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Stability and hyperbolicity of equilibria for a non-local quasilinear Chafee-Infante equation

Rafael de Oliveira Moura

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Rafael de Oliveira Moura

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Estabilidade e hiperbolicidade de equilíbrios para uma equação de Chafee-Infante quasilinear não-local

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *VERSÃO REVISADA*

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To my parents,
who raised me and worked hard
so that I could follow my dreams

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"The best way to predict the future
is to create it."
- Abraham Lincoln

ABSTRACT

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In this work we present the topics of spectral theory of operators, theory of semigroups and their generators and geometric theory of parabolic semilinear differential equations, and then apply these theories to analyze the qualitative aspects of the semilinear Chafee-Infante equation. Finally, we seek to study stability and hyperbolicity of equilibria for a non-local quasilinear Chafee-Infante equation, making use of a method of linearization for quasilinear problems, which has been developed in (CARVALHO; MOREIRA, 2021), in order to conclude that the equilibria of this complicated equation inherit some properties of stability and hyperbolicity from the classical semilinear equation.

Keywords: Spectral Analysis, Semigroups, Global Attractor, Gradient Semigroups, Semilinear Partial Differential Equations, Chafee-Infante Equation, Quasilinear Chafee-Infante Equation, Nonlocal Chafee-Infante Equation.

RESUMO

MOURA, R. O. **Estabilidade e hiperbolicidade de equilíbrios para uma equação de Chafee-Infante quasilinear não-local**. 2022. 144 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Neste projeto apresentamos os tópicos de teoria espectral de operadores, teoria de semigrupos e seus geradores e teoria geométrica de equações diferenciais parabólicas semilineares, e em seguida aplicamos tais conhecimentos para analisar os aspectos qualitativos da equação de Chafee-Infante semilinear. Por fim, busca-se estudar estabilidade e hiperbolicidade dos equilíbrios de uma equação de Chafee-Infante quasilinear não-local, utilizando-se um método de linearização para problemas quasilineares, desenvolvido em (CARVALHO; MOREIRA, 2021), a fim de se concluir que os equilíbrios dessa equação complicada herdam algumas propriedades de estabilidade e hiperbolicidade do caso semilinear clássico.

Palavras-chave: Análise Espectral, Semigrupos, Atrator Global, Semigrupos Gradientes, Equações Diferenciais Parciais Semilineres, Equação de Chafee-Infante, Equação de Chafee-Infante Quasilinear, Equação de Chafee-Infante Não-Local.

CONTENTS

CHAPTER

1

INTRODUCTION

Differential equations are widely studied in mathematics as they represent an essential tool for modeling phenomena of interest to several areas of study. Among them, we can mention the movement of bodies and particles (in physics), the dynamics of an ecosystem (in biology), the behavior of beams (in engineering), the growth of tumors (in medicine), and many others.

A large class of differential equations, in which we will focus our work in this project, is the class of semilinear partial differential equations. We are interested in studying the following initial value problem:

$$\begin{aligned} &\frac{d}{dt}u = Au + f(u,t), \quad t > t_0 \\ &u(t_0) = u_0 \in X, \end{aligned} \tag{1.1}$$

where X is a Banach space, $A : D(A) \subset X \to X$ is a closed and densely defined operator ($\overline{D(A)} = X$), and the function f has some regularity conditions.

Our study of (1.1) involves existence and uniqueness of solutions and global attractor, and stability and hyperbolicity of equilibria. Those concepts are essential to understand the asymptotic behavior of a semilinear differential equation, and their meaning will be explained soon. Therefore, throughout our study we will make use of several different mathematical theories, which will be presented along the chapters in this thesis. We organize the presentation of those theories as following.

In Chapter 2, we define the concepts of resolvent and spectrum of a closed operator, and present results about the resolvent of bounded and compact operators. Moreover, we present the numerical range, as a way to localize the resolvent of an operator. Finally, we present the basic theory of self-adjoint operators, and we use all the previous results to characterize the spectrum and eigenvectors of the Laplacian and Sturm-Liouville operators.

Let us denote by $\mathscr{L}(X)$ the space of all bounded linear maps from X into X, with the usual norm. Moreover, if $A : D(A) \subset X \to X$ is a linear operator, we denote by $R(A)$ its range and by $N(A)$ its kernel. In order to study the problem (1.1), it is crucial to analyze the spectrum and resolvent of the operator A, defined as following:

Definition 1 (Resolvent and spectrum)**.** Let $A : D(A) \subset X \to X$ be a closed linear operator in X. We define the **resolvent set** of A as following:

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) : D(A) \to X \text{ is a bijection}\}.$$

The **spectrum** of A is defined as $\sigma(A) := \mathbb{C} \setminus \rho(A)$, and it is decomposed in three disjoint parts:

1. The **point spectrum** of A is the set of eigenvalues of A, that is, $\sigma_p(A) := \{\lambda \in \sigma(A) : (\lambda - A) \text{ is not injective}\}$.
2. The **residual spectrum** of A is the set $\sigma_r(A) := \{\lambda \in \sigma(A) : (\lambda - A) \text{ is injective and } R(\lambda - A) \text{ is not dense in } X\}$.
3. The **continuous spectrum** of A is the set $\sigma_c(A) := \{\lambda \in \sigma(A) : (\lambda - A) \text{ is injective and } R(\lambda - A) \text{ is dense in } X \text{ but } R(\lambda - A) \neq X\}$.

If $\lambda \in \rho(A)$, $(\lambda - A)^{-1} \in \mathscr{L}(X)$. We call $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathscr{L}(X)$ the **resolvent** of A.

The study of the resolvent of the operator A is essential to develop the theory of fractional powers, as well as the theory of existence and uniqueness of solutions for (1.1). The analysis of the spectrum of A is also used to understand stability and hyperbolicity of equilibria, concepts that will be explained latter in this thesis.

For a moment consider the problem (1.1) with $t_0 = 0$, and a function f that does not depend on time (this is called the autonomous case, which will be more relevant in our work) Suppose that, for each $u_0 \in X$, (1.1) has an unique solution $u(\cdot, u_0) : \mathbb{R}^+ \to X$ such that $u(0, u_0) = u_0$, and this solutions depends continuously on t and u_0. Then we define, for each $t \geq 0$, the operator $T(t) : X \to X$ by $T(t)u_0 = u(t, u_0)$. It may be checked that the family $\{T(t) : t \geq 0\}$ satisfies the following properties:

- $T(0)x = x, \forall\, x \in X$.
- $T(t+s) = T(t)T(s), \forall\, t, s \geq 0$.
- $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.

A family with these properties is called a semigroup in X. It is in the language of semigroups that we study the asymptotic behavior of our differential equation, that is, the behavior of the solutions when $t \to \infty$. In Chapter 3, we define and study several objects that help us understand the asymptotic behavior of the solutions of a differential equation, including the global attractor, which is a compact subset of X that is invariant by the action of the semigroup and attracts bounded sets of X.

If $x^* \in X$ is such that $T(t)x^* = x^*$ for all $t \geq 0$, we call x^* an equilibrium for $\{T(t) : t \geq 0\}$. Let $\mathscr{E}$ denote the set of equilibria for $\{T(t) : t \geq 0\}$. A particular section of Chapter 3 will be dedicated to the study of gradient semigroups, for which there exists a function $V : X \to \mathbb{R}$ such that:

1. The map $\mathbb{R}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing, for each $x \in X$.

2. If $V(T(t)x) = V(x)$, $\forall\, t \geq 0$, then $x \in \mathscr{E}$.

For this kind of semigroup, the global attractor can be very well characterized, consisting of the equilibria and connections among them.

Also in Chapter 3, we study the relations between the semigroup associated to a linear differential equation $\dot{u} = Au$ and the operator A. It is important to understand the characteristics of the semigroup associated to this linear problem in order to extract properties of the semigroup associated to the semilinear problem (1.1). We present some results in this sense, and in particular, we prove that if $-A$ is a sectorial operator, the semigroup associated to A is analytic in a sector containing the positive real axis, and satisfies some estimates that are useful in the study of semilinear parabolic equations (this is Theorem 23).

Next, in Chapter 4, we present the theory of fractional powers of positive operators, which helps to develop the theory of parabolic differential equations in Chapter 5. Assuming that $-A$ is an operator of positive kind, we define $(-A)^{\alpha}$, for negative real α, by means of complex integration involving the resolvent operator, and $(-A)^{\alpha}$ for positive α taking the inverse. We study the properties of the domains of $(-A)^{\alpha}$ for positive α, studying inclusions and interpolation results, as well as the relation between fractional powers and semigroups, and with the perturbation of sectorial operators.

In Chapter 5, we define the semilinear differential equations of parabolic type, which are the main focus of the work in this thesis. A semilinear parabolic differential equation has the form of (1.1), but we strengthen the hypothesis on the operator A, asking $-A$ to be sectorial (see Definition 28) and positive (see Definition 29). Moreover, in a parabolic semilinear differential equation, the function f maps from $D((-A)^{\alpha})$ into X, where $D((-A)^{\alpha})$ is the domain of some fractional power of $-A$. We can define a solution for (1.1) starting at any point u_0 in $D((-A)^{\alpha})$, even if this point is not in $D(A)$, and the solution defined in $[t_0, t_1)$ will remain in $D(A)$ for all $t \in (t_0, t_1)$, which ensures extra regularity.

Also in Chapter 5, we will study existence and uniqueness of solutions and their continuous dependence with respect to the initial value $u_0 \in X$, presenting theorems that state that, under certain conditions on A and f, (1.1) has an unique solution $u(\cdot,t_0,u_0):[t_0,t_{max})\to X$ such that $u(t_0)=u_0$, for each initial value (t_0,u_0) in the domain of f, and we give conditions under which the solutions are defined for every $t\geq t_0$.

Finally, we give a lot of importance to the analysis of the equilibria of a differential equation. In Chapter 5, we study the concepts of stability and instability of an equilibrium point, which basically are related to the behavior of solutions that start near the equilibrium. If a solution starts in a point u_0 close enough to a stable equilibrium $x^*\in X$, then $T(t)u_0$ will remain close to x^* for all positive t. In the same chapter, we also study the concept of hyperbolicity, that is related to the structure of the global attractor near an equilibrium and its robustness under perturbation.

The concept of hyperbolicity of an equilibrium point is related to the existence of two "directions" in X, such that the equilibrium attracts in one direction and expels in another. We can prove that, under certain conditions, this behavior is inherited by a semilinear equation from the linear equation associated, and, near to the equilibrium, the directions of attraction and repulsion do not change much with the adding of the nonlinearity. The reader may see the precise meaning of these comments in Chapter 5.

Both stability and hyperbolicity for an equilibrium of a semilinear differential equation can be concluded using spectral analysis of the linearization operators around this point.

In the Chapter 6, we study the well known Chafee-Infante equation (CHAFEE; INFANTE, 1974).

$$\begin{aligned} &u_t=u_{xx}+\lambda f(u), \quad t>0,\ x\in(0,\pi)\\ &u(0,t)=u(\pi,t)=0,\\ &u(\cdot,0)=u_0\in H_0^1(0,\pi), \end{aligned} \tag{1.2}$$

where $\lambda>0$ is a parameter, $f\in\mathscr{C}^2(\mathbb{R})$ is odd (in particular, $f(0)=0$), $f'(0)=1$, and f satisfies:

$$f''(u)u<0,\ \forall\, u\neq 0,$$

and

$$\limsup_{|u|\to\infty}\frac{f(u)}{u}<0.$$

We apply the theory developed in the first chapters to show that for each $u_0\in H_0^1(0,\pi)$, there exists a unique solution $u(\cdot,u_0)\in\mathscr{C}\left(\mathbb{R}^+,H_0^1(0,\pi)\right)$ of (1.2), and the map defined by $\mathbb{R}^+\times H_0^1(0,\pi)\ni(t,u_0)\mapsto u(t,u_0)\in H_0^1(0,\pi)$ is continuous. If we define $T(t)u_0=u(t,u_0)$, the semigroup $\{T(t):t\in\mathbb{R}^+\}$ is gradient.

Next we study the equilibria of (1.2) in order to obtain a characterization for the gradient-kind global attractor of this semigroup. When $n^2<\lambda\leq(n+1)^2$, $n=0,1,2,...$ an analysis of the phase plane associated to the boundary problem

$$\begin{aligned} &w_{xx} + \lambda f(w) = 0, \quad 0 < x < \pi, \\ &w(0) = w(\pi) = 0 \end{aligned} \tag{1.3}$$

reveals that there exist 2n+1 equilibria $\{0, \phi_1^{\pm}, ..., \phi_n^{\pm}\}$ for (1.2).

We conclude from spectral analysis of the Sturm-Liouville operators associated to this equation that all equilibria are hyperbolic except for $0 \in H_0^1(0,\pi)$, which loses hyperbolicity for $\lambda = n^2$, for any $n \in \mathbb{N}$. Whenever $\lambda = n^2$, two new solutions bifurcate from the origin as λ increases. Moreover, we conclude that for $\lambda \leq 1$, $0 \in H_0^1(0,\pi)$ is the only equilibrium and is stable, and for $\lambda > 1$, only $\phi_1^{\pm}$ are stable.

Finally, in the Chapter 7, we study a non-local quasilinear Chafee-Infante equation, which is presented in (CARVALHO; MOREIRA, 2021):

$$\begin{aligned} &u_t = a(\|u_x\|^2)u_{xx} + \lambda f(u), \quad 0 < x < \pi, \ t > 0, \\ &u(0,t) = u(\pi,t) = 0, \ t > 0, \\ &u(\cdot,0) = u_0 \in H_0^1(0,\pi), \end{aligned} \tag{1.4}$$

where $\|u_x\|^2 = \int_0^\pi |u_x(s)|^2 ds$. The function f satisfies the same conditions as before, and we also ask $(0,\infty) \ni u \mapsto f(u)/u$ strictly decreasing. Also, $\lambda > 0$ is a parameter and $a : \mathbb{R}^+ \to [m,M] \subset (0,\infty)$ is a continuously differentiable, globally Lipschitz and non-decreasing function.

Non-local problems are important to model dynamical systems in which the behavior of a point of the function $u \in X$ depends on the value of the function in different points. Models of this type appear, for instance, in the study of the heating of ceramic. The reader may find several other examples of applications of non-local differential equations in (CHIPOT; VALENTE; CAFFARELLI, 2003), (DAVIDSON; DODDS, 2006) and (KRIEGSMANN, 1997).

With the aid of an auxiliary semilinear problem whose solutions are also solutions of the main problem through a solution dependent change in the time scale, we can use the theory of *semilinear* differential equations to conclude existence, uniqueness, continuous dependence and existence of a global attractor for this *quasilinear* non-local equation.

The semigroup associated to this quasilinear equation is also gradient, with the following Lyapunov function:

$$V(u) = \frac{1}{2}\int_0^{\|u_x\|^2} a(s)ds - \lambda \int_0^\pi \int_0^{u(x)} f(s)dsdx.$$

We refer to the results of (CARVALHO *et al.*, 2020) about the equilibria of this equations. For $n^2 < \lambda \leq (n+1)^2$, the authors of this paper construct a sequence of equilibria $\{0, \phi_1^{\pm}, ..., \phi_n^{\pm}\}$, and these equilibria have the same oscillatory properties of the equilibria of the semilinear equation.

The study of stability and hyperbolicity for this equation is done in the following way. The authors in (CARVALHO; MOREIRA, 2021) found a non-local operator that represents the linearization of the auxiliary semilinear problem around a given equilibrium. The local part of this operator is the linearization operator around some equilibrium for the Chafee-Infante classical problem, for a specific choose of the parameter λ, so that its spectrum is studied in Chapter 6, during the analysis of the Chafee-Infante classical equation. Finally, the spectrum of the non-local operator can be glanced if we know the spectrum of the local part (see for example (DAVIDSON; DODDS, 2006)), treating the non-local operator as a perturbation of its local part. We conclude that the stability and hyperbolicity of equilibria for the associated semilinear equation can be transferred to the quasilinear equation, obtaining results like the saddle point property and exponential attraction for the equilibria of (1.4).

In general, the study of hyperbolicity for non-local quasilinear equations is complicated, and a general approach to prove hyperbolicity for quasilinear problems with non-local coefficient is not yet available. However, the author in (LAPPICY, 2018) presents a method to study hyperbolicity in quasilinear local problems, based on linearization and *shooting*, and the monographs (LUNARDI, 1995, Section 9.1.2) and (YAGI, 2009, Section 6.8) present very interesting results about existence and uniqueness of solutions, and hyperbolicity of equilibria for quasilinear equations, but the equation (1.4) does not satisfy the conditions needed to apply these results.

In order to understand the mathematics in this thesis, the reader should have some knowledge about the basic theorems of functional analysis, like Hahn-Banach Theorem, the consequences of Baire Category Lemma, among others. Moreover, in the specific study of Laplacian and Sturm-Liouville operators, and in the study of the Chafee-Infante equation, we assume that the reader knows what are Sobolev spaces, L^p spaces and their basic properties. The particular sections about attractors for semigroups and gradient semigroups require less background in functional analysis, and are more focused on metric spaces. We also make extensive use of integration and differentiation of analytic functions $f : \Omega \subset \mathbb{C} \to X$ from a subset of $\mathbb{C}$ to a Banach space X, which retain some features of complex analytic functions (Cauchy Theorem, Maximum Modulus Theorem, among others). The reader may check the following books for reference about these subjects: (BREZIS, 2011), (KATO, 1995), (CARVALHO, 2012) and (TAYLOR; LAY, 1980) for functional analysis, and this last one also for integration of Banach space-valued functions; (SIMONS, 1963) and (CARVALHO, 2012) for metric spaces theory; (BREZIS, 2011) for Sobolev spaces.

It is our intention that this thesis will provide to the reader a detailed presentation of all the basic theory needed to study semilinear differential equations of parabolic type, as well as a practical demonstration of the application of those theories in a classical Chafee-Infante equation. Finally, we will show how the theory may be used to study a more complicated problem: the non-local quasilinear Chafee-Infante equation studied in (CARVALHO; MOREIRA, 2021).

CHAPTER

2

SPECTRAL ANALYSIS OF OPERATORS

2.1 The resolvent

In this section and beyond we study the resolvent and spectrum of an operator, which are essential to extract asymptotic properties in the Cauchy problems related to this operator. We focus ourselves on the theory needed to study the spectrum of the Laplace operator and the Sturm-Liouville operators, which are associated with the Chafee-Infante Equation, and also with a lot of partial differential equations in mathematical physics. For a more detailed approach to the subjects of spectral analysis, the reader may consult (BREZIS, 2011), (TAYLOR; LAY, 1980), (KATO, 1995), (CARVALHO, 2012), and for a detailed presentation of Sturm-Liouville operators and ordinary differential equations (ODEs), the reader may consult (BIRKHOFF; ROTA, 1989), (HALE, 1980).

In what follows, X will denote a Banach space over $\mathbb{C}$ and $X^* = \mathscr{L}(X,\mathbb{C})$ will be its dual space. We denote by $R(A)$ the range of an operator A and by $N(A)$ its kernel. Moreover, we denote by $\mathscr{L}(X) := \mathscr{L}(X,X)$ the set of bounded linear operators from X into X, with the usual norm.

Definition 2 (Resolvent and spectrum)**.** Let $A : D(A) \subset X \to X$ be a closed linear operator in X. We define the **resolvent set** of A as following:

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) : D(A) \to X \text{ is a bijection}\}.$$

The **spectrum** of A is defined as $\sigma(A) := \mathbb{C} \setminus \rho(A)$, and it is decomposed in three disjoint parts:

1. The **point spectrum** of A is the set of eigenvalues of A, that is, $\sigma_p(A) := \{\lambda \in \sigma(A) : (\lambda - A) \text{ is not injective}\}$.

2. The **residual spectrum** of A is the set $\sigma_r(A) := \{\lambda \in \sigma(A) : (\lambda - A)$ is injective and $R(\lambda - A)$ is not dense in $X\}$.

3. The **continuous spectrum** of A is the set $\sigma_c(A) := \{\lambda \in \sigma(A) : (\lambda - A)$ is injective and $R(\lambda - A)$ is dense in X but $R(\lambda - A) \neq X\}$.

Note that if $\lambda \in \rho(A)$, $(\lambda - A)^{-1} \in \mathscr{L}(X)$, from the Closed Graph Theorem. We call $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathscr{L}(X)$ the **resolvent** of A. Next we state some simple properties of the resolvent.

Theorem 1. Let $A : D(A) \subset X \to X$ be a closed linear operator. Then $\rho(A)$ is an open set, and $\sigma(A)$ is closed. More precisely, if $\mu \in \rho(A)$, and $\lambda \in \mathbb{C}$ is such that $|\mu - \lambda| \|(\mu - A)^{-1}\|_{\mathscr{L}(X)} < 1$, then $\lambda \in \rho(A)$ and

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\mu - A)^{-n-1}.$$

Proof. We only need to write $(\lambda - A) = (\mu - A)[I - (\mu - \lambda)(\mu - A)^{-1}]$, and the right-hand side can be inverted if $|\mu - \lambda| \|(\mu - A)^{-1}\|_{\mathscr{L}(X)} < 1$, yielding the desired expression for the operator $(\lambda - A)^{-1}$. □

Lemma 1. If $A : D(A) \subset X \to X$ is a closed linear operator and $\lambda, \mu \in \rho(A)$, then:

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}, \tag{2.1}$$

and

$$(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1}(\lambda - A)^{-1}. \tag{2.2}$$

Proof. Let $\lambda, \mu \in \rho(A)$, then:

$$\begin{aligned} (\mu - A)^{-1} &= (\mu - A)^{-1}(\lambda - A)(\lambda - A)^{-1} \\ &= (\mu - A)^{-1}[(\mu - A) + (\lambda - \mu)I](\lambda - A)^{-1} \\ &= (\lambda - A)^{-1} + (\lambda - \mu)(\mu - A)^{-1}(\lambda - A)^{-1}. \end{aligned}$$

And (2.2) follows from (2.1) □

Theorem 2. Let $A : D(A) \subset X \to X$ be a closed operator in X. Then, the function $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathscr{L}(X)$ is analytic.

Proof. For a fixed $\lambda_0 \in \rho(A)$, let $\lambda \in \rho(A)$ be such that $|\lambda - \lambda_0| \leq (2\|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)})^{-1}$. From (2.1),

$$\begin{aligned} \|(\lambda - A)^{-1} - (\lambda_0 - A)^{-1}\| &\leq |\lambda_0 - \lambda| \|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)} \|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \\ &\leq |\lambda_0 - \lambda| \|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)} \sum_{n=0}^{\infty} |\lambda_0 - \lambda|^n \|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)}^{n+1} \\ &\leq |\lambda_0 - \lambda| \|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)}^2 \sum_{n=0}^{\infty} \frac{1}{2^n}. \end{aligned}$$

Whence $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathscr{L}(X)$ is continuous. Also from (2.1), we get:

$$\begin{aligned} \frac{d}{d\lambda}(\lambda - A)^{-1}\big|_{\lambda_0} &= \lim_{\lambda \to \lambda_0} \frac{(\lambda - A)^{-1} - (\lambda_0 - A)^{-1}}{\lambda - \lambda_0} \\ &= \lim_{\lambda \to \lambda_0} -(\lambda_0 - A)^{-1}(\lambda - A)^{-1} \\ &= -(\lambda_0 - A)^{-2}, \end{aligned}$$

which shows the analyticity.

□

2.2 Bounded linear operators

In what follows we will see that the spectrum of a bounded linear operator is compact, and also define the concept of spectral radius.

Theorem 3. Let $A \in \mathscr{L}(X)$. If $\lambda > \|A\|_{\mathscr{L}(X)}$, then $\lambda \in \rho(A)$ and

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n, \tag{2.3}$$

and the series converges uniformly in $\{\lambda \in \mathbb{C} : |\lambda| \geq R\}$ for $R > \|A\|_{\mathscr{L}(X)}$.

As a consequence, $\sigma(A)$ is compact.

Proof. If $|\lambda| > 0$ and $\|\lambda^{-1}A\|_{\mathscr{L}(X)} < 1$, we use the Newman series to conclude that

$$(\lambda - A)^{-1} = [\lambda(I - \lambda^{-1}A)]^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1}A)^n = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n.$$

It is easy to see that the series converges uniformly in $\{\lambda \in \mathbb{C} : |\lambda| \geq R\}$ for $R > \|A\|_{\mathscr{L}(X)}$.

Theorem 1 shows that $\sigma(A)$ is closed. Since it is also bounded, it follows that $\sigma(A)$ is compact.

□

Definition 3 (Spectral radius)**.** Let $A \in \mathscr{L}(X)$. We define the **spectral radius** of A by

$$r_\sigma(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Theorem 4. The spectral radius of A is given by:

$$r_\sigma(A) = \limsup_{n \to \infty} \|A^n\|_{\mathscr{L}(X)}^{\frac{1}{n}}. \tag{2.4}$$

The series (2.3) converges for $|\lambda| > r_\sigma(A)$ and diverges if $|\lambda| < r_\sigma(A)$.

Proof. By uniqueness of the Laurent series for the analytic function $\mathbb{C} \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathscr{L}(X)$, we know that (2.3) holds for $|\lambda| > r_\sigma(A)$. Also, the series in (2.3) diverges if $|\lambda| < r_\sigma(A)$, otherwise it would converge for some point in the spectrum of A. Hence, $r_\sigma(A)$ is the radius of convergence of the Laurent series in (2.3), and satisfies the well-known formula for radius of convergence given by (2.4). □

The following theorem involves a simple calculation, hence its proof will be omitted.

Theorem 5. The sequence $\{\|A^n\|_{\mathscr{L}(X)}^{\frac{1}{n}}\}$ converges, and

$$r_\sigma(A) = \lim_{n\to\infty} \|A^n\|_{\mathscr{L}(X)}^{\frac{1}{n}} = \inf_{n\to\infty} \|A^n\|_{\mathscr{L}(X)}^{\frac{1}{n}}.$$

2.3 Compact operators

In this section, we present the definition and main results about compact operators. The spectrum of a compact operator A is very well characterized. In special, $\sigma(A) \setminus \{0\}$ is composed of isolated eigenvalues of A.

Definition 4 (Compact operator)**.** Let X, Y be Banach spaces over $\mathbb{C}$. We say that a linear operator $K : X \to Y$ is **compact** if $K(B)$ is precompact in Y whenever B is bounded in X. We denote by $\mathscr{K}(X,Y)$ the set of compact linear operators from X into Y, and $\mathscr{K}(X) := \mathscr{K}(X,X)$.

Example: Consider $X = \mathscr{C}([a,b],\mathbb{C})$ and $k \in \mathscr{C}([a,b] \times [a,b],\mathbb{C})$. In particular, k is uniformly continuous. Define $K \in \mathscr{L}(X)$ by

$$(Kx)(t) = \int_a^b k(t,s)x(s)ds.$$

It can be easily seen that in fact $K \in \mathscr{L}(X)$, and Arzelà-Ascoli's Theorem shows that $K \in \mathscr{K}(X)$.

Theorem 6. Let X, Y be Banach spaces over $\mathbb{C}$. Then $\mathscr{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$.

Proof. Let $\{K_n\}$ be a sequence in $\mathscr{K}(X,Y)$ such that $K_n \stackrel{n\to\infty}{\longrightarrow} K \in \mathscr{L}(X,Y)$. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$K(B_1^X(0)) \subset K_{n_\varepsilon}(B_1^X(0)) + B_{\frac{\varepsilon}{2}}^Y(0).$$

Let $\left\{B_{\frac{\varepsilon}{2}}^Y(x_i) : x_i \in \overline{K_{n_\varepsilon}(B_1^X(0))},\ i = 1,\ldots,k\right\}$ be a finite cover of $\overline{K_{n_\varepsilon}(B_1^X(0))}$, then the collection $\{B_\varepsilon^Y(x_i) : i = 1,\ldots,k\}$ is a covering for $K(B_1^X(0))$. It follows that $K(B_1^X(0))$ is totally bounded, whence it is precompact in Y.

□

Theorem 7. Let X, Y and Z be Banach spaces over $\mathbb{C}$, $A \in \mathscr{L}(X,Y)$, and $B \in \mathscr{L}(Y,Z)$. Then:

1. If either $A \in \mathscr{K}(X,Y)$ or $B \in \mathscr{K}(Y,Z)$, then $B \circ A \in \mathscr{K}(X,Z)$,

2. If $A \in \mathscr{K}(X,Y)$, and $R(A)$ is closed in Y, then $R(A)$ is finite dimensional.

Proof. The proof of 1 is trivial. Let us prove 2. Since $R(A) \subset Y$ is closed, it is a Banach space, and $A : X \to R(A)$ is surjective, hence it follows from the Open Mapping Theorem that $A(B_1^X(0))$ is an open set in $R(A)$ containing the origin. We conclude that there exists a ball in $R(A)$ which is contained in $A(B_1^X(0))$, so this ball is precompact. By a theorem from Riesz (BREZIS, 2011, Theorem 6.5), $R(A)$ has finite dimension. □

Theorem 8. Let X be a Banach space over $\mathbb{C}$ and $A \in \mathscr{K}(X)$. If $\lambda \in \mathbb{C} \setminus \{0\}$, $N((\lambda - A)^n)$ is a finite dimension Banach space, $n = 1,2,3,\ldots$.

Proof. It is easy to see that $N((\lambda - A)^n)$ is a closed vector subspace of X, hence it is a Banach space. Now, for $x \in N(\lambda - A)$, we have $Ix = \lambda^{-1}Ax$, whence $I : N(\lambda - A) \to N(\lambda - A)$ is compact, and $N(\lambda - A)$ is finite dimensional.

For $n \geq 2$, note that

$$(\lambda - A)^n = \lambda^n I + \sum_{k=1}^{n} \lambda^{n-k} \frac{n!}{k!(n-k)!}(-1)^k A^k. \tag{2.5}$$

For $x \in N((\lambda - A)^n)$, $Ix = -\lambda^{-n} A_\lambda x$, where A_λ is the compact operator given by the summation in (2.5). The same way as before, we conclude that $N((\lambda - A)^n)$ is finite dimensional. □

Next we present a part of the relevant theorem called Fredholm Alternative (BREZIS, 2011, Theorem 6.6).

Theorem 9. Let $A \in \mathscr{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then:

1. $R(\lambda - A)$ is closed.

2. $N(\lambda - A) = \{0\} \Rightarrow R(\lambda - A) = X$

Remark 1. In fact, $N(\lambda - A) = \{0\} \iff R(\lambda - A) = X$, as stated in Fredholm Alternative, but we will only prove and make use of the implication ($\Rightarrow$).

Proof. 1. Let $y_n = \lambda x_n - Ax_n \to y \in X$, and we will prove that $y \in R(\lambda - A)$. Let $d_n := \mathrm{dist}(x_n, N(\lambda - A))$. Now since $N(\lambda - A)$ is finite-dimensional (see Theorem 8), for a fixed $n \in \mathbb{N}$, we may face d_n as the distance between x_n and a compact set $B_r^X(0) \cap N(\lambda - A)$, for $r > 0$ big enough. Hence there exists $z_n \in N(\lambda - A)$ such that $d_n = \|x_n - z_n\|$.

Since $z_n \in N(\lambda - A)$, we have

$$y_n = \lambda(x_n - z_n) - A(x_n - z_n). \tag{2.6}$$

Claim: $\{\|x_n - z_n\|\}$ is bounded. Suppose not, and let $\{\|x_{n_k} - z_{n_k}\|\}$ be a subsequence such that $\|x_{n_k} - z_{n_k}\| \to \infty$ as $k \to \infty$. If $\omega_n = (x_n - z_n)/\|x_n - z_n\|$, it follows from (2.6) that $\lambda \omega_{n_k} - A\omega_{n_k} \to 0$ as $k \to \infty$, and we may assume (passing to a subsequence if needed), that $A\omega_{n_k} \to \omega \in X$ as $k \to \infty$. It follows that $\lambda \omega_{n_k} \to \omega$ as $k \to \infty$, and $\omega \in N(\lambda - A)$, so that $\operatorname{dist}(\omega_{n_k}, N(\lambda - A)) \to 0$ as $k \to \infty$. However, for any $n \in \mathbb{N}$:

$$\begin{aligned} \operatorname{dist}(\omega_n, N(\lambda - A)) &= \inf_{g \in N(\lambda - A)} \left\| \frac{x_n - z_n}{\|x_n - z_n\|} - g \right\| \\ &= \inf_{h \in N(\lambda - A)} \left\| \frac{x_n}{\|x_n - z_n\|} - \frac{h}{\|x_n - z_n\|} \right\| = \frac{\operatorname{dist}(x_n, N(\lambda - A))}{d_n} = 1, \end{aligned}$$

because the sets in which you take the infimum are the same. That is a contradiction, hence the claim is proved.

Since A is compact, we may pass to a subsequence if needed and suppose that $A(x_n - z_n) \to \ell$. From (2.6), $\lambda(x_n - z_n) \to y + \ell \stackrel{\text{def}}{=} \lambda p$. Using (2.6) again, and the fact that A is continuous, we get $y = (\lambda - A)p$.

This proves that $R(\lambda - A)$ is closed.

2. Assume, by contradiction, that

$$X_1 = R(\lambda - A) \neq X.$$

Then X_1 is closed in X, hence a Banach space, $A(X_1) \subset X_1$, and it is easy to see that $A|_{X_1} \in \mathscr{K}(X_1)$. From item 1, $X_2 := (\lambda - A)X_1$ is a closed subspace of X_1. Let $x \in X \setminus X_1$, then $(\lambda - A)x \in X_1$ but $(\lambda - A)x \notin (\lambda - A)X_1$ because $\lambda - A$ is injective, therefore $X_2 \subset X_1$ properly. Proceeding inductively, we set $X_0 = X$ and $X_n = (\lambda - A)^n X$ is a closed proper subspace of X_{n-1}, for $n = 1, 2, \ldots$. Using Riesz Lemma (BREZIS, 2011, Lemma 6.1), we can construct a sequence $\{x_n\}$ such that $x_n \in X_n$, $\|x_n\| = 1$, and $\operatorname{dist}(x_n, X_{n+1}) \geq 1/2$. On the other hand,

$$Ax_n - Ax_m = z - \lambda x_m,$$

where $z := -(\lambda x_n - Ax_n) + (\lambda x_m - Ax_m) + \lambda x_n$. Take $n > m$, so that $X_{n+1} \subset X_n \subset X_{m+1} \subset X_m$, and $z \in X_{m+1}$. It follows that $\|Ax_n - Ax_m\| \geq \operatorname{dist}(\lambda x_m, \lambda X_{m+1}) \geq |\lambda|/2$, whenever $n > m$. It contradicts the fact that A is compact.

□

Corollary 1. Let $A \in \mathscr{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $\lambda \in \sigma(A) \Rightarrow \lambda \in \sigma_p(A)$, that is, λ is an eigenvalue for A.

Remark 2. Let $T \in \mathscr{L}(X)$. It is easy to see that if $N(T^{n_0}) = N(T^{n_0+1})$ for some $n_0 \in \mathbb{N}$, then $N(T^n) = N(T^{n+1})$ for all $n \geq n_0$. *Hint:* $N(T^{n+1}) = \{x \in X : Tx \in N(T^n)\}$.

Theorem 10. Let $A \in \mathscr{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $N((\lambda - A)^{n+1}) = N((\lambda - A)^n)$, for all $n \geq n_0$.

Proof. The proof is left as an exercise for the reader. Hint: use Riesz Lemma as in the proof of the second part of Theorem 9. □

Definition 5 (Algebraic and geometric multiplicity)**.** Let $\lambda \in \mathbb{C}$ be an eigenvalue for $A \in \mathscr{K}(X)$. We say that $N(\lambda - A)$ is the **eigenspace** associated to λ and define the **geometric multiplicity** of λ as being the natural number $\dim N(\lambda - A)$. If n_0 is the lowest natural number such that $N((\lambda - A)^{n_0}) = N((\lambda - A)^{n_0+1})$, we call $N((\lambda - A)^{n_0})$ the **generalized eigenspace** associated to λ, and define the **algebraic multiplicity** of λ as the natural number $\dim N((\lambda - A)^{n_0})$

The next result summarizes the properties about the spectrum of a compact operator that we have seen so far.

Theorem 11. Let $A \in \mathscr{K}(X)$. Then $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$. Moreover, every point in $\sigma(A) \setminus \{0\}$ is isolated, that is, any sequence of different numbers $\{\lambda_n\}$ in $\sigma(A) \setminus \{0\}$ that converges has zero as limit. As a consequence, $\sigma(A)$ is countable.

Proof. Yet again we use the Riesz Lemma.

Let $\{\lambda_n\}$ be a sequence of different numbers in $\sigma(A) \setminus \{0\}$ such that $\lambda_n \to \lambda$. From Corollary 1, $\lambda_n \in \sigma_p(A)$, for all $n \in \mathbb{N}$. Let $x_n \in X$, $x_n \neq 0$, such that $(\lambda_n - A)x_n = 0$, and define $X_n = \operatorname{span}\{x_1, \ldots, x_n\}$. Now, since the eigenvalues λ_n are distinct, $\{x_1, \ldots, x_n\}$ are linearly independent and X_n has n dimensions, for each n, and $X_n \subset X_{n+1}$ with proper inclusion. Furthermore, $(\lambda_n - A)X_n \subset X_{n-1}$.

By the Riesz Lemma, there exists a sequence $\{y_n\}$, $y_n \in X_n$, such that $\|y_n\| = 1$, and $\operatorname{dist}(y_n, X_{n-1}) \geq 1/2$, for $n \geq 2$. If $2 \leq m < n$, we get $X_{m-1} \subset X_m \subset X_{n-1} \subset X_n$, and

$$\left\| \frac{Ay_n}{\lambda_n} - \frac{Ay_m}{\lambda_m} \right\| = \left\| \frac{(\lambda_m - A)y_m}{\lambda_m} - \frac{(\lambda_n - A)y_n}{\lambda_n} - y_m + y_n \right\| \geq \operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}. \tag{2.7}$$

Suppose $\lambda_n \to \lambda \neq 0$, then $\{\frac{y_n}{\lambda_n}\}$ is bounded, and $\{\frac{Ay_n}{\lambda_n}\}$ has a convergent subsequence, which contradicts (2.7). Then $\lambda = 0$.

For the last statement, just note that

$$\sigma(A) \setminus \{0\} = \bigcup_{n=1}^{\infty} \sigma(A) \cap \left\{ \lambda \in \mathbb{C} : \frac{1}{n} \leq |\lambda| \leq n \right\},$$

which is a countable union of finite sets. □

Compact operators appear sometimes as the inverse of an unbounded operator. Hence, we may use the theory developed in this section to characterize the spectrum of a much bigger class of operators. In particular, it will include the operators associated with the Chafee-Infante equation.

Definition 6 (Compact resolvent)**.** Let X be a Banach space over $\mathbb{C}$ and $A : D(A) \subset X \to X$ be a closed operator with $\rho(A) \neq \varnothing$. We say that A has **compact resolvent** if $(\lambda_0 - A)^{-1} \in \mathscr{K}(X)$ for some $\lambda_0 \in \rho(A)$.

Remark 3. If A has compact resolvent and $\lambda_0 \in \rho(A)$ with $(\lambda_0 - A)^{-1} \in \mathscr{K}(X)$, it follows that for any $\lambda \in \rho(A)$,

$$(\lambda - A)^{-1} = (\lambda_0 - A)^{-1} + (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}(\lambda - A)^{-1}$$

is compact as well (we used (2.1) and the fact that the composition of a linear operator with a compact operator is compact).

Proposition 1. Let $A : D(A) \subset X \to X$ have compact resolvent. Then $\sigma(A) = \sigma_p(A)$. Moreover, $\sigma(A)$ consists of a sequence of isolated eigenvalues.

Proof. There exists $\lambda_0 \in \rho(A)$ such that $(\lambda_0 - A)^{-1} \in \mathscr{K}(X)$. Since $0 \in \rho(\lambda_0 - A)$, $\sigma(\lambda_0 - A) = \sigma_p(\lambda_0 - A)$, and $\sigma(\lambda_0 - A)$ is a sequence of isolated eigenvalues. Then $\sigma(A)$ inherits the same properties by translation. □

The next result gives a practical way of finding operators with compact resolvent.

Proposition 2. Let $A : D(A) \subset X \to X$ be a closed linear operator and $0 \in \rho(A)$. Define the normed space $Y := (D(A), \|\cdot\| + \|A\ \cdot\|)$ (the domain of A endowed with the graph norm). Then, Y is a Banach space and if Y is compactly embedded in X, A has compact resolvent.

Proof. If a sequence $\{x_n\}$ is Cauchy in Y, then $\{x_n\}$ and $\{Ax_n\}$ are Cauchy in X, and both converge in X. Since A is closed, $(x_n, Ax_n) \to (x, Ax)$ in $X \times X$, which implies $x_n \to x$ in Y.

Suppose Y is compactly embedded in X, that is, B bounded in Y implies B precompact in X. It is easy to see that $A : Y \to X$ is closed, which implies that $A^{-1} : X \to Y$ closed, and since X and Y are Banach, the Closed Graph Theorem implies that $A^{-1} : X \to Y$ is bounded. If B is bounded in X, $A^{-1}B$ is bounded in Y and precompact in X. This shows that $A^{-1} : X \to X$ is compact. □

2.4 Numerical range

A very simple way to localize the spectrum of an operator is to use its numerical range.

Definition 7 (Numerical range)**.** Let X be a Banach space over $\mathbb{C}$ and $A : D(A) \subset X \to X$ be a linear operator. The **numerical range** of A is the subset of $\mathbb{C}$ given by:

$$W(A) := \{x^*(Ax) : x \in D(A),\ x^* \in X^*,\ \|x\| = \|x^*\| = \langle x, x^* \rangle = 1\}.$$

Remark 4. The set $W(A)$ is nonempty because of the Hahn-Banach Theorem. In the case X is a Hilbert space, we can rewrite:

$$W(A) = \{\langle Ax, x\rangle : x \in D(A), \|x\| = 1\}.$$

Theorem 12. Let $A : D(A) \subset X \to X$ be a closed, densely defined operator in the Banach space X. Let $W(A)$ be its numerical range.

1. If $\lambda \notin \overline{W(A)}$, then $\lambda - A$ is injective, has closed range, and satisfies

$$\|(\lambda - A)x\| \geq d(\lambda, W(A))\|x\|, \quad \forall\, x \in D(A). \tag{2.8}$$

   If, aditionally, $\lambda \in \rho(A)$, then

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, W(A))}. \tag{2.9}$$

2. Let Σ be an open and connected set in $\mathbb{C} \setminus W(A)$ and $\rho(A) \cap \Sigma \neq \varnothing$, then $\rho(A) \supset \Sigma$, and

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, W(A))}, \quad \forall \lambda \in \Sigma.$$

Proof. We start proving 1. Suppose $\lambda \notin \overline{W(A)}$. If $x \in D(A)$, with $\|x\| = 1$, there exists $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(x) = 1$, then:

$$0 < d(\lambda, W(A)) \leq |\lambda - x^*(Ax)| = |x^*(\lambda x - Ax)| \leq \|(\lambda - A)x\|.$$

If $y \in D(A)$, $y \neq 0$, we apply the previous reasoning to the normalization of y, given by $x = \frac{y}{\|y\|}$, which yields $\|(\lambda - A)y\| > 0$, and $\|(\lambda - A)y\| \geq d(\lambda, W(x))\|y\|$. The estimates (2.8) and (2.9) then follow imediately.

Now we prove that the range of $\lambda - A$ is closed. Indeed, if $y_n \to y \in X$, $y_n = (\lambda - A)x_n$, then $\|y_n - y_m\| \geq d(\lambda, W(A))\|x_n - x_m\|$, then $\{x_n\}$ converges in X to a limit $x \in X$. Since $\lambda - A$ is closed, $(\lambda - A)x = y$, and $y \in R(\lambda - A)$.

Now we prove 2. Let $\Sigma \subset \mathbb{C} \setminus W(A)$ be an open connected set such that $\rho(A) \cap \Sigma \neq \varnothing$. We will show that $\Sigma \cap \rho(A) = \Sigma$. Since Σ is connected and $\Sigma \cap \rho(A) \neq \varnothing$, we only need to show that $\Sigma \cap \rho(A)$ is closed and open in Σ.

Of course $\Sigma \cap \rho(A)$ is open in Σ. Now suppose $\lambda_n \in \rho(A) \cap \Sigma$, $\lambda_n \to \lambda \in \Sigma$. Since λ is in the open set $\Sigma \subset \mathbb{C} \setminus W(A)$, it is easy to see that

$$|\lambda - \lambda_n| < d(\lambda_n, W(A)), \quad \forall n \geq n_0,$$

for some $n_0 \in \mathbb{N}$. It follows that for $n \geq n_0$

$$|\lambda - \lambda_n|\|(\lambda_n - A)^{-1}\|_{\mathscr{L}(X)} < 1,$$

and we can use Theorem 1 with $\mu = \lambda_{n_0}$ to assure that $\lambda \in \rho(A) \cap \Sigma$. Therefore, $\rho(A) \cap \Sigma$ is also closed in Σ, and we are done. □

2.5 Adjoint and self-adjoint operators

The study of self-adjoint operators is very important in physics (specially in quantum mechanics). A self-adjoint operator has a very good characterization of its spectrum, namely the Minimax Theorem (see Theorem 16).

Let $(H, \langle\cdot,\cdot\rangle : H \times H \to \mathbb{C})$ be a Hilbert space over $\mathbb{C}$, and $A : D(A) \subset H \to H$ be a densely defined operator.

Theorem 13 (Representation Theorem from Riesz)**.** Let H be a Hilbert space. Every bounded linear functional $f \in H^* = \mathscr{L}(H,\mathbb{C})$ can be represented as an inner product, that is, there exists $z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$, where $z = z_f$ is unique with this property and $\|z\| = \|f\|_{H^*}$.

Definition 8 (Adjoint)**.** Let $A : D(A) \subset X \to X$ be a densely defined operator. The **adjoint** A^* of A is the operator defined by:

$$D(A^*) = \{u \in H : D(A) \ni v \overset{\phi_u}{\mapsto} \langle Av, u\rangle \in \mathbb{C} \text{ is bounded}\}.$$

For $u \in D(A^*)$, ϕ_u can be extended to a functional in H^* and we define A^*u as the unique representative of this functional. In other words, A^*u is the only element in H such that

$$\langle v, A^*u\rangle = \langle Av, u\rangle, \quad \forall\, v \in D(A).$$

Definition 9 (Symmetric and self-adjoint)**.** We say that a densely defined operator $A : D(A) \subset H \to H$ is **symmetric** (or **Hermitian**) if $A \subset A^*$ (that is, $\langle Ax, y\rangle = \langle x, Ay\rangle$, for all x, $y \in D(A)$). We say that A is **self-adjoint** if $A = A^*$, that is, $D(A^*) = D(A)$ and $\langle Ax, y\rangle = \langle x, Ay\rangle$, for all x, $y \in D(A)$.

Remark 5. It is simple to see that A^* is closed.

Proposition 3. If $A : D(A) \subset H \to H$ is a symmetric operator and λ is an eigenvalue for A, then λ is real and

$$\inf_{\|x\|=1} \langle Ax, x\rangle \leq \lambda \leq \sup_{\|x\|=1} \langle Ax, x\rangle. \tag{2.10}$$

Proof. Let $v \in D(A)$ be an eigenvector associated to λ. Then

$$\overline{\langle Av, v\rangle} = \langle v, Av\rangle = \langle Av, v\rangle = \lambda \langle v, v\rangle = \lambda \|v\|^2.$$

Therefore, $\lambda \|v\|^2$ must be real, and $\lambda \in \mathbb{R}$.

The estimate (2.10) follows from the fact that $\lambda = \langle Av, v\rangle$, for some $v \in D(A)$.

□

In what follows, we denote by $\Gamma(B) = \{(x, Bx) : x \in D(B)\}$ the graph of an operator $B : D(B) \subset X \to X$.

Lemma 2. Let $A : D(A) \subset H \to H$ be a densely defined linear operator in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Consider $H \times H$ with the inner product given by $\langle (a,b),(c,d) \rangle_{H\times H} = \langle a,c \rangle + \langle b,d \rangle$. Then

$$\Gamma(A^*) = \{(-Ax, x) : x \in D(A)\}^{\perp}.$$

Proof. The proof is direct. □

Proposition 4. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$, and $A : D(A) \subset H \to H$ be self-adjoint, injective and have dense range. Then A^{-1} is self-adjoint.

Proof. Since A is self-adjoint, using Lemma 2 we get $\Gamma(A) = \{(-Ax, x) : x \in D(A)\}^{\perp}$. With some adjustments, we get $\Gamma(A^{-1}) = \{(Ay, y) : y \in D(A)\} = \{(x, -Ax) : x \in D(A)\}^{\perp}$.

Since A^{-1} is densely defined, also from Lemma 2 we get

$$\Gamma((A^{-1})^*) = \{(-A^{-1}x, x) : x \in R(A)\}^{\perp} = \{(x, -Ax) : x \in D(A)\}^{\perp} = \Gamma(A^{-1}).$$

Whence $(A^{-1})^* = A^{-1}$. □

Theorem 14. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$, and $A : D(A) \subset H \to H$ be symmetric ($A \subset A^*$) and surjective. Then A is self-adjoint.

Proof. *Claim:* A and A^* are injective.

Indeed, if $x \in D(A)$, $Ax = 0$, then $0 = \langle Ax, y \rangle = \langle x, Ay \rangle$ for all $y \in D(A)$, and $\langle x, z \rangle = 0$ for all $z \in H$. From Hahn-Banach Theorem, x must be zero.

If $y \in D(A^*)$ and $A^*y = 0$, then $\langle Ax, y \rangle = \langle x, A^*y \rangle = 0$ for all $x \in D(A)$, and we continue as before to conclude that $y = 0$.

Now let us prove that A is closed. Indeed, if $D(A^*) \supset D(A) \ni x_n \to x \in H$, and $Ax_n = A^*x_n \to y \in H$, we get $x \in D(A^*)$ and $A^*x = y$ (using the fact that A^* is closed). Since A is surjective, there exists $w \in D(A)$ such that $Aw = A^*w = A^*x$, but since A^* is injective, $w = x \in D(A)$, and $Ax = A^*x = y$. Hence A is closed and $A^{-1} : H \to H$ is a closed operator between Banach spaces, whence it is bounded. It follows that $D((A^{-1})^*) = H = D(A^{-1})$, and since A is symmetric, A^{-1} is symmetric. Hence, A^{-1} is self-adjoint.

Using Proposition 4, and the fact that $R(A^{-1}) = D(A)$ is dense, we conclude that A is self-adjoint. □

In what follows we present the Friedrichs Extension Theorem, which is very useful to obtain self-adjoint operators.

Theorem 15 (Friedrichs)**.** Let X be a Hilbert space over $\mathbb{C}$ and $A : D(A) \subset X \to X$ be a symmetric linear operator for which there exists $\alpha \in \mathbb{R}$ such that

$$\langle Ax, x\rangle \le \alpha \|x\|^2, \quad \forall x \in D(A), \tag{2.11}$$

or

$$\langle Ax, x\rangle \ge \alpha \|x\|^2, \quad \forall x \in D(A). \tag{2.12}$$

Then A admits a surjective and self-adjoint extension $\tilde{A} : D(\tilde{A}) \subset X \to X$ for which the estimate still holds for all $x \in D(\tilde{A})$.

Proof. We only consider the case (2.12), because the other case will then follow considering $-A$. Furthermore, we assume $\alpha = 1$ because the general case follows using the operator $A + (1-\alpha)I$.

Consider the space $D(A)$ with inner product $D(A) \times D(A) \ni (x,y) \mapsto \langle x,y\rangle_{\frac{1}{2}} := \langle Ax, y\rangle \in \mathbb{C}$ and norm $D(A) \ni x \mapsto \|x\|_{\frac{1}{2}} = \langle Ax,x\rangle^{\frac{1}{2}} \in \mathbb{R}^+$, which satisfies $\|x\|_{\frac{1}{2}} \ge \|x\|$.

Denote by $(Y, \langle\cdot,\cdot\rangle_Y)$ some completion of $D(A)$ relatively to the inner product $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$, that is, Y is a Hilbert space and there exists a linear application $\phi : D(A) \to Y$ such that $\langle\phi(x),\phi(y)\rangle_Y = \langle x,y\rangle_{\frac{1}{2}}$, for all x, $y \in D(A)$, and $\phi(D(A))$ dense in Y.

We will show that there is a bijection between Y and a subset of X.

We define the application $T : Y \to X$ the following way. Let $y \in Y$, then there is a sequence $\{x_n\}$ in $D(A)$ such that $y = \lim_{n\to\infty} \phi(x_n)$. Then $\{\phi(x_n)\}$ is Cauchy in Y, and $\{x_n\}$ is Cauchy in X because

$$\|\phi(x_n) - \phi(x_m)\|_Y = \|x_n - x_m\|_{\frac{1}{2}} \ge \|x_n - x_m\|,$$

so we define $Ty = \lim_{n\to\infty} x_n$ (taking the limit in the X norm).

It is simple to see that T is well-defined (does not depend on the selection of $\{x_n\}$), and is linear.

In what follows we prove that T is injective, so that $T : Y \to TY \subset X$ is a bijective linear application.

Let $y \in Y$ with $y = \lim_{n\to\infty} \phi(x_n)$, which implies that $\{x_n\} \subset D(A)$ is Cauchy in the norm $\|\cdot\|_{\frac{1}{2}}$, and suppose

$$Ty = \lim_{n\to\infty} x_n = 0. \tag{2.13}$$

If $y \ne 0$, then $\|\phi(x_n)\|_Y = \|x_n\|_{\frac{1}{2}} \overset{n\to\infty}{\longrightarrow} a > 0$. Using the fact that A is symmetric, we get

$$\begin{aligned} 2\mathrm{Re}\langle Ax_n, x_m\rangle &= \langle Ax_n, x_n\rangle + \langle Ax_m, x_m\rangle - \langle A(x_n - x_m), (x_n - x_m)\rangle \\ &= \|x_n\|^2_{\frac{1}{2}} + \|x_m\|^2_{\frac{1}{2}} - \|x_n - x_m\|^2_{\frac{1}{2}}. \end{aligned}$$

It follows that

$$\|2\mathrm{Re}\langle Ax_n, x_m\rangle - 2a^2\| \le \left| \|x_n\|^2_{\frac{1}{2}} - a^2 + \|x_m\|^2_{\frac{1}{2}} - a^2 + \|x_n - x_m\|^2_{\frac{1}{2}} \right|,$$

whence $2\mathrm{Re}\langle Ax_n, x_m\rangle \overset{n,m\to\infty}{\longrightarrow} 2a^2 > 0$.

This is a contradiction because, from (2.13), we get

$$|\langle Ax_n, x_m\rangle| \le \|Ax_n\|\|x_m\| \overset{m\to\infty}{\longrightarrow} 0, \quad \forall n \in \mathbb{N}.$$

Hence $T : Y \to TY \subset X$ is a linear bijection.

Note that for any $y = \lim_{n\to\infty} \phi(x_n) \in Y$,

$$\|Ty\| = \lim_{n\to\infty} \|x_n\| \le \lim_{n\to\infty} \|x_n\|_{\frac{1}{2}} = \lim_{n\to\infty} \|\phi(x_n)\|_Y = \|y\|_Y,$$

so that $T : Y \to X$ is bounded, and $T^{-1} : TY \to Y$ is closed considering the norm of X in the domain.

Let $\tilde{D} = D(A^*) \cap TY$. Is is easy to see that $D(A) \subset \tilde{D} \subset D(A^*)$. Indeed, if $x \in D(A)$, $y = \phi(x) \in Y$, and $Ty = x$, and $x \in TY$.

Let $\tilde{A}$ be the restriction of A^* to $\tilde{D}$. Since A is symmetric, $\tilde{A}$ extends A, and we are left to show that $\tilde{A}$ is self-adjoint.

First note that $Z := TY$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_Z = \langle T^{-1}\cdot, T^{-1}\cdot\rangle_Y$. The norm induced is $\|z\|_Z = \|T^{-1}z\|_Y$. This norm makes Z complete. Indeed, let $\{z_n\}$ be a sequence in Z that is Cauchy, then

$$0 \longleftarrow \|z_n - z_m\|_Z = \|T^{-1}z_n - T^{-1}z_m\|_Y \ge \|z_n - z_m\|.$$

Where we used the boundedness of T. Therefore, $z_n \to z$ in X, $T^{-1}z_n \to f$ in Y, and $f = T^{-1}z$, because T^{-1} is closed, from which we conclude that $z \in Z$ and:

$$\|z_n - z\|_Z = \|T^{-1}z_n - T^{-1}z\|_Y \longrightarrow 0,$$

and we are done.

$\tilde{A}$ is symmetric: Let $x, y \in \tilde{D}$, then $x = T\tilde{x}$ and $y = T\tilde{y}$ with $\tilde{x}, \tilde{y} \in Y$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $D(A)$ such that $\phi(x_n) \to \tilde{x}$ and $\phi(y_n) \to \tilde{y}$. It follows from the definition of T that $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ (limits in the norm of X).

It follows from the continuity of the inner product with relation to the norm it induces that

$$\begin{aligned}
\langle x, y\rangle_Z &= \langle \tilde{x}, \tilde{y}\rangle_Y = \left\langle \lim_{n\to\infty} \phi(x_n), \lim_{m\to\infty} \phi(y_m) \right\rangle_Y \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle \phi(x_n), \phi(y_m)\rangle_Y = \lim_{n\to\infty} \lim_{m\to\infty} \langle x_n, y_m\rangle_{\frac{1}{2}} = \lim_{n\to\infty} \lim_{m\to\infty} \langle Ax_n, y_m\rangle.
\end{aligned}$$

Similarly, $\lim_{m\to\infty} \lim_{n\to\infty} \langle Ax_n, y_m\rangle = \langle x, y\rangle_Z$.

But calculating these limits in another way yields:

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle Ax_n, y_m\rangle = \lim_{n\to\infty}\langle Ax_n, y\rangle = \lim_{n\to\infty}\langle x_n, \tilde{A}y\rangle = \langle x, \tilde{A}y\rangle,$$

and

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle Ax_n, y_m\rangle = \langle \tilde{A}x, y\rangle.$$

From which we conclude that $\langle \tilde{A}x, y\rangle = \langle x, \tilde{A}y\rangle$, and $\tilde{A}$ is symmetric.

Now we need to show that $\tilde{A}$ is surjective. Let $y \in X$ and define the functional $f : D(A) \to \mathbb{C}$ given by $f(x) = \langle x, y\rangle$. It is clear that $|f(x)| \leq \|x\|\|y\| \leq \|x\|_Z\|y\|$, and since $D(A)$ is dense in Z, f can be extended to a continuous functional in the Hilbert space Z. From Theorem 13, there exists $z' \in Z$ such that:

$$f(x) = \langle x, y\rangle = \langle x, z'\rangle_Z = \langle Ax, z'\rangle, \quad \forall x \in D(A),$$

where the last equality is an exercise for the reader.

Then $z' \in D(A^*) \cap TY$ and $\tilde{A}z' = A^*z' = y$, and $\tilde{A}$ is surjective.

From Theorem 14, $\tilde{A}$ is self-adjoint, as desired.

□

Next we present an important theorem about the spectrum of compact self-adjoint operators. Its proof can be found in (CARVALHO, 2012), and involves weak topology and functional analysis.

Theorem 16 (Min-max Theorem)**.** Let H be a Hilbert space over $\mathbb{C}$ and $A \in \mathscr{K}(H)$ be a compact and self-adjoint operator such that $\langle Au, u\rangle \geq 0$ for all $u \in H$. Then:

1. $\lambda_1 = \sup\{\langle Au, u\rangle : \|u\| = 1\}$ is the largest eigenvalue of A, and there exists $v_1 \in H$, $\|v_1\| = 1$, such that $\lambda_1 = \langle Av_1, v_1\rangle$, and $Av_1 = \lambda_1 v_1$.

2. $\lambda_n = \sup\{\langle Au, u\rangle : \|u\| = 1$ and $u \perp v_j$, for $1 \leq j \leq n-1\}$ is an eigenvalue for A and there exists $v_n \in H$, $\|v_n\| = 1$, $v_n \perp v_j$ for all $1 \leq j \leq n-1$, such that $\lambda_n = \langle Av_n, v_n\rangle$ and $Av_n = \lambda_n v_n$.

3. If $\mathscr{V}_n = \{F \subset H : F$ is a $n-$dimensional linear subspace of $H\}$, then, for $n \geq 1$:

$$\lambda_n = \inf_{F\in\mathscr{V}_{n-1}} \sup\{\langle Au, u\rangle : \|u\| = 1, u \perp F\},$$

and

$$\lambda_n = \sup_{F\in\mathscr{V}_n} \inf\{\langle Au, u\rangle : \|u\| = 1, u \in F\}.$$

2.6 Laplacian and Sturm-Liouville operators

In this section, we study the spectrum and the eigenvectors of the Laplacian and Sturm-Liouville operators, using the theory we developed in this chapter.

Example 1 (Laplacian)**.** Let $X = L^2(0,\pi)$, with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, and $D(A_0) := \mathscr{C}_c^2((0,\pi),\mathbb{R})$ be the space of functions defined in $[0,\pi]$ and taking values in $\mathbb{R}$, twice differentiable and with compact support in $(0,\pi)$. Define $A_0 : D(A_0) \subset X \to X$ by:

$$(A_0\phi)(x) = -\phi''(x), \quad x \in (0,\pi),\ \phi \in D(A_0).$$

Using integration by parts, we can show that A_0 is symmetric. Moreover,

$$\langle A_0\phi,\phi\rangle = \int_0^\pi (-\phi'')\phi dx = \int_0^\pi \phi'(x)^2 dx = \|\phi\|^2_{H_0^1} \geq \frac{2}{\pi^2}\|\phi\|^2.$$

It follows from Theorem 15 that A_0 has a self-adjoint extension A, which satisfies

$$\langle A\phi,\phi\rangle \geq \frac{2}{\pi^2}\|\phi\|^2, \quad \forall\phi \in D(A). \tag{2.14}$$

In this particular case, we may assume $Y = H_0^1(0,\pi)$, $\phi : D(A_0) \to Y$ as the inclusion. Indeed, $\overline{D(A_0)}^{H_0^1} = H_0^1(0,\pi)$. Moreover, if ϕ, $\psi \in D(A_0)$,

$$\langle\phi,\psi\rangle_{H_0^1} = \int_0^\pi \phi'\psi' ds = \langle A\phi,\psi\rangle = \langle\phi,\psi\rangle_{\frac{1}{2}}.$$

And $H_0^1(0,\pi)$ is in fact a completion of $D(A_0)$ with relation to the inner product $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$.

Moreover,

$$D(A_0^*) = \{\phi \in L^2(0,\pi) : \exists\ \phi^* \in L^2(0,\pi) \text{ such that } \langle -u'',\phi\rangle = \langle u,\phi^*\rangle, \text{ for all } u \in D(A_0)\},$$

and this is precisely the space of functions in $L^2(0,\pi)$ that have a second weak derivative in $L^2(0,\pi)$. Then we conclude that $D(A) = H^2(0,\pi)\cap H_0^1(0,\pi)$, and A is given by $A\phi = -\phi''$, for all $\phi \in D(A)$, taking the derivative in the sense of distributions.

From Theorem 15, A is surjective, and from (2.14), A is injective, so that $0 \in \rho(A)$. Moreover, if x, $y \in [0,\pi]$, then, for any $\phi \in D(A)$,

$$|\phi(x)-\phi(y)| \leq |x-y|^{\frac{1}{2}}\|\phi'\|_{L^2} = |x-y|^{\frac{1}{2}}\langle A\phi,\phi\rangle^{\frac{1}{2}}.$$

Let B be a bounded subset of $D(A)$, with the norm of graph $\|\cdot\|_G = \|\cdot\| + \|A\cdot\|$. Then $\sup_{\phi\in B}\|\phi\| < \infty$ and $\sup_{\phi\in B}\|A\phi\| < \infty$.

It follows that $\sup_{\phi\in B}\sqrt{\langle A\phi,\phi\rangle} < \infty$, which implies $\sup_{\phi\in B}\|\phi'\| < \infty$. Using Arzelà-Ascoli Theorem, we conclude that B is precompact in $\mathscr{C}([0,\pi],\mathbb{R})$, therefore B is precompact in $L^2(0,\pi)$. From Proposition 2, A has compact resolvent, hence $\sigma(A) = \sigma_p(A)$.

Now consider the eigenvalue problem

$$A\phi = \lambda\phi \iff \phi'' = -\lambda\phi.$$

The general solution of this equation is $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, but the boundary conditions $\phi(0) = \phi(\pi) = 0$ yields $a = 0$ and $\lambda = n^2$ for some $n \in \mathbb{N}$. Then $\sigma(A) = \sigma_p(A) = \{\lambda_1, \lambda_2, \ldots\}$, with $\lambda_n = n^2$ associated to the normalized eigenvector $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$.

Example 2 (Sturm-Liouville operators)**.** The Sturm-Liouville operators are very important in physics, and they are also the linear operators associated to the Chafee-Infante equation.

We consider a particular case of Sturm-Liouville operators. Define $A : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$ by $Av = -v'' + q(x)v$, where $q : [0,\pi] \to \mathbb{R}$ is a continuous function.

Let $\alpha = |\inf_{x\in[0,\pi]} q(x)|$, and consider the operator $A + \alpha I$, given by $(A + \alpha I)v = -v'' + (q(x) + \alpha)v$. Define the new function $p : [0,\pi] \to \mathbb{R}$ by $p(x) = q(x) + \alpha \geq 0$. We will show that $B : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$, given by $Bv = -v'' + p(x)v$, is self-adjoint, which will imply that A is self-adjoint — because it is a translation by a real constant of B.

It is simple to see that B is symmetric, using that the Laplacian is symmetric. We only need to show that B is surjective, so consider the bilinear application between Hilbert spaces $a : H_0^1(0,\pi) \times H_0^1(0,\pi) \to \mathbb{R}$ given by:

$$a(u,v) = \int_0^\pi u_x(s)v_x(s)ds + \int_0^\pi p(s)u(s)v(s)ds.$$

Note that a is continuous — that is, there exists a constant $C \geq 0$ such that

$$|a(u,v)| \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}, \quad \text{for all } u,v \in H_0^1(0,\pi).$$

Moreover, a is *coercive* — that is, there exists some $\alpha > 0$ such that $a(v,v) \geq \alpha\|v\|_{H_0^1}^2$ for all $v \in H_0^1(0,\pi)$. Hence, from Lax-Milgram's Theorem (BREZIS, 2011, Corollary 5.8), for each $f \in L^2(0,\pi)$, there exists $u \in H_0^1(0,\pi)$ such that

$$\int_0^\pi u_x(s)v_x(s)ds + \int_0^\pi p(s)u(s)v(s)ds = \int_0^\pi f(s)v(s)ds, \quad \forall\, v \in H_0^1(0,\pi).$$

That is, the equation $-u_{xx} + p(x)u = f$ has a weak solution in $H_0^1(0,\pi)$. Next we need to conclude regularity for u and we do it the following way. Note that $f - p(x)u \in L^2(0,\pi)$, and since the Laplacian is a bijection (see Example 1), there exists $w \in H^2(0,\pi) \cap H_0^1(0,\pi)$ such that $-w_{xx} = f - p(x)u$. Now we define the continuous and coercive bilinear form $b : H^2(0,\pi) \times H_0^1(0,\pi) \to \mathbb{R}$ by

$$b(z,v) = \int_0^\pi z_x(s)v_x(s)ds,$$

and the function $h \in L^2(0,\pi)$, given by $h(x) = f(x) - p(x)u(x)$. Then $b(u,v) = b(w,v) = \langle v,h\rangle$, for all $v \in H_0^1(0,\pi)$. It follows from the uniqueness in Lax-Milgram's Theorem that $u = w \in$

$H^2(0,\pi)\cap H_0^1(0,\pi)$, and $Bu = f$. Therefore, B is surjective, hence self-adjoint. We conclude that A is self-adjoint.

Note that

$$\langle B\phi,\phi\rangle \geq \|\phi'\|^2 \geq \frac{2}{\pi^2}\|\phi\|^2.$$

Hence, B is injective, and $0 \in \rho(B)$. With the same reasoning as the one we used in Example 1, we may conclude that B has compact resolvent. It follows that A has compact resolvent.

The spectrum of A is a strictly increasing sequence of eigenvalues, which we denote by $\{\mu_j\}_{j\in\mathbb{N}^*}$, with $\mu_{j+1} > \mu_j$, and we claim that $\mu_j \to \infty$ as $j \to \infty$. Indeed, $\rho(B) \supset (-\infty, \frac{1}{\pi^2})$ because of Theorem 12, therefore the spectrum of B can only accumulate at $+\infty$, and it *does* accumulate there. Indeed, the normalized eigenvectors of $B^{-1} \in \mathscr{K}(L^2(0,\pi))$ are a Hilbert basis for $L^2(0,\pi)$ (BREZIS, 2011, Theorem 6.11), and each eigenvalue has finite geometric multiplicity, therefore B^{-1} must have infinitely many eigenvalues, accumulating at zero, so that B has infinitely many eigenvalues, accumulating at $+\infty$.

All the eigenvalues of the Sturm-Liouville operator are simple. Indeed, let v_1 and v_2 be solutions of

$$Av = \mu_j v \iff -v'' + (q(x) - \mu_j)v = 0,$$

for some $j \in \mathbb{N}^*$. This is a common ODE, and we calculate the Wronskian of v_1 and v_2 to determine whether they are linearly independent or dependent. But

$$\begin{vmatrix} v_1(0) & v_2(0) \\ v_1'(0) & v_2'(0) \end{vmatrix} = 0.$$

Then v_1 and v_2 are linearly dependent and the claim is proven. This fact is extremely important because it allows us to conclude information about symmetry of eigenvectors of A, which will be necessary in Chapter 7.

The next theorem is based on (BIRKHOFF; ROTA, 1989, Chapter 10, Section 7).

Theorem 17. Let $v_j \in H^2(0,\pi)\cap H_0^1(0,\pi)$ be the eigenvector of A associated to μ_j that satisfies $v_j'(0) = 1$, $j \in \mathbb{N}^*$. Then $v_1(x) > 0$ for $x \in (0,\pi)$, and v_j vanishes precisely $j+1$ times in $[0,\pi]$.

Proof. Consider the eigenvalue problem for the Sturm-Liouville operator:

$$-v''(x) + (q(x) - \lambda)v(x) = 0,$$

which may be written as

$$\begin{aligned} v' &= w \\ w' &= (q(x) - \lambda)v. \end{aligned} \tag{2.15}$$

Suppose $(v(\cdot,\lambda),v'(\cdot,\lambda))$ is a continuous solution of (2.15), for the initial conditions $(v(0,\lambda),v'(0,\lambda))=(0,1)$. Now we need to find the values of λ for which the associated solution satisfies the boundary condition $v(\pi,\lambda)=0$, and we will have found an eigenvector of the Sturm-Liouville operator.

Consider the change of variables

$$\begin{aligned} v(\cdot,\lambda) &= r(\cdot,\lambda)\sin\alpha(\cdot,\lambda) \\ w(\cdot,\lambda) &= r(\cdot,\lambda)\cos\alpha(\cdot,\lambda). \end{aligned}$$

With some calculations, we obtain the equivalent problem:

$$\begin{aligned} \alpha' &= \cos^2\alpha + (\lambda - q(x))\sin^2\alpha \\ r' &= r(1+q(x)-\lambda)\cos\alpha\,\sin\alpha, \end{aligned} \tag{2.16}$$

and initial conditions $(\alpha(0,\lambda),r(0,\lambda))=(0,1)$.

Notice that $r>0$, because if $r(x)=0$ for some $x\geq 0$, then $(v,w)=(0,0)$ is the origin in the phase plane of (2.15), a contradiction. Therefore, $v(\pi,\lambda)=r(\pi,\lambda)\sin\alpha(\pi,\lambda)=0 \iff \sin\alpha(\pi,\lambda)=0 \iff \alpha(\pi,\lambda)=j\pi$ for some $j\in\mathbb{Z}$.

It follows from simple comparison results in (BIRKHOFF; ROTA, 1989, Chapter 1, Section 11, Corollary 1) that for each $x\in(0,\infty)$, $\alpha(x,\lambda)$ is a continuously differentiable and strictly increasing function of $\lambda\in\mathbb{R}$.

The behavior of the function $\alpha(\cdot,\lambda)$ with relation to the first variable is not necessarily increasing, but it has the following important property: if for some $x_n>0$, $\alpha(x_n,\lambda)=n\pi$, where $n=0,1,2,...$, then $\alpha(x,\lambda)>n\pi$ for all $x>x_n$. Indeed, it follows from the differential equation that $\alpha'(x_n,\lambda)=1>0$. If there exists $c>x_n$ such that $\alpha(c,\lambda)\leq n\pi$, then the set $S:=\{x\in(x_n,c]:\alpha(x,\lambda)\leq n\pi\}$ has an infimum $x^*>x_n$ that satisfies

$$\begin{aligned} &\alpha(x,\lambda) > n\pi, \quad \text{for } x\in(x_n,x^*) \\ &\alpha(x^*,\lambda) = n\pi. \end{aligned} \tag{2.17}$$

From the differential equation, $\alpha'(x^*,\lambda)>0$, but from (2.17), and the Mean Value Theorem, $\alpha'(\cdot,\lambda)$ takes negative values arbitrarily close to x^*, and this is a contradiction. Therefore, we proved that if $x_n>0$, $\alpha(x_n,\lambda)=n\pi$, where $n\in\mathbb{N}$, then $\alpha(x,\lambda)>n\pi$ for all $x>x_n$. In particular, $\alpha(x,\lambda)>0$ for $x>0$.

Let us now prove that $\alpha(\pi,\lambda)\overset{\lambda\to-\infty}{\longrightarrow}0$. Indeed, let $0<\beta<\pi$ and $0<\varepsilon<\pi$. Consider the line segment in the $x\alpha$-plane joining the points $(0,\beta)$ and (π,ε). In this line segment, $\alpha>\min\{\varepsilon,\beta\}>0$ and $\alpha<\max\{\varepsilon,\beta\}<\pi$, so that we may take a λ sufficiently negatively large such that for any point (x,α) lying in the line segment and such that $x\in(0,\pi)$, the derivative $\alpha'(x,\lambda)=\cos^2\alpha+(\lambda-q(x))\sin^2\alpha$ is smaller than the slope of the line segment. It follows that

$\alpha(x,\lambda)$ remains below the line segment for all $x \in (0,\pi]$, and, in particular, $0 < \alpha(\pi,\lambda) \leq \varepsilon$. This shows that $\alpha(\pi,\lambda) \to 0$ as $\lambda \to -\infty$.

We already know from past observations that A has infinitely many distinct eigenvalues, so that there exists an infinite set $K \subset \mathbb{N}^*$ such that for each $k \in K$, there exists $\mu_k \in \mathbb{R}$ such that $\alpha(\pi,\mu_k) = k\pi$. From the Intermediate Value Theorem and the fact that $\alpha(\pi,\lambda) \to 0$ as $\lambda \to -\infty$, we conclude that for each $j \in \mathbb{N}^*$, there exists $\mu_j \in \mathbb{R}$ such that $\alpha(\pi,\mu_j) = j\pi$. Then, μ_j is an eigenvalue for A for each $j \in \mathbb{N}^*$, $\{\mu_j\}$ is increasing and is in fact the sequence of all eigenvalues of A. The associated eigenvector, $v_j(x) = r(x,\mu_j)\sin\alpha(x,\mu_j)$ has exactly $j+1$ zeros in $[0,\pi]$, in the points $x \in [0,\pi]$ such that $\alpha(x,\mu_j) = k\pi$, for some $k \in \{0,1,2,...,j\}$. This completes the proof.

□

CHAPTER

3

SEMIGROUPS

3.1 Global attractors for semigroups

As mentioned in the introduction, in order to unravel part of the asymptotic behavior of an autonomous differential equation, we make use of the theory of semigroups. In this section we present the definitions of a semigroup and the most important concepts related to it, and also the conditions that a semigroup must satisfy in order to have a global attractor. For a more thorough approach to the topics of this chapter — namely, semigroups, global attractors, gradient semigroups, generators of semigroups and spectral decomposition of semigroups —, the reader may consult: (BORTOLAN; CARVALHO; LANGA, 2020), (CARVALHO; LANGA; ROBINSON, 2013), (TEMAM, 1997).

Let X denote a metric space with metric d and $\mathscr{C}(X)$ be the set of continuous maps from X into itself. Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}^- = \{t \in \mathbb{R} : t \leq 0\}$.

Given $K \subset X$, $r > 0$, the **r-neighborhood** of K is the set defined by $\mathscr{O}_r(K) := \{x \in X : \mathrm{d}(x,K) < r\}$, where $\mathrm{d}(x,K) = \inf_{y \in K} \mathrm{d}(x,y)$.

Definition 10 (Semigroup)**.** A **semigroup** in X is a family $\mathscr{T} = \{T(t) : t \geq 0\} \subset \mathscr{C}(X)$ that satisfies:

- $T(0)x = x$, $\forall x \in X$.
- $T(t+s) = T(t)T(s)$, $\forall\, t,s \geq 0$.
- $\mathbb{R}^+ \times X \ni (t,x) \mapsto T(t)x \in X$ is continuous.

The space X is called the phase space of $\mathscr{T}$.

A semigroup $\mathscr{T} = \{T(t) : t \geq 0\}$ is used to study the evolution of a dynamical system in the phase space X, associating each initial position $x_0 \in X$ to a final position $T(t)x_0$ after a time

$t \geq 0$ is passed. The conditions imposed guarantee the compatibility of the semigroup with this interpretation.

Note that X has arbitrary dimension, and can represent some euclidean space $\mathbb{R}^n$, as in the case of an ordinary differential equation, or a space of functions, as in the case of some partial differential equations. The main restriction imposed is that the final position $T(t)x_0$ depends only on the initial position x_0 and the time t that has passed, not depending on the initial moment. Dynamical systems that satisfy this condition are called autonomous.

In order to study the asymptotics of a dynamical system, we first will define some previous concepts:

Definition 11 (ω-limit)**.** Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X, and for any $B \subset X$, let $T(t)B = \{T(t)b : b \in B\}$. We define:

- The **positive orbit** of B under $\mathscr{T}$:

$$\gamma^+(B) = \bigcup_{s \geq 0} T(s)B.$$

- The **partial orbit** of B under $\mathscr{T}$ starting on $t \in \mathbb{R}^+$, $t > 0$:

$$\gamma_t^+(B) = \bigcup_{s \geq t} T(s)B.$$

- The **ω-limit** set of $B \subset X$ under $\mathscr{T}$ is defined by

$$\omega(B) = \bigcap_{t \in \mathbb{R}^+} \overline{\gamma_t^+(B)}.$$

Definition 12 (Global solution and global orbit)**.** A **global solution** of $\mathscr{T} = \{T(t) : t \geq 0\}$ through $x \in X$ is a continuous function $\phi : \mathbb{R} \to X$ such that $T(t)\phi(s) = \phi(t+s)$, for all $t \geq 0$ and $s \in \mathbb{R}$, and $x = \phi(0)$. If $\phi(\mathbb{R})$ is bounded, we say that ϕ is a bounded global solution. Lastly, if ϕ is a constant global solution, it is called a **stationary solution** for $\mathscr{T}$, and its value is called an **equilibrium** for $\mathscr{T}$.

When a global solution $\phi : \mathbb{R} \to X$ through x exists, we define the **global orbit** of x relative to the global solution ϕ by:

$$\gamma_\phi(x) = \{\phi(t) \colon t \in \mathbb{R}\}.$$

If $t \in \mathbb{R}$, we write $(\gamma_\phi)_t^-(x) = \{\phi(s) \colon s \in \mathbb{R} \text{ and } s \leq t\}$.

Note that if ϕ is a global solution of $\mathscr{T}$ through $x \in X$, the value $\phi(-t)$ is a point that is taken by $T(t)$ to $T(t)\phi(-t) = x$. Because of this, we may say that a global solution is associated to a possible representation of the past history of a point $x \in X$. With that said, it is important to note that a global solution through $x \in X$ does not need to be unique for negative values of t.

Definition 13 (α-limit)**.** When a global solution $\phi : \mathbb{R} \to X$ through x exists, we define the **α-limit** set of x relative to ϕ as

$$\alpha_\phi(x) = \bigcap_{t \leq 0} \overline{(\gamma_\phi)_t^-(x)}.$$

The following characterization of the ω-limit and α-limit sets will be frequently used in the proofs of the results that follow.

Proposition 5. If $B \subset X$, then:

1. $\omega(B)$ is closed and

$$\begin{aligned} \omega(B) = \{y \in X : \ & \text{there are sequences } \{t_n\}_{n\in\mathbb{N}} \text{ in } \mathbb{R}^+ \text{ and } \{x_n\}_{n\in\mathbb{N}} \\ & \text{in } B \text{ such that } t_n \overset{n\to\infty}{\longrightarrow} \infty \text{ and } y = \lim_{n\to\infty} T(t_n)x_n\}. \end{aligned} \tag{3.1}$$

2. If $\phi : \mathbb{R} \to X$ is a global solution of $\mathscr{T}$ through $x \in X$, then $\alpha_\phi(x)$ is closed and

$$\begin{aligned} \alpha_\phi(x) = \{v \in X : \ & \text{there is a sequence } \{t_n\}_{n\in\mathbb{N}} \text{ in } \mathbb{R}^+ \text{ such that} \\ & t_n \overset{n\to\infty}{\longrightarrow} \infty \text{ and } v = \lim_{n\to\infty} \phi(-t_n)\}. \end{aligned} \tag{3.2}$$

Proof. We will prove the first claim, since the proof of the characterization of $\alpha_\phi(x)$ is analogous. It is obvious from definition that $\omega(B)$ is closed.

Let $y \in \omega(B) = \bigcap_{t\geq 0} \overline{\gamma_t^+(B)}$. Then, for each $n \in \mathbb{N}$, $y \in \overline{\gamma_n^+(B)}$. Therefore, there exist $x_n \in B$, and $t_n \geq n$ such that $\mathrm{d}(T(t_n)x_n, y) < \frac{1}{n}$. Obviously $t_n \to \infty$ and $y = \lim_{n\to\infty} T(t_n)x_n$, which concludes one inclusion.

Now, suppose that $y \in X$ and there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $\{x_n\}_{n\in\mathbb{N}} \subset B$ with $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and $y = \lim_{n\to\infty} T(t_n)x_n$. In this case, for any $\tau \geq 0$, we have $\{T(t_n)x_n\}_{t_n\geq\tau} \subset \gamma_\tau^+(B)$, and $y \in \overline{\gamma_\tau^+(B)}$. It proves that $y \in \omega(B)$ and the characterization of $\omega(B)$ is proved.

□

Definition 14 (Distance and Hausdorff semidistance)**.** Given two nonempty subsets A, $B \subset X$, we define their **Hausdorff semidistance** $\mathrm{d}_H(A,B)$ by:

$$\mathrm{d}_H(A,B) = \sup\left\{ \inf_{y\in B} \mathrm{d}(x,y) : x \in A \right\}.$$

Also, we denote by $\mathrm{d}(A,B)$ the usual distance between these sets, given by:

$$\mathrm{d}(A,B) = \inf\left\{ \inf_{y\in B} \mathrm{d}(x,y) : x \in A \right\}.$$

Definition 15 (Attraction and absorption)**.** Let A and B be nonempty subsets of X and $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X. We say that:

1. The set A $\mathscr{T}$**-attracts** B if $\lim_{t\to\infty} \mathrm{d}_H(T(t)B,A) = 0$.

2. A $\mathscr{T}$**-absorbs** B if there exists $t_0 \geqslant 0$ such that $T(t)B \subset A$, for all $t \geqslant t_0$.

Remark 6. It follows from this definition that if A $\mathscr{T}$-absorbs B, then A $\mathscr{T}$-attracts B, and if A $\mathscr{T}$-attracts B, then any ε-neighborhood of A $\mathscr{T}$-absorbs B.

Definition 16 (Invariance)**.** Let $\mathscr{T} = \{T(t)\colon t \in \mathbb{R}^+\}$ be a semigroup in X. The subset $A \subset X$ is said to be $\mathscr{T}$**-invariant** (resp. positively $\mathscr{T}$-invariant) if $T(t)A = A$ for all $t \geq 0$ (resp. if $T(t)A \subset A$ for all $t \geq 0$).

Now we are ready to define the global attractor of a semigroup.

Definition 17 (Global attractor)**.** Let $\mathscr{T}$ be a semigroup in X. A subset $\mathscr{A} \subset X$ is called its **global attractor** if it is nonempty, compact, $\mathscr{T}$-invariant, and $\mathscr{T}$-attracts any bounded subset of X.

Remark 7. If the semigroup $\mathscr{T} = \{T(t) : t \geq 0\}$ possesses a global attractor, it is unique. In fact, if $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are global attractors for $\mathscr{T}$, then $\mathscr{A}$ is bounded, so $\tilde{\mathscr{A}}$ $\mathscr{T}$-attracts $\mathscr{A}$. Moreover, $\mathscr{A} = T(t)\mathscr{A}$, for all $t \geq 0$, whence:

$$\mathrm{d}((\mathscr{A},\tilde{\mathscr{A}}) = \mathrm{d}((T(t)\mathscr{A},\tilde{\mathscr{A}}) \stackrel{t\to\infty}{\longrightarrow} 0,$$

which implies that $\mathscr{A} \subset \tilde{\mathscr{A}}$. The other inclusion follows analogously, and we are done.

We have the following characterization for global attractors.

Proposition 6. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X that possesses a global attractor $\mathscr{A}$. Then:

$$\mathscr{A} = \{x \in X\colon \text{ there exists a bounded global solution of } \mathscr{T} \text{ through } x\}.$$

Proof. Suppose $x \in X$ is such that there exists a bounded global solution of $\mathscr{T}$ through x, and call this solution $\phi : \mathbb{R} \to X$. Then, $\phi(\mathbb{R})$ is bounded and invariant, whence $\phi(\mathbb{R}) \subset \mathscr{A}$, and $x \in \mathscr{A}$.

Let $x \in \mathscr{A}$, and we will show that there exists a bounded global solution $\phi\colon \mathbb{R} \to \mathscr{A}$ through x. Indeed, define $\phi(t) = T(t)x$ for $t \geq 0$, and note that $\phi(\mathbb{R}^+)$ is bounded because $\mathscr{A}$ attracts $\{x\}$.

Since $x \in \mathscr{A} = T(1)\mathscr{A}$, there exists $x_{-1} \in \mathscr{A}$ such that $T(1)x_{-1} = x$ and proceding by induction, we can construct a sequence (not necessarily unique) $\{x_{-n}\}_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that $x_0 = x$ and $T(1)x_{-n-1} = x_{-n}$ for all $n \in \mathbb{N}$. Define

$$\phi(t) = \begin{cases} T(t)x, & t \geq 0 \\ T(j+t)x_{-j}, & t \in [-j,-j+1), \quad j \in \mathbb{N}^*. \end{cases} \tag{3.3}$$

Since $\mathscr{A}$ is invariant and bounded, $\phi : \mathbb{R} \to \mathscr{A}$ is a well defined bounded global solution of $\mathscr{T}$ through x, its continuity following from the continuity of the semigroup.

□

Proposition 7. Let $\mathscr{T}$ be a semigroup in X. Then $A \subset X$ is $\mathscr{T}$-invariant if, and only if, for each $x \in A$ there exists a global solution $\phi : \mathbb{R} \to A$ of $\mathscr{T}$ through x.

Proof. Repeat the construction in the proof of the last proposition. □

Now we intend to conclude some properties about the ω-limit of a set based on properties of the set and the semigroup.

Proposition 8. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X and K be a compact subset of X. Suppose K $\mathscr{T}$-attracts a nonempty compact set $K_1 \subset X$, then $\gamma^+(K_1)$ is precompact and $\varnothing \neq \omega(K_1) \subset K$.

Proof. First we prove the following claim: if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that

$$\mathrm{d}(x_n, K) \overset{n\to\infty}{\longrightarrow} 0,$$

then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence to a point in K.

Indeed, for any $m \in \mathbb{N}$, let $n_m \in \mathbb{N}$ be such that $\mathrm{d}(x_{n_m}, K) < 1/m$, so that there exists an $y_m \in K$ such that $\mathrm{d}(x_{n_m}, y_m) < \frac{1}{m}$. We can pass to a subsequence and assume $y_m \overset{m\to\infty}{\longrightarrow} y_0 \in K$, whence:

$$\mathrm{d}(x_{n_m}, y_0) \leq \mathrm{d}(x_{n_m}, y_m) + \mathrm{d}(y_m, y_0) \overset{m\to\infty}{\longrightarrow} 0,$$

and the claim is proven.

Now, note that for any $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that:

$$T(t)K_1 \subset \mathscr{O}_{\frac{\varepsilon}{2}}(K), \quad \text{for } t \geq t_0.$$

Consider a finite covering of K given by $\{B_{\frac{\varepsilon}{2}}(x_i) : i = 1, \ldots, N\}$ with x_1, ..., $x_N \in K$. It is easy to see that $\{B_\varepsilon(x_i) : i = 1, \ldots, N\}$ covers $\bigcup_{t\geq t_0} T(t)K_1$, so that this last set is totally bounded. Since K and $\bigcup_{0\leq t\leq t_0} T(t)K_1$ are compact, hence totally bounded, it follows that $\gamma^+(K_1) \cup K$ is totally bounded.

Now we will prove that $\gamma^+(K_1) \cup K$ is complete, which will imply that $\gamma^+(K_1) \cup K$ is compact. Indeed, let $\{x_n\} \subset \gamma^+(K_1) \cup K$ be a Cauchy sequence.

Suppose first that there exists $\tilde{t} \geq 0$ such that $\{x_n : n \in \mathbb{N}\} \subset K \cup \left(\bigcup_{0\leq t\leq \tilde{t}} T(t)K_1\right)$. Hence, $\{x_n\}$ is convergent to an element of $K \cup \left(\bigcup_{0\leq t\leq \tilde{t}} T(t)K_1\right)$, which is also an element of $\gamma^+(K_1) \cup K$.

Now suppose that for any $m \in \mathbb{N}$, there exists an $n_m \in \mathbb{N}$ such that $x_{n_m} \in \bigcup_{t>m} T(t)K_1$. Then, take $t_m > m$ and $y_m \in K_1$ such that $x_{n_m} = T(t_m)y_m$, and since K attracts K_1, the following holds:

$$\mathrm{d}(x_{n_m}, K) \leq \mathrm{d}_H(T(t_m)K_1, K) \overset{m\to\infty}{\longrightarrow} 0.$$

From the first claim, it follows that $\{x_{n_m}\}_m$ has a further subsequence that converges to an element of K. Since $\{x_n\}$ is Cauchy, it converges itself to an element of K, and we are done.

We just proved that $\gamma^+(K_1) \cup K$ is compact, and $\gamma^+(K_1)$ is precompact.

For all $t \geq 0$, $\overline{\gamma_t^+(K_1)}$ is compact and nonempty, and $\overline{\gamma_t^+(K_1)} \subset \overline{\gamma_s^+(K_1)}$ for $s \leq t$. It implies that the family $\{\overline{\gamma_t^+(K_1)}\}_{t\in\mathbb{R}^+}$ has the finite intersection property, and since it is a family of subsets of the compact $\overline{\gamma^+(K_1)}$, it follows from a basic topology theorem that:

$$\omega(K_1) = \bigcap_{t\geq 0} \overline{\gamma_t^+(K_1)} \neq \varnothing.$$

To prove that $\omega(K_1) \subset K$, suppose $y \in \omega(K_1)$, and for any $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that:

$$y \in \overline{\gamma_{t_0}^+(K_1)} \subset \mathscr{O}_\varepsilon(K),$$

so $\mathrm{d}(y,K) \leq \varepsilon$ and since ε is arbitrary, $y \in K$.

□

Lemma 3. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X and $B \subset X$. Then $\omega(B)$ is positively $\mathscr{T}$-invariant. If $\omega(B)$ is compact and $\mathscr{T}$-attracts B, then $\omega(B)$ is $\mathscr{T}$-invariant.

Proof. If $\omega(B) = \varnothing$ then there is nothing to prove. Assume then that $\omega(B) \neq \varnothing$ and fix $t \geq 0$. From Proposition 5, given $y \in \omega(B)$ there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and $\{x_n\}_{n\in\mathbb{N}}$ in B such that $y = \lim_{n\to\infty} T(t_n)x_n$. It follows from the continuity of $T(t)$ that $T(t)y = \lim_{n\to\infty} T(t+t_n)x_n$ and hence $T(t)y \in \omega(B)$. Hence $\omega(B)$ is positively $\mathscr{T}$-invariant.

Suppose now that $\omega(B)$ is compact and $\mathscr{T}$-attracts B, then we must show that $\omega(B) \subset T(t)\omega(B)$, for all $t \geq 0$. For $y \in \omega(B)$ there are sequences $\{t_n\}$ with $t_n \to \infty$ and $\{x_n\}$ in B such that $y = \lim_{n\to\infty} T(t_n)x_n$. If $t \in \mathbb{R}^+$, we can redefine $\{t_n\}$ and $\{x_n\}$ passing to subsequences so that $t_n \geq t$ for all $n \in \mathbb{N}$. Therefore, $T(t)T(t_n - t)x_n = T(t_n)x_n \to y \in \omega(B)$ as $n \to \infty$. Since $\omega(B)$ is compact and $\mathscr{T}$-attracts B we have $\mathrm{d}(T(t_n - t)x_n, \omega(B)) \overset{n\to\infty}{\longrightarrow} 0$, and from the first claim proved in Proposition 8, the sequence $\{T(t_n - t)x_n\}_{n\in\mathbb{N}}$ has a subsequence (which we denote the same) that converges to some $x \in \omega(B)$. Hence, by continuity of $T(t)$, we have $T(t)x = y$. Therefore $\omega(B) \subset T(t)\omega(B)$, for all $t \in \mathbb{R}^+$, and along with what was proved in the last paragraph, we get $T(t)\omega(B) = \omega(B)$ for all $t \in \mathbb{R}^+$, which completes the proof.

□

The following lemma presents a characterization for the α-limit of a point. Its proof is similar to the ones presented so far and will be omitted.

Lemma 4. Let $x \in X$ and suppose $\phi : \mathbb{R} \to X$ is a global solution through x such that $\overline{\phi(\mathbb{R}^-)}$ is compact. Then, $\alpha_\phi(x)$ is nonempty, compact, invariant and for every $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that $\phi(-t) \in \mathscr{O}_\varepsilon(\alpha_\phi(x))$ for all $t \geq t_0$.

Lemma 5. Suppose $\mathscr{T} = \{T(t) : t \geq 0\}$ is a semigroup in X, $B \subset X$ is connected and $\omega(B)$ is compact and attracts B. Then $\omega(B)$ is connected.

Proof. Since $\omega(B)$ attracts B, it follows that $\mathrm{d}_H(\overline{\gamma_t^+(B)}, \omega(B)) \xrightarrow{t\to\infty} 0$. Indeed, given $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that $T(t)B \subset \mathscr{O}_{\frac{\varepsilon}{2}}(\omega(B))$, for $t \geq t_0$, whence $\overline{\gamma_t^+(B)} \subset \mathscr{O}_\varepsilon(\omega(B))$.

Since $[0,\infty) \times X \ni (s,x) \mapsto T(s)x \in X$ is continuous, and $[t,\infty) \times B$ is connected, it follows that $\overline{\gamma_t^+(B)}$ is connected, for all $t \geq 0$.

Suppose that $\omega(B)$ is a disjoint union of nonempty compact sets ω_1 and ω_2, such that $\mathrm{d}(\omega_1, \omega_2) > 2\rho$. For some $t_0 \geq 0$, $\overline{\gamma_{t_0}^+(B)} \subset \mathscr{O}_\rho(\omega(B))$, but $\omega(B) \subset \overline{\gamma_{t_0}^+(B)}$, which contradicts the fact that $\overline{\gamma_{t_0}^+(B)}$ is connected. This concludes the proof.

□

The following concepts are used to characterize semigroups that have global attractors.

Definition 18 (Bounded and eventually bounded)**.** A semigroup $\mathscr{T}$ in X is said to be **eventually bounded** if for every $B \subset X$ bounded, there exists $t_B \geq 0$ such that $\gamma_{t_B}^+(B)$ is bounded. Also, $\mathscr{T}$ is said to be **bounded** if $\gamma^+(B)$ is bounded whenever $B \subset X$ is bounded.

Definition 19 (Asymptotically compact)**.** A semigroup $\mathscr{T}$ is said to be **asymptotically compact** if for any closed, bounded, nonempty and positively invariant subset $B \subset X$, there exists a compact $J \subset B$ that attracts B.

We have the following characterization for an asymptotically compact semigroup:

Proposition 9. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X and suppose that $\{T(t_n)x_n : n \in \mathbb{N}\}$ is precompact whenever $\{x_n : n \in \mathbb{N}\}$ and $\{T(t_n)x_n : n \in \mathbb{N}\}$ are bounded in X and $t_n \to \infty$. Then, $\mathscr{T}$ is asymptotically compact.

On the other hand, if $\mathscr{T}$ is asymptotically compact and eventually bounded, then $\{T(t_n)x_n : n \in \mathbb{N}\}$ is precompact whenever $\{x_n\}$ is a bounded sequence in X and $t_n \to \infty$.

Proof. Assume the first hypothesis and suppose that $B \subset X$ is closed, bounded, nonempty and positively invariant. We will prove that $\omega(B)$ is nonempty, compact, attracts B and $\omega(B) \subset B$.

Indeed, let $\{x_n\}$ be a sequence in B and $t_n \to \infty$, then $\{T(t_n)x_n\}$ is also in B, so $\{T(t_n)x_n\}$ has a subsequence that converges to $y \in X$, and by Proposition 5, $y \in \omega(B)$, and $\omega(B)$ is not empty.

Now let us prove that $\omega(B)$ is compact. Let $\{y_n\}$ be a sequence in $\omega(B)$, and note that, for each $n \in \mathbb{N}$, $y_n \in \overline{\gamma_n^+(B)}$ and it can be written as $y_n = \lim\limits_{k\to\infty} T(t_k^n)x_k^n$ with $t_k^n \geq n$ and $x_k^n \in B$, for all $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $d\left(y_n, T(t_{k_n}^n)x_{k_n}^n\right) < \frac{1}{n}$. The sequence $\left\{T(t_{k_n}^n)x_{k_n}^n\right\}_n$ satisfies $t_{k_n}^n \geq n$, and both $\left\{x_{k_n}^n\right\}_n$ and $\left\{T(t_{k_n}^n)x_{k_n}^n\right\}_n$ are contained in B, so it follows from the hypothesis that $\left\{T(t_{k_n}^n)x_{k_n}^n\right\}_n$ has a convergent subsequence, which we denote the same, converging to an element $y \in X$. It follows that:

$$d(y, y_n) \leq d\left(y_n, T\left(t_{k_n}^n\right)x_{k_n}^n\right) + d\left(T\left(t_{k_n}^n\right)x_{k_n}^n, y\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, $\{y_n\}$ has a convergent subsequence, from which follows that $\omega(B)$ is precompact, and also compact, since it is closed.

It follows from the positive invariance of B that $\omega(B) \subset B$, and we only are left to prove that $\omega(B)$ $\mathscr{T}$-attracts B. Indeed, suppose not, then, there exists an $\varepsilon_0 > 0$ and a sequence $t_n \to \infty$ such that $\mathrm{d}_H(T(t_n)B, \omega(B)) > \varepsilon_0$. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in B$ such that $\mathrm{d}(T(t_n)x_n, \omega(B)) > \varepsilon_0$. Yet, $\{T(t_n)x_n\}$ contains a subsequence converging to an element of $\omega(B)$, which is a contradiction.

Now, suppose $\mathscr{T}$ is asymptotically compact and eventually bounded. If $\{x_n\}$ is bounded in X and $t_n \to \infty$, there exists $t_0 \geq 0$ such that $B = \overline{\gamma_{t_0}^+(\{x_n : n \in \mathbb{N}\})}$ is bounded. Moreover, B is positively invariant, hence there exists a compact $J \subset B$ that attracts B. In particular, $\mathrm{d}(T(t_n)x_n, J) \overset{n\to\infty}{\longrightarrow} 0$, and since J is compact, $\{T(t_n)x_n\}$ has a convergent subsequence, and is precompact.

□

Finally, we present sufficient conditions to ensure that $\omega(B)$ is nonempty, compact and attracts B, and is invariant.

Lemma 6. If $\mathscr{T} = \{T(t) : t \geq 0\}$ is asymptotically compact and $B \subset X$ is a nonempty subset of X such that $\gamma_{t_0}^+(B)$ is bounded for some $t_0 \geq 0$, then $\omega(B)$ is nonempty, compact, invariant and attracts B.

Proof. From the continuity of $T(t)$, it follows that $T(t)\overline{\gamma_{t_0}^+(B)} \subset \overline{\gamma_{t_0}^+(B)}$, for $t \geq 0$, so there exists a compact $J \subset \overline{\gamma_{t_0}^+(B)}$ that attracts $\overline{\gamma_{t_0}^+(B)}$, hence there exists a sequence $t_n \to \infty$ such that

$$T(t)\overline{\gamma_{t_0}^+(B)} \subset \mathscr{O}_{\frac{1}{n}}(J), \quad \text{for} \quad t \geq t_n. \tag{3.4}$$

In particular, let $\{x_n\}$ be a sequence in B, then $\{T(t_n)T(t_0)x_n\}$ is bounded, $\{T(t_0)x_n\}$ is bounded, and $t_n \to \infty$, and from Proposition 9, $\{T(t_n)T(t_0)x_n\}$ has a subsequence that converges to an element of $\omega(B)$, hence $\omega(B) \neq \varnothing$.

Now let us prove that $\omega(B) \subset J$, so that $\omega(B)$ is compact. Indeed, let $y \in \omega(B)$. Then y can be written as $y = \lim_{n\to\infty} T(t_n)x_n$, for $t_n \geq t_0$ for all $n \in \mathbb{N}$, then $y = \lim_{n\to\infty} T(t_n - t_0)T(t_0)x_n$, and from (3.4), we get

$$\mathrm{d}(T(t_n - t_0)T(t_0)x_n, J) \stackrel{n\to\infty}{\longrightarrow} 0, \tag{3.5}$$

and since J is compact, it follows from a claim that has already been proved that $y \in J$.

Now we only need to show that $\omega(B)$ $\mathscr{T}$-attracts B, and it will follow from Lemma 3 that $\omega(B)$ is invariant. Suppose $\omega(B)$ does not attract B. Then, there exists an $\varepsilon_0 > 0$ and a sequence $t_n \to \infty$, $t_n \geq t_0$, such that $\mathrm{d}_H(T(t_n)B, \omega(B)) > \varepsilon_0$. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in B$ such that $\mathrm{d}(T(t_n)x_n, \omega(B)) > \varepsilon_0$. Since $\mathrm{d}(T(t_n - t_0)T(t_0)x_n, J) \stackrel{n\to\infty}{\longrightarrow} 0$, $\{T(t_n)x_n\}$ has a subsequence that converges, and its limit is in $\omega(B)$. This is a contradiction, and we are done.

□

Lemma 7. Suppose B is a nonempty subset of X and $\overline{\gamma^+_{t_0}(B)}$ is compact for some $t_0 \geq 0$. Then $\omega(B)$ is nonempty, compact, invariant and attracts B.

Proof. We know that $\left\{\overline{\gamma^+_t(B)} : t \geq t_0\right\}$ is a family of closed subsets of the compact set $\overline{\gamma^+_{t_0}(B)}$, and has the finite intersection property, hence $\omega(B) = \bigcap_{t\geq t_0} \overline{\gamma^+_t(B)}$ is nonempty and compact.

Now we only need to prove that $\omega(B)$ attracts B, and it will follow from Lemma 3 that $\omega(B)$ is invariant. Suppose not, so there exists an $\varepsilon_0 > 0$ and sequences $t_n \to \infty$, $x_n \in B$ such that $\mathrm{d}(T(t_n)x_n, \omega(B)) > \varepsilon_0$ for all $n \in \mathbb{N}$. However, there exists a $n_1 \in \mathbb{N}$ such that $\{T(t_n)x_n : n \geq n_1\} \subset \overline{\gamma^+_{t_0}(B)}$, which is compact. Hence $\{T(t_n)x_n\}$ has a convergent subsequence and the limit belongs to $\omega(B)$, which contradicts $\mathrm{d}(T(t_n)x_n, \omega(B)) > \varepsilon_0$ for all $n \in \mathbb{N}$.

□

Definition 20 (Conditionally eventually compact)**.** A semigroup $\mathscr{T} = \{T(t) : t \geq 0\}$ is said to be **conditionally eventually compact** if for any $B \subset X$ bounded and positively invariant, there exists a $t_B \in \mathbb{R}^+$ such that $\overline{T(t_B)B}$ is compact. Moreover, $\mathscr{T}$ is said to be **eventually compact** if for any $B \subset X$ bounded, there exists $t_B \in \mathbb{R}^+$ such that $\overline{T(t_B)B}$ is compact.

Theorem 18. A conditionally eventually compact semigroup is asymptotically compact.

Proof. Let $\{T(t) : t \geq 0\}$ be a conditionally eventually compact semigroup and suppose $B \subset X$ is nonempty, closed, bounded and $T(t)B \subset B$ for all $t \geq 0$. Then, there exists $t_B \geq 0$ such that $T(t_B)B$ is precompact. We claim that $\overline{\gamma^+_{t_B}}$ is compact. Indeed, if $\{x_n\}$ is a sequence in $\gamma^+_{t_B}$, for each $n \in \mathbb{N}$, there exist $t_n \geq 0$, $y_n \in B$ such that $x_n = T(t_n)T(t_B)y_n = T(t_B)T(t_n)y_n$ and since B is positively invariant, $\{x_n\}$ is a sequence in $T(t_B)B$ and has a convergent subsequence.

It follows from Lemma 7 that $\omega(B) \subset B$ is nonempty, compact and attracts B, which concludes the proof that $\{T(t) : t \geq 0\}$ is asymptotically compact.

□

Definition 21 (Point/bounded/compact-dissipative)**.** We say that a semigroup $\mathscr{T}$ is **point-dissipative** (resp. **bounded-dissipative**, **compact-dissipative**) if there exists a bounded subset $B \subset X$ which $\mathscr{T}$-attracts any point (resp. any bounded set, any compact set) of X.

Lemma 8. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a point-dissipative and asymptotically compact semigroup in X. Suppose that if K is compact, $\gamma^+_{t_K}(K)$ is bounded for some $t_K \in \mathbb{R}^+$. Then $\mathscr{T}$ is compact-dissipative.

Proof. Since $\mathscr{T}$ is point-dissipative, there exists a nonempty bounded set B which absorbs points of X. Define $U := \{x \in B : \gamma^+(x) \subset B\}$. It is easy to see that $U \neq \varnothing$, $\gamma^+(U) = U$, U is bounded and absorbs any point of X. Moreover, $\overline{\gamma^+(U)}$ is positively invariant. It follows that there exists a compact K, $K \subset \overline{\gamma^+(U)} = \overline{U}$, which attracts U. Since U absorbs points and K attracts U, it follows that K attracts points of X.

Now, let us prove that for some neighborhood V of K, $\gamma^+_t(V)$ is bounded for some $t \geq 0$. Suppose not, then for any $n \in \mathbb{N}$, $\gamma^+_n(\mathscr{O}_{\frac{1}{n}}(K))$ is unbounded, hence there exists $x_n \in \mathscr{O}_{\frac{1}{n}}(K)$, $t_n \geq n$, such that $\mathrm{d}(T(t_n)x_n, 0) > n$. The sequence $\{T(t_n)x_n\}$ is unbounded, $x_n \to y \in K$ and $t_n \to \infty$. It follows that the set $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact, but $\gamma^+_m(\overline{\{x_n : n \in \mathbb{N}\}})$ is not bounded for any $m \in \mathbb{N}$ because it contains $\{T(t_n)x_n : n \geq m\}$. This contradicts an hypothesis.

Let V be an ε-neighborhood of K and $t_V \geq 0$ be such that $\gamma^+_{t_V}(V)$ is bounded. If $x \in X$, there exists $t_x \geq 0$ such that $T(t)x \in V$ for $t \geq t_x$. Since $T(t_x)$ is continuous, there exists a neighborhood $\mathscr{O}_x$ of x such that $T(t_x)\mathscr{O}_x \subset V$. If $t \geq t_x + t_V$, $T(t)\mathscr{O}_x \subset \gamma^+_{t_V}(V)$. It follows that for any $x \in X$, $\gamma^+_{t_V}(V)$ absorbs some neighborhood of x, which implies that $\gamma^+_{t_V}(V)$ absorbs compact sets of X, hence $\mathscr{T}$ is compact-dissipative.

□

Proposition 10. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X. If K is compact and $\mathscr{T}$-attracts itself, then $\omega(K) = \bigcap_{t \geq 0} T(t)K$.

Proof. It follows from definition that $\bigcap_{t \geq 0} T(t)K \subset \omega(K)$. For the other inclusion, it follows from Proposition 8 that $\gamma^+(K)$ is precompact and $\varnothing \neq \omega(K) \subset K$. From Lemma 7, $\omega(K)$ is invariant, whence $\omega(K) = T(t)\omega(K) \subset T(t)K$ for all $t \geq 0$, which proves the other inclusion. □

The next theorem gives a characterization of semigroups that possess a global attractor.

Theorem 19. A semigroup $\mathscr{T} = \{T(t) : t \geq 0\}$ has a global attractor $\mathscr{A}$ if, and only if, $\mathscr{T}$ is point-dissipative, eventually bounded and asymptotically compact.

Proof. Suppose $\mathscr{T}$ is point-dissipative, eventually bounded and asymptotically compact, then by Lemma 8, $\mathscr{T}$ is compact-dissipative. Let C be a bounded subset of X which $\mathscr{T}$-absorbs compact sets of X, and $B := \{x \in C : \gamma^+(x) \subset C\}$. It is easy to see that B is nonempty, absorbs compacts of X and is positively invariant, hence $T(t)\overline{B} \subset \overline{B}$ for all $t \geq 0$. Since $\mathscr{T}$ is asymptotically compact, there exists a compact set $K \subset \overline{B}$ that attracts B, and, as a consequence, K attracts any compact set in X.

Consider the set $\mathscr{A} = \omega(K)$. Since K $\mathscr{T}$-attracts itself, it follows from Proposition 8 and Lemma 7 that $\mathscr{A}$ is nonempty, compact and invariant. If $J \subset X$ is compact, K $\mathscr{T}$-attracts J and it follows from the same lemma and proposition that $\omega(J) \subset K$, $\omega(J)$ $\mathscr{T}$-attracts J, and $\omega(J) = T(s)\omega(J) \subset T(s)K$ for all $s \geq 0$, and it follows from Proposition 10 that $\omega(J) \subset \omega(K)$. Since $\omega(J)$ $\mathscr{T}$-attracts J, $\omega(K)$ also $\mathscr{T}$-attracts J.

Let B be a bounded subset of X. Since $\mathscr{T}$ is asymptotically compact and eventually bounded, it follows from Lemma 6 that $\omega(B)$ is nonempty, compact, invariant and $\mathscr{T}$-attracts B. Hence, using $J = \omega(B)$ in the last paragraph claim, and the fact that $\omega(B)$ is invariant, we get $\omega(\omega(B)) = \omega(B) \subset \mathscr{A}$, from which $\mathscr{A}$ attracts B. Hence, $\mathscr{A}$ is the global attractor for $\mathscr{T}$.

Suppose now that $\mathscr{T}$ has a global attractor $\mathscr{A}$. Then it is easy to see that $\mathscr{T}$ is eventually bounded — since B bounded is attracted by $\mathscr{A}$, and $\mathscr{T}$ is point dissipative — since $\mathscr{A}$ attracts points. To see that $\mathscr{T}$ is asymptotically compact, we use Proposition 9. Suppose $\{x_n\}$ is bounded and $t_n \to \infty$, then $\mathscr{A}$ attracts $\{x_n : n \in \mathbb{N}\}$, and $\mathrm{d}(T(t_n)x_n, \mathscr{A}) \overset{n\to\infty}{\longrightarrow} 0$, and since $\mathscr{A}$ is compact, $\{T(t_n)x_n\}$ has a convergent subsequence, whence $\mathscr{T}$ is asymptotically compact.

□

3.2 Gradient semigroups

A very important kind of semigroup is the so called gradient semigroup. It arises from real world problems in which there is some kind of energy that is dissipated along the time, forcing the system to occupy states that are each time more basic. As it will be seen, the global attractor of a gradient semigroup can be very well characterized.

We recall that an equilibrium for a semigroup is an unitary invariant set. In this section we study a semigroup $\mathscr{T} = \{T(t) : t \geq 0\}$ that has a set of equilibria

$$\mathscr{E} = \{x \in X : T(t)x = x \,\forall\, t \geq 0\}. \tag{3.6}$$

Definition 22 (Gradient semigroup)**.** Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X and $\mathscr{E}$ be its set of equilibria. We say that $\mathscr{T}$ is **gradient** if there is a continuous function $V : X \to \mathbb{R}$ such that:

1. The map $\mathbb{R}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing, for each $x \in X$.
2. If $V(T(t)x) = V(x), \forall\, t \geq 0$, then $x \in \mathscr{E}$.

   The function V is called the Lyapunov function of $\mathscr{T}$.

In what follows, we characterize the asymptotic behavior and global attractor of a gradient semigroup.

Lemma 9. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a gradient semigroup and $\mathscr{E}$ be its set of equilibria. For any $x \in X$, $\omega(x) \subset \mathscr{E}$, and if there exists a global solution $\phi : \mathbb{R} \to X$ through x, then $\alpha_\phi(x)$ is a subset of $\mathscr{E}$.

If $\mathscr{T}$ has a global attractor and $\mathscr{E}$ only has isolated points, then $\mathscr{E} \subset \mathscr{A}$ is finite. In this case, for each $x \in X$, $\omega(x)$ is an unitary set, and if $\phi : \mathbb{R} \to X$ is a global solution through $x \in X$, $\alpha_\phi(x)$ is an unitary set.

Proof. We only consider the ω-limit, because the proofs for the α-limit are analogous. Suppose $\omega(x) \neq \varnothing$, and $y \in \omega(x)$. It follows from the characterization 5 that there exists an increasing sequence $\{t_n\}$, with $t_n \to \infty$ and $y = \lim_{n\to\infty} T(t_n)x$. The function $\mathbb{R} \ni t \mapsto V(T(t)x)$ is decreasing and $V(T(t_n)x) \overset{n\to\infty}{\longrightarrow} V(y) =: c \in \mathbb{R}$. It follows that $V(T(t)x) \overset{t\to\infty}{\longrightarrow} c$, and $V \equiv c$ in $\omega(x)$. Since $\omega(x)$ is positively invariant, $T(t)y \in \omega(x)$ and $V(T(t)y) = V(y)$, for all $t \in \mathbb{R}$.

If $\mathscr{T}$ has a global attractor $\mathscr{A}$, $\mathscr{E} \subset \mathscr{A}$ and since $\mathscr{A}$ is compact, $\mathscr{E}$ is finite. Since $\omega(x)$ is connected (Lemma 5), $\omega(x)$ is an unitary set. □

Definition 23 (Unstable and stable sets)**.** Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup in X and A be a subset of X. We define the **unstable set** of A by:

$$W^u(A) := \{y \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ through } y \text{ such that } \xi(t) \overset{t\to-\infty}{\longrightarrow} A\},$$

and the **stable set** of A is defined by:

$$W^s(A) := \{y \in X : T(t)y \overset{t\to\infty}{\longrightarrow} A\}.$$

Finally, given a δ-neighborhood $\mathscr{O}_\delta(A)$ of A, the **local unstable set** of A and **local stable set** of A are defined, respectively, by

$$\begin{aligned} W^{u,\delta}_{loc}(A) \;:=\; & \{y \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ through } y \text{ such that} \\ & \xi(t) \in \mathscr{O}_\delta(A), \forall t \leq 0, \text{ and } \xi(t) \overset{t\to-\infty}{\longrightarrow} A\}, \end{aligned}$$

and

$$W^{s,\delta}_{loc}(A) := \{y \in X : T(t)y \in \mathscr{O}_\delta(A), \forall t \geq 0, \text{ and } T(t)y \overset{t\to\infty}{\longrightarrow} A\}.$$

Theorem 20. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a gradient, eventually bounded and asymptotically compact semigroup and let $\mathscr{E}$, its set of equilibria, be bounded. Then $\mathscr{T}$ has a global attractor $\mathscr{A} = W^u(\mathscr{E})$. If $\mathscr{E} = \{e_1, \ldots, e_n\}$, then $\mathscr{A} = \bigcup_{i=1}^n W^u(e_i)$. Finally, if $\mathscr{A}$ is subset of a connected bounded set, then $\mathscr{A}$ is connected.

Proof. Since $\mathscr{T}$ is asymptotically compact and eventually bounded, we may use Lemma 6 to assure that $\omega(x)$ attracts x, and from the fact that $\mathscr{T}$ is gradient, it follows that $\omega(x) \subset \mathscr{E}$. Since $\mathscr{E}$ is bounded and attracts points, $\mathscr{T}$ is point-dissipative. From Theorem 19, $\mathscr{T}$ has a global attractor $\mathscr{A}$.

Let $x \in \mathscr{A}$, then there exists a bounded global solution ϕ of $\mathscr{T}$ through x (see Proposition 6). Since $\phi(\mathbb{R})$ is invariant and attracted by $\mathscr{A}$, we have $\phi(\mathbb{R}) \subset \mathscr{A}$. From Lemma 4 and Lemma 9, $\varnothing \neq \alpha_\phi(x) \subset \mathscr{E}$, and $\phi(t) \stackrel{t\to-\infty}{\longrightarrow} \mathscr{E}$. Hence $\mathscr{A} \subset W^u(\mathscr{E})$.

If $x \in W^u(\mathscr{E})$, there exists a global solution ϕ through x such that $\phi(t) \stackrel{t\to-\infty}{\longrightarrow} \mathscr{E}$, and since $\omega(x) \subset \mathscr{E}$ attracts x, we also have $\phi(t) \stackrel{t\to\infty}{\longrightarrow} \mathscr{E}$. Hence $\phi(\mathbb{R})$ is invariant and bounded, from which $\phi(\mathbb{R}) \subset \mathscr{A}$, and $x \in \mathscr{A}$. This proves the inverse inclusion, and $\mathscr{A} = W^u(\mathscr{E})$.

If $\mathscr{E} = \{e_1, ..., e_n\}$, we get the equality $W^u(\mathscr{E}) = \bigcup_{i=1}^n W^u(e_i)$ using the fact that α-limits of points are unitary.

Suppose now that $\mathscr{A} \subset B$ for B bounded and connected. Let $\mathscr{A}$ be the disjoint union of two compacts $\mathscr{A}_1$ and $\mathscr{A}_2$, such that $\mathrm{d}(\mathscr{A}_1, \mathscr{A}_2) = 2\rho > 0$. Since $\mathscr{A}$ $\mathscr{T}$-attracts B, there exists $t_0 \geq 0$, such that $T(t_0)B \subset \mathscr{O}_\rho(\mathscr{A})$, but $T(t_0)B \supset T(t_0)\mathscr{A} = \mathscr{A}$, and this contradicts the fact that $T(t_0)B$ is connected.

□

The following results are related to the concept of stability and hyperbolicity of equilibria.

Lemma 10. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a semigroup and y^* be an equilibrium for $\mathscr{T}$. Given $t \in \mathbb{R}^+$, $\varepsilon > 0$, there exists a $\delta > 0$ such that $\{T(s)y : 0 \leq s \leq t,\, y \in B_\delta(y^*)\} \subset B_\varepsilon(y^*)$.

Proof. It follows from the continuity of $\mathbb{R} \times X \ni (t,x) \mapsto T(t)x \in X$. □

Definition 24 (Stability)**.** Let $\{T(t) : t \geq 0\}$ be a semigroup in X and y^* be one of its equilibria. We say that y^* is **stable** if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathscr{O}_\delta(y^*)$, then $\gamma^+(x) \subset \mathscr{O}_\varepsilon(y^*)$. We say that y^* is **asymptotically stable** if it is stable and there exists $\eta > 0$ such that for any $x \in \mathscr{O}_\eta(y^*)$, we have $T(t)x \stackrel{t\to\infty}{\longrightarrow} y^*$.

Lemma 11. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a gradient semigroup with a global attractor $\mathscr{A}$ and a set of equilibria $\mathscr{E} = \{e_i : 1 \leq i \leq n\}$. Let $V : X \to \mathbb{R}$ be its Lyapunov function and $V(\mathscr{E}) = \{\alpha_1, ..., \alpha_p\}$, with $\alpha_i < \alpha_{i+1}$, $1 \leq i \leq p-1$.

If $1 \leq j \leq p-1$ and $\alpha_j \leq r < \alpha_{j+1}$, then $X_r = \{z \in X : V(z) \leq r\}$ is positively $\mathscr{T}$-invariant and $\{T_r(t) : t \geq 0\}$, the restriction of $\{T(t) : t \geq 0\}$ to X_r, has a global attractor $\mathscr{A}^j$ given by:

$$\mathscr{A}^j = \bigcup \{W^u(e_l) : V(e_l) \leq \alpha_j\}. \tag{3.7}$$

In particular, $V(z) \leq \alpha_j$ for $z \in \mathscr{A}^j$, $\alpha_1 = \min\{V(x) : x \in X\}$, and $\mathscr{A}^1 = \{e \in \mathscr{E} : V(e) = \alpha_1\}$ consists of asymptotically stable equilibria.

Proof. It follows from the definition of Lyapunov function that X_r is positively invariant.

The semigroup $\{T_r(t) : t \geq 0\}$ is point dissipative, asymptotically compact and eventually bounded, inheriting those properties from $\mathscr{T}$. It follows that $\{T_r(t) : t \geq 0\}$ has a global attractor $\mathscr{A}^j$, and is gradient with Lyapunov function $V_r = V|_{X_r}$. Hence, (3.7) holds.

Now we prove the asymptotical stability of the equilibria $\{e \in \mathscr{E} : V(e) = \alpha_1\}$. Define:

$$\delta_0 = \frac{1}{2}\min\{\mathrm{d}(x,y) : x,y \in \mathscr{A}^1, x \neq y\}.$$

Let $e \in \mathscr{E}$ be such that $V(e) = \alpha_1$ and suppose it is not stable. Then, there exists $0 < \varepsilon < \delta_0$ and sequences $\{x_k\}$ in X, $x_k \to e$, and $\{t_k\}$ in $\mathbb{R}^+$ with $\mathrm{d}(T(t_k)x_k, e) \geq \varepsilon$ and $\mathrm{d}(T(t)x_k, e) < \varepsilon$ for all $0 \leq t < t_k$. From Lemma 10, $t_k \to \infty$ and we use Proposition 9 to take a convergent subsequence of $\{T(t_k)x_k\}$, which we denote the same. Let y be its limit. It follows that

$$V(y) = \lim_{k\to\infty} V(T(t_k)x_k) \leq \lim_{k\to\infty} V(x_k) = \alpha_1.$$

Since α_1 is the global minimum of V, we get $V(T(t)y) = V(y) = \alpha_1$, whence $y \in \mathscr{A}^1$ and $\mathrm{d}(y,e) \geq \varepsilon$. However, $\{T(t_k - 1)x_k\}$ also has a convergent subsequence z, which belongs to $\mathscr{A}^1 \cap \overline{\mathscr{O}_\varepsilon(e)}$ (because $\mathrm{d}(T(t)x_k, e) < \varepsilon$ for all $0 \leq t < t_k$). Since $\mathrm{d}(z,e) < \delta_0$, $z = e$ and $e = T(1)e = T(1)z = y$, which is a contradiction. Hence, e is stable.

Now we only need to show that there is a neighborhood V of e such that e attracts each point in V. Indeed, using the stability, there exists $\delta > 0$ such that for $x \in V := \mathscr{O}_\delta(e)$, $\gamma^+(x) \in \mathscr{O}_{\delta_0}(e)$. Then $\omega(x) \subset \overline{\mathscr{O}_{\delta_0}(e)}$. Since $\omega(x)$ is a unitary set contained in $\mathscr{E}$, $\omega(x) = e$. Since $\omega(x)$ attracts x, we are done.

□

An important concept for the study of equilibria is the topological hyperbolicity, which is presented as following:

Definition 25 (Topological hyperbolicity)**.** We say that $\phi \in \mathscr{E}$ is **topologically hyperbolic** if $\{\phi\}$ is an isolated invariant set. Equivalently, there exists $\delta > 0$ such that if $\xi : \mathbb{R} \to X$ is a global solution that satisfies $\sup_{t\in\mathbb{R}}\|\xi(t) - \phi\| < \delta$, then $\xi(t) = \phi$ for all $t \in \mathbb{R}$.

A direct consequence of that is the following lemma.

Lemma 12. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be an asymptotically compact semigroup. Suppose U, V are open subsets of X with $\overline{U} \subset V$ and such that A is the maximal $\mathscr{T}$-invariant subset of V. If for some $u_0 \in X$, $t_0 \geq 0$, we have $T(t)u_0 \in U$ for all $t \geq t_0$, then $\mathrm{d}_H(T(t)u_0, A) \to 0$ as $t \to \infty$. Analogously, if ξ is a global solution for $\mathscr{T}$ and for some $t_0 \geq 0$ we have $\xi((-\infty, -t_0)) \in U$, then $\mathrm{d}_H(\xi(t), A) \to 0$ as $t \to -\infty$.

It follows that if $\phi \in \mathscr{E}$ is topologically hyperbolic and η is a global solution for $\mathscr{T}$ such that $\sup_{t\geq t_0}\|\eta(t) - \phi\| < \delta$, for some $t_0 \geq 0$, then $\eta(t) \overset{t\to\infty}{\longrightarrow} \phi$. And if $\sup_{t\leq -t_0}\|\eta(t) - \phi\| < \delta$, for some $t_0 \geq 0$, then $\eta(t) \overset{t\to-\infty}{\longrightarrow} \phi$.

Remark 8. For a gradient semigroup, if there is a finite number of equilibria, they are all topologically hyperbolic.

Indeed, let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a gradient semigroup with Lyapunov function V. Define $\delta := \frac{1}{2}\min\{d(y_1,y_2) : y_1,y_2 \in \mathscr{E}, y_1 \neq y_2\}$. Then, if $\xi : \mathbb{R} \to X$ is a global solution for $\mathscr{T}$ such that $\xi(t) \in \mathscr{O}_\delta(\phi)$, for all $t \in \mathbb{R}$, we have $\xi(\mathbb{R}) \subset \mathscr{A}$.

Let $y = \xi(0)$. Since $\mathscr{T}$ is asymptotically compact and eventually bounded, it follows from Lemma 6 that $\omega(y)$ is nonempty and attracts y. But $\omega(y) \subset \mathscr{E}$, so that $\omega(y) = \phi$, and $T(t)y = \xi(t) \stackrel{t\to\infty}{\longrightarrow} \phi$. Analogously, from Lemma 4, we have $\xi(t) \stackrel{t\to-\infty}{\longrightarrow} \phi$. For any $\tilde{t} \in \mathbb{R}$, by the properties of Lyapunov functions, we get

$$V(\xi(\tilde{t})) \leq \lim_{t\to-\infty} V(\xi(t)) = V(\phi) = \lim_{t\to\infty} V(\xi(t)) \leq V(\xi(\tilde{t})) \Rightarrow V(\xi(\tilde{t})) = V(\phi)$$

Hence V is constant in $\xi(\mathbb{R})$, so that for any $t_0 \in \mathbb{R}$, $V(T(t)\xi(t_0)) = V(\xi(t_0))$ for all $t \geq 0$, which implies $\xi(t_0) \in \mathscr{E}$, so that $\xi(t_0) = \phi$, because of the definition of δ.

3.3 Semigroups and their generators

In this section we study the association of a semigroup to the linear Cauchy problem $\dot{x} = Ax$, $x(0) = x_0$ in the phase space X, where $A : D(A) \subset X \to X$ is a linear operator. In particular, we study the relations between the properties of A and the properties of the semigroup associated to the Cauchy problem generated by it. The analysis in this section will be focused on characterizing the semigroups associated to an operator A when $-A$ is sectorial (see Definition 28), because the theory of parabolic diferential equations will be built using sectorial operators. The reader may find several results about semigroups associated to other kinds of operators in (BREZIS, 2011) and (CARVALHO, 2012).

First we present the definition of linear semigroup, which is a little bit different from the definition of semigroup we have worked with so far (see Definition 10). Let X be a Banach space with norm $\|\cdot\|$.

Definition 26 (Linear semigroup)**.** A **linear semigroup** in X is a family $\mathscr{T} = \{T(t) : t \geq 0\} \subset \mathscr{L}(X)$ of linear operators such that:

- $T(0) = I_X$,
- $T(t+s) = T(t)T(s)$, for all $t,s \geq 0$.

If $\mathscr{T}$ also satisfies

- $\|T(t) - I_X\|_{\mathscr{L}(X)} \to 0$ as $t \to 0^+$, we say that $\mathscr{T}$ is **uniformly continuous**.
- $\|T(t)x - x\| \to 0$ as $t \to 0^+$, $\forall x \in X$, we say that $\mathscr{T}$ is **strongly continuous**.

By simplicity, throughout this section, we may say only "semigroup" instead of "linear semigroup", always referring to the Definition 26. A strongly continuous semigroup has an exponential estimate, as stated in the following theorem.

Theorem 21. Suppose that $\{T(t) : t \geq 0\} \subset \mathscr{L}(X)$ is a strongly continuous semigroup. Then, there exist constants $M \geq 1$ and β such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq Me^{\beta t}, \quad \forall t \geq 0.$$

Proof. We claim that $\sup_{t\in[0,\eta]} \|T(t)\|_{\mathscr{L}(X)} < \infty$ for some $\eta > 0$. Indeed, for every sequence $\{t_n\}$ in $(0,\infty)$ such that $t_n \to 0$, we have $T(t_n)x \to x$ for every $x \in X$, so that $\{T(t_n)x\}$ is bounded for every $x \in X$, which implies (by the Uniform Boundedness Principle) that $\{T(t_n)\}$ is bounded in $\mathscr{L}(X)$. Suppose the claim is false, then for each $n \in \mathbb{N}$, there exists $t_n \in [0, \frac{1}{n}]$ such that $\|T(t_n)\|_{\mathscr{L}(X)} > n$, and the sequence $\{T(t_n)\}$ is not bounded, a contradiction.

Let $\eta > 0$ be such that $\sup\{\|T(t)\|_{\mathscr{L}(X)} : 0 \leq t \leq \eta\} = M < \infty$ and choose any $\beta \geq \frac{1}{\eta}\log(\|T(\eta)\|_{\mathscr{L}(X)})$, so that $\|T(\eta)\|_{\mathscr{L}(X)} \leq e^{\beta\eta}$. Then, if $s > 0$, we may write $s = k\eta + t$ with $k \in \mathbb{N}$, $t \in [0,\eta]$, and

$$\|T(k\eta+t)\|_{\mathscr{L}(X)} = \|T(\eta)^k T(t)\|_{\mathscr{L}(X)} \leq \|T(\eta)\|^k_{\mathscr{L}(X)} \|T(t)\|_{\mathscr{L}(X)} \leq Me^{|\beta|\eta} e^{\beta(k\eta+t)},$$

and the theorem follows. □

Definition 27 (Generator)**.** Let $\{T(t) : t \geq 0\} \subset \mathscr{L}(X)$ be a strongly continuous linear semigroup, we define its **generator** as the operator $A : D(A) \subset X \to X$, where

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}, \quad Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

It may be seen in (CARVALHO, 2012, Example 3.1.1 and Theorem 3.1.2) that a uniformly continuous linear semigroup $\{T(t) : t \geq 0\}$ is necessarily of the form

$$T(t) = e^{At} := \sum_0^\infty \frac{A^n t^n}{n!}, \quad t \geq 0,$$

for some $A \in \mathscr{L}(X)$. Moreover, $\frac{d}{dt}e^{At} = Ae^{At}$, $t > 0$, and $[0,\infty) \ni t \mapsto e^{At}x_0$ is the solution of the following Cauchy problem:

$$\dot{x} = Ax$$
$$x(0) = x_0 \in X.$$

Since in general we are interested in studying Cauchy problems with operators that are only closed, and not bounded, we will focus ourselves in the study of strongly continuous semigroups, instead of uniformly continuous semigroups.

The next theorem present some of the most important properties about strongly continuous semigroups.

Theorem 22. Suppose that $\{T(t) : t \geq 0\} \subset \mathscr{L}(X)$ is a strongly continuous semigroup, then:

1. For any $x \in X$, $[0,\infty) \ni t \mapsto T(t)x$ is continuous.

2. $[0,\infty) \ni t \mapsto \|T(t)\|_{\mathscr{L}(X)}$ is lower semicontinuous.

3. If A is the generator of $\{T(t) : t \geq 0\}$, then A is closed and densely defined. For $x \in D(A)$, $t \mapsto T(t)x$ is differentiable and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \quad t > 0.$$

4. Let β be as in Theorem 21, and $\operatorname{Re}\lambda > \beta$, then $\lambda \in \rho(A)$ and

$$(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)xdt, \quad \forall x \in X$$

Proof. (1) Let $x \in X$, $t > 0$, then:

$$\|T(t+h)x - T(t)x\| = \|(T(h) - I)T(t)x\| \overset{h\to 0^+}{\longrightarrow} 0,$$

$$\|T(t)x - T(t-h)x\| \leq \|(T(t-h)\|_{\mathscr{L}(X)}\|T(h)x - x\| \overset{h\to 0^+}{\longrightarrow} 0,$$

where we use Theorem 21 to estimate $\|(T(t-h)\|_{\mathscr{L}(X)}$.

(2) We will show that $\{t \geq 0 : \|T(t)\|_{\mathscr{L}(X)} > b\}$ is open in $[0,\infty)$ for each $b \geq 0$, which easily implies the lower semicontinuity. In fact, let $b \geq 0$ and $\|T(t_0)\|_{\mathscr{L}(X)} > b$, then there exists $x \in X$ with $\|x\| = 1$, such that $\|T(t_0)x\| > b$, and it follows from the continuity proved in the first part of this theorem that $\|T(t)x\| > b$ for t in a neighborhood of t_0, and $\|T(t)\|_{\mathscr{L}(X)} > b$ for t in a neighborhood of t_0, so that a neighborhood of t_0 is in $\{t \geq 0 : \|T(t)\|_{\mathscr{L}(X)} > b\}$, which completes the proof.

(3) Let $x \in X$ and $\varepsilon > 0$, and define $x_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon T(t)x\,dt$. It is easy to see that $x_\varepsilon \to x$ as $\varepsilon \to 0^+$, and if $h > 0$,

$$h^{-1}(T(h)x_\varepsilon - x_\varepsilon) = \frac{1}{\varepsilon h}\left\{\int_\varepsilon^{\varepsilon+h} T(t)xdt - \int_0^h T(t)xdt\right\} \overset{h\to 0^+}{\longrightarrow} \frac{1}{\varepsilon}(T(\varepsilon)x - x),$$

then $x_\varepsilon \in D(A)$, which proves that $D(A)$ is dense in X. The fact that A is closed will be a direct consequence of the proof of (4), because we will prove that for some $\lambda \in \mathbb{C}$, $\lambda - A : D(A) \to X$ is a bijection and $(\lambda - A)^{-1} \in \mathscr{L}(X)$, which implies that A is closed. If $x \in D(A)$, it follows that:

$$\frac{d^+}{dt}T(t)x = \lim_{h\to 0^+}\frac{1}{h}\{T(t+h)x - T(t)x\} \overset{def}{=} AT(t)x = T(t)Ax,$$

which is continuous by (1). It is left as an exercise for the reader to show that since $t \mapsto T(t)x$ is continuous and has continuous right derivative, $t \mapsto T(t)x$ is continuously differentiable.

(4) We will show that if $\operatorname{Re}\lambda > \beta$, $\lambda - A : D(A) \to X$ is a bijection and $(\lambda - A)^{-1} \in \mathscr{L}(X)$, which readily implies that A is closed, so that we can define the resolvent of A and $\lambda \in \rho(A)$. Define the operator $R(\lambda) : X \to X$ by:

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt.$$

Using the fact that $\|T(t)\|_{\mathscr{L}(X)} \leq Me^{\beta t}$, we can show that $R(\lambda)$ is well defined (the integral converges) and is bounded, with $\|R(\lambda)\|_{\mathscr{L}(X)} \leq \frac{M}{\operatorname{Re}\lambda - \beta}$. We will show that $\lambda - A$ is surjective. Indeed, let $x \in X$, and for $h > 0$,

$$\begin{aligned} h^{-1}(T(h) - I)R(\lambda)x &= R(\lambda)\frac{T(h)x - x}{h} \\ &= h^{-1}\left[\int_h^\infty e^{\lambda h - \lambda t} T(t)x dt - \int_0^\infty e^{-\lambda t} T(t)x dt\right] \\ &= h^{-1}\left[-\int_0^h e^{\lambda(h-t)} T(t)x dt + (e^{\lambda h} - 1)\int_0^\infty e^{-\lambda t} T(t)x dt\right] \\ &\overset{h \to 0^+}{\longrightarrow} -x + \lambda R(\lambda)x, \end{aligned} \tag{3.8}$$

so that $R(\lambda)x \in D(A)$ and $(\lambda - A)R(\lambda)x = x$, which proves that $\lambda - A$ is surjective. Moreover, if $x \in D(A)$, we have:

$$R(\lambda)\frac{T(h)x - x}{h} \overset{h \to 0^+}{\longrightarrow} R(\lambda)Ax,$$

and using (3.8) we obtain $x = R(\lambda)(\lambda - A)x$. Therefore, $(\lambda - A)R(\lambda)x = x = R(\lambda)(\lambda - A)x$, so that $\lambda - A$ is injective, and $R(\lambda) = (\lambda - A)^{-1} \in \mathscr{L}(X)$.

□

We now start the study of sectorial operators, which is the class of operators for which we will develope the theory of semilinear differential equations.

Definition 28 (Sectorial operator)**.** Let $A : D(A) \subset X \to X$ be a densely defined closed operator in X. We say that A is **sectorial** (with vertex $a \in \mathbb{R}$) if there exist constants $C \geq 0$, $\phi \in (\frac{\pi}{2}, \pi)$ such that $\Sigma = \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \phi\} \subset \rho(-A)$, and:

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{C}{|\lambda - a|}, \quad \forall \lambda \in \Sigma.$$

If we say only that A is sectorial, we mean it is sectorial with vertex 0.

Example 3. Consider the Laplacian operator $A : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$, given by $A\phi = -\phi''$. The operator A is sectorial with vertex 0. Indeed, $\sigma(-A) = \{-1^2, -2^2, -3^2, ...\}$, and $\mathbb{C} \setminus (-\infty, -1] \subset \rho(-A)$. Moreover, if $\phi \in D(-A)$, $\langle -A\phi, \phi \rangle = -\|\phi\|^2_{H_0^1} \leq -\frac{2}{\pi^2}\|\phi\|^2$, so that $W(-A) \subset (-\infty, -\frac{2}{\pi^2}]$. Consider the set $\Sigma_\varphi = \{\lambda : |\arg\lambda| < \varphi\}$, for any $\varphi \in (\frac{\pi}{2}, \pi)$, and note that if $\lambda \in \Sigma_\varphi$, $d(\lambda, W(A)) \geq |\lambda|\sin(\pi - \varphi)$, and by Theorem 12,

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{|\lambda|\sin(\pi - \varphi)}.$$

Next we will extract a lot of properties of the semigroup generated by the operator A when $-A$ is sectorial. In special, we will have a formula for $T(t)$ depending on the resolvent of A. This theorem will be essential in the study of fractional powers related to semigroups, and in the study of abstract theory of semilinear parabolic differential equations.

Theorem 23. Suppose $A : D(A) \subset X \to X$ is such that $-A$ is sectorial, that is, there exist constants $C \geq 0$ and $\phi \in (\frac{\pi}{2}, \pi)$ such that $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \phi\} \subset \rho(A)$,

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq C/|\lambda| \quad \forall \lambda \in \Sigma_\phi.$$

Then A generates a strongly continuous semigroup $\{T(t) : t \geq 0\} \subset \mathscr{L}(X)$, given by

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda t} (\lambda - A)^{-1} d\lambda, \tag{3.9}$$

where Γ_0 is the boundary of $\Sigma_\nu \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$, $\frac{\pi}{2} < \nu < \phi$, r small, and the curve Γ_0 is oriented in the direction of increasing imaginary part, that is, Γ_0 is going up in the complex plane. Moreover, $t \mapsto T(t)$ can be extended to an analytic function with domain $\{t \in \mathbb{C} : |\arg t| < \nu - \pi/2\}$, and for some $K > 0$,

$$\|T(t)\|_{\mathscr{L}(X)} \leq K, \quad \|AT(t)\|_{\mathscr{L}(X)} \leq Kt^{-1}$$

for all $t > 0$, and the operator

$$\frac{d}{dt} T(t) = AT(t)$$

is bounded for any $t > 0$.

Proof. Define $T(t)$ as in expression (3.9). In fact, $T(t)$ is well defined in $\mathscr{L}(X)$ because for $t > 0$, $\arg \lambda = \pm \nu$,

$$\|e^{\lambda t} (\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq C \frac{e^{-t|\lambda|k_1}}{|\lambda|}, \quad k_1 = |\cos \nu| > 0,$$

so the integral converges in $\mathscr{L}(X)$, for all $t > 0$ — note that $|\lambda|$ is the denominator is not a problem since the integration curve is away from zero. It is easy to see that the convergence of the integral is uniform for $t \in [\varepsilon, \infty)$ for all $\varepsilon > 0$, so that $t \mapsto T(t) \in \mathscr{L}(X)$ is continuous in $t > 0$. We can also guarantee the convergence of the integral for complex t. Indeed, if $|\arg(t)| < \nu - \frac{\pi}{2}$, we have:

$$\|e^{\lambda t} (\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq C \frac{e^{\mathrm{Re}(\lambda t)}}{|\lambda|},$$

so that if we are over the line $\arg \lambda = \pm \nu$, we have $\arg \lambda + \arg t \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and $\mathrm{Re}(\lambda t) = -|\lambda||t|\alpha$ for some $\alpha > 0$, and the integral converges the same way, uniformly in the region where $|\arg(t)| \leq \varepsilon_1 < \nu - \frac{\pi}{2}$ and $\varepsilon_0 \leq |t|$, for any $(\varepsilon_i > 0, i = 0, 1)$. Therefore, $t \mapsto T(t)$ is analytic in the region $|\arg t| < \nu - \frac{\pi}{2}$, which contains $\mathbb{R}^+ \setminus \{0\}$.

Let us prove that $\|T(t)\|_{\mathscr{L}(X)}$ and $t\|AT(t)\|_{\mathscr{L}(X)}$ are bounded for $t > 0$. If we change the variable to $\mu = \lambda t$,

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\mu}\left(\frac{\mu}{t} - A\right)^{-1}\frac{d\mu}{t},$$

and the integration curve is still Γ_0 because the argument of the integral is analytic, so we can use Cauchy's Theorem to transfer the integrals from $t\Gamma_0$ to Γ_0. We can estimate and obtain:

$$\|T(t)\|_{\mathscr{L}(X)} \leq \frac{1}{2\pi}\int_{\Gamma_0} e^{\mathrm{Re}\mu}\frac{C}{|\mu|/t}\frac{|d\mu|}{t} = K < \infty, \quad \forall t > 0.$$

Similarly, we have:

$$\begin{aligned}\mathscr{I}(A) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}A(\lambda - A)^{-1}d\lambda &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}[-I + \lambda(\lambda - A)^{-1}]d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}d\lambda + \frac{t^{-1}}{2\pi i}\int_{\Gamma_0} e^{\mu}\frac{\mu}{t}\left(\frac{\mu}{t} - A\right)^{-1}d\mu.\end{aligned}$$

It is simple to prove, using Cauchy's Theorem, that the first integral is zero, and the second one may be estimated the following way:

$$\left\|\frac{t^{-1}}{2\pi i}\int_{\Gamma_0} e^{\mu}\frac{\mu}{t}\left(\frac{\mu}{t} - A\right)^{-1}d\mu\right\|_{\mathscr{L}(X)} \leq \frac{1}{2\pi t}\int_{\Gamma_0} e^{\mathrm{Re}\mu}C|d\mu| = K_1 t^{-1} < \infty.$$

Note that $AT(t) = \mathscr{I}(A)$; indeed, if $x \in X$, $T(t)x$ is the limit of a sequence $\{y_n\}$ of Riemann sums, while $\mathscr{I}(A)x$ is the limit of the sequence $\{Ay_n\}$, and since A is closed, we have $T(t)x \in D(A)$, $AT(t)x = \mathscr{I}(A)x$ and:

$$AT(t)x = T(t)Ax, \quad \forall x \in D(A).$$

Therefore, $AT(t) \in \mathscr{L}(X)$, and $\|AT(t)\|_{\mathscr{L}(X)} \leq K_1 t^{-1}$.

Because of the uniform convergence, we can differentiate under the integration sign and obtain, for $t > 0$:

$$\frac{d}{dt}T(t) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\lambda(\lambda - A)^{-1}d\lambda,$$

and as we saw above, this is equal to $AT(t)$.

Next we will prove that $T(t)x \to x$ as $t \to 0^+$, for every $x \in X$. In fact, we use once again the identity $A(\lambda - A)^{-1} = -I + \lambda(\lambda - A)^{-1}$ to conclude that if $x \in D(A)$,

$$\begin{aligned}T(t)x &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\lambda\frac{(\lambda - A)^{-1}}{\lambda}x\,d\lambda \\ &= \left(\frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\frac{d\lambda}{\lambda}\right)x + \frac{t}{2\pi i}\int_{\Gamma_0} e^{\mu}\frac{\mu}{t}\left(\frac{\mu}{t} - A\right)^{-1}Ax\frac{d\mu}{\mu^2}.\end{aligned}$$

The first integral between parenthesis equals 1, so that:

$$\|T(t)x - x\| \leq \frac{t}{2\pi}\int_{\Gamma_0} e^{\mathrm{Re}\mu}C\|Ax\|\left|\frac{d\mu}{\mu^2}\right| \xrightarrow{t\to 0^+} 0,$$

and since $\|T(t)\|_{\mathscr{L}(X)}$ is bounded, $T(t)x \to x$ as $t \to 0^+$, for all $x \in X$. Finally, for $x \in X$, the application $[0,t] \ni s \mapsto T(t-s)T(s)x$ is continuous and is differentiable for $0 < s < t$, and a simple calculation shows that

$$\frac{d}{ds}(T(t-s)T(s)x) = -AT(t-s)T(s)x + T(t-s)AT(s)x = 0.$$

Hence this application is constant and

$$T(t-s)T(s)x = T(t)x, \quad \text{for} \quad 0 \leq s \leq t,\ x \in X.$$

This is the semigroup property, so that $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup. We are only left to show that its generator is A, but $T(t)x - x = \int_0^t T(s)Ax\,ds$ for $t > 0$, $x \in D(A)$, and using the strong continuity,

$$\frac{1}{t}(T(t)x - x) \stackrel{t\to 0^+}{\longrightarrow} Ax,$$

so that the generator B of $\{T(t) : t \geq 0\}$ extends A. However, $1 \in \rho(A)$ because $-A$ is sectorial, and $1 \in \rho(B)$ because of the fourth property in Theorem 22. If $z \in X$, $z = (1-A)a = (1-B)a$ for some $a \in D(A)$, so that $(1-B)^{-1}z = a$, and $D(B) \subset D(A)$, which completes the proof. □

Remark 9. The results obtained for sectorial operators with vertex 0 can be easily extended to operators that are sectorial with a vertex $a \neq 0$. For instance, consider Theorem 23; if $-A$ is sectorial with vertex $a \in \mathbb{R}$, then $-A + a$ is sectorial with vertex 0 and $A - a$ generates an analytic strongly continuous semigroup $\{T(t) : t \geq 0\}$ such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq K, \text{ and } \|AT(t)\|_{\mathscr{L}(X)} \leq Kt^{-1}, \quad \forall t \geq 0,$$

then, defining $T_1(t) = T(t)e^{at}$, it is easy to see that the strongly continuous analytic semigroup $\{T_1(t) : t \geq 0\}$ is generated by the operator A, and

$$\|T_1(t)\|_{\mathscr{L}(X)} \leq Ke^{at}, \text{ and } \|AT_1(t)\|_{\mathscr{L}(X)} \leq Kt^{-1}e^{at}, \forall t \geq 0.$$

3.4 Exponential dichotomy for linear semigroups

In this section we briefly present the theory of exponential dichotomy for linear semigroups, making use of some parts of the theories of spectral decomposition and operational calculus developed in (TAYLOR; LAY, 1980) and (CARVALHO, 2012) — the reader can easily catch up on these themes with the background we developed in Chapter 2. We will present the notion of exponential dichotomy for linear problems, which is basically when the spectrum of some operator in the semigroup does not intersect a circumference in $\mathbb{C}$. This separation of the spectrum induces a separation in the phase space $X = X_1 \oplus X_2$, and the restrictions of the semigroup to these spaces have either an attracting or expelling property of exponential type.

The exponential dichotomy for semigroups whose generator is bounded is relatively easy to obtain. Indeed, if $A \in \mathscr{L}(X)$, it follows from Theorem 3 that:

$$A^j = \int_\gamma \lambda^j (\lambda - A)^{-1} d\lambda,$$

where γ is a closed, rectifiable curve in the counterclockwise direction, which circles one time around the spectrum of A.

From this, and (CARVALHO, 2012, Theorem 3.1.2), we can conclude that:

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \int_\gamma e^{\lambda t} (\lambda - A)^{-1} d\lambda,$$

and it follows from the Spectral Mapping Theorem (TAYLOR; LAY, 1980, Theorem 5.71-A), that $\sigma(e^A) = e^{\sigma(A)}$, and if the spectrum of A is disjoint of $\mathrm{Re}\lambda = \alpha$ for some $\alpha \in \mathbb{R}$, then the spectrum of e^A is disjoint from $\{\lambda \in \mathbb{C} : |\lambda| = e^\alpha\}$, and we have the exponential dichotomy.

For semigroups generated by a more general operator, only closed, the Spectral Mapping Theorem does not apply in general, and the knowledge of the spectrum of A is not enough to understand the spectrum of e^A. However, it is easy to show dichotomy for the semigroup generated by an operator A if $-A$ is sectorial (with vertex α), as we will see in the next theorem.

First we present the following lemma about spectral decomposition (CARVALHO, 2012, Theorem 2.9.2):

Lemma 13. Let X be a Banach space over $\mathbb{C}$ and $A : D(A) \subset X \to X$ be a closed linear operator. Suppose $\sigma(A)$ contains a bounded set σ, and let D be a bounded Cauchy set such that $\partial D \subset \rho(A)$, $\sigma \subset D$, then

$$Q = \frac{1}{2\pi i} \int_{+\partial D} (\xi - A)^{-1} d\xi$$

is such that $Q \in \mathscr{L}(X)$, $Q^2 = Q$ and defining $X_\sigma = R(Q)$ and $X_{\sigma'} = N(Q)$, we have a decomposition $X = X_\sigma \oplus X_{\sigma'}$ in such a way that the spectra of the parts A_σ and $A_{\sigma'}$ in X_σ and $X_{\sigma'}$ are σ and σ', respectively, and $A_\sigma \in \mathscr{L}(X_\sigma)$.

Theorem 24 (Exponential dichotomy for sectorial operators)**.** Let L be a sectorial operator such that $\sigma(L)$ is disjoint from $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda = 0\}$, and define the projection:

$$P = \frac{1}{2\pi i} \int_{+\partial D} (\lambda - L)^{-1} d\lambda,$$

where D is a bounded Cauchy set such that $\partial D \subset \rho(L)$ and $\sigma_1 \subset D$, where $\sigma_1 = \sigma(L) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$ (or $P = 0$ if this intersection is empty). Define the phase space decomposition $X = X_1 \oplus X_2$, $X_1 = R(P)$, $X_2 = N(P)$. Then $\sigma_i = \sigma(L_i)$, where L_i is the restriction of L to X_i, $i = 1, 2$, and $L_1 \in \mathscr{L}(X)$.

Let $\{T(t) : t \geq 0\}$ be the analytic semigroup generated by $-L$. Then, we have $PT(t) = T(t)P$ for all $t \geq 0$, $T(t)|_{X_i} \in \mathscr{L}(X_i)$ is the semigroup generated by $-L_i$, $i = 1, 2$, and there exist

δ_1, $\delta_2 > 0$, $C \geq 0$ such that:

$$\|T(t)|_{X_2}\|_{\mathscr{L}(X_2)} \leq Ce^{-\delta_2 t}, \quad \forall t \geq 0. \tag{3.10}$$

The semigroup $\{T(t)|_{X_1} : t \geq 0\}$ can be extended to a group in $\mathscr{L}(X_1)$, with $T(t)|_{X_1} = (T(-t)|_{X_1})^{-1}$, for $t < 0$, and

$$\|T(t)|_{X_1}\|_{\mathscr{L}(X_1)} \leq Ce^{\delta_1 t}, \quad \forall t \leq 0. \tag{3.11}$$

Proof. It follows from Lemma 13 that $\sigma(-L_1) = \sigma(-L) \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$, which is a compact set, so there exists $\delta_1 > 0$ such that $\sigma(-L_1) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda > \delta_1\}$. By the Spectral Mapping Theorem, $\sigma(T(t)|_{X_1}) = e^{\sigma(-L_1 t)}$ is contained in $\{\lambda \in \mathbb{C} : |\lambda| > e^{\delta_1 t}\}$ for all $t > 0$. For each $t > 0$, $0 \in \rho(T(t)|_{X_1})$, and we denote by $T(-t)|_{X_1}$ the inverse of $T(t)|_{X_1}$. Now, the spectral radius of $T(-1)|_{X_1}$ satisfies

$$r_\sigma(T(-1)|_{X_1}) < e^{-\delta_1},$$

so that, using the formula for spectral radius in Theorem 5, we conclude that, for $n \in \mathbb{N}$ large enough:

$$\|T(-n)|_{X_1}\|_{\mathscr{L}(X_1)} < e^{-\delta_1 n},$$

hence, if $s < 0$, we write $s = -n + \tau$, with $\tau \in [0,1)$, and we get:

$$\|T(s)|_{X_1}\|_{\mathscr{L}(X_1)} < e^{\delta_1(-n)}\|T(+\tau)|_{X_1}\|_{\mathscr{L}(X_1)} \leq C_1 e^{\delta_1(-n+\tau)},$$

where $C_1 = \sup_{0 \leq \tau < 1} e^{-\delta_1 \tau}\|T(\tau)|_{X_1}\|_{\mathscr{L}(X_1)}$, and we have the exponential estimate (3.11).

Before we continue, since L is sectorial, there exist $\beta \in \mathbb{R}$, $\psi \in (\frac{\pi}{2}, \pi)$ and $K \geq 0$ such that $\Sigma_\psi = \{\lambda \in \mathbb{C} : |\arg(\lambda - \beta)| < \psi\} \in \rho(-L)$, and

$$\|(\lambda + L)^{-1}\|_{\mathscr{L}(X)} \leq \frac{K}{|\lambda - \beta|}, \quad \forall \lambda \in \Sigma_\psi.$$

For the semigroup in X_2, note that $\sigma(-L_2)$ is the (possibly unbounded) set $\{\lambda \in \sigma(-L) : \text{Re}\lambda < 0\}$.

The estimate (3.10) will follow from Theorem 23 — along with Remark 9 — if we prove that L_2 is sectorial with vertex in $-\delta_2$ (for a $\delta_2 > 0$ small enough). The fact that L is sectorial guarantees that $\sigma(-L_2)$ is at a positive distance of $\{\lambda \in \mathbb{C} : \text{Re}\lambda = 0\}$, even though $\sigma(-L_2)$ is not necessarily compact. Then, it is possible to choose $\delta_2 > 0$, $\phi \in (\frac{\pi}{2}, \pi]$, $\phi < \psi$, such that $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg(\lambda - (-\delta_2))| < \phi\} \subset \rho(-L_2)$. The estimate for $\|(\lambda + L_2)^{-1}\|_{\mathscr{L}(X_2)}$ in Σ_ϕ is trivial using the estimate for $\|(\lambda + L)^{-1}\|_{\mathscr{L}(X)}$ in Σ_ψ and using the fact that $\|(\lambda + L_2)^{-1}\|_{\mathscr{L}(X_2)}$ is bounded for λ in the compact region that is in Σ_ϕ but not in Σ_ψ. It follows that L_2 is sectorial with vertex $-\delta_2$, and we are done.

□

CHAPTER

4

FRACTIONAL POWERS

4.1 Definition and basic results

In this chapter we introduce the fractional powers, preparing the reader to develop the qualitative theory of parabolic semilinear differential equations in Chapter 5. The reader may find a very good presentation of this topic in (AMANN, 1995).

First of all, let us define the class of operators for which we can define the fractional powers.

Definition 29 (Operators of positive kind)**.** Let X be a Banach space. We say that a linear operator $A : D(A) \subset X \to X$ is of **positive kind** (or simply **positive**) with constant $M \geq 1$ if A is closed, densely defined, $\mathbb{R}^+ \subset \rho(-A)$, and

$$(1+s)\|(s+A)^{-1}\|_{\mathscr{L}(X)} \leq M, \quad s \in \mathbb{R}^+.$$

We denote by $\mathscr{P}_M = \mathscr{P}_M(X)$ the set of all operators of positive kind in X with constant $M \geq 1$.

An operator of positive kind has the following important property.

Theorem 25. Let $A \in \mathscr{P}_M$. If $\theta_M := \arcsin \frac{1}{2M}$, and

$$\Sigma_M := \{z \in \mathbb{C} : |\arg z| \leq \theta_M\} + \{z \in \mathbb{C} : |z| \leq 1/2M\}$$

Then $\Sigma_M \subset \rho(-A)$ and

$$(1+|\lambda|)\|(\lambda+A)^{-1}\|_{\mathscr{L}(X)} \leq 2M+1, \quad \lambda \in \Sigma_M \tag{4.1}$$

Proof. If $\lambda \in \mathbb{C}$ and $s \in \mathbb{R}$ is such that

$$|\lambda - s| \leq (1+s)/2M \tag{4.2}$$

Then $\lambda + A = (s + A)(1 + (\lambda - s)(s + A)^{-}1)$, and using the Newmann series we conclude that $\lambda \in \rho(-A)$ and

$$\begin{aligned}
\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} &\leq \|[1 + (\lambda - s)(s + A)^{-1}]^{-1}\|_{\mathscr{L}(X)}\|(s + A)^{-1}\|_{\mathscr{L}(X)} \\
&\leq 2M(1 + s)^{-1} \leq \frac{2M}{1 + |\lambda|}\frac{1 + s + |\lambda - s|}{1 + s} \\
&\leq \frac{2M}{1 + |\lambda|}\left(1 + \frac{1}{2M}\right) = \frac{2M + 1}{1 + |\lambda|},
\end{aligned}$$

For each $\lambda \in \Sigma_M$, there exists a positive $s \in \mathbb{R}^+$ such that (4.2) holds. The proof of this fact is simple, and the chose of s is represented in the figure below

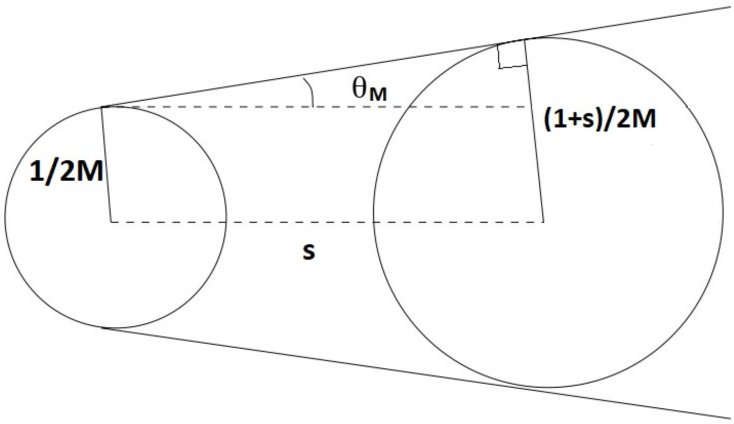

Figure 1

It follows that $\Sigma_M \in \rho(-A)$ and (4.1) holds.

□

We are ready to present the definition of negative fractional powers for operators of positive kind. This definition agrees with the one usually given to bounded operators, but is presented in a way that does not require a bounded spectrum to be calculated. The properties in Theorem 25 may be used to ensure that the integral converges. The reader can see more details in (CARVALHO, 2012).

Definition 30 (Negative fractional powers)**.** Let $A \in \mathscr{P}_M$, and $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha < 0$, then we define

$$A^{\alpha} := \frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{\alpha}(\lambda + A)^{-1}d\lambda = \frac{1}{2\pi i}\int_{-\Gamma}\lambda^{\alpha}(\lambda - A)^{-1}d\lambda, \tag{4.3}$$

where Γ is any simple, piecewise smooth curve in $\Sigma_M \setminus \mathbb{R}^+$ going from $\infty e^{-i\nu}$ until $\infty e^{i\nu}$, for some $\nu \in (0, \theta_M)$.

Remark 10. It follows from a standard complex integration argument using Cauchy's Theorem that this definition does not depend on Γ. Moreover, the integral converges in $\mathscr{L}(X)$, so that $A^{\alpha} \in \mathscr{L}(X)$, for $\operatorname{Re}\alpha < 0$.

Lemma 14. For all α, β with negative real part, $A^{\alpha}A^{\beta} = A^{\alpha+\beta}$.

Proof. Let α, $\beta \in \mathbb{C}$ be such that $\operatorname{Re}\alpha < 0$ and $\operatorname{Re}\beta < 0$, and choose the curves Γ_1 and Γ_2 in such a way that Γ_1 is on the left and Γ_2 on the right in the complex plane. Then:

$$\begin{aligned} A^{\alpha}A^{\beta} &= \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2}(-\lambda)^{\alpha}(-\mu)^{\beta}(\lambda+A)^{-1}(\mu+A)^{-1}d\mu d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2}(-\lambda)^{\alpha}(-\mu)^{\beta}(\lambda-\mu)^{-1}[(\mu+A)^{-1}-(\lambda+A)^{-1}]d\mu d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_2}(-\mu)^{\beta}(\mu+A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{(-\lambda)^{\alpha}}{\lambda-\mu}d\lambda\right)d\mu \\ &+ \frac{1}{2\pi i}\int_{\Gamma_1}(-\lambda)^{\alpha}(\lambda+A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{(-\mu)^{\beta}}{\mu-\lambda}d\mu\right)d\lambda \end{aligned}$$

For each $\mu \in \Gamma_2$, the complex function $\lambda \mapsto (-\lambda)^{\alpha}(\lambda-\mu)^{-1}$ is analytic to the left of Γ_1. Then Cauchy's Theorem implies that the integral on the first parenthesis is zero. For each $\lambda \in \Gamma_1$, however, the function $\mu \mapsto (-\mu)^{\beta}(\mu-\lambda)$ has a singularity in λ, and using Cauchy's Theorem again we conclude that the integral in the second parenthesis equals $(-\lambda)^{\beta}$. Then, we get:

$$A^{\alpha}A^{\beta} = \frac{1}{2\pi i}\int_{\Gamma_1}(-\lambda)^{\alpha+\beta}(\lambda+A)^{-1}d\lambda = A^{\alpha+\beta}$$

□

Theorem 26. Let $A \in \mathscr{P}_M$, then

$$A^{-z} = \frac{\sin(\pi z)}{\pi}\int_0^{\infty}s^{-z}(s+A)^{-1}ds, \quad 0 < \operatorname{Re}z < 1$$

Proof. The idea is to transfer the integral to the positive real axis. We choose $0 < \theta < \theta_M$ and $0 < r < \frac{1}{2M}$, and calculate the fractional power using the Definition (30), choosing Γ as the curve that comes from $\infty e^{-i\theta}$ until $re^{-i\theta}$ as a straight line, then circles the origin clockwisely until it reaches $re^{i\theta}$, and goes until $\infty e^{i\theta}$ as a straight line. After some calculations, we obtain

$$\begin{aligned} A^{-z} &= \frac{1}{2\pi i}\int_r^{\infty}s^{-z}e^{-i(-\pi+\theta)z}(se^{i\theta}+A)^{-1}e^{i\theta}ds \\ &-\frac{1}{2\pi i}\int_r^{\infty}s^{-z}e^{-i(\pi-\theta)z}(se^{-i\theta}+A)^{-1}e^{-i\theta}ds+ \\ &+\frac{1}{2\pi i}\int_{2\pi-\theta}^{\theta}(re^{i}(\xi-\pi))^{-z}(re^{i\xi}+A)^{-1}ire^{i\xi}d\xi. \end{aligned}$$

The absolute values of the arguments in the two first integrals are dominated by the function $Cs^{-\operatorname{Re}z}(1+s)^{-1}$, with $C \geq 0$ independent from θ and r, and the absolute value of the argument in the last integral is dominated by $dr^{-\mathrm{Re}z}r$ for $d \geq 0$ also independent of θ and r. Then, we can apply Lebesgue Dominated Convergence Theorem and making $\theta \to 0$, $r \to 0$, we obtain:

$$\begin{aligned} A^{-z} &= \frac{e^{i\pi z}}{2\pi i}\int_0^\infty s^{-z}(s+A)^{-1}ds - \frac{e^{-i\pi z}}{2\pi i}\int_0^\infty s^{-z}(s+A)^{-1}ds \\ &= \frac{\sin(\pi z)}{\pi}\int_0^\infty s^{-z}(s+A)^{-1}ds \end{aligned}$$

□

If $\mathrm{Re}\alpha > 0$, suppose $A^{-\alpha}x = 0$, for some $x \in X$, then we may choose $n \in \mathbb{N}$ such that $\mathrm{Re}\alpha \leq n$, and by Lemma 14,

$$A^{-n}x = A^{-(n-\alpha)-\alpha}x = A^{-(n-\alpha)}A^{-\alpha}x = 0,$$

and since A^{-n} is injective, $x = 0$, so that $A^{-\alpha}$ is injective for $\mathrm{Re}\alpha > 0$. Hence, $A^{-\alpha} : X \to R(A^{-\alpha})$ is a bijection, where $R(A^{-\alpha})$ is the range of $A^{-\alpha}$.

Definition 31 (Positive fractional powers)**.** Let $\mathrm{Re}\alpha > 0$, $A \in \mathscr{P}_M$, then we define the fractional power A^α by setting $D(A^\alpha) = R(A^{-\alpha})$, and $A^\alpha x = (A^{-\alpha})^{-1}x$, $x \in D(A^\alpha)$.

Let $\mathrm{Re}\alpha > 0$. As inverse of a closed operator, A^α is closed, which implies that $D(A^\alpha)$ is a Banach space with the norm of the graph $\|\cdot\| + \|A^\alpha \cdot\|$. Since $A^{-\alpha}$ is bounded, the graph norm is equivalent to the norm $\|A^\alpha \cdot\|$.

We define the **fractional power space** X^α as the Banach space $(D(A^\alpha), \|A^\alpha \cdot\|)$.

Lemma 15. Let $A \in \mathscr{P}_M$, and α, $\beta \in \mathbb{C}$ with $0 < \mathrm{Re}\alpha < \mathrm{Re}\beta$, then $X^\beta \subset X^\alpha \subset X$, and the inclusions are continuous.

Proof. If $x \in D(A^\beta)$, then

$$x = A^{-\beta}A^\beta x = A^{-\alpha-(\beta-\alpha)}A^\beta x = A^{-\alpha}A^{-(\beta-\alpha)}A^\beta x,$$

so that $x \in R(A^{-\alpha}) = D(A^\alpha)$, and we conclude that $D(A^\beta) \subset D(A^\alpha)$. Moreover,

$$\|A^\alpha x\| = \|A^{\alpha-\beta}A^\beta x\| \leq \|A^{\alpha-\beta}\|_{\mathscr{L}(X)}\|A^\beta x\|, \quad \forall x \in D(A^\beta),$$

and this implies that the inclusion $X^\beta \subset X^\alpha$ is continuous. Similarly, $X^\alpha \subset X$ is continuous. □

Lemma 16. Let $A \in \mathscr{P}_M$, α, $\beta \in \mathbb{C}$, with $\mathrm{Re}\alpha$, $\mathrm{Re}\beta$, $\mathrm{Re}(\alpha+\beta)$ all $\neq 0$. Then

$$A^\alpha A^\beta x = A^{\alpha+\beta}x, \quad \forall x \in D(A^u),$$

where $u \in \{\alpha, \beta, \alpha+\beta\}$, with $\mathrm{Re}u = \max\{\mathrm{Re}\alpha, \mathrm{Re}\beta, \mathrm{Re}(\alpha+\beta)\}$.

Proof. The proof is very simple yet extensive. The reader may check it in (CARVALHO, 2012). □

Proposition 11. If $A \in \mathscr{P}_M$ and $\alpha, \beta \in \mathbb{C}$, with $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, then $X^\beta \subset X^\alpha \subset X$ with dense inclusions.

Proof. By hypothesis, $D(A)$ is dense in X. We will show that $D(A^k)$ dense in X implies $D(A^{k+1})$ dense in X, for $k \in \mathbb{N}^*$. Indeed, let $x \in D(A^k)$, $\varepsilon > 0$, define $f = A^k x$ and since $D(A^k)$ is dense in X, there exists $u \in D(A^k)$ such that $\|u - f\| < \varepsilon / \|A^{-k}\|_{\mathscr{L}(X)}$, so that if $v = Au$,

$$\|A^{-k-1}v - x\| = \|A^{-k}u - A^{-k}f\| \leq \|A^{-k}\|_{\mathscr{L}(X)}\|u - f\| < \varepsilon.$$

Therefore, $D(A^k) \subset \overline{D(A^{k+1})}$, which implies $\overline{D(A^{k+1})} \supset \overline{D(A^k)} = X$, and $D(A^{k+1})$ is dense in X. By induction $D(A^k)$ is dense for all $k \in \mathbb{N}^*$.

If $\operatorname{Re}\alpha > 0$, $D(A^\alpha) \supset D(A^k)$ for $k \in \mathbb{N}^*$ large enough, so that $D(A^\alpha)$ is dense in X.

Now suppose that $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, and we will prove that $D(A^\beta)$ is dense in $D(A^\alpha)$. Let $x \in D(A^\alpha)$, $\varepsilon > 0$, and define $f = A^\alpha x$. Since $D(A^{\beta-\alpha})$ is dense in X, there exists $u \in D(A^{\beta-\alpha})$ such that $\|u - f\| < \varepsilon$, and if we set $v := A^{-\alpha}u \in D(A^\beta)$, we have

$$\|v - x\|_{X^\alpha} = \|A^\alpha(v - x)\| = \|u - f\| < \varepsilon,$$

and we are done.

□

Proposition 12. Let $A \in \mathscr{P}_M$ be an operator with compact resolvent, then $A^{-\alpha} \in \mathscr{K}(X)$ if $\operatorname{Re}\alpha > 0$, and if $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, then $X^\beta \subset X^\alpha \subset X$ with compact inclusions.

Proof. If $\operatorname{Re}\alpha < 0$, A^α is compact simply because it is defined as

$$A^\alpha := \frac{1}{2\pi i}\int_\Gamma (-\lambda)^\alpha(\lambda + A)^{-1}d\lambda = \frac{1}{2\pi i}\int_{-\Gamma} \lambda^\alpha(\lambda - A)^{-1}d\lambda,$$

which is the limit of a sequence of compact operators, and $\mathscr{K}(X)$ is closed in $\mathscr{L}(X)$.

Now let us prove that $X^\beta \subset X^\alpha$ is a compact inclusion (the proof that $X^\alpha \subset X$ is a compact inclusion is analogous). Assume $\{x_n\}$ is a bounded sequence in X^β, then $\{A^\beta x_n\}$ is bounded in X, and

$$A^\alpha x_n = A^\alpha A^{-\beta} A^\beta x_n = A^{\alpha-\beta} A^\beta x_n$$

that is, $\{A^\alpha x_n\}$ is the image by a compact operator of a bounded sequence, so that there is a subsequence $\{A^\alpha x_{n_k}\}$ which converges to $z \in X$. This implies that

$$\|A^\alpha x_{n_k} - z\| = \|A^\alpha(x_{n_k} - A^{-\alpha}z)\| \to 0.$$

Hence, $\{x_{n_k}\}$ converges in X^α. This completes the proof. □

Proposition 13. Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a self-adjoint operator with compact resolvent and such that $\langle Au, u\rangle \geq \delta \|u\|^2$, for all $u \in D(A)$, for some $\delta > 0$. Then, A is an operator of positive kind, and A^θ is self-adjoint for all $\theta \in \mathbb{R}$.

Proof. The numerical range of the operator $-A$ satisfies $W(-A) \subset (-\infty, -\delta]$. Consider the open connected set $\Sigma = \mathbb{C} \setminus (-\infty, -\delta]$. Since $-A$ has compact resolvent, its spectrum is countable, so that $\Sigma \cap \rho(-A) \neq \varnothing$, and by Theorem 12, $\Sigma \subset \rho(-A)$, so that $\mathbb{R}^+ \subset \rho(-A)$, and

$$\|(s+A)^{-1}\|_{\mathscr{L}(H)} \leq \frac{1}{\delta + s} \leq \frac{M}{1+s}, \quad \forall s \in \mathbb{R}^+$$

for some $M \geq 1$. Thus, A is an operator of positive kind.

Let $\theta \in (0,1)$, Theorem 26 implies:

$$A^{-\theta} = \frac{\sin(\pi\theta)}{\pi} \int_0^\infty s^{-\theta}(s+A)^{-1} ds,$$

and as a limit of symmetric operators, $A^{-\theta}$ is symmetric. This implies that $A^{-\theta}$ is symmetric for all $\theta > 0$. As inverse of a symmetric operator, A^θ is symmetric for $\theta > 0$, and since it is also surjective, it follows from Theorem 14 that A^θ is self-adjoint for $\theta > 0$. From Proposition 4, $A^{-\theta}$ are self-adjoint for $\theta > 0$. This completes the proof. □

Example 4 (Fractional powers of the Laplacian)**.** Consider the Laplace operator $A : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$, given by $A\phi = -\phi''$. We already know from Example 1 that A is self-adjoint, has compact resolvent, and satisfies:

$$\langle A\phi, \phi\rangle \geq \frac{2}{\pi^2}\|\phi\|^2, \quad \forall \phi \in D(A).$$

It follows from Proposition 13 that A is an operator of positive kind, and A^θ is self-adjoint for every $\theta \in \mathbb{R}$.

Next we calculate a fractional power space for the Laplacian, showing that $X^{\frac{1}{2}} = H_0^1(0,\pi)$. We have already seen that $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \subset D(A^{\frac{1}{2}}) \subset L^2(0,\pi)$ with dense inclusion, and the norm in $X^{\frac{1}{2}}$ in an element $\phi \in D(A)$ is given by:

$$\|A^{\frac{1}{2}}\phi\|^2 = \langle A^{\frac{1}{2}}\phi, A^{\frac{1}{2}}\phi\rangle = \langle A\phi, \phi\rangle = \|\phi'\|^2$$

So that $\|\phi\|_{X^{\frac{1}{2}}} = \|\phi\|_{H_0^1}$, for all $\phi \in D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$.

By the density, $X^{\frac{1}{2}} = \overline{D(A)}^{X^{\frac{1}{2}}}$. We claim that $\overline{D(A)}^{X^{\frac{1}{2}}} = \overline{D(A)}^{H_0^1}$. Indeed, let $x \in \overline{D(A)}^{X^{\frac{1}{2}}}$, then there exists a sequence $\{x_n\}$ in $D(A)$ such that $\|x - x_n\|_{X^{\frac{1}{2}}} \to 0$, then $\|x_n - x_m\|_{X^{\frac{1}{2}}} = \|x_n - x_m\|_{H_0^1} \overset{n,m\to\infty}{\longrightarrow} 0$, so that $\{x_n\}$ converges in $H_0^1(0,\pi)$, that is, there exists $y \in H_0^1(0,\pi)$ such that $\|y - x_n\|_{H_0^1} \to 0$. But both $X^{\frac{1}{2}}$ and $H_0^1(0,\pi)$ are included continuously in $L^2(0,\pi)$, so that y

and x are limits of $\{x_n\}$ in the norm of $L^2(0,\pi)$, which implies $x = y$, and $x \in \overline{D(A)}^{H_0^1}$. The other inclusion is proved analogously.

Therefore, $X^{\frac{1}{2}} = \overline{H^2(0,\pi) \cap H_0^1(0,\pi)}^{H_0^1}$, and using the fact that $C_c^\infty(0,\pi) \subset H^2(0,\pi) \cap H_0^1(0,\pi)$ and $\overline{C_c^\infty(0,\pi)}^{H_0^1} = H_0^1(0,\pi)$, we conclude that $X^{\frac{1}{2}} = H_0^1(0,\pi)$. To show that the norms in the two sets are equal, just notice that if $x \in X^{\frac{1}{2}}$, there exists a sequence $x_n \in D(A)$ such that $\|x - x_n\|_{X^{\frac{1}{2}}} \to 0$, and as before we can conclude that $\|x - x_n\|_{H_0^1} \to 0$. It follows that:

$$\|x\|_{H_0^1} = \lim_{n\to\infty} \|x_n\|_{H_0^1} = \lim_{n\to\infty} \|x_n\|_{X^{\frac{1}{2}}} = \|x\|_{X^{\frac{1}{2}}},$$

and we are done.

We can use this fact and the knowledge we have about the Laplacian to prove a very interesting result:

Proposition 14. Let $\|\cdot\|_{H_0^1}$ denote the norm of $H_0^1(0,\pi)$ and $\|\cdot\|$ denote the norm of $L^2(0,\pi)$. Then, $\|\phi\| \leq \|\phi\|_{H_0^1}$ for all $\phi \in H_0^1(0,\pi)$.

Proof. Let $A : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$ denote the Laplacian, defined by $A\phi = -\phi''$. We know from Example 1 that $A^{-1} \in \mathscr{K}(L^2(0,\pi))$, and if $\phi \in L^2(0,\pi)$, let $\psi = A^{-1}\phi \in H^2(0,\pi) \cap H_0^1(0,\pi)$, so we have:

$$\langle A^{-1}\phi, \phi \rangle = \langle \psi, A\psi \rangle = -\langle \psi, \psi'' \rangle = \|\psi'\|^2 \geq 0.$$

Finally, A^{-1} is self-adjoint, and all the hypothesis of Theorem 16 are satisfied. Therefore, the largest eigenvalue of A^{-1} is given by $\lambda_1 = \sup\{\langle A^{-1}u, u \rangle : \|u\| = 1\}$, but we have already seen in Example 1 that $\lambda_1 = 1$. Therefore,

$$\langle A^{-1}u, u \rangle \leq \|u\|^2, \quad \forall u \in L^2(0,\pi).$$

If $v \in H_0^1(0,\pi)$, we use $u = A^{\frac{1}{2}}v$ in the estimate above, and the fact that $A^{\frac{1}{2}}$ is self-adjoint, to conclude that $\|v\| \leq \|v\|_{H_0^1}$. □

4.2 Interpolation inequalities

In this section we present some inequalities relating the fractional powers of a positive operator A, which will be necessary to study semigroups and perturbation of sectorial operators in the sections that will follow.

Theorem 27. Let $A \in \mathscr{P}_M$. There exists a constant $K \geq 0$, which depends only on A, such that

$$\|A^\alpha x\| \leq K \|Ax\|^\alpha \|x\|^{1-\alpha}, \quad \forall\, 0 \leq \alpha \leq 1, x \in D(A). \tag{4.4}$$

and

$$\|A^\alpha x\| \le K\left[(1-\alpha)\mu^\alpha\|x\|+\alpha\mu^{\alpha-1}\|Ax\|\right], \quad \forall \mu>0, 0\le\alpha\le 1, x\in D(A). \tag{4.5}$$

Proof. It is trivial for $\alpha=0$ or $\alpha=1$. If $0<\alpha<1$, $x\in D(A)$, it follows from Theorem 26 that

$$A^\alpha x = A^{-(1-\alpha)}Ax = \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}A(s+A)^{-1}x ds$$

Therefore, if $\mu\in(0,\infty)$, using that $A(s+A)^{-1}=I-s(s+A)^{-1}$, and $\|s(s+A)^{-1}\|_{\mathscr{L}(X)}\le M$ for $s\ge 0$, we obtain:

$$\begin{aligned}
\|A^\alpha x\| &\le \frac{\sin\pi\alpha}{\pi}\left[\int_0^\mu s^{\alpha-1}(M+1)\|x\|ds+\int_\mu^\infty s^{\alpha-2}M\|Ax\|ds\right]\\
&\le \frac{\sin\pi\alpha}{\pi}(M+1)\left[\frac{\mu^\alpha}{\alpha}\|x\|+\frac{\mu^{\alpha-1}}{1-\alpha}\|Ax\|\right]\\
&\le \left(\frac{\sin\pi\alpha}{\pi}\frac{(M+1)}{\alpha(1-\alpha)}\right)\left[(1-\alpha)\mu^\alpha\|x\|+\alpha\mu^{\alpha-1}\|Ax\|\right],
\end{aligned}$$

and the coefficient between parenthesis has an upper bound for $0<\alpha<1$, so that (4.5) follows, and (4.4) follows if we use $\mu=\|Ax\|/\|x\|$. □

Proposition 15. Let $A\in\mathscr{P}(X)$ and $B:D(B)\subset X\to X$ be a closed linear operator such that $D(B)\supset D(A^\alpha)$, for some $\alpha>0$. Then there exist constants $C,C_1>0$ such that

$$\|Bx\|\le C\|A^\alpha x\|, \quad x\in D(A^\alpha)$$

and

$$\|Bx\|\le C_1(\mu^\alpha\|x\|+\mu^{\alpha-1}\|Ax\|), \quad \mu>0,\ x\in D(A).$$

Proof. The operator $BA^{-\alpha}:X\to X$ is well-defined because $D(B)\supset D(A^\alpha)$, and is closed, then by the Closed Graph Theorem, it is bounded, and the first estimate follows. Then, using also Theorem 27, we get, for $x\in D(A^\alpha)$:

$$\begin{aligned}
\|Bx\|=\|BA^{-\alpha}A^\alpha x\| &\le \|BA^{-\alpha}\|K\left[(1-\alpha)\mu^\alpha\|x\|+\alpha\mu^{\alpha-1}\|Ax\|\right]\\
&\le C_1(\mu^\alpha\|x\|+\mu^{\alpha-1}\|Ax\|), \quad \mu>0.
\end{aligned}$$

□

Theorem 28. Suppose that A and B are positive operators, with $D(A)=D(B)$ and $\operatorname{Re}\sigma(A)>0$, $\operatorname{Re}\sigma(B)>0$, and for some $\alpha\in[0,1)$, $(A-B)A^{-\alpha}\in\mathscr{L}(X)$. Then, for all $\beta\in[0,1]$, $A^\beta B^{-\beta}$ and $B^\beta A^{-\beta}$ are in $\mathscr{L}(X)$.

Proof. The cases $\beta=0$, $\beta=1$ are simple and are left to the reader as an exercise. For a fixed $\beta\in(0,1)$, it follows from Theorem 27 that, for $x\in X$, $\lambda\in[0,\infty)$, we have $\lambda\in\rho(-A)$ and:

$$\begin{aligned}
\|A^\beta(\lambda+A)^{-1}x\| &\le K\|A(\lambda+A)^{-1}x\|^\beta\|(\lambda+A)^{-1}x\|^{1-\beta}\\
&\le \|x-\lambda(\lambda+A)^{-1}x\|^\beta\|(\lambda+A)^{-1}x\|^{1-\beta}
\end{aligned} \tag{4.6}$$

so that $A^\beta(\lambda+A)^{-1}\in\mathscr{L}(X)$ for all $\lambda\geq 0$, and the same happens for $B^\beta(\lambda+B)^{-1}$. More than that, since both A and B are positive, we get:

$$\|A^\beta(\lambda+A)^{-1}\|_{\mathscr{L}(X)}\leq C_1(\lambda+1)^{\beta-1},\quad \lambda\in[0,\infty).$$

and

$$\|B^\beta(\lambda+B)^{-1}\|_{\mathscr{L}(X)}\leq C_2(\lambda+1)^{\beta-1},\quad \lambda\in[0,\infty).$$

Now, for $\beta\in(0,1)$,

$$B^{-\beta}-A^{-\beta}=\underbrace{\frac{\sin(\pi\beta)}{\pi}}_{=\gamma}\int_0^\infty \lambda^{-\beta}(\lambda+B)^{-1}(A-B)(\lambda+A)^{-1}d\lambda.$$

Therefore,

$$B^\beta A^{-\beta}=I-\gamma\int_0^\infty \lambda^{-\beta}B^\beta(\lambda+B)^{-1}(A-B)A^{-\alpha}A^\alpha(\lambda+A)^{-1}d\lambda$$

The integrals converge in $\mathscr{L}(X)$ because of the estimates we presented above, along with the fact that $(A-B)A^{-\alpha}\in\mathscr{L}(X)$, therefore, $B^\beta A^{-\beta}\in\mathscr{L}(X)$.

We prove that $A^\beta B^{-\beta}$ is bounded analogously, but in this case we need to show that $\|A^\alpha(\lambda+B)^{-1}\|=O(|\lambda|^{\alpha-1})$ as $|\lambda|\to\infty$. This is a consequence of the fact that:

$$[I+A^\alpha(\lambda+A)^{-1}(B-A)A^{-\alpha}]A^\alpha(\lambda+B)^{-1}=A^\alpha(\lambda+A)^{-1},$$

and we use the Newmann series to estimate $A^\alpha(\lambda+B)^{-1}$. □

Corollary 2. If A and B are like in the Theorem 28, then $D(A^\alpha)=D(B^\alpha)$, with equivalent norms $0\leq\alpha\leq 1$.

4.3 Semigroups and fractional powers

In this section we study the relation between a semigroup generated by an operator and its fractional powers, obtaining estimates that will be useful in the applications. We start with a consequence of an interpolation inequality of the last section.

Corollary 3 (of Theorem 27)**.** If A is a sectorial linear operator with vertex $a\in\mathbb{R}$ and $A\in\mathscr{P}_M$, let $\{e^{-At}:t\geq 0\}$ denote the analytic semigroup generated by $-A$. Then, if $\alpha\in[0,1]$,

$$\|A^\alpha e^{-At}\|_{\mathscr{L}(X)}\leq C_\alpha t^{-\alpha}e^{at},\quad t>0.$$

Proof. It is a simple application of Theorem 27, using the estimates in Theorem 23 and Remark 9. □

Theorem 29. Suppose A is a sectorial linear operator of positive kind, and let $\{e^{-At} : t \geq 0\}$ denote the analytic semigroup generated by $-A$. Then, the following holds:

1. If $t > 0$, $\alpha \geq 0$, $R(e^{-At}) \subset D(A^{\alpha})$, and

$$\|A^{\alpha}e^{-At}\|_{\mathscr{L}(X)} \leq M_{\alpha}t^{-\alpha}, \quad 0 < t \leq 1.$$

2. If $\alpha > 0$, we have $t^{\alpha}A^{\alpha}e^{-At}x \to 0$ as $t \to 0^{+}$, for each $x \in X$.

3. $\|(e^{-At} - I)A^{-\alpha}\|_{\mathscr{L}(X)} \leq M_{1-\alpha}\frac{t^{\alpha}}{\alpha}$ if $0 < \alpha \leq 1$, $0 \leq t \leq 1$.

Proof. 1) If $t > 0$, it follows from Theorem 23 that $R(e^{-At}) \subset D(A)$, $\|Ae^{-At}\|_{\mathscr{L}(X)} \leq Mt^{-1}$, $\|e^{-At}\|_{\mathscr{L}(X)} \leq M$. Then, for any $m \in \mathbb{Z}$, $R(e^{-At}) \subset D(A^{m})$ because $e^{-At} = (e^{-At/m})^{m}$ and if $y \in D(A^{k})$, then $e^{-At/m}y \in D(A^{k+1})$. If $0 \leq \alpha \leq 1$, from Theorem 27, we have:

$$\|A^{\alpha}e^{-At}\|_{\mathscr{L}(X)} \leq K\|Ae^{-At}\|^{\alpha}_{\mathscr{L}(X)}\|e^{-At}\|^{1-\alpha}_{\mathscr{L}(X)} \leq KMt^{-\alpha}.$$

It follows that, for $m = 0, 1, 2, \cdots$, $0 \leq \alpha \leq 1$, $0 < t \leq 1$,

$$\begin{aligned} \|A^{m+\alpha}e^{-At}\|_{\mathscr{L}(X)} &\leq \|A^{\alpha}e^{-At/(m+1)}\|_{\mathscr{L}(X)}\|Ae^{-At/(m+1)}\|^{m}_{\mathscr{L}(X)} \\ &\leq (KM)^{m+1}(m+1)^{m+\alpha}t^{-m-\alpha}, \end{aligned}$$

and the estimate is proved for any $\alpha \geq 0$.

2) If $\alpha > 0$, choose $m \in \mathbb{N}$ such that $m \geq \alpha > 0$, and if $x \in D(A^{m})$

$$\|t^{\alpha}A^{\alpha}e^{-At}x\| \leq t^{\alpha}\|A^{\alpha-m}\|_{\mathscr{L}(X)}\|e^{-At}\|_{\mathscr{L}(X)}\|A^{m}x\| \stackrel{t\to 0^{+}}{\longrightarrow} 0,$$

and $\|t^{\alpha}A^{\alpha}e^{-At}\|_{\mathscr{L}(X)} \leq M_{\alpha}$ for all $0 < t \leq 1$. Then, since $D(A^{m})$ is dense in X, the result follows for every $x \in X$.

3) Since $\frac{d}{ds}e^{-As} = -Ae^{-As}$, we have for all $x \in X$:

$$\|(e^{-At} - I)A^{-\alpha}x\| = \left\| -\int_0^t A^{1-\alpha}e^{-As}x\,ds \right\| \leq \int_0^t M_{1-\alpha}s^{\alpha-1}\|x\|ds = M_{1-\alpha}\frac{t^{\alpha}}{\alpha}\|x\|.$$

□

4.4 Perturbation of sectorial operators

In this section we will see that a nice perturbation of a sectorial operator is sectorial, possibly changing the vertex. This will help us use the theory of sectorial operators for semilinear differential equations, in which the linear part is perturbed by a non-linear function.

Theorem 30. Let $A : D(A) \subset X \to X$ be such that $-A$ is sectorial. There exists a $\delta > 0$ such that if $B : D(B) \subset X \to X$, $D(B) \supset D(A)$, is a linear operator satisfying

$$\|Bx\| \leq \delta \|Ax\| + K\|x\|, \quad \forall x \in D(A), \tag{4.7}$$

then $-(A+B)$ is sectorial with vertex $a \geq 0$, and $D(A+B) = D(A)$.

Proof. Since $-A$ is sectorial, there exist constants ϕ, C, with $\pi/2 < \phi \leq \pi$, such that for $|\arg(\lambda)| < \phi$, $\lambda \in \rho(A)$ and $\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq C/|\lambda|$. Choose $\delta > 0$ and θ such that $0 < \delta(C+1) < \theta < 1$, and suppose (4.7) holds. Let $|\arg(\lambda)| < \phi$, and we will show that $B(\lambda - A)^{-1} \in \mathscr{L}(X)$. Indeed $B(\lambda - A)^{-1}$ is well-defined in X because $D(A) \subset D(B)$, and if $x \in X$,

$$\begin{aligned}
\|B(\lambda - A)^{-1}x\| &\leq \delta\|A(\lambda - A)^{-1}x\| + K\|(\lambda - A)^{-1}x\| \\
&\leq \delta\| - x + \lambda(\lambda - A)^{-1}x\| + \frac{KC}{|\lambda|}\|x\| \\
&\leq \delta(1+C)\|x\| + \frac{KC}{|\lambda|}\|x\|,
\end{aligned}$$

and $\|B(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq \theta$ for $|\lambda| > R$ with R large enough. Note that we can write $\lambda - (A + B) = (I - B(\lambda - A)^{-1})(\lambda - A)$, so that we can use the Newmann Series to see that if $|\arg(\lambda)| < \phi$, $|\lambda| \geq R$ we have $\lambda \in \rho(A+B)$ and

$$\|[\lambda - (A+B)]^{-1}\|_{\mathscr{L}(X)} \leq \frac{C/(1-\theta)}{|\lambda|}.$$

It is easy to see then that $-(A+B)$ is sectorial with some vertex $a > R$. □

Corollary 4. Let A be a sectorial operator of positive kind and $B : D(B) \subset X \to X$ be a closed operator such that $D(B) \supset D(A^\alpha)$, for some $0 < \alpha < 1$. Then $A + B$ is sectorial with vertex $a \in \mathbb{R}$.

Proof. Since $D(B) \supset D(A^\alpha) \supset D(A)$, it follows from Proposition 15 that

$$\|Bx\| \leq C(\mu^\alpha \|x\| + \mu^{\alpha-1}\|Ax\|), \quad x \in D(A),\ \mu > 0,$$

then we can choose a big $\mu > 0$ and apply Theorem 30. □

CHAPTER

5

SEMILINEAR EVOLUTION EQUATIONS

5.1 Existence, uniqueness and continuous dependence

In this chapter we study semilinear partial differential equations, which arise in the study of partial differential equations that are a regular perturbation of a linear equation. For a thorough approach to this topic, including a more detailed discussion on continuous dependence, the reader may consult (HENRY, 2013) and (CARVALHO; LANGA; ROBINSON, 2013). In this chapter, we will make use of projections and spectral sets, which are presented in detail in (CARVALHO, 2012, section 2.9), and in (CARVALHO; LANGA; ROBINSON, 2013).

Consider the initial value problem:

$$\begin{aligned} &\frac{d}{dt}u = -Au + f(t,u), \quad t > t_0 \\ &u(t_0) = u_0 \in X, \end{aligned} \tag{5.1}$$

where X is a Banach space and $A : D(A) \subset X \to X$ is a sectorial (vertex 0) and positive operator (see definitions 28 and 29). Note that if we have a problem like (5.1) and A is sectorial with vertex $a \neq 0$, we can still apply this theory using the operator $B = \alpha + A$ as linear part and $g = f - \alpha$ as non-linear part — the equation remains unchanged, and the essential properties of f are preserved by g. By Theorem 23 and Corollary 3, $-A$ generates a strongly continuous semigroup, which we denote by $\{e^{-At} : t \geq 0\}$, and if $\alpha \in [0,1]$, there is a constant $M \geq 0$ such that $\|e^{-At}\|_{\mathscr{L}(X)} \leq M$ and $\|A^\alpha e^{-At}\|_{\mathscr{L}(X)} \leq Mt^{-\alpha}$, for all $t \geq 0$. Denote by $\|\cdot\|$ the norm of X.

Recall that, since A is positive, we define A^α, for $\alpha \in \mathbb{R}$, and $X^\alpha := (D(A^\alpha), \|A^\alpha \cdot\|)$.

Note that X^α is itself a Banach space and the semigroup $\{e^{-At} : t \geq 0\}$ may be restricted to a strongly continuous semigroup in X^α, because for $t > 0$, $\alpha \geq 0$, $x \in X^\alpha$, we have $R(e^{-At}) \subset D(A^\alpha)$ and $A^\alpha e^{-At}x = e^{-At}A^\alpha x$ (Theorem 23).

Definition 32 (Solution)**.** Let X be a Banach space and $A : D(A) \subset X \to X$ be a sectorial and positive operator. Assume $0 \leq \alpha < 1$, $U \subset \mathbb{R} \times X^\alpha$ is an open set and $f : U \to X$ is a continuous function. A **solution** of (5.1) in $[t_0,t_1)$ is a continuous function $u:[t_0,t_1)\to X$ that is differentiable in (t_0,t_1) and such that $(t,u(t))\in U$, for $t\in[t_0,t_1)$, $u(t)\in D(A)$, for $t\in(t_0,t_1)$, and $(t_0,t_1)\ni t\mapsto Au(t)\in X$ is continuous and (5.1) holds.

One advantage of studying this class of equations, named parabolic partial differential equations, is that we may take f mapping from fractional spaces of A into X. For instance, we may take $f:X^\alpha\to X$ given by $f(u)=A^\alpha u$, which is globally Lipschitz because of the definition of the norm in X^α. This allows for a larger class of non-linear functions that can be studied.

The next theorem states that the weak solution concept used for hyperbolic equations is not needed in the context of parabolic equations since weak solutions are also strong here.

Theorem 31. Let X,A,α,U,f be as above and assume f is locally Hölder continuous, that is, for any $x\in U$, there exists $V\subset U$ neighborhood of x in U such that:

$$\|f(\tau_1,u_1)-f(\tau_2,u_2)\|_X\le K\left(|\tau_1-\tau_2)|^\eta+\|u_1-u_2\|^\eta_{X^\alpha}\right),\quad \forall(\tau_1,u_1),(\tau_2,u_2)\in V.$$

If $u:[t_0,t_1]\to X^\alpha$ is continuous, $(t,u(t))\in U$ for $t\in[t_0,t_1]$, and u satisfies the Formula of Variation of Constants, given by:

$$u(t)=e^{-A(t-t_0)}u(t_0)+\int_{t_0}^t e^{-A(t-s)}f(s,u(s))ds,\quad t\ge t_0,$$

then u is a solution for (5.1).

Proof. Let us prove first that $u:(t_0,t_1]\to X^\alpha$ is locally Hölder continuous. Since $u:[t_0,t_1]\to X^\alpha$ is continuous, $\{(t,u(t)):t_0\le t\le t_1\}$ is compact in U and there exists a $B\ge 0$ such that $\sup_{t_0\le t\le t_1}\|f(t,u(t))\|_X\le B$. Hence, for $t_0<t\le t+h\le t_1$,

$$\begin{aligned}u(t+h)-u(t) &= (e^{-Ah}-I)\left[e^{-A(t-t_0)}u(t_0)+\int_{t_0}^t e^{-A(t-s)}f(s,u(s))ds\right]\\ &+ \int_t^{t+h}e^{-A(t+h-s)}f(s,u(s))ds.\end{aligned}$$

Choose a θ such that $0<\theta<1-\alpha$. Then, by Corollary 3 and Theorem 29, we get:

$$\begin{aligned}\|u(t+h)-u(t)\|_{X^\alpha} &\le Mh^\theta(t-t_0)^{-\theta}\|u(t_0)\|_{X^\alpha}+\int_{t_0}^t Mh^\theta(t-s)^{-\alpha-\theta}Bds\\ &+ \int_t^{t+h}M(t+h-s)^{-\alpha}Bds\end{aligned}$$

Since $-\alpha+1>0$ considering the second term we get

$$\int_{t_0}^t Mh^\theta(t-s)^{-\alpha-\theta}Bds=MB\frac{(t-t_0)^{-\alpha+1}}{-\alpha-\theta+1}h^\theta(t-t_0)^{-\theta}\le C_1h^\theta(t-t_0)^{-\theta}.$$

And for the third term, we get:

$$\begin{aligned}\int_t^{t+h} M(t+h-s)^{-\alpha} B ds &\leq MB\frac{h^{1-\alpha-\theta}}{1-\alpha}h^{\theta}\frac{(t-t_0)^{-\theta}}{(t_1-t_0)^{-\theta}}\\ &\leq C_2 h^{\theta}(t-t_0)^{-\theta}.\end{aligned}$$

Therefore, for a constant $C>0$,

$$\|u(t+h)-u(t)\|_{X^{\alpha}} \leq Ch^{\theta}(t-t_0)^{-\theta}.$$

It follows that $t \mapsto f(t,u(t)) \equiv g(t)$ is continuous in $[t_0,t_1]$ and satisfies a Hölder condition

$$\|g(t+h)-g(t)\| \leq K(t-t_0)^{-\delta}h^{\delta}, \quad t_0 < t \leq t+h \leq t_1,$$

for some $K,\delta > 0$ ($0<\delta<1-\alpha$, without loss of generality). It is enough to show that

$$G(t) = \int_{t_0}^{t} e^{-A(t-s)}g(s)ds$$

takes values in $D(A)$ with $t \to AG(t)$ continuous in $(t_0,t_1]$ — this will imply $u(t) \in D(A)$ for $t \in [t_0,t_1)$, $(t_0,t_1) \ni t \mapsto Au(t) \in X$ is continuous, and u is differentiable in (t_0,t_1). Therefore, we will show that $h^{-1}(e^{-Ah}-I)G(t)$ converges as $h \to 0^+$, uniformly in $t_0^* \leq t \leq t_1$, for any $t_0^* > t_0$. Now,

$$\begin{aligned}h^{-1}(e^{-Ah}-I)G(t) &= \int_{t_0}^{t} h^{-1}(e^{-Ah}-I)e^{-A(t-s)}(g(s)-g(t))ds\\ &\quad + h^{-1}\int_{t_0}^{t_0+h} e^{-A(t+h-s)}g(t)ds - h^{-1}\int_{t-h}^{t} e^{-A(t-s)}g(t)ds,\end{aligned}$$

and the two last terms converge uniformly in $t_0^* \leq t \leq t_1$. For the other term, not that

$$\int_{t_0}^{t} \|Ae^{-A(t-s)}\|_{\mathscr{L}(X)}\|g(t)-g(s)\|ds < \infty$$

because of the Hölder condition, and

$$\begin{aligned}&\|h^{-1}\int_{t_0}^{t}(e^{-Ah}-I+hA)e^{-A(t-s)}(g(s)-g(t))ds\| =\\ &= \|\int_{t_0}^{t} h^{-1}\int_0^h (I-e^{-A\sigma})d\sigma Ae^{-A(t-s)}(g(s)-g(t))ds\|\\ &\leq \int_{t_0}^{t} Mh^{\varepsilon}(t-s)^{-1-\varepsilon}K(s-t_0)^{-\delta}(t-s)^{\delta}ds \overset{h\to 0^+}{\longrightarrow} 0, \quad 0<\varepsilon<\delta\end{aligned}$$

uniformly for $t_0^* \leq t \leq t_1$.

Therefore,

$$h^{-1}(e^{-Ah}-I)G(t) \to -\int_{t_0}^{t} Ae^{-A(t-s)}(g(s)-g(t))ds + e^{-A(t-t_0)}g(t)-g(t)$$

as $h \to 0^+$ uniformly in $[t_0^*,t_1]$, which completes the proof.

□

In order to study the existence and uniqueness of solutions for (5.1), we need to use a version of Grönwall's Lemma that allows for singularities in the auxiliary functions. It is stated as following (HENRY, 2013):

Lemma 17. Suppose $a,b \geq 0$, $\alpha,\beta \in [0,1)$, $u:[0,T] \to \mathbb{R}$ is integrable, and the following holds:

$$0 \leq u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta} u(s)ds \tag{5.2}$$

Then, there exists a constant $K = K(b,\beta,T)$ such that:

$$u(t) \leq \frac{K}{1-\alpha} at^{-\alpha}$$

For almost every $t \in (0,T)$.

We are ready to prove the existence and uniqueness of solutions. As usual, we use the Banach Fixed Point Theorem.

Theorem 32 (Existence, uniqueness, extensions of solutions)**.** Suppose X is a Banach space, $A: D(A) \subset X \to X$ is sectorial and positive, $0 \leq \alpha < 1$, X^α is as defined before, and U is an open subset of $\mathbb{R} \times X^\alpha$. Assume $f: U \to X$ is locally Hölder continuous in its first argument and locally Lipschitz continuous in its second argument, that is, in a neighborhood of any point in U, this holds:

$$\|f(t,u_1) - f(s,u_2)\| \leq C\left(|t-s|^\theta + \|u_1 - u_2\|_{X^\alpha}\right), \tag{5.3}$$

for some $\theta, C > 0$ depending on the neighborhood. Then, given $(t_0,u_0) \in U$, there exists a unique solution $u: [t_0,t_1) \to X$, where the interval $[t_0,t_1)$ is maximal. Finally, if $u_0 \in D(A)$, the derivative of the solution is continuous as $t \to t_0^+$. If $t_1 < \infty$, then either $(t,u(t)) \overset{t\to t_1}{\longrightarrow} \partial U$ or

$$\frac{\|f(t,u(t))\|_X}{1 + \|u(t)\|_{X^\alpha}}$$

is not bounded in $[t_0,t_1)$, or both.

Note that u_0 does not need to belong to the domain of A, but only to a fractional domain X^α. However, the domain of A is dense in X^α, and we may find a solution $u: [t_0,t_1) \to X$ such that $u(t_0) = u_0$ and $u(t) \in D(A)$ for any $0 < t < t_1$, which is a regularization of the initial data.

Proof. Consider the operator G defined as following:

$$G(u)(t) = e^{-A(t-t_0)}u_0 + \int_{t_0}^t e^{-A(t-s)} f(s,u(s))ds, \quad t_0 \leq t \leq t_0 + T.$$

We will show that for T, r chosen correctly, this operator is a contraction in the ball of radius r and center u_0 of the space $\mathscr{C}([t_0,t_0+T],X^\alpha)$, namely,

$$B_r := \left\{ u: [t_0,t_0+T] \to X^\alpha : u \text{ is continuous and } \sup_{t\in[t_0,t_0+T]} \|u(t) - u_0\|_{X^\alpha} \leq r \right\}.$$

Choose r, $T > 0$ such that the set $V := [t_0, t_0 + T] \times \{u \in X^\alpha : \|u - u_0\|_{X^\alpha} \leq r\}$ is contained in the open set U. Since f satisfies (5.3), choose $B > 0$ so that $\|f(t,u)\| \leq B$ for $(t,u) \in V$, and let $L > 0$ be such that:

$$\|f(t,u_1) - f(t,u_2)\| \leq L\|u_1 - u_2\|_{X^\alpha}, \quad (t,u_i) \in V, \quad i = 1,2.$$

Using Theorem 23 and Corollary 3, let $M > 0$ be such that:

$$\begin{aligned}&\|e^{-As}z\|_{X^\alpha} \leq M\|z\|_{X^\alpha}, \quad 0 \leq s \leq T\\ &\|e^{-As}z\|_{X^\alpha} \leq Ms^{-\alpha}\|z\|, \quad 0 \leq s \leq T.\end{aligned}$$

Now, using that $\{e^{-At} : t \geq 0\}$ is strongly continuous, we may reduce the value of the chosen T so that:

$$\begin{aligned}&\|e^{-As}u_0 - u_0\|_{X^\alpha} \leq r/2, \quad 0 \leq s \leq T\\ &\frac{MBT^{1-\alpha}}{1-\alpha} \leq r/2\\ &\frac{MLT^{1-\alpha}}{1-\alpha} \leq 1/2,\end{aligned}$$

Now we prove that $G(B_r) \subset B_r$. Indeed, if $u \in B_r$:

$$\begin{aligned}\sup_{t\in[t_0,t_0+T]} \|G(u)(t) - u_0\|_{X^\alpha} &= \sup_{t\in[t_0,t_0+T]} \|e^{-A(t-t_0)}u_0 - u_0 + \int_{t_0}^t e^{-A(t-s)}f(s,u(s))ds\|_{X^\alpha}\\ &\leq \sup_{t\in[t_0,t_0+T]} \left(\frac{r}{2} + \int_{t_0}^t M(t-s)^{-\alpha}\|f(s,u(s))\|ds\right)\\ &\leq \sup_{t\in[t_0,t_0+T]} \left(\frac{r}{2} + M \sup_{s\in[t_0,t_0+T]} \|f(s,u(s))\| \frac{(t-t_0)^{1-\alpha}}{1-\alpha}\right)\\ &\leq \frac{r}{2} + \frac{MT^{1-\alpha}}{1-\alpha} \sup_{s\in[t_0,t_0+T]} \|f(s,u(s))\| \leq \frac{r}{2} + \frac{MBT^{1-\alpha}}{1-\alpha} \leq r\end{aligned}$$

With similar calculations we prove that if u, $\tilde{u} \in B_r$,

$$\sup_{t\in[t_0,t_0+T]} \|Gu(t) - G\tilde{u}(t)\|_{X^\alpha} \leq \frac{MLT^{1-\alpha}}{1-\alpha} \sup_{t\in[t_0,t_0+T]} \|u(s) - \tilde{u}(s)\|_{X^\alpha},$$

so that G is a contraction in B_r and possesses an unique fixed point $u \in B_r$, and by Theorem 31, $u(\cdot)$ is a solution of (5.1).

Now let us prove the uniqueness. Suppose u, $\tilde{u}$ are solutions of (5.1) defined in $[t_0, t_2]$. If $u(t) = \tilde{u}(t)$, for all $t \in [t_0, a]$, for some $a > t_0$, define $\alpha = \sup\{a \in (t_0, t_2] : u(t) = \tilde{u}(t) \,\forall\, t \in [t_0, a]\}$. If $\alpha = t_2$, we are done. Otherwise, consider instead the solutions u and $\tilde{u}$ defined in $[\alpha, t_2]$. This shows us that we can suppose that $u(t) \neq \tilde{u}(t)$ for t as close as we wish from t_0, $t > t_0$ (possibly redefining $t_0 = \alpha$).

Since u and $\tilde{u}$ are continuous, take $\tilde{T} > 0$ small enough so that $\tilde{T} < T$, and:

$$\sup_{t\in[t_0,t_0+\tilde{T}]} \|u(t)-u_0\|_{X^\alpha} \leq r$$
$$\sup_{t\in[t_0,t_0+\tilde{T}]} \|\tilde{u}(t)-u_0\|_{X^\alpha} \leq r,$$

But G is a contraction in the ball $\tilde{B}_r$ of radius r and center u_0 of the space $\mathscr{C}([t_0,t_0+\tilde{T}],X^\alpha)$, with norm of supremum, and $G(\tilde{B}_r) \subset \tilde{B}_r$, so G only has one fixed point in $\tilde{B}_r$. The restrictions of u and $\tilde{u}$ to $[t_0,t_0+\tilde{T}]$ are both fixed points of G in $\tilde{B}_r$, but $u(t) \neq \tilde{u}(t)$ for some $t \in [t_0,t_0+\tilde{T}]$. This is a contradiction, so $\alpha = t_2$ and $u(t) = \tilde{u}(t)$ for $t \in [t_0,t_2]$.

The maximal solution is constructed the following way: let

$$t_1 = \sup\{a > t_0 : \text{ there exists a solution of (5.1) defined in } [t_0,a)\}$$

and for $t \in [t_0,t_1)$, define $u(t) = \tilde{u}(t)$ where $\tilde{u}$ is a solution defined in $[t_0,a)$ for $a > t$. This solution u is well defined because of the uniqueness of solutions, and it is trivially maximal.

Now let us prove the last claim.

Suppose $t_1 < \infty$, and the limit $u_1 = \lim_{t\to t_1^-} u(t)$ exists. If $(t_1,u_1) \in U$, there exists a solution $\tilde{u} : [t_1,t_1+\delta) \to X$, with $\tilde{u}(t_1) = u_1$ for some $\delta > 0$. Then $\hat{u} : [t_0,t_1+\delta) \to X$ given by $\hat{u}(t) = u(t)$, for $t_0 \leq t < t_1$ and $\hat{u}(t) = \tilde{u}(t)$, for $t_1 \leq t < t_1+\delta$ satisfies the Formula of Variation of Constants and is a solution of (5.1). This contradicts the definition of t_1. So $(t_1,u_1) \in \partial U$.

Suppose now that:

$$\frac{\|f(t,u(t))\|}{1+\|u(t)\|_{X^\alpha}} \leq B < \infty, \quad t_0 \leq t < t_1 \tag{5.4}$$

We will show that in this case, the limit $u_1 = \lim_{t\to t_1^-} u(t)$ must exist, and the proof will be complete. First note that for $t \in [t_0,t_1)$, estimating in the Formula of Variation of Constants, and using the condition (5.4), we get:

$$1+\|u(t)\|_{X^\alpha} \leq [1+M\|u(t_0)\|_{X^\alpha}] + \int_{t_0}^{t} MB(t-s)^{-\alpha}(1+\|u(s)\|_{X^\alpha})ds.$$

By Grönwall's Lemma 17, we conclude that $\|u(t)\|_{X^\alpha}$ is bounded in $[t_0,t_1)$, and by the Lipschitz condition, $\|f(t,u(t))\| \leq B_1$, $t_0 \leq t < t_1$. Let us prove that $\|u(s)-u(r)\|_{X^\alpha} \to 0$ as s, $r \to t_1^-$, which will imply the existence of the limit.

Given $\varepsilon > 0$ choose $0 < \varepsilon_1 < t_1 - t_0$ with $\varepsilon_1 \leq \frac{\varepsilon}{4MB_1}$. Define $t^* := t_1 - \varepsilon_1$ and let $0 < \delta \leq \varepsilon_1$ be such that $\|(e^{-A(s-t^*)} - e^{-A(r-t^*)})u(t^*)\|_{X^\alpha} \leq \frac{\varepsilon}{4}$ if $|s-r| \leq \delta$. Let $s,r \in [t_1-\delta,t_1)$, and $s \leq r$ so that $t^* \leq t_1 - \delta \leq s \leq r < t_1$, we may use the Formula of Variation of Constants for $u(s)$ and $u(r)$ with $t_0 = t^*$, and we get:

$$\|u(s)-u(r)\|_{X^\alpha} \leq \frac{\varepsilon}{4} + 2\int_{t^*}^{s} MB_1 d\theta + \int_{s}^{r} MB_1 d\theta \leq \varepsilon,$$

□

which completes the proof.

Remark 11. Here we make a digression to talk about continuity of solutions in relation to the initial conditions (t_0, u_0). For non-autonomous problems (the case where f depends on t), this continuity is a consequence of the continuity of the contraction operator G in relation to t_0 and u_0 (HENRY, 2013, Corollary 3.4.6). However, we can show continuity in relation to initial data in a much simpler way if we are in the autonomous case (where f does not depend on t). Indeed, consider the problem:

$$\begin{aligned}&\frac{d}{dt}u = -Au + f(u), \quad t \geq 0\\ &u(0) = u_0 \in X^\alpha,\end{aligned}$$

where $A : D(A) \subset X \to X$ is sectorial, positive, $\alpha \in [0,1)$, $f : X^\alpha \to X$ is Lipschitz continuous in bounded sets of X^α. Then, Theorem 32 applies and for each $u_0 \in X^\alpha$, there exists a solution $u(\cdot,u_0) : [0,\tau) \to X^\alpha$ such that $u(0,u_0) = u_0$. Suppose further that for every $u_0 \in X^\alpha$, the solution $u(\cdot,u_0)$ is bounded in $[0,\tau)$ — in the applications, we usually show this boundedness using energy estimates, like the one we make in the first section of Chapter 6 — then the last part of Theorem 32 implies that $\tau = \infty$ for every initial condition $u_0 \in X^\alpha$. Now, let u_1, $u_2 \in X^\alpha$ be two initial conditions, and $T > 0$, then there exists $\rho \geq 0$ such that $\|u(s,u_1)\|_{X^\alpha} \leq \rho$ and $\|u(s,u_2)\|_{X^\alpha} \leq \rho$ for any $s \in [0,T]$ and let $k(\rho)$ be the Lipschitz constant of f in the ball of radius ρ in X^α. By the Formula of Variation of Constants, we have, for any $t \in [0,T]$:

$$\begin{aligned}\|u(t,u_1) - u(t,u_2)\|_{X^\alpha} \leq \|e^{-At}u_1 - e^{-At}u_2\|\\ \int_0^t \|e^{-A(t-s)}\left(f(u(s,u_1)) - f(u(s,u_2))\right)\|_{X^\alpha}ds\end{aligned}$$

which yields:

$$\begin{aligned}\|u(t,u_1) - u(t,u_2)\|_{X^\alpha} \leq M\|u_1 - u_2\|_{X^\alpha} +\\ \int_0^t Mk(\rho)(t-s)^{-\alpha}\|u(s,u_1) - u(s,u_2)\|_{X^\alpha}ds.\end{aligned}$$

From the Grönwall's Lemma 17,

$$\|u(t,u_1) - u(t,u_2)\|_{X^\alpha} \leq K(T)\|u_1 - u_2\|_{X^\alpha}, \quad \forall t \in [0,T].$$

Using the triangle inequality, we obtain that the application $\mathbb{R} \times X^\alpha \ni (t,u_0) \mapsto u(t,u_0) \in X^\alpha$ is continuous, which allows us to use the theory of semigroups and attractors developed in Chapter 3 to study parabolic semilinear differential equations when they are autonomous.

5.2 Stability of equilibria

In this section we study the behavior of a dynamical system in the neighborhood of an equilibrium point. We will show that, under certain conditions, we can approximate a semilinear parabolic equation by a linear equation, and analyze the spectrum of the linear operator associated in order to conclude information about the original semilinear equation.

From now on, let $0<\alpha<1$ and consider the sectorial operator A of positive kind in the Banach space X, and the function $f: U \to X$, where U is a neighborhood of $\mathbb{R}\times\{u_0\}$ in $\mathbb{R}\times X^\alpha$, such that there exists a fixed neighborhood of u_0 in X^α, say $V\subset X^\alpha$, satisfying $t\times V\subset U$ for all $t\in\mathbb{R}$. We say that $u_0\in X$ is an equilibrium for (5.1) if $u(t)\equiv u_0$ is a solution of:

$$\frac{d}{dt}u + Au = f(t,u), \quad t\in\mathbb{R}, \tag{5.5}$$

that is, $u_0\in D(A)$ and $Au_0 = f(t,u_0)$ for all $t\in\mathbb{R}$.

Definition 33 (Stability and uniform stability)**.** An equilibrium u^* of (5.1) is called **stable** for the equation (5.1) in X^α if for any $\varepsilon>0$, $t_0\in\mathbb{R}$, there exists $\delta=\delta(t_0)>0$ such that any solution u with $\|u(t_0)-u^*\|_{X^\alpha}<\delta$ is defined for $t\in[t_0,\infty)$ and satisfies $\|u(t)-u^*\|_{X^\alpha}<\varepsilon$ for all $t\geq t_0$.

An equilibrium u^* of (5.1) is called **uniformly stable** for equation (5.1) in X^α if for any $\varepsilon>0$, there exists $\delta>0$ for which if $\|\tilde{u}-u_0\|<\delta$, then for any $t_0\in\mathbb{R}$, $t\geq t_0$, the solution $u(\cdot,t_0,\tilde{u})$ such that $u(t_0,t_0,\tilde{u})=\tilde{u}$, $\|\tilde{u}-u^*\|<\delta$ is defined in $[t_0,\infty)$ and satisfies $\|u(t,t_0,\tilde{u})-u^*\|<\varepsilon$ for all $t\in[t_0,\infty)$.

The equilibrium u^* is **uniformly asymptotically stable** if it is uniformly stable and $\|u(t;t_0,\tilde{u})-u^*\|_{X^\alpha}\to 0$ as $t-t_0\to\infty$, uniformly for $t_0\in\mathbb{R}$ and $\|\tilde{u}-u^*\|_{X^\alpha}<\delta$, for some $\delta>0$.

We say that an equilibrium u^* is **unstable** if it is not stable.

Theorem 33 (Stability by linear approximation)**.** Let A and f be as in Theorem 32 and u^* an equilibrium for (5.1). Suppose that

$$f(t,u^*+z) = f(t,u^*) + Bz + g(t,z)$$

where $B\in\mathscr{L}(X^\alpha,X)$ and $\|g(t,z)\| = o(\|z\|_{X^\alpha})$ as $\|z\|_{X^\alpha}\to 0$, uniformly for $t\in\mathbb{R}$ and $f: U\to X$ is locally Hölder continuous in the first variable, and locally Lipschitz in the second.

If the spectrum of $L := A-B$ is a subset of $\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda>\beta\}$ for some $\beta>0$, then the equilibrium u^* is uniformly asymptotically stable in X^α. More than that, there exist $\rho>0$, $M\geq 1$ such that if $t_0\in\mathbb{R}$ and $\|u_0-u^*\|_{X^\alpha}\leq\frac{\rho}{2M}$ then there is a unique solution of

$$\frac{du}{dt} + Au = f(t,u), \quad t>t_0,\ u(t_0)=u_0, \tag{5.6}$$

defined in $[t_0,\infty)$ that satisfies

$$\|u(t;t_0,u_0)-u^*\|_{X^\alpha}\leq 2Me^{-\beta(t-t_0)}\|u_0-u^*\|_{X^\alpha}, \quad t\geq t_0. \tag{5.7}$$

Proof. From Corollary 4, we have that $L = A - B$ sectorial with vertex $a \in \mathbb{R}$, and since $\sigma(L) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \beta\}$, it is easy to see that L is sectorial with vertex 0 and $L \in \mathscr{P}$ is of positive kind.

Choose $0 < \beta < \beta' < \mathrm{Re}\sigma(L)$. Then Theorem 24 applies with $P = 0$ and $\delta_2 = \beta'$, and using also Corollary 2 and Corollary 3, we get:

$$\|e^{-Lt}z\|_{X^\alpha} = \|A^\alpha e^{-Lt}z\| \le C\|L^\alpha e^{-Lt}z\| \le Mt^{-\alpha}e^{-\beta' t}\|z\|,$$

$$\|e^{-Lt}z\|_{X^\alpha} = \|A^\alpha e^{-Lt}z\| = \|e^{-Lt}A^\alpha z\| \le Me^{-\beta' t}\|z\|_{X^\alpha}.$$

Now let $\sigma > 0$ be small enough so that

$$M\sigma \int_0^\infty s^{-\alpha}e^{-(\beta'-\beta)s}ds < 1/2, \tag{5.8}$$

and since $\frac{\|g(t,z)\|}{\|z\|_{X^\alpha}} \overset{\|z\|_{X^\alpha}\to 0}{\longrightarrow} 0$, uniformly in $t \in \mathbb{R}$, we choose $\rho > 0$ small enough so that

$$\|g(t,z)\| \le \sigma\|z\|_{X^\alpha} \text{ for } \|z\|_{X^\alpha} \le \rho \text{ and } t \in \mathbb{R}.$$

Let $z(t) = u(t;t_0,u_0) - u^*$, where $u(\cdot,t_0,u_0)$ is the (5.6), and $\|u_0 - u^*\|_{X^\alpha} \le \rho/2M$. By Theorem 32, z is defined in some maximal interval $[t_0,t_f)$, and $\|z(t)\|_{X^\alpha} \le \rho$ in some interval $[t_0,\tilde{t})$. It is easy to see that z is solution for the following problem:

$$\begin{aligned} &\frac{d}{dt}z = -Lz + g(t,z), \quad t \ge t_0 \\ &z(t_0) = u_0 - u^* \in X^\alpha. \end{aligned}$$

Suppose that $\|z(s)\|_{X^\alpha} < \rho$ for $t_0 < s < \tilde{t}$, then, by the Formula of Variation of Constants,

$$\begin{aligned} \|z(t)\|_{X^\alpha} &= \|e^{-L(t-t_0)}z(t_0) + \int_{t_0}^t e^{-L(t-s)}g(s,z(s))ds\|_{X^\alpha} \\ &\le Me^{-\beta'(t-t_0)}\|z(t_0)\|_{X^\alpha} + \sigma M\int_{t_0}^t (t-s)^{-\alpha}e^{-\beta'(t-s)}\|z(s)\|_{X^\alpha}ds \\ &\le \rho/2 + \rho\sigma M\int_{-\infty}^t (t-s)^{-\alpha}e^{-\beta'(t-s)}ds < \rho, \end{aligned} \tag{5.9}$$

where we used (5.8) in the last estimate. Then, if $\|z(s)\|_{X^\alpha} < \rho$ in $t_0 \le s < t_1$ with t_1 maximal with this property, we claim that $t_1 = \infty$. Suppose not, then either $t_1 = t_f$ or $t_1 < t_f$; the first case leads to a contradiction since the boundedness of z near t_f allows an extension of the interval $[t_0,t_f)$ of definition of z, contradicting the fact that z is defined in a maximal domain; in the second case, we would have $\|z(t_1)\|_{X^\alpha} = \rho$ because of the maximality, contradicting the estimate (5.9). It follows that $t_1 = \infty$.

Now we only need to prove (5.7). If $u(t) = \sup\{\|z(s)\|_{X^\alpha} e^{\beta(s-t_0)}, t_0 \le s \le t\}$, then

$$\begin{aligned} \|z(\tilde{t})\|_{X^\alpha} e^{\beta(\tilde{t}-t_0)} &\le M\|z(t_0)\|_{X^\alpha} + \sigma M \int_{t_0}^{\tilde{t}} (\tilde{t}-s)^{-\alpha} e^{-(\beta'-\beta)(\tilde{t}-s)} ds\, u(t) \\ &\le M\|z(t_0)\|_{X^\alpha} + \frac{1}{2} u(t), \quad \forall 0 \le \tilde{t} \le t \end{aligned}$$

where we used (5.8). Therefore, $u(t) \le M\|z(t_0)\|_{X^\alpha} + \frac{1}{2}u(t)$, and $u(t) \le 2M\|z(t_0)\|_{X^\alpha}$, which yields:

$$\|u(t;t_0,u_0) - u^*\|_{X^\alpha} \le 2Me^{-\beta(t-t_0)}\|u_0 - u^*\|_{X^\alpha}, \quad t \ge t_0.$$

□

The following lemma will be important to identify unstable equilibria, by stating, under certain conditions, the existence of a global solution which converges to the equilibrium as $t \to -\infty$.

Lemma 18. Let A and f be as above and u^* be an equilibrium for (5.6). Suppose that

$$f(t,u^*+z) = f(t,u^*) + Bz + g(t,z)$$

where $B \in \mathscr{L}(X^\alpha, X)$ and $\|g(t,z_1) - g(t,z_2)\| \le k(\rho)\|z_1 - z_2\|_{X^\alpha}$ for $\|z_1\|_{X^\alpha} \le \rho$ and $\|z_2\|_{X^\alpha} \le \rho$, and $k(\rho) \stackrel{\rho \to 0^+}{\longrightarrow} 0$.

Suppose that the spectrum of $L = A - B$ is disjoint of $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = 0\}$, and define the spectral sets $\sigma_1 = \sigma(L) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$ and $\sigma_2 = \sigma(L) \setminus \sigma_1$, and suppose that $\sigma_1 \neq \emptyset$. Let P be the projection from Theorem (24), associated to the part of the spectrum of L to the left of the imaginary axis, that is:

$$P = \frac{1}{2\pi i} \int_{\mathscr{C}} (\lambda - L)^{-1} d\lambda,$$

where $\mathscr{C}$ is a smooth curve wrapping σ_1, contained in $\mathrm{Re}\lambda < 0$, in the counterclockwise direction. Define the phase space decomposition $X = X_1 \oplus X_2$, $X_1 = R(P)$, $X_2 = N(P)$. Then $\sigma_i = \sigma(L_i)$, where L_i is the restriction of L to X_i, $i = 1,2$.

For $a \in X_1$, $\tau \in \mathbb{R}$, consider the following integral equation:

$$\begin{aligned} y(t) = e^{-L_1(t-\tau)}a + \int_\tau^t e^{-L_1(t-s)} P(g(s,y(s)))ds \\ + \int_{-\infty}^t e^{-L_2(t-s)}(I-P)(g(s,y(s)))ds, \quad t \le \tau. \end{aligned} \tag{5.10}$$

Then, there exists $\rho > 0$ such that if $\|a\|_{X^\alpha} \le \rho/2M$, (5.10) has a unique solution $y(t)$ over $-\infty < t \le \tau$, with $\|y(t)\|_{X^\alpha} \le \rho e^{2\beta(t-\tau)}$, and this solution to the integral equation is also a solution for the differential equation:

$$\dot{z} + Lz = g(t,z).$$

Proof. Due to the exponential dichotomy of the sectorial operator L (Theorem 24), we can choose $\beta > 0$, and $M \geq 0$ such that:

$$\|e^{-L_2 t}(I-P)u\|_{X^\alpha} \leq Me^{-\beta t}\|u\|_{X^\alpha}, \quad \|e^{-L_2 t}(I-P)u\|_{X^\alpha} \leq Mt^{-\alpha}e^{-\beta t}\|u\|, \quad t > 0$$

$$\|e^{-L_1 t}Pu\|_{X^\alpha} \leq Me^{3\beta t}\|u\|_{X^\alpha}, \quad \|e^{-L_1 t}Pu\|_{X^\alpha} \leq Me^{3\beta t}\|u\| \quad t \leq 0.$$

Where in the last estimate we used the fact that $(L_1)^\alpha \in \mathscr{L}(X_1)$. If $\mathscr{G}$ if the set of functions with domain $(-\infty, \tau]$ taking values in X_1^α and satisfying $\sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha} < \rho$, where ρ is small enough so that the following condition is satisfied:

$$Mk(\rho)(\|P\|\beta^{-1} + \|(I-P)\| \int_0^\infty u^{-\alpha}e^{-\beta u}du) \leq \frac{1}{4M} < \frac{1}{2}. \tag{5.11}$$

In $\mathscr{G}$, we define the metric:

$$d(y,\tilde{y}) = \sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s) - \tilde{y}(s)\|_{X^\alpha},$$

which makes it a complete metric space. Finally, define in $\mathscr{G}$ the following operator:

$$\begin{aligned}(Ty)(t) = e^{-L_1(t-\tau)}a + \int_\tau^t e^{-L_1(t-s)}P(g(s,y(s))ds \\ + \int_{-\infty}^t e^{-L_2(t-s)}(I-P)(g(s,y(s))ds, \quad t \leq \tau.\end{aligned} \tag{5.12}$$

We only need to show that the range of T is in $\mathscr{G}$ and $d(Ty, T\tilde{y}) \leq \frac{1}{2}d(y,\tilde{y})$ for all $y, \tilde{y} \in \mathscr{G}$, and the result will follow from Banach's Fixed Point Theorem. Indeed:

$$\begin{aligned}
&e^{-2\beta(t-\tau)}\|(Ty)(t)\|_{X^\alpha} \\
&\leq M\|a\|_{X^\alpha} + \int_t^\tau Me^{3\beta(t-s)}e^{-2\beta(t-s)}\|P\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha}ds \\
&+ \int_{-\infty}^t M(t-s)^{-\alpha}e^{-\beta(t-s)}\|(I-P)\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha}ds \\
&\leq M\|a\|_{X^\alpha} + \int_t^\tau Me^{\beta(t-s)}\|P\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha}ds \\
&+ \int_{-\infty}^t M(t-s)^{-\alpha}e^{-\beta(t-s)}\|(I-P)\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha}ds \\
&\leq M\|a\|_{X^\alpha} + Mk(\rho)[\|P\|\int_t^\tau e^{\beta(t-s)}ds \\
&+ \|(I-P)\|\int_{-\infty}^t (t-s)^{-\alpha}e^{-\beta(t-s)}ds] \sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha} \\
&\leq M\|a\|_{X^\alpha} + Mk(\rho)[\|P\|\beta^{-1} \\
&+ \|(I-P)\|\int_0^\infty s^{-\alpha}e^{-\beta s}ds] \sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha} \\
&\leq \rho/2 + \rho/2 = \rho,
\end{aligned} \tag{5.13}$$

So that T is well-defined and its range is in $\mathscr{G}$. To show that T is a uniform contraction in $\mathscr{G}$, we proceed as following:

$$\begin{aligned}
& e^{-2\beta(t-\tau)}\|(Ty)(t)-(T\tilde{y})(t)\|_{X^\alpha} \\
&\qquad \leq \int_t^\tau Me^{3\beta(t-s)}e^{-2\beta(t-s)}\|P\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha}ds \\
&\qquad\quad + \int_{-\infty}^t M(t-s)^{-\alpha}e^{-\beta(t-s)}\|(I-P)\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha}ds \\
&\qquad \leq \int_t^\tau Me^{\beta(t-s)}\|P\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha}ds \\
&\qquad\quad + \int_{-\infty}^t M(t-s)^{-\alpha}e^{-\beta(t-s)}\|(I-P)\|k(\rho)e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha}ds \\
&\qquad \leq Mk(\rho)[\|P\|\int_t^\tau e^{\beta(t-s)}ds \\
&\qquad\quad + \|(I-P)\|\int_{-\infty}^t (t-s)^{-\alpha}e^{-\beta(t-s)}ds]\sup_{t\leq\tau}e^{-2\beta(t-\tau)}\|y(t)-\tilde{y}(t)\|_{X^\alpha} \\
&\qquad \leq Mk(\rho)[\|P\|\beta^{-1} \\
&\qquad\quad + \|(I-P)\|\int_0^\infty s^{-\alpha}e^{-\beta s}ds]\sup_{s\leq\tau}e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha} \\
&\qquad \leq \frac{1}{2}\sup_{s\leq\tau}e^{-2\beta(s-\tau)}\|y(s)-\tilde{y}(s)\|_{X^\alpha}.
\end{aligned}$$

Therefore, there exists a unique fixed point $y\in\mathscr{G}$ of T, which is a solution of (5.10) satisfying the exponential estimate $\|y(t)\|_{X^\alpha}\leq\rho e^{2\beta(t-\tau)}$. Now we need to show that y is a solution of $\dot{z}+Lz=g(t,z)$. Define $\eta(s)=g(s,y(s))$ and $t_0\leq\tau$, and for $t_0\leq t\leq\tau$, we have:

$$\begin{aligned}
(I-P)y(t) &= \int_{-\infty}^t e^{-L_2(t-s)}(I-P)\eta(s)ds \\
&= e^{-L_2(t-t_0)}\int_{-\infty}^{t_0} e^{-L_2(t_0-s)}(I-P)\eta(s)ds \\
&+ \int_{t_0}^t e^{-L_2(t-s)}(I-P)\eta(s)ds \\
&= e^{-L_2(t-t_0)}(I-P)y(t_0)+\int_{t_0}^t e^{-L_2(t-s)}(I-P)\eta(s)ds
\end{aligned}$$

and

$$\begin{aligned}
Py(t) &= e^{-L_1(t-\tau)}a+\int_\tau^t e^{-L_1(t-s)}P\eta(s)ds \\
&= e^{-L_1(t-t_0)}e^{-L_1(t_0-\tau)}a \\
&+ e^{-L_1(t-t_0)}\int_\tau^{t_0} e^{-L_1(t_0-s)}P\eta(s)ds+\int_{t_0}^t e^{-L_1(t-s)}P\eta(s)ds.
\end{aligned}$$

Then, it is simple to see that $y=Py+(I-P)y$ satisfies the Formula of Variation of Constants, so that it is a solution of

$$\frac{dy}{dt}+Ly=\eta(t),\quad t_0<t<\tau,$$

and this concludes the proof. □

Theorem 34 (Instability by linear approximation)**.** Let A, f, B, g, L and u^* be as in Lemma 18. Then u^* is unstable. More precisely, there exists $t_0 \in \mathbb{R}$, $\varepsilon_0 > 0$ and $\{u_n : n \geq 1\}$ with $\|u_n - u^*\|_{X^\alpha} \to 0$ as $n \to 0$ such that, for all $n \in \mathbb{N}$,

$$\sup_t \|u(t;t_0,u_n) - u^*\|_{X^\alpha} \geq \varepsilon_0 > 0,$$

where the supremum is taken over the maximal existence interval of $u(\cdot,t_0,u_n)$.

Proof. Choose $\rho > 0$ as in Lemma 18, and $\|a\|_{X^\alpha} \leq \rho/2M$, $\tau \in \mathbb{R}$, then there exists an unique solution $y : (\infty, \tau] \to X$ of the integral equation (5.10), with $\|y(t)\|_{X^\alpha} \leq \rho e^{2\beta(t-\tau)}$. If we denote this solution by $y(t) = y^*(t;\tau,a)$, using an estimate in (5.13) and the fact that y is a fixed point for T, we get:

$$\sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha} \leq M\|a\|_{X^\alpha} + \frac{1}{2}\sup_{s\leq\tau} e^{-2\beta(s-\tau)}\|y(s)\|_{X^\alpha},$$

so that

$$\|y^*(t;\tau,a)\|_{X^\alpha} \leq 2M\|a\|_{X^\alpha} e^{2\beta(t-\tau)}, \quad t \leq \tau.$$

We will show that $\|y^*(\tau,\tau,a)\| \geq 1/2\|a\|_{X^\alpha}$. Indeed,

$$\begin{aligned}
\|y^*(\tau;\tau,a) - a\|_{X^\alpha} &\leq \left\| \int_{-\infty}^{\tau} e^{-L_2(\tau-s)}(I-P)g(s,y^*(s,\tau,a))ds \right\|_{X^\alpha} \\
&\leq \|(I-P)\|k(\rho)2M\|a\|_{X^\alpha} \int_{-\infty}^{\tau} M(\tau-s)^{-\alpha}e^{-\beta(\tau-s)}ds \\
&\leq \frac{1}{2}\|a\|_{X^\alpha}
\end{aligned}$$

e assim, $\|y^*(\tau;\tau,a)\|_{X^\alpha} \geq 1/2\|a\|_{X^\alpha}$.

Recall that $y^*(.;\tau,a)$ is a solution of

$$\frac{dz}{dt} + Lz = g(t,z), \quad t < \tau.$$

Now, let $t_0 \in \mathbb{R}$, if $z_n = y^*(t_0;t_0+n,a)$, the solution $z(\cdot,t_0,z_n)$ of the problem $\frac{dz}{dt} + Lz = g(t,z)$, $z(t_0,t_0,z_n) = z_n$ satisfies, by uniqueness, $z(t,t_0,z_n) = y^*(t,t_0+n,a)$ for $t_0 \leq t \leq t_0+n$, and

$$\begin{aligned}
\sup\{\|z(t;t_0,z_n)\|_{X^\alpha}, t \geq t_0\} &\geq \|z(t_0+n;t_0,z_n)\|_{X^\alpha} \\
&= \|y^*(t_0+n;t_0+n,a)\|_{X^\alpha} \geq 1/2\|a\|_{X^\alpha}.
\end{aligned}$$

Moreover, $\|z_n\|_{X^\alpha} \leq \rho e^{-2\beta n} \to 0$, as $n \to \infty$. □

5.3 Saddle Point Property

In this section we present a very important theorem about the behavior of the local stable and unstable sets of an equilibrium point.

Theorem 35 (Saddle Point Property)**.** Suppose that A, f, u^* are like in Theorem 33, with

$$f(t,u^*+z) = f(t,u^*) + Bz + g(t,z),$$

$B \in L(X^\alpha, X)$, $g(t,0)=0$ and $\|g(t,z_1)-g(t,z_2)\| \leq k(\rho)\|z_1-z_2\|_{X^\alpha}$ for $\|z_i\|_{X^\alpha} \leq \rho$, $i=1,2$, with $k(\rho) \overset{\rho\to 0^+}{\longrightarrow} 0$, and assume (without loss of generality), that $k(\cdot)$ is non-decreasing. Let $L = A - B$, and suppose that $\sigma(L)$ is disjoint from the imaginary axis. We define the spectral sets $\sigma_1 = \sigma(L) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$ and $\sigma_2 = \sigma(L) \setminus \sigma_1$. Let P be the spectral projection associated to σ_1, that is:

$$P = \frac{1}{2\pi i}\int_{\mathscr{C}} (\lambda - L)^{-1} d\lambda,$$

where $\mathscr{C}$ is a smooth curve wrapping σ_1, contained in $\mathrm{Re}\lambda < 0$, in the counterclockwise direction. Define the phase space decomposition $X = X_1 \oplus X_2$, $X_1 = R(P)$, $X_2 = N(P)$. Then $\sigma_i = \sigma(L_i)$, where L_i is the restriction of L to X_i, $i = 1,2$.Then there exist constants $\rho > 0$, $M \geq 1$ such that the following holds:

1. The local stable set of the semilinear problem $S(t_0,\rho)$, defined by

   $$S = \{z_0 : \|(I-P)z_0\|_{X^\alpha} \leq \rho/2M,\ \|z(t,t_0,z_0)\|_{X^\alpha} \leq \rho\ for\ t \geq t_0\}$$

   is homeomorphic under the homeomorphism $(I-P)\big|_S$ to the closed ball of radius $\rho/2M$ in X_2^α. Moreover, S is tangent to X_2^α in 0 and when $z_0 \in S$,

   $$\|z(t,t_0,z_0)\|_{X^\alpha} \to 0 \quad \text{as} \quad t\to\infty \quad \text{(exponentially)}$$

2. The unstable set of the semilinear problem $U = U(t_0,\rho)$, given by $U = \{z_0 : \|Pz_0\|_{X^\alpha} \leq \rho/2M$, $z(t,t_0,z_0)$ is a solution defined in $(-\infty,t_0)$, $\|z(t,t_0,z_0)\|_{X^\alpha} \leq \rho,\ t \leq t_0\}$ is homeomorphic under the homeomorphism $P\big|_U$ to the closed ball of radius $\rho/2M$ in X_1. Moreover, U is tangent to X_1 in 0 and when $z_0 \in U$, $z(t,t_0,z_0) \to 0$ as $t \to -\infty$ exponentially.

Proof. As before, using Theorem 24, let $M > 0$ and $\beta > 0$ be such that

$$\|A^\alpha e^{-L_1 t}\| \leq Me^{\beta t}, \quad \|e^{-L_1 t}\| \leq Me^{\beta t}, \quad \text{for } t \leq 0,$$
$$\|A^\alpha e^{-L_2 t}(I-P)A^{-\alpha}\| \leq Me^{-\beta t}, \quad \|A^\alpha e^{-L_2 t}\| \leq Mt^{-\alpha}e^{-\beta t}, \quad \text{for } t > 0.$$

Suppose that $z_0 \in S$, then $z(t,t_0,z_0) = z(t) = z_1(t) + z_2(t) \in X_1 \oplus X_2$

$$z_1(t) = e^{-L_1(t-t_0)}Pz_0 + \int_{t_0}^{t} e^{-L_1(t-s)}Pg(s,z(s))ds,$$

so that

$$e^{L_1 t} z_1(t) = e^{L_1 t_0} P z_0 + \int_{t_0}^{t} e^{L_1 s} P g(s, z(s)) ds \stackrel{t\to\infty}{\longrightarrow} 0.$$

We conclude that $Pz_0 = -\int_{t_0}^{\infty} e^{-L_1(t_0-s)} g(s,z(s))ds$, which means that for $t \geq t_0$,

$$z(t) = e^{-L_2(t-t_0)} a + \int_{t_0}^{t} e^{-L_2(t-s)} (I-P) g(s,z(s))ds - \int_{t}^{\infty} e^{-L_1(t-s)} P g(s,z(s))ds, \qquad (5.14)$$

where $a = (I-P)z(t_0)$.

Reciprocally, if $a \in X_2$, $\|a\|_{X^\alpha} \leq \rho/2M$, we will show that (for $\rho > 0$ small enough) there exists a unique solution $z(t) = z(t,t_0,a)$ of the integral equation (5.14) with $(I-P)z_0 = (I-P)z(t_0,t_0,a) = a$ and $\|z(t,t_0,a)\|_{X^\alpha} \leq \rho$ for all $t \geq t_0$.

In fact, if $\rho > 0$ is chosen in such a way that

$$MK(\rho)\{\|(I-P)\| \int_0^\infty u^{-\alpha} e^{-\beta u} du + \|P\| \int_0^\infty e^{-\beta u} du\} < 1/2,$$

then the right hand side of equation (5.14) defines an uniform contraction in the space of continuous functions $z : [t_0, \infty) \to X$ with $\sup \|z(t)\|_{X^\alpha} \leq \rho$ and $(I-P)z(t_0) = a$, provided that $\|a\|_{X^\alpha} \leq \rho/2M$ so that there is a solution of the integral equation $z(t,t_0,z)$ in this space. This solution is a Lipschitz function of $a \in X_2^\alpha$, $\|a\|_{X^\alpha} \leq \rho/2M$, in the norm $\|\cdot\|_{X^\alpha}$, and we can show that $t \to z(t,t_0,a)$ is locally Hölder continuous, and it follows from Theorem 31 that $z(t,t_0,a)$ is solution of $\frac{dz}{dt} + Lz = g(t,z)$, $t > 0$, with initial value

$$h(a) := z(t_0,t_0,a) = a - \int_{t_0}^{\infty} e^{-L_1(t_0-s)} P g(s, z(s,t_0,a))ds.$$

Then $(I-P)h(a) = a$, e $h(\cdot)$ is Lipschitz continuous, so that

$$S = \{h(a) : a \in X_2^\alpha,\ \|a\|_{X^\alpha} \leq \rho/2M\}$$

is the representation of the local stable set. Note that $(I-P)$ is injective in S because if $(I-P)b = 0$, b is the initial value of the unique solution of the integral equation with $a = 0$. Since the function $\phi \equiv 0$ is a solution of this integral equation we have $b = 0$.

Now note that

$$\|h(a) - a\|_{X^\alpha} \leq \int_{t_0}^{\infty} M e^{\beta(t_0-s)} \|P\| \, \|g(s, z(s,t_0,a))\| ds$$

and $\sup_{s \geq t_0} \|z(s,t_0,a)\| = O(\|a\|_{X^\alpha})$ as $\|a\|_{X^\alpha} \to 0$, $a \in X_2$, then $\|h(a) - a\|_{X^\alpha} = o(\|a\|_{X^\alpha}$, which proves that S is tangent to X_2^α at 0. Now we prove the exponential convergence. Let $z(t_0) \in S$; we can estimate in the Formula of Variation of Constants using that $z(t) = h(z_2(t))$ and h is Lipschitz with constant J, to obtain:

$$e^{\beta(t-t_0)} \|z_2(t)\|_{X^\alpha} \leq M\|z_0\|_{X^\alpha} + \int_{t_0}^{t} M\|I-P\| k(\rho)(t-s)^{-\alpha} e^{\beta(s-t_0)} J \|z_2(s)\|_{X^\alpha} ds.$$

Using Grönwall's Lemma in the expression above, we conclude that there exists some constant $K \geq 0$ such that:

$$\|z(t)\|_{X^\alpha} \leq Ke^{-\beta(t-t_0)}\|z_0\|_\alpha.$$

The proof for the unstable set is similar, using an integral equation as that one in the proof of Theorem 34. For the sake of conciseness, we omit the proof for this case. □

CHAPTER

6

SEMILINEAR CHAFEE-INFANTE EQUATION

6.1 Well-posedness and gradient structure

In this chapter we will study the classical Chafee-Infante equation (CHAFEE; INFANTE, 1974), given by:

$$\begin{aligned} &u_t = u_{xx} + \lambda f(u), \quad t > 0,\ x \in (0,\pi) \\ &u(0,t) = u(\pi,t) = 0, \quad t \geq 0 \\ &u(\cdot,0) = u_0 \in H_0^1(0,\pi), \end{aligned} \tag{6.1}$$

where $\lambda > 0$ is a parameter, $f \in \mathscr{C}^2(\mathbb{R})$ is odd (in particular, $f(0) = 0$), $f'(0) = 1$, and f satisfies:

$$f''(u)u < 0, \forall\, u \neq 0 \tag{6.2}$$

and

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0. \tag{6.3}$$

The dynamical system associated to this equation possesses a gradient attractor whose structure can be very well understood. In particular, we know how many equilibria it possesses for each value of λ, whether these equilibria are stable or unstable, hyperbolic or not, and how they connect to each other through global solutions.

We define $f^e : H_0^1(0,\pi) \to L^2(0,\pi)$, $f^e(u)(x) = f(u(x))$. Now we will apply the semilinear equations theory using $X_0 = L^2(0,\pi)$, $A = -\Delta$ with $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$, $\alpha = 1/2$. Recall that $X^{\frac{1}{2}} = H_0^1(0,\pi)$ (see Example 4), and this will be our phase space. We only need to show that $f^e : H_0^1(0,\pi) \to L^2(0,\pi)$ is Lipschitz in bounded sets of $H_0^1(0,\pi)$ to ensure local existence of solutions by Theorem 32.

Indeed, let $\|\cdot\|$ denote the $L^2(0,\pi)$ norm and suppose u, $v \in H_0^1(0,\pi)$, $\|u\|_{H_0^1} \le \rho$, $\|v\|_{H_0^1} \le \rho$. Then:

$$\begin{aligned} \|f^e(u) - f^e(v)\|^2 &= \int_0^\pi |f(u(x)) - f(v(x))|^2 dx \\ &= \int_0^\pi [f'(\theta(x)u(x) + (1-\theta(x))v(x))]^2 (u(x) - v(x))^2 dx \\ &\le C(\rho)\|u - v\|^2 \le C(\rho)\|u - v\|_{H_0^1}^2, \end{aligned}$$

where we used that f is continuously differentiable and for all $x \in [0,\pi]$:

$$|\theta(x)u(x) + (1-\theta(x))v(x)| \le |u(x)| + |v(x)| \le \sup_{x\in[0,\pi]} |u(x)| + \sup_{x\in[0,\pi]} |v(x)| \le 2\pi^{\frac{1}{2}}\rho.$$

Therefore, if u, $v \in H_0^1(0,\pi)$, $\|u\|_{H_0^1} \le \rho$, $\|v\|_{H_0^1} \le \rho$, there exists a constant $k(\rho) > 0$ such that:

$$\|f^e(u) - f^e(v)\| \le k(\rho)\|u - v\|_{H_0^1}, \tag{6.4}$$

so that Theorem 32 applies and given $u_0 \in H_0^1(0,\pi)$, there exist a local solution $u : [0,t_1) \to H_0^1(0,\pi)$ such that $u(0) = u_0$.

From now on, we may use the same notation for f and f^e, since the meaning will be given by the context.

Now consider the function $V : H_0^1(0,\pi) \to \mathbb{R}$ given by

$$V(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \int_0^\pi \lambda F(u(x))dx, \quad \text{where} \quad F(s) := \int_0^s f(\xi)d\xi.$$

This function is Lipschitz in bounded sets of $H_0^1(0,\pi)$, hence continuous. Moreover, V is differentiable. Now, let $u : [0,\pi] \times [0,t_1) \to \mathbb{R}$ be a solution of (6.1), then the composition $V \circ u$ is differentiable. Multiplying the differential equation by u_t and integrating yields:

$$\begin{aligned} &\int_0^\pi u_t(x,t)^2 dx = \int_0^\pi u_{xx}(x,t)u_t(x,t)dx + \lambda \int_0^\pi f(u(x,t))u_t(x,t)dx \Rightarrow \\ &\|u_t(\cdot,t)\|^2 = -\frac{1}{2}\frac{d}{dt}\|u_x(\cdot,t)\|^2 + \frac{d}{dt}\lambda \int_0^\pi F(u(x))dx = -\frac{d}{dt}V(u(\cdot,t)), \end{aligned}$$

that is, V is non-increasing along solutions and is only constant in $t \ge 0$ if $u_t = 0$, for $t \ge 0$, that is, if $u(x,t) \equiv u_0(x)$, where $u_0 \in H_0^1(0,\pi)$ is an equilibrium of (6.1).

This proves that V is a Lyapunov function for (6.1). Now, from condition (6.3), we can prove that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $f(s)s \le \varepsilon s^2 + C_\varepsilon$, for all $s \in \mathbb{R}$, and from this we conclude that there exists a constant $K_\varepsilon > 0$ such that $F(s) \le \varepsilon s^2 + K_\varepsilon$, for all $s \in \mathbb{R}$. Hence, if $u \in H_0^1(0,\pi)$,

$$\begin{aligned} &V(u) \ge \frac{1}{2}\|u\|_{H_0^1}^2 - \lambda\varepsilon\|u\|^2 - \lambda\pi K_\varepsilon \ge \left(\frac{1}{2} - \lambda\varepsilon\right)\|u\|_{H_0^1}^2 - \lambda\pi K_\varepsilon \Rightarrow \\ &V(u) \ge \frac{1}{4}\|u\|_{H_0^1}^2 - \lambda\pi K_{\frac{1}{4\lambda}}, \end{aligned}$$

where we used an appropriate ε and the property $\|u\| \leq \|u\|_{H_0^1}$ (see Proposition 14).

Then, for any solution $u : [0,t_1) \to H_0^1(0,\pi)$ with $u(0) = u_0 \in H_0^1$:

$$\|u(t)\|_{H_0^1}^2 \leq 4\lambda\pi K_{\frac{1}{4\lambda}} + 4V(u(t)) \leq 4\lambda\pi K_{\frac{1}{4\lambda}} + 4V(u_0). \tag{6.5}$$

This proves that the positive orbit of a solution is bounded in $H_0^1(0,\pi)$, and using the last claim in Theorem 32, we conclude that for any $u_0 \in H_0^1$, we have a maximal solution $u(\cdot,u_0) : [0,\infty) \to H_0^1(0,\pi)$ for which $u(0,u_0) = u_0$. Define the continuous mapping $T(t) : H_0^1(0,\pi) \to H_0^1(0,\pi)$ by $T(t)u_0 = u(t,u_0)$, for $t \geq 0$. It follows from Remark 11 that $\mathbb{R}^+ \times H_0^1(0,\pi) \ni (t,u_0) \mapsto T(t)u_0 \in H_0^1(0,\pi)$ is continuous, and $\mathscr{T} = \{T(t) : t \geq 0\}$ is a semigroup. With the Lyapunov function V, $\mathscr{T}$ is a gradient semigroup in $H_0^1(0,\pi)$.

6.2 Existence of the attractor and equilibria

In this section we prove that $\mathscr{T}$ possesses a global attractor of gradient kind, and show that the equation (6.1) has a finite number of equilibria, and this number depends on the value of $\lambda > 0$. For $\lambda \leq 1$, we show that the origin of $H_0^1(0,\pi)$ is the only equilibrium, and when λ becomes strictly larger than 1, two new equilibria bifurcate from the origin. New bifurcations from the origin occur whenever $\lambda = n^2$ for $n = 1,2,3,...$, so that equation (6.1) has precisely $2n+1$ equilibria if $n^2 < \lambda \leq (n+1)^2$, for $n = 0,1,2,3,...$.

In order to prove existence of the attractor, we start showing that $\mathscr{T}$ is bounded. Indeed, if B is bounded in $H_0^1(0,\pi)$, and $\sup_{u\in B}\|u\|_{H_0^1} \leq R$, by the estimate (6.5) and the fact that $V(B)$ is bounded for B bounded, we conclude that $\|T(s)u_0\|_{H_0^1} \leq C$, for all $s \in [0,\infty)$, $u_0 \in B$, where C only depends on R. Whence $\gamma^+(B)$ is bounded.

Now we show that $\mathscr{T}$ is eventually compact, hence asymptotically compact (see Theorem 18). More precisely, we show that $T(1) : H_0^1(0,\pi) \to H_0^1(0,\pi)$ is compact. Indeed, let B be bounded in $H_0^1(0,\pi)$, and $\sup_{u\in B}\|u\|_{H_0^1} \leq R$. We already know from the last paragraph that $\|T(s)u_0\|_{H_0^1} \leq C$, for all $s \in [0,1]$, $u_0 \in B$, where C only depends on R.

For any $u_0 \in B$, $\frac{1}{2} < \beta < 1$, we have $T(1)u_0 \in D(A) \subset D(A^\beta)$, and using Theorem 29 and (6.4) in the Variation of Parameters Formula, we get:

$$\begin{aligned} \|A^\beta T(1)u_0\| &= \left\| A^{\beta-\frac{1}{2}} e^{-A} A^{\frac{1}{2}} u_0 + \int_0^1 A^\beta e^{-A(1-s)} f(T(s)u_0)ds \right\| \\ &\leq M_{\beta-\frac{1}{2}} \|u_0\|_{H_0^1} + \int_0^1 M_\beta (1-s)^{-\beta} k(C)Cds, \end{aligned}$$

where $k(C)$ is the Lipschitz constant of f in the bounded set $\{u \in H_0^1(0,\pi) : \|u\|_{H_0^1} \leq C\}$.

It follows that $T(1)B$ is bounded in X^β, and precompact in $H_0^1(0,\pi)$ (see Proposition 12). This proves that $T(1)$ is a compact operator and $\mathscr{T}$ is eventually compact.

Now we prove that $\mathscr{T}$ is point dissipative. Indeed, let $u_0 \in H_0^1(0,\pi)$, then $\gamma_0^+(\{u_0\}) = \{T(t)u_0 : t \geq 0\}$ is bounded in $H_0^1(0,\pi)$. Since $\mathscr{T}$ is asymptotically compact, it follows from Lemma 6 that $\omega(u_0)$ is nonempty, compact, invariant and attracts u_0. Moreover, from Lemma 9 we conclude that $\omega(u_0) \subset \mathscr{E}$, where $\mathscr{E}$ is the set of equilibria of (6.1). Hence, $\mathscr{E}$ attracts u_0 and we only need to show that $\mathscr{E}$ is bounded. Indeed, if $u \in \mathscr{E}$,

$$u_{xx} + \lambda f(u) = 0 \Rightarrow \int_0^\pi u(s)u_{xx}(s)ds + \lambda \int_0^\pi u(s)f(u(s))ds = 0 \Rightarrow$$

$$\|u\|_{H_0^1}^2 \leq \lambda\varepsilon\|u\|^2 + \pi\lambda C_\varepsilon \Rightarrow \frac{\|u\|_{H_0^1}^2}{2} \leq (\lambda\varepsilon - \frac{1}{2})\|u\|^2 + \pi\lambda C_\varepsilon,$$

where we used that $f(s)s \leq \varepsilon s^2 + C_\varepsilon$ for all $s \in \mathbb{R}$. In particular, for $\varepsilon = \frac{1}{2\lambda}$, we get $\|u\|_{H_0^1}^2 \leq \pi\lambda C_{\frac{1}{2\lambda}}$.

Since $\mathscr{T}$ is point dissipative, asymptotically compact and bounded, it follows from Theorem 19 that $\mathscr{T}$ has a global attractor $\mathscr{A}$.

Now we need to find the equilibria of (6.1), that is, we need to find $u \in H_0^1(0,\pi)$ such that:

$$\begin{aligned} &u_{xx}(x) + \lambda f(u(x)) = 0, \quad x \in [0,\pi] \\ &u(0) = u(\pi) = 0. \end{aligned} \tag{6.6}$$

If we use the hypothesis over f and f', we conclude that the function $\mathbb{R} \ni s \mapsto f(s)s$ has a local minimum at the point 0. Moreover, $f(u)u$ is negative for large u because of Condition (6.3). Therefore, there exists a unique $\bar{a}$, $0 < \bar{a} < \infty$, such that:

$$f(u)u > 0, \quad \text{for} - \bar{a} < u < \bar{a}, \tag{6.7}$$

with $\bar{a}$ maximal with this property, which implies that $f(\bar{a}) = 0$. From the concavity hypothesis, $f(s)$ is negative for $s > \bar{a}$, and positive for $s < -\bar{a}$.

As before, we define $F(u) := \int_0^u f(\xi)d\xi$, $u \in \mathbb{R}$. It is obvious that F is non-negative and even in $(-\bar{a},\bar{a})$ and $F(0) = 0$. Also, F is strictly increasing in $[0,\bar{a})$, F reaches a local maximum in $\bar{a}$ and then decreases until positive infinite. Analogously, F is strictly decreasing in $(-\bar{a},0]$, F reaches a local maximum in $-\bar{a}$ and is strictly increasing in $(-\infty,-\bar{a})$. Now, define:

$$E_l := \lim_{u\to\bar{a}} F(u) = \lim_{u\to-\bar{a}} F(u).$$

In the interval $[0,\bar{a})$, F has a continuous inverse (from Inverse Function Theorem), which we call $U_+ : [0,E_l) \to [0,\bar{a})$. In $(-\bar{a},0]$, F has a continuous inverse $U_- : [0,E_l) \to (-\bar{a},0]$. It is obvious that $U_-(E) = -U_+(E)$, because F is even.

Now, in order to find functions that satisfy (6.6), we look for solutions of the following ordinary differential equation:

$$\begin{aligned} u_x &= v \\ v_x &= -\lambda f(u), \end{aligned} \tag{6.8}$$

It follows from simple differentiation that for any (u,v) solution of (6.8), the value of $\frac{1}{2}u'(x)^2 + \lambda F(u(x))$ is constant for every $x \in \mathbb{R}$.

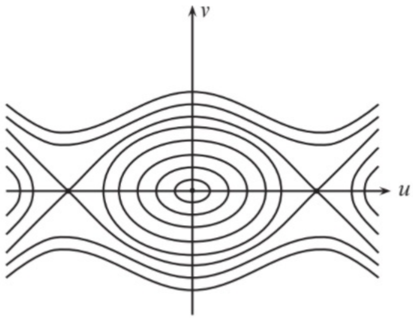

Figure 2 – Phase plane for (6.8) in the case $\lambda f(u) = \lambda u - bu^3$

For a fixed $\lambda > 0$, for any $v_0 \in \mathbb{R}$, consider the solution of (6.8) such that:

$$u(0) = 0, \quad u'(0) = v(0) = v_0.$$

If we take $E = \frac{1}{2}\lambda^{-1}v_0^2$, we get:

$$\lambda E = \frac{1}{2}u'(x)^2 + \lambda F(u(x)), \tag{6.9}$$

for all x in the domain of u.

We need u to satisfy the boundary condition in (6.6). It can only happen if:

$$0 < E < E_l, \tag{6.10}$$

because otherwise, E surpasses the maximum of F and, from the energy equation (6.9), u' cannot be zero for any x. That means u cannot become zero again to fulfill the boundary condition. So, we assume (6.10) holds. This means we are taking v_0 small enough so that the behavior of the solution (u,v) in the phase plane of (6.8) is oscillatory around the origin, and the condition $u(\pi) = 0$ may be fulfilled.

For this same reason, we need to assume that:

$$-(2\lambda E_l)^{\frac{1}{2}} < v_0 < (2\lambda E_l)^{\frac{1}{2}} \tag{6.11}$$

Indeed, if v_0 is negative, $u(x)$ is negative for small x, and in order to have $u(\pi) = 0$, we must ask E smaller than E_l, which means $v_0 = -\sqrt{2\lambda E} > -\sqrt{2\lambda E_l}$. The same reasoning applies for positive v_0.

It follows from a simple analysis of (6.8), considering (6.7), that if $v_0 \neq 0$, $u(0) = 0$, and $u'(0) = v_0$, we have a first $\alpha > 0$ such that $u(\alpha) = u_0$ and $v(\alpha) = 0$. Let us address the case $v_0 > 0$, $0 < E < E_l$. In this case, u increases and v decreases until the solution touches the u-axis, in the point $U_+(E)$ (see equation (6.9)). Through this trajectory, we have $u'(x) = \sqrt{2\lambda(E-F(u(x))}$, and from this we can calculate the first time $\Xi_+(E)$ such that $(u(\Xi_+(E)), v(\Xi_+(E))) = (U_+(E), 0)$:

$$\Xi_+(E) := \int_0^{U_+(E)} [2\lambda(E-F(u))]^{-\frac{1}{2}} du. \tag{6.12}$$

From symmetry of the problem, the first value of $x > 0$ such that $u(x) = 0$ is exactly $2\Xi+(E)$.

Analogously, if $v_0 < 0$ and $0 < E < E_l$, we can calculate the time $\Xi_-(E)$ such that $v(\Xi_-(E)) = 0$, which is:

$$\Xi_-(E) := \int_{U_-(E)}^{0} [2\lambda(E-F(u))]^{-\frac{1}{2}} du, \tag{6.13}$$

and $2\Xi_-(E)$ is the first positive value of x such that $u(x) = 0$.

Since $U_-(E) = -U_+(E)$, and F is even, it follows that $\Xi_+(E) = \Xi_-(E) \overset{\text{def}}{=} \Xi(E)$, for all $0 < E < E_l$. Then $\tau(E) = 2\Xi(E)$ is the time that a solution needs to perform a half translation around the origin, parting from $(0, v_0)$ and arriving at $(0, -v_0)$, $v_0 \in \mathbb{R}$.

We conclude that $u : [0, \pi] \to \mathbb{R}$ satisfies (6.6) if and only if u is the restriction to $[0, \pi]$ of the first coordinate of (u, v), where (u, v) satisfies (6.8) with $u(0) = 0$, $v(0) = v_0$, and defining $E = \frac{1}{2}\lambda^{-1}v_0^2$, we have $0 < E < E_l$, and the following equation holds:

$$k\tau(E) = \pi, \quad k = 1, 2, \ldots \tag{6.14}$$

Think about the phase plane of (6.8) with v as ordinate and u as abscissa. If $k\tau(E) = \pi$ with odd k, we have a solution (u, v) of (6.8) that starts at $(0, v_0)$, $v_0 > 0$ (or $v_0 < 0$), and circles in the clockwise direction going down (or up) $\frac{k+1}{2}$ times and going up (or down) $\frac{k-1}{2}$ times, until u vanishes in the instant π. Likewise, if $k\tau(E) = \pi$ with even k, we have a solution that starts at $(0, v_0)$ with $v_0 \in \mathbb{R}$ and performs $k/2$ complete translations around the origin until the moment π. All of those reveal equilibria of (6.1).

What is left is to find the solutions of (6.8) that have energies that satisfy (6.14). In order to do that, we need to study the behavior of the function τ, which depends on the value of λ. We do this using the following theorems, which come from (CHAFEE; INFANTE, 1974).

Consider $\tau(E) = 2\Xi(E) = 2\int_0^{U_+(E)} \{\sqrt{2\lambda(E-F(u))}\}^{-1} du$, for $0 < E < E_l$, and we change variables the following way: $Ey^2 = F(u)$, for $0 \le y \le 1$, $0 \le u \le U_+(E)$. Then, we obtain:

$$\tau(E) = 2\sqrt{\frac{2E}{\lambda}} \int_0^1 (1-y^2)^{-\frac{1}{2}} \frac{y}{f(u)} dy, \quad \text{where} \quad u = U_+(Ey^2). \tag{6.15}$$

Theorem 36. τ is continuous and:

$$\lim_{E \to 0^+} \tau(E) = \frac{\pi}{\sqrt{\lambda}}. \tag{6.16}$$

Proof. We only prove the second claim. Since $f(u) = f'(0)u + o(u) = u + o(u)$, given $0 < \varepsilon < 1$, there exists a $\delta > 0$ such that:

$$(1-\varepsilon)u \leq f(u) \leq (1+\varepsilon)u, \quad \text{if} \quad 0 \leq u \leq \delta. \tag{6.17}$$

Writing $F(u)$ as integral and using this last inequality, we get:

$$\frac{1}{2}(1-\varepsilon)u^2 \leq F(u) \leq \frac{1}{2}(1+\varepsilon)u^2, \quad \text{if} \quad 0 \leq u \leq \delta.$$

By continuity of U_+, there exist an η, $0 < \eta < E_l$, such that $U_+(E) \leq \delta$, for all $0 \leq E \leq \eta$. If $0 \leq E \leq \eta$, $0 \leq y \leq 1$, we have $Ey^2 = F(u)$ with $0 \leq u \leq \delta$, and the last inequality yields:

$$\sqrt{\frac{(1-\varepsilon)}{2E}}u \leq y \leq \sqrt{\frac{(1+\varepsilon)}{2E}}u, \quad \text{for} \quad 0 < E \leq \eta, \quad 0 \leq y \leq 1.$$

Using (6.17), we get:

$$\sqrt{\frac{1-\varepsilon}{2E(1+\varepsilon)^2}} \leq \frac{y}{f(u)} \leq \sqrt{\frac{1+\varepsilon}{2E(1-\varepsilon)^2}}, \quad \text{for} \quad 0 < E \leq \eta, \quad 0 \leq y \leq 1.$$

Using this estimate in (6.15), we get:

$$\frac{\pi}{\sqrt{\lambda}}\sqrt{\frac{1-\varepsilon}{(1+\varepsilon)^2}} \leq \tau(E) \leq \frac{\pi}{\sqrt{\lambda}}\sqrt{\frac{1+\varepsilon}{(1-\varepsilon)^2}}, \quad \text{for} \quad 0 < E \leq \eta \tag{6.18}$$

Whence $\lim\limits_{E \to 0^+} \tau(E) = \frac{\pi}{\sqrt{\lambda}}$.

□

Theorem 37. τ is differentiable in $(0, E_l)$, and:

$$\frac{d\tau}{dE}(E) > 0, \quad (0 < E < E_l). \tag{6.19}$$

Proof. It follows from an extensive yet straightforward differentiation of the expression in (6.15) that:

$$\frac{d\tau}{dE}(E) = \sqrt{\frac{2}{\lambda E}} \int_0^1 (1-y^2)^{-\frac{1}{2}} \frac{y}{f(u)} \left(1 - \frac{2f'(u)F(u)}{f(u)^2}\right) dy, \quad \text{with} \quad u = U_+(Ey^2). \tag{6.20}$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(u) = f(u)^2 - 2f'(u)F(u)$, and note that $g'(u) = -2f''(u)F(u)$. For $u > 0$, $f''(u) < 0$, and $g'(u) > 0$, and since $g(0) = 0$, we have $g(u) > 0$ for $u \in (0, \bar{a})$. It follows that $\frac{d\tau}{dE}(E) > 0$ for $0 < E < E_l$.

□

Theorem 38. $\lim\limits_{E \to E_l} \tau(E) = \infty$.

Therefore, the range of τ is $(\frac{\pi}{\sqrt{\lambda}}, \infty)$.

Proof. Suppose $\bar{a} < \infty$, so (6.8) has an equilibrium point $(\bar{a}, 0)$ and $E_l = F(\bar{a})$.

Consider the solution of (6.8) with $u(0) = 0$, $u'(0) = v(0) = v_0 > 0$, and $E = \frac{1}{2}\lambda^{-1}v_0^2$. This solution touches the u-axis at the point u_0 such that $F(u_0) = E$. If E approaches E_l, u_0 approaches $\bar{a}$, hence, it follows from Lemma 10 that given $T > 0$, there exists a $\delta > 0$ such that if $E_l - \delta < E < E_l$, $\tau(E) > T$, and the theorem is proven.

□

We are ready to show the last theorem of this section.

Theorem 39. Suppose $f \in \mathscr{C}^2(\mathbb{R})$ is odd, $f'(0) = 1$, and the conditions (6.2) and (6.3) hold. If $\lambda \le 1$, the origin $0 \in H_0^1(0,\pi)$ is the only equilibrium for (6.1). Let $N \in \mathbb{N}^*$, and $N^2 < \lambda \le (N+1)^2$, then there are $2N+1$ equilibria for (6.1), which we denote:

$$\{0\} \cup \{\phi_j^{\pm} : j = 1, ..., N\},$$

where ϕ_j^+ and ϕ_j^- have $j+1$ zeros in $[0,\pi]$, $\phi_j^- = -\phi_j^+$, and $\frac{d}{dx}\phi_j^+(0) > 0$, for $j = 1, ..., N$.

Proof. If $\lambda \le 1$, it follows from the last theorems that $\tau(E) > \frac{\pi}{\sqrt{\lambda}} \ge \pi$, for all $0 < E < E_l$ and $k\tau(E) = \pi$ can not be satisfied for any pair (E,k), with $0 < E < E_l$ and $k \in \mathbb{N}^*$. Hence, 0 is the only equilibrium of (6.1).

Now assume $1 < \lambda \le 2^2$, then τ has range $(\frac{\pi}{\sqrt{\lambda}}, \infty)$, and since $\frac{\pi}{2} \le \frac{\pi}{\sqrt{\lambda}} < \pi$, there exist an energy E_1 such that $\tau(E_1) = \pi$, and the pair $(E_1, 1)$ is the only one that satisfies $k\tau(E) = \pi$. The solution u with $u'(0) > 0$, and energy E_1 is called ϕ_1^+. It has already been mentioned that this solution only has two zeros in $[0,\pi]$. Analogously, the solution with $u'(0) < 0$ and energy E_1, is called ϕ_1^-, and from the fact that f is odd, we can conclude that $\phi_1^- = -\phi_1^+$.

More generally, if $N^2 < \lambda \le (N+1)^2$, the range of τ is $(\frac{\pi}{\sqrt{\lambda}}, \infty)$, and $\frac{\pi}{N+1} \le \frac{\pi}{\sqrt{\lambda}} < \frac{\pi}{N}$, and we have $0 < E_j < E_l$, such that $\tau(E_j) = \frac{\pi}{j}$, $j = 1, ..., N$. As before, the solutions with energy E_j are called ϕ_j^+ and ϕ_j^-, depending on the sign of $u'(0)$, $j = 1, ..., N$. The discussion about the number of zeros of $\phi_j^{\pm}$ has already been made, and $\phi_j^- = -\phi_j^+$ follows from the fact that f is odd.

This completes the proof.

□

Remark 12. Let $\phi \in \mathscr{E}$, then $\eta(\cdot) = \phi(\pi - \cdot)$ is another equilibrium for (6.1) with the same number of zeros as ϕ. It follows that $\eta = \pm\phi$, depending on the number of zeros of ϕ. It is easy to conclude that $\phi_j^{\pm}(\pi - x) = (-1)^{j-1}\phi_j^{\pm}(x)$, for $x \in [0,\pi]$, which is a symmetry property for $\phi_j^{\pm}$.

Moreover, if ϕ_j is an equilibrium for (6.1) with $j = 2r$ for an even natural number r, then we can consider $\psi_j = \phi_j|_{[0,\frac{\pi}{2}]}$, which is an equilibrium for the problem:

$$\begin{aligned} &u_t = u_{xx} + \lambda f(u), \quad t > 0,\ x \in (0, \tfrac{\pi}{2}) \\ &u(0,t) = u(\tfrac{\pi}{2},t) = 0, \quad t \geq 0 \\ &u(\cdot,0) = u_0 \in H_0^1(0, \tfrac{\pi}{2}), \end{aligned} \tag{6.21}$$

and the same reasoning as before may be applied to conclude that $\phi_j(\frac{\pi}{2} - x) = -\phi_j(x)$ for $x \in [0, \frac{\pi}{2}]$. Inductively, we conclude that if $k = 2^n(2j+1)$ with $n \geq 1$ and $j \geq 0$ integers, then

$$\phi_k\left(\frac{\pi}{2^i} - x\right) = -\phi_k(x), \quad \forall x \in \left[0, \frac{\pi}{2^i}\right], 1 \leq i < n.$$

These and other symmetry results will be important in the analysis of the quasilinear non-local Chafee-Infante equation in Chapter 7.

Let $\mathscr{E}$ be the finite set of isolated equilibria of (6.1). It follows from Theorem 20 that:

$$\mathscr{A} = \bigcup_{\phi \in \mathscr{E}} W^u(\phi). \tag{6.22}$$

For any bounded global solution $\xi : \mathbb{R} \to H_0^1(0,\pi)$, $\xi(\mathbb{R}) \subset \mathscr{A}$, so that $\alpha_\xi(\xi(0)) = \{\phi_1\}$ and $\omega(\xi(0)) = \{\phi_2\}$, for some ϕ_1, $\phi_2 \in \mathscr{E}$, and the following holds:

$$\phi_1 \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \phi_2. \tag{6.23}$$

And for any $u_0 \in H_0^1(0,\pi)$, there exists $\phi \in \mathscr{E}$ such that $T(t)u_0 \overset{t\to\infty}{\longrightarrow} \phi$.

6.3 Stability and hyperbolicity of the equilibria

In this section we study stability and hyperbolicity for the finite equilibria of equation (6.1). It is easy to see that the operator associated to the linearization of (6.1) around the equilibrium $\phi \in \mathscr{E}$ is given by the formula below.

$$\begin{aligned} &L_\phi : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi) \\ &L_\phi u = -u'' - \lambda f'(\phi) u \end{aligned}$$

It was proven in Example 2 that L_ϕ is self-adjoint and has compact resolvent, and its spectrum is an increasing sequence $\{\mu_j\}$ of eigenvalues such that $\mu_j \to \infty$.

From the theorems in Section 5.2 we know that if $\mu_1 > 0$, ϕ is stable, and if $\mu_1 < 0$, it is unstable. We will identify whether $\mu_1 > 0$ or $\mu_1 < 0$ for each equilibrium of (6.1).

First of all, consider the equilibrium $\phi_0 = 0$, whose linearization is $L_0 = -u'' - \lambda u$, with spectrum given by $\sigma(L_0) = \{n^2 - \lambda : n \in \mathbb{N}\}$, so that ϕ_0 is exponentially asymptotically stable if $\lambda < 1$ and unstable if $\lambda > 1$. Moreover, for any $\lambda \leq 1$, ϕ_0 is the only equilibrium of $\mathscr{T}$, then we have $\mathscr{A} = W^u(\phi_0)$. Suppose $W^u(\phi_0) \neq \{\phi_0\}$, then we would have a global solution $\xi : \mathbb{R} \to H_0^1(0,\pi)$ with $\xi(0) \neq \phi_0$ such that

$$\phi_0 \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \phi_0,$$

and we can show that this is a contradiction because $\mathscr{T}$ is gradient. So that $\mathscr{A} = \{\phi_0\}$, and ϕ_0 attracts any bounded set of $H_0^1(0,\pi)$.

For the nonzero equilibria, we will need a comparison result.

Theorem 40. Let $a \in \mathscr{C}([0,\pi],\mathbb{R})$, u, $v \in \mathscr{C}^2([0,\pi])$ with $u(0) = v(0) = 0$ and $u'(0) = v'(0) = 1$. Suppose that $u(x) \geq 0$ and $v(x) > 0$ for $x \in (0,x_1)$, for some $x_1 > 0$. If either

$$u''(x) + a(x)u(x) > v''(x) + a(x)v(x) = 0, \quad 0 < x < x_1$$

or

$$0 = u''(x) + a(x)u(x) > v''(x) + a(x)v(x), \quad 0 < x < x_1,$$

then $u(x) > v(x)$ for $0 < x \leq x_1$.

Proof. Suppose that the first estimate is true, then, for $x \in (0,x_1)$

$$\begin{aligned}\frac{d}{dx}(u'(x)v(x) - v'(x)u(x)) &= v(x)u''(x) - v''(x)u(x) \\ &> v(x)v''(x) + v(x)^2 a(x) - v(x)a(x)u(x) - u(x)v''(x) = 0.\end{aligned}$$

With similar calculations, we conclude the same for the other case. Then, since we have $u'(0)v(0) - v'(0)u(0) = 0$, we conclude that $u'(x)v(x) - v'(x)u(x) > 0$ for $x \in (0,x_1)$. Now,

$$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{u'(x)v(x) - v'(x)u(x)}{v(x)^2} > 0, \quad x \in (0,x_1)$$

Since $\lim_{x\to 0^+} u(x)/v(x) = 1$, we conclude $u(x) > v(x)$ for $x \in (0,x_1]$. ☐

Theorem 41. Let $\phi \in \mathscr{E}$ and consider the eigenvalue problem

$$\begin{aligned}&\theta'' + (\mu_1 + \lambda f'(\phi))\theta = 0 \iff L_\phi \theta = \mu_1 \theta \\ &\theta(0) = \theta(\pi) = 0, \quad \theta'(0) = 1\end{aligned} \tag{6.24}$$

Where μ_1 is the lowest real number such that the equation has a unique solution in $H_0^1(0,\pi) \cap H^2(0,\pi)$. It follows from Theorem 17 that $\theta(x) > 0$ in $(0,\pi)$.

Let v be the solution of the initial value ODE problem

$$\begin{aligned}&-v_{xx} = \lambda f'(\phi)v \\ &v(0) = 0, \quad v'(0) = 1.\end{aligned}$$

If v has no zero in $(0,\pi]$, then $\mu_1 > 0$. If v has a zero in $(0,\pi)$, then $\mu_1 < 0$.

Proof. If $v > 0$ in $(0,\pi]$, we cannot have a solution of (6.24) with $\mu_1 = 0$, because in this case θ and v would satisfy the same initial value problem in $[0,\pi)$, but would differ in $x = \pi$. If $v > 0$ in $(0,\pi]$ and $\mu_1 < 0$ then,

$$\theta'' + \lambda f'(\phi)\theta = -\mu_1\theta$$

And since $\theta(x) > 0$ for $x \in (0,\pi)$, we have:

$$\theta'' + \lambda f'(\phi)\theta > 0 = v'' + \lambda f'(\phi)v, \quad \text{for } x \in (0,\pi),$$

From Theorem 40, $\theta(x) > v(x)$ for all $x \in (0,\pi]$, so that $\theta(\pi) > v(\pi) > 0$, a contradiction.

Hence, we conclude $\mu_1 > 0$.

Now assume that v has a first zero x_1 in $(0,\pi)$, and $v(x) > 0$ in $x \in (0,x_1)$. Then, if $\mu_1 = 0$ we have that $v \equiv \theta$, and this is a contradiction with the fact that $\theta > 0$ in $(0,\pi)$. If, on the other hand, $\mu_1 > 0$, we have that

$$\theta'' + \lambda f'(\phi)\theta < 0 = v'' + \lambda f'(\phi)v$$

for $x \in (0,x_1)$. Then $v(x) > \theta(x)$ in $(0,x_1]$ and $\theta(x_1) < 0$, which is a contradiction. Then we must have $\mu_1 < 0$.

□

Now we analyze the stability for the nontrivial equilibria. Since f' is even, the solutions ϕ_j^+ and ϕ_j^- generate the same linearization operator L_ϕ, and we may restrict our analysis to ϕ_j^+, for each $j = 1,\ldots,n$, when $n^2 < \lambda \leq (n+1)^2$, $n \in \mathbb{N}^*$.

First consider $u(x) = \phi_1^+$ for $\lambda > 1$. We know that $u(0) = 0$, $u'(0) > 0$ and $f(u(x)) > 0$ for all $x \in (0,\pi)$. Define

$$w(x) := -\frac{1}{\lambda u'(0)}u''(x) = \frac{f(u(x))}{u'(0)}.$$

Then, $w(x) > 0$ for all $x \in (0,\pi)$, $w(0) = w(\pi) = 0$, $w'(0) = 1$, and

$$w'' + \lambda f'(u)w = \frac{f''(u(x))u'(x)^2}{u'(0)} < 0 = v'' + \lambda f'(u)v, \quad \forall x \in (0,\pi),$$

where we used the fact that $f''(u) < 0$ for $u > 0$.

Let us prove that $v(x) > 0$ for $x \in (0,\pi)$. Suppose not, then there exists a first zero $x_1 \in (0,\pi)$ such that $v(x_1) = 0$ and $v(x) > 0$ in $x \in (0,x_1)$. Then we use Theorem 40, and conclude that $v(x) > w(x) > 0$ in $(0,x_1]$, a contradiction. Hence, $v(x) > 0$ for $x \in (0,\pi)$, and we may apply Theorem 40 again to see that $v(x) > w(x) \geq 0$ for $x \in (0,\pi]$. Using Theorem 41, we conclude that $\mu_1 > 0$, and $\phi_1^\pm$ are asymptotically stable for (6.1).

Now, suppose that u is a nontrivial equilibrium of (6.1) satisfying $u'(0) > 0$ and vanishing somewhere in the interval $(0,\pi)$, say at $\bar{x}$ (that is the case of $u = \phi_j^+$ for some $1 < j \leq n$). From the way u was constructed, it has a negative minimum at some point $x^* \in (0,\pi)$, and we have $u(x^*) < 0$, $u'(x^*) = 0$, and $u''(x^*) > 0$. But both v and u' solve the ODE $-v_{xx} = \lambda f'(u)v$, so their Wronskian is constant, that is,

$$v'(x)u'(x) - v(x)u''(x) = const = u'(0) > 0.$$

Thus $-v(x^*)u''(x^*) = u'(0)$, and $v(x^*) < 0$. Thus $\mu_1 < 0$, and we conclude that $\phi_j^\pm$ are unstable for $1 < j \leq n$, if $n^2 < \lambda \leq (n+1)^2$.

So far we have proved the following theorem.

Theorem 42. If $\lambda \leq 1$, $\phi_0 = 0$ is the only equilibrium for (6.1) and is globally asymptotically stable. If $N^2 < \lambda \leq (N+1)^2$, for some $N \in \mathbb{N}^*$, then the equilibrium ϕ_0 is unstable, the equilibria $\phi_1^\pm$ are asymptotically stable, and the equilibria $\phi_j^\pm$ are unstable, for $0 < j \leq N$.

Note that ϕ_0 is hyperbolic provided that $\lambda \neq n^2$ for all $n \in \mathbb{N}$. Next we prove that a nontrivial equilibrium ϕ of the Chafee-Infante equation (6.1) is hyperbolic. We use properties of the function τ constructed in Section 6.2, which depends on the energy of an equilibrium, given by $E = \frac{1}{2}\lambda^{-1}u'(0)^2$.

Note that if E_ϕ is the energy associated to ϕ, there exists $n_\phi \in \mathbb{N}$ such that $n_\phi \tau(E_\phi) = \pi$, and

$$\begin{aligned} &\phi''(x) + \lambda f(\phi(x)) = 0, \\ &\phi(0) = \phi(\tau(E_\phi)) = \phi(\pi) = 0 \text{ and } \phi'(\frac{\tau(E_\phi)}{2}) = 0. \end{aligned} \tag{6.25}$$

In Section 6.2 we showed that for each $0 < E < E_l$, there is a solution of the following boundary value ODE problem:

$$\begin{aligned} &u''(x) + \lambda f(u(x)) = 0, \\ &u(0,E) = 0,\ u'(0,E) = \sqrt{2\lambda E} \text{ and } u(k\tau(E),E) = 0 \quad \forall\, k \in \mathbb{N}. \end{aligned} \tag{6.26}$$

In particular, $u(\cdot,E_\phi) = \phi$. Then both $\eta = \phi'$ and $\psi = \frac{\partial u}{\partial E}(\cdot,E)\Big|_{E=E_\phi}$ are solutions of the ODE $v'' + \lambda f'(\phi)v = 0$. Indeed,

$$\begin{aligned} \psi''(x) &= \frac{\partial u_{xx}}{\partial E}(x,E)\Big|_{E=E_\phi} = \frac{\partial}{\partial E}(-\lambda f(u(x,E)))\Big|_{E=E_\phi} \\ &= -\left[\lambda f'(u(x,E))\frac{\partial u}{\partial E}(x,E)\right]\Big|_{E=E_\phi} = -\lambda f'(\phi(x))\psi(x). \end{aligned}$$

We know that $\eta(0) \neq 0$, $\eta'(0) = \phi''(0) = -\lambda f(\phi(0)) = 0$ and $\psi(0) = 0$, $\psi'(0) = \frac{\lambda}{\sqrt{2\lambda E_\phi}} \neq 0$. Calculating the Wronskian at the point $x = 0$ shows that η and ψ are linearly independent and any solution of $v'' + \lambda f'(\phi)v = 0$ must be of the form

$$\omega = c_1\eta + c_2\psi.$$

Let us show that if $\omega(0) = \omega(\pi) = 0$ then we must have $\omega \equiv 0$, which implies that $L_\phi : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$ does not have zero as an eigenvalue.

In fact, $\psi(0) = 0$, $\eta(0) \neq 0$ and $c_1\eta(0) + c_2\psi(0) = 0$ implies $c_1 = 0$. Now, since $u(n_\phi\tau(E),E) = 0$ for all E, differentiating with respect to E yields

$$\frac{\partial u}{\partial x}(n_\phi\tau(E),E)n_\phi\tau'(E) + \frac{\partial u}{\partial E}(n_\phi\tau(E),E) = 0$$

It follows from the definition of u and $\tau(E)$ that $\frac{\partial u}{\partial x}(n_\phi\tau(E),E) \neq 0$ and since we have proved in Theorem 37 that $\tau'(E) > 0$, we must have $\psi(n_\phi\tau(E_\phi)) = \psi(\pi) = \frac{\partial u}{\partial E}(n_\phi\tau(E_\phi),E_\phi) \neq 0$. Hence, $0 = \omega(\pi) = c_2\psi(\pi)$ implies $c_2 = 0$, and the only solution ω of $L_\phi\omega = 0$ which satisfies $\omega(0) = \omega(\pi) = 0$ is $\omega \equiv 0$. This proves that 0 is not an eigenvalue of L_ϕ.

By Proposition 1, $0 \in \rho(L_\phi)$.

Hence we have proved the following:

Theorem 43. The equilibrium $\phi_0 = 0$ of (6.1) is hyperbolic if $\lambda \neq n^2$ for all $n \in \mathbb{N}$. The nonzero equilibria of (6.1) are all hyperbolic.

CHAPTER

7

A NON-LOCAL QUASILINEAR CHAFEE-INFANTE EQUATION

7.1 Well-posedness and gradient attractor

In this chapter we present the study developed in (CARVALHO; MOREIRA, 2021) about a non-local version of the Chafee-Infante equation. Non-local partial differential equations appear in several applications, from the heating of ceramic until population dynamics. The reader may find several examples of applications in (CHIPOT; VALENTE; CAFFARELLI, 2003), (DAVIDSON; DODDS, 2006) and (KRIEGSMANN, 1997). Consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = a(\|u_x\|^2)u_{xx} + \lambda f(u), \quad t>0,\ x\in(0,\pi)\\ &u(0,t)=u(\pi,t)=0, \quad t\geq 0 \\ &u(\cdot,0)=u_0\in H_0^1(0,\pi), \end{aligned} \tag{7.1}$$

where $\lambda > 0$ is a parameter, $a:\mathbb{R}^+ \to [m,M] \subset (0,\infty)$ is a continuosly differentiable, globally Lipschitzian and non-decreasing function; $f \in \mathscr{C}^2(\mathbb{R})$ is odd (in particular, $f(0)=0$), $f'(0)=1$, and f satisfies:

$$f''(u)u<0, \forall\, u\neq 0, \tag{7.2}$$

and

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0, \tag{7.3}$$

and finally

$$(0,\infty)\ni u \mapsto \frac{f(u)}{u} \text{ is strictly decreasing.} \tag{7.4}$$

We denote by $\|\cdot\|$ the usual norm in $L^2(0,\pi)$.

Throughout our study, we will work very often with the following auxiliary equation.

$$\begin{aligned} &w_\tau = w_{xx} + \frac{\lambda f(w)}{a(\|w_x\|^2)}, \quad \tau > 0,\ x \in (0,\pi) \\ &w(0,\tau) = w(\pi,\tau) = 0, \quad \tau \geq 0 \\ &w(\cdot,0) = u_0 \in H_0^1(0,\pi). \end{aligned} \tag{7.5}$$

The equation (7.1) has a non-local coefficient for u_{xx}, which makes it a quasilinear equation, instead of semilinear. Hence the results in Chapter 5 can not be applied directly. However, we will see that solutions for (7.1) can be obtained from solutions of (7.5) with a solution dependent change in time scale. Therefore, we may use the techniques we developed in Chapter 5 to conclude information about (7.5), and then transfer this information without great difficulty to the quasilinear equation.

First of all, we can conclude that (7.5) has solutions defined for all $t \geq 0$, which depend continuously on the initial condition, and (7.5) defines a gradient semigroup in $H_0^1(0,\pi)$ that has a global attractor. The calculations used to conclude this information resemble very much the ones we did for the classical Chafee-Infante equation (in Chapter 6), except for the fact that now the non-linearity contains a non-local term — which is nice to work with because the function a has strictly positive lower and upper bounds.

After that, we will study the bifurcation of equilibria for (7.1), as in (CARVALHO *et al.*, 2020). We note that these equilibria are also equilibria for (7.5), and then we may study the nonlocal linearization operator associated to equation (7.5) in order to conclude stability and hyperbolicty of equilibria for (7.5). At last, we may transfer the results to (7.1), concluding results about stability and hyperbolicity (in the sense of Definition 34) for the equilibria of the quasilinear equation (7.1).

Consider well-posedness for equation (7.5) in the phase space $H_0^1(0,\pi)$. Again, let $X_0 = L^2(0,\pi)$, $A = -\Delta$ in $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$, and recall that $D(A^{\frac{1}{2}}) = H_0^1(0,\pi)$. We need to show that $g^e : H_0^1(0,\pi) \to L^2(0,\pi)$, given by $g^e(w)(x) = \frac{\lambda f(w(x))}{a(\|w_x\|^2)}$, is Lipschitz in bounded sets of the domain.

Let $u, v \in H_0^1(0,\pi)$ be such that $\|u\|_{H_0^1} \leq \rho$ and $\|v\|_{H_0^1} \leq \rho$. Using the estimate (6.4) from the last chapter, and that a satisfies $|a(\|u_x\|^2) - a(\|v_x\|^2)| \leq k|\|u_x\|^2 - \|v_x\|^2|$ for some $k \geq 0$, we get:

$$\begin{aligned} \|g^e(u) - g^e(v)\| &= \left\| \frac{\lambda f(u) a(\|v_x\|^2) - \lambda f(v) a(\|u_x\|^2)}{a(\|u_x\|^2) a(\|v_x\|^2)} \right\| \\ &\leq \frac{|\lambda|}{m^2} \|f(u)a(\|v_x\|^2) - f(v)a(\|u_x\|^2)\| \\ &\leq \frac{|\lambda|}{m^2} \left[\|f(u)a(\|v_x\|^2) - f(v)a(\|v_x\|^2)\| + \|f(v)[a(\|v_x\|^2) - a(\|u_x\|^2)]\| \right] \\ &\leq C_1(\rho)\|u-v\|_{H_0^1} + C_2(\rho)\|v\|_{H_0^1} |\|v_x\|^2 - \|u_x\|^2| \\ &\leq C_1(\rho)\|u-v\|_{H_0^1} + C_2(\rho)\|v\|_{H_0^1} 2\rho \|u-v\|_{H_0^1} \leq C_3(\rho)\|u-v\|_{H_0^1}, \end{aligned}$$

where we used the inequality below

$$|\|v_x\|^2 - \|u_x\|^2| = |(\|v_x\| + \|u_x\|)(\|v_x\| - \|u_x\|)| \leq 2\rho \|u-v\|_{H_0^1}.$$

Then, Theorem 32 applies and for each $u_0 \in H_0^1$, there exists a local solution $w : [0,t_1) \to H_0^1(0,\pi)$ of (7.5) such that $w(0) = u_0$. Now consider the function $V : H_0^1(0,\pi) \to \mathbb{R}$ defined by:

$$V(u) = \frac{1}{2}\int_0^{\|u_x\|^2} a(s)ds - \lambda \int_0^{\pi} F(u(x))dx, \quad \text{where} \quad F(s) := \int_0^s f(\xi)d\xi.$$

It is easy to see that this function is Lipschitz in bounded sets of $H_0^1(0,\pi)$, hence continuous. Moreover, if $w : [0,t_1) \to H_0^1(0,\pi)$ is a solution for (7.5), the composition $V \circ w : [0,t_1) \to \mathbb{R}$ is differentiable, because both the functional V and the solution w are differentiable. For the solution w, we have:

$$a(\|w(\tau)\|^2_{H_0^1})w_\tau = a(\|w(\tau)\|^2_{H_0^1})w_{xx} + \lambda f(w).$$

We multiply both sides by w_τ and integrate in $[0,\pi]$, then we use the following facts

$$\int_0^{\pi} a(\|w(\tau)\|^2_{H_0^1})w_{xx}(x,\tau)w_\tau(x,\tau)dx = -\frac{d}{d\tau}\frac{1}{2}\int_0^{\|w(\tau)\|^2_{H_0^1}} a(s)ds,$$

and

$$\lambda \int_0^{\pi} f(w(x,\tau))w_\tau(x,\tau)dx = +\frac{d}{d\tau}\lambda \int_0^{\pi} F(w(x,\tau))dx.$$

In the end, we obtain:

$$\frac{d}{d\tau}V(w(\cdot,\tau)) = -a(\|w(\tau)\|^2_{H_0^1})\|w_\tau\|^2.$$

Which is always non-positive. If w is defined in $\tau \geq 0$ and $V(w(\tau)) = V(u_0)$ for all $\tau \geq 0$, then, $\frac{d}{d\tau}V(w(\cdot,\tau)) \equiv 0$, so that $w_\tau(\tau) \equiv 0$ for all $\tau \geq 0$ and we conclude that w is an equilibrium for (7.5). Then V has the properties of a Lyapunov function.

The same way we obtained (6.5) in the last chapter, we can show that if $w : [0,t_1) \to H_0^1(0,\pi)$ is a solution of (7.5) with initial value u_0, then:

$$\|w(\tau)\|^2_{H_0^1} \leq \frac{4\lambda\pi K_{\frac{m}{4\lambda}}}{m} + \frac{4}{m}V(w(\tau)) \leq \frac{4\lambda\pi K_{\frac{m}{4\lambda}}}{m} + \frac{4}{m}V(u_0), \quad \forall \tau \in [0,t_1), \tag{7.6}$$

where K_ε is a positive constant such that $F(u) \leq \varepsilon u^2 + K_\varepsilon$, for all $u \in \mathbb{R}$ (its existence follows from the condition (7.3)).

Using the last part of Theorem 32, we conclude that for every $u_0 \in H_0^1(0,\pi)$, there exists a solution $w(\cdot,u_0) : \mathbb{R}^+ \to H_0^1(0,\pi)$ such that $w(0,u_0) = u_0$. If we define $S(\tau) : H_0^1(0,\pi) \to H_0^1(0,\pi)$ as $S(\tau)u_0 = w(\tau,u_0)$, the mapping $\mathbb{R}^+ \times H_0^1(0,\pi) \ni (\tau,u_0) \mapsto S(\tau)u_0 \in H_0^1(0,\pi)$ is continuous by Remark 11, and $\mathscr{S} = \{S(\tau) : \tau \geq 0\}$ is a gradient semigroup with Lyapunov function V.

For each solution $w(\cdot, u_0)$ of (7.5), we define the solution dependent change in timescale $t = t_\tau = \int_0^\tau a(\|w(\theta, u_0)\|^2_{H^1_0})^{-1} d\theta$. Since $0 < m \leq a(s) \leq M < \infty$ for all $s \in \mathbb{R}^+$, the function $t_\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is a bijection and strictly increasing, satisfying $t_\tau \to \infty$ when $\tau \to \infty$. From now on we may write t instead of t_τ.

If we define $u(\cdot, u_0) : \mathbb{R}^+ \to H^1_0(0, \pi)$ by $u(t, u_0) = w(\tau, u_0)$, a direct calculation shows that $u(\cdot, u_0)$ is the unique solution of (7.1). The semigroup associated to (7.1) is $\mathscr{T} := \{T(t) : t \geq 0\}$, where $T(t)u_0 = S(\tau)u_0$ for each $t = \int_0^\tau a(\|S(\theta)u_0\|^2_{H^1_0})^{-1} d\theta$ (do note that the relation between t and τ depends on u_0, so we cannot simply write $T(t) = S(\tau)$). We conclude that $\{T(t) : t \geq 0\}$ is also gradient, with Lyapunov function V, and has the same equilibria as $\{S(\tau) : \tau \geq 0\}$ — we denote this set of equilibria by $\mathscr{E}$.

Remark 13. If $\eta : \mathbb{R} \to H^1_0(0, \pi)$ is a global solution for $\mathscr{S} = \{S(\tau) : \tau \geq 0\}$ through $u_0 \in H^1_0(0, \pi)$, then $\xi : \mathbb{R} \to H^1_0(0, \pi)$, given by $\xi(t) = \eta(\tau)$, where $t = \int_0^\tau a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta$, is a global solution for $\mathscr{T} = \{T(t) : t \geq 0\}$ through the same point. Indeed, for any $t \geq 0$, $s \in \mathbb{R}$, we have

$$T(t)\xi(s) = T(t)\eta(\sigma) = S(\tau)\eta(\sigma) = \eta(\tau + \sigma), \tag{7.7}$$

where $s = \int_0^\sigma a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta$, and $t = \int_0^\tau a(\|\eta(\sigma + \theta)\|^2_{H^1_0})^{-1} d\theta$. On the other hand,

$$\int_0^{\sigma+\tau} a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta = \int_0^\sigma a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta + \int_\sigma^{\sigma+\tau} a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta = s + t,$$

which implies, by definition, that $\xi(t + s) = \eta(\tau + \sigma)$, and by (7.7) we conclude that $T(t)\xi(s) = \xi(t + s)$.

Let $\mathscr{A}$ be a global attractor for $\mathscr{S}$, then it is also a global attractor for $\mathscr{T}$. Indeed, $\mathscr{A}$ is compact and nonempty. To show that $\mathscr{A}$ is $\mathscr{T}$-invariant, we use Proposition 7. First, since $\mathscr{A}$ is $\mathscr{S}$-invariant, for each point $x \in \mathscr{A}$ there exists a global solution $\eta : \mathbb{R} \to \mathscr{A}$ of $\mathscr{S}$ through x, then there exists a global solution $\xi : \mathbb{R} \to \mathscr{A}$ of $\mathscr{T}$ through x, which is defined as $\xi(t) = \eta(\tau)$ for $t = \int_0^\tau a(\|\eta(\theta)\|^2_{H^1_0})^{-1} d\theta$. It follows that $\mathscr{A}$ is $\mathscr{T}$-invariant.

Notice that $\mathscr{A}$ $\mathscr{S}$-attracts bounded sets of $H^1_0(0, \pi)$. Let B be a bounded set in $H^1_0(0, \pi)$, then for any $\varepsilon > 0$, there exists $\tau_0 \geq 0$ such that $S(\tau)B \subset \mathscr{O}_\varepsilon(\mathscr{A})$, for all $\tau \geq \tau_0$. If we define $t_0 = \frac{1}{m}\tau_0$, we have that for all $t \geq t_0$, $x \in B$,

$$t \geq t_0 = \frac{1}{m}\tau_0 \geq \int_0^{\tau_0} a(\|S(\theta)x\|^2_{H^1_0})^{-1} d\theta.$$

Therefore, $T(t)x = S(\tau)x$ for some $\tau \geq \tau_0$, and $T(t)x \in \mathscr{O}_\varepsilon(\mathscr{A})$, for all $t \geq t_0$. It follows that $T(t)B \subset \mathscr{O}_\varepsilon(\mathscr{A})$ for all $t \geq t_0$, so that $\mathscr{A}$ $\mathscr{T}$-attracts bounded sets in $H^1_0(0, \pi)$, and $\mathscr{A}$ is in fact the global attractor for $\mathscr{T}$.

We may apply the same calculations of the beginning of Section 6.2 to show that $\{S(\tau) : \tau \geq 0\}$ has a global attractor. To illustrate the similarity of ideas, we show that the set $\mathscr{E}$ is bounded, proceeding as following.

Let $u \in \mathscr{E}$. By Condition 7.3, given $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $f(s)s \leq \varepsilon s^2 + C_\varepsilon$, for all $s \in \mathbb{R}$. Then

$$a(\|u_x\|^2)\int_0^\pi u(s)u_{xx}(s)ds + \lambda\int_0^\pi u(s)f(u(s))ds = 0 \Rightarrow a(\|u_x\|^2)\|u\|^2_{H^1_0} \leq \lambda\varepsilon\|u\|^2 + \pi\lambda C_\varepsilon \Rightarrow$$

$$\|u\|^2_{H^1_0} \leq \frac{\lambda}{m}\varepsilon\|u\|^2 + \frac{\pi\lambda}{m}C_\varepsilon \Rightarrow \frac{\|u\|^2_{H^1_0}}{2} \leq \left(\frac{\lambda}{m}\varepsilon - \frac{1}{2}\right)\|u\|^2 + \frac{\pi\lambda}{m}C_\varepsilon,$$

and choosing an ε small enough, we have an uniform estimate for $\|u\|_{H^1_0}$, for $u \in \mathscr{E}$.

Therefore, we can use the same calculations as in last chapter to show that $\{S(\tau) : \tau \geq 0\}$ has a global attractor $\mathscr{A}$, which is also a global attractor for $\mathscr{T} = \{T(t) : t \geq 0\}$.

7.2 Bifurcation of equilibria

In this section we present the study developed in (CARVALHO *et al.*, 2020). We intend to construct equilibria for the quasilinear Chafee-Infante equation (7.1). To do this, we will first find a positive solution for an elliptical problem in the subinterval $[0, \frac{\pi}{j}]$, and then use the symmetry of this solution and of our problem to construct the sign-changing equilibria for (7.1). Consider the problem:

$$\begin{aligned} &a\left(\|u_x\|_j^2\right)u_{xx} + \lambda f(u) = 0, \quad x \in (0, \tfrac{\pi}{j}),\\ &u(0) = u(\tfrac{\pi}{j}) = 0, \end{aligned} \tag{7.8}$$

where $j \in \mathbb{N}^*$, $\|u_x\|_j^2 = \int_0^{\frac{\pi}{j}} |u_x(s)|^2 ds$ (the usual norm of the Banach space $L^2(0, \frac{\pi}{j})$), λ, a and f have the same conditions as before.

We say that $u \in H^1_0(0, \frac{\pi}{j})$ is a weak solution of the problem (7.8) if

$$a\left(\|u_x\|_j^2\right)\int_0^{\frac{\pi}{j}} u_x(s)v_x(s)ds - \lambda\int_0^{\frac{\pi}{j}} f(u(s))v(s)ds = 0, \quad \forall v \in H^1_0(0, \tfrac{\pi}{j}). \tag{7.9}$$

Note that a weak solution of (7.8) can be found as critical point of the energy functional $V_j : H^1_0(0, \frac{\pi}{j}) \to \mathbb{R}$ defined by

$$V_j(u) = \frac{1}{2}\int_0^{\|u_x\|_j^2} a(s)ds - \lambda\int_0^{\frac{\pi}{j}} F(u(s))ds, \quad \text{where} \quad F(s) := \int_0^s f(\xi)d\xi.$$

We use a standard calculation to show that

$$\langle V_j'(u), v\rangle = a\left(\|u_x\|_j^2\right)\int_0^{\frac{\pi}{j}} u_x(s)v_x(s)ds - \lambda\int_0^{\frac{\pi}{j}} f(u(s))v(s)ds, \quad \forall\, u, v \in H^1_0(0, \tfrac{\pi}{j}),$$

and if $V_j'(u) = 0$ for some $u \in H^1_0(0, \pi)$, then u is a weak solution for (7.8).

Lemma 19. If $\lambda > a(0)j^2$, there exists a nontrivial positive weak solution for the problem (7.8).

Proof. We restrict the domain of V_j to the space of positive functions in $H_0^1(0,\pi)$ which take values between 0 and $\bar{a}$, where $\bar{a}$ is the greatest real number (or infinite) such that $f(s)s > 0$ for $s \in (-\bar{a},0) \cup (0,\bar{a})$. Define

$$\mathscr{M} := \left\{ v \in H_0^1(0,\tfrac{\pi}{j}) : 0 \leq v(x) \leq \bar{a}, \forall x \in (0,\tfrac{\pi}{j}) \right\}.$$

As discussed before, $V_j(u) \geq C_1\|u_x\|_j^2 + C_2$ for constants $C_1 > 0$, $C_2 \in \mathbb{R}$. It follows that $V_j(u) \to \infty$ as $\|u_x\|_j \to \infty$.

Now we will prove that V_j is weakly lower semicontinuous on $H_0^1(0,\frac{\pi}{j})$, which means that for every sequence $\{u_n\}$ in $H_0^1(0,\frac{\pi}{j})$ such that $u_n \rightharpoonup u \in H_0^1(0,\frac{\pi}{j})$ (weak convergence in $H_0^1(0,\frac{\pi}{j})$), it holds

$$V_j(u) \leq \liminf_{n\to\infty} V_j(u_n).$$

Indeed, let $\{u_n\}$ be a sequence in $H_0^1(0,\frac{\pi}{j})$ that weakly converges to $u \in H_0^1(0,\frac{\pi}{j})$. Then, $\|u\|_{H_0^1(0,\frac{\pi}{j})} \leq \liminf_{n\to\infty} \|u_n\|_{H_0^1(0,\frac{\pi}{j})}$. By Fatou's Lemma:

$$\int_0^{\|u_x\|_j^2} a(s)ds \leq \int_0^{\left(\liminf_{n\to\infty}\|u_n'\|_j\right)^2} a(s)ds \leq \int_0^{\liminf_{n\to\infty}\|u_n'\|_j^2} a(s)ds \leq \liminf_{n\to\infty} \int_0^{\|u_n'\|_j^2} a(s)ds.$$

Since $H_0^1(0,\frac{\pi}{j})$ is compactly embedded in $L^2(0,\frac{\pi}{j})$, every subsequence of $\{u_n\}$ is bounded in $H_0^1(0,\frac{\pi}{j})$ and has a further subsequence $\{u_{n_k}\}$ which converges strongly to v in $L^2(0,\frac{\pi}{j})$, and since $\{u_{n_k}\}$ also converges weakly to u in $L^2(0,\frac{\pi}{j})$ (because of the weak convergence in $H_0^1(0,\frac{\pi}{j})$), we have $u = v$. Then every subsequence of $\{u_n\}$ has a further subsequence which converges strongly to u in $L^2(0,\frac{\pi}{j})$, and we conclude that $\{u_n\}$ converges strongly to u in $L^2(0,\frac{\pi}{j})$, which easily implies:

$$\int_0^{\pi} F(u(s))ds = \lim_{n\to\infty} \int_0^{\pi} F(u_n(s))ds.$$

It follows that $V_j(u) \leq \liminf_{n\to\infty} V_j(u_n)$, and V_j is weakly lower semicontinuous.

Clearly $\mathscr{M}$ is weakly closed, because it is convex and strongly closed. This means that $\mathscr{M}$ and V_j satisfy all the conditions of the Theorem 1.2 of (STRUWE, 2008). Then V_j attains a minimum u in $\mathscr{M}$. We will show that u is a weak solution for (7.8), by proving that $\langle V_j'(u), v\rangle = 0$ for all $v \in H_0^1(0,\pi)$. Let $\varphi \in C_c^\infty(0,\frac{\pi}{j})$ and $\varepsilon > 0$. We define $v_\varepsilon = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon$, where

$$\varphi^\varepsilon = \max\{0, u + \varepsilon\varphi - \bar{a}\} \geqslant 0 \qquad \text{and} \qquad \varphi_\varepsilon = \max\{0, -(u+\varepsilon\varphi)\} \geqslant 0.$$

Note that $\varphi^\varepsilon, \varphi_\varepsilon \in H_0^1(0,\frac{\pi}{j}) \cap L^\infty(0,\frac{\pi}{j})$, and $v_\varepsilon \in \mathscr{M}$.

Now, we have the following estimates

$$
\begin{aligned}
\langle V_j'(u),\varphi^\varepsilon\rangle &= a(\|u_x\|_j^2)\int_0^{\frac{\pi}{j}} u_x(s)\varphi_x^\varepsilon(s)ds-\lambda\int_0^{\frac{\pi}{j}} f(u(s))\varphi^\varepsilon(s)ds\\
&= a(\|u_x\|_j^2)\int_{\Omega^\varepsilon} u_x(s)(u_x+\varepsilon\varphi_x)(s)ds-\lambda\int_{\Omega^\varepsilon} f(u(s))(u+\varepsilon\varphi-\bar a)(s)ds\\
&\geqslant a(\|u_x\|_j^2)\int_{\Omega^\varepsilon} \varepsilon u_x(s)\varphi_x(s)ds-\lambda\int_{\Omega^\varepsilon} f(u(s))(u+\varepsilon\varphi-\bar a)(s)ds\\
&\geqslant a(\|u_x\|_j^2)\int_{\Omega^\varepsilon} \varepsilon u_x(s)\varphi_x(s)ds-\lambda\int_{\Omega^\varepsilon} f(u(s))\varepsilon\varphi(s)ds\\
&\geqslant -a(\|u_x\|_j^2)|\Omega^\varepsilon|\varepsilon\|u_x\|_{L^1(0,\frac{\pi}{j})}\|\varphi_x\|_{L^\infty(0,\frac{\pi}{j})}-\lambda\sup_{s\in[0,\frac{\pi}{j}]}|f(u(s))||\Omega^\varepsilon|\varepsilon\|\varphi\|_{L^\infty(0,\frac{\pi}{j})}\\
&= \left[-a(\|u_x\|_j^2)\|u_x\|_{L^1(0,\frac{\pi}{j})}\|\varphi_x\|_{L^\infty(0,\frac{\pi}{j})}-\lambda\sup_{s\in[0,\frac{\pi}{j}]}|f(u(s))|\|\varphi\|_{L^\infty(0,\frac{\pi}{j})}\right]\varepsilon|\Omega^\varepsilon|,
\end{aligned}
$$

where $\Omega^\varepsilon := \left\{x\in(0,\frac{\pi}{j}); u(x)+\varepsilon\varphi(x)\geqslant\bar a>u(x)\right\}$ satisfies $|\Omega^\varepsilon|\to 0$ when $\varepsilon\to 0^+$. Similarly

$$
\begin{aligned}
\langle V_j'(u),\varphi_\varepsilon\rangle &= a(\|u_x\|_j^2)\int_0^{\frac{\pi}{j}} u_x(s)(\varphi_\varepsilon)_x(s)ds-\lambda\int_0^{\frac{\pi}{j}} f(u(s))\varphi_\varepsilon(s)ds\\
&= -a(\|u_x\|_j^2)\int_{\Omega_\varepsilon} u_x(s)(u_x+\varepsilon\varphi_x)(s)ds+\lambda\int_{\Omega_\varepsilon} f(u(s))(u+\varepsilon\varphi)(s)ds\\
&\leqslant -a(\|u_x\|_j^2)\int_{\Omega_\varepsilon} \varepsilon u_x(s)\varphi_x(s)ds\\
&\leqslant a(\|u_x\|_j^2)\|u_x\|_{L^1(0,\frac{\pi}{j})}\|\varphi_x\|_{L^\infty(0,\frac{\pi}{j})}\varepsilon|\Omega_\varepsilon|,
\end{aligned}
$$

where $\Omega_\varepsilon := \left\{x\in(0,\frac{\pi}{j}); u(x)+\varepsilon\varphi(x)\leqslant 0<u(x)\right\}$ satisfies $|\Omega_\varepsilon|\to 0$ when $\varepsilon\to 0^+$.

Since u, $v_\varepsilon\in\mathscr{M}$, and $\mathscr{M}$ is convex, for any $h\in[0,1]$, $u+h(v_\varepsilon-u)\in\mathscr{M}$, so that $V_j(u)\le V_j(u+h(v_\varepsilon-u))$, and

$$
\langle V_j'(u),v_\varepsilon-u\rangle=\lim_{h\to 0^+}\frac{V_j(u+h(v_\varepsilon-u))-V_j(u)}{h}\ge 0.
$$

Noting that $v_\varepsilon-u=\varepsilon\varphi-\varphi^\varepsilon+\varphi_\varepsilon$, we conclude that

$$
\langle V_j'(u),\varphi\rangle\geqslant\frac{\langle V_j'(u),\varphi^\varepsilon\rangle-\langle V_j'(u),\varphi_\varepsilon\rangle}{\varepsilon}\geqslant C_1|\Omega^\varepsilon|+C_2|\Omega_\varepsilon|.
$$

For $C_1, C_2\in\mathbb{R}$. Since $\varepsilon>0$ can be arbitrarily small, we have $\langle V_j'(u),\varphi\rangle\geqslant 0$ for each $\varphi\in C_c^\infty(0,\frac{\pi}{j})$. But if $\varphi\in C_c^\infty(0,\frac{\pi}{j})$, we have $-\varphi\in C_c^\infty(0,\frac{\pi}{j})$ as well, so that $\langle V_j'(u),\varphi\rangle\le 0$. We conclude that $V_j'(u)$ vanishes in $C_c^\infty(0,\frac{\pi}{j})$, and since $V_j'(u):H_0^1(0,\frac{\pi}{j})\to\mathbb{R}$ is continuous and $C_c^\infty(0,\frac{\pi}{j})$ is dense in $H_0^1(0,\frac{\pi}{j})$, it follows that $V_j'(u)$ vanishes in $H_0^1(0,\frac{\pi}{j})$, and u is a positive weak solution for (7.8).

Finally, we will show that u is non-trivial. We do this by proving that V_j attains negative values in $\mathscr{M}$, so that $V_j(u)<0$ and consequently $u\neq 0$. In fact, consider the operator A :

$H^2(0,\frac{\pi}{j})\cap H_0^1(0,\frac{\pi}{j}) \to L^2(0,\frac{\pi}{j})$, given as $A\phi = -\phi''$, which has $\lambda_1 = j^2$ as its first (lowest) eigenvalue (see Example 1). Let ϕ be the eigenfunction associated to λ_1 such that $\phi'(0) = 1$. That is, $\phi \in H^2(0,\frac{\pi}{j})\cap H_0^1(0,\frac{\pi}{j})$ is solution to the following eigenvalue problem

$$\begin{aligned} -\phi_{xx} &= j^2\phi, \quad x \in (0,\tfrac{\pi}{j}), \\ \phi(0) &= \phi(\tfrac{\pi}{j}) = 0, \quad \phi'(0) = 1. \end{aligned}$$

Since $a(0)j^2 < \lambda$, it follows from the continuity of a, that there exist $\varepsilon > 0$, $\delta > 0$ such that $a(t)j^2 - \lambda < -\varepsilon$ for all $t \in \left[0, \delta^2 j^2 \|\phi\|_j^2\right]$. Note that ϕ is positive, and choosing δ small enough, we have $\delta\phi \in \mathscr{M}$. Now,

$$\|\phi_x\|_j^2 = \int_0^{\frac{\pi}{j}} \phi_x(s)^2 ds = -\int_0^{\frac{\pi}{j}} \phi_{xx}(s)\phi(s) ds = j^2\|\phi\|_j^2.$$

There exists a $c_\delta \in \left[0, \delta^2\|\phi_x\|_j^2\right] = \left[0, \delta^2 j^2\|\phi\|_j^2\right]$ such that

$$\int_0^{\|\delta\phi_x\|_j^2} a(s)ds = a(c_\delta)\|\delta\phi_x\|_j^2 = a(c_\delta)\delta^2 j^2\|\phi\|_j^2.$$

Moreover, $f(s) - s$ is quadratic near the origin, because $f'(0) = 1$, we write $f(s) - s = s^2 g(s)$, with g continuous in $\mathbb{R}^+$. Then we have:

$$\begin{aligned} V_j(\delta\phi) &= \frac{1}{2}\int_0^{\|\delta\phi_x\|_j^2} a(s)ds - \lambda \int_0^{\frac{\pi}{j}} \int_0^{\delta\phi(x)} f(\xi)d\xi dx \\ &= \frac{1}{2}\delta^2[a(c_\delta)j^2 - \lambda]\|\phi\|_j^2 - \lambda \int_0^{\frac{\pi}{j}} \int_0^{\delta\phi(x)} (f(\xi) - \xi)d\xi dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| V_j(\delta\phi) - \frac{1}{2}\delta^2\|\phi\|_j^2[a(c_\delta)j^2 - \lambda] \right| &\le \lambda \int_0^{\frac{\pi}{j}} \int_0^{|\delta\phi(x)|} \xi^2|g(\xi)|d\xi dx \\ &\le \lambda C \int_0^{\frac{\pi}{j}} \int_0^{|\delta\phi(x)|} \xi^2 d\xi dx = K\delta^3, \end{aligned} \tag{7.10}$$

where C and K are positive constants. If we take $\delta > 0$ small enough so that $K\delta < \frac{\|\phi\|_j^2\varepsilon}{4}$, the estimate (7.10) — along with the fact that $a(t)j^2 - \lambda < -\varepsilon$ for all $t \in [0, \delta^2 j^2\|\phi\|_j^2]$ — implies $V_j(\delta\phi) < 0$, and we are done.

□

Lemma 20. If $\lambda > a(0)j^2$, the nontrivial positive weak solution of the problem (7.8) is unique. Moreover, let u be the nontrivial positive weak solution of (7.8), then $u \in H^2(0,\frac{\pi}{j})\cap H_0^1(0,\frac{\pi}{j})$, $u \in \mathscr{C}^2(0,\frac{\pi}{j})$, and u satisfies (7.8) in the strong sense. Additionally, $u(x) = u(\frac{\pi}{j} - x)$, for all $x \in [0,\frac{\pi}{j}]$.

Proof. Let $u \in H_0^1(0,\frac{\pi}{j})$ be a weak solution of (7.8). Then $\frac{\lambda f(u(\cdot))}{a(\|u_x\|_j^2)} \in L^2(0,\frac{\pi}{j})$ because f and u are continuous functions. Since the Laplacian is surjective (see Example 1), it follows that there exists $w \in H^2(0,\frac{\pi}{j}) \cap H_0^1(0,\frac{\pi}{j})$ such that $-w_{xx} = \frac{\lambda f(u)}{a(\|u_x\|_j^2)}$. If we define the bilinear form $b : H_0^1(0,\frac{\pi}{j}) \times H_0^1(0,\frac{\pi}{j}) \to \mathbb{R}$ by

$$b(v,z) = a(\|u_x\|_j^2) \int_0^{\frac{\pi}{j}} v_x(s) z_x(s) ds,$$

it is easy to see that b is continuous and coercive, so Lax-Milgram's Theorem (BREZIS, 2011, Corollary 5.8) can be applied to $b(\cdot,\cdot)$. Now, note that

$$b(u,v) = b(w,v) = \int_0^{\frac{\pi}{j}} \lambda f(u(s)) v(s) ds, \quad \forall\, v \in H_0^1(0,\tfrac{\pi}{j}),$$

and the uniqueness given by Lax-Milgram's Theorem implies that $u = w$. Therefore, $u \in H^2(0,\frac{\pi}{j}) \cap H_0^1(0,\frac{\pi}{j})$, u satisfies (7.8), taking the derivatives in the sense of distributions, and $u \in \mathscr{C}^1(0,\frac{\pi}{j})$. Finally, $u_{xx} = -\frac{\lambda f(u)}{a(\|u_x\|_j^2)}$ is a continuous function, so that $u \in \mathscr{C}^2(0,\frac{\pi}{j})$, and u satisfies (7.8) with strong derivatives. Moreover, it follows from the Maximum Principle (PROTTER; WEINBERGER, 1984), that $u(x) > 0$ for $x \in (0,\frac{\pi}{j})$. Finally, note that $\eta(\cdot) = u(\frac{\pi}{j} - \cdot)$ is also nontrivial, positive and satisfies (7.8). So, if we prove the first claim, the second will follow.

Let u and v be two distinct nontrivial positive solutions of (7.8). By the observations above, we can show that $\frac{u^2}{v}, \frac{v^2}{u} \in H_0^1(0,\frac{\pi}{j})$. Thus, since a is non-decreasing,

$$\begin{aligned}
0 \;&\leqslant\; \left(a(\|u_x\|_j^2) - a(\|v_x\|_j^2)\right)\left(\|u_x\|_j^2 - \|v_x\|_j^2\right) + a(\|v_x\|_j^2) \int_0^{\frac{\pi}{j}} \left(u_x(s) - \frac{u(s)}{v(s)} v_x(s)\right)^2 ds + \\
&\quad a(\|u_x\|_j^2) \int_0^{\frac{\pi}{j}} \left(v_x(s) - \frac{v(s)}{u(s)} u_x(s)\right)^2 dx \\
&= a(\|u_x\|_j^2) \int_0^{\frac{\pi}{j}} u_x^2(s) ds - a(\|u_x\|_j^2) \int_0^{\frac{\pi}{j}} u_x(s) \left(\frac{v^2}{u}\right)_x (s) ds + a(\|v_x\|_j^2) \int_0^{\frac{\pi}{j}} v_x^2(s) ds \\
&\quad - a(\|v_x\|_j^2) \int_0^{\frac{\pi}{j}} v_x(s) \left(\frac{u^2}{v}\right)_x (s) ds \\
&= \lambda \int_0^{\frac{\pi}{j}} f(u(s)) \left(u - \frac{v^2}{u}\right)(s) ds + \lambda \int_0^{\frac{\pi}{j}} f(v(s)) \left(v - \frac{u^2}{v}\right)(s) ds \\
&= \lambda \int_0^{\frac{\pi}{j}} \left(\frac{f(u(s))}{u(s)} - \frac{f(v(s))}{v(s)}\right) (u(s)^2 - v(s)^2) ds.
\end{aligned}$$

If $u \neq v$, then we arrive at a contradiction because of Condition (7.4), and we are done.

□

We are ready to find the equilibria for equation (7.1), which are precisely the functions $u \in H^2(0,\pi) \cap H_0^1(0,\pi)$ that satisfy

$$\begin{aligned}
&a(\|u_x\|^2) u_{xx} + \lambda f(u) = 0 \quad x \in (0,\pi), \\
&u(0) = u(\pi) = 0.
\end{aligned} \tag{7.11}$$

First, if $0 < \lambda \leq a(0)$, let u be an equilibrium for (7.1), then u is a solution for the following problem:

$$\begin{aligned} &w_{xx} + \lambda_0 f(w) = 0 \quad x \in (0,\pi), \\ &w(0) = w(\pi) = 0. \end{aligned}, \tag{7.12}$$

where $\lambda_0 = \frac{\lambda}{a(\|u_x\|^2)} > 0$. Therefore, u is an equilibrium for the classical Chafee-Infante equation (6.1). Since a is non-decreasing, $0 < \lambda / a(\|u_x\|^2) \leq 1$, and it follows from Theorem 39 that $u = 0$. Hence, for $\lambda \leq a(0)$, the origin $0 \in H_0^1(0,\pi)$ is the only equilibrium for (7.1).

In the case $a(0) < \lambda$, we can apply Lemma 19 and Lemma 20 with $j = 1$, and find a positive nontrivial equilibrium for (7.1), which we denote ϕ_1^+. Observe that $\phi_1^- := -\phi_1^+$ is also an equilibrium for (7.1). The solutions $\phi_0 = 0$, ϕ_1^+ and ϕ_1^- are in fact the only equilibria for (7.1) provided that $a(0) < \lambda \leq 2^2 a(0)$. Indeed, if u is an equilibrium for (7.1), then u is a solution for the classical Chafee-Infante equilibrium problem (7.12), and since $\lambda / a(\|u_x\|^2) \leq 2^2$, it follows from Theorem 39 that u may be the origin, or $u > 0$ in $(0,\pi)$, or $u < 0$ in $(0,\pi)$. If $u > 0$ in $(0,\pi)$, it follows from the uniqueness in Lemma 20 that $u = \phi_1^+$. If $u < 0$ in $(0,\pi)$, then we may apply Lemma 20 to $v = -u$ to conclude that $v = \phi_1^+$, then $u = \phi_1^-$, and we are done.

Now suppose $a(0)2^2 < \lambda$, then we have the equilibria 0, ϕ_1^+ and ϕ_1^-, and we can also construct a pair of equilibria that change sign one time. For that, we are going to restrict ourselves to the following problem in $[0, \frac{\pi}{2}]$:

$$\begin{aligned} &a(2\|u_x\|_2^2) u_{xx} + \lambda f(u) = 0, \quad x \in (0, \tfrac{\pi}{2}) \\ &u(0) = u(\tfrac{\pi}{2}) = 0. \end{aligned} \tag{7.13}$$

It follows from Lemma 19 and Lemma 20 that (7.13) has a unique nontrivial positive solution $\phi_{\frac{\pi}{2}} \in H^2(0, \frac{\pi}{2}) \cap H_0^1(0, \frac{\pi}{2})$, with $\phi_{\frac{\pi}{2}}(0) = \phi_{\frac{\pi}{2}}(\frac{\pi}{2}) = 0$, and this solution satisfies $\phi_{\frac{\pi}{2}}(x) = \phi_{\frac{\pi}{2}}(\frac{\pi}{2} - x)$ for all $x \in (0, \frac{\pi}{2})$.

Then, we define $\phi_2^+(x) = \begin{cases} \phi_{\frac{\pi}{2}}(x), \text{ if } x \in [0, \frac{\pi}{2}], \\ -\phi_{\frac{\pi}{2}}(\pi - x), \text{ if } x \in [\frac{\pi}{2}, \pi]. \end{cases}$

Notice that ϕ_2^+ is continuous, $\phi_2^+ \in \mathscr{C}^2((0,\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi))$, and the derivative of ϕ_2^+ in $(0,\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ has coincident lateral limits in $\pi/2$. This may be used to show that $\phi_2^+ \in H^2(0,\pi) \cap H_0^1(0,\pi)$, and ϕ_2^+ is a solution for (7.11). Also, $\phi_2^+(\pi - x) = -\phi_2^+(x)$, for all $x \in (0,\pi)$ and $\phi_2^+(\frac{\pi}{2} - x) = \phi_2^+(x)$ for all $x \in [0, \frac{\pi}{2}]$. We also define $\phi_2^- = -\phi_2^+$, which is another equilibrium for (7.1). If $a(0)2^2 < \lambda \leq a(0)3^2$, any solution u of (7.11) is also a solution for the classical Chafee-Infante equilibrium equation (7.12) with parameter $\lambda / a(\|u_x\|^2) \leq 3^2$, and with a reasoning similar as before we conclude that u may only be one of the solutions of the set $\{0, \phi_1^-, \phi_1^+, \phi_2^-, \phi_2^+\}$ (in this case we also use the uniqueness of a positive nontrivial solution of (7.13)).

An inductive argument — finding a solution for a problem in $[0, \frac{\pi}{j}]$ and constructing an oscillatory equilibrium — can be applied and we can show that $a(0)j^2$ is a bifurcation point of the parameter $\lambda > 0$, and we can always show — using what we know for the classical Chafee-Infante equation – that these constructed equilibria are the only possible. We summarize the results in the following:

Theorem 44. If $\lambda \leq a(0)$, the origin $0 \in H_0^1(0,\pi)$ is the only equilibrium for (7.1). Let $N \in \mathbb{N}^*$, and $a(0)N^2 < \lambda \leq a(0)(N+1)^2$, then there are precisely $2N+1$ equilibria for (7.1), which we denote:

$$\{0\} \cup \{\phi_k^{\pm} : k = 1, \ldots, N\},$$

where ϕ_k^+ and ϕ_k^- have $k+1$ zeros in $[0,\pi]$, $\phi_k^- = -\phi_k^+$, and $\phi_k^+(x) > 0$ for $x \in (0, \frac{\pi}{k})$, for $k = 1, \ldots, N$.

7.3 Spectrum properties for a non-local operator

For $\phi \in \mathscr{E}$, we intend to find an expression for the linearization of equation (7.5) around ϕ. Let $h \in H_0^1(0,\pi)$, and let us calculate the Gateaux Derivative of $H_0^1(0,\pi) \ni w \mapsto \frac{f(w)}{a(\|w_x\|^2)} \in L^2(0,\pi)$ in the value ϕ and direction h:

$$\lim_{t\to 0^+} \frac{1}{t}\left[\frac{f(\phi+th)}{a(\|(\phi+th)_x\|^2)} - \frac{f(\phi)}{a(\|\phi_x\|^2)}\right] = \frac{f'(\phi)h}{a(\|\phi_x\|^2)} + \underbrace{\lim_{t\to 0^+} \frac{f(\phi)}{t}\left[\frac{a(\|\phi_x\|^2) - a(\|(\phi+th)_x\|^2)}{a(\|(\phi+th)_x\|^2)a(\|\phi_x\|^2)}\right]}_{=\gamma}$$

and this second limit yields:

$$\begin{aligned}
\gamma &= \lim_{t\to 0^+} -\frac{f(\phi)}{t}\frac{a'(\theta(t)\|\phi_x\|^2 + (1-\theta(t))\|(\phi+th)_x\|^2)}{a(\|(\phi+th)_x\|^2)a(\|\phi_x\|^2)}(\|(\phi+th)_x\|^2 - \|\phi_x\|^2)\\
&= \lim_{t\to 0^+} -\frac{f(\phi)}{t}\frac{a'\left(\int_0^\pi |\phi_x|^2 ds + (1-\theta(t))t\int_0^\pi (2\phi_x h_x + th_x^2)ds\right)}{a(\|(\phi+th)_x\|^2)a(\|\phi_x\|^2)}\left(t\int_0^\pi (2\phi_x h_x + th_x^2)ds\right)
\end{aligned}$$

where $\theta(t) \in [0,1]$ for each $t > 0$. Passing the limit in this expression, we conclude that:

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\left[\frac{f(\phi+th)}{a(\|(\phi+th)_x\|^2)} - \frac{f(\phi)}{a(\|\phi_x\|^2)}\right] = \frac{f'(\phi)h}{a(\|\phi_x\|^2)} - \frac{2f(\phi)a'(\|\phi_x\|^2)}{a(\|\phi_x\|^2)^2}\int_0^\pi \phi_x h_x ds\\
&= \frac{f'(\phi)h}{a(\|\phi_x\|^2)} - \frac{\lambda 2a'(\|\phi_x\|^2)f(\phi)}{a(\|\phi_x\|^2)^3}\int_0^\pi f(\phi(s))h(s)ds
\end{aligned}$$

where in the second equality we integrated by parts and used that ϕ is a solution of (7.5).

It follows that the linearization operator around ϕ for the semilinear equation (7.5) is given by the formula below — remember that we work with the operator with inverted sign, as in Chapter 5.

$$\begin{aligned}
&L_\varepsilon^\phi : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)\\
&L_\varepsilon^\phi u = -u'' - \frac{\lambda f'(\phi)}{a(\|\phi'\|^2)}u + \varepsilon f(\phi)\int_0^\pi f(\phi(s))u(s)ds,
\end{aligned}$$

in the particular case $\varepsilon = \frac{\lambda^2 2a'(\|\phi'\|^2)}{a(\|\phi'\|^2)^3} \geq 0$.

Note that since f is odd and f' is even, the linearization operator associated to an equilibrium $\phi \in \mathscr{E}$ is the same as the linearization operator associated to the equilibrium $-\phi$. Hence, we only need to study stability and hyperbolicity for the equilibria ϕ_k^+, $k = 1, 2, ..., N$ given by Theorem 44, and the results will be identical for their negative counterparts. We will denote for simplicity $\phi_k^+ = \phi_k$.

To obtain information about the spectrum of L_ε^ϕ, we interpret it as a bounded perturbation of the Sturm-Liouville operator

$$\begin{aligned} &L_0^\phi : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi) \\ &L_0^\phi u = -u'' - \frac{\lambda f'(\phi)}{a(\|\phi'\|^2)} u, \end{aligned} \tag{7.14}$$

whose spectrum is very well understood. In fact, an equilibrium ϕ for (7.5) is also an equilibrium for the following semilinear Chafee-Infante problem:

$$\begin{aligned} &w_\tau = w_{xx} + \frac{\lambda f(w)}{a(\|\phi'\|^2)}, \quad \tau > 0,\ x \in (0,\pi) \\ &w(0,\tau) = w(\pi,\tau) = 0, \quad \tau \geq 0 \\ &w(\cdot,0) = u_0 \in H_0^1(0,\pi). \end{aligned} \tag{7.15}$$

The linearization operator of ϕ for (7.15) is given by L_0^ϕ. Hence, from the results in Section 6.3, we know the following.

Lemma 21. Let ϕ be a nonzero equilibrium for (7.1), and define the operator L_0^ϕ as in (7.14). Then:

1. If $\phi(x) > 0$ in $(0,\pi)$, then L_0^ϕ has only strictly positive eigenvalues.
2. If ϕ has a zero in $(0,\pi)$, then L_0^ϕ has at least one strictly negative eigenvalue.
3. $0 \in \rho(L_0^\phi)$.

In order to discover relations between the spectrum of L_ε^ϕ and the spectrum of L_0^ϕ, we will use results from (DAVIDSON; DODDS, 2006) about the following non-local operator.

$$\begin{aligned} &L_\varepsilon : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi) \\ &L_\varepsilon u = -u'' + p(x)u + \varepsilon c(x) \int_0^\pi c(s)u(s)ds, \end{aligned} \tag{7.16}$$

where p, $c : [0,\pi] \to \mathbb{R}$ are continuous functions with $c \not\equiv 0$.

When $\varepsilon = 0$, $L_0 u = -u'' + p(x)u$ is a Sturm-Liouville operator, and by the results in Example 2, L_0 is a self-adjoint operator, with compact resolvent, and its sequence of eigenvalues accumulates at ∞. Let $\sigma(L_0) = \{\mu_j : j = 1,2,3,...\}$ denote its spectrum, with $\mu_j > \mu_{j-1}$ and $\mu_j \to \infty$ as $j \to \infty$.

Theorem 45. For any $\varepsilon \in \mathbb{R}$, the operator L_ε given by (7.16) is self-adjoint and has compact resolvent. Moreover, its sequence of eigenvalues $\{\mu_j(\varepsilon)\}_{j\in\mathbb{N}^*}$ accumulates at $+\infty$ and has a lower bound.

Proof. Let $\alpha = \left|\inf_{x\in[0,\pi]} p(x)\right|$ and $\beta = |\varepsilon|\|c\|^2$, and consider instead the operator $K_\varepsilon = L_\varepsilon + \alpha I + \beta I$. We will prove all the claims for K_ε, and L_ε will inherit the same properties. Define $r : [0,\pi] \to \mathbb{R}$ by $r(x) = p(x) + \alpha \geq 0$, and note that

$$K_\varepsilon u = -u'' + r(x)u + \varepsilon c(x) \int_0^\pi c(s)u(s)ds + \beta u.$$

Note that, for any $u \in D(K_\varepsilon) = H^2(0,\pi) \cap H_0^1(0,\pi)$,

$$\langle K_\varepsilon u, u\rangle = \|u\|_{H_0^1}^2 + \int_0^\pi r(x)u(x)^2 dx + \underbrace{\varepsilon \left(\int_0^\pi c(s)u(s)ds\right)^2 + \beta\|u\|^2}_{\geq 0} \geq \|u\|_{H_0^1}^2 \geq \|u\|^2. \quad (7.17)$$

It is easy to see that K_ε is symmetric. We will show that it is surjective. Indeed, consider the bilinear application $a : H_0^1(0,\pi) \times H_0^1(0,\pi) \to \mathbb{R}$ given by

$$a(u,v) = \int_0^\pi u_x v_x ds + \int_0^\pi r(s)u(s)v(s)ds + \varepsilon \int_0^\pi c(s)v(s)ds \int_0^\pi c(s)u(s)ds + \beta \int_0^\pi u(s)v(s)ds.$$

The application a is trivially continuous ($|a(u,v)| \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}$, for some $C \geq 0$, for all u, $v \in H_0^1(0,\pi)$). To see that it is also coercive, just note that:

$$a(u,u) = \langle K_\varepsilon u, u\rangle \geq \|u\|_{H_0^1}^2, \quad \forall u \in H_0^1(0,\pi).$$

It follows from Lax-Milgram's Theorem (BREZIS, 2011, Corollary 5.8) that for each $f \in L^2(0,\pi)$, there exists $u \in H_0^1(0,\pi)$ such that $a(u,v) = \langle f, v\rangle$ for all $v \in H_0^1(0,\pi)$. Next we need to show that $u \in H^2(0,\pi)$ and that $K_\varepsilon u = f$.

Let $b : H_0^1(0,\pi) \times H_0^1(0,\pi) \to \mathbb{R}$ be the continuous coercive bilinear application given by

$$b(w,v) = \int_0^\pi w_x v_x ds + \int_0^\pi r(s)w(s)v(s)ds,$$

and define $h \in L^2(0,\pi)$ by $h(x) = f(x) - \varepsilon c(x)\left(\int_0^\pi c(s)u(s)ds\right) - \beta u(x)$. If the operator $L : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$ is given by $Lw = -w'' + r(x)w$, it has been already proved that $0 \in \rho(L)$ (see Example 2). Therefore, there exists $w \in H^2(0,\pi) \cap H_0^1(0,\pi)$ such that $Lw = h$. But note that $b(u,v) = b(w,v) = \langle h, v\rangle$, for all $v \in H_0^1(0,\pi)$, and by the uniqueness assured by Lax-Milgram's Theorem, we conclude that $u = w$, $u \in H^2(0,\pi) \cap H_0^1(0,\pi)$, and $K_\varepsilon u = f$.

This proves that K_ε is surjective and self-adjoint. Since by (7.17) K_ε is also injective, we have $0 \in \rho(K_\varepsilon)$.

If B is a bounded set in $D(K_\varepsilon)$, with the norm of the graph, then it follows from (7.17) that $\sup_{\phi\in B}\|\phi'\| < \infty$. From Arzelà-Ascoli's Theorem, B is precompact in $\mathscr{C}([0,\pi],\mathbb{R})$, therefore B is precompact in $L^2(0,\pi)$. From Proposition 2, K_ε has compact resolvent.

The fact that the spectrum of K_ε accumulates at $+\infty$ and has a lower bound follows from the second part of Theorem 12, and from the fact that K_ε needs to have infinitely many eigenvalues because its inverse is compact, as we discussed in Example 2.

This completes the proof.

□

In what follows, $\{\mu_j\}$ will denote the sequence of eigenvalues of L_0, and $\{\mu_j(\varepsilon)\}$, the sequence of eigenvalues of L_ε, with $\mu_j(0) = \mu_j$ for each $j \in \mathbb{N}^*$. The reader may check (DAVIDSON; DODDS, 2006, Lemma 2.1), for results about continuity of the spectrum and eigenprojections of L_ε with respect to ε. In what follows, we present two results from this same article about the behavior and geometric multiplicity of the eigenvalues of L_ε.

Theorem 46. Suppose that, for some $\varepsilon \in \mathbb{R}$ and $j \in \mathbb{N}^*$, we have $\mu_j(\varepsilon) \neq \mu_k$ for every $k \in \mathbb{N}^*$. Then $\mu_j(\varepsilon)$ is a simple eigenvalue for L_ε.

Proof. Suppose $\mu_j(\varepsilon)$ is associated to two linearly independent eigenvectors, namely u and v. Then, for any a, $b \in \mathbb{R}$,

$$\begin{aligned}
\mu_j(\varepsilon)(au+bv) &= L_\varepsilon(au+bv)\\
&= L_0(au+bv)+\varepsilon c(x)\int_0^\pi c(s)[au(s)+bv(s)]ds\\
&= L_0(au+bv)+\varepsilon c(x)\left(a\int_0^\pi c(s)u(s)ds+b\int_0^\pi c(s)v(s)ds\right)
\end{aligned}$$

Then we may choose a and b, at least one of them being nonzero, such that this last term between parenthesis is equal to zero. For this pair a, b, we would have $au+bv \not\equiv 0$ and $\mu_j(\varepsilon)(au+bv) = L_0(au+bv)$, which implies that $\mu_j(\varepsilon)$ is an eigenvalue for L_0, contradicting the hypothesis.

□

It is proved in (DAVIDSON; DODDS, 2006) that the eigenvalues of L_ε have a unique corresponding eigenvector other than at points of intersection of a moving eigenvalue with a fixed eigenvalue (in relation to ε). If for some $\varepsilon^* \in \mathbb{R}$, $\mu_j(\varepsilon^*)$ has an unique normalized eigenvector, we denote it by $v_j(\varepsilon^*)$. If, other than that, we have an intersection of eigenvalues (that is, $\mu_j(\varepsilon^*) = \mu_k(\varepsilon^*)$, for some $k \neq j$), we can define

$$v_j(\varepsilon^*) = \lim_{\varepsilon \to \varepsilon^*} v_j(\varepsilon),$$

which is an eigenvector for $\mu_j(\varepsilon^*)$ by the continuity results in this same article.

Theorem 47. For each $j \in \mathbb{N}$, $\mu_j(\varepsilon)$ is continuously differentiable with respect to ε in $\mathbb{R}$. Moreover,

$$\mu_j'(\varepsilon) = +\frac{\left[\int_0^\pi c(x)[v_j(\varepsilon)](x)dx\right]^2}{\int_0^\pi [v_j(\varepsilon)]^2(x)dx}, \quad \forall \varepsilon \in \mathbb{R}, \tag{7.18}$$

and $\mu_j(\varepsilon)$ is non-decreasing in relation to ε.

Proof. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. Then

$$-[v_j(\varepsilon_1)]''(x) + p(x)[v_j(\varepsilon_1)](x) + \varepsilon_1 c(x) \int_0^\pi c(s)[v_j(\varepsilon_1)](s)ds = \mu_j(\varepsilon_1)[v_j(\varepsilon_1)](x),$$

and

$$-[v_j(\varepsilon_2)]''(x) + p(x)[v_j(\varepsilon_2)](x) + \varepsilon_2 c(x) \int_0^\pi c(s)[v_j(\varepsilon_2)](s)ds = \mu_j(\varepsilon_2)[v_j(\varepsilon_2)](x),$$

Multiplying these equations by $[v_j(\varepsilon_2)](x)$ and $[v_j(\varepsilon_1)](x)$, respectively, integrating between 0 and π and then subtracting yields:

$$(\varepsilon_1 - \varepsilon_2) \int_0^\pi c(s)[v_j(\varepsilon_1)](s)ds \int_0^\pi c(s)[v_j(\varepsilon_2)](s)ds = (\mu_j(\varepsilon_1) - \mu_j(\varepsilon_2)) \int_0^\pi [v_j(\varepsilon_1)][v_j(\varepsilon_2)]ds,$$

which implies

$$\frac{(\mu_j(\varepsilon_1) - \mu_j(\varepsilon_2))}{(\varepsilon_1 - \varepsilon_2)} = \frac{\int_0^\pi c(s)[v_j(\varepsilon_1)](s)ds \int_0^\pi c(s)[v_j(\varepsilon_2)](s)ds}{\int_0^\pi [v_j(\varepsilon_1)](s)[v_j(\varepsilon_2)](s)ds}$$

Passing to the limit and using the continuity results for the spectrum and eigenprojections of L_ε (DAVIDSON; DODDS, 2006) this expression yields (7.18). □

The last theorems are valid for L_ε^ϕ, because the functions $f'(\phi)$ and $f(\phi)$ satisfy what is asked.

Now consider the problem (7.5) with $a(0)N^2 < \lambda \leq a(0)(N+1)^2$. For each solution $\phi_k := \phi_k^+$, $k = 1, 2, ..., N$, remember that the linearization operator of the problem (7.5) around ϕ_k is given by

$$L_{\varepsilon_k}^{\phi_k} u = -u'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} u + \varepsilon_k f(\phi_k) \int_0^\pi f(\phi_k(s))u(s)ds,$$

where $\varepsilon_k = \frac{\lambda^2 2a'(\|\phi_k'\|^2)}{a(\|\phi_k'\|^2)^3}$.

Let $\{\mu_j^k\}_{j \in \mathbb{N}^*}$ denote the increasing sequence of eigenvalues of the Sturm-Liouville operator $L_0^{\phi_k}$. Let v_j denote the eigenvector associated to μ_j^k that satisfies $v_j'(0) = 1$.

Lemma 22. The eigenvectors of $L_0^{\phi_k}$ satisfy the following symmetry condition: for $x \in \left[0, \frac{\pi}{2}\right]$,

$$v_j(x+\tfrac{\pi}{2}) = \begin{cases} v_j(\frac{\pi}{2}-x), & \text{if } j \text{ is odd} \\ -v_j(\frac{\pi}{2}-x), & \text{if } j \text{ is even.} \end{cases} \tag{7.19}$$

Proof. It is easy to see that $\eta_j(\cdot) = v_j(\pi - \cdot)$ is another eigenvector associated to μ_j^k. Indeed,

$$\begin{aligned} \left(L_0^{\phi_k}\eta_j\right)(x) &= -\eta_j''(x) - \frac{\lambda f'(\phi_k(x))}{a(\|\phi_k'\|^2)}\eta_j(x) \\ &= -v_j''(\pi-x) - \frac{\lambda f'(\phi_k(\pi-x))}{a(\|\phi_k'\|^2)}v_j(\pi-x) \\ &= \left(L_0^{\phi_k}v_j\right)(\pi-x) = \mu_j^k v_j(\pi-x) = \mu_j^k\eta_j(x) \end{aligned}$$

Since the eigenvalues of Sturm-Liouville operators are simple (see Example 2), there exists a constant $c \in \mathbb{R}$ such that $v_j(\pi-x) = cv_j(x)$, for all $x \in [0,\pi]$. Then

$$\int_0^\pi [v_j(\pi-x)]^2 dx = c^2\int_0^\pi [v_j(x)]^2 dx \Rightarrow \|v\|^2 = c^2\|v\|^2 \Rightarrow c = \pm 1$$

Because of the behavior of Sturm-Liouville eigenvectors presented in Theorem 17, we conclude that for $s \in [0,\pi]$,

$$v_j(\pi-s) = \begin{cases} v_j(s), & \text{if } j \text{ is odd} \\ -v_j(s), & \text{if } j \text{ is even,} \end{cases}$$

which implies (7.19) if you take $s = \frac{\pi}{2} - x$.

□

Using the symmetry of the eigenvectors of the Sturm-Liouville operator, and the symmetry of the solutions of (7.1), we can prove a further relation between the spectrum of $L_{\varepsilon_k}^{\phi_k}$ and $L_0^{\phi_k}$.

Theorem 48 (From (CARVALHO; MOREIRA, 2021))**.** Let $\phi_k \in \mathscr{E}$. If $\{\mu_j^k\}_j$ denotes the increasing sequence of eigenvalues of $L_0^{\phi_k}$, the following holds:

1. If k is even, μ_{2j-1}^k is an eigenvalue for $L_{\varepsilon_k}^{\phi_k}$, and v_{2j-1} is an associated eigenvector, for all $j \in \mathbb{N}^*$.
2. If k is odd, μ_{2j}^k is an eigenvalue for $L_{\varepsilon_k}^{\phi_k}$, and v_{2j} is an associated eigenvector, for all $j \in \mathbb{N}^*$.

Proof. 1. Recall that f is odd, and since k is even, the solution ϕ_k has the following symmetry: $\phi_k(x+\frac{\pi}{2}) = -\phi_k(\frac{\pi}{2}-x)$, for all $x \in [0,\frac{\pi}{2}]$. Moreover, if j is odd, $v_j(x+\frac{\pi}{2}) = v_j(\frac{\pi}{2}-x)$, for all $x \in [0, \frac{\pi}{2}]$, then:

$$\begin{aligned}\int_0^{\pi} f(\phi_k(s))v_j(s)ds &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v_j(s)ds + \int_{\frac{\pi}{2}}^{\pi} f(\phi_k(s))v_j(s)ds \\ &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v_j(s)ds + \int_0^{\frac{\pi}{2}} f(\phi_k(\tfrac{\pi}{2}+s))v_j(\tfrac{\pi}{2}+s)ds \\ &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v_j(s)ds + \int_0^{\frac{\pi}{2}} f(-\phi_k(\tfrac{\pi}{2}-s))v_j(\tfrac{\pi}{2}-s)ds \\ &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v_j(s)ds - \int_0^{\frac{\pi}{2}} f(\phi_k(s))v_j(s)ds = 0.\end{aligned}$$

Therefore, $L^{\phi_k}_{\varepsilon_k} v_j = L^{\phi_k}_{0} v_j = \mu^k_j v_j$, and we are done.

2. The second claim can be proved the same way observing that if k is odd and j is even, then $\phi_k(\frac{\pi}{2}+x) = \phi_k(\frac{\pi}{2}-x)$ and $v_j(\frac{\pi}{2}+x) = -v_j(\frac{\pi}{2}-x)$, for all $x \in [0, \frac{\pi}{2}]$.

□

7.4 Stability of equilibria

Let us start by the origin. If $\lambda \leq a(0)$, it follows from Theorem 44 that ϕ_0 is the only equilibrium for (7.5), and also the only equilibrium for (7.1). Since $\mathscr{T}$ — the semigroup associated with equation (7.1) — is a gradient semigroup, the same argument as the one we used in the classical case may be used to show that $\mathscr{A} = \{\phi_0\}$, so that ϕ_0 attracts any bounded set of $H^1_0(0,\pi)$, and is asymptotically stable.

Since $f(0) = 0$ and $f'(0) = 1$, the linearization operator associated to $\phi_0 \equiv 0$ is given by $L^0_{\varepsilon_0} = -u'' - \frac{\lambda}{a(0)}u$, and its spectrum is $\sigma(L^0_{\varepsilon_0}) = \{n^2 - \frac{\lambda}{a(0)} : n \in \mathbb{N}^*\}$. If $\lambda < a(0)$, then ϕ_0 is exponentially stable for equation (7.5), and we will prove that it is also exponentially stable for (7.1). Indeed, there exists V neighborhood of $0 \in H^1_0(0,\pi)$ and constants K, $\beta > 0$ such that for every solution w of (7.5) with $w(0) = u_0 \in V$, we have

$$\|w(\tau) - 0\|_{H^1_0} \leq K e^{-\beta\tau}\|u_0 - 0\|_{H^1_0}, \quad \forall\, \tau \geq 0$$

The solution of (7.1) starting on u_0 is $u : \mathbb{R}^+ \to H^1_0(0,\pi)$ given by $u(t) = w(\tau)$, where $t = \int_0^{\tau} a(\|w(\theta,u_0)\|^2_{H^1_0})^{-1} d\theta$. Note that $\frac{\tau}{M} \leq t \leq \frac{\tau}{m}$, then:

$$\|u(t) - 0\|_{H^1_0} = \|w(\tau) - 0\|_{H^1_0} \leq K e^{-\beta\tau}\|u_0 - 0\|_{H^1_0} \leq K e^{-\beta m t}\|u_0 - 0\|_{H^1_0}.$$

Therefore, if $\lambda < a(0)$, ϕ_0 $\mathscr{T}$-attracts exponentially a neighborhood of itself in $H^1_0(0,\pi)$.

Now let $\lambda > a(0)$. Then ϕ_0 is exponentially unstable for (7.5), and by the constructions made in Chapter 5, there exists K, $\beta > 0$ and $\delta_0 > 0$ such that for each $\delta < \delta_0$, we may find a global solution $\eta:\mathbb{R}\to H^1_0(0,\pi)$ of $\{S(\tau):\tau\geq 0\}$ through $u_0\in H^1_0(0,\pi)$, $\|u_0\|_{H^1_0}<\delta$, that satisfies

$$\|\eta(\tau)-0\|_{H^1_0}\leq\delta,\quad \forall\tau\leq 0,$$
$$\|\eta(\tau)-0\|_{H^1_0}\leq Ke^{\beta\tau}\|u_0-0\|_{H^1_0},\quad \forall\tau\leq 0.$$

Then there exists a global solution $\xi:\mathbb{R}\to H^1_0(0,\pi)$ of $\{T(t):t\geq 0\}$ through u_0, given by $\xi(t)=\eta(\tau)$, for $t=\int_0^\tau a(\|\eta(\theta)\|^2_{H^1_0})^{-1}d\theta$. It satisfies:

$$\|\xi(t)-0\|_{H^1_0}=\|\eta(\tau)-0\|_{H^1_0}\leq Ke^{\beta\tau}\|u_0-0\|_{H^1_0}\leq Ke^{\beta mt}\|u_0-0\|_{H^1_0},\quad t\leq 0$$

Therefore, if $\lambda>a(0)$, then ϕ_0 is also unstable for (7.1).

Now let us study the stability of the equilibrium ϕ_1. The fact that the operator $L^{\phi_1}_0$ only has positive eigenvalues (because of the asymptotic stability of the positive equilibrium in Chafee-Infante classical problem), and the non-decreasing behavior of the eigenvalues of the non-local operator in relation to the parameter ε will guarantee that the positive equilibrium ϕ_1 is asymptotically stable, as in the classical case.

Theorem 49. Let $a(0)<\lambda$, then ϕ_1 is exponentially asymptotically stable for the problem (7.1).

Proof. Note that the linearization operator around this equilibrium is given by:

$$L^{\phi_1}_{\varepsilon_1}u=-u''-\frac{\lambda f'(\phi_1)}{a(\|\phi_1'\|^2)}u+\varepsilon_1 f(\phi_1)\int_0^\pi f(\phi_1(s))u(s)ds,$$

where $\varepsilon_1=\frac{2\lambda^2 a'(\|\phi_1'\|^2)}{a(\|\phi_1'\|^2)^3}\geq 0$, because a is non-decreasing. Denote by $\{\mu^1_j(\varepsilon_1)\}$ its eigenvalues.

From Lemma 21, $L^{\phi_1}_0$ only has strictly positive eigenvalues because it is the operator of linearization associated to the positive equilibrium of a Chafee-Infante classical problem. From Theorem 47, it follows that for each $j\in\mathbb{N}^*$, $\mu^1_j(\varepsilon)$ is non-decreasing in relation to ε, whence $\mu^1_j(\varepsilon_1)\geq\mu^1_j\geq\mu^1_1>0$. Then the eigenvalues of $L^{\phi_1}_{\varepsilon_1}$ have a strictly positive lower bound, and ϕ_1 is exponentially asymptotically stable in $H^1_0(0,\pi)$, for the equation (7.5). Proceeding exactly as before, with the change in time scale, we conclude that ϕ_1 is also exponentially asymptotically stable for equation (7.1).

□

For the equilibria ϕ_k with $k\geq 2$, the non-decreasing behavior of the eigenvalues in relation to ε do not actually say much, because an eigenvalue of $L^{\phi_k}_0$ can be negative and the correspondent eigenvalue of $L^{\phi_k}_{\varepsilon_k}$ can be positive. Then, we need to use Theorem 48 to spot negative eigenvalues of $L^{\phi_k}_{\varepsilon_k}$.

Theorem 50. The equilibria ϕ_k are unstable for the problem (7.1), for any $k\geq 2$.

Proof. Suppose first that k is even. Then, by Theorem 48, μ_1^k is an eigenvalue for $L_{\varepsilon_k}^{\phi_k}$. Note that μ_1^k is the lowest eigenvalue of $L_0^{\phi_k}$, and it is negative because $L_0^{\phi_k}$ is the linearization around a sign-changing equilibrium of the Chafee-Infante classical equation (as we stated in Lemma 21). Therefore, ϕ_k is unstable for the problem (7.5), and by consequence it is unstable for problem (7.1).

If k is odd, it follows from Theorem 48 that μ_2^k is an eigenvalue for $L_{\varepsilon_k}^{\phi_k}$. We will prove that μ_2^k is negative. Indeed, consider the auxiliary Sturm-Liouville operator defined by

$$
\begin{aligned}
&D_k : H^2(0,\tfrac{\pi}{2}) \cap H_0^1(0,\tfrac{\pi}{2}) \to L^2(0,\pi) \\
&D_k u = -u'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} u,
\end{aligned}
\tag{7.20}
$$

Note that if v_2 is the eigenvector of $L_0^{\phi_k}$ associated to μ_2^k such that $v_2'(0) = 1$, then $v_2(x) > 0$ for $x \in (0,\frac{\pi}{2})$, $v_2(\frac{\pi}{2}) = 0$, and $v_2|_{[0,\frac{\pi}{2}]}$ is the positive eigenvector for D_k, associated to the eigenvalue μ_2^k. It follows that μ_2^k is the first (lowest) eigenvalue of D_k.

Consider the problem of finding the lowest real number α such that

$$
\begin{aligned}
&\theta'' + \left(\alpha + \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)}\right)\theta = 0 \iff D_k\theta = \alpha\theta \\
&\theta(0) = \theta(\tfrac{\pi}{2}) = 0, \quad \theta'(0) = 1
\end{aligned}
\tag{7.21}
$$

has a solution $\theta \in H^2(0,\frac{\pi}{2}) \cap H_0^1(0,\frac{\pi}{2})$. Using Theorem 41, which still holds with the same proof for the operator D_k, we can analyze instead the ODE

$$
\begin{aligned}
&-v_{xx} = \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} v \\
&v(0) = 0, \quad v'(0) = 1.
\end{aligned}
\tag{7.22}
$$

If v has a zero in $(0,\frac{\pi}{2})$, then $\alpha = \mu_2^k < 0$. Note that since $k \geq 3$, ϕ_k has a local minimum $x^* \in [0,\frac{\pi}{2}]$. Then, since both v and ϕ_k' are solutions of $-v_{xx} = \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} v$, their Wronskian $\phi_k'(x)v'(x) - \phi_k''(x)v(x)$ is constant, and calculating it in zero and in x^*, we obtain:

$$
-\phi_k''(x^*)v(x^*) = \phi_k'(0) > 0,
$$

which implies $v(x^*) < 0$, and $v(y) = 0$ for some $y \in (0,\frac{\pi}{2})$. This completes the proof. □

7.5 Hyperbolicity of equilibria

In this section we will prove that all equilibria for (7.5) are hyperbolic, except for $0 \in H_0^1(0,\pi)$, which loses hyperbolicity when $\lambda = a(0)n^2$ for any $n \in \mathbb{N}$. After that, we will use change in time scale to see what information can be concluded for the local stable and unstable sets of the equilibria in the quasilinear problem.

It has already been said that $\sigma(L_{\varepsilon_0}^0)=\{n^2-\frac{\lambda}{a(0)}:n\in\mathbb{N}^*\}$, then the origin of (7.5) is always hyperbolic, except when $\lambda=a(0)n^2$ for some $n\in\mathbb{N}^*$, which are the points of bifurcation.

It was also proved that the spectrum of $L_{\varepsilon_1}^{\phi_1}$ has a strictly positive lower bound, then ϕ_1 is always hyperbolic, provided that $\lambda>a(0)$.

Now we prove the hyperbolicity of ϕ_2, which gives some intuition about what we will need to do in the general case ϕ_k where k is even.

Theorem 51. The equilibrium ϕ_2 is hyperbolic for equation (7.5).

Proof. The linearization operator around ϕ_2 is $L_{\varepsilon_2}^{\phi_2}$, which has compact resolvent from Theorem 45, then we only need to show that 0 is not an eigenvalue for $L_{\varepsilon_2}^{\phi_2}$. We do this by contradiction.

Suppose $0\in\sigma_p(L_{\varepsilon_2}^{\phi_2})$, so that there exists $0\neq v\in H^2(0,\pi)\cap H_0^1(0,\pi)$ such that $L_{\varepsilon_2}^{\phi_2}v=0$. From Lemma 21, 0 is not an eigenvalue for $L_0^{\phi_2}$, then by Theorem 46, 0 is a simple eigenvalue for $L_{\varepsilon_2}^{\phi_2}$.

Using that $\phi_2(\pi-x)=-\phi_2(x)$ for $x\in[0,\pi]$, one may easily check that $\eta(\cdot)=v(\pi-\cdot)$ is another eigenvector associated to the eigenvalue 0, then $v(\pi-x)=cv(x)$. Squaring each side and integrating yields $c=\pm1$. Suppose that, for each $x\in[0,\pi]$, $v(\pi-x)=v(x)$, then

$$\int_0^\pi f(\phi_2(x))v(x)dx=\int_0^\pi f(-\phi_2(\pi-x))v(\pi-x)dx=-\int_0^\pi f(\phi_2(x))v(x)dx$$

Therefore, $0=L_{\varepsilon_2}^{\phi_2}v=L_0^{\phi_2}v$, and this is a contradiction. We conclude, then, that $v(\pi-x)=-v(x)$ for $x\in[0,\pi]$, and $v(\frac{\pi}{2})=0$. Then

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} f(\phi_2(s))v(s)ds &= \int_0^{\frac{\pi}{2}} f(\phi_2(\tfrac{\pi}{2}+s))v(\tfrac{\pi}{2}+s)ds\\ &= \int_0^{\frac{\pi}{2}} f(\phi_2(\pi-s))v(\pi-s)ds\\ &= \int_0^{\frac{\pi}{2}} f(-\phi_2(s))[-v(s)]ds\\ &= \int_0^{\frac{\pi}{2}} f(\phi_2(s))v(s)ds\end{aligned}$$

Therefore, $\int_0^\pi f(\phi_2(s))v(s)ds=2\int_0^{\frac{\pi}{2}} f(\phi_2(s))v(s)ds$, and the equation $L_{\varepsilon_2}^{\phi_2}v=0$ becomes:

$$-v''-\frac{\lambda f'(\phi_2)}{a(\|\phi_2'\|^2)}v+2\varepsilon_2 f(\phi_2)\int_0^{\frac{\pi}{2}} f(\phi_2(s))v(s)ds=0,$$

where $\varepsilon_2=\frac{2\lambda^2a'(\|\phi_2'\|^2)}{a(\|\phi_2'\|^2)^3}\geq0$

For each $\varepsilon\in\mathbb{R}$, consider the operator $M_\varepsilon:H^2(0,\frac{\pi}{2})\cap H_0^1(0,\frac{\pi}{2})\to L^2(0,\frac{\pi}{2})$ given by

$$M_\varepsilon u=-u''-\frac{\lambda f'(\phi_2)}{a(\|\phi_2'\|^2)}u+\varepsilon f(\phi_2)\int_0^{\frac{\pi}{2}} f(\phi_2(s))u(s)ds.$$

Let $\tilde{v} = v|_{[0,\frac{\pi}{2}]}$, then $\tilde{v} \in D(M_{2\varepsilon_2})$ and $M_{2\varepsilon_2}\tilde{v} = 0$. Therefore, 0 is an eigenvalue for $M_{2\varepsilon_2}$, and since the eigenvalues of M_ε are non-decreasing with ε (Theorem 47 still holds), we have an eigenvalue $\gamma \leq 0$ for M_0. Now, consider the classical Chafee-Infante problem:

$$\begin{aligned} &u_t = u_{xx} + \frac{\lambda f(u)}{a(\|\phi_2'\|^2)}, \quad t > 0,\ x \in (0, \tfrac{\pi}{2}) \\ &u(0,t) = u(\tfrac{\pi}{2}, t) = 0, \quad t \geq 0 \\ &u(\cdot, 0) = u_0 \in H_0^1(0, \tfrac{\pi}{2}). \end{aligned} \tag{7.23}$$

It is easy to see that $\psi_1 = \phi_2|_{[0,\frac{\pi}{2}]}$ is the positive equilibrium for (7.23), and the operator M_0, which can be written as

$$M_0 u = -u'' - \frac{\lambda}{a(\|\phi_2'\|^2)} f'(\psi_1)u,$$

is the linearization around ψ_1. However, the analysis done in Chapter 6, Section 3, can be performed without any relevant change to guarantee that ψ_1 is exponentially stable for (7.23), and the spectrum of M_0 has a strictly positive lower bound. This contradicts the existence of the eigenvalue $\gamma \leq 0$, and we are done.

□

Theorem 52. The equilibrium ϕ_k is hyperbolic for equation (7.5), for any odd $k \geq 3$.

Proof. Suppose by contradiction that there exists $0 \neq v \in H^2(0,\pi) \cap H_0^1(0,\pi)$ such that $L_{\varepsilon_k}^{\phi_k} v = 0$. We know by Lemma 21 that 0 is not an eigenvalue for $L_0^{\phi_k}$, and by Theorem 46, 0 is a simple eigenvalue for $L_{\varepsilon_k}^{\phi_k}$. It may be checked that $\eta(\cdot) = v(\pi - \cdot)$ is another eigenvector associated to the eigenvalue 0, then $v(\pi - x) = cv(x)$ for all $x \in [0,\pi]$ for some constant $c \in \mathbb{R}$. Squaring each side and integrating between 0 and π yields $c = \pm 1$. Suppose $c = -1$, then:

$$\int_0^\pi f(\phi_k(s))v(s)ds = \int_0^\pi f(\phi_k(\pi - s))[-v(\pi - s)]ds = -\int_0^\pi f(\phi_k(s))v(s)ds,$$

and $0 = L_{\varepsilon_k}^{\phi_k} v = L_0^{\phi_k} v$, which is a contradiction because 0 is not an eigenvalue for $L_0^{\phi_k}$. It follows that $v(x) = v(\pi - x)$ for all $x \in [0,\pi]$. Now we need to consider two separated cases:

Case 1: $v(\frac{\pi}{k}) = 0$

In this case, we will prove that v has the same symmetry and oscillation properties that ϕ_k has. Indeed, define

$$v_1(x) = \begin{cases} v(\frac{\pi}{k} - x), & \text{if } x \in [0, \frac{\pi}{k}] \\ -v(x - \frac{\pi}{k}), & \text{if } x \in [\frac{\pi}{k}, \pi]. \end{cases}$$

We will show that $L_{\varepsilon_k}^{\phi_k} v_1 = 0$. Indeed, let $x \in [0, \frac{\pi}{k}]$, then $\phi_k(\frac{\pi}{k} - x) = \phi_k(x)$, and

$$L_{\varepsilon_k}^{\phi_k} v_1(x) = -v''(\tfrac{\pi}{k} - x) - \frac{\lambda f'(\phi_k(\frac{\pi}{k} - x))}{a(\|\phi_k'\|^2)} v(\tfrac{\pi}{k} - x) + \varepsilon_k f(\phi_k(\tfrac{\pi}{k} - x)) \int_0^\pi f(\phi_k(s)) v_1(s) ds.$$

And if $x \in [\frac{\pi}{k}, \pi]$, we use $\phi_k(x - \frac{\pi}{k}) = -\phi_k(x)$ and the fact that f' is even, to obtain:

$$L_{\varepsilon_k}^{\phi_k} v_1(x) = -\left[-v''(x - \tfrac{\pi}{k}) - \frac{\lambda f'(\phi_k(x - \frac{\pi}{k}))}{a(\|\phi_k'\|^2)} v(x - \tfrac{\pi}{k}) + \varepsilon_k f(\phi_k(x - \tfrac{\pi}{k})) \int_0^{\pi} f(\phi_k(s)) v_1(s) ds \right].$$

So that we are only left to show that $\int_0^{\pi} f(\phi_k(s)) v_1(s) ds = \int_0^{\pi} f(\phi_k(s)) v(s) ds$, which is done below

$$\begin{aligned}
\int_0^{\pi} f(\phi_k(s)) v_1(s) ds &= \int_0^{\frac{\pi}{k}} f(\phi_k(\tfrac{\pi}{k} - s)) v(\tfrac{\pi}{k} - s) ds + \int_{\frac{\pi}{k}}^{\pi} f(-\phi_k(s - \tfrac{\pi}{k}))[-v(s - \tfrac{\pi}{k})] ds \\
&= \int_0^{\frac{\pi}{k}} f(\phi_k(s)) v(s) ds + \int_0^{\pi - \frac{\pi}{k}} f(\phi_k(s)) v(s) ds \\
&= \int_{\pi - \frac{\pi}{k}}^{\pi} f(\phi_k(\pi - s)) v(\pi - s) ds + \int_0^{\pi - \frac{\pi}{k}} f(\phi_k(s)) v(s) ds \\
&= \int_0^{\pi} f(\phi_k(s)) v(s) ds,
\end{aligned}$$

where in the last equality we used that $v(\pi - x) = v(x)$, for $x \in [0, \pi]$.

Since 0 is a simple eigenvalue, $v_1 = \pm v$. Suppose $v_1 = -v$, then $v(x) = -v(\frac{\pi}{k} - x)$, for $x \in [0, \frac{\pi}{k}]$, and $v(x) = v(x - \frac{\pi}{k})$, for $x \in [\frac{\pi}{k}, \pi]$, and we can show that

$$\int_0^{\pi} f(\phi_k(s)) v(s) ds = 0,$$

which implies $L_0^{\phi_k} v = 0$, a contradiction. Then we must have $v_1 = v$, and v oscillates the same way that ϕ_k does, that is, $v(x) = -v(x - \frac{\pi}{k})$ and $\phi_k(x) = -\phi_k(x - \frac{\pi}{k})$, for $x \in [\frac{\pi}{k}, \pi]$. For each interval $[r\frac{\pi}{k}, (r+1)\frac{\pi}{k}]$, $r = 0, 1, \ldots, k-1$, we can prove, using an indution argument, that

$$\int_{r\frac{\pi}{k}}^{(r+1)\frac{\pi}{k}} f(\phi_k(s)) v(s) ds = \int_0^{\frac{\pi}{k}} f(\phi_k(s)) v(s) ds,$$

where so that $\int_0^{\pi} f(\phi_k(s)) v(s) ds = k \int_0^{\frac{\pi}{k}} f(\phi_k(s)) v(s) ds$.

We can apply the same argument that we applied for $k = 2$. Note that

$$-v'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} v + k\varepsilon_k f(\phi_k) \int_0^{\frac{\pi}{k}} f(\phi_k(s)) v(s) ds = 0,$$

where $\varepsilon_k = \frac{2\lambda^2 a'(\|\phi_k'\|^2)}{a(\|\phi_k'\|^2)^3} \geq 0$, and we define for $\varepsilon \in \mathbb{R}$ the operator $M_\varepsilon : H^2(0, \frac{\pi}{k}) \cap H_0^1(0, \frac{\pi}{k}) \to L^2(0, \frac{\pi}{k})$ by

$$M_\varepsilon u = -u'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} u + \varepsilon f(\phi_k) \int_0^{\frac{\pi}{k}} f(\phi_k(s)) u(s) ds.$$

Since $\tilde{v} = v|_{[0, \frac{\pi}{k}]}$ satisfies $M_{k\varepsilon_k} \tilde{v} = 0$, we have that 0 is an eigenvalue for $M_{k\varepsilon_k}$, and by Theorem 47 (which can be proved the same way if we work in the domain $[0, \frac{\pi}{k}]$), we obtain an eigenvalue $\gamma \leq 0$ for M_0.

However, we can consider the Chafee-Infante classical problem

$$\begin{aligned} u_t &= u_{xx} + \frac{\lambda f(u)}{a(\|\phi_k'\|^2)}, \quad t > 0,\ x \in (0, \tfrac{\pi}{k}) \\ u(0,t) &= u(\tfrac{\pi}{k}, t) = 0, \quad t \geq 0 \\ u(x,0) &= u_0(x),\ u_0 \in H_0^1(0, \tfrac{\pi}{k}), \end{aligned}$$

which has the positive equilibrium $\psi_1 = \phi_k|_{[0,\frac{\pi}{k}]}$, and M_0 represents the linearization around this positive equilibrium. The existence of γ is then a contradiction, and we are done with this case.

Case 2: $v(\frac{\pi}{k}) \neq 0$

First we define the following auxiliary functions:

$$v_1(x) = \begin{cases} v(x + \frac{\pi}{k}), & \text{if } x \in [0, \pi - \frac{\pi}{k}] \\ -v(x - (\pi - \frac{\pi}{k})), & \text{if } x \in [\pi - \frac{\pi}{k}, \pi], \end{cases}$$

and $v_2(x) = v_1(\pi - x)$, $x \in [0, \pi]$. It follows from this definition that $\int_0^\pi f(\phi_k(s))v(s)ds = -\int_0^\pi f(\phi_k(s))v_1(s)ds = -\int_0^\pi f(\phi_k(s))v_2(s)ds$. Note that v_1, $v_2 \in H^2(0,\pi)$, but do not vanish in 0 and π, then $v_1, v_2 \notin H_0^1(0,\pi)$. Moreover, it may be checked that $L_{\varepsilon_k}^{\phi_k} v_1 = L_{\varepsilon_k}^{\phi_k} v_2 = 0$.

Now we define $u_1 = v + v_1$, and $u_2 = v + v_2$. Note that

$$0 = L_{\varepsilon_k}^{\phi_k} u_1 = -u_1'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} u_1,$$

because the integral terms cancel. The same happens to u_2. We will show that $\{u_1, u_2\}$ is a linearly independent set, hence a fundamental set of solutions for the ODE:

$$-v'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} v = 0$$

Suppose not, then $u_1 = \alpha u_2$. Since

$$u_1(0) = \alpha u_2(0) \Rightarrow v(\tfrac{\pi}{k}) = -\alpha v(\tfrac{\pi}{k}),$$

we have $\alpha = -1$. Therefore, $u_1 + u_2 = 2v + v_1 + v_2 = 0$. Evaluating this equation in $\frac{j\pi}{k}$, for $j = 1, 2, ..., k-1$, and using the fact that $v(\pi - x) = v(x)$, for $x \in [0, \pi]$, we obtain:

$$v\left(\frac{(j-1)\pi}{k}\right) + 2v\left(\frac{j\pi}{k}\right) + v\left(\frac{(j+1)\pi}{k}\right) = 0, \quad j = 1, 2, ..., k-1.$$

This system of equations can be written in matrix form $\mathscr{L}_{k-1}V = 0$, where:

$$\mathscr{L}_{k-1} = \begin{bmatrix} 2 & 1 & 0 & \dots & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v(\frac{\pi}{k}) \\ v(\frac{2\pi}{k}) \\ v(\frac{3\pi}{k}) \\ v(\frac{4\pi}{k}) \\ \vdots \\ \vdots \\ v(\frac{(k-2)\pi}{k}) \\ v(\frac{(k-1)\pi}{k}) \end{bmatrix}.$$

We can show that $\det(\mathscr{L}_{k-1}) = k \neq 0$ by induction. Indeed, the identity is obviously true for $k = 3$ and $k = 4$. We suppose $k \geq 4$, and that both $\det(\mathscr{L}_{k-2}) = k-1$ and $\det(\mathscr{L}_{k-1}) = k$ hold, and we need to show that $\det(\mathscr{L}_k) = k+1$. If we use Laplace expansion with relation to the last line of $\mathscr{L}_k$, we get:

$$\det(\mathscr{L}_k) = 2\det(\mathscr{L}_{k-1}) - \det(B).$$

Where B is the matrix that we get by deleting the line k and column $k-1$ from $\mathscr{L}_k$. Note that $\det(B) = \det(\mathscr{L}_{k-2})$ (this follows from subtracting the column $k-1$ from column $k-2$ in B). Then we have $\det(\mathscr{L}_k) = 2\det(\mathscr{L}_{k-1}) - \det(\mathscr{L}_{k-2}) = 2k - (k-1) = k+1$, and we are done.

It follows that $V = 0$, so that $v(\frac{\pi}{k}) = 0$ and we have a contradiction. This implies that $\{u_1, u_2\}$ is in fact a fundamental set of solutions for

$$-v'' - \frac{\lambda f'(\phi_k)}{a(\|\phi_k'\|^2)} v = 0$$

It follows that there exist α and $\beta \in \mathbb{R}$ such that $\phi_k' = \alpha u_1 + \beta u_2$. Recall from the construction of the equilibria in Chapter 6 that $\phi_k'(0) = (-1)^j \phi_k'(\frac{j\pi}{k})$, for all $j \in \{1, 2, \dots k\}$, and $\phi_k'(x) = -\phi_k'(\pi - x)$, for all $x \in [0, \pi]$. Finally, note that $u_1(\pi - x) = v(\pi - x) + v_1(\pi - x) = v(x) + v_2(x) = u_2(x)$, for $x \in [0, \pi]$. Therefore,

$$\begin{aligned} &\phi_k'(x) = -\phi_k'(\pi - x) \Rightarrow \alpha u_1(x) + \beta u_2(x) = -(\alpha u_1(\pi - x) + \beta u_2(\pi - x)) = -\alpha u_2(x) - \beta u_1(x) \\ &\Rightarrow (\alpha + \beta)(u_1(x) + u_2(x)) = 0, \quad \forall x \in [0, \pi] \end{aligned}$$

Since $u_1 + u_2 \not\equiv 0$ because they are linearly independent, we have $\beta = -\alpha$, and $\phi_k' = \alpha(u_1 - u_2) = \alpha(v_1 - v_2)$. We can do as before and set a system of equations, for $1 \leq j \leq k-1$:

$$\begin{aligned} &\phi_k'(0) = 2\alpha v(\tfrac{\pi}{k}) \\ &\phi_k'(\tfrac{j\pi}{k}) = \alpha v\left(\frac{(j+1)\pi}{k}\right) - \alpha v\left(\frac{(j-1)\pi}{k}\right), \end{aligned}$$

and using $\phi_k'(0) = (-1)^j \phi_k'(\frac{j\pi}{k})$, we get

$$2(-1)^j v\left(\frac{\pi}{k}\right) - v\left(\frac{(j+1)\pi}{k}\right) + v\left(\frac{(j-1)\pi}{k}\right) = 0, \quad j = 1, \ldots, k-1 \tag{7.24}$$

This system of equations will imply $v(\frac{\pi}{k}) = 0$, and this will end the proof. For the case $k = 3$, simply note that (7.24) with $j = 1$ implies $v(\frac{\pi}{3}) = -\frac{1}{2}v(\frac{2\pi}{3}) = -\frac{1}{2}v(\frac{\pi}{3})$, so that $v(\frac{\pi}{3}) = 0$.

For $k = 5$, if we choose $j = 1, 2$, and use the symmetry of v, we get the system

$$\begin{bmatrix} -2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v(\frac{\pi}{5}) \\ v(\frac{2\pi}{5}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and since the determinant of the matrix is 5, we get $v(\frac{\pi}{5}) = 0$.

For $k \geq 7$, we can write $k = 2n+1$, $n \geq 3$, and the equations in (7.24) for $j = 1, \ldots, n$, along with the fact that $v(\pi - x) = v(x)$, generate the system $\mathscr{H}_n V = 0$, where

$$\mathscr{H}_n = \begin{bmatrix} -2 & -1 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -1 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2(-1)^{n-3} & 0 & 0 & 0 & \ldots & \ldots & 0 & -1 & 0 & 0 \\ 2(-1)^{n-2} & 0 & 0 & 0 & \ldots & \ldots & 1 & 0 & -1 & 0 \\ 2(-1)^{n-1} & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 & 0 & -1 \\ 2(-1)^{n} & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v(\frac{\pi}{k}) \\ v(\frac{2\pi}{k}) \\ v(\frac{3\pi}{k}) \\ v(\frac{4\pi}{k}) \\ \vdots \\ \vdots \\ v(\frac{(n-2)\pi}{k}) \\ v(\frac{(n-1)\pi}{k}) \\ v(\frac{n\pi}{k}) \end{bmatrix}$$

We will use induction to prove that $\det(\mathscr{H}_n) = (-1)^n(2n+1)$. The identity is obvious when $n = 2$, because the matrix is then

$$\mathscr{H}_2 = \begin{bmatrix} -2 & -1 \\ 3 & -1 \end{bmatrix}$$

Suppose $\det(\mathscr{H}_n) = (-1)^n(2n+1)$ for some $n \geq 2$, and we will prove that $\det(\mathscr{H}_{n+1}) = (-1)^{n+1}(2n+3)$.

If we add the n-th column of $\mathscr{H}_n$ to the $(n-1)$-th column, we obtain the following matrix:

$$B_n = \begin{bmatrix} -2 & -1 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -1 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2(-1)^{n-3} & 0 & 0 & 0 & \ldots & \ldots & 0 & -1 & 0 & 0 \\ 2(-1)^{n-2} & 0 & 0 & 0 & \ldots & \ldots & 1 & 0 & -1 & 0 \\ 2(-1)^{n-1} & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 & -1 & -1 \\ 2(-1)^{n} & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Note that, $\det(\mathscr{H}_{n+1}) = \det(B_{n+1})$, and B_{n+1} can be written as

$$B_{n+1} = \left[\begin{array}{c|c} \mathscr{H}_n & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{matrix} \\ \hline 2(-1)^{n+1}\ 0\ 0\ 0\ \dots\ 0\ 0\ 0 & -1 \end{array}\right] \qquad B_{n+1} = \left[\begin{array}{c|c} \begin{matrix} -2 \\ 3 \\ -2 \\ \vdots \\ 2(-1)^n \end{matrix} & T_n \\ \hline 2(-1)^{n+1} & 0\ 0\ 0\ \dots\ 0\ 0\ -1 \end{array}\right],$$

where T_n is the matrix:

$$T_n = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \dots & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 & -1 & -1 \end{bmatrix},$$

whose determinant can be calculated by Gauss elimination, adding the first line to the third, and then the second to the fourth, and so on, until what is left is a diagonal matrix. This yields $\det(T_n) = (-1)^n$.

Using Laplace expansion with relation to the last line of B_{n+1}, we get:

$$\begin{aligned} \det(\mathscr{H}_{n+1}) = \det(B_{n+1}) &= -\det(\mathscr{H}_n) - 2\det(T_n) \\ &= (-1)^{n+1}(2n+1) + 2(-1)^{n+1} = (-1)^{n+1}(2n+3). \end{aligned}$$

We proved that the matrix $\mathscr{H}_n$ is non-singular, which implies $v(\frac{\pi}{k}) = 0$.

This completes the proof that the equilibria ϕ_k are hyperbolic for k odd. □

Theorem 53. The equilibria ϕ_k are hyperbolic for k even.

Proof. Suppose that there exists $0 \neq v \in H^2(0,\pi) \cap H_0^1(0,\pi)$ such that $L_{\varepsilon_k}^{\phi_k} v = 0$. If we write $k = 2^n(2j+1)$, with $n \geq 1$ and $j \geq 0$ integers, we claim that the eigenvector v has the following oscillation result:

$$v\left(\frac{\pi}{2^i} - x\right) = -v(x), \quad \forall x \in \left[0, \frac{\pi}{2^i}\right], 1 \leq i < n \tag{7.25}$$

Suppose $n \geq 2$, otherwise there is nothing to prove.

I follows from the symmetric properties of the equilibria (see Remark 12) that

$$\phi_k\left(\frac{\pi}{2^i} - x\right) = -\phi_k(x), \quad \forall x \in \left[0, \frac{\pi}{2^i}\right], 1 \leq i < n.$$

From the simplicity of the zero eigenvalue for $L_{\varepsilon_k}^{\phi_k}$, we know that either $v(\pi - x) = v(x)$ for $x \in [0,\pi]$ or $v(\pi - x) = -v(x)$ for $x \in [0,\pi]$. The first possibility could not happen because it would imply $L^{\phi_k}_{\varepsilon_k} v = L^{\phi_k}_0 v = 0$ because the integral would vanish, and this is a contradiction. Hence $v(\pi - x) = -v(x)$, for $x \in [0,\pi]$, which implies $v(\frac{\pi}{2}) = 0$. Moreover,

$$\begin{aligned}\int_0^{\pi} f(\phi_k(s))v(s)ds &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds + \int_{\frac{\pi}{2}}^{\pi} f(\phi_k(s))v(s)ds \\ &= \int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds + \int_{\frac{\pi}{2}}^{\pi} f(-\phi_k(\pi - s))[-v(\pi - s)]ds \\ &= 2\int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds.\end{aligned}$$

Now we define the auxiliary function:

$$v_1(x) = \begin{cases} v(\frac{\pi}{2} - x), & \text{if } x \in [0, \frac{\pi}{2}] \\ -v(x - \frac{\pi}{2}), & \text{if } x \in [\frac{\pi}{2}, \pi]. \end{cases}$$

Proceeding the usual calculations, and using the fact that $\phi_k(\frac{\pi}{2} - x) = -\phi_k(x)$, for $x \in [0, \frac{\pi}{2}]$ and $\phi_k(x) = \phi_k(x - \frac{\pi}{2})$, for $x \in [\frac{\pi}{2}, \pi]$ (note that this depends on n being ≥ 2), we conclude that $L^{\phi_k}_{\varepsilon_k} v_1 = 0$. This implies $v = cv_1$, where $c = \pm 1$. Suppose that $c = 1$, then $v(x) = +v(\frac{\pi}{2} - x)$ for $x \in [0, \frac{\pi}{2}]$, and we obtain

$$\int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds = \int_0^{\frac{\pi}{2}} f(-\phi_k(\tfrac{\pi}{2} - s))v(\tfrac{\pi}{2} - s)ds = -\int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds,$$

that is, $\int_0^{\frac{\pi}{2}} f(\phi_k(s))v(s)ds = 0$ and $L^{\phi_k}_0 v = L^{\phi_k}_{\varepsilon_k} v = 0$, a contradiction.

Therefore, we conclude $v(x) = -v(\frac{\pi}{2} - x)$ for $x \in [0, \frac{\pi}{2}]$, and we proved (7.25) for $i = 1$.

As long as $i < n$, we can prove the next case using the previous one. In fact, now we have:

$$\int_0^{\pi} f(\phi_k(s))v(s)ds = 4\int_0^{\frac{\pi}{4}} f(\phi_k(s))v(s)ds.$$

We can define the auxiliary function:

$$v_2(x) = \begin{cases} v(\frac{\pi}{4} - x), & \text{if } x \in [0, \frac{\pi}{4}] \\ -v(x - \frac{\pi}{4}), & \text{if } x \in [\frac{\pi}{4}, \pi], \end{cases}$$

which satisfies $L^{\phi_k}_{\varepsilon_k} v_2 = 0$. Therefore, $v = cv_2$ with $c = \pm 1$, and if $c = 1$ we obtain

$$\int_0^{\frac{\pi}{4}} f(\phi_k(s))v(s)ds = 0,$$

which is a contradiction.

Then the claim is proved inductively, considering the auxiliary functions, for $1 \leq i < n$:

$$v_i(x) = \begin{cases} v(\frac{\pi}{2^i} - x), & \text{if } x \in [0, \frac{\pi}{2^i}] \\ -v(x - \frac{\pi}{2^i}), & \text{if } x \in [\frac{\pi}{2^i}, \pi], \end{cases}$$

Using the last claim, the similar oscillation properties of ϕ_k and v yield:

$$\int_0^{\pi} f(\phi_k(s))v(s)ds = 2^n \int_0^{\frac{\pi}{2^n}} f(\phi(s))v(s)ds$$

Then we can work with a problem in a reduced interval as we did in the case $k=2$. Let $\psi_k = \phi_k|_{[0,\frac{\pi}{2^n}]}$ and $u = v|_{[0,\frac{\pi}{2^n}]}$, and note that $0 \neq u \in H^2(0,\frac{\pi}{2^n}) \cap H_0^1(0,\frac{\pi}{2^n})$ satisfies

$$-u'' - \frac{\lambda f'(\psi_k)}{a(2^n\|\psi_k'\|_n^2)}u + 2^{n+1}\frac{\lambda^2 a'(2^n\|\psi_k'\|_n^2)}{a(2^n\|\psi_k'\|_n^2)^3} f(\psi_k)\int_0^{\frac{\pi}{2^n}} f(\psi_k(s))u(s)ds = 0, \tag{7.26}$$

where $\|u_x\|_n^2 = \int_0^{\frac{\pi}{2^n}} |u_x(s)|^2 ds$.

Now we only need to notice that $\psi_k = \phi_k|_{[0,\frac{\pi}{2^n}]}$ is an equilibrium for the system

$$\begin{aligned} &u_t = d(\|u_x\|_n^2)u_{xx} + \lambda f(u), \quad t>0,\ x\in(0,\tfrac{\pi}{2^n}) \\ &u(0,t) = u(\tfrac{\pi}{2^n},t) = 0, \quad t \geq 0 \\ &u(\cdot,0) = u_0 \in H_0^1(0,\tfrac{\pi}{2^n}), \end{aligned} \tag{7.27}$$

where $d = a(2^n\cdot)$. More precisely, ψ_k is one of the two equilibria with $2j+2$ zeros in $[0,\frac{\pi}{2^n}]$, therefore, we can use Theorem 52 with no significant change in the proof to assure that the linearization operator of (7.27) around ψ_k does not contain 0 as an eigenvalue, and this operator is given by

$$\begin{aligned} &L^{\psi_k} : H^2(0,\tfrac{\pi}{2^n}) \cap H_0^1(0,\tfrac{\pi}{2^n}) \\ &L^{\psi_k}v = -v'' - \frac{\lambda f'(\psi_k)}{d(\|\psi_k'\|_n^2)}v + \frac{2\lambda^2 d'(\|\psi_k'\|_n^2)}{d(\|\psi_k'\|_n^2)^3} f(\psi_k)\int_0^{\frac{\pi}{2^n}} f(\psi_k(s))v(s)ds. \end{aligned}$$

This contradicts (7.26), and we are done.

□

We only proved hyperbolicity for equation (7.5), in the sense discussed in Chapter 5. Now we need to pass the results for the quasilinear equation (7.1). To do this, we define a more general sense of hyperbolicity. Remember that an equilibrium $\phi \in \mathscr{E}$ is topologically hyperbolic (see Definition 25) if there exists $\delta > 0$ such that if $\xi : \mathbb{R} \to X$ is a global solution that satisfies $\sup_{t\in\mathbb{R}} \|\xi(t) - \phi\| < \delta$, then $\xi \equiv \phi$.

Let $B_\delta^{H_0^1}(\phi)$ denote the ball of radius δ centered in ϕ, in the norm of $H_0^1(0,\pi)$. Recall that we define the local stable and unstable sets of an equilibrium $\phi \in \mathscr{E}$, respectively, by:

$$\begin{aligned} W^{u,\delta}_{loc}(\phi) \ := \ &\{u \in H_0^1(0,\pi) :\ \text{there is a global solution } \xi : \mathbb{R} \to H_0^1(0,\pi) \text{ through } u \text{ such that} \\ &\xi(t) \in B_\delta^{H_0^1}(\phi), \forall t \leq 0, \text{ and } \xi(t) \overset{t\to-\infty}{\longrightarrow} \phi\}, \end{aligned}$$

and

$$W_{loc}^{s,\delta}(\phi) := \{u \in H_0^1(0,\pi) : T(t)u \in B_\delta^{H_0^1}(\phi), \forall t \geq 0, \text{ and } T(t)u \stackrel{t\to\infty}{\longrightarrow} \phi\}.$$

Where the convergence is in the norm of $H_0^1(0,\pi)$.

Definition 34 (Strict hyperbolicity)**.** We say that $\phi \in \mathscr{E}$ is **hyperbolic** if ϕ is topologically hyperbolic and there exist closed subspaces X_u and X_s of $H_0^1(0,\pi)$, with $H_0^1(0,\pi) = X_u \bigoplus X_s$ such that the local stable and unstable sets of ϕ are given as graphs of Lipschitz functions $\theta_u : X_u \to X_s$ and $\theta_s : X_s \to X_u$, with Lipschitz constants L_s, L_u both in $(0,1)$, $\theta_u(0) = \theta_s(0) = 0$, in the following sense: there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, there are $0 < \delta'' < \delta' < \delta$ with

$$\{\phi + x_u + \theta_u(x_u) : x_u \in X_u, \|x_u\|_{H_0^1} < \delta''\} \subset W_{loc}^{u,\delta'}(\phi) \subset \{\phi + x_u + \theta_u(x_u) : x_u \in X_u, \|x_u\|_{H_0^1} < \delta\}$$

and

$$\{\phi + \theta_s(x_s) + x_s : x_s \in X_s, \|x_s\|_{H_0^1} < \delta''\} \subset W_{loc}^{s,\delta'}(\phi) \subset \{\phi + \theta_s(x_s) + x_s : x_s \in X_s, \|x_s\|_{H_0^1} < \delta\}.$$

If an equilibrium of (7.5) has a linearization operator L whose spectrum does not contain 0, then we define the spectral projections P_u and P_s associated to the parts of $\sigma(L)$ to the left and to the right of the imaginary axis, respectively. Theorem 35 implies that ϕ is strictly hyperbolic with the sets $X_u = P_u(H_0^1(0,\pi))$ and $X_s = P_s(H_0^1(0,\pi))$. Moreover, the forward (resp. backward) attraction over the local stable (resp. unstable) set of ϕ is exponential. This is the so called saddle point property.

We will show that the hyperbolic equilibria of (7.5) are also hyperbolic for (7.1) in the sense of this last definition.

From Remark 8, we already know that all the equilibria for (7.1) are topologically hyperbolic.

Suppose the origin $\phi_0 = 0$ is a hyperbolic equilibrium for (7.5), then there is a $\delta_0 > 0$, and constants K, $\beta > 0$ in such a way that for all $0 < \delta < \delta_0$, there exists $0 < \delta'' < \delta' < \delta$ such that for all $x_u \in X_u$, $\|x_u\|_{H_0^1} < \delta''$, then $x_u + \theta_u(x_u) \in W_{loc}^{u,\delta'}(0)$, and there exists $\eta : \mathbb{R} \to H_0^1(0,\pi)$ global solution for (7.5) such that $\eta(0) = x_u + \theta_u(x_u)$, $\|\eta(\tau)\|_{H_0^1} < \delta'$ for $\tau \leq 0$, and

$$\eta(\tau) \in W_{loc}^{u,\delta'}(0) \subset \{x_u + \theta_u(x_u) : x_u \in X_u, \|x_u\|_{H_0^1} < \delta\}, \quad ,\forall \tau \leq 0,$$

so that $\eta(\tau) = P_u(\eta(\tau)) + \theta_u(P_u(\eta(\tau)))$, for $\tau \leq 0$. Moreover, we have the estimate

$$\|\eta(\tau)\|_{H_0^1} \leq K e^{\beta\tau} \|\eta(0)\|_{H_0^1}, \quad \forall \tau \leq 0.$$

We can change the time scale to $t = \int_0^\tau a(\|\eta(\theta)\|_{H_0^1}^2)^{-1} d\theta$, and define $\xi(t) = \eta(\tau)$. Then $\xi : \mathbb{R} \to H_0^1(0,\pi)$ is a global solution for (7.1). Therefore, for each $x_u \in X_u$, $\|x_u\|_{H_0^1} < \delta''$, we found a global solution ξ of (7.1) such that $\xi(0) = x_u + \theta_u(x_u)$, $\|\xi(t)\|_{H_0^1} < \delta'$ for all $t \leq 0$, and $\xi(t) = P_u(\xi(t)) + \theta_u(P_u(\xi(t)))$ for all $t \leq 0$. And the estimate translates into:

$$\|\xi(t)\|_{H_0^1} \leq Ke^{\beta mt}\|\xi(0)\|_{H_0^1}, \quad \forall t \leq 0.$$

Moreover, if $x_s \in X_s$, $\|x_s\|_{H_0^1} < \delta''$, then $x_s + \theta_s(x_s) \in W_{loc}^{s,\delta'}(0)$, and there exists $w : \mathbb{R} \to H_0^1(0,\pi)$, global solution for (7.5), such that $w(0) = x_s + \theta_s(x_s)$, $\|w(\tau)\|_{H_0^1} < \delta'$ for $\tau \geq 0$, and

$$w(\tau) \in W_{loc}^{s,\delta'}(0) \subset \{x_s + \theta_s(x_s) : x_s \in X_s, \|x_s\|_{H_0^1} < \delta\}, \quad ,\forall \tau \geq 0,$$

so that $w(\tau) = P_s(w(\tau)) + \theta_s(P_s(w(\tau)))$, for $\tau \geq 0$ and

$$\|w(\tau)\|_{H_0^1} \leq Ke^{-\beta\tau}\|w(0)\|_{H_0^1}, \quad \forall \tau \geq 0.$$

Defining the global solution $\alpha : \mathbb{R} \to H_0^1(0,\pi)$ by

$$\alpha(t) = w(\tau), \quad \text{for} \quad t = \int_0^\tau a(\|w(\theta)\|_{H_0^1}^2)^{-1} d\theta,$$

we have the graph characterization of the local stable set of 0, as well as the exponential estimate.

We can apply this very same reasoning to nonzero equilibria. It follows that the equilibria for (7.1) are all hyperbolic in the sense of Definition 34, except for the origin when $\lambda = a(0)n^2$ for some $n \in \mathbb{N}$. We were able to transfer the results of hyperbolicity from equation (7.5) to equation (7.1) essentially because the structure of the local stable and unstable sets does not change between the two equations, and because we could estimate the decay of $\|\xi(t)\|_{H_0^1}$ and $\|\alpha(t)\|_{H_0^1}$ using the uniform estimate for the variable change $\tau \leq mt$ for $\tau \leq 0$ and $\tau \geq mt$ for $\tau \geq 0$, which does not depend on the particular initial position $u_0 \in H_0^1(0,\pi)$.

CHAPTER

8

CONCLUSION

In this thesis, we presented several topics related to the study of parabolic partial differential equations, including spectral analysis, fractional powers, semigroups and exponential dichotomy, and then we applied this knowledge to study the classical semilinear Chafee-Infante equation and a non-local quasilinear Chafee-Infante equation (CARVALHO; MOREIRA, 2021).

In Chapter 2, we defined the resolvent and spectrum of closed operators in the Banach space X, and we showed that for a bounded operator $A \in \mathscr{L}(X)$ the spectrum is compact and the resolvent operator can be written as a series in $\mathscr{L}(X)$. Moreover, using the Fredholm Alternative, we showed that if $A : D(A) \subset X \to X$ has compact resolvent, its spectrum is a sequence of isolated eigenvalues of finite geometric multiplicity. Finally, we defined symmetric and self-adjoint operators, which are kinds of operators that appears in real-world applications with special kinds of symmetry, and we proved Friedrichs Theorem, which is a way to obtain self-adjoint operators from symmetric operators. Then, we used the results in this chapter to characterize the spectrum of the Sturm-Liouville operators of the form $L : H^2(0,\pi) \cap H_0^1(0,\pi) \to L^2(0,\pi)$, $Lv = -v'' + q(x)v$, where $q : [0,\pi] \to \mathbb{R}$ is a continuous function, showing that this operator is self-adjoint, has compact resolvent and its sequence of eigenvalues accumulates at $+\infty$. Furthermore, if we denote by $\{\mu_j\}$ the increasing sequence of eigenvalues of L, then μ_j is simple and an associated eigenvalue v_j has $j+1$ zeros in $[0,\pi]$.

Then, in Chapter 3, we presented the theory of linear semigroups associated to the differential equation

$$\dot{x} = Ax$$

when $-A$ is a sectorial operator with vertex $a \in \mathbb{R}$. We proved that the spectrum characteristics of this kind of operator allows us to construct a semigroup $\{e^{At} : t \geq 0\}$, generated by A, such that $\frac{d}{dt}e^{At} = Ae^{At} \in \mathscr{L}(X)$, and

$$\|e^{At}\|_{\mathscr{L}(X)} \leq Ke^{at}, \ \|Ae^{At}\|_{\mathscr{L}(X)} \leq Kt^{-1}e^{at}, \forall t \geq 0.$$

Moreover, if the spectrum of A is disjoint from $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = 0\}$, we have seen that, using the sectoriality of A, we can decompose the phase plane in two subspaces, such that in one subspace e^{At} decays exponentially as $t \to \infty$, and in the other subspace $\{e^{At} : t \geq 0\}$ can be extended to a group and e^{At} decays exponentially as $t \to -\infty$. This property is called exponential dichotomy. It is specially important to obtain this result for linear semigroups because under some conditions we can conclude similar properties for a semilinear equation, approximating it by a linear equation, as it is done to study stability and hyperbolicity of equilibria for parabolic semilinear equations in Chapter 5.

Also in Chapter 3, we introduce the theory of nonlinear semigroups as an elegant way to study the asymptotic behavior of a non-linear autonomous dynamical system. In particular, we give conditions under which the semigroup has a global attractor, and characterize the global attractor of gradient semigroups.

In Chapter 4, we introduce the fractional powers of a positive operator, along with interpolation and inclusion results about the fractional power spaces $X^\alpha = (D(A^\alpha), \|A^\alpha \cdot \|)$. We also use fractional powers to prove that a regular perturbation of a sectorial positive operator is still sectorial, with few assumptions on the perturbation (see Corollary 4).

The semilinear parabolic equations are defined in Chapter 5, being of the form

$$\begin{aligned} &\frac{d}{dt}u = -Au + f(t,u), \quad t > t_0 \\ &u(t_0) = u_0, \end{aligned} \tag{8.1}$$

where $A : D(A) \subset X \to X$ is a sectorial and positive operator, $(t_0, u_0) \in U$, which is an open set in $\mathbb{R} \times X^\alpha$, and $f : U \to X$ is continuous. Since we work in a fractional power space of A, we can use the interpolation and inclusion results about fractional powers, which are a useful tool. We study the existence, uniqueness and continuous dependence of solutions for (8.1), using the Banach Fixed Point, along with all the estimates we can obtain from the fractional powers formulation. Moreover, since A is sectorial and positive, we can approximate the function f near an equilibrium ϕ of (8.1) by a bounded linear operator B, and the operator $L = A - B$ will also be sectorial by perturbation results. We call L the linearization operator around ϕ, and the fact that L is sectorial help us conclude information about stability and hyperbolicity for the equilibrium ϕ, using the exponential dichotomy for the semigroup $\{e^{-Lt} : t \geq 0\}$.

Finally, we present an example of application for all these theories. In Chapter 6, we study the Chafee-Infante equation (CHAFEE; INFANTE, 1974), which is a parabolic semilinear differential equation in the phase space $H_0^1(0,\pi)$, given by:

$$\begin{aligned} &u_t = u_{xx} + \lambda f(u), \quad t > 0,\ x \in (0,\pi) \\ &u(0,t) = u(\pi,t) = 0, \quad t \geq 0 \\ &u(\cdot,0) = u_0 \in H_0^1(0,\pi). \end{aligned} \tag{8.2}$$

where $\lambda > 0$ is a parameter, $f \in \mathscr{C}^2(\mathbb{R})$ is odd (in particular, $f(0) = 0$), $f'(0) = 1$, and f satisfies:

$$f''(u)u < 0, \forall\, u \neq 0,$$

and

$$\limsup_{|u| \to \infty} \frac{f(u)}{u} < 0.$$

We show that this equation satisfies the conditions for local existence of solutions, then we find a Lyapunov functional and use it to estimate the solutions, concluding that a solution $u : [0, \tau) \to H^1_0(0, \pi)$ remains bounded in all its interval of existence, and can be extended to a solution defined for $t \geq 0$, so that for each point $u_0 \in H^1_0(0, \pi)$, there exists a bounded solution $u(\cdot, u_0) : \mathbb{R}^+ \to H^1_0(0, \pi)$ such that $u(0, u_0) = u_0$. Using the continuous dependence results in Chapter 5, we can conclude that the Chafee-Infante equation can be studied through a nonlinear semigroup, and we use the theory in Chapter 3 to show that the semigroup associated has a global attractor of gradient kind.

We study the bifurcation of equilibria for the Chafee-Infante equation, analyzing the phase plane associated to the equilibrium equation, and we conclude if $N^2 < \lambda \leq (N+1)^2$, there exist $2N+1$ equilibria for (8.2), which we denote $\{0\} \cup \{\phi_j^{\pm}\}$, $\phi_j^- = -\phi_j^+$, and ϕ_j^+ has $j+1$ zeros in $[0, \pi]$. Moreover, the equilibria have a lot of symmetry properties, for example $\phi_j^{\pm}(\pi - x) = (-1)^{j-1}\phi_j^{\pm}(x)$, for $x \in [0, \pi]$.

Using the spectral analysis of the linearization operators around the equilibria of Chafee-Infante equation — which are Sturm-Liouville operators as the ones we studied in Chapter 2 —, we conclude that $\phi_0 = 0$ is globally asymptotically stable for (8.2) for $\lambda \leq 1$, and if $\lambda > 1$, only $\phi_1^{\pm}$ are stable. Moreover, all the equilibria are hyperbolic except for ϕ_0 when $\lambda = n^2$ for some $n \in \mathbb{N}$.

Then, in Chapter 7, we study a non-local version of the Chafee-Infante equation, given by:

$$\begin{aligned} &u_t = a(\|u_x\|^2)u_{xx} + \lambda f(u), \quad 0 < x < \pi,\ t > 0, \\ &u(0,t) = u(\pi,t) = 0,\ t > 0, \\ &u(\cdot, 0) = u_0 \in H^1_0(0, \pi), \end{aligned} \tag{8.3}$$

where $\|u_x\|^2 = \int_0^{\pi} |u_x(s)|^2 ds$. The function f satisfies the same conditions as before, and we also ask $(0, \infty) \ni u \mapsto f(u)/u$ strictly decreasing. Also, $\lambda > 0$ is a parameter and $a : \mathbb{R}^+ \to [m, M] \subset (0, \infty)$ is a continuously differentiable, globally Lipschitz and non-decreasing function.

We see that this equation has some properties that resemble the ones of the classical semilinear Chafee-Infante equation. Namely, if $a(0)N^2 < \lambda \leq a(0)(N+1)^2$, then there are precisely $2N+1$ equilibria for (8.3), which we denote again by:

$$\{0\} \cup \{\phi_k^{\pm} : k = 1, \ldots, N\},$$

where ϕ_k^+ and ϕ_k^- have $k+1$ zeros in $[0,\pi]$, $\phi_k^- = -\phi_k^+$, and $\phi_k^+(x) > 0$ for $x \in (0, \frac{\pi}{k})$, for $k = 1,...,N$. Moreover, $\phi_0 = 0$ is stable for $\lambda \le a(0)$, and if $\lambda > a(0)$, $\phi_1^\pm$ are the only stable equilibria. All the equilibria are hyperbolic (in the sense of Definition 34) except for ϕ_0 when $\lambda = a(0)n^2$, for some $n \in \mathbb{N}^*$.

We prove this results for (8.3) with the aid of an auxiliary semilinear equation whose solutions are related to the solutions of (8.3) by a solution-dependent change in the time scale. All the theory of semilinear equations is applied to this semilinear equation to conclude stability and hyperbolicity of equilibria — the spectral analysis is complicated because the linearization operator is non-local. Then, the results can be transferred to the quasilinear equation, because the structure of the local stable and unstable sets does not change between the two equations, and because the change in time scale has a certain uniform estimate, because of which we can transfer the exponential decay results from an equation to the other.

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