pi_1^*X_1$ and $\pi_2^*X_2$ are p_1-related to X_1 and 0, respectively, it follows that $[\pi_1^*X_1, pi_2^*X_2] = [X_1, 0] = 0$, hence $[R_1, R_2] = 0$, and by uniqueness we conclude that R_1, R_2 are indeed the Reeb fields of the contact action. The Reeb distribution $\mathcal{R}$ can be seen as the span of either $\{R_1, R_2\}$ or $\{\pi_1^*X_1, \pi_2^*X_2\}$, hence its integral submanifolds are exactly the products of flowlines of M_1 and M_2.

The last example can be generalised to higher dimensional contact structures as long as both have the same dimension. All the computations involved are virtually the same.

Example 5 (Products of q-contact structures of same dimension)**.** Let $(M_1, \{\alpha_1, \cdots, \alpha_q\}, \mathcal{R}_1 \oplus \xi_1)$ and $(M_2, \{\beta_1, \cdots, \beta_q\}, \mathcal{R}_2 \oplus \xi_2)$ be q-contact manifolds. Then $M_1 \times M_2$ admits a $2q$-contact structure with adapted coframe given by the forms

$$\lambda_i := \pi_1^*\alpha_i + \pi_2^*\beta_i,$$
$$\eta_i := \pi_1^*\alpha_i - \pi_2^*\beta_i,$$

for $i = 1, \cdots, q$, and splitting $\mathrm{T}(M_1 \times M_2) = (\pi_1^*\mathcal{R}_1 \oplus \pi_2^*\mathcal{R}_2) \oplus (\pi_1^*\xi_1 \oplus \pi_2^*\xi_2)$. The leaves of the contact foliation are the products of the leaves in M_1 and M_2.

Example 6 (Products with flat Tori)**.** Let N be a q-contact manifold with adapted coframe $\{\lambda_i\}$ and Reeb fields R_i, and let $\mathbb{T}^l = \mathbb{R}^l/\mathbb{Z}^l$ be the l-torus. Then $M := N \times$

$\mathbb{T}^l$ admits a $(q+l)$-contact structure. To construct one, we begin by considering the canonical projections

$$\pi : M \to N \text{ and } \rho : M \to \mathbb{T}^l,$$

and the fields ∂_i on $\mathbb{T}^l$ descending from the coordinate vector fields on $\mathbb{R}^l$, i.e., for $\pi : \mathbb{R}^l \to \mathbb{T}^l$ we have $\partial_{x_i} = \pi^* \partial_i$. Write α_i to denote the 1-form on $\mathbb{T}^l$ which is π-related to $\mathrm{d}x_i$, so that $\alpha_i(\partial_j) = \delta_{ij}$ and $\mathrm{T}\mathbb{T}^l = \mathrm{Span}\{\partial_i\}$. By the naturality of the pullback operation, we have $\mathrm{d}\alpha_i = 0$.

We define on M, for any choice of indices $\{j_1, \cdots, j_l\} \subset \{1, \cdots, q\}^l$, a collection of $l+q$ differential 1-forms by setting

$$\begin{aligned} \eta_i &:= \pi^* \lambda_i, \qquad && \text{for } 1 \le i \le q; \\ \eta_{q+i} &:= \rho^* \alpha_i + \pi^* \lambda_{j_i}, \qquad && \text{for } 1 \le i \le l. \end{aligned}$$

Note that, for $i \in \{1, \cdots, q\}$, we have $\mathrm{d}\eta_i = \pi^* \mathrm{d}\lambda_i$ and

$$\ker \eta_i = (\pi^* \ker \lambda_i) \bigcup \left(\rho^* \mathrm{T}\mathbb{T}^l \right). \tag{2.2}$$

On the other hand, for $i \in \{1, \cdots, l\}$, we have $\mathrm{d}\eta_{q+i} = \pi^* \mathrm{d}\lambda_{j_i}$ and

$$\ker \eta_{q+i} = \left(\rho^* \mathrm{Span}\{\partial_1, \cdots, \widehat{\partial_i}, \cdots, \partial_l\} \right) \bigcup \left(\pi^* \ker \lambda_{j_i} \right) \bigcup \left(\mathrm{Span}\{\rho^* \partial_i - \pi^* R_{j_i}\} \right). \tag{2.3}$$

A straightforward calculation using Equation (2.2) shows that $\cap_{i=1}^{q} \ker \eta_i = (\pi^* \xi) \cup \left(\rho^* \mathrm{T}\mathbb{T}^l \right)$. Let us denote by $\vec{0}_{\mathbb{T}_l}$ the zero section of $\mathrm{T}\mathbb{T}^l$, so that

$$\bigcap_i \mathrm{Span}\{\partial_1, \cdots, \widehat{\partial_i}, \cdots, \partial_l\} = \{\vec{0}_{\mathbb{T}^l}\},$$

and therefore, from (2.3):

$$\bigcap_{i=1}^{l} \ker \eta_{q+i} = \left(\rho^* \{\vec{0}_{\mathbb{T}^l}\} \right) \cap \left(\bigcap_{i}^{l} \left(\pi^* \ker \lambda_{j_i} \right) \cup \left(\mathrm{Span}\{\rho^* \partial_i - \pi^* R_{j_i}\} \right) \right) \supset \rho^* \{\vec{0}_{\mathbb{T}^l}\} \cap \pi^* \xi.$$

Thus

$$\begin{aligned} \widetilde{\xi} := \bigcap_{i=1}^{q+l} \ker \eta_i &= \left(\bigcap_{i=1}^{q} \ker \eta_i \right) \cap \left(\bigcap_{i=1}^{l} \ker \eta_{q+i} \right) \\ &= \left((\pi^* \xi) \cup \left(\rho^* \mathrm{T}\mathbb{T}^l \right) \right) \cap \left(\bigcap_{i=1}^{l} \pi^* \ker \lambda_{q+i} \right) \\ &= \rho^* \{\vec{0}_{\mathbb{T}^l}\} \cap \pi^* \xi. \end{aligned}$$

In other words, the fibre of $\widetilde{\xi}$ at a point $p = (t, n) \in M$ is the set

$$\widetilde{\xi}_p = \{(0, X); X \in \xi_n\},$$

that is, $\widetilde{\xi} = \{0\} \oplus \xi$. It is clear that $\mathrm{d}\eta_i$ is non-degenerate on this intersection for every $1 \le i \le q+l$, and its kernel $\widetilde{\mathcal{R}}$ is is spawned by $\{\rho^*\partial_i\} \cup \{\pi^* R_j\}$, so that

$$\mathrm{T}M = \rho^*\mathbb{T}^l \oplus \pi^* N = \rho^*\mathbb{T}^l \oplus \pi^*\mathcal{R} \oplus \pi^*\xi = \widetilde{\mathcal{R}} \oplus \widetilde{\xi}.$$

This is to say $(M, \vec{\eta}, \widetilde{\mathcal{R}} \oplus \widetilde{\xi})$ is a $(q+l)$-contact manifold. Here the Reeb vector fields are

$$\begin{aligned} \widetilde{R}_i &:= (0, R_i), && \text{for } 1 \le i \le q; \\ \widetilde{R}_{q+i} &:= \frac{1}{2}(\partial_i, R_{j_i}), && \text{for } 1 \le i \le l. \end{aligned}$$

The contact foliation $\widetilde{\mathcal{F}}$ induced by this structure is just the product of the leaves of $\mathcal{F}$ with tori $\mathbb{T}^l$.

Example 7 (Flat $\mathbb{T}^l$-bundles over q-contact manifolds)**.** More generally, if M is a q-contact manifold and $E \xrightarrow{\pi} M$ is a principal $\mathbb{T}^l$-bundle supporting a flat connection, then E admits a $(q+l)$-contact structure. The construction, due to Almeida, is as follows. Denote, as usual, the q-contact structure on M by $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$. The connection induces a splitting $\mathrm{T}E = H \oplus V$ into horizontal and vertical bundles. If we let $X_1, \cdots, X_q$ be the commutative vector fields generating the $\mathbb{T}^l$-action on each fibre, then $V = \mathrm{Span}\{X_1, \cdots X_l\}$. Denote by α_i the 1-form defined on V by $\alpha_i(X_j) = \delta_{ij}$. Using the splitting $H \oplus V$, we extend the 1-forms α_i from the fibres to all of M by setting

$$\alpha_i(X_j) = \delta_{ij} \text{ and } \alpha_i|_H = 0.$$

We claim these 1-forms satisfy $\iota_{X_j}\mathrm{d}\alpha_i = 0$, for any i, j. Indeed, $[X_j, X_l] = 0$ for any choice of j, l. For any horizontal vector field W, the section $[X_j, W]$ is also horizontal (NESTEROV, 2000, Lemma 3.12), hence

$$\begin{aligned} \mathrm{d}\alpha_i(X_j, X_l) &= X_j\alpha_i(X_l) - X_l\alpha_i(X_j) - \alpha_i([X_j, X_l]) = 0 + 0 + 0 = 0, \\ \mathrm{d}\alpha_i(X_j, W) &= X_j\alpha_i(W) - W\alpha_i(X_j) - \alpha_i([X_j, W]) = 0 + 0 + 0 = 0. \end{aligned}$$

We also have $\mathrm{d}\alpha_i|_H = 0$, because H is involutive due to the flatness of the connection (TONDEUR, 1997, Lemma 3.1). Thus, for any $W_1, W_2 \in H$,

$$\mathrm{d}\alpha_i(W_1, W_2) = W_1\alpha_i(W_2) - W_2\alpha_i(W_1) - \alpha_i([W_1, W_2]) = 0 + 0 + 0 = 0.$$

Each fibre H_p of H is identifiable with the fibre $\mathrm{T}_{\pi(p)}M = \mathcal{R}_{\pi(p)} \oplus \xi_{\pi(p)}$. Thus H has a splitting $H = \overline{\mathcal{R}} \oplus \widetilde{\xi}$, where $\xi \approx \widetilde{\xi}$. Write $\widetilde{\mathcal{R}} := V \oplus \overline{\mathcal{R}}$ so that we have

$$\mathrm{T}E = \widetilde{\mathcal{R}} \oplus \widetilde{\xi},$$

and note that, since $\mathrm{d}\alpha_i|_H = 0$, we have $\overline{\mathcal{R}} \subset \ker \mathrm{d}\alpha_i$. For any choice of $\{j_1, \cdots, j_l\} \subset \{1, \cdots, q\}^l$, we set

$$\begin{aligned} \eta_i &:= \pi^* \lambda_i, \qquad \text{for } 1 \leq i \leq q; \\ \eta_{q+i} &:= \alpha_i + \pi^* \lambda_{j_i}, \qquad \text{for } 1 \leq i \leq l. \end{aligned}$$

These forms are such that, for $i = 1, \cdots, q$

$$\ker \eta_i = \pi^* \xi = \widetilde{\xi} \oplus \widetilde{\mathcal{R}}$$

and for $i = 1, \cdots, l$

$$\ker \widetilde{\eta}_{q+i} = \mathrm{Span}\{X_j\}_{j \neq i} \oplus \mathrm{Span}\{X - \lambda_{j_i}(\pi_* X) X_i;\ X \in \Gamma(H)\} \supset \widetilde{\xi}$$

It then follows that

$$\begin{aligned} \bigcap_{i=1}^{q+l} \eta_i &= \left(\bigcap_{i=1}^{q} \ker \eta_i \right) \cap \left(\bigcap_{i=1}^{l} \ker(\eta_{q+i}) \right) \\ &= \left(\widetilde{\xi} \oplus \widetilde{\mathcal{R}} \right) \cap \left(\bigcap_{i=i}^{l} \mathrm{Span}\{X - \lambda_{j_i}(\pi_* X) X_i;\ X \in \Gamma(H)\} \right) \\ &= \widetilde{\xi}, \end{aligned}$$

and $(E, \vec{\eta}, \widetilde{\mathcal{R}} \oplus \widetilde{\xi})$ is therefore a $(l+q)$-contact structure. Since the bundle $\overline{\mathcal{R}} \subset H$ is diffeomorphic to $\mathcal{R}$, we can think of the Reeb vector fields on M as vector fields on H. Hence the Reeb vector fields of the η_i are

$$\begin{aligned} \widetilde{R}_i &:= R_i, \qquad \text{for } 1 \leq i \leq q; \\ \widetilde{R}_{q+i} &:= \frac{1}{2}(X_i + R_{j_i}), \qquad \text{for } 1 \leq i \leq l. \end{aligned}$$

Note that the restriction of the flat connection on $\mathrm{T}E$ to $\widetilde{\mathcal{R}}$ is again flat, providing us with two complementary integrable sub-bundles V and $\mathcal{R}$ of $\widetilde{\mathcal{R}}$. This means the leaves of $\widetilde{\mathcal{F}}$ are the product of the leaves of $\mathcal{F}$ with the torus $\mathbb{T}^l$.

Example 8 (Mapping tori)**.** Another example similar to the above comes from mapping tori. Given a q-contact manifold$(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ and a structure-preserving diffeomorphism Φ of M (i.e. $\Phi^* \lambda_i = \lambda_i$) then its mapping torus

$$M_\Phi := {}^{M \times \mathbb{R}}\big/_{\mathbb{Z}},$$

where the $\mathbb{Z}$-action is given by

$$n \cdot (x, t) = (\Phi^n(x), t + n), \tag{2.4}$$

supports a $(q+1)$-contact structure. Indeed, in $M\times\mathbb{R}$ we define forms $\eta_1,\cdots,\eta_{q+1}$ by setting

$$\eta_i(X)=\pi^*\lambda_i(X),\quad \eta_i(\partial_t)=0,\ \text{for } i=1,\cdots,q;$$
$$\eta_{q+1}(X)=\pi^*\lambda_1(X),\quad \eta_{q+1}(\partial_t)=1;$$

where $\pi: M\times\mathbb{R}\to M$ is canonical projection, $X\in \mathrm{T}M$ and ∂_t is the standard coordinate field on $\mathbb{R}$. Thus, we obtain

$$\bigcap_i \ker\eta_i=\xi,$$

and since $\mathrm{d}\eta_i|_\xi=\mathrm{d}\lambda_i$ for $i=1,\cdots,q$ and $\mathrm{d}\eta_{q+1}|_\xi=\mathrm{d}\lambda_1$, the derivatives are all non-degenerate on the intersection. Hence $\vec{\eta}$ defines a $(q+1)$-contact structure on $M\times\mathbb{R}$. Now, since the structure is invariant under the $\mathbb{Z}$-action 2.4, it descends to a $(q+1)$-contact structure on M_Φ.

Remark 4. As pointed out in (GOERTSCHES; LOIUDICE, 2020b), it is unusual for a geometric structure to be preserved by mapping tori. It does not happen for contact manifolds (clearly, as the mapping torus has even dimension), nor for symplectic or Kähler manifolds. In this sense, q-contact structures are less rigid.

The manifolds constructed in Examples 4, 5, 6 and 7, and the mapping torus from Example 8 are all closed manifolds, provided the starting manifold M is closed. In particular, by taking M to be a contact manifold and choosing the dimension of the fibre appropriately, we can construct closed manifolds supporting q-contact structures for any q larger than 1.

2.3 Closed orbits and generalised Weinstein conjectures

A 1-contact manifold is simply a contact manifold (M,ξ) with a chosen defining 1-form λ for the contact structure ξ. One of the main questions in contact dynamics (that is, the study of Reeb vector fields and their flows) is the following conjecture due to Weinstein (WEINSTEIN, 1979).

Conjecture 1 (Weinstein Conjecture (WC))**.** The Reeb vector field of every closed contact manifold (M,λ) admits a closed orbit.

Thought not yet proven in its full generality, the conjecture is known to hold in several particular cases: in dimension three (TAUBES, 2007; HUTCHINGS, 2009); for overtwisted contact manifolds (ALBERS; HOFER, 2009; BORMAN; ELIASHBERG; MURPHY, 2015); and when the Reeb vector field is Killing with respect to some metric on

M (RUKIMBIRA, 1993; BANYAGA; RUKIMBIRA, 1995). The general "feeling" among researchers is that the conjecture holds in its full generality.

There are two different possibilities to generalise the one-dimensional Weinstein conjecture to higher dimensional contact foliations: one can ask if there is always an orbit which is periodic in at least one direction in $\mathcal{R}$ or if there is always an orbit which is periodic in every direction in $\mathcal{R}$, that is, a closed orbit. The latter is a much stronger condition on the foliation. There are topological obstructions to the existence of closed orbits for general foliations, in terms of cohomology classes and the *characteristic map* of the foliation (cf. (ARRAUT; DOS SANTOS, 1988; ARRAUT; DOS SANTOS, 1992; DOS SANTOS, 1994)). For a contact foliation, the characteristic map is given in terms of the adapted coframe $\{\lambda_1, \cdots, \lambda_q\}$ via the rule

$$\begin{aligned}\lambda_F : \mathbb{R}^q \times \cdots \times \mathbb{R}^q &\longrightarrow H^{4n+1}_{dR}(M)\\ (v_1, \cdots, v_{2n+1}) &\longmapsto [\eta_1 \wedge \mathrm{d}\eta_2 \wedge \cdots \mathrm{d}\eta_{2n+1}],\end{aligned}$$

where $v_i = (v^i_1, \cdots, v^i_q) \in \mathbb{R}^q$ and $\eta_i = v^i_1\lambda_1 + \cdots v^i_q\lambda_q \in \wedge^1(M)$.

As shown in (ARRAUT; DOS SANTOS, 1988), the non-vanishing of the characteristic map is an obstruction to the existence of closed orbits for the foliation. Such obstruction does not exist in the case of contact actions.

Proposition 6. The mapping λ_F vanishes for any contact action $F : \mathbb{R}^q \times M \to M$.

Proof. We consider the linear mapping $\overline{\lambda}_F : (\mathbb{R}^q)^{\otimes(2n+1)} \to H^{4n+1}_{dR}(M)$ induced by λ_F by means the universal property of the tensor product. The collection $\{e_{i_1} \otimes \cdots \otimes e_{i_{2n+1}};\ 1 \leq i_j \leq q\}$ is a basis for $(\mathbb{R}^q)^{\otimes(2n+1)}$, where $\{e_1, \cdots, e_q\}$ is the canonical basis of $\mathbb{R}^q$. Then

$$\overline{\lambda}_F(e_{i_1} \otimes \cdots \otimes e_{i_{2n+1}}) = [\lambda_{i_1} \wedge \mathrm{d}\lambda_{i_2} \wedge \cdots \wedge \mathrm{d}\lambda_{i_{2n+1}}].$$

Now, $\mathrm{d}\lambda_{i_2} \wedge \cdots \wedge \mathrm{d}\lambda_{i_{2n+1}}$ is a $4n$-form that vanishes whenever it is fed a vector field tangent to $\mathcal{F}$, whose dimension is q. Moreover, as ξ has rank $2n$, there are no linearly independent sets of $4n$ vector fields tangent to ξ, so that $\mathrm{d}\lambda_{i_2} \wedge \cdots \wedge \mathrm{d}\lambda_{i_{2n+1}}$ vanishes at every point. Thus $\overline{\lambda}_F$ evaluates to 0 at every vector in the basis of $(\mathbb{R}^q)^{\otimes(2n+1)}$, hence $\lambda_F = 0$.

□

This means that, *a priori*, there are no obstructions to the existence of closed orbits under the Reeb action of $\mathbb{R}^q$. With this in mind, we propose two generalisations for the one-dimensional Weinstein Conjecture.

Conjecture 2 (The Weak Generalised Weinstein Conjecture (**WGWC**))**.** A contact foliation on a closed manifold M cannot be a foliation by planes.

Conjecture 3 (The Strong Generalised Weinstein Conjecture (**SGWC**))**.** Every q-contact foliation has a closed leaf, that is, a leaf homeomorphic to a torus $\mathbb{T}^q$.

We remark that, being orbits of an action of the Euclidean space $\mathbb{R}^q$, every leaf of the foliation $\mathcal{F}$ is homeomorphic to $\mathbb{R}^{q-l} \times \mathbb{T}^l$ for some $0 \leq l \leq q$, which depends on the leaf. The **WGWC** states that $l \geq 1$ for some leaf, while the **SGWC** asks for the existence of a leaf for which $l = q$. Of course, in the contact case, when $q = 1$, a leaf that is not homeomorphic to $\mathbb{R}$ is automatically a closed curve homeomorphic to S^1. Therefore, the **WGWC** and the **SGWC** are equivalent to the Weinstein Conjecture on dimension 1 and comprise generalisations of this conjecture to higher dimensional contact foliations. They form a hierarchy

$$\text{SGWC} \implies \text{WGWC} \implies \text{WC},$$

with the converse implications holding when $q = 1$.

2.3.1 Anosov contact actions

There is a class of contact foliation for which it is relatively simple to show that **SGWC** holds: the Anosov contact foliations, which we briefly discuss in this section.

Definition 10 (*Anosov Elements and Anosov Actions*)**.** Consider an C^2 action $\varphi : \mathbb{R}^k \to \mathrm{Diff}(M)$ of $\mathbb{R}^k$ on a closed Riemannian manifold (M, g). A point $a \in \mathbb{R}^k$ is said to be an **Anosov element of the action** if $f = \varphi^a := \varphi(a, \cdot)$ acts **normally hyperbolically** on M, meaning that there are constants $A, C > 0$ and a $\mathrm{d}f$-invariant splitting

$$\mathrm{T}M = E_a^s \oplus \mathrm{T}\varphi \oplus E_a^u$$

of the tangent bundle of M such that:

(i) $\| \left(\mathrm{d}f|_{E_a^s}\right)^n \| \leq C e^{-An}$, for all $n > 0$;

(ii) $\| \left(\mathrm{d}f|_{E_a^u}\right)^n \| \leq C e^{An}$, for all $n < 0$.

The action φ is called an **Anosov action of** $\mathbb{R}^k$ if it has an Anosov element $a \in \mathbb{R}^k$.

In other words, the action is Anosov when there is a point $a \in \mathbb{R}^k$ whose action on M contracts a bundle E_a^s exponentially while simultaneously expanding another bundle E_a^u also exponentially. Anosov elements never come alone. In fact, each connected component of the set $\mathcal{A}(\varphi)$ of all Anosov elements of the action φ is an open cone in $\mathbb{R}^k$, called a **chamber** (BARBOT; MAQUERA, 2011). An open convex subcone in a chamber is called a **regular subcone of the action** φ.

Definition 11 (*Non-wandering subsets*)**.** A point $x \in M$ is non-wandering if for any open neighbourhood $U \subset M$ of x there exists a $v \in \mathbb{R}^k$ such that $g(v,v) > 1$ and such that

$$\varphi^v(U) \cap U \neq \emptyset.$$

The set of all non-wandering points is denoted by $\mathrm{nw}(\varphi)$.

More specifically, a point $x \in M$ is non-wandering *with respect to a regular subcone* $\mathcal{C}$ if for any open neighbourhood $U \subset M$ of x there exists a $v \in \mathcal{C}$ whose norm is greater than 1 and such that

$$\varphi^v(U) \cap U \neq \emptyset.$$

The set of all such points in M is denoted by $\mathrm{nw}(\mathcal{C})$ and called the **non-wandering subset of $\mathcal{C}$.**

Recall that an action of $\mathbb{R}^k$ is called **transitive** if it has a dense orbit. More generally, given subsets $N \subset M$, $E \subset \mathbb{R}^k$, we say N is E-transitive if N is φ-invariant and there is a point x in N such that

$$\overline{\{\varphi(v,x); v \in E\}} = N,$$

that is, the invariant set N contains a dense E-orbit.

Proposition 7 (Spectral Decomposition (BARBOT; MAQUERA, 2011))**.** Let M be a closed smooth manifold and φ be an Anosov action on M. Then there is a finite collection $\{\Lambda_i\}_{i=1}^l$ of pairwise disjoint, locally connected, closed φ-invariant subsets of M such that

(i) For every $1 \leq i \leq l$, the union of closed orbits of φ inside Λ_i is dense in Λ_i;

(ii) For any regular subcone $\mathcal{C}$ the set Λ_i is $\mathcal{C}$-transitive;

(iii) the non-wandering set of M decomposes

$$\mathrm{nw}(\varphi) = \bigcup_{i=1}^{l} \Lambda_i.$$

Theorem 1. If a contact action $\varphi : \mathbb{R}^q \to \mathrm{Diff}(M)$ is Anosov, then φ has a closed orbit.

Proof. In light of itens (i) and (iii) of the Spectral Decomposition 7, it is sufficient to show that $\mathrm{nw}(\varphi) \neq \emptyset$. To that end, given a point $x \in M$ and an open set U containing x, we consider an element $a \in \mathbb{R}^q$ with $|a| > 1$. The transformation $f = \varphi^a$ is volume-preserving (it preserves the forms $\mathrm{d}M_i$), hence, by Poincaré's recurrence theorem, there exists a positive natural number j such that $f^j(U) \cap U \neq \emptyset$. But $f^j = \varphi^{ja}$, where, by construction, $|ja| > 1$, which means exactly that x is a non-wandering point. Thus $\varphi = M$ and the theorem follows.

□

Note that the contact condition was used only to guarantee the existence of the invariant volume form so that what has been shown is that the non-wandering subset of volume-preserving Anosov actions is the entire manifold. Together with Theorem 7 and the connectedness of M, this yields an even stronger conclusion:

Corollary 1. If an Anosov action preserves a volume, then it is transitive.

Remark 5. If the Anosov action φ is also a contact action of $\mathbb{R}^q$, then $\mathrm{T}\varphi = \mathcal{R}$ and $E_a^s \oplus E_a^u \approx \xi$. In particular, each of these bundles inherits the φ-invariant symplectic forms $\mathrm{d}\lambda_i$. If X,Y are tangent to E_a^s then, for any $j \in N$ we have

$$\mathrm{d}\lambda_i(X,Y) = \mathrm{d}\lambda_i(\mathrm{d}f^j X, \mathrm{d}f^j Y) \xrightarrow{j\to\infty} 0,$$

and hence $\mathrm{d}\lambda_i(X,Y) = 0$, implying E_a^s is a Lagrangian sub-bundle of ξ. Similar arguments show the same for E_a^u. In particular, when φ is both contact and Anosov the bundles E_a^s and E_a^u have the same rank, namely n.

2.4 Constructions on contact foliations

Proposition 8 (Reparameterisation Lemma)**.** Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_q)$ be an adapted coframe for a q-contact structure on M, and $A(p) = \{a_{ij}(p)\}$ a mapping $A : M \to \mathcal{M}_q(\mathbb{R})$ from M to the space of real matrices. If A is sufficiently C^1-close to 0 and the functions a_{ij} are leaf-wise constants (with respect to the contact foliation), then

$$\vec{\eta} := (\mathrm{id} - A)\vec{\lambda}$$

is also an adapted coframe for a q-contact structure on M, and the splittings associated with these two structures are the same.

Proof. We have

$$\eta_i = \lambda_i - \sum_j a_{ij}\lambda_j,$$

and therefore $\xi = \cap_j \ker\lambda_j \subset \ker\eta_i$, for every i. On the other hand, by choosing A small enough so that $(\mathrm{id} - A)$ is invertible, we can write each λ_i as a linear combination of the η_j, thus obtaining that $\cap_j \ker\eta_j \subset \ker\lambda_i$, from where we obtain the equality

$$\cap_j \ker\eta_j = \cap_j \ker\lambda_j = \xi.$$

Moreover,

$$\mathrm{d}\eta_i = \mathrm{d}\lambda_i - \sum_j (\mathrm{d}a_{ij} \wedge \lambda_j + a_{ij}\,\mathrm{d}\lambda_j), \tag{2.5}$$

and, since by hypothesis the functions a_{ij} are leaf-wise constant, they satisfy the equations $a_{ij}(R_l) = 0$, for every i,j,l. Therefore, the Reeb vector fields R_i of the λ_i satisfy $\iota_{R_i}\mathrm{d}\eta_j = 0$.

In other words, $\mathcal{R} \subset \ker d\eta_i$, for every i. Now the splitting $\mathcal{R} \oplus \xi = TM$ and the non-degeneracy of $d\eta_i$ on ξ imply $\ker d\eta_i \subset \mathcal{R}$, and we have equality between these two bundles as well.

From Equation (2.5) it follows that

$$d\eta_i^n = d\lambda_i^n + \varepsilon_i,$$

where ε_i can get arbitrarily small if A is taken small enough. In particular, for ε_i sufficiently close to 0, the $2n$-form $d\eta_i^n$ is volume form on the bundle ξ, and $d\eta_i$ is therefore non-degenerate on ξ. Moreover, by employing the Determinant Theorem, we can derive the equality

$$\eta_1 \wedge \cdots \wedge \eta_q \wedge (d\eta_i)^n = \det(\mathrm{id} - A)\lambda_1 \wedge \cdots \lambda_q \wedge ((d\lambda_i)^n + \varepsilon_i),$$

where the RHS will be a volume form as long as ε_i is sufficiently small.

□

In general, the Reeb vector fields of the η_i are not the same as the R_i, though they spam the same invariant bundle, so the construction above might be thought of as a *reparameterisation of the contact action*. This is similar to what happens to a contact form when it is multiplied by a non-vanishing function, though in the contact case the product is always non-degenerate, so we do not need to require the function to be small. Note, however, that the bundle $\mathcal{R}$ remains the same if and only if the function is constant in the direction of the Reeb vector field.

Note that the hypothesis on the functions a_{ij} from Proposition 8 is used to ensure that $\mathcal{R}$ is contained in the kernels of the η_i. If we already have this property, then virtually the same arguments as the ones in the previous proof give the following.

Proposition 9. Suppose M is a manifold of dimension $2n+q$ and there are linearly independent non-vanishing 1- forms $\alpha_1, \cdots, \alpha_l, \lambda_1, \cdots, \lambda_{q-l}$ on M, together with a splitting $TM = \mathcal{R} \oplus \xi$ satisfying:

(i) $\operatorname{rank} \mathcal{R} = q$;

(ii) $\xi = \left(\cap \ker \alpha_j\right) \cap (\cap \ker \lambda_i)$;

(iii) $\ker d\lambda_i = \mathcal{R}$ and $d\lambda_i$ is non-degenerate on ξ, for $i = 1, \cdots, q-l$;

(iv) $\mathcal{R} \subset \ker d\alpha_j$, for $j = 1, \cdots, l$.

Then there is a mapping $B : M \to \mathrm{Gl}_l(\mathbb{R})$, C^1-close to the identity, such that the 1-forms $\eta_1, \cdots, \eta_l, \lambda_1, \cdots, \lambda_{q-l}$ define a q-contact structure on M with splitting $TM = \mathcal{R} \oplus \xi$, where

$$\eta_i := \sum_{j=1}^{l} b_{ij}\alpha_j.$$

Proof. Just choose $A : M \to \mathcal{M}_l(\mathbb{R})$ C^1-close to zero, set $B = \mathrm{id} - A$ and proceed as in the proof of Proposition 8.

□

When the action is transitive, the situation is considerably better, as we do not need the smallness conditions and actually have a dense subset of possible reparameterisations to choose from. Note that the existence of a dense leaf implies that any leaf-wise constant function is just a constant function, so the only reparameterisations are actual matrices in $\mathrm{Gl}_q(\mathbb{R})$.

Lemma 1. (ALMEIDA, 2018) Suppose the contact foliation $\mathcal{F}$ is transitive, and let $\mathcal{B} \subset \mathrm{Gl}_q(\mathbb{R})$ be the set of (constant) reparameterisations of the action; then the complement of $\mathcal{B}$ is closed and has empty interior. In particular, $\mathcal{B}$ is dense in $\mathrm{Gl}_q(\mathbb{R})$.

Proof. Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_q)$ be an adapted coframe of the action and $B \in \mathrm{Gl}_q(\mathbb{R})$. Then $\vec{\eta} = B\vec{\lambda}$ satisfy $\cap \ker \eta_i = \xi$ and $\mathcal{R} \subset \ker \mathrm{d}\eta_i$. In order to guarantee that $(\eta_1, \cdots, \eta_q)$ is also an coframe adapted to the action, it is sufficient to show that $\eta_1 \wedge \cdots \wedge \eta_q \wedge (\mathrm{d}\eta_i)^n$ is a volume form for every i.

Since $\mathrm{d}\eta_i = \sum_j b_{ij}\, \mathrm{d}\lambda_j$, we have

$$(\mathrm{d}\eta_i)^n = \sum_{|J|=n} c^i_J(B)(\mathrm{d}\lambda_1)^{J_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{J_q}, \tag{2.6}$$

where by J we mean the multi-index $J = (J_1, \cdots, J_q) \in \{1, \cdots n\}^q$ and by $|J|$ we denote the sum $J_1 + \cdots + J_q$ of its entries. The coefficients

$$c^i_J(B) = \frac{n!}{J_1! \cdots J_q!} \prod_{l=1}^{q} b_{il}^{J_l}$$

are monomials of degree n on the variables b_{ij}. The key step lies in noticing that each of the forms $(\mathrm{d}\lambda_1)^{J_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{J_q}$ is non degenerate on ξ, and therefore are related to the volume form $(\mathrm{d}\lambda_1)^n$ by an equation

$$(\mathrm{d}\lambda_1)^{J_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{J_q} = a_J(\mathrm{d}\lambda_1)^n, \tag{2.7}$$

where a_J is a non-vanishing function on M. However, since both $\mathrm{d}\lambda_1$ and $(\mathrm{d}\lambda_1)^{J_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{J_q}$ are leaf-wise invariant forms and $\mathcal{F}$ has a dense leaf, a_J must be a constant. Thus, it follows from Equations (2.6) and (2.7) that

$$p_i(B) := \sum_{|J|=n} a_J c^i_J(B),$$

is a homogeneous polynomial of degree n on the variables b_{ij}, and it is completely determined by the adapted coframe $(\lambda_1, \cdots, \lambda_q)$. In addition to that, p_i satisfies

$$\eta_1 \wedge \cdots \wedge \eta_q \wedge (\mathrm{d}\eta_i)^n = \det(B) p_i(B) \lambda_1 \wedge \cdots \wedge \lambda_q \wedge (\mathrm{d}\lambda_1)^n.$$

Therefore, $\eta_1 \wedge \cdots \wedge \eta_q \wedge (\mathrm{d}\eta_i)^n$ is non degenerate if and only if B is not an zero of p_i, and we see that $\mathcal{B}$ is exactly the complement of the set of zeros

$$\mathcal{Z} := \bigcup_i \{B \in \mathrm{Gl}_q(\mathbb{R}); p_i(B) = 0\}.$$

But each polynomial p_i is non zero, because $p_i(\mathrm{id}) = 1$, and therefore their sets of zeros are closed subsets of $\mathbb{R}^{q\times q}$ whose interiors are empty, as we wanted.

□

In Example 7, the foliation obtained on E is a product of the foliation on M by tori. There is a construction that allows us, under suitable hypotheses, to reduce the action by removing toric components from the leaves, effectively being a partial converse to the method of Example 7. First, let us recall that the topological type of a leaf is determined by the relation

$$\mathcal{F}(x) := \left. \mathbb{R}^q \middle/ \mathrm{Iso}(\mathcal{F}(x)) \right.,$$

where $\mathrm{Iso}(\mathcal{F}(x)) := \{a \in \mathbb{R}^q; F^a(x) = x\}$ is the isotropy group of the leaf $\mathcal{F}(x)$. The kernel of the action is a lattice in $\mathbb{R}^q$ consisting of the elements acting as the identity on M, and it equals the intersection of all the isotropy subgroups:

$$\bigcap_{x\in M} \mathrm{Iso}(\mathcal{F}(x)) = \ker F := \{a \in \mathbb{R}^q; F^a = \mathrm{id}\} \approx \mathbb{Z}^l,$$

hence every leaf is a cylinder $\mathbb{T}^s \times \mathbb{R}^{q-s}$ for some $s \geq l$. In particular, if the action's kernel is nontrivial, then no leaf is a plane.

Theorem 2 (Reduction of the action)**.** Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a closed q-contact manifold. Suppose the contact action F is transitive and has nontrivial kernel $\Gamma \approx \mathbb{Z}^l$, where $0 < l < q$. Then M is a principal $\mathbb{T}^l$-bundle over a closed $(2n+q-l)$-dimensional manifold M_0; moreover, the contact foliation on M induces a $(q-l)$-contact foliation on M_0.

Proof. We consider the vector spaces

$$G := \mathrm{Span}\, \Gamma \approx \mathbb{R}^l,$$
$$H := \left. \mathbb{R}^q \middle/ G \right. \approx \mathbb{R}^{q-l},$$

and the natural isomorphism $\mathbb{R}^q \approx G \oplus H$.

First, let the torus $\mathbb{T}^l \approx \Gamma/G$ act on M by

$$(a+\Gamma) \cdot x := F^a(x) = F(a,x).$$

This action is well defined because if $a+\Gamma = b+\Gamma$, then their difference belongs to the kernel of F. It is also a free action, since $(a+\Gamma)\cdot x = x$ means F^a acts as the identity on the leaf $\mathcal{F}(x)$, hence $a \in \Gamma$, the isotropy group of $\mathcal{F}(x)$. Finally, the action is proper due to the compactness of $\mathbb{T}^l$. It follows that the leaf space.

$$M_0 := {}^{M}/_{\mathbb{T}^l}$$

is a $(2n+q-l)$-dimensional closed manifold M_0, and $\mathbb{T}^l \hookrightarrow M \xrightarrow{\rho} M_0$ is a principal $\mathbb{T}^l$-bundle, where $\rho : M \to M_0$ is the canonical projection.

We define an action of $\mathbb{R}^{q-l}$ on M_0, via H, by

$$\begin{aligned} F_0 : H \times M_0 &\longrightarrow M_0 \\ (\overline{a}, \rho(x)) &\longmapsto \rho(F(a,x)). \end{aligned} \tag{2.8}$$

This clearly does not depend on the first representative a, since $\rho(F(a,x)) = \rho(F(b,x))$ for every $a,b \in \mathbb{R}^q$. It also does not depend on the second representative x. Indeed, if $\rho(x) = \rho(y)$, then $y = F(b,x)$ for some $b \in G$, and consequently $F(a,y) = F(a+b,x)$ belong to the same leaf as $F(a,x)$.

It remains to show that F_0 is a contact action. We will achieve this by using integration along the fibres, possibly after choosing a suitable reparameterisation of the action F. Since the bundle $\mathcal{R}$ is trivial, the isomorphism $\mathbb{R}^q \approx G \oplus H$ induces a splitting $\mathcal{R} = \mathcal{G} \oplus \mathcal{H}$. Note that $\mathcal{G}_x$ is composed of the directions tangent to the fibre $\rho^{-1}(\rho(x))$, hence $\mathcal{G}_x = \ker d\rho_x$. Similarly, the tangent space at $\rho(x)$ of the orbit $F_0(H, \rho(x))$ is exactly $\rho_*(\mathcal{H}_x)$, so it is sufficient to show that $\mathcal{H}_0 := \rho_*\mathcal{H}$ can be realised as the Reeb bundle of a $(q-l)$-contact structure on M_0.

We begin by decomposing the Reeb fields of $\vec{\lambda}$ as

$$R_i = R_i^{\mathcal{G}} \oplus R_i^{\mathcal{H}}.$$

Using Lemma 8 to find a suitable reparameterisation if necessary (recall $\mathcal{F}$ is a transitive foliation), we may assume without loss of generality that

$$\begin{gathered} \mathcal{H} = \mathrm{Span}\{R_1^{\mathcal{H}}, \cdots, R_{q-l}^{\mathcal{H}}\}; \\ \lambda_i(R_i^{\mathcal{H}}) \neq 0, \text{ for } i = 1, \cdots, q-l. \end{gathered}$$

Let us further replace $R_i^{\mathcal{H}}$ by a suitable multiple X_i as to assure we have $\lambda_i(X_i) \equiv 1$. In addition, we consider a fibre-wise volume form ω, normalised as to satisfy

$$\int_{\rho^{-1}(y)} \omega = 1,$$

for every $y \in M_0$. We define $q-l$ differential 1-forms on M_0 by $\eta_i := \rho_*(\lambda_i \wedge \omega)$, using the linear morphism $\rho_* : \wedge^*(M) \to \wedge^{*-l}(M_0)$, determined by ρ via integration along the fibres. To be more precise,

$$\eta_i|_y(Z) = \int_{\rho^{-1}(y)} \iota_{\widetilde{Z}}(\lambda_i \wedge \omega),$$

where the RHS is independent of the choice of lifting $\widetilde{Z}$. The forms η_i are non-vanishing, since $\eta_i(\rho_* X_i) \equiv 1$ for $i = i, \cdots q - l$; it is clear from their construction they are linearly independent forms whose restriction to $\xi_0 := \rho_* \xi$ is identically zero. Finally, the morphism ρ_* commutes with the exterior derivative, from where it follows that each $d\eta_i$ is non-degenerate on ξ_0, and has as its kernel the bundle $\rho_* \mathcal{R} = \rho_* \mathcal{H} =: \mathcal{H}_0$. Hence $(M_0, \vec{\eta}, \mathcal{H}_0 \oplus \xi_0)$ is a $(q-l)$-contact manifold, as we wanted.

□

It is clear that the **SGWC** and the **WC** do not permit minimal contact foliations to exist. It follows from Theorem 2 that this is also the case for the **WGWC**. Indeed, for minimal foliations, every leaf is dense, and the continuity of the action implies that every isotropy group is the same. Consequently, reducing a minimal contact action leaves us with a contact action by planes, which contradicts the **WGWC**.

Theorem 3. If a minimal contact foliation $(M, \mathcal{F})$ exists, then the Weak Generalised Weinstein Conjecture is invalid.

Proof. Since $(M, \mathcal{F})$ is a minimal q-contact foliation, all leaves are dense in M, and their isotropy groups are all the same, namely the lattice $\Gamma := \ker F \approx \mathbb{Z}^l$. None of these leaves can be closed due to minimality. Therefore $l < q$. From Theorem 2, $\rho : M \to M_0$ is a principal bundle over a $(q-l)$-contact manifold M_0. We claim that the contact action defined by (2.8) is a minimal action whose orbits are all planes. Indeed, the action is minimal: given $\rho(x), \rho(y) \in M_0$, the leaf $\mathcal{F}(x)$ accumulates on y, hence there is a sequence $a_n \in \mathbb{R}^q$ such that

$$F(a_n, x) \to y.$$

We write $a_n = a_n^G + a_n^H$, and note that $F(a_n^G, \cdot)$ gets arbitrarily close to the identity. We choose a sub-sequence a_{n_j} such that $F(a_{n_j}^G, x) \to x$, from where we conclude

$$F(a_{n_j}^H, F(a_{n_j}^G, x)) \to y,$$

which, in turn, implyies

$$F_0(\overline{a_{n_j}}, \rho(x)) = F_0(\overline{a_{n_j}}, \rho(F(a_{n_j}^G, x))) = \rho(F(a_{n_j}^H, F(a_{n_j}^G, x))) \to \rho(y).$$

Therefore, $\mathcal{F}_0(\rho(x))$ accumulates on $\rho(y)$, and every orbit of the action F_0 is dense. To see that the orbits are planes $\mathbb{R}^q$, it is simply a matter of noticing that F_0 is free. If $\rho(x) = F_0(\bar{a}, \rho(x)) := \rho(F(a, x))$, then there is $b \in G$ such that

$$F(a, x) = (b + \Gamma) \cdot x = F(b, x).$$

Thus a acts like an element of G, representing the identity on H. In particular, minimality implies that the isotropy group of any leaf is $\ker F_0 = \{0\}$. Hence every leaf is a plane.

We conclude that $(M_0, \mathcal{F}_0)$ is a minimal contact foliation by planes. Therefore the **WGWC** does not hold, as we wanted.

□

CHAPTER

3

THE CONTACT AND CHARACTERISTIC FOLIATIONS

3.1 Local representations

In a foliated chart $\psi : U \to \mathbb{R}^q \times \mathbb{R}^{2n}$ the tangent coordinates $(z_1, \cdots, z_q)$ can be chosen so that $\partial_{z_i} = R_i$ (cf (BOOTHBY, 1986, Theorem 8.3)). This gives rise to the following local characterisation of the adapted coframe $\vec{\lambda}$.

Proposition 10. Around each point $p \in M$ there are coordinates $(x, y, z) \in \mathbb{R}^{2n} \times \mathbb{R}^q$:

$$x = (x_1, \cdots, x_n)$$
$$y = (y_1, \cdots, y_n)$$
$$z = (z_1, \cdots, z_q),$$

and, for each $1 \le i \le q$ and $1 \le j \le n$, functions $f^i_j, g^i_j : \mathbb{R}^{2n} \to \mathbb{R}$ such that:

$$\lambda_i = \mathrm{d}z_i + \sum_{j=1}^{n} \left(-f^i_j(x,y)\,\mathrm{d}x_j + g^i_j(x,y)\,\mathrm{d}y_j \right).$$

Proof. Let $(U; x, y, z)$ be a foliated chart around p. As pointed out before, these coordinates can be chosen such that $\partial z_i = R_i$. In these coordinates, we write

$$\lambda_i = \sum_{l=1}^{q} h^i_l(x,y,z)\,\mathrm{d}z_l + \sum_{j=1}^{n} \left(f^i_j(x,y,z)\,\mathrm{d}x_j + g^i_j(x,y,z)\,\mathrm{d}y_j \right), \tag{3.1}$$

for appropriate functions f^i_j, g^i_j and h^i_l. Evaluating Equation 3.1 at the Reeb vector fields of $\mathcal{F}$ implies $h^i_l = \delta_{il}$, so that

$$\lambda_i = \mathrm{d}z_i + \sum_{j=1}^{n} \left(f^i_j(x,y,z)\,\mathrm{d}x_j + g^i_j(x,y,z)\,\mathrm{d}y_j \right). \tag{3.2}$$

Now we differentiate Equation (3.2) to get

$$\begin{aligned}\mathrm{d}\lambda_i = &\sum_{j,l=1}^{n}\left(\frac{\partial}{\partial x_j}g_l^i(x,y,z) - \frac{\partial}{\partial y_l}f_j^i(x,y,z)\right)\mathrm{d}x_j\wedge\mathrm{d}y_l\\ &+\sum_{l=1}^{q}\sum_{j=1}^{n}\left(\frac{\partial}{\partial z_l}f_j^i(x,y,z)\,\mathrm{d}z_l\wedge\mathrm{d}x_j + \frac{\partial}{\partial z_l}g_j^i(x,y,z)\,\mathrm{d}z_l\wedge\mathrm{d}y_j\right).\end{aligned}$$

Evaluating both sides of the above equation on the Reeb vector fields implies that all the partial derivatives $\frac{\partial}{\partial z_l}f_j^i(x,y,z)$ and $\frac{\partial}{\partial z_l}g_j^i(x,y,z)$ must vanish, hence f_j^i and g_j^i are functions depending only on the coordinates (x,y). Applying this to Equation (3.2) proves our assertion.

□

Motivated by the last proposition, one can pose oneself the question of whether q-contact structures are locally isomorphic or not, in the sense of a generalised Darboux Theorem. A partial result in this direction comes from the following proposition.

Proposition 11. Let M be a manifold supporting a contact foliation of codimension 2, that is, $\dim M = 2+q$. Then around each point of M there are coordinates $(U;x,y,z_1,\cdots,z_q)$ and leaf-wise constant functions $a_i : U \to \mathbb{R}$ such that $\partial_x a_i$ is non-vanishing and

$$\lambda_i = \mathrm{d}z_i + a_i\,\mathrm{d}y.$$

Moreover, we can take $a_1 = x$.

Proof. Around each point of M, one can find a neighbourhood U diffeomorphic to a product of discs $D^q \times D^2$. Let $\Phi : U \to D^2$ be the submersion defined by this diffeomorphism, the fibres of which are the plaques of $\mathcal{F}$ on U. Since the symplectic form $\mathrm{d}\lambda_i$ is holonomy-invariant, there is a well defined symplectic form ω_i on D^2 satisfying $\Phi^*\omega_i = \mathrm{d}\lambda_i$, and by Darboux's theorem there is a coordinate chart (x',y') on D^2 such that $\omega_1 = \mathrm{d}x'\wedge\mathrm{d}y'$. Moreover, for every other i there is a non-vanishing function b_i on D^2 such that $\omega_i = b_i\,\mathrm{d}x'\wedge\mathrm{d}y'$.

Consider on U the unique fields X and Y on M that are Φ-related to the coordinate vector fields $\partial_{x'}$ and $\partial_{y'}$ and have no components in any leaf-wise direction. More precisely, they satisfy the following:

$$\begin{cases} X = \Phi^*\partial_{x'}\\ Y = \Phi^*\partial_{y'}\\ \mathcal{L}_{R_i}X = \mathcal{L}_{R_i}Y = 0 \text{ for all } i.\end{cases}$$

Thus, $\{R_1,\cdots,R_q,X,Y\}$ is a local frame for the tangent bundle of M, satisfying $[R_i,R_j] = [X,R_j] = [Y,R_j] = 0$ for any i,j. As for $[X,Y]$, we have

$$\lambda_i([X,Y]) = X\lambda_i(Y) - Y\lambda_i(X) - \mathrm{d}\lambda_i(X,Y) = b_i\,\mathrm{d}x'\wedge\mathrm{d}y'(\partial_{x'},\partial_{y'}) = b_i\circ\Phi,$$

so that $[X,Y] = \sum_i (b_i \circ \Phi) R_i$. We define on D^2 functions

$$c_i(x', y') = \int_0^{x'} b_i(t, y')\, \mathrm{d}t$$

and note that, since X and $\partial_{x'}$ are Φ-related, these satisfy

$$X(c_i \circ \Phi) = \partial_{x'} c_i \circ \Phi = b_i \circ \Phi.$$

Thus, the vector field $\widetilde{Y} := Y - \sum_i (c_i \circ \Phi) R_i$ is linearly independent from X and the R_i; in addition, it is such that

$$[X, \widetilde{Y}] = [X,Y] - \sum_i [X, (c_i \circ \Phi) R_i] = \sum_i (b_i \circ \Phi) R_i - \sum_i (b_i \circ \Phi) R_i = 0.$$

One can, therefore, find coordinates $(x, y, z_1, \cdots, z_q)$ on M whose coordinate vector fields are $\{X, \widetilde{Y}, R_1, \cdots, R_q\}$, obtaining

$$\begin{cases} X = \partial_x \\ Y = \partial_y + \sum_i (c_i \circ \Phi) \partial_{z_i} \\ R_i = \partial_{z_i}. \end{cases}$$

It follows from construction that $\Phi^* \mathrm{d}x' = \mathrm{d}x$ and $\Phi^* \mathrm{d}y' = \mathrm{d}y$. Hence

$$\mathrm{d}\lambda_i = \Phi^* \omega_i = (b_i \circ \Phi)\, \mathrm{d}x \wedge \mathrm{d}y$$

and, since $\mathrm{d}(c_i \circ \Phi) = (b_i \circ \Phi)\, \mathrm{d}x + \partial_y (c_i \circ \Phi)\, \mathrm{d}y$,

$$d\left((c_i \circ \Phi)\, \mathrm{d}y\right) = (b_i \circ \Phi)\, \mathrm{d}x \wedge \mathrm{d}y.$$

Thus $\mathrm{d}(\lambda_i - (c_i \circ \Phi)\, \mathrm{d}y) = 0$, and there exists, on a possible smaller neighbourhood, a function f_i such that

$$\lambda_i = \mathrm{d}f_i - (c_i \circ \Phi)\, \mathrm{d}y.$$

Finally, we see that

$$\begin{aligned} \delta_{ij} &= \lambda_i(R_j) = \mathrm{d}f_i(\partial_{z_j}) - (c_i \circ \Phi)\, \mathrm{d}y(\partial_{z_j}) = \partial_{z_j} f_i; \\ 0 &= \lambda_i(X) = \mathrm{d}f_i(\partial_x) - (c_i \circ \Phi)\, \mathrm{d}y(\partial_x) = \partial_x f_i; \\ 0 &= \lambda_i(Y) = \mathrm{d}f_i(\partial_y) + \sum_j (c_j \circ \Phi)\, \mathrm{d}f_i(\partial_{z_j}) - (c_i \circ \Phi)\, \mathrm{d}y(\partial_y) - \sum_j (c_j \circ \Phi)\, \mathrm{d}y(\partial_{z_j}) \\ &= \partial_y f_i + (c_i \circ \Phi) - (c_i \circ \Phi) = \partial_y f_i, \end{aligned}$$

and therefore $\mathrm{d}f_i = \mathrm{d}f_i$, from where we conclude, setting $a_i := -(c_i \circ \Phi)$, that $\lambda_i = \mathrm{d}f_i + a_i\, \mathrm{d}y$, as we wanted. □

A crucial fact in the construction above is the existence of the non-vanishing functions b_i. When the codimension is greater than 2, one can not generally find a single basis for which all the symplectic forms are written canonically, so this argument does not hold. When, however, all the derivatives $\mathrm{d}\lambda_i$ coincide, then a similar construction yields for general codimension:

Proposition 12 ((BLAIR; TERLIZZI; KONDERAK, 2006))**.** If M is a uniform q-contact manifold, then around each point there are coordinates $(U; x, y, z_1, \cdots, z_q)$ such that

$$\lambda_i = \mathrm{d}z_i + \sum_j x_j \,\mathrm{d}y_j.$$

3.2 Transverse sections and holonomy transformations

Given a transverse section T to the contact foliation $\mathcal{F}$, note that T is a symplectic manifold. Indeed, given a field X tangent to T, there is a unique decomposition

$$X = X_{\mathcal{R}} + X_{\xi},$$

due to the splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$. Since T is transverse to $\mathcal{F}$, the component X_ξ is necessarily non-zero, except, of course, if $X = 0$. Hence the mapping $X \mapsto X_\xi$ is an isomorphism between $\mathrm{T}T$ and ξ, and since $\ker \mathrm{d}\lambda_i = \mathcal{R}$ for every i, we have the identities

$$\mathrm{d}\lambda_i(X,Y) = \mathrm{d}\lambda_i(X_\xi, Y_\xi), \quad \forall X, Y \in \mathrm{T}T, \ \forall i = 1, \cdots, q.$$

So each restriction $\mathrm{d}\lambda_i|_T$ is non-degenerate, hence a symplectic form. Thus, the pair $(T, \mathrm{d}\lambda_i)$ is an exact symplectic manifold for every transversal T and 1-form λ_i. In particular, the foliation $\mathcal{F}$ admits no closed transversal.

Proposition 13. There is no $2n$-dimensional closed submanifold of M everywhere transverse to $\mathcal{F}$.

Proof. Indeed, if T was one such manifold, then the exact $2n$-form

$$\mathrm{d}(\lambda_i)^n = \mathrm{d}(\lambda_i \wedge (\mathrm{d}\lambda_i)^{n-1})$$

would be an exact volume form on T, contradicting Stokes's theorem. □

Moreover, every holonomy transformation is a symplectomorphism between its domain and range.

Proposition 14. Let T be a complete transversal for the contact foliation $\mathcal{F}$. Then for every $i = 1, \cdots, q$, the holonomy maps of $\mathcal{F}$ are local symplectomorphisms of $(T, \mathrm{d}\lambda_i)$.

Proof. Let $h : D(h) \to R(h)$ be a holonomy transformation of $\mathcal{F}$. We are going to show that every point of $D(h)$ has a neighbourhood restricted to which h preserves the 2-form $\mathrm{d}\lambda_i$, and therefore $h : (D(h), \mathrm{d}\lambda_i) \to (R(h), \mathrm{d}\lambda_i)$ is a symplectomorphism. Now, since $\mathcal{F}$ is the orbit foliation of an action $F : \mathbb{R}^q \times M \to M$, given $x \in D(h)$, there is an open set $U \subset D(h)$ containing x, and a function $\tau : U \to \mathbb{R}^q$, such that $h(u) = F^{\tau(u)}(u)$ for every $u \in U$. Nevertheless, F is a composition of the flows of the Reeb field, each of which preserves the 2-form $\mathrm{d}\lambda_i$. Hence F, and therefore h, preserve the 2-form $\mathrm{d}\lambda_i$ on the open set U. □

For the characteristic foliations of the forms λ_i, on the other hand, there is no obstruction to the existence of closed transversals. As it happens to the contact foliation $\mathcal{F}$, their transversal also inherits geometrical structures.

Proposition 15. If T is a transversal of $\mathcal{C}_i$, then $\lambda_i|_T$ is a contact form.

Proof. Note that the equality $\mathfrak{C}_i = \mathrm{Span}\{R_j\}_{j \neq i}$ implies that each leaf of $\mathcal{C}_i$ is transverse to both R_i and ξ. Suppose that T is a transversal of $\mathcal{C}_i$. Then $\dim T = 2n+1$ and, as before, the fact that T is transverse to R_j for every j other than i means

$$\iota_{R_1}(\iota_{R_2}(\cdots \widehat{\iota_{R_i}}(\cdots \iota_{R_q}(\mathrm{d}M_i))\cdots) = \lambda_i \wedge \mathrm{d}\lambda_i^n$$

is a volume form on T. Hence $\lambda_i|_T$ is a contact form.

□

The contact structure associated is the intersection bundle

$$\ker \lambda_i \cap \mathrm{T}T = (\mathrm{T}\mathcal{C}_i \oplus \xi) \cap \mathrm{T}T = \xi \cap \mathrm{T}T.$$

Moreover, as in the case of Proposition 14, the holonomy maps of $\mathcal{C}_i$ connect different transversals by following the leaves of $\mathcal{C}_i$, effectively “flowing” along the vector fields R_j, $j \neq i$. As all these fields preserve the bundle ξ, so do the holonomy maps, implying that the holonomy transformations are contactomorphisms. Recall that a measure μ on a complete transversal T for $\mathcal{F}$ is *holonomy-invariant* if $\mu(h(U)) = \mu(U)$ for every measurable set $U \subset N$ and every holonomy transformation $h : D \to R$ such that $U \subset D$. As symplecto- and contactomorphisms preserve the volumes induced by the symplectic or contact form (i.e., the measures $\mu_i := |\,\mathrm{d}\lambda_i^n|$ or $\mu_i := |\lambda_i \wedge \mathrm{d}\lambda_i^{(n-1)}|$, respectively), we see that $\mathcal{H}$ and $\mathcal{H}_i$ consist of volume-preserving transformations. Rephrasing all in just one proposition, we get

Proposition 16. Let $(M, \mathcal{F})$ be a q-contact manifold.

(i) The holonomy pseudogroup $\mathcal{H}$ of the foliation $\mathcal{F}$ tangent to the orbits of the action consists of symplectomorphisms.

(ii) The holonomy pseudogroup $\mathcal{H}_i$ of each characteristic foliation $\mathcal{C}_i$ consists of contactomorphisms.

In particular, every contact and characteristic foliation admits holonomy-invariant measures.

Corollary 2. The Godbillon-Vey class $\mathrm{gv}(\mathcal{F})$ of any contact foliation $\mathcal{F}$ vanishes.

Proof. This is a consequence of the fact that $\mathrm{d}\lambda_i^n$ is a holonomy-invariant closed volume form (cf. Hurder's Vanishing theorem on (HURDER, 1986), and also (CANDEL; CONLON, 2000b, Section 7.1.E)). This can also be seen directly from the definition of $\mathrm{gv}(\mathcal{F})$. □

Remark 6. We emphasise that by $\lambda_i|_T$ we mean $j^*\lambda_i$, where $j : T \to M$ is an embedding. In particular, R_i need not coincide with the Reeb vector field of $\lambda_i|_N$. However, since both $\mathrm{T}\mathcal{C}_i \oplus \mathrm{T}T$ and $\mathrm{T}\mathcal{C}_i \oplus \mathrm{Span}\{R_i\} \oplus \xi$ are equal the whole $\mathrm{T}M$, there is a smooth bundle isomorphism $\Psi : \mathrm{Span}\{R_i\} \oplus \xi \to \mathrm{T}T$. This comes from the composition of linear isomorphisms of the quotient space $\mathrm{T}_pT \to \frac{Tan_pM}{\mathrm{T}_p\mathcal{C}_i}$ and $\mathrm{Span}\{R_i\} \oplus \xi \to \frac{Tan_pM}{\mathrm{T}_p\mathcal{C}_i}$, defined fibre-wise. In particular, the image ΨR_i is the unique vector field on T such that $\Psi R_i - R_i \in \mathrm{T}T$. If we write X_i for this difference, then

$$\lambda_i(\Psi R_i) = \lambda_i(R_i) + \lambda_i(X_i) = 1$$

and

$$\mathrm{d}\lambda_i(\Psi R_i, \cdot) = \mathrm{d}\lambda_i(R_i, \cdot) + \mathrm{d}\lambda_i(X_i, \cdot) = 0,$$

so that ΨR_i is indeed the Reeb vector field of $\lambda_i|_T$. Moreover, for any closed orbit $\gamma : \mathbb{R} \to T$ of ΨR_i of period τ we have

$$R_i(\gamma(s)) + X_i(\gamma(s)) = \Psi R_i(\gamma(s)) = \Psi R_i(\gamma(\tau + s)) = R_i(\gamma(\tau + s)) + X_i(\gamma(\tau + s))$$

and therefore

$$R_i(\gamma(s)) - R_i(\gamma(\tau + s)) = X_i(\gamma(\tau + s)) - X_i(\gamma(s)),$$

where the LHS belongs to $\mathrm{Span}\, R_i \oplus \xi$ while the LHS belongs to $\mathrm{T}\mathcal{C}_i$. Since these subspaces are complementary, it follows that γ is a closed orbit of period τ for both R_i and X_i. In particular, R_i admits a closed orbit in M.

Remark 7. If $q \geq 2$, then we can also see the symplectisation of $(T, \lambda_i|_T)$ as a submanifold of M, by flowing T along R_j for a small time, for some $j \neq i$. If we denote the submanifold thus obtained by W and write coordinates (t, x) for it, then the symplectic form is $\omega = \mathrm{d}(e^t\lambda_i|_T)$. In other words, the point $(t, x) = \exp(tR_j)(x)$. Then the form $\mathrm{d}t$ is equal to λ_j and ∂_t is just R_j. In particular, R_j is a *Liouville vector field* for (W, ω), that is

$$\mathcal{L}_{R_j}\omega = \omega.$$

So, for sufficiently small t, all the level sets $N_t := \{t\} \times T$ in a neighbourhood of T are contact manifolds, with contact forms $\iota_{R_j}\omega = e^t(\lambda_i|_T)$. Their Reeb vector fields are $e^{-t}R_i|_T$

3.3 Tautness

Let L be a leaf of $\mathcal{F}$, and consider a characteristic foliation $\mathcal{C}_i$. Then $\mathcal{C}_i$ induces a codimension 1 foliation of L, more specifically the one determined by the involutive bundle $\cup_{j\neq i}\mathbb{R}R_j$. It follows directly from the definition of the characteristic bundle $\mathfrak{C}_i$ that each flow line of R_i in L is a transversal of $\mathcal{C}_i$. Moreover, these are complete transversals, i.e., they intersect every leaf. Indeed, if $\mathcal{T}$ is the orbit of y under the flow of R_i and $\mathcal{C}_i(x)$ is a leaf of the characteristic foliation, then there is $a = (t_1, \cdots, t_q) \in \mathbb{R}^q$ such that $x = F^a(y)$. If F_i denotes the action of $\mathbb{R}^q$ whose underlying foliation is $\mathcal{C}_i$, then for $b = (-t_1, \cdots, -t_{i-1}, \widehat{t_i}, -t_{i+1}, \cdots, -t_q)$ we have $F_i(b,x) = \exp(t_i R_i)(y) \in \mathcal{T} \cap \mathcal{C}_i(x)$.

Recall that a codimension one foliation is *taut* if it has a complete closed transversal. It follows from the considerations above that

Proposition 17. The foliation $\mathcal{C}_i$ is taut on a leaf L if and only if R_i has a closed orbit in L.

The foliations $\mathcal{F}$ and $\mathcal{C}_i$ are taut as well, in the appropriate sense. Tautness generalises to higher codimension in several ways, all equivalent when the ambient manifold is compact. One of the definitions is given with respect to the minimal submanifolds (that is, a submanifold for which the mean curvature is zero in all normal directions) of the ambient manifold once a Riemannian metric is fixed.

Definition 12 (*Harmonic foliations*)**.** A foliation is **geometrically taut** (also called **harmonic**) if there is a Riemannian metric on the ambient manifold for which every leaf is a minimal submanifold.

Proposition 18. There is a Riemannian metric on M such that any submanifold of dimension $2 \leq d \leq q$ realisable as an orbit of the action of $\mathbb{R}^d$ induced by Reeb vector fields $R_{i_1}, \cdots, R_{i_d}$ is a minimal submanifold.

Proof. We begin with the case $d = q$; the others are analogous. Choose any metric $g^\perp$ on the q-contact distribution, and consider on $\mathcal{R}$ the adapted metric

$$g^\tau := \sum_{i=1}^{q} \lambda_i \otimes \lambda_i.$$

Then $g = g^\tau \oplus g^\perp$ is a metric on M such that $\mathcal{R} \perp \xi$ and $g(R_i, R_j) = \delta_{ij}$. In particular, the normal bundle to any leaf L is $\xi|_L$. The mean curvature of L in the normal direction $X \in \Gamma(\xi)$ is the trace of the Weingarten map $W^X(Y) := \pi_\xi \nabla_Y(X)$, where ∇ is the Levi-Civita connection of g.

Fix a normal field $X \in \Gamma(\xi)$. First, we observe that $\nabla_X R_j$ is orthogonal do R_j, for all j, since

$$0 = \nabla_X \underbrace{g(R_j, R_j)}_{\equiv 1} = 2g(\nabla_X R_j, R_j).$$

In particular,

$$g([X,R_j],R_j) = g(\nabla_X R_j, R_j) - g(\nabla_{R_j} X, R_j) = -g(W^X(R_j), R_j). \tag{3.3}$$

On the other hand, with respect to the characteristic form $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q$ we have

$$\lambda(R_1, \cdots, R_{j-1}, [X,R_j], R_{j+1}, \cdots, R_q) = \det(A^j_{il})_{il} = g([X,R_j],R_j), \tag{3.4}$$

where

$$A^j_{il} = \begin{cases} \lambda_i(R_l) = \delta_{il}, \text{ if } l \neq j \\ \lambda_i([X,R_j]) = g([X,R_j],R_i), \text{ if } l = j. \end{cases}$$

Finally, from Equations (3.3) and (3.4), it follows

$$\begin{aligned} \iota_X \, d\lambda(R_1, \cdots, R_q) &= -\sum_j \lambda(R_1, \cdots, R_{j-1}, [X,R_j], R_{j+1}, \cdots, R_q) \\ &= -\sum_j g([X,R_j],R_j) \\ &= \mathrm{tr} W^X, \end{aligned}$$

that is, the interior product $i_X \, d\lambda$, when evaluated on the global frame defined by the Reeb vector fields, yields the mean curvature in the X direction.

On the other hand, every normal vector is *foliate*, that is, their Lie bracket with any vector field tangent to $\mathcal{R}$ is again tangent to $\mathcal{R}$ (cf. Appendix A), implying

$$\mathrm{tr} W^X = \iota_X \, d\lambda(R_1, \cdots, R_q) = 0.$$

In particular, L is a minimal submanifold.

For L an orbit of the action generated by $R_{i_1}, \cdots, R_{i_d}$ just consider now the d-form

$$\eta = \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_d}$$

and notice that the normal bundle of L is now the direct sum of ξ with the spam of the other Reeb vector fields not appearing in the set $\{R_{i_1}, \cdots, R_{i_d}\}$. The same calculations as above will yield

$$\mathrm{tr} W^X = \iota_X \, d\eta(R_{i_1}, \cdots, R_{i_d}) = 0$$

for any vector field X normal to L.

□

Remark 8. Suppose that we take $d = 1$ in Proposition 18, so that L is a one-dimensional submanifold generated by a Reeb vector field, that is, a flow line of R_i for some i. Let γ denote one of such flow lines, and notice that it is automatically parameterised by arclength, since

$$\dot{\gamma}(s) := \frac{d}{dt}\Big|_{t=0} \gamma(s+t) = R_i(\gamma(s))$$

by definition and $g(R_i, R_i) = 1$ by construction. Since R_i has unit length, it follows that, for any normal vector field X, one has

$$0 = \nabla_{\dot{\gamma}} g(X, \dot{\gamma}) = g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) + g(X, \nabla_{\dot{\gamma}} \dot{\gamma}),$$

hence

$$g(W^X(\dot{\gamma}), \dot{\gamma}) = -g(X, \nabla_{\dot{\gamma}} \dot{\gamma}).$$

Thus, L is a minimal submanifold if and only if $\nabla_{\dot{\gamma}} \dot{\gamma}$ is tangent to L. But $\nabla_{\dot{\gamma}} \dot{\gamma}$ is already normal to L, since $0 = \nabla_{\dot{\gamma}} g(\dot{\gamma}, \dot{\gamma}) = 2g(\nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma})$. Therefore, the leaf L is minimal if and only if its arclength parametrisation γ satisfies $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, that is, γ is a geodesic.

To summarise the above discussion:

Corollary 3. There is a metric on M such that each Reeb vector field is of unit length, their flow lines are geodesics, and the foliations $\mathcal{F}$ and $\mathcal{C}_i$ are harmonic foliations.

Remark 9. Recall that a differential p-form η is called $\mathcal{F}$-closed if

$$\mathrm{d}\eta(X_1, \cdots, X_{p+1}) = 0,$$

whenever at least p fields among $X_1, \cdots, X_{p+1}$ are tangent to $\mathcal{F}$. The form $\eta = \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_d}$ is always $\mathcal{S}$-closed, where $\mathcal{S}$ denote the underlying foliation of the bundle spanned by the Reeb fields $R_{i_1}, \cdots, R_{i_d}$. In the case $d = q$, this means the characteristic form $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q$ is $\mathcal{F}$-closed. From this point of view, Proposition 18 is simply an application of Rummler's criterion for tautness (RUMMLER, 1979).

We recall the following definitions.

Definition 13 (*Harmonicity*)**.** Let $(M; g)$ be a C^2 Riemannian manifold, and $f : M \to \mathbb{R}$ a C^2 function. We say f is **harmonic** if it is a zero of the Laplace–Beltrami operator Δ, that is,

$$\Delta f = \nabla(\nabla f) = 0,$$

where ∇ is the Levi-Civita connection of (M, g). Similarly, a C^2-differential form ω is **harmonic** if it is a zero of the Laplace–de Rham operator Δ, that is,

$$\Delta\omega = \mathrm{d}(\delta\omega) + \delta(\mathrm{d}\omega) = (\delta + \mathrm{d})^2\omega,$$

where δ is the adjoint of the exterior differential d.

There are several links between harmonic foliations, harmonic functions, and harmonic differential forms (cf. (KAMBER; TONDEUR, 1982)). We are particularly interested in the following improved version of Proposition 10.

Proposition 19. Suppose M supports a q-contact structure of class C^2. Around each point $p \in M$ one can find coordinates $(x,y,z) \in \mathbb{R}^{2n} \times \mathbb{R}^q$ and, for each $1 \leq i \leq q$ and $1 \leq j \leq n$, functions $f^i_j, g^i_j : \mathbb{R}^{2n} \to \mathbb{R}$ such that:

(i) $\lambda_i = \mathrm{d}z_i + \sum_j \left(-f^i_j(x,y)\,\mathrm{d}x_j + g^i_j(x,y)\,\mathrm{d}y_j \right)$ for every i;

(ii) each coordinate function z_i and differential form $\mathrm{d}z_i$ is harmonic.

Proof. Given $p \in (M,g)$, let U be an open geodesic ball of radius ε around p, and choose a foliated chart $\psi : U \to \mathbb{R}^{2n} \times \mathbb{R}^q$ such that $R_i = \partial_{z_i}$ and $\psi(p) = 0$. Then the inclusion

$$\begin{aligned} z_j : (-\varepsilon, \varepsilon) &\longrightarrow M \\ t &\longmapsto (0, \cdots, 0, z_j(t), 0, \cdots, 0) \end{aligned}$$

is an arclength parameterisation of a segment of the flow line of R_j through the point p, and being so, it is a geodesic of g (see Remark 8 above). In particular, the function z_j is a Riemannian immersion, and its image is a minimal submanifold, hence z_j is harmonic (see the Proposition in Paragraph § 2 of (EELLS; SAMPSON, 1964)) and therefore so is its derivative $\mathrm{d}z_j$ (see also (EELLS; SAMPSON, 1964), Paragraph §3). This proves (ii). The proof of (i) is precisely the same as in Proposition 10.

□

3.4 On the topology of q-contact manifolds

It was proven by Tischler that if a closed manifold admits a closed non-vanishing 1-form, then it is a fibre bundle over S^1. From this, we have:

Proposition 20. If M is a closed q-contact manifold and $\mathrm{d}\lambda_i = \mathrm{d}\lambda_j$ for any pair i, j then M is a fibre bundle over S^1. If for any choice of indices $i_1, \cdots, i_l \subset \{1, \cdots, q\}$ one has $\mathrm{d}\lambda_{i_1} = \cdots = \mathrm{d}\lambda_{i_l}$ then $\dim H^1_{dR}(M) \geq l - 1$.

Proof. The 1-form $\alpha := \lambda_i - \lambda_j$ is nowhere vanishing, since $\alpha_p = 0$ would imply $R_i|_p = R_j|_p$, which in turn would mean the fibre $\mathcal{R}_p$ has dimension less than q, a contradiction. The form α is also closed by hypothesis; hence Tischler's theorem proves the first assertion. For the second one, note that each form $\alpha_j := \lambda_{i_1} - \lambda_{i_j}$ is closed and that they form an l.i. set. Moreover, the forms α_j can not be exact since any function $f : M \to \mathbb{R}$ has a critical point x. Hence, exactness of α_j would imply $\mathrm{d}f_x = \alpha_j|_x \equiv 0$, which can not happen as $\alpha_j(R_{i_1}) \equiv 1$. Hence $\{[\alpha_j]\}$ is an l.i. set of nontrivial cohomology classes.

□

Corollary 4. Let M be a closed, simply connected manifold. If M supports a q-contact structure with adapted coframe $\vec{\lambda}$, then $\lambda_i \neq \mathrm{d}\lambda_j$ for every $1 \leq i, j \leq q$. In particular, simply connected closed manifolds support no uniform q-contact structures for $q \geq 2$.

Corollary 5. A q-contact foliation defined by $\vec{\lambda} = (\lambda_1, \cdots, \lambda_q)$ on a sphere S^{2n+q} can not have $\mathrm{d}\lambda_{i_1} = \mathrm{d}\lambda_{i_2}$ for more than one pair of indices i_1, i_2. In particular, for $q > 2$, spheres admit no uniform q-contact structure.

We can improve Proposition 20 in the case of uniform q-contact structures, obtaining M as a fibration over the torus $\mathbb{T}^{q-1}$.

Theorem 4. Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a uniform q-contact manifold, $q \geq 2$. Suppose that M is of class at least C^2, and the splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$ is C^1, and consequently, so are the Reeb fields and the induced foliations. Then each characteristic foliation $\mathcal{C}_i$ is transverse to the leaves of a $(2n+1)$-dimensional C^2-foliation without holonomy.

Proof. We follow the steps of (TISCHLER, 1970, Corollary II). For simplicity, let us work with the characteristic foliation $\mathcal{C}_q$, associated with λ_q, noting that this choice implies no loss of generality. We define

$$\alpha_i := \lambda_i - \lambda_q, \quad i = 1, \cdots, q-1.$$

As in Proposition 20, these are linearly independent, non-vanishing closed forms. Moreover, α_i is a leaf-wise volume form for the foliation $\mathcal{L}_i$ defined by the flow of R_i, since $\alpha_i(R_i) \equiv 1$. In other words, the closed form α_i is *transverse* to $\mathcal{L}_i$ (cf. (SULLIVAN, 1976).

First, let us show that one can perturb the 1-forms α_i without losing linear independence. Given a finite atlas for M, we can define suitable C^r norms on $\wedge^1(M)$ as the maximum over all charts of the sum of C^r norms of coordinate functions. We consider on $\wedge^1(M)$ the topology obtained as direct limit of the C^r norms, and denote by $\mathcal{D}_1$ the corresponding topological $\mathbb{R}$-vector space (cf.(DE RHAM, 1973), (CANDEL; CONLON, 2000a). Due to the linear independence of $\{\alpha_i\}$, we can use the Hann-Banach theorem to find for each α_i a corresponding continuous dual (a *current*) $f_i : \mathcal{D}_1 \to \mathbb{R}$ in the strong dual space $\mathcal{D}_1^*$, satisfying

$$f_i(\alpha_j) = \delta_{ij}.$$

We define a mapping from the $(q-1)$-fold product of $\mathcal{D}_1$ with itself to $\mathbb{R}$ by

$$\begin{aligned} \Phi : \mathcal{D}_1 \times \cdots \times \mathcal{D}_1 &\longrightarrow \mathbb{R} \\ (\omega_1, \cdots, \omega_{q-1}) &\longmapsto \det\{f_i(\omega_j)\}. \end{aligned}$$

This mapping is continuous, and since $\Phi(\alpha_1, \cdots, \alpha_{q-1}) = 1$, we can find an open neighbourhood U of $(\alpha_1, \cdots, \alpha_{q-1})$ on which Φ is always positive. This means the matrix $\{f_i(\omega_j)\}$

is invertible for any choice of $(\omega_1, \cdots, \omega_{q-1}) \in U$, and consequently the set $\{\omega_1, \cdots, \omega_{q-1}\}$ is linearly independent for every element of U.

Now, each α_i can be arbitrarily well approximated by a C^2-closed 1-form ω_i with the properties:

(i) the foliation $\mathcal{N}_i$, integral to the bundle $\ker \omega_i$, comprises the fibres of a C^2 fibration $\rho_i : M \to S^1$, that is, $\mathcal{N}_i(p) = \rho_i^{-1}(\rho(p))$;

(ii) the leaves of $\mathcal{N}_i$ are transverse to $\mathcal{L}_i$.

See, for instance, (TISCHLER, 1970, Theorem 1) and (CANDEL; CONLON, 2000a, Theorem 9.4.2 and Proposition 10.3.14)) for better exposition of these results. Here we limit ourselves to emphasise that the C^1 property is necessary in order to apply such theorems to our setting.

Since the linear independence of a set of $(q-1)$ 1-forms is an open property, we can choose the ω_i to be linearly independent. We set

$$\begin{aligned} \rho : M &\longrightarrow \mathbb{T}^{q-1} \approx S^1 \times \cdots \times S^1 \\ x &\longmapsto (\rho_1(x), \cdots, \rho_{q-1}(x)). \end{aligned}$$

The mapping ρ is a C^2 submersion since the ω_i are linearly independent; moreover, it is proper, being a map between compact manifolds. Erehsmann Fibration Theorem shows that ρ is a (locally trivial) fibration. Each fibre $\mathcal{N}(p)$ of this fibration is an intersection

$$\mathcal{N}(p) := \bigcap_{i=1}^{q-1} \mathcal{N}_i(p).$$

By construction, the fibres are transverse to each of the Reeb fields $R_1, \cdots, R_{q-1}$, and consequently to the characteristic foliation $\mathcal{C}_q$. Being a fibration, the foliation $\mathcal{N}$ is without holonomy.

□

The most immediate and essential consequence of Theorem 4 is the existence of complete closed transversals for the characteristic foliations, i.e., any of the fibres of $\pi : M \to \mathbb{T}^{q-1}$.

Corollary 6. If $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ is a C^1 uniform q-contact manifold and $q \geq 2$, then every characteristic foliation $\mathcal{C}_i$ admits a complete transversal $\mathcal{T}_i$, that is, a closed $(2n+1)$-dimensional submanifold of M intercepting every leaf of $\mathcal{C}_i$ and satisfying

$$\mathrm{T}M = \mathrm{T}\mathcal{T}_i \oplus \mathrm{T}\mathcal{C}_i.$$

In light of Proposition 15, we have the following equivalence of Weinstein Conjectures:

Theorem 5. For closed uniform q-contact manifolds, the Weak Generalised Weinstein Conjecture is equivalent to the Weinstein Conjecture.

Proof. The implication **WGWC** $\implies$ **WC** is clear since a flow line that is not homeomorphic to $\mathbb{R}$ has to be closed. Conversely, if the Weinstein Conjecture is true, than $(\mathcal{T}_i, \lambda_i|_{\mathcal{T}_i})$ has a closed Reeb orbit, being a closed contact manifold (cf. Proposition 15). This, in turn, implies the existence of a closed orbit γ for $\mathcal{L}_i$, as pointed out in Remark 6. The leaf of $\mathcal{F}$ containing the closed orbit γ is not a plane; hence M satisfy the **WGWC**.

□

Another consequence of Corollary 5 is the following, a direct application of a famous result of Plante (PLANTE, 1975, Theorem 1.1).

Corollary 7. If a closed manifold M admits a uniform q-contact structure, then the homology group $H_{2n}(M;\mathbb{R})$ is non-zero.

Recall that an overtwisted contact 3-manifold is one which contains an embedded overtwisted disk. In higher dimensions, the role of the overtwisted disk can be filled by a *Plastikstufe* (NIEDERKRÜGER, 2006), or, more generally, by overtwisted $2n$-closed balls (BORMAN; ELIASHBERG; MURPHY, 2015; CASALS; MURPHY; PRESAS, 2019). In any case, it is known that such contact manifolds satisfy the **WC** (ALBERS; HOFER, 2009; BORMAN; ELIASHBERG; MURPHY, 2015). If we define a q-contact structure to be overtwisted when it "contains" an overtwisted contact structure in the classical sense, we can conclude that all such structures satisfy the **WGWC**. More precisely:

Definition 14 (*Overtwisted q-contact structures*)**.** A q-contact structure is **overtwisted** when, for some of the defining 1-forms λ_i, the characteristic foliation $\mathcal{F}_i$ admits a transversal $\mathcal{T}$ such that $(\mathcal{T}, \lambda_i)$ is an overtwisted contact manifold.

From Corollary 5 and Theorem 5 it follows:

Theorem 6. Every closed uniform overtwisted q-contact manifold satisfies the **WGWC**.

CHAPTER

4

INVARIANT METRICS

Last Chapter ended with a theorem giving conditions under which the **WGWC** holds, namely Theorem 5 and 6. In the present Chapter, we work on conditions to guarantee that the **SGWC** is satisfied. As we will see, the existence of an invariant metric for the action F implies strong recurrence properties for the contact foliation $\mathcal{F}$ and in the existence of $\mathcal{F}$-preserving toric actions on M, which we will be able to use to find closed leaves.

4.1 Isometric contact foliations

Following the work of Rukimbira for the contact case (cf. (RUKIMBIRA, 1991; RUKIMBIRA, 1993)), we define the following notion.

Definition 15 (*Isometric contact foliation*)**.** A foliation $\mathcal{F}$ induced by a q-contact structure is called a **isometric q-contact foliation** if there is a metric g [1] with respect to which the Reeb vector fields are Killing, that is

$$\mathcal{L}_{R_i} g = 0.$$

In other words, $\mathcal{F}$ is isometric if it is induced by an action of $\mathbb{R}^q$ through isometries. We say $(M, g, \vec{\lambda}, \xi \oplus \mathcal{R})$ is an **isometric q-contact manifold.**

There is always a metric on the bundle $\mathcal{R}$ for which the Reeb vector fields are Killing, namely the adapted metric

$$g^{\tau} = \sum_i \lambda_i \otimes \lambda_i,$$

[1] There is a similar notion in the contact topology literature, called a **K-contact manifold** (cf. (KON; YANO, 1985; BLAIR, 2010)). In a K-contact manifold (M, g, λ), the Reeb vector field R is Killing with respect to a **contact metric** g, which is a Riemannian metric satisfying a number of assumptions related to the non-degenerate form $d\lambda$. Here, we do not make any assumptions on the metric other than its invariance under the action.

that is, the metric defined by setting $g^\tau(R_i,R_j)=\delta_{ij}$, and extending it linearly. It satisfies

$$\mathcal{L}_{R_j}g^\tau=\sum_i(\mathcal{L}_{R_j}\lambda_i\otimes\lambda_i+\lambda_i\otimes\mathcal{L}_{R_j}\lambda_i)=0.$$

Therefore, asking for a foliation to be isometric amounts to assert the existence of a transverse invariant metric, that is, a metric $g^\perp$ on the bundle ξ such that $\mathcal{L}_{R_j}g^\perp=0$ for every Reeb vector field R_j. Since every transverse invariant metric is associated with a bundle-like metric (MOLINO, 1988, Proposition 3.3), it follows that, up to a choice of metric tensor, every Riemannian contact foliation is, in fact, isometric.

Theorem 7. For a q-contact manifold $(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ the following are equivalent

(i) The contact foliation $\mathcal{F}$ has a bundle-like metric;

(ii) The contact foliation $\mathcal{F}$ has an invariant metric;

(iii) There exists a metric g on M for which the Reeb vector fields R_i are Killing, and

$$\lambda_i(X)=g(R_i,X)$$

for any vector field on M.

Proof. (**i**) $\implies$ (**ii**): Suppose g_0 is a bundle-like metric for $\mathcal{F}$, and g^T is the associated $\mathcal{F}$-transverse metric. By definition, g^T is $\mathcal{F}$-basic, hence $\mathcal{L}_{R_i}g^T=0$ for every i. As before, we let

$$g^\tau=\sum_i\lambda_i\otimes\lambda_i$$

be the adapted metric of the collection $\vec{\lambda}$, so that $\mathcal{L}_{R_i}g^\tau=0$ for every i. It follows that $g:=g^\tau\oplus g^T$ is a Riemannian metric on M with respect to which every Reeb field is Killing.

(**ii**) $\implies$ (**iii**) If $\mathcal{F}$ is isometric and $\tilde{g}$ is a Riemannian metric with respect to which every Reeb field is Killing, then we define a new metric g on M by letting $g^\perp$ be the restriction of $\tilde{g}$ to the bundle ξ and setting

$$g=g^\tau\oplus g^\perp,$$

where $g^\tau=\sum_i\lambda_i\otimes\lambda_i$, as above. Then $\mathcal{L}_{R_i}g=0$ and $g(R_i,\cdot)=\lambda_i$.

(**iii**) $\implies$ (**i**) If every Reeb field is Killing then the contact action is an isometric action of $\mathbb{R}^q$, and it follows that g is bundle-like with respect to $\mathcal{F}$ (cf. (MOERDIJK; MRČUN, 2003, Remmark 2.7(8))).

□

Following the terminology of Rukimbira, we define

Definition 16 (*R-metric*)**.** On a q-contact manifold $(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ a metric g satisfying

(i) $\mathcal{L}_{R_i} g = 0$;

(ii) $g(R_i, \cdot) = \lambda_i$,

is called an R-metric of the foliation $\mathcal{F}$.

Remark 10. Note that Proposition 7 can be restated as "a contact foliation is isometric if and only if it admits an R-metric". *From now on, whenever we say the triple $(M, \mathcal{F}, g)$ is an isometric contact foliation, it is implicitly assumed that g is an R-metric.*

Example 9 (The canonical contact structure of $\mathbb{R}^{2n+q}$)**.** Recall that $\mathbb{R}^{2n+q}$ with coordinates $(x,y,z) = (x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_q)$ has a canonical contact structure given by the forms

$$\lambda_i := \mathrm{d}z_i + \sum_j x_j \, \mathrm{d}y_j.$$

The Reeb vector fields are ∂_{z_i}, and we can identify $\mathcal{R}$ with $\mathbb{R}^q \subset \mathbb{R}^{2n+q}$, the subspace of vectors with coordinates $x = y = 0$. Similarly, ξ is identified with $\mathbb{R}^{2n} \subset \mathbb{R}^{2n+q}$, realised as the set of vectors $\{z = 0\}$. We can consider on ξ the metric $g^{\perp}$, obtained as the restriction to $\mathbb{R}^{2n}$ of the canonical metric of $\mathbb{R}^{2n+q}$. Moreover, let $g^{\tau} = \sum_i \lambda_i \otimes \lambda_i$, as usual. Then the metric $g^{\perp}$ is invariant under the Reeb vector fields, as the ∂_{z_i} are Killing vector fields of the canonical metric on $\mathbb{R}^{2n+q}$, and therefore $g := g^{\tau} \oplus g^{\perp}$ satisfies

$$\mathcal{L}_{\partial_{z_i}} g = 0,$$

that is, g is an R-metric.

As for examples on closed manifolds, we have product-like constructions such as

Example 10 (Products of structures of same dimension)**.** Let $(M_1, \vec{\alpha}, \mathcal{R}_1 \oplus \xi_1, g_1)$ and $(M_2, \vec{\beta}, \mathcal{R}_2 \oplus \xi_2, g_2)$ be isometric q-contact manifolds (i.e., both have contact foliations of same dimension, namely q). Then $M := M_1 \times M_2$ is a $2q$-contact manifold with adapted coframe given by the forms

$$\lambda_i = \alpha_i + \beta_i,$$
$$\eta_i = \alpha_i - \beta_i,$$

as seen in Examples 4 and 5. We set on $\mathrm{T}M \equiv \mathrm{T}M_1 \oplus \mathrm{T}M_2$ a metric g by declaring $\mathrm{T}M_1$ orthogonal to $\mathrm{T}M_2$ and letting g restrict to g_i on $\mathrm{T}M_i$. In other words, let

$$g := g_1 \oplus g_2.$$

Now, if X_i is the Reeb field of α_i and Y_i the Reeb field of β_i, then the Reeb field of λ_i is

$$R_i = X_i + Y_i,$$

and the Reeb field of η_i is

$$S_i = X_i - Y_i.$$

So we have

$$\mathcal{L}_{R_i} g = \mathcal{L}_{R_i} g_1 + \mathcal{L}_{R_i} g_2 = \mathcal{L}_{X_i} g_1 + \mathcal{L}_{Y_i} g_1 + \mathcal{L}_{X_i} g_2 + \mathcal{L}_{Y_i} g_2 = 0,$$

and similarly for $\mathcal{L}_{S_i} g$. Thus g is an R-metric for M.

Example 11 (Toric extensions)**.** Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a q-contact manifold admitting an R-metric $g = g^\tau \oplus g^\perp$. Suppose $\pi : E \to M$ is a principal $\mathbb{T}^l$-bundle equipped with a flat connection $\mathrm{T}E = H \oplus V$. As it is shown in detail in Example 7, E admits a $(q+l)$-contact structure determined by 1-forms

$$\begin{aligned} \eta_i &= \pi^* \lambda_i, \text{ for } 1 \leq i \leq q; \\ \eta_{q+i} &= \alpha_i + \pi^* \lambda_{j_i}, \text{ for } 1 \leq i \leq l, \end{aligned}$$

where the α_i are 1-forms that are identically zero on the horizontal bundle and on the vertical bundle are given by

$$\alpha_i(X_j) = \delta_{ij},$$

with the X_i being the fields generating the toric action on the fibres. The identification of H with $\mathrm{T}M$ via π induces a splitting $H = \overline{\mathcal{R}} \oplus \widetilde{\xi}$, where $\mathcal{R} \approx \overline{\mathcal{R}}$ and $\xi \approx \widetilde{\xi}$, both diffeomorphisms being a restriction of $\mathrm{d}\pi$. Thus $\mathrm{T}E = \widetilde{\mathcal{R}} \oplus \xi$, where $\widetilde{\mathcal{R}} = V \oplus \overline{\mathcal{R}}$ is the bundle tangent to the $(q+l)$-dimensional contact foliation $\widetilde{\mathcal{F}}$ on E. The Reeb vector fields are

$$\begin{aligned} \widetilde{R}_i &= R_i, \text{ for } 1 \leq i \leq q; \\ \widetilde{R}_{q+i} &= \frac{1}{2}(X_i + R_{j_i}), \text{ for } 1 \leq i \leq l. \end{aligned} \tag{4.1}$$

We want to show that $(E, \vec{\eta}, \mathrm{T}E = \widetilde{\mathcal{R}} \oplus \widetilde{\xi})$ defines an isometric contact foliation for a suitable choice of metric $\widetilde{g}$ on E. For this, let g_0 be a metric on the tangent bundle $\widetilde{\mathcal{R}}$ defined by

$$g_0 := \sum_{i=1}^{q+l} \eta_i \otimes \eta_i.$$

This makes $\{\widetilde{R}_i\}$ into an orthonormal basis for $\overline{\mathcal{R}}$. Finally, we let

$$\widetilde{g} := g_0 \oplus g^\perp$$

be a Riemannian metric on $\mathrm{T}E = \widetilde{\mathcal{R}} \oplus \widetilde{\xi}$.

We claim that $\widetilde{g}$ is an R-metric for E. First, it is clear from the definition of $\widetilde{g}$ that

$$\widetilde{g}(X, \widetilde{R}_i) = g_0(X, \widetilde{R}_i) = \eta_i(X). \tag{4.2}$$

It is then straight forward that $\mathcal{L}_{\widetilde{R}_i} g_0 \equiv 0$, since

$$\begin{aligned}\left(\mathcal{L}_{\widetilde{R}_i} g_0\right)(\widetilde{R}_j, \widetilde{R}_l) &= \mathcal{L}_{\widetilde{R}_i}(\iota_{\widetilde{R}_l}\iota_{\widetilde{R}_j} g_0) - g_0([\widetilde{R}_i, \widetilde{R}_j], \widetilde{R}_l) - g_0(\widetilde{R}_j, [\widetilde{R}_i, \widetilde{R}_l]) \\ &= \mathcal{L}_{\widetilde{R}_i}\delta_{ij} - 0 - 0 \\ &= 0.\end{aligned}$$

Now, from the characterisation of the Reeb fields in Equation (4.1), we have

$$\begin{aligned}\mathcal{L}_{\widetilde{R}_i}\widetilde{g} = \mathcal{L}_{R_i}\widetilde{g} = \mathcal{L}_{R_i} g_0 + \mathcal{L}_{R_i} g^{\perp} = 0, &\quad \text{for } 1 \le i \le q; \\ \mathcal{L}_{\widetilde{R}_{q+i}}\widetilde{g} = \mathcal{L}_{\widetilde{R}_{q+i}} g^{\perp} = \frac{1}{2}\mathcal{L}_{X_i} g^{\perp} + \frac{1}{2}\mathcal{L}_{R_i} g^{\perp} = \frac{1}{2}\mathcal{L}_{X_i} g^{\perp}, &\quad \text{for } 1 \le i \le l.\end{aligned} \tag{4.3}$$

Recall, from Example 7, that for every generator field X_1 it holds that

$$[X_i, Z] \text{ is horizontal whenever } Z \text{ is horizontal.}$$

On the other hand, we know every vector field Z tangent to $\widetilde{\xi}$ is foliate with respect to $\widetilde{\mathcal{F}}$, that is, it satisfy $[\widetilde{R}_i, Z] \in \widetilde{\mathcal{R}}$ for every Reeb field. Together, these two conditions imply

$$[X_i, Z] \in \overline{\mathcal{R}} \text{ whenever } Z \in \Gamma(\widetilde{\xi}).$$

In particular, for Y, Z tangent to $\widetilde{\xi}$ we have

$$(\mathcal{L}_{X_i} g^{\perp})(Y, Z) = \mathcal{L}_{X_i}(\iota_Z \iota_Y g^{\perp}) - g^{\perp}([X_i, Y], Z) - g^{\perp}(Y, [X_i, Z]) = \mathcal{L}_{X_i}(\iota_Z \iota_Y g^{\perp}) = 0,$$

as the function $\iota_Z \iota_Y g^{\perp}$ can be thought as a lift to E of a leaf-wise function of $(M, \mathcal{F})$ to a leaf-wise function of $(E, \widetilde{\mathcal{F}})$, whose leaves all of the form $\widetilde{\mathcal{F}}(x) \approx \mathbb{T}^l \times \mathcal{F}(\pi(x))$, with the X_i being vectors on the $\mathbb{T}^l$ directions. Thus, Equation (4.3) reduces to $\mathcal{L}_{\widetilde{R}_i}\widetilde{g} \equiv 0$, which together with (4.2) means that $\widetilde{g}$ is an R-metric for $(E, \vec{\eta}, \mathrm{T}E = \widetilde{\mathcal{R}} \oplus \widetilde{\xi})$, and $\widetilde{\mathcal{F}}$ is an isometric contact foliation.

In particular, by taking products of manifolds supporting R-flows and products of such manifolds with tori, we can produce isometric contact foliations of any dimension $q \ge 2$. We remark that the resulting manifold is closed if we start with closed manifolds. Examples of R-flows include regular and almost regular contact manifolds (the canonical contact structures on odd spheres S^{2n+1} are specific examples of regular contact manifolds, cf. (BLAIR, 2010; GEIGES, 2008) for more details and examples), and also every

K-contact manifold. This last class includes, in particular, compact contact hypersurfaces (BANYAGA, 1993), Kähler manifolds of constant positive holomorphic sectional curvature, and Brieskorn manifolds (KON; YANO, 1985). A product of any such manifold with a torus $\mathbb{T}^{q-1}$ provides the desired isometric q-contact foliation.

Another class of examples is that of *metric f-K-contact manifolds*. These are f-manifolds with complemented frames $\{\lambda_1, \cdots, \lambda_q\}$ such that the Reeb fields are Killing for the associated metric

$$g = \omega \circ (f \times f) + \sum_i \lambda_i \otimes \lambda_i,$$

where ω is a 2-form satisfying $\mathrm{d}\lambda_i = \omega$ for every i, called the *fundamental form* (see (GOERTSCHES; LOIUDICE, 2020b) for more on such structures). In particular, every metric f-K-contact structure is a uniform isometric q-contact manifold.

4.2 The topology of isometric contact manifolds

Besides the orientability of the manifold and the suitable reduction of the structure group, when it comes to the existence of isometric contact foliations, the curvature of a metric also plays the role of an obstruction, which in turn restricts the topology of a manifold supporting such an action. Explicit examples of such phenomena are consequences of the fact that on an isometric contact manifold *every harmonic field most belong to* $\Gamma(\xi)$. Recall from Definition 13 that a differential form ω is harmonic if it belongs to the kernel of the Beltrami-de Rham operator Δ.

Definition 17 (*Harmonic fields*)**.** We say a field X is **harmonic** with respect to a metric g if its dual form $g(X,\cdot)$ is harmonic.

Remark 11. When a form ω has compact support harmonicity is equivalent to the condition that $\mathrm{d}\omega = \delta\omega = 0$. In particular, every harmonic form on a compact manifold is closed. It is a celebrated theorem due to Hodge that, for compact M, every de Rham class in $H^*_{dR}(M)$ has a harmonic representative.

Theorem 8. Let $(M, \mathcal{F}, g)$ be an isometric contact foliation on a closed manifold M. If a vector field is harmonic with respect to g, then X is tangent to the contact distribution ξ. In particular, every harmonic 1-form on M is basic.

Proof. The coordinate function of X in the direction R_i is

$$c_i := \lambda_i(X) = g(R_i, X) = \mu(R_i).$$

We wish to show that $c_i \equiv 0$ for every i. We divide the proof into two steps: first, we show that if we assume that $c_i \neq 0$ for some value of i, then there is no loss of generality in assuming $c_i \neq 0$ for a single value of i. Then we show that the latter case cannot happen.

Step I- *We may assume* $\mu(R_j) = c_l \delta_{lj}$ *for a single index* l. Since X is harmonic and R_i is a Killing field, it follows that $c_i = \mu(R_i)$ must be a constant for $i = 1, \cdots, q$ (cf. (POOR, 2015, Proposition 5.13)). Let us separate the components of X in the $\mathcal{R}$ directions and write $X = \sum_i c_i R_i + \widetilde{X}$, with $\widetilde{X}$ tangent to ξ. Taking duals with respect to g, let us write the harmonic form μ as a sum

$$\mu = \sum_{i=1}^{q} c_i \lambda_i + \widetilde{\mu}. \tag{4.4}$$

In particular, $\widetilde{\mu}(R_i) = 0$ for every Reeb vector field, by construction. As the manifold M is closed, harmonicity implies that μ is closed. Hence

$$\sum_{i=1}^{q} c_i \, \mathrm{d}\lambda_i = -\mathrm{d}\widetilde{\mu}. \tag{4.5}$$

Now, around any point $p \in M$, we can choose a foliated chart $(U; x_1, \cdots, x_{2n}, z_1, \cdots, z_q)$ as in Proposition 19. We expand both $\widetilde{\mu}$ and the forms λ_i in these coordinates so that μ is written as

$$\mu = \sum_{i=1}^{q} \left(c_i \, \mathrm{d}z_i + \sum_{j=1}^{2n} c_i f_j^i(x) \, \mathrm{d}x_j \right) + \underbrace{\sum_{j=1}^{2n} \alpha_j(x,z) \, \mathrm{d}x_j + \sum_{s=1}^{q} \beta_s(x,z) \, \mathrm{d}z_s}_{\widetilde{\mu}}.$$

By construction, $\widetilde{\mu}(R_i) = 0$ for every i. Since in these coordinates $R_i = \partial_{z_i}$, the functions β_s on the expansion above are actually constants equal to zero. Hence

$$\mu = \sum_{i=1}^{q} \left(c_i \, \mathrm{d}z_i + \sum_{j=1}^{2n} c_i f_j^i(x) \, \mathrm{d}x_j \right) + \sum_{j=1}^{2n} \alpha_j(x,z) \, \mathrm{d}x_j, \tag{4.6}$$

and, in these coordinates, the equality in (4.5) becomes

$$\sum_{j,l=1}^{2n} \left(\sum_i c_i \frac{\partial}{\partial x_l} f_j^i(x) \, \mathrm{d}x_l \wedge \mathrm{d}x_j \right) = -\sum_{j,l=1}^{2n} \frac{\partial}{\partial x_l} \alpha_j(x,z) \, \mathrm{d}x_l \wedge \mathrm{d}x_j - \sum_{j=1}^{2n} \sum_{l=1}^{q} \frac{\partial}{\partial z_l} \alpha_j(x,z) \, \mathrm{d}z_l \wedge \mathrm{d}x_j.$$

Evaluating both sides of the above equation at the Reeb vector fields implies that

$$\frac{\partial}{\partial z_l} \alpha_j(x,z) = 0 \quad \text{for every } 1 \leq l \leq q,$$

and therefore, the α_j are functions of the coordinate x, satisfying

$$-\frac{\partial}{\partial x_l} \alpha_j(x) = \frac{\partial}{\partial x_l} \sum_i c_i f_j^i(x) \quad \text{for every } 1 \leq l \leq 2n.$$

This means there are constants K_j such that

$$\alpha_j(x) = -\sum_i c_i f_j^i(x) + K_j.$$

Applying this to Equation (4.6) gives

$$\mu = \sum_{i=1}^{q} c_i \,\mathrm{d}z_i + \sum_{j=1}^{2n} K_j \,\mathrm{d}x_j.$$

Choose an index l such that $c_l \neq 0$, and write

$$\mu - \sum_{i \neq l} c_i \,\mathrm{d}z_i = c_l \lambda_l + \sum_{j=1}^{2n} K_j \,\mathrm{d}x_j.$$

The RHS is a harmonic form, for each $\mathrm{d}z_i$ is harmonic (cf. Proposition 19). Thus the 1-form

$$\mu_0 := c_l \lambda_l + \sum_{j=1}^{2n} K_j \,\mathrm{d}x_j$$

is a harmonic form on M satisfying

$$\mu_0(R_j) = c_l \delta_{jl},$$

as we wanted.

Step II - *Given a constant $c_l \neq 0$, there can be no harmonic form μ with $\mu(R_j) = c_l \delta_{lj}$.* Let us assume the existence of such a form. To simplify things a bit, let the non-zero coefficient be c_1. Write $X = c_1 R_1 + \widetilde{X}$ to denote the dual vector field of μ, where $\widetilde{X}$ is tangent to ξ. Then

$$\eta := \lambda_1 - \frac{1}{c_1}\mu$$

is a basic 1-form on M such that $\mathrm{d}\eta = \mathrm{d}\lambda_1$ (cf. Appendix A). This follows because, as $\mu(R_j) = c_1 \delta_{1j}$, we have

$$\eta(R_i) = 0 \text{ for every } i.$$

Moreover, since μ is harmonic, $\mathcal{L}_Y \mu = 0$ for every Killing field Y (cf (POOR, 2015, Proposition 5.13)). In particular,

$$\mathcal{L}_{R_i}\eta = \mathcal{L}_{R_i}\lambda_1 + \mathcal{L}_{R_i}\mu + \left(R_i \frac{1}{c_1}\right)\mu = 0 \text{ for every } i,$$

so that η is indeed basic. Finally, as M is closed, $\mathrm{d}\mu = 0$, hence $\mathrm{d}\eta = \mathrm{d}\lambda_1$. Consequently, we have

$$\mathrm{d}(\eta \wedge (\mathrm{d}\lambda_1)^{n-1}) = \mathrm{d}\eta \wedge (\mathrm{d}\lambda_1)^{n-1} = (\mathrm{d}\lambda_1)^n. \tag{4.7}$$

Recall that the characteristic form λ is the leaf-wise volume form dor the foliation $\mathcal{F}$. Using Equation (4.7) above we obtain a volume form on M written as

$$\lambda \wedge (\mathrm{d}\lambda_1)^n = \lambda \wedge \mathrm{d}(\eta \wedge (\mathrm{d}\lambda_1)^{n-1}) = (-1)^q \mathrm{d}\lambda \wedge \eta \wedge (\mathrm{d}\lambda_1)^{n-1} + (-1)^{q+1}\mathrm{d}(\lambda \wedge \eta \wedge (\mathrm{d}\lambda_1)^{n-1}).$$

Note that the $2n+q$-form $\mathrm{d}\lambda \wedge \eta \wedge (\mathrm{d}\lambda_1)^{n-1}$ is basic, being a product of basic forms. However, any choice of $2n+q$ fields on M cannot simultaneously be linearly independent and consist only of vector fields tangent to ξ. This means $\mathrm{d}\lambda \wedge \eta \wedge (\mathrm{d}\lambda_1)^{n-1} \equiv 0$, hence

$$\lambda \wedge (\mathrm{d}\lambda_1)^n = (-1)^{q+1}\mathrm{d}(\lambda \wedge \eta \wedge (\mathrm{d}\lambda_1)^{n-1}),$$

which can not be, for a closed manifold admits no exact volume form.

We conclude that there can be no harmonic vector field satisfying $X = c_1 R_1 + \widetilde{X}$, with $\widetilde{X}$ tangent to ξ, which implies, in turn, in light of Step (I), that any harmonic vector field with respect to an R-metric for a contact foliation $\mathcal{F}$ must be transverse to that foliation, as we wanted. In particular, since each R_i is Killing, any harmonic 1-form must satisfy

$$\begin{cases} \iota_{R_i}\mu = 0 \text{ for every } 1 \leq i \leq q; \\ \mathcal{L}_{R_i}\mu = 0 \text{ for every } 1 \leq i \leq q, \end{cases}$$

hence every harmonic 1-form is basic.

□

For any foliation $\mathcal{F}$ there is an injection $H^1_B(\mathcal{F}) \hookrightarrow H^1_{dR}(M)$ (cf. Proposition 35, Appendix A). Using our last result, we can show that for Riemannian contact foliations, this is an isomorphism.

Theorem 9. If $(M, \mathcal{F}, g)$ is an isometric contact foliation on a closed manifold M, there is an isomorphism

$$H^1_{dR}(M) \approx H^1_B(\mathcal{F})$$

between the first De Rham cohomology group of M and the first basic cohomology group of the foliation $\mathcal{F}$.

Proof. Recall that the basic functions are exactly those that are leaf-wise constant. If an exact 1 form $\eta = \mathrm{d}f$ is basic, then in particular $R_i f = \eta(R_i) = 0$ for every i, and f is automatically basic as well. This means any exact basic form is cohomologous to zero. Thus the inclusion $H^1_B(\mathcal{F}) \hookrightarrow H^1_{dR}(M)$ is injective. On the other hand, Hodge's Theorem states that every class in $H^1_{dR}(M)$ has a harmonic representative. Since every harmonic 1-form is basic, this gives an injection from $H^1_{dR}(M)$ into $H^1_B(\mathcal{F})$, which implies the desired isomorphism.

□

We apply this result in order to exclude the existence of isometric contact foliations on several closed manifolds:

Proposition 21. Let M be a closed orientable manifold of dimension $2n+q$, and suppose that, as a real vector space, $H^1_{dR}(M)$ also has dimension $2n+q$. Then M supports no isometric q-contact foliation, for every $q \geq 1$.

Proof. Suppose, by contradiction, that $\mathcal{F}$ is an isometric q-contact foliation on M, for any $q \geq 1$. Let $\eta_1, \cdots, \eta_{2n+q}$ be harmonic forms whose cohomology classes form a basis for the $\mathbb{R}$-vector space $H^1_{dR}(M)$. On one hand, since the classes $[\eta_i]$ are linearly independent, the product $\eta_1 \wedge \cdots \wedge \eta_{2n+q}$ is non-vanishing, hence its class in the top cohomology group $H^{2n+q}_{dR}(M)$ can also be represented a volume form ω on M. This means, in particular, that we can find a $(2n+q-1)$-form η such that

$$\eta_1 \wedge \cdots \wedge \eta_{2n+q} = \omega - \mathrm{d}\eta.$$

On the other hand, since $\mathcal{F}$ is an isometric contact foliation, it follows from Theorem 8 that each one the 1-forms η_i is also basic, hence $\eta_1 \wedge \cdots \wedge \eta_{2n+q} = 0$ (the maximum rank of basic form in $H^*_B(\mathcal{F})$ being the codimension $2n$, which is less than $2n+q$), and therefore the volume form $\omega = \mathrm{d}\eta$ is exact, a contradiction.

□

Corollary 8. There can be no isometric q-contact foliation on the torus $\mathbb{T}^{2n+q}$, whichever is $q \geq 1$. More generally, if $p : E \to \mathbb{T}^m$ is a $\mathbb{T}^l$-bundle, then E supports no isometric contact foliation.

Proof. For a torus $\mathbb{T}^{2n+q}$ the first cohomology group is isomorphic to $\mathbb{R}^{2n+q}$, hence it has the same dimension as the manifold, and the conclusion follows from Theorem 9 above.

As for a torus bundle $p : E \to \mathbb{T}^m$, it is a matter of noticing that the exact homotopy sequence of the fibration $\mathbb{T}^l \overset{i}{\hookrightarrow} E \overset{p}{\to} \mathbb{T}^m$ becomes

$$\cdots 0 \to \mathbb{Z}^l \overset{i_*}{\to} \pi_1(E) \overset{\pi_*}{\longrightarrow} \mathbb{Z}^m \to 0 \cdots,$$

hence

$$\pi_1(E) / \mathbb{Z}^l \approx \mathbb{Z}^m.$$

If $\{e_i\}$ and $\{f_i\}$ are spanning sets for $\mathbb{Z}^l$ and $\mathbb{Z}^m$, respectively, we set $E_i := i_*(e_i)$, and choose E_{n+j} such that the class $E_{n+j} + \mathbb{Z}^l$ corresponds to f_j, thus obtaining a spanning set $\{E_1, \cdots, E_{n+m}\}$ for the fundamental group $\pi_1(E)$. Hence

$$\pi_1(E) \approx \mathbb{Z}^{m+n}.$$

Now, the first de Rham cohomology group is isomorphic to $\mathrm{Hom}(\pi_1(E); (\mathbb{R}, +))$ via

$$[\omega] \mapsto \left([\gamma] \mapsto \int_\gamma \omega \right).$$

Since $\mathrm{Hom}(\pi_1(E); (\mathbb{R}, +))$ is generated (with $\mathbb{R}$ coefficients) by the mappings sending each generator E_i to 1, it follows that

$$H^1_{dR}(E) \approx \mathrm{Hom}(\pi_1(E); (\mathbb{R}, +)) \approx \mathbb{R}^{m+n},$$

and therefore $\dim H^1_{dR}(E) = m + n = \dim E$. According to Proposition 21, there can be no isometric contact foliation on E.

□

Another consequence of Theorem 8 is the following theorem, which imposes geometric restrictions on the curvature of a metric g making a contact foliation $\mathcal{F}$ isometric.

Theorem 10. If $(M,\mathcal{F},g)$ is an isometric foliation on a closed manifold M, then the sectional curvature of g is neither strictly positive nor strictly non-positive. In other words, there are points $p,q \in M$ and planes $\pi \subset \mathrm{T}_pM$ and $\sigma \subset \mathrm{T}_qM$ such that

$$\sec(\pi) > 0$$
$$\sec(\sigma) \leq 0.$$

Proof. We argue by contradiction. Assume that g is an R-metric for the contact foliation $\mathcal{F}$ whose sectional curvature is non-positive, and denote by ∇ the Levi-Civita connection. Since the action is locally free, for small enough t the flow $\exp(sR_1)$ provides an homotopy between a nontrivial isometry $\exp(tR_1)$ and the identity $\mathrm{id} = \exp(0R_1)$. It follows that M admits a *non-vanishing* parallel vector field X (BLAINE LAWSON Jr.; YAU, 1972, Proposition 2). Being a parallel field on a compact Riemannian manifold, X must be Killing and harmonic (POOR, 2015, Proposition 5.12). It follows from Theorem 8 that X is everywhere tangent to ξ, and its dual form $\mu = g(X,\cdot)$ is basic. We claim this implies $[R_i,X] = 0$ for every i. Indeed, given $Y \in \Gamma(M)$, we have

$$\begin{aligned} 0 = \mathcal{L}_{R_i}\mu(Y) &= R_i\mu(Y) - \mu([R_i,Y]) \\ &= R_i g(X,Y) - g(X,[R_i,Y]) \\ &= \mathcal{L}_{R_i} g(X,Y) - g(X,[R_i,Y]) + g([R_i,X],Y) + g(X,[R_i,Y]) \\ &= g([R_i,X],Y), \end{aligned}$$

from where it follows $[R_i,X] = 0$, as we wanted.

Moreover, since X is also Killing, for arbitrary $Y \in \Gamma(M)$ and $1 \leq i \leq q$, we have

$$\begin{aligned} \mathcal{L}_X\lambda_i(Y) &= X\lambda_i(Y) - \lambda_i([X,Y]) \\ &= Xg(R_i,Y) - g(R_i,[X,Y]) \\ &= \mathcal{L}_X g(R_i,Y) + g([X,R_i],Y) \\ &= g(R_i,[X,Y]) - g(R_i,[X,Y]) \\ &= 0. \end{aligned}$$

Finally, from the relations $\lambda_i(X) = \mathcal{L}_X\lambda_i = 0$ for every $1 \leq i \leq q$ it follows

$$0 = \mathcal{L}_X\lambda_i = \mathrm{d}\iota_X\lambda_i + \iota_X\,\mathrm{d}\lambda_i = \iota_X\,\mathrm{d}\lambda_i,$$

that is, X belongs to the kernel of $\mathrm{d}\lambda_i$, namely $\mathcal{R}$. This would imply $X = 0$, which contradicts the non-vanishing of X as given by (POOR, 2015, Proposition 5.12), proving that the curvature of g can not be strictly non-negative.

On the other hand, if the sectional curvature of g is strictly positive, then it follows from (PETERSEN, 2006, Theorem 8.3.5) that any two commuting Killing fields on M must be linearly dependent somewhere. We know, however, that $\{R_i\}$ is a global frame of commuting Killing fields for $\mathcal{R}$. Hence g can not have positive sectional curvature.

□

4.3 Equicontinuity

The existence of an invariant metric is a geometric property of the action, but it, in turn, implies a rather valuable dynamical property: equicontinuity. Equicontinuity means that close points remain close under the action, that is, close orbits can not grow too much apart as the dynamics progresses. It has substantial consequences on the dynamical behaviour of a system. Here we investigate some of the consequences equicontinuity has on the dynamics and define a stronger dynamical property, C^1-equicontinuity, which turns out to be equivalent to the existence of an invariant metric.

Definition 18 (*Equicontinuity*)**.** Let d be a metric on M, compatible with its manifold topology. The action $F : \mathbb{R}^q \to \mathrm{Diff}(M)$ is **(uniformly) equicontinuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(x,y) < \delta \implies d(F^a(x), F^a(y)) < \varepsilon \text{ for all } a \in \mathbb{R}^q.$$

Remark 12. The choice of metric d in Definition 18 is not crucial because we only deal with compact manifolds. In a compact metric space, any two metrics d_0, d_1 are *quasi-isometric*, meaning that there are positive constants c and C such that

$$cd_1(x,y) \leq d_0(x,y) \leq Cd_1(x,y)$$

for every x, y. It is easy to see quasi-isometries preserve that equicontinuity, so equicontinuity in compact spaces is independent of the choice of metric. This is not true for non-compact metric spaces, where equicontinuity is, in fact, a property of a family of functions with respect to a specific metric. In fact, equicontinuity can be defined more generally in terms of *uniformities*. However, since compact topological spaces have a single compatible uniformity, in such spaces, the notion of equicontinuity becomes a topological one.

Clearly, an isometric contact foliation is equicontinuous. However, only some contact foliations are equicontinuous, as the next examples show.

Example 12 (The standard overtwisted contact structure)**.** Consider $\mathbb{R}^3$ with cylindrical coordinates (ρ,θ,z). The 1-form

$$\lambda_{\mathrm{ot}} = \cos(\rho)\,\mathrm{d}z + \rho\sin(\rho)\,\mathrm{d}\theta$$

is the **standard overtwisted contact form**. It's derivative is

$$\begin{aligned}\mathrm{d}\lambda_{\mathrm{ot}} &= -\sin(\rho)\,\mathrm{d}\rho\wedge\mathrm{d}z + (\sin(\rho)+\rho\cos(\rho))\,\mathrm{d}\rho\wedge\mathrm{d}\theta\\ &= \mathrm{d}\rho\wedge\underbrace{[(\sin(\rho)+\rho\cos(\rho))\,\mathrm{d}\theta - \sin(\rho)\,\mathrm{d}z]}_{\alpha}\\ &= \mathrm{d}\rho\wedge\alpha.\end{aligned}$$

It's Reeb field $R_{\mathrm{ot}} = f\partial_\rho + g\partial_\theta + h\partial_z$ is uniquely defined by the relations

$$1 = \lambda_{\mathrm{ot}}(R_{\mathrm{ot}}) = \cos\rho h + \rho\sin\rho g$$

and

$$\begin{aligned}0 &= \mathrm{d}\lambda_{\mathrm{ot}}(R_{\mathrm{ot}},\cdot)\\ &= \mathrm{d}\rho\wedge\alpha(R_{\mathrm{ot}},\cdot)\\ &= \mathrm{d}\rho(R_{\mathrm{ot}})\alpha - \alpha(R_{\mathrm{ot}})\,\mathrm{d}\rho\\ &= f[(\sin\rho+\rho\cos\rho)\,\mathrm{d}\theta - \sin\rho\,\mathrm{d}z] - [(\sin\rho+\rho\cos\rho)g - \sin\rho h]\,\mathrm{d}\rho\\ &= [\sin\rho h - (\sin\rho+\rho\cos\rho)g]\,\mathrm{d}\rho + f(\sin\rho+\rho\cos\rho)\,\mathrm{d}\theta - f\sin\rho\,\mathrm{d}z.\end{aligned}$$

These equations are equivalent to the the system

$$\begin{cases}\cos\rho h + \rho\sin\rho g = 1\\ \sin\rho h = (\sin\rho+\rho\cos\rho)g\\ f(\sin\rho+\rho\cos\rho) = 0\\ f\sin\rho = 0\end{cases}$$

which, for $\rho > 0$, has as solutions

$$\begin{cases}f = 0\\ g = \frac{\sin\rho}{\rho+\sin(\rho)\cos(\rho)} = \frac{2\sin\rho}{2\rho+\sin(2\rho)}\\ h = \frac{\sin\rho+\rho\cos\rho}{\rho+\sin(\rho)\cos(\rho)} = \frac{2\sin\rho+2\rho\cos\rho}{2\rho+\sin(2\rho)},\end{cases}$$

where we made use of the relation $\sin(\rho)\cos(\rho) = \frac{\sin(2\rho)}{2}$. Thus

$$R_{\mathrm{ot}} = \frac{2}{2\rho+\sin(2\rho))}\left(\sin(\rho)\partial_\theta + (\sin\rho+\rho\cos\rho)\partial_z\right).$$

Note that the field does not depend on θ or z, nor does it have any radial components. This means each cylinder $\{\rho=\rho_0\}$ is invariant under the flow of R_{ot} and foliated by its flowlines, which are helices in $\mathbb{R}^3$. In particular, consider a value $\rho_0 > 0$ for which

$$\rho_0 + \mathrm{tg}(\rho_0) = 0.$$

Such value is an isolated zero of the function

$$\frac{2(\sin\rho+\rho\cos\rho)}{2\rho+\sin(2\rho))},$$

therefore the ∂_z component of R_{ot} vanishes at the cylinder C defined by $\rho=\rho_0$. The restriction of the Reeb flow $\exp(tR_{\text{ot}})$ to C is a closed foliation by circles parallel to the $\rho\theta$-plane. In particular, the z-component is constant along each flowline.

Fix a point $x_0=(\rho_0,\theta_0,z_0)\in C$. Given $\varepsilon>0$, in any δ-ball centred in x_0 we can find a point $y=(\rho_y,\theta_0,z_0)$ lying in a cylinder where the ∂_z component of R_{ot} *does not* vanish. This means the orbit of y under the flow of R_{ot} is a helix, and therefore the z component of $\exp(tR_{\text{ot}})y$ grows in module as t goes to infinity. The z-component of $\exp(tR_{\text{ot}})x_0$, on the other hand, is constant. Hence the distance between $\exp(tR_{\text{ot}})y$ and $\exp(tR_{\text{ot}})x_0$ will be greater than ε for large enough t, and the Reeb flow is not equicontinuous.

Example 13 (Anosov contact actions)**.** Anosov contact foliations can never be equicontinuous, for instance. To see this, consider a point $x\in M$ and a disc $W^u(x)$ tangent to the unstable distribution (which always exists because an Anosov contact action is, in particular, a partially hyperbolic dynamical system). Choose $\varepsilon>0$ small enough so that there is $y\in W^u(x)$ such that $d(x,y)>\varepsilon$. Now, for any $\delta>0$, the Anosov condition implies that there is an Anosov element $a\in\mathbb{R}^q$ and a natural number $N>0$ such that F^{-Na} satisfies

$$d(F^{-Na}(x),F^{-Na}(y))<\delta.$$

Thus the points $x_0=F^{-Na}(x)$ and $y_0=F^{-Na}(y)$ are δ-close, but their images under F^{Na} are further away than ε, meaning $F:\mathbb{R}^q\to\mathsf{Diff}(M)$ is not an equicontinuous contact action.

Example 14 (Contact structures on $\mathbb{T}^3$)**.** Consider the 1-form on $\mathbb{R}^3$, with coordinates (x,y,z), given by

$$\lambda=\cos(2\pi z)\,\mathrm{d}x+\sin(2\pi z)\,\mathrm{d}y.$$

One notices that $\lambda\wedge\mathrm{d}\lambda=-2\pi\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$. Therefore λ is a contact form on $\mathbb{R}^3$. A straightforward calculation shows that its Reeb vector field is

$$R_\alpha=\cos(2\pi z)\partial_x+\sin(2\pi z)\partial_y,$$

which can be integrated explicitly. The integral curve passing through a initial point $p_0(x_0,y_0,z_0)$ at time $t=0$ is

$$\phi(t)=(x_0+\cos(2\pi z_0)t,y_0+\sin(2\pi z_0)t,z_0).$$

Note that by slightly varying only the z-coordinate of the initial point p_0, one can obtain arbitrarily close distinct initial points whose orbits grow further and further apart as time passes.

Now, the 1-form λ and its Reeb field are invariant under the standard action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ by translations. Therefore, λ descends to a 1-form on the quotient $\mathbb{T}^3$, giving an example of a contact structure on a closed manifold which is not equicontinuous.

More generally, no expansive flow can be equicontinuous. Consider a flow ϕ defined on a Riemannian manifold (M,g).

Definition 19 (*Transverse geodesic ball*)**.** We define geodesic balls transverse to the flow ϕ by setting

$$H_\varepsilon(x) := \{\exp_x Y; g_x(Y,Y)^{\frac{1}{2}} < \varepsilon \text{ and } g_x(Y,X) = 0\}.$$

Informally, an expansive flow is one for which distinct orbits grow apart. To formalise this, we ask that, given distinct points x,y, any reparameterisation of the y orbit always ends up leaving a transverse ball of fixed radius around the orbit of x after finite time:

Definition 20 (*Expansive flow*)**.** The flow ϕ is **expansive** if there is $\delta > 0$ with the following property:

$$\text{if } \gamma:\mathbb{R}\to\mathbb{R} \text{ is smooth, increasing, surjective, } \gamma(0)=0 \text{ and } \phi^{\gamma(t)}(y) \in H_\delta(\phi^t(x)) \text{ for every } t\in\mathbb{R}, \text{ then } x=y.$$

Proposition 22. If ϕ is equicontinuous, then ϕ is not expansive.

Proof. Indeed, given $\delta > 0$ there is $\rho > 0$ such that $d(x,y) < \rho \implies d(\phi^t(x),\phi^t(y)) < \delta$ for every $t \in \mathbb{R}$, for every $x,y \in M$.

Given $x \in M$, consider $y \neq x$ in the transverse ball $H_\rho(x) \subset B_\rho(x)$. Consider a function $\gamma:\mathbb{R}\to\mathbb{R}$ defined as to satisfy the relation

$$\phi^{\gamma(t)(y)} = \mathcal{O}(y) \cap H_\delta(\phi^t(x)).$$

Intuitively, $\gamma(t)$ is the time it takes from the orbit of y to reach the transverse ball of radius δ around the point $\phi^t(x)$. A function γ defined as such is a reparameterisation of the orbit $\mathcal{O}(y)$, and by construction, it satisfies

$$\phi^{\gamma(t)}(y) \in H_\delta(\phi^t(x)) \ \forall t \in \mathbb{R}.$$

Since $x \neq y$ by choice, it follows that ϕ is not expansive.

□

This means an *expansive flow can never be equicontinuous.* It is known, due to the work of Paternain (PATERNAIN, 1993, Corollary 2), that if a 2-dimensional closed Riemannian manifold has no conjugated points and its Riemannian covering has no bi-asymptotic geodesics, then its geodesic flow is expansive. On the other hand, the *geodesic flow is of contact type*, as it can be seen as the Reeb flow of the canonical contact form on the unit tangent bundle $\mathrm{T}^1 M$ (which is, in this case, a 3-dimensional manifold). This provides yet another example of an entire family of contact foliations/actions which are not equicontinuous.

The class of equicontinuous contact foliations is somewhat restricted, as equicontinuity turns out to be equivalent to a strong regularity condition on the orbits of $\mathcal{F}$. More specifically, the underlying foliation of an equicontinuous action is *uniformly almost periodic*. This is a strong form of recurrence – the orbit returns to an arbitrary neighbourhood infinitely often. We recall the involved definitions.

Definition 21 (*Syndetic set*)**.** A subset $S \subset \mathbb{R}^q$ is **syndetic** if there is a compact subset $K \subset \mathbb{R}^q$ such that $\mathbb{R}^q = S + K$.

Basically, S is syndetic if it has "bounded gaps" inside $\mathbb{R}^q$. Note that a syndetic subset is necessarily unbounded. An example of syndetic set is the lattice $\mathbb{Z}^q$, since $\mathbb{Z}^q + [0,1]^q = \mathbb{R}^q$.

Definition 22 (*Almost periodicity*)**.** A point $x \in M$ is said to be **almost periodic** if for every $\varepsilon > 0$ there exists a syndetic subset $S_{x,\varepsilon}$ such that

$$d(x, F^s(x)) < \varepsilon \ \forall s \in S_\varepsilon.$$

The action F is **uniformly almost periodic** if for every $\varepsilon > 0$ there is a syndetic subset S_ε of $\mathbb{R}^q$ such that

$$d(x, F^s(x)) < \varepsilon$$

for every $x \in M$ and $s \in S_\varepsilon$.

In order to show that equicontinuity is equivalent to uniform almost periodicity, we follow (AUSLANDER, 1988) and begin with two lemmas.

Lemma 2. If F is equicontinuous and M compact, then $\mathcal{F}$ is point-wise almost periodic.

Proof. Let $x \in M$. We claim that $\overline{\mathcal{F}(x)}$ is a minimal subset of $\mathcal{F}$[2]. Indeed, for $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(x,y) < \delta \implies d(F^a(x), F^a(y)) < \varepsilon,$$

[2] Note that, for contact actions F, this does not contradict Theorem 3, because we're not saying that $\overline{\mathcal{F}(x)}$ is a minimal *contact foliation*, that is, there's no guarantee that the contact structure restricts to $\overline{\mathcal{F}(x)}$. Indeed, Theorem 3 prevents that from happening.

for every $x, y \in M, a \in \mathbb{R}^q$. Given $y \in \overline{\mathcal{F}(x)}$, there is $p \in \mathbb{R}^q$ such that $d(F^p(x), y) < \delta$, and therefore $d(x, F^{-p}(y)) = d(F^{-p+p}(x), F^p(y)) < \varepsilon \to 0$ as $\varepsilon \to 0$, implying that $x \in \overline{\mathcal{F}(y)}$. Hence $\overline{\mathcal{F}(x)}$ is minimal, and in particular, compact. For any open ball $B_\varepsilon(x)$ around x, $\overline{\mathcal{F}(x)} \subset \mathcal{F}(B_\varepsilon(x))$ (otherwise $\overline{cF(x)} \setminus \mathcal{F}(B_\varepsilon(x))$ is a closed invariant subset), and compactness implies that

$$\overline{\mathcal{F}(x)} \subset \bigcup_{j=1}^{l} F^{k_j}(B_\varepsilon(x))$$

for some finite collection of points $K = \{k_1, \cdots, k_l\} \subset \mathbb{R}^q$. For any $a \in \mathbb{R}^q$, the point $F^a(x)$ belongs to $F^{k_j}(B_\varepsilon(x))$ for some j, hence $F^{a-k_j}(x) \in B_\varepsilon(x)$. Let $S_{x,\varepsilon}$ be the set

$$S_{x,\varepsilon} := \{s \in \mathbb{R}^q; F^s(x) \in B_\varepsilon(x)\}.$$

Hence $a - k_j \in S_{x,\varepsilon}$ and consequently $\mathbb{R}^q = S_{x,\varepsilon} + K$. Thus $S_{x,\varepsilon}$ is syndetic, and x is almost periodic. □

Next, we prove a finite version of our equivalence.

Lemma 3. Let $F : M \to \mathrm{Diff}(M)$ be an equicontinuous action. Given $\varepsilon > 0$ and points $x_1, \cdots, x_l \in M$, there is a syndetic set S_ε on $\mathbb{R}^q$ such that

$$d(F^s(x_i), x_i) < \varepsilon, \ \forall s \in S_\varepsilon, \ \forall i = 1, \cdots, l.$$

Proof. We consider the Cartesian product M^l equipped with the metric

$$\overline{d}((y_1, \cdots, y_l), (z_1, \cdots, z_l)) := \max\{d(y_i, z_i); i = 1, \cdots, l\}.$$

Let $\overline{F}$ be the action of $\mathbb{R}^q$ on M^n given by acting via F on each coordinate. With respect to $\overline{d}$, this is an equicontinuous action. The point $x = (x_1, \cdots, x_l)$ from the statement is almost periodic due to Lemma 2, so there is a syndetic set $S_\varepsilon \subset \mathbb{R}^q$ such that

$$\overline{d}(\overline{F}^s(x), x) < \varepsilon$$

for every $s \in S_\varepsilon$. The set S_ε has the desired properties. □

Theorem 11. An action on a compact manifold is equicontinuous if and only if it is uniformly almost periodic.

Proof. We fix a metric d on M. Suppose first that F is uniformly almost periodic. Given $\varepsilon > 0$, let S be a syndetic set on $\mathbb{R}^q$ such that

$$d(x, F^s(x)) < \frac{\varepsilon}{3}$$

for every $x \in M$ and $s \in S$. By definition, there is a compact K such that $\mathbb{R}^q = S + K$. We consider the compact set

$$A = K \cup -K = \{k \in \mathbb{R}^q; k \in K \text{ or } -k \in K\}.$$

Since A is compact, there is a positive δ such that, for any $x,y \in M$,

$$d(x,y) < \delta \implies d(F^k(x), F^k(y)) < \frac{\varepsilon}{3}, \ \forall k \in A. \tag{4.8}$$

Given an arbitrary element $a \in \mathbb{R}^q$, there are $k \in K$ and $s \in S$ such that $-a = s + k$, and therefore $a + s = -k$. Now, since S is the syndetic set associated with $\frac{\varepsilon}{3}$ by the uniform almost periodicity, we have, for any $x,y \in M$,

$$\max\{d(F^a(x), F^{a+s}(x)), d(F^a(y), F^{a+s}(y))\} < \frac{\varepsilon}{3} \tag{4.9}$$

On the other hand, if x and y are δ-close, then condition in (4.8) implies

$$d(F^{a+s}(x), F^{a+s}(y) = d(F^{-k}(x), F^{-k}(y)) < \frac{\varepsilon}{3}.$$

The inequality above, together with the ones in (4.9) and the triangle inequality, yields

$$d(x,y) < \delta \implies d(F^a(x), F^a(y)) < \varepsilon \ \forall a \in \mathbb{R}^q,$$

hence F is equicontinuous.

For the converse, suppose that F is equicontinuous and let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that

$$d(x,y) < \delta \implies d(F^a(x), F^a(y)) < \frac{\varepsilon}{3} \ \forall a \in \mathbb{R}^q, \ \forall x,y \in M.$$

We can assume without loss of generality that $\delta < 3^{-1}\varepsilon$. Using the compactness of M we chose a finite collection $x_1, \cdots, x_l$ with the property that for every $y \in M$ there is an $1 \leq j \leq l$ such that $d(y, x_j) < \delta$. By Lemma 3 there is a syndetic set S_δ such that

$$d(F^s(x_i), x_i) < \delta \ \forall s \in S_\delta, \ \forall i = 1, \cdots, l.$$

Now, any $x \in M$ is δ-close to x_i for some i, thus

$$\begin{aligned} d(x, F^a(x)) &\leq d(x, x_i) + d(x_i, F^a(x_i) + d(F^a(x_i), F^a(x)) \\ &< \delta + \delta + \frac{\varepsilon}{3} \\ &< \varepsilon, \end{aligned}$$

which proves uniform almost periodicity.

□

4.3.1 The C^1-enveloping group and C^1-equicontinuity

Given an action $A : G \to \mathrm{Homeo}(X)$ of a topological group G on a compact topological space X, the enveloping semi-group (or Ellis semi-group) E_A is the closure of $A(G)$

in the set X^X of all functions $X \to X$, with respect to the topology of point-wise convergence. The famous Arzella-Ascolli Theorem (cf.(KELLEY, 2017, Theorem 7.16)) says that equicontinuity of a family of homeomorphisms of X is equivalent to the condition that this family have compact closure on the space $C(X)$ of continuous functions $X \to X$, with respect to the compact-open topology (which coincides with the topology of point-wise convergence for equicontinuous families). Moreover, such closure consists solely of homeomorphisms. In this sense, an action's equicontinuity is equivalent to the compactness of its enveloping group (AUSLANDER, 1988, Theorem 3.3).

We will define the C^1-enveloping group of a contact action in the same way but considering the C^1 compact-open topology on the space of diffeomorphisms. Note that the image $F(\mathbb{R}^q)$ of the contact action $F : \mathbb{R}^q \to \mathrm{Diff}(M)$ is the set spanned by all the flows of the Reeb fields R_i. In other words,

$$F(\mathbb{R}^q) = \mathrm{Span}\left\{\exp(tR_i); t \in \mathbb{R}, i = 1, \cdots, q\right\},$$

Now, $\mathrm{Diff}(M)$ is a Lie Group when equipped with the C^1 compact-open open topology τ_1, its Lie Algebra being that of vector fields on M.

Definition 23 (C^1*-enveloping group*)**.** Let $F : \mathbb{R}^q \to \mathrm{Diff}(M)$ be a contact action. The **C^1-enveloping group of F** is the closure

$$\mathrm{E}_F^1 := \overline{F(\mathbb{R}^q)}$$

in the Lie group $(\mathrm{Diff}(M), \tau_1)$.

If the contact action F is isometric, then $F(\mathbb{R}^q)$ is an equicontinuous family, since the C^1-enveloping group E_F^1 is a closed subset of the compact space $\mathrm{Iso}(M)$. With this in mind, we propose the following stronger version of equicontinuity.

Definition 24 (C^1*-equicontinuity*)**.** The action F is said to be **C^1-equicontinuous** if its C^1-enveloping group E_F^1 is compact.

The C^1-enveloping group acts on the manifold M in a natural way, and its orbits are exactly the closures of the leaves of $\mathcal{F}$.

Proposition 23. Given a C^1-equicontinuous action $F : \mathbb{R}^q \to \mathrm{Diff}(M)$ and $x \in M$, the orbit of x under the action of E_F^1 is exactly $\overline{\mathcal{F}(x)}$.

Proof. Let $y \in \overline{\mathcal{F}(x)}$. Then there is a sequence $a_n \in \mathbb{R}^q$ such that $F^{a_n}(x) \to y$ and since E_F^1 is compact, $F^{a_n} \to T \in \mathrm{E}_F^1$ (up to a sub-sequence). Hence $y = T(x)$ belongs to the orbit of x under E_F^1. Conversely, if y is in the orbit of x under E_F^1, then there is $T = \lim F^{a_n} \in \mathrm{E}_F^1$ such that $y = T(x) = \lim F^{a_n}(x)$, and therefore $y \in \overline{\mathcal{F}(x)}$. □

We remark that C^1-equicontinuity is a strong dynamical condition. It means the family $F(\mathbb{R}^q)$ is equicontinuous in the classic sense, and each of the families of derivatives $\{\mathrm{d}F_x^a; a \in \mathbb{R}^q\}$ is equicontinuous, for every $x \in M$.

It can be showed that every compact topological space M supporting an equicontinuous action F also supports a metric d (compatible with M's topology) which is invariant under F. Of course, even when M is a manifold, there is no *a priori* reason to expect the metric d to be associated with a Riemannian metric tensor in M, as it is a purely topological object of M. The main advantage of working with C^1-equicontinuity is precisely that, in this setting, there exists an F-invariant metric *coming from a Riemannian tensor on* M. In other words, being C^1-equicontinuous and being an isometric action are actually equivalent conditions.

Theorem 12. Let $F : \mathbb{R}^q \to \mathsf{Diff}(M)$ be a C^1-equicontinuous action. Then M supports an F-invariant Riemannian metric.

Proof. Let g_0 be any Riemannian metric on M. By hypothesis, E_F^1 is a compact Lie group. Let μ be a Haar measure defined on the Borel σ-algebra of E_F^1. For each $p \in M$ and $X, Y \in \mathrm{T}_pM$ we define a function

$$\begin{aligned} &\mathrm{E}_F^1 \to \mathbb{R} \\ &e \mapsto (e^* g_0)_p(X,Y) = g_0|_{e(p)}(\mathrm{d}e_pX, \mathrm{d}e_pY). \end{aligned} \tag{4.10}$$

Note that two elements of E_F^1 are close in the C^1 topology if they are close in the compact-open topology, and their derivatives are close as transformations. This, together with the fact that g_0 is smooth, implies that the function defined by the mapping (4.10) is continuous and, therefore, μ-measurable. We define a metric tensor g on M by averaging the metric g_0:

$$g_p(X,Y) = \int_{\mathrm{E}_F^1} (e^* g_0)_p(X,Y)\, \mathrm{d}\mu(e).$$

This is a Riemannian metric since it is smooth, bi-linear, and $g(X,X) > 0$ for every field X. Moreover, it is invariant under the action of E_F^1, as for any $f \in \mathrm{E}_F^1$ one has

$$\begin{aligned} (f^* g)_p(X,Y) &= g_0|f(p)(\mathrm{d}f_pX, \mathrm{d}f_pY) \\ &= \int_{e\in\mathrm{E}_F^1} e^* g_0|_{f(p)}(\mathrm{d}f_pX, \mathrm{d}f_pY) \\ &= \int_{e\in\mathrm{E}_F^1} (ef)^* g_0|_p(X,Y) \\ &= g_p(X,Y), \end{aligned}$$

thus $f^*g = g$. In particular, $(F^a)^*g = g$ for every $a \in \mathbb{R}^q$, that is, F is an isometric contact foliation. □

Thus, C^1-equicontinuity is simply a dynamical expression of the geometric property of preserving a metric. Using the famous result of Myers and Steenrod, we can characterise C^1-equicontinuous actions as precisely those which can be extended to a toric action on M.

Theorem 13. Let F be a action on a compact manifold M. Then F is C^1-equicontinuous if and only if E^1_F is a torus.

Proof. If $\mathrm{E}^1_F \approx \mathbb{T}^l$, then in particular E^1_F is compact and F is C^1-equicontinuous. Conversely, if F is C^1-equicontinuous then by Theorem 12 there is a metric tensor g invariant under F. Hence $F(\mathbb{R}^q)$ is a representation of the Lie Group $(\mathbb{R}^q,+)$ in $\mathrm{Iso}(M,g)$, which is a compact subset of $\mathrm{Diff}(M)$ in the C^1 topology (cf. (MYERS; STEENROD, 1939)). Thus $\mathrm{E}^1_F := \overline{F(\mathbb{R}^q)}$ is compact Abelian subgroup of $\mathrm{Iso}(M,g)$, and is therefore isomorphic to a torus $\mathbb{T}^l$. □

4.4 Toric actions on q-contact manifolds

We saw in the last section that contact foliations with invariant metrics are (C^1-) equicontinuous and, therefore, strongly recurrent, with every point returning arbitrarily close to itself infinitely often. In this section, we look for actually closed orbits for isometric contact foliations. To this end, we define a particular class of toric actions on a q-contact manifold.

Definition 25 (*Compatible toric action*)**.** Let $(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ be a q-contact foliation, and $\mathbb{T}^l \to \mathrm{Diff}(M)$ be an action of the torus $\mathbb{T}^l$ on M. We will say this action is **compatible** with the contact foliation $\mathcal{F}$ it it preserves the non-degenerate forms $\mathrm{d}\lambda_i$. To be more precise, let $X_1,\cdots,X_l$ be generators of the toric action. The action is compatible if

$$\mathcal{L}_{X_i}\mathrm{d}\lambda_j = 0 \quad \text{for any } i,j.$$

The most critical example of contact foliations supporting compatible toric actions is the class of isometric contact foliations.

Proposition 24. If $\mathcal{F}$ is C^1-equicontinuous q-contact foliation on a compact manifold M, then $\mathcal{F}$ admits a compatible action by isometries $\mathbb{T}^q \to \mathrm{Iso}(M,g)$.

Proof. Due to Proposition 7 and Theorem 12, we can assume without loss of generality that g is an R-metric, that is, that $\lambda_i(X) = g(R_i,X)$. Since g is F-invariant, the C^1-enveloping group E_F is the smallest compact Abelian subgroup containing the image $F(\mathbb{R}^q)$. Hence E^1_F is a torus $\mathrm{E}^1_F \approx \mathbb{T}^l$, with $l \geq q$.

Now, the inclusion $\mathbb{T}^q \hookrightarrow \mathrm{E}^1_F$ induces an $\mathbb{T}^q$-action on M. By construction this is an isometric action, so its generators $X_1, \cdots, X_q$ are Killing vector fields, that is, $\mathcal{L}_{X_i} g = 0$ for every i.

We claim this action is compatible. Indeed, since all the elements in E^1_F commute, it follows that $h \circ \exp(tR_i) = \exp(tR_i) \circ h$ for every $h \in \mathbb{T}^q$. Therefore, the action preserves the Reeb vector fields, that is, $\mathrm{d}h_p(R_i|_p) = R_i|_{h(p)}$. We can use these facts to conclude that the toric action also preserves the adapted coframe $\vec{\lambda}$:

$$\begin{aligned} h^*\lambda_i|_p(X) &= \lambda_i|_{h(p)}(\mathrm{d}h_p(X)) \\ &= g_{h(p)}(R_i|_{h(p)}, \mathrm{d}h_p(X)) \\ &= g_{h(p)}(\mathrm{d}h_p(R_i|_p), \mathrm{d}h_p(X)) \\ &= h^* g_p(R_i, X) \\ &= g_p(R_i, X) \\ &= \lambda_i|_p(X). \end{aligned}$$

It follows now, using the naturality of the pullback operation, that

$$h^* \mathrm{d}\lambda_i = \mathrm{d}\lambda_i,$$

so that the action is compatible, as we wished.

□

Compatible toric actions are helpful when searching for closed orbits because whenever a generator X_i is tangent to a leaf $\mathcal{F}(x)$, the leaf in question is not a plane.

Proposition 25. Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a closed q-contact manifold, and $\mathbb{T}^s \to \mathrm{Diff}(M)$ be a compatible action with generators $X_1, \cdots, X_s$. Then each generator is tangent to $\mathcal{R}$ in at least two points.

Proof. We consider on M a q-form $\overline{\lambda}$, obtained from the characteristic form λ by averaging it with respect to a Haar measure μ on $\mathbb{T}^s$, in the following sense:

$$\overline{\lambda}_p(Y_1, \cdots, Y_q) := \int_{g \in \mathbb{T}^s} g^* \lambda_p(Y_1, \cdots, Y_q) d\mu(g).$$

Note that g is being used to write both the element of $\mathbb{T}^l$ and the diffeomorphism of M it represents. Due to the invariance of μ under multiplication on $\mathbb{T}^s$, the form $\overline{\lambda}$ is invariant under the action of $\mathbb{T}^s$:

$$\begin{aligned}(h^*\overline{\lambda})_p(Y_1,\cdots,Y_q) &= \overline{\lambda}_{h(p)}(\mathrm{d}h_pY_1,\cdots,\mathrm{d}h_pY_q)\\ &= \int\limits_{g\in\mathbb{T}^q} g^*\lambda_{h(p)}(\mathrm{d}h_pY_1,\cdots,\mathrm{d}h_pY_q)\,\mathrm{d}\mu(g)\\ &= \int\limits_{g\in\mathbb{T}^q} \lambda_{g(h(p))}(\mathrm{d}g_{h(p)}\,\mathrm{d}h_pY_1,\cdots,\mathrm{d}g_{h(p)}\,\mathrm{d}h_pY_q)\,\mathrm{d}\mu(g)\\ &= \int\limits_{g\in\mathbb{T}^q} (hg)^*\lambda_p(Y_1,\cdots,Y_q)\,\mathrm{d}\mu(g)\\ &= \int\limits_{g\in\mathbb{T}^q} g^*\lambda_p(Y_1,\cdots,Y_q)\,\mathrm{d}\mu(g)\\ &= \overline{\lambda}_p(Y_1,\cdots,Y_q),\end{aligned}$$

that is, $h^*\overline{\lambda} = \overline{\lambda}$. In particular, for any $i \le q$, any $l \le s$ and any set of vector fields $Y_1,\cdots,Y_l$ we have

$$\mathcal{L}_{X_i}(\iota_{Y_1}\cdots\iota_{Y_l}\overline{\lambda}) = 0.$$

We also define $\overline{\lambda}_i$ by

$$\overline{\lambda}_i|_p(Y) := \int\limits_{g\in\mathbb{T}^q} g^*\lambda_i|_p(Y)d\mu(g).$$

We consider, for each $i \le s$ and $j \le q$, the functions

$$\varphi_{ij}(p) := (-1)\iota_{X_i}\overline{\lambda}_j|_p,$$

Note that, since $g^*X_i = X_i$ for any $g \in \mathbb{T}^q$ and generator X_i, these functions are constant along the orbits of the toric action. We claim that

$$d\varphi_{ij} = \iota_{X_i}d\lambda_j, \tag{4.11}$$

To see this, begin defining an auxiliary forms $\alpha_j = \lambda_j - \overline{\lambda}_j$, and notice that since $d\lambda_j$ is preserved by the action, the forms α_j is closed. Together with $\mathcal{L}_{X_i}\overline{\lambda}$, the closeness of α_j yields, for any generator X_i:

$$\iota_{X_i}d\lambda_j + d\iota_{X_i}\lambda_j = \mathcal{L}_{X_i}\lambda_j = \mathcal{L}_{X_i}\alpha_j + \mathcal{L}_{X_i}\overline{\lambda}_j = d\iota_{X_i}\alpha_j = d\iota_{X_i}\lambda_j - d\iota_{X_i}\overline{\lambda}_j,$$

that is

$$d\varphi_{ij} = -d\iota_{X_i}\overline{\lambda}_j = \iota_{X_i}d\lambda_j, \quad i,j = 1,\cdots,q,$$

which proves (4.11). Moreover, for any i,j,l, the relation $\iota_{[R_i,X_j]} = [\mathcal{L}_{R_i},\iota_{X_j}]$, together with Equation (4.11), yields

$$\begin{aligned}\iota_{[R_i,X_j]d\lambda_l} &= \mathcal{L}_{R_i}\iota_{X_j}d\lambda_l\\ &= \iota_{R_i}d(\iota_{X_j}d\lambda_l) + d(\iota_{R_i}\iota_{X_j}d\lambda_l)\\ &= \iota_{R_i}dd\varphi_{ij}\\ &= 0.\end{aligned}$$

Hence $[R_i, X_j] \in \ker d\lambda_l = \mathcal{R}$, that is, the generators X_i are *foliate vector fields* of $\mathcal{F}$. In particular, their flows preserve the leaves of $\mathcal{F}$. Finally, consider the function φ_{ij}. Since M is closed, φ_{ij} has at least two critical points, where φ_{ij} attains its maximum and minimum, respectively. At a critical point p, one has

$$0 = d\varphi_{ij}|_p = d\lambda_j|_p(X_i|_p, \cdot) \iff X_i|_p \in \mathcal{R}_p,$$

that is, the critical points of φ_{ij} are exactly the points where X_i is tangent to $\mathcal{R}$.

□

Theorem 14. If a closed q-contact manifold admits a compatible toric action, then the contact foliation can not be a foliation by planes. In other words, closed q-contact manifolds supporting compatible toric actions satisfy the WGWC.

Proof. As we saw in the last proposition, the generating fields X_i are foliate. Recall that the set of foliate vector fields is the normaliser

$$\mathfrak{B}(\mathcal{F}) := \{X \in \Gamma(M); [X,Y] \in \Gamma(\mathcal{F}) \text{ for all } Y \in \Gamma(\mathcal{F})\}$$

of the Lie algebra $\Gamma(\mathcal{F})$ of vector fields tangent to $\mathcal{F}$, that is, sections of $\mathcal{R}$. The Lie algebra of sections of $\mathcal{R}$ is clearly a sub-algebra of $\mathfrak{B}(\mathcal{F})$, and the quotient

$$\mathfrak{t}(\mathcal{F}) := {}^{\mathfrak{B}(\mathcal{F})}/_{\Gamma(\mathcal{F})}$$

is the Lie algebra of *transverse vector fields* (cf. Appendix A). We denote by $\overline{X}$ the image of a field X under the projection $\mathfrak{B}(\mathcal{F}) \to \mathfrak{t}(\mathcal{F})$. Each transverse field acts on the leaf space $M/\mathcal{F}$ via its flow. According to Proposition 25 above, if p is a critical point of the function φ_{ij}, then the generating field X_i is tangent to $\mathcal{F}$ at p, and consequently $\overline{X}_i$ is zero at the point (that is, the *leaf*) $\mathcal{F}(p)$. This means the flow of X_i fixes the entire leaf $\mathcal{F}(p)$. In particular, the entire orbit of p under X_i is contained in $\mathcal{F}(p)$.

Finally, since we can choose the generators of the toric action to be all periodic fields, it follows that $\mathcal{F}(p)$ contains an essential closed curve, and it is, therefore, not a plane.

□

In particular, when $q = 1$ last corollary is a result on circle invariant pre-symplectic forms by Banyaga and Rukimbira (BANYAGA; RUKIMBIRA, 1995).

Corollary 9. Every isometric contact foliation on a closed manifold satisfies the WGWC.

Proof. This is just Theorem 14 restated in light of Proposition 24.

□

Now we present a proof of Theorem A by showing that an isometric contact action of $\mathbb{R}^q$ must have at least two closed orbits, which generalises a result from Banyaga and Rukimbira. To that end, we first need the following Proposition, due to Caramello Jr. and Töben, which we state here for completeness:

Proposition 26 (Proposition 3.1 in (CARAMELLO Jr.; TÖBEN, 2019))**.** Let $(M,\mathcal{F})$ be a Riemannian foliation, and $\overline{X}$ be a transverse killing vector field. Then each connect component N of the set of zeros of $\overline{X}$ is an even-codimensional closed submanifold of M saturated by the leaves of $\mathcal{F}$. Moreover, N is horizontally totally geodesic, and if $\mathcal{F}$ is transversely orientable, so is $(N,\mathcal{F}|_N)$.

This is a generalisation to Riemannian foliations of a result for Riemannian manifolds regarding zero sets of Killing fields, that is, sets of points where the vector field vanishes (cf. (KOBAYASHI, 1995, Theorem 5.3, Chapter II), and also (KOBAYASHI, 1958)). We will need the following simple lemma as well.

Lemma 4. Let $(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ be a q-contact foliation. If $N\subset M$ is an $\mathcal{F}$-saturated submanifold of even codimension, then $(N,\vec{\lambda}|_N,\mathrm{T}N=\mathcal{R}|_N\oplus\xi\cap\mathrm{T}N)$ is a q-contact structure on N.

Proof. First note that since, N is saturated, that is, $\mathcal{F}(p)\subset N$ for every $p\in N$, we have $\dim N\geq q$, and therefore $\operatorname{codim} N=2m$ for some $m\leq n$. Moreover, we cannot have R_i transverse to N at any point. Hence all Reeb vector fields are everywhere tangent to N. In particular, $\mathcal{R}|_N\subset\mathrm{T}N$. One can also check that

$$\ker(\lambda_i|_N)=\ker\lambda_i\cap\mathrm{T}N,$$

for any i and, consequently,

$$\bigcap_i\ker(\lambda_i|_N)=\xi\cap\mathrm{T}N.$$

Moreover, the derivatives satisfy

$$\ker(\mathrm{d}\lambda_i|_N)=\mathcal{R}\cap\mathrm{T}N=\mathcal{R}|_N.$$

for whichever i we choose. Hence

$$\mathrm{T}N=(\mathrm{T}N\cap\mathcal{R})\oplus(\mathrm{T}N\cap\xi)=\mathcal{R}|_N\oplus(\mathrm{T}N\cap\xi)=\ker(\mathrm{d}\lambda_i|_N)\oplus\left(\bigcap_j\ker\left(\lambda_j|_N\right)\right),$$

and therefore $(N,\lambda_1|_N,\cdots,\lambda_k|_N,\mathrm{T}N=\mathcal{R}|_N\oplus\xi\cap\mathrm{T}N)$ is a q-contact structure on N, as claimed.

□

Theorem 15. Every isometric contact foliation on a closed manifold satisfies the **SGWC**, having at least 2 distinct closed orbits. Moreover, the set $\mathcal{C}$ of closed orbits consists of a union

$$\mathcal{C} = \bigcup N_i,$$

where each N_i is an even-codimensional totally geodesic closed submanifold carrying a closed contact foliation, that is, one where every leaf is closed.

Proof. Let $\mathbb{T}^q \to \mathrm{Iso}(M)$ be a compatible action by isometries as in Proposition 24. As remarked before in Proposition 25, since the generators $X_1, \cdots, X_q$ of the compatible toric action are foliated vector fields, to find a close orbit, it is sufficient to find a point where all the generators are tangent to $\mathcal{R}$. The main idea here is that the set of points where these generators are tangent to the foliation can be realised as a zero set of a Killing vector field. Hence it has even codimension and therefore carries a q-contact structure, according to Lemma 4.

Given a generator X_i, we denote by $\overline{X}_i$ its image under the projection onto the algebra of transverse fields. By construction, each $\overline{X}_i$ is *a transverse Killing vector field*, that is, one that preserves the metric $g^\perp$ on ξ.

Consider the set of zeros of $\overline{X}_1$, i.e.,

$$\{p \in M; \overline{X}_1|_p = 0\} \subset M.$$

This is exactly the set of points $p \in M$ where X_1 is tangent to the foliation, and, due to Proposition 25, it has at least two elements. Let M_1 be a connected component of this set. It follows from Proposition 26 that M_1 is a closed submanifold of even codimension saturated by the contact foliation $\mathcal{F}$. According to Lemma 4, there is on M_1 a q-contact structure inducing the same foliation as $\mathcal{R}|_{M_1}$, and whose Reeb vector fields are $R_i|_{M_1}$. By construction, X_1 is everywhere tangent to the leaves on M_1. Now, since M_1 is closed, for any $j = 1, \cdots, q$ the function $\varphi_{2j} = -\lambda_j(X_2)$ has a critical point on M_1, and hence the set

$$\{p \in M_1; \overline{X}_2|_p = 0\}$$

is non-empty. We choose a connected component M_2 on this set. It follows again from Proposition 26 that M_2 is a closed submanifold of even codimension (both in M and M_1), which is saturated by $\mathcal{F}$ and on which X_2 restricts to a vector field everywhere tangent to the leaves.

Iterating this procedure, we eventually arrive at a submanifold M_{q-1} of even codimension. By construction, the manifold M_{q-1} is $\mathcal{F}$-saturated, and on M_{q-1} the fields $X_1, \cdots, X_{q-1}$ are everywhere tangent to the leaves. On this closed submanifold the functions $\varphi_{q,j}$ have at least two distinct critical points p and q, corresponding the extrema of the function, so that both $X_q|_p$ and $X_q|_q$ are tangent to $\mathcal{R}$. By construction, all the X_i are tangent to $\mathcal{R}$ at p and q, hence, by Proposition 25, the leaves $\mathcal{F}(p)$ and $\mathcal{F}(q)$ are closed.

This proves that the subset $\mathcal{C}$ of points in M belonging to a closed leaf is non-empty. We know from Proposition 23 that the leaf closures are exactly the orbits of the action of E^1_F on M. Thus, if $p \in \mathcal{C}$, then p belongs to a closed leaf $\mathcal{F}(p) = \overline{\mathcal{F}(p)}$, which is the orbit of p under the action of E^1_F. Since $\mathcal{F}(p)$ is a torus $\mathbb{T}^q$, p is a fixed point of some element of E^1_F. Hence it is a fixed point of the elements of an entire isotropy subgroup of E^1_F (as is the entire leaf L to which p belongs). Now, being a compact Abelian Lie group, E^1_F admits only a finite number of such isotropy groups, say $I_1, \cdots, I_l$. Denote by $\mathfrak{I}_1, \cdots, \mathfrak{I}_l$ the corresponding Lie algebras of Killing fields. The leaf $\mathcal{F}(p)$ is a set of collective zeros of the Killing fields in one of the algebras $\mathfrak{I}_j$. Hence it follows from (KOBAYASHI, 1958, Corollary 1) that $\mathcal{F}(p)$ is contained in an even-codimensional totally geodesic submanifold N of M. The finite collection of all these submanifolds gives us the desired partition. Moreover, as pointed out before, $p \in N$ implies $\mathcal{F}(p) \subset N$. Hence each N is a saturated submanifold of even codimension so that Lemma 4 applies, allowing us to conclude that the restriction of $\vec{\lambda}$ to N gives rise to a q-contact foliation, the leaves of which are all closed by construction.

□

In the next section, we will see how to define the set $\mathcal{C}$ in terms of a Morse-Bott function and how to relate its properties to the basic cohomology of $\mathcal{F}$.

4.5 Counting closed leaves

In this section, we further generalise results from (RUKIMBIRA, 1995) and (GOERTSCHES; LOIUDICE, 2020b) to the context of isometric contact foliations. Originally, the results from this section were proved for K-contact manifolds and f-K-contact structures. However, as we show here, only the weaker hypothesis of the Reeb fields being Killing *for any metric* is sufficient for most of the results. In this section we work on a fixed isometric contact action $F : \mathbb{R}^q \to \mathrm{Iso}(M, g)$.

We begin by recalling that, given an arbitrary field K on a Riemannian manifold (M, g), one can use its dual form $\eta_K := g(K, \cdot)$ to rewrite Koszul's formula as

$$\begin{aligned} 2g(\nabla_X K, Y) &= Xg(K,Y) + Kg(X,Y) - Yg(K,X) \\ &\quad + g([X,K],Y) - g([X,Y],K) - g([K,Y],X) \\ &= Kg(X,Y) - g([K,X],Y) - g([K,Y],X) \\ &\quad + X\eta_K(Y) - Y\eta_K(X) - \eta_K([X,Y]) \\ &= \mathcal{L}_K g(X,Y) + d\eta_K(X,Y). \end{aligned}$$

If we suppose K is Killing, so that the first term in the RHS vanishes, this expression becomes

$$2g(\nabla_X K, Y) = d\eta(X,Y).$$

This means that for Killing fields K, the mapping $X \mapsto \nabla_X K$ defines a skew-symmetric $(1,1)$-tensor field on M. In the case of an isometric contact foliation, since each Reeb field is Killing, we can define:

Definition 26 (*Associate tensor field*)**.** For each Reeb field R_i of an isometric contact foliation $\mathcal{F}$, its *associate tensor field* is the $(1,1)$-tensor

$$f_i : X \mapsto \nabla_X R_i.$$

By construction, these tensors satisfy

$$g(f_i(X),Y) = \frac{1}{2} d\lambda_i(X,Y). \tag{4.12}$$

Next lemma follows immediately from Equation (4.12).

Lemma 5. For every i, the associated tensor field f_i is such that

(i) f_i is skew-symmetric with respect to g;

(ii) $f_i(R_j) = 0$ for every j;

(iii) $\mathrm{Im} f_i \cap \mathcal{R} = \{0\}$.

Remark the similarities between the properties (i) - (iii) of the tensor field associated with R_i and the properties satisfied by the defining tensor f of an f-structure. We do not know, however, that f_i satisfies the equality

$$f_i^3 + f_i = 0,$$

However, as we will see, such a condition is not necessary for our goals. Next, we show that using f_i to replace f, one can recover many of the results which hold for metric f-K-contact structures. First, we have the following lemma, which plays a role similar to that of Equation (2.6) in (GOERTSCHES; LOIUDICE, 2020b).

Lemma 6 (Kostant's Formula)**.** For a Reeb field R_i in an isometric contact manifold $(M, g, \vec{\lambda}, \xi \oplus \mathcal{R})$

$$\nabla_X f_i = R(X, R_i).$$

Proof. On the one hand, the tensor field $\mathcal{L}_{R_i}\nabla$ is zero since the flow of R_i consists of isometries, which preserve the connection. On the other hand,

$$\begin{aligned}
(\mathcal{L}_{R_i}\nabla)_X Y &= \mathcal{L}_{R_i}(\nabla_X Y) - \nabla_{\mathcal{L}_{R_i} X} Y - \nabla_X (\mathcal{L}_{R_i} Y) \\
&= [R_i, \nabla_X Y] - \nabla_{[R_i, X]} Y - \nabla_X [R_i, Y] \\
&= \nabla_{R_i} \nabla_X Y - \nabla_{\nabla_X Y} R_i - \nabla_{[R_i, X]} Y - \nabla_X \nabla_{R_i} Y + \nabla_X \nabla_Y R_i \\
&= \nabla_{R_i} \nabla_X Y - \nabla_{[R_i, X]} Y - \nabla_X \nabla_{R_i} Y + \nabla_X f_i(Y) - f_i(\nabla_X Y) \\
&= R(X, R_i) Y - (\nabla_X f_i) Y.
\end{aligned}$$

Hence $R(X,R_i)Y = (\nabla_X f_i)Y$ for any field Y, as we wanted. $\square$

We consider the Lie algebra $\mathfrak{e}$ of the C^1-enveloping group

$$\mathrm{E}_F^1 = \overline{\mathrm{Span}\{\exp(t_1R_1),\cdots,\exp(t_qR_q);(t_1,\cdots,t_q)\in\mathbb{R}^q\}},$$

noting that this is a subalgebra of $\mathfrak{iso}(M)$(that is, the family of Killing fields on M) containing the Reeb distribution $\mathcal{R}$.

For each point $p \in M$, the isotropy subalgebra $\mathfrak{I}^p$ is defined as the set of elements of $\mathfrak{e}$ whose value at p is zero, i.e., the fields in $\mathfrak{e}$ whose flow fixes p. Note that $\mathfrak{I}^p$ has dimension at most $\dim \mathrm{E}_F^1 - q$ since the Reeb fields never belong to it. We want to choose a vector field $Z : M \to \mathfrak{e}$ avoiding subspaces of non-maximal dimension, which is a generic choice in the following sense. We define

$$\widetilde{\mathfrak{I}}^p = \mathfrak{I}^p \oplus \mathcal{R} \subset \mathfrak{e}.$$

Due to M's compactness, there are only finitely many distinct subspaces $\widetilde{\mathfrak{I}}^p$. Let $\mathfrak{b}$ be the (finite) union of all the "bad" subspaces $\widetilde{\mathfrak{I}}^p \neq \mathfrak{e}$, that is, those whose dimension is non-maximal. We choose a Killing field Z to satisfy

$$Z \in \mathfrak{e} \setminus \mathfrak{b}.$$

Since $\mathcal{L}_Z\lambda_i = 0$ for any i, we have $d(\iota_Z\lambda_i) = -\iota_Z d\lambda_i$. Thus the critical point set $\mathcal{C}$ of the map

$$\begin{aligned} S : M &\to \mathbb{R} \\ p &\mapsto \iota_Z\lambda_i(p) \end{aligned}$$

is exactly the set of points where $d\lambda_i(Z_p,\cdot) \equiv 0$, there is, the set

$$\mathcal{C} = \{p \in M; Z_p \in \mathcal{R}_p\},$$

which, in light of our choice of Z, is the same as the set

$$\mathcal{C} = \{p \in M;\ \text{the dimension of the orbit of } p \text{ under the action of } \mathrm{E}_F^1 \text{ is exactly } q\}.$$

The latter is exactly the union of the closed leaves of the contact foliation $\mathcal{F}$. The set $\mathcal{C}$ is the union of fixed points of the subtori $T \subset \mathrm{E}_F^1$ whose dimension is $\dim \mathrm{E}_F^1 - q$, hence $\mathcal{C}$ is a manifold.

Lemma 7. Let N be a connected component of $\mathcal{C}$ and $p \in N$. Consider the Killing field

$$K = Z - \pi_{\mathcal{R}}Z,$$

and the tensor field

$$\Phi = \sum_{j=1}^{q} (\iota_Z\lambda_j) f_j.$$

Then for all $v, w \in T_pM$ perpendicular to N, we have:

(i) $\nabla_v K = \nabla_v Z + \Phi(v)$ is a non-zero vector perpendicular to N. In particular, $\nabla_v K \in \xi_p$.

(ii) $\mathrm{Hess}(S)|_p(v,w) = 2g(R(R_i,v)Z_p + f_i(\nabla_v Z), w)$.

(iii) The Hessian of S along N is non-degenerate in directions orthogonal to N. In particular, $S : M \to \mathbb{R}$ is a Morse-Bott function on M.

Proof. We remark that $\mathcal{R} \subset TN$, and since $\mathcal{R}$ and ξ are orthogonal as per choice of g, it follows that $T^{\perp}N \subset \xi$. To prove (i), first note that the Lie algebra of E^1_F has a decomposition $\mathfrak{e} = \mathfrak{I}^p \oplus \mathcal{R}$, and $Z = K + \pi_{\mathcal{R}} Z$ is simply the corresponding decomposition of Z. Evaluating the connection in the direction of v yields

$$
\begin{aligned}
\nabla_v Z &= \nabla_v K + \nabla_v \sum_{j_i}^{q} \lambda_j(Z) R_j \\
&= \nabla_v K + \sum_{j_i}^{q} (v\lambda_j(Z) R_j + \lambda_j(Z) f_j(v)) \\
&= \nabla_v K + \Phi(v) + \sum_{j=1}^{q} (v\lambda_j(Z)) R_j. \qquad (4.13)
\end{aligned}
$$

Now notice that on N the vector Z belongs to $\mathcal{R}$, and since $v \in \xi$, $\lambda_j([v,Z]) = 0$ for every j. Hence

$$v\lambda_j(Z) = d\lambda_j(v,Z) + Z\lambda_j(v) + \lambda_j([v,Z]) = d\lambda_j(v,Z) = 0,$$

and therefore Equation (4.13) becomes $\nabla_v K = \nabla_v Z + \Phi(v)$, as wanted.

We claim $\nabla_v K \neq 0$. To see this, suppose by contradiction that $\nabla_v K = 0$ and consider a geodesic γ starting at p with velocity vector v. Since K is Killing, it is a Jacobi field for γ, and since $K_p = 0$, it follows that $K|_\gamma = 0$. On the other hand, our choice of Z implies that in a neighbourhood of N, the only zeros of Z are those in N. Thus $\gamma \subset N$, contradicting the orthogonality of v with respect to N. Therefore $\nabla_v K \neq 0$.

Now, to see that $\nabla_v K$ is orthogonal to N, we note that N, being the zero set of a Killing field, is a totally geodesic submanifold. Because K is Killing and $K|_N$ is tangent to N, this implies $\nabla_X K$ tangent to N for any X tangent to N. Thus

$$g(\nabla_v K, X) = -g(\nabla_X K, v) = 0,$$

and since X was arbitrarily chosen, it follows that $\nabla_v K \in T^{\perp}N \subset \xi$.

As for (ii), we consider fields V and W extending v and w. Suppose these fields are obtained employing parallel transporting v and w along the geodesics starting at p. Observe that R_i and Z are commuting Killing fields, hence

$$\nabla_{R_i} Z = f_i(Z) + [R_i, Z] = f_i(Z).$$

Using the relation above, the properties of our choice of Z, and Lemma 6, we obtain

$$\begin{aligned}
\mathrm{Hess}(S)|_p(v,w) &= V(WS(p)) \\
&= V(W\lambda_i|_p(Z)) \\
&= V(Wg_p(R_i,Z) \\
&= V(g_p(f_i(W),Z)+g_p(R_i,\nabla_W Z)) \\
&= V(-g_p(f_i(Z),W)-g_p(f_i(Z),W) \\
&= -2Vg_p(f_i(Z),W) \\
&= -2(g_p(\nabla_V f_i(Z),W)+g_p(f_i(Z),\nabla_V W)) \\
&= -2(g_p((\nabla_V f_i)(Z)+f_i(\nabla_V Z),W)+\frac{1}{2}\underbrace{d\lambda_i(Z,\nabla_V W)}_{0} \\
&= -2g(R(v,R_i)Z+f_i(\nabla_v Z),w).
\end{aligned}$$

Finally, to see that $\mathrm{Hess}(S)|_p$ is non-degenerate in the normal bundle of N, we first observe that

$$R(X,R_i)R_j = (\nabla_X f_i)R_j = \nabla_X(f_i(R_j)) - f_i(\nabla_X R_j) = -f_i f_j(X).$$

Now we use item (ii) with $w = f_i(\nabla_v K)$, yielding:

$$\begin{aligned}
\mathrm{Hess}(S)|_p(v,f_i(\nabla_v K)) &= -2g(R(v,R_i)Z+f_i(\nabla_v Z),f_i(\nabla_v K)) \\
&= -2g\left(\sum_{j=1}^{q}\iota_Z\lambda_j R(v,R_i)R_j+f_i(\nabla_v Z),f_i(\nabla_v K)\right) \\
&= -2g\left(-\sum_{j=1}^{q}\iota_Z\lambda_j f_i f_j(v)+f_i(\Phi(v)+\nabla_v K),f_i(\nabla_v K)\right) \\
&= -2g(-f_i(\Phi(v))+f_i(\Phi(v))+f_i(\nabla_v K),f_i(\nabla_v K)) \\
&= -2|(f_i(\nabla_v K),f_i(\nabla_v K)|
\end{aligned}$$

Now note that $f_i(\nabla_v K)\neq 0$. In fact, suppose by contradiction that $f_i(\nabla_v K)=0$. Then, for every $X\in\xi$:

$$0 = g(X,f_i(\nabla_v K)) = -g(f_i(X),\nabla_v K) = -\frac{1}{2}d\lambda_i(X,\nabla_v K),$$

which can not happen since $d\lambda_i$ is non-degenerate on ξ. This means for each v normal to N there is another vector $f_i(\nabla_v K)$ such that $\mathrm{Hess}(S)|_p(v,f_i(\nabla_v K))\neq 0$, that is, $\mathrm{Hess}(S)|_p$ is non-degenerate in normal directions. □

Recall that the Betti number $b_i(M;\mathbb{R})$ is the dimension of the i-th cohomology group $H^i_{dR}(M)$ over $\mathbb{R}$. Similarly, we define the $\mathcal{F}$-basic Betti number $b_i(\mathcal{F})$ to be the real dimension of the basic cohomology group $H^i_B(\mathcal{F})$. The $\mathcal{F}$-basic Poincaré polynomial of the foliated manifold $(M,\mathcal{F})$ is

$$P_{\mathcal{F}}(t) = \sum_{i=0}^{\mathrm{codim}(\mathcal{F})} t^i b_i(\mathcal{F}).$$

We have just shown that $S : M \to \mathbb{R}$ is a $\mathcal{F}$-basic Morse-Bott function. We can then apply the results from (GOERTSCHES; TÖBEN, 2018) to relate the $\mathcal{F}$-basic Poincaré polynomials of $(M, \mathcal{F})$ and $(N, \mathcal{F}|_N)$, for each connected component N of the critical set $\mathcal{C}$, getting:

Theorem 16. The $\mathcal{F}$-basic Poincaré polynomials for M and N satisfy

$$P_{\mathcal{F}}(t) = \sum_N t^{i_N} P_{\mathcal{F}|_N}(t),$$

where N runs over all the connected components of $\mathcal{C}$, and i_N is the index of N, i.e., the rank of the negative normal bundle of N with respect to S.

In particular, by evaluating both sides at $t = 1$, we obtain

Theorem 17. Let $(M, g, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be an isometric contact foliation, and $\mathcal{C}$ be the critical point set of the Morse-Bott function $S : p \mapsto \lambda_i(Z_p)$. Then

$$\dim_{\mathbb{R}} H_B^*(\mathcal{F}) = \dim_{\mathbb{R}} H_B^*(\mathcal{F}|_{\mathcal{C}}).$$

In particular, if $\mathcal{F}$ has only finitely many closed orbits, then the dimension of the basic cohomology ring $H_B^*(\mathcal{F})$ is exactly the number of closed orbits of $\mathcal{F}$.

Theorem 17 allows us to estimate the number of closed orbits by studying the basic cohomology of the contact foliation $\mathcal{F}$. The first thing to note is that the even-dimensional basic cohomology groups are never zero because the exterior derivatives of the adapted coframe $\vec{\lambda}$ define non-zero basic forms. In fact, it is known that an invariant transversal volume form μ for a harmonic foliation $(M, \mathcal{F})$ represents a non-zero class on the top-dimensional basic cohomology space $H_B^{\operatorname{codim}\mathcal{F}}(\mathcal{F})$. This implies, in particular, that if $\mathcal{F}$ is a foliation of even codimension $2n$, and ω is an invariant transversal symplectic form, then $[\omega]^i \neq 0 \in H_B^{2i}(\mathcal{F})$ for $i = 1, \cdots, n$ (cf. (TONDEUR, 2012, Theorems 4.32 and 4.33)). In the case of q-contact structures, one notes that the operator

$$\omega \mapsto \int_M \lambda \wedge \omega,$$

where $\lambda := \lambda_1 \wedge \cdots \wedge \lambda_q$ is the characteristic form, descends to an operator $H_B^*(\mathcal{F}) \to \mathbb{R}$, because the q-form λ is $\mathcal{F}$-closed (cf. Remark 9). Such operator maps $[d\lambda_i]^n$ to a non-zero number, since $\lambda \wedge (d\lambda_i)^n$ is a volume form on M, hence $[d\lambda_i]^n \neq 0$, and consequently $[d\lambda_i]^j \neq 0$ for every $j = 1, \cdots, n$.

To better use this fact, we associate with each adapted coframe the following quantity.

Definition 27 (*Basic dimension of an adapted coframe*)**.** Let $(M, g, \vec{\lambda}, \mathcal{R} \oplus \xi, g)$ be an isometric contact foliation on the $(2n+q)$-dimensional manifold M. Define $\delta_0(\vec{\lambda}) := 1$, and for $i = 1, \cdots, n$, let $\delta_i(\vec{\lambda}))$ be the dimension of the linear subspace of $H_B^{2i}(\mathcal{F})$ spanned by $\{[d\lambda_1]^i, \cdots, [d\lambda_q]^i\}$. In other words

$$\delta_i(\vec{\lambda}) := \max\{\#L; L \subset \{[d\lambda_1]^i, \cdots, [d\lambda_q]^i\} \text{ is linearly independent}\}.$$

The natural number

$$\delta(\vec{\lambda}) := \sum_{i=0}^{n} \delta_i(\vec{\lambda}) = 1 + \sum_{i=1}^{n} \delta_i(\vec{\lambda})$$

is the **basic dimension** of the adapted coframe $\vec{\lambda}$.

Note that $\delta_i(\vec{\lambda})$ is bounded below by 1 and above by either q or the dimension of $H_B^{2i}(\mathcal{F})$. Hence the basic dimension satisfies the inequalities

$$n + 1 \leq \delta(\vec{\lambda}) \leq \min\{qn + 1, \dim H_B^*(\mathcal{F})\}. \tag{4.14}$$

We remark that for uniform contact foliations, the basic dimension is always minimal, i.e., equal to exactly $n+1$. Using Theorem 17, we obtain

Theorem 18. Let $\mathcal{F}$ be an isometric contact foliation on a closed manifold. Then the set $\mathcal{C}$ of closed orbits consists of a union

$$\mathcal{C} = \bigcup N_i,$$

where each N_i is an even-codimensional totally geodesic closed submanifold, and the restriction $\mathcal{F}|_{N_i}$ is a closed contact foliation. Moreover, the number of closed orbits of $\mathcal{F}$ is at least $\delta(\vec{\lambda}) + b_1(M;\mathbb{R})$. In particular, an isometric contact foliation of codimension $2n$ has no less than $n+1$ closed orbits.

Proof. The assertions regarding the set $\mathcal{C}$ are exactly the same as in Therem 15, so we only need to prove that the lower bound of $\delta(\vec{\lambda}) + b_1(M;\mathbb{R})$ closed orbits hold.

By definition, the basic dimension satisfies $\delta_i(\vec{\lambda}) \leq b_{2i}(\mathcal{F})$. Hence, it is a lower bound for $\dim_{\mathbb{R}} H_B^*(\mathcal{F})$. Note that this bound does not consider the dimensions of any of cohomology groups $H_B^i(\mathcal{F})$ for odd i. On the other hand, it was shown in Theorem 9 that every harmonic 1-form on an isometric q-contact manifold is also $\mathcal{F}$-basic, implying an isomorphism $H_{dR}^1(M) \approx H_B^1(\mathcal{F})$ (cf. Theorem 9). In particular, $b_1(M;\mathbb{R}) = b_1(\mathcal{F})$, and therefore $\delta(\vec{\lambda}) + b_1(M;\mathbb{R})$ is a lower bound for $\dim_{\mathbb{R}} H_B^*(\mathcal{F})$. □

As remarked above, the basic dimension does not consider any odd-dimensional basic cohomology groups. Hence, one should not expect it to be equal to the number of closed orbits.

Example 15 (Products of Stiefel manifolds)**.** We consider the 7-dimensional Stiefel manifold

$$V_{2,5} \approx {}^{\mathrm{SO}(5)}/_{\mathrm{SO}(3)}.$$

As shown in (GOERTSCHES; NOZAWA; TÖBEN, 2012, Section 8), $V_{2,5}$ supports a K-contact structure (α, g) whose Reeb field S has exactly

$$4 = \#\{0, [\mathrm{d}\alpha], [\mathrm{d}\alpha]^2, [\mathrm{d}\alpha]^3]\} = \delta(\alpha)$$

orbits. We consider the 14-dimensional manifold $M = V_{2,5} \times V_{2,5}$, and denote by $\pi_i : M \to V_{2,5}$ the projection on the i-th coordinate. For $i = 1, 2$, we write

$$g_i := \pi_i^* g,$$
$$\sigma_i := \pi_i^* \alpha,$$
$$X_i := \pi_i^* S.$$

Then the 1-forms

$$\lambda_+ = \sigma_1 + \sigma_2$$
$$\lambda_- = \sigma_1 - \sigma_2$$

define an non-uniform 2-contact structure on M whose Reeb fields are $R_+ = 2^{-1}(X_1 + X_2)$ and $R_- = 2^{-1}(X_1 - X_2)$, respectively, as in Example 10. The metric $g = g_1 + g_2$ on M is such that each Reeb field R_i is Killing, so that $(M, g, \{\lambda_+, \lambda_-\})$ defines an isometric 2-contact structure. The contact foliation $\mathcal{F}$ is the product of the contact flows in $(V_{2,5}, \alpha)$. In particular, the closed leaves are exactly the products of closed flowlines, hence $(M, \mathcal{F})$ has exactly 16 closed leaves, and $\dim_{\mathbb{R}} H_B^*(\mathcal{F})$ is 16, according to Theorem 17. On the other hand, the Stiefel manifold $V_{2,5}$ is a real cohomology sphere of dimension 7. Hence the Künneth formula implies that the first cohomology group of M is 0. Moreover, $\mathcal{F}$ has codimension $12 = 2 \cdot 6$, hence the minimal number of closed leaves of $\mathcal{F}$ as given by Theorem 18 is 7. Using Equation (4.14), we conclude that the basic dimension of the coframe $\{\lambda_+, \lambda_-\}$ is bounded above by $2 \cdot 6 + 1 = 13$, so that's the maximum number of closed orbits one could assume $\mathcal{F}$ has by using the estimates of Theorem 18 alone.

Observe that the basic dimension $\delta(\{\lambda_+, \lambda_-\})$ is not minimal. In general, for an adapted coframe $\{\lambda_1, \cdots, \lambda_q\}$, two basic classes $[d\lambda_i]$ and $[d\lambda_j]$ satisfy an equality

$$[d\lambda_i]^l = a[d\lambda_j]^l$$

for a non-zero real number, a if and only if there is a basic $(2l-1)$-form η such that

$$(d\lambda_i)^l - a(d\lambda_j)^l = d\eta,$$

and therefore

$$\lambda_i \wedge (d\lambda_i)^{l-1} - a\lambda_j \wedge (d\lambda_j)^{l-1} = \eta + \theta, \tag{4.15}$$

where θ is a closed $(2l-1)$-form. In particular, in the case of the manifold M of Example 10, if we assume $\delta_1(\{\lambda_+, \lambda_-\}) = 1$ and apply Equation (4.15) we get

$$\lambda_+ - a\lambda_- = df + \theta, \tag{4.16}$$

where f is a basic function and $d\theta = 0$, because $H^1_B(\mathcal{F}) \approx H^1_{dR}(M) = 0$ (cf. Theorem 9). Taking exterior derivatives in Equation (4.16) we obtain

$$(a-1)d\sigma_1 = -(a+1)d\sigma_2,$$

which can not happen for any real a. Thus $\delta_1(\{\lambda_+, \lambda_-\}) = 2$, and consequently

$$\delta(\{\lambda_+, \lambda_-\}) = 1 + 2 + \sum_{i=2}^{6} \delta_i(\{\lambda_+, \lambda_-\}) \geq 8$$

is strictly bigger than $2^{-1}\mathrm{codim}(\mathcal{F}) + 1 = 7$.

CHAPTER 5

C^1-EQUICONTINUITY, CONFORMALITY AND QUASICONFORMALITY

C^1-equicontinuity should be thought of as a dynamical way to express the geometrical property of having an invariant metric. Two other properties, one dynamical and one geometrical, generalise the concept of metric invariance: quasiconformality and conformality. When restricted to the normal bundle of a foliation, these properties are generalisations of the existence of a bundle-like metric. In other words, quasiconformal and conformal foliations generalise the class of Riemannian foliations. These generalisations are strict: there are conformal foliations that are not Riemannian and quasiconformal foliations that are not conformal.

We already saw that for contact foliations, the existence of a bundle-like metric is equivalent to the existence of an invariant metric, that is, the class of isometric contact foliations is the same as the class of Riemannian contact foliations, both being characterised by the dynamical condition of being C^1-equicontinuous. We will show in this chapter that the same occurs for the other two classes discussed above, i.e., that quasiconformal and conformal *contact foliations* are actually, up to a choice of metric, isometric contact foliations. In this sense, all of these properties are equivalent to being C^1-equicontinuous: quasiconformal, conformal, Riemannian and isometric contact foliations are all characterised by the fact that their C^1-enveloping groups are tori.

5.1 Conformal structures on vector bundles

On the n-dimensional Euclidean space, we say two inner products are **conformally equivalent** if one is a positive multiple of the other. Conformally equivalent inner products define the same (oriented) angles but not the same norms. A **conformal metric** on the Euclidean space is an equivalence class of conformally equivalent inner products.

The space of all inner products on $\mathbb{R}^n$ can be identified with the space $\mathrm{Sym}(n;\mathbb{R})$ of symmetric positive definite $n \times n$ matrices. This space is acted upon by $\mathrm{Gl}_n(\mathbb{R})$ via

$$X \cdot A = |\det X|^{\frac{2}{n}} X^t A X.$$

One can check (for instance, using Cholesky Decomposition) that every $A \in \mathrm{Sym}(n;\mathbb{R})$ belongs to the orbit of the identity matrix I. In other words, the action is transitive. Moreover, this action preserves conformal equivalence, so it descends to the space $C_n := \mathrm{Sym}(n;\mathbb{R})/\sim$ of conformal metrics. The isotropy subgroup of the identity (and hence of the entire action) is

$$\mathrm{Co}_n(\mathbb{R}) := \{X \in \mathrm{Gl}_n(\mathbb{R});\ X^t X = cI, c > 0\},$$

and we can then give the space C_n of conformal metrics the following description

$$C_n \approx {}^{\mathrm{Gl}_n(\mathbb{R})}\big/_{\mathrm{Co}_n(\mathbb{R})} \approx {}^{\mathrm{Sl}_n(\mathbb{R})}\big/_{\mathrm{So}_n(\mathbb{R})}.$$

Note that the last isomorphism is simply a consequence of the fact that conformal metrics can be re-scaled at will so that we can choose the representing matrices to have determinant 1. If an element of C_n is represented by the matrix A whose eigenvalues are $\alpha_1, \cdots, \alpha_n$, then the norm

$$d_0(A) := \frac{1}{2}\left(n\sum_{i=1}^{n}(\ln\alpha_i)^2\right)^{\frac{1}{2}}$$

defines an $\mathrm{Gl}_n(\mathbb{R})$-invariant metric on C_n, making it into a complete, simply connected, globally symmetric Riemannian manifold of non-compact type and negative curvature. Any linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ induces a linear isometry $C_n \to C_n$. We direct the interested reader to (TUKIA, 1986, Section D) for more details and further references.

Another norm d_1 can be defined on C_n, by setting

$$d_1(A) := \frac{n}{2}\max\left(\ln(\max\{\alpha_i\}), \ln\left(\frac{1}{\min\{\alpha_i\}}\right)\right),$$

and requiring invariance under the action of $\mathrm{Gl}_n(\mathbb{R})$. The following inequalities relate these two metrics

$$\frac{1}{\sqrt{n-1}}d_1 \le d_0 \le \begin{cases} d_1 & \text{if } n \text{ is even,} \\ \frac{1}{\sqrt{(n-1)n^{-1}}}d_1 & \text{if } n \text{ is odd} \end{cases} \tag{5.1}$$

(cf. (TUKIA, 1986, Section D)).

Suppose $E \to M$ is a vector bundle of constant rank n. On a fibre $E_x \approx \mathbb{R}^n$, we can consider the set C_x^E of all the conformal metrics, which has the structure of a Riemannian manifold, as discussed above. Thus, $C^E \to M$ becomes a fibre bundle of typical fibre C_n.

Definition 28 (*Conformal Structures*)**.** A **conformal structure** on E is a section $\gamma : M \to C^E$ of the bundle of conformal spaces. We say the conformal structure is *smooth* (resp. C^r, measurable) if the section γ is a smooth (resp. C^r, measurable) map.

Remark 13. At each point x, the conformal metric γ_x is represented by an inner product $g(\gamma)_x$ on the fibre E_x. This choice of representatives can be made so that $g(\gamma)_x$ varies smoothly on x (or varies in a C^r manner) given that the conformal structure is itself smooth (resp, C^r). Therefore, each smooth (resp. C^r) conformal structure $\gamma : M \to C^E$ defines a smooth (resp. C^r) Riemannian metric tensor g^γ on M. Conversely, given a Riemannian metric g on the bundle (M,E), it defines a unique conformal structure $\gamma(g)$ by defining $\gamma(g)_x$ as the conformal class of g_x. It is clear that $\gamma(g)$ defined in this manner is as regular as g.

Each fibre of the bundle of conformal metrics C^E has a canonically defined metric, as described earlier. Any linear isomorphism between two fibres of E induces an isometry between the associated fibres of C^E. So, if (Φ,ϕ) is a bundle isomorphism [1], it acts on a conformal structure via

$$(\Phi,\phi)\cdot\gamma := \Phi^{-1}\circ\gamma\circ\phi$$

(where Φ^{-1} is an abuse of notation denoting the isometry induced by $\Phi^{-1} : E \to E$ on C^E). If γ is represented by a metric tensor g, then $(\Phi,\phi)\cdot\gamma$ is represented by the metric

$$h_x := g_{\phi(x)} \circ (\Phi_x \times \Phi_x).$$

When $E = \mathrm{T}M$ and ϕ is a diffeomorphism, then $h = \phi^* g$. The same holds for any sub-bundle of $\mathrm{T}M$ preserved by $\mathsf{Diff}(M)$. In this case, instead of $(\Phi,\phi)\cdot\gamma$ we may write the action simply as $\phi\cdot\gamma$.

Specialising in the q-contact case, we have the following definitions.

Definition 29 (*Conformal contact action*)**.** The contact action $F : \mathbb{R}^q \to \mathsf{Diff}(M)$ is **conformal** if there is a smooth conformal structure $\gamma : M \to C^{\mathrm{T}M}$ such that

$$F^a\cdot\gamma = \gamma,$$

for all $a \in \mathbb{R}^q$.

Definition 30 (*Transversely conformal contact action*)**.** The contact action $F : \mathbb{R}^q \to \mathsf{Diff}(M)$ is **transversely conformal** if there is a smooth conformal structure $\gamma : M \to C^\xi$ such that

$$F^a\cdot\gamma = \gamma,$$

for all $a \in \mathbb{R}^q$.

That is, a contact foliation is (transversely) conformal when it preserves a conformal structure on (the contact distribution ξ) the manifold M . In other words, there is a

[1] that, is, $\Phi : E \to E$ is a fibre-wise linear isomorphism, $\phi : M \to M$ is a homeomorphism, and $\pi\circ\Phi = \phi\circ\pi$, where $\pi : E \to M$ is the natural projection

Riemannian metric tensor g on ξ (respectively, M) and a collection $\{\rho_a\}_{a\in\mathbb{R}^q}$ of positive-valued functions $\rho_a : M \to \mathbb{R}_+$ such that

$$(F^a)^* g = \rho_a g,$$

for all $a \in \mathbb{R}^q$. Smoothness of γ implies the family $\{\rho_a\}$ varies smoothly on a. We say the contact foliation $\mathcal{F}$ is (transversely) conformal if its underlying action is (transversely) conformal.

Generally, a vector field whose flow consists of conformal transformations is called a **conformal Killing field**. One possible characterisation for such fields is that they satisfy the following equation

$$\mathcal{L}_X g = \rho g,$$

for some non-negative function ρ, called the **conformal factor** of the field X. In particular, Killing fields are exactly the conformal ones whose conformal factor is identically zero. In the case of conformal contact actions, this translates into the following proposition.

Proposition 27. If $F : \mathbb{R}^q \to \mathrm{Diff}(M)$ is (transversely) conformal then each to each Reeb field corresponds a function σ_i such that

$$\mathcal{L}_{R_i} g = \sigma_i g.$$

Proof. The flow of R_i is exactly $\exp(tR_i) = F^{te_i}$, by definition. Hence, there are positive-valued functions ρ_{te_i} such that

$$(\exp(tR_i))^* g = \rho_{te_i} g.$$

Differentiating both sides with respect to t and evaluating at 0 gives

$$\begin{aligned} \mathcal{L}_{R_i} g := \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tR_i))^* g\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}(\rho_{te_i} g)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\rho_{te_i}\bigg|_{t=0} g. \end{aligned}$$

Letting $\sigma_i := \frac{\mathrm{d}}{\mathrm{d}t}\rho_{te_i}|_{t=0}$ gives the desired equality. □

Example 16 (Invariant metrics)**.** If M supports an $\mathcal{F}$-invariant metric g, then the conformal structure represented by g is also invariant. Indeed, for any positive function $f : M \to \mathbb{R}$,

$$(F^a)^* fg = (f \circ F^a) g.$$

So $\mathcal{F}$ is conformal. Similarly, if g is a bundle-like metric on ξ, then it represents an $\mathcal{F}$-invariant conformal structure on ξ. Thus every Riemannian foliation is transversely conformal.

Remark 14. Note that if g represents an invariant conformal structure on M, then it restriction to ξ induces an invariant conformal structure. On the other hand, contrary to the Riemannian case, there is no way to use the adapted metric g^τ to extend an $\mathcal{F}$-invariant conformal structure on ξ to the whole TM in a conformal way. Indeed, if such a conformal structure is represented by a metric g^ξ, we would have

$$(F^a)^*(g^\xi + g^\tau) = \rho_a g^\xi + g^\tau = f_a(g^\xi + g^\tau). \tag{5.2}$$

Evaluating this at a pair (R_i, R_i) yields $f_a \equiv 1$, and therefore $\rho_a \equiv 1$. In other words, the only way Equation (5.2) holds is if R_i is Killing with respect to g^ξ, so *a priori*, there is no clear way to extend an invariant conformal structure on ξ to TM, unless it is actually an invariant metric.

Now, let T be a complete transversal for $\mathcal{F}$. Since T$T \approx \xi$, the metric g^ξ pulls back to a metric on T, for which the Reeb fields are all conformal Killing fields. Using the same arguments as in Proposition 14, we note that the holonomy transformations induced by $\mathcal{F}$ on T can be seen, at least locally, in terms of the action of F on M, and are therefore conformal transformations of T.

Proposition 28. If the contact action $\mathcal{F}$ is transversely conformal, then the holonomy pseudogroup of $\mathcal{F}$ is a pseudogroup of conformal transformations.

Remark 15. In particular, if $(U_\alpha, f_\alpha, \gamma_{\alpha\beta})$ is a foliation cocycle for $\mathcal{F}$ on M, then the transition maps $\gamma_{\alpha\beta}$ are all conformal transformations, and therefore $\mathcal{F}$ is a **transversely flat conformal foliation**, in the sense of (ASUKE, 1996).

5.2 Quasiconformality

Suppose M is a manifold and $V \subset \mathrm{T}M$ is a subbundle equipped with a Riemannian metric tensor g. If $F : \mathbb{R} \to M$ is an action, then at each point $x \in M$ and element $a \in \mathbb{R}^q$, we consider the mappings

$$L_{F,V}(x,a) := \max\{|\mathrm{d}F^a_x v|;\ v \in V_x \text{ and } |v| = 1\},$$
$$l_{F,V}(x,a) := \min\{|\mathrm{d}F^a_x v|;\ v \in V_x \text{ and } |v| = 1\}.$$

These mappings measure, respectively, how much the unit ball in V is deformed in the maximum and minimum deformation directions under the map $\mathrm{d}F^a_x$, all with respect to the metric g, of course. Together, these quantities define the V**-eccentricity** of the action F as the ratio

$$E_{F,V}(x,a) := \frac{L_{F,V}(x,a)}{l_{F,V}(x,a)}.$$

This can be thought of as a measure of how much the unity ball in V_x is deformed into an ellipsoid in $V_{F^a(x)}$ by the transformation $\mathrm{d}F^a_x$. The action F is said to be K**-quasiconformal**

on V if the V-eccentricity map is globally bounded by K, that is, $E_{F,V}(x,a) < K$. The real number K is called a **quasiconformality constant** for the action. For simplicity, in the particular case when $V = \mathrm{T}M$ is the entire tangent bundle, we drop any references to the subbbundle V and adopt the notations l_F, l_F and E_F.

We remark that for compact M, the choice of metric is irrelevant since any two Riemannian metrics g, g' are quasi-isometric, that is

$$\frac{1}{c}g(v,v) \leq g'(v,v) \leq c^2 g(v,v)$$

for some fixed constant $c \in \mathbb{R}_+$ and every $v \in \mathrm{T}M$. The notions of conformality and quasiconformality, though very different in nature (the former is a geometric property, and the latter is dynamic), are related in many ways.

Proposition 29. The V-eccentricity map $E_{F,V}$ is constant equal to 1 if and only if F is conformal on V, i.e., preserves a conformal structure in C^V.

Proof. If F is conformal on V there is an $\mathbb{R}^q$ indexed family $\rho_a : M \to \mathbb{R}_+$ such that the unity ball at V_x is mapped by $\mathrm{d}F^a_x$ into a ball of radius $\rho_a(x)$ in $V_{F^a(x)}$. Hence $L_{F,V}(x,a) = l_{F,V}(x,a) = \rho_a(x)$, and $E_{F,V} \equiv 1$. Conversely, if $E_{F,V} \equiv 1$ then $L_{F,V} \equiv l_{F,V}$ and, for any x and a, the isomorphism $\mathrm{d}F^a_x$ maps the unity ball in V_x to a ball of radius $r_a(x) = L_{F,V}(x,a) = l_{F,V}(x,a)$ in $V_{F^a(x)}$, implying $(F^a)^* g = r_a(x) g$, as we wanted.

□

For contact actions, we define

Definition 31 (*Quasiconformal contact action*)**.** We say the contact action F is **quasiconformal** with respect to the metric g if its $\mathrm{T}M$-eccentricity mapping

$$E_F : M \times \mathbb{R}^q \to \mathbb{R}$$

is globally bounded. Similarly, the action is **transversely quasiconformal** if the ξ-eccentricity map $E_{F,\xi} : M \times \mathbb{R}^q \to \mathbb{R}$ is bounded.

We say the contact foliation $\mathcal{F}$ is (transversely) quasiconformal if the associated action is (transversely) quasiconformal.

Notice that quasiconformality restricts to subbundles, that is, if one has vector bundles $W \subset V \subset \mathrm{T}M$ and an action F is quasiconformal on V then it is also quasiconformal on W. In particular, any quasiconformal contact action is also transversely quasiconformal. The converse is generally not true, as we illustrate in Example 18 bellow. Akin to what happens in the conformal case, if the contact action is transversely quasiconformal, there is no clear way to use the adapted metric g^τ to extend the metric on ξ to the entire $\mathrm{T}M$ *quasiconformally.*

Indeed, let consider on $\xi, \mathcal{R}$ and $\mathrm{T}M$ the metrics $g^{\xi}, g^{\mathcal{R}}$ and $g^{\xi}+g^{\tau}$, respectively. From the fact that

$$L_{F,\xi}(x,a) \leq K l_{F,\xi}(x,a)$$

for all $x \in M$ and $a \in \mathbb{R}^q$, we would like to conclude that a similar inequality

$$L_F(x,a) \leq C l_F(x,a) \quad \forall x \in M,\ \forall a \in \mathbb{R}^q, \tag{5.3}$$

for some constant C. Now, since $\xi_x \subset \mathrm{T}_x M$, we immediately obtain that $l_F \leq l_{F,\xi}$. On the other hand, given a vector $v \in \mathrm{T}_x M$, we decompose it into $v = v_\xi + v_{\mathcal{R}}$, and notice that if $|v|+1$ then $|v_\xi|, |v_\xi| \leq 1$. Since the maximum of a linear map in the closed unit ball is always attained at its boundary, i.e., the unity sphere, we have

$$\begin{aligned} L_F(x,a) &= \max\{|\mathrm{d}F_x^a v|;\ |v|=1\} \\ &\leq \max\{|\mathrm{d}F_x^a v_\xi| + |\mathrm{d}F_x^a v_{\mathcal{R}}|;\ v_\xi \in \xi_x \text{ and } |v_\xi| \leq 1, v_{\mathcal{R}} \in \mathcal{R}_x \text{ and } |v_{\mathcal{R}}| \leq 1\} \\ &\leq \max\{|\mathrm{d}F_x^a v_\xi|;\ v_\xi \in \xi_x \text{ and } |v_\xi| \leq 1\} + \max\{|\mathrm{d}F_x^a v_{\mathcal{R}}|;\ v_{\mathcal{R}} \in \mathcal{R}_x \text{ and } |v_{\mathcal{R}}| \leq 1\} \\ &= \max\{|\mathrm{d}F_x^a v_\xi|;\ v_\xi \in \xi_x \text{ and } |v_\xi| = 1\} + \max\{|\mathrm{d}F_x^a v_{\mathcal{R}}|;\ v_{\mathcal{R}} \in \mathcal{R}_x \text{ and } |v_{\mathcal{R}}| = 1\} \\ &= L_{F,\xi}(x,a) + 1. \end{aligned}$$

Thus, we are left with three inequalities

$$\begin{aligned} L_F &\leq K l_F, \\ L_F &\leq L_{F,\xi} + 1, \\ L_F &\leq l_{F,\xi}, \end{aligned}$$

which are generally not enough to conclude the validity of (5.3).

More interesting is that whenever M is compact, quasiconformality can be made into conformality by a suitable change of Riemannian metrics. This can be showed adapting arguments from (TUKIA, 1986) and (SADOVSKAYA, 2005).

Theorem 19. If a contact action F on a compact manifold M is (transversely) quasiconformal, then there is a (transverse) bounded measurable conformal structure γ preserved by F.

Proof. We shall prove this only for quasiconformal actions, the arguments for the transverse case (or for any other subbundle of $\mathrm{T}M$) being completely analogous. Let γ_0 be a continuous conformal structure on $\mathrm{T}M$. For each $x \in M$, we consider the subset of $C^{\mathrm{T}M}(x)$ defined as

$$C(x) := \{(F^a \cdot \gamma_0)_x;\ a \in \mathbb{R}^q\}.$$

Quasiconformality of the action implies that $C(x)$ is always a bounded subset of $C^{\mathrm{T}M}(x)$ with respect to its $\mathrm{Gl}_{2n}(\mathbb{R})$-invariant metrics. Indeed, if $E_F \leq K$, then we have the following inclusion of spheres in $\mathrm{T}_{F^a(x)}M$

$$S^{2n}_{F^a(x)} \subset \frac{1}{l_F(x,a)} \mathrm{d}F_x^a S_x^{2n} \subset K S^{2n}_{F^a(x)}.$$

Now, if g_0 is a metric representing γ_0 and the eigenvalues of its corresponding matrix at $\mathrm{T}_x M$ are $\alpha_1, \cdots, \alpha_q$ (it doesn't matter which basis we choose for $\mathrm{T}_x M$ since the eigenvalues are invariant), then an element $(F^a \cdot \gamma_0)_x$ of $C(x)$ is represented by the metric $(F^a)^* g$, and at $\mathrm{T}_x M$ its eigenvalues $\beta_1, \cdots, \beta_n$ can be indexed as to satisfy

$$\frac{1}{l_F(x,a)} \alpha_i \leq \beta_i < K \alpha_i.$$

It follows from the inequalities in (5.1) that $(F^a \cdot \gamma_0)_x$ is at bounded distance from $(\gamma_0)_x$, for every $a \in \mathbb{R}^n$, and hence $C(x)$ is bounded. Note that the choice of γ_0 does not impact this fact since M is compact and γ_0 is continuous. Finally, since $C^{\mathrm{T}} M(x)$ is a space of non-positive curvature, there is a unique ball of minimal radius containing $C(x)$. It can be shown that the centre of this ball is a bounded, measurable, F-invariant conformal structure (TUKIA, 1986).

□

Definition 32 (*Good measures*)**.** A holonomy invariant measure μ is said to be **good** if it is non-atomic and the union of all the leaves in its support is the entire ambient manifold M.

Proposition 30. The measure $\mu_i = |\mathrm{d}\lambda_i^n|$ is a good measure for $\mathcal{F}$.

Proof. Indeed, μ_i is a volume form on T, hence given $x \in M$ and $y \in \mathcal{F}(x) \cap T$, any open set $U \subset T$ containing y is such that $\mu_i(U) > 0$. Thus $\mathcal{F}(x) \in \mathrm{supp}(\mu_i)$, and since x is arbitrary it follows that

$$\bigcup_{\mathcal{F}(x) \in \mathrm{supp}(\mu_i)} \mathcal{F}(x) = M,$$

and therefore, μ_i is a good measure. □

Theorem 20. Let $(M, \mathcal{F}, g_0)$ be a quasiconformal contact foliation. Then there is a Riemannian metric g on M for which $\mathcal{F}$ is isometric.

Proof. Indeed, by Proposition 19 there is a Riemannian metric g_1 such that $(M, \mathcal{F}, g_1)$ is a conformal contact foliation. On the other hand, since $\mathcal{F}$ admits a good measure μ_i, it follows from (ASUKE, 1998) that $\mathcal{F}$ supports a transversely invariant Riemannian metric, which implies $\mathcal{F}$ to be an isometric contact foliation, as in Proposition 7. □

Corollary 10. Every quasiconformal contact foliation on a compact manifold is uniformly almost periodic.

Proof. Quasi-isometries preserve equicontinuity, and any two Riemannian metrics on a compact manifold are quasi-isometric. Since any isometric action is equicontinuous, any quasiconformal contact action is uniformly almost periodic (cf. Theorem 11). □

Putting together Theorems 7, 12, 13, 19, and 20, we obtain

Theorem B. Let $\mathcal{F}$ be a contact foliation on a compact manifold M. The following are equivalent.

(i) $\mathcal{F}$ is C^1-equicontinuous;

(ii) the C^1-enveloping group $\mathrm{E}^1_{\mathcal{F}}$ is a torus;

(iii) $\mathcal{F}$ admits a bundle-like metric;

(iv) $\mathcal{F}$ admits an invariant metric;

(v) $\mathcal{F}$ is quasiconformal;

(vi) $\mathcal{F}$ admits an invariant conformal structure.

As a first consequence, notice that if a contact action if transversely quasiconformal (in particular, transversely conformal), even though generally there is no way to quasiconformally (or conformally) extend the metric in ξ to the entire $\mathrm{T}M$, there is always *another metric tensor* on M with respect to which the action is quasiconformal (conformal) on the entire tangent bundle.

Example 17 (Regular contact forms)**.** A contact form on M is regular if every point of M has a neighbourhood U such that each integral curve of the Reeb field intersecting U passes passes through U only once (cf. (BLAIR, 2010, page 24)). It is known that for regular contact forms on closed manifolds all the Reeb orbits are closed, as the action of the Reeb field is actually and S^1-action (cf. (BOOTHBY; WANG, 1958)). In particular, it follows that C^1-enveloping group associated with the action is S^1, therefore the contact structures involved are C^1-equicontinuous, and in particular quasiconformal.

Similarly, every q-contact structure obtained from a regular contact structure via a “product-like” construction (cf. Examples 5,7,8) is also quasiconformal.

Example 18 (Anosov contact actions)**.** An Anosov action (cf. Section 2.3.1) is said to be **u-quasiconformal** if it is quasiconformal with respect to the unstable bundle E^u. Similarly, the action is **s-quasiconformal** when it is quasiconformal with respect to the stable bundle E^s. When the action is both u-quasiconformal and s-quasiconformal, we simply say it is an quasiconformal Anosov action. Recently, quasiconformal Anosov contact actions have been subject to intense research (cf. (FANG, 2004; SADOVSKAYA, 2005; NEPOMUCENO, 2022)).

It follows from Theorem B that a quasiconformal Anosov contact action on a compact manifold can never be *transversely* quasiconformal. Indeed, first we note that the Anosov property of an action is preserved by quasi-isometries, hence, if an Anosov action on a compact manifold is Anosov no matter which metric tensor is considered. If a quasiconformal Anosov contact action $F : M \to \mathrm{Diff}(M)$ were quasiconformal on $\xi = E^u + E^s$, then according to Theorem B there would be a metric tensor g on ξ with respect to which F is simultaneously both Anosov (i.e., hyperbolic) and isometric, a contradiction.

Another way of showing this is by remarking that Anosov actions are not equicontinuous (cf. Example 13), and therefore not C^1-equicontinuous.

In particular, there is no way to *quasiconformaly* extend the metrics on E^u or E^s to the entire transverse bundle ξ.

The following theorem is a direct application of Theorems B and 18.

Theorem C. Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a closed quasiconformal q-contact manifold. The contact foliation $\mathcal{F}$ satisfies the **SGWC**. Moreover, $\mathcal{F}$ has at least $\delta(\vec{\lambda}) + b_1(M;\mathbb{R}) \geq 2^{-1}\,\mathrm{codim}\,\mathcal{F} + 1$ closed orbits.

In particular, we obtain the following corollary, a novel affirmative answer for the Weinstein Conjecture on contact manifolds.

Theorem 21. Let (M, λ) be a closed contact manifold of dimension $(2n+1)$. If the Reeb field R is quasiconformal (i.e., its flow is a quasiconformal contact action) then it has at least $n+1$ closed orbits.

There is a theorem due to Banyaga stating that a contact form which is C^2-close to a regular contact form satisfies the Weinstein conjecture. As a corollary to Theorem 21 above, we obtain the following new proof for this fact.

Corollary 11 (Theorem 1 in (BANYAGA, 1990))**.** Let M be a closed manifold and λ a regular contact form on M. If λ' is C^2-close to λ, then (M, λ') satisfies the Weinstein conjecture.

Proof. The contact condition for 1-forms is simply that $\lambda \wedge \mathrm{d}\lambda > 0$, which is clearly open in the C^1-topology. On the other hand, quasiconformality is also an open property. If λ and λ' are sufficiently C^2-close, then their contact actions F_λ and $F_{\lambda'}$ are C^1-close enough to guarantee that $F_{\lambda'}$ is also quasiconformal, and therefore satisfies the Weinstein conjecture. □

Before ending this section, we expand briefly on the discussion of Example 12. Recall that a contact structure is said to be **tight** if it is not overtwisted. We have the following result.

Proposition 31. If a Reeb flow is quasiconformal, then the associated contact structure is tight..

Proof. According to geometrical characterisation of overtwistedness presented in (CASALS; MURPHY; PRESAS, 2019), it is sufficient to prove that the model

$$(\mathbb{R}^3 \times D^{2n-4}, \lambda_{\text{ot}} + \lambda_{\text{std}})$$

is not equicontinuous. Here, $\mathbb{R}^3$ is given cylindrical coordinates (ρ, θ, z) and D^{2n-4} is a ball of a fixed radius with coordinates $(x_1, y_1, \cdots, x_{n-2}, y_{n-2})$. The contact form is the sum of

$$\lambda_{\text{ot}} = \cos\rho\, \mathrm{d}z + \rho\cos\rho\, \mathrm{d}\theta,$$
$$\lambda_{\text{std}} = \frac{1}{2}\sum_{i=1}^{n-2}(x_i\, \mathrm{d}y_i - y_i\, \mathrm{d}x_i).$$

The Reeb field R associated to such form is simply $R_{\text{ot}} + \vec{0}$, which is not equicontinuous. Indeed, if its flow were equicontinuous, then its projection on $\mathbb{R}^3$ would be equicontinuous as well, but we know from Example 12 this is does not happen.

Now, it follows from (CASALS; MURPHY; PRESAS, 2019, Theorem 1.1) that any overtwisted contact manifold admits an contact embedding of $(\mathbb{R}^3 \times D^{2n-4}(r), \lambda_{\text{ot}} + \lambda_{\text{std}})$ for some $r > 0$.Thus, its Reeb flow is not equicontinuous, and therefore not quasiconformal. □

If the converse to Proposition 31 were true, then Theorem 21 together with the dichotomy overtwisted/tight would prove the Weinstein Conjecture. However, this is not the case. Indeed, the standard contact form on the 3-torus (cf. Example 14) is tight (even more, it is *strongly fillable* (ELIASHBERG, 1996)) but it is not equicontinuous, and therefore not quasiconformal. Similar constructions hold in higher dimensions (MASSOT; NIEDERKRüGER; WENDL, 2013).

5.3 Basic cohomology of quasiconformal contact foliations

As shown in Example 15, the lower bound of $2^{-1}\mathrm{codim}(\mathcal{F}) - 1$ need not be the exact number of closed orbits, even when there are only finitely many of them. However, in the case when the number of closed orbits is finite and *minimal*, i.e., equal to exactly

$2^{-1}\text{codim}(\mathcal{F})-1$, then substantial topological restrictions are imposed on the ambient manifold. This is true for metric f-K-contact structures, and since we showed in Section 4.5 that the function $S: M \to \mathbb{R}$ is Morse-Bott even without the presence of the tensor f, those topological restrictions carry over to the more general class of C^1-equicontinuous uniform q-contact manifolds. All the proofs in (GOERTSCHES; LOIUDICE, 2020b) apply, *mutatis mutandis*, to the case of q-contact manifolds, giving rise to the results in this section.

The Reeb fields are pair-wise commutative, hence the bundle spanned by any choice of them is integrable. Denote by $\mathcal{F}_i$ the integral foliation of the bundle

$$\bigoplus_{j=1}^{i} \mathbb{R}R_j \subset \mathcal{R},$$

that is, the bundle $\text{Span}\{R_1, \cdots, R_i\}$. This is exactly the orbit foliation of the action

$$\begin{aligned} F_i : \mathbb{R}^i &\longrightarrow \text{Iso}(M,g) \\ (t_1, \cdots, t_i) &\longmapsto (\exp t_1 R_1) \circ \cdots \circ (\exp t_i R_i). \end{aligned}$$

Of course, $F_q = F$ is simply the contact action, and for $s=0$ we defined $\mathcal{F}_0$ to be the trivial foliation by points. As for the contact action F, we can define C^1-enveloping groups for each action F_i as the closure of the subgroup spanned by the flows of the first i Reeb fields, i.e.:

$$\text{E}_i^1 = \overline{\text{Span}\{\exp(t_1 R_1), \cdots, \exp(t_i R_i); (t_1, \cdots, t_i) \in \mathbb{R}^i\}}.$$

Note that each foliation $\mathcal{F}_i$ isometric because the Reeb fields are all Killing, though generally, it will not be a contact foliation. Regardless, the arguments of Theorem 13 apply, and we conclude each E_i^1 is a torus, and each $\mathcal{F}_i$ is C^1-equicontinuous. Again, $\text{E}_q^1 = \text{E}_F^1$ is just the C^1-enveloping group of F.

To see how the q-contact structure affects the topology of the ambient manifold, we begin by considering, for $s = 1, 2, \cdots, q$, the following sets of basic forms invariant under the compactified action of $\mathbb{R}^i$.

$$\wedge^*(\mathcal{F}_s)^{\text{E}_{s+1}^1} := \{\omega \in \wedge^*(M); \omega \text{ is } \mathcal{F}_s\text{-basic and } T^*\omega = \omega \text{ for every } T \in \text{E}_{s+1}^1\}.$$

Since the operator d preserves basic forms and commutes with pullbacks, the set $\wedge^*(\mathcal{F}_s)^{\text{E}_{s+1}^1}$ is a differential complex for every $s = 1, \cdots, q-1$. We denote its cohomology by $H^*(\mathcal{F}_s)^{\text{E}_{s+1}^1}$. The natural inclusion $j : \wedge^*(\mathcal{F}_s)^{\text{E}_{s+1}^1} \hookrightarrow \wedge^*(\mathcal{F}_s)$ induces an injective morphism

$$j^* : H^*(\mathcal{F}_s)^{\text{E}_{s+1}^1} \to H_B^*(\mathcal{F}_s).$$

There is also, for $s = 1, \cdots, q-1$, an averaging operator

$$\text{av} : \wedge^*(\mathcal{F}_s) \to \wedge^*(\mathcal{F}_s)^{\text{E}_{s+1}^1}$$

given by

$$\mathrm{av}(\omega) = \int_{\mathrm{E}^1_{s+1}} T^*\omega \, \mathrm{d}\mu(T),$$

where μ, as usual, is the Haar measure of the torus E^1_{s+1}. This is well defined since the averaging procedure produces in fact E^1_{s+1}-invariant forms, just like in Theorem 12 and Proposition 25. Note also that av is a chain map, that is, $\mathrm{d} \circ \mathrm{av} = \mathrm{av} \circ \mathrm{d}$, since the exterior derivative commutes with pullbacks. Hence it induces a morphism in cohomology

$$\mathrm{av}^* : H^*_B(\mathcal{F}_s) \to H^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}.$$

Lemma 8. The mapping $\mathrm{av}^* : H^*_B(\mathcal{F}_s) \to H^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}$ is an isomorphism.

Proof. The mapping av^* is the same as the identity. As argued in (BAZZONI; GOERTSCHES, 2019, Lemma 5.3), there is a chain homotopy between av and the identity map, which preserves the property of being $\mathcal{F}_s$-basic. Thus av^* and j^* are inverses to one another. □

Lemma 9. The set $\wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}$ contains all the $\mathcal{F}_{s+1}$-basic forms. In other words,

$$\wedge^*(\mathcal{F}_{s+1}) \subset \wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}.$$

Proof. Indeed, if ω is $\mathcal{F}_{s+1}$-basic and $T \in \mathrm{E}^1_{s+1}$, then there is a sequence $a_n = \sum_{i=1}^{s+1} a^i_n e_i$ in $\mathbb{R}^q$ such that

$$T = \lim_n F^{a_n} = \lim_n (F^{a^{s+1}_n e_{s+1}} \circ \cdots \circ F^{a^1_n e_1}).$$

Hence

$$\lim_n (F^{a_n})^* \omega = \lim_n (F^{a^1_n e_1})^* \cdots (F^{a^{s+1}_n e_{s+1}})^* \omega = \omega,$$

and by continuity $T^*\omega = \omega$. So $\wedge^*(\mathcal{F}_{s+1}) \subset \wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}$. □

These lemmas allow us to construct an analogue of the Gysin sequence for pairs of foliations, which gives us exact sequences such as below:

Proposition 32. (GOERTSCHES; LOIUDICE, 2020b, Proposition 4.4) For $s = 0, \cdots, q-2$ there are short exact sequences

$$0 \longrightarrow H^*_B(\mathcal{F}_{s+1}) \longrightarrow H^*_B(\mathcal{F}_s) \longrightarrow H^{*-1}_B(\mathcal{F}_{s+1}) \longrightarrow 0,$$

as well as a long exact sequence

$$\cdots \to H^*_B(\mathcal{F}) \to H^*_B(\mathcal{F}_{q-1}) \to H^{*-1}_B(\mathcal{F}) \xrightarrow{\delta} H^{*+1}_B(\mathcal{F}) \to \cdots$$

where $\delta([\sigma]) = [d\lambda_{q-1} \wedge \sigma]$.

Proof. Suppose $0 \leq s \leq q-1$. We consider the following sequence of complexes

$$0 \to \wedge^*(\mathcal{F}_{s+1}) \to \wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}} \xrightarrow{\iota_{R_{s+1}}} \wedge^{*-1}(\mathcal{F}_{s+1}) \to 0, \tag{5.4}$$

where the first map is simply the inclusion of Lemma 9. On the other hand, given any $\mathcal{F}_{s+1}$-basic form in $\wedge^{*-1}(\mathcal{F}_{s+1})$, the form $\lambda_{s+1} \wedge \omega$ belongs to $\wedge^*(\mathcal{F}_{s+1}) \subset \wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}}$, and

$$\iota_{R_{s+1}}(\lambda_{s+1} \wedge \omega) = \omega.$$

Thus, $\iota_{R_{s+1}} : \wedge^*(\mathcal{F}_s)^{\mathrm{E}^1_{s+1}} \to \wedge^{*-1}(\mathcal{F}_{s+1})$ is surjective, and Sequence 5.4 is exact. It, therefore, descends to a long exact sequence in cohomology

$$\cdots \to H^*_B(\mathcal{F}_{s+1}) \to H^*_B(\mathcal{F}_s) \to H^{*-1}_B(\mathcal{F}_{s+1}) \xrightarrow{\delta} H^{*+1}_B(\mathcal{F}_{s+1}) \to \cdots$$

To better understand the connecting homomorphism δ, observe that given a closed form $\sigma \in \wedge^{*-1}(\mathcal{F}_{s+1})$, the form $\lambda_{s+1} \wedge \sigma$ is a pre-image for σ under $\iota_{R_{s+1}}$, hence

$$\delta([\sigma]) = [d(\lambda_{s+1} \wedge \sigma)] = [d\lambda_{s+1} \wedge \sigma].$$

Now, for any $s < q-1$, the 1-form λ_{s+1} belongs to $\wedge^*(\mathcal{F}_{s+1})$, hence $d\lambda_{s+1}$ is closed in this complex, and therefore $\delta \equiv 0$. We thus obtain

$$0 \longrightarrow H^*_B(\mathcal{F}_{s+1}) \longrightarrow H^*_B(\mathcal{F}_s) \longrightarrow H^{*-1}_B(\mathcal{F}_{s+1}) \longrightarrow 0,$$

for $s = 1, \cdots, q-2$, as we wanted. As for the case $s = q-1$, the long exact sequence becomes

$$\cdots \to H^*_B\mathcal{F}) \to H^*_B(\mathcal{F}_{q-1}) \to H^{*-1}_B(\mathcal{F}) \xrightarrow{\delta} H^{*+1}_B(\mathcal{F}) \to \cdots$$

with $\delta[(\sigma)] = [d\lambda_{q-1} \wedge \sigma]$, as claimed.

□

Now, if the isometric q-contact structure is also uniform, that is, if $d\lambda_i = \omega$ for every i, then the first de Rham cohomology group of M has dimension at least $q-1$, since each form $\lambda_i - \lambda_q$ is closed and non-exact (cf. Proposition 20). In particular, we can obtain a homomorphism

$$\wedge(\mathbb{R}^{q-1}) \to H^1_{dR}(M),$$

where $\wedge(\mathbb{R}^{q-1})$ is the exterior algebra on $q-i$ generators, by sending the i-th generator to the class $[\lambda_i - \lambda_q]$. This gives $H^*_{dR}(M)$ the structure of an $\wedge(\mathbb{R}^{q-1})$-algebra, and then the exact sequences of Lemma 32 allow us to conclude the existence of the following isomorphism (cf. (GOERTSCHES; LOIUDICE, 2020b, Theorem 1.1))

$$H^*_{dR}(M) \approx \wedge(\mathbb{R}^{q-1}) \otimes H^*_B(\mathcal{F})$$

Using the isomorphism above, we obtain the following.

Theorem 22. (GOERTSCHES; LOIUDICE, 2020b, Theorem 6.4) For a q-dimensional uniform quasiconformal contact foliation $\mathcal{F}$ on a $(2n+q)$-dimensional closed manifold M, the following are equivalent.

(i) $\mathcal{F}$ has exactly $n+1$ closed orbits;

(ii) The basic cohomology of $\mathcal{F}$ is the same as the de Rham cohomology of $\mathbb{C}P^n$;

(iii) The basic cohomology of $\mathcal{F}_{q-1}$ is the same as the de Rham cohomology of the sphere S^{2n+1};

(iv) The de Rham cohomology of M is the same as that of $S^{2n+1} \times \mathbb{T}^{q-1}$.

In relation to item (iv), we remark that a general uniform q-contact foliation is known to be a fibration over the torus $\mathbb{T}^{q-1}$ (cf. Theorem 4), while a similar result of Goertsches and Loiudice (GOERTSCHES; LOIUDICE, 2020a) states that every metric f-K-contact manifold can be constructed from a K-contact manifold by taking mapping tori and applying certain deformations.

CHAPTER 6

CONCLUSION

6.1 Final thoughts and comments

The theory of q-contact structures, as dealt with here, started with Almeida's work, inspired by Bolle, Bande and Hadjar, and other authors. Almeida's focus was mainly on the contact action, and at that, in the particular case when such action is partially hyperbolic. In this work, we focus instead on the perspective of the *contact foliation*.

In Chapters 2 and 3, we dived into the fundamental theory of contact foliations, greatly expanding the foundations laid out by Almeida. In particular, we provided new examples, local descriptions for these foliations (Propositions 10 and 19), canonical forms for the adapted coframe in particular cases (Propositions 11 and 12), and pointed out some basic properties of the holonomy groups of various distinct foliations related to a q-contact structure (Proposition 16).

Two exciting results of ours in these first chapters are the reduction procedure of Theorem 2 and the characterisation of uniform contact manifolds in Theorem 4. The former is a partial converse to the construction of contact structures in flat toric bundles introduced by Almeida. It allowed us to prove that non-trivial minimal contact foliations can not take place if any of the Weinstein conjectures are valid (cf. Theorem 3), a result not all obvious at first sight. The latter imposes substantial restrictions on the existence of uniform q-contact structures for q greater than 2, hinting that such foliations might not occur so frequently. It is also our first result specifically concerning contact *foliations*, in that it does not apply for contact flows, i.e., for the classic case. It is a result nicely related to a Theorem of Goertsches and Loiudice stating that every metric f-K-contact manifold can be constructed from a K-contact manifold utilising mapping tori and the so-called "type II" deformations (cf. (GOERTSCHES; LOIUDICE, 2020a, Theorem 4.4)).

Theorem 4 has other interesting aspects to it. When comparing the results of

Propositions 13 and 15, we see a crucial distinction between the contact and characteristic foliations, in that the former does not support complete transversals. In particular, a contact foliation can never be a suspension. In this sense, Theorem 4 hints at some sort of *maximality* of the characteristic foliations concerning being a suspension, or, equivalently, *minimality* of the contact foliation with respect to the property of not being a suspension.

It is in Chapter 4 that we begin tackling the main problem in our work: finding closed leaves for contact foliations. From a dynamical point of view, a closed leaf in a non-singular foliation is simply a trivial minimal set. Our first strategy was to restrict the foliation to a minimal set and then use a procedure like the one from Theorem 2 to reduce the dimension of the acting Euclidean space. This would hopefully lead us to a situation where we could use one of the proven versions of the Weinstein conjecture, allowing us to prove the validity of, at least, the **WGWC**. Though the restriction of a contact action to a non-trivial minimal subset can not be a *contact action*, we were looking for hypotheses that would allow us to guarantee that the *reduced action* was of contact nature. Unfortunately, minimal sets are generally very pathological, so this approach was unsuccessful.

There is, however, a class of foliations for which the closure of leaves is actually very well-behaved: Riemannian foliations. Leaf closures in Riemannian foliations are manifolds with all the necessary properties needed to guarantee that the contact foliation can be restricted (cf. Proposition 26 and Lemma 4). Moreover, the adapted metric g^{τ} makes it possible to consider *isometric* actions instead of Riemannian foliations. Working with an isometric action not only makes the theory a lot simpler but also implies strong recurrence properties for the foliation, creating further evidence to support the existence of closed leaves.

Making use of all these outstanding properties of Riemannian contact foliations, we developed a procedure to reduce the contact foliation. Such procedure could be iterated to a point where we find ourselves in a familiar situation, for which the Weinstein conjecture is known to be valid: Riemannian contact flows (cf. (BANYAGA, 1993; RUKIMBIRA, 1993)). Thus Theorem 15 came to be.

We end Chapter 4 with Theorem 18, which is an improved version of Theorem 15 obtained by means of Morse Theory. Theorem 18 generalises results of Goertsches and Loiudice and Rukimbira, specially (GOERTSCHES; LOIUDICE, 2020a, Corollary 6.2). We note that in these works, the authors extensively use the properties of the tensor f associated with the (generalised) K-contact structure in question and also that the Reeb fields are Killing for a *contact metric*. What is remarkable about Theorem 18 is our discovery that the theory of Goertsches and Loiudice and Rukimbira does not really depend in any way on the properties of the tensor f, but only on the fact that the Reeb fields are Killing with respect to *any metric* (not necessarily a contact one), a result which is not at all clear at first sight.

With the **SGWC** settled for the Riemannian case, we wanted to check what would happen under the weaker hypotheses that the holonomy pseudogroup preserve not a metric, but a conformal structure, or more generally, that they had bounded transverse distortion, i.e., the foliation is quasiconformal. It was clear from the results for the isometric case that the closure $\overline{F(\mathbb{R}^q)}$ of the action was the crucial object to be analysed. While looking for conditions that would let us conclude the compactness of $\overline{F(\mathbb{R}^q)}$ for quasiconformal and conformal contact actions F, we discovered Theorems 19 and 20, and ultimately Theorem B. This is a surprising result because, generally, the hierarchy

$$\{\text{isometric}\} \subset \{\text{Riemannian}\} \subset \{\text{conformal}\} \subset \{\text{quasiconformal}\}$$

is a strict one. This made clear to us that the critical dynamical condition involved was the possibility of compactifying the contact action, and it led to the definition of C^1-equicontinuity. In particular, applying our results to the contact case proves the Weinstein conjecture for conformal and quasiconformal Reeb flows (cf Theorem 21), a result which was not known before, to the best of the author's knowledge.

Essential to the proof of Theorem B is the result of Asuke on the existence of a smooth bundle-like metric for conformal foliations supporting good measures. The techniques Asuke uses in his work are more in line with what is usual to Foliation Theory than with the classical tools of the study of flows. Theorem B and its corollaries are, in this sense, an example of how looking at more general objects (in our case, foliations) can yield results regarding particular ones (flows) which would not be so readily accessible by the examination of the simpler cases. We defend the position that this further enriches and justifies the study of contact foliations.

6.2 Open questions and further work

We briefly discussed in the Introduction the question of whether or not our generalised Weinstein conjectures are equivalent to one another. However, we did not address this problem in the present work. Of course, the distinction between the **WGWC** and the **SGWC** does not matter in the 1-dimensional case, which is arguably the most crucial instance of the Weinstein conjecture. However, when working with higher dimensional contact foliations on closed manifolds, one would expect the problem of showing that no such foliation is a foliation by planes to be much simpler than actually finding closed leaves.

Equivalence between the **WGWC** and the **SGWC** could even prove helpful in finding new classes of manifolds where the conjecture holds since the existence of foliation by planes on a compact manifold usually imposes strong topological restrictions on the ambient manifold (cf. (BIASI; MAQUERA, 2012; ROSENBERG, 1968)).

Question 1. Are the **WGWC** and **SGWC** equivalent?

Another question left without answer in Chapter 2, coming from the basic theory of q-contact manifolds, regards their local structure. As we saw in Propositions 11 and 12, there are canonical forms for each 1-form in the adapted coframe $\vec{\lambda}$ when the q-contact structure is either uniform or 2-codimensional. We think it likely to be true that such canonical forms also exist in the general case, though we do not yet know how to prove this. In general, of course, a q-contact structure need not be uniform, so a general Darboux-like theorem for q-contact manifolds, if true, should most likely be like the one in Proposition 11, i.e.

Conjecture 4 (Darboux's Theorem for q-contact structures)**.** Given a q-contact manifold $(M,\vec{\lambda},\mathcal{R}\oplus\xi)$ and a point $p\in M$, there are coordinates $(U;x_1,y_1,\cdots,x_n,y_n,z_1,\cdots,z_q)$ and leaf-wise constant functions $a_{ij}:U\to\mathbb{R}$ such that $\partial_{x_i}a_{ij}$ is non vanishing for every $1\leq i,j\leq n$, and

$$\lambda_i = \mathrm{d}z_i + \sum_{j=1}^{n} a_{ij}\,\mathrm{d}y_j.$$

One of the fundamental question in the development of Symplectic Geometry was the problem of *symplectic rigidity.* Shortly, this was related to the relative size of the group $\mathrm{Symp}(M)$ – diffeomorphisms of M preserving a symplectic form – inside $\mathrm{Diff}_{\mathrm{vol}}(M)$, in the sense of whether of not the C^0-closure of $\mathrm{Symp}(M)$ is the entire $\mathrm{Diff}_{\mathrm{vol}}(M)$. Gromov formulated this problem in terms of a dichotomy *rigidity* vs. *flexibility*, the former meaning $\mathrm{Symp}(M)$ is closed in the C^0-topology, while the latter means the C^0-closure of $\mathrm{Symp}(M)$ is $\mathrm{Diff}_{\mathrm{vol}}(M)$ (see (MCDUFF; SALAMON, 2017) for a more thorough discussion). Rigidity has been shown to hold by Eliashberg (ELIASHBERG, 1982; ELIASHBERG, 1987), Gromov (GROMOV, 2013), and Ekeland and Hofer (EKELAND; HOFER, 1989). In particular, their work shows that merely continuous transformations can preserve symplectic structures, which are, *a priori*, intrinsically differentiable objects. This marks the beginning of the theory of Symplectic Topology.

The deep connections between Contact and Symplectic geometries would suggest that some sort of structural rigidity also holds for contact manifolds. This is the case indeed, was shown, for instance, by Müller and Spaeth in their work (MÜLLER; SPAETH, 2014; MÜLLER, 2019). Their techniques are based on the existence of symplectic invariants defined in terms of cohomology classes. For contact manifolds, similar invariants can be defined by means of symplectisations. This leads us to believe that the same ideas can be used to define invariants for q-contact structures, making use of the constructions from Appendix B. Once this is done, one expect to use such invariants, together with Symplectic Rigidity, to prove results akin to those of Müller and Spaeth:

Conjecture 5 (C^0-rigidity for q-contact structures)**.** The group of automorphisms of a q-contact structure is closed with respect to the C^0-topology.

The following is another question on the basic theory for q-contact structures regarding symplectisations.

Question 2. Suppose $\vec{\eta} = A\vec{\lambda}$ is a reparameterisation. How are the symplectisations of $\vec{\eta}$ and $\vec{\lambda}$ related?

In Chapter 4, we defined the concept of C^1-equicontinuity as a smooth analogue to classical equicontinuity in terms of the compacity of the group of transformations acting on the manifold. In general, one should expect the closure of a family concerning the C^0-topology to be different from the C^1-closure. However, since the family $F(\mathbb{R}^q)$ exists under so many heavy hypotheses, like simultaneously preserving all the forms $\mathrm{d}\lambda_i$, volume forms $\mathrm{d}M_i$ and all the foliations $\mathcal{F}_i$ and $\mathcal{C}_i$, one might suspect that equicontinuity alone, together with the said regularity of the family $F(\mathbb{R}^q)$, is enough to impose C^1-equicontinuity. This suspicion is further encouraged by the facts that all the examples of non-equicontinuous contact actions known to the author are either Anosov or expansive and that this sort of "*bootstrapping*" phenomena (where convergence in the C^0 topology implies C^∞-convergence) are recurrent in Symplectic topology.

Question 3. In the class of contact actions, does equicontinuity imply C^1-equicontinuity?

Even if that is not true, it would be interesting to study the dynamics of a contact action $F : \mathbb{R}^q \to M$ which is equicontinuous but not C^1-equicontinuous. In such a setting, the action would still preserve a metric $d : M \times M \to \mathbb{R}$, but such a metric could not possibly come from a Riemannian metric tensor on M. It makes one wonder how the reasoning and techniques used throughout this work could be modified to prove the validity of the SGWC in this case.

As for further investigation of the Weinstein conjectures, there are a few promising cases, especially those where it should be possible to reduce the action to a flow and apply Taubes' theorem (TAUBES, 2007). For instance:

Question 4. Let M be a 4-dimensional closed manifold equipped with a 2-dimensional contact foliation. If the underlying contact action F is unfaithful, does the SGWC hold?

Here, one possibility is to try and find conditions implying the transitivity of the action, as it would then be possible to apply a reduction procedure as in Theorem 2 to conclude the SGWC from the WC. For example, is the volume-preservation property of F enough to guarantee transitivity in the contact setting, as it happens for Anosov actions (cf. Corollary 1)?

In the same line of reasoning, we remark that in his doctoral dissertation, Arakawa showed that codimension 2 Anosov actions on closed manifolds are essentially toric extensions of Anosov flows, contributing to the efforts towards a general classification of Anosov systems. Though the work of Arakawa extensively uses the Anosov property of the action and his techniques have no direct analogue to the contact case, his results raise the question of whether or not a similar assertion holds for a contact action.

Question 5. Is a codimension 2 contact action a toric extension of a contact flow?

If this holds, even in a smaller class of contact actions, a direct application of Taubes' Theorem could prove the **SGWC** for such 2-codimensional contact foliations.

It is worth pointing out that all of our results regarding the generalised versions of the Weinstein conjecture are similar in that they impose further restrictions on the dynamics, like having an invariant structure (a hyperbolic splitting, a Riemannian metric or conformal metric) or having bounded distortion (in the quasiconformal case). Such restriction, when combined with the contact foliation's properties, yields closed orbits' existence. However, the most profound and meaningful results regarding the Weinstein conjecture are usually centred only on the properties of the contact structure itself. Such is the case, for instance, in Taubes' proof of the **WC** for 3-dimensional closed manifolds, where he uses Seiberg-Witten invariants of 4-manifolds, together with embedded contact homology (cf. (TAUBES, 2007; HUTCHINGS, 2009)).

In this spirit, we could consider the action functional

$$\begin{aligned} \mathcal{A} : C^\infty(\mathbb{T}^q, M) &\to \mathbb{R} \\ \gamma &\mapsto \int_\gamma \lambda, \end{aligned} \tag{6.1}$$

where $C^\infty(\mathbb{T}^q, M)$ is the set of smooth q-tori embedded in M, and λ is the characteristic form. Almeida pointed out to the author in personal communications that the critical points of such a functional are the closed orbits of the associated contact foliation, at least in the uniform case (showing this already requires some effort). One would hope that, by deploying the functional $\mathcal{A}$, it would be possible to develop some sort of Floer homology theory for q-contact manifolds. This is, *a priori*, a viable strategy to try and find closed orbits without relying on additional dynamical hypotheses. However, there are myriad technical difficulties regarding the gradient equation one obtains when trying to reproduce the usual steps of a Floer theory. Namely, most of the problems arise because the equations one gets from the gradient of the operator (6.1) and a suitable associated almost complex structure are non-linear.

As for possible applications of the concept of contact action, we again turn to the question of classifying Anosov systems. There is a famous result due to Benoist, Foulon and Labourie which says, essentially, that

BFL's Theorem. Every contact Anosov flow with smooth bundles is, up to finite coverings and change of parameters, isomorphic to the geodesic flow of a smooth manifold of strictly negative curvature.

This result was exactly what motivated Almeida to define the notion of q-contact structure: he was after a geometrical entity which could replicate in higher dimensions the properties of the Reeb flow in order to generalise BFL's Theorem to Anosov actions. He was partially successful in this task. Indeed, contact Actions with smooth bundles are not generally algebraic, but only *quasi-algebraic* (cf. (ALMEIDA, 2018)).

Not many years ago, Fang used BFL's Theorem to classify quasiconformal Anosov flows, obtaining the following

Theorem 23. Let M be a closed manifold, and ϕ a C^∞ uniformly quasiconformal volume-preserving Anosov flow on M. If $E^u \oplus E^s$ is smooth and the invariant bundles have rank at least 2, then up to a change of parameters and finite coverings, the flow ϕ is either the suspension of a hyperbolic automorphism of the torus or a perturbation of the geodesic flow of a hyperbolic manifold.

BFL's theorem plays a central role in Fang results, as the conclusions of Theorem 23 suggest. Since q-contact structures were designed to be the higher dimensional analogues to the contact flow in BFL's Theorem, it is only natural that we make the following conjecture.

Conjecture 6. Let M be a closed manifold, and $F : \mathbb{R}^k \to \mathsf{Diff}(M)$ a C^∞ uniformly quasiconformal volume-preserving Anosov action on M. Suppose $E^u \oplus E^s$ is smooth and the invariant bundles have rank at least 2. Then the action is, up to reparameterisations and finite coverings, either the suspension of a hyperbolic automorphism of the torus or a perturbation of quasi-algebraic contact action.

The most straightforward strategy for proving this conjecture would be to generalise the arguments of Fang's original proof (FANG, 2004). His is a very technical demonstration, relying heavily on notions from the Ergodic Theory of flows (all of which have the proper analogues in Foliation Theory) and also of results of Sadovskaya. The latter we have already shown to remain valid for contact actions in Theorem 19, and the same arguments can be used to show that they are also valid for Anosov actions. Even so, if Fang's techniques turn out not to work in the setting of actions, one may hope that the process of trying to adapt them reveals a counter-example to the conjecture, further enriching the theory of generalised contact manifolds.

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APPENDIX

A

TRANSVERSE GEOMETRY OF FOLIATIONS

In this section, we deal with an arbitrary q-dimensional smooth foliation $\mathcal{F}$ on a $(q+r)$-dimensional manifold M. We write $\mathrm{T}\mathcal{F}$ for the tangent bundle to the leaves of $\mathcal{F}$, and $\Gamma(\mathcal{F}) := \Gamma(\mathrm{T}\mathcal{F})$ for the Lie Algebra of fields tangent to $\mathcal{F}$. The vector space of all k-forms on M is denoted by $\wedge^k(M)$.

Definition 33 (*Transverse bundle*)**.** The **transverse bundle** $\mathcal{T}$ of a foliation $\mathcal{F}$ is the bundle of rank $r = \operatorname{codim}\mathcal{F}$ for which there is an exact sequence

$$0 \to \mathrm{T}\mathcal{F} \to \mathrm{T}M \xrightarrow{\pi} \mathcal{T} \to 0.$$

Up to isomorphism, all transverse bundles are the same, isomorphic to the quotient

$$\pi : \mathrm{T}M \to {}^{\mathrm{T}M}\!/_{\mathrm{T}\mathcal{F}}.$$

In the case of contact foliation, the non-integrable bundle ξ is a natural choice of a transverse bundle.

Definition 34 (*Foliate vector fields.*)**.** Let $X \in \Gamma(M)$ be a vector field whose local description in the foliated chart $(U; x_1, \cdots, x_q, y_1, \cdots, y_r)$ is

$$X = \sum_{i=1}^{q} a_i(x,y)\partial_{x_i} + \sum_{l=1}^{r} b_l(x,y)\partial_{y_l}.$$

We say X is **foliate** if

$$\frac{\partial}{\partial x_i} b_l = 0$$

for all $i = 1, \cdots, q$ and $l = 1, \cdots, r$. In other words, the functions $b_l(y)$ are independent of x, for every l, i.e., last l coordinate functions are *leaf-wise constants.*

A straightforward calculation shows that the property of being foliate does not depend on the choice of foliated chart. Instead of conducting such a calculation, we want to provide a coordinate-free characterisation of foliate fields.

Definition 35 (*Infinitesimal symmetries*)**.** A vector field $X \in \Gamma(M)$ is an **infinitesimal symmetry** of $\mathcal{F}$ if its flow preserves the bundle $T\mathcal{F}$. In other words, X is an infinitesimal symmetry of $\mathcal{F}$ if, for every field $Y \in \Gamma(\mathcal{F})$, one has

$$\mathcal{L}_X Y = [X,Y] = 0.$$

Proposition 33. The foliate vector fields of a foliation $\mathcal{F}$ are exactly its infinitesimal symmetries.

Proof. Being an infinitesimal symmetry is a local property, so it is sufficient to prove this in a local foliated chart $(U; x_1, \cdots, x_q, y_1, \cdots, y_r)$. Suppose we can write X and Y in such coordinates as

$$X = \sum_{i=1}^{q} a_i(x,y)\partial_{x_i} + \sum_{l=1}^{r} b_l(x,y)\partial_{y_l}$$
$$Y = \sum_{j=1}^{q} c_j(x,y)\partial_{x_j}.$$

Then

$$\begin{aligned}
[X,Y] &= \sum_i [a_i\partial_{x_i}, Y] + \sum_l [b_l\partial_{y_l}, Y] \\
&= \sum_i [a_i\partial_{x_i}, Y] + \sum_l\sum_j [b_l\partial_{y_l}, c_j\partial_{x_j}] \\
&= \sum_i [a_i\partial_{x_i}, Y] + \sum_l\sum_j \left(\left(b_l\frac{\partial}{\partial y_l}c_j\right)\partial_{x_j} - \left(c_j\frac{\partial}{\partial x_j}b_l\right)\partial_{y_l}\right) \\
&= Z - \sum_l\sum_j \left(c_j\frac{\partial}{\partial x_j}b_l\right)\partial_{y_l}, \qquad \text{(A.1)}
\end{aligned}$$

where

$$Z := \sum_i [a_i\partial_{x_i}, Y] + \sum_l\sum_j \left(b_l\frac{\partial}{\partial y_l}c_j\right)\partial_{x_j}$$

is, by Fröbenius' Theorem, a vector field tangent to $T\mathcal{F}$. Therefore, Z is of the form

$$Z = \sum_{i=1}^{q} d_i(x,y)\partial_{x_i}.$$

Thus, using Equation (A.1) we get that $[X,Y] = 0$ if and only if the following equality holds

$$\sum_l\sum_j \left(c_j\frac{\partial}{\partial x_j}b_l\right)\partial_{y_l} = \sum_{i=1}^{q} d_i(x,y)\partial_{x_i},$$

which can only happen if

$$\frac{\partial}{\partial x_j}b_l \equiv 0$$

for every j and l, as wanted. □

Using any of the definitions above, one can check by an immediate calculation that the set of foliated fields is closed under Lie brackets and form a subalgebra of $\Gamma(M)$.

Definition 36 (*Algebra of foliate fields*)**.** We denote by

$$\mathfrak{B}(\mathcal{F}) := \{X \in \Gamma(M); [X,Y] = 0 \quad \forall Y \in \Gamma(\mathcal{F})\}$$

the Lie algebra of foliate fields.

In a foliated chart $(U; x_1, \cdots, x_q, y_1, \cdots, y_r)$, the bundle $\mathcal{T}|_U$ has as global frame the set $\{\pi\partial_{y_1}, \cdots, \pi\partial_{y_r}\}$. For a foliate field $X = \sum_l b_l(y)\partial_{y_l}$ defined U, its image under the projection $\pi : \mathrm{T}U \to \mathcal{T}|_U$ is (point-wise)

$$\pi X = \sum_l b_l(y)\pi\partial_{y_l}. \tag{A.2}$$

Hence, the foliate fields on U are exactly the fields which are π-related to fields on the quotient ${}^{U}/_{\mathcal{F}|_U}$. In the particular case when $\mathcal{F}$ is the foliation defined by the fibres of a smooth fibration $f : M \to B$ (this is usually called, in the literature, a **simple foliation**), the foliate fields are exactly those f-related to fields in the base manifold B. For this reason, foliate fields are also referred to as *projectable fields, basic fields* and *base-like fields*.

Remark that $\Gamma(\mathcal{F}) \subset \mathfrak{B}(\mathcal{F})$ is an ideal. Indeed, if a foliate vector is described locally as one for which the last r coordinate functions b_l are leaf-wise constants, a field tangent to $\mathcal{F}$ is exactly one for which the functions b_l are constant and equal to 0. The"truly transverse" fields are the foliated fields whose the first q coordinate functions a_i are zero: the fields with no components in the $\mathrm{T}\mathcal{F}$ directions. To describe this property in a coordinate-free manner, we define the following notion.

Definition 37 (*Transverse fields*)**.** The elements of the set

$$\mathfrak{t}(\mathcal{F}) := {}^{\mathfrak{B}(\mathcal{F})}/_{\Gamma(\mathcal{F})}$$

are called the **transverse fields** of $\mathcal{F}$.

We can extend the projection $\pi : \mathrm{T}M \to \mathcal{T}$ to a map between $\Gamma(M)$ and $\Gamma(\mathcal{T})$ by applying π point-wise on each field. Let us denote this extension by π as well. It follows from Equation (A.2) that the restriction $\pi : \mathfrak{B}(\mathcal{F}) \to \Gamma(\mathcal{T})$ is surjective. It is immediate, also from Equation (A.2), that the kernel of π is $\Gamma(\mathcal{F})$. Therefore, there is an exact sequence of vector spaces

$$0 \to \Gamma(\mathcal{F}) \hookrightarrow \mathfrak{B}(\mathcal{F}) \xrightarrow{\pi} \mathfrak{t}(\mathcal{F}) \to 0.$$

Since $\Gamma(\mathcal{F}) \subset \mathfrak{B}(\mathcal{F})$ is an ideal, we can equip $\mathfrak{t}(\mathcal{F})$ with a Lie algebra structure by setting $[\pi X, \pi Y] = \pi([X,Y])$, so that the sequence above becomes an exact sequence of Lie Algebras.

For general foliations, the leaf space $M/_{\mathcal{F}}$ need not even be a manifold, but we can still regard $\mathfrak{B}(\mathcal{F})$ as an algebra of "derivations".

Definition 38 (*Basic function*)**.** A smooth function $f : M \to \mathbb{R}$ is called **basic** if $\mathrm{d}f(X) = 0$ for every $X \in \Gamma(\mathcal{F})$. In other words, the basic functions of $(M, \mathcal{F})$ are the leaf-wise constant functions.

In this sense, we could define the foliate fields of $\mathcal{F}$ as those for which the coordinate functions in the transverse directions are basic. Note that if X is foliate and f is basic, then fX is still foliate, and the function $Xf = \mathrm{d}f(X)$ is still basic. So $\mathfrak{B}(\mathcal{F})$ is a module over the ring of basic functions, and the elements of $\mathfrak{B}(\mathcal{F})$ act as derivations on this ring. Again, the basic functions correspond to pullbacks of functions on the base manifold for simple foliations.

Following the same reasoning, we define "transverse" differential forms.

Definition 39 (*Basic form*)**.** A differential form ω is said to be $\mathcal{F}$**-basic** (or simply **basic** when the context permits), if it satisfies

$$\iota_X \omega = \mathcal{L}_X \omega = 0, \tag{A.3}$$

for every $X \in \Gamma(\mathcal{F})$. We write $\wedge^k(\mathcal{F})$ for the set of all $\mathcal{F}$-basic differential k-forms on M.

In a foliated chart $(U; x_1, \cdots, x_q, y_1, \cdots, y_r)$ this translates to fact that a $\mathcal{F}$-basic k-form ω can be written as a sum of terms

$$\omega_{I_k} \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k},$$

where $\omega_{I_k} : U \to \mathbb{R}$ is independent of the first q-variables. We note immediately from this description that any basic k-form for $k > \operatorname{codim} \mathcal{F}$ must be zero. It is also clear that $\mathcal{F}$-basic 0-forms are the leaf-wise constant functions, i.e., basic functions.

$$\begin{aligned} &\wedge^0(\mathcal{F}) = \{\text{basic functions } f : M \to \mathbb{R}\}, \\ &\wedge^k(\mathcal{F}) = 0, \text{ for every } k \text{ greater than } \operatorname{codim} \mathcal{F}. \end{aligned} \tag{A.4}$$

It is clear from the coordinate-free description given by the equations in A.3 that $\wedge^k(M)$ is a vector space and that the wedge product turns $\wedge^*(\mathcal{F})$ into a graded algebra.

Proposition 34. The set $\wedge^*(\mathcal{F})$ is a differential graded algebra.

Proof. In fact, we restrict the usual exterior derivative d from $\wedge^*(M)$ to $\wedge^*(\mathcal{F})$. If ω is basic, then, for $X \in \Gamma(\mathcal{F})$,

$$\mathcal{L}_X(\mathrm{d}\omega) = \mathrm{d}(\mathcal{F}_X\omega) = 0,$$
$$\iota_X(\mathrm{d}\omega) = \mathrm{d}(\iota_X\omega) - \mathcal{L}_X\omega = 0.$$

□

We can therefore consider the cohomology of $(\wedge^*(M), \mathrm{d})$.

Definition 40 (*Basic cohomology*)**.** The cohomology of the pair $(\wedge^*(\mathcal{F}), \mathrm{d})$ is denoted by $H_B^*(\mathcal{F})$ and called **basic cohomology of** $\mathcal{F}$.

We can easily calculate $H_B^0(\mathcal{F})$ for connected M: the 0-cycles are the leaf-wise constant functions $f : M \to \mathbb{R}$ such that $\mathrm{d}f = 0$, that is, the constant functions on M. Hence

$$H_B^0(\mathcal{F}) = \mathbb{R}.$$

Similarly, using Equations A.4 we see that

$$H_B^k(\mathcal{F}) = 0 \quad \forall k > \operatorname{codim}\mathcal{F}.$$

In this sense, the basic cohomology of $\mathcal{F}$ plays the role of the De Rham cohomology for the leaf space ${}^{M}/_{\mathcal{F}}$.

In some way, basic functions, foliate fields and basic differential forms provide a rudimentary way to define a theory of calculus on the transverse distribution. For more results in this sense, the reader is referred to the much interesting survey by Vitagliano (VITAGLIANO, 2018).

Contrary to what happens for the De Rham cohomology of manifolds, in general, the basic cohomology groups can be infinite-dimensional. It can be shown that for Riemannian foliations on closed manifolds, these groups are always finite-dimensional (cf. (TONDEUR, 1997, Theorems 7.22 and 7.51)). For general foliations, at least $H_B^1(\mathcal{F})$ is always finite.

Proposition 35. The inclusion $\wedge^1(\mathcal{F}) \hookrightarrow \wedge^1(M)$ induces an injective morphism

$$H_B^1(\mathcal{F}) \to H_{dR}^1(M).$$

Proof. Suppose $[\omega]_B$ and $[\omega']_B$ are two basic classes in $H_B^1(\mathcal{F})$ such that $[\omega]_{dR} = [\omega']_{dR}$. Then there is a function $f : M \to \mathbb{R}$ such that

$$\omega - \omega' = \mathrm{d}f.$$

But both ω and ω' are $\mathcal{F}$-basic, hence, for $X \in \Gamma(\mathcal{F})$, we have

$$\mathcal{L}_X f = \iota_X \, \mathrm{d}f = \iota_X \omega - \iota_X \omega' = 0,$$

and therefore $f \in \wedge^0(\mathcal{F})$. In other words, ω and ω' differ only by an $\mathcal{F}$-basic function, which means exactly that $[\omega]_B = [\omega']_B$ □

APPENDIX

B

SYMPLECTISATIONS

Let M be a compact q-contact manifold of dimension $2n+q$. For each form λ_i, let R_i be the associated Reeb vector field and $\xi = \cap \ker \lambda_i$ the sub-bundle where the 2-forms $\mathrm{d}\lambda_i$ are non-degenerated. The complementary sub-bundle $\mathcal{R} := \mathrm{Span}\{R_i\}$ is the shared kernel of the forms $\mathrm{d}\lambda_i$ and we have a splitting $\mathrm{T}M = \mathcal{R} \oplus \xi$. For each i the product $\lambda_1 \wedge \cdots \wedge \lambda_q \wedge (\mathrm{d}\lambda_i)^n$ is a volume form, which we denote by $\mathrm{d}M_i$.

Now consider in $\mathbb{R}^q$ an open ball $B^q_\varepsilon(e_1)$ around the point $e_1 = (1,0,\cdots,0)$ and its closure $\overline{B^q_\varepsilon(e_1)}$. We define W to be the product manifold $\overline{B^q_\varepsilon(e_1)} \times M$. We consider on W coordinates $(t_1,t_2,\cdots,t_q,x)$ and the following differential forms

$$\sigma := \sum_{i=1}^{q} t_i \lambda_i$$
$$\omega := \mathrm{d}\sigma = \sum_i \mathrm{d}t_i \wedge \lambda_i + \sum_i t_i \,\mathrm{d}\lambda_i,$$

where by λ_i and $\mathrm{d}\lambda_i$ we actually mean the pullback under the projection $\pi_M : W \to M$ onto the second coordinate, that is, if $v = v_1 \oplus v_2 \in \mathrm{T}W \approx \mathbb{R}^q \oplus \mathrm{T}M$ then $\lambda_i(v) := \lambda_i(v_2)$, and so forth.

The 2 -form ω is non-degenerated for sufficiently small choices of ε, and hence a symplectic form on W. Showing that the top form ω^{n+q} is nowhere vanishing is sufficient. Let $\alpha := \sum_i \mathrm{d}t_i \wedge \lambda_i$ and $\beta := \sum_i t_i \,\mathrm{d}\lambda_i$, so that $\omega = \alpha + \beta$ and hence

$$\omega^{n+q} = \sum_{l=0}^{n+q} \binom{n+q}{l} \alpha^l \wedge \beta^{n+q-l}.$$

Now, on the one hand, we have

$$\alpha^l = \begin{cases} q!\,\mathrm{d}t_1 \wedge \lambda_1 \wedge \cdots \wedge \mathrm{d}t_q \wedge \lambda_q, & \text{if } l = q \\ 0, & \text{if } l > q. \end{cases}$$

Moreover, $dt_1 \wedge \lambda_1 \wedge \cdots \wedge dt_q \wedge \lambda_q = (-1)^{q+1} dt_1 \wedge dt_2 \wedge \cdots \wedge dt_l \wedge \lambda_1 \wedge \cdots \lambda_q$. On the other hand, as the product of 2-forms is commutative, we can use the multinomial theorem to get

$$\begin{aligned}\beta^l &= \sum_{|j|=l} \left(\frac{l!}{j_1! \cdots j_q!}\right) t_1^{j_1} (d\lambda_1)^{j_1} \wedge \cdots \wedge t_q^{j_q} (d\lambda_q)^{j_q} \\ &= \sum_{|j|=l} \left(\frac{l! t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!}\right) (d\lambda_1)^{j_1} \wedge \cdots \wedge (d\lambda_q)^{j_q},\end{aligned}$$

where by j we mean the multi-index $j = (j_1, \cdots, j_q) \in \{1, \cdots n\}^q$ and by $|j|$ we denote the sum $j_1 + \cdots + j_q$ of its entries. Note that the product $(d\lambda_1)^{j_1} \wedge \cdots \wedge (d\lambda_q)^{j_q}$ is a $(2|j|)$-form, and therefore it is equal to 0 whenever $|j| > n$, each $d\lambda_i$ being non-degenerated only the $2n$-dimensional sub-bundle ξ. Hence we have $\beta^l = 0$ for any exponent l greater than n, and therefore

$$\omega^{n+q} = \sum_{l=0}^{n+q} \binom{n+q}{l} \alpha^l \wedge \beta^{n+q-l} = \binom{n+q}{q} \alpha^q \wedge \beta^n.$$

From there one has

$$\begin{aligned}\omega^{n+q} &= \frac{(n+q)!}{n!q!} \alpha^q \wedge \beta^n \\ &= (-1)^{q+1} (n+q)! \, dt_1 \wedge \cdots \wedge \lambda_q \wedge \left(\sum_{|j|=n} \frac{t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!} (d\lambda_1)^{j_1} \wedge \cdots \wedge (d\lambda_q)^{j_q} \right) \\ &= (-1)^{q+1} (n+q)! \, dt_1 \wedge \cdots \wedge \lambda_q \wedge \left(\frac{t_1^n}{n!} (d\lambda_1)^n + \sum_{\substack{|j|=n \\ j_1 \neq n}} \frac{t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!} (d\lambda_1)^{j_1} \wedge \cdots \wedge (d\lambda_q)^{j_q} \right) \\ &= (-1)^{q+1} \frac{t_1^n (n+q)!}{n!} dt_1 \wedge \cdots \wedge dt_q \wedge dM_1 + \eta,\end{aligned}$$

where

$$\eta := (-1)^{q+1} (n+q)! \, dt_1 \wedge \cdots \wedge \lambda_q \wedge \left(\sum_{\substack{|j|=n \\ j_1 \neq n}} \frac{t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!} (d\lambda_1)^{j_1} \wedge \cdots \wedge (d\lambda_q)^{j_q} \right).$$

The product $dt_1 \wedge \cdots \wedge dt_q \wedge dM_1$ is a volume form on W, and for sufficiently small choices of ε the coordinate t_1 can be taken arbitrarily close to 1 while all the other entries will be arbitrarily closed to 0, and so will be the $2(n+q)$-form η. Hence $\omega^{n+q} \neq 0$ and (W, ω) is a symplectic manifold.

Definition 41 (*Symplectisation*)**.** If M is a q-contact manifold of dimension $2n+q$, then the $2(n+q)$-dimensional manifold $W = \overline{B_\varepsilon^q(e_1)} \times M$, equipped with the symplectic form $\omega = \sum_i d(t_i \lambda_i)$, is called a **symplectisation** of M.

The compact manifold W is also a Liouville domain.

Proposition 36. (W,ω,σ) is a Liouville domain.

Proof. The 1-form σ is a primitive for ω by definition, so we only need to show that $\sigma|_{\partial W}$ is a contact form on $\partial W \approx S^{q-1}_{\varepsilon} \times M$, that is, that $\sigma \wedge (\mathrm{d}(\sigma)|_{\partial W})^{n+q-1}$ is a volume form. First, recall that the canonical volume form for the unit sphere $r : S^{q-1} \hookrightarrow \mathbb{R}^K$ is $dS := r^* \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_q = \sum_i (-1)^{i+1} t_i \,\mathrm{d}t_1 \wedge \cdots \wedge \widehat{(\mathrm{d}t_i)} \wedge \cdots \mathrm{d}t_q$. Now, as we pointed out before, $\mathrm{d}\sigma = \alpha + \beta$ where $\alpha^l = 0$ for $l > 0$ and $\beta^l = 0$ for $l > n$. Hence

$$\mathrm{d}\sigma^{n+q-1} = \sum_l \binom{n+q-1}{l} \alpha^l \wedge \beta^{n+q-(l+1)} = \binom{n+q-1}{q} \alpha^q \wedge \beta^{n-1} + \binom{n+q-1}{q-1} \alpha^{q-1} \wedge \beta^n,$$

and therefore

$$\sigma \wedge \mathrm{d}\sigma^{n+q-1} = \frac{(n+q-1)!}{(n-1)!q!} \sum_i t_i \lambda_i \alpha^q \wedge \beta^{n-1} + \frac{(n+q-1)!}{n!(q-1)!} \sum_i t_i \lambda_i \alpha^{q-1} \wedge \beta^n.$$

The first term of sum above vanishes since $\lambda_i \wedge \alpha^q = q!\lambda_i \wedge \mathrm{d}t_1 \wedge \lambda_1 \wedge \cdots \wedge \mathrm{d}t_q \wedge \lambda_q = 0$. On the other hand,

$$\begin{aligned}
\alpha^{q-1} &= \sum_{|j|=q-1} \frac{(q-1)!}{j_1! \cdots j_q!} (\mathrm{d}t_1 \wedge \lambda_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}t_q \wedge \lambda_q)^{j_q} \\
&= (q-1)! \sum_{l=1}^{q} (\mathrm{d}t_1 \wedge \lambda_1) \wedge \cdots \wedge \widehat{(\mathrm{d}t_l \wedge \lambda_l)} \wedge \cdots \wedge (\mathrm{d}t_q \wedge \lambda_q),
\end{aligned}$$

hence

$$\begin{aligned}
\sum_i t_i \lambda_i \alpha^{q-1} &= (q-1)! \sum_{i=1}^{q} t_i \lambda_i \wedge \mathrm{d}t_1 \wedge \lambda_1 \wedge \cdots \wedge \widehat{\mathrm{d}t_i} \wedge \cdots \wedge \mathrm{d}t_q \wedge \lambda_q \\
&= (-1)^{q+1} (q-1)! \left(\sum_i (-1)^{i+1} \mathrm{d}t_1 \wedge \cdots \wedge \widehat{\mathrm{d}t_i} \wedge \cdots \wedge \mathrm{d}t_q \right) \wedge (\lambda_1 \wedge \cdots \wedge \lambda_q) \\
&= (-1)^{q+1} (q-1)! dS \wedge \lambda_1 \wedge \cdots \wedge \lambda_q.
\end{aligned}$$

Finally, using $\beta^n = t_1^n (\mathrm{d}\lambda_1)^n + n! \sum_{\substack{|j|=n \\ j_1 \neq n}} \frac{t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!} (\mathrm{d}\lambda_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{j_q}$, as we computed before, we get:

$$\begin{aligned}
\sigma \wedge (\mathrm{d}\sigma)^{n+q-1} &= \frac{(n+q-1)!}{n!(q-1)!} \sum_i t_i \lambda_i \alpha^{q-1} \wedge \beta^n \\
&= \frac{(n+q-1)!}{n!} (-1)^{q+1} dS \wedge \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \beta^n \\
&= \frac{(n+q-1)! t_1^n}{n!} (-1)^{q+1} dS \wedge \mathrm{d}M_1 \\
&\quad + (n+q-1)!(-1)^{q+1} dS \wedge \lambda_1 \wedge \cdots \wedge \lambda_q \left(\sum_{\substack{|j|=n \\ j_1 \neq n}} \frac{t_1^{j_1} \cdots t_q^{j_q}}{j_1! \cdots j_q!} (\mathrm{d}\lambda_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}\lambda_q)^{j_q} \right).
\end{aligned}$$

The product $dS \wedge \mathrm{d}M_1$ is a volume form on $S^{q+1} \times M \approx \partial w$, and the second term is gets arbitrarily small for small choices of ε. Thus, for sufficiently small choices of ε, the 1-form σ is a Liouville form for the symplectic manifold (W, ω), as we wanted.

□

We can consider M as a submanifold of W by means of the inclusion $M \hookrightarrow W$ sending x to (e_1, x). In this regard, M is a compact co-isotropic submanifold of W. Indeed, the restriction of ω to M is $\sum_i \mathrm{d}t_i \wedge \lambda_i + \mathrm{d}\lambda_1$. Thus, if $X \in \mathrm{T}W$ and $u \in \mathrm{T}M$, we have

$$\omega|_{\mathrm{T}M}(X, u) = \sum_i \mathrm{d}t_i(X)\lambda_i(u) + \mathrm{d}\lambda_1(X, u).$$

If X belongs to symplectic complement of $\mathrm{T}M$, that is, if $\omega(X, u) = 0$ for all $u \in \mathrm{T}M$, then, in particular, for $u = R_j$ one has

$$0 = \omega|_{\mathrm{T}M}(X, R_j) = \mathrm{d}t_j(X),$$

and therefore $\mathrm{d}t_i(X) = 0$ for every i, meaning X belong to $\mathrm{T}M$.

Recall the following definition, due to Bolle.

Definition 42 (*C_p-Condition*)**.** Let (M, ω) be a $2n$-dimensional closed symplectic manifold. For $1 \leq p \leq n$, we say a submanifold S satisfies the C_p-condition if

(i) S is a compact co-isotropic submanifold of codimension p. In other words, $\ker(\omega|_S)$ is a vector bundle of constant rank p over S;

(ii) There are 1-forms $\lambda_1, \cdots, \lambda_q$ on S such that $\mathrm{d}\lambda_1 = \cdots = \mathrm{d}\lambda_q = \omega|_S$;

(iii) the application $X \mapsto (\lambda_1(X), \cdots, \lambda_q(X))$ is a bundle isomorphism between $\ker(\omega|_S)$ and $\mathbb{R}^q$.

Proposition 37. Every uniform q-contact manifold satisfies the C_q-condition.

Proof. Indeed, suppose M is a uniform q-contact manifold and let (W, ω) be its symplectisation. The, since $\mathrm{d}\lambda_i = \mathrm{d}\lambda_j$ for every i, j, then

$$\omega = \mathrm{d}\sigma = \sum_i (\mathrm{d}t_i \wedge \lambda_i) + \left(\sum_i t_i\right) \mathrm{d}\lambda_1.$$

Then $M \approx \{e_1\} \times M$ is co-isotropic, and the restriction of the kernel of

$$\omega|M = \sum_i (\mathrm{d}t_i \wedge \lambda_i) + \mathrm{d}\lambda_1,$$

is exactly the Reeb bundle $\mathcal{R}$. Hence it has rank q. Moreover, for any $X, Y \in \Gamma(M)$, we have

$$\omega|_M(X, Y) = \sum_i (\mathrm{d}t_i \wedge \lambda_i)(X, Y) + \mathrm{d}\lambda_1(X, Y) = \mathrm{d}\lambda_1(X, Y),$$

because $\mathrm{d}t_i(X) = \mathrm{d}t_i(Y) = 0$ for every i, as argued above. Finally, isomorphism in (iii) is simply the decomposition of X in the frame $\{R_1, \cdots, R_q\}$. □

This means the concept of uniform q-contact structure and C_p-submanifold are the same.

On (W, ω) we consider the projection $H_i : (t_1, \cdots, t_q, x) \mapsto t_i$. The vector field $X_i := 0 \oplus R_i$ satisfies:

$$\begin{aligned}
\iota_X \omega &= \sum_j \mathrm{d}t_j \wedge \lambda_j(X, \cdot) + \sum_j t_j \, \mathrm{d}\lambda_j(X, \cdot) \\
&= \sum_j (\mathrm{d}t_j(0)\lambda_j - \lambda_j(R_i)\,\mathrm{d}t_j) + \sum_j t_j \, \mathrm{d}\lambda_j(R_i, \cdot) \\
&= -\mathrm{d}t_i = -dH_i,
\end{aligned}$$

and is, therefore, the Hamiltonian vector field of H. Similarly, the vector field $0 \oplus R_i$ is the Hamiltonian of the projection on the i-th $\mathbb{R}^q$ coordinate.

Analogously to the relation between Legendrian submanifolds of a contact manifold and Lagrangian submanifolds in its symplectisation, we have the following.

Definition 43 (λ_i-*Legendrian submanifold*)**.** We say a *n-dimensional* submanifold $L \subset M$ is **λ_i-Legendrian** if for every $p \in L$ the tangent space T_pL is a co-isotropic subset of the symplectic vector field $(\xi_p, \mathrm{d}\lambda_i|_p)$. In other words, for every $p \in L$, we have inclusions

$$\mathrm{T}_pL^{\perp_i} \subset \mathrm{T}_pL \subset \xi_p,$$

where $\mathrm{T}_pL^{\perp_i} := \{v \in \xi_p;\ \mathrm{d}\lambda_i|_p(v, u) = 0 \text{ for all } u \in \mathrm{T}_pL\} \subset \mathrm{T}_pL$.

It follows from a direct verification of the definitions that

Proposition 38. A submanifold $L \subset W$ is Lagrangian if and only if $\pi_M(L)$ is λ_i-Legendrian for every i.

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