UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

The kinematic space of special relativity and its hyperbolic geometry

Rafael Ferreira Pereira

Dissertação de Mestrado do Programa de Pós-Graduação em Matemática (PPG-Mat)

ICMC USP
SÃO CARLOS

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: ______________________

Rafael Ferreira Pereira

The kinematic space of special relativity and its hyperbolic geometry

Master dissertation submitted to the Institute of Mathematics and Computer Sciences – ICMC-USP, in partial fulfillment of the requirements for the degree of the Master Program in Mathematics. *EXAMINATION BOARD PRESENTATION COPY*

Concentration Area: Mathematics

Advisor: Prof. Dr. Carlos Henrique Grossi Ferreira

USP – São Carlos
February 2021

Rafael Ferreira Pereira

O espaço cinemático da relatividade restrita e sua geometria hiperbólica

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Carlos Henrique Grossi Ferreira

USP – São Carlos
Fevereiro de 2021

Dedico esta dissertação ao meu avô Jardel José Ferreira.

ACKNOWLEDGEMENTS

Gostaria de agradecer ao meu orientador, Carlos Grossi, que de todos os professores impactantes em minha formação foi de longe a principal inspiração, boa parte de meus ideais acerca do fazer científico foram influenciados por você.

Minha família foi essencial em todo este processo. Em especial, gostaria de prestar minha gratidão à minha avó Maria Luiza, à minha namorada Gabrielle, aos meus pais Fábio e Simone, e aos meus avós Regina e Norivaldo e ao meu primo Leonardo, sem vocês seria, sob todos os efeitos, impossível concluir este trabalho.

Aos eternos amigos do reino 11, João, Murilo, Christian, Uirá e Philipy, que me acompanharam nesta jornada desde a graduação e aos grandes amigos que fiz ou me tornei próximo durante o mestrado, Matheus, Gabriel, Rafaela, Vinicius e Alexandre, muito obrigado por todas as discussões matemáticas, físicas e/ou filosóficas. Aos amigos de longa data Kenneth, Henrique, Pedro, Vitor e João obrigado pelas discussões não-físicas e não-matemáticas, sempre importantes para que o indivíduo não se torne um completo alienígena.

Por último, agradeço à CAPES por financiar este trabalho.

"The principle of relativity corresponds to the hypothesis that the kinematic space is a space of constant negative curvature, the space of Lobachevski and Bolyai. The value of the radius of curvature is the speed of light."

Émile Borel

RESUMO

PEREIRA, R. F. **O espaço cinemático da relatividade restrita e sua geometria hiperbólica**. 2021. 74 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

O espaço classificador de referenciais inerciais em relatividade restrita é naturalmente hiperbólico. Há uma interação notável entre os elementos centrais de geometria hiperbólica e aqueles da relatividade especial — os quais, até certo ponto, já foram observados no passado — que apresentamos e discutimos mais profundamente nessa dissertação. Nosso objetivo é uma geometrização da relatividade restrita no nível do espaço cinemático, dando aos conceitos/fenômenos físicos definições/descrições puramente geométricas. Dessa forma, as diferenças entre a relatividade restrita e a mecânica clássica podem ser vistas como manifestação das naturezas geométricas distintas de seus espaços cinemáticos. A dissertação tem 4 capítulos; os dois primeiros apresentam alguns dos pré-requisitos, bem como um pouco da "história" que culminou no preprint apresentado no Capítulo 4. Esse preprint arXiv, submetido para publicação, contém os principais resultados da dissertação. No Capítulo 3 discutimos brevemente os espaços gyrovetoriais, que constituem um outro ambiente algébrico para lidar com a geometria hiperbólica e a relatividade restrita.

Palavras-chave: Geometria Hiperbólica, Relatividade Restrita, Espaço Cinemático, Geometrias Clássicas Livres de Coordenadas.

ABSTRACT

PEREIRA, R. F. **The kinematic space of special relativity and its hyperbolic geometry**. 2021. 74 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

The classifying space of inertial reference frames in special relativity is naturally hyperbolic. There is a remarkable interplay between central elements of hyperbolic geometry and those of special relativity — which, to a certain extent, have already been observed in the past — that we present and further discuss in this dissertation. We aim at a geometrization of special relativity at the level of kinematic space by giving to physical concepts/phenomena purely geometric definitions/descriptions. In this way, the differences between special relativity and classical mechanics can be seen as a manifestation of the distinct geometric natures of their kinematic spaces. The dissertation has 4 chapters; the first two present some of the prerequisites, as well as a little bit of the "history" that culminated in the preprint that constitutes Chapter 4. This arXiv preprint, submitted for publication, contains the main results of the dissertation. In Chapter 3, we briefly discuss gyrovector spaces which constitute another algebraic framework to deal with hyperbolic geometry and special relativity.

Keywords: Hyperbolic Geometry, Special Relativity, Kinematic Space, Coordinate-free Classic Geometries.

LIST OF FIGURES

CONTENTS

CHAPTER

1

INTRODUCTION

The group of orientation-preserving isometries of the hyperbolic space is the restricted Lorentz group $\mathrm{SO}^+(1,3)$, which is precisely the group of symmetries in special relativity. This fact turns out to be not just a coincidence, but rather the first hint of a meaningful interplay between hyperbolic geometry and special relativity.

1.1. Ancient history. Just a few years after Albert Einstein's "Annus Mirabilis", the rich interaction between hyperbolic geometry and special relativity started being explored by some authors (VARIĆAK, 1910), (BOREL, 1913b), (BOREL, 1913a). In (VARIĆAK, 1910), the relative rapidity

$$w := \tanh^{-1}\left(\frac{v}{c}\right)$$

is defined and its role further discussed (v stands for the relative velocity and c for the speed of light in the vacuum). Note that $w \to \infty$ when $v \to c$, and also that

$$cw = v + \frac{v^3}{3c^2} + \frac{v^5}{5c^4} + O(v^7)$$

which means that the quantity cw, called true velocity by Varićak, approximates v for small values of v/c.

In terms of the rapidity, the well-known Lorentz transformation from an inertial reference frame $\mathscr{O}$ to an inertial reference frame $\mathscr{O}'$ moving with velocity v along the x-axis with respect to $\mathscr{O}$ can be written as

$$\begin{cases} ct' = ct\cosh(w) - x\sinh(w) \\ x' = -ct\sinh(w) + x\cosh(w) \\ y' = y,\ z' = z \end{cases}$$

since $\cosh(w) = \gamma$ and $\sinh(w) = \beta\gamma$, where $\gamma := 1/\sqrt{1 - v^2/c^2}$ stands for the Lorentz factor and $\beta := v/c$.

Let $u = dx/dt$ and $u' = dx'/dt'$ be the velocities of an object as measured respectively by $\mathscr{O}$ and $\mathscr{O}'$; also, let $\phi = \tanh^{-1}(u/c)$ and $\phi' = \tanh^{-1}(u'/c)$ be the corresponding rapidities.

Using the above form of a Lorentz transformation, one can write

$$u' = \big(u\cosh(w) - c\sinh(w)\big)\frac{dt}{dt'} = c\frac{u\cosh(w) - c\sinh(w)}{c\cosh(w) - u\sinh(w)}$$

which implies

$$\tanh(\phi') = \frac{\tanh(\phi) - \tanh(w)}{1 - \tanh(\phi)\tanh(w)} = \tanh(\phi - w).$$

Therefore, in one dimension, rapidities are additive. When v, u and u' are not collinear, it is known that the general relativistic law for composition of velocities implies the hyperbolic law of cosines. That is, given a geodesic triangle (p,q,r) with vertices p,q,r in the hyperbolic disk $\mathbb{H}^2_{\mathbb{R}}$ and sides of lengths $w = \tanh^{-1}(|v|/c)$, $\phi = \tanh^{-1}(|u|/c)$ and $\phi' = \tanh^{-1}(|u'|/c)$, the formula for the norm of the composition of velocities is equivalent to

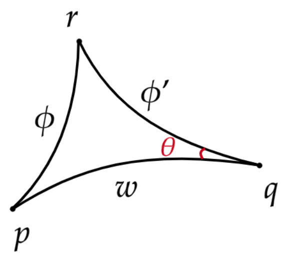

Figure 1 – Addition of rapidities.

$$\cosh(\phi) = \cosh(w)\cosh(\phi') + \sinh(w)\sinh(\phi')\cos(\theta),$$

where $\pi - \theta$ is the angle between v and u'. The velocity u is often called the relativistic addition of the velocities v and u' and is denoted by $u =: v \oplus u'$. The noncommutativity of this addition can be geometrically seen as follows.

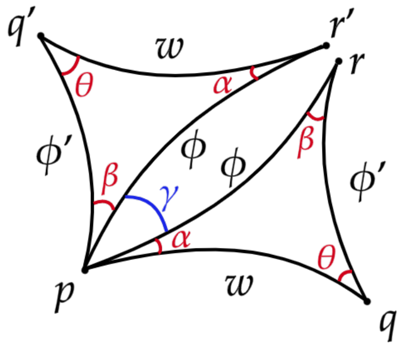

Figure 2 – Noncommutativity of the relativistic velocities addition.

In Figure 2, α is the angle between the velocities v and u; β is the angle between u and u'; and $\alpha+\beta+\gamma$ is the velocity between v and u' which equals $\pi - \theta$. Hence, $\gamma = \pi - (\alpha+\beta+\theta)$. But $\alpha+\beta+\theta < \pi$ because, in hyperbolic geometry, $\pi - (\alpha+\beta+\theta)$ is precisely the area of the triangle (p,q,r) which is bounded above by π. Therefore, $\gamma \neq 0$, indicating that the constant negative sectional curvature of the hyperbolic plane is directly related to the noncommutativity of the relativistic velocity addition.

In (BOREL, 1913b) and (BOREL, 1913a), E. Borel defined the kinematic space and suggested a formula, using the Gauss-Bonnet theorem, for the phenomenon that later would be called Thomas precession. Borel's kinematic space is nothing but the unitary open ball $\mathbb{B} := \{v \in \mathbb{R}^3 : |v| < 1\}$, each vector $v \in \mathbb{B}$ corresponding to a velocity u, measured in a chosen reference frame regarded as the origin of $\mathbb{B}$ and such that $v = u/c$. In this way, the boundary $\partial\mathbb{B}$ of the kinematic space corresponds to the velocities of photons. Note that, in kinematic space, inertial trajectories correspond to points while accelerated ones are represented by non-constant curves, each point of such a curve being the velocity as measured by the chosen frame at the origin in a given instant of time.

According to Borel, the kinematic space of classical mechanics is Euclidean (actually, it would be more accurate to say that it is a vector space with no preferred inner product, see ???). That being said, it is only natural to expect the existence of a limit velocity (the speed of light) to impose conditions on the geometry of the kinematic space; as Borel himself quoted "the principle of relativity corresponds to the hypothesis that the kinematic space is a space with a constant negative curvature, the space of Lobachevsky and Bolyai. The value of the radius of curvature is the speed of light."

The Thomas precession is the well-known phenomenon in relativistic kinematics in which the angular momentum of a body precesses when the body is in an accelerated trajectory. Suppose a body is in a uniform circular motion with angular velocity ω in relation to a given inertial reference frame. Then the angular momentum of the body, measured in this reference frame, precesses with angular velocity Ω given by

$$\Omega = \left(1 - \gamma^{-1}\right)\omega \tag{1.1.1}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ (RINDLER, 2006, p. 2000, formula (9.31)). Borel not only predicted the existence of such phenomenon but, postulating the Thomas Precession as a direct consequence of the parallel transport along a (plane) closed path in the kinematic space, used the Gauss-Bonnet theorem to estimate the area enclosed by such path and arrived at $\Omega = \left(1 - \gamma^{-2}\right)\omega$, which approximates formula $(1.1.1)$ when $v \to c$.

1.2. Gyroworld. A more recent way to formulate special relativity in terms of hyperbolic geometry appears in (UNGAR, 2008), where the axiomatization of the "weak commutativity and associativity" properties satisfied by the addition of relativistic velocities, also called Einstein's addition in this context, gives rise to the concepts of gyrogroup and gyrovector space, which are presented in more detail in Chapter 3. In the sense of this algebraic structure, the addition of velocities

$$v \oplus w := \frac{1}{1 + \frac{\langle v,w\rangle}{c^2}}\left(v + \frac{w}{\gamma_v} + \frac{1}{c^2}\,\frac{\gamma_v}{1+\gamma_v}\langle v,w\rangle v\right)$$

for all $v, w \in \mathrm{B}_0(c) := \{v \in \mathbb{R}^3 : |v| < c\}$, where $\gamma_v := 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor, is shown to be equivalent to the Möbius addition

$$a \oplus b := \frac{a+b}{1+\overline{a}b}$$

for all $a, b \in \mathbb{D}$. Here, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the unitary complex disk, and $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. Besides the addition, a gyrovector space is equipped with a scalar product which, in the case of Einstein gyrovector space, is given by

$$r \otimes v := c \tanh\left(r \tanh^{-1} \frac{|v|}{c} \right) \frac{v}{|v|}$$

for all $v \in \mathrm{B}_0(c)$ and $r \in \mathbb{R}$.

Perhaps the most interesting thing about gyrovector spaces is that, when endowed with an inner product, they provide an algebraic framework to hyperbolic geometry in a similar way that inner product vector spaces provide for Euclidean geometry. For instance, while straight lines (Euclidean geodesics) in a vector space can be parameterized as curves of the type $c(t) = p + tv$, hyperbolic geodesics in Einstein (or, equivalently , Möbius) gyrovector space can be parameterized as curves of the type $c(t) = p \oplus t \otimes v$. Also, in a normed gyrovector space G, the hyperbolic geodesics are the minimizing curves of the *gyrometric* $d_\oplus(p,q) := |p \ominus q|$, $p,q \in G$, in the way as straight lines are the minimizing curves of the metric $d(p,q) := |p - q|$, $p,q \in V$, in a normed vector space V.

Kinematic space in the spotlight. In (FERREIRA; REIS; GROSSI, 2020), we propose yet another framework to explore the interplay between special relativity and hyperbolic geometry. The mathematical tools we use come mainly from (ANAN'IN; GROSSI, 2011b), where a synthetic and coordinate-free frameset for several of the so-called classic geometries (including, say, hyperbolic, Fubini-Study, de Sitter and anti-de Sitter) is presented; Chapter 2 is devoted to a more detailed description of such methods.

Consider the Minkowski spacetime $\mathbb{M}$, *i.e.*, a real vector space endowed with a nondegenerate symmetric bilinear form $\langle -, - \rangle$ of signature $-+++$, and let $\mathbb{P}$ be the projectivization of $\mathbb{M}$. We define the kinematic space as the classifying space of inertial massive observers in Minkowski spacetime, that is, the kinematic space is the space $\mathscr{K} := \{p \in \mathbb{P} \mid \langle p, p \rangle < 0\}$. Given a point $p \in \mathscr{K}$, the tangent space $T_p\mathscr{K}$ is isomorphic to the space of linear transformations from the subspace generated by p to its orthogonal $p^\perp$,

$$T_p\mathscr{K} \simeq \mathrm{Lin}(\mathbb{R}p, p^\perp).$$

Using the above identification, we provide $\mathscr{K}$ with the hyperbolic metric

$$\langle \varphi_1, \varphi_2 \rangle_p := -\frac{\langle \varphi_1(p), \varphi_2(p) \rangle}{\langle p, p \rangle} \tag{1.2}$$

for all $\varphi_1, \varphi_2 \in T_p\mathscr{K}$. An interesting and useful fact is that, in this way, the hyperbolic space is naturally compactified by the de Sitter space, which is $\mathbb{P} \setminus \overline{\mathscr{K}}$ endowed with metric (1.2), where $\overline{\mathscr{K}}$ is the topological closure of $\mathscr{K}$ in $\mathbb{P}$.

Here, the quote by Borel can be interpreted as the fact that, when working with Minkowski spacetime, the kinematic space is a topological ball naturally provided with a hyperbolic metric. If one takes a look at Newtonian spacetime, that is, a real vector space V with a symmetric bilinear form of signature $0+++$, the classifying space of inertial observers turns out to be a real vector space with no distinguished inner product. To observe this, one just have to note that, choosing $v \in V$ such that $\langle v, v\rangle = 0$, each inertial observer (straight line in V) can be mapped to a vector in the vector subspace of signature $+++$.

We arrived at a geometrization of special relativity at the level of kinematic space, translating physical quantities (Lorentz factor, addition of velocities, Doppler effect) into purely geometric definitions and objects:

- The Lorentz factor is given by the square root of the *tance*, which is the simplest algebraic (projective) invariant involving two points (see (FERREIRA; REIS; GROSSI, 2020), Subsection 3.1). The square root of the tance is a monotonic function of the hyperbolic distance;
- The relative velocity between inertial observers $\boldsymbol{p},\boldsymbol{q} \in \mathscr{K}$ appears as a natural algebraic expression for the tangent vector to the geodesic segment joining $\boldsymbol{p},\boldsymbol{q}$ (see (FERREIRA; REIS; GROSSI, 2020), Definition 3.2.4);
- Rapidity and the closely related concept of scaled rapidity are shown to have distinct geometric origins. While rapidity comes from the hyperbolic metric on $\mathscr{K}$ and measures the hyperbolic distance between inertial reference frames, scaled rapidity comes from the hyperbolic metric on a velocity space $\mathscr{V}_{\boldsymbol{p}}$ for $\boldsymbol{p} \in \mathscr{K}$, and measures the hyperbolic distance between relative velocities at $\boldsymbol{p}$ (see (FERREIRA; REIS; GROSSI, 2020), Section 3.2);
- Parallel transport gives rise to the relativistic velocity addition in a straightforward generalization of the classical velocity addition (see (FERREIRA; REIS; GROSSI, 2020), Definition 3.2.1);
- Hypercycles (that is, curves equidistant from a geodesic in $\mathscr{K}$) allow one to write a “parallelogram law” for the relativistic velocity addition (see (FERREIRA; REIS; GROSSI, 2020), end of Subsection 3.2);
- The general relativistic Doppler effect can be described by a natural expression involving the Busemann function related to a photon or, equivalently, to a point in the ideal boundary of $\mathscr{K}$ ((FERREIRA; REIS; GROSSI, 2020), see Proposition 3.3.2). Moreover, horospheres appear as level surfaces of energy/frequency ((FERREIRA; REIS; GROSSI, 2020), see Corollary 3.3.3). There is a striking resemblance between such geometric form of the relativistic Doppler effect and the study of probability measures in the context of Patterson-Sullivan theory (see (QUINT., 2006, Section 1.2 and Proposition 3.9) for the Patterson-Sullivan perspective);
- A basic algebraic invariant involving two inertial observers in $\mathscr{K}$ and a pair of space-like separated events determines whether the observers agree or disagree on the order of occurrence of the events (see (FERREIRA; REIS; GROSSI, 2020), Subsection 3.5);

- Curves in $\mathscr{K}$ can be seen as describing the inertial reference frames occupied by an observer at each instant of its proper time and a tangent vector to such a curve gives the instantaneous 4-acceleration of the observer. Hence, dynamics can also be modelled at the level of the kinematic space (see (FERREIRA; REIS; GROSSI, 2020), Subsection 3.6).

We expect that this geometrization could help to study special relativity in other geometries, such as de Sitter and anti-de Sitter spaces, that also can be modelled by the same approach, presented on Chapter 2. We believe, as well, that some techniques could be extended to general relativity, studying for instance the classifying space of massive inertial observers in the Schwarzschild metric.

CHAPTER

2

PROJECTIVE CLASSIC GEOMETRIES

This chapter is devoted to the study of a coordinate free approach to the so-called classic geometries (see Subsection 2.2 for examples of classic geometries). These spaces share a common algebraic structure related to their geometry which is sometimes simpler to work with than the usual Riemannian/pseudo-Riemannian concepts; this is the viewpoint that is applied in Chapter 4 to the study of special relativity. The chapter is entirely based in (ANAN'IN; GROSSI, 2011b) and in (ANAN'IN; GROSSI, 2011a).

2.1 Projective Spaces

2.1.1. Definition. Let V be a $\mathbb{K}$-vector space of dimension $n+1$ endowed with a nondegenerate hermitian form $\langle -,-\rangle : V\times V\to\mathbb{K}$, where the field $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$. The *projective space* of dimension n over $\mathbb{K}$ is defined as the quotient:

$$\mathbb{P}_{\mathbb{K}}V := \mathbb{P}^n_{\mathbb{K}} := V^\bullet/\mathbb{K}^\bullet$$

where $V^\bullet := V\setminus\{0\}$ and $\mathbb{K}^\bullet := \mathbb{K}\setminus\{0\}$. That means $\mathbb{P}^n_{\mathbb{K}}$ is the quotient of $V^\bullet$ by the equivalence relation $u\sim v \Leftrightarrow u = rv$, for some $r\in\mathbb{K}^\bullet$. Of course $\mathbb{P}_{\mathbb{K}}V$ is provided with the quotient topology and its usual C^∞ structure.

From now on, in moments with no ambiguity, we are going to use the same letter to denote a point $p\in\mathbb{P}_{\mathbb{K}}V$ and a representative of this point $p\in V$. We are also going to use the notation $\pi : V^\bullet\to\mathbb{P}_{\mathbb{K}}V$ for the quotient projection.

2.1.2. Definition. Given a point $p\in\mathbb{P}_{\mathbb{K}}V$, we define its sign as the sign of $\langle p,p\rangle$ ($-$, $+$ or 0). The sign is well defined since $\langle kp,kp\rangle = |k|^2\langle p,p\rangle$, $k\in\mathbb{K}^\bullet$, that is, it does not depend on the choice of representatives.

The sign divides $\mathbb{P}_{\mathbb{K}}V$ in three disjoint components. The points in $\mathbb{P}_{\mathbb{K}}V$ can be *negative,* *positive* or *isotropic*:

$$\mathrm{B}V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p,p\rangle < 0\}, \quad \mathrm{E}V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p,p\rangle > 0\}, \quad \mathrm{S}V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p,p\rangle = 0\}.$$

We call *absolute* the space $\mathrm{S}V$ of isotropic points.

The sets $B := \{v \in V^\bullet | \langle v,v\rangle < 0\}$ and $E := \{v \in V^\bullet | \langle v,v\rangle > 0\}$ are open, since they're inverse images of open sets by the continuous map $V^\bullet \to \mathbb{R}$, $v \mapsto \langle v,v\rangle$, and therefore $\mathrm{B}V = \pi(B)$ and $\mathrm{E}V = \pi(E)$ are also open since the quotient map is open. The set $\mathrm{S}V$ is immediately closed, because $\mathrm{S}V = \mathbb{P}_{\mathbb{K}}V \setminus (\mathrm{B}V \sqcup \mathrm{E}V)$.

Given a non-isotropic point $p \in \mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$, we are going to use, for orthogonal decomposition, the following notation:

$$V = \mathbb{K}p \oplus p^\perp, \qquad v = \pi[p]v + \pi'[p]v$$

where

$$p^\perp \ni \pi[p]v := v - \frac{\langle v,p\rangle}{\langle p,p\rangle}p$$

$$\mathbb{K}p \ni \pi'[p]v := \frac{\langle v,p\rangle}{\langle p,p\rangle}p$$

2.1.3. Proposition. There exists a natural isomorphism:

$$T_p\mathbb{P}_{\mathbb{K}}V \simeq \mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p) \tag{2.1.4}$$

where $\mathbb{K}p$ is the subspace spanned by $p \in V$ and $\mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p)$ is the space of $\mathbb{K}$-linear maps from $\mathbb{K}p$ to $V/\mathbb{K}p$.

Proof. Given a smooth function $f \in C^\infty(\mathbb{P}_{\mathbb{K}}V)$, let $\tilde f$ be the map $\tilde f := f \circ \pi : V^\bullet \to \mathbb{R}$. Given a linear map $\varphi \in \mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p)$, let $\tilde\varphi : \mathbb{K}p \to V$ be a lift of φ. Consider the map $\mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p) \to T_p\mathbb{P}_{\mathbb{K}}V$, $\varphi \mapsto v_\varphi$, where:

$$v_\varphi f := \frac{d}{dt}\Big|_{t=0} \tilde f\big(p + t\tilde\varphi(p)\big) \tag{2.1.5}$$

for all $f \in C^\infty(\mathbb{P}_{\mathbb{K}}V)$. Note that $\tilde f(kv) = \tilde f(v)$, for all $k \in \mathbb{K}^\bullet$, $v \in V^\bullet$, so our map doesn't depend on the choice of representative $p \in V^\bullet$. Besides, given $k \in \mathbb{K}^\bullet$,

$$\frac{d}{dt}\Big|_{t=0} \tilde f\big(p + t(\tilde\varphi(p) + kp)\big) = \frac{d}{dt}\Big|_{t=0} \tilde f\big((1+tk)p + t\tilde\varphi(p)\big) =$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde f\big(p + \frac{t}{1+tk}\tilde\varphi(p)\big) = \frac{d}{dt}\Big|_{t=0} \tilde f\big(p + t\tilde\varphi(p)\big)$$

So, v_φ also doesn't depend on the choice of $\tilde\varphi$.

The map $\varphi \mapsto v_\varphi$ is linear due to the linearity of the directional derivative and to the fact that $\tilde f$ is invariant under the choice of representative. It's not hard to see that the map $\mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p) \to T_p\mathbb{P}_{\mathbb{K}}V$ is injective. Indeed, if we suppose $v_\varphi \equiv 0$, we necessarily have

$\varphi(p) \in \mathbb{K}p$ because, otherwise, we would be able to find a smooth function $f \in C^\infty(\mathbb{P}_\mathbb{K}V)$ such that the directional derivative $v_\varphi f$ wouldn't vanish. Now, the surjectivity comes from the fact that $\dim(\mathrm{Lin}(\mathbb{K}p, V/\mathbb{K}p)) = \dim(\mathbb{P}_\mathbb{K}V)$. ■

2.1.6. Corollary. Given a non-isotropic point $p \in \mathbb{P}_\mathbb{K}V \setminus \mathrm{S}V$, there exists a natural identification

$$T_p\mathbb{P}_\mathbb{K}V \simeq \mathrm{Lin}(\mathbb{K}p, p^\perp). \tag{2.1.7}$$

Proof. It's easy to see that the map $V/\mathbb{K}p \to p^\perp$, $[v] \mapsto \pi[p]v$ is well defined and is a linear isomorphism. ■

2.2 Metric

Using the notation $\langle -, p\rangle v : V \to V$ for the linear map $x \mapsto \langle x, p\rangle v$, we have, for a non-isotropic $p \in \mathbb{P}_\mathbb{K}V$,

$$T_p\mathbb{P}_\mathbb{K}V = \left\{\langle -, p\rangle v \mid v \in p^\perp\right\}.$$

Given $v \in p^\perp$, we're going to use the notation $t_{p,v} := \langle -, p\rangle v$. We equip $T_p\mathbb{P}_\mathbb{K}V$ with a hermitian form in a natural way:

$$\langle\langle t_{p,v}, t_{p,w}\rangle\rangle_p := \pm\frac{\langle t_{p,v}(p), t_{p,w}(p)\rangle}{\langle p, p\rangle} = \pm\langle p, p\rangle\langle v, w\rangle$$

for all $v, w \in p^\perp$. Clearly, we have the associated bilinear symmetric form

$$\langle t_{p,v}, t_{p,v}\rangle_p = \mathrm{Re}\langle\langle t_{p,v}, t_{p,v}\rangle\rangle_p, \tag{2.2.1}$$

where $\mathrm{Re}(z)$ stands for the real part of $z \in \mathbb{C}$. In this way, we obtain in $\mathbb{P}^n_\mathbb{R} \setminus \mathrm{S}V$ a pseudo-Riemannian metric, since one can readily see that this formula depends smoothly on $p \in \mathbb{P}^n_\mathbb{R} \setminus \mathrm{S}V$.

2.2.2. Examples. With this metric we can model many classic geometries, depending on the choice of field $\mathbb{K}$, sign in the equation (2.2.1) and signature of the hermitian form.

- Let $\mathbb{K} = \mathbb{R}$, $-++$ be the signature of $\langle -, -\rangle$ and $-$ the sign in (2.2.1). In this case, $\mathrm{B}V$ is topologically an open disk endowed with a Riemannian metric. Such manifold is known as *Beltrami-Klein's disk*, and it's a projective model for *plane hyperbolic geometry*, which is going to be a central case in this work. The component $\mathrm{E}V$ is a Möbius strip with a *Lorentzian*[1]metric. The space $\mathbb{P}^2_\mathbb{R}$ provided with these two geometries is called *Möbius-Beltrami-Klein's projective plane.*

- Let $\mathbb{K} = \mathbb{R}$, $-+++$ be the signature of $\langle -, -\rangle$ and $-$ the sign in (2.2.1). Here, $\mathrm{B}V$ is the *real hyperbolic space* of dimension 3. The manifold $\mathrm{E}V$ is Lorentzian and is known as *de Sitter* space (an important space in general relativity).

[1] A pseudo-Riemannian metric (on a smooth manifold) is called *Lorentzian* when its signature is $-+\ldots+$.

- Let $\mathbb{K}=\mathbb{R}$, $--++$ be the signature of $\langle -,-\rangle$ and $-$ the sign in (2.2.1). Now, $\mathrm{B}V$ is provided with a pseudo-Riemannian metric known as *anti-de Sitter* metric (also important in general relativity and in the adS/CFT correspondence).

- Let $\mathbb{K}=\mathbb{C}$, $++$ be the signature of $\langle -,-\rangle$ and $+$ the sign in (2.2.1). We get the usual sphere of dimension 2 with constant positive curvature.

- Let $\mathbb{K}=\mathbb{C}$, $-+$ be the signature of $\langle -,-\rangle$ and $-$ the sign in (2.2.1). In this case, $\mathrm{B}V$ and $\mathrm{E}V$ are two Poincaré disks glued together along the absolute $\mathrm{S}V$.

- Let $\mathbb{K}=\mathbb{C}$, $-++$ be the signature of $\langle -,-\rangle$ and $-$ the sign in (2.2.1). Then, $\mathrm{B}V$ is the *complex hyperbolic plane* $\mathbb{H}^2_{\mathbb{C}}$.

- Let $\mathbb{K}=\mathbb{C}$, $+...+$ be the signature of $\langle -,-\rangle$ and $+$ the sign in (2.2.1). This one is known as *Fubini-Study* metric and its use in physics is in the geometry of quantum information.

Although the aim of this dissertation is to use this approach to describe special relativity in the light of hyperbolic geometry, some other examples of classic geometries above, as cited, appear in physics, indicating that this *framework* could be useful in other areas of physics as well. So this section can also be seen as an invitation to physicists to become aware of this approach.

An interesting and useful fact is that, in this language, important geometric objects can be expressed in a "linear" way, in some sense. For instance, geodesics are projectivizations of certain $\mathbb{R}$-linear subspaces of dimension 2. But before going in that direction, we need to understand the Levi-Civita connection.

2.3 Levi-Civita Connection

Given a tangent vector $t\in T_p\mathbb{P}_{\mathbb{K}}V\simeq \mathrm{Lin}(\mathbb{K}p,p^{\perp})$, we can extend by zeroes and see it as an element of $\mathrm{Lin}(V,V)$. Conversely, given $t\in\mathrm{Lin}(V,V)$, we can produce a tangent vector $t_p\in\mathbb{P}_{\mathbb{K}}V$ by composing with projectors:

$$t_p := \pi[p]t\pi'[p] \tag{2.3.1}$$

2.3.2. Definition. Let $U\subset V$, $\mathbb{P}_{\mathbb{K}}U\cap \mathrm{S}V=\emptyset$, be a saturated open set (i.e., $\pi^{-1}(\pi(U))=U$). A *lifted* field on U is a smooth map $X:U\to\mathrm{Lin}(V,V)$ such that $X(p)_p=X(p)$ and $X(kp)=X(p)$ for all $p\in U$, $k\in\mathbb{K}^{\bullet}$.

2.3.3. Definition. Given $t\in\mathrm{Lin}(V,V)$, the lifted field T spread from t is defined by $T(p):=t_p$ for all $p\in\mathbb{P}_{\mathbb{K}}V$, $p\notin\mathrm{S}V$.

2.3.4. Definition. Let $X, T : U \to \mathrm{Lin}(V,V)$ be lifted fields on a saturated open set $U \subset V$ without isotropic points and let $t \in \mathrm{Lin}(V,V)$ be such that $T(p) = t$. We define:

$$\nabla_T X(p) := \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X\big(p + \varepsilon t(p)\big) \right)_p .$$

The proof that ∇ is an affine connection is routine (it comes directly from the properties of the derivative as a linear map).

2.3.5. Lemma. Let $p \in \mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ and $t \in T_p\mathbb{P}_{\mathbb{K}}V$. Then:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi'[p + \varepsilon t(p)] = -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi[p + \varepsilon t(p)] = t + t^*,$$

where t^* stands for the adjoint of t.

Proof. From the definition (2.3.4), we get:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi'[p + \varepsilon t(p)] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\langle -, p + \varepsilon t(p) \rangle}{\langle p + \varepsilon t(p), p + \varepsilon t(p) \rangle} \big(p + \varepsilon t(p)\big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\langle -, p \rangle + \varepsilon \langle -, t(p) \rangle}{\langle p, p \rangle + \varepsilon^2 \langle t(p), t(p) \rangle} \big(p + \varepsilon t(p)\big) =$$

$$= \frac{\langle -, t(p) \rangle}{\langle p, p \rangle} p + \frac{\langle -, p \rangle}{\langle p, p \rangle} t(p).$$

Since t is a tangent vector to the point p, we can write $t\pi'[p] = t$. Let $t' := \frac{\langle -, t(p) \rangle}{\langle p, p \rangle} p$; then, for arbitrary $v, w \in V$,

$$\langle t(v), w \rangle = \left\langle \frac{\langle v, p \rangle}{\langle p, p \rangle} t(p), w \right\rangle = \frac{\langle v, p \rangle}{\langle p, p \rangle} \langle t(p), w \rangle =$$

$$= \left\langle v, \frac{\langle t(p), w \rangle^*}{\langle p, p \rangle} p \right\rangle = \left\langle v, \frac{\langle w, t(p) \rangle}{\langle p, p \rangle} p \right\rangle = \langle v, t'(w) \rangle$$

Therefore, $t' = t^*$. ■

2.3.6. Lemma. Let $p \in \mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ and $s, t \in T_p\mathbb{P}_{\mathbb{K}}V$. Let S, T be the fields spread respectively from s and t. Then

$$\nabla_T S(x) = (s\pi[x]t - t\pi'[x]s)_x$$

for all non-isotropic $x \in \mathbb{P}_{\mathbb{K}}V$.

Proof. By definition (2.3.4), lemma (2.3.5), and using that $(t_x)^*\pi'[x] = \pi[x](t_x)^* = 0$, we have

$$\nabla_T S(x) = \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S\big(x + \varepsilon t_x x\big) \right)_x = \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi[x + \varepsilon t_x x] s \pi'[x + \varepsilon t_x x] \right)_x =$$

$$= \Big(\pi[x]s(t_x + (t_x)^*) - (t_x + (t_x)^*)s\pi'[x] \Big)_x = (s\pi[x]t - t\pi'[x]s)_x$$

■

2.3.7. Proposition. The affine connection ∇ in (2.3.4) is the Levi-Civita connection for the (pseudo-)Riemannian metric defined in (2.2.1) (for each component, $\mathrm{B}V$ and $\mathrm{E}V$ of $\mathbb{P}_{\mathbb{K}}V$).

Proof. To show that ∇ is symmetric we recall that, because tensor fields are linear over $C^\infty(\mathbb{P}_\mathbb{K}V)$, the torsion tensor $\tau(S,T) := \nabla_S T - \nabla_T S - [S,T]$ in a point only depends on the value of the vector fields S,T on that point. Therefore, given $p \in \mathbb{P}_\mathbb{K}V \setminus \mathrm{S}V$ and S and T lifted fields, we can assume that S and T are spread respectively from $t := T(p)$ and $s := S(p)$. Then, from Lemma 2.3.6, we have $\nabla_S T(p) = \nabla_T S(p) = 0$. Now, it remains to check that $[S,T](p) = 0$. By definition, given $f \in C^\infty(\mathbb{P}_\mathbb{K}V)$:

$$T(x)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\big(p + \varepsilon\pi[x]t(x)\big)$$

where $\tilde{f} = f \circ \pi$. Then,

$$S(Tf)(p) = \frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\big(p + \delta s(p) + \varepsilon\pi[p+\delta s(p)]t(p+\delta s(p))\big)\right) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\big(p + \delta s(p) + \varepsilon t(p) - \varepsilon\delta\frac{\langle s(p),t(p)\rangle}{\langle p,p\rangle + \delta^2\langle s(p),s(p)\rangle}(p+\delta s(p))\big)\right) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\left(\left(1 - \varepsilon\delta\frac{\langle s(p),t(p)\rangle}{\langle p,p\rangle}\right)(p+\delta s(p)) + \varepsilon t(p)\right)\right) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\left(p + \delta s(p) + \frac{\varepsilon t(p)}{1 - \varepsilon\delta\langle s(p),t(p)\rangle\langle p,p\rangle^{-1}}\right)\right).$$

In the last equality above we used that $\tilde{f}(kp) = \tilde{f}(p)$ for any $k \in \mathbb{K}^\bullet$. Now, being f smooth, we end up with:

$$S(Tf)(p) = \frac{d}{d\delta}\Big|_{\delta=0}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{f}\big(p + \delta s(p) + \varepsilon t(p)\big)\right).$$

Therefore, since the above expression is symmetric in t and s, we have $S(Tf)(p) = T(Sf)(p)$, *i.e.*, $[S,T](p) = 0$ and $\tau(S,T) = 0$.

To show the compatibility with the metric, we note that, given smooth vector fields $S,T,X \in \mathfrak{X}(\mathbb{P}_\mathbb{K}V \setminus \mathrm{S}V)$, the quantity $C(X,S,T) := X\big(\langle S,T\rangle\big) - \langle\nabla_X S,T\rangle - \langle S,\nabla_X T\rangle$ is also a tensor field. Therefore, given a non isotropic $p \in \mathbb{P}_\mathbb{K}V$, in order to calculate $C(X,S,T)(p)$ we can assume that X,S,T are respectively spread from $x,s,t \in T_p\mathbb{P}_\mathbb{K}V$. We have

$$X\big(\langle S,T\rangle\big)(p) = \pm\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{\Big\langle S_{p+\varepsilon x(p)}\big(p+\varepsilon x(p)\big), T_{p+\varepsilon x(p)}\big(p+\varepsilon x(p)\big)\Big\rangle}{\langle p+\varepsilon x(p), p+\varepsilon x(p)\rangle} =$$

$$= \pm\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{\big\langle \pi[p+\varepsilon x(p)]s\big(p+\varepsilon x(p)\big), \pi[p+\varepsilon x(p)]t\big(p+\varepsilon x(p)\big)\big\rangle}{\langle p,p\rangle + \varepsilon^2\langle x(p),x(p)\rangle} =$$

$$= \pm\frac{1}{\langle p,p\rangle}\left(\Big\langle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+\varepsilon x(p)]s(p), t(p)\Big\rangle + \Big\langle s(p), \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+\varepsilon x(p)]t(p)\Big\rangle\right) =$$

$$= \pm\frac{1}{\langle p,p\rangle}\Big(\big\langle -(x^*+x)s(p), t(p)\big\rangle + \big\langle s(p), -(x^*+x)t(p)\big\rangle\Big)$$

where in the last equality we used Lemma 2.3.5. Then, knowing that $x^* = \frac{\langle -,x(p)\rangle}{\langle p,p\rangle}p$, we end up with $X\big(\langle S,T\rangle\big)(p) = 0$. From Lemma 2.3.6, we have $\nabla_X S(p) = \nabla_X T(p) = 0$ and therefore, $C(X,S,T)(p) = 0$. ■

2.4 Geodesics, Tance and Parallel Transport

The simplest algebraic invariant of two non-isotropic points $p,q \in \mathbb{P}_{\mathbb{K}}V$ is the **tance** (see (ANAN'IN; GROSSI, 2011b)), defined as:

$$\operatorname{ta}(p,q) := \frac{\langle p,q\rangle\langle q,p\rangle}{\langle p,p\rangle\langle q,q\rangle}. \tag{2.4.1}$$

It's immediate that the tance doesn't depend on the choice of representatives. As a convention, when one of the points p or q is isotropic, we define $\operatorname{ta}(p,q) := +\infty$ if $\langle p,q\rangle \neq 0$, and $\operatorname{ta}(p,q) = 1$ if $\langle p,q\rangle = 0$.

It turns out that the tance is a monotonic function of the distance, as will be shown in the future (see Propositions 2.4.9 and 2.4.11). This makes the tance extremely useful since we can use it instead of the distance on many occasions; this is desirable because the tance is algebraic while the distance is a transcendental function of the tance.

2.4.2. Lemma. Let $\gamma : [a,b] \to \mathbb{P}_{\mathbb{K}}V$ be a smooth curve and $\gamma_0 : [a,b] \to V$ be a smooth lift of γ, *i.e.* a smooth curve such that $\pi \circ \gamma_0 = \gamma$. Then, for $\gamma(t_0)$ non isotropic, the tangent vector $\dot\gamma(t_0) : \mathbb{K}\gamma_0(t_0) \to \gamma_0(t_0)^{\perp}$ to γ at the point $\gamma(t_0)$ is given by:

$$\dot\gamma(t_0) = \frac{\langle -,\gamma_0(t_0)\rangle}{\langle \gamma_0(t_0),\gamma_0(t_0)\rangle}\pi[\gamma_0(t_0)]\dot\gamma_0(t_0)$$

Proof. Given $f \in C^{\infty}(\mathbb{P}_{\mathbb{K}}V)$ and $\tilde f := f \circ \pi$, we can write:

$$\dot\gamma(t_0)f := \frac{d}{dt}\Big|_{t=t_0}(f\circ\gamma) = \frac{d}{dt}\Big|_{t=t_0}(\tilde f\circ\gamma_0) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde f\big(\gamma_0(t_0)+\varepsilon\dot\gamma_0(t_0)\big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde f\big(\gamma_0(t_0)+\varepsilon\pi'[\gamma_0(t_0)]\dot\gamma_0(t_0)+\varepsilon\pi[\gamma_0(t_0)]\dot\gamma_0(t_0)\big).$$

Now, being $k := \langle\dot\gamma_0(t_0),\gamma_0(t_0)\rangle/\langle\gamma_0(t_0),\gamma_0(t_0)\rangle$, we have:

$$\dot\gamma(t_0)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde f\big((1+\varepsilon k)\gamma_0(t_0)+\varepsilon\pi[\gamma_0(t_0)]\dot\gamma_0(t_0)\big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde f\big(\gamma_0(t_0)+\frac{\varepsilon}{1+\varepsilon k}\pi[\gamma_0(t_0)]\dot\gamma_0(t_0)\big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde f\big(\gamma_0(t_0)+\varepsilon\pi[\gamma_0(t_0)]\dot\gamma_0(t_0)\big).$$

■

2.4.3. Definition (Geodesic). Let $W \leq V$ be a two dimensional $\mathbb{R}$-linear subspace of V such that $\langle -,-\rangle\big|_W$ is real and non-null. We will call $\mathbb{P}_{\mathbb{K}}W := \pi(W) \subset \mathbb{P}_{\mathbb{K}}V$ a **geodesic**.

The above definition of geodesic (restricted to $\mathrm{B}V$ or $\mathrm{E}V$) coincides with the usual one as we will see in Corollary 2.4.16. Since $\mathbb{K}p \cap W = \mathbb{R}p$ for all $0 \neq p \in W$, we have $\mathbb{P}_{\mathbb{K}}W = \mathbb{P}_{\mathbb{R}}W$.

Hence, $\mathbb{P}_{\mathbb{K}}W \simeq \mathbb{S}^1$, *i.e*, every geodesic is topologically a circle. Of course, when we restrict a geodesic to BV or to EV, in order to get an actual pseudo-Riemannian geodesic, we may no longer have a circle.

2.4.4. Lemma. The geodesics $\mathbb{P}_{\mathbb{K}}W$ and $\mathbb{P}_{\mathbb{K}}W'$ coincide iff $W = kW'$ for some $k \in \mathbb{K}^\bullet$.

Proof. One direction is immediate. Now, let $\mathbb{P}_{\mathbb{K}}W = \mathbb{P}_{\mathbb{K}}W'$ be two coincident geodesics. Let $w_1, w_2 \in W$ be such that $\langle w_1, w_2\rangle \neq 0$ and $W = \mathbb{R}w_1 + \langle w_1, w_2\rangle\mathbb{R}w_2$. It follows from $\mathbb{P}_{\mathbb{K}}W = \mathbb{P}_{\mathbb{K}}W'$ that there exist $w_1', w_2' \in W'$ and $k_1, k_2 \in \mathbb{K}^\bullet$ such that $w_1 = k_1w_1'$ and $w_2 = k_2w_2'$. So,

$$W = \mathbb{R}k_1w_1' + k_1\overline{k}_2\langle w_1', w_2'\rangle\mathbb{R}k_2w_2' = k_1(\mathbb{R}w_1' + \mathbb{R}w_2') = k_1W' \qquad \blacksquare$$

2.4.5. Definition. Let $W \leq V$ be a real linear subspace. A point $p \in W$ is said to be projectively smooth in W if $\dim_{\mathbb{R}}(\mathbb{K}p \cap W) = \min\limits_{0\neq w\in W} \dim_{\mathbb{R}}(\mathbb{K}w \cap W)$.

2.4.6. Lemma. Let $W \leq V$ be a real linear subspace, $p \in W$ a non-isotropic projectively smooth point in W and $\varphi \in \operatorname{Lin}(\mathbb{K}p, V)$. Then $\varphi \in T_p\mathbb{P}_{\mathbb{K}}W$ if, and only if $\varphi(p) \in W \cap p^\perp$.

Proof. We begin by defining:

$$d := \min_{0\neq w\in W} \dim_{\mathbb{R}}(\mathbb{K}w \cap W), \qquad D := \{w \in W \mid \dim_{\mathbb{R}}(\mathbb{K}w \cap W) = d\}$$

Note that given $0 \neq w \in W$ we have either $\mathbb{K}w \cap W = \mathbb{R}w$ or $\mathbb{K}w \cap W = \mathbb{K}w$, which implies that D is open in W. When $d = 2$, this is immediate; when $d = 1$ is just a matter of noticing that $D = \{w \in W \mid iw \notin W\}$, and therefore that $W \setminus D$ is a $\mathbb{C}$-linear subspace of V contained in W.0

Now, suppose that $\varphi(p) \in W \cap p^\perp$. Since D is open in W, for a sufficiently small $\varepsilon > 0$, the curve $\gamma_0(\varepsilon) := p + \varepsilon\varphi(p)$ is contained in D and the curve $\gamma := \pi \circ \gamma_0$, which by Lemma 2.4.2 has tangent vector φ at p, is contained in $\mathbb{P}_{\mathbb{K}}D$. So, we proved one direction, $\varphi \in T_p\mathbb{P}_{\mathbb{K}}W$. To finish, we have:

$$\dim_{\mathbb{R}}\mathbb{P}_{\mathbb{K}}D = \dim_{\mathbb{R}}W - d = \dim_{\mathbb{R}}(W \cap p^\perp)$$

And therefore $T_p\mathbb{P}_{\mathbb{K}}W \simeq W \cap p^\perp$. $\blacksquare$

2.4.7. Lemma. Let $p, q, r \in \mathbb{P}_{\mathbb{K}}V$ with $p \notin \mathrm{S}V$, $q \neq r$ and $\langle q, r\rangle \neq 0$. Let $t \in T_p\mathbb{P}_{\mathbb{K}}V$. Then:

1) There exists a unique geodesic containig q and r;

2) There exists a unique geodesic passing through p and with tangent vector t at p. Such geodesic is the projectivization of the subspace $W = \mathbb{R}p + \mathbb{R}t(p)$.

Proof. 1) Let $W := \mathbb{R}q + \mathbb{R}\langle q, r\rangle r$. It's clear that $q, r \in \mathbb{P}_{\mathbb{K}}W$. The hermitian form restricted to W is non-null and real since $\big\langle q, \langle q, r\rangle r\big\rangle = \overline{\langle q, r\rangle}\langle q, r\rangle \in \mathbb{R}$. Let $\mathbb{P}_{\mathbb{K}}W'$ be another geodesic passing through q and r. Then $W' = \mathbb{R}kq + \mathbb{R}k'r$ for some $k, k' \in \mathbb{K}$. We have $\langle kq, k'r\rangle = k\overline{k'}\langle q, r\rangle \in \mathbb{R}^\bullet$. So, $W' = \mathbb{R}kq + \mathbb{R}k\overline{k'}\langle q, r\rangle k'r = kW$ which, by Lemma 2.4.4, implies that $\mathbb{P}_{\mathbb{K}}W = \mathbb{P}_{\mathbb{K}}W'$.

2) By Lemma 2.4.6, t is tangent to the geodesic $\mathbb{P}_{\mathbb{K}}W$ at p (remember that $t(p) \in p^\perp$). Now, let $\mathbb{P}_{\mathbb{K}}W'$ be another geodesic passing through p with tangent vector t at p. We can

choose, by Lemma 2.4.4, a subspace W' such that $p \in W'$. Again by Lemma 2.4.6, we have $t(p) \in W' \cap p^\perp$ and therefore $W' = \mathbb{R}p + \mathbb{R}t(p)$. ■

2.4.8. Proposition. Let $p, q \in \mathbb{P}_\mathbb{K}V$ be two distinct and non orthogonal points, being p non isotropic. Then $t := \frac{\langle -,p\rangle \pi[p]q}{\langle q,p\rangle}$ is a tangent vector to the geodesic passing through p and q at the point p.

Proof. The geodesic passing through p and q is given by the subspace $W = \mathbb{R}p + \mathbb{R}\langle p,q\rangle q$. It's immediate that $t(p) \in p^\perp$. We have:

$$t(p) = \frac{\langle p,p\rangle}{\langle q,p\rangle}\pi[p]q = \frac{\langle p,p\rangle}{|\langle q,p\rangle|^2}\langle p,q\rangle q - p$$

So, we also have $t(p) \in W$. Therefore, by Lemma 2.4.6, $t \in T_p\mathbb{P}_\mathbb{K}W$. ■

2.4.9. Spherical Geodesics. A geodesic $\mathbb{P}_\mathbb{K}W$ is called spherical if the subspace W has signature $++$. We shall parameterize $\mathbb{P}_\mathbb{K}W$ and calculate its length. Let $\{p,q\}$ be an orthonormal basis for W. The curve:

$$\gamma_0 : [0,a] \to V, \qquad t \mapsto \cos(t)p + \sin(t)q$$

where $a \in [0, \frac{\pi}{2}]$, is a lift of the parameterization $\gamma := \pi \circ \gamma_0$ of a segment of geodesic connecting p and $r := \gamma(a)$ contained in $\mathbb{P}_\mathbb{K}W$. From Lemma 2.4.2 we can write

$$\dot\gamma(t) = \frac{\langle -,\gamma_0(t)\rangle}{\langle \gamma_0(t),\gamma_0(t)\rangle}\pi[\gamma_0(t)]\dot\gamma_0(t) = \langle -,\gamma_0(t)\rangle\dot\gamma_0(t)$$

because $\langle \gamma_0(t),\gamma_0(t)\rangle = 1$ and $\langle \dot\gamma_0(t),\gamma_0(t)\rangle = 0$ for all $t \in [0,a]$. Choosing the $+$ sign in equation (2.2.1), we have:

$$l(\gamma) = \int_0^a \sqrt{\langle \dot\gamma(t),\dot\gamma(t)\rangle_{\gamma(t)}}\,dt = \int_0^a \sqrt{\langle \dot\gamma_0(t),\dot\gamma_0(t)\rangle\langle \gamma_0(t),\gamma_0(t)\rangle}\,dt = a$$

By Sylvester's criterion (Appendix A, Theorem A.1.7), $\langle p,p\rangle\langle r,r\rangle - \langle p,r\rangle\langle r,p\rangle \geq 0$. Therefore $0 \leq \mathrm{ta}(p,r) \leq 1$. Calculating the tance we obtain $\mathrm{ta}(p,r) = \cos^2 a$ which gives us

$$l(\gamma) = \arccos\sqrt{\mathrm{ta}(p,r)} \tag{2.4.10}$$

■

2.4.11. Hyperbolic Geodesics. A geodesic $\mathbb{P}_\mathbb{K}W$ is called hyperbolic if the subspace W has signature $-+$. Following a similar procedure as we did in the spherical case, we take an orthonormal basis $\{p,q\}$ for W, where $\langle p,p\rangle = -1$. Given $a > 0$, the curve

$$\gamma_0 : [0,a] \to V, \qquad \gamma_0(t) := \cosh(t)p + \sinh(t)q$$

is a lift of the parameterization $\gamma := \pi \circ \gamma_0$ of a segment of geodesic passing through p and $r := \gamma(a)$ and contained in $\mathbb{P}_\mathbb{K}W$. We have

$$\langle \gamma_0(t),\gamma_0(t)\rangle = -\cosh^2 t + \sinh^2 t = -1, \qquad \langle \dot\gamma_0(t),\dot\gamma_0(t)\rangle = -\sinh^2 t + \cosh^2 t = 1$$

and $\langle\dot{\gamma}_0(t),\gamma_0(t)\rangle = 0$. Therefore, by Lemma 2.4.2, we have $\dot{\gamma}(t) = -\langle -,\gamma_0(t)\rangle\dot{\gamma}_0(t)$. Choosing the $-$ sign in equation (2.2.1) and, as in the spherical case, calculating the length of γ, we end up with $l(\gamma) = a$. By Sylvester's criterion (Appendix A, Theorem A.1.7), we have $\langle p,p\rangle\langle r,r\rangle - \langle p,r\rangle\langle r,p\rangle \leq 0$, which implies $\operatorname{ta}(p,r) \geq 1$. If we calculate the tance, we gain $\operatorname{ta}(p,r) = \cosh^2 a$, and then the following relation holds:

$$l(\gamma) = \operatorname{arccosh}\sqrt{\operatorname{ta}(p,r)}. \tag{2.4.12}$$

■

Our next goal is to describe parallel transport along geodesics and to show that our geodesics, outside isotropic points, are actual geodesics of the Levi-Civita connection. Given a geodesic $G = \mathbb{P}_{\mathbb{K}}W$, we define the *projective line* of the geodesic as $L := \mathbb{P}_{\mathbb{K}}(\mathbb{K}W)$, where $\mathbb{K}W := W$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{K}W := W \oplus iW$ if $\mathbb{K} = \mathbb{C}$.

2.4.12. Definition. Let $p \in \mathbb{P}_{\mathbb{K}}V$ be a non-isotropic point, $t \in T_p\mathbb{P}_{\mathbb{K}}V$ be a tangent vector at p and T be the spread (lifted) field from t. We define the smooth field

$$\operatorname{Tn}(t)(x) := \frac{T(x)}{\operatorname{ta}(x,p)}$$

for all $x \in \mathbb{P}_{\mathbb{K}}V \setminus (\mathbb{P}_{\mathbb{K}}p^{\perp} \cup \mathrm{S}V)$.

2.4.13. Proposition. Let $p \in \mathbb{P}_{\mathbb{K}}V$ be a non-isotropic point, $t \in T_p\mathbb{P}_{\mathbb{K}}V$ a tangent vector at p and G a geodesic passing through p with tangent vector t at p. Then the field $\operatorname{Tn}(t)$ is non-null and tangent to G in the points of G where it's defined.

Proof. Let $x \in G$, $x \notin p^{\perp}$. It's clear that $\operatorname{Tn}(t)(x) \neq 0$, otherwise we would have $x \in p^{\perp}$. By Lemma 2.4.7, we know that $G = \mathbb{P}_{\mathbb{K}}W$, where $W = \mathbb{R}p + \mathbb{R}t(p)$. We choose a representative such that $x \in W$ and write

$$\operatorname{Tn}(t)(x) = \pi[x]t(x) = \pi[x]t\left(\frac{\langle x,p\rangle}{\langle p,p\rangle}p\right) = \frac{\langle x,p\rangle}{\langle p,p\rangle}t(p) - \frac{\langle x,p\rangle}{\langle p,p\rangle}\frac{\langle t(p),x\rangle}{\langle x,x\rangle}x$$

(since $t \in T_p\mathbb{P}_{\mathbb{K}}V \simeq \operatorname{Lin}(V,V)$, we have $t(\mathbb{K}p) \subset p^{\perp}$ and $t(p^{\perp}) = \{0\}$). Therefore $\operatorname{Tn}(t)(x) \in W \cap x^{\perp}$, which by Lemma 2.4.6 concludes the proof. ■

2.4.14. Proposition. Let $p \in \mathbb{P}_{\mathbb{K}}V$ be a non-isotropic point, $t \in T_p\mathbb{P}_{\mathbb{K}}V$ a tangent vector at p and T be the spread field from t. Then

$$T(x)\operatorname{ta}(-,p) = -2\operatorname{ta}(x,p)\operatorname{Re}\frac{\langle t(x),x\rangle}{\langle x,x\rangle}$$

for all $x \in \mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$.

Proof. It's routine:

$$T(x)\operatorname{ta}(-,p) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\frac{\langle p,x+\varepsilon\pi[x]tx\rangle\langle x+\varepsilon\pi[x]tx,p\rangle}{\langle p,p\rangle\langle x+\varepsilon\pi[x]tx,x+\varepsilon\pi[x]tx\rangle} =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\big\langle p, x+\varepsilon\pi[x]tx\big\rangle\big\langle x+\varepsilon\pi[x]tx, p\big\rangle}{\langle p,p\rangle\Big(\langle x,x\rangle+\varepsilon^2\big\langle \pi[x]tx,\pi[x]tx\big\rangle\Big)} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\big\langle p, x+\varepsilon\pi[x]tx\big\rangle\big\langle x+\varepsilon\pi[x]tx, p\big\rangle}{\langle p,p\rangle\langle x,x\rangle} =$$

$$= \frac{\big\langle p,\pi[x]tx\big\rangle\langle x,p\rangle+\langle p,x\rangle\big\langle \pi[x]tx,p\big\rangle}{\langle p,p\rangle\langle x,x\rangle} = -\frac{\langle p,x\rangle\langle x,p\rangle\overline{\langle tx,x\rangle}-\langle tx,x\rangle\langle p,x\rangle\langle x,p\rangle}{\langle p,p\rangle\langle x,x\rangle^2} =$$

$$= -2\mathrm{ta}(x,p)\mathrm{Re}\frac{\langle t(x),x\rangle}{\langle x,x\rangle}$$

■

2.4.15. Proposition. Let G be a geodesic with tangent vector $t\in T_pG$ at a point $p\in G$, $p\notin \mathrm{S}V$, and let $h\in T_pL$, where $L=\mathbb{P}_{\mathbb{K}}(\mathbb{K}W)$ is the projective line of G. Then,

$$\nabla_{\mathrm{Tn}(t)}\mathrm{Tn}(h)(x)=0$$

for all non-isotropic $x\in G$, $x\notin \mathbb{P}_{\mathbb{K}}p^{\perp}$.

Proof. We can write, by Lemma 2.4.7, that $G=\mathbb{P}_{\mathbb{K}}W$ where $W=\mathbb{R}p+\mathbb{R}t(p)$. It suffices to show that $\nabla_T\mathrm{Tn}(h)(x)=0$, being T the spread field from t. If H is the spread field from h, we have:

$$\nabla_T\mathrm{Tn}(h)(x)=\nabla_T\left(\frac{1}{\mathrm{ta}(p,-)}H\right)(x)=$$

$$=T(x)\left(\frac{1}{\mathrm{ta}(p,-)}\right)H(x)+\frac{1}{\mathrm{ta}(p,x)}\nabla_TH(x)=$$

$$=-\frac{1}{\mathrm{ta}(p,x)^2}T(x)\big(\mathrm{ta}(p,-)\big)H(x)+\frac{1}{\mathrm{ta}(p,x)}\nabla_TH(x)=$$

$$=\frac{1}{\mathrm{ta}(p,x)}\left(2\mathrm{Re}\frac{\langle t(x),x\rangle}{\langle x,x\rangle}H(x)+\nabla_TH(x)\right)$$

where, in the last equality, we used Proposition 2.4.14. We can choose a representative $x\in W$ of the form $x=p+rt(p)$, for some $r\in\mathbb{R}$, which gives us $t(x)=t(p)$. From Lemma 2.4.6 we know that $h(p)\in\mathbb{K}W\cap p^{\perp}$, being $\mathbb{K}W=\mathbb{K}p+\mathbb{K}t(p)$. Therefore $h(p)=kt(p)$, for some $k\in\mathbb{K}^{\bullet}$, and $h(x)=kt(x)$. Now, from Lemma 2.3.6, we have

$$\big(\nabla_T\mathrm{Tn}(h)(x)\big)x=\frac{1}{\mathrm{ta}(p,x)}\left(2\frac{\langle t(x),x\rangle}{\langle x,x\rangle}h+h\pi[x]t-t\pi'[x]h\right)_x(x)=$$

$$=\frac{1}{\mathrm{ta}(p,x)}\left(2\frac{\langle t(x),x\rangle}{\langle x,x\rangle}\pi[x]h(x)+\pi[x]h\left(t(x)-\frac{\langle t(x),x\rangle}{\langle x,x\rangle}x\right)-\frac{\langle h(x),x\rangle}{\langle x,x\rangle}\pi[x]t(x)\right)=0$$

■

2.4.16. Corollary. In each connected component of $\mathbb{P}_{\mathbb{K}}V\setminus \mathrm{S}V$ the geodesics introduced in Definition 2.4.3 are geodesics of the Levi-Civita connection. Conversely, all pseudo-Riemannian geodesics are of this kind.

2.4.17. Definition. Let $p\in\mathbb{P}_{\mathbb{K}}V\setminus \mathrm{S}V$, $t\in T_p\mathbb{P}_{\mathbb{K}}V$ and T be the spread (lifted) field from t. We define the field:

$$\mathrm{Ct}(t)(x):=\frac{T(x)}{\sqrt{\mathrm{ta}(p,x)}}$$

for all non-isotropic $x \in \mathbb{P}_{\mathbb{K}}V \setminus \mathbb{P}_{\mathbb{K}}p^{\perp}$, such that x is in the same connected component of $\mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ as the point p.

2.4.18. Proposition. Let G be a geodesic, $p \in G$ a non-isotropic point and $0 \neq t \in T_pG$. Let L be the projective line of G and let $v \in (T_pL)^{\perp}$. Then

$$\nabla_{\mathrm{Tn}(t)}\mathrm{Ct}(v)(x) = 0$$

for all non-isotropic $x \in G \setminus \mathbb{P}_{\mathbb{K}}p^{\perp}$ such that x is in the same connected component of $\mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ as the point p.

Proof. Let W and T be the spread (lifted) fields from v and t respectively. We shall proceed similarly to what we did in Proposition 2.4.15, choosing a representative $x \in W$ of the form $x = p + rt(p)$ for some $r \in \mathbb{R}$. It suffices to check that $\nabla_T\mathrm{Ct}(v)(x) = 0$. We have

$$\nabla_T\mathrm{Ct}(v)(x) = T(x)\left(\frac{1}{\sqrt{\mathrm{ta}(p,-)}}\right)W(x) + \frac{1}{\sqrt{\mathrm{ta}(p,x)}}\nabla_T W(x) =$$

$$= -\frac{1}{2\mathrm{ta}(p,x)^{\frac{3}{2}}}T(x)\big(\mathrm{ta}(p,-)\big)\pi[x]v\pi'[x] + \frac{1}{\sqrt{\mathrm{ta}(p,x)}}\nabla_T W(x) =$$

$$= \frac{1}{\sqrt{\mathrm{ta}(p,x)}}\left(\frac{\langle t(x),x\rangle}{\langle x,x\rangle}v + v\pi[x]t - t\pi'[x]v\right)_x$$

where, in the last equality, we used Proposition 2.4.14 and Lemma 2.3.6. We know that $t(x) = t(p)$ and, by Lemma 2.4.6, $t(p) \in W \cap p^{\perp}$. By the definition of the metric 2.2.1 one can immediately see that $v(p) \in \big(\mathbb{K}p + \mathbb{K}tp\big)^{\perp} \cap p^{\perp}$, which implies that $t\pi'[x]v(x) = \frac{\langle vx,x\rangle}{\langle x,x\rangle}t(x) = 0$. And therefore:

$$\nabla_T\mathrm{Ct}(v)(x) = \frac{1}{\sqrt{\mathrm{ta}(p,x)}}\left(\frac{\langle tx,x\rangle}{\langle x,x\rangle}v + v\pi[x]t\right)_x = 0$$

■

Let $G = \mathbb{P}_{\mathbb{K}}W$ be a geodesic, $L = \mathbb{P}_{\mathbb{K}}(\mathbb{K}W)$ be the projective line of the geodesic G and $p \in G \setminus \mathrm{S}V$. If the real linear subspace $W \leq V$ is nondegenerate one can readily see that we have the decomposition $T_p\mathbb{P}_{\mathbb{K}}V = T_pL \oplus (T_pL)^{\perp}$.

2.4.19. Corollary. Let $c : (-a,a) \to \mathbb{P}_{\mathbb{K}}W$ be a geodesic[2] with $c(0) = p$ nonisotropic and $W \leq V$ nondegenerate. Given $t \in T_p\mathbb{P}_{\mathbb{K}}V$, let $t = h + v$, where $h \in T_pL$, $v \in (T_pL)^{\perp}$ and $L = \mathbb{P}_{\mathbb{K}}(\mathbb{K}W)$ is the projective line of $\mathbb{P}_{\mathbb{K}}W$. The parallel transport of t along the geodesic c is given by $\mathrm{Tn}(h)\big(c(\varepsilon)\big) + \mathrm{Ct}(v)\big(c(\varepsilon)\big)$ for every $c(\varepsilon) \notin p^{\perp}$.

Proof. Follows immediately from Propositions 2.4.15 and 2.4.18. ■

Clearly, when dealing with geodesics $G = \mathbb{P}_{\mathbb{K}}W$, where $W \leq V$ is degenerate, we cannot proceed as in Corollary 2.4.19. In this case, we need another special lifted field. Given $p \in$

[2] Here we are referring to geodesics in the usual way, as parameterized curves with null covariant derivative. This ambiguity is not a source of any problem due to Corollary 2.4.16.

$\mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ and $t \in T_p\mathbb{P}_{\mathbb{K}}$, where T stands for the spread field from t, we define the smooth (lifted) field

$$\mathrm{Eu}(t)(x) := \frac{1}{2}\big(\pi[p]\pi'[x]t\big)_x - T(x) \tag{2.4.20}$$

for all non-isotropic $x \in \mathbb{P}_{\mathbb{K}}V$.

2.4.21. Proposition. Let $G = \mathbb{P}_{\mathbb{K}}W$ be a geodesic, where the real subspace $W \leq V$ is degenerate. Let $p \in G$ be a non-isotropic point and let $0 \neq t \in T_pG$ and $h \in T_p\mathbb{P}_{\mathbb{K}}V$. Then $\nabla_{\mathrm{Tn}(t)}\mathrm{Eu}(h)(x) = 0$ for all $x \in G \setminus \mathrm{S}V$.

Proof. We can assume W has signature $+0$ (the case of signature -0 is totally analogous). By Lemma 2.4.7 we know that we can choose representatives such that $W = \mathbb{R}p + \mathbb{R}t(p)$. Choosing a representative such that $\langle p, p\rangle = 1$, by Sylvester's law of inertia (see Appendix A, Theorem A.1.6) it's immediate that $\big\langle t(p), t(p)\big\rangle = 0$. It suffices to show that $\nabla_T\mathrm{Eu}(h)(x) = 0$, where T is the spread field from t and $x \in G \setminus \mathrm{S}V$. Then

$$\nabla_T\mathrm{Eu}(h)(x) = \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathrm{Eu}(h)(x+\varepsilon t_x x)\right)_x =$$

$$= \left(\frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[x+\varepsilon t_x x]\pi[p]\pi'[x+\varepsilon t_x x]h\pi'[x+\varepsilon t_x x] + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[x+\varepsilon t_x x]h\pi'[x+\varepsilon t_x x]\right)_x.$$

By Lemma 2.3.5,

$$\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[x+\varepsilon t_x x]\pi[p]\pi'[x+\varepsilon t_x x]h\pi'[x+\varepsilon t_x x]\right)_x =$$

$$= \Big(-\big(t_x + (t_x)^*\big)\pi[p]\pi'[x]h\pi'[x] + \pi[x]\pi[p]\big(t_x + t_x^*\big)h\pi'[x] + \pi[x]\pi[p]\pi'[x]h\big(t_x + (t_x)^*\big)\Big)_x =$$

$$= \Big(\pi[x]\pi[p]\big(t_x + (t_x)^*\big)h\pi'[x]\Big)_x$$

since $(t_x)^* = \frac{\langle -, t_x x\rangle}{\langle x, x\rangle}x$ which implies $t_x^* x = \pi[x]t_x^* = 0$. It follows from

$$\pi[x]\pi[p]\pi'[x]h\pi[x]t(x) = \pi[x]\pi[p]\pi'[x]h\left(tp - \frac{\langle tp, p\rangle}{\langle x, x\rangle}x - r\frac{\langle tp, tp\rangle}{\langle x, x\rangle}x\right) =$$

$$= \pi[x]\pi[p]\pi'[x]ht(p) = 0,$$

where $x = p + rt(p)$ for some $r \in \mathbb{R}$, and from

$$\pi[x]t\pi'[x]\pi[p]\pi'[x]h(x) = \pi[x]t\pi'[x]\left(\frac{\langle hx, x\rangle}{\langle x, x\rangle}x - \frac{\langle hx, x\rangle\langle x, p\rangle}{\langle x, x\rangle\langle p, p\rangle}p\right) =$$

$$= \pi[x]t\pi'[x]\pi[p]\pi'[x]h(x) = \pi[x]t\left(\frac{\langle hx, x\rangle}{\langle x, x\rangle}x - \frac{\langle hx, x\rangle}{\langle x, x\rangle}\mathrm{ta}(p, x)x\right) = 0$$

that

$$\Big(\nabla_T\mathrm{Eu}(h)(x)\Big)x = \frac{1}{2}\pi[x]\pi[p]\big(t_x + (t_x)^*\big)hx - \pi[x]h\pi[x]tx - \pi[x]t\pi'[x]hx$$

$$= \frac{1}{2}\pi[x]\pi[p]\left(\frac{\langle hx, t_x x\rangle}{\langle x, x\rangle}x + \frac{\langle hx, x\rangle}{\langle x, x\rangle}t_x x\right) - \pi[x]h\left(tx - \frac{\langle tx, x\rangle}{\langle x, x\rangle}x\right) - \pi[x]t\left(\frac{\langle hp, x\rangle}{\langle x, x\rangle}x\right) =$$

$$= \pi[x]\left(r\frac{\langle hp, tp\rangle}{\langle x, x\rangle}tp - r^3\frac{\langle hp, tp\rangle\langle tp, tp\rangle}{\langle x, x\rangle^2}tp + \frac{\langle tx, x\rangle}{\langle x, x\rangle}hx - \frac{\langle tx, x\rangle\langle hx, x\rangle}{\langle x, x\rangle^2}x - \frac{\langle hp, x\rangle}{\langle x, x\rangle}tx\right) = 0.$$

∎

2.5 Curvature Tensor

We begin this section by expressing the curvature tensor in terms of the hermitian form. Then we will proceed to the calculation of the sectional curvature and finish showing that in $\mathbb{P}^n_{\mathbb{R}}$ and in $\mathbb{P}^1_{\mathbb{C}}$ the sectional curvature is constant in every connected component.

Let $p \in \mathbb{P}_{\mathbb{K}}V$ be a non-isotropic point and S, T and U be lifted smooth vector fields defined in a neighborhood of p. To express the curvature tensor $R(S,T)U := \nabla_S \nabla_T U - \nabla_T \nabla_S U - \nabla_{[S,T]} U$ we recall that, due to the fact that tensor fields are linear over smooth functions, the curvature tensor at a point only depends on the value of the fields at the point. Therefore, we can assume that S, T and U are spread fields respectively from s, t and v, where $s, t, v \in T_p \mathbb{P}_{\mathbb{K}}V$. As we have shown in the proof of Proposition 2.3.7, $[S,T](p) = 0$, and therefore $\nabla_{[S,T]}U(p) = 0$. From Lemma 2.3.6, we have

$$\nabla_S \nabla_T U(p) = \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \big(v\pi[p+\varepsilon sp]t - t\pi'[p+\varepsilon sp]v\big)_{p+\varepsilon sp} \right)_p .$$

So, by Lemma 2.3.5, we can write

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi[p+\varepsilon sp]v\pi[p+\varepsilon sp]t\pi'[p+\varepsilon sp] =$$

$$= -(s+s^*)v\pi[p]t\pi'[p] + \pi[p](-vs^*t - vst)\pi'[p] + \pi[p]v\pi[p]t(s+s^*)$$

and

$$-\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi[p+\varepsilon sp]t\pi'[p+\varepsilon sp]v\pi'[p+\varepsilon sp] =$$

$$= (s+s^*)t\pi'[p]v\pi'[p] + \pi[p](-ts^*v - tsv)\pi'[p] - \pi[p]t\pi'[p]v(s+s^*).$$

Now, from the fact that $\pi'[p]v = st = sv = vt = 0$ and $\pi[p]t = t$ we end up with

$$\nabla_S \nabla_T U(p) = -vs^*t - ts^*v$$

Similarly,

$$\nabla_T \nabla_S U(p) = -vt^*s - st^*v,$$

which implies

$$R(S,T)U = vt^*s + st^*v - vs^*t - ts^*v. \tag{2.5.1}$$

■

Let $p \in \mathbb{P}_{\mathbb{K}}V \setminus \mathrm{S}V$ and $W \leq T_p\mathbb{P}_{\mathbb{K}}V$ be a 2-dimensional real linear subspace such that $\langle -, - \rangle_p\big|_W$ is nondegenerate. Given $t, s \in W$ two $\mathbb{R}$-linearly independent tangent vectors, we define the sectional curvature of W by

$$KW := K(s,t) := \frac{\big\langle R(s,t)t, s \big\rangle_p}{\langle s,s\rangle_p \langle t,t\rangle_p - \langle t,s\rangle_p^2} \tag{2.5.2}$$

We can express the tangent vectors s,t as $s = \langle -, p\rangle v$ and $t = \langle -, p\rangle w$ for some $v, w \in p^{\perp}$. Since the formula (2.5.2) doesn't depend on the choice of representatives v and w, we can assume $\langle v, v\rangle = \sigma$ and $\langle w, w\rangle = \delta$, where $\delta, \sigma \in \{-1, 0, 1\}$. It is easy to see that $s^* = \langle -, v\rangle p$ and $t^* = \langle -, w\rangle p$. Let $k := \langle v, w\rangle$. From equations (2.2.1) and (2.5.1), we have

$$\langle R(s,t)t, s\rangle_p = \pm\frac{1}{\langle p,p\rangle}\mathrm{Re}\Big(\big\langle tt^*s(p), s(p)\big\rangle + \big\langle st^*t(p), s(p)\big\rangle - 2\big\langle ts^*t(p), s(p)\big\rangle\Big) =$$

$$= \pm\langle p,p\rangle^2\,\mathrm{Re}\big(\langle v,w\rangle\langle w,v\rangle + \langle v,v\rangle\langle w,w\rangle - 2\langle w,v\rangle\langle w,v\rangle\big) =$$

$$\pm\langle p,p\rangle^2\big(|k|^2 + \sigma\delta - 2\mathrm{Re}(k^2)\big).$$

Hence,

$$KW = \pm\frac{|k|^2 + \sigma\delta - 2\mathrm{Re}(k^2)}{\sigma\delta - (\mathrm{Re}k)^2} = \pm\left(1 + 3\frac{(\mathrm{Re}k)^2 - \mathrm{Re}(k^2)}{\sigma\delta - (\mathrm{Re}k)^2}\right) =$$

$$= \pm\left(1 + \frac{3}{4}\frac{|k - \bar{k}|^2}{\sigma\delta - (Rek)^2}\right). \tag{2.5.3}$$

2.5.4. Corollary. In each connected component of $\mathbb{P}^n_{\mathbb{R}}$, for arbitrary $n \in \mathbb{N}$, and in $\mathbb{P}^1_{\mathbb{C}}$, the sectional curvature is constant.

Proof. Clearly, when $\mathbb{K} = \mathbb{R}$, the sectional curvature satisfies $KW = \pm 1$. When $\mathbb{K} = \mathbb{C}$, and $v, w \in W$ are $\mathbb{C}$-linearly dependent,

$$|k|^2 - \sigma\delta = \langle v,w\rangle\langle w,v\rangle - \langle v,v\rangle\langle w,w\rangle = 0.$$

Since, for $p \in \mathbb{P}^1_{\mathbb{C}}$, we have $\dim_{\mathbb{C}} p^{\perp} = 1$, then $v, w \in W$ are always $\mathbb{C}$-linearly dependent. Therefore, we end up with

$$KW = \pm\left(1 + 3\frac{(\mathrm{Im}k)^2}{|k|^2 - (\mathrm{Re}k)^2}\right) = \pm 4.$$

■

2.6 The Real Hyperbolic Space

In this work we are particularly interested in the case where the vector space V is real, of dimension $n+1$, and it's endowed with a bilinear form $\langle -, -\rangle$ of signature $- + \ldots +$. If we choose $n = 3$, the vector space V is the well known Minkowski spacetime of special relativity.[3]

We will take the $-$ sign in equation (2.2.1). For all $p \in \mathrm{B}V$, the subspace $p^{\perp}$ has signature $+ \ldots +$ and therefore the metric in $\mathrm{B}V$ is Riemannian. Let $B = \{b_j | 1 \leq j \leq n\}$ be an orthonormal basis for V, with $\langle b_1, b_1\rangle = -1$ and $\langle b_j, b_j\rangle = 1$ for $j \neq 1$. Given $p \in \mathrm{B}V$, it's clear that, for any

3 More precisely, the Minkowski spacetime is an affine space with V as its underlying vector space (see (GOURGOULHON, 2013)).

representative of p, its component in the direction of b_1 cannot be null, so there is a unique representative of the form $p = b_1 + \sum_j x_j b_j$, for some $x_j \in \mathbb{R}$. From $\langle p, p\rangle < 0$, we have $\sum_j x_j^2 < 1$ and, therefore, $\mathrm{B}V$ is diffeomorphic to the open ball $\mathbb{B}^n := \{v \in \mathbb{R}^n : |v| < 1\}$. We will call $\mathrm{B}V$ the real hyperbolic space and denote it by $\mathbb{H}^n_{\mathbb{R}}$. Similarly, given $p \in \mathrm{S}V$, we can choose a representative of the form $p = b_1 + \sum_j x_j b_j$, and $\langle p, p\rangle = 0$ implies the relation $\sum_j x_j^2 = 1$, *i.e.*, $\mathrm{S}V$ is diffeomorphic to the sphere $\mathbb{S}^{n-1}$ and it's the boundary of $\mathbb{H}^n_{\mathbb{R}}$.

For all $p \in \mathrm{E}V$, the subspace $p^{\perp}$ has signature $- + \ldots +$ and therefore the metric in $\mathrm{E}V$ is Lorentzian. We will call $\mathrm{E}V$ the de Sitter space. In the case where $n = 2$, the manifold $\mathrm{E}V$ is diffeomorphic to an open möbius strip, since it's the complement of a closed disk in $\mathbb{P}^2_{\mathbb{R}}$.

In order to give consistency to the name we gave to $\mathrm{B}V$ we shall show that this space is isometric to the well-known hyperboloid model for hyperbolic geometry. So let $w \in V$ be a negative vector. The space

$$H := \{v \in V \,|\, \langle v, v\rangle = -1, \langle v, w\rangle < 0\} \tag{2.6.1}$$

is the hyperboloid of one sheet. We provide the tangent space to a point $p \in H$, $T_pH = p^{\perp}$ with the Riemannian metric $\langle\!\langle -, - \rangle\!\rangle_p : T_pH \times T_pH \to \mathbb{R}$ given by:

$$\langle\!\langle v, w \rangle\!\rangle_p := \langle v, w\rangle \tag{2.6.2}$$

for all $v, w \in p^{\perp}$.

2.6.3. Proposition. The map $I : H \to \mathbb{H}^n_{\mathbb{R}}$, $v \mapsto \pi(v)$ is an isometry.

Proof. Given $f \in C^{\infty}(\mathbb{P}^n_{\mathbb{R}})$, $p \in H$ and $v \in T_pH$, we can write

$$d_pI(v)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(I(p+\varepsilon v)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (f \circ \pi)\big(p + \varepsilon v\big) = \varphi f$$

where $\varphi = -\langle -, p\rangle v \in T_p\mathbb{H}^n_{\mathbb{R}}$. Now, for $v, w \in p^{\perp}$, we have

$$\langle d_pI(v), d_pI(w)\rangle_{I(p)} = \big\langle -\langle -, p\rangle v, -\langle -, p\rangle w\big\rangle_{I(p)} = -\langle p, p\rangle\langle v, w\rangle = \langle\!\langle v, w\rangle\!\rangle_p$$

■

If we take $n = 2$, in coordinates $\mathbb{H}^2_{\mathbb{R}}$ is identified with the unitary disk in $\mathbb{R}^2$, as we discussed above. The induced metric in this disk is the famous Beltrami-Klein disk metric for hyperbolic geometry. Indeed, take the chart $\phi : \mathbb{B}^2 \to \mathbb{H}^2_{\mathbb{R}}$, $(x, y) \mapsto p = [1, x, y]$. Given $f \in C^{\infty}(\mathbb{H}^2_{\mathbb{R}})$, $(a, b) = v \in T_{(x,y)}\mathbb{B}^2 \simeq \mathbb{R}^2$, and $(0, a, b) = \tilde{v} \in V$, we have

$$\begin{aligned} d_{(x,y)}\phi(v)f &= \frac{d}{dt}\Big|_{t=0} (f \circ \phi)(x + ta, y + tb) = \frac{d}{dt}\Big|_{t=0} \tilde{f}(1, x + ta, y + tb) = \\ &= \frac{d}{dt}\Big|_{t=0} \tilde{f}(p + t\tilde{v}) = \frac{d}{dt}\Big|_{t=0} \tilde{f}\Big(p + t\frac{\langle \tilde{v}, p\rangle}{\langle p, p\rangle} p + t\pi[p]\tilde{v}\Big) = \\ &= \frac{d}{dt}\Big|_{t=0} \tilde{f}\Big(p + \frac{t}{1 + t\langle \tilde{v}, p\rangle / \langle p, p\rangle} \pi[p]\tilde{v}\Big) = \frac{d}{dt}\Big|_{t=0} \tilde{f}\big(p + t\pi[p]\tilde{v}\big) \end{aligned}$$

where $\tilde{f} = f \circ \pi$. Therefore, the differential $d\phi_{(x,y)}(v)$ is given by the linear map $\mathbb{R}p \to p^{\perp}$, $p \mapsto \pi[p]\tilde{v}$, and the pullback of the metric $g_{(x,y)}(v,w) := \left\langle d\phi_{(x,y)}(v), d\phi_{(x,y)}(w)\right\rangle_p$, for all $v,w \in T_{(x,y)}\mathbb{B}^2$, by

$$g_{(x,y)}(v,w) = -\frac{\langle \pi[p]\tilde{v}, \pi[p]\tilde{w}\rangle}{\langle p,p\rangle} = \frac{1}{\langle p,p\rangle}\left(\frac{\langle \tilde{v},p\rangle\langle p,\tilde{w}\rangle}{\langle p,p\rangle} - \langle \tilde{v},\tilde{w}\rangle\right).$$

Hence, one can readily see that, expressing the above formula in coordinates, we arrive at the well known Beltrami-Klein metric:

$$g = \frac{dx^2+dy^2}{x^2+y^2-1} + \frac{x^2dx^2+2dxdy+y^2dy^2}{(x^2+y^2-1)^2}.$$

2.7 The Complex Hyperbolic Disk

Take a $\mathbb{C}$-vector space V of dimension 2, endowed with a nondegenerate hermitian form of signature $-+$. Both $\mathrm{B}V$ and $\mathrm{E}V$ are diffeomorphic, respectively by the maps in homogeneous coordinates $[1,z] \mapsto z$ and $[z,1] \mapsto z$, to the unitary complex disk $\mathbb{D} := \{z \in \ : |z| < 1\}$. Moreover, choosing the minus sign in the definition of the metric (2.2.1), the spaces $\mathrm{B}V$ and $\mathrm{E}V$ are in fact isometric. We will call $\mathrm{B}V$, provided with such metric, the complex hyperbolic disk and denote it by $\mathbb{H}^1_{\mathbb{C}}$.

2.7.1. Proposition. The map $\phi : \mathbb{H}^1_{\mathbb{C}} \to \mathrm{E}V$, $p \mapsto p^{\perp}$, is an isometry (due to dimensional reasons, we abuse notation and write $p^{\perp}$ instead of $\mathbb{P}_{\mathbb{C}}p^{\perp}$).

Proof. The map ϕ is well defined because of the signature of the hermitian form. Given $\varphi \in T_p\mathbb{H}^1_{\mathbb{C}}$, let $\tilde{\phi} : \pi^{-1}(\mathbb{H}^1_{\mathbb{C}}) \to \pi^{-1}(\mathrm{E}V)$ be defined by $\tilde{\phi}\big(p+t\varphi(p)\big) = p' - t\varphi^*(p')$ for small t, where φ^* stands for the adjoint of φ and p', for a representative of $p^{\perp}$. Note that

$$\big\langle p+t\varphi(p), p^{\perp} - t\varphi^*(p)\big\rangle = t\big\langle \varphi(p), p^{\perp}\big\rangle - t\big\langle p, \varphi^*(p^{\perp})\big\rangle = 0.$$

Therefore, $\pi \circ \tilde{\phi} = \phi \circ \pi$. Given $f \in C^{\infty}(\mathrm{E}V)$, the differential of ϕ at p is given by

$$d\phi_p(\varphi)f = \varphi(f \circ \phi) = \frac{d}{dt}\Big|_{t=0}(f \circ \phi \circ \pi)\big(p+t\varphi(p)\big) =$$

$$= \frac{d}{dt}\Big|_{t=0}(f \circ \pi \circ \tilde{\phi})\big(p+t\varphi(p)\big) = \frac{d}{dt}\Big|_{t=0}(f \circ \pi)\big(p' - t\varphi^*(p')\big).$$

In other words, $d\phi_p(\varphi) = -\varphi^*$. Writing $\varphi = \langle -,p\rangle v$, with $\langle p,v\rangle = 0$, we have $\varphi^* = \langle -,v\rangle p$ by Lemma 2.3.5. Hence $\langle \varphi,\varphi\rangle = -\langle p,p\rangle\langle v,v\rangle = \langle -\varphi^*, -\varphi^*\rangle$. ■

The metric in $\mathbb{D}$ induced by the metric in $\mathbb{H}^1_{\mathbb{C}}$ is the famous Poincaré disk metric (rescaled by a real factor). To show this, we shall proceed similarly as we did in the last section.

Consider the map in homogeneous coordinates $\psi : \mathbb{D} \to \mathbb{H}^1_{\mathbb{C}}$, $z \mapsto [1,z]$. Given $f \in C^{\infty}(\mathbb{H}^1_{\mathbb{C}})$, and $v \in T_z\mathbb{D} \simeq \mathbb{C}$,

$$d\psi_z(v)f = \frac{d}{dt}\Big|_{t=0}(f \circ \psi)(z+tv) = \frac{d}{dt}\Big|_{t=0}\tilde{f}\big((1,z)+t(0,v)\big) =$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{f}\Big((1,z) + t\frac{\langle (0,v),(1,z)\rangle}{\langle (1,z),(1,z)\rangle}(1,z) + t\pi[(1,z)](0,v)\Big) =$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{f}\left((1,z) + \frac{t\pi[(1,z)](0,v)}{1+t\langle (0,v),(1,z)\rangle/\langle (1,z),(1,z)\rangle}\right) =$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{f}\big((1,z) + t\pi[(1,z)](0,v)\big)$$

where $\tilde{f} = f\circ\pi$. Hence, the differential $d\psi_z(v)$ is the linear map $(1,z)\mapsto \pi[(1,z)](0,v)$, and the pullback metric $g_z(v,w) := \langle d\psi_z(v), d\psi_z(w)\rangle_z$, for all $v,w\in T_z\mathbb{D}$, is given by

$$g_z(v,w) = -\mathrm{Re}\frac{\langle \pi[(1,z)](0,v), \pi[1,z](0,w)\rangle}{\langle (1,z),(1,z)\rangle} =$$

$$= -\frac{1}{(1-|z|^2)^2}\mathrm{Re}\frac{\langle (\bar{z}v,v),(\bar{z}w,w)\rangle}{\langle (1,z),(1,z)\rangle} = -\frac{\mathrm{Re}(v\overline{w})}{(1-|z|^2)^2} =$$

$$= \frac{\mathrm{Re}(v)\mathrm{Re}(w) + \mathrm{Im}(v)\mathrm{Im}(w)}{(1-|z|^2)^2}$$

Therefore, from the above formula, it is easy to see that in coordinates,

$$g = \frac{dz d\bar{z}}{(1-|z|^2)^2} \tag{2.7.2}$$

2.7.3. Remark. The usual metric h for the Poincaré disk $\mathbb{D}$ differs from equation $(2.7.2)$ by a factor 4, being expressed in coordinates by

(2.7.4). $$h = \frac{4dzd\bar{z}}{(1-|z|^2)^2}$$

In order to obtain the Poincaré metric (2.7.4), one would have to define the metric

$$\langle \varphi_1, \varphi_2\rangle_p := -4\mathrm{Re}\,\frac{\langle \varphi_1(p), \varphi_2(p)\rangle}{\langle p,p\rangle}$$

for all $p\in\mathbb{H}^1_{\mathbb{C}}$, and $\varphi_1,\varphi_2\in T_p\mathbb{H}^1_{\mathbb{C}}$, in place of (2.2.1). This way we would have constant sectional curvature equal to -1 instead of -4 (see Corollary 2.5.4).

CHAPTER

3

GYROVECTOR SPACES AND SPECIAL RELATIVITY

Unlike in Newtonian mechanics, in special relativity the addition of velocities is neither associative nor commutative; in fact, it satisfies a "weak associativity and commutativity" condition. This motivated A. A. Ungar in (UNGAR, 2008) to develop the concepts of gyrovector spaces and gyrogroups, and here lies one link between relativity and hyperbolic geometry.

It turns out that gyrovector spaces provide a framework for the study of hyperbolic geometry. In some sense, they play the same role in hyperbolic geometry as vector spaces do in Euclidean geometry. So, one can notice that this whole section is about one of these examples of mathematical structures that were discovered inspired by physics problems.

We shall begin by defining gyrogroups, since gyrovector spaces are gyrogroups with more structure.

3.1 Gyrogroups

3.1.1. Definition (Gyrogroups). A gyrogroup is a nonempty set G with a binary operation $\oplus : G \times G \to G$ satisfying:

$G1$) There exists an element $e \in G$ such that $e \oplus a = a$, $\forall a \in G$. (left identity)

$G2$) For every $a \in G$ exists an element $\ominus a \in G$ such that $\ominus a \oplus a = e$. (left inverse)

$G3$) Given $a, b, c \in G$, there exists a unique element $\{a,b\}c \in G$ such that:

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \{a,b\}c \qquad \text{(Gyroassociative law)}$$

$G4$) The map $\{a,b\} : G \to G$, $c \mapsto \{a,b\}c$ is a gyrogroup automorphism, which we denote by $\{a,b\} \in \mathrm{Aut}(G, \oplus)$; this means that

$$\{a,b\}(c \oplus d) = \{a,b\}c \oplus \{a,b\}d$$

for all $c,d \in G$.

$G5$) The relation $\{a,b\} = \{a \oplus b, b\}$ holds for all $a,b \in G$.

3.1.2. Definition (Gyrocommutative gyrogroups). A gyrogroup $(G,\oplus)$ is gyrocommutative if the automorphism of "gyroassociativity" is also involved in a commutation rule:

$$a \oplus b = \{a,b\}(b \oplus a)$$

for all $a,b \in G$.

Just as with ordinary groups, we will use the notation $a \ominus b := a \oplus (\ominus a)$.

Exploring the details of the gyrogroup structure is outside the scope of this dissertation, so we will show only some basic algebraic properties that may be useful for our purposes. For more details concerning the study of gyrogroups see (UNGAR, 2008).

3.1.3. Proposition Let $(G,\oplus)$ be a gyrogroup. For elements $a,b,c,e \in G$, being e a left identity and $\ominus a$ the left inverse of a with respect to the identity e, then:

1) $a \oplus b = a \oplus c \Rightarrow b = c$

2) $\{e,a\} = Id$

3) $\{a,a\} = Id$

4) $\{\ominus a, a\} = Id$

5) There exists $e' \in G$ a left identity that is also a right identity.

6) There is only one left identity.

7) Every left inverse is a right inverse.

8) The left inverse of an element is unique.

Proof. 1) We know that $\ominus a \oplus (a \oplus b) = \ominus a \oplus (a \oplus c)$. Then, by gyroassociativity,

$$(\ominus a \oplus a) \oplus \{\ominus a, a\}b = (\ominus a \oplus a) \oplus \{\ominus a, a\}c.$$

So, $\{\ominus a, a\}b = \{\ominus a, a\}c$. Since $\{\ominus a, a\}$ is bijective, $a = c$.

2) From gyroassociativity,

$$e \oplus (a \oplus c) = (e \oplus a) \oplus \{e,a\}c$$
$$\Rightarrow a \oplus c = a \oplus \{e,a\}c.$$

Now, from 1), we get $\{e,a\}c = c$.

3) and 4) are immediate from 2) and from the property G5.

5) Let $a' \in G$ be the left inverse of a with respect to e'. From gyroassociativity, 4) and 1) we can write

$$a' \oplus (a \oplus e') = (a' \oplus a) \oplus \{a', a\}e' = e' = a' \oplus a$$
$$\Rightarrow a \oplus e' = a.$$

6) From 5), we have $e = e \oplus e' = e'$.

7) From gyroassociativity, 4) and 1) we have:

$$\ominus a \oplus (a \ominus a) = (\ominus a \oplus a) \oplus \{\ominus a, a\}(\ominus a) = e \ominus a = \ominus a \oplus e$$

$$\Rightarrow a \ominus a = e.$$

8) From 1) and 7), we have $a \oplus a' = a \ominus a \Rightarrow a' = \ominus a$. ■

There are two gyrogroups that are specially important to us. The first is the obvious one, the gyrogroup related to the sum of velocities in special relativity; the second is a gyrogroup inspired by Möbius transformations. The latter is important because it is isomorphic to the first, in the sense of its gyrogroup/gyrovector space structures, fact that is going to be particularly important to the task of finding a geometric construction of the relativistic velocity addition.

3.1.4. Definition (Möbius Gyrogroup). Inspired by Möbius transformations of the complex unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, we define the Möbius addition $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ by

$$a \oplus b := \frac{a+b}{1+\overline{a}b}$$

for all $a, b \in \mathbb{D}$.

Given $a, b \in \mathbb{D}$, it is clear that we can correct the non commutativity of $\oplus$ by the factor:

$$\{a,b\} := \frac{a \oplus b}{b \oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b}. \tag{3.2.2}$$

It's immediate that $\{a,b\} \in \mathrm{Aut}(\mathbb{D}, \oplus)$. Also notice that $|1+a\overline{b}|/|1+\overline{a}b| = 1$; hence, the non commutativity is corrected by a rotation around the origin of the disk. Now, given $a, b, c \in \mathbb{D}$, we can write

$$a \oplus (b \oplus c) = \left(a + \frac{b+c}{1+\overline{b}c}\right)\left(1 + \frac{\overline{a}(b+c)}{1+\overline{b}c}\right)^{-1} = \frac{a + a\overline{b}c + b + c}{1 + \overline{b}c + \overline{a}b + \overline{a}c}$$

$$(a \oplus b) \oplus \{a,b\}c = \left(\frac{a+b}{1+\overline{a}b} + \frac{1+a\overline{b}}{1+\overline{a}b}c\right)\left(1 + \frac{\overline{a}+\overline{b}}{1+\overline{a}b}c\right)^{-1} = \frac{a+b+c+a\overline{b}c}{1+\overline{a}b+\overline{a}c+\overline{b}c}.$$

Therefore, the automorphism also corrects the associativity. Finally,

$$\{a \oplus b, b\} = \left(1 + \frac{a+b}{1+\overline{a}b}\overline{b}\right)\left(1 + \frac{\overline{a}+\overline{b}}{1+a\overline{b}}b\right)^{-1} = \frac{1+a\overline{b}}{1+\overline{a}b} = \{a,b\}$$

So, $(\mathbb{D}, \oplus)$ satisfies the gyrogroup axioms.

Inspired now by special relativity, we provide the space of admissible velocities, *i.e.*, $\mathrm{B}_0(c) := \{v \in \mathbb{R}^3 : |v| < c\}$, where $c \in \mathbb{R}$ is the speed of light in the vacuum, with a sum that, in physics, means the relativistic addition of relative velocities. What we mean by that is, given an observer $\mathscr{O}$, another observer $\mathscr{O}'$ moving with velocity $v \in \mathrm{B}_0(c)$ relative to $\mathscr{O}$ and an object moving with velocity $w \in \mathrm{B}_0(c)$ relative to $\mathscr{O}'$, we are going to denote the velocity of the object as measured by $\mathscr{O}$ as $v \oplus w$.

3.1.5. Definition (Einstein Gyrogroup). We define Einstein's addition $\oplus : \mathrm{B}_0(c) \times \mathrm{B}_0(c) \to \mathrm{B}_0(c)$ by the rule:

$$v \oplus w := \frac{1}{1 + \frac{\langle v,w \rangle}{c^2}} \left(v + \frac{w}{\gamma_v} + \frac{1}{c^2} \frac{\gamma_v}{1+\gamma_v} \langle v,w \rangle v \right)$$

for all $v, w \in \mathrm{B}_0(c)$. Here, c is the speed of light in the vacuum and $\gamma_v := 1/\sqrt{1 - |v|^2/c^2}$ is called the Lorentz factor; it appears in most special relativity formulas.

According to Ungar in (UNGAR, 2008), it can be proved by computer algebra that this addition satisfies the gyrogroup axioms and the gyrocommutativity condition, therefore $(\mathrm{B}_0(c), \oplus)$ is a gyrocommutative gyrogroup.

Now, the next step is to provide our gyrogroups with a scalar multiplication and an inner product in order to set an algebraic framework to hyperbolic geometry in a similar way to what we do with usual inner product vector spaces and Euclidean geometry.

3.2 Gyrovector Spaces

3.2.1. Definition (Gyrovector Spaces). Let V be a real vector space provided with an inner product $\langle -,- \rangle$ and $G \subset V$ be a subset such that $(G, \oplus)$ is a gyrocommutative gyrogroup. If G is provided with a scalar multiplication $\otimes : \mathbb{R} \times G \to G$ satisfying, for all $v_1, v_2, u, w \in G$ and $r, k \in \mathbb{R}$:

$V0)$ $\big\langle \{u,w\}v_1, \{u,w\}v_2 \big\rangle = \langle v_1, v_2 \rangle$;

$V1)$ $1 \otimes u = u$;

$V2)$ $(r+k) \otimes u = r \otimes u \oplus k \otimes u$;

$V3)$ $(rk) \otimes u = r \otimes (k \otimes u)$;

$V4)$ $\dfrac{|r| \otimes u}{|r \otimes u|} = \dfrac{u}{|u|}, \quad u \neq 0, r \neq 0$;

$V5)$ $\{v_1, v_2\}(r \otimes u) = r \otimes \{v_1, v_2\}u$;

$V6)$ $\{r \otimes u, k \otimes u\} = Id$;

$VV)$ The set $|G| := \{\pm |u| : u \in G\}$ is provided with an addition $\oplus$ and a scalar multiplication $\otimes$ that turn $|G|$ into a vector space;

$V7)$ $|r \otimes u| = |r| \otimes |u|$;

$V8)$ $|u \oplus w| \leq |u| \oplus |w|$.

we will call $(G, \oplus, \otimes)$ a real inner product gyrovector space, or just gyrovector space.

It is easy to verify that $(-1) \otimes u = \ominus u$, $0 \otimes u = 0$ and $|\ominus u| = |u|$. Also, an important remark is that we are using the same notation for the gyrovector operations in G and for the vector ones in $|G|$, since there is not much chance of ambiguity.

In order to define a scalar multiplication for Einstein gyrogroup we will use the fact that, for a gyrovector space $(G, \oplus, \otimes)$, given $v \in G$ and $n \in \mathbb{N}$, the equality

$$n \otimes v = \underbrace{v \oplus \ldots \oplus v}_{n \text{ times}}$$

holds immediately from $V2)$ by induction.

Taking $(\mathrm{B}_0(c), \oplus)$ to be an Einstein gyrogroup, we are going to define scalar multiplication motivated by the equality above and, given $v \in \mathrm{B}_0(c)$ and $n \in \mathbb{N}$, show that:

$$n \otimes v := \underbrace{v \oplus \ldots \oplus v}_{n \text{ times}} = c \cdot \frac{(1+|v|/c)^n - (1-|v|/c)^n}{(1+|v|/c)^n + (1-|v|/c)^n} \frac{v}{|v|}. \tag{3.2.2}$$

To show the second equality in (3.2.2) we shall proceed by induction. For $n = 1$ it is immediate that the equality holds. Now assuming it holds for $n \in \mathbb{N}$, and using the reduced addition formula, valid for parallel velocities (see (RINDLER, 2006), section 3.6), we have a pretty straightforward calculation:

$$(n+1) \otimes v = n \otimes v \oplus v = \frac{n \otimes v + v}{1 + \frac{\langle n \otimes v, v \rangle}{c^2}} =$$

$$= \left(c \cdot \frac{(1+|v|/c)^n - (1-|v|/c)^n}{(1+|v|/c)^n + (1-|v|/c)^n} \frac{v}{|v|} + v \right) \left(1 + \frac{(1+|v|/c)^n - (1-|v|/c)^n}{(1+|v|/c)^n + (1-|v|/c)^n} \frac{|v|}{c} \right)^{-1} =$$

$$= c \cdot \frac{\left(1+\frac{|v|}{c}\right)^n - \left(1-\frac{|v|}{c}\right)^n + \frac{|v|}{c}\left(1+\frac{|v|}{c}\right)^n + \frac{|v|}{c}\left(1-\frac{|v|}{c}\right)^n}{\left(1+\frac{|v|}{c}\right)^n + \left(1-\frac{|v|}{c}\right)^n + \frac{|v|}{c}\left(1+\frac{|v|}{c}\right)^n - \frac{|v|}{c}\left(1-\frac{|v|}{c}\right)^n} \frac{v}{|v|} =$$

$$= c \cdot \frac{\left(1+\frac{|v|}{c}\right)^{n+1} - \left(1-\frac{|v|}{c}\right)^{n+1}}{\left(1+\frac{|v|}{c}\right)^{n+1} + \left(1+\frac{|v|}{c}\right)^{n+1}} \frac{v}{|v|}$$

■

3.2.3. Definition (Einstein Gyrovector Space). Let $(\mathrm{B}_0(c), \oplus)$ be Einstein's gyrogroup. Motivated by equation (3.2.2) we define the scalar multiplication $\otimes : \mathbb{R} \times \mathrm{B}_0(c) \to \mathrm{B}_0(c)$ by:

$$r \otimes v := c \cdot \frac{(1+|v|/c)^r - (1-|v|/c)^r}{(1+|v|/c)^r + (1-|v|/c)^r} \frac{v}{|v|} = c \tanh\left(r \tanh^{-1} \frac{|v|}{c} \right) \frac{v}{|v|}$$

for all $r \in \mathbb{R}$ and $v \in \mathrm{B}_0(c)$. The second equality is just a straightforward calculation. We also provide the set $|\mathrm{B}_0(c)| := \{\pm|v| : v \in \mathrm{B}_0(c)\}$ with the operations

$$|u| \oplus |v| := \frac{|u| + |v|}{1 + |u||v|/c^2}$$

$$r \otimes |v| := c \tanh\left(r \tanh^{-1} \frac{|v|}{c} \right)$$

for all $u, v \in \mathrm{B}_0(c)$ and $r \in \mathbb{R}$.

The step by step proof that the above operations satisfy all the axioms of a gyrovector space can be found in (UNGAR, 2008); it is just a matter of either performing direct calculations or using computer algebra.

3.2.4. Definition (Gyrovector Space Isomorphisms). Let $(G, \oplus, \otimes)$ and $(H, \oplus, \otimes)$ be two gyrovector spaces. A bijective map $\varphi : G \to H$ is a gyrovector space isomorphism if

1) $\varphi(u \oplus v) = \varphi(u) \oplus \varphi(v)$;

2) $\varphi(r \otimes v) = r \otimes \varphi(v)$;

3) $\dfrac{\langle \varphi(u), \varphi(v) \rangle}{|\varphi(u)||\varphi(v)|} = \dfrac{\langle u, v \rangle}{|u||v|}, \quad u \neq 0, v \neq 0,$

for all $u, v \in G$ and $r \in \mathbb{R}$.

3.2.5. Definition. Let $(V, +, \cdot)$ be a real vector space with inner product $\langle -, - \rangle$, and let $\mathrm{B}_0(s) := \{v \in V : |v| < s\}$, where $s > 0$ is a positive constant. We define the generalized Möbius addition $\hat{\oplus} : \mathrm{B}_0(s) \times \mathrm{B}_0(s) \to \mathrm{B}_0(s)$ by

$$u \hat{\oplus} v := \frac{\left(1 + 2\langle u, v \rangle / s^2 + |v|^2/s^2\right) u + \left(1 - |u|^2/s^2\right) v}{\left(1 + 2\langle u, v \rangle / s^2 + |u|^2 |v|^2 / s^4\right)}$$

for all $u, v \in \mathrm{B}_0(c)$. We define the generalized Einstein addition $\oplus : \mathrm{B}_0(c) \times \mathrm{B}_0(c) \to \mathrm{B}_0(c)$ using the same expression of Definition 3.1.5:

$$u \oplus v := \frac{1}{1 + \frac{\langle u, v \rangle}{s^2}} \left(u + \frac{v}{\gamma_u} + \frac{1}{s^2} \frac{\gamma_u}{1 + \gamma_u} \langle u, v \rangle u \right)$$

for all $u, v \in \mathrm{B}_0(s)$, where $\gamma_u := 1/\sqrt{1 - |u|^2/s^2}$ is the Lorentz factor; the scalar multiplication $\otimes : \mathbb{R} \times \mathrm{B}_0(s) \to \mathrm{B}_0(s)$ is also given by a familiar expression (from Definition 3.2.3):

$$r \otimes v := s \tanh\left(r \tanh^{-1} \frac{|v|}{s} \right) \frac{v}{|v|}$$

for all $v \in \mathrm{B}_0(s)$ and $r \in \mathbb{R}$.

Both $(\mathrm{B}_0(c), \oplus, \otimes)$ and $(\mathrm{B}_0(c), \hat{\oplus}, \otimes)$ are gyrovector spaces, as shown in (UNGAR, 2008). We will call the first the generalized Einstein gyrovector space and the latter the generalized Möbius gyrovector space.

When $V = \mathbb{C}$ and $s = 1$, the generalized Möbius addition is reduced to the known one, defined in 3.1.4. Given $u, v \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,we have

$$\frac{(u + v)(1 + u\bar{v})}{(1 + u\bar{v})(1 + \bar{u}v)} = \frac{u + u^2\bar{v} + |u|^2 v + v + u|v|^2 - |u|^2 v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} =$$

$$= \frac{\left(1 + 2\langle u, v \rangle + |v|^2\right) u + \left(1 - |u|^2\right) v}{\left(1 + 2\langle u, v \rangle + |u|^2 |v|^2\right)}.$$

According to (UNGAR, 2008), the operations $\oplus$ and $\hat{\oplus}$, defined above, are related by the expression

$$u\hat{\oplus}v = \frac{1}{2}\otimes(2\otimes u\oplus 2\otimes v) \tag{3.2.6}$$

for all $u,v\in \mathrm{B}_0(s)$. This relation gives us a hint on how to define an isomorphism between $(B_0(s),\hat{\oplus},\otimes)$ and $(B_0(s),\oplus,\otimes)$.

3.2.7. Proposition. The generalized Möbius gyrovector space and the generalized Einstein gyrovector space are isomorphic via the map $\varphi : (\mathrm{B}_0(s),\hat{\oplus},\otimes)\to(\mathrm{B}_0(s),\oplus,\otimes)$, $v\mapsto 2\otimes v$.

Proof. The map φ is immediately bijective. Given $u,v\in \mathrm{B}_0(s)$, we write

$$\varphi(u\hat{\oplus}v) = 2\otimes(u\hat{\oplus}v) = 2\otimes\left(\frac{1}{2}\otimes(2\otimes u\oplus 2\otimes v)\right) =$$

$$= 2\otimes u\oplus 2\otimes v = \varphi(u)\oplus\varphi(v).$$

Given $r\in\mathbb{R}$, it follows that

$$\varphi(r\otimes v) = 2\otimes(r\otimes v) = (2r)\otimes v = r\otimes(2\otimes v) = r\otimes\varphi(v).$$

Finally, if $u\neq 0$ and $v\neq 0$,

$$\frac{\langle\varphi(u),\varphi(v)\rangle}{|\varphi(u)||\varphi(v)|} = \frac{\langle 2\otimes u, 2\otimes v\rangle}{\sqrt{\langle 2\otimes u,2\otimes u\rangle}\cdot\sqrt{\langle 2\otimes v,2\otimes v\rangle}} =$$

$$= \frac{\left\langle s\tanh\left(2\tanh^{-1}(|u|/s)\right), s\tanh\left(2\tanh^{-1}(|v|/s)\right)\right\rangle}{s\tanh\left(2\tanh^{-1}(|u|/s)\right)|u|\cdot s\tanh(2\tanh^{-1}\left(|v|/s)\right)|v|} = \frac{\langle u,v\rangle}{|u||v|}.$$

■

Our next aim is to see how hyperbolic geometry takes place in a gyrovector space. In the process, the similarities with the usual relation between vector spaces and Euclidean geometry are going to become clearer.

3.3 Gyrometric and Gyrolines

3.3.1. Definition (Gyrometric). Let $(G,\oplus,\otimes)$ be a gyrovector space. We will call a gyrometric the function $d_\oplus : G\times G\to\mathbb{R}$ given by

$$d_\oplus(x,y) := |y\ominus x|$$

for all $x,y\in G$.

3.3.2. Proposition. Given a gyrovector space $(G,\oplus,\otimes)$, its gyrometric $d_\oplus : G\times G\to\mathbb{R}$ satisfies:

1) $d_\oplus(x,y)\geq 0$, and $d_\oplus(x,y)=0$ if and only if $x=y$;

2) $d_\oplus(x,y) = d_\oplus(y,x)$;

3) $d_\oplus(x,z) \leq d_\oplus(x,y) \oplus d_\oplus(y,z)$

for all $x,y,z \in G$.

Proof. The first item is immediate from the definition of norm in a vector space. To prove 2), we will use the identity:

$$\{x,y\}(\ominus x \ominus y) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus (\ominus y \ominus x))) = \ominus(x \oplus y) \oplus (x \oplus \{y, \ominus y\}(\ominus x)) =$$

$$= \ominus(x \oplus y) \tag{3.2.6}$$

From the identity (3.2.6), $V0$) in the Definition 3.2.1 and from gyrocommutativity,

$$|y \ominus x| = \big| \ominus (y \ominus x)\big| = \big|\{x,y\}(\ominus y \oplus x)\big| = |\ominus y \oplus x| = |x \ominus y|.$$

To show 3), we will use the identity $\ominus x \oplus z = (\ominus x \oplus y) \oplus \{\ominus x, y\}(\ominus y \oplus z)$ (see (UNGAR, 2008)) and the inequality $V8$ in 3.2.1:

$$|\ominus x \oplus z| = \big|(\ominus x \oplus y) \oplus \{\ominus x, y\}(\ominus y \oplus z)\big| \leq \big|(\ominus x \oplus y)\big| \oplus \big|\{\ominus x, y\}(\ominus y \oplus z)\big| =$$

$$= \big|(\ominus x \oplus y)\big| \oplus \big|(\ominus y \oplus z)\big|.$$

■

3.3.3. Definition (Gyroline). Let $x,y \in G$ be distinct points in a gyrovector space $(G, \oplus, \otimes)$. The gyroline in G that passes through x and y is the curve $\gamma : \mathbb{R} \to G$ defined by

$$\gamma(t) := x \oplus t \otimes (\ominus x \oplus y).$$

It turns out that, in full analogy with straight lines in vector spaces, in Einstein gyrovector space, these gyrolines are the geodesics of the Beltrami-Klein model for hyperbolic geometry, and in Möbius gyrovector space they are the Poincaré disk geodesics.

3.3.4. Proposition. Let $(\mathrm{B}_0(c), \oplus, \otimes)$ be the Einstein gyrovector space defined in 3.2.3, equipped with its gyrometric $d_\oplus : \mathrm{B}_0(c) \times \mathrm{B}_0(c) \to \mathbb{R}$ (see Definition 3.3.1). Let $d : \mathrm{B}_0(c) \times \mathrm{B}_0(c) \to \mathbb{R}$ be the usual hyperbolic distance in the Beltrami-Klein model in the ball of radius c and curvature $-1/c^2$. Then

$$d_\oplus(x,y) = c \tanh \frac{d(x,y)}{c}$$

for all $x,y \in \mathrm{B}_0(c)$.

Proof. In Remark 4.3.2.9 we show that $x \ominus y$ is obtained geometrically by reflecting x (in the sense of the hyperbolic metric) in the middle point between x and the origin 0, which implies that $d(x \ominus y, 0) = d(x,y)$. The gyrometric $d_\oplus(x,y) = |x \ominus y|$ gives the norm of the relative velocity between objects moving with velocities x and y as measured by a observer at rest with respect to the origin (this fact comes immediately from the identity $x \oplus (\ominus x \oplus y) = (x \ominus x) \oplus \{x, \ominus x\} y$, which essentially means that x sees y with velocity $\ominus x \oplus y$). The hyperbolic distance gives the norm of the relative scaled rapidity, as shown in equation (4.3.2.6), thus the desired result comes directly from the well-known relation between the scaled rapidity and the velocity. ■

The role of the gyrometric in the context of gyrovectors is similar to the role of the tance (see equation (2.4.1)) in the projective model for hyperbolic geometry presented in Chapter 2. Both are algebraic monotonic functions of the hyperbolic distance, so they can be used in place of the latter, which is harder to calculate.

3.3.5. Proposition. Let $\mathbb{H}^2_{\mathbb{R}}$ be the Einstein gyrovector space $(\mathrm{B}_0(1), \oplus, \otimes)$ for the open 2-ball of radius 1 (see Definition 3.2.5) provided with the Beltrami-Klein hyperbolic distance $d : \mathrm{B}_0(1) \times \mathrm{B}_0(1) \to \mathbb{R}$. Also, let $\mathbb{H}^1_{\mathbb{C}}$ be the Möbius gyrovector space $(\mathbb{D}, \hat{\oplus}, \otimes)$ defined in 3.2.5, equipped with the Poincaré disk metric $d' : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$. Any isometry $\varphi : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^1_{\mathbb{C}}$ that fixes the origin is a gyrovector space isomorphism.

Proof. Let $x, y \in \mathbb{H}^2_{\mathbb{R}}$, $I : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{R}}$ be the hyperbolic isometry that stabilizes the geodesic G passing through 0 and x and such that $I(0) = x$, and $I' : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^1_{\mathbb{C}}$ the hyperbolic isometry that stabilizes the geodesic G' connecting 0 and $\varphi(x)$ and such that $I'(0) = \varphi(x)$. By Theorem B.2, the proof reduces to show that $\varphi(I(y)) = I'(\varphi(y))$. Indeed, the isometry φ maps the hypercycle H of G that passes through y onto the hypercycle H' of G' that contains $\varphi(y)$, therefore $\varphi(I(y)), I'(\varphi(y)) \in H'$ and $d'\big(\varphi(I(y)), \varphi(x)\big) = d'\big(I'(\varphi(y)), \varphi(y)\big)$, which concludes the proof. ■

3.3.6. Corollary The relation

$$d'_{\hat{\oplus}}(x,y) = \left| x \hat{\ominus} y \right| = \tanh d'(x,y)$$

holds for all $x, y \in \mathbb{H}^1_{\mathbb{C}}$.

3.3.6. Proposition. Every gyroline $\gamma : \mathbb{R} \to \mathbb{H}^2_{\mathbb{R}}$, $t \mapsto x \oplus t \otimes v$, where $x, v \in \mathbb{H}^2_{\mathbb{R}}$, is a geodesic parameterized with constant velocity of norm $|v|_{\mathbb{H}}$, where $|.|_{\mathbb{H}}$ stands for the hyperbolic norm in each tangent space. Moreover, every geodesic can be written in this way for some $x, v \in \mathbb{H}^2_{\mathbb{R}}$.

Proof. Clearly the curve $\widetilde{\gamma} : t \mapsto t \otimes v = c \tanh\left(t \tanh^{-1} \frac{|v|}{c}\right) \frac{v}{|v|}$ is a geodesic, since it is a straight line passing through the origin. By the expression for the hyperbolic distance we have:

$$d(t \otimes v, 0) = c \operatorname{arctanh} \frac{|t \otimes v|}{c} = ct \tanh^{-1} \frac{|v|}{c} = t d(v, 0).$$

Therefore, $\widetilde{\gamma}$ is parameterized with constant velocity of norm $|v|_{\mathbb{H}}$. The fact that $\gamma = I \circ \widetilde{\gamma}$, where I is the hyperbolic isometry that preserves the geodesic containing 0 and x and maps 0 to x (see Theorem B.2), proves the first part of the proposition.

Conversely, given a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2_{\mathbb{R}}$, it's easy to see that it can be written in the form $\gamma(t) = \gamma(0) \oplus t \otimes dI_{\gamma(0)}(\dot{\gamma}(0))$, where I is the hyperbolic isometry that stabilizes the geodesic that passes through 0 and $\gamma(0)$ and sends $\gamma(0)$ to the origin. ■

Note that, when $\gamma(t) = x \oplus t \otimes (\ominus x \oplus y)$, $x, y \in \mathbb{H}^2_{\mathbb{R}}$, then $\gamma(1) = y$ and therefore γ is the geodesic connecting x and y.

CHAPTER

4

ON THE GEOMETRY OF THE KINEMATIC SPACE IN SPECIAL RELATIVITY

This chapter presents the arXiv preprint "On the geometry of the kinematic space in special relativity" which is submitted for publication. It contains the main results that we obtained during the Master's project and is therefore to be seen as the culmination of the project.

On the geometry of the kinematic space in special relativity

Rafael Ferreira João dos Reis Junior Carlos H. Grossi

Abstract

The classifying space of inertial reference frames in special relativity is naturally hyperbolic. There is a remarkable interplay between central elements of hyperbolic geometry and those of special relativity — which, to a certain extent, have already been observed in the past — that we present and further discuss in the paper. We aim at a geometrization of special relativity at the level of kinematic space by giving to physical concepts/phenomena purely geometric definitions/descriptions. In this way, the differences between special relativity and classical mechanics can be seen as a manifestation of the distinct geometric natures of their kinematic spaces.

1 Introduction

A major conceptual difference between Newtonian mechanics and special relativity is that the kinematic space $\mathcal{K}$ of the first is Euclidean[1] while that of the former is hyperbolic, a fact already observed by V. Varićak in 1910 [13] and E. Borel in 1913 [6], [5]. Here, *kinematic space* is to be understood as the classifying space of all inertial reference frames (see Subsection 2.1).

The hyperbolic nature of special relativity has been explored by several authors from distinct perspectives. Some are based on the role played by *rapidity,* introduced by Varićak and called *true velocity* by E. Borel. Rapidity appears naturally in the context of special relativity because it is simply the hyperbolic distance between inertial reference frames, that is, it is the hyperbolic distance in $\mathcal{K}$. Another hyperbolic view on special relativity involves the use of gyrovector spaces, introduced by A. Ungar (see, for instance, [12]), which constitute an algebraic framework for hyperbolic geometry that builds upon an axiomatization of the (noncommutative and nonassociative) relativistic velocity addition.

The path we take in this paper focuses on some simple geometric invariants related to finite configurations of points in kinematic space. (It comes mainly from [2], where a coordinate-free toolbox that suits several "classic" geometries — including, for instance, hyperbolic, spherical, Fubini-Study, de Sitter, and anti de Sitter geometries — is developed.) A first example of such a geometric invariant is the *tance* (see (2.1.5) for the definition) which is, in a certain sense, the simplest algebraic invariant of a pair of points in $\mathcal{K}$. The square root of the tance is a fundamental quantity in hyperbolic geometry because distance is a monotonic function of it. Curiously, when translated into the context of special relativity, the square root of the tance between two inertial observers in $\mathcal{K}$ is simply the Lorentz factor related to the observers (see Remark 3.1.1). Keeping up with this idea of translating into special relativity some natural concepts and geometric invariants in hyperbolic geometry, we obtain the following:

- The relative velocity between inertial observers $\boldsymbol{p},\boldsymbol{q}\in\mathcal{K}$ appears as a natural algebraic expression for the tangent vector to the geodesic segment joining $\boldsymbol{p},\boldsymbol{q}$ (see Definition 3.2.4);
- Rapidity and the closely related concept of scaled rapidity are shown to have distinct geometric origins; while rapidity measures the hyperbolic distance between inertial reference frames, scaled rapidity measures the hyperbolic distance between relative velocities (see Section 3.2);
- Parallel transport gives rise to the relativistic velocity addition in a straightforward generalization of the classical velocity addition (see Definition 3.2.1);
- Hypercycles (that is, curves equidistant from a geodesic in $\mathcal{K}$) allow one to write a "parallelogram law" for the relativistic velocity addition (see the end of Subsection 3.2);
- The general relativistic Doppler effect can be described by a natural expression involving the Busemann function related to a photon or, equivalently, to a point in the ideal boundary of $\mathcal{K}$ (see Proposition 3.3.2); moreover, horospheres appear as level surfaces of energy/frequency (see Corollary 3.3.3). There is a striking resemblance between such geometric form of the relativistic Doppler effect and the study of probability measures in the context of Patterson-Sullivan theory (see [10, Section 1.2 and Proposition 3.9] for the Patterson-Sullivan perspective);

[1] It would be more accurate to say that it is just a vector space (with no distinguished metric), see [8].

- A basic algebraic invariant involving two inertial observers in $\mathcal{K}$ and a pair of space-like separated events determines whether the observers agree or disagree on the order of occurrence of the events (see Subsection 3.5);
- Curves in $\mathcal{K}$ can be seen as describing the inertial reference frames occupied by an observer at each instant of its proper time and a tangent vector to such a curve gives the instantaneous 4-acceleration of the observer. Hence, dynamics can also be modelled at the level of the kinematic space (see Subsection 3.6).

We arrive at what seems to be an effective geometrization of special relativity: physical concepts and phenomena (like the Lorentz factor, velocity, velocity addition, the Doppler effect, among others) gain a purely geometric description which does not depend on their actual definitions in physics. Moreover, the techniques that are used in the paper directly extended to Grassmannians [3], [1] and this allows one to deal in a similar fashion with special relativity in other Einstein geometries like anti de Sitter and de Sitter spacetimes.

It is worthwhile mentioning that, in our construction, kinematic space is naturally compactified by the de Sitter space as they are are glued along their common ideal boundaries. The interplay between these geometries, which are linked by the geometry of Minkowski space, is very rich. For instance, in the case of 4-dimensional Minkowski space, there is a duality between points in the de Sitter component (which correspond to the sometimes called tachyonic inertial reference frames) and circles in the ideal boundary (which correspond to families of photons whose velocities, as measured by certain inertial observers, are all coplanar), see Remark 2.1.4.

In spite of the emphasis we give on the geometric point of view, the synthetic and coordinate-free methods that we use provide simple explicit formulae for all the involved concepts (say, geodesics, parallel transport, Riemannian connection, curvature tensor, among others [2]). These methods are essentially "linear" and they are also applicable to several other geometries which are common in physics; in this regard, see Subsection 2.2 and Example 2.2.3.

Finally, developing a similar approach to classical mechanics requires one to take as spacetime a vector space equipped with a *degenerate* symmetric bilinear form (of signature $0++\cdots+$) in place of Minkowski space [8]. In a certain sense, special relativity and classical mechanics arise from their kinematic spaces in the same way; however, being very different from each other, the geometric natures of such kinematic spaces give rise to completely distinct phenomenologies.

2 Preliminaries

2.1 Kinematic space

Let $\mathbb{M}^{n+1}$ be Minkowski $(n+1)$-space, that is, an $\mathbb{R}$-vector space equipped with a symmetric bilinear form $\langle -,-\rangle : \mathbb{M}^{n+1} \to \mathbb{R}$ of signature $-+\cdots+$. As usual, the *light cone* consists of the *lightlike* vectors $v \in \mathbb{M}^{n+1}$ which satisfy $v \neq 0$ and $\langle v, v\rangle = 0$. Minkowski space is divided by the light cone into *timelike* and *spacelike* vectors, respectively characterized by $\langle v, v\rangle < 0$ and $\langle v, v\rangle > 0$. We also assume that one of the light cone sheets is chosen as the future light cone.

The 1-dimensional subspace $\mathbb{R}v \subset \mathbb{M}^{n+1}$, where v is a timelike vector, can be seen as the worldline of an inertial reference frame. The space of all such worldlines consists of an open subspace of the real projective space $\mathbb{P}^n_{\mathbb{R}}$ and, topologically, this subspace is an open n-ball called the (open) *kinematic space* $\mathcal{K}$. The boundary $\partial\mathcal{K}$ of $\mathcal{K}$ is an $(n-1)$-sphere consisting of the projectivization of the light cone; in other words, each point in $\partial\mathcal{K}$, an *isotropic* point, represents the worldline of a photon. We call $\overline{\mathcal{K}} := \mathcal{K} \cup \partial\mathcal{K}$ the *closed* kinematic space and the entire projective space, the *extended* kinematic space. Moreover, we denote by $\mathcal{G}$ the complement $\mathbb{P}^n_{\mathbb{R}} \setminus \overline{\mathcal{K}}$.

A point in projective space will be denoted by a bold letter and a representative of this point in Minkowski space, by the same roman letter; so, $\boldsymbol{p} \in \mathbb{P}^n_{\mathbb{R}}$ stands for the equivalence class $\mathbb{R}p$ of a point $p \in \mathbb{M}^{n+1}$. Strictly speaking, the points in kinematic space represent the worldlines of inertial observers that synchronised their clocks at a same point in spacetime (the vertex of the lightcone which corresponds to coordinate time $t = 0$ for every inertial observer). By choosing a representative $p \in \mathbb{M}^{n+1}$ of a point $\boldsymbol{p} \in \mathcal{K}$, we therefore pick a specific coordinate time $t = \pm|p|/c$ in the frame of the corresponding inertial observer (c denotes the speed of light in vacuum). However, we will typically abuse nomenclature and refer to a point in $\mathcal{K}$ simply as an inertial observer (or inertial reference frame).

2.1.1. Remark. When dealing with 3 inertial reference frames or, equivalently, with three points in $\mathcal{K}$ (a configuration that will be considered several times in the paper), we can assume that $n = 2$ because the vector space generated by these frames (equipped with the induced form) is precisely $\mathbb{M}^3$. In this case, the extended kinematic space is the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ and the worldlines corresponding to photons give rise to a topological circle $\mathbb{S}^1$ which divides $\mathbb{P}^2_{\mathbb{R}}$ into the open disk $\mathcal{K}$ and the open Möbius band $\mathcal{G} = \mathbb{P}^2_{\mathbb{R}} \setminus \overline{\mathcal{K}}$.

Tangent space and metric. The symmetric bilinear form in $\mathbb{M}^{n+1}$ canonically induces a Riemannian metric in the open kinematic space $\mathcal{K}$ as well as a Lorentzian metric in $\mathcal{G}$. Indeed, there is a natural identification

$$\mathrm{T}_{\boldsymbol{p}}\mathbb{P}^n_{\mathbb{R}} = \mathrm{Lin}(\mathbb{R}p, p^{\perp}) \tag{2.1.2}$$

between the tangent space to $\mathbb{P}^n_{\mathbb{R}}$ at a nonisotropic point $\boldsymbol{p} \in \mathbb{P}^n_{\mathbb{R}}$ and the space of linear maps from $\mathbb{R}p$ to its orthogonal complement $p^{\perp}$ with respect to the symmetric bilinear form. This identification may be interpreted in the following way (for a formal proof see, for instance, [4, Subsection A.1.1]). A tangent vector $\varphi \in \mathrm{T}_{\boldsymbol{p}}\mathbb{P}^n_{\mathbb{R}}$ can be seen as representing a movement in its direction. When the point $\boldsymbol{p}$ starts moving in the direction of φ, the corresponding subspace $\mathbb{R}p$ rotates around the origin of $\mathbb{M}^{n+1}$ and such a rotation can be described in terms of a linear map $\mathbb{R}p \to p^{\perp}$ as in Figure 1.

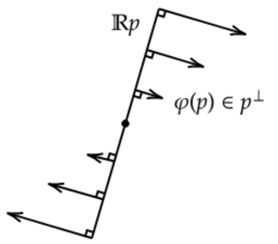

Figure 1: Tangent vector

In view of the identification (2.1.2), given tangent vectors $\varphi_1, \varphi_2 \in \mathrm{Lin}(\mathbb{R}p, p^{\perp})$ at a nonisotropic point $\boldsymbol{p} \in \mathbb{P}^n_{\mathbb{R}}$, we define

$$\langle \varphi_1, \varphi_2 \rangle_{\boldsymbol{p}} := -\frac{\big\langle \varphi_1(p), \varphi_2(p) \big\rangle}{\langle p, p \rangle}. \tag{2.1.3}$$

This provides a semi-Riemannian metric in extended kinematic space outside isotropic points (note that the above formula does not depend on the choice of the representative for $\boldsymbol{p}$). This metric is actually Riemannian in the open kinematic space $\mathcal{K}$ because, in this case, the symmetric bilinear form, restricted to $p^{\perp}$, is positive-definite. It is called the *hyperbolic* metric and endows $\mathcal{K}$ with a geometric structure equivalent to Klein's model of the hyperbolic n-ball. One can similarly see that (2.1.3) is a Lorentzian metric in $\mathcal{G}$, called the *de Sitter* metric. The extended kinematic space is therefore the gluing, along isotropic points, of the kinematic space with the de Sitter space. In order to explore the interplay between $\mathcal{K}$ and $\mathcal{G}$, we need to introduce (extended) geodesics.

Extended geodesics and duality. An *extended geodesic* is a projective line, that is, the projectivization $\mathbb{P}_{\mathbb{R}}W$ of a 2-dimensional linear subspace $W \subset \mathbb{M}^{n+1}$. In particular, there exists a unique extended geodesic, denoted by $\mathrm{G}\wr\boldsymbol{p},\boldsymbol{q}\wr$, that contains a pair of distinct points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}^n_{\mathbb{R}}$. Topologically, an extended geodesic is always a circle. The intersection of $\mathbb{P}_{\mathbb{R}}W$ with $\mathcal{K}$ (respectively, with $\mathcal{G}$) is, if non-empty, a usual geodesic in hyperbolic space (respectively, in de Sitter space). Moreover, all the geodesics in hyperbolic space, as well as in de Sitter space, appear in this way [2]. The possible signatures of the symmetric bilinear form restricted to W are $-+$, $+0$, and $++$. The first case provides all the geodesics in $\mathcal{K}$ and it is easy to see that each such geodesic has a pair of isotropic points, called its *vertices.* In the case of de Sitter space, all the admissible signatures of W appear: when W is respectively of signatures $-+$, $0+$, or $++$, the corresponding geodesics have spacelike, lightlike, or timelike tangent vectors with respect to the Lorentzian metric (2.1.3). Moreover, a geodesic has a pair of distinct isotropic vertices in the first case, a single isotropic vertex in the second case, and no isotropic points in the last case.

We can now see, by means of a simple duality, that the de Sitter space is nothing but the space of all geodesics in kinematic space when $n = 2$. Indeed, given a point $\boldsymbol{p} \in \mathcal{G}$, we obtain the geodesic $\mathbb{P}_{\mathbb{R}}p^{\perp} \cap \mathcal{K}$ due to $p^{\perp}$ being of signature $-+$. The point $\boldsymbol{p}$ is called the *polar point* of the geodesic $\mathbb{P}_{\mathbb{R}}p^{\perp} \cap \mathcal{K}$. Conversely, given a geodesic $\mathbb{P}_{\mathbb{R}}W \cap \mathcal{K}$, we obtain the point $\mathbb{P}_{\mathbb{R}}W^{\perp} \in \mathcal{G}$. (Clearly, the kinematic space itself can be seen as the space of all timelike geodesics in $\mathcal{G}$ and the extended kinematic space, as the space of all geodesics in $\mathcal{G}$.) For arbitrary n, the de Sitter space is the space of all totally geodesic *hyperplanes* in the kinematic space (a totally geodesic hyperplane in $\mathcal{K}$ is given by $\mathbb{P}_{\mathbb{R}}W \cap \mathcal{K}$ when W is a codimension 1 linear subspace of $\mathbb{M}^{n+1}$ of signature $- + \cdots +$).

2.1.4. Remark. Let $n = 3$. Given $\boldsymbol{p} \in \mathcal{G}$, the totally geodesic plane $P := \mathbb{P}_{\mathbb{R}}p^{\perp} \cap \mathcal{K}$ intersects the ideal boundary $\partial\mathcal{K}$ in a circle C. It follows from Definition 3.2.4, Proposition 3.2.5, and from the fact that P is totally geodesic that any observer in P agrees that the velocities of the photons corresponding to the points in the circle C are coplanar. In other words, under the mentioned duality, one can see an inertial "reference frame" corresponding to a point in $\mathcal{G}$ (sometimes called a tachyonic worldline) as being equivalent to such a family of photons.

Tance. The length of the geodesic segment joining two inertial reference frames $\boldsymbol{p},\boldsymbol{q} \in \mathcal{K}$ is the hyperbolic distance $d(\boldsymbol{p},\boldsymbol{q})$ between $\boldsymbol{p}$ and $\boldsymbol{q}$. It is given by $d(\boldsymbol{p},\boldsymbol{q}) = \operatorname{arccosh}\sqrt{\mathrm{ta}(\boldsymbol{p},\boldsymbol{q})}$, where

$$\mathrm{ta}(\boldsymbol{p},\boldsymbol{q}) := \frac{\langle p, q \rangle \langle q, p \rangle}{\langle p, p \rangle \langle q, q \rangle} \tag{2.1.5}$$

is the *tance* between $\boldsymbol{p},\boldsymbol{q}$ [2]. (In the next subsection, we will also refer to the tance in the case of a non-degenerate Hermitian form in a complex vector space; this is why we write its definition in this way.) The hyperbolic distance, also known in the context of special relativity as *rapidity,* is therefore a monotonic function of (the square root of) the tance.

In a certain way, (the square root of) the tance can be seen as being more fundamental than the distance: it is the simplest algebraic invariant of two non-isotropic points in projective space while the distance involves applying to such algebraic invariant a transcendental function. Unlike the distance, the tance is well-defined for any pair of non-isotropic points. For instance, in view of the above duality, the tance in $\mathcal{G}$ allows to determine the relative position of the dual hyperplanes (or geodesics, when $n = 2$) in $\mathcal{K}$ and to calculate the corresponding Riemannian quantities (distances and angles between hyperplanes). Similarly, the tance between a point $\boldsymbol{p} \in \mathcal{K}$ and a point $\boldsymbol{q} \in \mathcal{G}$ allows to calculate the distance between $\boldsymbol{p}$ and the dual hyperplane $\mathbb{P}_{\mathbb{R}} q^\perp \cap \mathcal{K}$. Curiously, the square root of the tance is exactly the Lorentz factor γ corresponding to a pair of inertial reference frames $\boldsymbol{p}, \boldsymbol{q}$ (see Subsection 3.1).

Isometries. The restricted Lorentz group $\mathrm{SO}^+(1,n)$ of all linear, orientation and future-preserving isometries of $\mathbb{M}^{n+1}$ naturally acts on $\mathcal{K}$ by orientation-preserving isometries ($\mathrm{SO}^+(1,n)$ is in fact isomorphic to the group $\mathrm{PSO}(1,n)$ of orientation-preserving isometries of $\mathcal{K}$). The non-identical orientation-preserving isometries of $\mathcal{K}$ can be *elliptic, parabolic,* or *hyperbolic.* Consider $n = 2$. In this case, the elliptic isometries have exactly one fixed point (its center) in $\mathcal{K}$ and, geometrically, they are rotations around the center. The orbit of a point under a one-parameter group generated by an elliptic isometry is a *metric circle,* that is, a locus equidistant from the center. A parabolic isometry has a unique isotropic fixed point $\boldsymbol{v}$ and the orbit of a point under a one-parameter group generated by such an isometry is a *horocycle,* that is, a curve containing $\boldsymbol{v}$ that is orthogonal to every geodesic that has $\boldsymbol{v}$ as a vertex.

Finally, a hyperbolic isometry I has exactly a pair of fixed isotropic points $\boldsymbol{v}_1, \boldsymbol{v}_2$. The geodesic $G := \mathrm{G}\wr v_1, v_2 \wr \cap \mathcal{K}$ is I-stable and, moreover, the orbit of a point under a one-parameter group generated by I is a *hypercycle,* that is, a locus equidistant from G. Note that, at the level of Minkowski space, I is what is called a *boost.* Indeed, the geodesic G can be interpreted as a family of inertial observers such that any of these observers sees all the others with velocities in a same direction (see Subsection 3.2). Now, given inertial observers $\boldsymbol{p}, \boldsymbol{q} \in G$ and a hyperbolic isometry stabilizing $G\wr \boldsymbol{p}, \boldsymbol{q}\wr$, the relative velocity between $\boldsymbol{p}, \boldsymbol{q}$ and that between $\boldsymbol{p}, \boldsymbol{I}(\boldsymbol{q})$ (as measured by $\boldsymbol{p}$) have the same direction.

As we will see, elliptic, hyperbolic, and parabolic isometries play a major role respectively in the Wigner rotation, the relativistic velocities addition, and the Doppler effect.

The above construction endowing (open subspaces of) the projective space with a geometric structure arising from a non-degenerate form on a vector space does not depend on the choice of the signature of the form nor on the field of real numbers. In fact, many other geometries that are relevant in physics can be approached in this manner. This includes Fubini-Study geometries (quantum information theory), anti-de Sitter space (adS/CFT correspondence), and complex hyperbolic geometry (complex Minkowski space). For this reason, in what follows, we will briefly discuss how the above works in more general settings.

2.2 Classic geometries

Let V be an $(n+1)$-dimensional $\mathbb{K}$-vector space, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ (it is also possible to take a module over the quaternions in place of V, see [2]). We endow V with a nondegenerate symmetric bilinear (respectively, Hermitian) form $\langle -, - \rangle : V \times V \to \mathbb{K}$ when $\mathbb{K} = \mathbb{R}$ (respecitvely, $\mathbb{K} = \mathbb{C}$). As in the previous subsection, we will denote by $\boldsymbol{p}$ a point in projective space $\mathbb{P}_{\mathbb{K}} V$ and by $p \in V \setminus \{0\}$ a representative of $\boldsymbol{p}$.

The *signature* of a point $\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}} V$ is the sign of $\langle p, p \rangle$ (which can be $-$, $+$, or 0). The signature is well defined because $\langle kp, kp \rangle = |k|^2 \langle p, p \rangle$ for all $0 \neq k \in \mathbb{K}$. It divides $\mathbb{P}_{\mathbb{K}} V$ into *negative, positive,* and *isotropic* points:

$$\mathrm{B}V := \{\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}} V \mid \langle p, p \rangle < 0\}, \qquad \mathrm{E}V := \{\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}} V \mid \langle p, p \rangle > 0\}, \qquad \mathrm{S}V := \{\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}} V \mid \langle p, p \rangle = 0\}.$$

The space $\mathrm{S}V$ of isotropic points is called the *absolute.* Note that $\mathcal{K}$, $\partial\mathcal{K}$, and $\mathcal{G}$ in the previous subsection, where V is taken as the Minkowski space $\mathbb{M}^{n+1}$, correspond respectively to $\mathrm{B}V$, $\mathrm{S}V$, and $\mathrm{E}V$.

Let $\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}} V \setminus \mathrm{S}V$ be a nonisotropic point. Then

$$V = \mathbb{K}p \oplus p^\perp, \qquad v = \pi[\boldsymbol{p}]v + \pi'[\boldsymbol{p}]v$$

where

$$\pi[\boldsymbol{p}] : v \mapsto v - \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in p^\perp, \qquad \pi'[\boldsymbol{p}] : v \mapsto \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in \mathbb{K}p \tag{2.2.1}$$

are the orthogonal projectors.

As in (2.1.2), we have a natural identification $\mathrm{T}_{\boldsymbol{p}} \mathbb{P}_{\mathbb{K}} V \simeq \mathrm{Lin}_{\mathbb{K}}(\mathbb{K}p, p^\perp)$ of the tangent space to $\mathbb{P}_{\mathbb{K}} V$ at a nonisotropic point $\boldsymbol{p}$ with the space of $\mathbb{K}$-linear maps from $\mathbb{K}p$ to $p^\perp$. Using this identification, we define the pseudo-Riemannian metric

$$\langle \varphi_1, \varphi_2 \rangle_{\boldsymbol{p}} := \pm \mathrm{Re} \frac{\big\langle \varphi_1(p), \varphi_2(p) \big\rangle}{\langle p, p \rangle} \tag{2.2.2}$$

where $\boldsymbol{p}$ is a nonisotropic point and $\varphi_1, \varphi_2 \in \mathrm{T}_{\boldsymbol{p}}\mathbb{P}_{\mathbb{K}}V$. Clearly, when $\mathbb{K} = \mathbb{C}$, this pseudo-Riemannian metric comes from a Hermitian metric (simply do not take the real part in the above expression; the imaginary part of the Hermitian metric is the Kähler form).

Let W be a 2-dimensional real linear subspace $W \subset V$ such that the restriction of the form to W is non-null; in the complex case, we also require the Hermitian form restricted to W to be real. The projectivization $\mathbb{P}_{\mathbb{K}}W$ is called an *extended* geodesic (note that, in the complex case, we take the *complex* projectivization of the *real* subspace W). Extended geodesics are always topological circles and their intersections with $\mathrm{B}V$ and $\mathrm{E}V$ provide all the usual geodesics of the corresponding (pseudo-)Riemannian metric connection [2].

2.2.3. Example. Besides the extended (real) hyperbolic space constructed in Subsection 2.1 (a hyperbolic ball glued with de Sitter space along their absolutes), we point out a few other examples:

- Let $\mathbb{K} = \mathbb{C}$, let $- + + \cdots +$ be the signature of the Hermitian form $\langle -, - \rangle$, and take the sign $-$ in (2.2.2). In this case, $\mathrm{B}V =: \mathbb{H}^n_{\mathbb{C}}$ is the *complex hyperbolic* space. Complex hyperbolic space is to complex Minkowski space as the real hyperbolic space is to real Minkowski space. Note that, when $\dim_{\mathbb{C}} V = 2$, both $\mathrm{B}V$ and $\mathrm{E}V$ are Poincaré hyperbolic discs isometric to the kinematic space $\mathcal{K}$ (see Subsection 2.1).

- Let $\mathbb{K} = \mathbb{R}$, let $- - + \cdots +$ be the signature of the symmetric bilinear form of $\langle -, - \rangle$, and take the $-$ sign in (2.2.2). Now, $\mathrm{B}V =: ad\mathbb{S}^n$ is the *anti-de Sitter* space (which appears, say, in relativity and in the adS/CFT correspondence). Note that there is a natural map $ad\mathbb{S}^{2n+1} \to \mathbb{H}^n_{\mathbb{C}}$, the *anti-Hopf fibration*: when V is an $(n+1)$-dimensional complex vector space with a Hermitian form of signature $- + \cdots +$, its decomplexification is a $2(n+1)$-dimensional real vector space with a symmetric bilinear form of signature $- - + \cdots +$ (the real part of the Hermitian form); the fibers of the map $\mathbb{P}_{\mathbb{R}}V \to \mathbb{P}_{\mathbb{C}}V$, $\mathbb{R}p \mapsto \mathbb{C}p$, are circles. In particular, the fibration $ad\mathbb{S}^3 \to \mathbb{H}^1_{\mathbb{C}}$ can be relevant to special relativity (see the previous item).

- Let $\mathbb{K} = \mathbb{C}$, let $+ ... +$ be the signature of the Hermitian form $\langle -, - \rangle$ and take the sign $+$ in (2.2.2). In this case, we obtain the *Fubini-Study* metric on the complex projective space $\mathrm{E}V = \mathbb{P}_{\mathbb{C}}V$. The Fubini-Study metric is widely used in the geometry of quantum information (the Bloch sphere corresponds to the case $\dim_{\mathbb{C}} V = 2$).

Following this approach, it is possible to express many other important (pseudo-)Riemmannian concepts (say, curvature tensor, metric connection, parallel transport) in a similar coordinate-free fashion [2]. Moreover, all the geometries obtained in this way, including their natural generalization to grassmannians, are Einstein manifolds [3].

3 The physics of kinematic space

3.1 Tance and Lorentz factor

Let us first describe the Lorentz factor, the time dilation, and the length contraction at the level of the kinematic space $\mathcal{K}$ introduced in Subsection 2.1.

Let $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{K}$ be inertial reference frames, and let p be an event that happened at time $t_0 = |p|/c$ for $\boldsymbol{p}$. Hence, p happened at time $t = \big|\pi'[\boldsymbol{q}]p\big|/c$ for $\boldsymbol{q}$ and we obtain

$$\frac{t}{t_0} = \sqrt{\frac{\Big\langle \pi'[\boldsymbol{q}]p, \pi'[\boldsymbol{q}]p \Big\rangle}{\langle p, p \rangle}} = \sqrt{\frac{\Big\langle \frac{\langle p,q\rangle q}{\langle q,q\rangle}, \frac{\langle p,q\rangle q}{\langle q,q\rangle} \Big\rangle}{\langle p, p \rangle}} = \sqrt{\mathrm{ta}(p,q)} =: \gamma_{\boldsymbol{p},\boldsymbol{q}},$$

where $\mathrm{ta}(\boldsymbol{p}, \boldsymbol{q})$ is the tance defined in (2.1.5). Clearly, $\gamma_{\boldsymbol{p},\boldsymbol{q}}$ is the usual Lorentz factor and $t = \gamma_{\boldsymbol{p},\boldsymbol{q}} t_0$ is nothing but the time dilation (see Proposition 3.1.3).

3.1.1. Remark. The usual formula for the Lorentz factor in terms of the relative scalar velocity between $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{K}$ can be obtained as follows. Take homogeneous coordinates $[c, x_1, \dots, x_n]$ with $\sum x_i^2 \leqslant c^2$ that identify the closed kinematic space with a closed n-ball $\overline{\mathbb{B}}^n$ of radius c centred at $\boldsymbol{p} = [c, 0, 0, \dots, 0]$. Then, if $\boldsymbol{q} = [c, v_1, \dots, v_n]$, the relative scalar velocity v between $\boldsymbol{p}$ and $\boldsymbol{q}$ is given by the Euclidean distance in $\overline{\mathbb{B}}^n$ between the observers, that is, $v = \sqrt{\sum v_i^2}$. Hence, we have

$$\gamma_{\boldsymbol{p},\boldsymbol{q}} = \sqrt{\mathrm{ta}(p,q)} = \sqrt{\frac{c^4}{-c^2(-c^2 + \sum v_i^2)}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

In particular, in terms of the tance, the relative scalar velocity between $\boldsymbol{p},\boldsymbol{q}$ is given by

$$v=c\sqrt{1-\frac{1}{\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})}}. \tag{3.1.2}$$

(For a coordinate-free form of this remark, see Subsection 3.2.)

3.1.3. Proposition (time dilation). *Let $\boldsymbol{p},\boldsymbol{q}\in\mathcal{K}$ be inertial observers and let $w\in\mathbb{M}^{n+1}\setminus\{0\}$ be an event that happened at time $t_{\boldsymbol{p}}\neq 0$ for $\boldsymbol{p}$ and at time $t_{\boldsymbol{q}}$ for $\boldsymbol{q}$. Then*

$$\frac{t_{\boldsymbol{q}}}{t_{\boldsymbol{p}}}=\sqrt{\frac{\operatorname{ta}(\boldsymbol{q},\boldsymbol{w})}{\operatorname{ta}(\boldsymbol{p},\boldsymbol{w})}}.$$

(*Note that the formula is also well-defined when w is lightlike because the term $\langle w,w\rangle$ cancels out.*) *In particular, when $\boldsymbol{w}=\boldsymbol{p}$, we obtain $t_{\boldsymbol{q}}=\gamma_{\boldsymbol{q},\boldsymbol{p}}t_{\boldsymbol{p}}$.*

Proof. Follows directly from $t_{\boldsymbol{p}}^2=\big\langle\pi'[p]w,\pi'[p]w\big\rangle=\frac{\langle p,w\rangle\langle w,p\rangle}{\langle p,p\rangle}$ and $t_{\boldsymbol{q}}^2=\big\langle\pi'[q]w,\pi'[q]w\big\rangle=\frac{\langle q,w\rangle\langle w,q\rangle}{\langle q,q\rangle}$. □

Taking $\frac{t_{\boldsymbol{p}}}{t_{\boldsymbol{q}}},\frac{t_{\boldsymbol{q}}}{t_{\boldsymbol{p}}}$ as projective coordinates, one can think of time dilation as a function $\mathbb{M}^{n+1}\setminus\{0\}\to\mathbb{P}^1_{\mathbb{R}}$; this allows to accommodate the cases when the event w happens at time $t=0$ for (exactly) one of the inertial reference frames.

3.1.4. Proposition (length contraction). *Let $\boldsymbol{p},\boldsymbol{q}\in\mathcal{K}$ be inertial observers and assume that $\boldsymbol{p}$ observes a rigid rod at rest as having length $\ell_{\boldsymbol{p}}$. We represent the rod by a spacelike vector $w\in p^{\perp}\setminus\{0\}$. Then,*

$$\frac{\ell_{\boldsymbol{q}}}{\ell_{\boldsymbol{p}}}=\sqrt{1+\frac{\operatorname{ta}(\boldsymbol{q},\boldsymbol{w})}{\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})}},$$

where $\ell_{\boldsymbol{q}}$ stands for the length of the rod as measured by $\boldsymbol{q}$. In particular, if p,q,w are coplanar (*that is, the rod is in the direction of the relative velocity between $\boldsymbol{p}$ and $\boldsymbol{q}$*)*, then $\ell_{\boldsymbol{p}}=\gamma_{\boldsymbol{p},\boldsymbol{q}}\ell_{\boldsymbol{q}}$.*

Proof. We have $\ell_{\boldsymbol{p}}=|w|$ and $\ell_{\boldsymbol{q}}=|w'|$, where $w':=w-\frac{\langle w,q\rangle}{\langle p,q\rangle}p$ (note that $w'\in q^{\perp}$ and that w' belongs to the straight line through w parallel to $\mathbb{R}p$). Therefore,

$$\ell_{\boldsymbol{q}}^2=\left\langle w-\frac{\langle w,q\rangle}{\langle p,q\rangle}p,w-\frac{\langle w,q\rangle}{\langle p,q\rangle}p\right\rangle=\langle w,w\rangle\left(1+\frac{\langle w,q\rangle\langle q,w\rangle\langle p,p\rangle}{\langle p,q\rangle\langle q,p\rangle\langle w,w\rangle}\cdot\frac{\langle q,q\rangle}{\langle q,q\rangle}\right)=\ell_{\boldsymbol{p}}^2\left(1+\frac{\operatorname{ta}(\boldsymbol{q},\boldsymbol{w})}{\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})}\right).$$

When p,q,w are coplanar, the determinant $\det\begin{bmatrix}\langle p,p\rangle & \langle p,q\rangle & 0\\ \langle q,p\rangle & \langle q,q\rangle & \langle q,w\rangle\\ 0 & \langle w,q\rangle & \langle w,w\rangle\end{bmatrix}$ vanishes, that is, $\operatorname{ta}(w,q)+\operatorname{ta}(p,q)=1$ which implies the result. □

Given $\boldsymbol{p},\boldsymbol{q}\in\overline{\mathcal{K}}$, the geometric configuration corresponding to the coplanar case in the above proposition is unique. Indeed, $\boldsymbol{w}$ must be the point orthogonal to $\boldsymbol{p}$ in the extended geodesic $\mathrm{G}\wr\boldsymbol{p},\boldsymbol{q}\wr$ (because $w\in p^{\perp}$ and the coplanarity of p,q,w means that $\boldsymbol{w}$ belongs to $\mathrm{G}\wr\boldsymbol{p},\boldsymbol{q}\wr$). Similarly, $\boldsymbol{w}'$ must be the point in $\mathrm{G}\wr\boldsymbol{p},\boldsymbol{q}\wr$ orthogonal to $\boldsymbol{q}$. Moreover, it is curious to note that the formula $\ell_{\boldsymbol{p}}=\gamma_{\boldsymbol{p},\boldsymbol{q}}\ell_{\boldsymbol{q}}$ is actually a direct consequence of the geometric identity $\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})=\operatorname{ta}(\boldsymbol{w},\boldsymbol{w}')$ (whose proof is a straightforward calculation). Indeed, we have

$$\gamma_{\boldsymbol{p},\boldsymbol{q}}^2=\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})=\operatorname{ta}(\boldsymbol{w},\boldsymbol{w}')=\frac{\langle w,w'\rangle\langle w',w\rangle}{\langle w,w\rangle\langle w',w'\rangle}=\frac{\langle w,w\rangle\langle w,w\rangle}{\langle w,w\rangle\langle w',w'\rangle}=\frac{|w|^2}{|w'|^2}=\frac{\ell_{\boldsymbol{p}}^2}{\ell_{\boldsymbol{q}}^2}$$

since $w'=w-\frac{\langle w,q\rangle}{\langle p,q\rangle}p$ and $w\in p^{\perp}$.

3.2 Rapidity, velocity, and parallel transport

Rapidity and rapidity addition. Given an inertial observer $\boldsymbol{p}\in\mathcal{K}$, we call the tangent space $\mathrm{T}_{\boldsymbol{p}}\mathcal{K}$ the *space of rapidities* at $\boldsymbol{p}$. A tangent vector $w\in\mathrm{T}_{\boldsymbol{p}}\mathcal{K}$ is the *relative rapidity,* as measured by $\boldsymbol{p}$ (or, simply, at $\boldsymbol{p}$), between $\boldsymbol{p}$ and the inertial observer $\boldsymbol{q}:=\exp_{\boldsymbol{p}}w$, where $\exp$ stands for the Riemannian exponential map. Hence, the hyperbolic distance between $\boldsymbol{p},\boldsymbol{q}$ is $d(\boldsymbol{p},\boldsymbol{q})=|w|$.

There is a natural way to sum rapidities at $\boldsymbol{p}\in\mathcal{K}$ that takes into account the geometry of the kinematic space. After introducing it, we will relate rapidity and velocity in order to show that the geometric sum of rapidities leads to the relativistic velocities addition.

3.2.1. Definition. Let $\boldsymbol{p} \in \mathcal{K}$ be an inertial observer and let $w_1, w_2 \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ be rapidities. Take $\boldsymbol{q} := \exp_{\boldsymbol{p}} w_1$, let $w_2' \in \mathrm{T}_{\boldsymbol{q}}\,\mathcal{K}$ be the parallel transport of w_2 along the geodesic segment joining $\boldsymbol{p}$ and $\boldsymbol{q}$, and let $\boldsymbol{r} := \exp_{\boldsymbol{q}} w_2'$. We define the *sum of rapidities* $w_1 \oplus w_2 \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ as the unique rapidity $w \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ such that $\exp_{\boldsymbol{p}} w = \boldsymbol{r}$. Equivalently, $w_1 \oplus w_2 := \exp_{\boldsymbol{p}}^{-1} \boldsymbol{r}$ (see Figure 2).

Clearly, the above definition works in any Riemannian manifold with infinite injectivity radius and, in the particular case of an Euclidean vector space, it coincides with the vector space sum. (In fact, Definition 3.2.1 can be seen as a straightforward generalization of the vector sum in an Euclidean vector space.)

Scaled rapidity. While rapidities live in the tangent spaces to points in the kinematic space, *scaled rapidities* (a.k.a *hyperbolic velocities*) appear naturally as tangent vectors to points in the *scaled kinematic space* $\mathcal{K}^c$. In order to introduce the scaled kinematic space we will use the following remark.

3.2.2. Remark. Once a representative $p \in \mathbb{M}^{n+1}$ of $\boldsymbol{p} \in \mathcal{K}$ is chosen, we identify $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ with $p^\perp$ via (2.1.2), that is, via the map $\varphi \mapsto \varphi(p) \in p^\perp$, $\varphi \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K} \simeq \mathrm{Lin}(\mathbb{R}p, p^\perp)$. (There is, however, a *natural* identification $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K} \simeq p^\perp$, see Remark 3.2.3.)

The (open) scaled kinematic space is the manifold $\mathcal{K}$ endowed with a different Riemannian metric as follows. Given $\boldsymbol{p} \in \mathcal{K}$, we take the future-directed representative $p \in \mathbb{M}^{n+1}$ such that $\langle p, p\rangle = -c^2$ and identify $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K} \simeq p^\perp$ as in Remark 3.2.2. Now, we equip $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ with the inner product in $p^\perp$ (that is, the restriction of the symmetric bilinear form in $\mathbb{M}^{n+1}$ to $p^\perp$). Provided with such Riemannian metric, the manifold $\mathcal{K}$ is called the *scaled kinematic space* $\mathcal{K}^c$. The scaled kinematic space $\mathcal{K}^c$ is a hyperbolic space of constant curvature $-1/c^2$ because it is isometric to the future sheet of the hyperboloid $\langle x, x\rangle = -c^2$ with the induced metric from Minkowski space. The concepts of *space of scaled rapidities,* of *relative scaled rapidity,* and of *sum of scaled rapidities* are analogous to their rapidity counterparts.

Let $w \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$ be a relative rapidity at $\boldsymbol{p}$ which correponds to the relative scaled rapidity $w_c \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c$. It follows from (2.1.3) that $|w_c| = c|w|$, where the left-hand side (respectively, the right-hand side) norm is the one in $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c$ (respectively, in $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}$).

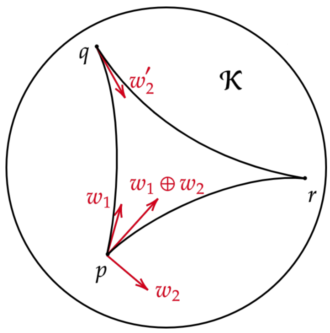

Figure 2: Rapidity addition

3.2.3. Remark. Let $\boldsymbol{p} \in \mathcal{K}$. There is a natural identification $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K} \simeq p^\perp \subset \mathbb{M}^{n+1}$ because, given $\varphi \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K} = \mathrm{Lin}(\mathbb{R}p, p^\perp)$, there exists a unique future-oriented representative $p \in \mathbb{M}^{n+1}$ such that $u := \varphi(p) \in p^\perp$ satisfies $\langle u, u\rangle = \langle \varphi, \varphi\rangle_{\boldsymbol{p}}$. Clearly, $\langle p, p\rangle = -1$. Analogously, there is a natural identification $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c \simeq p^\perp \subset \mathbb{M}^{n+1}$ and the corresponding representative of p in this case satisfies $\langle p, p\rangle = -c^2$.

Velocity. Velocity and (relative) rapidity are concepts of different natures because velocity is algebraic. Let us introduce the space of velocities at a point $\boldsymbol{p} \in \mathcal{K}^c$ and endow it with its natural geometric structure.

Given $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{K}^c$, we define the *relative velocity* between $\boldsymbol{p}, \boldsymbol{q}$ at $\boldsymbol{p}$ as the simplest algebraic expression (in the sense that it does not depend on the choice of representatives) for a tangent vector $v \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c$ that is tangent to the geodesic $\mathrm{G}\wr\boldsymbol{p}, \boldsymbol{q}\wr$ at $\boldsymbol{p}$:

3.2.4. Definition. Given $\boldsymbol{p} \in \mathcal{K}^c$, the relative velocity $v \in \mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c \simeq \mathrm{Lin}(\mathbb{R}p, p^\perp)$ between $\boldsymbol{p}$ and $\boldsymbol{q} \in \overline{\mathcal{K}}^c$ at $\boldsymbol{p}$ is defined as the linear map $v = \langle -, p\rangle \dfrac{\pi[\boldsymbol{p}]q}{\langle q, p\rangle}$, where $\langle -, p\rangle$ stands for the linear functional $x \mapsto \langle x, p\rangle$, $x \in \mathbb{M}^{n+1}$.

By [2, Lemma 5.2], the relative velocity between $\boldsymbol{p}$ and $\boldsymbol{q}$ at $\boldsymbol{p}$ is tangent to the geodesic $G\wr\boldsymbol{p}, \boldsymbol{q}\wr$. So, the relative (scaled) rapidity and the corresponding relative velocity between inertial observers $\boldsymbol{p}, \boldsymbol{q}$ at $\boldsymbol{p}$ have the same direction.

3.2.5. Proposition. *Under the identification $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c \simeq p^\perp$ in Remark 3.2.3, the above definition of relative velocity coincides with the usual one.*

Proof. Let $\boldsymbol{p} \in \mathcal{K}^c$ and let $\boldsymbol{q} \in \overline{\mathcal{K}}^c$. At the level of Minkowski space, the usual relative velocity between $\mathbb{R}p, \mathbb{R}q$ as measured by $\mathbb{R}p$ has the norm given in equation (3.1.2) and the direction of the projection $\pi[\boldsymbol{p}]q \in p^\perp$ for a future-oriented q. On the other hand, the tangent vector $\langle -, p\rangle \frac{\pi[\boldsymbol{p}]q}{\langle q, p\rangle}$ corresponds, via the identication $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c \simeq p^\perp$, to $-c^2 \frac{\pi[\boldsymbol{p}]q}{\langle q, p\rangle}$. It remains to observe that $\langle p, q\rangle < 0$ (since both are future-oriented) and that

$$\left\langle -c^2 \frac{\pi[\boldsymbol{p}]q}{\langle q, p\rangle}, -c^2 \frac{\pi[\boldsymbol{p}]q}{\langle q, p\rangle} \right\rangle = c^2 \left(1 - \frac{1}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{q})}\right). \qquad \square$$

The symmetric bilinear form restricted to $\mathbb{R}p + \mathbb{R}q$, where $\boldsymbol{p}, \boldsymbol{q}$ are as in the proof above, has signature $-+$. Hence, the determinant of the Gram matrix $\left[\begin{smallmatrix} \langle p,p\rangle & \langle p,q\rangle \\ \langle q,p\rangle & \langle q,q\rangle \end{smallmatrix}\right]$ is negative which implies that $\operatorname{ta}(\boldsymbol{p}, \boldsymbol{q}) \geqslant 1$. The norm of the velocity in Definition 3.2.4 is therefore always less or equal than c. So, the relative velocities at $\boldsymbol{p}$ constitute the closed n-ball $\mathcal{V}_{\boldsymbol{p}} \subset \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$ of radius c centered at $0 \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$. Such closed ball is called the *space of velocities* $\mathcal{V}_p$ at $\boldsymbol{p}$. (This definition can be seen as a coordinate-free form of Remark 3.1.1.)

Hyperbolic structure on $\mathcal{V}_{\boldsymbol{p}}$. Besides the inner product inherited from $p^\perp$, the space of velocities $\mathcal{V}_{\boldsymbol{p}}$ has a natural hyperbolic structure induced from $\mathcal{K}^c$: we simply send a velocity $v \in \mathcal{V}_{\boldsymbol{p}}$ to the inertial observer $\boldsymbol{q} \in \mathcal{K}^c$ such that the relative velocity between $\boldsymbol{p}, \boldsymbol{q}$ at $\boldsymbol{p}$ equals v and equip $\mathcal{V}_{\boldsymbol{p}}$ with the pullback metric. From the perspective of Minkowski space (see Figure 3), this is nothing but (1) associating a vector $v \in p^\perp$ satisfying $\langle v, v\rangle < c^2$ to the inertial observer $\mathbb{R}(p + v)$, where p is the future-oriented representative of $\boldsymbol{p}$ with $\langle p, p\rangle = -c^2$, and (2) equipping $p + \mathbb{B}^n \simeq \mathcal{V}_{\boldsymbol{p}}$ with the hyperbolic metric that comes from the stereographic projection onto the hyperboloid $\langle x, x\rangle = -c^2$, where $\mathbb{B}^n \subset p^\perp$ stands for the open ball of radius c centred at the origin. Note that, while rapidity is intended to measure the distance between inertial reference frames, the role of scaled rapidity is to measure the "distance between velocities" in a velocity space $\mathcal{V}_{\boldsymbol{p}}$.

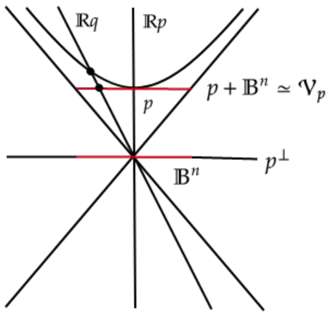

Figure 3: Hyperbolic structure on $\mathcal{V}_{\boldsymbol{p}}$ (at the level of Minkowski space)

Relativistic velocity addition. The relative velocity $v \in \mathcal{V}_{\boldsymbol{p}}$ between $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{K}^c$ at $\boldsymbol{p}$ and the corresponding relative scaled rapidity $w_c \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$ are related by

$$v = v(w_c) = c\Big(\tanh\big(|w_c|/c\big)\Big)\frac{w_c}{|w_c|} \tag{3.2.6}$$

because those tangent vectors have the same direction and

$$|v|^2 = c^2\left(1 - \frac{1}{\operatorname{ta}(\boldsymbol{p}, \boldsymbol{q})}\right) = c^2\left(1 - \frac{1}{\cosh^2\big(d^c(\boldsymbol{p}, \boldsymbol{q})/c\big)}\right) = c^2 \tanh^2 \frac{d^c(\boldsymbol{p}, \boldsymbol{q})}{c} = c^2 \tanh^2 \frac{|w_c|}{c}$$

by Remark 3.1.1, where $d^c(\boldsymbol{p}, \boldsymbol{q})$ stands for the distance function in $\mathcal{K}^c$. In particular, $v = v(w) = c\big(\tanh|w|\big)\frac{w}{|w|}$, where $w \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}$ stands for the rapidity between $\boldsymbol{p}, \boldsymbol{q}$ at $\boldsymbol{p}$.

3.2.7. Definition. Let $v_1, v_2 \in \mathcal{V}_{\boldsymbol{p}}$ be velocities and let $w_1, w_2 \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$ be the corresponding scaled rapidities. We define $v_1 \oplus v_2$ simply as the velocity that corresponds to $w_1 \oplus w_2$, that is, $v_1 \oplus v_2 := v(w_1 \oplus w_2)$. (One can also take rapidities instead of scaled rapidities here.)

3.2.8. Proposition. *The above definition of velocity addition coincides with the usual relativistic velocity addition.*

Proof. Let $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in \mathcal{K}^c$ be inertial observers, let $v_1 \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$ be the relative velocity between $\boldsymbol{p}, \boldsymbol{q}$ at $\boldsymbol{p}$, and let $v_2' \in \mathrm{T}_{\boldsymbol{q}} \mathcal{K}^c$ be the relative velocity between $\boldsymbol{q}, \boldsymbol{r}$ at $\boldsymbol{q}$. The parallel transport $v_2 \in \mathrm{T}_{\boldsymbol{p}} \mathcal{K}^c$ of v_2' along the geodesic segment joining $\boldsymbol{q}$ and $\boldsymbol{p}$ can be interpreted as the relative velocity between $\boldsymbol{q}, \boldsymbol{r}$ as measured by $\boldsymbol{p}$. Indeed, let I be the hyperbolic isometry that stabilizes $G\wr\boldsymbol{p}, \boldsymbol{q}\wr$ and satisfies $I(\boldsymbol{q}) = \boldsymbol{p}$. It is easy to see that $I_*(w_c') = w_c$, where I_* stands for the differential of I and w_c, w_c' denote respectively the scaled rapidities corresponding to v_2, v_2'. By the naturality of the exponential map (see [9, Proposition 5.20], for instance), $\exp_{\boldsymbol{p}} w_c = \exp_{\boldsymbol{p}}\big(I_*(w_c')\big) = I(\exp_{\boldsymbol{q}}(w_c')) = I(r)$. At the level of Minkowski space, the boost $\widetilde{I}$ corresponding to I sends the pair of inertial observers $\mathbb{R}q, \mathbb{R}r$ to $\mathbb{R}p, \mathbb{R}\widetilde{I}(r)$ and the relative velocity between the last two observers, as measured by $\mathbb{R}p$, is therefore exactly the relative velocity between the first two ones as measured by $\mathbb{R}p$. $\square$

"Parallelogram" law. Let us take a closer look at the geometry of the sum of velocities. Given velocities $v_1, v_2 \in \mathcal{V}_{\boldsymbol{p}}$, where $\boldsymbol{p} \in \mathcal{K}^c$, we can assume that $\mathcal{V}_{\boldsymbol{p}}$ is an open disk in the two-dimensional subspace of $\mathrm{T}_{\boldsymbol{p}}\,\mathcal{K}^c$ generated by v_1, v_2. Now, the sum $v_1 \oplus v_2$ is obtained simply by applying to v_2 the hyperbolic isometry I (in the sense of the hyperbolic structure of $\mathcal{V}_{\boldsymbol{p}}$) that sends the null vector 0 to v_1 and stabilizes the geodesic $G := \mathrm{G}\wr 0, v_1 \wr$. Note that the sum of velocities is noncommutative because, if we apply to v_1 the hyperbolic isometry I' that sends 0 to v_2 and stabilizes the geodesic $\mathrm{G}\wr 0, v_2 \wr$ then, in general, $I(v_2) \neq I'(v_1)$. In other words, at a first glance, it seems that there is no "parallelogram" law for the relativistic addition of velocities. However, this is the case only if we require the parallelogram to be *geodesic*; substituting one of the sides for a hypercycle, that is, for a curve that is equidistant from a geodesic, there is indeed a "parallelogram law" where the "parallelogram" has vertices $0, v_1, v_1 \oplus v_2, v_2$ and the sides are the geodesic segment joining $0, v_1$, the geodesic segment joining $v_1, v_1 \oplus v_2$, the segment of the hypercycle H of G joining $v_1 \oplus v_2, v_2$, and the geodesic segment joining $v_2, 0$. In other words, $v_1 \oplus v_2$ is obtained by the geometric construction that follows. Draw: the geodesic G joining $0, v_1$; the geodesic G' joining $0, v_2$; the geodesic G'' through v_1 such that the oriented angle from G to G'' at v_1 equals that from G to G' at 0; the hypercycle H of G through v_2. Then, $v_1 \oplus v_2$ is given by the intersection $H \cap G''$.

3.2.9. Remark. This construction of the relativistic velocity addition can also be seen as a geometric realization of the Möbius addition discussed by A. Ungar; this follows from the above considerations and from the fact that Poincaré's hyperbolic disk $\mathbb{H}^1_{\mathbb{C}}$ (see Example 2.2.3) is isometric to $\mathcal{K}$ when $\dim \mathcal{K} = 2$. More precisely, given $\boldsymbol{o}, \boldsymbol{p}, \boldsymbol{q} \in \mathbb{H}^1_{\mathbb{C}}$, we define $\boldsymbol{p} \oplus_{\boldsymbol{o}} \boldsymbol{q} := I(\boldsymbol{q})$, where I stands for the hyperbolic isometry that stabilizes the geodesic $G\wr \boldsymbol{o}, \boldsymbol{p} \wr$ and satisfies $I(\boldsymbol{o}) = \boldsymbol{p}$. This is a coordinate-free geometric form of the Möbius addition formula in [12, Section 3.4]: take the unitary disk $\mathbb{D}$ in $\mathbb{C}$ centered at the origin (which plays the role of $\boldsymbol{o}$) and define $a \oplus_M b := (a+b)/(1+\overline{a}b)$ for all $a, b \in \mathbb{D}$.

Similarly, one can give a geometric description of the Möbius subtraction $a \ominus_M b := a \oplus_M (-b)$ by defining $-\boldsymbol{q} := R(\boldsymbol{o})\boldsymbol{q}$ and $\boldsymbol{p} \ominus_{\boldsymbol{o}} \boldsymbol{q} := R(\boldsymbol{m})\boldsymbol{q}$, where $R(\boldsymbol{o})$ and $R(\boldsymbol{m})$ stand respectively for the reflection in $\boldsymbol{o}$ and in the middle point $\boldsymbol{m}$ of the geodesic segment joining $\boldsymbol{o}$ and $\boldsymbol{p}$. Indeed, the hyperbolic isometry I that stabilizes the geodesic $G\wr \boldsymbol{o}, \boldsymbol{p} \wr$ and satisfies $I(\boldsymbol{o}) = \boldsymbol{p}$ can be written as $I = R(\boldsymbol{m})R(\boldsymbol{o})$. Now, $\boldsymbol{p} \oplus_{\boldsymbol{o}} (-\boldsymbol{q}) = I(-\boldsymbol{q}) = I\big(R(\boldsymbol{o})\boldsymbol{q}\big) = R(\boldsymbol{m})R(\boldsymbol{o})R(\boldsymbol{o})\boldsymbol{q} = R(\boldsymbol{m})\boldsymbol{q}$.

Another geometric way to look at the relativistic velocities addition is the following. In order to obtain $v_1 \oplus v_2$, we first project v_2 orthogonally (in the hyperbolic sense) over the direction of v_1 thus obtaining the *horizontal component* v of v_2. Now, if v and v_1 have the same direction, we add v_1 and v by simply taking the velocity $v_1 \oplus v = v \oplus v_1 \in \mathcal{V}_{\boldsymbol{p}}$ that lies in the geodesic $G := \mathrm{G}\wr 0, v_1 \wr$ and satisfies $d^c(0, v_1 \oplus v) = d^c(0, v_1) + d^c(0, v)$, where d^c stands for the hyperbolic distance in $\mathcal{V}_{\boldsymbol{p}}$ (the case when v and v_1 have opposite directions is handled similarly). Finally, it remains to take the unique velocity $v_1 \oplus v_2 \in \mathcal{V}_{\boldsymbol{p}}$ that is on the same side of G as v_2, whose orthogonal projection onto G is $v_1 \oplus v$, and whose distance to G equals that of v_2 (in other words, the *vertical component* of $v_1 \oplus v_2$ is the same as that of v_2).

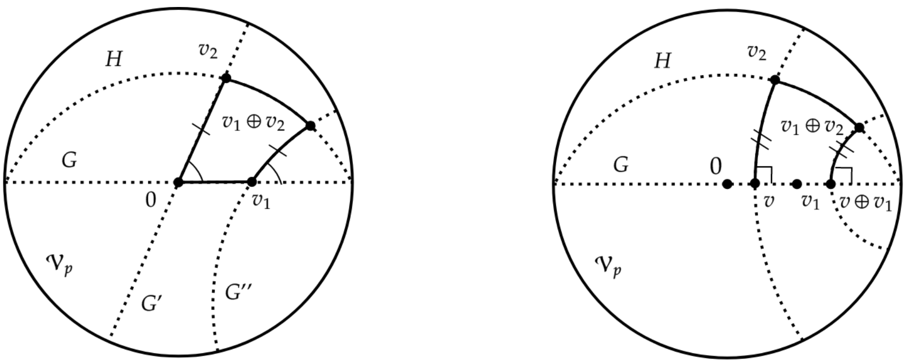

Figure 4: "Parallelogram" law and component sum

3.3 Relativistic Doppler effect

The relativistic Doppler effect can also be seen in a geometric way.[2] In this section, we can assume (without loss of generality) that $\dim \mathcal{K} = 2$.

A metric circle C in $\mathcal{K}$ is the locus of inertial observers that see a given inertial observer $\boldsymbol{q} \in \mathcal{K}$ (the center of the circle) with a same given energy. Indeed, $C = \{\boldsymbol{p} \in \mathcal{K} \mid \mathrm{ta}(\boldsymbol{p}, \boldsymbol{q}) = r\}$, $r > 0$, and the energy of $\boldsymbol{q}$ as measured by $\boldsymbol{p}$ is

[2] We thank J. A. Hoyos for suggesting that horocycles should be related to the relativistic Doppler effect.

determined by $\gamma_{\boldsymbol{p},\boldsymbol{q}} = \sqrt{\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})}$. In the limit where $\boldsymbol{q}$ goes to the absolute (and r is fixed) this metric circle turns into a horocycle tangent to the absolute at a point $\boldsymbol{f} \in \partial\mathcal{K}$ and the energy being measured by the inertial observers corresponding to points in this horocycle becomes that of the photon $\boldsymbol{f}$. In other words, the function that assigns to each inertial observer in $\mathcal{K}$ the energy (or, equivalently, the frequency) that it measures for the photon $\boldsymbol{f}$ is constant along horocycles (in fact, horocycles will be the level curves of this function, see Corollary 3.3.3). Let us formalize this argument.

3.3.1. Lemma. *Let $\boldsymbol{f} \in \partial\mathcal{K}$ and let $\boldsymbol{r},\boldsymbol{r}' \in \mathcal{K}$ be inertial observers in a same horocycle containing $\boldsymbol{f}$. Then, $\nu_{\boldsymbol{r}} = \nu_{\boldsymbol{r}'}$, where $\nu_{\boldsymbol{r}}, \nu_{\boldsymbol{r}'}$ stand for the frequencies of $\boldsymbol{f}$ as measured respectively by $\boldsymbol{r},\boldsymbol{r}'$.*

Proof. Let $I : \overline{\mathcal{K}} \to \overline{\mathcal{K}}$ be the parabolic isometry that fixes $\boldsymbol{f}$ and maps $\boldsymbol{r}'$ to $\boldsymbol{r}$. It is well-known that the energy $E_{\boldsymbol{r}}$ of the photon $\boldsymbol{f}$ as measured by $\boldsymbol{r}$ is given by the magnitude of the projection of the $(n+1)$-momentum of the photon in the direction of $\mathbb{R}r$ divided by c. Similarly, one can express the energy $E_{\boldsymbol{r}'}$ of the photon $\boldsymbol{f}$ as measured by $\boldsymbol{r}'$, which leads to

$$\left(\frac{E_{\boldsymbol{r}'}}{E_{\boldsymbol{r}}}\right)^2 = \frac{\langle \pi'[\boldsymbol{r}']f, \pi'[\boldsymbol{r}']f\rangle}{\langle \pi'[\boldsymbol{r}]f, \pi'[\boldsymbol{r}]f\rangle} = \frac{\langle f, r'\rangle^2\langle r, r\rangle}{\langle f, r\rangle^2\langle r', r'\rangle} = \frac{\langle f, r'\rangle^2\langle \widetilde{I}(r'), \widetilde{I}(r')\rangle}{\langle f, \widetilde{I}(r')\rangle^2\langle r', r'\rangle} = \frac{\langle f, r'\rangle^2\langle \widetilde{I}(r'), \widetilde{I}(r')\rangle}{\langle \widetilde{I}(f), \widetilde{I}(r')\rangle^2\langle r', r'\rangle} = 1,$$

where $\widetilde{I}$ stands for the element in $\mathrm{SO}^+(1,2)$ corresponding to I; it satisfies $\widetilde{I}(f) = f$ because I is parabolic (see, for instance, [7]). $\square$

Now, consider the case of two inertial observers $\boldsymbol{r},\boldsymbol{s} \in \mathcal{K}$ which are respectively considered as the receiver and the source of a photon $\boldsymbol{f} \in \partial\mathcal{K}$ such that $\boldsymbol{r},\boldsymbol{s},\boldsymbol{f}$ are in a same geodesic G. Assume that the inertial observers are moving away from each other (it is easy to see that, in order to reach the receiver, the photon that has to be sent by the source is such that $\boldsymbol{r}$ is in the geodesic segment joining $\boldsymbol{s}$ and $\boldsymbol{f}$). Let $\nu_{\boldsymbol{s}}$ (respectively, $\nu_{\boldsymbol{r}}$) be the frequency of $\boldsymbol{f}$ as measured by $\boldsymbol{s}$ (respectively, by $\boldsymbol{r}$). Then (see, for example, [11, Section 4.3])

$$\frac{\nu_{\boldsymbol{s}}}{\nu_{\boldsymbol{r}}} = \sqrt{\frac{1+v/c}{1-v/c}} = \sqrt{\frac{1+(e^w-e^{-w})/(e^w+e^{-w})}{1-(e^w-e^{-w})/(e^w+e^{-w})}} = e^w = e^{d(\boldsymbol{r},\boldsymbol{s})},$$

where v and w are respectively the scalar relative velocity and relative rapidity between $\boldsymbol{r}$ and $\boldsymbol{s}$. When the inertial observers are moving towards each other (in this case, the photon $\boldsymbol{f}'$ to be sent corresponds to the other vertex of G) we have $\nu_{\boldsymbol{s}}/\nu_{\boldsymbol{r}} = e^{-d(\boldsymbol{r},\boldsymbol{s})}$. We are now able to prove the following proposition (for the definition of Busemann function see, for instance, [10, Section 1.2]).

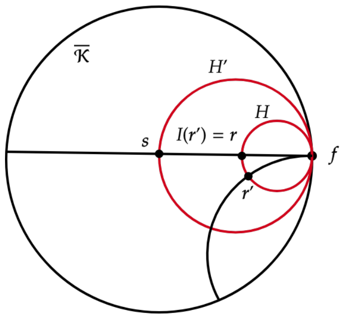

Figure 5: Horocycles and the relativistic Doppler effect

3.3.2. Proposition (relativistic Doppler effect). *Let $\boldsymbol{p},\boldsymbol{q} \in \mathcal{K}$ be inertial observers and let $\boldsymbol{f} \in \partial\mathcal{K}$ be a photon. Let $\nu_{\boldsymbol{p}}$ and $\nu_{\boldsymbol{q}}$ be respectively the frequencies of $\boldsymbol{f}$ as measured by $\boldsymbol{p}$ and $\boldsymbol{q}$. We have*

$$\frac{\nu_{\boldsymbol{p}}}{\nu_{\boldsymbol{q}}} = e^{b_{\boldsymbol{f}}(\boldsymbol{p},\boldsymbol{q})},$$

where $b_{\boldsymbol{f}}$ stands for the Busemann function determined by $\boldsymbol{f}$.

Proof. By Lemma 3.3.1, the ratio $\nu_{\boldsymbol{p}}/\nu_{\boldsymbol{q}}$ can be obtained in terms of the distance between the horocycles H, H' containing $\boldsymbol{f}$ and passing respectively through $\boldsymbol{p},\boldsymbol{q}$. Now the proof follows from the case of collinear $\boldsymbol{p},\boldsymbol{q},\boldsymbol{f}$ which was already considered above. $\square$

A direct consequence of Lemma 3.3.1 and Proposition 3.3.2 is the following Corollary.

3.3.3. Corollary. *Let $\boldsymbol{f} \in \partial\mathcal{K}$ and let $\boldsymbol{p},\boldsymbol{q} \in \mathcal{K}$ be inertial observers. Then $\nu_{\boldsymbol{p}} = \nu_{\boldsymbol{q}}$ if and only if $\boldsymbol{p},\boldsymbol{q}$ belong to a same horocycle containing $\boldsymbol{f}$, where $\nu_{\boldsymbol{p}}, \nu_{\boldsymbol{q}}$ stand for the frequencies of $\boldsymbol{f}$ as measured respectively by $\boldsymbol{p},\boldsymbol{q}$.*

3.4 Wigner rotation

Let $\boldsymbol{p},\boldsymbol{q},\boldsymbol{r} \in \mathcal{K}$ be inertial observers. A well-known fact in special relativity is that the composition of boosts $\mathbb{R}p \to \mathbb{R}q \to \mathbb{R}r \to \mathbb{R}p$ is a spatial rotation called the *Wigner rotation.* Let us give a coordinate-free proof of this phenomenon at the level of the kinematic space $\mathcal{K}$. In the next proposition we consider, without loss of generality, that $\dim\mathcal{K} = 2$ and that the kinematic space is (arbitrarily) oriented.

3.4.1. Proposition (Wigner Rotation). *Let $\boldsymbol{p}_i \in \mathcal{K}$, $i = 1,2,3$, be inertial observers and let $G_{ij} := G\wr\boldsymbol{p}_i,\boldsymbol{p}_j\wr$ be the geodesic connecting $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$. Let $h_1, h_2, H : \mathcal{K} \to \mathcal{K}$ stand for the hyperbolic isometries such that h_1 stabilizes G_{12} and $h_1(\boldsymbol{p}_1) = \boldsymbol{p}_2$; h_2 stabilizes G_{23} and $h_2(\boldsymbol{p}_2) = \boldsymbol{p}_3$; H stabilizes G_{13} and $H(\boldsymbol{p}_1) = \boldsymbol{p}_3$. Then $h_2h_1 = e_\theta H$, where $e_\theta : \mathcal{K} \to \mathcal{K}$ is the elliptic isometry that fixes $\boldsymbol{p}_3$ and whose angle of rotation $\theta \in [-\pi, \pi]$ is minus the oriented area of the triangle with vertices $(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$.*

Proof. Let $\boldsymbol{q}_1 \in G_{12}$ be the middle point of the geodesic segment joining $\boldsymbol{p}_1,\boldsymbol{p}_2$ and let $\boldsymbol{q}_2 \in G_{23}$ the middle point of the geodesic segment joining $\boldsymbol{p}_2,\boldsymbol{p}_3$. We have $h_1 = R_2R_1$ where R_1 stands for the reflection in the geodesic orthogonal to G_{12} passing through $\boldsymbol{q}_1$ and R_2, for the reflection in the geodesic orthogonal to G_{12} passing through $\boldsymbol{p}_2$. Similarly, $h_2 = R_4R_3$ where R_3 denotes the reflection in the geodesic orthogonal to G_{23} passing through $\boldsymbol{p}_2$ and R_4, the reflection in the geodesic orthogonal to G_{23} passing through $\boldsymbol{q}_2$. Lastly, let R_5 and R_6 be the reflections in the geodesics orthogonal to $G\wr\boldsymbol{q}_1,\boldsymbol{q}_2\wr$ passing respectively through $\boldsymbol{q}_2$ and $\boldsymbol{q}_1$ and let $h_3 : \mathcal{K} \to \mathcal{K}$, $h_3 := R_5R_6$, be a hyperbolic isometry that stabilizes the geodesic $G\wr\boldsymbol{q}_1,\boldsymbol{q}_2\wr$.

Note that R_1R_6, R_3R_2, and R_5R_4 are elliptic isometries such that $R_1R_6 = \sigma_2\sigma_1$, $R_3R_2 = \sigma_3\sigma_2$, and $R_5R_4 = \sigma_1\sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ stand respectively for the reflections in the geodesics $G\wr\boldsymbol{q}_1,\boldsymbol{q}_2\wr$, $G\wr\boldsymbol{q}_1,\boldsymbol{p}_2\wr$, and $G\wr\boldsymbol{p}_2,\boldsymbol{q}_2\wr$. Hence,

$$R_5h_2h_1R_6 = (R_5R_4)(R_3R_2)(R_1R_6) = (\sigma_1\sigma_3)(\sigma_3\sigma_2)(\sigma_2\sigma_1) = 1$$

which implies $h_2h_1 = R_5R_6 = h_3$. Now, note that $h_3H^{-1}(\boldsymbol{p}_3) = h_3(\boldsymbol{p}_1) = h_2\big(h_1(\boldsymbol{p}_1)\big) = \boldsymbol{p}_3$, and h_3 is obviously not the inverse of H, so h_3H^{-1} has to be an elliptic isometry e_θ fixing $\boldsymbol{p}_3$. In other words, $h_2h_1 = e_\theta H$.

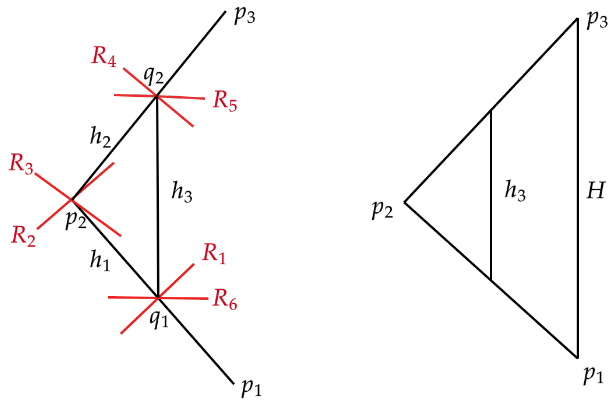

Figure 6: Proof of Proposition 3.4.1

The differential of a hyperbolic isometry, being applied to a vector tangent at a point of its stable geodesic, coincides with the parallel transport along this geodesic. So, since $h_2h_1H^{-1} = e_\theta$, we conclude that θ is minus the oriented area of the triangle $(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$ (the minus sign comes from the fact that the sum of the internal angles of a geodesic triangle in $\mathcal{K}$ is less than π or, equivalently, from the Gauss-Bonnet theorem). $\square$

3.4.2. Remark. Wigner rotation can also be seen as a measure of the non-commutativity of the rapidity addition (see Definition 3.2.1) as follows. Let $w_1, w_2 \in \mathrm{T}_{\boldsymbol{p}_1}\mathcal{K}$ be rapidities at $\boldsymbol{p}_1 \in \mathcal{K}$. Moreover, define $\boldsymbol{p}_2 := \exp_{\boldsymbol{p}_1} w_1$, $\boldsymbol{p}_3 := \exp_{\boldsymbol{p}_1}(w_1 \oplus w_2)$, $\boldsymbol{q}_2 := \exp_{\boldsymbol{p}_1} w_2$, and $\boldsymbol{q}_3 := \exp_{\boldsymbol{p}_1}(w_2 \oplus w_1)$. The triangles $(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$ and $(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)$ are clearly congruent and it is straightforward to see that the angle θ at $\boldsymbol{p}_1$ between the geodesic ray joining $\boldsymbol{p}_1,\boldsymbol{p}_3$ and the geodesic ray joining $\boldsymbol{p}_1,\boldsymbol{q}_3$ is given by $\theta = \pi - \sum_i \alpha_i = \mathrm{Area}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$, where the α_i's stand for the internal angles of the triangle $(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$.

3.5 An invariant of three points and causality

Let us take a look at a relativistic interpretation of the algebraic invariant

$$\eta(\boldsymbol{p},\boldsymbol{q},\boldsymbol{u}) := \frac{\langle u,p\rangle\langle p,q\rangle\langle q,u\rangle}{\langle p,p\rangle\langle q,q\rangle\langle u,u\rangle} \tag{3.5.1}$$

of two inertial observers $\boldsymbol{p},\boldsymbol{q} \in \mathcal{K}$ and a point $\boldsymbol{u} \in \mathcal{G}$ in de Sitter space.

The invariant $\eta(\boldsymbol{p},\boldsymbol{q},\boldsymbol{u})$ determines whether $\boldsymbol{p}$ and $\boldsymbol{q}$ agree or disagree on the order of occurrence of an event that happened at time $t=0$ and a space-like event $u\in\mathbb{M}^{n+1}$. Indeed, the observers agree or disagree respectively when the sign of

$$\frac{\big\langle \pi'[\boldsymbol{p}]u,\pi'[\boldsymbol{q}]u\big\rangle}{\langle u,u\rangle}=\frac{\Big\langle \frac{\langle u,p\rangle}{\langle p,p\rangle}p,\frac{\langle u,q\rangle}{\langle q,q\rangle}q\Big\rangle}{\langle u,u\rangle}=\eta(\boldsymbol{p},\boldsymbol{q},\boldsymbol{u})$$

is negative or positive. At the level of the extended kinematic space, this can be translated as follows: the observers agree/disagree exactly when $\boldsymbol{p},\boldsymbol{q}$ lie in the same/in distinct components of $\mathcal{K}\setminus G$, where G is the geodesic with polar point u (this can be inferred by looking at the relative position between $\mathbb{R}p$, $\mathbb{R}q$, and $u^\perp$). A usual way of saying that there will always exist observers that do not agree on the occurrence order of spacelike separated events is that causality is not well defined for this kind of events.

3.6 Dynamics

At a first glance it may seem that, when passing from Minkowski space to kinematic space, one loses information, obtaining a space that models well kinematic phenomena but is not suited to described dynamics. This subsection is intended to illustrate that this is not the case.

Let $\xi:I\to\mathbb{M}^{n+1}$ be a smooth curve such that $\xi(0)=0$, $\big\langle\dot\xi(\tau),\dot\xi(\tau)\big\rangle=-c^2$ (that is, ξ is parameterized by proper time), and $\dot\xi(\tau)$ is future-oriented for every $\tau\in I$. It gives rise to the curve $\zeta(\tau)=\mathbb{P}_\mathbb{R}\dot\xi(\tau)$ in the scaled kinematic space $\mathcal{K}^c$, where $\mathbb{P}_\mathbb{R}\dot\xi(\tau)$ stands for the image of $\dot\xi(\tau)$ under the canonical projection $\mathbb{M}^{n+1}\to\mathbb{P}^n_\mathbb{R}$. Conversely, given a smooth curve $\zeta:I\to\mathcal{K}^c$, there exists a unique lift $\zeta_0:I\to\mathbb{M}^{n+1}$ of ζ to $\mathbb{M}^{n+1}$ such that $\big\langle\zeta_0(\tau),\zeta_0(\tau)\big\rangle=-c^2$ and $\zeta_0(\tau)$ is future-oriented for every $\tau\in I$. Now, there exists a unique smooth curve $\xi:I\to\mathbb{M}^{n+1}$ such that $\xi(0)=0$ and $\dot\xi(\tau)=\zeta_0(\tau)$ for every $\tau\in I$.

Let us see that a tangent vector to the curve ζ is nothing but the $(n+1)$-acceleration of ξ in view of the identification $\mathrm{T}_{\zeta(\tau)}\mathcal{K}^c\simeq\zeta(\tau)^\perp$ (see Remark 3.2.3). On one hand, as a linear map $\mathrm{T}_{\zeta(\tau)}\mathcal{K}^c=\mathrm{Lin}\big(\mathbb{R}\zeta(\tau),\zeta(\tau)^\perp\big)$,

$$\dot\zeta(\tau):\zeta_0(\tau)\mapsto\pi\big[\zeta(\tau)\big]\dot\zeta_0(\tau)=\pi\big[\zeta(\tau)\big]\ddot\xi(\tau)$$

by [4, Lemma A.1]. On the other hand, $\pi\big[\zeta(\tau)\big]\ddot\xi(\tau)=\ddot\xi(\tau)$ since $\big\langle\dot\xi(\tau),\dot\xi(\tau)\big\rangle$ is constant.

The curve ζ can be interpreted as the list of inertial frames occupied by the observer with worldline ξ (that is, $\zeta(\tau)$ is the inertial frame occupied at the instant τ). Note that, if ζ is constant, $\zeta(\tau)=\boldsymbol{p}$ for every τ, then ξ is a straight line in $\mathbb{M}^{n+1}$ passing through the origin (the worldline $\mathbb{R}p$ of an inertial observer, as expected); when ζ is a geodesic, ξ is a hyperbola that represents a motion with constant $(n+1)$-acceleration (a.k.a. hyperbolic motion).

Finally, let $A=A(\boldsymbol{p},\tau)$, $\boldsymbol{p}\in\mathcal{K}$, $\tau\in\mathbb{R}$, be a smooth time-dependent vector field in $\mathcal{K}$. Let ζ be the maximal integral curve of A corresponding to the initial conditions $\boldsymbol{p}_0\in\mathcal{K}$ and $\tau_0\in\mathbb{R}$, that is, $\dot\zeta(\tau)=A\big(\zeta(\tau),\tau\big)$ and $\zeta(\tau_0)=\boldsymbol{p}_0$ (such an integral curve exists and is unique by [9, Theorem 9.48]). The ξ obtained from ζ as above is nothing but the dynamics associated to the time-dependent force field $F=mA$, where m is the rest mass of an observer whose worldline is ξ.

References

[1] S. Anan'in, E. C. B. Gonçalves, and Grossi. C. H. Grassmannians and conformal structure on absolutes. *Advances in Applied Clifford Algebras*, 29(5), 2019.

[2] S. Anan'in and C. H. Grossi. Coordinate-free classic geometry. *Moscow Mathematical Journal*, 11(4):633–655, 2011.

[3] S. Anan'in and C. H. Grossi. Differential geometry of grassmannians and the plücker map. *Central European Journal of Mathematics*, 10(3):873–884, 2012.

[4] S. Anan'in, C. H. Grossi, and N. Gusevskii. Complex hyperbolic structures on disc bundles over surfaces. *International Mathematics Research Notices*, 2011(19):4285–4375, 2011.

[5] E. Borel. La cinématique dans la théorie de la relativité. *Comptes Rendus des Séances de l'Académie des Sciences*, 157:703–705, 1913.

[6] E. Borel. La théorie de la relativité et la cinématique. *Comptes Rendus des Séances de l'Académie des Sciences*, 156:215–218, 1913.

[7] V. Charette and T. Drumm. Complete lorentzian 3-manifolds. *Contemporary Mathematics*, 639:43–72, 2015.

[8] R. Ferreira and C. H. Grossi. A note on the kinematic space of classical mechanics. In preparation, 2020.

[9] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2013.

[10] J.-F. Quint. An overview of Patterson-Sullivan theory. *Workshop "The barycenter method", FIM, Zurich*, 2006.

[11] W. Rindler. *Relativity: Special, General, and Cosmological*. Oxford University Press, 2 edition, 2006.

[12] A. Ungar. *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, 2008.

[13] V. Varićak. Anwendung der lobatschefskijschen geometrie in der relativtheorie. *Physikalische Zeitschrift*, 11:93–96, 1910.

RAFAEL FERREIRA
DEPARTAMENTO DE MATEMÁTICA, ICMC, UNIVERSIDADE DE SÃO PAULO, SÃO CARLOS, BRAZIL
rafael.ferreira.pereira@usp.br

JOÃO DOS REIS JUNIOR
DEPARTAMENTO DE MATEMÁTICA, ICMC, UNIVERSIDADE DE SÃO PAULO, SÃO CARLOS, BRAZIL
joao.reis.reis@usp.br

CARLOS H. GROSSI
DEPARTAMENTO DE MATEMÁTICA, ICMC, UNIVERSIDADE DE SÃO PAULO, SÃO CARLOS, BRAZIL
grossi@icmc.usp.br

BIBLIOGRAPHY

ANAN'IN, S.; GROSSI, C. H. Basic coordinate-free non-euclidean geometry. Available on arXiv:1107.0346. 2011. Citation on page 25.

____. Coordinate-free classic geometry. **Moscow Mathematical Journal**, v. 11, n. 4, p. 633–655, 2011. Citations on pages 22, 25, and 31.

BOREL, E. La cinématique dans la théorie de la relativité. **Comptes Rendus des Séances de l'Académie des Sciences**, v. 157, p. 703–705, 1913. Citations on pages 19 and 20.

____. La théorie de la relativité et la cinématique. **Comptes Rendus des Séances de l'Académie des Sciences**, v. 156, p. 215–218, 1913. Citations on pages 19 and 20.

FERREIRA, R.; REIS, J. dos; GROSSI, C. H. On the geometry of the kinematic space in special relativity. 2020. Available at arXiv:2012.00681. Citations on pages 22, 23, and 24.

GOURGOULHON, E. **Special Relativity in General Frames: From Particles to Astrophysics**. [S.l.]: Springer, 2013. Citation on page 39.

QUINT., J.-F. An overview of Patterson-Sullivan theory. **Workshop "The barycenter method", FIM, Zurich**, 2006. Citation on page 23.

RINDLER, W. **Relativity: Special, General, and Cosmological**. 2. ed. [S.l.]: Oxford University Press, 2006. Citations on pages 21 and 47.

UNGAR, A. **Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity**. [S.l.]: World Scientific, 2008. Citations on pages 21, 43, 44, 46, 48, 49, and 50.

VARIĆAK, V. Anwendung der lobatschefskijschen geometrie in der relativtheorie. **Physikalische Zeitschrift**, v. 11, p. 93–96, 1910. Citation on page 19.

APPENDIX

A

HERMITIAN FORMS

A.1 Hermitian Forms

This section is intended to present the basic tools of Hermitian algebra, which are necessary in Chapters 2 and 4. In what follows, $\mathbb{K}$ denotes the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$.

A.1.1. Definition. Let V be a vector space over $\mathbb{K}$. A *Hermitian form* in V is a map $\langle -,-\rangle : V \times V \to \mathbb{K}$, $(v,w) \mapsto \langle v,w\rangle$, linear in the first entry and such that $\langle v,w\rangle = \overline{\langle w,v\rangle}$ for all $v,w \in V$.

In particular, if $\langle -,-\rangle$ is a Hermitian form, then

$$\langle u,v+w\rangle = \overline{\langle v+w,u\rangle} = \overline{\langle v,u\rangle + \langle w,u\rangle} = \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle} = \langle u,v\rangle + \langle u,w\rangle$$

and

$$\langle v,kw\rangle = \overline{\langle kw,v\rangle} = \overline{k\langle w,v\rangle} = \overline{k} \cdot \overline{\langle w,v\rangle} = \overline{k} \cdot \langle v,w\rangle$$

for all $u,v,w \in V$ and $k \in \mathbb{K}$. Clearly, when $\mathbb{K} = \mathbb{R}$, a Hermitian form is nothing but a symmetric bilinear form. A $\mathbb{K}$-vetor espace endowed with a Hermitian form is called a *Hermitian space.*

A.1.2. Definition. Let V be a Hermitian space and let $W \leqslant V$ be a linear subspace. We define

$$W^{\perp} := \big\{v \in V | \langle v,W\rangle = 0\big\},$$

where $\langle v,W\rangle$ denotes the set $\big\{\langle v,w\rangle \mid w \in V\big\} \subset \mathbb{K}$. In this way, the kernel of the form is nothing but $V^{\perp}$. When $V^{\perp} = 0$, we say that V is *nondegenerate.* Let $U,W \leqslant V$ be linear subspaces of V. We define the *orthogonal of W relatively to U* as $W^{\perp U} := W^{\perp} \cap U$.

A.1.3. Definition. A basis $\beta = (b_1,b_2,...,b_n)$ for a Hermitian space is *orthonormal* if $\langle b_i,b_i\rangle \in \{-1,0,1\}$ for all i and if $\langle b_i,b_j\rangle = 0$ for all i,j, $i \neq j$. Let β_-,β_0, e β_+ be the number of vectors in the basis β such that $\langle b_i,b_i\rangle$ is, respectively, $-1,0,1$. The triple $(\beta_-,\beta_0,\beta_+)$ is called the *signature* of the basis.

A.1.4. Definition. Let $(v_1, v_2, ..., v_k)$ be a k-tuple of vectors in a Hermitian space V. The *Gram matrix* of this k-tuple is defined as $G := [g_{ij}]$, where $g_{ij} := \langle v_i, v_j \rangle$.

It is immediate from the properties of the Hermitian form that the Gram matrix satisfies $G^* = G$, where G^* is the transpose conjugate matrix of G, *i.e.*, G is Hermitian. Thus, the Gram matrix of an orthonormal basis is diagonal with entries -1, 0 and 1.

A.1.5. Lemma. Let $\beta = (b_1, b_2, ..., b_n)$ be an orthonormal basis in a Hermitian space V. Then $\beta_0 = \dim V^{\perp}$.

Proof. It is sufficient to show that $\beta' := \{b_j \in \beta \mid \langle b_j, b_j \rangle = 0\}$ is a basis of $V^{\perp}$. Let $v \in V^{\perp}$. We write $v = \sum_{i=1}^{n} \alpha_i b_i$, $\alpha_i \in \mathbb{K}$, $1 \leqslant i \leqslant n$. If $\langle b_j, b_j \rangle \neq 0$, then $\alpha_j = 0$ because $0 = \langle v, b_j \rangle = \alpha_j \langle b_j, b_j \rangle$. ■

The main result in Hermitian algebra, known as *Sylvester's Law of Inertia*, states that the signature is an intrinsic quantity of the Hermitian space. In other words, the signature of any orthonormal basis in a Hermitian space is always the same (and, besides that, every Hermitian space admits an orthonormal basis). This way, we can refer to the signature of a Hermitian space without mentioning any particular orthonormal basis.

A.1.6. Theorem (Sylvester's Law of Inertia). The signature doesn't depend on the choice of orthonormal basis.

Proof. Let $\beta = (b_1, \ldots, b_n)$ and $\beta' = (b'_1, \ldots, b'_n)$ be orthonormal basis of signatures $(\beta_-, \beta_0, \beta_+)$ and $(\beta'_-, \beta'_0, \beta'_+)$, respectively. By Lemma A.1.5 we know that $\beta_0 = \beta'_0 = \dim V^{\perp}$ and, therefore, it doesn't depend on the choice of basis. Taking $V/V^{\perp}$ in place of V, we can assume that V is a nondegenerate Hermitian space (it is easy to see that $V = V^{\perp} \oplus V/V^{\perp}$).

Now, we proceed by induction on $\dim V$. If $\beta_- = 0$, we have $\langle v, v \rangle \geqslant 0$ for all $v \in V$ and, hence, $\beta'_- = 0$. Similarly, $\beta_+ = 0$ implies $\beta'_+ = 0$. In this way, we can assume $\langle b_n, b_n \rangle = 1$ and $\langle b'_n, b'_n \rangle = -1$. We define

$$W := \mathbb{K}b_n + \mathbb{K}b'_n, \quad U := (\mathbb{K}b_n)^{\perp}, \quad U' := (\mathbb{K}b'_n)^{\perp}.$$

Note that $U = \mathbb{K}b_1 + \mathbb{K}b_2 + ... + \mathbb{K}b_{n-1}$ and $U' = \mathbb{K}b'_1 + \mathbb{K}b'_2 + ... + \mathbb{K}b'_{n-1}$. Thus, the signatures of the basis indicated in U and U' are respectively $(\beta_-, 0, \beta_+ - 1)$ and $(\beta'_- - 1, 0, \beta'_+)$. Moreover, a direct calculation shows that W is nondegenerate and, therefore, $W^{\perp}$ is also nondegenerate.

By the relation $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$, which holds for any subspaces W_1 and W_2 of a Hermitian space, we have $W^{\perp} = U \cap U'$. Since $U \cap U'$ is nondegenerate, we have the following orthogonal decompositions:

$$U = (U \cap U') \oplus (U \cap U')^{\perp U}$$

$$U' = (U \cap U') \oplus (U \cap U')^{\perp U'}.$$

Let α, γ and γ' be orthonormal basis respectively in $U \cap U'$, $(U \cap U')^{\perp U}$, and $(U \cap U')^{\perp U'}$. So, we have that $\alpha \cup \gamma$ and $\alpha \cup \gamma'$ are orthonormal basis in U and U' respectively. And since, by the induction hypothesis, the signature in U and U' doesn't depend on the basis, we can write the following relations:

$$(\beta_-, 0, \beta_+ - 1) = \big((\alpha \cup \gamma)_-, (\alpha \cup \gamma)_0, (\alpha \cup \gamma)_+\big) = (\alpha_-, 0, \alpha_+) + (\gamma_-, \gamma_0, \gamma_+)$$

$$(\beta'_- - 1, 0, \beta'_+) = \big((\alpha \cup \gamma')_-, (\alpha \cup \gamma')_0, (\alpha \cup \gamma')_+\big) = (\alpha_-, 0, \alpha_+) + (\gamma'_-, \gamma'_0, \gamma'_+).$$

Now, it remains to prove that $(U \cap U')^{\perp U} = (\mathbb{K}b_n)^{\perp W}$ and that $(U \cap U')^{\perp U'} = (\mathbb{K}b'_n)^{\perp W}$, because this implies that $(\gamma_-, \gamma_0, \gamma_+) = (1,0,0)$ and that $(\gamma'_-, \gamma'_0, \gamma'_+) = (0,0,1)$.

Being W and V nondegenerate, we have

$$(U \cap U')^{\perp U} = (U \cap U')^{\perp} \cap U = W^{\perp\perp} \cap U = W \cap (\mathbb{K}b_n)^{\perp} = (\mathbb{K}b_n)^{\perp W}.$$

Similarly, $(U \cap U')^{\perp U'} = (\mathbb{K}b'_n)^{\perp W}$. ■

Finally, it remains to solve the following problem: if we have a (reasonably) arbitrary basis in V, how can we measure, using this basis, the signature of V without having to explicitly finding an orthonormal basis? The answer of that question is Sylvester's criterion.

A.1.7. Theorem (Sylvester's Criterion). Let V be a nondegenerate Hermitian space, let $\gamma = (b_1, ..., b_n)$ be a basis in V, and let G be the Gram matrix of γ. We will assume that, for all k, the submatrix[1] G_k of G has non vanishing determinant. Then the signature of V is given by $(n_-, 0, n_+)$, where n_- and n_+ are respectively the amount of negative and positive numbers in the sequence

$$\det G_1, \quad \frac{\det G_2}{\det G_1}, \quad \frac{\det G_3}{\det G_2}, \quad \ldots, \quad \frac{\det G_n}{\det G_{n-1}}.$$

The proof of the Sylvester's criterion consists of observing that, by applying to the basis γ an orthonormalization process similar to Gram Schmidt's, the signs of the determinants $\det G_k$ don't change; clearly, when we arrive at an orthonormal basis (with diagonal Gram matrix), the criterion measures the signature correctly.

[1] the submatrix G_k is the $k \times k$-matrix formed by the first k rows and by the first k columns of G.

APPENDIX

B

A PROOF OF REMARK 4.3.2.9

Here we present a proof, using the tools shown in Chapter 2, of the fact that Möbius addition satisfies the same geometric construction as the relativistic addition of velocities, and to do so, we begin by the following lemma.

B.1. Lemma. Let $a,b \in \mathbb{D} \cap \mathbb{R}$. Möbius addition $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ satisfies the rule:

$$d(a \oplus b, 0) = d(a,0) + d(b,0)$$

where $d : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ is the hyperbolic distance in Poincaré's disk model.

Proof. The value of $d(a \oplus b, 0)$ in Poincaré's disk model is given by:

$$d(a \oplus b, 0) = \ln \frac{1 + \frac{a+b}{1+ab}}{1 - \frac{a+b}{1+ab}} = \ln \frac{1+a}{1-a} + \ln \frac{1+b}{1-b} = d(a,0) + d(b,0)$$

where ln is the natural logarithm. ■

B.2. Theorem (Geometric Construction of Möbius Addition). Let $a,b \in \mathbb{H}^1_{\mathbb{C}}$, where $\mathbb{H}^1_{\mathbb{C}}$ is the disk $\mathbb{D}$ provided with Poincaré's model hyperbolic distance d (see Chapter 2, Subsection 2.7). Let G be the geodesic passing through a and the origin O. Let b_G be the hyperbolic projection of b in the geodesic G (*i.e*, the intersection between G and the geodesic orthogonal to G passing through b). Now take H to be the hypercycle of G passing through b, and G' to be the geodesic orthogonal to G and passing trough $a \oplus b_G$. Then $\{a \oplus b\} = G' \cap H$.

Proof. Let $v_1 = (1,-1)$ and $v_2 = (1,1)$ be the vertices of the geodesic G (in homogeneous coordinates in $\mathbb{H}^1_{\mathbb{C}}$). It suffices to show that $a \oplus b = Ib$, where I is the hyperbolic isometry that stabilizes G and sends O to a (indeed, by Lemma B.1, $Ib_G = a \oplus b_G$). Let $a = (1,r)$. In the basis v_1, v_2 we take the representatives $a = \sqrt{\frac{1-r}{1+r}}\, v_1 + \sqrt{\frac{1+r}{1-r}}\, v_2$ and $O = v_1 + v_2$. It is now easy to see that, in the basis v_1, v_2,

$$I = \begin{bmatrix} \sqrt{\frac{1-r}{1+r}} & 0 \\ 0 & \sqrt{\frac{1+r}{1-r}} \end{bmatrix}.$$

We have $b = zv_1 + \frac{1}{z}v_2 \simeq \left(1, \frac{1-z^2}{1+z^2}\right)$ for some $z \in \mathbb{C}$. So, on one hand,

$$a \oplus b = \left(1, \frac{r + \frac{1-z^2}{1+z^2}}{1 + r \cdot \frac{1-z^2}{1+z^2}}\right) = \left(1, \frac{r(1+z^2)+1-z^2}{1+z^2+r(1-z^2)}\right).$$

On the other hand,

$$Ib = z\sqrt{\frac{1-r}{1+r}}v_1 + \frac{1}{z}\sqrt{\frac{1+r}{1-r}}v_2 \simeq \left(1, \frac{1-z^2\left(\frac{1-r}{1+r}\right)}{1+z^2\left(\frac{1-r}{1+r}\right)}\right) =$$

$$= \left(1, \frac{1+r-z^2(1-r)}{1+r+z^2(1-r)}\right) = \left(1, \frac{r(1+z^2)+1-z^2}{1+z^2+r(1-z^2)}\right)$$

which completes the proof. ■

ICMC USP
SÃO CARLOS