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Instituto de Ciências Matemáticas e de Computação

# On Betti numbers for symmetric powers of modules and some applications

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**Johnny Albert dos Santos Lima**

# Sobre números de Betti para potências simétricas de módulos e aplicações

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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**Agosto de 2022**

*Este trabalho é dedicado a todos os pesquisadores brasileiros que,*
*mesmo enfrentando épocas tempestuosas, não perderam a determinação para realizarem seus sonhos.*

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*“No great discovery was ever made without a bold guess.”*

*(Isaac Newton)*

# RESUMO

LIMA, J.A. **Sobre números de Betti para potências simétricas de módulos e aplicações**. 2022. 67 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Seja $M$ um módulo finitamente gerado sobre um anel local $(R,\mathfrak{m})$. Por $S_j(M)$, denotamos a $j$-ésima potência simétrica de $M$($j$-ésima componente graduada da álgebra simétrica $S_R(M)$). O propósito desta tese é investigar a resolução livre minimal de $S_j(M)$ como $R$-módulo para cada $j \geq 2$ e determinar os números de Betti de $S_j(M)$ em termos dos números de Betti de $M$. Isso tem algumas aplicações, por exemplo para ideais de tipo linear $I$, obtemos fórmulas dos números de Betti de $I^j$ em termos dos números de Betti de $I$. Além disso, estabelecemos cotas superiores e inferiores para os números de Betti de $S_j(M)$ em termos dos números de Betti de $M$. Em particular, obtemos algumas aplicações sobre a famosa conjectura de Buchsbaum-Eisenbud-Horrocks.

**Palavras-chave:** Álgebra simétrica, Potência simétrica, Tipo linear, Betti numbers, Resoluções livres minimais.

# ABSTRACT

LIMA, J.A. **On Betti numbers for symmetric powers of modules and some applications**. 2022. 67 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Let $M$ a finitely generated module over a local ring $(R, \mathfrak{m})$. By $S_j(M)$, we denote the $j$th symmetric power of $M$ ($j$th graded component of the symmetric algebra $S_R(M)$). The purpose of this thesis is to investigate the minimal free resolutions $S_j(M)$ as $R$-module for each $j \geq 2$ and determine the Betti numbers of $S_j(M)$ in terms of the Betti numbers of $M$. This has some applications, for example for linear type ideals $I$, we obtain formulas of the Betti numbers $I^j$ in terms of the Betti numbers of $I$. In addition, we establish upper and lower bounds of Betti numbers of $S_j(M)$ in terms of Betti numbers of $M$. In particular, obtain some applications about the famous Buchsbaum-Eisenbud-Horrocks conjecture.

**Keywords:** Symmetric algebra, Symmetric power, Linear type, Betti numbers, Minimal free resolutions.

# LIST OF TABLES

# LIST OF SYMBOLS

$S_j(M)$ — $j$th symmetric power of $M$

$S_R(M)$ — Symmetric algebra of $M$

$\mathscr{T}_R(M)$ — Torsion module of $M$

$\mathbf{F}_\bullet$ — Free or minimal free resolution of a module

$\beta_n^R(M)$ — $n$th Betti number of $M$

$\operatorname{grade}(I)$ — Lenght of maximal $R$-sequence in $I$

$\operatorname{pd}_R M$ — projective dimension of $M$

$\operatorname{Ass} M$ — Associated primes of $M$

$D_j(F)$ — $j$th divided power of a free module $F$

$\dim R$ — Krull dimension of $R$

$\mathscr{R}_R M$ — Rees algebra of $M$

$\mathscr{R}_j(M)$ — $j$th graded component of Rees algebra of $M$

$S \times_k T$ — Fiber product

# CONTENTS

CHAPTER

# 1

# INTRODUCTION

Throughout this thesis, we assume that $(R,\mathfrak{m})$ be a Noetherian local ring with identity and every $R$-module $M$ is finitely generated over $R$. For an $R$-module $M$, we denoted $S_j(M)$ as the $j$th symmetric power of $M$ or $j$th graded component of symmetric algebra of $S_R(M)$.

Our main goal in this thesis is to examine some homological properties related to the symmetric powers of a finitely generated module. More precisely, we are interested in computing the minimal free resolution and the Betti numbers of the symmetric powers of a finitely generated module. The motivation for this investigation came from the works (WEYMAN, 1979), (AVRAMOV, 1981) and (MOLICA; RESTUCCIA, 2002), where they study the acyclicity of the complexes $S_j\mathbf{F}_\bullet$ that are associated with the symmetric powers, $S_j(M)$ . Due to the acyclicity criteria established in (WEYMAN, 1979; AVRAMOV, 1981; MOLICA; RESTUCCIA, 2002), we noticed that not all finite projective dimension modules allow their symmetric powers to have a minimal free resolution coming from the minimal resolution of $M$. Motivated by this, the following question naturally arise:

**Question A**: If the projective dimension of $M$ is finite, then is the projective dimension of $S_j(M)$ finite for all $j \geq 2$?

**Question B**: Is it possible to determine the Betti numbers of $S_j(M)$ knowing the Betti numbers of $M$?

A summary of the content of this work is:

In chapter 2 we present general facts of the theory used in the other chapters, as well as fix the notation.

Chapter 3 is the heart of this work. In sections 3.1 and 3.2 we emphasize the construction of the complex $S_j\mathbf{F}_\bullet$ and later, in Theorem 10, we show that it produces a minimal free resolution for $j$th symmetric power of a finitely generated module $M$. As a consequence of this fact, we obtain the finiteness of the projective dimension of $S_j(M)$ in the case where $M$ has a finite projective dimension and we explain, Corollary 4, $\mathrm{pd}_R\, S_j(M)$ as a function of the Betti numbers of $M$. This last fact also shows that there is a finite amount of powers $j$, such that $S_j\mathbf{F}_\bullet$ is a minimal free resolution of $S_j(M)$ over a local ring of dimension $d$, Remark 15. And the most important consequence of this work, Corollary 8, where we establish a formula, in particular a criterion, that expresses the Betti numbers of $S_j(M)$ as a function of the Betti numbers of $M$.

Chapter 4 is dedicated to two applications. The first one is the Buchsbaum-Eisenbud-Horrocks conjecture or as we denote in this work, conjecture (BEH). This conjecture says that if $M$ is a finite-length, finite-dimensional $R$-module over a local Noetherian ring of dimension $d$, then for all $i \geq 0$,

$$\beta_i^R(M) \geq \binom{d}{i}.$$

The conjecture is not yet solved but it already has a positive answer for local rings where $d \leq 4$, see (AVRAMOV; BUCHWEITZ, 1993). What we observe here is that the symmetric powers of $M$, satisfy the same inequality, for modules of projective dimension 1 such that the $(SW_j)$ and $\beta_1^R(M) \geq d$ conditions are satisfied. In other words, we get that

$$\beta_t^R(S_j(M)) \geq \binom{d}{t} \text{ for all } t = 0,\ 1, \ldots,\ \min\{\beta_1^R(M), j\},$$

if $M$ is a $R$-module of projective dimension 1, which satisfies the $(SW_j)$ and $\beta_1^R(M) \geq d$ conditions.

The second part of this chapter, where we use the results of (AVRAMOV, 1981) and (FUKUMURO; KUME; NISHIDA, 2015), is motivated to find a class of modules that satisfy the $(SW_j)$ condition. Here we can verify that modules of projective dimension 1 and that are of linear type, that is, their Rees Algebra is isomorphic to its Symmetric Algebra, satisfy the $(SW_j)$ condition for all $j = 1, \ldots,\ \beta_1^R(M)$. In particular, we get the equality

$$\beta_t^R(I^j) = \binom{\beta_0^R(I) + j - t - 1}{j - t}\binom{\beta_1^R(I)}{t}, \text{ for all } t = 0,\ 1, \ldots,\ \min\{\beta_1^R(M), j\}.$$

A more general version of the above equality is given in the Proposition 16, where we assume $\mathrm{pd}_R\, M > 1$ and that the ideal satisfies the $(SW_j)$ condition. The reason for this is that linear ideals that have a projective dimension greater than 1 do not always satisfy the condition $(SW_j)$, as shown in the example 6.

In the thesis, examples are also given and are calculated with help of MACAULAY2.

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CHAPTER 2

# PRELIMINARIES

## 2.1 The symmetric algebra of a finitely generated module

In this section, we will present some basic tools that involve the symmetric algebra of a finitely generated module, found in the literature. In addition to its construction, we will see that it is possible to interpret it when we are dealing with free modules or even finitely generated modules. For example, using the fact that the symmetric algebra is a covariant functor, every finitely generated $R$-module $M$ can be seen as a quotient of a ring of polynomials of the form

$$S_R(M) \cong R[X_1,\, X_2,\, X_3, \ldots,\, X_n]/J, \text{ where } J = \left( \sum_{1=1}^{n} a_{i_1} X_i, \ldots, \sum_{i=1}^{n} a_{i_m} X_i \right).$$

Let be $M$ a non-zero finitely generated $R$-module and suppose that $M$ is generated by $m_1,\, m_2, \ldots,\, m_l$, where $m_i \in M \ \ \forall i = 1, \ldots,\, l$. Let $j$ an arbitrary integer non negative, we define

$$T^j(M) = \begin{cases} R, & \text{if} \quad j = 0; \\ M, & \text{if} \quad j = 1; \\ \bigotimes_{i=1}^{j} M, & \text{if} \quad j > 1. \end{cases} \tag{2.1}$$

and

$$\mathfrak{t}_j(M) = \begin{cases} \{0\} \subset R, & \text{if} \quad j = 0; \\ \{0\} \subset M, & \text{if} \quad j = 1; \\ \text{the submodule of } T^j(M) & \\ \text{generated by elements of form} & \\ m_{i_1} \otimes m_{i_2} \cdots \otimes m_{i_j} - m_{i_{\sigma(1)}} \otimes m_{i_{\sigma(2)}} \otimes \cdots m_{i_{\sigma(j)}}, & \text{if} \quad j > 1, \\ \text{where } \sigma \text{ denotes a permutation on } \{1, \ldots, l\}. & \end{cases} \tag{2.2}$$

Then consider the $R$-module defined by

$$S_R(M) := \bigoplus_{j \geq 0} S_j(M)$$

where $S_j(M) = T^j(M)/\mathfrak{t}_j(M)$. Then $S_R(M)$ with the multiplication defined by:

$$g_1 \otimes g_2 \otimes \cdots \otimes g_r \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_s$$

for each $g_1 \otimes g_2 \otimes \cdots \otimes g_r \in S_r(M)$ and $h_1 \otimes h_2 \otimes \cdots \otimes h_s \in S_s(M)$ it is an graded $R$-algebra.

**Definition 1.** The $R$-algebra $S_R(M)$ is called *symmetric algebra* of $M$ and $S_j(M)$ is called *jth component* of symmetric algebra (or *symmetric power*) of $M$.

When $M$ is a non-zero finitely generated $R$-module on a Noetherian ring we one could ask if $S_R(M)$ and its symmetric powers also acquire the same property. To answer such a question we may observe at the following proposition.

**Proposition 1.** Let $M$ be a finitely generated $R$-module. Then

a) For any $B$ $R$-algebra and all $\phi : M \to B$ $R$-module homomorphism there exists a unique $R$-algebra homomorphism $\psi : S_R(M) \to B$ such that it commutes the following diagramm:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & B \\ {\scriptstyle \mathfrak{i}}\downarrow & \nearrow_{\psi} & \\ S_R(M) & & \end{array}$$

Where $\mathfrak{i} : M \to S_R(M)$ denotes the inclusion.

b) If $B$ is a $R$-algebra then

$$S_R(M) \otimes_R B \cong S_B(M \otimes B).$$

c) For any M and N $R$-modules, it holds that

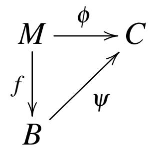
$$S_R(M \oplus N) = S_R(M) \otimes S_R(N).$$

*Demonstration.* See (D.EISENBUD, 1995), Appendix 2. □

**Proposition 2.** Let $M$ be a finitely generated $R$-module. Suppose that $B$ be a $\mathbb{N}$-graded $R$-algebra and $f : M \to B$ be a $R$-module injective homomorphism such that:

a) $B$ is generated by $f(M)$ as $R$-algebra.

b) For every $C$ $R$-algebra and any $\phi : M \to C$ $R$-module homomorphism there exists a unique $R$-algebra homomorphism $\psi : B \to C$ such that it commmutes the diagramm:




Then $S_R(M) \cong B$.

*Demonstration.* See (D.EISENBUD, 1995), Appendix 2.

□

The next two results give us properties that characterize symmetric algebras of free and finitely generated modules.

**Proposition 3.** Let $M$ be a free $R$-module with rank $n$. Then $S_R(M) \cong R[X_1, X_2, \ldots, X_n]$.

*Demonstration.* Let $e_1, \ldots, e_n$ be a basis for $M$. Consider the $R$-module homomorphism given by

$$\phi : M \to R[X_1, \ldots, X_n]$$
$$e_i \mapsto X_i, \ \ \forall i = 1, 2, \ldots, n.$$

As $M$ is free $\phi$ is well defined. Note that $\phi$ is injective hommomorphism and that $\phi(M)$ generate the $R$-algebra $R[X_1, \ldots, X_n]$. Let $B$ be a $R$-algebra and $\psi : M \to B$ be a $R$-modulo homomorphism. Define

$$\widetilde{\psi} : R[X_1, X_2, \ldots, X_n] \to B$$
$$\widetilde{\psi}(f(X_1, X_2, \ldots, X_n)) = f(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)).$$

By definition of $\widetilde{\psi}$ and $\phi$ we have that $\widetilde{\psi} \circ \phi = \psi$. Therefore $\widetilde{\psi}$ it is uniquely determined. Now using the proposition 2 we will be have to $S_R(M) \cong R[X_1, \ldots, X_n]$. □

**Corollary 1.** Let $M$ be a free $R$-module with rank $n$. Then $S_R(M)$ is the polynomial ring on the "variables" $X_i$, and $S_j(M)$ is the free $R$-module of rank $\binom{n+j-1}{n-1}$.

Although free modules are finitely generated, we will see in the proposition below that the condition of linear independence is very crucial when we are computing symmetric algebras of these modules. Such cruciality can generate interesting questions such as, for example, the symmetric algebra of a free module $M$ over a Noetherian domain $R$ is always a domain. But this is not always the case when $M$ is finitely generated $R$-module.

**Theorem 1.** Let $M$ be a finitely generated $R$-module and suppose that $\phi = (a_{ij})_{m \times n}$ is the presentation matrix of $M$. Then

$$S_R(M) \cong R[X_1, X_2, \ldots, X_n]/J$$

where $J = (\sum_{1=1}^{n} a_{i_1}X_i, \ldots, \sum_{i=1}^{n} a_{i_m}X_i)$.

*Demonstration.* Suppose that $M =< m_1, m_2, \ldots, m_n >$ is a non-zero finitely generated $R$-module. Consider a free presentation of $M$, i.e, a exact sequence

$$R^m \xrightarrow{\phi} R^n \xrightarrow{\pi} M \longrightarrow 0$$

For each $j \geq 1$ we consider the $R$-module homomorphism given by

$$\Pi : T^j(R^n) \longrightarrow T^j(M)$$

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_j} \mapsto \pi(m_{i_1}) \otimes \pi(m_{i_2}) \otimes \cdots \otimes \pi(m_{i_j})$$

We have that Ker($\Pi$) is generated by products $x_1 \otimes x_2 \otimes \cdots \otimes x_j$ such that $x_i \in \text{Ker}(\pi) = \text{Im}(\phi) = \left(\sum_{i=1}^{n} a_{i_1}e_i, \ldots, \sum_{i=1}^{n} a_{i_m}e_i\right)$ for some $1 \leq i \leq j$. Follow that the kernel of induced homomorphism

$$S_j(R^n) \longrightarrow S_j(M) \tag{2.3}$$

is generated by elements of form

$$\left(\sum_{i=1}^{n} a_{i_1}e_i, \ldots, \sum_{i=1}^{n} a_{i_m}e_i\right).$$

So the $R$-algebra homomorphism

$$\Gamma : R[X_1, X_2, \ldots, X_n] \cong S(R^n) \longrightarrow S_R(M)$$

induced of 2.3 will have kernel the ideal

$$J = \left(\sum_{1=1}^{n} a_{i_1}X_i, \ldots, \sum_{i=1}^{n} a_{i_m}X_i\right). \tag{2.4}$$

□

The elements of ideal $J$ above are called of *symmetric algebra definition equations* $S_R(M)$.

In particular, it follows from the above characterizations that $S_R(M)$ is always Noetherian when $M$ is Noetherian $R$-module.

**Remark 1.** Several authors have investigated the case where $S_R(I)$ is a domain when $I$ is an ideal. And, some conditions on the $R$ ring, (HUNEKE, 1981, p. 113) showed that when $S_R(M)$ it is domain is equivalent to grade$(I_t(A)) \geq m+2-t$ for $1 \leq t \leq m$, for $R$-modules $M$ having having finite free resolution given by

$$0 \longrightarrow R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0, \qquad A = (a_{ij}).$$

**Definition 2.** Let be $R$ a ring with total fraction ring $Q$. The *torsion* of $M$ with respect to $R$ is the kernel of aplication

$$\tau : M \longrightarrow M \otimes_R Q,$$

which will be denoted by $\mathscr{T}_R(M)$. When $\mathscr{T}_R(M) = 0$ we say that the $R$-module $M$ is *torsion-free* module. If $\mathscr{T}_R(M) = M$ we say that $M$ is an of *torsion* module.

In particular, the torsion $\mathscr{T}_R(M)$ of torsion-free $R$-module $M$ over a domain $R$ it is a submodule of $M$. We will see that over domains, the symmetric algebra is an example of a $R$-module that is not necessarily torsion-free but that its torsion is still a prime ideal.

**Lemma 1.** Let be $R$ a domain and $M$ an finitely generated $R$-module. Then $\mathscr{T}_R(S_R(M))$ is a prime ideal.

*Demonstration.* Let be $Q = S^{-1}R$ the fraction field of $R$, where $S = R - \{0\}$. Since $M$ is an finitely generated $R$-module, then $M \otimes_R Q$ is a vector space of finite dimension. Therefore, let us suppose that $M \otimes_R Q$ has dimension $n$. So, by Proposition 3,

$$S_R(M \otimes Q) \cong Q[X_1, \ldots, X_n].$$

Now consider the aplication

$$\tau : S_R(M) \longrightarrow S_R(M)_R \otimes Q.$$

We obtain that $\mathscr{T}(S_R(M)) = \mathrm{Ker}(\tau)$. Hence

$$S_R(M)/\mathscr{T}(S_R(M)) \cong Q[X_1, \ldots, X_n].$$

Since $Q[X_1, \ldots, X_n]$ is a domain it yields that $\mathscr{T}(S_R(M))$ is a prime ideal. As we wanted to demonstrate.

□

**Remark 2.** $R$-modules $M$, in particular ideals, that satisfy the property of having torsion-free symmetric algebras belong to a class of modules that have been extensively studied today. An interesting characterization due to (AVRAMOV, 1981, p. 249) and later, redone in a more elementary way, in the paper (FUKUMURO; KUME; NISHIDA, 2015, p. 106) characterizes the condition for $M$ belonging to that class in terms of the determinantal ideal of a certain matrix. The $R$-modules $M$ that belong to this class will be called *Linear Type* modules. In the chapter 5, we will go into more detail about these modules.

## 2.2 A modest theory of homological algebra

In this section we introduce basic concepts of homological algebra, such as $\mathrm{Tor}_n^R(M,N)$ and $\mathrm{Ext}_R^n(M,N)$. Next, we present a result that characterizes the exactness of a complex in terms of the determinants of the submatrices of its maps. Finally, we exhibit the classic Hilbert-Burch theorem that characterizes grade two perfect ideals via projective resolutions of length one.

**Definition 3.** Let $R$ be a ring, $M$ an $R$-module and $Q$ be the total ring of fractions of $R$. We say that $M$ has *rank* $r$ if $M \otimes Q$ is a free $Q$-module of rank $r$. If $\phi : M \longrightarrow N$ is a homomorphism of $R$-modules, then $\phi$ has *rank* $r$ if $\mathrm{Im}(\phi)$ has rank $r$.

**Definition 4.** Let $R$ be a ring and $M$ an $R$-module. The *projective resolution* of $M$ is an exact complex of $R$-modules

$$\mathbf{F} : \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where $F_0,\ F_1, \ldots$ are projective modules. If $F_0,\ F_1, \ldots$ are all free modules, then the resolution is a *free resolution* of $M$. If there exists some $n \geq 0$ such that $F_k = 0$ for all $k > n$ (and $F_n \neq 0$), then the resolution is said to be *finite of lenght* $n$.

**Remark 3.** Sometimes a resolution of $M$ is written in the following way:

$$\mathbf{F}_\bullet : \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

so the complex is exact everywhere except at $F_0$, with the homology at $F_0$ being:

$$H_0(\mathbf{F}_\bullet) := \mathrm{Ker}(\phi_0)/\mathrm{Im}(\phi_1) = F_0/\mathrm{Im}(\phi_1) = F_0/\mathrm{Ker}(\phi_0) \cong \mathrm{Im}(\phi_0) = M.$$

In this case we say that $\mathbf{F}_\bullet$ is a *deleted resolution* of $M$.

**Definition 5.** Let $R$ be a ring and $M$ an $R$-module, for each $R$-module $N$, we define $\mathrm{Tor}_n^R(M,N)$ to be the $n$ th homology module of the complex

$$\mathbf{F}_\bullet \otimes_R N : \cdots \longrightarrow F_2 \otimes_R N \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow 0,$$

i.e.,

$$\mathrm{Tor}_n^R(M,N) := H_n(\mathbf{F}_\bullet \otimes_R N) \text{ for all } n \geq 0,$$

where $\mathbf{F}_\bullet$ is a projective resolution of $M$.

**Definition 6.** Let $R$ be a ring and $M$ an $R$-module, for each $R$-module $N$, we define $\mathrm{Ext}_R^n(M,N)$ to be the $n$ th homology module of the complex

$$\mathrm{Hom}_R(\mathbf{F}_\bullet, N) : 0 \longrightarrow \mathrm{Hom}(F_0,N) \longrightarrow \mathrm{Hom}(F_1,N) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(F_n,N) \longrightarrow \cdots,$$

i.e.,

$$\mathrm{Ext}_R^n(M,N) := H_n(\mathrm{Hom}_R(\mathbf{F}_\bullet,N)) \text{ for all } n \geq 0,$$

where $\mathbf{F}_\bullet$ is a projective resolution of $M$.

**Proposition 4.** Any module $M$ over a given ring $R$ possesses a free resolution.

*Demonstration.* Firstly, we can choose a free $R$-module $F_0$ and a map a set of generators at $F_0$ to a set of generators of $M$, giving a surjective map $\phi_0 : F_0 \longrightarrow M$. Now the "obstruction to exactness" of the sequence $0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ is that homology at $F_0$ is $\text{Ker}(\phi_0)$, rather 0, so it is of interest to look at this submodule at $F_0$.

As for $M$, we can choose a free module $F_1$ and a homomorphism $\phi_1$ which maps a set of generators of $F_1$ to a set of generators of $\text{Ker}(\phi_0)$. Continuing in this way, we build the free resolution:

$$\mathbf{F} : \cdots \longrightarrow F_{n+1} \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where each $\phi_n$ surjectively maps $F_n$ onto $\text{Ker}(\phi_{n-1})$ (for $n > 0$) and $\phi_0$ surjectively maps $F_0$ onto $M$. □

**Remark 4.** Note that it includes which are not necessarily finitely generated.

**Remark 5.** Set $M_0 = M$ and $M_n = \text{Ker}(\phi_{n-1})$ for $n \geq 1$. The modules $M_i$ depend obvioulsy $\mathbf{F}$. However, $M$ determines $M_i$ up to projective equivalence (HOTMAN, 1979, Theorem 9.4), and therefore it is justified to call $M_i$ the *i-th syzygy* of $M$.

**Definition 7.** Let $R$ be a ring and an finitely generated $R$-module $M$, the *projective dimension* (or *homological dimension*) of $M$ (denoted $\text{pd}_R\, M$) is the smallest non-negative integer $n$ for which there exists a projective resolution of $M$ of lenght $n$, i.e. the minimal $n$ such that the projective modules $F_0, \ldots, F_n$ exist turning the following complex an exact one:

$$\mathbf{F} : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

In general, free modules are projective modules over a ring $R$, hence free resolutions are projective resolutions. Now when $R$ is a local ring, these concepts are equivalent. For this reason, whenever we work with projective dimensions on local rings, we will consider free resolutions.

Among the free resolutions of a finitely generated $R$-module $M$ over a local ring, we will highlight throughout this work those that provide properties related to the minimal number of generators of $M$. Such resolutions are called *minimal free resolutions* and can be constructed using the Proposition 4 argument along with Nakayama's Lemma.

**Definition 8.** Let $(R, \mathfrak{m})$ be a local ring, a free resolution of $R$-modules (such as below) is *minimal* if $\text{Im}(\phi_n) \subseteq \mathfrak{m}\phi_{n-1}$ for all $n$.

$$\mathbf{F} : \cdots \longrightarrow F_{n+1} \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

The number $\beta_n^R(M) := \text{rank}\, F_n$ is called the $n$-th *Betti number* of $M$.

The following proposition shows that minimal free resolutions are quite useful when we need to calculate homologies $\text{Tor}_n^R(M,k)$ and $\text{Ext}_R^n(M,k)$.

**Proposition 5.** Let $(R,\mathfrak{m},k)$ be a Noetherian local ring, $M$ an finitely generated $R$-module, and

$$\mathbf{F}:\cdots\longrightarrow F_{n+1}\xrightarrow{\phi_{n+1}}F_n\xrightarrow{\phi_n}F_{n-1}\xrightarrow{\phi_{n-1}}\cdots\xrightarrow{\phi_3}F_2\xrightarrow{\phi_2}F_1\xrightarrow{\phi_1}F_0\xrightarrow{\phi_0}M\longrightarrow 0$$

a free resolution of $M$. Then the following are equivalent:

(a) $\mathbf{F}$ is minimal;

(b) $\phi_n(F_n)\subset\mathfrak{m}F_{n-1}$ for all $n\geq 1$;

(c) rank $F_n=\dim_k\mathrm{Tor}_n^R(M,k)$ for all $n\geq 0$;

(d) rank $F_n=\dim_k\mathrm{Ext}_R^n(M,k)$ for all $n\geq 0$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Proposition 1.3.1). □

**Corollary 2.** Let $(R,\mathfrak{m},k)$ be a Noetherian local ring, $M$ an finitely generated $R$-module. Then $\beta_n^R(M)=\dim_k\mathrm{Tor}_n^R(M,k)$ for all $n$ and

$$\mathrm{pd}_R\,M=\sup\{n|\mathrm{Tor}_n^R(M,k)\neq 0\}.$$

*Demonstration.* See (BRUNS; HERZOG, 1993, Corollary 1.3.2). □

The following Theorem, although elementary, is one of the key results for the development of this work. It assures us that the Betti numbers of a finitely generated module are isomorphism invariant.

**Theorem 2.** Let $M,N$ be finitely generated modules over a Noetherian local ring $R$. Let

$$\mathbf{F}:\cdots\xrightarrow{\phi_3}F_2\xrightarrow{\phi_2}F_1\xrightarrow{\phi_1}F_0\xrightarrow{\phi_0}M\longrightarrow 0$$

$$\mathbf{F}':\cdots\xrightarrow{\psi_3}F_2'\xrightarrow{\psi_2}F_1'\xrightarrow{\psi_1}F_0'\xrightarrow{\psi_0}N\longrightarrow 0$$

be mimimal free resolutions of $M$ and $N$, respectively. If $M\cong N$, then $F_i\cong F_i'$ for each $i\in\{0,1,2,3,\ldots\}$.

*Demonstration.* See (LEE; SONG, 2018, Theorem 3.6)

□

**Remark 6.** One sees from the above Corollary 2 that $\mathrm{Tor}_n^R(M,k)=0$ implies that $F_n=0$, and therefore $\mathrm{pd}_R\,M<n$, so that $\mathrm{Tor}_m^R(M,k)=0$ for $m>n$. It is conjectured that this holds in more generality, or more precisely:

*Rigidty conjecture*. Let $R$ be a Noethering ring, $M$ and $N$ finitely generated $R$-modules. Suppose that $\mathrm{pd}_R\,M<\infty$. Then $\mathrm{Tor}_n^R(M,N)=0$ implies that $\mathrm{Tor}_m^R(M,N)=0$ for all $m>n$.

This has been proved by (LICHTENBAUM, 1966, Theorem 3), if $R$ is a regular ring, but is unsolved in general.

Another essential invariant for the development of this thesis, which we will define next, is the grade of a module.

**Definition 9.** Let $R$ be a ring and an finitely generated $R$-module $M$. We say that $x \in R$ is a $M$*-regular element* if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if $x$ is not a zero-divisor on $M$. Regular sequences are composed of successively regular elements:

A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of $R$ is called an $M$*-regular sequence* or simply an $M$*-sequence* if the following conditions are satisfied:

(a) $x_i$ is an $M/(x_1, \ldots, x_{i-1})$ $M$-regular element for $i = 1, 2, \ldots, n$;

(b) $M/\mathbf{x}M \neq 0$.

A *regular sequence* is an $R$-sequence.

Let $R$ be a Noetherian ring and $M$ an $R$-module. Then we obtain the strict ascendance of the sequence $(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n)$, if $\mathbf{x} = x_1, \ldots, x_n$ is an $M$-sequence. Therefore an $M$-sequence can be extended to a maximal such sequence, i.e., an $M$-sequence $\mathbf{x}$ is *maximal*, if $x_1, \ldots, x_n, x_{n+1}$ is not an $M$-sequence for any $x_{n+1} \in R$.

The following Rees Theorem shows that all maximal $M$-sequences in an ideal $I$ with $IM \neq M$ have the same lenght if $M$ is finitely generated.

**Theorem 3** (Rees)**.** Let $R$ be a Noetherian ring and $M$ a finitely generated $R$-module, and $I$ an ideal such that $IM \neq M$. Then all maximal $M$-sequences in $I$ have the same lenght $n$ given by

$$n = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

The Rees Theorem above allows us to introduce the fundamental notions of grade and depth.

**Definition 10.** Let $R$ be a Noetherian ring and an finitely generated $R$-module $M$, and $I$ an ideal such that $IM \neq M$. The common lenght of the maximal $M$-sequences in $I$ is called the *grade* of $I$ on $M$, denoted by

$$\operatorname{grade}(I, M).$$

When $(R, \mathfrak{m})$ is a Noetherian local ring, the grade of $\mathfrak{m}$ on $M$ is called the *depth* of $M$, denoted by

$$\operatorname{depth} M.$$

**Definition 11.** Let $R$ be a Noetherian ring and an finitely generated $R$-module $M$. We define the grade of $M$ by

$$\text{grade}(M) = \text{grade}(\text{Ann}M,\ R).$$

**Remark 7.** It is customary to set

$$\text{grade}(I) = \text{grade}(R/I) = \text{grade}(I,R)$$

for an ideal $I \lhd R$, and we follow this convention.

The Auslander-Buchsbaum theorem below, in addition to being an effective formula for calculating the depth of a module, expresses an upper bound for finite projective dimension modules.

**Theorem 4** (Auslander-Buchsbaum)**.** Let $(R,\mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated $R$-module. If $\text{pd}_R M < \infty$, then

$$\text{pd}_R\, M + \text{depth}\, M = \text{depth}\, R.$$

*Demonstration.* See (BRUNS; HERZOG, 1993, Theorem 1.3.3). □

**Definition 12.** Let $R$ be a ring. We say $R$ has *finite global dimension* if there exists an $n \in \mathbb{Z}$ such that $\text{pd}_R\, M \leq n$ for all $R$-modules $M$. The smallest such $n$ is the *global dimension* of $R$, which we will denote by gl dim R.

**Theorem 5** (Serre)**.** Let $R$ be a Noetherian local ring. Then

$$R \text{ is regular} \Longleftrightarrow \text{gl dim}\, R = \dim R \Longleftrightarrow \text{gl dim}\, R < \infty.$$

*Demonstration.* See (MATSUMURA, 1989, Theorem 19.2) □

We saw in the Proposition 4 that every $R$-module $M$ finitely generated module has a free resolution (minimal or not) and in the case where $R$ is a regular ring, by Theorem 5, such resolution stops. What we need to know now is how to decide the exactness of a free resolution of a finitely generated $R$-module $M$. Below we will present some results, made by Buchsbaum; Eisenbud (1973), that help us to make this decision.

**Definition 13.** Let $A$ be a $m \times n$ matrix over $R$ where $m,n \geq 0$. For $t = 1,2,\ldots,\min\{m,n\}$ we then denote by $I_t(A)$ the ideal generated by the $t$-minors of $A$(the determinants of $t \times t$ submatrices). For systematic reasons one sets $I_t(A) = R$ for $t \leq 0$ and $I_t(A) = 0$ for $t > \min\{m,n\}$. If $\phi : F \longrightarrow G$ is a homomorphism de finite free $R$-modules, then $\phi$ is given by a matrix $A$ with respect to bases of $F$ and $G$. Therefore we may put $I_t(\phi) = I_t(A)$.

**Proposition 6.** Let $R$ be a Noetherian ring, and $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ an exact sequence of finitely generated $R$-modules. If two of $U$, $M$, $N$, have a rank, then so does the third, and $\operatorname{rank}(M) = \operatorname{rank}(U) + \operatorname{rank}(N)$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Proposition 1.4.5). □

**Corollary 3.** Let $R$ be a Noetherian ring, and $M$ an $R$-module with a finite free resolution

$$\mathbf{F}_\bullet : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0.$$

Then $\operatorname{rank}(M) = \sum_{j=0}^{n} (-1)^j \operatorname{rank}(F_j)$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Corollary 1.4.6). □

**Proposition 7.** Let $R$ be a Noetherian ring, and let $\phi : F \longrightarrow G$ be a homomorphism of finite free $R$-modules. Then $\operatorname{rank}(\phi) = r$ if and only if $\operatorname{grade}(I_r(\phi)) \geq 1$ and $I_{r+1}(\phi) = 0$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Proposition 1.4.11) □

The following theorem exhibit a criterion that relates the exactness of a complex $F_\bullet$ with the ideals generated by certain minors of the homomorphisms $\phi_n$.

**Theorem 6.** Let $\mathbf{F}_\bullet : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$ be a complex of free $R$-modules. Then $\mathbf{F}_\bullet$ is exact if and only if two following conditions are satisfied:

(a) $\operatorname{rank}(\phi_{k+1}) + \operatorname{rank}(\phi_k) = \operatorname{rank} F_k$ for all $k$;

(b) $\operatorname{grade}(I(\phi_k) \geq k$ for all $k = 1, 2, \ldots, n$.

*Demonstration.* See (BUCHSBAUM; EISENBUD, 1973). □

**Theorem 7** (Peskine-Szpiro)**.** Let $\mathbf{F}_\bullet : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$ be a complex of free $R$-modules. Then $\mathbf{F}_\bullet$ is exact if and only if $\mathbf{F}_\bullet \otimes R_\mathfrak{p}$ is exact for all $\mathfrak{p}$ with depth $R_\mathfrak{p} < n$.

*Demonstration.* See (BUCHSBAUM; EISENBUD, 1974, Corollary 1.3). □

Let $R$ be a Noetherian ring, and $M$ a finite $R$-module. Since it is possible to compute $\operatorname{Ext}_R^i(M, R)$ from a projective resolution of $M$ (BRUNS; HERZOG, 1993, Theorem 1.3.3), we have the following inequality that relates the grade of a module to its projective dimension

$$\operatorname{grade}(M) \leq \operatorname{pd}_R M.$$

This motivates the following definition.

**Definition 14.** Let $R$ be a Noetherian ring. A non-zero finite $R$-module $M$ is perfect if $\operatorname{pd}_R M = \operatorname{grade}(M)$. An ideal $I$ is called perfect if $R/I$ is a perfect module, in which case the *type* of $I$ is defined to be the value of the last (nonzero) Betti number of $I$. We will use $\operatorname{Type}(I)$ to denote the type of $I$.

**Proposition 8.** Let $R$ be a Noetherian ring, and $M$ a perfect $R$-module. For a prime $\mathfrak{p} \in \operatorname{Supp} M$ the following are equivalent:

(a) $\mathfrak{p} \in \operatorname{Ass} M$;

(b) depth $R_{\mathfrak{p}} = \operatorname{grade}(M)$. Furthermore $\operatorname{grade}(\mathfrak{p}) = \operatorname{grade}(M)$ for all ideals $\mathfrak{p} \in \operatorname{Ass} M$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Proposition 1.4.16) □

We and this chapter with the theorem of Hilbert-Burch, where it gives us a characterization of grade two perfect ideals. In particular, it states that their Betti numbers are consecutive integers.

**Theorem 8** (Hilbert-Burch)**.** Let $R$ be a Noetherian ring, and $I$ an ideal with a free resolution

$$\mathbf{F}_\bullet : 0 \longrightarrow R^n \xrightarrow{\phi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists an $R$-regular element $a$ such that $I = aI_n(\phi)$. If $I$ is projective, then $I = (a)$, and if $\operatorname{pd}_R I = 1$, then $I_n(\phi)$ is perfect ideal of grade 2.

Conversely, if $\phi : R^n \longrightarrow R^{n+1}$ is an $R$-linear map with $\operatorname{grade}(I_n(\phi)) \geq 2$, then $I = I_n(\phi)$ has the free resolution $\mathbf{F}_\bullet$.

*Demonstration.* See (BRUNS; HERZOG, 1993, Theorem 1.4.17). □

CHAPTER

3

# SYMMETRIC POWERS AND THEIR MINIMAL FREE RESOLUTIONS

The main objective of this chapter is to study the minimality of the complex $S_j\mathbf{F}_\bullet$, in order to give affirmative answers to the Questions **A** and **B**. For example, we show that if $M$ is a module that satisfies the $(SW_j)$ condition and has a minimal free resolution, then $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$. As a consequence of this fact we obtain two important invariants in homological algebra: the projective dimension and the Betti numbers of $S_j(M)$. Furthermore, we show that such invariants depend only on the power $j$ and the homological information of $M$, i.e., from the projective dimension of $M$ and its Betti numbers.

But before introducing the complex $S_j\mathbf{F}_\bullet$ we need to define the $j$th *Divided power* of a $F$ free $R$-module, $D_j(F)$ (for more details see BUCHSBAUM; EISENBUD (1975)).

**Definition 15** (Divided power)**.** Let $F$ be a free $R$-module of rank finite, and $j \geq 0$ integer non negative. The $j$th *Divided power* $D_j(F)$ is defined as the set of symmetric tensors in $T^j(F)$, that is,

$$D_j(F) := \{\omega \in T^j(F) : \sigma(\omega) = \omega \text{ for all } \sigma \in \mathfrak{S}_j\},$$

where $\mathfrak{S}_j$ is set the permutation of order $j$.

By definition $D_j(F)$ is a $R$-module. Now suppose that $F$ be a finite free $R$-module generated by $f_1, f_2, \ldots, f_l$. To get a basis for $D_j(F)$ we first consider the orbits

$$\mathscr{O}_{a_1,\, a_2, \ldots,\, a_l} := \mathfrak{S}_j . f_1^{\otimes a_1} \otimes f_2^{\otimes a_2} \otimes \cdots \otimes f_l^{\otimes a_l}$$

and for $a_1 + a_2 + \cdots + a_l = j$ consider the *Divided power monomials*

$$f_1^{(a_1)} \cdots f_l^{(a_l)} := \sum_{\omega \in \mathscr{O}_{a_1,\, a_2, \ldots,\, a_l}} \omega$$

they form a basis for $D_j(F)$. In other words,

$$D_j(F) = \langle \{\prod_i f_i^{(a_i)} | \sum a_i = j\}\rangle \tag{3.1}$$

is a free $R$-module.

**Remark 8.** For convention we will use $D_0(F) = R$ and $D_1(F) = F$ for all $F$ free $R$-module.

As we will see in the following lemma, the Equation 3.1 gives us a formula for the number of generated $D_j(F)$.

**Lemma 2.** Let $F$ be a free $R$-module and $j$ be a integer non-negative. If $\operatorname{rank}(F) = l$, then $D_j(F)$ has rank $\binom{j+l-1}{l-1}$.

*Demonstration.* By Equation 3.1, the minimum number of generators of $D_j(F)$ can be seen how the number of solutions distinct with non-negative integers satisfying the equation

$$a_1 + a_2 + \cdots + a_l = j$$

where $l = \operatorname{rank} F$, i.e. $\binom{l+j-1}{l-1}$. Therefore we get the result. □

## 3.1 Free resolution for symmetric powers

It can be seen in the paper above mentioned that the construction of such a complex is produced from a free resolution of $M$. What we add here is that if $M$ has a minimal free resolution then the Lemma 3 tells us that the complex $S_j\mathbf{F}_\bullet$, with some assumptions, is a minimal free resolution.

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $M$ be a finitey generated $R$-module. Assume that

$$\mathbf{F}_\bullet : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0$$

is a finite free resolution of $M$ where $\operatorname{pd}_R M = p$. Let $a_0,\ a_1, \ldots,\ a_p$ be a sequence of non-negative integers. We define the functors

$$S(a_0, \ldots, a_p; \mathbf{F}_\bullet) := \begin{cases} D_{a_0}F_0 \otimes \overset{a_1}{\Lambda} F_1 \otimes D_{a_2}F_2 \otimes \cdots \otimes \overset{a_{p-1}}{\Lambda} F_{p-1} \otimes D_{a_p}F_p, & \text{for} \quad p \quad \text{even;} \\ D_{a_0}F_0 \otimes \overset{a_1}{\Lambda} F_1 \otimes D_{a_2}F_2 \otimes \cdots \otimes D_{a_{p-1}}F_{p-1} \otimes \overset{a_p}{\Lambda} F_p, & \text{for} \quad p \quad \text{odd ,} \end{cases}$$

and the differential maps as follows:

$$d^i : S(a_0, \ldots,\ a_p : \mathbf{F}_\bullet) \to S(b_0, \ldots,\ b_p; \mathbf{F}_\bullet)$$

is zero when $(b_0, \ldots,\ b_p) \neq (a_0, \ldots, a_i + 1, a_{i+1} - 1, \ldots, a_p)$ for all $i$, and in the case $(b_0, \ldots,\ b_p) = (a_0, \ldots,\ a_i + 1,\ a_{i+1} - 1, \ldots,\ a_p)$,

$$d^i = \begin{cases} \pm 1 \otimes \cdots \otimes 1 \otimes A_{a_{i+1}, a_i} \phi_{i+1} \otimes 1 \cdots, & \text{for} \quad i \quad \text{odd;} \\ \pm 1 \otimes \cdots \otimes 1 \otimes B_{a_{i+1}, a_i} \phi_{i+1} \otimes 1 \cdots, & \text{for} \quad i \quad \text{even ,} \end{cases} \tag{3.2}$$

where $\pm$ denotes $(-1)^{\sigma}; \sigma = a_0 + 2a_1 + \cdots + (i+1)a_i$ and the homomorphisms $A_{a_{i+1},a_i}\phi_{i+1}$ and $B_{a_{i+1},a_i}\phi_{i+1}$ are defined as follows: Suppose that $f_1, f_2, \ldots, f_r$ and $g_1, g_2, \ldots, g_s$ form a basis for $F_{i+1}$ and $F_i$ respectively. Let

$$A_{a_{i+1},a_i}\phi_{i+1} : D_{a_{i+1}}F_{i+1} \otimes \overset{a_i}{\Lambda} F_i \longrightarrow D_{a_{i+1}-1}F_{i+1} \otimes \overset{a_i+1}{\Lambda} F_i$$

$$A_{a_{i+1},a_i}\phi_{i+1}(f_1^{(a_{i+1})_1} \ldots f_r^{(a_{i+1})_r} \otimes v) = \sum_{l=1}^{r} f_1^{(a_{i+1})_1} \cdots f_l^{(a_{i+1})_l-1} \cdots f_r^{(a_{i+1})_r} \otimes \phi_{i+1}(f_l) \wedge v$$

and

$$B_{a_{i+1},a_i}\phi_{i+1} : \overset{a_{i+1}}{\Lambda} F_{i+1} \otimes D_{a_i}F_i \longrightarrow \overset{a_{i+1}-1}{\Lambda} F_{i+1} \otimes D_{a_i+1}F_i$$

$$B_{a_{i+1},i}\phi_{i+1}(f_{(a_{i+1})_1} \wedge \cdots \wedge f_{(a_{i+1})_s} \otimes w) = \sum_{l=1}^{r} (-1)^l f_{(a_{i+1})_1} \wedge \cdots \wedge \widehat{f_{(a_{i+1})_l}} \wedge \cdots \wedge f_{i_s} \otimes \phi_{i+1}(f_{(a_{i+1})_l}) \cup w,$$

where $g_i \cup g_1^{(i_1)} \cdots g_s^{(i_s)} = g_1^{(i_1)} \cdots g_i^{(i_i+1)} \cdots g_s^{(i_s)}$. Here $\widehat{f_{i_l}}$ means that $f_{i_l}$ is omitted. Thus, we define

$$(S_j\mathbf{F}_\bullet)_t = \bigoplus_{\substack{(a_0,\ldots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} S(a_0, \ldots, a_p; \mathbf{F}_\bullet) \quad \text{for all } \; t \geq 0.$$

and the differentials $d_t$ are given by

$$d_t : (S_j\mathbf{F}_\bullet)_t \longrightarrow (S_j\mathbf{F}_\bullet)_{t-1} \;\text{ where }\; d_t := (d_t^{j_1}, d_t^{j_2}, \ldots, d_t^{j_r}), \text{ for all } \; t \geq 1.$$

Thus, we get the complex

$$S_j\mathbf{F}_\bullet : \cdots \longrightarrow (S_j\mathbf{F}_\bullet)_{t+1} \xrightarrow{d_{t+1}} (S_j\mathbf{F}_\bullet)_t \xrightarrow{d_t} \cdots \xrightarrow{d_2} (S_j\mathbf{F}_\bullet)_1 \xrightarrow{d_1} (S_j\mathbf{F}_\bullet)_0. \tag{3.3}$$

Observe that, the notation $d_t^{j_r}$ is to indicate the differential $d^{j_r}$ on the $t$th level of the complex $S_j\mathbf{F}_\bullet$ with $r$ the $r$th solution of equation system

$$\begin{cases} \sum_{i=0}^{p} i a_i = t \\ \sum_{i=0}^{p} a_i = j \end{cases} \tag{3.4}$$

**Remark 9.** Summarizing the construction above, each component of the complex $S_j\mathbf{F}_\bullet$ is given by:

$$(S_j\mathbf{F}_\bullet)_t = \begin{cases} \bigoplus_{\substack{(a_0,\ldots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} D_{a_0}F_0 \otimes \overset{a_1}{\Lambda} F_1 \otimes D_{a_2}F_2 \otimes \overset{a_3}{\Lambda} F_3 \otimes \cdots \otimes D_{a_p}(F_p), & \text{if } p \text{ is even,} \\ \bigoplus_{\substack{(a_0,\ldots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} D_{a_0}F_0 \otimes \overset{a_1}{\Lambda} F_1 \otimes D_{a_2}F_2 \otimes \overset{a_3}{\Lambda} F_3 \otimes \cdots \otimes \overset{a_p}{\Lambda} F_p, & \text{if } p \text{ is odd.} \end{cases}$$

See (WEYMAN, 1979, p. 335) for more details.

**Remark 10.** Observe that, $S_0\mathbf{F}_\bullet = R$ and $S_1\mathbf{F}_\bullet = \mathbf{F}_\bullet$. In particular, if $\mathbf{F}_\bullet$ is a minimal free resolution of $M$, then $\beta_i^R(S_1(M)) = \beta_i^R(M)$, for all $i = 0,\dots, p$. For this reason, we will always be considering the symmetric powers $S_j(M)$ and $S_j\mathbf{F}_\bullet$ for $j \geq 2$.

An important fact that will be used a lot in the next chapters is that the complex $S_j\mathbf{F}_\bullet$ is a bounded complex when $\mathbf{F}_\bullet$ is. With this information in mind, we will be able to calculate the projective dimension of symmetric power $S_j(M)$.

**Remark 11.** (WEYMAN, 1979, p. 336). Each complex $S_j\mathbf{F}_\bullet$ in 3.3 is a bounded complex as $R$-module and its lenght is given by

$$\lambda(S_j\mathbf{F}_\bullet) = \begin{cases} jp, & \text{for} \quad p \quad \text{even;} \\ j(p-1) + \min\{\text{rank } F_p, j\}, & \text{for} \quad p \quad \text{odd} \end{cases}$$

where $\lambda(-)$ denote $\lambda(\mathbf{F}_\bullet) := \sup\{i | F_i \neq 0\}$ for some complex $\mathbf{F}_\bullet$.

In the next Remark, we highlight the complex $S_j\mathbf{F}_\bullet$ for modules of projective dimension 1.

**Remark 12.** Suppose that $p = 1$, then for each $j \geq 2$ the solutions of the system of equations 3.4 are given by $S = \{(a_0,\, a_1) = (j-t,\, t) | t = 0,\, 1, \dots,\, \lambda(S_j\mathbf{F}_\bullet)\}$. Thus

$$(S_j\mathbf{F}_\bullet)_t = \bigoplus_{\substack{(a_0,\, a_1) \\ a_0 + a_1 = j \\ a_1 = t}} D_{a_0}F_0 \otimes \overset{a_1}{\Lambda} F_1 = D_{j-t}F_0 \otimes \overset{t}{\Lambda} F_1;$$

for all $t = 0, 1, \dots,\, \lambda(S_j\mathbf{F}_\bullet)$. Hence $S_j\mathbf{F}_\bullet$ is given by

$$S_j\mathbf{F}_\bullet : 0 \longrightarrow D_{j-l}(F_0) \otimes \overset{l}{\Lambda} F_1 \xrightarrow{d_l} D_{j-l+1}(F_0) \otimes \overset{l-1}{\Lambda} F_1 \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_2} D_{j-1}(F_0) \otimes \overset{1}{\Lambda} F_1 \xrightarrow{d_1} D_j(F_0) \otimes \overset{0}{\Lambda} F_1,$$

where $l = \lambda(S_j\mathbf{F}_\bullet) = \min\{\text{rank } F_1,\ j\}$.

After these constructions, we point out below the theorem that answers Question A. Theorem that establishes a criteria for the exactness of $S_j\mathbf{F}_\bullet$ in terms of the determinant ideals of the maps that make up the free resolution (or minimal free resolution) $\mathbf{F}_\bullet$.

**Theorem 9.** (WEYMAN, 1979, Theorem 1) Let

$$\mathbf{F}_\bullet : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0$$

be a finite free resolution with $\text{coker}(\phi_1) = M$ and $r_i = \sum_{n=i}^{p} (-1)^{n-i} \text{rank}(F_n)$. Then, $S_j\mathbf{F}_\bullet$ is exact if and only if

(a) $\text{grade}(I_{r_i}(\phi_i)) \geq ji$, for all $i$ even, where $1 \leq i \leq p$;

(b) $\operatorname{grade}(I_{r_i-j+1}(\phi_i)) \geq ji$, $\operatorname{grade}(I_{r_i-j+2}(\phi_i)) \geq ji-1$, $\operatorname{grade}(I_{r_i}(\phi_i)) \geq (i-1)j+1$, for all $i$ odd, where $1 \leq i \leq p$.

If $S_j\mathbf{F}_\bullet$ is exact for each $j$, it is a finite free resolution of the symmetric power $S_j(M)$ for each $j$.

Motivated by the theorem above, (MOLICA; RESTUCCIA, 2002) defines the condition $(SW_j)$ to simplify the criteria in items (a) and (b) of Theorem 3.

**Definition 16.** (MOLICA; RESTUCCIA, 2002) Let $M$ be a $R$-module and let

$$\mathbf{F}_\bullet : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0$$

be a finite minimal free resolution of $M$. For each $j \geq 2$. We say that $M$ satisfies the $(SW_j)$ *condition* if $S_j\mathbf{F}_\bullet$ is a finite free resolution of $S_j(M)$.

Thus, by Theorem 9, we say that $M$ satisfies $(SW_j)$ condition if and only if

(a) $\operatorname{grade}(I_{r_i}(\phi_i)) \geq ji$ for all $i$ even, where $1 \leq i \leq p$;

(b) $\operatorname{grade}(I_{r_i-j+1}(\phi_i)) \geq ji$, $\operatorname{grade}(I_{r_i-j+2}(\phi_i)) \geq ji-1$, $\operatorname{grade}(I_{r_i}(\phi_i)) \geq (i-1)j+1$ for all $i$ odd, where $1 \leq i \leq p$.

The example below illustrates an ideal $I$ that satisfies the $(SW_2)$ condition.

**Example 1.** Let $R = k[[x,\ y,\ z]]$ be a ring of formal power series over a field $k$ and let the ideal of $R$ given by $I = (yz^2,\ x^2z,\ x^3y^2)$. By MACAULAY2, a minimal free resolution for $I$ is given by

$$\mathbf{F}_\bullet : 0 \longrightarrow R^2 \xrightarrow{\phi_1} R^3 \longrightarrow I \longrightarrow 0.$$

Where the map $\phi_1$ is given by matrix $3 \times 2$

$$[\phi_1] = \begin{pmatrix} -yz & -xz^2 \\ x^2 & 0 \\ 0 & z \end{pmatrix}.$$

We get that $I_1(\phi_1) = (-yz,\ x^2, -xz^2,\ z)$ and $I_2(\phi_1) = (x^3z^2,\ -yz^2,\ x^2z)$ implying that $\operatorname{grade}(I_1(\phi_1)) = 2$ and $\operatorname{grade}(I_2(\phi_1)) = 1$. Hence, the ideal $I$ satisfy $(SW_2)$ condition.

**Remark 13.** As we said above, the Theorem gives us a criterion for the complex $S_j\mathbf{F}_\bullet$ to be a free resolution of the symmetric power $S_j(M)$. In particular, it says that we can build a free resolution of $S_j(M)$ from a free resolution of $M$. So we are interested to know, if the minimality of $\mathbf{F}_\bullet$ is transferred to $S_j\mathbf{F}_\bullet$. That is, if $\mathbf{F}_\bullet$ is minimal and $S_j\mathbf{F}_\bullet$ is exact, then $S_j\mathbf{F}_\bullet$ is also minimal? (WEYMAN, 1979) gives an affirmative answer to this fact, although he does not give explicit proof. For this reason, we will present a demonstration of this fact. And to start with the proof, we need the following lemma.

**Lemma 3.** Let $(R,\mathfrak{m},k)$ be a Noetherian local ring and $d^i$ be a map of free $R$-modules defined in 3.2. Suppose that $\mathbf{F}_\bullet$ is a minimal free resolution of $M$, then $d^i \otimes 1_k = 0$.

*Demonstration.* Let $1_k : k \to k$ be a map defined by $1_k(\bar{y}) = \bar{y}$. By definition of $d^i$, if $(b_0,\ b_1,\dots,\ b_p) \neq (a_0, a_1, \dots, a_i+1,\ a_{i+1}-1, \dots,\ a_p)$ for all $i$, then $d^i = 0 \Rightarrow d^i \otimes 1_k = 0$. And in this case, the result follows. Now suppose that $(b_0,\ b_1,\dots,\ b_p) = (a_0,\ a_1,\dots,\ a_i+1,\ a_{i+1}-1,\dots,\ a_p)$ for some $i$. Therefore, we need consider the following cases:

**Case**$(1)$**:** ***i* is odd**. Let $d^i = \pm 1 \otimes \cdots \otimes 1 \otimes A_{a_{i+1},a_i}\phi_{i+1} \otimes 1 \otimes \cdots$, where

$$A_{a_{i+1},a_i}\phi_{i+1} : D_{a_{i+1}}F_{i+1} \otimes \overset{a_i}{\Lambda} F_i \longrightarrow D_{a_{i+1}-1}F_{i+1} \otimes \overset{a_i+1}{\Lambda} F_i$$

is given by

$$A_{a_{i+1},a_i}\phi_{i+1}(f_1^{(a_{i+1})_1} \dots f_r^{(a_{i+1})_r} \otimes v) = \sum_{l=1}^{r} f_1^{(a_{i+1})_1} \cdots f_l^{(a_{i+1})_{l-1}} \cdots f_r^{(a_{i+1})_r} \otimes \phi_{i+1}(f_l) \wedge v.$$

Let $\bar{y} \in k$. By hypothesis $\mathbf{F}_\bullet$ is a minimal free resolution which implies that $\phi_{i+1}(f_l) = xf$ for some $x \in \mathfrak{m}$ and $f \in F_i$. Now by the linearity of the tensor product, we get

$$d^i \otimes 1_k(f_1^{(a_{i+1})_1} \cdots f_r^{(a_{i+1})_r} \otimes v \otimes \bar{y}) = d^i(f_1^{(a_{i+1})_1} \cdots f_r^{(a_{i+1})_r} \otimes v) \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes A_{a_{i+1},a_i}\phi_{i+1}(f_1^{(a_{i+1})_1} \dots f_r^{(a_{i+1})_r} \otimes v) \otimes 1 \otimes \cdots \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} f_1^{(a_{i+1})_1} \cdots f_l^{(a_{i+1})_{l-1}} \cdots f_r^{(a_{i+1})_r} \otimes \phi_{i+1}(f_l) \wedge v \otimes 1 \otimes \cdots \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} f_1^{(a_{i+1})_1} \cdots f_l^{(a_{i+1})_{l-1}} \cdots f_r^{(a_{i+1})_r} \otimes xf \wedge v \otimes 1 \otimes \cdots \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} f_1^{(a_{i+1})_1} \cdots f_l^{(a_{i+1})_{l-1}} \cdots f_r^{(a_{i+1})_r} \otimes f \wedge v \otimes 1 \otimes \cdots \otimes 1_k(\overline{xy})$$

$$= 0.$$

**Case**$(2)$**:** ***i* is even**. Let $d^i = \pm 1 \otimes \cdots \otimes 1 \otimes B_{a_{i+1},a_i}\phi_{i+1} \otimes 1 \otimes \cdots$, with

$$B_{a_{i+1},a_i}\phi_{i+1} : \overset{a_{i+1}}{\Lambda} F_{i+1} \otimes D_{a_i}F_i \longrightarrow \overset{a_{i+1}-1}{\Lambda} F_{i+1} \otimes D_{a_i+1}F_i$$

$$B_{a_{i+1},i}\phi_{i+1}(f_{(a_{i+1})_1} \wedge \cdots \wedge f_{(a_{i+1})_r} \otimes w) = \sum_{l=1}^{r}(-1)^l f_{(a_{i+1})_1} \wedge \cdots \wedge \widehat{f_{(a_{i+1})_l}} \wedge \cdots \wedge f_{(a_{i+1})_r} \otimes \phi_{i+1}(f_{(a_{i+1})_l}) \cup w.$$

Using again that $\mathbf{F}_\bullet$ is a minimal free resolution exist $x' \in \mathfrak{m}$ and $f' \in F_i$ such that $\phi_{i+1}(f_{(a_{i+1})_l}) = x'f'$. Now by linearity of tensor

$$d^i \otimes 1_k(f_{(a_{i+1})_1} \wedge \cdots \wedge f_{(a_{i+1})_r} \otimes w \otimes \bar{y}) =$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} (-1)^l f_{(a_{i+1})_1} \wedge \cdots \wedge \widehat{f_{(a_{i+1})_l}} \wedge \cdots \wedge f_{i_r} \otimes \phi_{i+1}(f_{(a_{i+1})_l}) \cup w \otimes 1 \otimes \cdots \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} (-1)^l f_{(a_{i+1})_1} \wedge \cdots \wedge \widehat{f_{(a_{i+1})_l}} \wedge \cdots \wedge f_{(a_{i+1})_r} \otimes x' f' \cup w \otimes 1 \otimes \cdots \otimes 1_k(\bar{y})$$

$$= \pm 1 \otimes \cdots \otimes 1 \otimes \sum_{l=1}^{r} (-1)^l f_{(a_{i+1})_1} \wedge \cdots \wedge \widehat{f_{(a_{i+1})_l}} \wedge \cdots \wedge f_{(a_{i+1})_s} \otimes f' \cup w \otimes 1 \otimes \cdots \otimes 1_k(\overline{x'y})$$

$$= 0.$$

completing the proof. □

Now, with Lemma 3 in the next theorem, we show that $S_j\mathbf{F}_\bullet$ is a minimal free resolution of $S_j(M)$.

**Theorem 10.** Let $M$ be a finitely generated $R$-module with $\operatorname{pd}_R M < \infty$. If $M$ satisfies the $(SW_j)$ condition and $\mathbf{F}_\bullet$ is a minimal free resolution of $M$, then $S_j\mathbf{F}_\bullet$ is a minimal free resolution of $S_j(M)$ and $\operatorname{pd}_R S_j(M) < \infty$.

*Demonstration.* Let $\mathbf{F}_\bullet : 0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0$ be a minimal free resolution of $M$ where $\operatorname{pd}_R M = p$. Since $M$ satisfies the $(SW_j)$ condition, by Theorem 9, the complex $S_j\mathbf{F}_\bullet$ is a free resolution for $S_j(M)$. To show that $S_j\mathbf{F}_\bullet$ is a minimal free resolution it is enough to show that $d_t \otimes 1_k = 0$ for all $t \geq 1$, where $d_t$ is a map defined by

$$d_t : (S_j\mathbf{F}_\bullet)_t \longrightarrow (S_j\mathbf{F}_\bullet)_{t-1},\ \ d_t = (d_t^{j_1},\, d_t^{j_2}, \ldots, d_t^{j_r}).$$

Now, let

$$f_* = (f_*^{j_1}, f_*^{j_2}, \ldots,\ f_*^{j_r}) \in \bigoplus_{\substack{(a_0,\ldots,\ a_p) \\ \sum a_i = j \\ \sum i a_i = t}} S(a_0, \ldots,\ a_p; \mathbf{F}_\bullet)\ ,$$

where each $f_*^{j_r} \in S(a_0^r,\ a_1^r, \ldots,\ a_p^r)$ with $(a_0^r,\ a_1^r, \ldots,\ a_p^r)$ the $r$-th non negative integer solution of system $\sum a_i = j, \sum i a_i = j$. Now as $\mathbf{F}_\bullet$ is a minimal free resolution, by Lemma 3 we obtain that $d_t^{j_r} \otimes 1_k = 0$, for all $r$. Therefore,

$$\begin{aligned} d_t \otimes 1_k(f^* \otimes \bar{y}) &= d_t(f^*) \otimes 1_k(\bar{y}) \\ &= (d_t^{j_1}(f_*^{j_1}),\ d_t^{j_2}(f_*^{j_2}), \ldots,\ d_t^{j^r}(f_*^{j_r})) \otimes 1_k(\bar{y}) \\ &= (d_t^{j_1}(f_*^{j_1}) \otimes 1_k(\bar{y}),\ d_t^{j_2}(f_*^{j_2}) \otimes 1_k(\bar{y}), \ldots,\ d_t^{j^r}(f_*^{j_r}) \otimes 1_k(\bar{y})) \\ &= 0 \end{aligned}$$

for all $\bar{y} \in k$. Hence $d_t \otimes 1_k = 0$ and this show that $S_j\mathbf{F}_\bullet$ is minimal free resolution. Now $\mathrm{pd}_R\, S_j(M) < \infty$ follows by construction of complex $S_j\mathbf{F}_\bullet$ (see 3.3).

□

**Remark 14.** From now on, for the sake of simplicity, whenever we say that $M$ has a finite projective dimension, we are assuming $\mathbf{F}_\bullet$ is a minimal free resolution of $M$.

As an immediate application from Theorem 10 and Remark 11, we got a formula to calculate the projective dimension of $S_j(M)$.

**Corollary 4.** Let $M$ be a $R$-module finitely generated with $\mathrm{pd}_R\, M < \infty$. If $M$ satisfies the $(SW_j)$ condition , then

$$\mathrm{pd}_R\, S_j(M) = \begin{cases} j\mathrm{pd}_R\, M, & \text{for} \quad \mathrm{pd}_R\, M \quad \text{even;} \\ j(\mathrm{pd}_R\, M - 1) + \min\{\beta^R_{\mathrm{pd}_R M}(M), j\}, & \text{for} \quad \mathrm{pd}_R\, M \quad \text{odd.} \end{cases}$$

*Demonstration.* Since $\mathrm{pd}_R\, M < \infty$, we can consider $\mathbf{F}_\bullet$ a minimal free resolution of $M$. As $M$ satisfies the $(SW_j)$ condition (Definition 16), we get $S_j\mathbf{F}_\bullet$ is exact. Now, by Theorem 10, $S_j\mathbf{F}_\bullet$ is a minimal free resolution of $S_j(M)$. Thus, $\mathrm{pd}_R\, S_j(M) = \lambda(S_j\, \mathbf{F}_\bullet)$. Therefore, by Remark 11, we obtain the result. □

**Remark 15.** Let $M$ a finitely generated $R$-module with $0 < \mathrm{pd}_R\, M < \infty$. If $M$ satisfies $(SW_j)$ condition, by Theorem 10 and Auslander-Buchsbaum formula, we get $\mathrm{pd}_R\, S_j(M) \leq \dim R$. Thus, by Corollary 4, we have the following cases:

(a) Case $\mathrm{pd}_R\, M$ is even: $j \leq \dfrac{\dim R}{\mathrm{pd}_R\, M}$.

(b) Case $\mathrm{pd}_R\, M$ is odd: If $\min\{\beta^R_{\mathrm{pd}_R\, M}(M), j\} = \beta^R_{\mathrm{pd}_R M}(M)$, then $j \leq \dfrac{\dim R - \beta^R_{\mathrm{pd}_R\, M}(M)}{\mathrm{pd}_R\, M - 1}$, for $\mathrm{pd}_R\, M \neq 1$ and for $\mathrm{pd}_R\, M = 1$, we get $\min\{\beta^R_1(M), j\} \leq \dim R$. Now, if $\mathrm{pd}_R\, M \neq 1$ and $\min\{\beta^R_1(M), j\} = j$, then $j \leq \dfrac{\dim R}{\mathrm{pd}_R\, M}$.

This means that the complexes $S_j\mathbf{F}_\bullet$, over a local ring of dimension $d$, do not always produce minimal free resolutions of $S_jM$ for all $j$ (see Example 2).

**Example 2.** Let $R = k[[x,\, y,\, z,\, w]]$ be a ring of formal power series over a field $k$ and let $I = (xw,\, xz,\, yw,\, yz)$ be a ideal of $R$. Since $j = 3 > \dfrac{\dim R}{\mathrm{pd}_R\, I}$, the complex $S_3\mathbf{F}_\bullet$ (in 3.3) not produces a minimal free resolution for $S_3(I)$. Firstly, computing in MACAULAY2, the minimal free resolutions of $I$ and $S_3(I)$ are given respectively by

$$\mathbf{F}_\bullet : 0 \longrightarrow R \xrightarrow{\phi_2} R^4 \xrightarrow{\phi_1} R^4 \longrightarrow I \longrightarrow 0$$

and

$$S_3(I)_\bullet : 0 \longrightarrow R^4 \longrightarrow R^{16} \longrightarrow R^{33} \longrightarrow R^{40} \longrightarrow R^{20} \longrightarrow S_3(I) \longrightarrow 0,$$

where

$$[\phi_1] = \begin{pmatrix} -y & 0 & -w & 0 \\ x & 0 & 0 & -w \\ 0 & -y & z & 0 \\ 0 & x & 0 & z \end{pmatrix}, \ [\phi_2] = \begin{pmatrix} w \\ -z \\ -y \\ x \end{pmatrix}.$$

Thus, $\mathrm{pd}_R\, S_3(I) = 4 = \dim R$.

On the other hand, now using the complex $S_3\mathbf{F}_\bullet$ (in 3.3), we get

$$S_3\mathbf{F}_\bullet : 0 \longrightarrow D_3(R) \longrightarrow R^4 \otimes D_2(R) \longrightarrow \overset{2}{\Lambda} R^4 \otimes R \oplus R^4 \otimes D_2(R) \longrightarrow R^4 \otimes R \otimes R^4 \oplus \overset{3}{\Lambda} R^4 \longrightarrow$$

$$\longrightarrow R^4 \otimes \overset{2}{\Lambda} R^4 \oplus R^4 \otimes R \longrightarrow D_2(R^4) \otimes R^4 \longrightarrow D_3(R^4) \longrightarrow 0.$$

Since $\mathrm{grade}(I_1(\phi_2)) = 4 < 6$, by Theorem 9, the ideal $I \lhd R$ does not satisfies $(SW_3)$ condition. Hence $S_3\mathbf{F}_\bullet$ is not an exact complex and consequently $S_3\mathbf{F}_\bullet$ is not a minimal free resolution of $S_3(I)$.

According to Remark 15 and Theorem 10, a natural question arises. Are there intervals of $j$ where $S_j\mathbf{F}_\bullet$ is minimal free resolution for $S_j(M)$? The next example illustrates that this can happen.

**Example 3.** (HUNEKE, 1982, Example 1.3) The “generic" ideal of projective dimension one is given by the ideal defined by the exact sequence

$$\mathbf{F}_\bullet : 0 \longrightarrow R^n \xrightarrow{\phi_1} R^{n+1} \longrightarrow I \longrightarrow 0$$

where $[\phi_1] = (x_{rs})$ is a generic $n$ by $n+1$ matrix over a field $k$, and let $R = k[x_{rs}]$ be the polynomial ring over a field $k$. Let $I$ be a ideal of $R$ generated by the $n$ by $n$ minors of $[\phi_1]$. So, by (EAGON; HOCHSTER, 1971, Corollary 4), we have

$$\mathrm{grade}(I_t(\phi_1)) = (n-t+1)(n+2-t), \ t = 1, \ldots, n. \tag{3.5}$$

Checking the condition $(b)$ of the Theorem 9, with $n = r_1$, and using the equality 3.5, we get

$$\mathrm{grade}(I_{r_1-j+1}(\phi_1)) \geq j, \ \mathrm{grade}(I_{r_1-j+2}(\phi_1)) \geq j-1 \ \text{and} \ \mathrm{grade}(I_{r_1}(\phi_1)) \geq 1.$$

Thus, the ideal $I$ satisfies the condition $(b)$. Now, if $\mathbf{F}_\bullet$ is a minimal free resolution, by Theorem 10 and Remark 15 item $(b)$, $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(I)$ for $j = 1, \ldots, n$

The elegance of the Example 3 above is that it gives us a class of ideals with a projective dimension 1 that satisfy the $(SW_j)$ condition .

**Remark 16.** Note that, by the equality 3.5, the ideal generated by the $t$ by $t$ minors of $[\phi_1]$ satisfies

$$\text{grade}(I_t(\phi_1)) \geq n-t+1, \text{ for all } t=0,1,\ldots, n. \tag{3.6}$$

Therefore, by Example 3, we show that the ideals of projective dimension 1 that satisfy the inequality 3.6 above also satisfy the $(SW_j)$ condition. In the following corollary, we will see that we can obtain the same result in a more general context.

**Corollary 5.** Let $M$ be a finitely generated $R$-module with $\text{pd}_R\, M = 1$. Suppose that $\text{grade}(I_j(\phi_1)) \geq \beta_1^R(M) - j + 1$ hold for $j = 1, \ldots, \beta_1^R(M)$, then the complex $S_j\mathbf{F}_\bullet$ is minimal free resolution for $S_j(M)$ for $j = 2, \ldots, \beta_1^R(M)$.

*Demonstration.* Since $\text{pd}_R\, M = 1$, by (AVRAMOV, 1981, Proposition 3), the $(SW_j)$ condition is equivalent to $\text{grade}(I_j(\phi_1)) \geq \beta_1^R(M) - j+1$ for $j = 1, \ldots, \beta_1^R(M)$. Now, from Theorem 10, we have $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$ for $j = 2, \ldots, \beta_1^R(M)$.

□

Let $R$ be a Cohen Macaulay Noetherian domain and $M$ be a finitely generated $R$-module satisfying the grade conditions of the Corollary 5. For such modules, (HUNEKE, 1981) shows that the grade conditions can be replaced by a relation involving the minimal number of generators of $M_\mathfrak{p}$ and the height$(\mathfrak{p})$ for all non-zero $\mathfrak{p} \in \text{Spec}(R)$. In other words, we have the following corollary.

**Corollary 6.** Let $R$ be a local Cohen-Macaulay domain and $M$ an $R$-module with a minimal free resolution,

$$\mathbf{F}_\bullet : 0 \longrightarrow R^m \xrightarrow{\phi_1} R^n \longrightarrow M \longrightarrow 0$$

Let $I_t(\phi_1)$ denote the ideal in $R$ generated by the $t \times t$ minors of $[\phi_1]$ and $\mu(M_\mathfrak{p}, R_\mathfrak{p}) \leq n - m + \text{height}(\mathfrak{p}) - 1$ for all non-zero primes $\mathfrak{p}$ in $R$. Then, the complex $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$ for each $j = 1, \ldots, m$.

*Demonstration.* Suppose that $\mu(M_\mathfrak{p}, R_\mathfrak{p}) \leq n-m+\text{height}(\mathfrak{p}) - 1$ for all non-zero primes $\mathfrak{p}$ in $R$. By (HUNEKE, 1981, Theorem 1.1), we get

$$\text{height}(I_t(\phi_1)) \geq m+2-t \text{ for } t = 1, \ldots, m.$$

Now, since $R$ is Cohen-Macaulay (BRUNS; HERZOG, 1993, Corollary 2.1.4),

$$\text{grade}(I_t(\phi_1)) = \text{height}(I_t(\phi_1)) \geq m+2-t \text{ for } t = 1, \ldots, m. \tag{3.7}$$

Checking the condition $(b)$ of the Theorem 9, with $m = r_1$ and using the inequality 3.7, we get the following inequalities

$$\text{grade}(I_{r_1-j+1}(\phi_1)) \geq j, \ \text{grade}(I_{r_1-j+2}(\phi_1)) \geq j-1 \text{ and } \text{grade}(I_{r_1}(\phi_1)) \geq 1.$$

Thus, $M$ satisfies the condition $(b)$. Now, by Theorem 10 and Remark 15 item $(b)$, $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$ for $j = 1, \ldots, m$. As we wanted to demonstrate. □

**Corollary 7.** Let $R$ be a local Cohen-Macaulay domain and $I$ an ideal of $R$ with a minimal free resolution

$$\mathbf{F}_\bullet : 0 \longrightarrow R^n \xrightarrow{\phi_1} R^{n+1} \longrightarrow I \longrightarrow 0$$

If $\mu(I_\mathfrak{p}, R_\mathfrak{p}) \leq \text{height}(\mathfrak{p})$ for every non-zero prime $\mathfrak{p}$ in $R$, then the complex $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(I)$ for each $j = 1, \ldots, n$.

*Demonstration.* Follow immediately of Corollary 6. □

Once the minimality of free resolution $S_j\mathbf{F}_\bullet$ is known, by the Corollary 2, we will be able to extract information related to the Betti numbers of $S_j(M)$. And it is with this idea that we start the next section.

## 3.2 On Betti numbers for symmetric powers

In the previous section, we showed that the free resolution $S_j\mathbf{F}_\bullet$ carries the minimality of the free resolution $\mathbf{F}_\bullet$. With this, it is to be expected that something similar will happen with the Betti numbers of $M$. And indeed, this is what we will see in the following proposition.

**Proposition 9.** Let $M$ be a finitely generated $R$-module with $\text{pd}_R\, M = p < \infty$. Suppose that $S_j\mathbf{F}_\bullet$ is a free resolution to $S_j(M)$ then $S_j\mathbf{F}_\bullet$ is a minimal free resolution if and only if

(a) for $p$ even,

$$\beta_t^R(S_j(M)) = \sum_{\substack{(a_0,\ldots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} \binom{\beta_0^R(M) + a_0 - 1}{a_0} \binom{\beta_1^R(M)}{a_1} \binom{\beta_2^R(M) + a_2 - 1}{a_2} \binom{\beta_3^R(M)}{a_3} \cdots \binom{\beta_p^R(M) + a_p - 1}{a_p};$$

(b) for $p$ odd,

$$\beta_t^R(S_j(M)) = \sum_{\substack{(a_0,\ldots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} \binom{\beta_0^R(M) + a_0 - 1}{a_0} \binom{\beta_1^R(M)}{a_1} \binom{\beta_2^R(M) + a_2 - 1}{a_2} \binom{\beta_3^R(M)}{a_3} \cdots \binom{\beta_p^R(M)}{a_p};$$

for all $t = 0,\ 1, \ldots,\ l := \text{pd}_R\, S_j(M)$.

*Demonstration.* In fact, let $\mathbf{F}_\bullet : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0$ be a minimal free resolution for $M$. From (3.3) we obtain the following free finite complex

$$S_j\mathbf{F}_\bullet : 0 \longrightarrow (S_j\mathbf{F}_\bullet)_l \xrightarrow{d_l} (S_j\mathbf{F}_\bullet)_{l-1} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_2} (S_j\mathbf{F}_\bullet)_1 \xrightarrow{d_1} (S_j\mathbf{F}_\bullet)_0$$

for each integer $j \geq 2$. Now, by Remark 9 and Lemma 2, we get that for $p$ even

$$\operatorname{rank}(S_j\mathbf{F}_\bullet)_t = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+a_p-1}{a_p}$$

for all $t = 0,1,\dots,l$. Similarly, for $p$ odd, we obatin that

$$\operatorname{rank}(S_j\mathbf{F}_\bullet)_t = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)}{a_p}$$

for all $t = 0,\ 1,\dots,\ l$.

Finally, the proof follows from the following fact: The free resolution $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$ if and only if $\dim_k(\operatorname{Tor}_t^R(k, S_j(M))) = \operatorname{rank}(S_j\mathbf{F}_\bullet)_t$ $t = 0,\dots,\ l$. □

Most prominent in the above Proposition 9 is that we show that the Betti numbers of $S_j(M)$ depend on the Betti numbers of $M$. This leads us to think that certain properties regarding the Betti numbers of $M$ can be conveyed to the Betti numbers of $S_j\mathbf{F}_\bullet$.

Since the length of the free resolution $S_j\mathbf{F}_\bullet$ depends on the projective dimension of $M$, by the Corollary 4, we notice that a certain difficulty when trying to explain in a better way the sum that appears in the Proposition 9 when $\operatorname{pd}_R M > 2$. Now when $\operatorname{pd}_R M \leq 2$, we get some interesting formulas. But before stating them, let us go to the following Corollary which immediately follows from Theorem 10 and Proposition 9.

**Corollary 8.** Let $M$ be a finitely generated $R$-module with $\operatorname{pd}_R M = p < \infty$. If $M$ satisfies $(SW_j)$, then

(a) for $p$ even,

$$\beta_t^R(S_j(M)) = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+a_p-1}{a_p};$$

(b) for $p$ odd,

$$\beta_t^R(S_j(M)) = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)}{a_p};$$

for all $t = 0,\ 1,\dots,\ \operatorname{pd}_R S_j(M)$.

**Corollary 9.** Let $M$ be a finitely generated $R$-module with projective dimension 1 such that $\operatorname{grade}(I_j(\phi_1)) \geq \beta_1^R(M) - j + 1$, for all $j = 1,\dots,\ \beta_1^R(M)$. Then

$$\beta_t^R(S_j(M)) = \binom{\beta_0^R(M)+j-t-1}{j-t}\binom{\beta_1^R(M)}{t},\ \text{for all } t = 0,\ 1,\dots,\ \operatorname{pd}_R S_j(M).$$

*Demonstration.* Let $\mathbf{F}_\bullet$ be a minimal free resolution. By assumptions and by Corollary 5, the complex

$$S_j\mathbf{F}_\bullet : 0 \longrightarrow D_{j-l}(F_0)\otimes \overset{l}{\Lambda} F_1 \longrightarrow D_{j-l+1}(F_0)\otimes \overset{l-1}{\Lambda} F_1 \longrightarrow \cdots \longrightarrow D_{j-1}(F_0)\otimes \overset{1}{\Lambda} F_1 \longrightarrow D_j(F_0)\otimes \overset{0}{\Lambda} F_1$$

is a minimal free resolution for $S_j(M)$, with $l = \mathrm{pd}_R\, S_j(M)$, for all $j = 1,\ldots,\ \beta_1^R(M)$. Now, by Proposition 9 we obtain that

$$\beta_t^R(S_j(M)) = \binom{\beta_0^R(M)+j-t-1}{j-t}\binom{\beta_1^R(M)}{t},\ \text{for all } t = 0,\ 1,\ldots,\ l.$$

As we wanted to demonstrate. □

**Remark 17.** Since, by the Corollary 4, $\mathrm{pd}_R\, S_j(M) = \min\{\beta_1^R(M), j\}$ we get that the binomial $\binom{\beta_1^R(M)}{t}$ is well defined for all $t = 0,\ 1,\ldots,\ \mathrm{pd}_R\, S_j(M)$.

The Corollary 9 above describes the Betti numbers of $S_j(M)$ in the case $\mathrm{pd}_R\, M = 1$. For this case, we get a simple formula due to the complex $S_j\mathbf{F}_\bullet$ in the Remark 12. Now for the case where $\mathrm{pd}_R\, M = 2$, the complex $S_j\mathbf{F}_\bullet$ already starts to present some difficulties, in the sense that we need to unsolve the sum in 9 from a system of equations. In this case, we obtain the following Corollary.

**Corollary 10.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R\, M = 2$. If $M$ satisfies $(SW_j)$ condition, then

(a) for $j \geq t$,

$$\beta_t^R(S_j(M)) = \sum_{r=0}^{\lfloor \frac{t}{2}\rfloor}\binom{\beta_2^R(M)+r-1}{r}\binom{\beta_1^R(M)}{t-2r}\binom{\beta_0^R(M)+j-t+r-1}{j-t+r};$$

(b) for $j < t$,

$$\beta_t^R(S_j(M)) = \sum_{r=t-j}^{\min\{j,\lfloor \frac{t}{2}\rfloor\}}\binom{\beta_2^R(M)+r-1}{r}\binom{\beta_1^R(M)}{t-2r}\binom{\beta_0^R(M)+j-t+r-1}{j-t+r};$$

for all $t = 0,\ 1,\ldots,\ \mathrm{pd}_R\, S_j(M)$.

*Demonstration.* Observe that, by Proposition 9, is enough to calculate the non-negative integers solutions of system

$$\begin{cases} a_1 + 2a_2 = t \\ a_0 + a_1 + a_2 = j \end{cases} \tag{3.8}$$

For this we consider the following cases: $j \geq t$ and $j < t$.

(a) Case $j \geq t$. Fixing $a_2$ the system above is equivalent to

$$\begin{cases} a_0 + a_1 = j - a_2 \\ a_1 = t - 2a_2 \end{cases}$$

Whose solutions are given by $a_1 = t - 2a_2$ and $a_0 = j - t + a_2$. The conditions $0 \leq a_0 \leq j$ and $0 \leq a_1 \leq n$ imply that $t - j \leq a_2 \leq t$, $0 \leq a_2 \leq \lfloor \frac{t}{2} \rfloor$ and therefore the solutions of system (3.8) above are given by triple

$$(a_0,\, a_1,\, a_2) = (k - t + a_2,\, t - 2a_2,\, a_2) \text{ where } 0 \leq a_2 \leq \lfloor \frac{t}{2} \rfloor.$$

Since $M$ satisfy $(SW_j)$ condition, by Proposition 9, we obtain that

$$\beta_t^R(S_j(M)) = \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \binom{\beta_2^R(M) + r - 1}{r} \binom{\beta_1^R(M)}{t - 2r} \binom{\beta_0^R(M) + j - t + r - 1}{j - t + r}.$$

(b) Case $j < t$. Analogously, fixing $a_2$, the conditions $0 \leq a_0 \leq j$ and $0 \leq a_1 \leq j$ imply that $t - j \leq a_2 \leq \min\{j, \lfloor \frac{t}{2} \rfloor\}$. In this case, the solutions of system (3.8) above are given by triple

$$(a_0,\, a_1,\, a_2) = (j - t + a_2,\, t - a_2,\, a_2) \text{ where } t - j \leq a_2 \leq \min\{j, \lfloor \frac{t}{2} \rfloor\}.$$

Follow that

$$\beta_t^R(S_j(M)) = \sum_{r=t-j}^{\min\{j, \lfloor \frac{t}{2} \rfloor\}} \binom{\beta_2^R(M) + r - 1}{r} \binom{\beta_1^R(M)}{t - 2r} \binom{\beta_0^R(M) + j - t + r - 1}{j - t + r}.$$

As we wanted to demonstrate.

□

Note that we are always using the condition that $M$ satisfies the condition $(SW_j)$. But what happens if we do not make that assumption? We end this chapter by illustrating this in the example below.

**Example 4.** Let $R = k[[x_1,\, x_2,\, x_3,\, y_1,\, y_2]]$ be a ring of formal power series over a field $k$ and let $I = (x_1x_2y_1,\, x_1x_3y_1,\, x_2x_3y_1,\, x_1x_2y_2,\, x_1x_3y_2,\, x_2x_3y_2)$ be an ideal of $R$. By MACAULAY2, we obtain a minimal free resolution of $I$ given by

$$\mathbf{F}_\bullet : 0 \longrightarrow R^2 \xrightarrow{\phi_2} R^7 \xrightarrow{\phi_1} R^6 \longrightarrow I \longrightarrow 0$$

where $\phi_1$ and $\phi_2$ have a matrix representation given by

$$[\phi_1] = \begin{pmatrix} -x_3 & 0 & 0 & 0 & -y_2 & 0 & 0 \\ x_2 & -x_2 & 0 & 0 & 0 & -y_2 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_2 \\ 0 & 0 & -x_3 & 0 & y_1 & 0 & 0 \\ 0 & 0 & -x_2 & x_2 & 0 & y_1 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & y_1 \end{pmatrix}; \quad [\phi_2] = \begin{pmatrix} y_2 & y_2 \\ 0 & y_2 \\ -y_1 & -y_1 \\ 0 & -y_1 \\ -x_3 & -x_3 \\ x_2 & 0 \\ 0 & x_1 \end{pmatrix}.$$

Notice that grade$(I_4(\phi_1)) > 2$, grade$(I_5(\phi_1)) > 1$ and grade$(I_2(\phi_2)) = 4$. So, the ideal $I$ satisfies $(SW_2)$ condition. Now, by Theorem 10, we get that $S_2\mathbf{F}_\bullet$ is a minimal free resolution of $S_2(I)$ given by

$$S_2\mathbf{F}_\bullet : D_2(R^2) \longrightarrow R^7 \otimes R^2 \longrightarrow \overset{2}{\Lambda} R^7 \oplus R^6 \otimes R^2 \longrightarrow R^6 \otimes R^7 \longrightarrow D_2(F_0).$$

On the other hand, by MACAULAY2, we get the minimal free resolution of $S_2(I)$

$$S_2(I)_\bullet : 0 \longrightarrow R^3 \longrightarrow R^{14} \longrightarrow R^{33} \longrightarrow R^{42} \longrightarrow R^{21} \longrightarrow S_2(I) \longrightarrow 0.$$

We can see that the ranks of these two minimal free resolution are the same, i. e., the Betti numbers are the same ( see Table 1).

Similarly, we compute the minimal free resolution of $S_3(I)$. In fact, the complex $S_3\mathbf{F}_\bullet$ is given

$$S_3\mathbf{F}_\bullet : 0 \longrightarrow D_3(R^2) \longrightarrow R^7 \otimes D_2(R^2) \longrightarrow \overset{2}{\Lambda} R^7 \otimes R \oplus R^6 \otimes D_2(R^2) \longrightarrow R^7 \otimes R^2 \otimes R^6 \oplus \overset{3}{\Lambda} R^7 \longrightarrow$$

$$\longrightarrow D_2(R^6) \otimes R^2 \oplus R^6 \otimes \overset{2}{\Lambda} R^7 \longrightarrow D_2(R^6) \otimes R^7 \longrightarrow D_3(R^6) \longrightarrow 0.$$

Note that, by Theorem 9, the complex $S_3\mathbf{F}_\bullet$ is not a free resolution for $S_3(I)$, because the ideal $I_2(\phi_2)$ satisfy grade$(I_2(\phi_2)) < 6$ (does not satisfy the $(SW_3)$ condition). On the other hand, by MACAULAY2, we get a minimal free resolution of $S_3(I)$

$$S_3(I)_\bullet : 0 \longrightarrow R^4 \longrightarrow R^{32} \longrightarrow R^{97} \longrightarrow R^{160} \longrightarrow R^{146} \longrightarrow R^{56} \longrightarrow S_3(I) \longrightarrow 0.$$

We see that the ranks coming from $S_3\mathbf{F}_\bullet$ do not coincide with the Betti numbers of a minimal free resolution of $S_3(I)$( see Table 1 ).

Table 1

| $t$-th Betti Number | $S_2(I)$ | $S_3(I)$ |
|---|---|---|
| 0 | 21 | 56 |
| 1 | 42 | 127 |
| 2 | 33 | 147 |
| 3 | 14 | 119 |
| 4 | 3 | 60 |

Source: Research data.

CHAPTER

4

# BOUNDS AND APPLICATIONS

The purpose of this chapter is to use the results obtained previously. First we show that the Betti numbers of $S_j(M)$ are bounded from above by a binomial expression, for modules of projective dimension $p \geq 1$ that satisfy the condition $(SW_j)$. In particular, i.e. when $p = 1$, we show that $\beta_t^R(S_j(M))$ is bounded below and above. Later we make some applications for modules of linear type and point out a similarity with the famous Buchsbaum-Eisenbud-Horrocks conjecture.

## 4.1 Lower and upper bounds

In the following Propositions, we show that there are upper bounds for the numbers of Betti $\beta_t^R(S_j(M))$, which are independent of $t$. Furthermore, such bounds have a general expression in the sense that they serve any projective dimension $p$ of $M$.

**Proposition 10.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = p \geq 1$. If $M$ satisfies $(SW_j)$, then

(a) for $p$ even,

$$\beta_t^R(S_j(M)) \leq \binom{\sum_{i=0}^{p} \beta_i^R(M) + j(\frac{p+2}{2})}{j};$$

(b) for $p$ odd,

$$\beta_t^R(S_j(M)) \leq \binom{\sum_{i=0}^{p} \beta_i^R(M) + j(\frac{p+1}{2})}{j};$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_j(M)$.

*Demonstration.* Suppose that $p$ is even, in the case $p$ odd the proof follows similarly. By hypothesis $M$ satisfies $(SW_j)$ condition. So, by Corollary 8, we obtain the following equality

$$\begin{aligned}
\beta_t^R(S_j(M)) &= \sum_{\substack{(a_0,\ldots,\,a_p)\\ \sum a_i=j\\ \sum ia_i=t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+a_p-1}{a_p}\\
&\leq \sum_{\substack{(a_0,\ldots,\,a_p)\\ \sum a_i=j}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+a_p-1}{a_p},
\end{aligned} \tag{4.1}$$

where $t = 0,\ 1,\ldots,\ \mathrm{pd}_R\, S_j(M)$. Since $a_i \leq j$, for all even integer positive $i$ between 0 and $p$, we get that

$$\binom{\beta_i^R(M)+a_i-1}{a_i} \leq \binom{\beta_i^R(M)+j}{a_i}. \tag{4.2}$$

Of inequalities 4.1 and 4.2 we obtain that

$$\beta_t^R(S_j(M)) \leq \sum_{\substack{(a_0,\ldots,\,a_p)\\ \sum a_i=j}} \binom{\beta_0^R(M)+j}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+j}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+j}{a_p}.$$

Now, using the Generalized Vandermonde's identity, follow that

$$\beta_t^R(S_j(M)) \leq \binom{\sum_{i=0}^{p}\beta_i^R(M)+j(\frac{p+2}{2})}{j}.$$

As we wanted to demonstrate. □

As an immediate consequence of the Proposition 10 above, we have the following corollary.

**Corollary 11** (Bound Low-Upp)**.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = 1$ satisfying $(SW_j)$ condition. Then the following inequalities hold

$$\binom{\beta_1^R(M)}{t} \leq \beta_t^R(S_j(M)) \leq \binom{\sum_{i=0}^{p}\beta_i^R(M)+j}{j}, \text{ for all } t = 0,\ 1,\ldots,\ \mathrm{pd}_R S_j(M).$$

*Demonstration.* It follows immediately from the Corollary 9 Proposition 10 item (b) with $p = 1$. □

We saw in the Corollary 9 that, for modules of projective dimension 1, the $(SW_j)$ condition can be replaced when the grade of the ideals $I_j(\phi_1)$ satisfies a certain inequality. In this case, the following corollary shows that $M$ satisfies the inequalities obtained in Corollary 11 for all $j = 1,\ldots,\ \beta_1^R(M)$.

**Corollary 12.** Let $R$ be a local ring and $M$ be a finitely generated $R$-module with $\mathrm{pd}_R(M) = 1$ such that $\mathrm{grade}(I_j(\phi_1)) \geq \beta_1^R(M) - j + 1$, for all $j = 1, \ldots, \beta_1^R(M)$. Then,

$$\binom{\beta_1^R(M)}{t} \leq \beta_t^R(S_j(M)) \leq \binom{\sum_{i=0}^{p} \beta_i^R(M) + j}{j}, \text{ for all } t = 0, 1, \ldots, \mathrm{pd}_R S_j(M).$$

*Demonstration.* It follows immediately from the Corollary 9. □

When we consider modules of projective dimension 2, we observe that something similar happens to the inequality obtained in the Corollary 11. As we will see in the following Corollary.

**Corollary 13.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = 2$. If $M$ satisfy $(SW_j)$ condition, then

(a) If $j \geq t$, then $\beta_t^R(S_j(M)) \geq \binom{\beta_1^R(M)}{t}$, for all $t = 0, 1, \ldots, j$;

(b) If $j < t$, then $\beta_t^R(S_j(M)) \geq \binom{\beta_1^R(M)}{2j - t}$, for all $t = j+1, \ldots, \mathrm{pd}_R S_j(M)$.

*Demonstration.* Since $M$ satisfies $(SW_j)$ condition, by Theorem 10, $S_j\mathbf{F}_\bullet$ is a minimal free resolution for $S_j(M)$. So, by Corollary 10, we need consider two cases $j \geq t$ and $j < t$.

(a) For $j \geq t$, we obtain that

$$\begin{aligned}
\beta_t^R(S_j(M)) &= \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \binom{\beta_2^R(M) + r - 1}{r} \binom{\beta_1^R(M)}{t - 2r} \binom{\beta_0^R(M) + j - t + r - 1}{j - t + r} \\
&\geq \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \binom{\beta_1^R(M)}{t - 2r} \\
&\geq \binom{\beta_1^R(M)}{t}, \text{ for all } t = 0,\ 1, \ldots,\ j.
\end{aligned}$$

(b) Similarly, for $j < t$, we get that

$$\begin{aligned}
\beta_t^R(S_j(M)) &= \sum_{r=t-j}^{\min\{j, \lfloor \frac{t}{2} \rfloor\}} \binom{\beta_2^R(M) + r - 1}{r} \binom{\beta_1^R(M)}{t - 2r} \binom{\beta_0^R(M) + j - t + r - 1}{j - t + r} \\
&\geq \binom{\beta_1^R(M)}{2j - t}, \text{ for all } t = j+1, \ldots, \mathrm{pd}_R S_j(M).
\end{aligned}$$

□

The Corollary 11 and Corollary 13 show that we can find lower bounds for the Betti numbers $\beta_t^R(S_j(M))$ for all $0 \leq t \leq \mathrm{pd}_R(S_j(M))$, when $M$ has projective dimension 1 or 2. On the other hand for $\mathrm{pd}_R M \geq 3$, the job of getting lower bounds becomes computationally more difficult. This is due to the fact that, when considering such dimensions, the positive integer solutions of the system of equations in 3.4.

In addition to obtaining lower and upper bounds, we will see in the following proposition that the Betti numbers of two arbitrary symmetric powers satisfy the order relation $\beta_t^R(S_j(M)) < \beta_t^R(S_{j+1}(M))$ for any $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_{j+1}(M)$. In particular, $\mu(S_j(M)) < \mu(S_{j+1}(M))$.

**Proposition 11.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = 1$. If $M$ satisfies the $(SW_j)$ and $(SW_{j+1})$ conditions, then

$$\beta_t^R(S_j(M)) < \beta_t^R(S_{j+1}(M)),$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_{j+1}(M)$.

*Demonstration.* As $M$ has a projective dimension 1 and satisfies the $(SW_j)$ and $(SW_{j+1})$ conditions, by the Theorem 10, we obtain that the complexes $S_j\mathbf{F}_\bullet$ and $S_{j+1}\mathbf{F}_\bullet$ are finite minimal free resolutions of $S_j(M)$ and $S_{j+1}(M)$, respectively. Let $l = \mathrm{pd}_R\, S_{j+1}(M)$. By the Corollary 4, we have that $l = \min\{\beta_1^R(M), j+1\}$ and thus $\mathrm{pd}_R\, S_j(M) \leq l$. Note that if $l = j+1$ then $\mathrm{pd}_R\, S_j(M) = j$ and therefore $\beta_{j+1}^R(S_j(M)) < \beta_{j+1}^R(S_{j+1}(M))$. Now, for all $t = 0,\ 1, \ldots,\ j$, by the Corollary 8 we get that

$$\begin{aligned}
\beta_t^R(S_{j+1}(M)) &= \binom{\beta_0^R(M)+j-t}{j+1-t}\binom{\beta_1^R(M)}{t} = \frac{\beta_0^R(M)+j-t}{j+1-t}\binom{\beta_0^R(M)+j-t-1}{j-t}\binom{\beta_1^R(M)}{t} \\
&= \frac{\beta_0^R(M)+j-t}{j+1-t}\beta_t^R(S_j(M)) \\
&> \beta_t^R(S_j(M)).
\end{aligned}$$

Soon,

$$\beta_t^R(S_j(M)) < \beta_t^R(S_{j+1}(M)),$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_{j+1}(M)$. The case where $l = \beta_1^R(M)$ follows analogously. □

**Corollary 14.** Let $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = 1$ and $n \geq 2$ be a integer. If $M$ satisfies $(SW_j)$ condition for all $j = 2, \ldots,\ n$, then

$$\beta_t^R(S_j(M)) < \beta_t^R(S_n(M)),$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_n(M)$.

*Demonstration.* It follows by applying the Proposition 11 inductively. □

**Corollary 15.** Let $R$ be a local Cohen-Macaulay domain and $I$ an ideal of $R$ having a minimal free resolution

$$\mathbf{F}_\bullet : 0 \longrightarrow R^n \xrightarrow{\phi_1} R^{n+1} \longrightarrow I \longrightarrow 0$$

If the ideal $I$ satisfy $\mu(I_\mathfrak{p}, R_\mathfrak{p}) \leq \text{height}(\mathfrak{p})$ for every non-zero prime $\mathfrak{p}$ in $R$, by Corollary 9 and 14, we obtain that the Betti numbers of all symmetric powers, between 2 and $n$, satisfy the following decreasing relation

$$\beta_t^R(S_2(I)) < \beta_t^R(S_3(I)) < \cdots < \beta_t^R(S_n(I)),$$

$t = 2, \ldots,\ \text{pd}_R\, S_n(I)$. For these ideals, as we will see in the next section, this same order of descent is also satisfied when we consider the powers $I^j$. That is,

$$\beta_t^R(I^2) < \beta_t^R(I^3) < \cdots < \beta_t^R(I^n),$$

for all $t = 2, \ldots,\ \text{pd}_R\, I^n$.

## 4.2 Applications

This section is dedicated to some applications. Here we show that the class of modules of linear type of projective dimension 1 satisfies the $(SW_j)$ condition and consequently we obtain formulas for the Betti numbers of the $j$th ordinary power of an ideal of linear type. Next, although immediate, we highlight a fact similar to the Buchsbaum-Eisenbud-Horrocks conjecture, which is satisfied for symmetric powers $S_j(M)$.

### 4.2.1 Linear type modules and their Betti numbers

**Definition 17.** (FUKUMURO; KUME; NISHIDA, 2015) Let $R$ be a Noetherian ring with total ring $Q$ and let $M$ be a finitely generated $R$-module with rank $r$. Suppose that $M$ is torsion-free. Then, there exists an embedding $\sigma : M \hookrightarrow F$, where $F$ is a finitely free generated $R$-module. We define $\mathscr{R}_R(M)$ to be the image of the homomorphism

$$S_R(\sigma) : S_R(M) \longrightarrow S_R(F)$$

of $R$-algebras and call it the *Rees algebra* of $M$.

Let us notice that $S_R(F)$ is a polynomial ring and $\mathscr{R}_R(M)$ is its subalgebra. Since the $R$-torsion part of $S_R(M)$, which is denoted by $\mathscr{T}_R(S_R(M))$, coincides with de kernel of $S_R(\sigma)$ (SIMIS; ULRICH; VASCONCELOS , 2003), we have $\mathscr{R}_R(M) \cong S_R(M)/\mathscr{T}_R(S_R(M))$, which means that $\mathscr{R}(M)$ does not depend of choice of $\sigma$. We say $M$ is an $R$-module of *linear type* if $\mathscr{T}_R(S_R(M)) = 0$, that is $S_R(M) \cong \mathscr{R}_R(M)$.

**Remark 18.** The above definition generalizes the idea of Rees algebra of ideals. In the context of ideals, we can also define its Rees algebra as follows.

**Definition 18.** Let $R$ be a ring and $I \subseteq R$ an ideal. The *Rees Algebra* of $I$, denoted by $\mathscr{R}_R(I)$ is a subring of $R[t]$

$$\mathscr{R}_R(I) = \bigoplus_{n=0}^{\infty} I^n t^n = R + It + \cdots + I^n t^n + \cdots \subseteq R[t]$$

where $t$ is an indeterminates.

When $I$ is finitely generated ideal, for example $I = (f_1, \dots, f_k)$ then

$$\mathscr{R}_R(I) = R[f_1 t, \dots, f_k t] \subseteq R[t].$$

This implies that the following sequence is exact

$$R[t_1, \dots, t_k] \xrightarrow{\varphi} \mathscr{R}_R(I) \longrightarrow 0.$$

The Kernel of $\varphi$, which we denote by $\mathscr{J}$, is a homogeneous ideal with standard graduation $\deg(t_j) = 1$ for all $j = 1, \dots, k$. In this case $\mathscr{J}$ can be defined by

$$\mathscr{J} = \{F(t_1, \dots, t_k) \in R[t_1, \dots, t_k] | F(f_1 t, \dots, f_k t) = 0\}.$$

The ideal $\mathscr{J}$ is said *presentation ideal* of $\mathscr{R}_R(I)$ with relation to $f_1, \dots, f_k$.

Now consider the $R$-modules homorphism

$$\begin{array}{cccc} \phi: & R^k & \longrightarrow & I \\ & (a_1, \dots, a_k) & \longmapsto & \sum_{i=1}^{k} a_i f_i \end{array}$$

We have that $\phi$ induces a $R$-algebras onto homomorphism

$$\beta : R[t_1, \dots, t_k] \longrightarrow S_R(I)$$

Hence

$$S_R(I) \cong R[t_1, \dots, t_k]/\mathrm{Ker}(\beta)$$

On the other hand, as $\mathscr{J} = \mathrm{Ker}(\beta)$ is homogeneous ideal, follow that $\mathscr{J}$ is generated by homogeneous $F(f_1, \dots, f_k)$ such that $F(t_1, \dots, t_k) = 0$. So, we can factor $\phi$ through the following commutative diagram

$$\begin{array}{ccc} R[t_1, \dots, t_k] & \xrightarrow{\phi} & \mathscr{R}_R(I) \\ \beta \downarrow & \nearrow_{\alpha} & \\ S_R(I) & & \end{array}$$

Where $\alpha$ is defined by $\overline{F(t_1, \dots, t_k)} \mapsto F(f_1 t, \dots, f_k t)$. Hence we get the following relations

$$S_R(I) \cong R[t_1, \dots, t_k]/\mathrm{Ker}(\beta) \longrightarrow R[t_1, \dots, t_k]/\mathscr{J} \cong \mathscr{R}_R(I).$$

**Definition 19.** We say that an ideal $I$ is of *linear type* if $\alpha$ is an isomorphism.

If $R$ is a Noetherian domain, the lemma below shows that linear type ideals can be characterized via the freeness of torsion of its symmetric algebra.

**Lemma 4.** Let be $R$ an Noetherian domain and $I \neq 0$ ideal of $R$. Are equivalent:

(a) $S_R(I)$ is a domain;

(b) $S_R(I)$ is torsion free;

(c) $I$ is of linear type.

*Demonstration.* See (MICALI, 1964). □

**Remark 19.** In this work, we will denote the $j$th graded component of Rees algebra from $M$ by $\mathscr{R}_j(M)$.

**Remark 20.** If $M$ is of linear type module then $\mathscr{R}_j(M) \cong S_j(M)$ for all $j$. Therefore, we can explain the Betti numbers of $j$th graded component of $\mathscr{R}_j(M)$ through $j$th symmetric power $S_j(M)$. In other words, we have the following proposition and our first application.

**Proposition 12.** Let $M$ be a linear type module of $R$ with $\operatorname{pd}_R M = p < \infty$. If $M$ satisfy $(SW_j)$ condition, then

(a) for $p$ even,

$$\beta_t^R(\mathscr{R}_j(M)) = \sum_{\substack{(a_0,\dots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(M)}{a_3}\cdots\binom{\beta_p^R(M)+a_p-1}{a_p};$$

(b) for $p$ odd,

$$\beta_t^R(\mathscr{R}_j(M)) = \sum_{\substack{(a_0,\dots,\, a_p) \\ \sum a_i = j \\ \sum i a_i = t}} \binom{\beta_0^R(M)+a_0-1}{a_0}\binom{\beta_1^R(M)}{a_1}\binom{\beta_2^R(M)+a_2-1}{a_2}\binom{\beta_3^R(I)}{a_3}\cdots\binom{\beta_p^R(M)}{a_p};$$

for all $t = 0,\ 1, \dots,\ \operatorname{pd}_R \mathscr{R}_j(M)$.

*Proof.* Since $M$ is a linear type module, we get the isomorphism $\mathscr{R}_R(M) \cong S_R(M)$ of graded algebra which implies that $S_j(M) \cong \mathscr{R}_j(M)$ for all $j$. Thus, by Corollary 8, we obtain the result. □

**Remark 21.** In the case where $M = I$ is an ideal, the isomorphism mentioned in Remark 20 give us the isomorphism $I^j \cong S^j(I)$. Therefore, through the invariance of the Betti numbers by isomorphism, we obtain the following corollary.

**Corollary 16.** Let $I$ be a linear type ideal of $R$ with $\operatorname{pd}_R I = p < \infty$. If $I$ satisfy $(SW_j)$ condition, then

(a) for $p$ even,

$$\beta_t^R(I^j) = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(I)+a_0-1}{a_0}\binom{\beta_1^R(I)}{a_1}\binom{\beta_2^R(I)+a_2-1}{a_2}\binom{\beta_3^R(I)}{a_3}\cdots\binom{\beta_p^R(I)+a_p-1}{a_p};$$

(b) for $p$ odd,

$$\beta_t^R(I^j) = \sum_{\substack{(a_0,\dots,\, a_p)\\ \sum a_i = j\\ \sum i a_i = t}} \binom{\beta_0^R(I)+a_0-1}{a_0}\binom{\beta_1^R(I)}{a_1}\binom{\beta_2^R(I)+a_2-1}{a_2}\binom{\beta_3^R(I)}{a_3}\cdots\binom{\beta_p^R(I)}{a_p};$$

for all $t = 0,\ 1, \dots,\ \mathrm{pd}_R\, I^j$.

As already stated in the Remark 2 (AVRAMOV, 1981, p. 249), and later (FUKUMURO; KUME; NISHIDA, 2015, p. 106), characterize linear type modules with projective dimension 1 as a function of determinantal ideals. With this idea in mind, we show in the following proposition that such modules satisfy the $(SW_j)$ condition for a finite amount of $j$. In particular, we get a class of modules that satisfy the $(SW_j)$ condition. Without further ado, we have the following proposition.

**Proposition 13.** Let $M$ be an $R$-module of linear type with projective dimension 1. Then

$$\beta_t^R(\mathscr{R}_j(M)) = \binom{\beta_0^R(M)+j-t-1}{j-t}\binom{\beta_1^R(M)}{t}$$

for all $t = 0,\ 1, \dots,\ \mathrm{pd}_R\, \mathscr{R}_j(M)$.

*Demonstration.* Since $\mathrm{pd}_R\, M = 1$, we get a minimal free resolution

$$\mathbf{F}_\bullet : 0 \longrightarrow R^{\beta_1^R(M)} \xrightarrow{\ \phi\ } R^{\beta_0^R(M)} \longrightarrow M \longrightarrow 0,$$

such that $\mathrm{rank}\, M = \beta_0^R(M) - \beta_1^R(M)$. Furthermore, since $M$ is of linear type, $\mathscr{T}_R(S_R(M)) = 0$. Thus, by (AVRAMOV, 1981, Proposition 3), we obtain that $\mathrm{grade}(I_j(\phi)) \geq \beta_1^R(M) - j + 2$ for all $1 \leq j \leq \beta_1^R(M)$. Therefore, by Corollary 8 and Proposition 12 follow the result. □

In particular, if $I$ is a linear ideal with a projective dimension 1, the $j$th ordinary power of the ideal $I$ has its Betti numbers explained by the formula

$$\beta_t^R(I^j) = \binom{\beta_0^R(I)+j-t-1}{j-t}\binom{\beta_1^R(I)}{t},$$

for all $t = 0,\ 1, \dots,\ \mathrm{pd}_R\, I^j$.

Computing minimal free resolutions of powers of ideals is not an easy task. As well as getting your Betti numbers. In this sense, the above Proposition 13 summarizes this work by just calculating a minimal free resolution of $I$.

**Remark 22.** Note that the above proposition was obtained through the fact that linear type modules of projective dimension 1 satisfy the $(SW_j)$ condition. But the reciprocal does not always happen. Below is an example of an ideal that has a projective dimension 1, and that satisfies the condition $(SW_2)$ but is not linear type ideal.

**Example 5.** Let $R = k[[x,\ y,\ z]]$ be a ring of formal power series over a field $k$ and let the ideal of $R$ given by $I = (yz^2,\ x^2z,\ x^3y^2)$ as in the Example 1. We have already seen that $I$ satisfies the condition $(SW_2)$. But, by Macaulay2, $I$ is not linear type ideal.

The Example 5 above also justifies that there must be a larger class of modules that satisfy the $(SW_j)$ condition.

**Remark 23.** Unfortunately, the result obtained in the Proposition 13 cannot be generalized to any finite projective dimension $p$. This is due to the fact that there are linear type ideals that have a projective dimension greater than 1 and that do not satisfy the condition $(SW_j)$. The following example illustrates this well.

**Example 6.** Let $R = k[[x,\ y,\ z,\ w,\ s]]$ be a ring of formal power series over a field $k$ and linear type ideal $I = (zws,\ xyz,\ xyw,\ xys)$. By Macaulay2, we get a minimal free resolution of $I$ given by

$$\mathbf{F}_\bullet : \longrightarrow R^1 \xrightarrow{\phi} R^4 \longrightarrow R^4 \longrightarrow I \longrightarrow 0$$

where the map $\phi$ have matrix representation

$$[\phi] = \begin{pmatrix} s \\ -w \\ z \\ 0 \end{pmatrix}.$$

Thus, $\mathrm{pd}_R\, I = 2$. As $I_1(\phi) = (s,\ w,\ z)$, which implies that $\mathrm{grade}(I_1(\phi)) < 4$, we obtain that the ideal $I$ does not satisfy the $(SW_2)$ condition.

Due to the fact that linear type modules behave well under the $(SW_j)$ condition, when they have a projective dimension 1, the corollary below shows that the inequalities obtained in the Corollary 11.

**Corollary 17.** Let $M$ be an $R$-module of linear type with projective dimension 1. Then

$$\binom{\beta_1^R(M)}{t} \leq \beta_t^R(\mathscr{R}_j(M)) \leq \binom{\sum_{i=0}^{p} \beta_i^R(M) + j}{j},\ \text{for all } t = 0,\ 1, \ldots,\ \mathrm{pd}_R \mathscr{R}_j(M).$$

*Demonstration.* It follows immediately from the Proposition 13 and Corollary 12. □

In particular, by Corollary 17 above, if $I$ is a linear ideal with a projective dimension 1 we obtain that

$$\binom{\beta_1^R(I)}{t} \leq \beta_t^R(I^j) \leq \binom{\sum_{i=0}^{p} \beta_i^R(I)+j}{j}, \text{ for all } t=0,\ 1,\dots,\ \mathrm{pd}_R\, I^j.$$

To end this subsection, we leave the following question.

**Question C:** Is there a class larger than the class of modules of linear type so that the condition $(SW_j)$ is satisfied?

### 4.2.2 Buchsbaum-Eisenbud-Horrocks conjecture Versus symmetric powers

In order to obtain some applications let us remember the famous Buchsbaum-Eisenbud-Horrocks conjecture (BEH).

**Buchsbaum-Eisenbud-Horrocks conjecture (BEH)**: Let $(R,\mathfrak{m},k)$ be a $d$-dimensional Noetherian local ring, and let $M$ be a finitely generated nonzero $R$-module. If $M$ has finite length and finite projective dimension, then for all $i \geq 0$, the Betti numbers of $M$ over $R$ satisfy the inequality

$$\beta_i^R(M) \geq \binom{d}{i}.$$

This conjecture has a positive answer for local rings with dimension $\leq 4$ (see (AVRAMOV; BUCHWEITZ, 1993)), but for larger dimensions the problem still open. Some positive answers in certain cases are provided, for instance, in (EVANS; GRIFFITHS, 1985), (CHARALAMBOUS, 1991), (SANTONI, 1990) and (CHANG, 1997).

Despite being an immediate consequence of the results obtained in the previous sections, we would like to highlight here an inequality similar to the inequality in the (BEH) conjecture. When the module has projective dimension 1 and carries certain additional properties, the following proposition states that the Betti numbers of its $j$th symmetric power satisfy the inequality in the (BEH) conjecture.

**Proposition 14.** Let $R$ be a local ring of dimension $d$ and $M$ be a finitely generated $R$-module with $\mathrm{pd}_R M = 1$. If $M$ satisfies $(SW_j)$ condition and $\beta_1^R(M) \geq d$, then

$$\beta_t^R(S_j(M)) \geq \binom{d}{t}, \text{ for all } t=0,\ 1,\dots,\ \mathrm{pd}_R\, S_j(M).$$

*Demonstration.* Since $\mathrm{pd}_R\, M = 1$ and satisfying $(SW_j)$ condition, by Corollary 8, we obtain

$$\beta_t^R(S_j(M)) \geq \binom{\beta_1^R(M)}{t}, \text{ for all } t=0,\ 1,\dots,\ \mathrm{pd}_R\, S_j(M). \tag{4.3}$$

Now by hypothesis $\beta_1^R(M) \geq d$, thus $\binom{\beta_1^R(M)}{t} \geq \binom{d}{t}$. Therefore, by inequality 4.3, we obtain the result. □

The next corollary easily follows from Propositions 11 and 14.

**Corollary 18.** Let $R$ be a local ring of dimension $d$ and $I$ be an ideal of linear type with projective dimension 1 such that $\beta_1^R(I) \geq d$. Then, for all $j = 1, \ldots, \beta_1^R(I)$,

$$\beta_t^R(I^j) \geq \binom{d}{t}$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, I^j$.

Note that in the Proposition 14, in addition to the condition $(SW_j)$, we need to impose that $\beta_1^R(M) \geq d$ to obtain the sought inequality. This motivates us to look for which types of modules have the first Betti number greater than or equal to the ring dimension. An answer to this was found within the fiber product which we will mention below.

**Definition 20.** Let $(S,\ \mathfrak{s},\ k)$ and $(T,\ \mathfrak{t},\ k)$ be commutative local rings, and let $S \overset{\pi_S}{\twoheadrightarrow} k \overset{\pi_T}{\twoheadleftarrow} T$ be surjective homomorphisms of rings. The fiber product

$$S \times_k T = \{(s,\ t) \in S \times T \ \mid\ \pi_S(s) = \pi_T(t)\},$$

is a Noetherian local ring with maximal ideal $\mathfrak{s} \oplus \mathfrak{t}$, residual field $k$, and it is a subring of the usual direct product $R \times S$ see (ANANTHNARAYAN; AVRAMOV; MOORE , 2012, Lemma 1.2). The fiber product is deemed *non-trivial* provided neither $S$ nor $T$ is equal to $k$.

**Proposition 15.** Let $S \times_k T$ be a $d$-dimensional local ring. Let $M$ be a finitely generated $S$-module with $\mathrm{pd}_{S\times_k T}\, M = 1$. If $M$ satisfies $(SW_j)$ condition, then

$$\beta_t^{S\times_k T}(S_j(M)) \geq \binom{\beta_1^T(k)}{t}.$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_{S\times_k T}\, S_j(M)$.

*Demonstration.* Since $\mathrm{pd}_{S\times_k T}\, M = 1$ and satisfying $(SW_j)$, by Corollary 8, we get

$$\beta_t^{S\times_k T}(S_j(M)) \geq \binom{\beta_1^{S\times_k T}(M)}{t}, \text{ for all } t = 0,\ 1, \ldots,\ \mathrm{pd}_{S\times_k T}\, S_j(M). \tag{4.4}$$

Now, by (MOORE, 2009, Theorem 1.8), one obtain that $\beta_1^{R\times_k S}(M) = \beta_0^S(M)\beta_1^T(k) + \beta_1^S(M)$. Thus, $\beta_1^{R\times_k S}(M) \geq \beta_1^T(k)$. Therefore, by inequality 4.4, obtain the result. □

We showed in the Proposition 15 above that, although we are considering a minimal free resolution on the fiber product $S \times_k T$, we only need to take a minimal free resolution of $k$ on one of the pieces of the fiber product to lower the numbers $\beta_t^{S\times_k T}(S_j(M))$. In particular, we will see in the following corollary that when we consider one of the pieces of the fiber product $S \times_k T$ to be regular, we also obtain a similar inequality in the (BEH) conjecture.

**Corollary 19.** Let $S \times_k T$ be a $d$-dimensional local ring with $d := \dim(T) \geq \dim(S)$ and $T$ is a regular local ring. Let $M$ be a finitely generated $S$-module with $\mathrm{pd}_{S\times_k T}\, M = 1$. If $M$ satisfies $(SW_j)$ condition, then

$$\beta_t^{S\times_k T}(S_j(M)) \geq \binom{d}{t}.$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_{S\times_k T}\, S_j(M)$.

*Demonstration.* In fact, since $T$ is a regular local ring, then $\beta_1^T(k) \geq d$. Now, by Proposition 15 the result is immediate. □

**Remark 24.** By Corollary 19, suppose there is an ideal of linear type $I$ in $S \times_k T$, with $\mathrm{pd}_R\, I = 1$ and $d := \dim(T) \geq \dim(S)$ where $T$ is a regular local ring. Then,

$$\beta_t^{S\times_k T}(I^j) \geq \binom{d}{t}.$$

for all $t = 0,\ 1, \ldots,\ \mathrm{pd}_{S\times_k T}\, I^j$.

Finally, note that the inequality $\beta_1^R(M) \geq d$ occurs for modules that satisfy the (BEH) conjecture. Adding the $(SW_j)$ condition and $\mathrm{pd}_R\, M = 1$, by Proposition 14, we get that the inequality in the conjecture is also satisfied for the module $S_j(M)$. This is,

$$\beta_t^R(S_j(M)) \geq \binom{\beta_1^R(M)}{t}, \text{ for all } t = 0,\ 1, \ldots,\ \mathrm{pd}_R\, S_j(M).$$

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APPENDIX

# A

# COMPUTATIONAL RESULTS

Here are some independent results used throughout the text.

Below, we list two theorems that helped us in building examples of linear-type ideals.

**Theorem 11.** (LA BARBIERA; STAGLIANÒ, 2014, Theorem 3.1) Let $I_t \subset R = k[x_1,\ldots,x_n], n > 1$. $I_t$ is of linear type if and only if $t = n-1$.

**Theorem 12.** (LA BARBIERA; STAGLIANÒ, 2014, Theorem 3.2) Let $S = k[x_1,\ldots,x_n;y_1,\ldots,y_m]$, $n,m > 1$. The following conditions hold:

(a) $L = I_sJ_r$ is of linear type if and only if $s = n-1$ and $r = m$ or $s = 1$ and $r = m$ (resp. $s = n$ and $r = m-1$ or $r = 1$).

(b) $L = I_sJ_r + I_{s+1}J_{r-1}$ is of linear type if and only if $s = n-1$ and $r = m$.

(c) $L = J_r + I_sJ_t$ is of linear type if and only if $r = m, s = n, t = 1$ and $m = n+1$.

Although the theorem below is the same theorem produced by (AVRAMOV, 1981, p. 249), we chose to reference it here because it presents a more elementary proof and with a different perspective.

**Theorem 13.** (FUKUMURO; KUME; NISHIDA, 2015, Theorem 1.1) The following conditions are equivalent:

(a) $\operatorname{grade}(I_j(A)) \geq m-j+2$ for all $j = 1,2,\ldots,m$.

(b) $M$ has rank $n-m$ and $\mathscr{T}_R(S(M)) = 0$.

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