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A new approach to the differential geometry of frontals in the Euclidean space

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Uma nova abordagem da geometria diferencial de frontais no espaço euclidiano

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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This work is dedicated to my parents and grandparents,
who supported me since I was a child to delve into
science and pursue my dreams.
Especially to my grandmother who prays and motivates me every day from far away.

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"Mathematics reveals its secrets only to those
who approach it with pure love, for its own beauty."
(Archimedes)

RESUMO

MEDINA-TEJEDA, T. A. **Uma nova abordagem da geometria diferencial de frontais no espaço euclidiano**. 2021. 90 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Neste trabalho investigamos a geometria diferencial de superfícies singulares conhecidas como frontais. Provamos um resultado semelhante ao teorema fundamental das superfícies regulares na geometria diferencial clássica, que estende o teorema clássico aos frontais no espaço Euclidiano. Além disso, caracterizamos de forma simples essas superfícies singulares e suas formas fundamentais com propriedades locais na diferencial de sua parametrização e decomposições nas matrizes associadas às formas fundamentais. Em particular, introduzimos novos tipos de curvaturas que podem ser usadas para caracterizar as frentes de onda. Por outro lado, investigamos as condições necessárias e suficientes para estender e delimitar a curvatura Gaussiana, curvatura média e curvaturas principais perto de todos os tipos de singularidades das frentes. Além disso, estudamos a convergência para limites infinitos desses invariantes geométricos e mostramos como isso está estreitamente relacionado a uma propriedade de aproximação de frentes por superfícies paralelas.

Palavras-chave: Frontal, Frente, Curvatura Gaussiana , Curvatura Média, Curvaturas Principais.

ABSTRACT

MEDINA-TEJEDA, T. A. **A new approach to the differential geometry of frontals in the Euclidean space**. 2021. 90 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

In this work we investigate the differential geometry of singular surfaces known as frontals. We prove a similar result to the fundamental theorem of regular surfaces in classical differential geometry, which extends the classical theorem to the frontals in Euclidean 3-space. Also, we characterize in a simple way these singular surfaces and its fundamental forms with local properties in the differential of its parametrization and decompositions in the matrices associated to the fundamental forms. In particular we introduce new types of curvatures which can be used to characterize wave fronts. On the other hand, we investigate necessary and sufficient conditions for the extendibility and boundedness of Gaussian curvature, Mean curvature and principal curvatures near all types of singularities of fronts. Furthermore, we study the convergence to infinite limits of these geometrical invariants and we show how this is tightly related to a property of approximation of fronts by parallel surfaces.

Keywords: Frontal, Front, Gaussian curvature, Mean curvature, Principal curvatures.

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

RCE	Relative compatibility equations
SCE	Singular compatibility equations
tmb	tangent moving basis

LIST OF SYMBOLS

$\Sigma(\mathbf{x})$ — Singular set of $\mathbf{x}$

$D\mathbf{f}$ — Differential of $D\mathbf{f}$ as a map

$D\mathbf{f}_{x_i}$ — Partial derivative of $D\mathbf{f}$ with respect to x_i

$D\mathbf{f}(\mathbf{p})$ — Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}$

$\mathbb{I}_n$ — Identity matrix

$\mathcal{M}_{n\times m}(\mathbb{R})$ — Matrices $n \times m$ over $\mathbb{R}$

$\mathbf{A}_{(i)}$ — i^{th}-row of $\mathbf{A}$

$\mathbf{A}^{(j)}$ — j^{th}-column of $\mathbf{A}$

$\boldsymbol{\Omega}$ — Tangent moving basis

$adj(\mathbf{A})$ — The adjoint of a matrix $\mathbf{A}$

$()^T$ — The operation of transposing a matrix

$\mathbf{I}$ — Matrix of the first fundamental form

$\mathbf{II}$ — Matrix of the second fundamental form

$\boldsymbol{\Gamma}_i$ — Matrix of Christoffel symbols

$\boldsymbol{\alpha}$ — Weingarten matrix

$GL(n)$ — Invertible matrices $n \times n$

$[,]$ — Lie bracket operation for matrices

$[\mathbf{f}]$ — Map germ of $\mathbf{f}$

$\mathcal{E}_n$ — Germs of smooth functions at $0 \in \mathbb{R}^n$

$\mathfrak{I}_\Omega(U)$ — Principal ideal generated by λ_Ω in $C^\infty(U,\mathbb{R})$

$C^\infty(U,\mathbb{R})$ — Smooth real functions on U

$tr(\mathbf{A})$ — The trace of $\mathbf{A}$

K — Gaussian curvature

H — Mean curvature

κ_-, κ_+ — Principal curvatures

CONTENTS

CHAPTER 1

INTRODUCTION

In recent years, there is a great interest in the differential geometry of a special type of singular surface, namely, frontal. Many papers are dedicated to the study of frontals from singularity theory and geometry viewpoints (see (SAJI, 2010; ISHIKAWA, 2018; ISHIKAWA, 2020; MEDINA-TEJEDA, 2019) and the references therein), in particular wavefronts, a subclass of these (ARNOL'D; GUSEIN-ZADE; VARCHENKO, 2012; ARNOL'D, 1990; ISHIKAWA, 2018; MARTINS *et al.*, 2016; MEDINA-TEJEDA, 2019; MURATA; UMEHARA, 2009; KOSSOWSKI, 2004; SAJI; UMEHARA; YAMADA, 2009; TERAMOTO, 2016; TERAMOTO, 2019a; TERAMOTO, 2019b). The word "front" comes from physical fronts, bounding a domain in which a physical process propagates at a fixed moment in time. For instance, a wave propagating in the 3-Euclidean space with constant speed starting from each point of an ellipsoid in direction of the interior of this (the initial domain to be perturbed) creates a equidistant surface at time t bounding an interior part of the ellipsoid that it has not been perturbed at time t. In this case, the complete equidistant surface is called the wavefront, this changes as time passes leading to the formation of singularities along the whole equidistant surface in any time (ARNOL'D, 1990). The notion of "frontal" emerged as a natural generalization of wavefront in the case of hypersurfaces and a generalized definition with equivalences can be found in (ISHIKAWA, 2018).

Much of the existing work focuses on a generic set (in Whitney's topology) of these singular surfaces that have certain good types of singularities. The geometric properties of the generic surfaces are not necessarily satisfied for the entire class of singular surfaces. The methods to study generic singular surfaces rely on results from the theory of singularities and differential geometry, and in many cases depend upon special coordinate systems called adapted coordinate systems. In this work we introduce tools that allow us to use arbitrary coordinate systems and frames to investigate the geometry of singular surfaces in a neighborhood of a singular point. Our results do not depend on genericity assumptions and apply to any proper frontals or wavefronts. We are interested in exploring the geometrical behavior near the most degenerate types of singularities.

The behavior of Gaussian curvature, mean curvature and principal curvatures near non-degenerate singularities on wavefronts have been widely studied in (SAJI; UMEHARA; YAMADA, 2009; MARTINS *et al.*, 2016; TERAMOTO, 2016; TERAMOTO, 2019b). However, in the degenerate case this is unknown, as well as the convergence to infinite limits of these invariants has been little explored. For this reason it is natural to wonder which properties of wavefronts determine one behavior or another on general types of singularities. Also, there is a lack of literature about the geometry of singularities of rank 0 (or corank 2) on wavefronts and our approach here allow us to study them.

In chapter 2 we present the notation, classical terminology and basic results that we use most and are present in books of differential geometry and singularities. Also we introduce new additional terminology and symbols which are very related to the classical ones of the differential geometry and can be defined on singularities without problems. In chapter 3 we see how the fundamental forms, Christoffel symbols and classical curvatures in frontals are related with the new symbols and how these determine properties to characterize wavefronts. Additionally, we use these properties to obtain some formulas of representation for wavefronts. In chapter 4 we present a fundamental theorem for frontals similar to the classical one for regular surfaces. At the last chapter we give necessary and sufficient condition to the boundedness, convergence to infinite limits and extendibility of the classical invariant near all types of singularities of wavefronts.

CHAPTER

2

PRELIMINARIES

2.1 Fixing notation and some definitions

A smooth map $\mathbf{x}: U \to \mathbb{R}^3$ defined in an open set $U \subset \mathbb{R}^2$ is called a *frontal* if, for all $\mathbf{p} \in U$ there exists a unit normal vector field $\mathbf{n}: V_p \to \mathbb{R}^3$ along $\mathbf{x}$, where V_p is an open set of U, $\mathbf{p} \in V_p$. This means, $|\mathbf{n}| = 1$ and it is orthogonal to the partial derivatives of $\mathbf{x}$ for each point $(u,v) \in V_p$. If also the singular set $\Sigma(\mathbf{x}) = \{\mathbf{p} \in U : \mathbf{x} \text{ is not immersive at } \mathbf{p}\}$ has empty interior we call $\mathbf{x}$ *proper frontal*, in another case we say that $\mathbf{x}$ is a *non-proper frontal*. Since $\Sigma(\mathbf{x})$ is closed, this is equivalent to have the complement $\Sigma(\mathbf{x})^c = U - \Sigma(\mathbf{x})$ being dense and open in U. We call a point $\mathbf{p} \in \Sigma(\mathbf{x})$ a *singularity* or *singular point* and a point in the complement $\Sigma(\mathbf{x})^c$ a *regular point*. A frontal $\mathbf{x}$ is a *wave front* or simply *front* if the pair $(\mathbf{x}, \mathbf{n})$ is an immersion for all $\mathbf{p} \in U$. There are many examples of frontals which are not wave fronts, cuspidal S_k singularities for instance (SAJI, 2010). The existence of a smooth normal vector field on these singular surfaces determines planes (the orthogonal spaces) at singularities that can be understood as limiting planes of the tangent planes on regular points around them (see Figure 1).

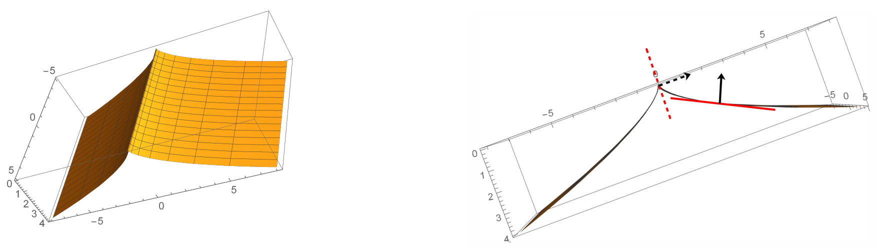

Figure 1 – The cuspidal edge $(\mathbf{x}(u,v) = (u, v^2, v^3), \mathbf{n} = (0, -3v, 2)(4+9v^2)^{-\frac{1}{2}})$ and the limiting tangent planes.

The cuspidal edge and the swallowtail (see Figure 1 and 2) are two types of singular points that represent the generic singularities in the space of wave fronts with the Whitney

C^∞-topology (ARNOL'D; GUSEIN-ZADE; VARCHENKO, 2012). For this reason, all the re-parametrizations and diffeomorphic singular surfaces to these are the most studied and there exist criterias to recognize them (KOKUBU *et al.*, 2005; ISHIKAWA, 2020). However, these singularities are not generic in the space of all frontals (in fact proper frontals are not generic either)(ISHIKAWA, 2018). There are some non-proper frontals which are not "surfaces", $\mathbf{x}(u,v) = (uv, 0, 0)$ for instance and others whose entire image is a surface but locally at some singular points the image of a neighborhood at these is a constant (see example 2.5 (ISHIKAWA, 2018)). Here we treat frontals in general, but our main result aim to proper frontals.

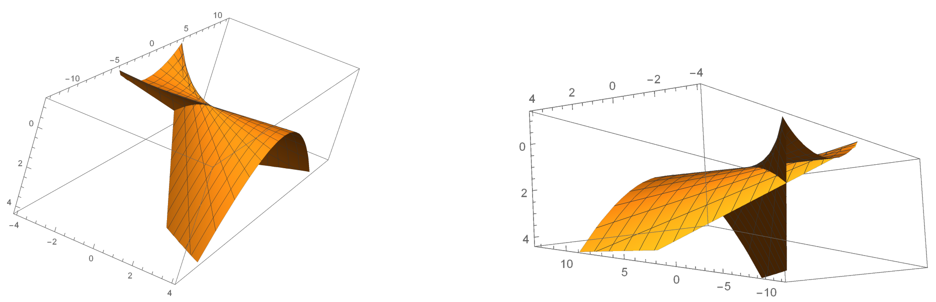

Figure 2 – The swallowtail $(\mathbf{x}(u,v) = (3u^4 + u^2 v, 4u^3 + 2uv, v), \mathbf{n} = (1, -u, u^2)(1 + u^2 + u^4)^{-\frac{1}{2}})$, an example of front.

From now on, we denote U and V in this work open sets in $\mathbb{R}^2$. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, and as we are interested in exploring local properties of frontals, restricting the domain if necessary, we can suppose that we have a global normal vector field $\mathbf{n}: U \to \mathbb{R}^3$. There are two possible choices of normal vector fields along $\mathbf{x}$ ($\mathbf{n}$ and $-\mathbf{n}$). We are always assuming that we have chosen one of them and we hold fixed this for all the concepts defined using a normal vector field along $\mathbf{x}$. Let $\mathbf{f}: U \to \mathbb{R}^n$ be a smooth map, we denote by $D\mathbf{f} := (\frac{\partial \mathbf{f}_i}{\partial x_j})$, the differential of $\mathbf{f}$ and we consider it as a smooth map $D\mathbf{f}: U \to \mathcal{M}_{n\times 2}(\mathbb{R})$. We write $D\mathbf{f}_{x_1}$, $D\mathbf{f}_{x_2}$ the partial derivatives of $D\mathbf{f}$ and $D\mathbf{f}(\mathbf{p}) := (\frac{\partial \mathbf{f}_i}{\partial x_j}(\mathbf{p}))$ for $\mathbf{p} \in U$. We denote $\mathbb{I}_n$ the identity matrix $n \times n$. Also, a vector in $\mathbb{R}^n$ is identified as a column vector in $\mathcal{M}_{n\times 1}(\mathbb{R})$ and if $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$, $\mathbf{A}_{(i)}$ is the i^{th}-row and $\mathbf{A}^{(j)}$ is the j^{th}-column of $\mathbf{A}$.

Definition 2.1.1. We call *moving basis* a smooth map $\mathbf{\Omega}: U \to \mathcal{M}_{3\times 2}(\mathbb{R})$ in which the columns $\mathbf{w}_1, \mathbf{w}_2 : U \to \mathbb{R}^3$ of the matrix $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ are linearly independent smooth vector fields.

Definition 2.1.2. We call a *tangent moving basis* (tmb) of $\mathbf{x}$ a moving basis $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ such that $\mathbf{x}_u, \mathbf{x}_v \in \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, where $\langle , \rangle$ denotes the linear span vector space.

Example 2.1.1. For the cuspidal edge $(\mathbf{x}(u,v) = (u, v^2, v^3))$, we have $\mathbf{x}_u = (1,0,0)$ and

$\mathbf{x}_v = (0, 2v, 3v^2) = (0, 2, 3v)v$, then denoting $\mathbf{w}_1 := (1, 0, 0)$ and $\mathbf{w}_2 := (0, 2, 3v)$,

$$\boldsymbol{\Omega} := \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3v \end{pmatrix}$$

is a tmb of $\mathbf{x}$. Observe that multiplying $\boldsymbol{\Omega}$ by a matrix-valued smooth map $\mathbf{B} : \mathbb{R}^2 \to \mathcal{M}_{2\times 2}(\mathbb{R})$ with $det(\mathbf{B}) \neq 0$, we have that $\boldsymbol{\Omega}\mathbf{B}$ is another tmb of $\mathbf{x}$, because the columns are still linearly independent and generate the same vector space of $\mathbf{w}_1, \mathbf{w}_2$.

Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal with a global normal vector field $\mathbf{n} : U \to \mathbb{R}^3$. Denoting the inner product by $(\cdot)$, $adj(\mathbf{A})$ the adjoint of a matrix $\mathbf{A}$ (i.e $\mathbf{A}adj(\mathbf{A}) = adj(\mathbf{A})\mathbf{A} = det(\mathbf{A})\mathbb{I}_2$) and $()^T$ the operation of transposing a matrix, we set the matrices:

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_u \cdot \mathbf{x}_v & \mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix} \tag{2.1a}$$

$$\mathbf{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} := \begin{pmatrix} \mathbf{n} \cdot \mathbf{x}_{uu} & \mathbf{n} \cdot \mathbf{x}_{uv} \\ \mathbf{n} \cdot \mathbf{x}_{uv} & \mathbf{n} \cdot \mathbf{x}_{vv} \end{pmatrix} \tag{2.1b}$$

$$\boldsymbol{\Gamma}_1 = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} \\ \Gamma^1_{21} & \Gamma^2_{21} \end{pmatrix} := \begin{pmatrix} \frac{1}{2}E_u & (F_u - \frac{1}{2}E_v) \\ \frac{1}{2}E_v & \frac{1}{2}G_u \end{pmatrix} \mathbf{I}^{-1} \tag{2.1c}$$

$$\boldsymbol{\Gamma}_2 = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} \end{pmatrix} := \begin{pmatrix} \frac{1}{2}E_v & \frac{1}{2}G_u \\ (F_v - \frac{1}{2}G_u) & \frac{1}{2}G_v \end{pmatrix} \mathbf{I}^{-1} \tag{2.1d}$$

$$\boldsymbol{\alpha} := -\mathbf{II}^T \mathbf{I}^{-1} \tag{2.1e}$$

The matrices $\mathbf{I}$ and $\mathbf{II}$ in a non-singular point $\mathbf{p} \in U$ coincide with the matrix representation of the *first fundamental form* and of the *second fundamental form* respectively. $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$ and $\boldsymbol{\alpha}$ are defined in $\Sigma(\mathbf{x})^c$, they are the Christoffel symbols and the Weingarten matrix. Also observe that, we can compute these matrices in this way:

$$\mathbf{I} = D\mathbf{x}^T D\mathbf{x} \tag{2.2a}$$

$$\mathbf{II} = -D\mathbf{x}^T D\mathbf{n} \tag{2.2b}$$

$$\boldsymbol{\Gamma}_1 = (D\mathbf{x}_u^T D\mathbf{x})\mathbf{I}^{-1} \tag{2.2c}$$

$$\boldsymbol{\Gamma}_2 = (D\mathbf{x}_v^T D\mathbf{x})\mathbf{I}^{-1} \tag{2.2d}$$

2.2 The new symbols

Here we present the definitions of the new symbols that can be defined even on singularities and allow us to obtain information about the classical ones near singularities of frontals. Let $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ be a moving basis, we denote by $\mathbf{n} := \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}$ and we set the matrices:

$$\mathbf{I}_\Omega = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} := \mathbf{\Omega}^T\mathbf{\Omega} \tag{2.3a}$$

$$\mathbf{II}_\Omega = \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix} := -\mathbf{\Omega}^T D\mathbf{n} \tag{2.3b}$$

$$\mathcal{T}_1 = \begin{pmatrix} \mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^2 \end{pmatrix} := (\mathbf{\Omega}_u^T\mathbf{\Omega})\mathbf{I}_\Omega^{-1} \tag{2.3c}$$

$$\mathcal{T}_2 = \begin{pmatrix} \mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^2 \end{pmatrix} := (\mathbf{\Omega}_v^T\mathbf{\Omega})\mathbf{I}_\Omega^{-1} \tag{2.3d}$$

$$\boldsymbol{\mu}_\Omega := -\mathbf{II}_\Omega^T\mathbf{I}_\Omega^{-1} \tag{2.3e}$$

Since $\mathbf{n}\cdot\mathbf{w}_1 = 0$ and $\mathbf{n}\cdot\mathbf{w}_2 = 0$, then we have $-\mathbf{n}_u\cdot\mathbf{w}_1 = \mathbf{n}\cdot\mathbf{w}_{1u}$, $-\mathbf{n}_v\cdot\mathbf{w}_1 = \mathbf{n}\cdot\mathbf{w}_{1v}$, $-\mathbf{n}_u\cdot\mathbf{w}_2 = \mathbf{n}\cdot\mathbf{w}_{2u}$ and $-\mathbf{n}_v\cdot\mathbf{w}_2 = \mathbf{n}\cdot\mathbf{w}_{2v}$. Therefore,

$$\mathbf{II}_\Omega = \begin{pmatrix} \mathbf{n}\cdot\mathbf{w}_{1u} & \mathbf{n}\cdot\mathbf{w}_{1v} \\ \mathbf{n}\cdot\mathbf{w}_{2u} & \mathbf{n}\cdot\mathbf{w}_{2v} \end{pmatrix} \tag{2.4}$$

Also, as $\mathbf{n}_u, \mathbf{n}_v \in \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, there exist real functions $(\bar{\mu}_{ij})$ $i,j \in \{1,2\}$ defined on U, such that:

$$\mathbf{n}_u = \bar{\mu}_{11}\mathbf{w}_1 + \bar{\mu}_{12}\mathbf{w}_2$$
$$\mathbf{n}_v = \bar{\mu}_{21}\mathbf{w}_1 + \bar{\mu}_{22}\mathbf{w}_2$$

Then, $D\mathbf{n} = \mathbf{\Omega}\bar{\boldsymbol{\mu}}^T$, where $\bar{\boldsymbol{\mu}} = (\bar{\mu}_{ij})$. Thus, using (2.3b) $\mathbf{II}_\Omega = -\mathbf{\Omega}^T D\mathbf{n} = -\mathbf{\Omega}^T\mathbf{\Omega}\bar{\boldsymbol{\mu}}^T = -\mathbf{I}_\Omega\bar{\boldsymbol{\mu}}^T$, therefore $\bar{\boldsymbol{\mu}} = -\mathbf{II}_\Omega^T\mathbf{I}_\Omega^{-1} = \boldsymbol{\mu}_\Omega$ and we have:

$$D\mathbf{n} = \mathbf{\Omega}\boldsymbol{\mu}_\Omega^T \tag{2.6}$$

By last, $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, the positive-definite quadratic form $(\cdot)$ restricted to $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$ has $\mathbf{I}_\Omega = \mathbf{\Omega}^T\mathbf{\Omega}$ as its matrix representation in the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ and therefore $det(\mathbf{I}_\Omega) > 0$. Notice that, given a frontal $\mathbf{x}$, on a small neighborhood V of a regular point, $\mathbf{\Omega} = D\mathbf{x}$ is a tmb of $\mathbf{x}|_V$, then the matrices $\mathbf{I}_\Omega, \mathbf{II}_\Omega, \mathcal{T}_1, \mathcal{T}_2$ and $\boldsymbol{\mu}_\Omega$ coincide with $\mathbf{I}, \mathbf{II}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2$ and $\boldsymbol{\alpha}$. At singular points $D\mathbf{x}$ is not a tmb of $\mathbf{x}$, but we will see in chapter 3 that for frontals there exists tmb locally at singularities.

2.3 The Frobenius Theorem

The following is a particular version of Frobenius theorem that can be found in ((STOKER, 1969) appendix B) or (TERNG, 2005).

Theorem 2.3.1 (Frobenius)**.** Let $\mathbf{\Theta}, \mathbf{\Xi} : U \times V \to \mathbb{R}^n$ be smooth vector fields, where $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^n$ are open sets. Let $(u_0, v_0) \in U$ be a fixed point. Then for each point $\mathbf{p} \in V$ the system of partial differential equations:

$$\frac{\partial \mathbf{x}}{\partial u} = \mathbf{\Theta}(u, v, \mathbf{x}(u, v)),$$
$$\frac{\partial \mathbf{x}}{\partial v} = \mathbf{\Xi}(u, v, \mathbf{x}(u, v)),$$
$$\mathbf{x}(u_0, v_0) = \mathbf{p},$$

has a unique smooth solution $\mathbf{x} : U_0 \to \mathbb{R}^n$ defined on a neighborhood U_0 of $(u_0, v_0) \in U_0$ if and only if, it satisfies the compatibility condition:

$$\frac{\partial \mathbf{\Theta}}{\partial v} + \frac{\partial \mathbf{\Theta}}{\partial \mathbf{x}}\mathbf{\Xi} = \frac{\partial \mathbf{\Xi}}{\partial u} + \frac{\partial \mathbf{\Xi}}{\partial \mathbf{x}}\mathbf{\Theta} \tag{2.8}$$

Corollary 2.3.1. Let $\mathbf{S}, \mathbf{T} : U \to \mathcal{M}_{n\times n}(\mathbb{R})$ be smooth vector fields, where U is an open set in $\mathbb{R}^2$. Let $(u_0, v_0) \in U$ be a fixed point. Then for each point $\mathbf{A} \in GL(n)$ the system of partial differential equations:

$$\frac{\partial \mathbf{G}}{\partial u} = \mathbf{SG},$$
$$\frac{\partial \mathbf{G}}{\partial v} = \mathbf{TG},$$
$$\mathbf{G}(u_0, v_0) = \mathbf{A},$$

has a unique smooth solution $\mathbf{G} : U_0 \to GL(n)$ defined on a neighbourhood U_0 of $(u_0, v_0) \in U_0$ if and only if, it satisfies the compatibility condition:

$$\frac{\partial \mathbf{S}}{\partial v} - \frac{\partial \mathbf{T}}{\partial u} + [\mathbf{S}, \mathbf{T}] = 0, \tag{2.10}$$

where $[\mathbf{S}, \mathbf{T}] = \mathbf{ST} - \mathbf{TS}$ is the Lie bracket.

Proof. Identifying $\mathcal{M}_{n\times n}(\mathbb{R}) \equiv \mathbb{R}^{n^2}$ and defining $\mathbf{\Theta}(u, v, \mathbf{X}) := \mathbf{SX}$ and $\mathbf{\Xi}(u, v, \mathbf{X}) := \mathbf{TX}$ for $\mathbf{X} \in \mathcal{M}_{n\times n}(\mathbb{R})$, the compatibility condition (2.8) is equivalent to (2.10) and by theorem 2.3.1 follows the result. □

2.4 The Hadamard Lemma

In the following we establish one useful fact sometimes called the Hadamard Lemma (GIBSON, 1979).

Lemma 2.4.1 (Hadamard)**.** Let U be a convex neighbourhood of 0 in $\mathbb{R}^n$ and let f be a smooth function defined on $U \times \mathbb{R}^q$ which vanishes on $0 \times \mathbb{R}^q$. Then, there exist smooth functions $g_1, g_2, \cdots, g_n$ on $U \times \mathbb{R}^q$ with

$$f = x_1 g_1 + \cdots + x_n g_n$$

where $x_1, \cdots, x_n$ are the standard co-ordinate functions on $\mathbb{R}^n$.

Proof. Denoting $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_q)$

$$f(\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y}) - f(0) = \int_0^1 \frac{d}{dt}\{f(t\mathbf{x},\mathbf{y})\}dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t\mathbf{x},\mathbf{y})x_i dt$$

$$= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(t\mathbf{x},\mathbf{y})dt = \sum_{i=1}^{n} x_i g_i(\mathbf{x},\mathbf{y}),$$

where $g_i(\mathbf{x},\mathbf{y}) = \int_0^1 \frac{\partial f}{\partial x_i}(t\mathbf{x},\mathbf{y})dt$ □

2.5 Map-Germs

Let X and Y be two subsets of $\mathbb{R}^n$ containing a point $\mathbf{p} \in \mathbb{R}^n$. We say that X is equivalent to Y if there exists an open set $U \subset \mathbb{R}^n$ containing $\mathbf{p}$ such that $X \cap U = Y \cap U$. This defines an equivalence relation among subsets of $\mathbb{R}^n$ containing the point $\mathbf{p}$. The equivalence class of a subset X is called the *germ* of X at $\mathbf{p}$ and is denoted by $(X, \mathbf{p})$.

Let U and V be two open subsets of $\mathbb{R}^n$ containing a point $\mathbf{p} \in \mathbb{R}^n$, and let $\mathbf{f}$: $U \to \mathbb{R}^m$ and $\mathbf{g} : V \to \mathbb{R}^m$ be two smooth maps. We say that $\mathbf{f} \sim \mathbf{g}$ if there exists an open set $W \subset U \cap V$ containing $\mathbf{p}$ such that $\mathbf{f} = \mathbf{g}$ on W, that is $\mathbf{f}|_W = \mathbf{g}|_W$.

The relation $\sim$ is an equivalence relation and a germ at $\mathbf{p}$ of a smooth map is by definition an equivalent class under this equivalence relation. A map-germ at $\mathbf{p}$ is denoted by

$$[\mathbf{f}] : (\mathbb{R}^n, \mathbf{p}) \to \mathbb{R}^m$$

where $\mathbf{f} : U \to \mathbb{R}^m$ is a representative of germ $[\mathbf{f}]$ in a neighbourhood U of $\mathbf{p}$. However, at most of the cases we omit the brackets [ ] at germs, when there are no risk of confusion.

Sometimes we require that all the elements of the equivalence classes have the same value at $\mathbf{p}$, say $\mathbf{q}$. Then we write

$$\mathbf{f} : (\mathbb{R}^n, \mathbf{p}) \to (\mathbb{R}^m, \mathbf{q}) \ .$$

Let $\mathcal{E}_n$ denote the set of germs, at the origin 0 in $\mathbb{R}^n$, of smooth functions $(\mathbb{R}^n, 0) \to \mathbb{R}$, $\mathcal{E}_n = \{f : (\mathbb{R}^n, 0) \to \mathbb{R} | f$ is the germ of a smooth function$\}$.

With the addition and multiplication operations, $\mathcal{E}_n$ becomes a commutative $\mathbb{R}$-algebra with a unit. It has a maximal ideal $\mathcal{M}_n$ which is the subset of germs of functions that vanish at the origin. We have

$$\mathcal{M}_n = \{[f] \in \mathcal{E}_n | f(0) = 0\}.$$

Since $\mathcal{M}_n$ is the unique maximal ideal of $\mathcal{E}_n, \mathcal{E}_n$ is a local algebra. If $x_1, \cdots, x_n$ are the standard co-ordinate functions on $\mathbb{R}^n$, then by Hadamard lemma $\mathcal{M}_n$ is generated by the germs of functions x_i, $i = 1, \cdots, n$.

The set of all smooth map-germs $\mathbf{f} : (\mathbb{R}^n, 0) \to \mathbb{R}^m$ is denoted by $\mathcal{E}(n,m)$. It is the direct product of m-copies of $\mathcal{E}_n$, that is,

$$\mathcal{E}(n,m) = \underbrace{\mathcal{E}_n \times \cdots \times \mathcal{E}_n}_{m \text{ times}}$$

2.6 Left-Right Equivalence

Definition 2.6.1. Let $\mathbf{f}_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0), i = 1,\ 2$ be germs of smooth maps between Euclidean spaces. They are

1. *right-equivalent*, if there exists a germ of diffeomorphism $\mathbf{h} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\mathbf{f}_2 = \mathbf{f}_1 \circ \mathbf{h}^{-1}$;

2. *left-equivalent*, if there exists a germ of diffeomorphism $\mathbf{k} : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $\mathbf{f}_2 = \mathbf{k} \circ \mathbf{f}_1$;

3. *left-right-equivalent*, if there exist germs of diffeomorphism $\mathbf{h} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\mathbf{k} : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $\mathbf{f}_2 = \mathbf{k} \circ \mathbf{f}_1 \circ \mathbf{h}^{-1}$.

The advantage when the source and target are fixed is that the equivalences can be seen as group actions. Let $\mathscr{A} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^m, 0)$ be the group of pairs of diffeomorphisms. We have an action of $\mathscr{A}$ on the set of germs $\mathbf{f}$: $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, given by

$$(\mathbf{h}, \mathbf{k}) \cdot \mathbf{f} = \mathbf{k} \circ \mathbf{f} \circ \mathbf{h}^{-1}.$$

Analogously, we can consider the groups $\mathscr{R} = Diff(\mathbb{R}^n, 0)$ and $\mathscr{L} = Diff(\mathbb{R}^p, 0)$ and the corresponding actions. If $\mathscr{G} = \mathscr{R}, \mathscr{L}$ or $\mathscr{A}$, we say that $\mathbf{f}_1, \mathbf{f}_2$ are *$\mathscr{G}$-equivalent* if they are in the same $\mathscr{G}$-orbit. In this situation, we will use the terms $\mathscr{R}, \mathscr{L}$ or $\mathscr{A}$-equivalences instead of right, left or right-left equivalences, respectively. The interested reader in the study of singularities of map-germs can consult (GIBSON, 1979; MOND; NUÑO-BALLESTEROS, 2020; IZUMIYA *et al.*, 2015).

CHAPTER

3

DECOMPOSITIONS OF THE FUNDAMENTAL FORMS

In this chapter we characterize a frontal $\mathbf{x}$ in terms of the differential of $\mathbf{x}$, its fundamental forms through a decomposition of matrices and wave fronts in terms of two new curvatures which are related with the Gaussian and mean curvature. As in our corollary 3.3.1, T. Fukunaga and M. Takahashi in (FUKUNAGA; TAKAHASHI, 2019) also characterized wave fronts in terms of curvatures. The curvatures introduced in (FUKUNAGA; TAKAHASHI, 2019), are a particular case of the relative curvatures presented here.

3.1 Characterizing a frontal and its fundamental forms

Proposition 3.1.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a smooth map with $U \subset \mathbb{R}^2$ an open set.Then, the following statements are equivalent:

(i) The map $\mathbf{x}$ is a frontal.

(ii) For all $\mathbf{p} \in U$ there is a tangent moving basis $\mathbf{\Omega}: V_p \to \mathcal{M}_{3\times 2}(\mathbb{R})$ of $\mathbf{x}$ with $V_p \subset U$ a neighborhood of $\mathbf{p}$.

(iii) For all $\mathbf{p} \in U$ there are smooth maps $\mathbf{\Omega}: V_p \to \mathcal{M}_{3\times 2}(\mathbb{R})$ and $\mathbf{\Lambda}: V_p \to \mathcal{M}_{2\times 2}(\mathbb{R})$ with $rank(\mathbf{\Omega}) = 2$, $V_p \subset U$ a neighbourhood of $\mathbf{p}$, such that $D\mathbf{x}(\mathbf{q}) = \mathbf{\Omega}\mathbf{\Lambda}^T$ for all $\mathbf{q} \in V_p$.

Proof.

- $(i \Leftrightarrow ii)$ If $\mathbf{x}$ is a frontal, then for all $\mathbf{p} \in U$ there exists a unitary vector field $\mathbf{n}: V_p \to \mathbb{R}^3$ with $\mathbf{x}_u \cdot \mathbf{n} = 0$, $\mathbf{x}_v \cdot \mathbf{n} = 0$, $\mathbf{n} = (n_1, n_2, n_3)$, V_p a neighborhood of $\mathbf{p}$ which we can reduce in order to get $n_i \neq 0$ on V_p for some $i \in \{1,2,3\}$. Without loss of generality let us suppose that $n_1 \neq 0$ and define $\mathbf{\Omega} := \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ with $\mathbf{w}_1 = (n_2, -n_1, 0)$ and $\mathbf{w}_2 = (n_3, 0, -n_1)$. Since $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, orthogonal to $\mathbf{n}$ and $dim(\mathbf{n}^\perp) = 2$ ($\mathbf{n}^\perp$ orthogonal space to $\mathbf{n}$), we have that $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \mathbf{n}^\perp$. Therefore, $\mathbf{\Omega} : V_p \to \mathcal{M}_{3\times 2}(\mathbb{R})$ is a tangent moving basis of $\mathbf{x}$. The converse, just define $\mathbf{n} := \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}$ taking $\mathbf{w}_1$ and $\mathbf{w}_2$ the columns from a tangent moving basis $\mathbf{\Omega} : V_p \to \mathcal{M}_{3\times 2}(\mathbb{R})$. Then, $\mathbf{n}$ is orthogonal to $\mathbf{x}_u$ and $\mathbf{x}_v$ which belong to $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$.

- ($ii \Leftrightarrow iii$) If we suppose (ii), for all $\mathbf{p} \in U$ there is a tangent moving basis $\mathbf{\Omega} : V_p \to \mathcal{M}_{3\times 2}(\mathbb{R})$ of $\mathbf{x}$ with $V_p \subset U$ a neighborhood of $\mathbf{p}$. Thus, there are coefficients λ_{ij} such that $\mathbf{x}_u = \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2$ and $\mathbf{x}_v = \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2$. Therefore, $D\mathbf{x}(\mathbf{q}) = \mathbf{\Omega}\mathbf{\Lambda}^T$ for all $\mathbf{q} \in V_p$ where $\mathbf{\Lambda} = (\lambda_{ij})$. Multiplying the equality by $\mathbf{\Omega}^T$ and as $\mathbf{I}_\Omega$ is invertible, we have that $\mathbf{I}_\Omega^{-1}\mathbf{\Omega}^T D\mathbf{x}(\mathbf{q}) = \mathbf{\Lambda}^T$. Then, $\mathbf{\Lambda} : V_p \to \mathcal{M}_{2\times 2}(\mathbb{R})$ is smooth. Reciprocally, if we have $D\mathbf{x}(\mathbf{q}) = \mathbf{\Omega}\mathbf{\Lambda}^T$ for all $\mathbf{q} \in V_p$, then $\mathbf{x}_u = \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2$ and $\mathbf{x}_v = \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2$. Hence $\mathbf{x}_u, \mathbf{x}_v \in \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$ and as $rank(\mathbf{\Omega}) = 2$, $\mathbf{\Omega}$ is a tangent moving basis of $\mathbf{x}$.

□

Remark 3.1.1. In the proof of proposition 3.1.1, observe that $\mathbf{\Lambda} = D\mathbf{x}^T\mathbf{\Omega}(\mathbf{I}_\Omega^T)^{-1}$, then $\mathbf{\Lambda}$ is determined by a local tangent moving basis of $\mathbf{x}$. Also having a decomposition $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$ with $rank(\mathbf{\Omega}) = 2$ implies that $\mathbf{\Omega}$ is a tangent moving basis of $\mathbf{x}$.

Example 3.1.1. For the cuspidal edge ($\mathbf{x}(u,v) = (u, v^2, v^3)$) observe that the Jacobian matrix decomposes as in proposition 3.1.1:

$$D\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \\ 0 & 3v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

where,

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3v \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix},$$

being $\mathbf{\Omega}$ a tmb of $\mathbf{x}$ as we had seen in example 2.1.1.

Notice that, the proof of proposition 3.1.1 did not use that the set $\Sigma(\mathbf{x}) = \{\mathbf{p} \in U : \mathbf{x}$ is not immersive at $\mathbf{p}\}$ has empty interior, therefore this is valid for even non-proper frontals. However we need this condition at the moment that we consider to relate different tangent moving bases. If we have a proper frontal $\mathbf{x} : U \to \mathbb{R}^3$ with a tmb $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$, inducing a normal vector field $\mathbf{n}$ and $\bar{\mathbf{\Omega}} = \begin{pmatrix} \bar{\mathbf{w}}_1 & \bar{\mathbf{w}}_2 \end{pmatrix}$ is another tmb of $\mathbf{x}$, we have that for $\mathbf{p} \in \Sigma(\mathbf{x})^c$, $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = \langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2 \rangle$, then we have $\mathbf{n}_i \cdot \bar{\mathbf{w}}_i = 0$, for $i = 1, 2$ on $\Sigma(\mathbf{x})^c$. By continuity and density of the regular points, it is also satisfied on U, thus $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2 \rangle$ on U ensured that every tmb generate the same vector space at each point. From this, we have that for all $\mathbf{p}\in U$, $\mathbf{\Omega}(\mathbf{p})=\bar{\mathbf{\Omega}}(\mathbf{p})\mathbf{B}(\mathbf{p})$, where $\mathbf{B}(\mathbf{p})$ is a non-singular 2×2 matrix, which seen as a map is smooth because $\mathbf{B}=\mathbf{I}_{\bar{\mathbf{\Omega}}}^{-1}\bar{\mathbf{\Omega}}^T\mathbf{\Omega}$. For non-proper frontals, the latter is not always valid, for example the non-proper frontal $\mathbf{x}=(uv,0,0)$ has the following different tangent moving bases which not generate the same vector space at every point:

$$\mathbf{\Omega}=\begin{pmatrix}1&0\\0&1\\0&u\end{pmatrix},\ \bar{\mathbf{\Omega}}=\begin{pmatrix}1&0\\0&v\\0&1\end{pmatrix}.$$

From now on, as we want to describe local properties and tangent moving bases exist locally, we can suppose that we have a global tangent moving basis for a frontal restringing the domain if necessary.

Definition 3.1.1. Let $\mathbf{x}$ be a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, we denote $\mathbf{\Lambda}_\Omega=(\lambda_{ij}):=D\mathbf{x}^T\mathbf{\Omega}(\mathbf{I}_\Omega)^{-1}$, $\lambda_\Omega:=det(\mathbf{\Lambda}_\Omega)$ and $\mathfrak{I}_\Omega(U)$ as the principal ideal generated by λ_Ω in the ring $C^\infty(U,\mathbb{R})$ (smooth real functions on U).

Thus, we have globally $D\mathbf{x}=\mathbf{\Omega}\mathbf{\Lambda}_\Omega^T$, $\Sigma(\mathbf{x})=\lambda_\Omega^{-1}(0)$ and $rank(D\mathbf{x})=rank(\mathbf{\Lambda}_\Omega)$. With a given tangent moving basis $\mathbf{\Omega}=\begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$, we always choose as unit normal vector field along $\mathbf{x}$, the induced by $\mathbf{\Omega}$ (i.e $\mathbf{n}=\frac{\mathbf{w}_1\times\mathbf{w}_2}{\|\mathbf{w}_1\times\mathbf{w}_2\|}$). Also, we are going to write simply $\mathfrak{I}_\Omega$, $\mathbf{\Lambda}=(\lambda_{ij})$ and $\boldsymbol{\mu}=(\mu_{ij})$ instead of $\mathfrak{I}_\Omega(U)$, $\mathbf{\Lambda}_\Omega$ and $\boldsymbol{\mu}_\Omega$ when there is no risk of confusion.

Definition 3.1.2. Let $\mathbf{x}:U\to\mathbb{R}^3$ be a frontal, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p}\in\Sigma(\mathbf{x})$, we say that $\mathbf{p}$ is a *non-degenerate* singularity if $D\lambda_\Omega(\mathbf{p})\neq(0,0)$, in another case it is called degenerate.

Remark 3.1.2. This definition does not depend on the chosen tmb $\mathbf{\Omega}$. If $\mathbf{x}$ is proper frontal with another tmb $\bar{\mathbf{\Omega}}$, we have $\lambda_{\bar{\Omega}}=\rho\lambda_\Omega$ with $\rho=det(\mathbf{B})$, being $\mathbf{B}$ the non-singular matrix that satisfies $\mathbf{\Omega}=\bar{\mathbf{\Omega}}\mathbf{B}$. Thus, $D\lambda_\Omega(\mathbf{p})\neq(0,0)$ if and only if $D\lambda_{\bar{\Omega}}(\mathbf{p})\neq(0,0)$. If $\mathbf{x}$ is non-proper and $\mathbf{p}$ is a non-degenerate singularity, then by the Implicit function theorem, the singular set is locally at $\mathbf{p}$ a regular curve, therefore on a neighborhood of $\mathbf{p}$, $\mathbf{x}$ is proper frontal and we can apply the argument discussed before.

In the literature (see for example (MARTINS; SAJI; TERAMOTO, 2019; SAJI; UMEHARA; YAMADA, 2009)) is quite used a function λ called *signed area density function* for the last definition instead of λ_Ω, this is defined by

$$\lambda:=det(\begin{pmatrix}\mathbf{x}_u & \mathbf{x}_v & \mathbf{n}\end{pmatrix}),$$

where $\mathbf{n}$ is the normal vector field induced by a tmb $\mathbf{\Omega}=\begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$. Since, $\mathbf{x}_u=\lambda_{11}\mathbf{w}_1+\lambda_{12}\mathbf{w}_2$, $\mathbf{x}_v=\lambda_{21}\mathbf{w}_1+\lambda_{22}\mathbf{w}_2$, then

$$\mathbf{x}_u\times\mathbf{x}_v=\lambda_\Omega\mathbf{w}_1\times\mathbf{w}_2$$

and thus using the Lagrange's identity we have

$$\lambda = (\mathbf{x}_u \times \mathbf{x}_v)\cdot \mathbf{n} = \lambda_\Omega |\mathbf{w}_1 \times \mathbf{w}_2| = \lambda_\Omega \sqrt{E_\Omega G_\Omega - F_\Omega^2} = \lambda_\Omega det(\mathbf{I}_\Omega)^{\frac{1}{2}}.$$

As $det(\mathbf{I}_\Omega)^{\frac{1}{2}} > 0$, then the definition of non-degenerate singularity does not depend if the function used is λ_Ω or λ.

Definition 3.1.3. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega} = \begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$ and $\bar{\mathbf{\Omega}} = \begin{pmatrix}\bar{\mathbf{w}}_1 & \bar{\mathbf{w}}_2\end{pmatrix}$ tangent moving bases of $\mathbf{x}$, where $\mathbf{\Omega} = \bar{\mathbf{\Omega}}\mathbf{B}$. We say that $\mathbf{\Omega}$ and $\bar{\mathbf{\Omega}}$ are *compatibles* if $\mathbf{w}_1 \times \mathbf{w}_2 \cdot \bar{\mathbf{w}}_1 \times \bar{\mathbf{w}}_2 > 0$. Also, $\mathbf{\Omega}$ is an *orthonormal* tangent moving basis if $|\mathbf{w}_1| = |\mathbf{w}_2| = 1$ and $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$.

In the following theorem we show the properties that first and second fundamental forms of frontals always satisfy. In chapter 4, we will see that having fundamentals forms satisfying these properties and the Gauss and Mainardi-Codazzi equations on the regular set, we can get a frontal.

Theorem 3.1.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, then the matrices defined by equations (2.1a) and (2.1b) have the following decomposition:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}^T \tag{3.1a}$$

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix}, \tag{3.1b}$$

in which all the components are smooth real functions defined on U, $E_\Omega > 0$, $G_\Omega > 0$, $E_\Omega G_\Omega - F_\Omega^2 > 0$, $rank(D\mathbf{x}) = rank(\mathbf{\Lambda})$, $\Sigma(\mathbf{x}) = \lambda_\Omega^{-1}(0)$ and

$$\mathbf{\Lambda}_{(1)u}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u}^T + E_v - F_u \in \mathfrak{T}_\Omega \tag{3.2a}$$

$$\mathbf{\Lambda}_{(1)v}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)v}^T + F_v - G_u \in \mathfrak{T}_\Omega \tag{3.2b}$$

where $\mathbf{\Lambda} = (\lambda_{ij})$.

Proof. We have $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, then using (2.2a) $\mathbf{I} = D\mathbf{x}^T D\mathbf{x} = \mathbf{\Lambda}\mathbf{\Omega}^T\mathbf{\Omega}\mathbf{\Lambda}^T = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T$. Also, using (2.2b) $\mathbf{II} = -D\mathbf{x}^T D\mathbf{n} = \mathbf{\Lambda}(-\mathbf{\Omega}^T D\mathbf{n}) = \mathbf{\Lambda}\mathbf{II}_\Omega$. Now, let us set the skew-symmetric matrices:

$$\mathbf{A}_1 := \begin{pmatrix} 0 & -(E_v - F_u) \\ E_v - F_u & 0 \end{pmatrix},\ \mathbf{B}_1 := \begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix} := \mathbf{\Omega}_u^T\mathbf{\Omega} - \mathbf{\Omega}^T\mathbf{\Omega}_u.$$

From (2.1c) and (2.2c) we have $D\mathbf{x}_u^T D\mathbf{x} - \frac{1}{2}\mathbf{I}_u = \frac{1}{2}\mathbf{A}_1$, then using that $\mathbf{I} = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T$, $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$ and developing derivatives,

$$(\mathbf{\Lambda}\mathbf{\Omega}_u^T + \mathbf{\Lambda}_u\mathbf{\Omega}^T)\mathbf{\Omega}\mathbf{\Lambda}^T = \frac{1}{2}(\mathbf{\Lambda}_u\mathbf{I}_\Omega\mathbf{\Lambda}^T + \mathbf{\Lambda}\mathbf{I}_{\Omega u}\mathbf{\Lambda}^T + \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}_u^T) + \frac{1}{2}\mathbf{A}_1.$$

Substituting $\mathbf{I}_\Omega = \mathbf{\Omega}^T\mathbf{\Omega}$ and $\mathbf{I}_{\Omega u} = \mathbf{\Omega}_u^T\mathbf{\Omega} + \mathbf{\Omega}^T\mathbf{\Omega}_u$, we can group and cancel similar terms, getting

$$\mathbf{\Lambda}\mathbf{B}_1\mathbf{\Lambda}^T = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}_u^T - \mathbf{\Lambda}_u\mathbf{I}_\Omega\mathbf{\Lambda}^T + \mathbf{A}_1.$$

multiplying the equality by left side with $\begin{pmatrix}1 & 0\end{pmatrix}$ and by the right side with $\begin{pmatrix}0 & 1\end{pmatrix}^T$, we obtain,

$$-\tau_1\lambda_\Omega = \mathbf{\Lambda}_{(1)}\begin{pmatrix}0 & -\tau_1\\ \tau_1 & 0\end{pmatrix}\mathbf{\Lambda}_{(2)}^T = \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u}^T - \mathbf{\Lambda}_{(1)u}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - (E_v - F_u)$$

and from it follows (3.2a). Setting the matrices:

$$\mathbf{A}_2 := \begin{pmatrix}0 & -(F_v - G_u)\\ F_v - G_u & 0\end{pmatrix},\ \mathbf{B}_2 = \begin{pmatrix}0 & -\tau_2\\ \tau_2 & 0\end{pmatrix} := \mathbf{\Omega}_v^T\mathbf{\Omega} - \mathbf{\Omega}^T\mathbf{\Omega}_v$$

Observing that, $D\mathbf{x}_v^T D\mathbf{x} - \frac{1}{2}\mathbf{I}_v = \frac{1}{2}\mathbf{A}_2$ and proceeding similarly as before, it follows (3.2b).

□

The conditions (3.2a) and (3.2b) in theorem 3.1.1 may seem kind of strange, but we will see in proposition 3.2.3 why these are so important. Also these expressions can be reduced depending on the type of tmb $\mathbf{\Omega}$. If we have a tangent moving basis of a frontal, we always can construct an orthonormal one applying Gram-Schmidt orthonormalization, then the decompositions in theorem 3.1.1 are reduced and follows easily the corollary:

Corollary 3.1.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega}$ an orthonormal tangent moving basis of $\mathbf{x}$, then the matrices defined by equations (2.1a) and (2.1b) have the following decomposition:

$$\begin{pmatrix}E & F\\ F & G\end{pmatrix} = \begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}^T$$
$$\begin{pmatrix}L & M\\ M & N\end{pmatrix} = \begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}\begin{pmatrix}L_\Omega & M_{1\Omega}\\ M_{2\Omega} & N_\Omega\end{pmatrix},$$

in which all the components are smooth real functions defined on U, $rank(D\mathbf{x}) = rank(\mathbf{\Lambda})$, $\Sigma(\mathbf{x}) = \lambda_\Omega^{-1}(0)$ and

$$(\mathbf{\Lambda}_{(1)}\mathbf{\Lambda}_{(1)}^T)_v - 2\mathbf{\Lambda}_{(1)}\mathbf{\Lambda}_{(2)u}^T \in \mathfrak{T}_\Omega$$
$$2\mathbf{\Lambda}_{(1)v}\mathbf{\Lambda}_{(2)}^T - (\mathbf{\Lambda}_{(2)}\mathbf{\Lambda}_{(2)}^T)_u \in \mathfrak{T}_\Omega$$

where $\mathbf{\Lambda} = (\lambda_{ij})$.

Remark 3.1.3. If $\mathbf{x} : U \to \mathbb{R}^3$ is a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, we can find a tangent moving basis $\hat{\mathbf{\Omega}}$ having one of the following forms:

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ g_1 & g_2\end{pmatrix},\ \begin{pmatrix}1 & 0\\ g_1 & g_2\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 1 & 0\\ g_1 & g_2\end{pmatrix},$$

$$\begin{pmatrix} g_1 & g_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} g_1 & g_2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ g_1 & g_2 \\ 1 & 0 \end{pmatrix},$$

with $g_1, g_2 : U \to \mathbb{R}$ smooth functions and the matrix $\hat{\mathbf{\Lambda}}^T$ as an *exact differential*, it means, there is a smooth map $(a,b) : U \to \mathbb{R}^2$ such that $D(a,b) = \hat{\mathbf{\Lambda}}^T$. To see this, as the columns of $\mathbf{\Omega}$ are linearly independent, then applying reduction of Gauss-Jordan with a finite number of operations by columns, it can be reduced to one of the forms above. Without loss of generality, let us suppose it is reduced to the first one. If we denote $\mathbf{E}_1, \mathbf{E}_2, .., \mathbf{E}_m$ the elementary 2×2 matrices corresponding to the operations by columns, we have:

$$D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T = \mathbf{\Omega}\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_m\mathbf{E}_m^{-1} \cdots \mathbf{E}_2^{-1}\mathbf{E}_1^{-1}\mathbf{\Lambda}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & g_2 \end{pmatrix} \mathbf{E}_m^{-1} \cdots \mathbf{E}_2^{-1}\mathbf{E}_1^{-1}\mathbf{\Lambda}^T$$

Denoting $\hat{\mathbf{\Lambda}}^T := \mathbf{E}_m^{-1} \cdots \mathbf{E}_2^{-1}\mathbf{E}_1^{-1}\mathbf{\Lambda}^T$ and $\mathbf{x} = (a,b,c)$, we can multiply the last equality by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ to get:

$$D(a,b) = \begin{pmatrix} a_u & a_v \\ b_u & b_v \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} D\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & g_2 \end{pmatrix} \hat{\mathbf{\Lambda}}^T = \mathbb{I}_2\hat{\mathbf{\Lambda}}^T = \hat{\mathbf{\Lambda}}^T.$$

On the other hand, a simple computation leads to

$$\mathbf{I}_{\hat{\Omega}} = \begin{pmatrix} 1+g_1^2 & g_1g_2 \\ g_1g_2 & 1+g_2^2 \end{pmatrix}, \mathbf{II}_{\hat{\Omega}} = \begin{pmatrix} g_{1u} & g_{1v} \\ g_{2u} & g_{2v} \end{pmatrix} (1+g_1^2+g_2^2)^{-\frac{1}{2}},$$

and since $D\mathbf{n} = \hat{\mathbf{\Omega}}\boldsymbol{\mu}^T$ with $\mathbf{n} = (-g_1, -g_2, 1)det(\mathbf{I}_\Omega)^{-\frac{1}{2}}$, reasoning as before we get that $D(-g_1det(\mathbf{I}_\Omega)^{-\frac{1}{2}}, -g_2det(\mathbf{I}_\Omega)^{-\frac{1}{2}}) = \boldsymbol{\mu}^T$.

By this fact and theorem 3.1.1, follows the result:

Corollary 3.1.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ with the form of remark 3.1.3, then the matrices defined by equations (2.1a) and (2.1b) have a decomposition in this form:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} a_u & b_u \\ a_v & b_v \end{pmatrix} \begin{pmatrix} 1+g_1^2 & g_1g_2 \\ g_1g_2 & 1+g_2^2 \end{pmatrix} \begin{pmatrix} a_u & b_u \\ a_v & b_v \end{pmatrix}^T$$
$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a_u & b_u \\ a_v & b_v \end{pmatrix} \begin{pmatrix} g_{1u} & g_{1v} \\ g_{2u} & g_{2v} \end{pmatrix} (1+g_1^2+g_2^2)^{-\frac{1}{2}}$$

in which g_1, g_2, a and b are smooth real functions defined in U. In particular, $(a,b)_u \cdot (g_1,g_2)_v = (a,b)_v \cdot (g_1,g_2)_u$.

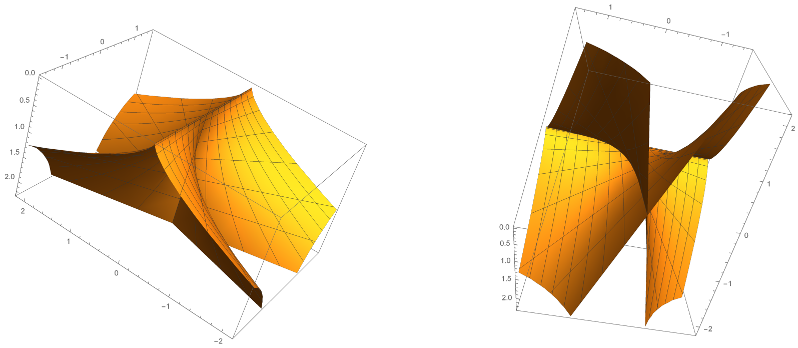

Figure 3 – The cuspidal cross-cap ($\mathbf{x}(u,v) = (u, v^2, uv^3)$), an example of a proper frontal which is not a front (FUJIMORI *et al.*, 2008).

Example 3.1.2. The cuspidal cross-cap (see Figure 3) can be decomposed in this way:

$$D\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v^3 & \frac{3}{2}uv \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix} = \mathbf{\Omega}\mathbf{\Lambda}^T, \text{ where } \mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v^3 & \frac{3}{2}uv \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix} \begin{pmatrix} 1+v^6 & \frac{3}{2}uv^4 \\ \frac{3}{2}uv^4 & 1+\frac{9}{4}u^2v^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix}^T$$

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix} \begin{pmatrix} 0 & 3v^2 \\ \frac{3}{2}v & \frac{3}{2}u \end{pmatrix} \frac{1}{\sqrt{1+v^6+\frac{9}{4}u^2v^2}}$$

Theorem 3.1.2. Let $\mathbf{I} : U \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be a smooth map, with $\mathbf{I}$ decomposing in this form:

$$\mathbf{I} = \begin{pmatrix} a_u & b_u \\ a_v & b_v \end{pmatrix} \begin{pmatrix} 1+g_1^2 & g_1g_2 \\ g_1g_2 & 1+g_2^2 \end{pmatrix} \begin{pmatrix} a_u & b_u \\ a_v & b_v \end{pmatrix}^T$$

in which g_1, g_2, a and b are smooth real functions defined in U, satisfying $(a,b)_u \cdot (g_1,g_2)_v = (a,b)_v \cdot (g_1,g_2)_u$. Then, for each $(u_0,v_0) \in U$ and $\mathbf{p} \in \mathbb{R}^3$, there is a frontal $\mathbf{x} : V \to \mathbb{R}^3$, $V \subset U$, V a neighborhood of (u_0,v_0) with first fundamental form $\mathbf{I}$ and second fundamental form $D(a,b)^T D(g_1,g_2)(1+g_1^2+g_2^2)^{-\frac{1}{2}}$.

Proof. Setting the matrices:

$$\mathbf{\Omega} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & g_2 \end{pmatrix}, \mathbf{\Lambda}^T := \begin{pmatrix} a_u & a_v \\ b_u & b_v \end{pmatrix}, \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $(a,b)_u \cdot (g_1,g_2)_v = (a,b)_v \cdot (g_1,g_2)_u$, then

$$\mathbf{\Omega}_u \mathbf{\Lambda}^T \mathbf{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ g_{1u} & g_{2u} \end{pmatrix} \begin{pmatrix} a_v \\ b_v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ g_{1v} & g_{2v} \end{pmatrix} \begin{pmatrix} a_u \\ b_u \end{pmatrix} = \mathbf{\Omega}_v \mathbf{\Lambda}^T \mathbf{e}_1$$

on the other hand, since $\mathbf{\Lambda}^T$ is an exact differential, $\mathbf{\Lambda}_u^T \mathbf{e}_2 = \mathbf{\Lambda}_v^T \mathbf{e}_1$. Thus, $\mathbf{\Omega\Lambda}_u^T \mathbf{e}_2 = \mathbf{\Omega\Lambda}_v^T \mathbf{e}_1$ and adding this equality to the above one, we get:

$$(\mathbf{\Omega\Lambda}^T)_u \mathbf{e}_2 = \mathbf{\Omega}_u \mathbf{\Lambda}^T \mathbf{e}_2 + \mathbf{\Omega\Lambda}_u^T \mathbf{e}_2 = \mathbf{\Omega}_v \mathbf{\Lambda}^T \mathbf{e}_1 + \mathbf{\Omega\Lambda}_v^T \mathbf{e}_1 = (\mathbf{\Omega\Lambda}^T)_v \mathbf{e}_1$$

Denoting by $\mathbf{z}_1$ and $\mathbf{z}_2$ the first and second columns of $\mathbf{\Omega\Lambda}^T$ respectively, fixing $(u_0, v_0) \in U$ and $\mathbf{p} \in \mathbb{R}^3$ the last equality is equivalent to $\mathbf{z}_{2u} = \mathbf{z}_{1v}$, which is the compatibility condition of the system:

$$\mathbf{x}_u = \mathbf{z}_1 \tag{3.7a}$$

$$\mathbf{x}_v = \mathbf{z}_2 \tag{3.7b}$$

$$\mathbf{x}(u_0, v_0) = \mathbf{p}, \tag{3.7c}$$

By theorem 2.3.1, this system of partial differential equations has a solution $\mathbf{x} : V \to \mathbb{R}^3$, $V \subset U$, V a neighborhood of (u_0, v_0). Therefore $D\mathbf{x} = \mathbf{\Omega\Lambda}^T$ and by proposition 3.1.1, $\mathbf{x} : V \to \mathbb{R}^3$ is a frontal. Now, the first fundamental form is $D\mathbf{x}^T D\mathbf{x} = \mathbf{\Lambda\Omega}^T \mathbf{\Omega\Lambda}^T = \mathbf{I}$ as we wished. Using that $\mathbf{n} = (-g_1, -g_2, 1)(1 + g_1^2 + g_2^2)^{-\frac{1}{2}}$ and (2.4), the second fundamental form is $\mathbf{\Lambda II}_\Omega = D(a,b)^T D(g_1, g_2)(1 + g_1^2 + g_2^2)^{-\frac{1}{2}}$. □

3.2 The new symbols and its relations with the Christoffel symbols

In this section we study how the symbols $\mathcal{T}_i$ (defined in 2.3c and 2.3d) are related with the Christoffel symbols on the regular set $(\Sigma(\mathbf{x})^c)$. In proposition 3.2.3, we see how the decomposition in matrices (as in theorem 3.1.1) is tightly connected with the extendibility of $\mathcal{T}_i$ to the singularities from its expression in terms of Christoffel symbols.

Proposition 3.2.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, then the matrices $\mathcal{T}_1, \mathcal{T}_2$ satisfy $\mathbf{I}_\Omega \mathcal{T}_1^T + \mathcal{T}_1 \mathbf{I}_\Omega = \mathbf{I}_{\Omega u}$ and $\mathbf{I}_\Omega \mathcal{T}_2^T + \mathcal{T}_2 \mathbf{I}_\Omega = \mathbf{I}_{\Omega v}$.

Proof. Using (2.3a), (2.3c) we have $\mathbf{I}_{\Omega u} = \mathbf{\Omega}_u^T \mathbf{\Omega} + \mathbf{\Omega}^T \mathbf{\Omega}_u = \mathbf{\Omega}_u^T \mathbf{\Omega} \mathbf{I}_\Omega^{-1} \mathbf{I}_\Omega + \mathbf{I}_\Omega \mathbf{I}_\Omega^{-1} \mathbf{\Omega}^T \mathbf{\Omega}_u = \mathcal{T}_1 \mathbf{I}_\Omega + \mathbf{I}_\Omega \mathcal{T}_1^T$. For $\mathbf{I}_{\Omega v}$ is analogous. □

Proposition 3.2.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, then the Christoffel symbols defined on $U - \lambda_\Omega^{-1}(0)$ have the following decomposition:

$$\mathbf{\Gamma}_1 = (\mathbf{\Lambda}\mathcal{T}_1 + \mathbf{\Lambda}_u)\mathbf{\Lambda}^{-1} \text{ and } \mathbf{\Gamma}_2 = (\mathbf{\Lambda}\mathcal{T}_2 + \mathbf{\Lambda}_v)\mathbf{\Lambda}^{-1}$$

Proof. $\mathbf{\Gamma}_1 = (D\mathbf{x}_u^T D\mathbf{x})\mathbf{I}^{-1} = ((\mathbf{\Omega}_u \mathbf{\Lambda}^T + \mathbf{\Omega\Lambda}_u^T)^T \mathbf{\Omega\Lambda}^T)(\mathbf{\Lambda}^T)^{-1} \mathbf{I}_\Omega^{-1} \mathbf{\Lambda}^{-1}$
$= (\mathbf{\Lambda\Omega}_u^T + \mathbf{\Lambda}_u \mathbf{\Omega}^T)\mathbf{\Omega\Lambda}^T (\mathbf{\Lambda}^T)^{-1} \mathbf{I}_\Omega^{-1} \mathbf{\Lambda}^{-1} = (\mathbf{\Lambda\Omega}_u^T \mathbf{\Omega I}_\Omega^{-1} + \mathbf{\Lambda}_u \mathbf{\Omega}^T \mathbf{\Omega I}_\Omega^{-1})\mathbf{\Lambda}^{-1}$. Since $\mathcal{T}_1 = \mathbf{\Omega}_u^T \mathbf{\Omega I}_\Omega^{-1}$ and $\mathbf{I}_\Omega = \mathbf{\Omega}^T \mathbf{\Omega}$ we have the result. For $\mathbf{\Gamma}_2$ it is analogous. □

Remark 3.2.1. With this decomposition of the Christoffel symbols, by the density of non-singular points and smoothness of $\mathcal{T}_i$ on U, we get that $\mathcal{T}_1$ and $\mathcal{T}_2$ can be expressed by:

- For $\mathbf{p} \in \Sigma(\mathbf{x})^c$,

$$\mathcal{T}_1 = \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u) \text{ and } \mathcal{T}_2 = \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_2\mathbf{\Lambda} - \mathbf{\Lambda}_v).$$

- For $\mathbf{p} \in \Sigma(\mathbf{x})$,

$$\mathcal{T}_1 = \lim_{(u,v)\to p} \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u) \text{ and } \mathcal{T}_2 = \lim_{(u,v)\to p} \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_2\mathbf{\Lambda} - \mathbf{\Lambda}_v).$$

The right hand sides of the above equations are restricted to the open set $\Sigma(\mathbf{x})^c$. As $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are expressed in terms of E, F, G and these by (3.1a) are expressed in terms of E_Ω, F_Ω, G_Ω and λ_{ij}, then $\mathcal{T}_1$ and $\mathcal{T}_2$ can be expressed just using E_Ω, F_Ω, G_Ω and λ_{ij} on $\Sigma(\mathbf{x})^c$. By density, these are completely determined by E_Ω, F_Ω, G_Ω and λ_{ij} on U.

Proposition 3.2.3. Let $\mathbf{I}, \mathbf{I}_\Omega, \mathbf{\Lambda} : U \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be arbitrary smooth maps, $\mathbf{I}_\Omega$ symmetric non-singular, $\lambda_\Omega = det(\mathbf{\Lambda})$ and $\mathfrak{T}_\Omega$ the principal ideal generated by λ_Ω in the ring $C^\infty(U,\mathbb{R})$. If we have,

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T \tag{3.8}$$

with $int(\lambda_\Omega^{-1}(0)) = \emptyset$ and if we define $\mathbf{\Gamma}_1$ by (2.1c) and $\mathbf{\Gamma}_2$ by (2.1d) on $U - \lambda_\Omega^{-1}(0)$, then the maps

$$\mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u), \tag{3.9a}$$
$$\mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_2\mathbf{\Lambda} - \mathbf{\Lambda}_v), \tag{3.9b}$$

defined on $U - \lambda_\Omega^{-1}(0)$, have unique C^∞ extensions to U if and only if,

$$\mathbf{\Lambda}_{(1)u}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u}^T + E_v - F_u \in \mathfrak{T}_\Omega \tag{3.10a}$$
$$\mathbf{\Lambda}_{(1)v}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)v}^T + F_v - G_u \in \mathfrak{T}_\Omega \tag{3.10b}$$

Proof. For the necessary condition, let us set the skew-symmetric matrix

$$\mathbf{A}_1 := \begin{pmatrix} 0 & -(E_v - F_u) \\ E_v - F_u & 0 \end{pmatrix}$$

and suppose that $\mathcal{T}_1$ is the C^∞ extension of $\mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u)$, then

$$\mathbf{\Lambda}\mathcal{T}_1 = \mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u$$

on $U-\lambda_\Omega^{-1}(0)$, hence using (2.1c) we have

$$\mathbf{\Lambda}\mathcal{T}_1 = (\frac{1}{2}\mathbf{I}_u + \frac{1}{2}\mathbf{A}_1)\mathbf{I}^{-1}\mathbf{\Lambda} - \mathbf{\Lambda}_u.$$

Substituting $\mathbf{I}$ and $\mathbf{I}_u$ in the last equality using (3.8) and multiplying by the right side with $2\mathbf{I}_\Omega\mathbf{\Lambda}^T$, operating some terms we can get,

$$\mathbf{\Lambda}(2\mathcal{T}_1\mathbf{I}_\Omega - \mathbf{I}_{\Omega u})\mathbf{\Lambda}^T = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}_u^T - \mathbf{\Lambda}_u\mathbf{I}_\Omega\mathbf{\Lambda}^T + \mathbf{A}_1. \tag{3.11}$$

Observe that, the right side of (3.11) is skew-symmetric, then $2\mathcal{T}_1\mathbf{I}_\Omega - \mathbf{I}_{\Omega u}$ as well and since $U-\lambda_\Omega^{-1}(0)$ is dense, this is also true on U. Thus,

$$2\mathcal{T}_1\mathbf{I}_\Omega - \mathbf{I}_{\Omega u} = \begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix}$$

for any $\tau_1 \in C^\infty(U,\mathbb{R})$ and since the equality (3.11) is valid on U by density, then multiplying this by left side with $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and by the right side with $\begin{pmatrix} 0 & 1 \end{pmatrix}^T$, we obtain,

$$-\tau_1\lambda_\Omega = \mathbf{\Lambda}_{(1)}\begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix}\mathbf{\Lambda}_{(2)}^T = \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u}^T - \mathbf{\Lambda}_{(1)u}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - (E_v - F_u)$$

and from it follows (3.10a). Setting the matrix:

$$\mathbf{A}_2 := \begin{pmatrix} 0 & -(F_v - G_u) \\ F_v - G_u & 0 \end{pmatrix}$$

and observing that $\mathbf{\Gamma}_2 = (\frac{1}{2}\mathbf{I}_v + \frac{1}{2}\mathbf{A}_2)\mathbf{I}^{-1}$, proceeding similarly as before, it follows (3.10b). For the sufficient condition, if we have (3.10a), (3.10b), as $U-\lambda_\Omega^{-1}(0)$ is dense then there exist unique $\tau_1, \tau_2 \in C^\infty(U,\mathbb{R})$ such that,

$$\mathbf{\Lambda}_{(1)u}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u}^T + E_v - F_u = \lambda_\Omega\tau_1,$$

$$\mathbf{\Lambda}_{(1)v}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)v}^T + F_v - G_u = \lambda_\Omega\tau_2.$$

Defining the smooth maps on U,

$$\mathcal{T}_1 := \frac{1}{2}(\begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix} + \mathbf{I}_{\Omega u})\mathbf{I}_\Omega^{-1} \text{ and } \mathcal{T}_2 := \frac{1}{2}(\begin{pmatrix} 0 & -\tau_2 \\ \tau_2 & 0 \end{pmatrix} + \mathbf{I}_{\Omega v})\mathbf{I}_\Omega^{-1}, \tag{3.12}$$

we have that

$$\mathbf{\Lambda}(2\mathcal{T}_1\mathbf{I}_\Omega - \mathbf{I}_{\Omega u})\mathbf{\Lambda}^T = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}_u^T - \mathbf{\Lambda}_u\mathbf{I}_\Omega\mathbf{\Lambda}^T + \mathbf{A}_1,$$

$$\mathbf{\Lambda}(2\mathcal{T}_2\mathbf{I}_\Omega - \mathbf{I}_{\Omega v})\mathbf{\Lambda}^T = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}_v^T - \mathbf{\Lambda}_v\mathbf{I}_\Omega\mathbf{\Lambda}^T + \mathbf{A}_2,$$

which leads to $\mathcal{T}_1$ and $\mathcal{T}_2$ be equal to (3.9a) and (3.9b) respectively on $U-\lambda_\Omega^{-1}(0)$. Thus, by density and smoothness of $\mathcal{T}_1$ and $\mathcal{T}_2$, these are unique C^∞-extensions. □

Remark 3.2.2. By proposition 3.2.3, we always can define the matrices $\mathcal{T}_1, \mathcal{T}_2$ by (3.12) from a smooth map $\mathbf{I}: U \to \mathcal{M}_{2\times 2}(\mathbb{R})$ satisfying a decomposition as in (3.8) with the conditions (3.10a) and (3.10b). These maps $\mathcal{T}_1, \mathcal{T}_2$ automatically satisfy the relations of proposition 3.2.1 as they also are the unique C^∞ extension of (3.9a) and (3.9b). It is natural the question if a decomposition as in (3.8) implies the conditions (3.10a), (3.10b) and the answer is not. For example the matrix $\mathbf{I}$ associated to the first fundamental form of $(u,v) \to (u,v^2,uv)$ (the cross-cap) is singular at $(0,0)$ and have a rank ≥ 1 on the entire $\mathbb{R}^2$, then you can obtain the Cholesky decomposition $\mathbf{I} = \mathbf{\Lambda}\mathbf{\Lambda}^T$ (here $\mathbf{I}_\Omega$ can be chosen as $\mathbb{I}_2$), where $\mathbf{\Lambda}: \mathbb{R}^2 \to \mathcal{M}_{2\times 2}(\mathbb{R})$ is smooth and a lower triangular matrix. It is not difficult to check that in this case the condition (3.10a) and (3.10a) are not satisfied for all neighborhood of $(0,0)$.

3.3 The relative curvatures

Definition 3.3.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, we define the Ω-*relative curvature* $K_\Omega := det(\boldsymbol{\mu}_\Omega)$ and the Ω-*relative mean curvature* $H_\Omega := -\frac{1}{2}tr(\boldsymbol{\mu}_\Omega adj(\mathbf{\Lambda}_\Omega))$, where $tr()$ is the trace and $adj()$ is the adjoint of a matrix.

Remark 3.3.1. Recall that $\mathbf{\Lambda}_\Omega = (\lambda_{ij}) = D\mathbf{x}^T\mathbf{\Omega}(\mathbf{I}_\Omega)^{-1}$ and $\boldsymbol{\mu}_\Omega = (\mu_{ij}) = -\mathbf{II}_\Omega^T\mathbf{I}_\Omega^{-1}$ we have that $H_\Omega = -\frac{1}{2}(\lambda_{22}\mu_{11} - \lambda_{21}\mu_{12} + \lambda_{11}\mu_{22} - \lambda_{12}\mu_{21})$ and $K_\Omega = \frac{L_\Omega N_\Omega - M_{1\Omega}M_{2\Omega}}{E_\Omega G_\Omega - F_\Omega^2}$.

We are going to use K_Ω and H_Ω to characterize wave fronts in theorem 3.3.1 and colorally 3.3.1, but first we need to prove some propositions. The reason why we call these functions curvatures is in the following result.

Proposition 3.3.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, K_Ω, H_Ω, K and H the $\mathbf{\Omega}$-relative curvature, the $\mathbf{\Omega}$-relative mean curvature, the Gaussian curvature and the mean curvature of $\mathbf{x}$ respectively.Then,

- for $\mathbf{p} \in \Sigma(\mathbf{x})^c$, $K_\Omega = \lambda_\Omega K$ and $H_\Omega = \lambda_\Omega H$,
- for $\mathbf{p} \in \Sigma(\mathbf{x})$, $K_\Omega = \lim_{(u,v)\to p} \lambda_\Omega K$ and $H_\Omega = \lim_{(u,v)\to p} \lambda_\Omega H$,

where the right sides are restricted to the open set $\Sigma(\mathbf{x})^c$ and $\lambda_\Omega = det(\mathbf{\Lambda}_\Omega)$.

Proof. By theorem 3.1.1, $\mathbf{I} = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T$ and $\mathbf{II} = \mathbf{\Lambda}\mathbf{II}_\Omega$, then for $\mathbf{p} \in \Sigma(\mathbf{x})^c$, using (2.1e), $\boldsymbol{\alpha} = -\mathbf{II}^T\mathbf{I}^{-1} = -\mathbf{II}_\Omega^T\mathbf{\Lambda}^T(\mathbf{\Lambda}^T)^{-1}\mathbf{I}_\Omega^{-1}\mathbf{\Lambda}^{-1} = \boldsymbol{\mu}\mathbf{\Lambda}^{-1}$. Thus, $\boldsymbol{\alpha}\mathbf{\Lambda} = \boldsymbol{\mu}$ and $K_\Omega = det(\boldsymbol{\mu}) = det(\boldsymbol{\alpha})det(\mathbf{\Lambda}) = \lambda_\Omega K$. Also, we have $\boldsymbol{\alpha}\lambda_\Omega = \boldsymbol{\mu}adj(\mathbf{\Lambda})$, then $H_\Omega = -\frac{1}{2}tr(\boldsymbol{\mu}adj(\mathbf{\Lambda})) = -\frac{1}{2}\lambda_\Omega tr(\boldsymbol{\alpha}) = \lambda_\Omega H$. By density of $\Sigma(\mathbf{x})^c$ and the smoothness of K_Ω and H_Ω we have the result for $\mathbf{p} \in \Sigma(\mathbf{x})$. □

Example 3.3.1. In the example 3.1.1 we saw that the cuspidal edge $\mathbf{x} = (u, v^2, v^3)$ with normal vector field $\mathbf{n} = (0, -3v, 2)(4+9v^2)^{-\frac{1}{2}}$ had as tmb

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3v \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix},$$

then

$$\mathbf{I}_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 4+9v^2 \end{pmatrix}, \ \mathbf{II}_\Omega = \begin{pmatrix} \mathbf{n}\cdot\mathbf{w}_{1u} & \mathbf{n}\cdot\mathbf{w}_{1v} \\ \mathbf{n}\cdot\mathbf{w}_{2u} & \mathbf{n}\cdot\mathbf{w}_{2v} \end{pmatrix} = (4+9v^2)^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix},$$

$$\boldsymbol{\mu}_\Omega = -\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1} = -(4+9v^2)^{-\frac{3}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix},$$

therefore $K_\Omega = 0 = K$, $H_\Omega = 3(4+9v^2)^{-\frac{3}{2}}$ and $H = 3v^{-1}(4+9v^2)^{-\frac{3}{2}}$. Observe that $H_\Omega \neq 0$, even on the singular set $(v = 0)$. We will see in theorem 3.3.1 that this is related to the fact that $\mathbf{x}$ besides being frontal it is wavefront.

Proposition 3.3.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$. The zeros of K_Ω and H_Ω do not depend on the chosen tangent moving basis $\mathbf{\Omega}$ of $\mathbf{x}$. Also, the signs are preserved if we restrict $\mathbf{\Omega}$ to compatibles tangent moving bases.

Proof. Let $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ and $\bar{\mathbf{\Omega}} = \begin{pmatrix} \bar{\mathbf{w}}_1 & \bar{\mathbf{w}}_2 \end{pmatrix}$ be tmb of $\mathbf{x}$, $\mathbf{\Lambda} = D\mathbf{x}^T\mathbf{\Omega}(\mathbf{I}_\Omega^T)^{-1}$ and $\bar{\mathbf{\Lambda}} = D\mathbf{x}^T\bar{\mathbf{\Omega}}(\mathbf{I}_{\bar{\Omega}}^T)^{-1}$. Since $\mathbf{\Omega} = \bar{\mathbf{\Omega}}\mathbf{C}$, where $\mathbf{C}$ is a non-singular matrix-valued map, then $\mathbf{\Lambda} = D\mathbf{x}^T\bar{\mathbf{\Omega}}\mathbf{C}(\mathbf{C}^T\bar{\mathbf{\Omega}}^T\bar{\mathbf{\Omega}}\mathbf{C})^{-1} = D\mathbf{x}^T\bar{\mathbf{\Omega}}(\mathbf{I}_{\bar{\Omega}}^T)^{-1}(\mathbf{C}^T)^{-1} = \bar{\mathbf{\Lambda}}(\mathbf{C}^T)^{-1}$. On the other hand, $\boldsymbol{\mu} = -\mathbf{II}_\Omega^T\mathbf{I}_\Omega^{-1} = -D\mathbf{n}^T\mathbf{\Omega}\mathbf{I}_\Omega^{-1}$
$= -D\mathbf{n}^T\bar{\mathbf{\Omega}}\mathbf{C}(\mathbf{C}^T\bar{\mathbf{\Omega}}^T\bar{\mathbf{\Omega}}\mathbf{C})^{-1} = -\mathbf{II}_{\bar{\Omega}}^T\mathbf{I}_{\bar{\Omega}}^{-1}(\mathbf{C}^T)^{-1} = \bar{\boldsymbol{\mu}}(\mathbf{C}^T)^{-1}$. Now, $K_{\bar{\Omega}} = det(\bar{\boldsymbol{\mu}}) = det(\boldsymbol{\mu})det(\mathbf{C}) = det(\mathbf{C})K_\Omega$ and $H_{\bar{\Omega}} = -\frac{1}{2}tr(\bar{\boldsymbol{\mu}}adj(\bar{\mathbf{\Lambda}})) = -\frac{1}{2}tr(\boldsymbol{\mu}\mathbf{C}^T adj(\mathbf{C}^T)adj(\mathbf{\Lambda}))$
$= -\frac{1}{2}tr(\boldsymbol{\mu}adj(\mathbf{\Lambda}))det(\mathbf{C}) = det(\mathbf{C})H_\Omega$, then $K_\Omega = 0$ if and only if, $K_{\bar{\Omega}} = 0$ and $H_\Omega = 0$ if and only if, $H_{\bar{\Omega}} = 0$. For the last assertion, observe that, if $\mathbf{\Omega}$ and $\bar{\mathbf{\Omega}}$ are compatibles, as $\mathbf{\Omega} = \bar{\mathbf{\Omega}}\mathbf{C}$, then $\mathbf{w}_1 \times \mathbf{w}_2 = det(\mathbf{C})\bar{\mathbf{w}}_1 \times \bar{\mathbf{w}}_2$ and thus $det(\mathbf{C}) = (\mathbf{w}_1 \times \mathbf{w}_2 \cdot \bar{\mathbf{w}}_1 \times \bar{\mathbf{w}}_2)|\bar{\mathbf{w}}_1 \times \bar{\mathbf{w}}_2|^{-2} > 0$, therefore K_Ω and H_Ω have the same sign of $K_{\bar{\Omega}}$ and $H_{\bar{\Omega}}$. □

Proposition 3.3.3. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal, $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ tangent moving bases of $\mathbf{x}$. Then, $K_{\Omega_2} = det(\mathbf{C})K_{\Omega_1}$ and $H_{\Omega_2} = \pm det(\mathbf{C})H_{\Omega_1}$ (+ if $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are compatibles, − if these are not compatible), where $\mathbf{C}$ is the non-singular matrix-valued map satisfying $\mathbf{\Omega}_1 = \mathbf{\Omega}_2\mathbf{C}$.

Proof. Reasoning like in the proof of the above proposition and observing that the normal vector fields $\mathbf{n}_1$ and $\mathbf{n}_2$ induced by $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ respectively are equal if these tangent moving bases are compatible and opposite if they are not, from these results that $\boldsymbol{\mu}_{\Omega_2} = \pm\boldsymbol{\mu}_{\Omega_2}\mathbf{C}^T$ (+ if $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are compatibles, − if these are not compatible) and we can get the result. □

Remark 3.3.2. Propositions 3.3.2 and 3.3.3 are valid also for non-proper frontals, but just if we consider tangent moving bases generating the same vector space at every point. It means, each pair $\mathbf{\Omega}_1$, $\mathbf{\Omega}_2$ of tangent moving bases of $\mathbf{x}$ is related by $\mathbf{\Omega}_1 = \mathbf{\Omega}_2\mathbf{C}$, where $\mathbf{C}$ is a non-singular matrix-valued map.

If we have a frontal $\mathbf{x} : U \to \mathbb{R}^3$ with a tangent moving basis $\mathbf{\Omega}$ and we compose $\mathbf{x}$ with a diffeomorphism $\mathbf{h} : V \to U$, this composition results a frontal ($D(\mathbf{x} \circ \mathbf{h}) = (\mathbf{\Omega} \circ \mathbf{h})(\mathbf{\Lambda} \circ \mathbf{h})^T D\mathbf{h}$) with $\mathbf{\Omega} \circ \mathbf{h}$ being a tangent moving basis of $\mathbf{x} \circ \mathbf{h}$. Similarly, if we compose $\mathbf{x}$ with a diffeomorphism $\mathbf{k} : W \to Z$, $\mathbf{x}(U) \subset W$, where W, Z are open sets of $\mathbb{R}^3$, this composition results a frontal ($D(\mathbf{k} \circ \mathbf{x}) = D\mathbf{k}(\mathbf{x})\mathbf{\Omega}\mathbf{\Lambda}^T$) with $D\mathbf{k}(\mathbf{x})\mathbf{\Omega}$ being a tangent moving basis of $\mathbf{x} \circ \mathbf{h}$. Also, it is not difficult to see that if we have a front $\mathbf{x} : U \to \mathbb{R}^3$, then $\mathbf{x} \circ \mathbf{h}$ and $\boldsymbol{\phi} \circ \mathbf{x}$ are fronts when $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ and $\mathbf{h} : V \to U$ is a diffeomorphism.

Proposition 3.3.4. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal, $\mathbf{h} : V \to U$ a diffeomorphism, $\boldsymbol{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ an isometry of $\mathbb{R}^3$, $\bar{\mathbf{x}} := \boldsymbol{\phi} \circ \mathbf{x} \circ \mathbf{h}$ the composite frontal, $\mathbf{\Omega}$ and $\bar{\mathbf{\Omega}}$ tangent moving bases of $\mathbf{x}$ and $\bar{\mathbf{x}}$ respectively. If K_Ω, H_Ω are the relative curvatures for $\mathbf{x}$ and $\bar{K}_{\bar{\Omega}}$, $\bar{H}_{\bar{\Omega}}$ are the relative curvatures for $\bar{\mathbf{x}}$, then

- $K_\Omega(\mathbf{h}(x,y)) = 0$ if and only if $\bar{K}_{\bar{\Omega}}(x,y) = 0$.
- $H_\Omega(\mathbf{h}(x,y)) = 0$ if and only if $\bar{H}_{\bar{\Omega}}(x,y) = 0$.

Proof. It is sufficient to prove the cases in which $\boldsymbol{\phi}$ and $\mathbf{h}$ are the identities respectively. If $\boldsymbol{\phi}$ is the identity, $\bar{\mathbf{x}} = \mathbf{x} \circ \mathbf{h}$, thus for the first item, as $\hat{\mathbf{\Omega}} := \mathbf{\Omega}(\mathbf{h})$ is a tangent moving basis of $\bar{\mathbf{x}}(x,y)$, by proposition (3.3.2) $\bar{K}_{\hat{\Omega}}(x,y) = 0$ if and only if $\bar{K}_{\bar{\Omega}}(x,y) = 0$, but observe that $\hat{\mathbf{n}} = \mathbf{n} \circ \mathbf{h}$, then $\hat{\boldsymbol{\mu}} = -\mathbf{II}_{\hat{\Omega}}^T \mathbf{I}_{\hat{\Omega}}^{-1} = -D\hat{\mathbf{n}}^T \hat{\mathbf{\Omega}} \mathbf{I}_{\hat{\Omega}}^{-1} = -D\mathbf{h}^T D\mathbf{n}^T(\mathbf{h}) \hat{\mathbf{\Omega}} \mathbf{I}_{\hat{\Omega}}^{-1} = -D\mathbf{h}^T \mathbf{II}_\Omega^T(\mathbf{h}) \mathbf{I}_\Omega^{-1}(\mathbf{h}) = D\mathbf{h}^T \boldsymbol{\mu}(\mathbf{h})$, therefore $\bar{K}_{\hat{\Omega}}(x,y) = det(D\mathbf{h}) K_\Omega(\mathbf{h}(x,y))$ which proves the item. On the other hand $\hat{\mathbf{\Lambda}} = D\bar{\mathbf{x}}^T \hat{\mathbf{\Omega}} (\mathbf{I}_{\hat{\Omega}})^{-1} = D\mathbf{h}^T \mathbf{\Lambda}(\mathbf{h})$, thus

$$\bar{H}_{\hat{\Omega}} = -\frac{1}{2} tr(\hat{\boldsymbol{\mu}} adj(\hat{\mathbf{\Lambda}})) = -\frac{1}{2} tr(adj(\hat{\mathbf{\Lambda}})\hat{\boldsymbol{\mu}}) = -det(D\mathbf{h}) \frac{1}{2} tr(\boldsymbol{\mu}(\mathbf{h}) adj(\mathbf{\Lambda}(\mathbf{h}))) = det(D\mathbf{h}) H_\Omega(\mathbf{h})$$

and therefore $H_\Omega(\mathbf{h}(x,y)) = 0$ if and only if $\bar{H}_{\hat{\Omega}}(x,y) = 0$. By proposition 3.3.2 it follows the second item. In the last case $\mathbf{h}$ is the identity, $\bar{\mathbf{x}} = \boldsymbol{\phi} \circ \mathbf{x}$. As $\boldsymbol{\phi}$ is an isometry, then we can write it in this form $\boldsymbol{\phi}(\mathbf{p}) = \mathbf{O}\mathbf{p} + \mathbf{a}$, where $\mathbf{O} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is an orthogonal matrix and $\mathbf{a} \in \mathbb{R}^3$ is a fixed vector. Thus, $\hat{\mathbf{\Omega}} := \mathbf{O}\mathbf{\Omega}$ is a tangent moving basis of $\bar{\mathbf{x}}$ and $\hat{\mathbf{n}} = \pm\mathbf{O}\mathbf{n}$ ($+$ if $det(\mathbf{O}) = 1$ and $-$ if $det(\mathbf{O}) = -1$), then $\mathbf{II}_{\hat{\Omega}} = \pm(-\mathbf{\Omega}^T \mathbf{O}^T \mathbf{O} D\mathbf{n}) = \pm\mathbf{II}_\Omega$, $\mathbf{I}_{\hat{\Omega}} = \mathbf{\Omega}^T \mathbf{O}^T \mathbf{O} \mathbf{\Omega} = \mathbf{I}_\Omega$ and $\hat{\mathbf{\Lambda}} = \mathbf{\Lambda}$. Therefore, $\hat{\boldsymbol{\mu}} = \pm\boldsymbol{\mu}$ which implies $K_\Omega = K_{\hat{\Omega}}$ and $H_\Omega = \pm H_{\hat{\Omega}}$. By proposition 3.3.2 it follows both items. □

Proposition 3.3.5. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{h} : V \to U$ a diffeomorphism and $\boldsymbol{\phi}(\mathbf{p}) = \mathbf{O}\mathbf{p} + \mathbf{a}$, where $\mathbf{O} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is an orthogonal matrix and $\mathbf{a} \in \mathbb{R}^3$ is a fixed vector. Denoting $\mathbf{x}_1 = \mathbf{x} \circ \mathbf{h}$, $\mathbf{\Omega}_1 = \mathbf{\Omega} \circ \mathbf{h}$ (tmb of $\mathbf{x}_1$), $\mathbf{x}_2 = \boldsymbol{\phi} \circ \mathbf{x}$ and $\mathbf{\Omega}_2 = \mathbf{O}\mathbf{\Omega}$ (tmb of $\mathbf{x}_2$), then

- $K^1_{\Omega_1} = det(\mathbf{h})K_\Omega \circ \mathbf{h}$ and $H^1_{\Omega_1} = det(\mathbf{h})H_\Omega \circ \mathbf{h}$,
- $K^2_{\Omega_2} = K_\Omega$ and $H^2_{\Omega_2} = \pm H_\Omega$ ($+$ if $det(\mathbf{O}) = 1$ and $-$ if $det(\mathbf{O}) = -1$),

where $K^1_{\Omega_1}$, $H^1_{\Omega_1}$ and $K^2_{\Omega_2}$, $H^2_{\Omega_2}$ are the relative curvatures of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.

Proof. It is contained in the proof of proposition 3.3.4. □

Proposition 3.3.6. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal and $\boldsymbol{\Omega}$ a tangent moving basis of $\mathbf{x}$, then $\mathbf{x}$ is a front if and only if,

$$\begin{pmatrix} \boldsymbol{\Lambda}^T \\ \boldsymbol{\mu}^T \end{pmatrix} \tag{3.13}$$

has a 2×2 minor different of zero, for each $\mathbf{p} \in \Sigma(\mathbf{x})$.

Proof. Let $\mathbf{n}$ be the normal vector field along $\mathbf{x}$. By definition of front and using that $D\mathbf{n} = \boldsymbol{\Omega}\boldsymbol{\mu}^T$ (see equation (2.6)), $\mathbf{x}$ is a front if and only if,

$$2 = rank(\begin{pmatrix} D\mathbf{x} \\ D\mathbf{n} \end{pmatrix}) = rank(\begin{pmatrix} \boldsymbol{\Omega}\boldsymbol{\Lambda}^T \\ \boldsymbol{\Omega}\boldsymbol{\mu}^T \end{pmatrix}) = rank(\begin{pmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Lambda}^T \\ \boldsymbol{\mu}^T \end{pmatrix}) = rank(\begin{pmatrix} \boldsymbol{\Lambda}^T \\ \boldsymbol{\mu}^T \end{pmatrix})$$

which is equivalent to have a 2×2 minor of the matrix (3.13) different of zero. □

The propositions 3.3.2 and 3.3.4 now allow us to explore in which point any of K_Ω and H_Ω turns zero making change of coordinates, applying isometries of $\mathbb{R}^3$ and switching tangent moving bases. In the following theorem the necessary condition of the first item was proved in ((MARTINS; SAJI; TERAMOTO, 2019),Proposition 2.4) considering a C^∞ extension of λH for fronts with singular set having empty interior. We are going to prove also the reciprocal and the case in which the singularity has rank 0, using the relative curvatures.

Theorem 3.3.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\boldsymbol{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$. Then,

- $\mathbf{x} : U \to \mathbb{R}^3$ is a front on a neighborhood V of $\mathbf{p}$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$ if and only if $H_\Omega(\mathbf{p}) \neq 0$.
- $\mathbf{x} : U \to \mathbb{R}^3$ is a front on a neighborhood V of $\mathbf{p}$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$ if and only if $H_\Omega(\mathbf{p}) = 0$ and $K_\Omega(\mathbf{p}) \neq 0$.

Proof. For the first item, we can apply a change of coordinates $\mathbf{h}$ and an isometry $\boldsymbol{\phi}$ of $\mathbb{R}^3$ (making the line $D\mathbf{x}(\mathbf{p})(\mathbb{R}^2)$ parallel to $(1,0,0)$) such that $\bar{\mathbf{x}} = \boldsymbol{\phi} \circ \mathbf{x} \circ \mathbf{h} = (u, b(u,v), c(u,v))$, $\mathbf{h}(0,0) = \mathbf{p}$, $b_u(0,0) = b_v(0,0) = c_u(0,0) = 0$ and having a tangent moving basis $\bar{\boldsymbol{\Omega}}$ in the form of remark 3.1.3. Thus, $D\bar{\mathbf{x}} = \bar{\mathbf{\Omega}}\bar{\mathbf{\Lambda}}^T$, $\bar{\mathbf{\Lambda}}^T = D(u,b)$, $\bar{\boldsymbol{\mu}}^T = D(-g_1 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}}, -g_2 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}})$ and $(u,b)_u \cdot (g_1,g_2)_v = (u,b)_v \cdot (g_1,g_2)_u$ (by corollary 3.1.2). Hence, $c_u = g_1 + g_2 b_u$ and $g_{1v} + b_u g_{2v} = b_v g_{2u}$ which implies that $g_1(0,0) = g_{1v}(0,0) = 0$. Since $\bar{\mathbf{x}}$ is wave front locally at $(0,0)$, by proposition 3.3.6 the matrix

$$\begin{pmatrix} D(u,b) \\ \bar{\boldsymbol{\mu}}^T \end{pmatrix}$$

has a 2×2 minor different from zero at $(0,0)$ and therefore $(-g_2 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}})_v(0,0) \neq 0$. A simple computation using the definition leads to $\bar{H}_{\bar{\Omega}}(0,0) = -\frac{1}{2}(-g_2 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}})_v(0,0) \neq 0$, hence $H_\Omega(\mathbf{p}) \neq 0$. Now, if we suppose that $H_\Omega(\mathbf{p}) \neq 0$, as $H_\Omega(\mathbf{p}) = -\frac{1}{2}(\lambda_{22}\mu_{11} - \lambda_{21}\mu_{12} + \lambda_{11}\mu_{22} - \lambda_{12}\mu_{21})(\mathbf{p})$, then $(\lambda_{12}\mu_{21} - \lambda_{22}\mu_{11})(\mathbf{p}) \neq 0$ or $(\lambda_{11}\mu_{22} - \lambda_{21}\mu_{12})(\mathbf{p}) \neq 0$, which are two 2×2 minors of (3.13) and also $\mathbf{\Lambda}(\mathbf{p}) \neq \mathbf{0}$. Thus, $rank(D\mathbf{x}(\mathbf{p})) = rank(\mathbf{\Lambda}(\mathbf{p})) = 1$ and there exists a neighborhood V of $\mathbf{p}$, where any of these two 2×2 minors is different of zero, therefore by proposition 3.3.6 $\mathbf{x}$ is a front on V. For the second item, if $\mathbf{x}$ is a front and $rank(D\mathbf{x}(\mathbf{p})) = rank(\mathbf{\Lambda}(\mathbf{p})) = 0$, then $\mathbf{\Lambda}(\mathbf{p}) = \mathbf{0}$, $H_\Omega(\mathbf{p}) = 0$ and by proposition 3.3.6 $K_\Omega(\mathbf{p}) = det(\boldsymbol{\mu}^T) \neq 0$. Now, if $K_\Omega(\mathbf{p}) \neq 0$ and $H_\Omega(\mathbf{p}) = 0$, there exist a neighborhood V of $\mathbf{p}$ where $K_\Omega \neq 0$ and by proposition 3.3.6 $\mathbf{x}$ is a front on V. By the first item, $rank(D\mathbf{x}(\mathbf{p})) \neq 1$ because $H_\Omega(\mathbf{p}) = 0$, then $rank(D\mathbf{x}(\mathbf{p})) = 0$. □

From theorem 3.3.1 follows immediately the following corollary.

Corollary 3.3.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, this is a front if and only if, $(K_\Omega, H_\Omega) \neq \mathbf{0}$ on $\Sigma(\mathbf{x})$.

Example 3.3.2. Let $\mathbf{x}: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(u,v) := (u^2, v^2, v^3 + u^3)$, this is a frontal with $rank(D\mathbf{x}(\mathbf{p})) = 0$ at $\mathbf{p} = (0,0)$ (Figure 4). We have the decomposition:

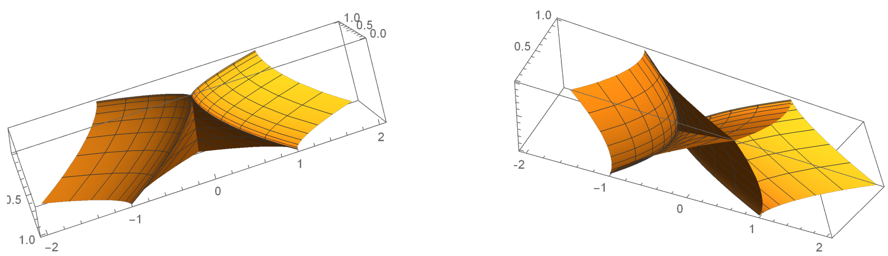

Figure 4 – A front with $rank(D\mathbf{x}(0,0)) = 0$.

$$D\mathbf{x} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 3u & 3v \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \mathbf{\Omega}\mathbf{\Lambda}^T, \text{ where } \mathbf{\Omega} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 3u & 3v \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

being $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, then we have $\mathbf{n} = (-6u, -6v, 4)\varepsilon^{-\frac{1}{2}}$, $\mathbf{w}_{1u} = (0,0,3)$, $\mathbf{w}_{1v} = (0,0,0)$, $\mathbf{w}_{2u} = (0,0,0)$ and $\mathbf{w}_{2v} = (0,0,3)$. Thus

$$\mathbf{I}_\Omega = \mathbf{\Omega}^T\mathbf{\Omega} = \begin{pmatrix} 4+9u^2 & 9uv \\ 9uv & 4+9v^2 \end{pmatrix}, \mathbf{II}_\Omega = \begin{pmatrix} \mathbf{n}\cdot\mathbf{w}_{1u} & \mathbf{n}\cdot\mathbf{w}_{1v} \\ \mathbf{n}\cdot\mathbf{w}_{2u} & \mathbf{n}\cdot\mathbf{w}_{2v} \end{pmatrix} = \begin{pmatrix} 12\varepsilon^{-\frac{1}{2}} & 0 \\ 0 & 12\varepsilon^{-\frac{1}{2}} \end{pmatrix}$$

$$\boldsymbol{\mu}_\Omega = -\mathbf{II}_\Omega^T\mathbf{I}_\Omega^{-1} = -12\varepsilon^{-\frac{3}{2}} \begin{pmatrix} 4+9u^2 & -9uv \\ -9uv & 4+9v^2 \end{pmatrix}$$

where $\varepsilon = 36u^2+36v^2+16$. Also, $K_\Omega(u,v) = 144(36u^2+36v^2+16)^{-2} \neq 0$ and $H_\Omega(0,0) = -\frac{1}{2}(\lambda_{22}\mu_{11} - \lambda_{21}\mu_{12} + \lambda_{11}\mu_{22} - \lambda_{12}\mu_{21})(0,0) = 0$, then by corollary 3.3.1, $\mathbf{x}$ is a front.

Example 3.3.3. Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(u,v) := (ue^u, v^2, (\frac{u^2}{2}+u)v^3)$, this is a frontal with $rank(D\mathbf{x}(\mathbf{p})) = 0$ at $\mathbf{p} = (-1,0)$ (Figure 5). We have the decomposition:

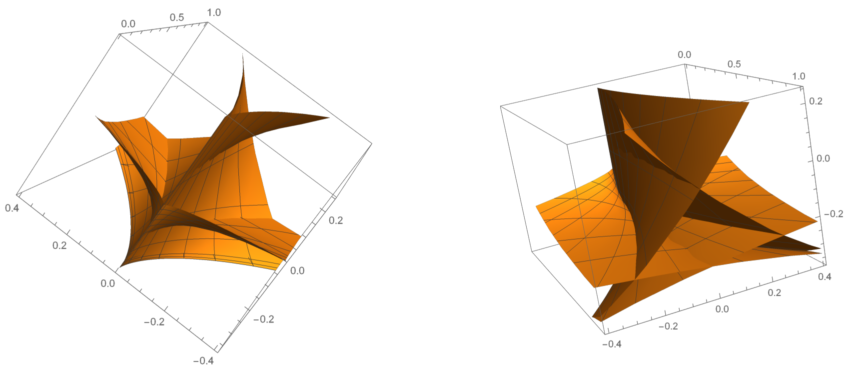

Figure 5 – A frontal with $rank(D\mathbf{x}(-1,0)) = 0$.

$$D\mathbf{x} = \begin{pmatrix} e^u & 0 \\ 0 & 2 \\ v^3 & (\frac{u^2}{2}+u)3v \end{pmatrix} \begin{pmatrix} 1+u & 0 \\ 0 & v \end{pmatrix} = \mathbf{\Omega}\mathbf{\Lambda}^T,$$

$$\text{where } \mathbf{\Omega} = \begin{pmatrix} e^u & 0 \\ 0 & 2 \\ v^3 & (\frac{u^2}{2}+u)3v \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} 1+u & 0 \\ 0 & v \end{pmatrix},$$

being $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, then we have $\mathbf{n} = (-2v^3, -e^u(\frac{u^2}{2}+u)3v, 2e^u)\delta^{-\frac{1}{2}}$, $\mathbf{w}_{1u} = (e^u,0,0)$, $\mathbf{w}_{1v} = (0,0,3v^2)$, $\mathbf{w}_{2u} = (0,0,(u+1)3v)$ and $\mathbf{w}_{2v} = (0,0,3(\frac{u^2}{2}+u))$. Thus

$$\mathbf{I}_\Omega = \begin{pmatrix} e^{2u}+v^6 & 3(\frac{u^2}{2}+u)v^4 \\ 3(\frac{u^2}{2}+u)v^4 & 4+9(\frac{u^2}{2}+u)^2v^2 \end{pmatrix}, \mathbf{II}_\Omega = \begin{pmatrix} -2v^3e^u & 6e^uv^2 \\ 6(1+u)e^uv & 6e^u(\frac{u^2}{2}+u) \end{pmatrix}\delta^{-\frac{1}{2}}$$

where $\delta = 4v^6 + e^{2u}(9(\frac{u^2}{2}+u)^2v^2+4)$. Also, $K_\Omega(-1,0) = 0$ and $H_\Omega(-1,0) = 0$, then by corollary 3.3.1, $\mathbf{x}$ is not a front at $\mathbf{p} = (-1,0)$.

3.4 Representation formulas of Wavefronts

In this section we obtain formulas to construct all the local parametrizations of wavefronts on a neighborhood of singularities of rank 0 and 1. These formulas are in terms of some functions as parameters and they can be freely chosen. They are very useful to give examples and counterexamples with desired characteristics.

Proposition 3.4.1 (Formula for rank 1)**.** Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a germ of a wavefront, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $0 \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(0)) = 1$. Then, up to an isometry $\mathbf{x}$ is $\mathscr{R}$-equivalent to $\mathbf{y}(w,z) = (w, \int_0^z \lambda_{\hat{\Omega}}(w,t)dt + f_1(w), \int_0^z t\lambda_{\hat{\Omega}}(w,t)dt + f_2(w))$ which has as tangent moving basis

$$\hat{\mathbf{\Omega}} = \begin{pmatrix} & 0 \\ \mathbf{y}_w & 1 \\ & z \end{pmatrix}, \mathbf{\Lambda}_{\hat{\Omega}} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_{\hat{\Omega}} \end{pmatrix}$$

where $\lambda_{\hat{\Omega}}(w,z)$, $f_1(w)$, $f_2(w)$ are smooth functions with $\lambda_{\hat{\Omega}}(0) = 0$. In particular, $\mathbf{x}$ is $\mathscr{A}$-equivalent to $(w, \int_0^z \lambda_{\hat{\Omega}}(w,t)dt, \int_0^z t\lambda_{\hat{\Omega}}(w,t)dt)$.

Proof. We can apply a change of coordinates $\mathbf{h_1}$ and an isometry $\boldsymbol{\phi}$ of $\mathbb{R}^3$ (making the line $D\mathbf{x}(0)(\mathbb{R}^2) \subset \mathbf{\Omega}(0)(\mathbb{R}^2)$ parallel to $(1,0,0)$ and the plane $\mathbf{\Omega}(0)(\mathbb{R}^2)$ coincide with $\mathbb{R}^2 \times 0$) such that $\bar{\mathbf{x}} = \boldsymbol{\phi} \circ \mathbf{x} \circ \mathbf{h_1} = (u, b(u,v), c(u,v))$, $b_u(0,0) = b_v(0,0) = 0$ and having a tangent moving basis $\bar{\mathbf{\Omega}}$ in the form:

$$\bar{\mathbf{\Omega}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & g_2 \end{pmatrix}$$

with $g_1(0) = g_2(0) = 0$. Thus, $D\bar{\mathbf{x}} = \bar{\mathbf{\Omega}}\bar{\mathbf{\Lambda}}^T$, $\bar{\mathbf{\Lambda}}^T = D(u,b)$, $\bar{\boldsymbol{\mu}}^T = D(-g_1 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}}, -g_2 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}})$. Since $\bar{\mathbf{x}}$ is wave front locally at $(0,0)$, by theorem 3.3.1 $\bar{H}_{\bar{\Omega}}(0,0) = -\frac{1}{2}(-g_2 det(\mathbf{I}_{\bar{\Omega}})^{-\frac{1}{2}})_v(0,0) \neq 0$, hence $g_{2v} \neq 0$. Then, by the local form of the submersion, there exist a diffeomorphism with the form $\mathbf{h_2}(w,z) = (w, l(w,z))$ such that $g_2 \circ \mathbf{h_2} = z$, therefore setting $\mathbf{y}(w,z) := \bar{\mathbf{x}} \circ \mathbf{h_2}(w,z) = (w, \tilde{b}(w,z), \tilde{c}(w,z))$, $\tilde{\mathbf{\Omega}} := \bar{\mathbf{\Omega}}(\mathbf{h_2})$ and $\tilde{g_1} = g_1 \circ \mathbf{h_2}$ we have

$$D\mathbf{y} = \bar{\mathbf{\Omega}}(\mathbf{h_2})\bar{\mathbf{\Lambda}}^T(\mathbf{h_2})D\mathbf{h_2} = \bar{\mathbf{\Omega}}(\mathbf{h_2})D(u,b)(\mathbf{h_2})D\mathbf{h_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \tilde{g_1} & z \end{pmatrix} D(w,\tilde{b})$$

and thus $\tilde{c}_z = z\tilde{b}_z$, $\lambda_{\tilde{\Omega}} = \tilde{b}_z$. Integrating we get $\tilde{c} = \int_0^z t\lambda_{\tilde{\Omega}}(w,t)dt + \tilde{c}(w,0)$, $\tilde{b} = \int_0^z \lambda_{\tilde{\Omega}}(w,t)dt + \tilde{b}(w,0)$. Observe that the tangent moving basis $\hat{\mathbf{\Omega}}$ and $\mathbf{\Lambda}_{\hat{\Omega}}$ given in the statement of the proposition give a decomposition of this last $\mathbf{y}$ in the proof, $\lambda_{\tilde{\Omega}} = \tilde{b}_z = \lambda_{\hat{\Omega}}$ and from this follows the result. □

Remark 3.4.1. The formula in proposition 3.4.1 can be rewritten in the form

$$(u, b(u,v), \int_0^v tb_v(u,t)dt + f_2(u)),$$

where b is a smooth function and $b_v = \lambda_{\hat{\Omega}}$.

Example 3.4.1 (Arbitrary singular set)**.** Let $C \subset \mathbb{R}^2$ be a closed set, by theorem (2.29,(LEE, 2013)) there exist a smooth nonnegative function $\lambda_\Omega : \mathbb{R}^2 \to \mathbb{R}$ such that $\lambda_\Omega^{-1}(0) = C$, then $\mathbf{y} = (w, \int_0^z \lambda_\Omega(w,t)dt, \int_0^z t\lambda_\Omega(w,t)dt)$ is a wavefront with singular set $\Sigma(\mathbf{y}) = C$.

Example 3.4.2 (Wavefronts with vanishing Gaussian curvature)**.** Because $K_\Omega(\mathbf{p}) \neq 0$ on singularities $\mathbf{p}$ of rank 0 and $\lim\limits_{(u,v)\to p} |K| = \frac{|K_\Omega|}{|\lambda_\Omega|} = \infty$, then a wavefront with vanishing Gaussian curvature $\mathbf{x}$ only has singularities of rank 1. Without loss of generality, let us suppose $(0,0)$ is a singularity, thus up to an isometry this is $\mathscr{R}$-equivalent to the formula in remark 3.4.1 at $(0,0)$. Then taking the tangent moving basis in proposition 3.4.1, since $L_\Omega N_\Omega - M_{1\Omega} M_{2\Omega} = 0$, a simple computation leads to $-vb_{uu} + \int_0^v tb_{uuv}(u,t)dt + f_{2uu}(u) = 0$. Therefore $f_{2uu}(u) = 0$ and taking derivative in v we get $b_{uu} = 0$. We conclude that $\mathbf{x}$, up to an isometry is $\mathscr{R}$-equivalent to

$$(u, ur_1(v) + r_2(v), \int_0^v tur_1'(t) + r_2'(t)dt + uc_1 + c_2),$$

where r_1, r_2 are smooth functions with $r_2'(0) = 0$ and c_1, c_2 constants. In particular $\mathbf{x}$ is a ruled surface locally at $(0,0)$ with a directrix curve $(0, r_2(v), r_2(v) + c_2)$ having a singularity at $v = 0$.

Proposition 3.4.2. Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a germ of a wavefront, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $0 \in \Sigma(\mathbf{x})$ with $K_\Omega(0) \neq 0$. Then, up to an isometry $\mathbf{x}$ is $\mathscr{R}$-equivalent to $\mathbf{y} = (a, b, \int_0^u t_1 a_u(t_1,v) + vb_u(t_1,v)dt_1 + \int_0^v t_2 b_v(0,t_2)dt_2)$, where a, b are smooth functions and $a_v = b_u$. In particular, $\mathbf{y} = (a, \int_0^u a_v(t,v)dt + f_1(v), \int_0^u ta_u(t,v) + va_v(t,v)dt + \int_0^v tf_{1v}(t)dt)$, where $f_1(v)$ is a smooth function.

Proof. Applying an isometry we always can choose a tangent moving basis of $\mathbf{x} = (a,b,c)$ in the form

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & g_2 \end{pmatrix}, \mathbf{\Lambda}_\Omega^T = D(a,b).$$

We have that $K_\Omega(0) \neq 0$ if and only if $det(\mathbf{II}_\Omega(0)) \neq 0$ and by corollary 3.1.2 this is equivalent to have $det(D(g_1,g_2) \neq 0)$, therefore by the Inverse function theorem there exist a diffeomorphism $\mathbf{h}(w,z)$ such that $(g_1,g_2) \circ \mathbf{h} = (w,z)$. Setting $\mathbf{y} := \mathbf{x} \circ \mathbf{h} = (\hat{a}, \hat{b}, \hat{c})$ we have

$$D\mathbf{y} = \mathbf{\Omega}(\mathbf{h})\mathbf{\Lambda}_\Omega^T(\mathbf{h})D\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ w & z \end{pmatrix} D(\hat{a}, \hat{b}),$$

thus by corollary 3.1.2 $(\hat{a},\hat{b})_w \cdot (w,z)_z = (\hat{a},\hat{b})_z \cdot (w,z)_w$, it means $\hat{b}_w = \hat{a}_z$. Also, $\hat{c}_w = w\hat{a}_w + z\hat{b}_w$ and $\hat{c}_z = w\hat{a}_z + z\hat{b}_z$. Then, $\hat{c} = \int_0^w t_1 \hat{a}_w(t_1,z) + z\hat{a}_z(t_1,z)dt_1 + \hat{c}(0,z)$, but $\hat{c}(0,z) =$

$\int_0^z t_2 \hat{b}_z(0,t_2)dt_2$, thus we get the first part. For the last part just define $f_1(z) := \hat{b}(0,z)$, observe that $\hat{b} = \int_0^w \hat{a}_z(t,z)dt + f_1(z)$ and substitute these in the last formula. □

Corollary 3.4.1 (Formula for rank 0)**.** Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a germ of a wavefront, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $0 \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(0)) = 0$. Then, up to an isometry $\mathbf{x}$ is $\mathscr{R}$-equivalent to $\mathbf{y} = (a, b, \int_0^u t_1 a_u(t_1,v) + v b_u(t_1,v)dt_1 + \int_0^v t_2 b_v(0,t_2)dt_2)$, where a, b are smooth functions, $a_v = b_u$ and $D(a,b)(0) = 0$. In particular, $\mathbf{y} = (a, \int_0^u a_v(t,v)dt + f_1(v), \int_0^u t a_u(t,v) + v a_v(t,v)dt + \int_0^v t f_{1v}(t)dt)$, where $f_1(v)$ is a smooth function with $a_u(0) = a_v(0) = f_{1v}(0) = 0$.

Proof. By theorem 3.3.1 $K_\Omega(0) \neq 0$ and applying the proposition 3.4.2 we get the result. □

Corollary 3.4.2 (Local form for general rank)**.** Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a germ of a wavefront, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $0 \in \Sigma(\mathbf{x})$. Then, $\mathbf{x}$ is $\mathscr{A}$-equivalent to $\mathbf{y} = (a, b, \int_0^u t_1 a_u(t_1,v) + v b_u(t_1,v)dt_1 + \int_0^v t_2 b_v(0,t_2)dt_2)$, where a, b are smooth functions and $a_v = b_u$.

Proof. The case $rank(D\mathbf{x}(0)) = 0$ is the last colollary. If $rank(D\mathbf{x}(0)) = 1$ by proposition 3.4.1 $\mathbf{x}$ is $\mathscr{A}$-equivalent to $(w, \int_0^z \lambda_{\hat{\Omega}}(w,t)dt, \int_0^z t\lambda_{\hat{\Omega}}(w,t)dt)$ which is $\mathscr{A}$-equivalent to $(w, \int_0^z \lambda_{\hat{\Omega}}(w,t)dt, \int_0^z t\lambda_{\hat{\Omega}}(w,t)dt + w^2)$. By a simple computation for this last wavefront $K_{\hat{\Omega}}(0) \neq 0$ and applying proposition 3.4.2 we get the result. □

CHAPTER

4

THE FUNDAMENTAL THEOREM

In classical differential geometry, the fundamental theorem of regular surfaces (see(CARMO, 1976; STOKER, 1969)) states that if we have E,F,G,L,M,N smooth functions defined in an open set $U \subset \mathbb{R}^2$, with $E>0$, $G>0$, $EG-F^2>0$ and the given functions satisfy formally the Gauss and Mainardi-Codazzi equations, then for each $\mathbf{p} \in U$ there exist a neighborhood $V \subset U$ of p and a diffeomorphism $\mathbf{x}: V \to \mathbf{x}(V) \subset \mathbb{R}^3$ such that the regular surface $\mathbf{x}(U)$ has E,F,G and L,M,N as coefficients of the first and second fundamental forms, respectively. Furthermore, if U is connected and if $\bar{\mathbf{x}}: U \to \bar{\mathbf{x}}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exist a translation $\mathbf{T}$ and a proper linear orthogonal transformation $\boldsymbol{\rho}$ in $\mathbb{R}^3$ such that $\bar{\mathbf{x}} = \mathbf{T} \circ \boldsymbol{\rho} \circ \mathbf{x}$.
Gauss equation:

$$\Gamma^2_{12u} - \Gamma^2_{11v} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -EK$$

Mainardi-Codazzi equations:

$$\begin{aligned} L_v - M_u &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \\ M_v - N_u &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12} \end{aligned}$$

where K is the Gaussian Curvature and Γ^j_{ik} are the Christoffel symbols.

This theorem realizes the first and the second fundamental forms compatible as a regular surface in the euclidean 3-space. In (KOSSOWSKI, 2004) M. Kossowski gave sufficient conditions for a singular first fundamental form (metrics admitting only non-degenerate singularities) to be realized as a wave front. Saji, Umehara and Yamada in (SAJI; UMEHARA; YAMADA, 2011) also consider this question of the realization of frontals and they give a theorem in terms of “coherent tangent bundles” a new concept they introduced. They show that a coherent tangent bundle induces compatible first and second fundamental forms. In addition in (HASEGAWA *et al.*, 2015), M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada proved that a Kossowski metrics induces uniquely a coherent tangent bundle. However the set of metrics associated to frontals is bigger than Kossowski metrics and it was not shown what properties satisfy explicitly these metrics and how they induce a coherent tangent bundle. In theorem 3.1.1 we describe what are the properties that every metric of a frontal satisfies, which allow us to realize all the proper frontals. In this chapter, we present our main result theorem 4.4.1 in terms of the classical fundamental forms satisfying the properties of decomposition in their associated matrices. This result generalizes the fundamental theorem of regular surfaces mentioned before including now all the proper frontals, with the possibility to distinguish wave fronts from its fundamental forms.

4.1 The Relative Compatibility Equations

There are two groups of equations, which are present in all frontals and guarantee the integrability conditions for the system of partial differential equations that we consider in theorem 4.4.1. In this section we show one group of these and we prove that they are equivalent essentially to 3 equations that curiously seem very similar to the Gauss equation and Mainardi-Codazzi equations. The other group is presented in the next section.

Let $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix} : U \to \mathcal{M}_{3\times 2}(\mathbb{R})$ be a moving base and $\mathbf{n} = \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}$. We have that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{n}$ is a base of $\mathbb{R}^3$, then there are real functions (p_{ij}) and (q_{ij}) defined in U, $i, j \in \{1,2,3\}$ such that:

$$
\begin{aligned}
\mathbf{w}_{1u} &= p_{11}\mathbf{w}_1 + p_{12}\mathbf{w}_2 + p_{13}\mathbf{n} \\
\mathbf{w}_{2u} &= p_{21}\mathbf{w}_1 + p_{22}\mathbf{w}_2 + p_{23}\mathbf{n} \\
\mathbf{n}_u &= p_{31}\mathbf{w}_1 + p_{32}\mathbf{w}_2 + p_{33}\mathbf{n} \\
\mathbf{w}_{1v} &= q_{11}\mathbf{w}_1 + q_{12}\mathbf{w}_2 + q_{13}\mathbf{n} \\
\mathbf{w}_{2v} &= q_{21}\mathbf{w}_1 + q_{22}\mathbf{w}_2 + q_{23}\mathbf{n} \\
\mathbf{n}_v &= q_{31}\mathbf{w}_1 + q_{32}\mathbf{w}_2 + q_{33}\mathbf{n}
\end{aligned}
$$

If we set the matrix $\mathbb{W} := \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{n} \end{pmatrix} \in GL(3)$ whose columns are $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{n}$. Also, denoting by $\mathbb{P} := (p_{ij})$ and $\mathbb{Q} := (q_{ij})$, we have:

$$\mathbb{W}_u = \mathbb{W}\mathbb{P}^T \tag{4.3a}$$

$$\mathbb{W}_v = \mathbb{W}\mathbb{Q}^T \tag{4.3b}$$

which is equivalent to:

$$\mathbb{W}_u^T = \mathbb{P}\mathbb{W}^T \tag{4.4a}$$

$$\mathbb{W}_v^T = \mathbb{Q}\mathbb{W}^T \tag{4.4b}$$

then, we have that $\mathbb{P} = \mathbb{W}_u^T(\mathbb{W}^T)^{-1} = \mathbb{W}_u^T\mathbb{W}\mathbb{W}^{-1}(\mathbb{W}^T)^{-1} = \mathbb{W}_u^T\mathbb{W}(\mathbb{W}^T\mathbb{W})^{-1}$ and $\mathbb{Q} = \mathbb{W}_v^T(\mathbb{W}^T)^{-1} = \mathbb{W}_v^T\mathbb{W}\mathbb{W}^{-1}(\mathbb{W}^T)^{-1} = \mathbb{W}_v^T\mathbb{W}(\mathbb{W}^T\mathbb{W})^{-1}$. Considering $\mathbb{W} = \begin{pmatrix}\boldsymbol{\Omega} & \mathbf{n}\end{pmatrix}$ as a block matrix, we have:

$$
\begin{aligned}
\mathbb{P} = \mathbb{W}_u^T\mathbb{W}(\mathbb{W}^T\mathbb{W})^{-1} &= \begin{pmatrix}\boldsymbol{\Omega}_u^T \\ \mathbf{n}_u^T\end{pmatrix}\begin{pmatrix}\boldsymbol{\Omega} & \mathbf{n}\end{pmatrix}(\begin{pmatrix}\boldsymbol{\Omega}^T \\ \mathbf{n}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{\Omega} & \mathbf{n}\end{pmatrix})^{-1} \\
&= \begin{pmatrix}\boldsymbol{\Omega}_u^T\boldsymbol{\Omega} & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\boldsymbol{\Omega} & \mathbf{n}_u^T\mathbf{n}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Omega}^T\boldsymbol{\Omega} & \boldsymbol{\Omega}^T\mathbf{n} \\ \mathbf{n}^T\boldsymbol{\Omega} & \mathbf{n}^T\mathbf{n}\end{pmatrix}^{-1} = \begin{pmatrix}\boldsymbol{\Omega}_u^T\boldsymbol{\Omega} & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\boldsymbol{\Omega} & 0\end{pmatrix}\begin{pmatrix}\mathbf{I}_\Omega & \mathbf{0} \\ \mathbf{0} & 1\end{pmatrix}^{-1} \\
&= \begin{pmatrix}\boldsymbol{\Omega}_u^T\boldsymbol{\Omega} & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\boldsymbol{\Omega} & 0\end{pmatrix}\begin{pmatrix}\mathbf{I}_\Omega^{-1} & \mathbf{0} \\ \mathbf{0} & 1\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Omega}_u^T\boldsymbol{\Omega}\mathbf{I}_\Omega^{-1} & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\boldsymbol{\Omega}\mathbf{I}_\Omega^{-1} & 0\end{pmatrix}
\end{aligned}
$$

from (2.6), we have $\mathbf{n}_u^T = \boldsymbol{\mu}_{(1)}^T\boldsymbol{\Omega}^T$ and $\mathbf{n}_v^T = \boldsymbol{\mu}_{(2)}^T\boldsymbol{\Omega}^T$. Then,

$$
\mathbb{P} = \begin{pmatrix}\boldsymbol{\Omega}_u^T\boldsymbol{\Omega}\mathbf{I}_\Omega^{-1} & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\boldsymbol{\Omega}\mathbf{I}_\Omega^{-1} & 0\end{pmatrix} = \begin{pmatrix}\mathcal{T}_1 & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \boldsymbol{\mu}_{(1)}^T\boldsymbol{\Omega}^T\boldsymbol{\Omega}\mathbf{I}_\Omega^{-1} & 0\end{pmatrix} = \begin{pmatrix}\mathcal{T}_1 & \boldsymbol{\Omega}_u^T\mathbf{n} \\ \boldsymbol{\mu}_{(1)}^T & 0\end{pmatrix}
$$

Finally, using (2.4) and by analogy with the same procedure for $\mathbb{Q}$, we get:

$$
\mathbb{P} = \begin{pmatrix}\mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 & L_\Omega \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^2 & M_{2\Omega} \\ \mu_{11} & \mu_{12} & 0\end{pmatrix} \tag{4.5}
$$

$$
\mathbb{Q} = \begin{pmatrix}\mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 & M_{1\Omega} \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^2 & N_\Omega \\ \mu_{21} & \mu_{22} & 0\end{pmatrix} \tag{4.6}
$$

now, as $\mathbb{W}_{uv}^T = \mathbb{W}_{vu}^T$, then $\mathbb{P}_v\mathbb{W}^T + \mathbb{P}\mathbb{W}_v^T = \mathbb{Q}_u\mathbb{W}^T + \mathbb{Q}\mathbb{W}_u^T$. Using (4.4a) and (4.4b) in the last equality, $\mathbb{P}_v\mathbb{W}^T + \mathbb{P}\mathbb{Q}\mathbb{W}^T = \mathbb{Q}_u\mathbb{W}^T + \mathbb{Q}\mathbb{P}\mathbb{W}^T$, then $(\mathbb{P}_v - \mathbb{Q}_u + \mathbb{P}\mathbb{Q} - \mathbb{Q}\mathbb{P})\mathbb{W}^T = 0$ and finally we get:

$$
\mathbb{P}_v - \mathbb{Q}_u + [\mathbb{P}, \mathbb{Q}] = 0 \tag{4.7}
$$

which is the compatibility condition of the system (4.4) by corollary 2.3.1.

Using (4.5) and (4.6) to compute each component (i, j) of (4.7) we obtain the following equations that we call the $\mathbf{\Omega}$*-relative compatibility equations (RCE)*:

$$
\begin{aligned}
&(1,1) \quad (\mathcal{T}^1_{11})_v - (\mathcal{T}^1_{12})_u = \mathcal{T}^1_{12}\mathcal{T}^1_{11} - \mathcal{T}^1_{11}\mathcal{T}^1_{12} + \mathcal{T}^2_{12}\mathcal{T}^1_{21} - \mathcal{T}^1_{22}\mathcal{T}^2_{11} + \mu_{11}M_{1\Omega} - \mu_{21}L_\Omega && (4.8a)\\
&(1,2) \quad (\mathcal{T}^2_{11})_v - (\mathcal{T}^2_{12})_u = \mathcal{T}^1_{12}\mathcal{T}^2_{11} + \mathcal{T}^2_{12}\mathcal{T}^2_{21} - \mathcal{T}^1_{11}\mathcal{T}^2_{12} - \mathcal{T}^2_{11}\mathcal{T}^2_{22} + \mu_{12}M_{1\Omega} - \mu_{22}L_\Omega && (4.8b)\\
&(2,1) \quad (\mathcal{T}^1_{21})_v - (\mathcal{T}^1_{22})_u = \mathcal{T}^1_{22}\mathcal{T}^1_{11} + \mathcal{T}^2_{22}\mathcal{T}^1_{21} - \mathcal{T}^1_{21}\mathcal{T}^1_{12} - \mathcal{T}^2_{21}\mathcal{T}^1_{22} + \mu_{11}N_\Omega - \mu_{21}M_{2\Omega} && (4.8c)\\
&(2,2) \quad (\mathcal{T}^2_{21})_v - (\mathcal{T}^2_{22})_u = \mathcal{T}^2_{22}\mathcal{T}^2_{21} - \mathcal{T}^2_{21}\mathcal{T}^2_{22} + \mathcal{T}^2_{11}\mathcal{T}^1_{22} - \mathcal{T}^2_{12}\mathcal{T}^1_{21} + \mu_{12}N_\Omega - \mu_{22}M_{2\Omega} && (4.8d)\\
&(1,3) \quad \mu_{11v} - \mu_{21u} = \mathcal{T}^1_{11}\mu_{21} + \mathcal{T}^1_{21}\mu_{22} - \mathcal{T}^1_{12}\mu_{11} - \mathcal{T}^1_{22}\mu_{12} && (4.8e)\\
&(2,3) \quad \mu_{12v} - \mu_{22u} = \mathcal{T}^2_{11}\mu_{21} + \mathcal{T}^2_{21}\mu_{22} - \mathcal{T}^2_{12}\mu_{11} - \mathcal{T}^2_{22}\mu_{12} && (4.8f)\\
&(3,1) \quad (L_\Omega)_v - (M_{1\Omega})_u = L_\Omega\mathcal{T}^1_{12} + M_{2\Omega}\mathcal{T}^2_{12} - M_{1\Omega}\mathcal{T}^1_{11} - N_\Omega\mathcal{T}^2_{11} && (4.8g)\\
&(3,2) \quad (M_{2\Omega})_v - (N_\Omega)_u = L_\Omega\mathcal{T}^1_{22} + M_{2\Omega}\mathcal{T}^2_{22} - M_{1\Omega}\mathcal{T}^1_{21} - N_\Omega\mathcal{T}^2_{21} && (4.8h)\\
&(3,3) \quad L_\Omega\mu_{21} + M_{2\Omega}\mu_{22} - M_{1\Omega}\mu_{11} - N_\Omega\mu_{12} = 0 && (4.8i)
\end{aligned}
$$

Using that the Ω-relative curvature $K_\Omega = det(\boldsymbol{\mu}) = \frac{det(\mathbf{II}_\Omega)}{det(\mathbf{I}_\Omega)} = \frac{L_\Omega N_\Omega - M_{1\Omega}M_{2\Omega}}{E_\Omega G_\Omega - F^2_\Omega}$ and $\boldsymbol{\mu} = -\mathbf{II}^T_\Omega \mathbf{I}^{-1}_\Omega$ in (4.8b) we get:

$$(\mathcal{T}^2_{12})_u - (\mathcal{T}^2_{11})_v + \mathcal{T}^1_{12}\mathcal{T}^2_{11} + \mathcal{T}^2_{12}\mathcal{T}^2_{21} - \mathcal{T}^2_{11}\mathcal{T}^2_{22} - \mathcal{T}^1_{11}\mathcal{T}^2_{12} = -E_\Omega K_\Omega. \tag{4.9}$$

4.2 The Singular Compatibility Equations

In the following group of equations that we present here, the functions (λ_{ij}) are involved. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal and $\mathbf{\Omega} = \begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$ a tangent moving base of $\mathbf{x}$. Then, $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$ and we have that,

$$
\begin{aligned}
\mathbf{x}_u &= \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2\\
\mathbf{x}_v &= \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2
\end{aligned}
$$

where $\mathbf{\Lambda} = (\lambda_{ij})$. Setting,

$$\bar{\mathbf{\Lambda}} := \begin{pmatrix}\lambda_{11} & \lambda_{12} & 0\\ \lambda_{21} & \lambda_{22} & 0\\ 0 & 0 & 1\end{pmatrix}, \ \mathbb{X} := \begin{pmatrix}D\mathbf{x} & \mathbf{n}\end{pmatrix},$$

we have $\mathbb{X} = \mathbb{W}\bar{\mathbf{\Lambda}}^T$, where $\mathbb{W} = \begin{pmatrix}\mathbf{\Omega} & \mathbf{n}\end{pmatrix}$. Denoting $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ the canonical base of $\mathbb{R}^3$, the compatibility condition $\mathbf{x}_{uv} = \mathbf{x}_{vu}$ is equivalent to

$$\mathbb{X}_u\hat{\mathbf{j}} = \mathbb{X}_v\hat{\mathbf{i}}, \tag{4.11}$$

using (4.3) with (4.11) we have

$$\mathbb{W}\mathbb{P}^T\bar{\mathbf{\Lambda}}^T\hat{\mathbf{j}} + \mathbb{W}\bar{\mathbf{\Lambda}}^T_u\hat{\mathbf{j}} = \mathbb{W}_u\bar{\mathbf{\Lambda}}^T\hat{\mathbf{j}} + \mathbb{W}\bar{\mathbf{\Lambda}}^T_u\hat{\mathbf{j}} = \mathbb{W}_v\bar{\mathbf{\Lambda}}^T\hat{\mathbf{i}} + \mathbb{W}\bar{\mathbf{\Lambda}}^T_v\hat{\mathbf{i}} = \mathbb{W}\mathbb{Q}^T\bar{\mathbf{\Lambda}}^T\hat{\mathbf{i}} + \mathbb{W}\bar{\mathbf{\Lambda}}^T_v\hat{\mathbf{i}},$$

then (4.11) is equivalent to

$$\mathbb{P}^T\bar{\mathbf{\Lambda}}^T\hat{\mathbf{j}}+\bar{\mathbf{\Lambda}}_u^T\hat{\mathbf{j}}=\mathbb{Q}^T\bar{\mathbf{\Lambda}}^T\hat{\mathbf{i}}+\bar{\mathbf{\Lambda}}_v^T\hat{\mathbf{i}}. \tag{4.12}$$

Computing each component of 4.12, we get the following equations that we call *singular compatibility equations (SCE)*:

$$\lambda_{11v}-\lambda_{21u}=\mathcal{T}_{11}^1\lambda_{21}+\mathcal{T}_{21}^1\lambda_{22}-\mathcal{T}_{12}^1\lambda_{11}-\mathcal{T}_{22}^1\lambda_{12} \tag{4.13a}$$

$$\lambda_{12v}-\lambda_{22u}=\mathcal{T}_{11}^2\lambda_{21}+\mathcal{T}_{21}^2\lambda_{22}-\mathcal{T}_{12}^2\lambda_{11}-\mathcal{T}_{22}^2\lambda_{12} \tag{4.13b}$$

$$\lambda_{11}M_{1\Omega}+\lambda_{12}N_{\Omega}=\lambda_{21}L_{\Omega}+\lambda_{22}M_{2\Omega} \tag{4.13c}$$

4.3 Equivalences between equations

In this section we prove that (RCE) are equivalent to the equations (4.9), (4.8g), (4.8h) which have a structure similar to the Gauss equation and Mainardi-Codazzi equations. If we set the matrices:

$$\mathbf{e}_1:=\begin{pmatrix}1\\0\end{pmatrix},\mathbf{e}_2:=\begin{pmatrix}0\\1\end{pmatrix},$$

we can rewrite RCE and SCE with the following very useful compact notation.
Equations (4.13a) and (4.13b):

$$\mathbf{e}_2^T(\mathbf{\Lambda}\mathcal{T}_1+\mathbf{\Lambda}_u)=\mathbf{e}_1^T(\mathbf{\Lambda}\mathcal{T}_2+\mathbf{\Lambda}_v). \tag{4.14}$$

Equation (4.13c):

$$\mathbf{\Lambda}_{(1)}\mathbf{II}_\Omega^{(2)}=\mathbf{\Lambda}_{(2)}\mathbf{II}_\Omega^{(1)}, \tag{4.15}$$

that is, $\mathbf{\Lambda}\mathbf{II}_\Omega$ is symmetric.
Equations (4.8a), (4.8b), (4.8c) and (4.8d):

$$\mathcal{T}_{1v}-\mathcal{T}_{2u}+\mathcal{T}_1\mathcal{T}_2-\mathcal{T}_2\mathcal{T}_1+\mathbf{II}_\Omega^{(1)}\mathbf{e}_2^T\boldsymbol{\mu}-\mathbf{II}_\Omega^{(2)}\mathbf{e}_1^T\boldsymbol{\mu}=0. \tag{4.16}$$

Equations (4.8e) and (4.8f):

$$\mathbf{e}_2^T(\boldsymbol{\mu}\mathcal{T}_1+\boldsymbol{\mu}_u)=\mathbf{e}_1^T(\boldsymbol{\mu}\mathcal{T}_2+\boldsymbol{\mu}_v). \tag{4.17}$$

Equations (4.8g) and (4.8h):

$$\mathbf{e}_2^T(\mathbf{II}_\Omega^T\mathcal{T}_1^T-\mathbf{II}_{\Omega u})=\mathbf{e}_1^T(\mathbf{II}_\Omega^T\mathcal{T}_2^T-\mathbf{II}_{\Omega v}). \tag{4.18}$$

Equation (4.8i):

$$\boldsymbol{\mu}_{(1)}\mathbf{II}_\Omega^{(2)}=\boldsymbol{\mu}_{(2)}\mathbf{II}_\Omega^{(1)}, \tag{4.19}$$

that is, $\boldsymbol{\mu}\mathbf{II}_\Omega$ is symmetric.

Proposition 4.3.1. Let $\mathbf{I}_\Omega, \mathbf{II}_\Omega, \mathcal{T}_1, \mathcal{T}_2 : U \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be arbitrary smooth maps with $\mathbf{I}_\Omega$ symmetric positive definite. If we set $\boldsymbol{\mu} = -\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}$, $K_\Omega = det(\boldsymbol{\mu})$ and we have that

$$\mathbf{I}_\Omega \mathcal{T}_1^T + \mathcal{T}_1 \mathbf{I}_\Omega = \mathbf{I}_{\Omega u}, \tag{4.20}$$
$$\mathbf{I}_\Omega \mathcal{T}_2^T + \mathcal{T}_2 \mathbf{I}_\Omega = \mathbf{I}_{\Omega v}, \tag{4.21}$$

denoting

$$\mathbf{I}_\Omega = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix}, \mathbf{II}_\Omega = \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix},$$

then:

(i) The equation (4.16) is satisfied if and only if, the equation (4.9) is satisfied.

(ii) The equation (4.18) is satisfied if and only if, the equation (4.17) is satisfied.

Proof.

- (i) The equation (4.16) is satisfied if and only if, the resulting equation of multiplying this by the right side with $\mathbf{I}_\Omega$ is satisfied

$$\mathcal{T}_{1v}\mathbf{I}_\Omega - \mathcal{T}_{2u}\mathbf{I}_\Omega + \mathcal{T}_1\mathcal{T}_2\mathbf{I}_\Omega - \mathcal{T}_2\mathcal{T}_1\mathbf{I}_\Omega - \mathbf{II}_\Omega^{(1)}\mathbf{II}_\Omega^{(2)T} + \mathbf{II}_\Omega^{(2)}\mathbf{II}_\Omega^{(1)T} = 0. \tag{4.22}$$

  Observe that $-\mathbf{II}_\Omega^{(1)}\mathbf{II}_\Omega^{(2)T} + \mathbf{II}_\Omega^{(2)}\mathbf{II}_\Omega^{(1)T}$ is skew-symmetric. Let us set,

$$\mathbf{A} := \mathcal{T}_{1v}\mathbf{I}_\Omega - \mathcal{T}_{2u}\mathbf{I}_\Omega + \mathcal{T}_1\mathcal{T}_2\mathbf{I}_\Omega - \mathcal{T}_2\mathcal{T}_1\mathbf{I}_\Omega, \tag{4.23}$$
$$\mathbf{B} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} := -\mathbf{II}_\Omega^{(1)}\mathbf{II}_\Omega^{(2)T} + \mathbf{II}_\Omega^{(2)}\mathbf{II}_\Omega^{(1)T}, \tag{4.24}$$

  we have that

$$-\mathbf{A}^T = -\mathbf{I}_\Omega\mathcal{T}_{1v}^T + \mathbf{I}_\Omega\mathcal{T}_{2u}^T - \mathbf{I}_\Omega\mathcal{T}_2^T\mathcal{T}_1^T + \mathbf{I}_\Omega\mathcal{T}_1^T\mathcal{T}_2^T. \tag{4.25}$$

  On the other hand, deriving (4.20) by v, (4.21) by u and subtracting the results we get:

$$\mathbf{I}_\Omega\mathcal{T}_{2u}^T - \mathbf{I}_\Omega\mathcal{T}_{1v}^T - \mathcal{T}_1\mathbf{I}_{\Omega v} + \mathcal{T}_2\mathbf{I}_{\Omega u} = \mathcal{T}_{1v}\mathbf{I}_\Omega - \mathcal{T}_{2u}\mathbf{I}_\Omega - \mathbf{I}_{\Omega u}\mathcal{T}_2^T + \mathbf{I}_{\Omega v}\mathcal{T}_1^T. \tag{4.26}$$

  Substituting in (4.26), $\mathbf{I}_{\Omega u}$ and $\mathbf{I}_{\Omega v}$ by (4.20) and (4.21), we obtain canceling similar terms that, the right side of (4.23) is equal to the right side of (4.25). Then, $\mathbf{A}$ is skew-symmetric having the form

$$\mathbf{A} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix},$$

hence (4.22) is satisfied if and only if, $a+b=0$ as also, (4.16) can be expressed in this form,

$$\begin{pmatrix} 0 & -a-b \\ a+b & 0 \end{pmatrix} \mathbf{I}_\Omega^{-1} = \mathbf{0}. \tag{4.27}$$

Computing the component $(1,2)$ of (4.27) we have $E_\Omega(-a-b)\det(\mathbf{I}_\Omega)^{-1} = 0$, then $a+b=0$ if and only if, the component $(1,2)$ of (4.16) is satisfied, which is the equation (4.8b) that is simplified to (4.9).

- (ii) Using that $\boldsymbol{\mu} = -\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}$, we substitute $\boldsymbol{\mu}$, $\boldsymbol{\mu}_u$ and $\boldsymbol{\mu}_v$ in (4.17), we get $\mathbf{e}_2^T(-\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}\mathcal{T}_1 - \mathbf{II}_{\Omega u}^T \mathbf{I}_\Omega^{-1} - \mathbf{II}_\Omega^T(\mathbf{I}_\Omega^{-1})_u) = \mathbf{e}_1^T(-\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}\mathcal{T}_2 - \mathbf{II}_{\Omega v}^T \mathbf{I}_\Omega^{-1} - \mathbf{II}_\Omega^T(\mathbf{I}_\Omega^{-1})_v)$ that is satisfied if and only if, the resulting equation of multiply this by the right side with $\mathbf{I}_\Omega$ is satisfied, in which we can later substitute $(\mathbf{I}_\Omega^{-1})_u \mathbf{I}_\Omega = -\mathbf{I}_\Omega^{-1}\mathbf{I}_{\Omega u}$, $(\mathbf{I}_\Omega^{-1})_v \mathbf{I}_\Omega = -\mathbf{I}_\Omega^{-1}\mathbf{I}_{\Omega v}$, factorize similar terms in both sides and get $\mathbf{e}_2^T(\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}(\mathbf{I}_{\Omega u} - \mathcal{T}_1\mathbf{I}_\Omega) - \mathbf{II}_{\Omega u}^T) = \mathbf{e}_1^T(\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}(\mathbf{I}_{\Omega v} - \mathcal{T}_2\mathbf{I}_\Omega) - \mathbf{II}_{\Omega v}^T)$. Since $\mathbf{I}_{\Omega u} - \mathcal{T}_1\mathbf{I}_\Omega = \mathbf{I}_\Omega\mathcal{T}_1^T$ and $\mathbf{I}_{\Omega v} - \mathcal{T}_2\mathbf{I}_\Omega = \mathbf{I}_\Omega\mathcal{T}_2^T$ by hypothesis, substituting these, the equation becomes in (4.18).

□

Remark 4.3.1. Since that equation (4.19) is always satisfied by definition of $\boldsymbol{\mu}$ and as every frontal satisfy (4.20) and (4.21) (proposition 3.2.1), by this last proposition (RCE) are equivalent to (4.9), (4.8g) and (4.8h).

4.4 The Fundamental Theorem

Theorem 4.4.1. Let E,F,G,L,M,N smooth functions defined in an open set $U \subset \mathbb{R}^2$, with $E \geq 0$, $G \geq 0$ and $EG - F^2 \geq 0$. Assume that the given functions have the following decomposition:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}^T \tag{4.28a}$$

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix} \tag{4.28b}$$

in which all the components are smooth real functions defined in U, $E_\Omega > 0$, $G_\Omega > 0$, $E_\Omega G_\Omega - F_\Omega^2 > 0$, $\lambda_\Omega^{-1}(0)$ has empty interior and

$$\boldsymbol{\Lambda}_{(1)u} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \boldsymbol{\Lambda}_{(2)}^T - \boldsymbol{\Lambda}_{(1)} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \boldsymbol{\Lambda}_{(2)u}^T + E_v - F_u \in \mathfrak{I}_\Omega \tag{4.29a}$$

$$\boldsymbol{\Lambda}_{(1)v} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \boldsymbol{\Lambda}_{(2)}^T - \boldsymbol{\Lambda}_{(1)} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \boldsymbol{\Lambda}_{(2)v}^T + F_v - G_u \in \mathfrak{I}_\Omega, \tag{4.29b}$$

where $\boldsymbol{\Lambda} = (\lambda_{ij})$, $\lambda_\Omega = det(\boldsymbol{\Lambda})$ and $\mathfrak{T}_\Omega$ is the principal ideal generated by λ_Ω in the ring $C^\infty(U,\mathbb{R})$. Assume also that E,F,G,L,M,N formally satisfy the Gauss and Mainardi-Codazzi equations for all $(u,v) \in U - \lambda_\Omega^{-1}(0)$. Then,

- (Existence) for each $(u_0,v_0) \in U$ there exists a neighborhood $V \subset U$ of (u_0,v_0) and a frontal $\mathbf{x} : V \to \mathbf{x}(V) \subset \mathbb{R}^3$ with a tangent moving base $\boldsymbol{\Omega}$ such that $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}^T$,

$$\mathbf{I}_\Omega = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix},\ \mathbf{II}_\Omega = \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix}$$

  and the frontal $\mathbf{x}$ has E,F,G and L,M,N as coefficients of the first and second fundamental forms, respectively.

- (Rigidity) If U is connected and if

$$\bar{\mathbf{x}} : U \to \mathbb{R}^3 \text{ and } \bar{\boldsymbol{\Omega}} : U \to \mathbb{R}^3$$

  are another frontal and a tangent moving base satisfying the same conditions, then there exist a translation $\mathbf{T}$ and a proper linear orthogonal transformation $\boldsymbol{\rho}$ in $\mathbb{R}^3$ such that $\bar{\boldsymbol{\Omega}} = \boldsymbol{\rho}\boldsymbol{\Omega}$ and $\bar{\mathbf{x}} = \mathbf{T} \circ \boldsymbol{\rho} \circ \mathbf{x}$.

In order to prove theorem 4.4.1, we shall divide the proof in two parts, existence and rigidity separately. We are also going to use the following lemma.

Lemma 4.4.1. If we have:

$$\bar{\boldsymbol{\Gamma}}_1\bar{\boldsymbol{\Lambda}} - \bar{\boldsymbol{\Lambda}}_u = \bar{\boldsymbol{\Lambda}}\bar{\mathcal{T}}_1 \tag{4.30a}$$
$$\bar{\boldsymbol{\Gamma}}_2\bar{\boldsymbol{\Lambda}} - \bar{\boldsymbol{\Lambda}}_v = \bar{\boldsymbol{\Lambda}}\bar{\mathcal{T}}_2 \tag{4.30b}$$

in which $\bar{\boldsymbol{\Lambda}}, \bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2 : U \to \mathcal{M}_{n\times n}(\mathbb{R})$ and $\bar{\boldsymbol{\Gamma}}_1, \bar{\boldsymbol{\Gamma}}_2 : U - det^{-1}(\bar{\boldsymbol{\Lambda}})(0) \to \mathcal{M}_{n\times n}(\mathbb{R})$ are smooth maps with $int(det^{-1}(\bar{\boldsymbol{\Lambda}})(0)) = \emptyset$. Then,

$\bar{\boldsymbol{\Gamma}}_{1v} - \bar{\boldsymbol{\Gamma}}_{2u} + [\bar{\boldsymbol{\Gamma}}_1, \bar{\boldsymbol{\Gamma}}_2] = 0$ is equivalent to $\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2] = 0$ in U. Futhermore, if

$$\bar{\mathbf{I}} = \bar{\boldsymbol{\Lambda}}\bar{\mathbf{I}}_\Omega\bar{\boldsymbol{\Lambda}}^T \tag{4.31}$$

where $\bar{\mathbf{I}}, \bar{\mathbf{I}}_\Omega$ are smooth maps with $det(\bar{\mathbf{I}}_\Omega) \neq 0$. Then,

- $\bar{\mathbf{I}}\bar{\boldsymbol{\Gamma}}_1^T + \bar{\boldsymbol{\Gamma}}_1\bar{\mathbf{I}} = \bar{\mathbf{I}}_u$ if and only if, $\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega = \bar{\mathbf{I}}_{\Omega u}$ on U.
- $\bar{\mathbf{I}}\bar{\boldsymbol{\Gamma}}_2^T + \bar{\boldsymbol{\Gamma}}_2\bar{\mathbf{I}} = \bar{\mathbf{I}}_v$ if and only if, $\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\bar{\mathbf{I}}_\Omega = \bar{\mathbf{I}}_{\Omega v}$ on U.

Proof. For the first part, deriving (4.30a) in v, (4.30b) in u we get:

$$\bar{\mathbf{\Lambda}}_v\bar{\mathcal{T}}_1+\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_{1v}=\bar{\mathbf{\Gamma}}_{1v}\bar{\mathbf{\Lambda}}+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}_v-\bar{\mathbf{\Lambda}}_{uv} \tag{4.32a}$$

$$\bar{\mathbf{\Lambda}}_u\bar{\mathcal{T}}_2+\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_{2u}=\bar{\mathbf{\Gamma}}_{2u}\bar{\mathbf{\Lambda}}+\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}}_u-\bar{\mathbf{\Lambda}}_{vu} \tag{4.32b}$$

Subtracting (4.32b) from (4.32a)

$$\bar{\mathbf{\Lambda}}(\bar{\mathcal{T}}_{1v}-\bar{\mathcal{T}}_{2u})+\bar{\mathbf{\Lambda}}_v\bar{\mathcal{T}}_1-\bar{\mathbf{\Lambda}}_u\bar{\mathcal{T}}_2=(\bar{\mathbf{\Gamma}}_{1v}-\bar{\mathbf{\Gamma}}_{2u})\bar{\mathbf{\Lambda}}+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}_v-\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}}_u \tag{4.33a}$$

Substituting (4.30a) and (4.30b) in (4.33a) on right side

$$\begin{aligned}\bar{\mathbf{\Lambda}}(\bar{\mathcal{T}}_{1v}-\bar{\mathcal{T}}_{2u})+\bar{\mathbf{\Lambda}}_v\bar{\mathcal{T}}_1-\bar{\mathbf{\Lambda}}_u\bar{\mathcal{T}}_2=(\bar{\mathbf{\Gamma}}_{1v}-\bar{\mathbf{\Gamma}}_{2u})\bar{\mathbf{\Lambda}}+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}}-\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}-\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_2\\+\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_1\end{aligned}$$

Then,

$$\bar{\mathbf{\Lambda}}(\bar{\mathcal{T}}_{1v}-\bar{\mathcal{T}}_{2u})+(\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}-\bar{\mathbf{\Lambda}}_u)\bar{\mathcal{T}}_2+(\bar{\mathbf{\Lambda}}_v-\bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}})\bar{\mathcal{T}}_1=(\bar{\mathbf{\Gamma}}_{1v}-\bar{\mathbf{\Gamma}}_{2u})\bar{\mathbf{\Lambda}}+[\bar{\mathbf{\Gamma}}_1,\bar{\mathbf{\Gamma}}_2]\bar{\mathbf{\Lambda}}$$

Using (4.30a) and (4.30b) on the left side

$$\bar{\mathbf{\Lambda}}(\bar{\mathcal{T}}_{1v}-\bar{\mathcal{T}}_{2u})-\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_2\bar{\mathcal{T}}_1+\bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_1\bar{\mathcal{T}}_2=(\bar{\mathbf{\Gamma}}_{1v}-\bar{\mathbf{\Gamma}}_{2u})\bar{\mathbf{\Lambda}}+[\bar{\mathbf{\Gamma}}_1,\bar{\mathbf{\Gamma}}_2]\bar{\mathbf{\Lambda}}$$

Therefore, we have

$$\bar{\mathbf{\Lambda}}(\bar{\mathcal{T}}_{1v}-\bar{\mathcal{T}}_{2u}+[\bar{\mathcal{T}}_1,\bar{\mathcal{T}}_2])=(\bar{\mathbf{\Gamma}}_{1v}-\bar{\mathbf{\Gamma}}_{2u}+[\bar{\mathbf{\Gamma}}_1,\bar{\mathbf{\Gamma}}_2])\bar{\mathbf{\Lambda}}$$

As $U-det^{-1}(\bar{\mathbf{\Lambda}})(0)$ is dense in U and $\bar{\mathbf{\Lambda}}$ is invertible there, we have the result.

For the last part, the proof of the second item is analogous to the first one, so we are going to prove just the first. For $\mathbf{p}\in U-det^{-1}(\bar{\mathbf{\Lambda}})(0)$, by (4.30a) we have,

$$\begin{aligned}\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T+\bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega&=\bar{\mathbf{I}}_\Omega(\bar{\mathbf{\Lambda}}^T\bar{\mathbf{\Gamma}}_1^T-\bar{\mathbf{\Lambda}}_u^T)(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{\Lambda}}^{-1}(\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}-\bar{\mathbf{\Lambda}}_u)\bar{\mathbf{I}}_\Omega\\&=\bar{\mathbf{I}}_\Omega\bar{\mathbf{\Lambda}}\bar{\mathbf{\Gamma}}_1^T(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}}\bar{\mathbf{I}}_\Omega-\bar{\mathbf{I}}_\Omega\bar{\mathbf{\Lambda}}_u^T(\bar{\mathbf{\Lambda}}^T)^{-1}-\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Lambda}}_u\bar{\mathbf{I}}_\Omega\end{aligned} \tag{4.38}$$

On the other hand, $\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Lambda}}=\mathbb{I}_n$, then $\bar{\mathbf{\Lambda}}_u^T(\bar{\mathbf{\Lambda}}^T)^{-1}=-\bar{\mathbf{\Lambda}}^T((\bar{\mathbf{\Lambda}}^T)^{-1})_u$, $\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Lambda}}_u=-(\bar{\mathbf{\Lambda}}^{-1})_u\bar{\mathbf{\Lambda}}$. Also, from (4.31) $\bar{\mathbf{I}}_\Omega\bar{\mathbf{\Lambda}}^T=\bar{\mathbf{\Lambda}}^{-1}\mathbf{I}$, $\bar{\mathbf{\Lambda}}\bar{\mathbf{I}}_\Omega=\bar{\mathbf{I}}(\bar{\mathbf{\Lambda}}^T)^{-1}$ substituting the last four equalities in (4.38) we get:

$$\begin{aligned}\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T+\bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega&=\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_1^T(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Gamma}}_1\bar{\mathbf{I}}(\bar{\mathbf{\Lambda}}^T)^{-1}-\bar{\mathbf{I}}_\Omega\bar{\mathbf{\Lambda}}_u^T(\bar{\mathbf{\Lambda}}^T)^{-1}-\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{\Lambda}}_u\bar{\mathbf{I}}_\Omega\\&=\bar{\mathbf{\Lambda}}^{-1}(\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_1^T+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{I}})(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{I}}_\Omega\bar{\mathbf{\Lambda}}^T((\bar{\mathbf{\Lambda}}^T)^{-1})_u+(\bar{\mathbf{\Lambda}}^{-1})_u\bar{\mathbf{\Lambda}}\bar{\mathbf{I}}_\Omega\\&=\bar{\mathbf{\Lambda}}^{-1}(\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_1^T+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{I}})(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{I}}((\bar{\mathbf{\Lambda}}^T)^{-1})_u+(\bar{\mathbf{\Lambda}}^{-1})_u\bar{\mathbf{I}}(\bar{\mathbf{\Lambda}}^T)^{-1}\end{aligned}$$

By hypothesis $\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_1^T+\bar{\mathbf{\Gamma}}_1\bar{\mathbf{I}}=\bar{\mathbf{I}}_u$, then

$$\begin{aligned}\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T+\bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega&=\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{I}}_u(\bar{\mathbf{\Lambda}}^T)^{-1}+\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{I}}((\bar{\mathbf{\Lambda}}^T)^{-1})_u+(\bar{\mathbf{\Lambda}}^{-1})_u\bar{\mathbf{I}}(\bar{\mathbf{\Lambda}}^T)^{-1}\\&=(\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{I}}(\bar{\mathbf{\Lambda}}^T)^{-1})_u=\bar{\mathbf{I}}_{\Omega u}\end{aligned}$$

By density of $U-det^{-1}(\bar{\mathbf{\Lambda}})(0)$, $\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T+\bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega=\bar{\mathbf{I}}_{\Omega u}$ holds on U. The converse is obtained in the same way. □

Proof. Theorem 4.4.1(Existence). By proposition 3.2.3 there exist $\mathcal{T}_1, \mathcal{T}_2 : U \to \mathcal{M}_{2\times 2}(\mathbb{R})$ smooth maps such that on $(\lambda_\Omega^{-1}(0))^c$,

$$\mathcal{T}_1 = \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_u), \tag{4.39a}$$

$$\mathcal{T}_2 = \mathbf{\Lambda}^{-1}(\mathbf{\Gamma}_2\mathbf{\Lambda} - \mathbf{\Lambda}_v). \tag{4.39b}$$

Let us construct $\bar{\mathcal{T}}_1$ and $\bar{\mathcal{T}}_2$ as the matrices $\mathbb{P}$ and $\mathbb{Q}$ in (4.5) and (4.6) respectively, using (2.3e), (2.3a) and (2.3b). By (4.39a), (4.39b) and since $\boldsymbol{\alpha}\mathbf{\Lambda} = \boldsymbol{\mu}$ on $(\lambda_\Omega^{-1}(0))^c$ (caused by (4.28a) and (4.28b)) we have for all $(u,v) \in (\lambda_\Omega^{-1}(0))^c$,

$$\bar{\mathbf{\Gamma}}_1\bar{\mathbf{\Lambda}} - \bar{\mathbf{\Lambda}}_u = \bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_1 \text{ and } \bar{\mathbf{\Gamma}}_2\bar{\mathbf{\Lambda}} - \bar{\mathbf{\Lambda}}_v = \bar{\mathbf{\Lambda}}\bar{\mathcal{T}}_2$$

where,

$$\bar{\mathbf{\Gamma}}_1 = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{21}^1 & \Gamma_{21}^2 & M \\ \alpha_{11} & \alpha_{12} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u & (F_u - \frac{1}{2}E_v) & L \\ \frac{1}{2}E_v & \frac{1}{2}G_u & M \\ -L & -M & 0 \end{pmatrix} \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{4.41}$$

$$\bar{\mathbf{\Gamma}}_2 = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{22}^1 & \alpha_{21} \\ \Gamma_{12}^2 & \Gamma_{22}^2 & \alpha_{22} \\ M & N & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_v & \frac{1}{2}G_u & -M \\ (F_v - \frac{1}{2}G_u) & \frac{1}{2}G_v & -N \\ M & N & 0 \end{pmatrix} \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{4.42}$$

$$\bar{\mathbf{\Lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $(u_0, v_0) \in U$, $\mathbf{q} \in \mathbb{R}^3$ be fixed points and since $E_\Omega G_\Omega - F_\Omega^2 > 0$ we can find $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ fixed vectors of $\mathbb{R}^3$ linearly independent and positively oriented such that $\mathbf{z}_1 \cdot \mathbf{z}_1 = E_\Omega(u_0, v_0)$, $\mathbf{z}_1 \cdot \mathbf{z}_2 = F_\Omega(u_0, v_0)$, $\mathbf{z}_2 \cdot \mathbf{z}_2 = G_\Omega(u_0, v_0)$, $\mathbf{z}_3 \cdot \mathbf{z}_3 = 1$ and $\mathbf{z}_3 \cdot \mathbf{z}_i = 0$ for $i = 1, 2$. Consider the system of partial differential equations,

$$\mathbb{W}_u^T = \bar{\mathcal{T}}_1 \mathbb{W}^T \tag{4.43a}$$

$$\mathbb{W}_v^T = \bar{\mathcal{T}}_2 \mathbb{W}^T \tag{4.43b}$$

$$\mathbb{W}(u_0, v_0) = \begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \end{pmatrix} \tag{4.43c}$$

It is known in classical differential geometry that, the Gauss and Mainardi-Codazzi equations are equivalent to $\bar{\mathbf{\Gamma}}_{1v} - \bar{\mathbf{\Gamma}}_{2u} + [\bar{\mathbf{\Gamma}}_1, \bar{\mathbf{\Gamma}}_2] = 0$, then as this is satisfied, by lemma 4.4.1 $\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2] = 0$ in U which is the compatibility condition of the above system of equations. By corollary 2.3.1, this system has a unique solution $\mathbb{W} : \bar{V} \to GL(3)$, where $\bar{V}$ is a neighborhood of (u_0, v_0). Since $det(\mathbb{W}(u_0, v_0)) > 0$, restricting $\bar{V}$ if it is necessary, we can suppose that $det(\mathbb{W}) > 0$ on $\bar{V}$. Setting the matrices,

$$\bar{\mathbf{I}} := \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}, \bar{\mathbf{I}}_\Omega := \begin{pmatrix} E_\Omega & F_\Omega & 0 \\ F_\Omega & G_\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{Y} := \mathbb{W}^T\mathbb{W} \tag{4.44}$$

We want to prove that $\bar{\mathbf{I}}_\Omega = \mathbb{Y}$. Consider the following system of partial differential equations.

$$\mathbb{Y}_u = \mathbb{Y}\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\mathbb{Y} \tag{4.45a}$$

$$\mathbb{Y}_v = \mathbb{Y}\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\mathbb{Y} \tag{4.45b}$$

$$\mathbb{Y}(u_0,v_0) = \bar{\mathbf{I}}_\Omega(u_0,v_0) \tag{4.45c}$$

Defining $\mathbf{\Theta}(u,v,\mathbf{X}) := \mathbf{X}\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\mathbf{X}$ and $\mathbf{\Xi}(u,v,\mathbf{X}) := \mathbf{X}\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\mathbf{X}$ for $\mathbf{X} \in \mathcal{M}_{3\times3}(\mathbb{R})$, we can compute the compatibility condition 2.8 and we get:

$$\begin{aligned} &\mathbf{X}\bar{\mathcal{T}}_{1v}^T + \bar{\mathcal{T}}_{1v}\mathbf{X} + (\mathbf{X}\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\mathbf{X})\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1(\mathbf{X}\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\mathbf{X}) \\ =&\mathbf{X}\bar{\mathcal{T}}_{2u}^T + \bar{\mathcal{T}}_{2u}\mathbf{X} + (\mathbf{X}\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\mathbf{X})\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2(\mathbf{X}\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\mathbf{X}) \end{aligned}$$

Eliminating common terms and grouping we have:

$$\begin{aligned} &\mathbf{X}(\bar{\mathcal{T}}_{1v}^T - \bar{\mathcal{T}}_{2u}^T) + (\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u})\mathbf{X} \\ =&\mathbf{X}(\bar{\mathcal{T}}_1^T\bar{\mathcal{T}}_2^T - \bar{\mathcal{T}}_2^T\bar{\mathcal{T}}_1^T) + (\bar{\mathcal{T}}_2\bar{\mathcal{T}}_1 - \bar{\mathcal{T}}_1\bar{\mathcal{T}}_2)\mathbf{X} \end{aligned}$$

then,

$$\mathbf{X}(\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1,\bar{\mathcal{T}}_2])^T + (\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1,\bar{\mathcal{T}}_2])\mathbf{X} = 0 \tag{4.46}$$

As $\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1,\bar{\mathcal{T}}_2] = 0$, (4.46) is satisfied for all $\mathbf{X} \in \mathcal{M}_{3\times3}(\mathbb{R})$, then by theorem 2.3.1 the system of partial differential equations (4.45) has unique solution. On the other hand, using (4.43a) and (4.43b), it can be verified easily that $\mathbb{X}$ defined in 4.44 is a solution of the system 4.45. Also by (4.41) and (4.42) we have $\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_1^T + \bar{\mathbf{\Gamma}}_1\bar{\mathbf{I}} = \bar{\mathbf{I}}_u$ and $\bar{\mathbf{I}}\bar{\mathbf{\Gamma}}_2^T + \bar{\mathbf{\Gamma}}_2\bar{\mathbf{I}} = \bar{\mathbf{I}}_v$ on $(\lambda^{-1}(0))^c$, then by lemma 4.4.1, $\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_1^T + \bar{\mathcal{T}}_1\bar{\mathbf{I}}_\Omega = \bar{\mathbf{I}}_{\Omega u}$ and $\bar{\mathbf{I}}_\Omega\bar{\mathcal{T}}_2^T + \bar{\mathcal{T}}_2\bar{\mathbf{I}}_\Omega = \bar{\mathbf{I}}_{\Omega v}$ on U, it means, $\bar{\mathbf{I}}_\Omega$ is also a solution of the system 4.45, therefore by uniqueness $\bar{\mathbf{I}}_\Omega = \mathbb{Y}$ on any neighborhood $\hat{V}$ of (u_0,v_0). Now, as $\bar{\mathbf{I}}_\Omega = \mathbb{W}^T\mathbb{W}$, we have that $\mathbf{w}_3$ is orthogonal to $\mathbf{w}_1$,$\mathbf{w}_2$ and $\mathbf{w}_3\cdot\mathbf{w}_3 = 1$. Since $det(\mathbb{W}) > 0$, $\mathbf{n} := \frac{\mathbf{w}_1\times\mathbf{w}_2}{\|\mathbf{w}_1\times\mathbf{w}_2\|} = \mathbf{w}_3$ and if we define $\mathbf{\Omega} := \begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$ then,

$$\mathbf{\Omega}^T\mathbf{\Omega} = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} = \mathbf{I}_\Omega$$

from (4.43a) and (4.43b) we have,

$$\begin{pmatrix} \mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 & L_\Omega \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^2 & M_{2\Omega} \\ \mu_{11} & \mu_{12} & 0 \end{pmatrix} = \mathbb{W}_u^T\mathbb{W}\bar{\mathbf{I}}_\Omega^{-1} = \begin{pmatrix} \mathbf{\Omega}_u^T\mathbf{\Omega}\mathbf{I}_\Omega^{-1} & \mathbf{\Omega}_u^T\mathbf{n} \\ \mathbf{n}_u^T\mathbf{\Omega}\mathbf{I}_\Omega^{-1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 & M_{1\Omega} \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^2 & N_\Omega \\ \mu_{21} & \mu_{22} & 0 \end{pmatrix} = \mathbb{W}_v^T\mathbb{W}\bar{\mathbf{I}}_\Omega^{-1} = \begin{pmatrix} \mathbf{\Omega}_v^T\mathbf{\Omega}\mathbf{I}_\Omega^{-1} & \mathbf{\Omega}_v^T\mathbf{n} \\ \mathbf{n}_v^T\mathbf{\Omega}\mathbf{I}_\Omega^{-1} & 0 \end{pmatrix}$$

then,

$$\mathcal{T}_1 = \begin{pmatrix} \mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^2 \end{pmatrix} = (\mathbf{\Omega}_u^T \mathbf{\Omega})\mathbf{I}_\Omega^{-1} \text{ and } \mathcal{T}_2 = \begin{pmatrix} \mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^2 \end{pmatrix} = (\mathbf{\Omega}_v^T \mathbf{\Omega})\mathbf{I}_\Omega^{-1}$$

$$\mathbf{II}_\Omega = \begin{pmatrix} \mathbf{n}\cdot\mathbf{w}_{1u} & \mathbf{n}\cdot\mathbf{w}_{1v} \\ \mathbf{n}\cdot\mathbf{w}_{2u} & \mathbf{n}\cdot\mathbf{w}_{2v} \end{pmatrix} = \begin{pmatrix} L_\Omega & M_{1\Omega} \\ M_{2\Omega} & N_\Omega \end{pmatrix}$$

Let us consider the system of partial differential equations restricted to $\hat{V}$,

$$\mathbf{x}_u = \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2 \tag{4.49a}$$

$$\mathbf{x}_v = \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2 \tag{4.49b}$$

$$\mathbf{x}(u_0, v_0) = \mathbf{q} \tag{4.49c}$$

As,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (\mathbf{\Lambda}\mathcal{T}_1 + \mathbf{\Lambda}_u) = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{\Gamma}_1\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{\Gamma}_2\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{\Lambda}\mathcal{T}_2 + \mathbf{\Lambda}_v)$$

for $(u,v) \in (\lambda^{-1}(0))^c$, then by density

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (\mathbf{\Lambda}\mathcal{T}_1 + \mathbf{\Lambda}_u) = \begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{\Lambda}\mathcal{T}_2 + \mathbf{\Lambda}_v)$$

on the entire U, as also, by (4.28b) $\mathbf{\Lambda}\mathbf{II}_\Omega$ is symmetric, then the singular compatibility equations (4.13a), (4.13b) and (4.13c) are satisfied, which are the compatibility condition of the system (4.49). Therefore by theorem 2.3.1, this system has a solution $\mathbf{x} : V \to \mathbf{x}(V) \subset \mathbb{R}^3$, where $V \subset \hat{V}$ is a neighborhood of (u_0, v_0). As $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, by proposition 3.1.1, $\mathbf{x}$ is a frontal with $\mathbf{\Omega}$ being a tangent moving base of it, satisfying what we wished. □

Proof. Theorem 4.4.1(Rigidity). Let $\bar{\mathbf{x}} : U \to \bar{\mathbf{x}}(U) \subset \mathbb{R}^3$ be a frontal, U connected, with $\bar{\mathbf{\Omega}}$ a tangent moving base of $\bar{\mathbf{x}}$ satisfying the same conditions of $\mathbf{x}$ and $\mathbf{\Omega}$. As $\mathbf{I}_\Omega = \mathbf{I}_{\bar{\Omega}}$, exists a rotation $\boldsymbol{\rho} \in SO(3)$ such that $\boldsymbol{\rho}\mathbf{\Omega}(u_0,v_0) = \bar{\mathbf{\Omega}}(u_0,v_0)$. Set $\mathbf{a} := \bar{\mathbf{x}}(u_0,v_0) - \boldsymbol{\rho}\mathbf{x}(u_0,v_0)$, $\hat{\mathbf{x}} := \boldsymbol{\rho}\mathbf{x} + \mathbf{a}$ and $\hat{\mathbf{\Omega}} := \boldsymbol{\rho}\mathbf{\Omega}$. Observe that, $\bar{\mathbf{x}}(u_0,v_0) = \hat{\mathbf{x}}(u_0,v_0)$, $\hat{\mathbf{\Omega}}(u_0,v_0) = \bar{\mathbf{\Omega}}(u_0,v_0)$, $D\hat{\mathbf{x}} = \hat{\mathbf{\Omega}}\mathbf{\Lambda}^T$, $\mathbf{I}_\Omega = \mathbf{I}_{\hat{\Omega}}$ and $\mathbf{II}_\Omega = \mathbf{II}_{\hat{\Omega}}$ (caused by $\boldsymbol{\rho}\mathbf{w}_1 \times \boldsymbol{\rho}\mathbf{w}_2 = \boldsymbol{\rho}(\mathbf{w}_1 \times \mathbf{w}_2)$). Also by remark 3.2.1 $\mathcal{T}_i = \bar{\mathcal{T}}_i = \hat{\mathcal{T}}_i$. We want to prove that $\bar{\mathbf{x}} = \hat{\mathbf{x}}$ on U, so, let us define the set,

$$\mathcal{B} := \{(u,v) \in U : \bar{\mathbf{\Omega}}(u,v) = \hat{\mathbf{\Omega}}(u,v)\}$$

$\mathcal{B}$ is not empty and closed by continuity. For each $(\bar{u},\bar{v}) \in \mathcal{B}$, as we had seen before, $\begin{pmatrix} \bar{\mathbf{\Omega}} & \bar{\mathbf{n}} \end{pmatrix}$ is a solution of the system:

$$\mathbb{W}_u^T = \mathbb{P}\mathbb{W}^T$$

$$\mathbb{W}_v^T = \mathbb{Q}\mathbb{W}^T$$

$$\mathbb{W}(\bar{u},\bar{v}) = \begin{pmatrix} \bar{\mathbf{\Omega}}(\bar{u},\bar{v}) & \bar{\mathbf{n}}(\bar{u},\bar{v}) \end{pmatrix}$$

As the matrices $\mathbb{P}$ (4.5) and $\mathbb{Q}$ (4.6) are constructed with the coefficients of $\mathbf{I}_\Omega$, $\mathbf{II}_\Omega$ and $\mathcal{T}_i$, then $\begin{pmatrix} \hat{\mathbf{\Omega}} & \hat{\mathbf{n}} \end{pmatrix}$ is solution of the system as well and by uniqueness, $\hat{\mathbf{\Omega}} = \bar{\mathbf{\Omega}}$ on a neighborhood of $(\bar{u},\bar{v})$. We have that $\mathcal{B}$ is open and since U is connected, $\mathcal{B} = U$. Therefore, $D\bar{\mathbf{x}} = \bar{\mathbf{\Omega}}\mathbf{\Lambda}^T = \hat{\mathbf{\Omega}}\mathbf{\Lambda}^T = D\hat{\mathbf{x}}$ and since $\bar{\mathbf{x}}(u_0,v_0) = \hat{\mathbf{x}}(u_0,v_0)$, $\bar{\mathbf{x}} = \hat{\mathbf{x}}$ on U. □

Remark 4.4.1. In theorem 4.4.1 can be switched the hypothesis of E,F,G,L,M,N satisfying the Gauss and Mainardi-Codazzi equations for all $(u,v) \in U - \lambda_\Omega^{-1}(0)$ by hypothesis of $E_\Omega,F_\Omega,G_\Omega,L_\Omega,M_{1\Omega},M_{2\Omega},N_\Omega$ satisfying the equations (4.9), (4.8g) and (4.8h) on U, where $\mathcal{T}_1,\mathcal{T}_2$ are defined as in proposition 3.2.3 (see remark 3.2.2). Since $\bar{\mathcal{T}}_{1v} - \bar{\mathcal{T}}_{2u} + [\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2] = 0$ is equivalent to (4.9), (4.8g) and (4.8h), using lemma 4.4.1 these two different hypothesis are equivalent, then we obtain the same result in the theorem. By last, the frontal obtained is going to be a wavefront if $(K_\Omega, H_\Omega) \neq (0,0)$ on the domain, where K_Ω, H_Ω are computed with the given coefficients $E_\Omega, F_\Omega, G_\Omega, L_\Omega, M_{1\Omega}, M_{2\Omega}, N_\Omega$ and λ_{ij}.

CHAPTER 5

BEHAVIOR OF THE CLASSICAL INVARIANTS

In this chapter we introduce the relative principal curvatures which give us geometrical information near singularities and are defined even on them. After that, we study singularities of rank 1 both degenerate and non-degenerate of wavefronts. The theorems 5.2.1, 5.2.2 give equivalent conditions for boundedness and extendibility of the Gaussian curvature which generalize the one found in (SAJI; UMEHARA; YAMADA, 2009) and similarly theorem 5.2.5 for the principal curvatures. We also study the convergence to infinite limits of the classical invariants and show how this is tightly related to a particular property of uniform approximation of fronts by parallel surfaces.

For a singularity $\mathbf{p}$ of a wavefront $\mathbf{x}$, there exists $l > 0$ and a neighborhood U_l of $\mathbf{p}$ such $\mathbf{y}_l = \mathbf{x} + l\mathbf{n}$ is an immersion, also this l can be chosen as small as we wish (see lemma 5.1.1). The neighborhood U_l may shrink as l is smaller, then is natural to ask when U_l can be hold fixed for l arbitrarily small, in this case we say that $\mathbf{x}$ is *parallelly smoothable at* $\mathbf{p}$. We will see that, this last property is determined by the convergence to infinite limits of the classical invariants at each type of singularity and also is related with the extendibility of the principal curvatures at singularities. The theorems 5.2.3, 5.2.4 and 5.2.6 characterize when a wavefront is parallelly smoothable at all types of singularities.

Finally, we study the behavior of the invariants at singularities of rank 0 of wavefronts, obtaining results quite different from those obtained in the rank 1 case. The example 5.2.6 shows explicitly a wavefront with singularity of rank 0 and mean curvature vanishing everywhere, like a minimal surface in the regular case, which make these types of singular surfaces very interesting.

5.1 The relative principal curvatures

Definition 5.1.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$, for each $\mathbf{p} \in U$ we define the Ω-*relative Weingarten matrix* as follows:

$$\boldsymbol{\alpha}_\Omega := \boldsymbol{\mu}_\Omega adj(\boldsymbol{\Lambda}_\Omega)$$

Proposition 5.1.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal and $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$. We have the following equality, $\boldsymbol{\alpha}_\Omega = \boldsymbol{\alpha}\lambda_\Omega$ on $\Sigma(\mathbf{x})^c$. In particular $\boldsymbol{\alpha}_\Omega$ has real eigenvalues.

Proof. By theorem 3.1.1, $\mathbf{I} = \boldsymbol{\Lambda}\mathbf{I}_\Omega\boldsymbol{\Lambda}^T$ and $\mathbf{II} = \boldsymbol{\Lambda}\mathbf{II}_\Omega$, then for $\mathbf{p} \in \Sigma(\mathbf{x})^c$, $\boldsymbol{\alpha} = -\mathbf{II}^T\mathbf{I}^{-1} = -\mathbf{II}_\Omega^T\boldsymbol{\Lambda}^T(\boldsymbol{\Lambda}^T)^{-1}\mathbf{I}_\Omega^{-1}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\mu}_\Omega\boldsymbol{\Lambda}^{-1}$. Thus, we have $\boldsymbol{\alpha}\lambda_\Omega = \boldsymbol{\mu}_\Omega adj(\boldsymbol{\Lambda}) = \boldsymbol{\alpha}_\Omega$. The eigenvalues of $\boldsymbol{\alpha}_\Omega$ are real if $tr(\boldsymbol{\alpha}_\Omega)^2 - 4det(\boldsymbol{\alpha}_\Omega) \geq 0$. As $K_\Omega = det(\boldsymbol{\mu}_\Omega)$ and $H_\Omega = -\frac{1}{2}tr(\boldsymbol{\alpha}_\Omega)$ this is equivalent to have $H_\Omega^2 - \lambda_\Omega K_\Omega \geq 0$ and by proposition 3.3.1 $H_\Omega^2 - \lambda_\Omega K_\Omega = \lambda_\Omega^2(H^2 - K) \geq 0$ on $\Sigma(\mathbf{x})^c$, then by continuity and the density of regular points, it follows the result on U. □

Denoting the eigenvalues of $\boldsymbol{\alpha}_\Omega$ by $-k_{1\Omega}$, $-k_{2\Omega}$, then $k_{1\Omega}$, $k_{2\Omega}$ satisfy the equation $k^2 + tr(\boldsymbol{\alpha}_\Omega^T)k + det(\boldsymbol{\alpha}_\Omega^T) = 0$. Since $K_\Omega = det(\boldsymbol{\mu}_\Omega)$ and $H_\Omega = -\frac{1}{2}tr(\boldsymbol{\alpha}_\Omega)$, we have $k^2 - 2H_\Omega k + \lambda_\Omega K_\Omega = 0$. Thus,

$$k = H_\Omega \pm \sqrt{H_\Omega^2 - \lambda_\Omega K_\Omega}$$

Definition 5.1.2. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$, we call the functions $k_{1\Omega} := H_\Omega - \sqrt{H_\Omega^2 - \lambda_\Omega K_\Omega}$ and $k_{2\Omega} := H_\Omega + \sqrt{H_\Omega^2 - \lambda_\Omega K_\Omega}$ the *relative principal curvatures*. We also define for a proper frontal $\mathbf{x}$ the following functions on $\Sigma(\mathbf{x})^c$:

$$k_1 := \begin{cases} H - \sqrt{H^2 - K} & \text{if } \lambda_\Omega > 0, \\ H + \sqrt{H^2 - K} & \text{if } \lambda_\Omega < 0. \end{cases}$$

$$k_2 := \begin{cases} H + \sqrt{H^2 - K} & \text{if } \lambda_\Omega > 0, \\ H - \sqrt{H^2 - K} & \text{if } \lambda_\Omega < 0. \end{cases}$$

We clarify that, the principal curvatures of $\mathbf{x}$ are the functions defined by $\kappa_- := H - \sqrt{H^2 - K}$ and $\kappa_+ := H + \sqrt{H^2 - K}$ on $\Sigma(\mathbf{x})^c$.

Remark 5.1.1. It follows from the above definition that the relative principal curvatures satisfy $k_{1\Omega} + k_{2\Omega} = 2H_\Omega$ and $k_{1\Omega}k_{2\Omega} = \lambda_\Omega K_\Omega$. The smooth functions k_1 and k_2 defined on $\Sigma(\mathbf{x})^c$ have similar properties to the classical principal curvatures. Also their definitions do not depend on the chosen tmb $\boldsymbol{\Omega}$ inducing the same orientation of the normal vector field $\mathbf{n}$. If another tmb $\hat{\boldsymbol{\Omega}}$ induces an opposite orientation of $\mathbf{n}$, then the signs of these functions are opposite as well. Observe that $k_1k_2 = K$ and $\frac{k_1+k_2}{2} = H$ on $\Sigma(\mathbf{x})^c$. In the case of non-degenerate singularities, if we make a suitable change of coordinates k_1, k_2 coincide with those functions defined in ((TERAMOTO, 2016), equation (2.6)).

Proposition 5.1.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal, $\boldsymbol{\Omega}$ a tangent moving basis of $\mathbf{x}$. Then,

1. for $\mathbf{p} \in \Sigma(\mathbf{x})^c$, $k_{1\Omega} = \lambda_\Omega k_1$ and $k_{2\Omega} = \lambda_\Omega k_2$,
2. for $\mathbf{p} \in \Sigma(\mathbf{x})$, $k_{1\Omega} = \lim_{(u,v)\to p} \lambda_\Omega k_1$ and $k_{2\Omega} = \lim_{(u,v)\to p} \lambda_\Omega k_2$.

Proof. We have that $k_{1\Omega} = \lambda_\Omega H - \sqrt{\lambda_\Omega^2 H^2 - \lambda_\Omega^2 K} = \lambda_\Omega H - |\lambda_\Omega|\sqrt{H^2 - K} = \lambda_\Omega k_1$ and similarly $k_{2\Omega} = \lambda_\Omega k_2$ on $\Sigma(\mathbf{x})^c$. For $\mathbf{p} \in \Sigma(\mathbf{x})$, by smoothness of $k_{1\Omega}$, $k_{2\Omega}$ and density of $\Sigma(\mathbf{x})^c$, $k_{1\Omega} = \lim_{(u,v)\to p} \lambda_\Omega k_1$ and $k_{2\Omega} = \lim_{(u,v)\to p} \lambda_\Omega k_2$. □

Example 5.1.1. For the cuspidal edge $\mathbf{x} = (u, v^2, v^3)$ in example 3.3.1 we saw that $K_\Omega = 0 = K$, $H_\Omega = 3(4+9v^2)^{-\frac{3}{2}}$ and $H = 3v^{-1}(4+9v^2)^{-\frac{3}{2}}$. Since $\lambda_\Omega = v$, then $k_{1\Omega} = 0$, $k_{2\Omega} = 6(4+9v^2)^{-\frac{3}{2}}$,

$$k_1 := \begin{cases} 3v^{-1}(4+9v^2)^{-\frac{3}{2}} - |3v^{-1}(4+9v^2)^{-\frac{3}{2}}| & \text{if } v > 0 \\ 3v^{-1}(4+9v^2)^{-\frac{3}{2}} + |3v^{-1}(4+9v^2)^{-\frac{3}{2}}| & \text{if } v < 0, \end{cases}$$

$$k_2 := \begin{cases} 3v^{-1}(4+9v^2)^{-\frac{3}{2}} + |3v^{-1}(4+9v^2)^{-\frac{3}{2}}| & \text{if } v > 0 \\ 3v^{-1}(4+9v^2)^{-\frac{3}{2}} - |3v^{-1}(4+9v^2)^{-\frac{3}{2}}| & \text{if } v < 0, \end{cases}$$

therefore $k_1 = 0$ and $k_2 = 6v^{-1}(4+9v^2)^{-\frac{3}{2}}$.

In the proof of proposition 3.3.4 was observed that making change of coordinates $\mathbf{h}$ on a frontal $\mathbf{x}$ and taking $\hat{\boldsymbol{\Omega}} := \boldsymbol{\Omega} \circ \mathbf{h}$ as tmb of $\mathbf{x} \circ \mathbf{h}$, it results with new different relative curvatures $det(\mathbf{h})(K_\Omega \circ \mathbf{h})$ and $det(\mathbf{h})(H_\Omega \circ \mathbf{h})$. However, if we choose the tmb $\boldsymbol{\Omega}^h := (\boldsymbol{\Omega} \circ \mathbf{h})D\mathbf{h}$ instead of $\hat{\boldsymbol{\Omega}}$ when we make a change of coordinates, they remain invariant.

Proposition 5.1.3 (Invariance property)**.** Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{h} : V \to U$ diffeomorphism, then the new relative curvatures of $\mathbf{x} \circ \mathbf{h}$ are $K_{\Omega^h} = K_\Omega \circ \mathbf{h}$ and $H_{\Omega^h} = H_\Omega \circ \mathbf{h}$. In particular, $k_{1\Omega^h} = k_{1\Omega} \circ \mathbf{h}$, $k_{2\Omega^h} = k_{2\Omega} \circ \mathbf{h}$ and $\lambda_{\Omega^h} = \lambda_\Omega \circ \mathbf{h}$.

Proof. Observe that, the matrix $\boldsymbol{\Lambda}_{\Omega^h}$ induced by $\boldsymbol{\Omega}^h$ is $(D\mathbf{h})^{-1}(\boldsymbol{\Lambda}_\Omega \circ \mathbf{h})D\mathbf{h}$, then $\lambda_{\Omega^h} = \lambda_\Omega \circ \mathbf{h}$ and since $D(\mathbf{n} \circ \mathbf{h}) = \boldsymbol{\Omega}^h \boldsymbol{\mu}_{\Omega^h}^T$, we have $(D\mathbf{h})^{-1}\boldsymbol{\mu}_\Omega^T(D\mathbf{h}) = \boldsymbol{\mu}_{\Omega^h}^T$ and therefore $K_{\Omega^h} = K_\Omega \circ \mathbf{h}$. Also, $\boldsymbol{\mu}_{\Omega^h} adj(\boldsymbol{\Lambda}_{\Omega^h}) = det(D\mathbf{h})D\mathbf{h}^{-1}\boldsymbol{\mu}_\Omega adj(\boldsymbol{\Lambda}_\Omega)adj(D\mathbf{h}^{-1})$ and using that $tr(\mathbf{ABC}) = tr(\mathbf{CAB})$ we get $H_{\Omega^h} = H_\Omega \circ \mathbf{h}$. Since, $k_{1\Omega^h}$ and $k_{2\Omega^h}$ are written in terms of K_{Ω^h} and H_{Ω^h}, the result follows for them. □

The following two lemmas give a tmb in connection with parallel surfaces for wavefronts and they will be used to prove theorems 5.2.1 and 5.2.2 in the next section.

Lemma 5.1.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, for each $\mathbf{p} \in \Sigma(\mathbf{x})$ there exist locally an embedding $\mathbf{y}_l : V \to \mathbb{R}^3$, $\mathbf{p} \in V$, such that $D\mathbf{y}_l$ is a tmb of $\mathbf{x}$, the matrix $\mathbf{\Lambda}_{D\mathbf{y}_l}$ determined for this tmb is $\mathbb{I}_2 - l\boldsymbol{\alpha}_l$, where $l \in \mathbb{R}^+$ and $\boldsymbol{\alpha}_l$ is the Weingarden matrix of $\mathbf{y}_l$ and $\mathbb{I}_2$ is the identity matrix.

Proof. For each $t \in \mathbb{R}$, consider $\mathbf{y}_t = \mathbf{x} + t\mathbf{n}$, as $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$ and $D\mathbf{n} = \mathbf{\Omega}\boldsymbol{\mu}^T$ we have $D\mathbf{y}_t = \mathbf{\Omega}\mathbf{\Lambda}^T + t\mathbf{\Omega}\boldsymbol{\mu}^T$, then $\mathbf{y}_t$ has a singularity at $\mathbf{q}$ if and only if $det(\mathbf{\Lambda}^T + t\boldsymbol{\mu}^T)(\mathbf{q}) = 0$. Making a direct computation $det(\mathbf{\Lambda}^T + t\boldsymbol{\mu}^T) = \lambda_\Omega - 2tH_\Omega + t^2K_\Omega$ and now taking $\mathbf{p} \in \Sigma(\mathbf{x})$, by corollary (3.3.1), there exist $l \in \mathbb{R}^+$ such that $det(\mathbf{\Lambda}^T + l\boldsymbol{\mu}^T)(\mathbf{p}) = -2lH_\Omega(\mathbf{p}) + l^2K_\Omega(\mathbf{p}) \neq 0$. Thus, there exists a neighborhood V of $\mathbf{p}$ such that $\mathbf{y}_l : V \to \mathbb{R}^3$ is an embedding. Since, $D\mathbf{y}_l = \mathbf{\Omega}(\mathbf{\Lambda} + l\boldsymbol{\mu})$, $D\mathbf{y}_l$ is a tmb of $\mathbf{x}$. We can assume $D\mathbf{y}_l$ and $\mathbf{\Omega}$ induce the same normal vector $\mathbf{n}$ (i.e $det(\mathbf{\Lambda}^T + l\boldsymbol{\mu}^T) > 0$ on V), otherwise we can change the order of columns in $\mathbf{\Omega}$ from the beginning. Therefore, we have $D\mathbf{y}_l(\mathbb{I}_2 - l\boldsymbol{\alpha}_l^T) = D\mathbf{y}_l - lD\mathbf{n} = D\mathbf{x} = D\mathbf{y}_l\mathbf{\Lambda}_{D\mathbf{y}_l}^T$, thus $\mathbf{\Lambda}_{D\mathbf{y}_l} = \mathbb{I}_2 - l\boldsymbol{\alpha}_l$. □

Lemma 5.1.2. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a wavefront, $\mathbf{p} \in \Sigma(\mathbf{x})$ and $\mathbf{\Omega} = D\mathbf{y}_l$ a tmb of $\mathbf{x}$ as above, then

1. $\mathbf{I}_\Omega = \mathbf{I}_l$, $\mathbf{II}_\Omega = \mathbf{II}_l$, $\boldsymbol{\mu} = \boldsymbol{\alpha}_l$ and $\boldsymbol{\mu}adj(\mathbf{\Lambda}) = \boldsymbol{\alpha}_l - lK_l\mathbb{I}_2$.

2. $K_\Omega = K_l$, $H_\Omega = H_l + K_l l$ and $\lambda_\Omega = 1 + 2H_l l + K_l l^2$.

3. $k_{1\Omega} = k_{1l}(1 + lk_{2l})$ and $k_{2\Omega} = k_{2l}(1 + lk_{1l})$.

where $\mathbf{I}_l$, $\mathbf{II}_l$, K_l, H_l, k_{1l}, k_{2l} are first fundamental form, second fundamental form, Gaussian curvature, mean curvature and principal curvatures of $\mathbf{y}_l$ respectively. Additionally, $rank(D\mathbf{x}(\mathbf{p})) = 1$ if and only if $\mathbf{y}_l$ is free of umbilical point on a neighborhood of $\mathbf{p}$. Similarly, $rank(D\mathbf{x}(\mathbf{p})) = 0$ if and only if $\mathbf{y}_l$ has a umbilical point at $\mathbf{p}$ of positive Gaussian curvature.

Proof.

1. Applying the definition directly we get the first three equalities. By lemma 5.1.1 $\mathbf{\Lambda} = \mathbb{I}_2 - l\boldsymbol{\alpha}_l$, then $\boldsymbol{\mu}adj(\mathbf{\Lambda}) = \boldsymbol{\alpha}_l(\mathbb{I}_2 - l\,adj(\boldsymbol{\alpha}_l)) = \boldsymbol{\alpha}_l - lK_l\mathbb{I}_2$.

2. Using item (1), $K_\Omega = det(\boldsymbol{\mu}) = det(\boldsymbol{\alpha}_l) = K_l$, $H_\Omega = -\frac{1}{2}tr(\boldsymbol{\mu}adj(\mathbf{\Lambda})) = -\frac{1}{2}tr(\boldsymbol{\alpha}_l - lK_l\mathbb{I}_2) = H_l + K_l l$ and $\lambda_\Omega = det(\mathbf{\Lambda}) = det(\mathbb{I}_2 - l\boldsymbol{\alpha}_l) = 1 + 2H_l l + K_l l^2$.

3. Using the formulas in definition 5.1.2, item (2) and knowing that $k_{1l} = H_l - \sqrt{H_l^2 - K_l}$, $k_{2l} = H_l + \sqrt{H_l^2 - K_l}$ and $K_l = k_{1l}k_{2l}$, a simple computation leads to item (3).

For the last part, by proposition 3.3.1 $rank(D\mathbf{x}(\mathbf{p})) = 0$ if and only if $H_\Omega(\mathbf{p}) = 0$, $\lambda_\Omega(\mathbf{p}) = 0$ and $K_\Omega(\mathbf{p}) \neq 0$. On the other hand these conditions are equivalent to $H_l(\mathbf{p}) = -K_l(\mathbf{p})l$

and $0 = 1 - 2l^2K_l(\mathbf{p}) + l^2K_l(\mathbf{p})$ which is equivalent to $K_l(\mathbf{p}) = \frac{1}{l^2}$ and $H_l(\mathbf{p}) = -\frac{1}{l}$. Then $\mathbf{y}_l$ has an umbilical point at $\mathbf{p}$ of positive Gaussian curvature. Conversely, we have $0 < K_l(\mathbf{p}) = H_l(\mathbf{p})^2$, then $0 = 1 + 2lH_l(\mathbf{p}) + l^2H_l(\mathbf{p})^2$ which imply $H_l(\mathbf{p}) = -\frac{1}{l}$ and therefore $H_\Omega(\mathbf{p}) = -\frac{1}{l} + \frac{1}{l^2}l = 0$, by proposition 3.3.1 $rank(D\mathbf{x}(\mathbf{p})) = 0$. Equivalently $rank(D\mathbf{x}(\mathbf{p})) = 1$ if and only if $\mathbf{p}$ is not a umbilical point which is equivalent to have $\mathbf{y}_l$ free of umbilical point on a neighborhood of $\mathbf{p}$. □

5.2 Extendibility and boundedness

5.2.1 Singularities of rank 1

In this section, we study the behavior at a singular point of rank 1 of the classical invariant of wavefronts, using the relative principal curvatures defined in previous section. The non-degenerate case was investigated in (SAJI; UMEHARA; YAMADA, 2009; MARTINS *et al.*, 2016; TERAMOTO, 2016; TERAMOTO, 2019b).

Proposition 5.2.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, then for every $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, the following is always satisfied:

1. $(k_{1\Omega}(\mathbf{p}), k_{2\Omega}(\mathbf{p})) \neq (0,0)$. In particular, if $k_{1\Omega}(\mathbf{p}) \neq 0$ (resp. $k_{2\Omega}(\mathbf{p}) \neq 0$), then $k_{2\Omega}(\mathbf{p}) = 0$ (resp. $k_{1\Omega}(\mathbf{p}) = 0$). Also, $H_\Omega(\mathbf{p}) < 0$ (resp. $H_\Omega(\mathbf{p}) > 0$) if and only if $k_{1\Omega}(\mathbf{p}) \neq 0$ (resp. $k_{2\Omega}(\mathbf{p}) \neq 0$).

2. There is an open neighborhood $V \subset U$ of $\mathbf{p}$ in which one of the functions k_1, k_2 has a C^∞ extension to V. More precisely, k_1 (resp. k_2) has a C^∞ extension if only if $k_{1\Omega}(\mathbf{p}) = 0$ (resp. $k_{2\Omega}(\mathbf{p})$).

3. One of the functions k_1, k_2 in module diverge to ∞. More precisely, $\lim\limits_{(u,v)\to p} |k_1| = \infty$ (resp. $|k_2|$) if and only if $k_{1\Omega}(\mathbf{p}) \neq 0$ (resp. $k_{2\Omega}(\mathbf{p})$).

4. $\lim\limits_{(u,v)\to p} |H| = \infty$.

5. If $K_\Omega(\mathbf{p}) \neq 0$ then $\lim\limits_{(u,v)\to p} |K| = \infty$.

Proof.

1. Observe that, for $\mathbf{p} \in \Sigma(\mathbf{x})$, $k_{1\Omega}(\mathbf{p}) = H_\Omega(\mathbf{p}) - |H_\Omega(\mathbf{p})|$ and $k_{2\Omega}(\mathbf{p}) = H_\Omega(\mathbf{p}) + |H_\Omega(\mathbf{p})|$. By proposition 3.3.1, $H_\Omega(\mathbf{p}) \neq 0$, then just one of $k_{1\Omega}(\mathbf{p})$, $k_{2\Omega}(\mathbf{p})$ is different of zero. Thus, the sub index of $k_{i\Omega}(\mathbf{p})$ corresponding to the non-zero value is determined bijectively by the sign of $H_\Omega(\mathbf{p})$.

2. By item (1), without loss of generality, we can assume that $k_{1\Omega}(\mathbf{p}) \neq 0$. Let V be a neighborhood of $\mathbf{p}$ such that $k_{1\Omega} \neq 0$ on V, then by proposition 3.3.1, $k_2 = \frac{\lambda_\Omega k_1 k_2}{\lambda_\Omega k_1} = \frac{K_\Omega}{k_{1\Omega}}$ on $V - \Sigma(\mathbf{x})$. Thus, $\frac{K_\Omega}{k_{1\Omega}}$ is a C^∞ extension of k_2 to V.

3. By item (1), without loss of generality, we can assume that $k_{1\Omega}(\mathbf{p}) \neq 0$. Let V be a neighborhood of $\mathbf{p}$ such that $k_{1\Omega} \neq 0$ on V, then $k_1 = \frac{k_{1\Omega}}{\lambda_\Omega}$ on $V - \Sigma(\mathbf{x})$. Thus, for every $\mathbf{p} \in \Sigma(\mathbf{x}) \cap V$, $\lim_{(u,v)\to p} |k_1| = \lim_{(u,v)\to p} \frac{|k_{1\Omega}|}{|\lambda_\Omega|} = \infty$.

4. Since $H_\Omega(\mathbf{p}) \neq 0$ and by proposition 3.3.1, $\lim_{(u,v)\to p} |H| = \lim_{(u,v)\to p} \frac{|H_\Omega|}{|\lambda_\Omega|} = \infty$.

5. Since $K_\Omega(\mathbf{p}) \neq 0$ and by proposition 3.3.1, $\lim_{(u,v)\to p} |K| = \lim_{(u,v)\to p} \frac{|K_\Omega|}{|\lambda_\Omega|} = \infty$.

□

We shall use the following two lemmas to prove theorems 5.2.1 and 5.2.2 about boundedness and extendibility of the Gaussian curvature.

Lemma 5.2.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $W \subset U$ a compact set, $V \subset U$ an open set, $a: W \to \mathbb{R}$ a continuous function, $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ tangent moving bases of $\mathbf{x}$, then we have:

1. If there exist a constant $C_1 > 0$ such that $|a| \leq C_1 |\lambda_{\Omega_1}|$ on W then there exist a constant $C_2 > 0$ such that $|a| \leq C_2 |\lambda_{\Omega_2}|$ on W.

2. $\mathfrak{T}_{\Omega_1}(V) = \mathfrak{T}_{\Omega_2}(V)$

Proof.

1. Setting $\mathbf{A} = \mathbf{I}_{\Omega_1}^{-1} \mathbf{\Omega}_1^T \mathbf{\Omega}_2$ (change of basis matrix) and $\rho = det(\mathbf{A})$, we have $\mathbf{\Omega}_2 = \mathbf{\Omega}_1 \mathbf{A}$, therefore $\mathbf{\Lambda}_{\Omega_1} = \mathbf{A} \mathbf{\Lambda}_{\Omega_2}$ and $\lambda_{\Omega_1} = \rho \lambda_{\Omega_2}$. Since $|a| \leq C_1 |\lambda_{\Omega_1}| = C_1 |\rho| |\lambda_{\Omega_2}|$ and choosing C_2 as the maximum of $C_1 |\rho|$ on W, we get the result.

2. Using the proof of item (1), $\lambda_{\Omega_1} = \rho \lambda_{\Omega_2}$ with $\rho \neq 0$, then we have the equality.

□

Lemma 5.2.2. Let $\mathbf{x}_1 : U \to \mathbb{R}^3$ be a proper wavefront with U open connected, $W \subset U$ a compact set, $V \subset U$ an open set, $\mathbf{h}: Z \to U$ a diffeomorphism and $\mathbf{\Omega}$ a tmb of $\mathbf{x}$. Setting $\mathbf{x}_2 := \mathbf{x}_1 \circ \mathbf{h}$ and choosing $\mathbf{\Omega}^h = (\mathbf{\Omega} \circ \mathbf{h}) D\mathbf{h}$ as tmb of $\mathbf{x}_2$ we have:

1. There exist a constant $C_1 > 0$ such that $|L_1| \leq C_1 |\lambda_\Omega|$, $|M_1| \leq C_1 |\lambda_\Omega|$, $|N_1| \leq C_1 |\lambda_\Omega|$ on W if and only if there exist a constant $C_2 > 0$ such that $|L_2| \leq C_2 |\lambda_{\Omega^h}|$, $|M_2| \leq C_2 |\lambda_{\Omega^h}|$, $|N_2| \leq C_2 |\lambda_{\Omega^h}|$ on $\mathbf{h}^{-1}(W)$. Where L_1, M_1, N_1 and L_2, M_2, N_2 are the coefficients of the second fundamental form of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.

2. $L_1, M_1, N_1 \in \mathfrak{T}_\Omega(V)$ if and only if $L_2, M_2, N_2 \in \mathfrak{T}_{\Omega^h}(\mathbf{h}^{-1}(V))$.

Proof. Let us denote by $\mathbf{II}_1$, $\mathbf{II}_2$ the matrices of the second fundamental forms of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. If $det(D\mathbf{h}) > 0$, then $\mathbf{n}_2 = \mathbf{n}_1 \circ \mathbf{h}$ (in the case $det(D\mathbf{h}) < 0$, $\mathbf{n}_2 = -\mathbf{n}_1 \circ \mathbf{h}$ and it is analogous) therefore $\mathbf{II}_2 = -D\mathbf{x}_2^T D\mathbf{n}_2 = -D\mathbf{h}^T D\mathbf{x}_1^T D\mathbf{n}_1 D\mathbf{h} = D\mathbf{h}^T \mathbf{II}_1 D\mathbf{h}$. This last equality expresses the coefficients L_2, M_2, N_2 as sum of multiples of the coefficients L_1, M_1, N_1 and vice versa. Since $\lambda_{\Omega^h} = \lambda_\Omega \circ \mathbf{h}$ (see proposition 5.1.3) we get items (1) and (2) easily. □

Theorem 5.2.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper wavefront with just singularities of rank 1, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$. Let $W \subset U$ be a compact neighborhood of $\mathbf{p}$ in which the relative principal curvature $k_{i\Omega} \neq 0$ does not vanish. Let k_j be the function that admits a C^∞ extension to W and K the Gaussian curvature, then the following statements are equivalent:

1. K is bounded on $W - \Sigma(\mathbf{x})$.
2. There exist a constant $C > 0$ such that $|K_\Omega| \leq C|\lambda_\Omega|$ on W.
3. There exist a constant $C > 0$ such that $|k_j| \leq C|\lambda_\Omega|$ on W.
4. There exist a constant $C > 0$ such that $|L| \leq C|\lambda_\Omega|$, $|M| \leq C|\lambda_\Omega|$ and $|N| \leq C|\lambda_\Omega|$ on W.

Proof.

- $(1 \Leftrightarrow 2)$ As $|K| = \frac{|K_\Omega|}{|\lambda_\Omega|}$ on $W - \Sigma(\mathbf{x})$ and by the density of $W - \Sigma(\mathbf{x})$ in W, it follows the equivalence.
- $(2 \Leftrightarrow 3)$ Since W is compact, k_j, K_Ω, $k_{i\Omega}$ are continuous on W and $k_j = \frac{K_\Omega}{k_{i\Omega}}$, from this last equality follows the equivalence.
- $(4 \Rightarrow 1)$ We have that $|LN - M^2| \leq 2C^2|\lambda_\Omega|^2$ on W, but by proposition 3.1.1, $EG - F^2 = (E_\Omega G_\Omega - F_\Omega^2)\lambda_\Omega^2$, then $|K| \leq \frac{2C^2}{E_\Omega G_\Omega - F_\Omega^2}$ on $W - \Sigma(\mathbf{x})$. Since $\frac{2C^2}{E_\Omega G_\Omega - F_\Omega^2}$ is continuous on U and W is compact, K is bounded on $W - \Sigma(\mathbf{x})$.
- $(2 \Rightarrow 4)$ If we prove (4) locally on W, we can choose an open covering B_k (open sets with the induced topology) of W in which (4) is satisfied in each compact $\bar{B}_k$ with constants C_k. Reducing this covering to a finite one, we have finite constants $C_{k_1}, .., C_{k_n}$ and choosing C as the maximum of these constants, (4) is satisfied globally on W.

  To prove this locally, first, for each $\mathbf{q} \in W$ let us take a tmb $D\mathbf{y}_l$ as in lemma 5.1.1 on a neighborhood V of $\mathbf{q}$ with $\mathbf{y}_l$ free of umbilical point (5.1.2) on V. Shrinking

V if it is necessary, there exist a diffeomorphism $\mathbf{h}: V' \to V$, such that $\mathbf{y}_l \circ \mathbf{h}$ has derivatives as principal directions. By lemmas 5.2.1 and 5.2.2, we can assume that $\boldsymbol{\Omega} = D\mathbf{y}_l$ being $\mathbf{y}_l$ an embedding with derivatives as principal directions. Thus, by lemmas 5.1.1 and 5.1.2 $\mathbf{I}_\Omega$, $\mathbf{II}_\Omega$, $\boldsymbol{\alpha}_l$ and $\boldsymbol{\Lambda} = (\lambda_{ij}) = \mathbb{I} - l\boldsymbol{\alpha}_l$ are diagonal matrices. If $rank(\boldsymbol{\Lambda}(\mathbf{q})) = 1$, without loss of generality shrinking V to a compact neighborhood, we can suppose that $\lambda_{22}(\mathbf{q}) = 1 - l\alpha_{l22}(\mathbf{q}) = 0$, with $\lambda_{11} \neq 0$ and $-\frac{N_\Omega}{G_\Omega} = \alpha_{l22} \neq 0$ on V. By proposition 3.1.1, $M = 0$, $L = \lambda_{11}L_\Omega$, $N = \lambda_{22}N_\Omega = \lambda_\Omega \frac{N_\Omega}{\lambda_{11}}$ and by hypothesis (2) $|\frac{N_\Omega L_\Omega}{E_\Omega G_\Omega}| \leq C|\lambda_\Omega|$ on $V' = V \cap W$, thus $|L| \leq C|\lambda_{11}||\frac{E_\Omega G_\Omega}{N_\Omega}||\lambda_\Omega|$. If we choose C' as the biggest maximum of the functions $C|\lambda_{11}||\frac{E_\Omega G_\Omega}{N_\Omega}|$ and $|\frac{N_\Omega}{\lambda_{11}}|$ on V', we get that $|L| \leq C'|\lambda_\Omega|$, $|M| \leq C'|\lambda_\Omega|$ and $|N| \leq C'|\lambda_\Omega|$ on V'.

On the other hand, if $rank(\boldsymbol{\Lambda}(\mathbf{q})) = 2$, shrinking V to a compact neighborhood, we can suppose that $\lambda_{11} \neq 0$ and $\lambda_{22} \neq 0$ on V. Thus, $M = 0$, $L = \lambda_{11}L_\Omega = \lambda_\Omega \frac{L_\Omega}{\lambda_{22}}$, $N = \lambda_{22}N_\Omega = \lambda_\Omega \frac{N_\Omega}{\lambda_{11}}$, then choosing C' as the biggest maximum of the functions $|\frac{L_\Omega}{\lambda_{22}}|$ and $|\frac{N_\Omega}{\lambda_{11}}|$ on $V' = V \cap W$, we have $|L| \leq C'|\lambda_\Omega|$, $|M| \leq C'|\lambda_\Omega|$ and $|N| \leq C'|\lambda_\Omega|$ on V'.

□

It is known that the boundedness and extendibility of the Gaussian curvature are equivalent in the non-degenerate case (see proof of theorem 3.1 in (SAJI; UMEHARA; YAMADA, 2009)), however in the degenerate case the following example shows that boundedness does not implies extendibility. Theorem 5.2.2 characterizes extendibility in the general case.

Example 5.2.1. The wavefront $\mathbf{x}(u,v) = (u, 2v^3 + u^2v, 3v^4 + u^2v^2)$ (cuspidal lips) with normal vector $\mathbf{n} = (2uv^2, -2v, 1)(4u^2v^4 + 4v^2 + 1)^{-\frac{1}{2}}$ has an isolated (then, degenerated) singularity at $(0,0)$ of rank 1 and Gaussian curvature $K = -\frac{4v^2}{(4u^2v^4+4v^2+1)^2(u^2+6v^2)}$ with $|K| \leq 1$. Observe that K does not converge when $(u,v) \longrightarrow (0,0)$, then it is not extendable.

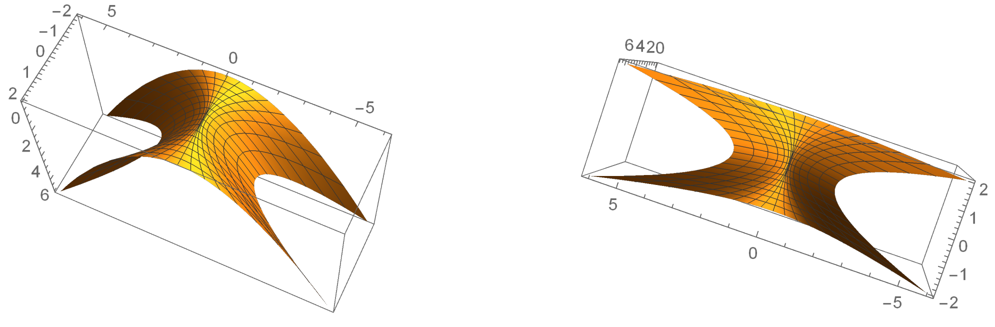

Figure 6 – A wavefront with degenerate singularity of rank 1 at the origin and Gaussian curvature bounded but non-extendable.

Theorem 5.2.2. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront with just singularities of rank 1, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$. Let $V \subset U$ be an open neighborhood of $\mathbf{p}$ in which the relative principal curvature $k_{i\Omega} \neq 0$ does not vanish. Let k_j be the function that admits a C^∞ extension to V and K the Gaussian curvature, then the following statements are equivalent:

1. The Gaussian curvature K admits a C^∞ extension to V.
2. $K_\Omega \in \mathfrak{I}_\Omega(V)$.
3. $k_j \in \mathfrak{I}_\Omega(V)$.
4. $L, M, N \in \mathfrak{I}_\Omega(V)$.

where $\mathfrak{I}_\Omega(V)$ is the principal ideal generated by λ_Ω in the ring $C^\infty(V, \mathbb{R})$.

Proof.

- $(1 \Leftrightarrow 2)$ As $K_\Omega = K\lambda_\Omega$ on $V - \Sigma(\mathbf{x})$ and by the density of $V - \Sigma(\mathbf{x})$ in V, it follows the equivalence.

- $(2 \Leftrightarrow 3)$ Since $k_j = \frac{K_\Omega}{k_{i\Omega}}$, from this last equality follows the equivalence.

- $(4 \Rightarrow 1)$ we have that $LN - M^2 = \phi\lambda_\Omega^2$ with $\phi \in C^\infty(V, \mathbb{R})$, but by proposition 3.1.1, $EG - F^2 = (E_\Omega G_\Omega - F_\Omega^2)\lambda_\Omega^2$, then $K = \frac{\phi}{E_\Omega G_\Omega - F_\Omega^2}$ on $V - \Sigma(\mathbf{x})$. Since $\frac{\phi}{E_\Omega G_\Omega - F_\Omega^2}$ is smooth on V, K has a C^∞ extension to V.

- $(2 \Rightarrow 4)$ if we prove (4) locally on V, we can choose an locally finite open covering $B_k \subset V$ (open balls) of V, $k \in \mathbb{N}$ with a partition of the unity ψ_k subordinated to this open cover in which $L, M, N \in \mathfrak{I}_\Omega(B_k)$ for every B_k. For each $k \in \mathbb{N}$ there exist $f_{1k}, f_{2k}, f_{3k} \in C^\infty(V, \mathbb{R})$ such that $L = f_{1k}\lambda_\Omega$, $M = f_{2k}\lambda_\Omega$, $N = f_{3k}\lambda_\Omega$ on B_k. Since the supports of $f_{sk}\psi_k$ form families locally finite for $s = 1, 2, 3$, we have that $f_s := \sum_k f_{sk}\psi_k \in C^\infty(V, \mathbb{R})$ for $s = 1, 2, 3$, therefore $L = f_1\lambda_\Omega$, $M = f_2\lambda_\Omega$, $N = f_3\lambda_\Omega$ on V.

  To prove this locally, first, for each $\mathbf{q} \in V$ let us take a tmb $D\mathbf{y}_l$ as in lemma 5.1.1 on a neighborhood $Z \subset V$ of $\mathbf{q}$ with $\mathbf{y}_l$ free of umbilical point on Z. Shrinking Z if it is necessary, there exist a diffeomorphism $\mathbf{h}: Z' \to Z$, such that $\mathbf{y}_l \circ \mathbf{h}$ has derivatives as principal directions. By lemmas 5.2.1 and 5.2.2, we can assume that $\boldsymbol{\Omega} = D\mathbf{y}_l$ being $\mathbf{y}_l$ an embedding with derivatives as principal directions on Z. Thus, by lemmas 5.1.1 and 5.1.2 $\mathbf{I}_\Omega$, $\mathbf{II}_\Omega$, $\boldsymbol{\alpha}_l$ and $\boldsymbol{\Lambda} = (\lambda_{ij}) = \mathbb{I} - l\boldsymbol{\alpha}_l$ are diagonal matrices. If $rank(\boldsymbol{\Lambda}(\mathbf{q})) = 1$, without loss of generality shrinking Z to a open neighborhood V', we can suppose that $\lambda_{22}(\mathbf{q}) = 1 - l\alpha_{l22}(\mathbf{q}) = 0$, with $\lambda_{11} \neq 0$ and $-\frac{N_\Omega}{G_\Omega} = \alpha_{l22} \neq 0$ on V'. By proposition 3.1.1, $M = 0$, $L = \lambda_{11}L_\Omega$, $N = \lambda_{22}N_\Omega = \lambda_\Omega\frac{N_\Omega}{\lambda_{11}}$ and by hypothesis (2) $\frac{N_\Omega L_\Omega}{E_\Omega G_\Omega} = \phi\lambda_\Omega$ for some $\phi \in C^\infty(V,\mathbb{R})$, then $L = \phi\lambda_{11}\frac{E_\Omega G_\Omega}{N_\Omega}\lambda_\Omega$. Thus, we get that $L \in \mathfrak{T}_\Omega(V')$, $M \in \mathfrak{T}_\Omega(V')$ and $N \in \mathfrak{T}_\Omega(V')$.

On the other hand, if $rank(\mathbf{\Lambda}(\mathbf{q})) = 2$, shrinking Z to a open neighborhood V', we can suppose that $\lambda_{11} \neq 0$ and $\lambda_{22} \neq 0$ on V'. Thus, $M = 0$, $L = \lambda_{11}L_\Omega = \lambda_\Omega\frac{L_\Omega}{\lambda_{22}}$, $N = \lambda_{22}N_\Omega = \lambda_\Omega\frac{N_\Omega}{\lambda_{11}}$, then we have $L \in \mathfrak{T}_\Omega(V')$, $M \in \mathfrak{T}_\Omega(V')$ and $N \in \mathfrak{T}_\Omega(V')$.

□

Let $[\mathbf{x}] : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a germ of a frontal $\mathbf{x}$, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $0 \in \Sigma(\mathbf{x})$, we define the following ideals:

- J as the ideal in $\mathcal{E}_2$ generated by the germ $[\lambda_\Omega]$.
- $\hat{J} := \{[g] \in \mathcal{E}_2 : \text{there exist } C > 0 \text{ such that } |g| \leq C|\lambda_\Omega| \text{ on some neighborhood of } \mathbf{0}\}$.
- $J_\Sigma := \{[g] \in \mathcal{E}_2 : \text{for some neighborhood } U \text{ of } \mathbf{0}, g \text{ vanish on } U \cap \lambda_\Omega^{-1}(0)\}$.

These ideals satisfy $J \subset \hat{J} \subset J_\Sigma$, their definitions do not depend on the chosen tmb $\mathbf{\Omega}$ and when $\mathbf{0}$ is a non-degenerate singularity these three ideals are equal. To see that, making a change of coordinates, we can assume that λ_Ω is equal to u or v, then applying the Hadamard lemma, we obtain the result. From this and theorems 5.2.1 and 5.2.2 we have the following corollary.

Corollary 5.2.1. Let $[\mathbf{x}] : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a germ of a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $0 \in \Sigma(\mathbf{x})$ a non-degenerate singularity with $rank(D\mathbf{x}(0)) = 1$. Let k_j be the function that admits a local C^∞ extension at 0 and K the Gaussian curvature, then the following statements are equivalent:

1. The Gaussian curvature K admits a local C^∞ extension at 0.
2. The Gaussian curvature K is locally bounded on some neighborhood of 0.
3. $[K_\Omega] \in J_\Sigma$.
4. $[k_j] \in J_\Sigma$.
5. $[L],[M],[N] \in J_\Sigma$.

The equivalences between (1), (2) and (5) were obtained by K. Saji, M. Umehara, and K. Yamada (see proof of theorem 3.1 in (SAJI; UMEHARA; YAMADA, 2009)).

Example 5.2.2. The wavefront $(u, \sin(ku)\frac{v^{k+1}}{k+1}, \sin(ku)\frac{v^{k+2}}{k+2})$, k a positive natural number, has as a tangent moving basis:

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ \cos(ku)k\frac{v^{k+1}}{k+1} & 1 \\ \cos(ku)k\frac{v^{k+2}}{k+2} & v \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \sin(ku)v^k \end{pmatrix}$$

$$\mathbf{II}_\Omega = \begin{pmatrix} \sin(ku)k^2v^{k+2}(\frac{1}{k+1} - \frac{1}{k+2}) & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{\varepsilon}}$$

where $\varepsilon = 1 + v^2 + \cos^2(ku)k^2v^{2k+4}(\frac{1}{k+1} - \frac{1}{k+2})^2$. Then, $\lambda_\Omega = \sin(ku)v^k$, $(L_\Omega N_\Omega - M_{1\Omega}M_{2\Omega}) \in \mathfrak{I}_\Omega$ therefore $K_\Omega \in \mathfrak{I}_\Omega$ and by theorem 5.2.2 the Gaussian curvature K admit a C^∞ extension to $\mathbb{R}^2$. Observe that, since $\mathbf{II} = \mathbf{\Lambda}\mathbf{II}_\Omega$ (see theorem 3.1.1), we have $L, M, N \in \mathfrak{I}_\Omega$.

Remark 5.2.1. The boundedness and extendibility of Gaussian curvature are conserved under changes of coordinates in the domain, however they are not conserved making changes at the target. The wavefront $\mathbf{x}(u,v) = (u, 2v^3 + u^2v, 3v^4 + u^2v^2 + u^2)$ with Gaussian curvature unbounded can be obtained from example 5.2.1 whose Gaussian curvature is bounded, applying at the target the diffeomorphism $F(X,Y,Z) = (X,Y,Z+X^2)$. The same situation occurs with the example 5.2.2 and $(u, \sin(ku)\frac{v^{k+1}}{k+1}, \sin(ku)\frac{v^{k+2}}{k+2} + u^2)$ which have extendable and non-extendable Gaussian curvatures respectively.

In the following, we study the convergence to infinite limits of the classical invariants. Also in the next definition we introduce a notion which is tightly related with this behavior.

Definition 5.2.1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a wavefront and $\mathbf{p} \in \Sigma(\mathbf{x})$. We say that $\mathbf{x}$ is *parallelly smoothable at* $\mathbf{p}$ if there exist $\varepsilon > 0$ and an open neighborhood V of $\mathbf{p}$ such that $rank(D(\mathbf{x} + l\mathbf{n})(\mathbf{q})) = 2$ for every $(\mathbf{q}, l) \in V \times (0, \varepsilon)$ or every $(\mathbf{q}, l) \in V \times (-\varepsilon, 0)$.

Example 5.2.3. The wavefront $\mathbf{x} = (u, \frac{v^3}{3}, \frac{v^4}{4})$ has as singular set the axis u $(v = 0)$ and normal vector field $\mathbf{n} = (0, -v, 1)\rho$, where $\rho = (1 + v^2)^{-\frac{1}{2}}$. Thus, if we consider $\mathbf{y}_l = \mathbf{x} + l\mathbf{n}$ we have

$$D\mathbf{y}_l = \begin{pmatrix} 1 & 0 \\ 0 & v^2 - l\rho - l\rho_v v \\ 0 & v^3 + l\rho_v \end{pmatrix}.$$

Since $v^2 - l\rho - l\rho_v v > 0$ for every $(\mathbf{q}, l) \in \mathbb{R}^2 \times (-\infty, 0)$, then $\mathbf{x}$ is parallelly smoothable at every point of $\Sigma(\mathbf{x})$.

Theorem 5.2.3. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, $K_\Omega(\mathbf{p}) \neq 0$ and $H_\Omega(\mathbf{p}) < 0$ (resp. $H_\Omega(\mathbf{p}) > 0$) then the following statements are equivalents:

1. $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.

2. $\lim\limits_{(u,v)\to p} H = \pm\infty$.

3. $\lim\limits_{(u,v)\to p} k_1 = \pm\infty$ (resp. k_2).

4. $\lim\limits_{(u,v)\to p} K = \pm\infty$.

5. There exist an open neighborhood V of $\mathbf{p}$ in which λ_Ω does not change sign.

6. There exist an open neighborhood V of $\mathbf{p}$ in which $k_{2\Omega}$ (resp. $k_{1\Omega}$) does not change sign.

Proof.

- $(1 \Rightarrow 5)$ if $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$ if and only if there exist $\varepsilon > 0$ and an open neighborhood V of $\mathbf{p}$ such that $\mathbf{y}_t = \mathbf{x} + t\mathbf{n}|_V$ is an immersion for every $t \in (0,\varepsilon)$ (or every $t \in (-\varepsilon, 0)$) if and only if $det(\mathbf{\Lambda}^T + t\boldsymbol{\mu}^T)(\mathbf{q}) = \lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ for every $t \in (0,\varepsilon)$ (or every $t \in (-\varepsilon,0)$) and $\mathbf{q} \in V$. Shrinking V we can suppose this is connected, then we have that $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) > 0$ (or < 0) for every $(\mathbf{q},t) \in V \times (0,\varepsilon)$, thus taking the limit in both sides of this inequality when t tends to 0, we get that $\lambda_\Omega \geq 0$ on V.

- $(5 \Leftrightarrow 2)$ as $H = \frac{H_\Omega}{\lambda_\Omega}$ on $\Sigma(\mathbf{x})^c$ and $H_\Omega(\mathbf{p}) \neq 0$, follows the equivalence.

- $(5 \Leftrightarrow 4)$ as $K = \frac{K_\Omega}{\lambda_\Omega}$ on $\Sigma(\mathbf{x})^c$ and $K_\Omega(\mathbf{p}) \neq 0$, follows the equivalence.

- $(2 \Leftrightarrow 3)$ by proposition 5.2.1, there exist a neighborhood V of $\mathbf{p}$ such that k_2 has a C^∞ extension and since $k_1 = 2H - k_2$ follows the equivalence.

- $(3 \Leftrightarrow 6)$ there exist a neighborhood W of $\mathbf{p}$ such that $H_\Omega < 0$ and $K_\Omega \neq 0$ on W, then $k_1 \neq 0$ on $W - \Sigma(\mathbf{x})$. Since $\frac{K_\Omega}{k_1} = k_{2\Omega}$ on $W - \Sigma(\mathbf{x})$ and $k_{2\Omega} = 0$ on $\Sigma(\mathbf{x})$(by proposition 5.2.1), follows the equivalence.

- $(6 \Rightarrow 1)$ there exist a neighborhood W of $\mathbf{p}$ such that $H_\Omega < 0$ and $K_\Omega \neq 0$ on W. By proposition 5.2.1 $k_{1\Omega}(\mathbf{p}) \neq 0$, thus $\frac{k_{1\Omega}(\mathbf{p})}{K_\Omega(\mathbf{p})} \neq 0$, then using item (6) there exist $\varepsilon > 0$ and an open connected V of $\mathbf{p}$ such that $\frac{k_{2\Omega}}{K_\Omega}$ does not change sign and $|\frac{k_{1\Omega}}{K_\Omega}| > \varepsilon$ on V. Thus, if $\frac{k_{2\Omega}}{K_\Omega} \geq 0$ (resp. $\frac{k_{2\Omega}}{K_\Omega} \leq 0$) then $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ for every $(\mathbf{q},t) \in V \times (-\varepsilon, 0)$ (resp. $V \times (0,\varepsilon)$) because $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) = 0$ if and only if $t = \frac{k_{1\Omega}}{K_\Omega}(\mathbf{q})$ or $t = \frac{k_{2\Omega}}{K_\Omega}(\mathbf{q})$.

□

Theorem 5.2.4. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, $K_\Omega(\mathbf{p}) = 0$ and $H_\Omega(\mathbf{p}) < 0$ (resp. $H_\Omega(\mathbf{p}) > 0$) then the following statements are equivalents:

1. $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.

2. $\lim\limits_{(u,v)\to p} H = \pm\infty$.

3. $\lim\limits_{(u,v)\to p} k_1 = \pm\infty$ (resp. k_2).

4. There exist an open neighborhood V of $\mathbf{p}$, in which λ_Ω does not change sign.

Proof. The proof of $(1 \Rightarrow 4)$, $(2 \Leftrightarrow 3)$, $(2 \Leftrightarrow 4)$ is equal to corresponding ones in theorem 5.2.3 because this does not use the hypothesis of $K_\Omega(\mathbf{p}) \neq 0$. To prove $(3 \Rightarrow 1)$, let us define $A := \{\mathbf{q} \in U : K_\Omega(\mathbf{q}) \neq 0\}$. If $\mathbf{p} \notin \bar{A}$, there exist a neighborhood W of $\mathbf{p}$ in which $K_\Omega \equiv 0$, thus $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) = \lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q})$ on W and since $H_\Omega(\mathbf{p}) \neq 0$, using that λ_Ω does not change sign on a neighborhood of $\mathbf{p}$, shrinking W we have that $\frac{\lambda_\Omega}{H_\Omega}$ does not change sign, then if $\frac{\lambda_\Omega}{H_\Omega} \geq 0$ (resp. $\frac{\lambda_\Omega}{H_\Omega} \leq 0$) we have that $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) \neq 0$ for every $(\mathbf{q}, t) \in W \times (-\varepsilon, 0)$ (resp. $W \times (0, \varepsilon)$) with an arbitrary $\varepsilon > 0$ and it follows the result. If $\mathbf{p} \in \bar{A}$, by hypothesis there exist a open neighborhood W of $\mathbf{p}$ such that $k_1 > 0$ and $H > 0$ (resp. or < 0) on $W - \Sigma(\mathbf{x})$ and as $k_{1\Omega}(\mathbf{p}) \neq 0$, we have that $\lim\limits_{(u,v)\to p} |\frac{k_{1\Omega}}{K_\Omega}|\big|_A = \infty$. Let $\varepsilon > 0$ be given, there exist a open ball B such that $|\frac{k_{1\Omega}}{K_\Omega}| > \varepsilon$ on $B \cap A$ and $k_1 > 0$ (resp. < 0), $H > 0$ (resp. < 0), $\frac{\lambda_\Omega}{H_\Omega} = \frac{1}{H} > 0$ (resp. < 0) on $B - \Sigma(\mathbf{x})$. Since $\frac{k_{2\Omega}}{K_\Omega} = \frac{1}{k_1}$ on $(B \cap A) - \Sigma(\mathbf{x})$, we have that $\frac{k_{2\Omega}}{K_\Omega} \geq 0$ (resp. ≤ 0) on $B \cap A$. Now, if $(\mathbf{q}, t) \in B \times (-\varepsilon, 0)$ (resp. $B \times (0, \varepsilon)$) and $K_\Omega(\mathbf{q}) = 0$ then $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) = \lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) \neq 0$ because $\frac{\lambda_\Omega}{H_\Omega} \geq 0$ (resp. ≤ 0) on B. The another option is that $K_\Omega(\mathbf{q}) \neq 0$, then $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ because this is 0 if and only if $t = \frac{k_{1\Omega}}{K_\Omega}(\mathbf{q})$ or $t = \frac{k_{2\Omega}}{K_\Omega}(\mathbf{q})$ which is impossible since that $|\frac{k_{1\Omega}}{K_\Omega}(\mathbf{q})| > \varepsilon$ and $\frac{k_{2\Omega}}{K_\Omega}(\mathbf{q}) \geq 0$ (resp. ≤ 0). It follows the result. □

Corollary 5.2.2. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, we have that $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$ if and only if λ_Ω does not change sign on a neighborhood of $\mathbf{p}$.

Corollary 5.2.3. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, if $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$, then $\mathbf{p}$ is a degenerate singularity.

Proof. If we suppose that $\mathbf{p} = (p_1, p_2)$ is a non-degenerate singularity, then $\lambda_{\Omega u}(\mathbf{p}) \neq 0$ or $\lambda_{\Omega v}(\mathbf{p}) \neq 0$ and therefore $\lambda_\Omega(u, p_2)$ or $\lambda_\Omega(p_1, v)$ is strictly monotone as function of one variable on every sufficient small neigborhood of $\mathbf{p}$, which is contradictory, because $\lambda_\Omega(\mathbf{p}) = 0$ and this does not change sign by corollary 5.2.2. □

Corollary 5.2.4. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$, if $\mathbf{p}$ is an isolated singularity then $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.

Proof. If $\mathbf{p}$ is an isolated singularity, then there exist an open connected neighborhood V of $\mathbf{p}$, such that $\lambda_\Omega \neq 0$ on $V - \{\mathbf{p}\}$ and since that $V - \{\mathbf{p}\}$ is arc-connected, λ_Ω does not change sign on V. By corollary 5.2.2, it follows the result. □

Corollary 5.2.5 (Representation formula of wavefronts parallelly smoothable rank 1)**.** Let $[\mathbf{x}] : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a germ of a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $0 \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(0)) = 1$. If $\mathbf{x}$ is parallelly smoothable at 0 then, up to an isometry $\mathbf{x}$ is $\mathscr{R}$-equivalent to $\mathbf{y}(u,v) = (u, \int_0^v \lambda_\Omega(u,t)dt + f_1(u), \int_0^v t\lambda_\Omega(u,t)dt + f_2(u))$ where λ_Ω does not change sign on some neighborhood of 0.

Proof. Using proposition 3.4.1 and corollary 5.2.2 we get the result. □

Example 5.2.4. The wavefront $(u, 2v^3 + u^2v, 3v^4 + u^2v^2)$ (cuspidal lips) has an isolated singularity at $(0,0)$, then by corollary 5.2.4 it is parallelly smoothable at $(0,0)$. On the other hand, $(u, 2v^3 - u^2v, 3v^4 - u^2v^2)$ (cuspidal beaks) is not parallelly smoothable at $(0,0)$ by corollary 5.2.2, because taking as a tangent moving basis:

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ -2uv & 1 \\ -2uv^2 & 2v \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 6v^2 - u^2 \end{pmatrix}$$

we get $\lambda_\Omega = 6v^2 - u^2$, which changes of sing on every neighborhood of $(0,0)$. By the same argument, $\mathbf{x}(u,v) = (u, v^2, v^3)$ (cuspidal edge) and $\mathbf{x}(u,v) = (3u^4 + u^2v, 4u^3 + 2uv, v)$ (swallowtail) are not parallelly smoothable at $(0,0)$, because can be chosen tmb's $\mathbf{\Omega}$ in which λ_Ω is v and $12u^2 + 2v$ respectively.

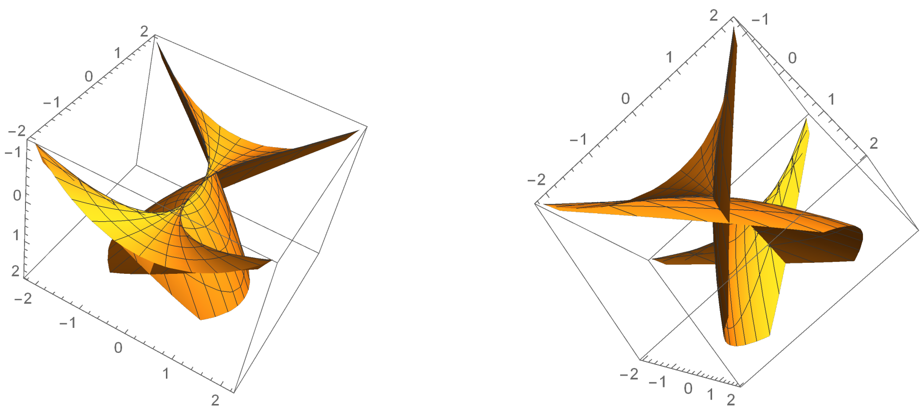

Figure 7 – A wavefront (cuspidal beaks) non-parallelly smoothable at $(0,0)$.

Theorem 5.2.5. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$. One of the principal curvatures κ_-, κ_+ has a C^∞-extension to an open neighborhood of $\mathbf{p}$ if and only if $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.

Proof. If $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$, then by corollary 5.2.2 $\lambda_\Omega \geq 0$ (or $\lambda_\Omega \leq 0$, this case is analogous) on an open neighborhood V of $\mathbf{p}$, thus $k_1 = \kappa_-$, $k_2 = \kappa_+$ on $V - \Sigma(\mathbf{x})$ and by proposition 5.2.1 one of these function has a C^∞-extension to an open neighborhood of $\mathbf{p}$. Conversely, without loss of generality let us suppose that κ_- has a C^∞-extension to an open neighborhood W of $\mathbf{p}$, then λ_Ω does not change sign on some neighborhood of $\mathbf{p}$, otherwise there are sequences $\mathbf{a}_n \longrightarrow \mathbf{p}$, $\mathbf{b}_n \longrightarrow \mathbf{p}$ such that $\lambda_\Omega(\mathbf{a}_n) > 0$ and $\lambda_\Omega(\mathbf{b}_n) < 0$ for every $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} |k_1(\mathbf{a}_n)| = \lim_{n\to\infty} |\kappa_-(\mathbf{a}_n)| = |\kappa_-(\mathbf{p})| = \lim_{n\to\infty} |\kappa_-(\mathbf{b}_n)| = \lim_{n\to\infty} |k_2(\mathbf{b}_n)|$ which is contradictory, because by proposition 5.2.1 one of the limits $\lim_{n\to\infty} |k_1(\mathbf{a}_n)|$, $\lim_{n\to\infty} |k_2(\mathbf{b}_n)|$ is ∞. □

Corollary 5.2.6. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$. If κ_+ (or κ_-) have a C^∞-extension locally at $\mathbf{p}$, then the other one diverges to $\pm\infty$ at $\mathbf{p}$.

Proof. If κ_+ have a C^∞-extension locally at $\mathbf{p}$, by theorem 5.2.5 $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$, then by corollary 5.2.2 $\lambda_\Omega \geq 0$ (or $\lambda_\Omega \leq 0$, this case is similar) on a neighborhood V of $\mathbf{p}$. Thus, $k_1 = \kappa_-$, $k_2 = \kappa_+$ on $V - \Sigma(\mathbf{x})$ and by items (3)s of theorems 5.2.3 and 5.2.4, one of the functions k_1, k_2 diverges to $\pm\infty$. Since $k_2 = \kappa_+$ extends, $k_1 = \kappa_-$ diverges to $\pm\infty$. □

Corollary 5.2.7. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$. One of the principal curvatures κ_-, κ_+ diverges to $\pm\infty$ at $\mathbf{p}$ if and only if $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.

Proof. If κ_- (with κ_+ is analogous) diverges to $\pm\infty$ and we suposse that $\mathbf{x}$ is not parallelly smoothable at $\mathbf{p}$ then λ_Ω changes sign on every neighborhood of $\mathbf{p}$. Thus, there exists sequences $\mathbf{a}_n \longrightarrow \mathbf{p}$, $\mathbf{b}_n \longrightarrow \mathbf{p}$ such that $\lambda_\Omega(\mathbf{a}_n) > 0$ and $\lambda_\Omega(\mathbf{b}_n) < 0$ for every $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} |k_1(\mathbf{a}_n)| = \lim_{n\to\infty} |\kappa_-(\mathbf{a}_n)| = \pm\infty = \lim_{n\to\infty} |\kappa_-(\mathbf{b}_n)| = \lim_{n\to\infty} |k_2(\mathbf{b}_n)|$, which is contradictory, because by proposition 5.2.1 one of the functions k_1, k_2 extends at $\mathbf{p}$. The converse follows immediately from theorem 5.2.5 and corollary 5.2.6. □

Corollary 5.2.8. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 1$. The principal curvature κ_- (resp. κ_+) is bounded locally at $\mathbf{p}$ if and only if κ_- (resp. κ_+) have a C^∞-extension locally at $\mathbf{p}$.

Proof. If κ_- is bounded locally at $\mathbf{p}$, using the same reasoning of the proof in theorem 5.2.5, we have that $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$, then by theorem 5.2.5 and corollary

5.2.6 one principal curvature has a C^∞-extension locally at $\mathbf{p}$ and the other one diverge. Since κ_- is bounded locally at $\mathbf{p}$, this is the extendable one. □

5.2.2 Singularities of rank 0

Proposition 5.2.2. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$, then:

1. $(k_{1\Omega}, k_{2\Omega})(\mathbf{p}) = (0,0)$.
2. $\lim_{(u,v)\to p} |K| = \infty$.
3. $\frac{1}{k_1}$ and $\frac{1}{k_2}$ have continuous extensions on a neighborhood V of $\mathbf{p}$, which are of class C^∞ except possibly at umbilical points and singularities of rank 0 of $\mathbf{x}$.
4. $\lim_{(u,v)\to p} |k_1| = \infty$ and $\lim_{(u,v)\to p} |k_2| = \infty$.
5. $\frac{1}{k_1} + \frac{1}{k_2}$ has a C^∞-extension on a neighborhood V of $\mathbf{p}$.

Proof.

1. By theorem 3.3.1, $H_\Omega(\mathbf{p}) = 0$, then $k_{1\Omega}(\mathbf{p}) = H_\Omega(\mathbf{p}) - \sqrt{H_\Omega^2(\mathbf{p}) - \lambda_\Omega(\mathbf{p})K_\Omega(\mathbf{p})} = 0$. Similarly $k_{2\Omega}(\mathbf{p}) = 0$.
2. By theorem 3.3.1, $K_\Omega(\mathbf{p}) \neq 0$ and then it follows by proposition 3.3.1 that $\lim_{(u,v)\to p} |K| = \lim_{(u,v)\to p} |\frac{K_\Omega}{\lambda_\Omega}| = \infty$
3. There exist a neighborhood V of $\mathbf{p}$ such that $K_\Omega(\mathbf{p}) \neq 0$ and since $K_\Omega = \lambda_\Omega k_1 k_2$ on $\Sigma(\mathbf{x})^c$ then $k_1 \neq 0, k_2 \neq 0$ on $V - \Sigma(\mathbf{x})$, therefore $\frac{1}{k_1} = \frac{k_2}{k_1 k_2} = \frac{k_{2\Omega}}{K_\Omega}$ and similarly $\frac{1}{k_2} = \frac{k_{1\Omega}}{K_\Omega}$ which are well defined on V. Notice that $k_{1\Omega}$ and $k_{2\Omega}$ may not be differentiable at umbilical points and singularities of rank 0 of $\mathbf{x}$ (because $H_\Omega^2(\mathbf{p}) - \lambda_\Omega(\mathbf{p})K_\Omega(\mathbf{p}) = 0$ at those points and this expression is under a radical sign in the relative principal curvatures).
4. By item (2), there exist a neighborhood V of $\mathbf{p}$ such that $k_1 \neq 0, k_2 \neq 0$ on $V - \Sigma(\mathbf{x})$, therefore $k_{1\Omega} \neq 0, k_{2\Omega} \neq 0$ as well. Then, $k_1 = \frac{K_\Omega}{k_{2\Omega}}$ and $k_2 = \frac{K_\Omega}{k_{1\Omega}}$ on $V - \Sigma(\mathbf{x})$ and using that $K_\Omega(\mathbf{p}) \neq 0$ and item (1) we get (3).
5. Since $K_\Omega(\mathbf{p}) \neq 0$, there exists an open neighborhood V of $\mathbf{p}$ such that K_Ω does not vanish on V, then $\frac{1}{k_1} + \frac{1}{k_2} = \frac{2H}{K} = \frac{2H_\Omega}{K_\Omega}$ on $V - \Sigma(\mathbf{x})$ and as $\frac{2H_\Omega}{K_\Omega}$ is well defined on V, this is a C^∞-extension.

□

Theorem 5.2.6. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$, then the following statements are equivalent:

1. $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$.
2. $\lambda_\Omega K_\Omega \geq 0$ and H_Ω does not change sign on a neighborhood V of $\mathbf{p}$.
3. $\lim_{(u,v)\to p} K = \infty$ and $\lim_{(u,v)\to p} H = \pm\infty$.
4. $\lim_{(u,v)\to p} k_1 = \lim_{(u,v)\to p} k_2 = \infty$ or $\lim_{(u,v)\to p} k_1 = \lim_{(u,v)\to p} k_2 = -\infty$.

Proof.

- $(1 \Rightarrow 2)$ If $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$, then there exist $\varepsilon > 0$ and an open connected neighborhood V of $\mathbf{p}$ such that $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ for every $(\mathbf{q},t) \in V \times (0,\varepsilon)$ (or $V \times (-\varepsilon, 0)$, this case is analogous). As $K_\Omega \neq 0$ and does not change sign on V (shrinking V if it is necessary), $k_{1\Omega} = k_{2\Omega} = 0$ and since $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ if and only if $t = \frac{k_{1\Omega}}{K_\Omega}(\mathbf{q})$ or $t = \frac{k_{2\Omega}}{K_\Omega}(\mathbf{q})$, we have that $\frac{k_{1\Omega}}{K_\Omega} \leq 0$ and $\frac{k_{2\Omega}}{K_\Omega} \leq 0$ on V. Then, $k_{1\Omega}k_{2\Omega} \geq 0$ on V, but $-k_{1\Omega}$ and $-k_{2\Omega}$ are the eigenvalues of $\boldsymbol{\alpha}_\Omega^T = (\boldsymbol{\mu}_\Omega adj(\boldsymbol{\Lambda}))^T$, then $\lambda_\Omega K_\Omega = k_{1\Omega}k_{2\Omega} \geq 0$ on V. Observe that $k_{1\Omega}$ and $k_{2\Omega}$ do not change sign on V, then H_Ω neither.

- $(2 \Rightarrow 3)$ Since $\lambda_\Omega K_\Omega = \lambda_\Omega^2 K$ on $\Sigma(\mathbf{x})^c$ and using that $\lim_{(u,v)\to p} |K| = \infty$ we get that $\lim_{(u,v)\to p} K = \infty$. On the other hand, $H^2 \geq K$, then $\lim_{(u,v)\to p} |H| = \infty$. As H_Ω and λ_Ω do not change sign on a neighborhood of $\mathbf{p}$, $H = \frac{H_\Omega}{\lambda_\Omega}$ neither and we get the result.

- $(3 \Rightarrow 4)$ As K is positive near to $\mathbf{p}$ then $\lambda_\Omega K_\Omega = \lambda_\Omega^2 K \geq 0$ and $K_\Omega \neq 0$ on a neighborhood Z of $\mathbf{p}$. Shrinking Z, λ_Ω does not change sign and H neither on $Z - \Sigma(\mathbf{x})$. Without loss of generality, let us suppose $\lambda_\Omega \geq 0$ on Z, then $k_1 = H - \sqrt{H^2 - K}$ and $k_2 = H + \sqrt{H^2 - K}$ and since H does not change sign on $Z - \Sigma(\mathbf{x})$, one of the functions k_1, k_2 neither. By this last and since that $K > 0$, we have that $k_1 > 0, k_2 > 0$ or $k_1 < 0, k_2 < 0$ on $Z - \Sigma(\mathbf{x})$, then using item (3) of proposition 5.2.2 we get the result.

- $(4 \Rightarrow 1)$ There exists a neighborhood V of $\mathbf{p}$ such that $k_1 > 0, k_2 > 0$ (or $k_1 < 0, k_2 < 0$, this case is analogous) on $V - \Sigma(\mathbf{x})$ and $K_\Omega \neq 0$ on V, then $\frac{k_{1\Omega}}{K_\Omega} = \frac{1}{k_2} > 0$ and $\frac{k_{2\Omega}}{K_\Omega} = \frac{1}{k_1} > 0$ on $V - \Sigma(\mathbf{x})$, thus by density the of $V - \Sigma(\mathbf{x})$ $\frac{k_{1\Omega}}{K_\Omega}, \frac{k_{2\Omega}}{K_\Omega} \geq 0$ on V. Choose $\varepsilon > 0$ arbitrary and we have that $\lambda_\Omega(\mathbf{q}) - 2tH_\Omega(\mathbf{q}) + t^2K_\Omega(\mathbf{q}) \neq 0$ for every $(\mathbf{q},t) \in V \times (-\varepsilon, 0)$. It follows (1).

□

Corollary 5.2.9. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$. If there exist a neighborhood V of $\mathbf{p}$ in which the only singularity of rank 0 is $\mathbf{p}$, then $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$ if and only if $\lambda_\Omega K_\Omega \geq 0$ on a neighborhood W of $\mathbf{p}$.

Proof. If $\lambda_\Omega K_\Omega \geq 0$ on W, shrinking if it is necessary we can suppose that $K \neq 0$ on $W - \Sigma(\mathbf{x})$ and since $\lambda_\Omega K_\Omega = \lambda_\Omega^2 K$, then $H^2 > K > 0$ on $W - \Sigma(\mathbf{x})$. As $H_\Omega \neq 0$ on singularities of rank 1, then H_Ω has a isolated zero on $W \cap V$ and therefore H_Ω does not change sign. Applying the last theorem we get the result. □

Observe that, if we have a wavefront $\mathbf{x}: U \to \mathbb{R}^3$, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$ and this is parallelly smoothable at $\mathbf{p}$, since $K_\Omega(\mathbf{p}) \neq 0$ and $\lambda_\Omega K_\Omega \geq 0$ on a neighborhood of $\mathbf{p}$, then λ_Ω does not change sign on a neighborhood of $\mathbf{p}$. However, this condition is not sufficient as it happened in the case of singularities of rank 1. The next example shows this.

Example 5.2.5. The wavefront $\mathbf{x} := (u^k, \pm v^k, \frac{k}{k+1}u^{k+1} \pm \frac{k}{k+1}v^{k+1})$, with $k \in \mathbb{N}$, $k \geq 2$ has as tmb:

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u & v \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} ku^{k-1} & 0 \\ 0 & \pm kv^{k-1} \end{pmatrix},$$

then $\lambda_\Omega = \pm k^2 u^{k-1} v^{k-1}$, $K_\Omega = \frac{1}{(1+u^2+v^2)^2}$. By corollary 5.2.9, $\mathbf{x}$ is parallelly smoothable at $(0,0)$ when we choose k odd and the sign $+$. If k is even or the sign is $-$, this is not parallelly smoothable at $(0,0)$, even when k is odd with sing $-$ in the expression, in which λ_Ω does not change sign.

Corollary 5.2.10. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$ and $\Sigma(\mathbf{x})_0 = \{\mathbf{q} \in \Sigma(\mathbf{x}) : rank(D\mathbf{x}(\mathbf{q})) = 0\}$. If $\mathbf{x}$ is parallelly smoothable at $\mathbf{p}$ then:

1. There exists an open neighborhood V of $\mathbf{p}$ in which one of the functions k_1, k_2 has a C^∞ extension to $V - \Sigma(\mathbf{x})_0$. More precisely, k_1 (resp. k_2) has a C^∞ extension to $V - \Sigma(\mathbf{x})_0$ if only if $H_\Omega \leq 0$ (resp. $H_\Omega \geq 0$) on V.
2. There exists an open neighborhood V of $\mathbf{p}$ in which one of the functions k_1, k_2 diverge to $\pm\infty$ (just one sign globally) near the singularities to $\mathbf{p}$. More precisely, $\lim\limits_{(u,v)\to\Sigma(\mathbf{x})\cap V} k_1 = \pm\infty$ (resp. k_2) if and only if $H_\Omega \leq 0$ (resp. $H_\Omega \geq 0$) on V.

Proof.

1. Since that H_Ω does not change sign on a neighborhood V of $\mathbf{p}$ by item (2) of proposition 5.2.6 and applying proposition 5.2.1 we get the result.

2. By items (2) and (4) of proposition 5.2.6 H_Ω and k_1 do not change sign on a neighborhood V of $\mathbf{p}$ and applying proposition 5.2.1 we get the result.

□

Proposition 5.2.3. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper wavefront, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$. If H is bounded on a neighborhood of $\mathbf{p}$ then we have:

1. There exists a neighborhood V of $\mathbf{p}$ such that $K < 0$ on $V - \Sigma(\mathbf{x})$ and $\lim_{(u,v)\to p} K = -\infty$.
2. λ_Ω does not change sign on a neighborhood V of $\mathbf{p}$.
3. One of the functions k_1, k_2 diverges to ∞ and the another one to $-\infty$ at $\mathbf{p}$. More precisely, if $\lambda_\Omega \geq 0$ (resp. $\lambda_\Omega \leq 0$) on a neighborhood V of $\mathbf{p}$ then $\lim_{(u,v)\to p} k_1 = -\infty$ (resp. ∞) and $\lim_{(u,v)\to p} k_2 = \infty$ (resp. $-\infty$).
4. There is no singularity of rank 1 on a neighborhood of $\mathbf{p}$.
5. There exists a neighborhood V of $\mathbf{p}$ such that $\lim_{(u,v)\to\Sigma(\mathbf{x})\cap V} \frac{k_1}{k_2} = -1$

Proof.

1. We know that $K \neq 0$ near to $\mathbf{p}$. If there exists a sequence $\mathbf{a}_n \longrightarrow \mathbf{p}$ with $K(\mathbf{a}_n) > 0$ for every $n \in \mathbb{N}$, as $H^2 \geq K$ and $\lim_{(u,v)\to p} K(\mathbf{a}_n) = \infty$ we have that $\lim_{(u,v)\to p} |H(\mathbf{a}_n)| = \infty$ which is contradictory, then we have (1).
2. By (1) $\lambda_\Omega K_\Omega = \lambda_\Omega^2 K < 0$ near to $\mathbf{p}$ and since $K_\Omega(\mathbf{p}) \neq 0$, then λ_Ω does not change sign on a neighborhood V of $\mathbf{p}$.
3. If $\lambda_\Omega \geq 0$ near p, $k_1 = H - \sqrt{H^2 - K}$ and $k_2 = H + \sqrt{H^2 - K}$ on a neighborhood of $\mathbf{p}$ and since $K < 0$ near $\mathbf{p}$, then $k_2 > 0 > k_1$ on a neighborhood of $\mathbf{p}$. By (3) of proposition 5.2.2 we have the result.
4. If H is bounded on a neighborhood V of $\mathbf{p}$ and suppose that there exists a singularity $\mathbf{q}$ of rank 1 in V, by (4) of proposition 5.2.1 $\lim_{(u,v)\to p} |H| = \infty$ witch is contradictory.
5. Let V be a bounded neighborhood of $\mathbf{p}$ with just singularities of rank 0 with H bounded. There exists $C > 0$ such that $|k_1 + k_2| < C$, then $|1 + \frac{k_1}{k_2}| < \frac{C}{|k_2|}$ and by (3) of proposition 5.2.2 $\lim_{(u,v)\to\mathbf{q}} \frac{k_1}{k_2} = -1$ for every $\mathbf{q} \in \Sigma(\mathbf{x}) \cap V$. Since $\Sigma(\mathbf{x}) \cap V$ is compact we have the result.

□

Proposition 5.2.4. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$ and let us choose W a compact neighborhood of $\mathbf{p}$ in which $K_\Omega \neq 0$. The following statements are equivalent:

1. The mean curvature H is bounded on $W - \Sigma(\mathbf{x})$.
2. There exists $C > 0$ such that $|H_\Omega| \leq C|\lambda_\Omega|$ on W.
3. There exists $C > 0$ such that $|LG + NE - 2MF| \leq C|\lambda_\Omega^2|$ on W.
4. There exists $C > 0$ such that $|\frac{1}{k_1} + \frac{1}{k_2}| \leq C|\lambda_\Omega|$ on W.
5. $\frac{1}{k_{1\Omega}} + \frac{1}{k_{2\Omega}}$ is bounded on W.

Proof.

- $(1 \Leftrightarrow 2)$ Using that $H_\Omega = \lambda_\Omega H$ on $W - \Sigma(\mathbf{x})$ which is dense in W, we get the equivalence.
- $(1 \Leftrightarrow 3)$ Using that $H(EG - F^2) = LG + NE - 2MF$ on $W - \Sigma(\mathbf{x})$ and $EG - F^2 \in \mathfrak{T}_\Omega^2(W)$ (see proposition 3.1.1), by compactness of W we get the equivalence.
- $(2 \Leftrightarrow 4)$ by proposition 5.2.2 $\frac{1}{k_1} + \frac{1}{k_2}$ has a C^∞-extension to W and this is equal to $\frac{2H_\Omega}{K_\Omega}$. From this equality follows the equivalence.
- $(4 \Leftrightarrow 5)$ Since that $k_{1\Omega} = \lambda_\Omega k_1$ and $k_{2\Omega} = \lambda_\Omega k_2$ on $W - \Sigma(\mathbf{x})$ which is dense in W, we get the equivalence.

□

Proposition 5.2.5. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper wavefront, $\mathbf{\Omega}$ a tmb of $\mathbf{x}$, $\mathbf{p} \in \Sigma(\mathbf{x})$ with $rank(D\mathbf{x}(\mathbf{p})) = 0$ and let us choose V an open neighborhood of $\mathbf{p}$ in which $K_\Omega \neq 0$. The following statements are equivalent:

1. The mean curvature H has a C^∞-extension to the neighborhood V of $\mathbf{p}$.
2. $H_\Omega \in \mathfrak{T}_\Omega(V)$.
3. $LG + NE - 2MF \in \mathfrak{T}_\Omega^2(V)$.
4. $\frac{1}{k_1} + \frac{1}{k_2} \in \mathfrak{T}_\Omega(V)$.
5. $\frac{1}{k_{1\Omega}} + \frac{1}{k_{2\Omega}}$ has a C^∞-extension to the neighborhood V of $\mathbf{p}$.

Proof. The proof of proposition 5.2.4 can be reproduced here to prove the corresponding equivalences. □

Example 5.2.6. The wavefront $\mathbf{x} := \left(\frac{1}{2}\log\left(v^2+1\right) - \frac{1}{2}\log\left(u^2+1\right), \frac{uv}{v^2+1}, \frac{uv^2}{v^2+1} - u + \tan^{-1}(u)\right)$ (see figure 8) has as tmb:

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u & v \end{pmatrix}, \ \boldsymbol{\Lambda} = \begin{pmatrix} \frac{-u}{1+u^2} & \frac{v}{1+v^2} \\ \frac{v}{1+v^2} & \frac{u(1-v^2)}{(1+v^2)^2} \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \frac{-(1+v^2)}{(1+u^2+v^2)^{\frac{3}{2}}} & \frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}} \\ \frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}} & \frac{-(1+u^2)}{(1+u^2+v^2)^{\frac{3}{2}}} \end{pmatrix},$$

then $\lambda_\Omega = \frac{-(u^2+v^2)}{(1+u^2)(1+v^2)^2}$, $H_\Omega = -\frac{1}{2}(\lambda_{22}\mu_{11} - \lambda_{21}\mu_{12} + \lambda_{11}\mu_{22} - \lambda_{12}\mu_{21}) = 0$ and therefore the mean curvature is extendable, with $H = 0$ on $\mathbb{R}^2$.

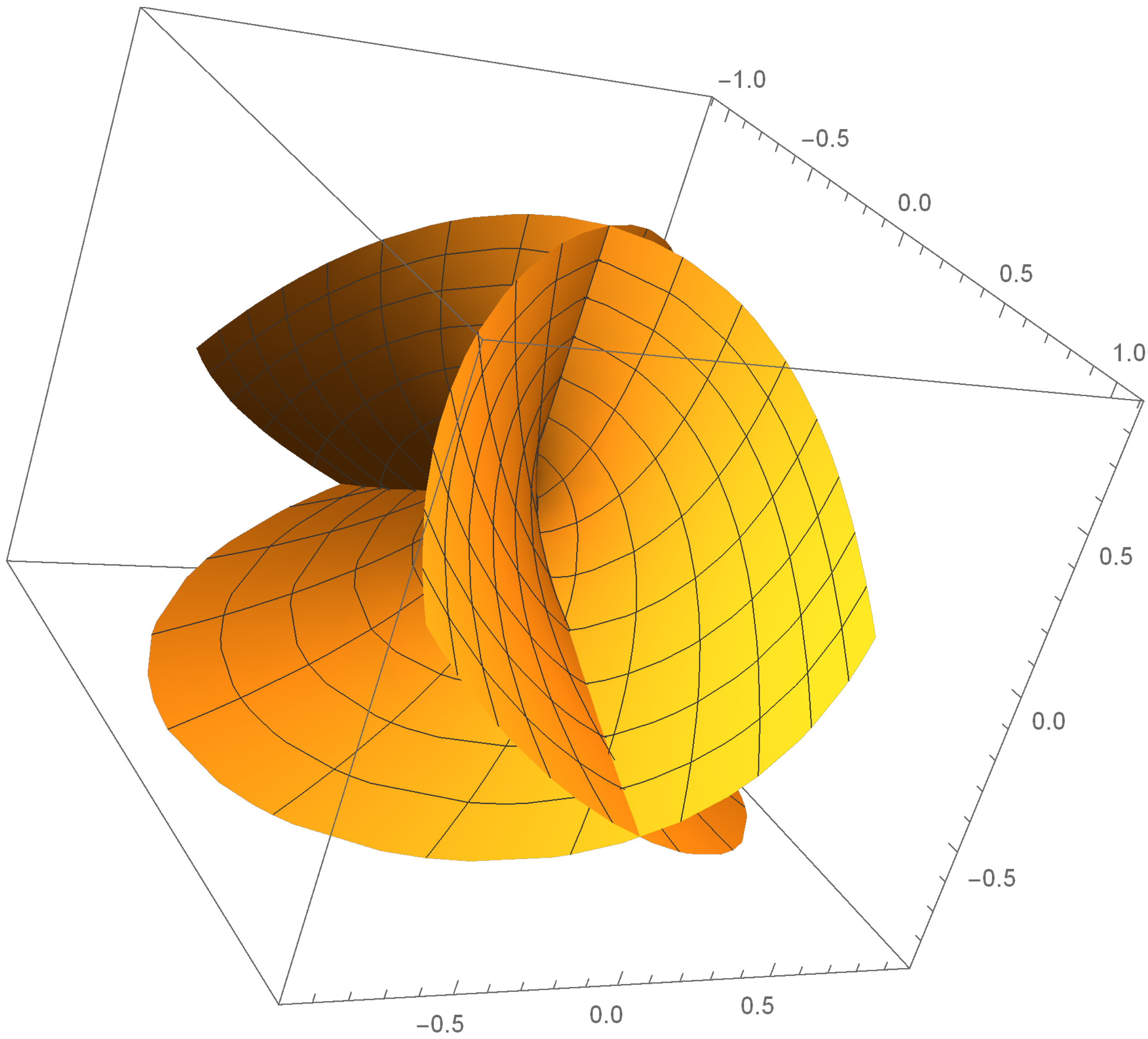

Figure 8 – A wavefront with singularity of rank 0 at the origin and mean curvature vanishing everywhere.

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