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Surfaces of Enneper type

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Surfaces of Enneper type

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Superfícies do tipo Enneper

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RESUMO

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Superfícies no espaço Euclidiano 3-dimensional sem pontos umbílicos cujas linhas de curvatura correspondentes a uma de suas curvaturas principais estão contidas em esferas ou planos são chamadas de *superfícies do tipo Enneper*. Neste trabalho, apresentamos uma parametrização para as superfícies com linhas de curvatura planares e descrevemos como uma superfície do tipo Enneper arbitrária pode ser obtida a partir de uma superfície do tipo Enneper com linhas de curvatura planares. Apresentaremos ainda uma nova descrição da classe especial das superfícies em que os centros das esferas que contêm as linhas de curvatura correspondentes a uma de suas curvaturas principais estão todos em uma mesma reta. Em particular, esta última classe contém as superfícies do tipo Enneper com curvatura Gaussiana constante não nula. Discutimos ainda a classificação das superfícies mínimas do tipo Enneper.

Palavras-chave: superfícies, linhas de curvatura, linha de curvatura planar, linha de curvatura esférica.

ABSTRACT

FRANÇA, A. S. **Surfaces of Enneper type**. 2023. 98 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

We study surfaces free of umbilical points in the 3-dimensional Euclidean space with the lines of curvature of one family contained in planes or spheres, called *surfaces of Enneper type*. We present a parametrization of the surfaces with planar lines of curvature and describe how an arbitrary surface of Enneper type can be constructed from a surface of Enneper type with one family of planar lines of curvature. We obtain a new description of the special class of surfaces of Enneper type with the property that the spheres that contain the lines of curvature correspondent to one of its principal curvatures are all centered on a common straight line. In particular, this class includes the nonzero constant Gaussian curvature ones. We also discuss the classification of the minimal surfaces of Enneper type.

Keywords: surfaces, lines of curvature, planar line of curvature, spherical line of curvature.

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INTRODUCTION

Surfaces in $\mathbb{R}^3$ with the property that the lines of curvature correspondent to one of its principal curvatures are contained in planes or spheres have been an object of study by many classical geometers as Dobriner (1887), Bianchi (1922), Eisenhart (1909) and Darboux (1993). Since there was a book on the subject by Enneper (1878) in 1878, some authors call any surface satisfying this geometric condition an *Enneper surface* or a *surface of Enneper type*. We will use the latter along this work.

Bianchi and Eisenhart first consider the case where all the lines of curvature are contained in planes. Since the tangents to a line of curvature and to its spherical representation at corresponding points are parallel, a line of curvature is contained in a plane if and only if its spherical representation is a circle on the unit sphere. Then, in this case, the Gauss map transforms the two families of lines of curvature into an orthogonal system of circles. The orthogonal systems of circles on the sphere are known, and this fact is used to give parametric equations for such surfaces. A parametrization for the surfaces for which only one family of lines of curvature are contained in planes is also presented, which is used by Bianchi to show how an arbitrary surface of Enneper type can be constructed by means of a surface in this class.

If one family of lines of curvature on a minimal surface consists of curves contained in planes, then so does the other family. Besides the Catenoid and Enneper minimal surface, there is a one-parameter family of such surfaces, see for Example (NITSCHE, 1989) and (CHO; OGATA, 2017). Using a similar approach as Nitche, Leite (2015) obtained the classification of the maximal surfaces with planar lines of curvature in the Minkowski space. By definition, a *maximal surface* is a space-like surface with zero mean curvature.

The interest in the surfaces of Enneper type has been renewed since the construction of immersed constant mean curvature tori in $\mathbb{R}^3$ by Wente (1986). These examples solved the long standing Hopf-Conjecture (dating back to the early 1800s): Is it possible to immerse a compact surface of positive genus in $\mathbb{R}^3$ with constant mean curvature? A theorem due to Alexandrov (1956) states that such a surface can not be embedded. The Wente tori are surfaces of Enneper type, as shown in (ABRESCH, 1987) and (SPRUCK, 1988).

Chion and Tojeiro (2021) in a recent work introduced the notion of Ribaucour partial tubes and generalized the results described in Bianchis's book for the corresponding hypersurfaces of Enneper type in $\mathbb{R}^{n+1}$. The authors obtained in particular a new description of the special class of surfaces of Enneper type with the property that the spheres that contain the lines of curvature correspondent to one of its principal curvatures are all centered on a common straight line, called *Joachimsthal surfaces*, based on the conformal diffeomorphism of $\mathbb{H}^2 \times \mathbb{R}$ onto $\mathbb{R}^3 \setminus \mathbb{R}$.

In this work, we revisit the subject of surfaces of Enneper type in $\mathbb{R}^3$ by providing a modern treatment of some classical results on this topic. We first present a parametrization for the class of surfaces of Enneper type for which the lines of curvature of one family are contained in planes, then we obtain parametric equations for the surfaces with this property in both families by using the analysis of the orthogonal system of circles on $\mathbb{S}^2$.

We show that, except for some special cases, a general surface of Enneper type can be constructed in terms of a surface in the latter class. For that, we show how to parametrize any surface of Enneper type with the lines of curvature of one family contained in spheres in terms of its Gauss map and a triple (γ, α, β), where γ is a smooth curve in $\mathbb{R}^3$ and α, β are smooth functions defined on an open interval. Then, we determine all the triples $(\bar{\gamma}, \bar{\alpha}, \bar{\beta})$ that give rise to surfaces of Enneper type that share the same Gauss map with a given one. It turns out that, among them, there always exists a surface for which the spheres containing the lines of curvature all pass through a common point. Since the inversion with respect to a sphere centered at that point maps such spheres into planes, we may conclude that the composition of the surface with this inversion is a surface of Enneper type with the lines of curvature of one family contained in planes.

Inspired by the recent results obtained in (CHION; TOJEIRO, 2021) and (TASSI; TOJEIRO, In preparation.) we give an explicit description of surfaces of Enneper type for which the lines of curvature of one family are contained either in concentric spheres, parallel planes, or planes that intersect along a common straight line. This result leads to a new description of all Joachimsthal surfaces and since every surface of nonzero constant Gaussian curvature with the lines of curvature of one family contained in planes is a Joachimsthal surface, then we also obtain a new description for such surfaces.

Moreover, a more detailed treatment of the known classification of minimal surfaces of Enneper type with the lines of curvature of one family contained in planes is presented.

Dissertation outline

Chapter 1 has the basic concepts of surfaces in $\mathbb{R}^3$. Except for the first section of Chapter 2 and the analysis of the orthogonal system of circles on $\mathbb{S}^2$ presented in the second section, all the other chapters are independent and all of them contain main results on the subject.

The first section of Chapter 2 presents some basic facts about lines of curvature. In particular, we see that every surface of Enneper type can be locally parametrized by lines of curvature. In the next sections, we show how to recover surfaces of Enneper type in terms of its Gauss map and support functions, which yields parametrizations for these surfaces.

In Chapter 3 we turn to give a complete description of some special surfaces of Enneper type. We present a proof of the fact that every surface of nonzero constant Gaussian curvature with the lines of curvature of one family contained in planes is a Joachimsthal surface. At the end, we use our results to construct some explicit examples of Joachimsthal surfaces.

Chapter 4 contains the classification of minimal surfaces of Enneper type with the lines of curvature of one family contained in planes. Here the orthogonal system of circles on $\mathbb{S}^2$ makes a comeback. Furthermore, in order to carry out our approach we need to derive the representation formulas for minimal surfaces.

CHAPTER

1

SURFACES IN THE EUCLIDEAN THREE SPACE

In this chapter, we establish some basic facts of the theory of surfaces in $\mathbb{R}^3$ that will be used throughout this work. The starting point for the results presented here is to introduce the second fundamental form by means of the Gauss and Weingarten formulas. Then we derive the Gauss and Codazzi equations, and as an application of the Gauss equation, we recover the celebrated *Theorema Egregium*. A more detailed treatment can be found in the books of Dajczer and Tojeiro (2019) and Spivak (1975).

1.1 The second fundamental form

Let M^2 be a two-dimensional manifold and let $f : M^2 \to \mathbb{R}^3$ be a smooth map. We say that f is an *immersion* if the differential $f_*(p) : T_pM^2 \to T_{f(p)}\mathbb{R}^3 \equiv \mathbb{R}^3$ is injective for all $p \in M^2$. Here the Euclidean space $\mathbb{R}^3$ is endowed with its usual metric denoted by $\langle , \rangle^\sim$.

The *induced metric* by f on M^2 is given by

$$\langle X,Y \rangle = \langle f_*(p)X, f_*(p)Y \rangle^\sim,$$

for all $p \in M^2$ and $X,Y \in T_pM^2$, with respect to which $f : M^2 \to \mathbb{R}^3$ becomes an isometric immersion. We refer to $f : M^2 \to \mathbb{R}^3$, or to $f(M^2)$, as an *immersed surface* in $\mathbb{R}^3$, or simply a *surface*.

A smooth map $V : U \to \mathbb{R}^3$ defined on an open subset $U \subset M^2$ such that $V(p) \in T_{f(p)}\mathbb{R}^3$ for all $p \in U$ is called a *vector field along* f. Considering $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{R}^3$, it is a standard result of Riemannian manifolds that $\tilde{\nabla}$ naturally induces a unique connection ∇^f on the set of all vector fields along f, called the *induced connection along* f.

Let $V : U \subset M^2 \to \mathbb{R}^3$ be a vector field along f which admits an extension to a vector field $\tilde{V}$ defined on some open subset $\tilde{U} \subset \mathbb{R}^3$, that is, $V = \tilde{V} \circ f$ where $\tilde{V} : \tilde{U} \to \mathbb{R}^3$ is a smooth map. Then the induced connection satisfies

$$(\nabla^f_X V)(p) = (\tilde{\nabla}_{f_*(p)X(p)}\tilde{V})(f(p)),$$

for all $p \in U$ and $X \in \mathfrak{X}(M^2)$. Since the Levi-Civita connection of $\mathbb{R}^3$ for the Euclidean metric is the usual differentiation of vector fields, we have

$$\nabla^f_X V(p) = d\tilde{V}_{f(p)} \cdot f_*(p)X(p) = \frac{d}{dt}\tilde{V}(x(t))\Big|_{t=0} := V_*X(p), \tag{1.1}$$

where $x : I \to \tilde{U}$ is a smooth curve with $x(0) = f(p)$ and $x'(0) = f_*(p)X(p)$. From now on we always identify ∇^f with $\tilde{\nabla}$ and use the same notation $\tilde{\nabla}$.

The following linear subspaces of $\mathbb{R}^3$

$$T_p f := f_*(p)(T_pM^2) \quad \text{and} \quad N_f M(p) := [f_*(p)(T_pM^2)]^\perp$$

are called the *tangent space* and *normal space* of f at $p \in M$, respectively. Given $V : M^2 \to \mathbb{R}^3$ a vector field along f, we can consider the decomposition

$$\tilde{\nabla}_X V = (\tilde{\nabla}_X V)^T + (\tilde{\nabla}_X V)^\perp,$$

with respect to the orthogonal decomposition $\mathbb{R}^3 = T_p f \oplus N_f M(p)$ for all $p \in M^2$. Thus, vector fields along f lying in the tangent space are given by

$$f_*X(p) := f_*(p)X(p), \quad p \in M^2$$

for some $X \in \mathfrak{X}(M^2)$.

Next, we present an important property of the induced connection along f, see (GORODSKI, 2016) for a proof of this result.

Proposition 1.1.1. Let $X, Y \in \mathfrak{X}(M^2)$ be vector fields on M^2 and let $U, V : M^2 \to \mathbb{R}^3$ be vector fields along f. Then the following identities hold

$$\tilde{\nabla}_X f_*Y - \tilde{\nabla}_Y f_*X = f_*[X,Y], \quad \text{and} \quad X\langle U,V\rangle^\sim = \langle \tilde{\nabla}_X U, V\rangle^\sim + \langle U, \tilde{\nabla}_X V\rangle^\sim$$

where $[,]$ stands for the Lie bracket of X and Y.

Given $X, Y \in \mathfrak{X}(M^2)$, let $C(X,Y)$ be the unique vector field on M^2 such that $f_*C(X,Y) = (\tilde{\nabla}_X f_*Y)^T$. One can easily check that

$$C : \mathfrak{X}(M^2) \times \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$$

defines an affine connection on M^2. Moreover, by the above proposition, it follows that C is symmetric and compatible with the induced metric by f. Therefore,

Proposition 1.1.2. Let $f : M^2 \to \mathbb{R}^3$ be a surface, then the Levi-Civita connection ∇ of M^2 is given by

$$\nabla_X Y = f_*^{-1}(\tilde{\nabla}_X f_*Y)^T$$

If we now consider $c : I \to M$ a smooth curve on M^2, then Proposition (1.1.2) means that

$$f_* \left(\frac{Dc'}{dt} \right) = \left(\frac{d^2}{dt^2} (f \circ c) \right)^T,$$

where $\frac{D}{dt}$ stands for the covariant derivative along c and $\frac{d}{dt}$ for the usual derivative of curves in $\mathbb{R}^3$. Hence, c is a geodesic of M^2 if and only if $(f \circ c)''(t)$ is normal to $T_{c(t)} f$, for all $t \in I$. In particular, if $f \circ c : I \to \mathbb{R}^3$ is a straight line then c is a geodesic.

Definition 1.1.1. The map α that assigns to the pair of vector fields $X, Y \in \mathfrak{X}(M^2)$ the vector field $\alpha(X,Y)$ along f given by

$$\alpha(X,Y) = [\tilde{\nabla}_X f_* Y]^{\perp}$$

is called the *second fundamental form* of f.

Using the Proposition 1.1.2, we can write the following basic formula of a surface

$$\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + \alpha(X,Y),$$

which is called the *Gauss formula*.

We now consider the following situation: Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a conformal diffeomorphism with $\lambda \in C^\infty(\mathbb{R}^3)$ the corresponding conformal factor; thus $\tilde{f} = \varphi \circ f : M^2 \to \mathbb{R}^3$ is also a surface. Since, for all $X, Y \in \mathfrak{X}(M^2)$,

$$\langle \tilde{f}_* X, \tilde{f}_* Y \rangle^{\sim} = \langle \varphi_* \circ f_* X, \varphi_* \circ f_* Y \rangle^{\sim} = \lambda^2 \circ f \langle f_* X, f_* Y \rangle^{\sim},$$

it follows that the induced metric by $\tilde{f}$ and the induced metric by f are conformal metrics on M^2 with conformal factor $\lambda \circ f$.

It is a known fact that if $g_2 = \lambda^2 g_1$ are conformal metrics on a manifold M, then the Levi-civita connections ∇^1 and ∇^2 of g_1 and g_2, respectively, are related by

$$\nabla^2_X Y = \nabla^1_X Y + \frac{1}{\lambda} \left(Y(\lambda) X + X(\lambda) Y - g_1(X,Y) \mathrm{grad}_1 \lambda \right),$$

for all $X, Y \in \mathfrak{X}(M)$, where $\mathrm{grad}_1 \lambda$ denotes the gradient of λ with respect to g_1. Now, applying this to the Gauss formula for f and $\tilde{f}$, we obtain the following relation between its second fundamental forms α^f and $\alpha^{\tilde{f}}$, respectively,

$$\alpha^{\tilde{f}}(X,Y) = \alpha^f(X,Y) - \frac{\langle f_* X, f_* Y \rangle^{\sim}}{\lambda \circ f} [(\mathrm{grad}\, \lambda) \circ f]^{\perp}, \tag{1.2}$$

for all $X, Y \in \mathfrak{X}(M^2)$, where $\mathrm{grad}\, \lambda$ is the gradient of λ with respect to $\langle , \rangle^{\sim}$. In particular, if $\lambda \equiv 1$ (i.e. φ is an isometry of $\mathbb{R}^3$) we see that the corresponding second fundamental forms agree.

From the properties of linearity of a connection, it is immediate that

$$\alpha(X+Z,Y) = \alpha(X,Y) + \alpha(Z,Y), \quad \alpha(X,Y+Z) = \alpha(X,Y) + \alpha(X,Z)$$

and

$$\alpha(\varphi X, Y) = \varphi\alpha(X, Y), \quad \alpha(X, \varphi Y) = \varphi\alpha(X, Y)$$

for all $X, Y, Z \in \mathfrak{X}(M^2)$ and $\varphi \in C^\infty(M^2)$, that is, the map α is $C^\infty(M^2)$-bilinear. Hence the value $\alpha(X,Y)(p) \in N_f M(p)$ depends only on the values $X(p), Y(p) \in T_p M^2$, see (LEE, 2018, Lemma B.6, p. 398).

The normal space $N_f M(p)$ contains precisely one unit vector up to sign. If $n_p \in N_f M(p)$ is one of these, we can write the Gauss formula at p as

$$(\tilde{\nabla}_X f_* Y)(p) = (f_* \nabla_X Y)(p) + \langle \alpha(X,Y)(p), n_p \rangle^{\sim} n_p. \tag{1.3}$$

Moreover, since

$$\begin{aligned}\alpha(X,Y) - \alpha(Y,X) &= \tilde{\nabla}_X f_* Y - f_* \nabla_X Y - \left(\tilde{\nabla}_Y f_* X - f_* \nabla_Y X\right) \\ &= \tilde{\nabla}_X f_* Y - \tilde{\nabla}_Y f_* X - f_*[X,Y] = 0\end{aligned}$$

taking Proposition 1.1.1 into account, we conclude that the map $B_{n_p} : T_p M \times T_p M \to \mathbb{R}$ given by

$$B_{n_p}(X,Y) = \langle \alpha(X,Y)(p), n_p \rangle^{\sim}, \quad X, Y \in T_p M,$$

is a symmetric bilinear form. Sometimes we also refer to B_{n_p} as the second fundamental form of f at p.

Definition 1.1.2. The *shape operator* of f at p is the self-adjoint operator $A_{n_p} : T_p M^2 \to T_p M^2$ corresponding to the second fundamental form, which is characterized by

$$\langle A_{n_p} X, Y \rangle = B_{n_p}(X,Y), \quad \text{for all } X, Y \in T_p M.$$

Notice that, since the shape operator of f at $p \in M^2$ is a self-adjoint operator, there exists an orthonormal basis of eigenvectors $\{e_1, e_2\}$ of $T_p M^2$ with real eigenvalues k_1, k_2. We say that e_i are *principal directions* and that k_i are *principal curvatures* of f at p.

Definition 1.1.3. The determinant $K := \det(A_{n_p})$ and the half of the trace $H := tr(A_{n_p})/2$ are the *Gaussian curvature* and the *mean curvature* of f at p, respectively.

In terms of the principal curvatures k_1 e k_2, we have

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}.$$

At every $p \in M^2$, we have a choice of two unit normal vectors $\pm n_p$. The principal directions do not depend on this choice, and only the signs of the principal curvatures do. Then, the sign of H depends on the choice of the unit normal vector, but the sign of K does not. If the mean curvature of f vanishes everywhere, the surface is said to be *minimal*.

Definition 1.1.4. A point $p \in M^2$ is called

- *umbilical* if A_{n_p} is a multiple of the identity.
- *planar* if $A_{n_p} = 0$.
- *flat* if $K = 0$.

Notice that p is an umbilical point if and only if all the unit vectors of T_pM are principal directions with respect to the same principal curvature, which is equivalent to $K = H^2$.

1.2 Gauss map

Naturally, it is not useful to choose the unit normal vector n_p at random for each point $p \in M^2$.

Definition 1.2.1. Let $N : U \subset M^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ be a smooth map defined on an open subset $U \subset M^2$. We say that N is a *unit normal vector field* or a *Gauss map* of f if $N(p) = n_p$, for all $p \in U$.

If $N : U \to \mathbb{S}^2$ is a Gauss map of f, then for all $X, Y \in \mathfrak{X}(U)$ we can write

$$\langle AX, Y\rangle = \langle \alpha(X,Y), N\rangle^{\sim},$$

where A stands for the shape operator with respect to N, that is, $A(p) = A_{N(p)}$ and $AX \in \mathfrak{X}(U)$ is given by $AX(p) := A(p)X(p) \in T_pM$, for all $p \in U$. Thus, the Gauss formula becomes

$$\tilde{\nabla}_X f_*Y = f_*\nabla_X Y + \langle AX, Y\rangle N.$$

Proposition 1.2.1. If $f : M^2 \to \mathbb{R}^3$ is a surface and $N : U \to \mathbb{S}^2$ is a Gauss map defined on the open subset $U \subset M$, then

$$\tilde{\nabla}_X N = -f_*AX, \tag{1.4}$$

for all $X \in \mathfrak{X}(U)$.

Proof. Given $X, Y \in \mathfrak{X}(U)$ we have $\langle f_*Y, N\rangle^{\sim} = 0$. Then

$$\begin{aligned} 0 = X\langle f_*Y, N\rangle^{\sim} &= \langle \tilde{\nabla}_X f_*Y, N\rangle^{\sim} + \langle f_*Y, \tilde{\nabla}_X N\rangle^{\sim} \\ &= \langle AX, Y\rangle + \langle f_*Y, \tilde{\nabla}_X N\rangle^{\sim}, \end{aligned}$$

and hence

$$\langle -f_*AX, f_*Y\rangle^{\sim} = \langle \tilde{\nabla}_X N, f_*Y\rangle^{\sim}. \tag{1.5}$$

Now, for every $p \in U$, T_pf is the orthogonal complement to $N(p) \in \mathbb{S}^2$, so we can identify $T_pf = T_{N(p)}\mathbb{S}^2$. Since Y has been chosen arbitrarily, the conclusion follows from (1.5). □

In addition to the Gauss formula, the equation (1.4) is another basic formula of the theory of surfaces, known as the *Weingarten formula*. In terms of the differential of N, the Weingarten formula becomes

$$i_* \circ N_* = -f_* \circ A,$$

where $i : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map. By the inverse Function Theorem for Manifolds, we see that if $p \in U$ is not a flat point, then N is a diffeomorphism in a neighborhood of p.

Next, we see that a Gauss map can always be defined locally. Let $(U, x = (u_1, u_2))$ be a chart on M^2 and consider the coordinate vector fields

$$\partial_{u_1}(p) := \frac{\partial}{\partial u_1}(p) \;\text{ and }\; \partial_{u_2}(p) := \frac{\partial}{\partial u_2}(p),$$

and also the coordinate vector fields along f

$$\frac{\partial f}{\partial u_1}(p) := f_* \frac{\partial}{\partial u_1}(p) \;\text{ and }\; \frac{\partial f}{\partial u_2}(p) := f_* \frac{\partial}{\partial u_2}(p),$$

for all $p = x^{-1}(u_1, u_2) \in U$. Let $\{du_1, du_2\}$ be the dual 1-forms of the basis $\{\partial_{u_1}, \partial_{u_2}\}$. Then, with respect to the coordinates (u_1, u_2), the metric of M^2 is given by

$$ds^2 = g_{11} du_1^2 + 2g_{12} du_1 du_2 + g_{22} du_2^2,$$

where we have set

$$g_{ij} := \langle \partial_{u_i}, \partial_{u_j} \rangle = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle^{\sim},$$

called the *coefficients of the first fundamental form*. We say that (u_1, u_2) is a local system of *orthogonal* coordinates if $g_{12} = 0$. If, in addition, $g_{11} = g_{22}$, we say that u_1 and u_2 are *isothermal parameters* of f.

Since

$$\tilde{N} := \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2}$$

defines a vector field along f everywhere normal to the tangent space of f, we obtain that $N := \tilde{N}/\|\tilde{N}\|$ is a Gauss map of f defined on U. The corresponding *coefficients of the second fundamental form* are given by

$$b_{ij} := \langle \alpha(\partial_{u_i}, \partial_{u_j}), N \rangle^{\sim} = \langle \tilde{\nabla}_{\partial_{u_i}} f_* \partial_{u_j}, N \rangle^{\sim} = \left\langle \frac{\partial^2 f}{\partial u_i \partial u_j}, N \right\rangle^{\sim}.$$

Taking another chart $(\bar{U}, \bar{x} = (\bar{u}_1, \bar{u}_2))$, we have

$$\frac{\partial f}{\partial \bar{u}_1} \wedge \frac{\partial f}{\partial \bar{u}_2} = \frac{\partial(u_1, u_2)}{\partial(\bar{u}_1, \bar{u}_2)} \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2}.$$

Thus N preserves its sign or changes it, depending on whether the Jacobian of the coordinate change $\partial(u_1, u_2)/\partial(\bar{u}_1, \bar{u}_2)$ is positive or negative, respectively. Therefore, if M^2 is oriented, we can define globally a Gauss map $N : M^2 \to \mathbb{S}^2$.

Let us now compute the matrix of the shape operator with respect to the basis $\{\partial_{u_1}, \partial_{u_2}\}$. We first observe that

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} := \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}, \tag{1.6}$$

where $()^{-1}$ means the inverse matrix of $()$. Writing

$$A\partial_{u_i} = \sum_{l=1}^{2} a_{il}\partial_{u_l},$$

with as yet undetermined coefficients a_{il}, it follows from the Gauss formula that

$$b_{ij} = \langle A\partial_{u_i}, \partial_{u_j}\rangle = \sum_{l=1}^{2} a_{il} g_{lj},$$

whence

$$a_{ik} = \sum_{j=1}^{2}\sum_{l=1}^{2} a_{il} g_{lj} g^{jk} = \sum_{j=1}^{2} b_{ij} g^{jk}.$$

Therefore, this and (1.6) give

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}. \tag{1.7}$$

From (1.7) we immediately obtain

$$K = \det(a_{ij}) = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

and

$$H = \frac{1}{2} tr(a_{ij}) = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}.$$

In particular, we have just proved that the Gaussian curvature K and the mean curvature H are smooth functions in M^2.

Since k_1 and k_2 are the eigenvalues of the shape operator, it follows that k_1 and k_2 satisfy the equation

$$k^2 - 2Hk + K = 0,$$

which implies that

$$k = H \pm \sqrt{H^2 - K}.$$

Thus, if we choose $k_1(p) \geq k_2(p)$, $p \in M^2$, the functions k_1 and k_2 are continuous in M^2, and are also smooth, except possibly at the umbilical points of f. Moreover, the set of nonumbilical points is precisely the set where $k_1 > k_2$, and then this is an open subset of M^2.

It is also useful to compute the vectors $N_*\partial_{u_i}$ in terms of the basis $\{f_*\partial_{u_1}, f_*\partial_{u_1}\}$. By the Weingarten formula we have

$$\langle N_*\partial_{u_i}, f_*\partial_{u_j}\rangle^\sim = -\langle f_*A\partial_{u_i}, f_*\partial_{u_j}\rangle^\sim = \langle A\partial_{u_i}, \partial_{u_j}\rangle = -b_{ij}.$$

By a straightforward computation, we obtain

$$N_*\partial_{u_1} = \frac{1}{g_{11}g_{22}-g_{12}^2}\left((g_{12}b_{12}-g_{22}b_{11})f_*\partial_{u_1} + (g_{12}b_{11}-g_{11}b_{12})f_*\partial_{u_2}\right)$$

and

$$N_*\partial_{u_2} = \frac{1}{g_{11}g_{22}-g_{12}^2}\left((g_{12}b_{22}-g_{22}b_{12})f_*\partial_{u_1} + (g_{12}b_{12}-g_{11}b_{22})f_*\partial_{u_2}\right).$$

Hence, it further holds that

$$\langle N_*\partial_{u_i}, N_*\partial_{u_j}\rangle^\sim = 2Hb_{ij} - Kg_{ij}. \tag{1.8}$$

Remark 1. Assuming that $N : U \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a local diffeomorphism, we infer from (1.8) that. if $H \equiv 0$, that is, if $f : U \to \mathbb{R}^3$ is minimal, then the metric induced by N on U is conformal to the induced metric by f.

A smooth local orthonormal frame $\{e_1, e_2\}$ of M^2 is said to be *principal* if each $e_i(p)$ is a principal direction of f at p.

Proposition 1.2.2. Let $f : M^2 \to \mathbb{R}^3$ be a surface and let $p \in M^2$ a nonumbilical point of f. Then there is a principal orthonormal frame defined on a neighborhood of p.

Proof. We start with coordinate vector fields on a neighborhood U of p free of umbilical points and consider $N : U \to \mathbb{S}^2$ a Gauss map of f on U with corresponding shape operator A. Then we apply the Gram-Schmidt algorithm simultaneously over U to obtain a smooth orthonormal frame $\{E_1, E_2\}$.

Since A is a self-adjoint operator we can write

$$AE_1 = aE_1 + bE_2 \quad \text{and} \quad AE_2 = bE_1 + cE_2,$$

for some $a, b, c \in C^\infty(U)$. Hence

$$K = ac - b^2, \quad \text{and} \quad H = \frac{1}{2}(a+c).$$

Using that A at p is not a multiple of the identity, we can assume that $E_i(p)$ is not a principal direction, after applying orthogonal matrices to E_1 and E_2, if necessary. Thus, $b(p) \neq 0$ and the same holds in a smaller neighborhood $\tilde{U}$ of p. Let k_1 and k_2 the principal curvatures of f in $\tilde{U}$. Since $k_i^2 - 2Hk_i + K = 0$, we obtain

$$b^2 + c(k_1 - a) = k_1(k_1 - a), \quad \text{and} \quad b^2 + a(k_2 - c) = k_2(k_2 - c). \tag{1.9}$$

Now, define the new vector fields $\tilde{e}_1$ and $\tilde{e}_2$ in $\tilde{U}$ by

$$\tilde{e}_1 = bE_1 + (k_1 - a)E_2 \text{ and } \tilde{e}_2 = (k_2 - c)E_1 + bE_2.$$

Since $b \neq 0$, we see that $\tilde{e}_i$ is never zero in $\tilde{U}$. Moreover, we have

$$\begin{aligned} A\tilde{e}_1 &= bAE_1 + (k_1 - a)AE_2 \\ &= b(aE_1 + bE_2) + (k_1 - a)(bE_1 + cE_2) \\ &= k_1 bE_1 + (b^2 + c(k_1 - a))E_2 \\ &= k_1 \tilde{e}_1, \end{aligned}$$

taking (1.9) into account. Similarly, we obtain that $A\tilde{e}_2 = k_2\tilde{e}_2$. Finally, setting $e_i = \tilde{e}_i/\|\tilde{e}_i\|$, we see that $\{e_1, e_2\}$ is a principal orthonormal frame on $\tilde{U}$. □

We consider again the surface $\tilde{f} = \varphi \circ f : M^2 \to \mathbb{R}^3$. If $N : U \subset M^2 \to \mathbb{S}^2$ is a Gauss map of f, then $\tilde{N} = N/(\lambda \circ f)$ is clearly a Gauss map of $\tilde{f}$ on U. Denoting by A and $\tilde{A}$ the shape operator of f and $\tilde{f}$ with respect to N and $\tilde{N}$, respectively, we can use (1.2) to obtain

$$\begin{aligned} \langle \tilde{A}X, Y \rangle &= \left\langle \alpha^{\tilde{f}}(X,Y), \frac{N}{\lambda \circ f} \right\rangle^{\sim} \\ &= \left\langle \frac{1}{\lambda \circ f} AX - \frac{\langle (\operatorname{grad} \lambda) \circ f, N \rangle^{\sim}}{\lambda^2 \circ f} X, Y \right\rangle, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(U)$. Hence

$$\tilde{A}X = \frac{1}{\lambda \circ f} AX - \frac{\langle (\operatorname{grad} \lambda) \circ f, N \rangle^{\sim}}{\lambda^2 \circ f} X. \tag{1.10}$$

Then, we see from (1.10) that $X(p)$ is an eigenvector of A_p if and only if $X(p)$ is an eigenvector of $\tilde{A}_p$, and we express this fact by saying that the conformal diffeomorphism φ preserves principal curvatures.

Before we present some examples, notice that any immersed submanifold $M^2 \subset \mathbb{R}^3$ is a surface with respect to the inclusion map $i : M^2 \to \mathbb{R}^3$.

Example 1.2.1 (Plane)**.** For any plane in $\mathbb{R}^3$ the Gauss map is constant, so its shape operator is zero by the Weingarten formula. Thus all points are umbilical and planar and $K \equiv H \equiv 0$.

Example 1.2.2 (Sphere)**.** Let $\mathbb{S}^2(p_0, r)$ be a sphere with center $p_0 \in \mathbb{R}^3$ and radius $r > 0$ oriented by its inward pointing unit position vector field. Since

$$N(p) = -\frac{p - p_0}{r},$$

we obtain that the corresponding shape operator is $A = (1/r)\mathrm{Id}$, and we see that every point is umbilical with $k_1 = k_2 = 1/r$, hence $K \equiv 1/r^2$ and $H \equiv 1/r$.

Example 1.2.3 (Vertical Cylinder over a plane curve)**.** Let $g : I \to \mathbb{R}^2$ be a regular curve defined on an open interval I. The *cylinder over* g is the set $g(I) \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} \equiv \mathbb{R}^3$. Here a parametrization is given by

$$f(s,t) = (g(s),t), \quad (s,t) \in I \times \mathbb{R}.$$

It is easy to see that the Gauss map N is always parallel to $\mathbb{R}^2$ and $N_* \partial/\partial t = 0$, then by Weingarten formula $\partial/\partial t$ is a principal direction at every point with $k_1 = 0$, hence $K \equiv 0$.

Example 1.2.4 (Cone)**.** Let $h : I \to \mathbb{R}^3$ be a regular curve and $V \in \mathbb{R}^3$, consider the map $f : I \times \mathbb{R} \to \mathbb{R}^3$ given by

$$f(s,t) = V + t(h(s) - V).$$

We say that f is a *cone over h with vertex at V*. We have

$$\frac{\partial f}{\partial s} = th'(s), \quad \text{and} \quad \frac{\partial f}{\partial t} = h(s) - V,$$

hence the regular points of f occur when $t \neq 0$ (V can not be in the surface) and $h'(s)$ is linearly independent of $h(s) - V$. Thus, on the open subset of regular points, f defines a surface. Moreover, since the tangent space of f at (s,t) is spanned by $h'(s)$ and $h(s) - V$, the Gauss map is constant along the coordinate curves $t \mapsto (s_0, t)$, thus $N_* \partial/\partial t = 0$ and, consequently, $K \equiv 0$.

Example 1.2.5 (Surfaces of Revolution)**.** These surfaces are obtained by rotating a regular curve $s \in I \mapsto (g_1(s), g_2(s))$ (the *profile* curve) contained in a plane about an axis that does not intersect the curve. If the curve intersects the rotation axis, it must do so at a right angle. Taking $y = 0$ as the plane of the curve and the z-axis as the rotation axis, the surface can be parametrized by

$$f(s,t) = (g_1(s)\cos t, g_1(s)\sin t, g_2(s)), \; (s,t) \in I \times (0, 2\pi).$$

Here g_1 is positive everywhere. The image by f of the coordinate curves $t \mapsto (s_0, t)$ and $s \mapsto (s, t_0)$ are called *parallels* and *meridians*, respectively. Note that the parallels are contained in planes that are parallel to the plane $z = 0$ and the meridians are contained in planes that intersect along the z-axis.

Assuming that the profile curve is parametrized by arclength, a straightforward calculation shows that

$$K = -\frac{g_1''}{g_1}, \quad \text{and} \quad H = \frac{1}{2}\frac{-g_2' + g_1(g_2' g_1'' - g_2'' g_1')}{g_1},$$

with respect to the inward pointing Gauss map. One can deduce by this explicit expression for the Gaussian curvature that the only surfaces of revolution for which $K \equiv 0$ are the cylinders over a circle, the cones over a circle, and planes.

Next, we show that spheres and planes are essentially the only surfaces all of whose points are umbilical.

Proposition 1.2.3. Let $f : M^2 \to \mathbb{R}^3$ be a surface. If M^2 is connected and all of its points are umbilical, then $f(M^2)$ is either contained in a sphere or in a plane.

Proof. Let $(U, x = (u_1, u_2))$ be a chart on M^2 with U connected. Using that all the points are umbilical, it follows by the Weingarten formula that

$$\frac{\partial N}{\partial u_i} = \lambda \frac{\partial f}{\partial u_i}, \tag{1.11}$$

for some $\lambda \in C^\infty(U)$. By differentiating the preceding equation, we obtain

$$\frac{\partial \lambda}{\partial u_i}\frac{\partial f}{\partial u_j} + \lambda \frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial^2 N}{\partial u_i \partial u_j} = \frac{\partial^2 N}{\partial u_j \partial u_i} = \frac{\partial \lambda}{\partial u_j}\frac{\partial f}{\partial u_i} + \lambda \frac{\partial^2 f}{\partial u_j \partial u_i},$$

hence

$$\frac{\partial \lambda}{\partial u_i}\frac{\partial f}{\partial u_j} = \frac{\partial \lambda}{\partial u_j}\frac{\partial f}{\partial u_i}. \tag{1.12}$$

Since $\partial f / \partial u_1$ and $\partial f / \partial u_2$ are linearly independent, we see from (1.12) that the partial derivatives of λ vanish, so λ must be constant on U.

If $\lambda = 0$, we infer from (1.11) that N is constant on U, thus $f(U)$ is contained in a plane. For $\lambda \neq 0$, we see that $f - N/\lambda \equiv p_0$ must be constant on U, hence

$$\|f - p_0\| = \frac{1}{|\lambda|},$$

and this implies that $f(U) \subset \mathbb{S}^2(p_0, |\lambda|^{-1})$. This proves the claim in the case $M^2 = U$, but we may conclude the proof using the connectedness of M^2. □

1.3 The Gauss and Codazzi equations

We now turn to relate the Gaussian curvature of a surface defined extrinsically by means of the second fundamental form, with the sectional curvature of the immersed manifold.

Let (M, g) be a general Riemannian manifold and denote by ∇ its Levi-Civita connection. In this work, we are considering the following sign convention for the curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Now, let $f : M^2 \to \mathbb{R}^3$ be a surface. Since the curvature tensor of $\mathbb{R}^3$ is identically zero, we have

$$\tilde{\nabla}_X \tilde{\nabla}_Y f_* Z - \tilde{\nabla}_Y \tilde{\nabla}_X f_* Z - \tilde{\nabla}_{[X,Y]} f_* Z = 0, \tag{1.13}$$

for all $X, Y, Z \in \mathfrak{X}(M^2)$.

Proposition 1.3.1 (Gauss Equation)**.** For vector fields X,Y,Z,W on M^2,

$$\langle R(X,Y)Z,W\rangle = \langle \alpha(X,W),\alpha(Y,Z)\rangle^{\sim} - \langle \alpha(X,Z),\alpha(Y,W)\rangle^{\sim},$$

where R denotes the curvature tensor of M^2 with respect to the induced metric by f.

Proof. By the Gauss formula, we obtain

$$\begin{aligned}\langle \tilde{\nabla}_X\tilde{\nabla}_Y f_*Z, f_*W\rangle^{\sim} &= X\langle \tilde{\nabla}_Y f_*Z, f_*W\rangle^{\sim} - \langle \tilde{\nabla}_Y f_*Z, \tilde{\nabla}_X f_*W\rangle^{\sim} \\ &= X\langle f_*\nabla_Y Z, f_*W\rangle^{\sim} - \langle f_*\nabla_Y Z, f_*\nabla_X W\rangle^{\sim} \\ &\quad - \langle \alpha(Y,Z),\alpha(X,W)\rangle^{\sim} \\ &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \alpha(Y,Z),\alpha(X,W)\rangle^{\sim} \\ &= \langle \nabla_X\nabla_Y Z, W\rangle - \langle \alpha(Y,Z),\alpha(X,W)\rangle^{\sim}\end{aligned}$$

and by interchanging X and Y,

$$\langle \tilde{\nabla}_Y\tilde{\nabla}_X f_*Z, f_*W\rangle^{\sim} = \langle \nabla_Y\nabla_X Z, W\rangle - \langle \alpha(X,Z),\alpha(Y,W)\rangle^{\sim}.$$

We further have

$$\langle \tilde{\nabla}_{[X,Y]} f_*Z, f_*W\rangle^{\sim} = \langle f_*\nabla_{[X,Y]}Z, f_*W\rangle^{\sim} = \langle \nabla_{[X,Y]}Z, W\rangle.$$

Now, using (1.13), it follows that

$$\begin{aligned}0 &= \langle \tilde{\nabla}_X\tilde{\nabla}_Y f_*Z - \tilde{\nabla}_Y\tilde{\nabla}_X f_*Z - \tilde{\nabla}_{[X,Y]} f_*Z, f_*W\rangle^{\sim} \\ &= \langle R(X,Y)Z,W\rangle - \langle \alpha(Y,Z),\alpha(X,W)\rangle^{\sim} + \langle \alpha(X,Z),\alpha(Y,W)\rangle^{\sim}.\end{aligned}$$

□

If we now chose $(U, x = (u_1,u_2))$ a chart on M^2, by the Gauss equation it follows that

$$\frac{\langle R(\partial_{u_1},\partial_{u_2})\partial_{u_2},\partial_{u_1}\rangle}{|\partial_{u_1}|^2|\partial_{u_2}|^2 - \langle \partial_{u_1},\partial_{u_2}\rangle^2} = \frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2} = K.$$

Thus we have:

Corollary 1.3.0.1. The sectional curvature of M^2 and the Gaussian curvature of f coincide at each point $p \in M^2$.

In the case where $M^2 \subset \mathbb{R}^3$, we recover from the previous Corollary the celebrated *Theorem Egregium*, which asserts that the Gaussian curvature of a regular surface in $\mathbb{R}^3$ is an intrinsic invariant.

Next, we see that the normal component of (1.13) gives another equation for a surface.

Proposition 1.3.2 (Codazzi Equation)**.** Let $N : U \subset M^2 \to \mathbb{R}^3$ be a Gauss map of f. Then, for vector fields defined on U,

$$\nabla_X AY - A\nabla_X Y = \nabla_Y AX - A\nabla_Y X.$$

Proof. By the Gauss and Weingarten formula, we obtain

$$\begin{aligned}
\langle\tilde{\nabla}_X\tilde{\nabla}_Y f_*Z, N\rangle^{\sim} &= X\langle\tilde{\nabla}_Y f_*Z, N\rangle^{\sim} - \langle\tilde{\nabla}_Y f_*Z, \tilde{\nabla}_X N\rangle^{\sim} \\
&= X\langle\alpha(Y,Z), N\rangle^{\sim} - \langle\tilde{\nabla}_Y f_*Z, -f_*AX\rangle^{\sim} \\
&= X\langle AY, Z\rangle + \langle f_*\nabla_Y Z, f_*AX\rangle^{\sim} \\
&= \langle\nabla_X AY, Z\rangle + \langle AY, \nabla_X Z\rangle + \langle\nabla_Y Z, AX\rangle,
\end{aligned}$$

and by interchanging X and Y,

$$\langle\tilde{\nabla}_Y\tilde{\nabla}_X f_*Z, N\rangle^{\sim} = \langle\nabla_Y AX, Z\rangle + \langle AX, \nabla_Y Z\rangle + \langle\nabla_X Z, AY\rangle.$$

It also holds that

$$\begin{aligned}
\langle\tilde{\nabla}_{[X,Y]} f_*Z, N\rangle^{\sim} &= \langle\alpha([X,Y],Z), N\rangle^{\sim} = \langle A[X,Y], Z\rangle \\
&= \langle A\nabla_X Y - A\nabla_Y X, Z\rangle,
\end{aligned}$$

which, together with (1.13), give

$$\begin{aligned}
0 &= \langle\tilde{\nabla}_X\tilde{\nabla}_Y f_*Z - \tilde{\nabla}_Y\tilde{\nabla}_X f_*Z - \tilde{\nabla}_{[X,Y]} f_*Z, N\rangle^{\sim} \\
&= \langle\nabla_X AY - A\nabla_X Y - \nabla_Y AX + A\nabla_Y X, Z\rangle,
\end{aligned}$$

and since Z has been chosen arbitrarily we obtain the desired equation. □

Remark 2. Let $\{e_1, e_2\}$ be a principal orthonormal frame defined in a neighborhood U of M^2 with corresponding principal curvatures k_1 and k_2. Since

$$\begin{aligned}
\langle\nabla_{e_i} Ae_j - A\nabla_{e_i} e_j, e_l\rangle &= \langle k_j\nabla_{e_i} e_j + e_i(k_j)e_j, e_l\rangle - \langle\nabla_{e_i} e_j, Ae_l\rangle \\
&= \langle(k_j - k_l)\nabla_{e_i} e_j + e_i(k_j)e_j, e_l\rangle,
\end{aligned}$$

we see that, on U, the Codazzi equation is equivalent to the following two equations:

$$e_1(k_2) = (k_1 - k_2)\langle\nabla_{e_2} e_1, e_2\rangle \quad \text{and} \quad e_2(k_1) = (k_2 - k_1)\nabla_{e_1} e_2, e_1\rangle.$$

CHAPTER 2

SURFACES OF ENNEPER TYPE

We present a parametrization for the class of surfaces of Enneper type for which the lines of curvature of one family are contained in planes, then we obtain parametric equations for the surfaces with this property in both families. We also describe how a general surface of Enneper type can be constructed in terms of a surface in the latter class.

2.1 Lines of curvature

In this section, we establish some definitions and basic facts about the class of regular curves on a surface that are tangent to a principal direction at any point. We will concentrate only on those facts which are of importance to us later on.

Let $f : M^2 \to \mathbb{R}^3$ be a surface and let $c : I \to M^2$ be a smooth curve. The curve c is said to be a *line of curvature* of f if c' always points along a principal direction of f. By the Weingarten formula, this means that

$$N_* c' = -k f_* c', \tag{2.1}$$

where N is a Gauss map of f defined in a neighborhood of $c(I)$ and $k(t)$ is a principal curvature at $c(t)$. The following result is useful to find lines of curvature.

Theorem 2.1.1 (Joachimsthal's theorem)**.** Let $f : M_1 \to \mathbb{R}^3$ and $g : M_2 \to \mathbb{R}^3$ be surfaces. Consider $c_1 : I \to M_1$ and $c_2 : I \to M_2$ regular curves with

$$f(c_1(t)) = g(c_2(t)) := c(t),$$

for all $t \in I$. Suppose that the intersection of f and g along c be transversal, that is, $T_{c_1(t)} f \neq T_{c_2(t)} g$. Then each two of the following statements imply the third:

(a) c_1 is a line of curvature of f;

(b) c_2 is a line of curvature of g;

(c) $f(M_1)$ and $g(M_2)$ intersect at a constant angle along c.

Proof. The result is local in nature, so we assume that both M_1 and M_2 are oriented. Let $N_i : M_i \to \mathbb{S}^2$, $1 \leq i \leq 2$, be the Gauss map that determines the corresponding surface. Then

$$\begin{aligned} \frac{d}{dt}\langle N_1 \circ c_1, N_2 \circ c_2\rangle^{\sim} &= \langle \tilde{\nabla}_{c_1'} N_1, N_2\rangle^{\sim} + \langle N_1, \tilde{\nabla}_{c_2'} N_2\rangle^{\sim} \\ &= \langle -f_* A_1 c_1', N_2\rangle^{\sim} + \langle N_1, -g_* A_2 c_2'\rangle^{\sim}, \end{aligned} \tag{2.2}$$

where A_i, $1 \leq i \leq 2$, is the shape operator with respect to N_i. Since

$$c'(t) = f_* c_1'(t) = g_* c_2'(t),$$

we also have

$$\langle f_* c_1'(t), N_2(c_2(t))\rangle^{\sim} = \langle g_* c_2'(t), N_1(c_1(t))\rangle^{\sim} = 0. \tag{2.3}$$

If the statements (a) and (b) hold, then (2.2) and (2.3) imply that

$$\langle N_1(c_1(t)), N_2(c_2(t))\rangle^{\sim} = const,$$

and (c) follows. Suppose now that (a) and (c) hold. By (2.2) and (2.3), we see that $g_* A_2 c_2'(t)$ is perpendicular to $N_1(c_1(t))$. On the other hand, it is also perpendicular to $N_2(c_1(t))$. Since $\langle N_1(c_1(t)), N_2(c_2(t))\rangle^{\sim}$ is constant and $N_1(c_1(t)) \neq \pm N_2(c_2(t))$, we conclude that $N_1(c_1(t))$ and $N_2(c_2(t))$ are linearly independent, hence $g_* A_2 c_2'(t)$ must be a multiple of $c'(t) = g_* c_2'(t)$, and consequently c_2 is a line of curvature of g. The proof that (b) and (c) imply (a) is analogous to the latter case. □

Let $\Sigma \subset \mathbb{R}^3$ denote either a plane or a sphere of $\mathbb{R}^3$. Since every regular curve in Σ is a line of curvature of $i : \Sigma \to \mathbb{R}^3$, we have the following restatement of the previous theorem.

Corollary 2.1.1.1. Let $f : M^2 \to \mathbb{R}^3$ be a surface and let $c : I \to M^2$ be a regular curve such that $f(c(I))$ is contained in either a plane or a sphere Σ of $\mathbb{R}^3$. Then c is a line of curvature of f if and only if Σ intersects $f(M^2)$ at a constant angle along $f(c(I))$.

Joachimsthal's theorem will be used mostly in the last form. A smooth curve $c : I \to M^2$ is said to be *planar* or *spherical* if the corresponding curve $f(c(I))$ is contained in a plane or a sphere of $\mathbb{R}^3$. From (2.1) a line of curvature of a surface is planar if and only if its image by the Gauss map in the unit sphere is a planar curve, that is, it is an arc of a circle.

Although all the planar curves are lines of curvature of its planes, only straight lines are geodesics. In the next Corollary, we state a result in the converse direction.

Corollary 2.1.1.2. Let $f : M^2 \to \mathbb{R}^3$ be a surface and let $c : I \to M^2$ be a geodesic. Suppose that $f \circ c$ has nowhere vanishing curvature (as a curve in $\mathbb{R}^3$) and that it lies in a plane normal to a nonzero vector $b \in \mathbb{R}^3$. Then, c is a line of curvature of f.

Proof. Clearly, $\langle (f\circ c)'', b\rangle^{\sim} = 0$ and, since c is a geodesic, $(f\circ c)''$ is a multiple of $N\circ c$. Thus

$$\langle N\circ c, b\rangle^{\sim} = 0,$$

which implies that c is a line of curvature. □

Given a chart $(U, x = (u_1, u_2))$ on M^2, we say that $f\circ x^{-1}$ is a local parametrization of f *by lines of curvature* if the coordinate curves $u_1 \mapsto x^{-1}(u_1, u_2^0)$ and $u_2 \mapsto x^{-1}(u_1^0, u_2)$ are lines of curvature of f, or equivalently, if the coordinate vector fields ∂_{u_1} and ∂_{u_1} are eigenvectors of the shape operator of f. If we assume that the coordinates (u_1, u_2) are orthogonal, then ∂_{u_1} and ∂_{u_2} are eigenvectors of the shape operator A of f if and only if $b_{12} = \langle A\partial_{u_1}, \partial_{u_2}\rangle \equiv 0$. In this case, we also say that such coordinates are *principal coordinates* or *parameters of lines of curvature*.

Remark 3. Let $(U, x = (u_1, u_2))$ be a local system of principal coordinates and suppose that f is free of flat points in U. Consider the surface $h = i\circ N : U \to \mathbb{R}^3$, where $i : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map. Since

$$\langle h_*\partial_{u_1}, h_*\partial_{u_2}\rangle^{\sim} = \langle f_*A\partial_{u_1}, f_*A\partial_{u_2}\rangle^{\sim} = \langle A\partial_{u_1}, A\partial_{u_2}\rangle = 0,$$

we see that (u_1, u_2) are also orthogonal coordinates with respect to the metric induced by h.

Let us now consider $p \in M^2$ a nonumbilical point of f and $\{e_1, e_2\}$ a principal orthonormal frame defined on a neighborhood V of p (see Proposition 1.2.2). Hence, the lines of curvature of f in V are the integral curves of these vector fields, up to reparametrization. Next, we observe a general fact about any two-dimensional manifold.

Proposition 2.1.1. ((SPIVAK, 1975, ADDENDUM 2)) Let X_1 and X_2 be linearly independent vector fields in a neighborhood of a point p in a two-dimensional manifold M^2. Then there is a local chart $(U, x = (u_1, u_2))$ on M^2 with $p \in U$ such that the coordinate curves $u_1 \mapsto x^{-1}(u_1, u_2^0)$ and $u_2 \mapsto x^{-1}(u_1^0, u_2)$ lie along the integral curves of X_1 and X_2, respectively.

Therefore, applying the above proposition to e_1 and e_2 we obtain a local system of principal coordinates $(U, x = (u_1, u_2))$ for f such that $U \subset V$.

Definition 2.1.1. A surface $f : M^2 \to \mathbb{R}^3$ is said to be of *Enneper type* if it is free of umbilical points and the family of lines of curvature correspondent to one of its principal curvatures are planar or spherical.

In the case of a surface of revolution, Corollary 2.1.1.1 shows that meridians and parallels must be lines of curvature, hence surfaces of revolution free of umbilical points are surfaces of Enneper type.

We end this section by presenting an intrinsic notion of curvature for regular curves in a Riemannian manifold. Let $(M^2, \langle , \rangle)$ be an oriented Riemannian manifold, and let $c : I \to M^2$ be a regular curve. Denote $e := c'/\|c'\|$ the unit field of directions, and consider n the vector field along c such that $\{e(t), n(t)\}$ is an orthonormal positive basis of $T_{c(t)}M^2$. We define the *signed geodesic curvature* of c by the number

$$k_g(t) := \frac{1}{\|c'(t)\|^2}\left\langle \frac{Dc'}{dt}(t), n(t)\right\rangle,$$

for all $t \in I$. It follows immediately from the compatibility of the Levi-Civita connection that

$$\frac{De}{dt} = \|c'\|k_g n \quad \text{and} \quad \frac{Dn}{dt} = -\|c'\|k_g e.$$

Thus, the curve c has vanishing signed geodesic curvature if and only if it is a geodesic, up to reparametrization.

Example 2.1.1 (Curves with constant geodesic curvature on the Sphere)**.** Let $\mathbb{S}^2$ be the unit sphere oriented by its outward pointing unit normal vector field, and let $c : I \to \mathbb{S}^2$ be a circle. In order to calculate the geodesic curvature of c, note that the geodesic curvature is invariant under isometries. Thus, we can assume that the circle lies in a plane $z = \sqrt{1-r^2}$, where $r \leq 1$ is the radius of the circle, after applying a rotation, if necessary. Then, the parametrization of c is

$$c(t) = \left(r\cos\frac{t}{r}, r\sin\frac{t}{r}, \sqrt{1-r^2}\right)$$

which implies that

$$c'(t) = \left(-\sin\frac{t}{r}, \cos\frac{t}{r}, 0\right)$$

and

$$c''(t) = \left(-\frac{1}{r}\cos\frac{t}{r}, -\frac{1}{r}\sin\frac{t}{r}, 0\right).$$

Since c is a unit speed curve, then $c(t) \wedge c'(t)$ is a unit normal vector field along c. Then, a straightforward computation gives

$$k_g(t) = \langle \frac{Dc'}{dt}(t), c(t) \wedge c'(t)\rangle^{\sim} = \langle c''(t), c(t) \wedge c'(t)\rangle^{\sim} = \frac{\sqrt{1-r^2}}{r}.$$

Conversely, let $c : I \to \mathbb{S}^2$ be a unit speed curve with constant geodesic curvature d, and consider $b(t) = n(t) + dc(t)$, where $n(t) = c(t) \wedge c'(t)$. Since $\langle n'(t), c(t)\rangle^{\sim} = 0$, we obtain

$$b'(t) = n'(t) + dc'(t) = \frac{Dn}{dt}(t) + dc'(t) = -k_g c'(t) + dc'(t) = 0.$$

This shows that $b(t) \equiv b$ where b is a nonzero vector in $\mathbb{R}^3$. Furthermore, we have

$$\langle c(t), b\rangle^{\sim} = \langle c(t), n(t) + dc(t)\rangle^{\sim} = d\|c(t)\|^2 = d,$$

and therefore c must be planar.

2.2 Surfaces with planar lines of curvature

We start our study of surfaces of Enneper type by those surfaces for which one family of lines of curvature are contained in planes. It turns out that, except for some special cases, surfaces of Enneper type with spherical lines of curvature, which are treated in the next section, can be constructed in terms of a surface in this class.

Proposition 2.2.1. Let $N : M^2 \to \mathbb{S}^2$ be a local diffeomorphism and let $\gamma \in C^\infty(M^2)$ be a smooth function. Consider the surface $h = i \circ N : M^2 \to \mathbb{R}^3$, and define $f : M^2 \to \mathbb{R}^3$ by

$$f = \gamma h + h_* grad \gamma, \tag{2.4}$$

where the gradient $grad \gamma$ is computed with respect to the metric induced by h. Then, on the open subset of regular points, f defines a surface in $\mathbb{R}^3$ having N as a Gauss map.

Conversely, any oriented surface in $\mathbb{R}^3$ free of flat points can be parametrized in this way.

Proof. First, note that, by the Weingarten formula, the shape operator of h with respect to N is equal to $-I$, where I is the identity. Differentiating (2.4) and using the Gauss formula we obtain

$$\begin{aligned} f_*X &= X(\gamma)h + \gamma h_*X + \tilde{\nabla}_X h_* \text{grad } \gamma \\ &= X(\gamma)h + \gamma h_*X + h_*\nabla_X \text{grad } \gamma - \langle X, \text{grad } \gamma \rangle N \\ &= X(\gamma)h + \gamma h_*X + h_*(\text{Hess } \gamma)X - X(\gamma)h \\ &= h_*(\gamma I + \text{Hess } \gamma)X, \end{aligned}$$

for all $X \in \mathfrak{X}(M^2)$, where Hess γ is the Hessian operator computed with respect to the metric induced by h. Thus, setting $P = \gamma I + \text{Hess } \gamma$, the differential of f becomes

$$f_* = i_* N_* P. \tag{2.5}$$

We conclude that, on the open subset where P is invertible, the map f defines a surface having N as a Gauss map and $A = -P^{-1}$ as the corresponding shape operator.

Conversely, let $f : M^2 \to \mathbb{R}^3$ be an oriented surface free of flat points and let $N : M^2 \to \mathbb{S}^2$ be its Gauss map, which is a local diffeomorphism. Define $\gamma := \langle f, h \rangle^\sim$, and denote by $\langle , \rangle^*$ the metric induced by h. Then

$$\begin{aligned} \langle h_* \text{grad } \gamma, h_*X \rangle^\sim &= \langle \text{grad } \gamma, X \rangle^* = X(\gamma) \\ &= \langle f_*X, h \rangle^\sim + \langle f, h_*X \rangle^\sim \\ &= \langle f, h_*X \rangle^\sim, \end{aligned}$$

since $h = i \circ N$ is normal to f. Therefore, we can decompose f as

$$f = \gamma h + h_* \text{grad } \gamma. \tag{2.6}$$

□

The parametrization given by (2.6) is called the *Gauss parametrization* of f.

Next, we show how all surfaces of Enneper type with planar lines of curvature can be parametrized in terms of its Gauss map and a support function in the sense of (2.6). Before we state and prove this result, we need to calculate the geodesic curvature of a coordinate curve in the sphere.

Let $N : I\times J \to \mathbb{S}^2$ be a local diffeomorphism, where $I,J \subset \mathbb{R}$ are open intervals. Assume that the metric induced by N is given by

$$ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2.$$

Since (u_1,u_2) are orthogonal coordinates of $I\times J$, it is an elementary fact that, with respect to this metric, we have

$$\nabla^N_{\partial_{u_1}}\partial_{u_2} = \frac{\partial(\log v_1)}{\partial u_2}\partial_{u_1} + \frac{\partial(\log v_2)}{\partial u_1}\partial_{u_2},$$

and hence

$$\left\langle \nabla^N_{\partial_{u_1}}\partial_{u_1}, \frac{\partial_{u_2}}{v_2}\right\rangle = -\frac{1}{v_2}\langle \nabla^N_{\partial_{u_1}}\partial_{u_2}, \partial_{u_1}\rangle = -\frac{1}{v_2}\frac{\partial(\log v_1)}{\partial u_2}v_1^2.$$

Thus, the geodesic curvature (up to sign) of a coordinate curve $u_1 \in I \mapsto (u_1,u_2^0), u_2^0 \in J$ is given by $\varphi(u_1,u_2^0)$, where

$$\varphi := -\frac{1}{v_2}\frac{\partial(\log v_1)}{\partial u_2}. \tag{2.7}$$

Moreover, taking into account that N is a local isometry, we see that the coordinate curves have constant geodesic curvature, or equivalently, that φ depends only on u_2, if and only if the curves $u_1 \in I \mapsto N(u_1,u_2^0), u_2^0 \in J$ are arcs of circles in $\mathbb{S}^2$ (see Example 2.1.1).

Theorem 2.2.1. Let $N : I\times J \to \mathbb{S}^2$ be a local diffeomorphism, defined on a product of open intervals $I,J \subset \mathbb{R}$, whose induced metric is

$$ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2.$$

Suppose that the curves $u_1 \mapsto N(u_1,u_2^0)$, $u_2^0 \in J$, are arcs of circles in $\mathbb{S}^2$. Given $U \in C^\infty(I)$ and $V \in C^\infty(J)$, let $\gamma \in C^\infty(I\times J)$ be defined by

$$\gamma(u_1,u_2) = v_1(u_1,u_2)\left(U(u_1) + \int_{u_2^0}^{u_2}\frac{V(\tau)v_2(u_1,\tau)}{v_1(u_1,\tau)}d\tau\right). \tag{2.8}$$

Then the map $f : I\times J \to \mathbb{R}^3$ given by

$$f(u_1,u_2) = \gamma(u_1,u_2)N(u_1,u_2) + \frac{1}{v_1^2}\frac{\partial\gamma}{\partial u_1}\frac{\partial N}{\partial u_1} + \frac{1}{v_2^2}\frac{\partial\gamma}{\partial u_2}\frac{\partial N}{\partial u_2} \tag{2.9}$$

defines, on the open subset of its regular points, a surface parametrized by lines of curvature whose coordinate curves $u_1 \mapsto f(u_1,u_2^0)$, $u_2^0 \in J$, are contained in planes.

Conversely, any surface of Enneper type free of flat points with one family of planar lines of curvature can be locally parametrized in this way.

Proof. Note that (2.9) can be written as

$$f = \gamma(i \circ N) + i_* N_* \mathrm{grad}\, \gamma,$$

where grad γ is computed with respect to the metric induced by $i \circ N$. Then, by Proposition 2.2.1, on the open subset of regular points, f defines a surface free of flat points having N as a Gauss map. Moreover, the differential of f is

$$f_* = i_* N_* P, \tag{2.10}$$

and the shape operator of f with respect to N is

$$A = -P^{-1}, \tag{2.11}$$

where $P = \mathrm{Hess}\, \gamma + \gamma I$, with the Hessian being also computed with respect to the metric induced by $i \circ N$.

We claim that Hess $\gamma(\partial_{u_1}, \partial_{u_2}) = 0$ if and only if γ is given by (2.8) for some $U \in C^\infty(I)$ and $V \in C^\infty(J)$. Indeed, we have

$$\nabla^N_{\partial_{u_1}} \partial_{u_2} = \frac{\partial(\log v_2)}{\partial u_1} \partial_{u_2} - v_2 \varphi \partial_{u_1},$$

where φ is given by (2.7). Hence

$$\begin{aligned} \mathrm{Hess}\, \gamma(\partial_{u_1}, \partial_{u_2}) &= \frac{\partial}{\partial u_2}\left(\frac{\partial \gamma}{\partial u_1}\right) - (\nabla^N_{\partial_{u_1}} \partial_{u_2})(\gamma) \\ &= \frac{\partial}{\partial u_2}\left(\frac{\partial \gamma}{\partial u_1}\right) - \frac{\partial(\log v_2)}{\partial u_1}\frac{\partial \gamma}{\partial u_2} + v_2 \varphi \frac{\partial \gamma}{\partial u_1}, \end{aligned}$$

and therefore Hess $\gamma(\partial_{u_1}, \partial_{u_2}) = 0$ if and only if

$$\frac{\partial}{\partial u_2}\left(\frac{\partial \gamma}{\partial u_1}\right) + v_2 \varphi \frac{\partial \gamma}{\partial u_1} = \frac{\partial(\log v_2)}{\partial u_1}\frac{\partial \gamma}{\partial u_2}. \tag{2.12}$$

Since φ depends only on u_2, we obtain

$$\frac{\partial(\varphi v_2 \gamma)}{\partial_{u_1}} = \varphi v_2 \frac{\partial(\log v_2)}{\partial u_1}\gamma + \varphi v_2 \frac{\partial \gamma}{\partial u_1},$$

and, consequently, (2.12) can be written as

$$\frac{\partial}{\partial u_1}\left(\frac{\partial \gamma}{\partial u_2} + \varphi v_2 \gamma\right) = \left(\frac{\partial \gamma}{\partial u_2} + \varphi v_2 \gamma\right)\frac{\partial(\log v_2)}{\partial u_1}. \tag{2.13}$$

If $\gamma = 0$, there is nothing to prove. Suppose that $\gamma \neq 0$ and that

$$\frac{\partial \gamma}{\partial u_2} = -\varphi v_2 \gamma = \gamma \frac{\partial(\log v_1)}{\partial u_2}.$$

Then $\gamma = v_1 U$ for some $U \in C^\infty(I)$, and the claim follows. Suppose now that

$$\left(\frac{\partial \gamma}{\partial u_2} + \varphi v_2 \gamma\right) \neq 0.$$

From (2.13), we obtain

$$\left(\frac{\partial\gamma}{\partial u_2}+\varphi v_2\gamma\right)=v_2V \tag{2.14}$$

for some $V\in C^\infty(J)$. Taking into account that

$$\begin{aligned}\frac{\partial(\gamma v_1^{-1})}{\partial u_2}&=\frac{\partial\gamma}{\partial u_2}v_1^{-1}-\gamma v_1^{-2}\frac{\partial v_1}{\partial u_2}\\&=v_1^{-1}\left(\frac{\partial\gamma}{\partial u_2}+\varphi v_2\gamma\right),\end{aligned} \tag{2.15}$$

(2.14) can be written as

$$\frac{\partial(\gamma v_1^{-1})}{\partial u_2}=v_2Vv_1^{-1},$$

which proves our claim.

Denoting by $\langle,\rangle^*$ the metric induced by $i\circ N$, it follows from the claim that

$$\langle P\partial_{u_1},\partial_{u_2}\rangle^*=\text{Hess }(\partial_{u_1},\partial_{u_2})+\gamma\langle\partial_{u_1},\partial_{u_2}\rangle^*=0,$$

and this also implies that ∂_{u_1} and ∂_{u_2} are eigenvectors of P; hence

$$\langle P\partial_{u_1},P\partial_{u_2}\rangle^*=0.$$

Now, using (2.10) and (2.11) gives

$$\begin{aligned}\langle\partial_{u_1},\partial_{u_2}\rangle&=\langle f_*\partial_{u_1},f_*\partial_{u_2}\rangle^\sim=\langle i_*N_*P\partial_{u_1},i_*N_*P\partial_{u_2}\rangle^\sim\\&=\langle P\partial_{u_1},P\partial_{u_2}\rangle^*=0\end{aligned}$$

and

$$\begin{aligned}\langle A\partial_{u_1},\partial_{u_2}\rangle&=\langle -f_*P^{-1}\partial_{u_1},f_*\partial_{u_2}\rangle^\sim=\langle -i_*N_*\partial_{u_1},i_*N_*P\partial_{u_2}\rangle^\sim\\&=\langle\partial_{u_1},P\partial_{u_2}\rangle^*=\langle P\partial_{u_1},\partial_{u_2}\rangle^*=0,\end{aligned}$$

which shows that f is parametrized by lines of curvature. Finally, since N is the Gauss map of f and the curves $u_1\mapsto N(u_1,u_2^0)$, $u_2^0\in J$, are circles in $\mathbb{S}^2$, it follows that the images by f of the u_1-coordinate curves are contained in planes.

For the converse, let $f:M^2\to\mathbb{R}^3$ be a surface of Enneper type free of flat points with the lines of curvature correspondent to one of its principal curvatures being contained in planes. Since f is free of umbilical points, we can consider f locally parametrized by lines of curvature with principal coordinates (u_1,u_2) ranging on a product $I\times J$ of open intervals $I,J\subset\mathbb{R}$. Let $N:I\times J\to\mathbb{S}^2$ be the Gauss map of f. Then N is a local diffeomorphism and (u_1,u_2) are also orthogonal coordinates with respect to the metric induced by N (see Remark 3).

Since the u_1-lines of curvature are planar, the curves $u_1\in I\mapsto N(u_1,u_2^0), u_2^0\in J$ are arcs of circles of $\mathbb{S}^2$. Now, the Gauss parametrization allows to recover f in terms of N and a support function $\gamma \in C^\infty(I \times J)$ by means of (2.9). Furthermore, the shape operator of f is $A = -P^{-1}$, where $P = \gamma I + \text{Hess}\ \gamma$. We have

$$\langle P\partial_{u_1}, \partial_{u_2}\rangle^* = \langle -\partial_{u_1}, A\partial_{u_2}\rangle = 0,$$

whence

$$\text{Hess}\ \gamma(\partial_{u_1}, \partial_{u_2}) = \langle P\partial_{u_1}, \partial_{u_2}\rangle^* - \gamma\langle \partial_{u_1}, \partial_{u_2}\rangle^* = 0.$$

Therefore, the support function γ must be given by (2.8) for some $U \in C^\infty(I)$ and $V \in C^\infty(J)$, as shown in the proof of the direct statement. □

For the corresponding surface given by (2.9), we have

$$K = -\frac{1}{\det(\text{Hess}\ \gamma + \gamma I)}, \quad \text{and} \quad H = -\frac{1}{2}\frac{\text{tr}\,(\text{Hess}\ \gamma + \gamma I)}{\det(\text{Hess}\ \gamma + \gamma I)} = -\frac{1}{2}\frac{\Delta\gamma + 2\gamma}{\det(\text{Hess}\ \gamma + \gamma I)},$$

where Δ stands for the Laplacian of γ. If f is free of umbilical points ($H^2 - K \neq 0$), then f is a surface of Enneper type.

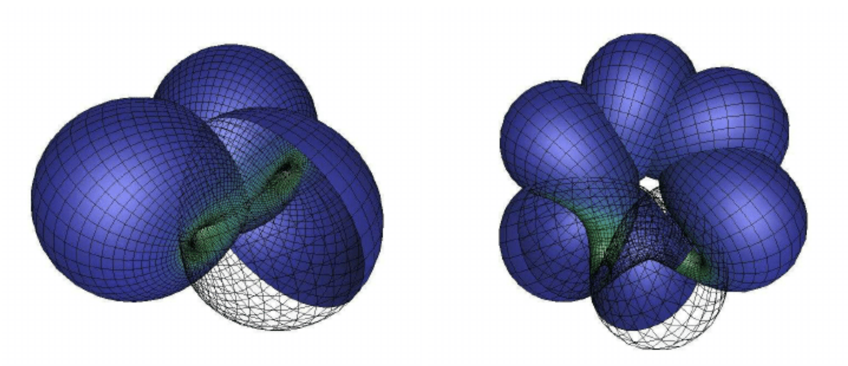
The *Wente tori* are a family of compact surfaces in $\mathbb{R}^3$ of genus 1 with constant mean curvature discovered by Wente (1986). These examples solved the famous Hopf-Conjecture: Is it possible to immerse a compact surface of positive genus in $\mathbb{R}^3$ with constant mean curvature? A theorem due to Alexandrov (1956) states that such a surface can not be embedded. Abresch (1987) obtained a classification of all Wente tori for which one family of lines of curvature are contained in planes using elliptic integrals. In fact, the original method of Wente yields exactly those tori in Abresch's classification, as shown by Spruck (1988); see also (STERLING, 1991). Explicit parametric equations of these tori in terms of elliptic and theta functions of Jacobi type were obtained by Walter (1987) who also remarked that each line of curvature is either planar or spherical. (Fig. 1).

We now concentrate on the surfaces of Enneper Type with planar lines of curvature in both families. Let $f : I \times J \to \mathbb{R}^3$ be a surface in this class and assume that f is free of flat points. An *orthogonal system of circles* on $\mathbb{S}^2$ is a pair of two families of circles, with the property that at each point where two circles from distinct families intersect, their tangent vectors are orthogonal. Since the lines of curvature of f intersect orthogonally, we obtain that the two families of planar lines of curvature are transformed by the Gauss map $N : I \times J \to \mathbb{S}^2$ into an orthogonal system of circles. Thus, we must find orthogonal systems of circles on the unit sphere.

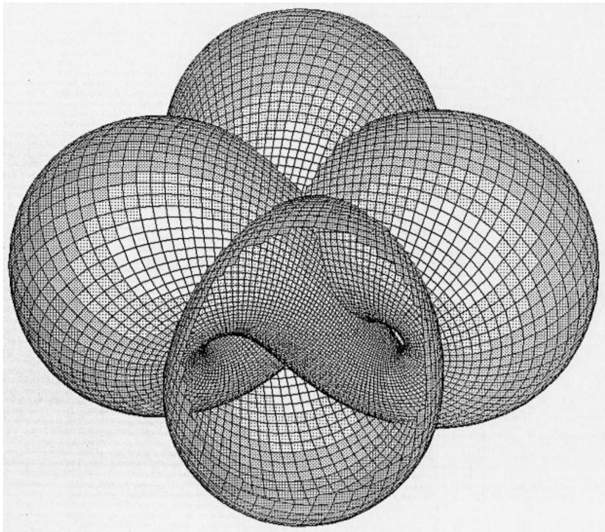
A *pencil of planes* is the set of planes through a given straight line in $\mathbb{R}^3$, called the *axis of the pencil*. A subtle property is that a pencil is determined by any two of its planes. According to Eisenhart (1909) and Leite (2015), the orthogonal systems of circles on $\mathbb{S}^2$ consist of the intersections of the sphere with two pencils of planes, where the pencils' axes are reciprocal polars for the sphere, in the sense described as follows.

Let r_1 and r_2 be the axes of such pencils. By a suitable choice of the coordinate axes of $\mathbb{R}^3$, we can assume that r_1 is a line parallel to the y-axis through the point $(0,0,a)$, with

Figure 1 – Wente Tori



(a) Wente torus (left), twisted torus (right).



(b) Wente's Twisted Torus

Source: (MCINTOSH, 2008) and (STERLING, 2022)

$0 \le a \le 1$. Then, the line r_2 must be parallel to the x-axis, cutting the z-axis at $(0,0,1/a)$, where the limiting case $a = 0$ is viewed as a line at infinity, and so this corresponds to the system of meridians and parallels, with respect to the y-axis (Fig. 2). We will discuss the cases $a = 1$, $a = 0$, and $0 < a < 1$ separately.

In the case $a = 1$, the two axes r_1 and r_2 meet at $(0,0,1)$ and, clearly, the planes $x = 0$ and $z - 1 = 0$ intersect in r_1. Thus, any plane in the pencil with respect to r_1, except $z - 1 = 0$, can be obtained by

$$x + u_1(z-1) = 0, \; u_1 \in \mathbb{R}, \tag{2.16}$$

and any plane in the second pencil, except $z - 1 = 0$, can be obtained by

$$y + u_2(z-1) = 0, \; u_2 \in \mathbb{R}. \tag{2.17}$$

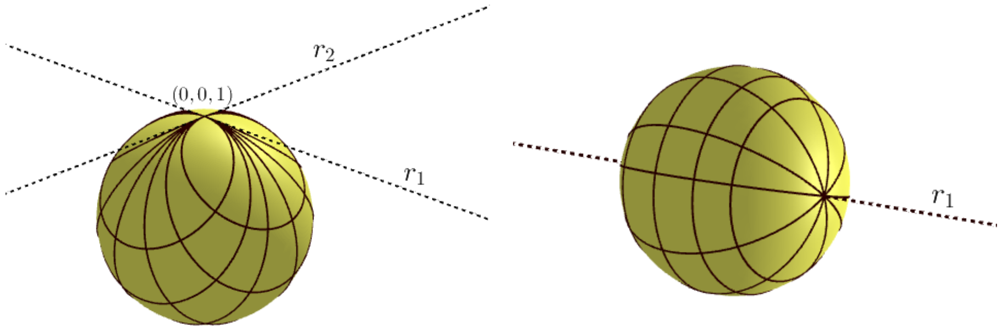
Thus, solving the equations in (2.16) and (2.17) simultaneously with the equation of the sphere $x^2 + y^2 + z^2 = 1$, we obtain the following expression of the Gauss map with respect to the parameters of lines of curvature (u_1, u_2)

$$N(u_1,u_2) = \frac{1}{1+u_1^2+u_2^2}(2u_1, 2u_2, u_1^2+u_2^2-1),$$

that is, N is the inverse of the stereographic projection from the North pole, whose induced metric is given by

$$ds^2 = \frac{4}{(1+u_1^2+u_2^2)^2}(du_1^2+du_2^2)$$

Figure 2 – Orthogonal systems of circles on $\mathbb{S}^2$.



(a) $a = 1$ (b) $a = 0$

Source: Elaborated by the author.

Similarly, in the case where $0 < a < 1$, the equations of the planes with respect to $r_1(a)$ and $r_2(a)$ are given by

$$x - \lambda(z-a) = 0, \quad \text{and} \quad y - \mu(z - \frac{1}{a}) = 0, \;\; \lambda, \mu \in \mathbb{R},$$

respectively. Using that the distance from $(0,0)$ to the line $Ax+By+C=0$ is given by $d = |C|/\sqrt{A^2+B^2}$, we see that each plane in the first pencil is secant to $\mathbb{S}^2$, and only the planes of the second pencil for which $|\mu| < a/\sqrt{1-a^2}$ are secant to $\mathbb{S}^2$. Then, we can reparametrize the parameters (λ, μ) and express the pencils of planes in the form

$$x - \frac{\tan u_1}{\sqrt{1-a^2}}(z-a) = 0 \quad \text{and} \quad y - \frac{a \tanh u_2}{\sqrt{1-a^2}}(z - \frac{1}{a}) = 0, \tag{2.18}$$

where $-\pi/2 < u_1 < \pi/2$ and $u_2 \in \mathbb{R}$. This new parametrization of the pencils will be useful in the computations below.

Hence, solving the equations in (2.18) simultaneously with the equation of the sphere $x^2+y^2+z^2 = 1$ we obtain that, with respect to the parameters of lines of curvature (u_1, u_2), the Gauss map is given by

$$N(u_1,u_2) = \left(\frac{\sqrt{1-a^2}\sin u_1}{\cosh u_2 + a\cos u_1}, -\frac{\sqrt{1-a^2}\sinh u_2}{\cosh u_2 + a\cos u_1}, \frac{a\cosh u_2 + \cos u_1}{\cosh u_2 + a\cos u_1} \right), \tag{2.19}$$

By differentiating N, we have

$$\frac{\partial N}{\partial u_1} = b(u_1,u_2)\left(\cos u_1 \cosh u_2 + a, -a\sinh u_2 \sin u_1, -\sqrt{1-a^2}\sin\sigma\cosh u_2\right) \tag{2.20}$$

and

$$\frac{\partial N}{\partial u_2} = -b(u_1,u_2)\left(\sin u_1 \sinh u_2, (1+a\cosh u_2\cos u_1), \sqrt{1-a^2}\sinh u_2\cos u_1\right), \tag{2.21}$$

where

$$b(u_1,u_2) = \frac{\sqrt{1-a^2}}{(\cosh u_2 + a\cos u_1)^2}.$$

It follows that

$$\begin{aligned}
\left\langle \frac{\partial N}{\partial u_1}, \frac{\partial N}{\partial u_1}\right\rangle^{\sim} =&(1-a^2)(\cosh u_2 + a\cos u_1)^{-4}((\cos u_1\cosh u_2 + a)^2 \\
&+ (1-a^2)\sin^2 u_1 \cosh^2 u_2 + a^2\sinh^2 u_2 \sin^2 u_1) \\
=&(1-a^2)(\cosh u_2 + a\cos u_1)^{-4}(\cos^2 u_1 \cosh^2 u_2 + \sin^2 u_1\cosh^2 u_2 \\
&+ a^2 + 2a\cosh u_1\cosh u_2 + a^2\sin^2 u_1(\sinh^2 u_2 - \cosh^2 u_2)) \\
=&(1-a^2)(\cosh u_2 + a\cos u_1)^{-4}(\cosh^2 u_2 \\
&+ 2a\cosh u_2 \cos u_1 + a^2\cos^2 u_1) \\
=&(1-a^2)(\cosh u_2 + a\cos u_1)^{-2},
\end{aligned}$$

where we used the fundamental equations $\sin^2 u_1 + \cos^2 u_1 = 1$ and $\cosh^2 u_2 - \sinh^2 u_2 = 1$. A similar computation gives

$$\left\langle \frac{\partial N}{\partial u_2}, \frac{\partial N}{\partial u_2}\right\rangle^{\sim} = (1-a^2)(\cosh u_2 + a\cos u_1)^{-2}.$$

Hence the induced metric by N is

$$ds^2 = (1-a^2)(\cosh u_2 + a\cos u_1)^{-2}(du_1^2 + du_2^2).$$

Now, if we put $a = 0$ in 2.19, we obtain

$$N(u_1,u_2) = \left(\frac{\sin u_1}{\cosh u_2}, -\frac{\sinh u_2}{\cosh u_2}, \frac{\cos u_1}{\cosh u_2}\right),$$

whence the coordinate curves $u_1 \mapsto N(u_1,u_2^0)$ are contained in the parallel planes $y = const.$ and the coordinate curves $u_2 \mapsto N(u_1^0,u_2)$ are contained in planes of the pencil generated by the y-axis. Thus, we recover the case corresponding to the system of meridians and parallels. Since the planes containing the coordinate curves $u_1 \mapsto f(u_1,u_2^0)$ are parallel to the planes containing the curves $u_1 \mapsto N(u_1,u_2^0)$, we see that, in the case $a = 0$, the u_1-lines of curvature of f are contained in parallel planes (see Theorem 3.1.2 in Chapter 3).

Therefore, in virtue of Theorem 2.2.1, we obtain the following result.

Theorem 2.2.2. Let $f : M^2 \to \mathbb{R}^3$ be a surface of Enneper type free of flat points with planar lines of curvature in both families. Then, up to isometries of $\mathbb{R}^3$, f is locally parametrized by

$$f(u_1,u_2) = \gamma(u_1,u_2)N(u_1,u_2) + \frac{1}{v_1(u_1,u_2)^2}\left(\frac{\partial\gamma}{\partial u_1}\frac{\partial N}{\partial u_1} + \frac{\partial\gamma}{\partial u_2}\frac{\partial N}{\partial u_2}\right),$$

where either

$$N(u_1,u_2) = \frac{1}{1+u_1^2+u_2^2}(2u_1, 2u_2, u_1^2+u_2^2-1)$$

and

$$v_1(u_1,u_2) = \frac{2}{(1+u_1^2+u_2^2)},$$

or

$$N(u_1,u_2) = \left(\frac{\sqrt{1-a^2}\sin u_1}{\cosh u_2 + a\cos u_1}, -\frac{\sqrt{1-a^2}\sinh u_2}{\cosh u_2 + a\cos u_1}, \frac{a\cosh u_2 + \cos u_1}{\cosh u_2 + a\cos u_1}\right)$$

and

$$v_1(u_1,u_2) = \frac{\sqrt{1-a^2}}{\cosh u_2 + a\cos u_1},$$

where $0 \le a < 1$.

In either case, $\gamma(u_1,u_2) = v_1(u_1,u_2)(U(u_1)+V(u_2))$, where U and V are smooth functions of u_1 and u_2, respectively.

For minimal surfaces, the principal parameters (u_1,u_2) in Theorem 2.2.2 are also conformal with respect to the metric induced by f. This fact, together with the Codazzi equation, will enable us to find explicit parametric equations for these surfaces.

2.3 The general case

In order to describe all surfaces of Enneper type, we now treat the case in which the lines of curvature of one family are contained in spheres. For a surface in this class, we are able to present a parametrization in terms of its Gauss map and a triple (γ,α,β), where γ is a smooth curve in $\mathbb{R}^3$ and α,β are smooth functions defined on an open interval.

Let $f : M^2 \to \mathbb{R}^3$ be a surface of Enneper type free of flat points with the lines of curvature correspondent to one of its principal curvatures being contained in spheres. Consider f locally parametrized by lines of curvature with principal coordinates (u_1,u_2) ranging on a product $I\times J$ of open intervals $I,J \subset \mathbb{R}$. Let $N : I\times J \to \mathbb{S}^2$ be the Gauss map of f, and let

$$ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2$$

be the metric induced by N on $I\times J$.

Let us assume that the u_1-lines of curvature are spherical. Thus, fixing an arbitrary $u_2^0 \in J$, the image by f of the coordinate curve $u_1 \in I \mapsto (u_1,u_2^0)$ lies in a sphere $\mathbb{S}^2(\gamma(u_2^0),R(u_2^0))$ of

$\mathbb{R}^3$ with center $\gamma(u_2^0) \in \mathbb{R}^3$ and radius $R(u_2^0)$. Denoting by $e_i := \partial_{u_i}/\|\partial_{u_i}\|$ be the unit coordinate vector fields with respect to the metric induced by f and using that $f_*\partial_{u_1}(u_1, u_2^0)$ is tangent to $\mathbb{S}^2(\gamma(u_2^0), R(u_2^0))$ for all $u_1 \in I$, we can write the corresponding position vector field as

$$\frac{f(u_1,u_2^0)-\gamma(u_2^0)}{R(u_2^0)} = \cos\theta(u_1,u_2^0)N(u_1,u_2^0)+\sin\theta(u_1,u_2^0)f_*e_2(u_1,u_2^0),$$

where θ is the angle between this position vector field and N, which depends only on u_2^0 by Joachimsthal's Theorem.

Now, note that

$$\begin{aligned} v_2^2 &= \langle N_*\partial_{u_2}, N_*\partial_{u_2}\rangle^{\sim} = \langle f_*A\partial_{u_2}, f_*A\partial_{u_2}\rangle^{\sim} \\ &= k_2^2\langle f_*\partial_{u_2}, f_*\partial_{u_2}\rangle^{\sim} = k_2^2\|\partial_{u_2}\|^2, \end{aligned}$$

and, after changing e_2 by $-e_2$, if necessary, it follows that

$$f_*e_2 = \frac{1}{k_2\|\partial_{u_2}\|}N_*\partial_{u_2} = \frac{1}{v_2}N_*\partial_{u_2}.$$

Then, we can write

$$f = \gamma + \alpha N + \beta v_2^{-1} N_*\partial_{u_2}, \tag{2.22}$$

where $\alpha = \alpha(u_2) = R(u_2)\cos\theta(u_1,u_2)$ and $\beta = \beta(u_2) = R(u_2)\sin\theta(u_1,u_2)$.

Differentiating (2.22) with respect to u_2 gives

$$\begin{aligned} f_*\partial_{u_2} &= \gamma' + \alpha' N + \alpha N_*\partial_{u_2} \\ &\quad + v_2^{-1}\left(\beta' - \beta\frac{\partial(\log v_2)}{\partial u_2}\right)N_*\partial_{u_2} + \beta v_2^{-1}\tilde{\nabla}_{\partial_{u_2}}N_*\partial_{u_2}, \end{aligned}$$

where the prime means derivative with respect to u_2. Furthermore, by the Gauss formula,

$$\tilde{\nabla}_{\partial_{u_2}}N_*\partial_{u_2} = N_*\nabla^N_{\partial_{u_2}}\partial_{u_2} - v_2^2 N,$$

where ∇^N denotes the Levi-Civita connection of $I\times J$ with respect to the metric induced by $i\circ N$. Now we see that $\langle f_*\partial_{u_2}, N\rangle^{\sim} = 0$ is equivalent to

$$\langle \gamma', N\rangle^{\sim} + \alpha' - \beta v_2 = 0. \tag{2.23}$$

We have thus proved the converse statement of the following result.

Theorem 2.3.1. Let $N : I\times J \to S^2$ be a local diffeomorphism, defined on a product of open intervals $I, J \subset \mathbb{R}$, whose induced metric is

$$ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2.$$

If there exist $\alpha, \beta \in C^\infty(J)$ with no common zeros and a smooth curve $\gamma : J \to \mathbb{R}^3$ such that (2.23) holds, then the map $f : I\times J \to \mathbb{R}^3$ given by (2.22) defines a surface parametrized by lines of curvature having N as a Gauss map such that the coordinate curves $u_1 \in I \mapsto f(u_1, u_2^0), u_2^0 \in J$ are contained in spheres.

Conversely, any surface of Enneper type free of flat points with one family of spherical lines of curvature can be locally parametrized in this way.

Proof. Let $f : I \times J \to \mathbb{R}^3$ be given by (2.22) in terms of N and (γ, α, β). Using that ∂_{u_1} and ∂_{u_2} are orthogonal with respect to the metric induced by $i \circ N$, it follows by the Gauss formula that

$$\begin{aligned}\tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} &= \tilde{\nabla}_{\partial_{u_1}} N_* \partial_{u_2} = N_* \nabla^N_{\partial_{u_1}} \partial_{u_2} \\ &= \frac{\partial(\log v_1)}{\partial u_2} N_* \partial_{u_1} + \frac{\partial(\log v_2)}{\partial u_1} N_* \partial_{u_2}.\end{aligned}$$

Note that the triple (γ, α, β) only depends on u_2, hence

$$\begin{aligned}f_* \partial_{u_1} &= \alpha N_* \partial_{u_1} + v_2^{-1} \beta \left(-\frac{\partial(\log v_2)}{\partial u_1} N_* \partial_{u_2} + \tilde{\nabla}_{\partial_{u_1}} N_* \partial_{u_2} \right) \\ &= \left(\alpha + v_2^{-1} \beta \frac{\partial(\log v_1)}{\partial u_2} \right) N_* \partial_{u_1}.\end{aligned} \tag{2.24}$$

Differentiating (2.23) with respect to u_1 gives

$$\begin{aligned}\langle \gamma', N_* \partial_{u_1} \rangle^\sim &= \beta \frac{\partial v_2}{\partial u_1} = \beta v_2^{-1} v_2^2 \frac{\partial(\log v_2)}{\partial u_1} = \beta v_2^{-1} \langle N_* \partial_{u_2}, \tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} \rangle^\sim \\ &= \langle f, \tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} \rangle^\sim - \langle \gamma, \tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} \rangle^\sim,\end{aligned}$$

where we used the definition of f in the last equality. On the other hand, differentiating

$$\langle f, N_* \partial_{u_1} \rangle^\sim = \langle \gamma, N_* \partial_{u_1} \rangle^\sim$$

with respect to u_2 we obtain

$$\begin{aligned}\langle f_* \partial_{u_2}, N_* \partial_{u_1} \rangle^\sim &= -\langle f, \tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} \rangle^\sim + \langle \gamma', N_* \partial_{u_1} \rangle^\sim + \langle \gamma, \tilde{\nabla}_{\partial_{u_2}} N_* \partial_{u_1} \rangle^\sim \\ &= 0.\end{aligned} \tag{2.25}$$

Now, taking into account that the condition (2.23) is equivalent to $\langle f_* \partial_{u_2}, N \rangle^\sim = 0$, it follows by (2.24) and (2.25) that

$$\langle f_* \partial_{u_1}, f_* \partial_{u_2} \rangle^\sim = 0.$$

Since α and β have no common zeros, we conclude that f defines a surface having N as a Gauss map and that (u_1, u_2) are also orthogonal coordinates with respect to the metric induced by f.

To see that the coordinate curves $u_1 \in I \mapsto (u_1, u_2^0), u_2^0 \in J$ are spherical lines of curvature of f, just observe that (2.22) gives

$$\| f(u_1, u_2^0) - \gamma(u_2^0) \|^\sim = \sqrt{\alpha(u_2^0)^2 + \beta(u_2^0)^2} = const$$

and

$$\langle N(u_1,u_2^0), \frac{f(u_1,u_2^0)-\gamma(u_2^0)}{\|f(u_1,u_2^0)-\gamma(u_2^0)\|^{\sim}}\rangle^{\sim} = \frac{\alpha(u_2^0)}{\sqrt{\alpha(u_2^0)^2+\beta(u_2^0)^2}} = const,$$

and therefore the result is a consequence of Joachimsthal's Theorem. □

Let $f: I\times J\to\mathbb{R}^3$ to be a surface of Enneper type as in the previous theorem. Assume that f is parametrized by 2.22 in terms of its Gauss map $N: I\times J\to\mathbb{S}^2$ and a triple (γ,α,β) satisfying (2.23). Suppose further that γ is a smooth regular curve and that no coordinate curve $u_1\in I\mapsto f(u_1,u_2^0), u_2^0\in J$ is an arc of a circle. Under these assumptions, in the following result we are able to determine all surfaces of Enneper type with spherical lines of curvature corresponding to the same family that share the same Gauss map with f.

Proposition 2.3.1. Let $f: I\times J\to\mathbb{R}^3$ be a surface of Enneper type as above. Then, any other surface $\bar f: I\times J\to\mathbb{R}^3$ of Enneper type free of flat points having N as a Gauss map and such that the coordinate curves $u_1\in I\mapsto(u_1,u_2^0), u_2^0\in J$ are spherical lines of curvature is parametrized by 2.22 in terms of a triple $(\bar\gamma,\bar\alpha,\bar\beta)$ which is related to (γ,α,β) by

$$\bar\gamma' = \lambda\gamma',\ \ \bar\alpha' = \lambda\alpha' \text{ and } \bar\beta = \lambda\beta$$

for some $\lambda\in C^\infty(J)$.

Proof. First, we claim that β must be nowhere vanishing. Indeed, if β vanishes at some $u_2^0\in J$, condition (2.23) becomes

$$\langle\gamma'(u_2^0), N(u_1,u_2^0)\rangle^{\sim} + \alpha'(u_2^0) = 0,$$

and since γ' is nowhere vanishing, this implies that the curve $u_1\in I\mapsto N(u_1,u_2^0), u_2^0\in J$ is planar. Hence the curve $u_1\in I\mapsto f(u_1,u_2^0), u_2^0\in J$ is also planar, and therefore it is an arc of a circle, contradicting our assumption.

By Theorem (2.3.1), after possibly reducing I and J, $\bar f: I\times J$ can be parametrized by (2.22) in terms of N and a triple $(\bar\gamma,\bar\alpha,\bar\beta)$, where $\bar\alpha,\bar\beta\in C^\infty(J)$ are smooth functions without common zeros and $\bar\gamma: J\to\mathbb{R}^3$ is a smooth curve satisfying

$$\langle\bar\gamma', N\rangle^{\sim} + \bar\alpha' - \bar\beta v_2 = 0. \tag{2.26}$$

Now, we can write $\bar\beta = \lambda\beta$ for some $\lambda\in C^\infty(J)$, since β is nowhere vanishing. From (2.23) and (2.26) we obtain

$$\langle\bar\gamma' - \lambda\gamma', N\rangle^{\sim} + \bar\alpha' - \lambda\alpha' = 0. \tag{2.27}$$

If $\bar\gamma' - \lambda\gamma'$ was nonzero for some u_2^0, then, arguing as before, we would conclude that the curve $u_1\in I\mapsto f(u_1,u_2^0), u_2^0\in J$, would be an arc of a circle, a contradiction. Therefore $\bar\gamma'-\lambda\gamma'$ must vanish everywhere, and by (2.27) the same holds for $\bar\alpha'-\lambda\alpha'$. □

At this point, let us review some properties of the inversion in a sphere. Let $\mathbb{S}^2(x_0,r)$ be the sphere centered at $x_0 \in \mathbb{R}^3$ with radius r. The *inversion* with respect to $\mathbb{S}^2(x_0,r)$ is the map $\mathscr{I} : \mathbb{R}^3 \setminus \{x_0\} \to \mathbb{R}^3 \setminus \{x_0\}$ defined by

$$\mathscr{I}(x) = x_0 + r^2 \frac{x - x_0}{\|x - x_0\|^2}.$$

Note that x and $\mathscr{I}(x)$ are on the same ray emanating from x_0, and that

$$\|\mathscr{I}(x) - x_0\| \|x - x_0\| = r^2.$$

It also follows immediately that $\mathscr{I}$ maps $\mathbb{R}^3 \setminus \{x_0\}$ onto itself, and that $\mathscr{I}^2 = I$. Thus, by the chain rule, the matrix of $\mathscr{I}_*(\mathscr{I}(x)) \circ \mathscr{I}_*(x)$ is the identity, and, taking determinants, we see that $\mathscr{I}_*(x)$ is a linear isomorphism, hence the inverse function theorem guarantees that $\mathscr{I}$ is a diffeomorphism.

Lemma 2.3.2. The inversion in a sphere is a conformal diffeomorphism.

Proof. Without loss of generality, take x_0 to be the origin and $r = 1$. Then

$$\mathscr{I}(x) = \frac{x}{\|x\|^2} = \frac{1}{\|x\|^2}(x^1, x^2, x^3)$$

and

$$\mathscr{I}_*(x) = \frac{1}{\|x\|^4} \begin{bmatrix} \|x\|^2 - 2(x^1)^2 & -2x^1x^2 & -2x^1x^3 \\ -2x^2x^1 & \|x\|^2 - 2(x^2)^2 & -2x^2x^3 \\ -2x^3x^1 & -2x^3x^2 & \|x\|^2 - 2(x^3)^2 \end{bmatrix}.$$

Applying this to a vector $v = (v^1, v^2, v^3) \in \mathbb{R}^3$ we obtain

$$\begin{bmatrix} v^1\|x\|^2 - 2x^1\langle x, v\rangle \\ v^2\|x\|^2 - 2x^2\langle x, v\rangle \\ v^3\|x\|^2 - 2x^3\langle x, v\rangle \end{bmatrix}.$$

Therefore

$$\langle \mathscr{I}_*(x)v, \mathscr{I}_*(x)v\rangle = \frac{1}{\|x\|^8} \sum_{i=1}^{3} (v^i\|x\|^2 - 2x^i\langle x, v\rangle)^2 = \frac{\langle v, v\rangle}{\|x\|^4},$$

for any $x \in \mathbb{R}^3 \setminus \{0\}$ and any $v \in \mathbb{R}^3$, and this gives the result. □

A plane and a sphere in $\mathbb{R}^3$ are given by equations of the form

$$\langle u, x\rangle = A, \quad x \in \mathbb{R}^3 \tag{2.28}$$

and

$$B\langle x, x\rangle - \langle v, x\rangle = C, \quad x \in \mathbb{R}^3 \tag{2.29}$$

respectively, where $u, v \in \mathbb{R}^3$ are nonzero vectors and $A, B, C \in \mathbb{R}$ with $B \neq 0$. Observe that the planes and the spheres given by (2.29) all pass through the origin if and only if $A = C = 0$.

Considering the inversion with respect to $\mathbb{S}^2$, if $y = \mathscr{I}(x)$ we have

$$\frac{1}{\|y\|^2} = \|x\|^2.$$

Then, if $A \neq 0$, $\mathscr{I}$ maps the planes given by equation (2.28) into

$$\langle u, \frac{y}{\|y\|^2}\rangle = A \Leftrightarrow A\langle y,y\rangle - \langle u,y\rangle = 0,$$

which are spheres passing through the origin. Similarly, taking $C = 0$, the spheres given by equation (2.29) are mapped to

$$\frac{B}{\|y\|^2} - \langle v, \frac{y}{\|y\|^2}\rangle = 0 \Leftrightarrow \langle v,y\rangle = B,$$

which are planes not passing through the origin. We then have the following result.

Lemma 2.3.3. Under inversion with respect to a sphere centered at x_0, a plane that does not contain x_0 is mapped to a sphere that contains x_0. Conversely, a sphere containing x_0 is mapped to a plane that does not contain x_0.

We now describe how a surface of Enneper type with spherical lines of curvature given as in Theorem 2.3.1 can be constructed by means of a surface of Enneper type with planar lines of curvature given as in Theorem 2.2.1.

Theorem 2.3.4. Let $\bar{f} : I \times J \to \mathbb{R}^3$ be a surface such that the u_1-coordinate curves are spherical lines of curvature. Assume that its Gauss map $\bar{N} : I \times J \to \mathbb{S}^2$ is a local diffeomorphism whose induced metric is

$$ds^2 = v_1^2 du_1^2 + v_2^2 du_2^2$$

and that $\bar{f}$ is parametrized by (2.22) in terms of $\bar{N}$, a smooth regular curve $\bar{\gamma} : J \to \mathbb{R}^3$ and $\bar{\alpha}, \bar{\beta} \in C^\infty(J)$ without common zeros satisfying (2.23). Then $\bar{f}$ can be constructed in terms of a surface $\tilde{f}$ whose u_1-coordinate curves are planar lines of curvature.

Proof. Since the triple $(\bar{\gamma} = (\bar{a},\bar{b},\bar{c}), \bar{\alpha}, \bar{\beta})$ satisfies (2.23), then the same holds for the new triple $(\gamma = (a,b,c), \alpha, \beta)$ defined by

$$\gamma' = \lambda\bar{\gamma}', \;\; \alpha' = \lambda\bar{\alpha}' \text{ and } \beta = \lambda\bar{\beta} \tag{2.30}$$

for some $\lambda \in C^\infty(J)$. By Theorem 2.3.1, after reducing I and J if necessary so that $\alpha^2 + \beta^2 > 0$, the map $f : I \times J \to \mathbb{R}^3$ parametrized by (2.22) in terms of $\bar{N}$ and (γ, α, β) is a surface such that the coordinate curves $u_1 \in I \mapsto f(u_1, u_2^0), u_2^0 \in J$ are lines of curvature of f contained in spheres with center $\gamma(u_2^0) \in \mathbb{R}^3$ and radius $\sqrt{\alpha^2(u_2^0) + \beta^2(u_2^0)}$.

Next, we see that, given $\lambda \in C^\infty(J)$, it is always possible to choose the functions a,b,c,α,β satisfying the equations (2.30). Clearly, we need to solve the ordinary differential equations

$$(a',b',c') = \lambda(\bar{a}',\bar{b}',\bar{c}') \text{ and } \alpha' = \lambda\bar{\alpha}'. \tag{2.31}$$

Consider the functions A,B,C and D given by one of the infinitely many solutions of the homogeneous system of linear equations

$$\begin{cases} A\bar{a}' + B\bar{b}' + C\bar{c}' + D\bar{\alpha}' & = 0 \\ A\bar{a}'' + B\bar{b}'' + C\bar{c}'' + D\bar{\alpha}'' & = 0 \\ A\bar{a}''' + B\bar{b}''' + C\bar{c}''' + D\bar{\alpha}''' & = 0. \end{cases}$$

By differentiating the first equation twice and using the other equations, we obtain

$$\begin{cases} A'\bar{a}' + B'\bar{b}' + C'\bar{c}' + D'\bar{\alpha}' & = 0 \\ A''\bar{a}' + B''\bar{b}' + C''\bar{c}' + D''\bar{\alpha}' & = 0. \end{cases}$$

Taking into account (2.31), we see that

$$\begin{cases} Aa' + Bb' + Cc' + D\alpha' & = 0 \\ A'a' + B'b' + C'c' + D'\alpha' & = 0 \\ A''a' + B''b' + C''c' + D''\alpha' & = 0. \end{cases} \tag{2.32}$$

By taking the auxiliary function $u := Aa + Bb + Cc + D\alpha$, we conclude from (2.32) that

$$\begin{cases} A'a + B'b + C'c + D'\alpha & = u' \\ A''a + B''b + C''c + D''\alpha & = u'' \\ A'''a + B'''b + C'''c + D'''\alpha & = u''', \end{cases}$$

so that a,b,c and α can be recovered from linear combinations of u,u',u'' and u'''. Since $\bar{\gamma}' = (\bar{a}',\bar{b}',\bar{c}') \neq 0$, it follows that λ can be recovered from linear combinations of u,u',u'',u''' and u'''', whence u must be a solution of a 4th order linear differential equation, which always exists for all values of $u_2 \in J$.

If we now impose

$$\|\gamma\|^2 = \alpha^2 + \beta^2, \tag{2.33}$$

by differentiating we obtain

$$\lambda(a\bar{a}' + b\bar{b}' + c\bar{c}' + \alpha\bar{\alpha}') - \lambda\bar{\beta}(\lambda'\bar{\beta} + \lambda\bar{\beta}') = 0,$$

or equivalently,

$$a\bar{a}' + b\bar{b}' + c\bar{c}' + \alpha\bar{\alpha}' - \bar{\beta}(\lambda'\bar{\beta} + \lambda\bar{\beta}') = 0, \tag{2.34}$$

hence u is a solution of a 5th order linear differential equation. Thus, given $\lambda \in C^\infty(J)$, there always exist infinitely many solutions of (2.30) satisfying (2.33).

Choosing one of these solutions, the condition (2.33) implies that the spheres containing the lines of curvature of the surface f all pass through the origin. Therefore, the composition $\tilde{f} = \mathscr{I} \circ f$ of f with an inversion with respect to a sphere centered at the origin is a surface such that the coordinate curves $u_1 \in I \mapsto \tilde{f}(u_1,u_2^0), u_2^0 \in J$ are lines of curvature of $\tilde{f}$ contained in planes. □

The argument used in the previous theorem does not apply to surfaces of Enneper type for which the lines of curvature are contained in concentric spheres, which corresponds to the case $\bar{\gamma} = const$. We will present a description of these surfaces in the next chapter.

CHAPTER

3

ON SOME SPECIAL SURFACES OF ENNEPER TYPE

In the last chapter, we showed that surfaces of Enneper type can be parametrized essentially in terms of its Gauss map. We now turn to give a complete description of some special classes of such surfaces. More precisely, we will see how any surface of Enneper type for which the lines of curvature of one family are contained either in concentric spheres, parallel planes, or planes that intersect along a common line arises.

In particular, those in first class are ruled out in Theorem 2.3.4. Furthermore, we will prove that the last property is satisfied by a surface of nonzero constant Gaussian curvature with one family of planar lines of curvature.

3.1 Joachimsthal surfaces

Let $\mathbb{Q}^2_\varepsilon$ denote either $\mathbb{S}^2, \mathbb{R}^2$ or $\mathbb{H}^2$, according as $\varepsilon = 1$, $\varepsilon = 0$ or $\varepsilon = -1$, respectively. We will first recall some conformal maps between $\mathbb{R}^3$ and the Riemannian product spaces $\mathbb{Q}^2_\varepsilon \times \mathbb{R}$. Then, a complete description of some surfaces in $\mathbb{Q}^2_\varepsilon \times \mathbb{R}$ obtained by Tojeiro (2010) leads to an approach to carry out our main results. We remark that the theory developed in Chapter 1 for surfaces in $\mathbb{R}^2 \times \mathbb{R} \equiv \mathbb{R}^3$ can be extended with minor modifications to surfaces in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, see the book of Dajczer and Tojeiro (2019) for more details.

Let $\mathbb{R}^{n+2}_\mu$ denote either the Euclidean space $\mathbb{R}^{n+2}$ or the Lorentzian space $\mathbb{R}^{n+2}_1$, according as $\mu = 0$ or $\mu = 1$, respectively. We always regard

$$\mathbb{R}^k_\mu = \{(x_1, x_2, \cdots, x_{n+2}) \in \mathbb{R}^{n+2}_\mu : x_{k+1} = \cdots = x_{n+2} = 0\}.$$

Here, the spaces $\mathbb{Q}^2_\varepsilon \times \mathbb{R} \subset \mathbb{R}^3_\mu \times \mathbb{R} = \mathbb{R}^4_\mu$ are endowed with its standard metrics, where $\mu = 0$ if $\varepsilon = 0$ or $\varepsilon = 1$, and $\mu = 1$ if $\varepsilon = -1$.

Given a surface $F : M^2 \to \mathbb{Q}^2_\varepsilon \times \mathbb{R}$, we denote by N a unit normal vector field along F. If $\bar{\partial}$ denotes the unit coordinate vector field of the factor $\mathbb{R}$ (up to sign), we use the same notation for its horizontal lift with respect to the projection $\pi : \mathbb{Q}^2_\varepsilon \times \mathbb{R} \to \mathbb{R}$. With this notation, a vector field $T \in \mathfrak{X}(M^2)$ and a smooth function ν on M^2 are defined by

$$\bar{\partial} = F_*T + \nu N.$$

Note that F_*T is the orthogonal projection of the constant vector field $\bar{\partial} \in \mathbb{R}^4_\mu$ onto the tangent space to F.

Tojeiro in 2010 presented a classification of the surfaces for which T always lies in the direction of a principal vector field of F. A trivial class arises when ν vanishes identically, in which case $F(M^2)$ is an open subset of a product $M^1 \times \mathbb{R}$, where M^1 is the image set of a smooth regular curve in $\mathbb{Q}^2_\varepsilon$. Other examples arise as follows.

Let $\gamma : I \to \mathbb{Q}^2_\varepsilon$ be a smooth regular curve, where $I \subset \mathbb{R}$ is an open interval. Now let $\gamma_s : I \to \mathbb{Q}^2_\varepsilon$ be the family of its parallel curves given by

$$\gamma_s(t) = C_\varepsilon(s)\gamma(t) + S_\varepsilon(s)n(t), \quad t \in I,$$

where $n : I \to \mathbb{R}^3_\mu$ is a unit normal vector field along γ and

$$C_\varepsilon(s) = \begin{cases} \cos s, & \text{if } \varepsilon = 1 \\ 1, & \text{if } \varepsilon = 0 \\ \cosh s, & \text{if } \varepsilon = -1 \end{cases} \quad \text{and} \quad S_\varepsilon(s) = \begin{cases} \sin s, & \text{if } \varepsilon = 1 \\ s, & \text{if } \varepsilon = 0 \\ \sinh s, & \text{if } \varepsilon = -1. \end{cases}$$

Define $F : I \times J \to \mathbb{Q}^2_\varepsilon \times \mathbb{R} \subset \mathbb{R}^4_\mu$ by

$$F(t,s) = \gamma_s(t) + a(s)(0,0,0,1), \tag{3.1}$$

for some smooth function $a : J \to \mathbb{R}$ with positive derivative on an open interval $J \subset \mathbb{R}$.

Theorem 3.1.1 ((TOJEIRO, 2010))**.** The map F given by (3.1) defines, at regular points, a surface in $\mathbb{Q}^2_\varepsilon \times \mathbb{R}$ that has T as an eigenvector of its shape operator. Conversely, any surface $F : M^2 \to \mathbb{Q}^2_\varepsilon \times \mathbb{R}$ with correspondent nowhere vanishing function ν such that T lies in the direction of a principal vector field of F is locally given in this way.

At this point, let us recall that there exists a conformal diffeomorphism $\Psi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$ given by

$$\Psi(x,t) = e^t x. \tag{3.2}$$

It is immediate that Ψ is a diffeomorphism with inverse given by

$$\Psi^{-1}(y) = \left(\frac{y}{\|y\|}, \log \|y\| \right). \tag{3.3}$$

Since $\Psi_*(x,t) = e^t I$ for all $(x,t) \in \mathbb{S}^2 \times \mathbb{R}$, where I is the identity matrix of order 3, we see that Ψ is indeed a conformal diffeomorphism whose conformal factor is e^t. Notice that (3.3) implies that Ψ^{-1} takes spheres centered at the origin onto slices $\mathbb{S}^2 \times \{s\}$ of $\mathbb{S}^2 \times \mathbb{R}$, and takes each ray through the origin onto a slice $\{x\} \times \mathbb{R}$, $x \in \mathbb{S}^2$.

Next we present a conformal diffeomorphism between $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{R}^3 \setminus \mathbb{R}$. For that, let $\{u_1, u_2, u_3\}$ be the usual orthonormal basis of $\mathbb{R}^3_1$, that is, $\langle u_1, u_1 \rangle^\sim = -1$, $\langle u_2, u_2 \rangle^\sim = \langle u_3, u_3 \rangle^\sim = 1$ and $\langle u_i, u_j \rangle^\sim = 0$ if $i \neq j$, and define the new vectors

$$v_1 = \frac{u_1 + u_3}{2}, \;\; v_2 = u_2 \;\text{ and }\; v_3 = \frac{u_1 - u_3}{2}.$$

Thus $\{v_1, v_2, v_3\}$ is a new basis of $\mathbb{R}^3_1$ satisfying

$$\langle v_1, v_1 \rangle^\sim = \langle v_3, v_3 \rangle^\sim = 0, \;\; \langle v_1, v_3 \rangle^\sim = -\frac{1}{2} \;\text{ and }\; \langle v_2, v_j \rangle^\sim = \delta_{2j}$$

and also

$$\|x_1 v_1 + x_2 v_2 + x_3 v_3\|^2 = -\frac{x_1 x_3}{2} + x_2^2 - \frac{x_3 x_1}{2} = -x_1 x_3 + x_2^2.$$

On the other hand, $-x_1 x_3 + x_2^2 = -1 \Leftrightarrow x_1 x_3 = 1 + x_2^2$, hence x_1 and x_3 have the same sign. Then, in terms of this new basis, the hyperbolic plane can be identified with its hyperboloid model given by

$$\mathbb{H}^2 = \{x_1 v_1 + x_2 v_2 + x_3 v_3 \in \mathbb{R}^3_1 \,:\, -x_1 x_3 + x_2^2 = -1, x_1 > 0\}.$$

We now consider the maps $\Phi : \mathbb{H}^2 \times \mathbb{S}^1 \to \mathbb{R}^3 \setminus \mathbb{R}$ and $\bar{\Phi} : \mathbb{R}^3 \setminus \mathbb{R} \to \mathbb{H}^2 \times \mathbb{S}^1$ defined by

$$\Phi\left(x_1 v_1 + x_2 v_2 + x_3 v_3, (y_1, y_2)\right) = \frac{1}{x_1}(x_2, y_1, y_2)$$

and

$$\bar{\Phi}(y_1, y_2, y_3) = \frac{1}{\sqrt{y_2^2 + y_3^2}} \left(v_1 + y_1 v_2 + (y_1^2 + y_2^2 + y_3^2) v_3, (y_2, y_3)\right). \tag{3.4}$$

It is easy to see that these maps are well defined and differentiable. Furthermore, a straightforward computation gives that $\bar{\Phi}$ is the inverse map of Φ.

The partial derivatives of $\bar{\Phi}$ are given by

$$\begin{aligned}
\bar{\Phi}_{y_1}(y_1, y_2, y_3) &= \frac{1}{\sqrt{y_2^2 + y_3^2}} \left(v_2 + 2 y_1 v_3, (0,0)\right). \\
\bar{\Phi}_{y_2}(y_1, y_2, y_3) &= -\frac{y_2}{(y_2^2 + y_3^2)^{\frac{3}{2}}} \left(v_1 + y_1 v_2 + (y_1^2 + y_2^2 + y_3^2) v_3, (y_2, y_3)\right) + \\
&\quad + \frac{1}{\sqrt{y_2^2 + y_3^2}} \left(2 y_2 v_3, (1,0)\right). \\
\bar{\Phi}_{y_3}(y_1, y_2, y_3) &= -\frac{y_3}{(y_2^2 + y_3^2)^{\frac{3}{2}}} \left(v_1 + y_1 v_2 + (y_1^2 + y_2^2 + y_3^2) v_3, (y_2, y_3)\right) + \\
&\quad + \frac{1}{\sqrt{y_2^2 + y_3^2}} \left(2 y_3 v_3, (0,1)\right).
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
\|\bar{\Phi}_{y_1}(y_1,y_2,y_3)\|^2 &= \frac{1}{y_2^2+y_3^2}.\\
\|\bar{\Phi}_{y_2}(y_1,y_2,y_3)\|^2 &= \frac{y_2^2}{(y_2^2+y_3^2)^3}\left(-(y_1^2+y_2^2+y_3^2)+y_1^2+(y_2^2+y_3^2)\right)\\
&\quad +2\left(\frac{y_2^2}{(y_2^2+y_3^2)^2}-\frac{y_2^2}{(y_2^2+y_3^2)^2}\right)+\frac{1}{y_2^2+y_3^2}.\\
&= \frac{1}{y_2^2+y_3^2}\\
\|\bar{\Phi}_{y_3}(y_1,y_2,y_3)\|^2 &= \frac{1}{y_2^2+y_3^2}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\langle\bar{\Phi}_{y_1}(y_1,y_2,y_3),\bar{\Phi}_{y_2}(y_1,y_2,y_3)\rangle^{\sim} &= -\frac{y_2y_1}{(y_2^2+y_3^2)^2}+\frac{y_2y_1}{(y_2^2+y_3^2)^2}=0\\
\langle\bar{\Phi}_{y_1}(y_1,y_2,y_3),\bar{\Phi}_{y_3}(y_1,y_2,y_3)\rangle^{\sim} &= 0\\
\langle\bar{\Phi}_{y_2}(y_1,y_2,y_3),\bar{\Phi}_{y_3}(y_1,y_2,y_3)\rangle^{\sim} &= \frac{y_2y_3}{(y_2^2+y_3^2)^3}(-(y_1^2+y_2^2+y_3^2)+(y_1^2+y_2^2+y_3^2))\\
&\quad -\frac{y_2}{(y_2^2+y_3^2)^2}(-y_3+y_3)-\frac{y_3}{(y_2^2+y_3^2)^2}(-y_2+y_2)\\
&= 0.
\end{aligned}
$$

Therefore, $\bar{\Phi}$ is a conformal diffeomorphism whose conformal factor is $1/\sqrt{y_2^2+y_3^2}$.

Note that, given $\alpha\in\mathbb{R}$ and $\beta\in(0,\infty)$, we can choose $\lambda=(\alpha^2+1)/\beta>0$ such that $\beta v_1+\alpha v_2+\lambda v_3\in\mathbb{H}^2$, whence

$$\Phi\left(\mathbb{H}^2\times\{(y_1^0,y_2^0)\}\right)=\{(\alpha,\beta y_1^0,\beta y_2^0)\,:\,\alpha\in\mathbb{R},\beta>0\}. \tag{3.5}$$

The last set in (3.5) is clearly contained in the half-plane given by

$$\begin{cases} yy_2^0-zy_1^0=0,\\ yy_1^0+zy_2^0>0.\end{cases} \tag{3.6}$$

Conversely, given a point (x,y,z) in the half-plane given by (3.6), it follows from the second equation that $y^2+z^2>0$. Setting $\beta=\sqrt{y^2+z^2}$ and $\alpha=x$, we can use the first equation to obtain that

$$|\beta y_1^0|=\sqrt{y^2+z^2}|y_1^0|=\sqrt{(yy_1^0)^2+(zy_1^0)^2}=\sqrt{(yy_1^0)^2+(yy_2^0)^2}=|y|.$$

On the other hand,

$$yy_1^0(yy_1^0+zy_2^0)=(yy_1^0)^2+(zy_1^0)(yy_2^0)=(y_1^0)^2(y^2+z^2),$$

which implies that y and y_1^0 have the same sign, and hence $y=\beta y_1^0$. Similarly, we obtain that $z=\beta y_2^0$.

This shows that $\Phi\left(\mathbb{H}^2 \times \{(y_1^0, y_2^0)\}\right)$ is a half-plane of the plane $yy_2^0 - zy_1^0 = 0$ in $\mathbb{R}^3$ containing $\mathbb{R}$, but since each half-plane of a plane in $\mathbb{R}^3$ containing $\mathbb{R}$ is given by (3.6) for some $(y_1^0, y_2^0) \in \mathbb{S}^1$, we may conclude that the map Φ^{-1} takes each half-plane of a plane containing $\mathbb{R}$ onto a slice $\mathbb{H}^2 \times \{x\}$ of $\mathbb{H}^2 \times \mathbb{S}^1$. Moreover, arguing as above we can easily see that Φ takes slices $\{x\} \times \mathbb{S}^1$, $x \in \mathbb{H}^2$, onto circles centered at $\mathbb{R}$ lying in planes orthogonal to $\mathbb{R}$, and each such circle in $\mathbb{R}^3$ is mapped by Φ^{-1} onto a slice $\{x\} \times \mathbb{S}^1$ of $\mathbb{H}^2 \times \mathbb{S}^1$.

Finally, composing $\Phi : \mathbb{H}^2 \times \mathbb{S}^1 \to \mathbb{R}^3 \setminus \mathbb{R}$ with the Riemannian covering map

$$\begin{aligned} \pi : \mathbb{H}^2 \times \mathbb{R} &\to \mathbb{H}^2 \times \mathbb{S}^1 \\ (x,t) &\mapsto (x, (\cos t, \sin t)) \end{aligned} \tag{3.7}$$

gives rise to a conformal covering map, which we still denote by $\Phi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \mathbb{R}$, given by

$$\Phi(x_1 v_1 + x_2 v_2 + x_3 v_3, t) = \frac{1}{x_1}(x_2, \cos t, \sin t). \tag{3.8}$$

We are now ready to prove the main result of this section. To simplify the notation, we denote by Φ either the conformal diffeomorphism $\Phi : \mathbb{Q}_\varepsilon^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$ given by (3.2), if $\varepsilon = 1$, the conformal covering map $\Phi : \mathbb{Q}_\varepsilon^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \mathbb{R}$ given by (3.8), if $\varepsilon = -1$, or the standard isometry $\Phi : \mathbb{Q}_\varepsilon^2 \times \mathbb{R} \to \mathbb{R}^3$, if $\varepsilon = 0$.

Theorem 3.1.2. Let $f : I \times J \to \mathbb{R}^3$ be a surface parametrized by lines of curvature. Assume that the coordinates curves $t \in I \mapsto f(t, s_0), s_0 \in J$, are contained in either

(a) concentric spheres,

(b) parallel planes,

(c) planes intersecting along a common line.

Then one of the following possibilities holds, up to isometries of $\mathbb{R}^3$:

(i) f is a cone over a regular curve $\gamma : I \to \mathbb{S}^2$ in case (a), the cylinder over a regular curve $\gamma : I \to \mathbb{R}^2$ in case (b), or a surface of revolution obtained by rotating a regular curve $\gamma : I \to \mathbb{R}_+^2$, the latter regarded as the half-plane model of $\mathbb{H}^2$, in case (c);

(ii) there exists $F : I \times J \to \mathbb{Q}_\varepsilon^2 \times \mathbb{R}$ given by (3.1) in terms of a regular curve $\gamma : I \to \mathbb{Q}_\varepsilon^2$, with $\varepsilon = 1$ in case (a), $\varepsilon = 0$ in case (b) and $\varepsilon = -1$ in case (c), such that $f = \Phi \circ F$.

Proof. We first observe that, up to isometries of $\mathbb{R}^3$, we may assume that the spheres in (a) are centered at the origin, the planes in (b) are parallel to $\mathbb{R}^2$ and that the planes in (c) intersect along $\mathbb{R}$.

Consider the surface $F = \Phi^{-1} \circ f : I \times J \to \mathbb{Q}^2_\varepsilon \times \mathbb{R}$, where $\varepsilon = 1$ in case (a) and $\varepsilon = 0$ in case (b). Let N be a unit normal vector field along F and consider the decomposition

$$\bar{\partial} = F_* T + \nu N.$$

Suppose that condition (a) holds. Since the spheres containing the coordinate curves $t \in I \mapsto f(t, s_0), s_0 \in J$ are centered at the origin and Φ^{-1} takes such spheres onto slices $\mathbb{S}^2 \times \{s\}$ of $\mathbb{S}^2 \times \mathbb{R}$, we then have that the height function

$$(t,s) \mapsto \langle F(t,s), \bar{\partial} \rangle^{\sim} \tag{3.9}$$

depends only on $s \in J$. By differentiating (3.9) with respect to the coordinate t we obtain

$$0 = \langle F_* \partial_t, \bar{\partial} \rangle^{\sim} = \langle F_* \partial_t, F_* T + \nu N \rangle^{\sim} = \langle F_* \partial_t, F_* T \rangle^{\sim},$$

which implies that

$$\langle \partial_t, T \rangle = 0,$$

taking into account that the metrics induced by f and F are conformal. This means that T is a multiple of the coordinate vector field ∂_s. Moreover, since conformal diffeomorphisms preserve principal directions and the integral curves of ∂_s are lines of curvature of f, we conclude that T lies in the direction of a principal vector field of F.

We now assume that condition (b) holds. In this case, the coordinate curves $t \in I \mapsto F(t, s_0), s_0$ are contained in slices $\mathbb{R}^2 \times \{s\}$ of $\mathbb{R}^2 \times \mathbb{R}$. Thus, one can argue as in the preceding paragraph to conclude that T lies in the direction of a principal vector field of F.

Finally, assume that condition (c) is satisfied. Let $\bar{\Phi}$ be the conformal diffeomorphism given by (3.4) and consider the surface $\bar{F} = \bar{\Phi} \circ f : I \times J \to \mathbb{H}^2 \times \mathbb{S}^1$. Then the coordinate curves $t \in I \mapsto \bar{F}(t, s_0), s_0 \in J$ are contained in slices $\mathbb{H}^2 \times \{x\}$ of $\mathbb{H}^2 \times \mathbb{S}^1$. Let $F : I \times J \to \mathbb{H}^2 \times \mathbb{R}$ be such that $\bar{F} = \pi \circ F$, where π is the covering map given by (3.7). Then the coordinate curves $t \in I \mapsto F(t, s_0), s_0 \in J$ are contained in slices $\mathbb{H}^2 \times \{s\}$ of $\mathbb{H}^2 \times \mathbb{R}$. Now we can argue as in case (a) in order to conclude that T lies in the direction of a principal vector field of F.

In either of the preceding cases, the map F is either given by

$$F(t,s) = \gamma(t) + s(0,0,0,1), \tag{3.10}$$

where $\gamma : I \to \mathbb{Q}^2_\varepsilon$ is a smooth regular curve, which corresponds to the case where ν vanishes identically, or it is given by (3.1) in terms of such a curve, with $\varepsilon = 1$ in case (a), $\varepsilon = 0$ in case (b) and $\varepsilon = -1$ in case (c). In any case, we have $f = \Phi \circ F$.

If we now regard $\mathbb{H}^2$ as the half-plane model, the map F given by (3.10) is a vertical cylinder over γ. Since Φ takes slices $\{x\} \times \mathbb{R}$ of $\mathbb{S}^2 \times \mathbb{R}$ onto rays through the origin, we obtain that, in case (a), $f = \Phi \circ F$ is the cone over γ with vertex at the origin. On the other hand, when $\varepsilon = -1$, the map Φ takes slices $\{x\} \times \mathbb{R}$ of $\mathbb{H}^2 \times \mathbb{R}$ onto circles centered at $\mathbb{R}$ that are contained in planes orthogonal to $\mathbb{R}$. Then, the corresponding $f = \Phi \circ F$ is a surface of revolution with γ as a profile curve and $\mathbb{R}$ as the rotation axis, in case (c). $\square$

Let $\gamma : I \to \mathbb{H}^2$ be written as $\gamma(t) = a(t)v_1 + b(t)v_2 + c(t)v_3$, for some smooth functions $a, b, c \in C^\infty(I)$ with $a > 0$. Then the corresponding surface of revolution $f = \Phi \circ F : I \times J \to \mathbb{R}^3 \setminus \mathbb{R}$ is given by

$$f(t,s) = \Phi(a(t)v_1 + b(t)v_2 + c(t)v_3, s) = \left(\frac{b(t)}{a(t)}, \frac{1}{a(t)}\cos s, \frac{1}{a(t)}\sin s\right).$$

Conversely, any surface of revolution $f : I \times \mathbb{R} \to \mathbb{R}^3$ obtained by rotating a regular planar curve $t \in I \mapsto (g_1(t), g_2(t))$ about the x-axis can be written in the form

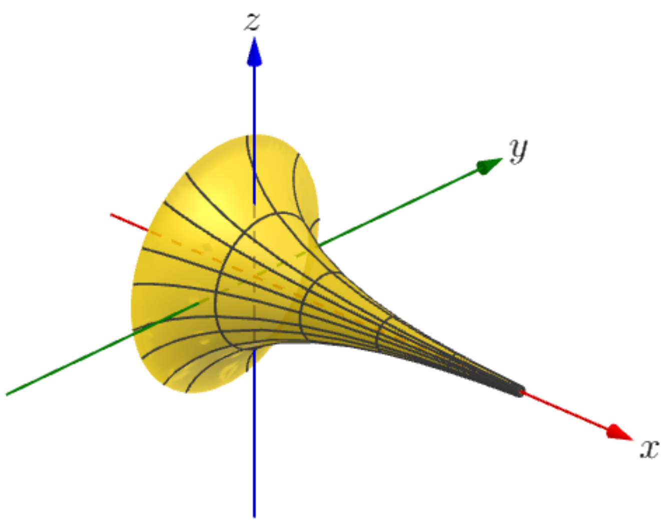
$$f(t,s) = (g_1(t), g_2(t)\cos s, g_2(t)\sin s),$$

with $g_2 > 0$. Then, we see that $f = \Phi \circ F$, where F is given by (3.10) with respect to the regular curve $\gamma : I \to \mathbb{H}^2$ given by

$$\gamma(t) = \frac{1}{g_2(t)}v_1 + \frac{g_1(t)}{g_2(t)}v_2 + \frac{g_1(t)^2 + g_2(t)^2}{g_2(t)}v_3.$$

A surface $f : M^2 \to \mathbb{R}^3$ of Enneper type with one family of planar lines of curvature such that the planes containing every line of curvature of this family satisfy the condition (c) is known as a *Joachimsthal surface*. Clearly, every surface of revolution free of umbilical points is a Joachimsthal surface (see Fig. 3).

Figure 3 – Pseudosphere



The pseudosphere is a surface of revolution with constant negative Gaussian curvature obtained by rotation of a tractrix about the x-axis.

Source: Elaborated by the author.

Corollary 3.1.2.1. Let $\gamma : I \to \mathbb{H}^2$ be a unit speed curve and let $\gamma_s : I \to \mathbb{H}^2$ be the family of its parallel curves, given by

$$\gamma_s(t) = \cosh s\,\gamma(t) + \sinh s(\gamma(t) \wedge \gamma'(t)),$$

where $\wedge$ stands for the Lorentzian cross-product. Define $F : I \times J \to \mathbb{H}^2 \times \mathbb{R} \subset \mathbb{R}^4_1$ by

$$F(t,s) = \gamma_s(t) + h(s)(0,0,0,1), \tag{3.11}$$

where $J \subset \mathbb{R}$ is an open interval and $h \in C^\infty(J)$ has positive derivative. Then, on the open subset $M^2 \subset I \times J$ of its regular points, the map $f = \Phi \circ F : M^2 \to \mathbb{R}^3 \setminus \mathbb{R}$, where $\Phi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \mathbb{R}$ is the conformal covering map given by (3.8), defines a surface parametrized by lines of curvature, such that the coordinate curves $t \in I \mapsto f(t,s_0), s_0 \in J$ are contained in planes intersecting along a common line, whereas the coordinate curves $s \in J \mapsto f(t_0,s), t_0 \in I$ lie on spheres centered on that line.

Conversely, any Joachimsthal surface in $\mathbb{R}^3$ locally can either be parametrized in this way or is a surface of revolution, up to isometries of $\mathbb{R}^3$.

Proof. It suffices to prove the direct statement. By Theorem 3.1.1, the map F given by (3.11) defines on the open subset $M^2 \subset I \times J$ of its regular points a surface $F : M^2 \to \mathbb{H}^2 \times \mathbb{R}$ for which T lies in the direction of a principal vector field. Furthermore, a point (t,s) is regular for F if and only if $\gamma_s'(t) \neq 0$, in which case $\langle F_*\partial_t, F_*\partial_s\rangle^\sim = 0$. Arguing as in the proof of Theorem 3.1.2, we obtain that T lies in the direction of ∂_s, and since Φ is a conformal diffeomorphism, we conclude that $f = \Phi \circ F : M^2 \to \mathbb{R}^3 \setminus \mathbb{R}$ defines a surface parametrized by lines of curvature.

Since the lines of curvature $t \in I \mapsto F(t,s_0), s_0 \in J$, are clearly contained in slices $\mathbb{H}^2 \times \{s\}$ of $\mathbb{H}^2 \times \mathbb{R}$, it follows that the t-lines of curvature of f are contained in planes intersecting along $\mathbb{R}$. It remains to show that the s-lines of curvature lie on spheres centered on that line. To see this, write

$$\gamma(t) = a(t)v_1 + b(t)v_2 + c(t)v_2, \ \gamma'(t) = a'(t)v_1 + b'(t)v_2 + c'(t)v_2.$$

We have

$$\gamma \wedge \gamma' = (ab' - a'b)v_1 + \frac{1}{2}(ac' - a'c)v_2 + (bc' - b'c)v_3$$

and hence that

$$f(t,s) = \begin{bmatrix} \dfrac{b(t)\cosh s + 1/2(a(t)c'(t) - a'(t)c(t))\sinh s}{a(t)\cosh s + (a(t)b'(t) - a'(t)b(t))\sinh s} \\ \dfrac{\cos(h(s))}{a(t)\cosh s + (a(t)b'(t) - a'(t)b(t))\sinh s} \\ \dfrac{\sin(h(s))}{a(t)\cosh s + (a(t)b'(t) - a'(t)b(t))\sinh s} \end{bmatrix}. \tag{3.12}$$

From this expression for f, a long but straightforward computation shows that

$$\left\| f(t_0,s) - \left(\frac{b'(t_0)}{a'(t_0)},0,0\right) \right\| = \frac{1}{|a'(t_0)|}.$$

If $a'(t_0) = 0$, the sphere is identified with the plane $x = 0$.

□

The above complete description of all Joachimsthal surfaces in $\mathbb{R}^3$ in terms of the conformal covering map $\Phi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \mathbb{R}$ was first obtained by Chion and Tojeiro (2021), and yields a new description for such surfaces. Furthermore, Corollary 3.1.2.1 allows us to recover the classical fact that, for surfaces free of umbilical points, the lines of curvature in one family lie in planes passing through a common line r if and only if the lines of curvature in the second family lie on spheres whose centers are on r and which cut the surface orthogonally, see (EISENHART, 1909, p. 308) and (BIANCHI, 1922, §193).

Motivated by the above description, Tassi and Tojeiro (In preparation.) investigated the pairs (γ, h) for which the corresponding Joachimsthal surface $f = \Phi \circ F$ given by (3.12) has nonzero constant Gaussian curvature. First, if $\gamma : I \to \mathbb{H}^2$ is a unit speed curve given by $\gamma(t) = a(t)v_1 + b(t)v_2 + c(t)v_3$, then

$$-a(t)c(t) + b^2(t) = -1, \;\text{ and }\; -a'(t)c'(t) + (b'(t))^2 = 1$$

whence a simple computation gives

$$a'(t) = a(t)\frac{b(t)b'(t) \pm \phi(t)}{1 + b^2(t)}, \;\text{ and }\; c(t) = \frac{1 + b^2(t)}{a(t)},$$

where $\phi(t) := \sqrt{1 + b^2(t) - (b'(t))^2}$. Thus, solving this ODE we can recover the coordinates $a(t)$ and $c(t)$ from $b(t)$ by

$$a(t) = de^{B_\sigma(t)}, \; d > 0$$

where

$$B_\sigma(t) = \int_{t_0}^{t} \frac{b(s)b'(s) + \sigma\phi(s)}{1 + b^2(s)} ds, \; t_0 \in I, \sigma \in \{1, -1\}.$$

The following result was obtained.

Proposition 3.1.1 ((TASSI; TOJEIRO, In preparation.)). The surface $f = \Phi \circ F : I \times J \to \mathbb{R}^3 \setminus \mathbb{R}$ given in Corollary 3.1.2.1 has nonzero constant Gaussian curvature if and only if one of the following possibilities holds:

(i) $h(s) = ls + m$, for some $l, m \in \mathbb{R}$ with $l \neq 0$, and b is a solution of the following ODE

$$((b')^2 - b^2)b'' = b + \sigma b'\phi$$

(ii)

$$h(s) = d_2 \pm \int \frac{e^{P(s)}}{\sqrt{d_1^2 - e^{2P(s)}}} ds,$$

for some $d_1, d_2 \in \mathbb{R}$, where

$$P(s) = \frac{1}{2}\log(a_1 - a_3 - (a_1 + a_3)\cosh(2s) + 2a_2\sinh(2s)),$$

with $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$, and b is a solution of the ODE

$$a_1(1 + b^2)\phi + a_2(\sigma(b'' - b)(1 + b^2) + (b' - \sigma b\phi)\phi) + a_3\sigma(b'' - b)(b' - \sigma b\phi) = 0.$$

We will see in the next section that every surface of Enneper type with nonzero constant Gaussian curvature for which one family of lines of curvature are contained in planes must be a Joachimsthal surface. Granting this for now, we can combine the previous result and Corollary 3.1.2.1 to provide the following more explicit description.

Corollary 3.1.2.2. Let $\gamma : I \to \mathbb{H}^2$ be a unit speed curve given by $\gamma(t) = a(t)v_1 + b(t)v_2 + c(t)v_3$ and assume that $b \in C^\infty(I)$ and $h \in C^\infty(J)$ are given by one of the items of Proposition 3.1.1. Then, on the open subset $M^2 \subset I \times J$ of its regular points, the map $f : M^2 \to \mathbb{R}^3 \setminus \mathbb{R}$ given by (3.12), defines a surface of Enneper type of nonzero constant Gaussian curvature with one family of planar lines of curvature.

Conversely, any surface of Enneper type of nonzero constant Gaussian curvature with one family of planar lines of curvature locally can either be parametrized in this way or is a surface of revolution, up to isometries of $\mathbb{R}^3$.

3.2 Surfaces with nonzero constant Gaussian curvature

Our goal in this section is to prove that if a surface of nonzero constant Gaussian curvature has the property that the lines of curvature of one family are contained in planes, then all these planes intersect along a common straight line, that is, it must be a Joachimsthal surface.

Let $f : M^2 \to \mathbb{R}^3$ be a surface free of umbilical and flat points. We assume without loss of generality that M^2 is oriented. From now on, we identify $N : M^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, the Gauss map of f, with the surface $i \circ N : M^2 \to \mathbb{R}^3$, and denote by $\langle , \rangle^*$ its induced metric. We first note that the Gauss and Weingarten formulas enable us to relate the Levi-Civita connections ∇ and ∇^* of $\langle , \rangle$ and $\langle , \rangle^*$, respectively. Indeed, for all $X, Y \in \mathfrak{X}(M^2)$ we have

$$\begin{aligned} \tilde{\nabla}_X N_* Y &= N_* \nabla^*_X Y + \langle N_* A^* X, N_* Y \rangle^\sim N \\ &= -f_* A \nabla^*_X Y + \langle -N_* X, N_* Y \rangle^\sim N \\ &= -f_* A \nabla^*_X Y - \langle AX, AY \rangle N. \end{aligned} \tag{3.13}$$

On the other hand,

$$\tilde{\nabla}_X N_* Y = -\tilde{\nabla}_X f_* AY = -(f_* \nabla_X AY + \langle AX, AY \rangle N). \tag{3.14}$$

Thus, comparing (3.13) and (3.14) we obtain

$$A \nabla^*_X Y = \nabla_X AY.$$

Now we consider $\{e_1, e_2\}$ a principal orthonormal frame for f defined on some open subset $U \subset M^2$ with correspondent principal curvatures $k_1, k_2 \in C^\infty(U)$. Notice that the vector fields $e_1^* := (1/k_1)e_1$ and $e_2^* := (1/k_2)e_2$ form an orthonormal frame with respect to the metric induced by N. If we denote by $c : I \to U$ an e_1-line of curvature of f, we see that the geodesic curvature (up to sign) of c is $\alpha \circ c$, where $\alpha \in C^\infty(U)$ is given by

$$\alpha := \langle \nabla_{e_1} e_1, e_2 \rangle.$$

Similarly, the geodesic curvature of the e_2-lines of curvature is given by $\beta \circ c \in C^\infty(U)$, where

$$\beta := \langle \nabla_{e_2} e_2, e_1 \rangle.$$

Moreover, the compatibility of ∇ with the metric gives

$$\nabla_{e_1} e_1 = \alpha e_2,\ \nabla_{e_2} e_2 = \beta e_1,\ \nabla_{e_1} e_2 = -\alpha e_1,\ \text{ and } \ \nabla_{e_2} e_1 = -\beta e_2$$

Lemma 3.2.1. The e_1-lines of curvature of f are planar if and only if $e_1(k_1)\alpha = k_1 e_1(\alpha)$.

Proof. The geodesic curvature (up to sign) of the e_1-lines of curvature with respect to the metric induced by N is given by

$$\begin{aligned}\frac{1}{k_1^2}\langle \nabla^*_{e_1} e_1, e_2^* \rangle^* &= \frac{1}{k_1^2}\langle A\nabla^*_{e_1} e_1, Ae_2^* \rangle = \frac{1}{k_1^2}\langle \nabla_{e_1} Ae_1, e_2 \rangle \\ &= \frac{1}{k_1^2}\langle k_1 \nabla_{e_1} e_1 + e_1(k_1)e_1, e_2 \rangle = \frac{1}{k_1^2}\langle k_1 \nabla_{e_1} e_1, e_2 \rangle \\ &= \frac{\alpha}{k_1}.\end{aligned}$$

We further have that

$$e_1(k_1)\alpha = k_1 e_1(\alpha) \Leftrightarrow e_1\left(\frac{\alpha}{k_1}\right) = 0,$$

which is equivalent to the geodesic curvature α/k_1 being constant along the e_1-lines of curvature. Since $N : U \to \mathbb{S}^2$ is a local isometry, this is also equivalent to the image by N of the e_1-lines of curvature being arcs of circles in $\mathbb{S}^2$, and the result follows. □

Lemma 3.2.2. A line of curvature c of f has constant geodesic curvature α if and only if it is contained in either a plane or a sphere that intersects $f(M^2)$ orthogonally along $f(c(I))$, according to whether α is zero or not, respectively.

Proof. Let $c : I \to U$ be an e_1-line of curvature and suppose first that c is contained in a plane that intersects $f(M)$ orthogonally along c. This means that there exists a unit vector $b \in \mathbb{R}^3$ such that

$$\langle f \circ c, b \rangle^\sim = const,\ \text{ and } \langle N \circ c, b \rangle^\sim = 0. \tag{3.15}$$

Differentiating the first equation in (3.15) we obtain

$$\langle f_* e_1 \circ c, b \rangle^\sim = 0,$$

which, together with the second equation in (3.15), allows to conclude that f_*e_2 must be constant along c. Thus, by the Gauss formula

$$\begin{aligned} 0 &= (\tilde{\nabla}_{e_1} f_*e_2)\circ c = (f_*\nabla_{e_1}e_2 + \langle Ae_1, e_2\rangle N)\circ c \\ &= (f_*\nabla_{e_1}e_2)\circ c = (-\alpha f_*e_1)\circ c, \end{aligned}$$

the geodesic curvature $\alpha\circ c$ is zero. Suppose now that c is contained in a sphere with center P_0 and radius r that intersects $f(M^2)$ orthogonally along c. We then have

$$\langle f\circ c - P_0, f\circ c - P_0\rangle^{\sim} = r^2, \text{ and } \left\langle N\circ c, \frac{f\circ c - P_0}{r}\right\rangle^{\sim} = 0. \tag{3.16}$$

Arguing as before, we see that $f - rf_*e_2$ must be constant along c, whence

$$\begin{aligned} 0 &= (f_*e_1 - r\tilde{\nabla}_{e_1} f_*e_2)\circ c = (f_*e_1 + r\alpha f_*e_1)\circ c \\ &= ((1+r\alpha)f_*e_1)\circ c, \end{aligned}$$

and therefore the geodesic curvature $\alpha\circ c$ is equal to the constant $-1/r$.

For the converse statement, assume that c is an e_1-line of curvature with constant geodesic curvature α. If $\alpha = 0$, it follows that

$$(\tilde{\nabla}_{e_1} f_*e_2)\circ c = (f_*\nabla_{e_1}e_2 + \langle Ae_1, e_2\rangle N)\circ c = 0,$$

since $(\nabla_{e_1}e_2)\circ c = (-\alpha e_1)\circ c = 0$. Hence f_*e_2 is constant in $\mathbb{R}^3$ along c, and this implies that $f(c(I))$ is contained in a plane that intersects $f(M)$ orthogonally along $f(c(I))$, taking into account that $(f\circ c)' = f_*e_1$ is orthogonal to f_*e_2. In the case where $\alpha \neq 0$ we obtain

$$(\tilde{\nabla}_{e_1} f_*e_2)\circ c = (f_*\nabla_{e_1}e_2 + \langle Ae_1, e_2\rangle N)\circ c = (-\alpha f_*e_1)\circ c,$$

thus the map $g = f + (1/\alpha)f_*e_2$ satisfies

$$(g_*e_1)\circ c = (f_*e_1)\circ c + \frac{1}{\alpha}(-\alpha f_*e_1)\circ c = 0.$$

This shows that g has a constant value $P_0 \in \mathbb{R}^3$ along c, and therefore $f(c(I))$ is contained in a sphere centered at P_0 with radius $1/|\alpha|$. Furthermore, since the normal vector of such sphere along c is given by f_*e_2, we conclude that this sphere intersects $f(M)$ orthogonally along $f(c(I))$.

Clearly, the same proof holds for a e_2-line of curvature.

□

The proof of the Theorem 3.2.4 below also relies on the next lemma.

Lemma 3.2.3. If $f : M^2 \to \mathbb{R}^3$ has nonzero constant Gaussian curvature, then the e_1-lines of curvature are planar if and only if the e_2-lines of curvature have constant geodesic curvature.

Proof. By Remark 2, the Codazzi equation for f is equivalent to the equations

$$e_1(k_2) = (k_2 - k_1)\beta \quad \text{and} \quad e_2(k_1) = (k_1 - k_2)\alpha. \tag{3.17}$$

Note that the e_2-lines of curvature have constant geodesic curvature if and only if $e_2(\beta) = 0$. Furthermore, by the first equation in (3.17), this is also equivalent to

$$e_2(e_1(k_2))(k_2 - k_1) = e_1(k_2)(e_2(k_2) - e_2(k_1)). \tag{3.18}$$

Since $K = k_1 k_2$ is constant, we have

$$e_1(k_1 k_2) = 0 = e_2(k_1 k_2) = e_2(k_1)k_2 + k_1 e_2(k_2), \tag{3.19}$$

which implies that

$$e_2(k_2) - e_2(k_1) = -\frac{k_2 + k_1}{k_1} e_2(k_1).$$

Substituting in (3.18) yields

$$k_1(k_2 - k_1)e_2(e_1(k_2)) + (k_1 + k_2)e_2(k_1)e_1(k_2) = 0,$$

and, using again (3.17), we see that (3.18) is equivalent to

$$-k_1^2 e_2(e_1(k_2)) + (k_2^2 - k_1^2)\alpha\beta k_1 = 0. \tag{3.20}$$

It further holds that

$$\begin{aligned} e_2(e_1(k_2)) - e_1(e_2(k_2)) &= [e_2, e_1](k_2) = (\nabla_{e_2} e_1 - \nabla_{e_1} e_2)(k_2) \\ &= -\beta e_2(k_2) + \alpha e_1(k_2). \end{aligned} \tag{3.21}$$

On the other hand, from (3.19) and (3.17) we obtain

$$\begin{aligned} -k_1^2 e_1(e_2(k_2)) &= k_1^2 e_1\left(\frac{e_2(k_1)k_2}{k_1}\right) \\ &= k_1 k_2 e_1(e_2(k_1)) + k_1^2 e_2(k_1) e_1\left(\frac{k_2}{k_1}\right) \\ &= k_1 k_2 e_1(\alpha(k_1 - k_2)) + k_1^2(\alpha(k_1 - k_2))\left(\frac{k_1 e_1(k_2) - k_2 e_1(k_1)}{k_1^2}\right) \\ &= (k_1 - k_2)k_2(e_1(\alpha)k_1 - \alpha e_1(k_1)) - 2\alpha k_1 k_2 e_1(k_2) + \alpha k_1 e_1(k_1 k_2) \\ &= (k_1 - k_2)k_2(e_1(\alpha)k_1 - \alpha e_1(k_1)) - 2\alpha k_1 k_2 (k_2 - k_1)\beta \end{aligned}$$

and

$$\begin{aligned} -k_1^2(-\beta e_2(k_2) + \alpha e_1(k_2)) &= -k_1^2\left(-\beta\frac{k_2 e_2(k_1)}{k_1} + \alpha(k_2 - k_1)\beta\right) \\ &= -k_1^2\beta\left(\frac{k_2}{k_1}(k_2 - k_1)\alpha + \alpha(k_2 - k_1)\right) \\ &= \alpha\beta k_1(k_1 - k_2)^2. \end{aligned}$$

Now, (3.21) gives

$$\begin{aligned}-k_1^2 e_2(e_1(k_2)) &= (k_1-k_2)k_2(e_1(\alpha)k_1-\alpha e_1(k_1))-2\alpha k_1k_2(k_2-k_1)\beta\\ &\quad+\alpha\beta k_1(k_1-k_2)^2\\ &=(k_1-k_2)k_2(e_1(\alpha)k_1-\alpha e_1(k_1))+\alpha\beta k_1(k_1^2-k_2^2).\end{aligned}$$

Therefore, (3.20) holds if and only if $e_1(\alpha)k_1-\alpha e_1(k_1)=0$, which is equivalent to the e_1-lines of curvature being planar by Lemma 3.2.1. □

We are now able to prove the wished result.

Theorem 3.2.4. Every surface of Enneper type $f: M^2\to\mathbb{R}^3$ of nonzero constant Gaussian curvature whose lines of curvature correspondent to one of its principal curvatures are planar is a Joachimsthal surface.

Proof. Given $p\in M^2$, let $\{e_1,e_2\}$ be a principal orthonormal frame for f defined on a neighborhood U of p with correspondent principal curvatures k_1 and k_2. Assume that the e_1-lines of curvature are planar. Then the e_2-lines of curvature have constant geodesic curvature by Lemma 3.2.3. Let $c_2: J\to U$ denote an e_2-line of curvature and let $\beta\circ c_2\equiv\beta$ be its geodesic curvature. We will discuss the cases where $\beta=0$ and $\beta\neq 0$ separately.

In the case where $\beta=0$, from Lemma 3.2.2 we have that $f(c_2(J))$ is contained in a plane that intersects $f(M^2)$ orthogonally. This means that the e_2-lines of curvature are also planar, hence the e_1-lines of curvature have constant geodesic curvature α, again by Lemma 3.2.3. We claim that α must be nonzero. Indeed, if α and β are identically zero along the e_1-lines of curvature and the e_2-lines of curvature, respectively, we obtain that

$$\nabla_{e_1}e_1=\nabla_{e_2}e_2=\nabla_{e_1}e_2=\nabla_{e_2}e_1=[e_1,e_2]=0,$$

whence

$$R(e_1,e_2)e_2=\nabla_{e_1}\nabla_{e_2}e_2-\nabla_{e_2}\nabla_{e_1}e_2-\nabla_{[e_1,e_2]}e_2=0.$$

Then M^2 would be flat, a contradiction. Therefore, if $c_1: I\to M^2$ is an e_1-line of curvature, it follows that c_1 is an arc of a circle in a sphere that intersects $f(M^2)$ orthogonally. Since each plane containing an e_2-line of curvature passing through $c_1(t)$ is orthogonal to such sphere, for all $t\in I$, we conclude that all these planes must intersect along a common straight line r. Moreover, this implies that the planes containing the e_1-lines of curvature must be parallel to each other. Therefore, $f(M^2)$ is a surface of revolution with r as its axis of rotation.

We now analyze the case where $\beta\neq 0$. By Lemma 3.2.2, $f(c_2(J))$ is contained in a sphere whose normal vector along c_2 is given by f_*e_1. But since $f_*e_1(c_2(t))$ belongs to the plane that contains the e_1-line of curvature passing through $c_2(t)$, for all $t\in J$, we see that every such plane passes through the center of that sphere, which is parametrized by the map

$g = f + (1/\beta) f_* e_1$. On the other hand,

$$\begin{aligned} g_* e_1 &= f_* e_1 - \frac{e_1(\beta)}{\beta^2} f_* e_1 + \frac{1}{\beta} \tilde{\nabla}_{e_1} f_* e_1 \\ &= \left(1 - \frac{e_1(\beta)}{\beta^2}\right) f_* e_1 + \frac{1}{\beta}(f_* \nabla_{e_1} e_1 + \langle A e_1, e_1 \rangle N) \\ &= \left(1 - \frac{e_1(\beta)}{\beta^2}\right) f_* e_1 + \frac{1}{\beta}(\alpha f_* e_2 + k_1 N), \end{aligned} \tag{3.22}$$

and since k_1 is nowhere zero, this implies that the map g can not be constant along the e_1-lines of curvature, that is, the spheres containing each e_2-line of curvature can not be concentric. Therefore, all the planes containing the e_1-lines of curvature intersect along a common straight line. □

3.3 Examples

We end this chapter by presenting some examples of Joachimsthal surfaces in $\mathbb{R}^3$. The computations here were made by using a computational software.

Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be the unit speed curve given by

$$\gamma(t) = \left(\frac{2}{r_0} \cosh t\right) v_1 + \left(\frac{x_0}{r_0} \cosh t + \sinh t\right) v_2 + \left(\frac{r_0^2 + x_0^2}{2 r_0} \cosh t + x_0 \sinh t\right) v_3,$$

where $r_0 > 0$ and $x_0 \in \mathbb{R}$ are constants. This family of unit speed curves are the geodesics corresponding to the half circles with centers on the x-axis in the hyperbolic upper half-plane model. Let $h : J \to \mathbb{R}$ a smooth function with positive derivative defined on some open interval $J \subset \mathbb{R}$. By Corollary 3.1.2.1, the corresponding Joachimsthal surface $f : M^2 \subset \mathbb{R} \times J \to \mathbb{R}^3$ is given by

$$f(t,s) = \begin{bmatrix} \dfrac{x_0 \cosh s \cosh t + x_0 \sinh s + r_0 \cosh s \sinh t}{2 \cosh s \cosh t + 2 \sinh s} \\ \dfrac{r_0 \cos(h(s))}{2 \cosh s \cosh t + 2 \sinh s} \\ \dfrac{r_0 \sin(h(s))}{2 \cosh s \cosh t + 2 \sinh s} \end{bmatrix},$$

see Figure 4. The Gaussian curvature of f is

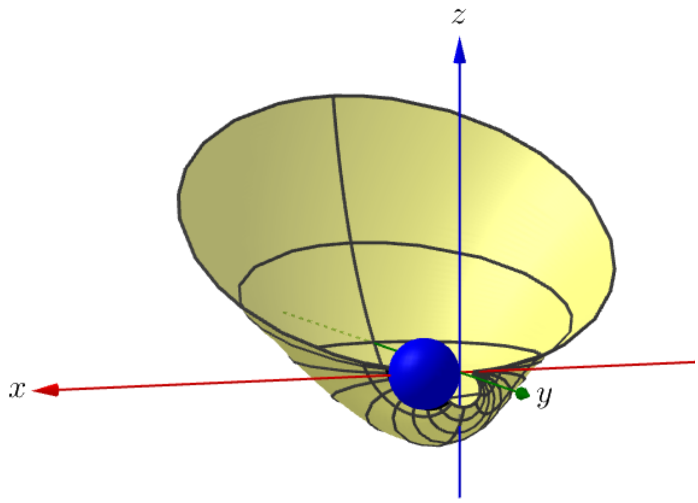
$$K(t,s) = \frac{4 h'(s)((1 + \cosh t \tanh s)(h'(s) + h'(s)^3) - h''(s)(\cosh t + \tanh s))}{r_0^2 (1 + h'(s)^2)^2}.$$

Now, let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be the unit speed curve given by

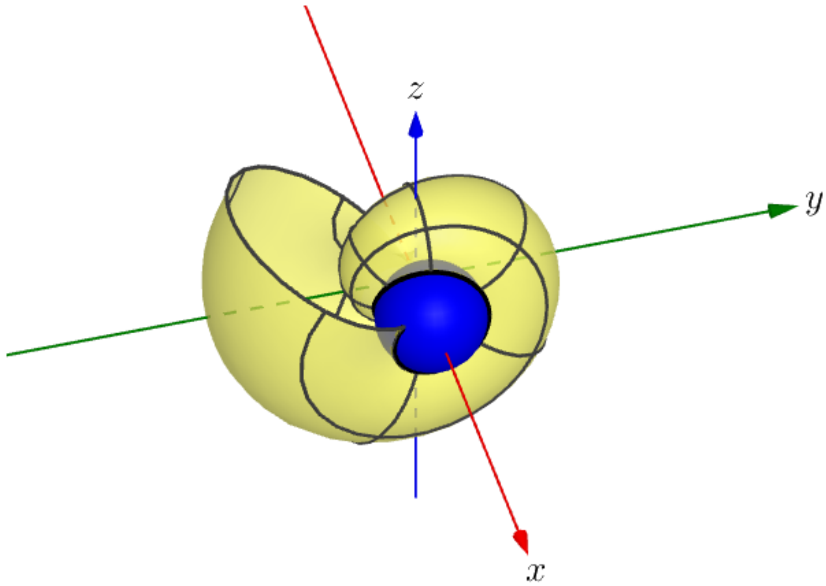
$$\gamma(t) = \left(\frac{2}{e^t}\right) v_1 + \left(\frac{x_0}{e^t}\right) v_2 + \left(\frac{x_0^2 + e^{2t}}{2 e^t}\right) v_3,$$

where $x_0 \in \mathbb{R}$ is a constant. This is the family of geodesics corresponding to the straight lines through the x-axis in the hyperbolic upper half-plane model. In this case, the corresponding Joachimsthal surface $f : M^2 \subset \mathbb{R} \times J \to \mathbb{R}^3$ is given by

Figure 4 – Joachimsthal Surfaces ($x_0 = 0$ and $r_0 = 1$)



(a) $h(s) = s$



(b) $h(s) = \sinh s$

Source: Elaborated by the author.

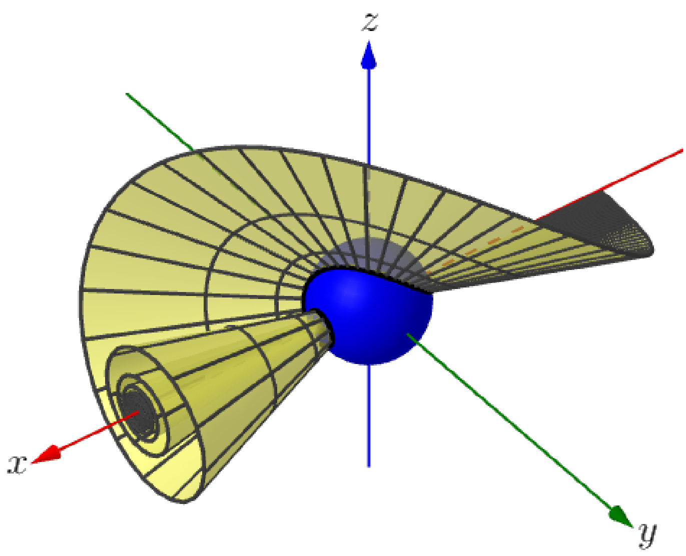
$$f(t,s) = \frac{1}{2}\left(x_0 + e^t \tanh s, \frac{e^t}{\cosh s}\cos(h(s)), \frac{e^t}{\cosh s}\sin(h(s))\right)$$

and the Gaussian curvature of f is identically zero, see Figure 5.

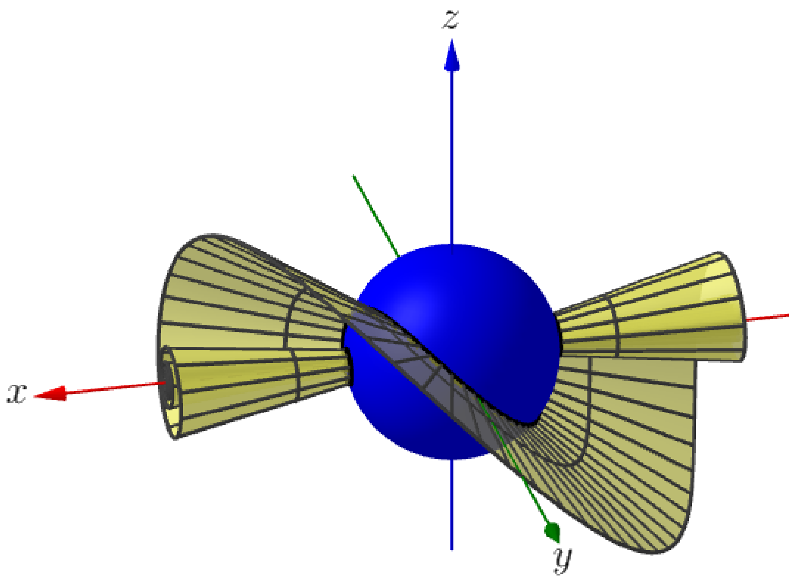
The next surfaces are classical Joachimsthal surfaces of nonzero constant Gaussian curvature found in the literature.

The *Dini's Helicoid* is a surface obtained by twisting a pseudosphere along its axis (Fig. 6).

Figure 5 – Joachimsthal Flat Surfaces ($x_0 = 0$)



(a) $h(s) = e^s$



(b) $h(s) = \sinh s$

Source: Elaborated by the author.

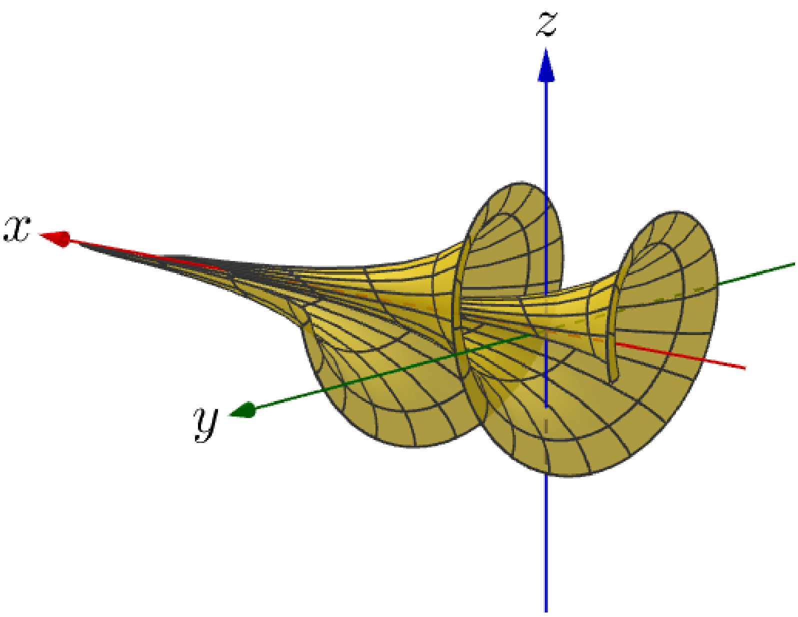
It can be parametrized by

$$f(u_1,u_2) = \left(m(u_1 - \tanh u_1) + nu_2, m\frac{\cos u_2}{\cosh u_1}, m\frac{\sin u_2}{\cosh u_1} \right),$$

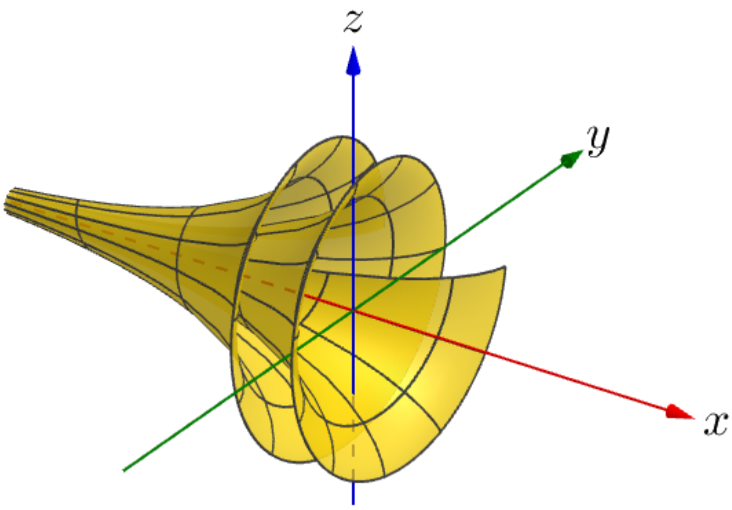
where m and n are constants with $m \neq 0$. If $U = \{(u_1,u_2) \in \mathbb{R}^2 \,|\, u_1 \neq 0\}$, then $f : U \to \mathbb{R}^3$ defines a surface satisfying

$$K = -\frac{1}{m^2+n^2}.$$

Figure 6 – Dini's Helicoid



(a) $m = 6, n = 1.4, 0 < u_1 < \pi, -2\pi < u_2 < 2\pi$



(b) $m = 6, n = 0.4, -2\pi < u_1 < 0, -2\pi < u_2 < 2\pi$

Source: Elaborated by the author.

We have

$$f_*\partial_{u_1}(u_1,u_2) = m\left(\tanh^2 u_1, -\frac{\tanh u_1}{\cosh u_1}\cos u_2, -\frac{\tanh u_1}{\cosh u_1}\sin u_2\right)$$

$$f_*\partial_{u_2}(u_1,u_2) = \left(n, -m\frac{\sin u_2}{\cosh u_1}, m\frac{\cos u_2}{\cosh u_1}\right)$$

$$N(u_1,u_2) = \frac{1}{\sqrt{m^2+n^2}}\left(\frac{m}{\cosh u_1}, m\cos u_2 \tanh u_1 + n\sin u_2, m\sin u_2\tanh u_1 - n\cos u_2\right),$$

whence

$$N_*\partial_{u_1}(u_1,u_2)=\frac{m}{\sqrt{m^2+n^2}}\left(-\frac{\tanh u_1}{\cosh u_1},\frac{\cos u_2}{\cosh^2 u_1},\frac{\sin u_2}{\cosh^2 u_1}\right).$$

Then, we see that

$$N_*\partial_{u_1}(u_1,u_2^0)=-\frac{1}{\sqrt{m^2+n^2}\sinh u_1}f_*\partial_{u_1}(u_1,u_2^0),$$

and this implies that the u_1-coordinate curves coincide with one family of lines of curvature of the surface. On the other hand, we have

$$\langle f(u_1,u_2^0),(0,\sin u_2^0,-\cos u_2^0)\rangle^\sim=0,$$

hence the u_1-lines of curvature are contained in planes that intersect along the x-axis. Except for $n=0$, in which case the surface reduces to the Pseudosphere, the u_2-coordinate curves are not lines of curvature of f, that is, f is not parametrized by lines of curvature.

If $n\neq 0$, the map

$$x(u_1,u_2)=\left(u_1+\frac{n}{m}u_2,-\frac{n}{m}u_2\right)$$

is a diffeomorphism and its inverse is given by

$$x^{-1}(t,s)=\left(t+s,-\frac{m}{n}s\right).$$

Thus, on $M^2=x(U)=\{(t,s)\in\mathbb{R}^2\,|\,t+s\neq 0\}$, the surface $f=f\circ x^{-1}:M^2\to\mathbb{R}^3$ is a parametrization of f given by

$$f(t,s)=\left(m(t-\tanh(t+s)),\frac{m\cos(\frac{m}{n}s)}{\cosh(t+s)},-\frac{m\sin(\frac{m}{n}s)}{\cosh(t+s)}\right).$$

This new parametrization is by lines of curvature. Now, assuming, for simplicity, that $m=n=1$ in terms of Corollary 3.1.2.1, a straightforward computation gives that the surface $\bar{F}=\bar{\Phi}\circ f: M^2\to\mathbb{H}^2\times\mathbb{S}^1$ is given by

$$\begin{bmatrix}(\cosh(t+s))v_1+(t\cosh(t+s)-\sinh(t+s))v_2+(-2t\sinh(t+s)+(1+t^2)\cosh(t+s))v_3\\ \cos s\\ \sin s\end{bmatrix}$$

whence the corresponding surface $F:M^2\to\mathbb{H}^2\times\mathbb{R}$ becomes

$$\begin{bmatrix}(\cosh(t+s))v_1+(t\cosh(t+s)-\sinh(t+s))v_2+(-2t\sinh(t+s)+(1+t^2)\cosh(t+s))v_3\\ s\end{bmatrix}.$$

Then, the corresponding curve in $\mathbb{H}^2$ becomes

$$\gamma(t)=F(t,0)=(\cosh t)v_1+(t\cosh t-\sinh t)v_2+(-2t\sinh t+(1+t^2)\cosh t)v_3.$$

We have

$$\gamma'(t)=(\sinh t)v_1+(t\sinh t)v_2+((t^2-1)\sinh t)v_3$$

and hence

$$\langle \gamma'(t), \gamma'(t) \rangle^{\sim} = -(t^2-1)\sinh^2 t + t^2 \sinh^2 t = \sinh^2 t,$$

so that the arc length is $u(t) = \cosh t$, and consequently $t = \operatorname{arccosh} u$. Therefore, in this case we obtain $b(t) = t \operatorname{arccosh} t - \sinh(\operatorname{arccosh} t) = t \operatorname{arccosh} t - \sqrt{t^2-1}$ and $h(s) = s$ for which the condition (i) of Proposition 3.1.1 is satisfied when $\sigma = -1$.

The *Kuen's Surface* is parametrized by

$$f(t,s) = \frac{1}{1+t^2\sin^2 s} \begin{bmatrix} \log(\tan(s/2))(1+t^2\sin^2 s) + 2\cos s \\ 2(\cos t + t\sin t)\sin s \\ 2(\sin t - t\cos t)\sin s \end{bmatrix}.$$

This parametrization yields on $M^2 = (0, 2\pi) \times (0, \pi)$ a surface parametrized by lines of curvature of constant Gaussian curvature -1 (Fig. 7). Since

$$\langle f(t_0,s), (0, \sin t_0 - t_0\cos t_0, -\cos t_0 - t_0 \sin t_0) \rangle^{\sim} = 0$$

we see that Kuen's Surface is indeed a Joachimsthal surface. Now, arguing as in the preceding example we may obtain that

$$b(t) = -\frac{t}{2}\left(8 + \sqrt{16 + \frac{1}{t^2}} \log\left(4t + t\sqrt{16 + \frac{1}{t^2}}\right)\right)$$

and

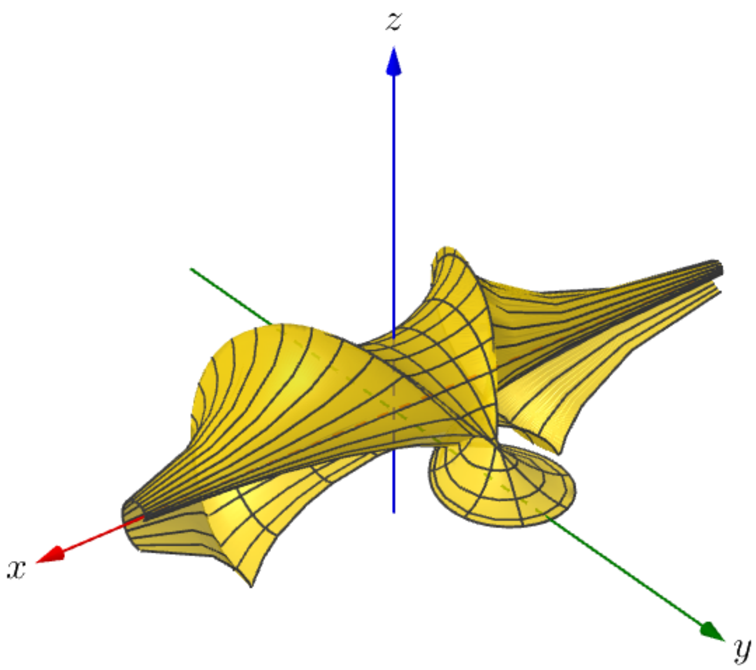
$$h(s) = \arctan\left(\frac{\sin(\theta(s)) - \theta(s)\cos(\theta(s))}{\cos(\theta(s)) + \theta(s)\sin(\theta(s))}\right),$$

where

$$\theta(s) = \sqrt{\frac{-2\sinh s}{\cosh s + \sinh s}}.$$

In this case, the functions b and h satisfy the condition (ii) of Proposition 3.1.1 for $a_1 = 1$, $a_2 = -1/2$, $a_3 = 0$ and $\sigma = -1$.

Figure 7 – Kuen's Surface



This surface was obtained by Theodor Kuen in 1984. It comes from the Pseudosphere through the Bianchi Transform (a geometric transformation that preserves the Gaussian curvature).

Source: Elaborated by the author.

CHAPTER 4

MINIMAL SURFACES OF ENNEPER TYPE

So far, we have been able to give an explicit description of some special classes of surfaces of Enneper type. The aim of this chapter is to give a full classification of minimal surfaces of Enneper type with one family of planar lines of curvature. In this case, we will see that the other family also consists of planar lines of curvature. Then, by analyzing the orthogonal systems of circles on $\mathbb{S}^2$, we are able to recover the corresponding minimal surfaces.

The theory of minimal surfaces is one of the most developed subjects of differential geometry. The condition $H \equiv 0$ is necessarily satisfied by surfaces which minimize area with a given boundary configuration, and this explains why we use the word *minimal* for such surfaces. We refer the reader to the books of Dierkes, Hildebrandt and Sauvigny (2010) and Nitsche (1989) for a discussion of this subject, as well as for many other results on minimal surfaces.

4.1 Minimal surfaces

Minimal surfaces can be constructed from a careful choice of complex functions, using the Weierstrass-Enneper Representation formula. In the first two sections, we develop the necessary tools to study this formula and then derive the representation formula of Weierstrass. Using the latter we can easily introduce principal coordinates that are also conformal parameters on a minimal surface, which play a key role in the wished classification.

Let $f : U \to \mathbb{R}^3$ be a surface defined on the open subset $U \subset \mathbb{R}^2$ and let $N : U \to \mathbb{S}^2$ be its Gauss map given by

$$N = \left\| \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2} \right\|^{-1} \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2},$$

where (u_1, u_2) denote the coordinates of U.

Recall that the coordinates (u_1, u_2) are isothermal if

$$\left\langle \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_1} \right\rangle^{\sim} = \left\langle \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_2} \right\rangle^{\sim} \quad \text{and} \quad \left\langle \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2} \right\rangle^{\sim} = 0, \tag{4.1}$$

or equivalently

$$ds^2 = g_{11}(du_1^2 + du_2^2),$$

which means that the metric induced by f is conformal to the Euclidean metric on U.

Next, we see how harmonic functions are related to isothermal coordinates on minimal surfaces.

Proposition 4.1.1. Let $f : U \to \mathbb{R}^3$ be a surface given by isothermal coordinates. Then,

$$\frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2} = (2g_{11}H)N.$$

In particular, f is a minimal surface if and only if its coordinate functions are harmonic.

Proof. Differentiating the first equation of (4.1) with respect to u_1 and the second one with respect to u_2, we obtain

$$\left\langle \frac{\partial^2 f}{\partial u_1^2}, \frac{\partial f}{\partial u_1} \right\rangle^{\sim} = \left\langle \frac{\partial^2 f}{\partial u_2 \partial u_1}, \frac{\partial f}{\partial u_2} \right\rangle^{\sim} = -\left\langle \frac{\partial f}{\partial u_1}, \frac{\partial^2 f}{\partial u_2^2} \right\rangle^{\sim}.$$

Combining these equations gives

$$\left\langle \frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2}, \frac{\partial f}{\partial u_1} \right\rangle^{\sim} = 0.$$

Similarly, differentiating the first equation of (4.1) with respect to u_2 and the second one with respect to u_1, we conclude that

$$\left\langle \frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2}, \frac{\partial f}{\partial u_2} \right\rangle^{\sim} = 0.$$

Thus, the vector $\partial^2 f/\partial u_1^2 + \partial^2 f/\partial u_2^2$ must be a multiple of N. Since the coordinates are isothermal, we have

$$H = \frac{1}{2}\frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{g_{11}g_{22} - g_{12}^2} = \frac{1}{2}\frac{b_{11} + b_{22}}{g_{11}} = \frac{1}{2g_{11}}\left\langle \frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2}, N \right\rangle^{\sim}$$

and hence

$$\frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2} = (2g_{11}H)N.$$

□

In order to study results that enable us to construct minimal surfaces, we need to link ideas from functions of a complex variable to minimal surfaces. Let $R : U \subset \mathbb{C} \to \mathbb{C}$ be a complex function. Here we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ by setting $w = u_1 + iu_2, w \in \mathbb{C}, (u_1, u_2) \in \mathbb{R}^2$. Recall that R is holomorphic when, by writing

$$R(w) = R_1(u_1, u_2) + iR_2(u_1, u_2),$$

the real functions R_1 and R_2 have continuous partial derivatives of first order and satisfy the Cauchy-Riemann equations:

$$\frac{\partial R_1}{\partial u_1} = \frac{\partial R_2}{\partial u_2}, \quad \frac{\partial R_1}{\partial u_2} = -\frac{\partial R_2}{\partial u_1},$$

or, in terms of the differential operators

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\right),$$

if and only if

$$\frac{\partial R}{\partial \bar{w}} = 0.$$

In this case, we can write the differential of R as

$$R_*(w)X = \frac{\partial R}{\partial w} \cdot X := R'(w) \cdot X, \quad w \in U, X \in \mathbb{C},$$

where the dot stands for the complex number multiplication.

Returning now to the surface $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, it is useful to associate with f the complex map $\varphi : U \to \mathbb{C}^3$ given by

$$\varphi(w) = \frac{1}{2}\left(\frac{\partial f}{\partial u_1} - i\frac{\partial f}{\partial u_2}\right) := \frac{\partial f}{\partial w},$$

with coordinate complex functions

$$\varphi_1(w) = \frac{1}{2}\left(\frac{\partial x}{\partial u_1} - i\frac{\partial x}{\partial u_2}\right), \ \varphi_2(w) = \frac{1}{2}\left(\frac{\partial y}{\partial u_1} - i\frac{\partial y}{\partial u_2}\right), \ \varphi_3(w) = \frac{1}{2}\left(\frac{\partial z}{\partial u_1} - i\frac{\partial z}{\partial u_2}\right),$$

where x, y and z are the coordinate functions of f. We say that φ is the *associated phi function* of f. It is easy to see that

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \frac{1}{4}(g_{11} - g_{22} + 2ig_{12}),$$

hence, the coordinates (u_1, u_2) are isothermal, if and only if

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0. \tag{4.2}$$

Proposition 4.1.2. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface given by isothermal coordinates and let φ be its associated phi function. Then f is a minimal surface if and only if φ_1, φ_2 and φ_3 are holomorphic functions.

Proof. We have from Proposition 4.1.1 that f is minimal if and only if

$$\frac{\partial^2 f}{\partial u_1^2} + \frac{\partial^2 f}{\partial u_2^2} = 0. \tag{4.3}$$

This means that

$$\begin{aligned}
\frac{\partial}{\partial u_1}\left(\frac{\partial x}{\partial u_1}\right) &= \frac{\partial}{\partial u_2}\left(-\frac{\partial x}{\partial u_2}\right),\\
\frac{\partial}{\partial u_1}\left(\frac{\partial y}{\partial u_1}\right) &= \frac{\partial}{\partial u_2}\left(-\frac{\partial y}{\partial u_2}\right),\\
\frac{\partial}{\partial u_1}\left(\frac{\partial z}{\partial u_1}\right) &= \frac{\partial}{\partial u_2}\left(-\frac{\partial z}{\partial u_2}\right),
\end{aligned} \tag{4.4}$$

which is one of the Cauchy-Riemann equations for φ_1, φ_2 and φ_3, respectively. Since the other equation follows directly from

$$\frac{\partial}{\partial u_1}\left(\frac{\partial f}{\partial u_1}\right) = \frac{\partial}{\partial u_2}\left(\frac{\partial f}{\partial u_1}\right),$$

we obtain that (4.3) is equivalently to the complex functions φ_1, φ_2 and φ_3 being holomorphic. □

We present next some examples of minimal surfaces in $\mathbb{R}^3$. The two first ones were in fact the first nonplanar minimal surfaces to be discovered.

Example 4.1.1 (Catenoid)**.** The *Catenoid* is the revolution surface obtained by rotating a catenary $u_1 \in \mathbb{R} \mapsto (\alpha\cosh(u_1/\alpha), u_1)$, about the z-axis, where $\alpha \in \mathbb{R}$ is a nonzero constant. Then, the catenoid is given by

$$f(u_1,u_2) = (\alpha\cosh u_1 \cos u_2, \alpha\cosh u_1 \sin u_2, \alpha u_1). \tag{4.5}$$

We have

$$\frac{\partial f}{\partial u_1} = \alpha(\sinh u_1 \cos u_2, \sinh u_1 \sin u_2, 1), \;\frac{\partial^2 f}{\partial u_1^2} = \alpha(\cosh u_1 \cos u_2, \cosh u_1 \sin u_2, 0)$$

and

$$\frac{\partial f}{\partial u_2} = \alpha(-\cosh u_1 \sin u_2, \cosh u_1 \cos u_2, 0), \;\frac{\partial^2 f}{\partial u_2^2} = -\frac{\partial^2 f}{\partial u_1^2}$$

whence

$$ds^2 = \alpha^2\cosh^2 u_1 (du_1^2 + du_2^2)$$

and $\partial^2 f/\partial u_1^2 + \partial^2 f/\partial u_2^2 \equiv 0$. Thus, the catenoid is a minimal surface by Proposition 4.1.1. It is straightforward to check that

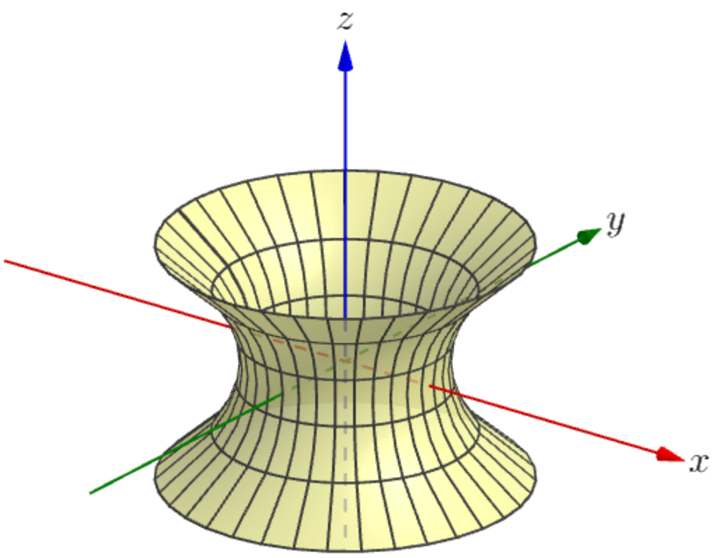
$$K = -\frac{1}{\alpha^2\cosh^4 u_1}.$$

which implies that f is free of umbilical points. Thus, the catenoid is a surface of Enneper type with the two families of lines of curvature being planar (Fig. 8).

Example 4.1.2 (Helicoid)**.** The *Helicoid* is the surface parametrized by

$$f(u_1,u_2) = (\alpha\sinh u_1 \cos u_2, \alpha\sinh u_1 \sin u_2, \alpha u_2).$$

Figure 8 – Catenoid



The Catenoid is the only nonplanar minimal surface of revolution.

Source: Elaborated by the author.

Similarly to the catenoid, it is easy to check that

$$ds^2 = \alpha^2 \cosh^2 u_1 (du_1^2 + du_2^2)$$

and that $\partial^2 f/\partial u_1^2 + \partial^2 f/\partial u_2^2 \equiv 0$. Thus, the helicoid is a minimal surface (Fig. 9).

Example 4.1.3 (Enneper Minimal Surface)**.** The *Enneper minimal surface* is given by

$$f(u_1,u_2) = \left(u_1 - \frac{u_1^3}{3} + u_1 u_2^2, u_2 - \frac{u_2^3}{3} + u_2 u_1^2, u_1^2 - u_2^2 \right).$$

We have

$$\frac{\partial f}{\partial u_1} = (1 - u_1^2 + u_2^2, 2u_1 u_2, 2u_1), \ \frac{\partial^2 f}{\partial u_1^2} = (-2u_1, 2u_2, 2)$$

and

$$\frac{\partial f}{\partial u_2} = (2u_1 u_2, 1 + u_1^2 - u_2^2, -2u_2), \ \frac{\partial^2 f}{\partial u_2^2} = -\frac{\partial^2 f}{\partial u_1^2}.$$

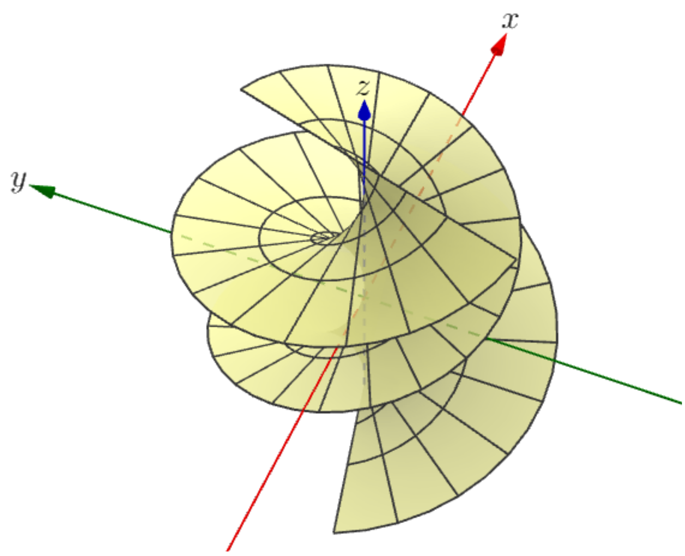
Hence the induced metric is

$$ds^2 = (1 + u_1^2 + u_2^2)^2 (du_1^2 + du_2^2),$$

and since $\partial^2 f/\partial u_1^2 + \partial^2 f/\partial u_2^2 \equiv 0$, we conclude that f is a minimal surface. We also obtain that

$$N = \frac{1}{1 + u_1^2 + u_2^2} (-2u_1, 2u_2, 1 - (u_1^2 + u_2^2))$$

Figure 9 – Helicoid



The helicoid is generated by a screw motion of some straight line meeting the z-axis perpendicularly.

Source: Elaborated by the author.

and

$$\frac{\partial^2 f}{\partial u_1 \partial u_2} = (2u_2, 2u_1, 0)$$

which implies that $b_{12} \equiv 0$. Moreover, we easily compute that

$$K = -\frac{4}{(1+u_1^2+u_2^2)^4}.$$

Thus, f is free of umbilical points and it is parametrized by lines of curvature. Now, we see that

$$\begin{aligned}\langle f(u_1^0, u_2), (1, 0, u_1^0)\rangle &= u_1^0 - \frac{(u_1^0)^3}{3} + u_1^0 u_2^2 + u_1^0((u_1^0)^2 - u_2^2) \\ &= u_1^0 + \frac{2}{3}(u_1^0)^2 = const\end{aligned}$$

and

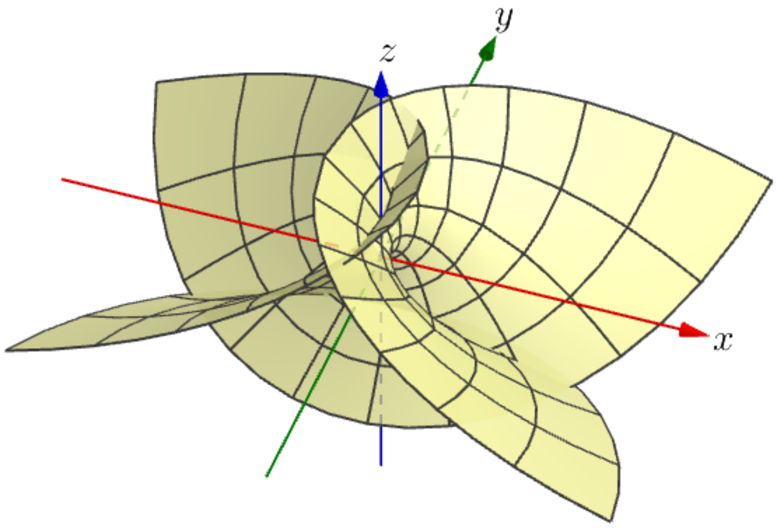
$$\langle f(u_1, u_2^0), (0, 1, u_2^0)\rangle = -u_2^0 - \frac{2}{3}(u_2^0)^2 = const.$$

Therefore, the two families of lines of curvature consist of planar curves.

4.2 Representation formulas

Let $f : U \to \mathbb{R}^3$ be a minimal surface free of umbilical points, or equivalently, free of flat points. Since its Gauss map $N : U \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a local diffeomorphism and $H \equiv 0$, it follows

Figure 10 – Enneper Minimal Surface



The Enneper minimal surface is a self-intersecting surface.

Source: Elaborated by the author.

from Remark 1 that the metric induced by N is conformal to the metric induced by f on U. Now, let (V,x) be a chart of $\mathbb{S}^2$ by stereographic projection. Then $x : V \to \tilde{U}$ is a conformal diffeomorphism onto an open subset $\tilde{U} \subset \mathbb{R}^2$ with respect to the Euclidean metric. By restricting to an open subset if necessary, we may assume that $N : U \to N(U)$ is a diffeomorphism and that $N(U) \subset V$. Therefore, the surface $\tilde{f} = f \circ (x \circ N)^{-1} : \tilde{U} \to \mathbb{R}^3$ is a reparametrization of f by isothermal coordinates.

A well-known result in Geometry guarantees that any surface in $\mathbb{R}^3$ can be locally parametrized by isothermal coordinates; however, the proof of this fact in the C^∞ case is significantly more complicated, see (SPIVAK, 1999, p. 345) for example. From now on, when we say that a surface $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is minimal we are meaning that f is minimal and parametrized by isothermal coordinates.

Theorem 4.2.1. Let $U \subset \mathbb{C}$ be a simply connected open subset, let $(x_0, y_0, z_0) \in \mathbb{R}^3$ and let $w_0 \in U$. Suppose that $\varphi(w) = (\varphi_1(w), \varphi_2(w), \varphi_3(w))$ is an holomorphic map of U into $\mathbb{C}^3$ which is never zero and satisfies

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0. \tag{4.6}$$

Then the formula

$$f(u_1, u_2) = (x_0, y_0, z_0) + 2\mathrm{Re} \int_{w_0}^{w} \varphi(\omega) d\omega, \ \ w = u_1 + iu_2 \in U \tag{4.7}$$

defines a minimal surface $f : U \to \mathbb{R}^3$.

Conversely, any minimal surface $f: U \to \mathbb{R}^3$ defined on a simply connected open subset $U \subset \mathbb{C}$ can be parametrized in this way.

Proof. We can write $f(w) = 2\text{Re } \Phi(w)$, where $\Phi: U \to \mathbb{C}$ is a holomorphic map with derivative

$$\Phi' = \varphi.$$

On the other hand, using $\partial\Phi/\partial\bar{w} = 0$ and the Cauchy-Riemann equations we obtain

$$\begin{aligned} \frac{\partial\Phi}{\partial w} &= \frac{\partial \text{Re}\Phi}{\partial u_1} + i\frac{\partial \text{Im}\Phi}{\partial u_1} = \frac{\partial \text{Re}\Phi}{\partial u_1} - i\frac{\partial \text{Re}\Phi}{\partial u_2} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial u_1} - i\frac{\partial f}{\partial u_2}\right), \end{aligned} \tag{4.8}$$

which implies that

$$\varphi = \frac{\partial f}{\partial w}. \tag{4.9}$$

Note that (4.6) shows that Re $\varphi \wedge$ Im $\varphi \neq 0$, since φ is never zero. Thus, by (4.9) we have

$$\frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2} = (2\text{Re } \varphi) \wedge (-2\text{Im } \varphi) \neq 0.$$

This shows that $f: U \to \mathbb{R}^3$ defines a surface with φ as its associated phi function. Therefore, f is a minimal surface by Proposition 4.1.2.

For the converse statement, just take φ to be the associated phi function of f and the result follows immediately from (4.2) and Proposition 4.1.2.

□

Let us now study a minimal surface $f: U \to \mathbb{R}^3$ in terms of the parametrization given by (4.7). Since

$$\frac{\partial f}{\partial u_1} = 2\text{Re } \varphi, \quad \frac{\partial f}{\partial u_2} = -2\text{Im } \varphi$$

it follows that $2\langle \text{Re } \varphi, \text{Re } \varphi\rangle^{\sim} = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 := |\varphi|^2$, whence

$$ds^2 = 2|\varphi|^2(du_1^2 + du_2^2).$$

Furthermore, the Gauss map of f becomes

$$N = \frac{(2\text{Re } \varphi) \times (-2\text{Im } \varphi)}{2|\varphi|^2} = 2|\varphi|^{-2}\text{Im } (\varphi_2\bar{\varphi}_3, \varphi_3\bar{\varphi}_1, \varphi_1\bar{\varphi}_2). \tag{4.10}$$

By differentiating φ and using that f is harmonic we have

$$2\varphi' = \left(\frac{\partial^2 f}{\partial u_1^2} - \frac{\partial^2 f}{\partial u_1 \partial u_2} i\right) = -\frac{\partial^2 f}{\partial u_1^2} - \frac{\partial^2 f}{\partial u_1 \partial u_2} i,$$

and taking the inner product of the real and imaginary parts of the preceding equation with N we obtain

$$\langle 2\varphi', N\rangle^{\sim} = b_{11} - b_{12}i = -b_{22} - b_{12}i.$$

In particular, we see that $b_{11} = -b_{22}$, which agrees with the fact that $g_{11} = g_{22}, g_{12} = 0$ and $H = 0$.

Denoting $v_1^2 := 2|\varphi|^2$ and using the isothermal conditions, we obtain

$$\nabla_{\partial_{u_1}} \partial_{u_2} = \frac{\partial(\log v_1)}{\partial u_2}\partial_{u_1} + \frac{\partial(\log v_1)}{\partial u_1}\partial_{u_2},$$
$$\nabla_{\partial_{u_1}} \partial_{u_1} = \frac{\partial(\log v_1)}{\partial u_1}\partial_{u_1} - \frac{\partial(\log v_1)}{\partial u_2}\partial_{u_2}.$$

Thus, it follows that

$$\begin{aligned}\langle A\partial_{u_1}, \nabla_{\partial_{u_2}}\partial_{u_1}\rangle - \langle A\partial_{u_2}, \nabla_{\partial_{u_1}}\partial_{u_1}\rangle &= \frac{\partial(\log v_1)}{\partial u_2}b_{11} + \frac{\partial(\log v_1)}{\partial u_1}b_{12} \\ &\quad - \left(\frac{\partial(\log v_1)}{\partial u_1}b_{12} - \frac{\partial(\log v_1)}{\partial u_2}b_{22}\right) \\ &= 0,\end{aligned} \tag{4.11}$$

since $b_{11} = -b_{22}$. The Codazzi equation for ∂_{u_1} and ∂_{u_2} gives

$$\nabla_{\partial_{u_1}} A\partial_{u_2} = \nabla_{\partial_{u_2}} A\partial_{u_1},$$

and taking the inner product with ∂_{u_1} yields

$$\langle \nabla_{\partial_{u_1}} A\partial_{u_2}, \partial_{u_1}\rangle = \langle \nabla_{\partial_{u_2}} A\partial_{u_1}, \partial_{u_1}\rangle. \tag{4.12}$$

On the other hand, since

$$\frac{\partial b_{ij}}{\partial u_i} = \partial_{u_i}(\langle A\partial_{u_i}, \partial_{u_j}\rangle) = \langle \nabla_{\partial_{u_i}} A\partial_{u_i}, \partial_{u_j}\rangle + \langle A\partial_{u_i}, \nabla_{\partial_{u_i}}\partial_{u_j}\rangle,$$

it follows from (4.12) that

$$\frac{\partial b_{11}}{\partial u_2} - \frac{\partial b_{12}}{\partial u_1} = \langle A\partial_{u_1}, \nabla_{\partial_{u_2}}\partial_{u_1}\rangle - \langle A\partial_{u_2}, \nabla_{\partial_{u_1}}\partial_{u_1}\rangle = 0, \tag{4.13}$$

taking (4.11) into account. Similarly, we obtain

$$\frac{\partial b_{11}}{\partial u_1} + \frac{\partial b_{12}}{\partial u_2} = 0. \tag{4.14}$$

Remark 4. In virtue of (4.13) and (4.14), for a minimal surface defined on a connected domain such that $b_{12} = 0$, the Codazzi equation implies that $b_{11} = -b_{22}$ is a nonzero constant.

Next, we introduce the complex function $l : U \to \mathbb{C}$ defined by

$$l(w) := b_{11}(w) - ib_{12}(w) = \langle 2\varphi', N\rangle^{\sim}.$$

We have

$$\begin{aligned}\frac{\partial l}{\partial \bar{w}} &= \frac{1}{2}\left(\frac{\partial b_{11}}{\partial u_1} - i\frac{\partial b_{12}}{\partial u_1} + i\left(\frac{\partial b_{11}}{\partial u_2} - i\frac{\partial b_{12}}{\partial u_2}\right)\right)\\ &= \frac{1}{2}\left(\frac{\partial b_{11}}{\partial u_1} + \frac{\partial b_{12}}{\partial u_2} + i\left(\frac{\partial b_{11}}{\partial u_2} - \frac{\partial b_{12}}{\partial u_1}\right)\right)\\ &= 0,\end{aligned} \tag{4.15}$$

on account of (4.13) and (4.14), hence l is holomorphic in U.

Moreover, the Gaussian curvature of f becomes

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{-(b_{11}^2 + b_{12}^2)}{4|\varphi|^4} = -\frac{|l|^2}{4|\varphi|^4}. \tag{4.16}$$

We then conclude from (4.16) that the umbilical points of f must either be isolated or else $b_{11} \equiv b_{22} \equiv b_{12} \equiv 0$, which implies that f is a planar surface. In other words, the umbilical points of a nonplanar minimal surface are isolated.

Finally, it turns out that the lines of curvature of f can be obtained by means of the holomorphic function l. Indeed, let $w(t) = (u_1(t), u_2(t)), t \in I$, be a curve in U. Then $w(t)$ is a line of curvature of f if and only if $N_* w'(t) = -k(t) f_* w'(t)$, for some $k \in C^\infty(I)$, or equivalently,

$$u_1'(t)N_*\partial_{u_1} + u_2'(t)N_*\partial_{u_2} = -k(t)(u_1'(t)f_*\partial_{u_1} + u_2'(t)f_*\partial_{u_2}) \tag{4.17}$$

Taking the inner product of both sides of (4.17) first with $f_*\partial_{u_1}$ and then with $f_*\partial_{u_1}$, and using the isothermal condition, yields the following equivalent system of equations

$$\begin{aligned} u_1'(t)\langle N_*\partial_{u_1}, f_*\partial_{u_1}\rangle^\sim + u_2'(t)\langle N_*\partial_{u_2}, f_*\partial_{u_1}\rangle^\sim &= -k(t)u_1'(t)g_{11},\\ u_1'(t)\langle N_*\partial_{u_1}, f_*\partial_{u_2}\rangle^\sim + u_2'(t)\langle N_*\partial_{u_2}, f_*\partial_{u_2}\rangle^\sim &= -k(t)u_2'(t)g_{11}.\end{aligned}$$

Using that $\langle N_*\partial_{u_i}, f_*\partial_{u_j}\rangle^\sim = \langle -f_*A\partial_{u_2}, f_*\partial_{u_2}\rangle^\sim = -\langle A\partial_{u_i}, \partial_{u_j}\rangle = -b_{ij}$ and that $b_{11} = -b_{12}$ the above system becomes

$$\begin{aligned} -u_1'(t)b_{11} - u_2'(t)\langle b_{12} &= -k(t)u_1'(t)g_{11},\\ -u_1'(t)b_{12} + u_2'(t)b_{11} &= -k(t)u_2'(t)g_{11}.\end{aligned} \tag{4.18}$$

Now, multiplying the first equation with $u_2'(t)$, the second with $-u_1'(t)$ and adding the resulting equations, we obtain the *differential equation of the lines of curvature*,

$$(u_1'(t)^2 - u_2'(t)^2)b_{12} - 2u_1'(t)u_2'(t)b_{11} = 0. \tag{4.19}$$

Here b_{12} and b_{11} have to be understood as $b_{12}(w(t))$ and $b_{11}(w(t))$. If we now consider the holomorphic quadratic differential

$$l(w)(dw)^2, \quad dw = du_1 + idu_2$$

the differential equation (4.19) transforms into

$$\text{Im } l(w)(dw)^2 = 0. \tag{4.20}$$

Once one determines all the holomorphic functions φ_1, φ_2 and φ_3 satisfying the condition $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$, the Theorem above yields all minimal surfaces defined on simply connected domains. This is achieved by the following elementary result whose proof uses only basic facts of complex functions and can be found on (DIERKES; HILDEBRANDT; SAUVIGNY, 2010, p. 111).

Lemma 4.2.2. Let $\mu(w)$ be a holomorphic function and $\nu(w)$ be a meromorphic function in a domain U in $\mathbb{C}$ such that $\mu\nu^2$ is holomorphic. Furthermore, assume that if w is a pole of order n of ν, then w is a zero of order $2n$ of μ, and that these are the only zeros of μ. Then the complex map

$$\varphi = \left(\frac{1}{2}\mu(1-\nu^2), \frac{i}{2}\mu(1+\nu^2), \mu\nu\right)$$

is holomorphic in U and satisfies the conditions of Theorem 4.2.1.

Conversely, every such φ satisfying the conditions of Theorem 4.2.1 can be written in the form above if and only if $\varphi_1 - i\varphi_2 \not\equiv 0$.

If we now consider a minimal surface f given as in Theorem 4.2.1, and suppose that $\varphi_1 - i\varphi_2 \equiv 0$, it follows that

$$-\varphi_3^2 = \varphi_1^2 + \varphi_2^2 = (\varphi_1 - i\varphi_2)(\varphi_1 + i\varphi_2) = 0.$$

Hence

$$l = \langle 2\varphi', N\rangle^{\sim} = 2\langle(\varphi_1', \varphi_2', 0), (0, 0, 2|\varphi|^{-2}\text{Im } \varphi_1\bar{\varphi}_2)\rangle^{\sim} = 0,$$

which shows that $K \equiv 0$, and consequently f must be planar.

Therefore, combining Theorem 4.2.1 with Lemma 4.2.2 we obtain the *Weirstrass-Enneper Representation Formula*:

Theorem 4.2.3. For every nonplanar minimal surface $f : U \to \mathbb{R}^3$ defined on the simply connected open $U \subset \mathbb{C}$, there exist a holomorphic function μ and a meromorphic function ν in U such that $\mu\nu^2$ is holomorphic in U, μ and $\mu\nu^2$ have no common zeros, and such that the formula

$$f(w) = f(w_0) + \text{Re} \int_{w_0}^{w} \mu(\omega)\left(1-\nu(\omega)^2, i(1+\nu(\omega)^2), 2\nu(\omega)\right) d\omega \tag{4.21}$$

holds for arbitrary $w, w_0 \in U$.

Conversely, two complex functions μ and ν defined on a simply connected domain U in $\mathbb{C}$ as above define by means of (4.21) a nonplanar minimal surface $f : U \to \mathbb{R}^3$.

Let $f : U \to \mathbb{R}^3$ be a minimal surface given in the form (4.21). By virtue of (4.10) and using that

$$\varphi_1 = \frac{1}{2}\mu(1-\nu^2), \ \varphi_2 = \frac{i}{2}\mu(1+\nu^2), \ \varphi_3 = \mu\nu,$$

a straightforward computation yields the formula

$$N = \frac{1}{1+|\nu|^2}\left(2\text{Re } \nu, 2\text{Im } \nu, |\nu|^2 - 1\right) \tag{4.22}$$

for the Gauss map of f.

Next, we address the problem of giving a representation formula that only involves an arbitrary complex function instead of two as in (4.21). We start by considering $\bar{\mathbb{C}}$ as the compactified complex plane, that is, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Now, let $\sigma : \mathbb{S}^2 \to \bar{\mathbb{C}}$ be the stereographic projection from the north pole $P = (0,0,1)$, where P is mapped to ∞.

The formula for the stereographic projection is

$$\sigma(x,y,z) = \frac{x+iy}{1-z}, \ \ (x,y,z) \neq P,$$

and

$$\sigma^{-1}(\omega) = \frac{1}{1+|\omega|^2}\left(2\text{Re } \omega, 2\text{Im } \omega, |\omega|^2 - 1\right), \ \omega \neq \infty.$$

Note that, if we compare the formula 4.22 with the expression of σ^{-1}, we see that

$$N(w) = \sigma^{-1}(\nu(w)),$$

hence

$$\nu(w) = \sigma(N(w)). \tag{4.23}$$

In other words, the meromorphic function ν is the stereographic projection of the Gauss map N, hence $w \in U$ is either a singularity or a zero of ν, according to whether the point $N(w) \in \mathbb{S}^2$ is the north or the south pole, respectively.

In a neighborhood of a nonumbilic point of f, after possibly restricting U to a smaller subset and relabeling the coordinate axes if necessary, we can assume that $N : U \to \mathbb{S}^2$ is a diffeomorphism and that $N(w)$ is not parallel to the z-axis, for all $w \in U$. Then, by (4.23) and the inverse function theorem for holomorphic functions, we infer that $\nu : U \to \Omega$ is biholomorphic of U onto some simply connected open $\Omega \subset \mathbb{C} \setminus \{0\}$.

Before we proceed, let us note that if (u_1, u_2) are isothermal coordinates of f and R is a holomorphic function, with $R' \neq 0$, then $R \circ (u_1, u_2)$ are also isothermal coordinates of f, since R is a conformal map.

Then, $\tilde{f} = f \circ \nu^{-1} : \Omega \to \mathbb{R}^3$ is a reparametrization of f in Ω, which is again a minimal surface, and we can assume that the Jacobian of the coordinate change is positive, by interchanging u_1 and u_2, if necessary.

Finally, using the rules for the change of variables for integrals, we obtain from (4.21) the following *representation formula of Weirstrass*:

$$\tilde{f}(\omega)=\tilde{f}(\omega_0)+\mathrm{Re}\int_{\omega_0}^{\omega}F(\zeta)\left(1-\zeta^2,i(1+\zeta^2),2\zeta\right)d\zeta,\ \ \omega,\omega_0\in\Omega, \tag{4.24}$$

where

$$F(\omega):=\frac{\mu(\nu^{-1}(\omega))}{\nu'(\nu^{-1}(\omega))}=(\nu^{-1})'(\omega)\mu(\nu^{-1}(\omega)),\quad \omega\in\Omega.$$

The Gaus map of $\tilde{f}$ is therefore given by

$$\tilde{N}(\omega)=N(\nu^{-1}(\omega))=\sigma^{-1}(\omega)=\frac{1}{1+|\omega|^2}\left(2\mathrm{Re}\ \omega,2\mathrm{Im}\ \omega,|\omega|^2-1\right).$$

Furthermore, we have

$$\begin{aligned}\left\langle\varphi'(\omega),\tilde{N}(\omega)\right\rangle^{\sim}&=\frac{F'(\omega)}{1+|\omega|^2}\left(2\mathrm{Re}\ \omega(1-\omega^2)+2i\mathrm{Im}\ \omega(1+\omega^2)+2\omega(|\omega|^2-1)\right)\\&+\frac{F(\omega)}{1+|\omega|^2}\left(2\mathrm{Re}\ \omega(-2\omega)+2\mathrm{Im}\ \omega(2i\omega)+2(|\omega|^2-1)\right)\\&=\frac{F'(\omega)}{1+|\omega|^2}2\omega\left(1+\omega^2-2\mathrm{Re}\ \omega(\omega)+|\omega|^2-1\right)\\&+\frac{F(\omega)}{1+|\omega|^2}-2\left(1+|\omega|^2\right)\\&=-2F(\omega),\end{aligned}$$

hence

$$\tilde{l}(\omega)=\left\langle 2\varphi'(\omega),\tilde{N}(\omega)\right\rangle^{\sim}=-4F(\omega), \tag{4.25}$$

and, using (4.20), the differential equation of the lines of curvature of $\tilde{f}$ becomes

$$\mathrm{Im}\ F(\omega)(d\omega)^2=0.$$

Conversely, for every nowhere vanishing holomorphic function F defined on a simply connected open subset $\Omega\subset\mathbb{C}\setminus\{0\}$, the formula

$$f(\omega)=\mathrm{Re}\int_{\omega_0}^{\omega}F(\zeta)\left(1-\zeta^2,i(1+\zeta^2),2\zeta\right)d\zeta,\ \ \omega,\omega_0\in\Omega \tag{4.26}$$

defines a minimal surface, since we can choose $\mu(\omega)=F(\omega)$ and $\nu(\omega)=\omega$ in (4.21).

Remark 5. Consider the holomorphic function given by

$$F(\zeta)\equiv k,$$

where $k\in\mathbb{R}$ is a nonzero constant. Substituting F in (4.26) we have

$$\begin{aligned}f(\omega)&=k\mathrm{Re}\left(\omega-\frac{\omega^3}{3},i\left(\omega+\frac{\omega^3}{3}\right),\omega^2\right)+(x_0,y_0,z_0)\\&=k\left(u_1-\frac{u_1^3}{3}+u_1u_2^2,-\left(u_2-\frac{u_2^3}{3}+u_2u_1^2\right),u_1^2-u_2^2\right)+(x_0,y_0,z_0),\end{aligned}$$

where $\omega = u_1 + iu_2$. Hence, the corresponding minimal surface f is, up to an isometry and a homothety of $\mathbb{R}^3$, a piece of the Enneper minimal surface.

Remark 6. We now consider the holomorphic function

$$F(\zeta) = \frac{k}{\zeta^2},$$

for some nonzero constant $k \in \mathbb{R}$. Looking at (4.26), we obtain

$$f(\omega) = k\mathrm{Re}\,\left(-\omega^{-1} - \omega, i(-\omega^{-1} + \omega), 2\log\omega\right) + (x_0, y_0, z_0).$$

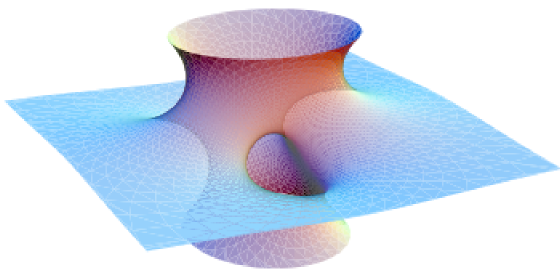
By introducing the new variable $w = \log\omega$, the expression for f can be rewritten as

$$\begin{aligned} f(w) &= k\mathrm{Re}\,\left(-(e^w + e^{-w}), i(e^w - e^{-w}), 2w\right) + (x_0, y_0, z_0) \\ &= k\mathrm{Re}\,\left(-2\cosh w, i2\sinh w, 2w\right) + (x_0, y_0, z_0) \\ &= 2k\left(-\cosh u_1 \cos u_2, -\cosh u_1 \sin u_2, u_1\right) + (x_0, y_0, z_0) \end{aligned}$$

where $w = u_1 + iu_2$. Therefore, in this case, f is a reparametrization of a piece of a catenoid, up to isometries of $\mathbb{R}^3$.

Nitsche (1989, p. 148) lists several other specific examples for the use of the representation formula of Weierstrass. For example, the choice $F(\zeta) = \wp(\zeta)$ (the Weierstrass $\wp$ function) leads to Costa's Surface (Fig. 11).

Figure 11 – Costa's Surface



The Costa surface is a complete minimal embedded surface in $\mathbb{R}^3$. This surface was discovered in 1984 by the Brazilian mathematician Celso José da Costa.

Source: (WEISSTEIN, 2022)

4.3 The classification

Let $f : M^2 \to \mathbb{R}^3$ be a minimal surface free of umbilical points. After composing f with an isometry of $\mathbb{R}^3$, if necessary, the representation formula of Weierstrass allows us to consider

$f:\Omega\to\mathbb{R}^3$ locally represented on the simply connected open subset $\Omega\subset\mathbb{C}\setminus\{0\}$ by means of a nowhere vanishing holomorphic function $F:\Omega\to\mathbb{C}$ in the form

$$f(\omega)=f(\omega_0)+\operatorname{Re}\int_{\omega_0}^{\omega}F(\zeta)\left(1-\zeta^2,i(1+\zeta^2),2\zeta\right)d\zeta,\ \ \omega,\omega_0\in\Omega. \tag{4.27}$$

As we saw in the last section, for a surface given in this form we have

$$N(\omega)=\sigma^{-1}(\omega)\ \text{ or }\ \sigma(N(\omega))=\omega, \tag{4.28}$$

where $\sigma:\mathbb{S}^2\to\mathbb{C}$ is the stereographic projection from the north pole. Moreover,

$$l(\omega)=b_{11}(\omega)-ib_{12}(\omega)=-4F(\omega), \tag{4.29}$$

and the differential equation of the lines of curvature becomes

$$\operatorname{Im}F(\omega)(d\omega)^2=0,$$

which is also equivalent to

$$\operatorname{Re}\sqrt{F(\omega)}d\omega\ \operatorname{Im}\sqrt{F(\omega)}d\omega=0,$$

since $\operatorname{Im}\omega=2\operatorname{Re}\sqrt{\omega}\ \operatorname{Im}\sqrt{\omega}$, for a complex number ω. Thus, by integrating $\operatorname{Re}\sqrt{F(\omega)}d\omega=0$ and $\operatorname{Im}\sqrt{F(\omega)}d\omega=0$, we conclude that the lines of curvature of f are given by the equations

$$\operatorname{Re}\int_{\omega_0}^{\omega}\sqrt{F(\zeta)}d\zeta=const,\ \operatorname{Im}\int_{\omega_0}^{\omega}\sqrt{F(\zeta)}d\zeta=const. \tag{4.30}$$

The preceding equation leads us to introduce principal coordinates which are also isothermal on the minimal surface. Indeed, fix some $\omega_0\in\Omega$ and set

$$R(\omega):=\int_{\omega_0}^{\omega}\sqrt{F(\zeta)}d\zeta,\ \ \omega\in\Omega. \tag{4.31}$$

This defines a holomorphic function

$$\zeta=R(\omega),\ \ \omega\in\Omega.$$

Since $R'(\omega)=\sqrt{F(\omega)}\neq 0$, we may assume that $R:\Omega\to R(\Omega)$ is biholomorphic, after possibly restricting Ω to a smaller subset. Then, (ζ_1,ζ_2) defined by $\zeta=\zeta_1+i\zeta_2$ are isothermal coordinates of $f\circ R^{-1}$, and by (4.30) the coordinate curves $\zeta_1=const$ and $\zeta_2=const$ are its lines of curvature.

We point out that local parameterizations that are both isothermal and by lines of curvature at the same time do not exist on most surfaces. Besides minimal surfaces, other examples of surfaces with this property are quadrics and surfaces with constant mean curvature, among others, see (CANEVARI, 2004). Making use of these special coordinates for minimal surfaces, we are able to prove the following key fact.

Theorem 4.3.1. Let $f : M^2 \to \mathbb{R}^3$ be a minimal surface free of umbilical points. If the lines of curvature of one family are contained in planes, then the same holds for those of the other family.

Proof. Let f be locally parametrized by lines of curvature with isothermal coordinates (u_1, u_2). Then the induced metric can be written as

$$ds^2 = v_1^2(du_1^2 + du_2^2).$$

We have

$$\begin{aligned}\left\langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \frac{\partial f}{\partial u_k} \right\rangle^{\sim} &= \left\langle \tilde{\nabla}_{\partial_{u_i}} f_* \partial_{u_j}, f_* \partial_{u_k} \right\rangle^{\sim} = \left\langle f_* \nabla_{\partial_{u_i}} \partial_{u_j}, f_* \partial_{u_k} \right\rangle^{\sim} \\ &= \langle \nabla_{\partial_{u_i}} \partial_{u_j}, \partial_{u_k} \rangle.\end{aligned}$$

It follows that

$$\left\langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \frac{\partial f}{\partial u_k} \right\rangle^{\sim} = \begin{cases} -\frac{\partial(\log v_1)}{\partial u_k} v_1^2 & \text{if } i = j \neq k, \\ \frac{\partial(\log v_1)}{\partial u_i} v_1^2 & \text{if } j = k, \\ \frac{\partial(\log v_1)}{\partial u_j} v_1^2 & \text{if } k = i \neq j. \end{cases}$$

We also have

$$\left\langle \frac{\partial N}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle^{\sim} = -\langle f_* A \partial_{u_i}, f_* \partial_{u_j} \rangle^{\sim} = -\langle A \partial_{u_i}, \partial_{u_j} \rangle = -b_{ij}.$$

Since ∂_{u_1} and ∂_{u_2} are eigenvectors of the shape operator, we obtain $b_{12} = 0$, and consequently the Codazzi equation implies that $b_{11} = -b_{22}$ is a nonzero constant (see Remark 4). Thus, the Gauss and Weingarten formulas become

$$\begin{aligned}-\frac{\partial^2 f}{\partial u_2^2} = \frac{\partial^2 f}{\partial u_1^2} &= \frac{\partial(\log v_1)}{\partial u_1} \frac{\partial f}{\partial u_1} - \frac{\partial(\log v_1)}{\partial u_2} \frac{\partial f}{\partial u_2} + b_{11} N, \\ \frac{\partial^2 f}{\partial u_1 \partial u_2} &= \frac{\partial(\log v_1)}{\partial u_2} \frac{\partial f}{\partial u_1} + \frac{\partial(\log v_1)}{\partial u_1} \frac{\partial f}{\partial u_2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial N}{\partial u_1} &= -\frac{b_{11}}{v_1^2} \frac{\partial f}{\partial u_1}, \\ \frac{\partial N}{\partial u_2} &= \frac{b_{11}}{v_1^2} \frac{\partial f}{\partial u_2}.\end{aligned}$$

A straightforward computation now yields

$$\begin{aligned}\frac{\partial^3 f}{\partial u_1^3} &= \left[\frac{\partial^2(\log v_1)}{\partial u_1^2} + \left(\frac{\partial(\log v_1)}{\partial u_1} \right)^2 - \left(\frac{\partial(\log v_1)}{\partial u_2} \right)^2 - \frac{b_{11}^2}{v_1^2} \right] \frac{\partial f}{\partial u_1} \\ &+ \left[-\frac{\partial^2(\log v_1)}{\partial u_1 \partial u_2} - 2 \frac{\partial(\log v_1)}{\partial u_1} \frac{\partial(\log v_1)}{\partial u_2} \right] \frac{\partial f}{\partial u_2} + b_{11} \frac{\partial(\log v_1)}{\partial u_1} N\end{aligned}$$

and

$$\frac{\partial^3 f}{\partial u_2^3} = \left[\frac{\partial^2(\log v_1)}{\partial u_2^2} - \left(\frac{\partial(\log v_1)}{\partial u_1}\right)^2 + \left(\frac{\partial(\log v_1)}{\partial u_2}\right)^2 - \frac{b_{11}^2}{v_1^2}\right]\frac{\partial f}{\partial u_2}$$
$$+\left[-\frac{\partial^2(\log v_1)}{\partial u_1 \partial u_2} - 2\frac{\partial(\log v_1)}{\partial u_1}\frac{\partial(\log v_1)}{\partial u_2}\right]\frac{\partial f}{\partial u_1} - b_{11}\frac{\partial(\log v_1)}{\partial u_2}N$$

Fixing an orientation we obtain

$$\frac{\partial f}{\partial u_1} \wedge \frac{\partial^2 f}{\partial u_1^2} = -b_{11}\frac{\partial f}{\partial u_2} - \frac{\partial(\log v_1)}{\partial u_2}v_1^2 N, \quad \frac{\partial f}{\partial u_2} \wedge \frac{\partial^2 f}{\partial u_2^2} = -b_{11}\frac{\partial f}{\partial u_1} + \frac{\partial(\log v_1)}{\partial u_1}v_1^2 N.$$

Now, a simple calculation shows that

$$\left\langle \frac{\partial f}{\partial u_i} \wedge \frac{\partial^2 f}{\partial u_i^2}, \frac{\partial^3 f}{\partial u_i^3} \right\rangle^{\sim} = b_{11} v_1^2 \left(\frac{\partial(\log v_1)}{\partial u_1}\frac{\partial(\log v_1)}{\partial u_2} + \frac{\partial^2(\log v_1)}{\partial u_1 \partial u_2}\right). \tag{4.32}$$

Therefore, the u_1-lines of curvature are planar if and only if its torsion (as a curve in $\mathbb{R}^3$) are everywhere vanishing, which is by (4.32) equivalent to

$$\frac{\partial(\log v_1)}{\partial u_1}\frac{\partial(\log v_1)}{\partial u_2} + \frac{\partial^2(\log v_1)}{\partial u_1 \partial u_2} \equiv 0,$$

and, again by (4.32), this last equation holds if and only if the u_2-lines of curvature are planar. □

We now address the mentioned classification. Let $f : M^2 \to \mathbb{R}^3$ be a minimal surface of Enneper type with one family of planar lines of curvature, and sufficient for the following considerations, let us assume that f is locally represented on some simply connected open subset $\Omega \subset \mathbb{C} \setminus \{0\}$ by (4.27).

By virtue of the preceding Theorem, the Gauss map of f transforms the lines of curvature into an orthogonal system of circles on the unit sphere, see the analysis of the orthogonal system of circles in Chapter 2 and Theorem 2.2.2. Moreover, considering that the Gauss map is given by (4.28), the image by the stereographic projection σ of such circles yield the lines of curvature of f in Ω.

In the case $a = 1$, the equations of the pencils of planes are given by

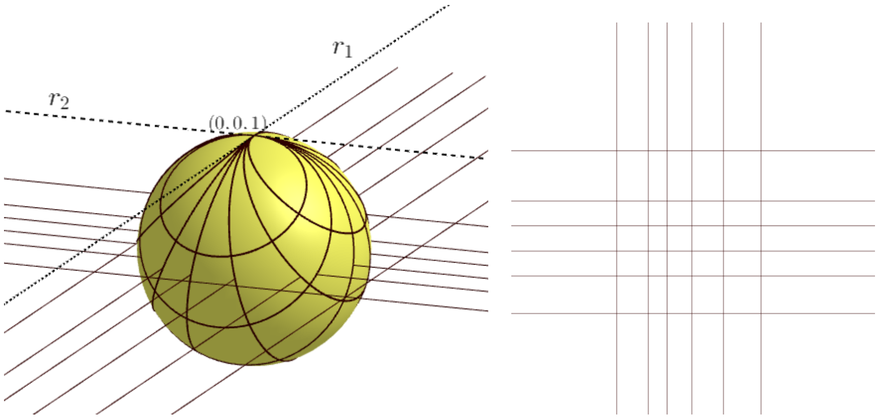
$$x + \lambda(z-1) = 0, \quad \text{and} \quad y + \mu(z-1) = 0, \quad \lambda, \mu \in \mathbb{R}. \tag{4.33}$$

Intersecting the planes in (4.33) with $\mathbb{S}^2$, it follows by applying σ that

$$\omega_1 = \frac{x}{1-z} = \lambda, \quad \text{and} \quad \omega_2 = \frac{y}{1-z} = \mu,$$

whence the lines of curvature corresponding to this orthogonal system of circles are the coordinate curves $\omega_1 = const$ and $\omega_2 = const$ (Fig. 12). Hence, using (4.29) we obtain $\operatorname{Im} F(\omega) = (1/4)b_{12}(\omega) = 0$, which implies that the holomorphic function F must be identically equal to

Figure 12 – Orthogonal system of circles in case $a = 1$.



(a) Orthogonal circles on $\mathbb{S}^2$ (b) Stereographic projection

Source: Elaborated by the author.

a real constant. We then conclude by Remark 5 that the corresponding minimal surface f is a piece of the Enneper minimal surface.

In the case $a = 0$, the line r_1 coincides with the y-axis, and the second pencil, the planes through the line at infinity, must be all the planes parallel to the plane $y = 0$. Since the planes $x = 0$ and $z = 0$ pass through r_1, in this case we have the pencils of planes $x - \lambda z = 0$ and $y = \mu$, or, by relabeling the coordinate axes of $\mathbb{R}^3$,

$$x - \lambda y = 0, \quad \text{and} \quad z = \mu, \ \lambda, \mu \in \mathbb{R}. \tag{4.34}$$

Again, intersecting the planes given in (4.34) with $\mathbb{S}^2$ and by applying σ we obtain

$$\omega_1 = \frac{x}{1-z} = \lambda \frac{y}{1-z} = \lambda \omega_2$$

for the system of circles with respect to r_1, and

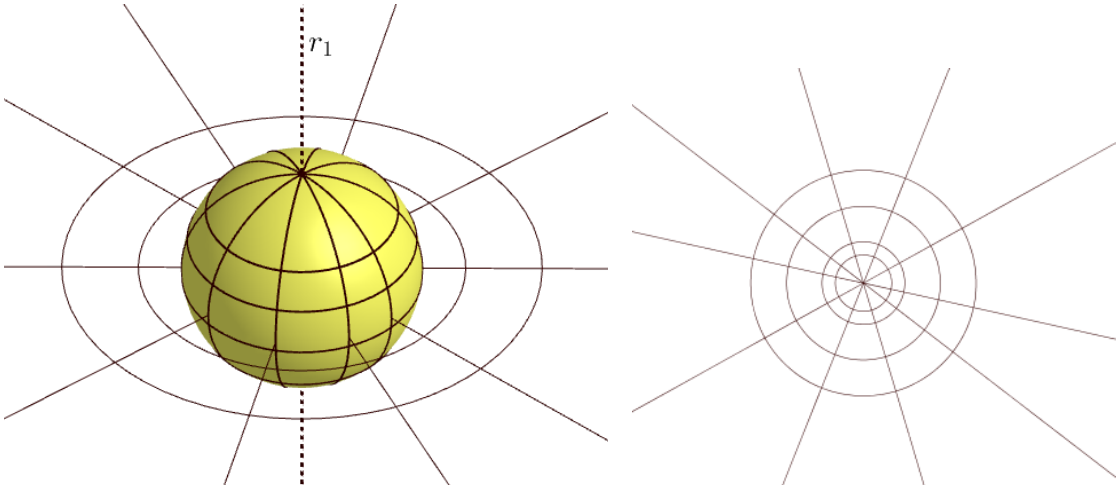
$$\omega_1^2 + \omega_2^2 = \frac{1-\mu^2}{(1-\mu)^2}$$

for the second system (Fig. 13). In other words, the lines of curvature corresponding to this system are the straight lines through the origin and the concentric circles $\omega_1^2 + \omega_2^2 = const$ in Ω.

Recall that these lines of curvature are also described by $\text{Re } R(\omega) = const$ and $\text{Im } R(\omega) = const$, where $R : \Omega \to \mathbb{C}$ is the holomorphic function defined by (4.31). Moreover, since $R' \neq 0$, we obtain that R is a conformal map that takes the straight lines through the origin and the concentric circles centered at the origin that are contained in Ω into coordinate curves.

Next, we consider $L : \Omega \to \mathbb{C}$ a branch of the complex logarithm defined on Ω and the composition $G = R \circ \exp : L(\Omega) \to \mathbb{C}$ of R with the complex exponential. Then, the coordinate curves of $L(\Omega)$ are mapped by the conformal map G into the coordinate curves of Ω. This implies

Figure 13 – Orthogonal system of circles in case $a = 0$.



(a) Meridians and parallels (b) Stereographic projection

Source: Elaborated by the author.

that

$$G(\zeta_1,\zeta_2) = (G_1(\zeta_1), G_2(\zeta_2)), \quad \text{or} \quad G(\zeta_1,\zeta_2) = (G_1(\zeta_2), G_2(\zeta_1)).$$

We have $|\partial G/\partial \zeta_1| = |\partial G/\partial \zeta_2|$, and consequently $G_1'(\zeta_1) = G_2'(\zeta_2)$ or $G_1'(\zeta_2) = \pm G_2'(\zeta_1)$. It follows that

$$G(\zeta_1,\zeta_2) = (k\zeta_1 + \zeta_1^0, \pm k\zeta_2 + \zeta_2^0), \quad \text{or} \quad G(\zeta_1,\zeta_2) = (k\zeta_2 + \zeta_2^0, \pm k\zeta_1 + \zeta_1^0),$$

for some $k, \zeta_1^0, \zeta_2^0 \in \mathbb{R}$. Since G is holomorphic, it remains only the following two possibilities

$$G(\zeta_1,\zeta_2) = (k\zeta_1 + \zeta_1^0, k\zeta_2 + \zeta_2^0), \quad \text{or} \quad G(\zeta_1,\zeta_2) = (-k\zeta_2 + \zeta_2^0, k\zeta_1 + \zeta_1^0),$$

and this can be written as

$$G(\zeta) = k\zeta + \zeta_0, \quad \text{or} \quad G(\zeta) = ki\zeta + \zeta_0.$$

Hence

$$R(\omega) = kL(\omega) + \zeta_0, \quad \text{or} \quad R(\omega) = kiL(\omega) + \zeta_0.$$

Taking into account the fact that $R'(\omega) = \sqrt{F(\omega)}$, we conclude that

$$F(\omega) = \pm\frac{k^2}{\omega^2},$$

for all $\omega \in \Omega$. Therefore, by Remark 6, the corresponding minimal surface f is a piece of a Catenoid.

In the case where $0 < a < 1$, with respect to the parameters of lines of curvature (u_1, u_2) the Gauss map has the expression

$$N(u_1,u_2) = \left(\frac{\sqrt{1-a^2}\sin u_1}{\cosh u_2 + a\cos u_1}, -\frac{\sqrt{1-a^2}\sinh u_2}{\cosh u_2 + a\cos u_1}, \frac{a\cosh u_2 + \cos u_1}{\cosh u_2 + a\cos u_1}\right),$$

and the metric induced by N is

$$ds^2 = (1-a^2)(\cosh u_2 + a\cos u_1)^{-2}(du_1^2 + du_2^2).$$

Since f is a minimal surface and $g_{12} = b_{12} = 0$, we have

$$\frac{\partial N}{\partial u_1} = -\frac{b_{11}}{g_{11}}\frac{\partial f}{\partial u_1},$$
$$\frac{\partial N}{\partial u_2} = \frac{b_{11}}{g_{11}}\frac{\partial f}{\partial u_2},$$

whence it follows that (u_1, u_2) are also conformal parameters and the Codazzi equation now implies that $b_{11} = -b_{22} \equiv \alpha$ is a nonzero constant. Furthermore, we have

$$(1-a^2)(\cosh u_2 + a\cos u_1)^{-2} = \left\langle \frac{\partial N}{\partial u_1}, \frac{\partial N}{\partial u_1} \right\rangle^{\sim} = \frac{-b_{11}}{g_{11}}(-b_{11}),$$

and hence

$$\frac{\partial f}{\partial u_1} = \frac{\alpha}{1-a^2}(\cosh u_2 + a\cos u_1)^2 \frac{\partial N}{\partial u_1}, \quad \frac{\partial f}{\partial u_2} = \frac{-\alpha}{1-a^2}(\cosh u_2 + a\cos u_1)^2 \frac{\partial N}{\partial u_2}.$$

Now, using the expressions for $\partial N/\partial u_1$ and $\partial N/\partial u_2$ obtained in (2.20) and 2.21, respectively, we then conclude that

$$\frac{\partial f}{\partial u_1} = \frac{\alpha}{\sqrt{1-a^2}}\left(\cos u_1 \cosh u_2 + a, -a\sinh u_2 \sin u_1, -\sqrt{1-a^2}\sin u_1 \cosh u_2\right),$$
$$\frac{\partial f}{\partial u_2} = \frac{\alpha}{\sqrt{1-a^2}}\left(\sin u_1 \sinh u_2, 1 + a\cosh u_2 \cos u_1, \sqrt{1-a^2}\sinh u_2 \cos u_1\right).$$

Finally, by integrating and choosing the constant of integration appropriately, we arrive at the following explicit parametric equation for f:

$$f(u_1, u_2) = \frac{\alpha}{\sqrt{1-a^2}} \begin{bmatrix} au_1 + \sin u_1 \cosh u_2 \\ u_2 + a\cos u_1 \sinh u_2 \\ \sqrt{1-a^2}\cosh u_2 \cos u_1 \end{bmatrix}.$$

For $a = 0$, this surface reduces to the Catenoid (here the catenary is rotated about the y-axis), and this agrees with the analysis we made in Theorem 2.2.2. We have therefore proved the following theorem.

Theorem 4.3.2. Let $f : M^2 \to \mathbb{R}^3$ be a minimal surface of Enneper type with one family of planar lines of curvature. Then f is locally, up to isometries and homotheties of $\mathbb{R}^3$, a piece of one, and only one, of

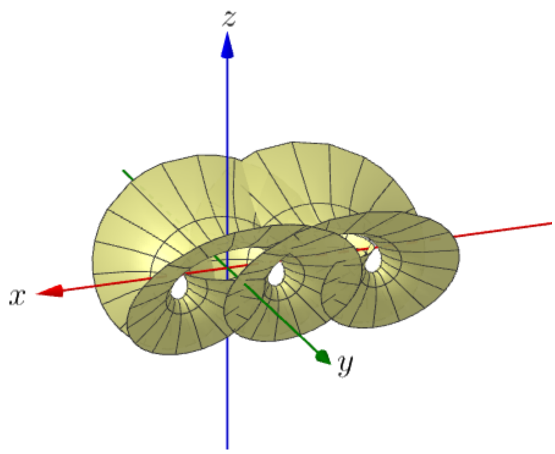
- Enneper minimal surface,
- Catenoid, or

- one surface of the family $\{B_a : \mathbb{R}^2 \to \mathbb{R}^3 ; 0 < a < 1\}$ given by

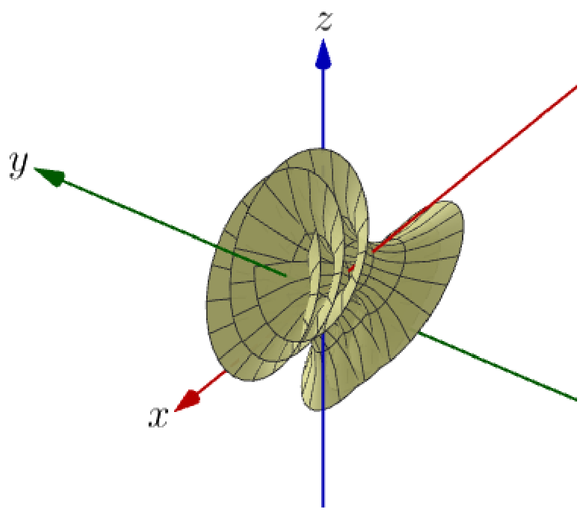
$$B_a(u_1, u_2) = \frac{1}{\sqrt{1-a^2}} \begin{bmatrix} au_1 + \sin u_1 \cosh u_2 \\ u_2 + a \cos u_1 \sinh u_2 \\ \sqrt{1-a^2} \cosh u_2 \cos u_1 . \end{bmatrix} \tag{4.35}$$

According to Nitsche, the family of surfaces (4.35) was discovered by Bonnet (1855) (Fig. 14).

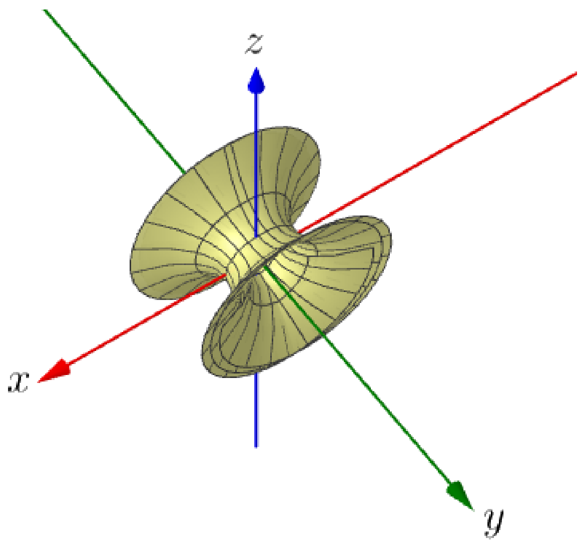
Figure 14 – Bonnet family B_a



(a) $a = 0.5$



(b) $a = 0.2$



(c) $a = 0.05$

Source: Elaborated by the author.

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