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Orbibundles, complex hyperbolic manifolds and geometry over algebras

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Orbibundles, variedades hiperbólicas complexas e geometria sobre álgebras

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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In memory of Sasha Anan′in

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"Whatever happens,
happens"
Spike Spiegel - Cowboy Bebop

RESUMO

BOTÓS, H. C. **Orbibundles, variedades hiperbólicas complexas e geometria sobre álgebras**. 2022. 130 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Esta tese consiste dos trabalhos originais

- Hugo C. Botós, *Orbifolds and orbibundles in complex hyperbolic geometry,* arXiv:2011.09372;
- Hugo C. Botós, Carlos H. Grossi. *Quotients of the holomorphic 2-ball and the turnover,* arXiv:2109.08753;
- Hugo C. Botós, *Geometry over algebras,* arXiv:2203.05101

bem como de uma análise dos principais resultados de cada um deles.

O primeiro estabelece ferramentas básicas sobre orbifolds e orbibundles do ponto de vista da difeologia. O foco é desenvolver ferramentas a serem aplicadas à construção de variedades hiperbólicas complexas.

No segundo trabalho, vários novos exemplos de fibrados de disco (sobre superfícies fechadas) com estruturas hiperbólicas complexas são construídos. Esses fibrados originam-se de orbibundles de discos sobre esferas com três pontos cônicos e, como tais, admitem estrutura hiperbólica complexa não-rígida (deformável). Todos os exemplos obtidos suportam a conjectura de Gromov-Lawson-Thurston.

O último estabelece a teoria de geometrias clássicas para álgebras além dos números reais, complexos e quaternions. Utilizamos tais geometrias para descrever os espaços de geodésicas orientadas do plano hiperbólico, do plano Euclidiano e da 2-esfera redonda. Finalmente, apresentamos uma transição geométrica natural entre tais espaços e construímos um modelo projetivo para a geometria do bidisco hiperbólico (o produto Riemanniano de dois planos hiperbólicos).

Palavras-chave: Geometria hiperbólica complexa, Invariantes discretos, Orbifolds, Difeologia, Álgebras reais.

ABSTRACT

BOTÓS, H. C. **Orbibundles, complex hyperbolic manifolds and geometry over algebras**. 2022. 130 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

This thesis consists of the original works

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as well as an analysis of the main results of each one of them.

The first work introduced basic tools to deal with orbifolds and orbibundles from a diffeological viewpoint. The focus is on developing tools applicable to the construction of complex hyperbolic manifolds.

In the second work, several new examples of disc bundles (over closed surfaces) admitting complex hyperbolic structures are constructed. They originate from disc orbibundles over spheres with three cone points and, as such, admit a non-rigid (deformable) complex hyperbolic structure. All the examples obtained support the Gromov-Lawson-Thurston conjecture.

The latter establishes the theory of classic geometries over algebras beyond real numbers, complex numbers, and quaternions. We use these geometries to describe the spaces of oriented geodesics in the hyperbolic plane, the Euclidean plane, and the round 2-sphere. Finally, we present a natural geometric transition between such spaces and build a projective model for the geometry of the hyperbolic bidisc (the Riemannian product of two hyperbolic planes).

Keywords: Complex hyperbolic geometry, Discrete invariants, Orbifolds, Diffeology, Real algebras.

CONTENTS

CHAPTER

1

INTRODUCTION

The Erlangen program idealized by F. Klein proposes describing geometry, roughly speaking, as spaces endowed with a transitive group action. A **model geometry** is a simply connected manifold on which a Lie group acts transitively with compact stabilizers. The hyperbolic plane, the Euclidean plane, and the 2-sphere are models of two-dimensional geometries; the classification of closed 3-manifolds relies on 8 models (the Thurston geometries), as established in the geometrization conjecture. In dimension four, focusing on the negative curvature case, there are two obvious model geometries: the real hyperbolic 4-space $\mathbb{H}^4_{\mathbb{R}}$, which models the complete manifolds with negative constant curvature, and the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$.

The Kähler structure of the Poincaré disc $\mathbb{H}^1_{\mathbb{C}}$ is the natural geometry having the disc biholomorphisms as its group of orientation preserving isometries. More generally, taking as orientation preserving isometries the biholomorphisms of the unit open ball in $\mathbb{C}^n$, we obtain **the *n*-dimensional complex hyperbolic space** $\mathbb{H}^n_{\mathbb{C}}$. The complex hyperbolic space models complete Kähler manifolds with negative constant holomorphic curvature.

An important question concerning uniformization of 4-manifolds is the **Gromov-Lawson-Thurston conjecture** (GROMOV; LAWSON; THURSTON, 1988) which links the topology and geometry of an oriented disc bundle $M \to S$ over a compact oriented surface S with genus $g > 1$. The conjecture states that the bundle $M \to S$ is real hyperbolic if, and only if, its Euler number e satisfies $|e/\chi| \leq 1$, where $\chi := 2-2g$ is the Euler characteristic of the surface S. The **Euler number** e is the oriented intersection number of two transversal sections and describes the disc bundle up to isomorphism, where the orientation of a section is provided by the base S. More precisely, two transversal sections S_1, S_2 of the disc bundle intersect in a finite number of points $q_1, \ldots, q_k$. If the orientation of $T_{q_i}S_1 \oplus T_{q_i}S_2$ agrees with the orientation of $T_{q_i}M$, then we assign the value $\operatorname{ind}_{q_i} = 1$; otherwise, we define $\operatorname{ind}_{q_i} = -1$. The Euler number e is the sum of all ind_{q_i}, and it does not depend on the choice of transversal sections.

Both sides of the conjecture are still open. The maximal $|e/\chi|$ reached so far is $3/5$, obtained by S. Anan'in and P. V. Chiovetto (ANAN'IN; CHIOVETTO, 2020); for a real hyperbolic disc bundle, it is known that $|e| \leq \exp(\exp(10^8 \chi))$, a result due to M. Kapovich (KAPOVICH, 1993).

The GLT conjecture also seems to hold when one replaces the real hyperbolic 4-space with the complex hyperbolic plane. On the latter, disc bundles $M \to S$ have been found with **relative Euler number** e/χ well distributed in the interval $[-1,1]$, including the cases $e/\chi = -1,0,1$. Thus, there are cotangent, trivial and tangent bundles $M \to S$ which are complex hyperbolic (see (GOLDMAN; KAPOVICH; LEEB, 2001), (ANAN'IN; GROSSI; GUSEVSKII, 2011), and (BOTÓS; GROSSI, 2021)). Strengthening the idea that both conjectures are related is the fact that complex hyperbolic disc bundles with $|e/\chi| \leq 1/3$ are also real hyperbolic (KUIPER, 1988).

A third "hyperbolic" model in dimension four is the hyperbolic bidisc $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$ which has non-positive sectional curvature. There are trivial and tangent bundles $M \to S$ uniformized by the bidisc (COSTA; GROSSI, 2022), suggesting that the GLT-conjecture holds for the hyperbolic bidisc as well.

This thesis consists of the three original articles (Chapters 6, 7, 8) described below as well as of a brief summary of their contents (Chapters 3, 4, 5). We also have a short presentation on classic geometries (Chapter 2) which is an approach (used along the papers) to several model geometries.

The works **Orbifolds and orbibundles in complex hyperbolic geometry** (BOTÓS, 2020) and **Quotients of the holomorphic 2-ball and the turnover** (BOTÓS; GROSSI, 2021) revolve around the complex hyperbolic GLT conjecture. The first lays down a framework for the construction of complex hyperbolic disc orbibundles over 2-orbifolds. In the latter, we construct wide families of complex hyperbolic orbibundles which can be pulled back to examples supporting the complex GLT conjecture. It is important to point out that, generically, the orbibundles obtained in (BOTÓS; GROSSI, 2021) are non-rigid; this means its complex hyperbolic structure can be deformed without changing its topological/smooth structure.

The work **Geometry over algebras** (BOTÓS, 2022) goes beyond complex hyperbolic geometry. It extends the theory of (projective) classic geometries, as introduced in (ANAN'IN; GROSSI, 2011), to non-division algebras. We are therefore able to describe natural geometric structures on the spaces of oriented geodesics in the hyperbolic plane, the Euclidean plane, and the round 2-sphere. Moreover, it is possible to transit between these three spaces of oriented geodesics by naturally embedding them into the split-quaternionic projective line. We also found a natural projective model for the hyperbolic bidisc (the Riemannian product of two hyperbolic planes). So, the real hyperbolic 4-space, the complex hyperbolic plane, and the hyperbolic bidisc admit projective/linear models.

We believe that the three mentioned versions of the GLT conjecture are somehow linked. Transiting between the corresponding projective models of $\mathbb{H}^4_{\mathbb{R}}$, $\mathbb{H}^2_{\mathbb{C}}$, and $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$ inside an ambient classic geometry may provide insight into such possible links.

CHAPTER

2

CLASSIC GEOMETRIES

A classic geometry arises from $\mathbb{K}$-linear space V endowed with a Hermitian $\langle \cdot, \cdot \rangle$, where the standard $\mathbb{K}$'s are $\mathbb{R}$ and $\mathbb{C}$ (and, sometimes, quaternions). Examples of classic geometries include the hyperbolic (real, complex, quaternionic), Fubini-Study, de Sitter and anti-de Sitter geometries. One of the advantages of the classic geometries approach is that usual Riemannian concepts/objects like geodesics, distance, Levi-Civita connection, curvature, area, etc can be described using only linear algebra, thus providing a simple and effective computational environment (see, for instance, (ANAN'IN; GROSSI, 2011) and (GOLDMAN, 1999)).

2.1 Basic definitions and results

Consider the projective space $\mathbb{P}_{\mathbb{K}}(V)$, where we respectively denote by $\boldsymbol{p}$ and p a point $\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}}(V)$ and a representative $p \in V$. The Hermitian form divides the projective space into three parts:

$$B(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p \rangle < 0\}, \quad E(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p \rangle > 0\},$$

$$S(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p \rangle = 0\}.$$

The choice of letters representing these spaces comes from the hyperbolic model (and special relativity), where $B(V)$ is a ball, $S(V)$ is a sphere, and $E(V)$ is "elsewhere". The points in $S(V)$ are usually called isotropic or singular points.

The tangent space at a point $\boldsymbol{p} \notin S(V)$ is naturally identified with $\mathrm{lin}_{\mathbb{K}}(\mathbb{K}p, p^{\perp})$, as it is done in (ANAN'IN; GROSSI, 2011, Remark 2.3), and the Hermitian form on V induces a Hermitian metric

$$\langle t_1, t_2 \rangle_{\boldsymbol{p}} = -\frac{\langle t_1(p), t_2(p) \rangle}{\langle p, p \rangle},$$

a pseudo-Riemannian metric

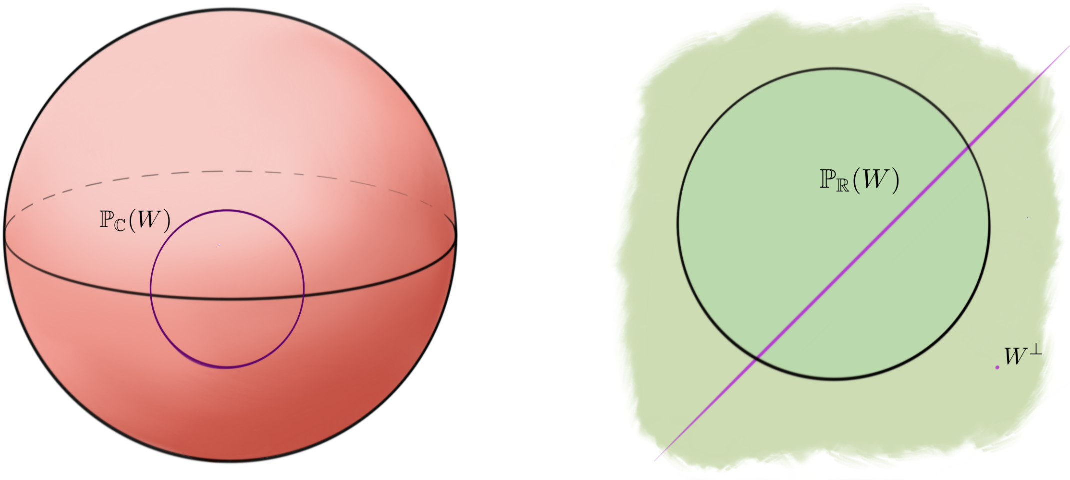
$$g(t_1, t_2)_{\boldsymbol{p}} = \mathrm{Re}\langle t_1, t_2 \rangle_{\boldsymbol{p}},$$

and, in the complex case, a symplectic form

$$\omega(t_1,t_2)_{\boldsymbol{p}} = \operatorname{Im}\langle t_1,t_2\rangle_{\boldsymbol{p}},$$

where $t_1,t_2 \in T_{\boldsymbol{p}}\mathbb{P}(V)$ (depending on the case, one can also take the plus sign in the definition of the Hermitian metric). The Hermitian metric, the pseudo-Riemannian metric and the symplectic form are defined on $B(V) \sqcup E(V)$.

Every geodesic of the pseudo-Riemannian metric is a component of $\mathbb{P}_{\mathbb{K}}(W) \simeq \mathbb{P}_{\mathbb{R}}(W)$, where W is a real 2-dimensional vector subspace of V such $\langle -,-\rangle|_{W\times W}$ is real-valued and non-zero (see (ANAN'IN; GROSSI, 2011, Section 3)).

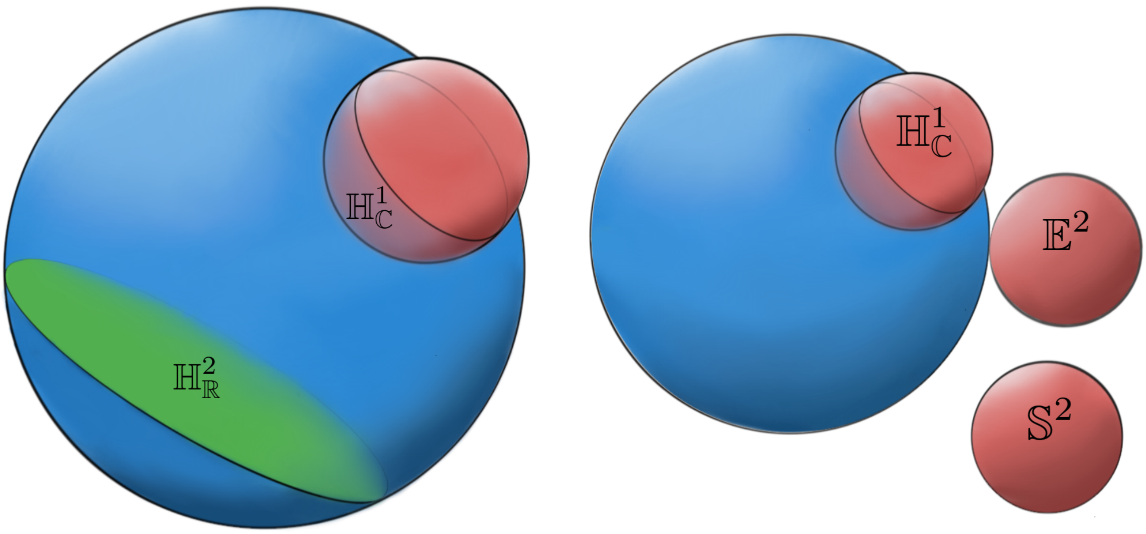


(a) (b)

Figure 1 – **(a)** The Riemann-Poincaré sphere and **(b)** the Beltrami-Klein plane.

Figure 1 (a) shows the **Riemann-Poincaré sphere** obtained from a 2-dimensional $\mathbb{C}$-linear space V endowed with a Hermitian form of signature $-+$. Both $B(V)$ and $E(V)$ are models for the Poincaré disc $\mathbb{H}^1_{\mathbb{C}}$ with curvature -4. The absolute $S(V)$ is the circle forming the "equator" where the two discs are glued. The purple circle is a geodesic of the model, half of it being a geodesic of $B(V)$ and, the other half, a geodesic of $E(V)$. Figure 1 (b) is the **Beltrami-Klein plane** obtained from a 3-dimensional $\mathbb{R}$-linear space with a Hermitian form of signature $-++$. The disc $B(V)$ is the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ with curvature -1 and $E(V)$ is a Möbius strip with a Lorentzian metric, called the de Sitter plane. The purple line is a geodesic of the model and the point with the same color is the associated point obtained by the line-plane duality in V (the plane W is dual to the line $W^{\perp}$). From this observation, we conclude that the space of non-oriented geodesics on the hyperbolic plane is the de Sitter plane.

The projective model for the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ is $B(V)$ for a 3-dimensional $\mathbb{C}$-linear space V with a Hermitian form of signature $-++$, see Figure 2 (a). Again, by the line-plane duality in V, we can see that $E(V)$ is the space of all complex projective lines intersecting $\mathbb{H}^2_{\mathbb{C}}$. Each of these projective lines is a Riemann-Poincaré disc whose component in $\mathbb{H}^2_{\mathbb{C}}$ is a Poincaré disc called a **complex geodesic**. The Beltrami-Klein model $\mathbb{H}^2_{\mathbb{R}}$ embeds in $\mathbb{H}^2_{\mathbb{C}}$ as the projectivizations of real three-dimensional subspaces W of V such that $\langle -,-\rangle|_{W\times W}$ is real-valued of signature $-++$. Complex geodesics and Beltrami-Klein planes are the only non-trivial totally geodesic subspaces of the complex hyperbolic plane. Note that, despite being isometric (up to scaling), they behave quite differently: the first is an embedded Riemann surface while the latter is a Lagrangian submanifold.



(a) (b)

Figure 2 – **(a)** $\mathbb{H}^2_{\mathbb{C}}$ with two types of hyperbolic planes and **(b)** the transition between hyperbolic, Euclidean and elliptic geometries.

In general, the point dual to a projective hyperplane will be called the **polar** of the hyperplane. From the duality between points and projective lines in $\mathbb{H}^2_{\mathbb{C}}$, it is easy to see the transition between the three 2-dimensional model geometries inside $E(V)$. A projective line is a Riemann-Poincaré sphere if its polar point is positive and a round sphere if its polar point is negative. In the case where the polar point $\boldsymbol{p}$ is isotropic, the corresponding projective line $L := \mathbb{P}_{\mathbb{C}}(p^{\perp})$ is a 2-sphere tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at $\boldsymbol{p}$ (note that $\boldsymbol{p} \in L$). Despite the metric restricted to the plane $L \setminus \{\boldsymbol{p}\}$ being null, we think of such plane as a Euclidean plane (see (GROSSI, 2006, Observação 1.5.4)) because the group of Euclidean isometries acts naturally on such space (see Figure 2 (b)).

In general, the real (complex) hyperbolic n-dimensional space is obtained via projectization of a $n+1$ dimensional real (complex) vector space V endowed with a Hermitian form of signature $-+\cdots+$. Note that, for the complex case, the number n stands for the complex dimension and, therefore, the real dimension is $2n$. The real and the complex hyperbolic spaces are denoted by $\mathbb{H}^n_{\mathbb{R}}$ and $\mathbb{H}^n_{\mathbb{C}}$, respectively. The distance between two points $\boldsymbol{p},\boldsymbol{q}$ in the real (complex) hyperbolic space is given by the formula

$$d(\boldsymbol{p},\boldsymbol{q}) = \operatorname{arccosh}(\sqrt{\operatorname{ta}(\boldsymbol{p},\boldsymbol{q})}),$$

where the **tance** $\mathrm{ta}(\cdot,\cdot)$ is defined as

$$\mathrm{ta}(\boldsymbol{p},\boldsymbol{q}) := \frac{\langle p,q\rangle\langle q,p\rangle}{\langle p,p\rangle\langle q,q\rangle}$$

(see (ANAN'IN; GROSSI; GUSEVSKII, 2011, Subsection 2.1) or (ANAN'IN; GROSSI, 2011, Subsection 3.2)).

Since distance is a monotone function of tance, we can use the latter in order to compare distances. Moreover, being algebraic, the tance is easier to manipulate and implement computationally than the distance. The tance appears in several places in the study of classic geometries, making it the perfect substitute for the distance function.

The following result provides the relative position of the lines in terms of the tance of their polar points (see (ANAN'IN; GROSSI; GUSEVSKII, 2011, Lemma 4.1.7)). Consider the vector space $\mathbb{K}^3$ over $\mathbb{K}=\mathbb{R},\mathbb{C}$ endowed with the canonical Hermitian form of signature $-++$. Given projective lines $L_1 := \mathbb{P}_{\mathbb{K}}(q_1^{\perp})$ and $L_2 := \mathbb{P}_{\mathbb{K}}(q_2^{\perp})$ with positive polar points $\boldsymbol{q}_1,\boldsymbol{q}_2 \in E(V)$ (i.e., the projective lines intersect the hyperbolic plane $\mathbb{H}^2_{\mathbb{K}}$), we have the following configurations:

- If $\mathrm{ta}(\boldsymbol{q}_1,\boldsymbol{q}_2) > 1$, the lines L_1, L_2 are ultraparallel, that is, they do not intersect in $\mathbb{H}^2_{\mathbb{K}}$ and their distance is positive; this distance equals $\operatorname{arccosh}(\sqrt{\mathrm{ta}(\boldsymbol{q}_1,\boldsymbol{q}_2)})$;
- If $\mathrm{ta}(\boldsymbol{q}_1,\boldsymbol{q}_2) = 1$, the lines L_1, L_2 are asymptotic, that is, they do not intersect in $\mathbb{H}^2_{\mathbb{K}}$ and their distance is zero;
- If $\mathrm{ta}(\boldsymbol{q}_1,\boldsymbol{q}_2) < 1$, the lines L_1, L_2 intersect at exaclty one point in $\mathbb{H}^2_{\mathbb{K}}$ and their intersection angle equals $\arccos(\sqrt{\mathrm{ta}(\boldsymbol{q}_1,\boldsymbol{q}_2)})$.

For $\mathbb{K}=\mathbb{R}$, the open segment $L_i \cap \mathbb{H}^2_{\mathbb{R}}$ is a geodesic in the real hyperbolic plane. For $\mathbb{K}=\mathbb{C}$, the open disc $L_i \cap \mathbb{H}^2_{\mathbb{R}}$ is a complex geodesic, a Poincaré disc.

2.2 Curvature

The Levi-Civita connection and its Riemann curvature tensor admit very elegant algebraic expressions in the classic geometry framework. To provide them, we need a special type of vector field called spread vector field. These vector fields simplify many computations and can be seen as being analogous to left-invariant vector fields in Lie groups.

For $\boldsymbol{x} \in \mathbb{P}_{\mathbb{K}}(V) \setminus S(V)$, we have the projection in the direction of x

$$\pi'[\boldsymbol{x}]v := \frac{\langle v,x\rangle}{\langle x,x\rangle}x,$$

and the projection on the hyperplane perpendicular to x

$$\pi'[\boldsymbol{x}]v := v - \pi'[\boldsymbol{x}]v.$$

Consider a tangent vector $t : \mathbb{K}p \to p^\perp$ at $\boldsymbol{p} \in \mathbb{P}_{\mathbb{K}}(V) \setminus S(V)$. By defining t as being null on $p^\perp$, we extend t to a map $t : V \to V$. Conversely, every linear map $t : V \to V$ such that $t(\mathbb{K}p) \subset p^\perp$ and $t(p^\perp) = 0$ can be seen as a tangent vector at $\boldsymbol{p}$. Given a tangent vector $t : V \to V$ at $\boldsymbol{p}$, define the smooth vector field

$$T(\boldsymbol{x}) = \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}]$$

on $\mathbb{P}(V) \setminus S(V)$. Indeed, $T(\boldsymbol{x})$ maps $\mathbb{K}x$ to $x^\perp$ and vanishes over $x^\perp$. We say that T is the vector field **spread from** t. Observe that $T(\boldsymbol{p}) = t$.

If t_1, t_2 are tangent vectors at $\boldsymbol{p}$, then the commutator of their spread vector fields T_1, T_2 vanishes at $\boldsymbol{p}$, i.e., $[T_1, T_2](\boldsymbol{p}) = 0$ (see (ANAN'IN; GROSSI, 2011, Proposition 4.4) and (BOTÓS, 2022, Proposition 36)). For the Levi-Civita connection we have the expression

$$\nabla_T S(\boldsymbol{x}) = \pi[\boldsymbol{x}] \circ s \circ \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}] - \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}] \circ s \circ \pi'[\boldsymbol{x}],$$

where T, S are the vector fields spread from the vectors $t, s \in T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{K}}(V)$ (see (ANAN'IN; GROSSI, 2011, Lemma 4.3) and (BOTÓS, 2022, Proposition 35)). Additionally, $\nabla_T S(\boldsymbol{p}) = 0$.

Now we analyse the Riemann curvature tensor. First, given $t \in T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{K}}(V)$, we extend it to a linear endomorphism of V as discussed above. Since V is endowed with a Hermitian form, the linear map $t : V \to V$ admits an adjoint[1] $t^* : V \to V$ given by

$$t^*(v) = \frac{\langle v, p \rangle}{\langle p, p \rangle} t(p)$$

which satisfies the identities $\langle tu, v \rangle = \langle u, t^* v \rangle$ for every $u, v \in V$.

For $t_1, t_2, s \in T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{K}}(V)$, the Riemann curvature tensor[2] is given by

$$R(t_1, t_2)s = -s \circ (t_1^* \circ t_2 - t_2^* \circ t_1) + (t_1 \circ t_2^* - t_2 \circ t_1^*) \circ s.$$

For details, see (ANAN'IN; GROSSI, 2011, Subsection 4.5).

Finally, using the metric $g := \mathrm{Re}\langle \cdot, \cdot \rangle$, we have the sectional curvature

$$K(W) := \frac{g(R(t_1, t_2)t_2, t_1)}{g(t_1, t_1)g(t_2, t_2) - g(t_1, t_2)^2},$$

where W is a real two-dimensional subspace of the tangent space $T_{\boldsymbol{p}}\mathbb{P}(V)$ of $\boldsymbol{p}$ such that $g|_{W \times W}$ is non-degenerate and t_1, t_2 is a basis of W.

Using these tools, we have, for instance, that the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ always has curvature -1, the Poincaré disc $\mathbb{H}^1_{\mathbb{C}}$ has curvature -4 and the other complex hyperbolic spaces have curvature varying in the interval $[-4, -1]$. Furthermore, in the complex hyperbolic case, the curvature is -1 exactly when W is a tangent plane of a Beltrami-Klein plane $\mathbb{H}^2_{\mathbb{R}}$ and -4 when it is tangent to a Poincaré disc $\mathbb{H}^1_{\mathbb{C}}$. For more details, see (ANAN'IN; GROSSI, 2011, Subsection 4.6).

[1] If the Hermitian form is non-degenerate, the adjoint is unique.

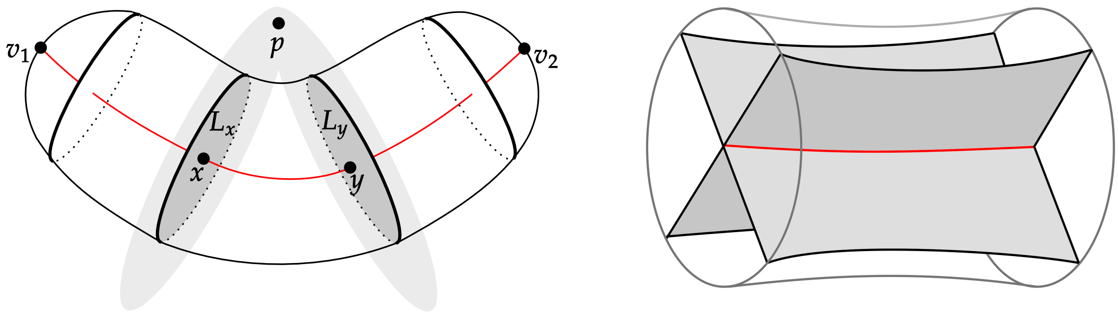
[2] We adopt the convention $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

2.3 Bisectors

The usual way of constructing complex hyperbolic manifolds is via tesselation of the corresponding hyperbolic space by copies of a fundamental domain for the action of some discrete group of isometries. In (BOTÓS; GROSSI, 2021), the fundamental domains we construct in $\mathbb{H}^2_{\mathbb{C}}$ are bounded by bisectors, which we discuss in what follows.

Geometrically, a bisector is a hypersurface in $\mathbb{H}^2_{\mathbb{C}}$ which is equidistant from a pair of given points. However, we will not work with such definition and will use a more algebraic one instead.

Let $G = \mathbb{P}_{\mathbb{C}}(W) \cap \mathbb{H}^2_{\mathbb{C}}$ be a geodesic in the complex hyperbolic plane and $\boldsymbol{p}$ be the polar point of the complex geodesic $\mathbb{P}_{\mathbb{C}}(\mathbb{C}W) \cap \mathbb{H}^2_{\mathbb{C}}$, i.e., $p^{\perp} = \mathbb{C}W$. The **bisector** generated by the geodesic G is the hypersurface $B := \mathbb{P}_{\mathbb{C}}(W + \mathbb{C}p) \cap \mathbb{H}^2_{\mathbb{C}}$. The set B is a topological cylinder foliated by the complex geodesics $L_{\boldsymbol{x}}$ orthogonal to G (see Figure 3 (a)) at $\boldsymbol{x} \in G$. More precisely, $L_{\boldsymbol{x}} := \mathbb{P}_{\mathbb{C}}(\mathbb{C}x + \mathbb{C}p) \cap \mathbb{H}^2_{\mathbb{C}}$, $\boldsymbol{x} \in G$. These complex geodesics are called the **slices** of the bisector. Alternatively, the bisector can be seen as the set of Beltrami-Klein discs (Figure 3 (b)) sharing the geodesic G; each of these discs is called a **meridian** of the bisector.



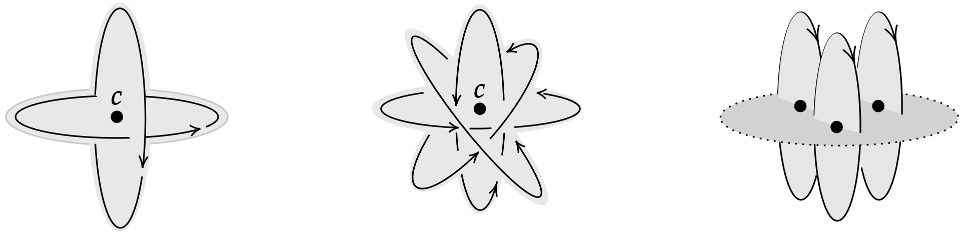
(a) (b)

Figure 3 – **(a)** The bisector foliated by complex geodesics and **(b)** the meridional decomposition.

Two ultraparallel complex geodesics C_1, C_2 determine a unique bisector B which contains them as slices, and the segment of bisector connecting C_1, C_2 is denoted $B[C_1, C_2]$.

2.4 Elliptic isometries

The orientation preserving isometries of the complex hyperbolic plane form the subgroup $\mathrm{PU}(2,1)$ of $\mathrm{PGL}(3,\mathbb{C})$. An **elliptic** isometry is a non-identity element in $\mathrm{PU}(2,1)$ with a fixed point in $\mathbb{H}^2_{\mathbb{C}}$. An elliptic isometry is **regular elliptic** if, as an element of $\mathrm{SU}(2,1)$, it has pairwise distinct eigenvalues; otherwise, the isometry is **special elliptic**. Observe that, by definition, the regular case is the generic one. Furthermore, only two eigenvalues of a special isometry can be equal, because an elliptic isometry cannot be the identity.



(a) (b) (c)

Figure 4 – **(a)** Rotations about c on two orthogonal complex geodesics by distinct angles, **(b)** Rotation about point, and **(c)** Rotation about complex line.

Geometrically, if an isometry I is regular elliptic with fixed points $\boldsymbol{c}, \boldsymbol{p}, \boldsymbol{q}$, where $\boldsymbol{c}$ is negative and the other two are positive, we have exactly two stable projective lines, $\mathbb{P}(\mathbb{C}c \oplus \mathbb{C}p)$ and $\mathbb{P}(\mathbb{C}c \oplus \mathbb{C}q)$. In this case, the isometry restricted to each of the corresponding complex geodesics is a rotation about the center $\boldsymbol{c}$ (see Figure 4 (a)). On the other hand, if the isometry is special elliptic, we have two cases:

- If the eigenvalues of any two positive fixed points of I are equal, then the isometry has only one negative fixed point $\boldsymbol{c}$, every complex geodesic passing through $\boldsymbol{c}$ is stable (but not fixed), and the angle of rotation of I restricted to any of these complex geodesics is the same. This type of special elliptic isometry is called a rotation about a point. See Figure 4 (b);
- If there are two distinct positive fixed points with distinct eigenvalues, then there is an I-fixed complex geodesic. Furthermore, the isometry acts as a rotation about this complex geodesic. We say that the described special elliptic isometry is a rotation about a complex geodesic. See Figure 4 (c).

In all cases, the angles of rotations can be computed directly from the eigenvalues (BOTÓS; GROSSI, 2021, Subsection 2.4).

CHAPTER

3

QUOTIENTS OF THE HOLOMORPHIC 2-BALL AND THE TURNOVER

3.1 The turnover and its PU(2,1)-character variety

In this chapter, we briefly present the results discussed in (BOTÓS; GROSSI, 2021), where the $\mathrm{PU}(2,1)$-character variety of the simplest good 2-orbifolds with negative Euler characteristics is analyzed. Moreover, several families of disc bundles supporting the complex GLT conjecture are constructed; generically, these families come from non-rigid examples of complex hyperbolic disc orbibundles. Note that, in (BOTÓS; GROSSI, 2021), we use several constructions at the level of orbifolds, like orbibundles, Euler number of disc orbibundles, and Toledo invariant for representations of orbifolds groups. These tools are defined and thoroughly studied in (BOTÓS, 2020), discussed in Chapter 4.

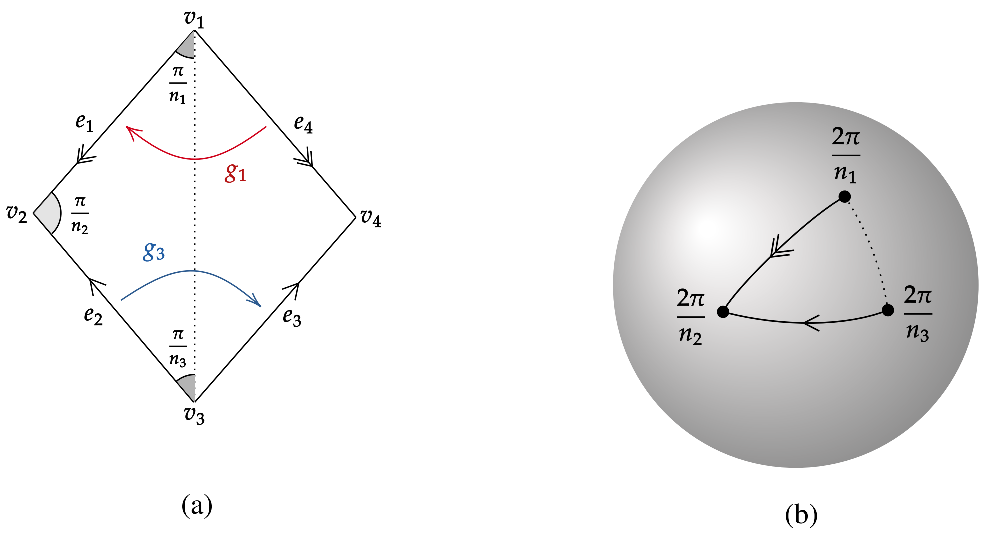


Figure 5 – **(a)** Fundamental domain for G and **(b)** Orbifold $\mathbb{S}^2(n_1,n_2,n_3)$

Consider the hyperbolic orbifold $\mathbb{S}^2(n_1,n_2,n_3)$, the sphere with cone points of angles $2\pi/n_1$, $2\pi/n_2$, $2\pi/n_3$. By Gauss-Bonnet, its Euler characteristic is

$$\chi = -1 + \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3},$$

which must be negative.

This is the simplest compact oriented hyperbolic 2-orbifold; it can be obtained by tesselating the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ with the fundamental domain given by the quadrilateral in Figure 5 under the action of the Fuchsian group

$$G(n_1,n_2,n_3) := \langle g_1, g_2, g_3 | g_3 g_2 g_1 = g_1^{n_1} = g_2^{n_2} = g_3^{n_3} = 1 \rangle$$

called the **turnover group**. Note that g_i is a rotation about v_i of angle $2\pi/n_i$.

The $\mathrm{PU}(2,1)$-character variety $\mathscr{R}(n_1,n_2,n_3)$ of the turnover group $G(n_1,n_2,n_3)$ is the space of all faithful representations $\rho : G(n_1,n_2,n_3) \to \mathrm{PU}(2,1)$ modulo $\mathrm{PU}(2,1)$-conjugation. Whenever possible, we write G and $\mathscr{R}$ instead of $G(n_1,n_2,n_3)$ and $\mathscr{R}(n_1,n_2,n_3)$.

Let $\rho : G \to \mathscr{R}$ be a faithful representation and let $I_k := \rho(g_k)$. The isometries I_1, I_2, I_3 are elliptic, i.e., they have a fixed point in $\mathbb{H}^2_{\mathbb{C}}$, because they have finite order. In what follows, we think of an orientation-preserving isometry of $\mathbb{H}^2_{\mathbb{C}}$ isometries as an element of $\mathrm{SU}(2,1)$. Remember that an elliptic isometry is regular if its eigenvalues are pairwise distinct; otherwise, it is special (see Section 2.4). Generically, the region in the character variety $\mathscr{R}$ corresponding to faithful representations where I_1, I_2, I_3 are regular elliptic is 2-dimensional.

Definition 3.1. Let ρ be a faithful representation where none of the I_k's is special elliptic. We call the representation *generic* if there exists $i \neq j$ such that the fixed points of I_i and I_j are pairwise non-orthogonal.

Proposition 3.2 (Proposition 8, (BOTÓS; GROSSI, 2021))**.** *Let $\rho : G \to \mathrm{PU}(2,1)$ be a faithful representation. If exactly one of the I_j's is special elliptic, then ρ is rigid. Assume that none of the I_j's is special elliptic. If ρ is generic, the corresponding component of $\mathscr{R}$ has dimension* 2*; otherwise, the dimension is bounded by* 1.

If two of the isometries I_j's are special, the relation $I_3 I_2 I_1 = 1$ implies that the third one is necessarily special as well; in this case, the representation has a stable complex projective line.

Let us sketch the proof of Proposition 3.2. We will need the following well-known Goldman's result: An isometry $I \in \mathrm{SU}(2,1)$ is regular elliptic only when $f(\mathrm{tr}(I)) < 0$, where $f(z) := |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27$ (see (GOLDMAN, 1999, p. 204, Theorem 6.2.4)). Therefore, an isometry is regular elliptic if it has the trace of a regular elliptic isometry.

Consider a faithful representation $\rho : G(n_1,n_2,n_3) \to \mathrm{PU}(2,1)$ and let $I_j := \rho(g_j)$. Up to choosing representatives, we may assume that I_1, I_2, and I_3 are elements of $\mathrm{SU}(2,1)$ satisfying

$I_3I_2I_1 = 1$. Proposition 3.2 assumes that two of the I_j's are regular elliptic. Thus we take I_1 and I_3 to be regular elliptic.

Let $\alpha_i, \beta_i, \gamma_i^{-1}$, with $i = 1,2,3$, be the eigenvalues of I_1, I_2, I_3 respectively, where $\alpha_1, \beta_1, \gamma_1^{-1}$ are the eigenvalues corresponding to negative eigenvectors. The assumption that I_1 and I_3 are regular elliptic means that the α_i's are pairwise distinct; the same holds for the γ_i's. Since we are interested in representations modulo $\mathrm{SU}(2,1)$-conjugation (our concern is describing the character variety), we may assume that I_1 is diagonal. Besides, $I_3 = I_1^{-1}I_2^{-1}$ expresses I_3 as a function of I_1, I_2.

Therefore, the representation ρ modulo $\mathrm{PU}(2,1)$ is described by an elliptic isometry $I_2 \in \mathrm{SU}(2,1)$ with eigenvalues $\beta_1, \beta_2, \beta_3$ such that I_2I_1 is regular elliptic with eigenvalues $\gamma_1, \gamma_2, \gamma_3$, where I_1 is the diagonal matrix $\operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$. By Goldman's theorem, the condition over I_2I_1 can be replaced by the equation $\operatorname{tr}(I_2I_1) = \gamma_1 + \gamma_2 + \gamma_3$. By continuity, every representation near ρ is prescribed by the same equation, since the eigenvalues as functions of the representation are locally constant. Therefore, finding representations near ρ is the same as finding I_2's with eigenvalues $\beta_1, \beta_2, \beta_3$ satisfying the trace equation.

If the I_j's are not all regular, then I_2 is special elliptic, since I_1, I_3 are already taken to be regular elliptic. In this case, there are at most finitely many I_2's solving the above trace equation, and the representation ρ is therefore rigid (an isolated point in the character variety).

Otherwise, in the case when I_2 is regular elliptic, we have two variants. If the given representation is not generic, then there is at most a one-dimensional family of I_2's satisfying the trace equation. Otherwise, if the given representation is generic, then there is a two-dimensional family of I_2's solving the trace equation, proving the proposition.

The I_2 matrices, in the generic case, are parameterized by two real positive numbers s, t. These numbers have geometrical meaning. Let $L := \mathbb{P}(\mathbb{C} \times 0 \times \mathbb{C})$ and $L' = \mathbb{P}(\mathbb{C} \times \mathbb{C} \times 0)$ be the two positive complex geodesics stable under I_1 and let $\boldsymbol{p}$ be the fixed point of I_2 in $\mathbb{H}^2_{\mathbb{C}}$. We have $1 + s = \operatorname{ta}(\boldsymbol{p}, L)$ and $1 + t = \operatorname{ta}(\boldsymbol{p}, L')$. Thus, s and t measure the “distances" between the center of the isometry I_2 and the stable complex geodesics of I_1 (see (BOTÓS; GROSSI, 2021, Section 5)).

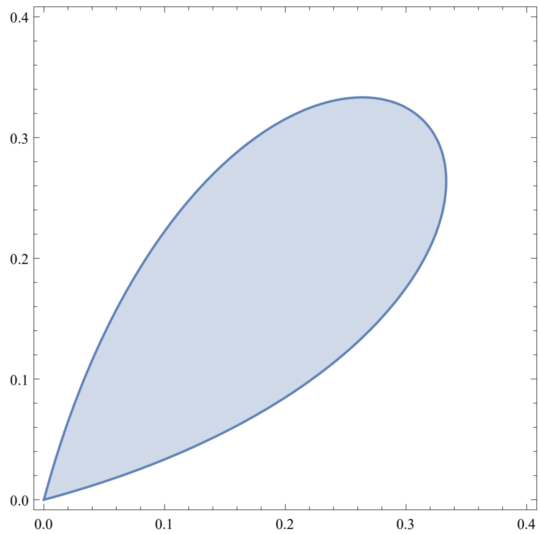
3.2 Computational results

Before we discuss how the disc orbibundles over $\mathbb{S}^2(n_1, n_2, n_3)$ are constructed in (BOTÓS; GROSSI, 2021), we present a few important characteristics of such bundles which were discovered via computer-assisted procedures.

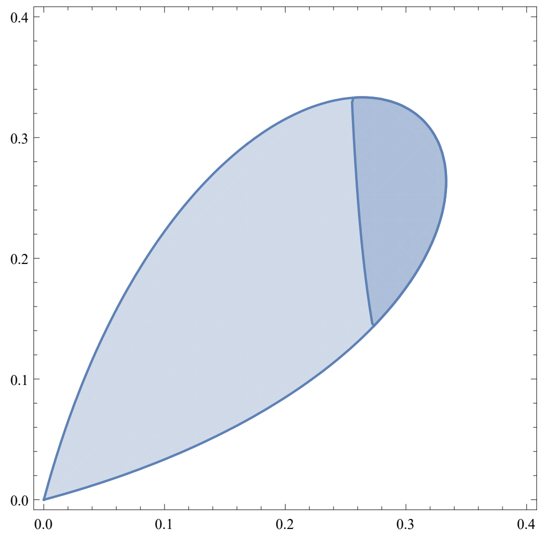
We searched for generic representations $G(n_1, n_2, n_3) \to \mathrm{PU}(2,1)$ with $3 \le n_1, n_2, n_3 \le 12$. For 533 triplets (n_1, n_2, n_3), we established the existence of the two dimensional region described in Proposition 3.2. For each of these regions, we explicitly found two-dimensional families of pairwise non-isometric complex hyperbolic disc orbibundles over $\mathbb{S}^2(n_1,n_2,n_3)$. The corresponding relative Euler numbers $e_r := e/\chi$ (see Chapter 4) vary in the interval $[-1,0.5)$. They assume values which are well-spread in such interval and include the trivial bundle $e_r = 0$ and the cotangent bundle $e_r = -1$.

By Selberg's lemma, every compact hyperbolic orbifold is finitely covered by a surface. So, each such complex hyperbolic disc orbibundle pullbacks to examples of complex hyperbolic disc bundles over compact hyperbolic surfaces. Note that the relative Euler number is unchanged under pullbacks. Every previously known example supporting the complex Gromov-Lawson-Thurston conjecture (see Chapter 1) have non-negative relative Euler number. Furthermore, to the best of our knowledge, these are the first non-rigid examples of disc orbibundles with complex hyperbolic structures. The existence of trivial complex hyperbolic disc bundles has been conjectured in several places (see (ELIASHBERG, 1992), (SCHWARTZ, 2007), (GOLDMAN, 1983)); it has first been solved in (ANAN'IN; GUSEVSKII, 2005), where a single rigid example is constructed (in our work, the complex hyperbolic trivial bundle appears as a part of a large family of examples).

Figure 6 (a) depicts the two-dimensional region of the $\mathrm{PU}(2,1)$-character variety $\mathscr{R}(3,3,4)$ corresponding to generic representations. The coordinates here are the positive numbers s,t described in the end of Section 3.1.



(a)



(b)

Figure 6 – **(a)** Generic representations in $\mathscr{R}(3,3,4)$ and **(b)** some disc orbibundles over $\mathbb{S}^2(3,3,4)$

The constructed complex hyperbolic disc orbibundles over $\mathbb{S}^2(3,3,4)$ lie in the shaded region of Figure 6 (b).

At the beginning of our research, we (naively) conjectured that all these generic faithful representations were discrete. This turned out to be false as we established the existence of representations in the region in Figure 6 (a) with elliptic isometries of infinite order. More precisely, remember from Section 3.1 that an isometry I is regular elliptic if, and only if, $f(\mathrm{tr}(I)) < 0$, where $f(z) := |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27$ is Goldman's function. Plotting $f(\mathrm{tr}([I_1, I_2])) < 0$, we obtain the shaded region in Figure 7. The commutator $[I_1, I_2]$ cannot have finite order everywhere in the described region, because the value of $f(\mathrm{tr}([I_1, I_2]))$ varies continuously and there are only countable many possible values for the eigenvalues of a finite order isometry. Therefore, there exists a representation such that $[I_1, I_2]$ is elliptic of infinite order.

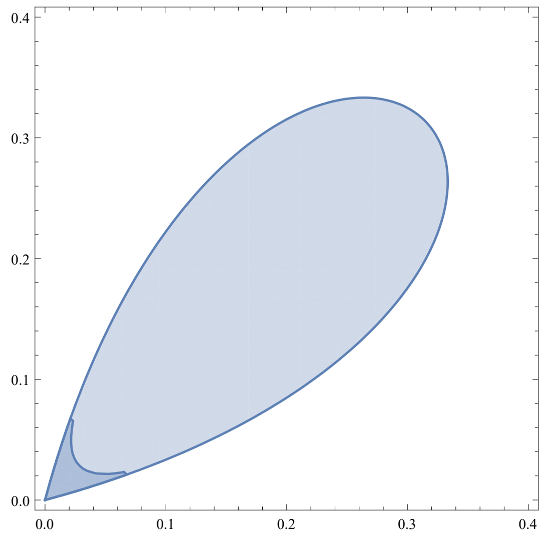


Figure 7 – Region where $[I_1, I_2]$ is elliptic.

For representations where one, and only one, of the isometries I_j's is special elliptic, we found 17.368 examples, with $e_r \in [-1, 0.5]$. As in the generic case, we obtained $e_r = 0$ and $e_r = -1$. We also found the case $e_r = 0.5$, which did not appear in the generic case. As stated in Proposition 3.2, all of these examples are rigid.

3.3 Complex hyperbolic disc orbibundles

The prototypical fundamental domain for a disc bundle over a sphere $\mathbb{S}(n_1, n_2, n_3)$ with three cone points, to be obtained from a faithful discrete representation $G(n_1, n_2, n_3) \to \mathrm{PU}(2,1)$, imitates that for the turnover group acting on the hyperbolic plane (see Figure 5). In Figure 8, the complex geodesics C_1, C_2, C_3 are stable under the action of the elliptic isometries I_1, I_2, I_3 respectively, and $C_4 := I_1^{-1} C_2 = I_3 C_2$. More precisely, if $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are positive fixed points of I_1, I_2, I_3, respectively, and $\boldsymbol{p}_4 := I_1^{-1} \boldsymbol{p}_2 = I_3 \boldsymbol{p}_2$, we define C_i to be the complex geodesic polar to $\boldsymbol{p}_i$, i.e., $C_i := \mathbb{P}_{\mathbb{C}}(p_i^{\perp}) \cap \mathbb{H}^2_{\mathbb{C}}$. The hypersurfaces connecting the complex geodesics are segments of bisectors (see Section 2.3), which exist if we assume that the discs C_i's are ultraparallel (the tance between different p_i's is greater than 1). Remember that these hypersurfaces form cylinders.

Under certain conditions, this configuration of bisectors bounds a topological 4-ball which tesselates $\mathbb{H}^2_{\mathbb{C}}$ under the action of the turnover (see (BOTÓS; GROSSI, 2021, Subsection 7.1)).

The construction of a (smooth) disc fibration for this fundamental domain is rather technical and revolves around a deformation lemma (BOTÓS; GROSSI, 2021, Lemma 20). In this way, we obtain the disc orbibundles described in Section 3.2.

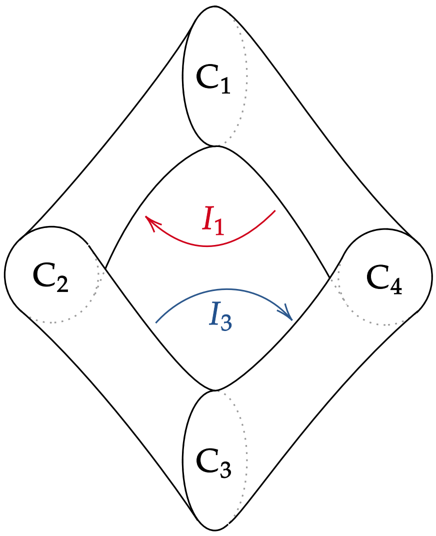


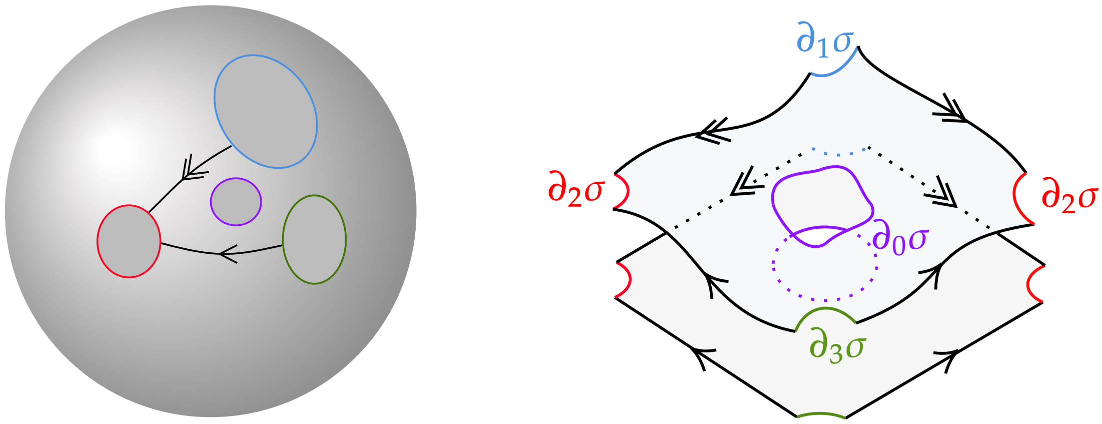
Figure 8 – Fundamental domain

The disc fibration of the fundamental domain is constructed as follow: consider a complex geodesic $D = \mathbb{P}_{\mathbb{C}}(q^{\perp}) \cap \mathbb{H}^2_{\mathbb{C}}$, where $\boldsymbol{q}$ is positive, and over it a copy P' of the quadrilateral described in Figure 5. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ be the vertices of P'. The region

$$\bigsqcup_{\boldsymbol{x} \in P'} \mathbb{P}_{\mathbb{C}}(\mathbb{C}x + \mathbb{C}q) \cap \mathbb{H}^2_{\mathbb{C}}$$

is a 4-ball foliated by discs and bounded by the bisectors $B[C_1', C_2']$, $B[C_2', C_3']$, $B[C_3', C_4']$ and $B[C_4', C_1']$, where $C_i' := \mathbb{P}_{\mathbb{C}}(\mathbb{C}v_i + \mathbb{C}q) \cap \mathbb{H}^2_{\mathbb{C}}$. The deformation lemma (BOTÓS; GROSSI, 2021, Lemma 20) proves that the described foliation by discs can be deformed (via isotopy) to the fundamental domain we are interested in, thus providing it with a disc bundle structure. The deformation also carries the quadrilateral P' to a quadrilateral P embedded in the fundamental domain with vertices $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are the centers of the isometries I_1, I_2, I_3 and $\boldsymbol{c}_4 := I_1^{-1}\boldsymbol{c}_2 = I_3\boldsymbol{c}_2$. Additionally, the quadrilateral P is transversal to the obtained disc fibers of the fundamental domain. The quotient of $\mathbb{H}^2_{\mathbb{C}}$ by the turnover is obtained by gluing the sides of the fundamental domain; by doing so, we glue the sides of the quadrilateral P as well, thus providing a sphere with three cone points. We have just obtained the desired complex hyperbolic disc orbibundle over the sphere with three cone points. After the quotient, the singular points of the orbifold arising from the quadrilateral P are $\boldsymbol{c}_1$, $\boldsymbol{c}_3$ and $\boldsymbol{c}_2 \simeq \boldsymbol{c}_4$, with corresponding angles $2\pi/n_1$, $2\pi/n_3$ and $2\pi/n_2$.

The Euler number of a disc orbibundle is computed using techniques developed in (BOTÓS, 2020, Section 3), which we discuss in Chapter 4. The calculation goes as follows: consider a disc orbibundle $L \to B$ as above, where $B = \mathbb{S}^2(n_1, n_2, n_3)$. Since L and B are oriented, each disc fiber is oriented as well. Considering the $\mathbb{S}^1$-orbibundle $M \to B$ associated to the disc orbibundle, each $\mathbb{S}^1$ fiber is oriented because the discs fibering M are oriented. The Euler number of L is defined to be the Euler number of M.



(a) (b)

Figure 9 – **(a)** Surface B', and **(b)** Section $\sigma : B' \to M'$

Remove from B small discs centered at the three singular points as well as a fourth disc centered at an arbitrary regular point. The surface with boundary obtained from removing such discs is denoted by B' and the disc bundle $M|_{B'} \to B'$ has a section σ, because B' is homotopically equivalent to a graph. The Euler number $e(M)$ of M is the number satisfying

$$\sigma|_{\partial B'} = -e(M)s$$

in $H_1(M,\mathbb{Q})$, where s is a positively oriented $\mathbb{S}^1$ fiber of $M \to B$ over a regular point. For details, look at (BOTÓS; GROSSI, 2021, Subsection 7.5).

3.4 Toledo invariant and holomorphic section

Each representation $\rho : G(n_1,n_2,n_3) \to \mathrm{PU}(2,1)$ has its **Toledo invariant** τ: given a $G(n_1,n_2,n_3)$-equivariant map $f : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{C}}$, the pullback $f^*\omega$ of the symplectic form ω of $\mathbb{H}^2_{\mathbb{C}}$ by f is $G(n_1,n_2,n_3)$-equivariant. The Toledo invariant of ρ is the number

$$\tau := \frac{4}{2\pi}\int_P f^*\omega,$$

where $P \subset \mathbb{H}^2_{\mathbb{R}}$ is a fundamental domain of the turnover group, as in Figure 5. The number τ does not depend on the choice of the equivariant map. Section 4.4 has more details about the Toledo invariant for orbifolds (see also (BOTÓS, 2020, Section 6) and (BOTÓS; GROSSI, 2021, Section 8).

We have a simple expression for the Toledo invariant mod 2 in the case of representations providing disc orbibundles:

Proposition 3.3 (Proposition 34, (BOTÓS; GROSSI, 2021)). *Let* $\rho : G(n_1, n_2, n_3) \to \mathrm{PU}$ *be a representation satisfying the necessary conditions for the tesselation outlined in Section 3.3. Then* $\tau \equiv \dfrac{\operatorname{Arg}(\alpha_1\beta_1\gamma_1^{-1})}{\pi} \mod 2$*, where* τ *stands for the Toledo invariant of* ρ*.*

Here, $\alpha_1, \beta_1, \gamma_1^{-1}$ are the eigenvalues of I_1, I_2, I_3 corresponding to the negative eigenvectors.

For all complex hyperbolic disc orbibundles constructed in (ANAN'IN; GROSSI; GUSEVSKII, 2011), (ANAN'IN; GUSEVSKII, 2005), (BOTÓS; GROSSI, 2021), the equality $3\tau = 2(e+\chi)$ holds. This identity is a necessary condition for the existence of a holomorphic section (see Proposition 4.7). It is an open question whether this condition is also sufficient. Misha Kapovich proved that, for the examples in (ANAN'IN; GROSSI; GUSEVSKII, 2011), this is indeed the case (see (KAPOVICH, 2019)). Nevertheless, his proof relies on the rigidity of the representations and, therefore, does not work for the examples in (BOTÓS; GROSSI, 2021).

CHAPTER

4

ORBIFOLDS AND ORBIBUNDLES IN COMPLEX HYPERBOLIC GEOMETRY

We now discuss (BOTÓS, 2020), where we establish a language to deal with orbibundles over orbifolds as those in (BOTÓS; GROSSI, 2021). These spaces are modeled using diffeology, a generalization of manifolds but with good categorical properties, which is very useful when dealing with quotients and infinite-dimensional spaces.

4.1 Diffeology 101

A diffeological space is a set X equipped with a special set $\mathscr{F}$, called a diffeology of X (in the same way that a topological space is a set equipped with a topology). The set $\mathscr{F}$ is formed by maps from Euclidean open sets to X, where a Euclidean open set is an open set in some $\mathbb{R}^n$ (n can vary). Furthermore, to be a diffeology, the family $\mathscr{F}$ must satisfy three axioms:

- $\mathscr{F}$ contains all constant maps $U \to X$, where U is an Euclidean open set;
- if $\phi : U \to X$ belongs to $\mathscr{F}$ and $f : V \to U$ is a smooth map between Euclidean open sets, then $\phi \circ f$ belongs to $\mathscr{F}$;
- if we have a function $\phi : U \to X$, where U is an Euclidean open set, and every $p \in U$ admits a neighborhood V such that $\phi|_V \in \mathscr{F}$, then $f \in \mathscr{F}$.

The elements of $\mathscr{F}$ are called **plots**. So the first axiom states that constant maps are plots and the third axiom states that if a function is locally a plot, then it is a plot. A map $f : X \to Y$ between diffeological spaces is smooth if, for any plot ϕ of X, the map $f \circ \phi$ is a plot of Y. Thus, diffeological spaces form a category with smooth maps as morphisms. Note as well that the plots of a diffeological space are smooth themselves.

Manifolds are naturally diffeological spaces. The diffeology of a smooth manifold X is the family $\mathscr{F}$ of all smooth maps from Euclidean open sets to X, where the smoothness here is the usual one. If X and Y are smooth manifolds, then a map $f : X \to Y$ is smooth in the traditional sense if, and only if, it is also smooth in the diffeological one.

Diffeological spaces also have a canonical topology, the smallest topology that makes all plots continuous. With this topology, we can define manifolds using only diffeology. An n-dimensional manifold is a diffeological space that is locally diffeomorphic to n-dimensional Euclidean open sets (plus Hausdorff and second countable). The same goes for orbifolds, as we will see in the next section.

Every set X admits two extreme diffeologies[1]: the **discrete diffeology**, where the only plots are the locally constant ones (that is, constant in the connected components of the Euclidean open sets); and the **indiscrete diffeology**, formed by all functions from Euclidean open sets to X. We say that a diffeological space endowed with the discrete diffeology is discrete.

Quotients of diffeological spaces are also diffeological spaces. Given a diffeological space X and an equivalence relation $\sim$ on X, consider the quotient map $\pi : X \to X/{\sim}$. The quotient $X/{\sim}$ admits a natural diffeology, the smallest diffeology containing all maps $\pi \circ \phi$, where ϕ is a plot of X. Note that this construction makes sense because $X/{\sim}$ admits the indiscrete diffeology, the largest possible one. Observe that this simple construction allows us to give smooth meaning to arbitrary quotients of manifolds, among them, good orbifolds. Similarly, it is possible to construct diffeological structures for the product and coproduct of diffeological spaces.

The category of diffeological spaces forms a bicomplete category. Furthermore, this category also has exponential objects, that is, spaces of smooth maps between diffeological spaces are naturally diffeological.

A differential n-form ω on a diffeological space X is a function that maps a plot $\phi : U \to X$ to a differential n-form $\phi^*\omega$ on U such that $g^*(\phi^*\omega) = (\phi \circ g)^*\omega$ for any plot $\phi : U \to X$ and any smooth map $g : V \to U$, where U and V are Euclidean open sets. The exterior derivative of ω is a differential $(n+1)$-form $d\omega$ defined by $\phi^*(d\omega) := d(\phi^*\omega)$. As expected, $d^2 = 0$. Furthermore, if we have a smooth map $f : Y \to X$ and a differential n-form ω on X, then the pullback $f^*\omega$ is defined by the formula $\phi^*(f^*\omega) := (f \circ \phi)^*\omega$. Thus, the theory of differential forms is very well behaved in diffeology, even allowing the definition of de Rham cohomology groups.

4.2 Orbibundles, orbigoodles and the Euler number

Following the work of Patrick Iglesias-Zemmour (see (IGLESIAS-ZEMMOUR; KARSHON; ZADKA, 2010) and (IGLESIAS-ZEMMOUR, 2013)), an n-dimensional orbifold is a Hausdorff and second countable diffeological space locally modeled by $\mathbb{B}^n/\Gamma$, where $\mathbb{B}^n$ is the

[1] The same happens in topology, where there are the discrete and the chaotic topologies.

n-dimensional unit open ball and Γ is a finite subgroup of $\mathrm{O}(n)$ (the subgroup Γ can vary). The maps that locally give the structure of the orbifold are called **orbifold charts**; more precisely, they are diffeomorphisms of the form $\mathbb{B}^n/\Gamma \to U$, where U is an open subset of the orbifold. We deal only with locally oriented orbifolds, meaning that $\Gamma \subset \mathrm{SO}(n)$. The novelty of our work is that we take advantage of considering orbibundles as diffeological spaces.

Consider a (diffeological) space F, a finite group $\Gamma \subset \mathrm{SO}(n)$, and an action of Γ on $\mathbb{B}^n \times F$ of the form $g(x,f) = (gx, a(x,g)f)$, i.e., each $a(x,g) : F \to F$ is a diffeomorphism and these diffeomorphisms depend smoothly on x. The local model for an orbibundle with fiber F over an n-orbifold is

$$\begin{tikzcd} (\mathbb{B}^n \times F)/\Gamma \arrow[d, "\mathrm{pr}_1"] \\ \mathbb{B}^n/\Gamma \end{tikzcd}$$

where $\mathrm{pr}_1([x,f]) = [x]$; thus, an **orbibundle** over an n-orbifold B is a smooth map $\zeta : M \to B$ locally trivialized according to the diagram

$$\begin{tikzcd} (\mathbb{B}^n \times F)/\Gamma \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_1"'] & \zeta^{-1}(D) \arrow[d, "\zeta"] \\ \mathbb{B}^n/\Gamma \arrow[r, "\phi"] & D \end{tikzcd}$$

where ϕ is an orbifold chart. For discrete fibers, we obtain the standard definition of orbifold covering. If F is $\mathbb{R}^n$ as a vector space, each fiber of ζ is a real vector space, the maps $a(x,g)$ are linear, and each trivialization of the bundle is a linear isomorphism on fibers, then we obtain the concept of a vector orbibundle. Similarly, we define G-orbibundles, where G is a Lie group.

We now restrict ourselves to compact, connected, oriented 2-orbifolds. Consider an $\mathbb{S}^1$-orbibundle $\zeta : M \to B$, where B is a 2-orbifold. We define the Euler number for ζ in the following manner: the action of $\mathbb{S}^1$ on M induces an orientation of the fibers of ζ. Let s be an oriented fiber over a regular (non-conic) point. The fiber s generates $H_1(M,\mathbb{Q}) \simeq \mathbb{Q}$. Let $x_1,\dots,x_n$ be the singular points of B and x_0 be a regular one. Removing small discs D_i centered on each x_i, we obtain a surface B' with boundary. The bundle $\zeta|_{B'}$ is trivial because $\mathbb{S}^1$-bundles over graphs are trivial and B' is homotopically equivalent to a graph. Hence, there is a section σ for $\zeta|_{B'}$. The **Euler number** $e(M)$ of ζ is defined as the rational number $e(M)$ satisfying

$$\sigma|_{\partial B'} = -e(M)s$$

in $H_1(M,\mathbb{Q})$.

Proposition 4.1 (Theorem 22, (BOTÓS, 2020)). *Consider a compact, connected, and oriented 2-orbifold B with isolated singularities $x_1,\dots,x_n$, where m_k is the order of x_k. The Euler number of an $\mathbb{S}^1$-orbibundle $\zeta : M \to B$ belongs to $\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \cdots + \frac{1}{m_n}\mathbb{Z}$.*

In the above proposition, when we say that x_k has order m_k, we mean that its conic angle is $2\pi/m_k$.

If we have an oriented real vector orbibundle of rank 2 over a 2-orbifold then, up to the choice of a metric, we obtain an $\mathbb{S}^1$-orbibundle. The Euler number of the vector orbibundle is by definition the Euler number of the associated $\mathbb{S}^1$-orbibundle. This definition of the Euler number is inspired by the Poincaré-Hopf theorem. The Euler number of the tangent orbibundle of B is the Euler characteristic $\chi(B) = \chi(\tilde{B}) + \sum_i(-1+1/m_i)$, where $\tilde{B}$ is the topological surface underlying B, a well-known formula. For $\mathbb{S}^2(n_1,n_2,n_3)$, we obtain the number described in Section 3.1. The relative Euler number of an $\mathbb{S}^1$-orbibundle $\zeta : M \to B$ is $e(M)/\chi(B)$.

Analogous to the concept of a good orbifold, we have the good orbibundle (that we call orbigoodle). If we have a simply connected manifold $\mathbb{H}$ and a group G acting properly discontinuously on $\mathbb{H}$, then $\mathbb{H}/G$ is a good orbifold. Furthermore, if we have a space F and an action of G on $\mathbb{H}\times F$ of the form $g(x,f) = (gx, a(x,g)f)$, where $a(x,g)$ is an automorphism of F depending smoothly on x, then the natural map $(\mathbb{H}\times F)/G \to \mathbb{H}/G$ is an orbibundle with fiber F which we call an **orbigoodle**. Similar to the definition of orbibundles, we have vector and principal orbigoodles.

The examples in (BOTÓS; GROSSI, 2021) are disc/plane orbigoodles and a reason for their importance is that they behave well under pullbacks. More precisely, if G' is a subgroup of G, then the map $\mathbb{H}/G' \to \mathbb{H}/G$ is an orbifold covering. Furthermore, $(\mathbb{H}\times F)/G' \to \mathbb{H}/G'$ is an orbigoodle called the pullback by the orbifold covering; it gives rise to the commutative diagram

$$\begin{tikzcd}
(\mathbb{H}\times F)/G' \arrow[r] \arrow[d] & (\mathbb{H}\times F)/G \arrow[d] \\
\mathbb{H}/G' \arrow[r] & \mathbb{H}/G
\end{tikzcd}$$

Taking $\mathbb{H}$ as the real hyperbolic plane, G as a cocompact Fuchsian group, and $\mathbb{S}^1$-orbigoodles, we have the following proposition (the proposition also holds for oriented compact good 2-orbifolds):

Proposition 4.2 (Theorem 25, (BOTÓS, 2020))**.** *The relative Euler number is unchanged by pullbacks of orbigoodles under finite orbifold coverings.*

This explains why for every complex hyperbolic orbigoodle found in (BOTÓS; GROSSI, 2021), there exists a complex hyperbolic disc bundle over a surface with the same relative Euler number: every compact, oriented, connected hyperbolic orbifold is finitely covered by a surface (this follows from Selberg's Lemma[2]), thus we can pullback the complex hyperbolic disc

[2] Selberg's lemma: Every finitely generated Fuchsian group admits a normal torsion-free subgroup of finite index.

orbigoodle to a complex hyperbolic disc bundle over a surface.

4.3 Chern-Weil theory

Consider an oriented vector orbibundle $\zeta : L \to B$ of rank 2 over a compact, connected, and oriented 2-orbifold. These vector bundles admit a connection ∇. The Pfaffian of the curvature tensor R is a 2-form, denoted by $\mathrm{pf}(R)$. The definition of the Pffafian can be found in (TU, 2017, Section 25.3) and integration over orbifolds is discussed, for instance, in (BOTÓS, 2020, Section 5).

Proposition 4.3 (Proposition 31, (BOTÓS, 2020)). *The integral of $\frac{1}{2\pi}\mathrm{pf}(R)$ over the orbifold is the Euler number $e(L)$.*

$$e(L) = \frac{1}{2\pi}\int_B \mathrm{pf}(R).$$

If L is a complex line orbigoodle, then we define the first Chern class $\frac{1}{2\pi i}\mathrm{tr}(R)$ and the first Chern number

$$c_1(L) := \frac{1}{2\pi i}\int_B \mathrm{tr}(R)$$

as well.

Proposition 4.4 (Proposition 32, (BOTÓS, 2020)). *The identity $c_1(L) = e(L)$ holds.*

4.4 Applications to complex hyperbolic geometry

Let G be a cocompact Fuchsian group and $B = \mathbb{H}^1_{\mathbb{C}}/G$ be the corresponding hyperbolic orbifold, where $\mathbb{H}^1_{\mathbb{C}}$ is the Poincaré disc.

Given a representation $\rho : G \to \mathrm{PU}(2,1)$, the Toledo invariant $\tau(\rho)$ of ρ is defined as

$$\tau(\rho) = \frac{4}{2\pi}\int_B f^*\omega,$$

where ω is the symplectic form of $\mathbb{H}^2_{\mathbb{C}}$ and $f : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$ is an arbitrary ρ-equivariant smooth map. This definition is analogous to the one given by D. Toledo in (TOLEDO, 1989) in the context of surfaces. In order to the orbifold definition of the Toledo invariant to work, we establish the existence of equivariant maps (a non-trivial fact) and that any two equivariant maps are equivariantly homotopic (see (BOTÓS, 2020, Lemma 34)).

For surface groups,[3] there is the integrality property of the Toledo invariant, i.e., the Toledo invariant belongs to $\frac{2}{3}\mathbb{Z}$ (see (GOLDMAN; KAPOVICH; LEEB, 2001)). We give an alternative

[3] Meaning that the group G is a torsion free cocompact Fuchsian group (the fundamental group of a compact, connected, oriented hyperbolic surface).

proof of this fact using tools from classic geometries (the algebraic formula for the Riemann curvature in $\mathbb{H}^2_{\mathbb{C}}$). Additionally, we establish the integrality for orbifolds:

Proposition 4.5 (Corollary 41, (BOTÓS, 2020))**.** *For orbifolds, the Toledo invariant belongs to*

$$\frac{2}{3}\Big(\mathbb{Z}+\frac{1}{m_1}\mathbb{Z}+\cdots\frac{1}{m_n}\mathbb{Z}\Big),$$

where $2\pi/m_1,\ldots,2\pi/m_n$ are the angles of the cone points of B.

Domingo Toledo proves his famous rigidity theorem in (TOLEDO, 1989): the inequality $|\tau/\chi|\leq 1$ holds for any representation $G\to\mathrm{PU}(2,1)$, where G is a surface group. This inequality is called **Toledo inequality**. Furthermore, $|\tau/\chi|=1$ if, and only if, there exists a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ stable under action of G. This second statement is called **Toledo rigidity**.

Proposition 4.6 (Theorem 44, (BOTÓS, 2020))**.** *Toledo rigidity and inequality hold for orbifolds in the context of $\mathbb{H}^2_{\mathbb{C}}$.*

The inequality $|\tau/\chi|\leq 1$ follows from Toledo's result for surfaces. The non-trivial part is to show that, when $|\tau|=|\chi|$, the representation has a stable complex geodesic. Here we combine the Toledo rigidity for surfaces with Goldman's Theorem A (see (GOLDMAN, 1980)).

When the representation ρ corresponds to a disc orbibundle, the Euler number e is an invariant as well. We also proved the folkloric result below, concerning holomorphic sections.

Proposition 4.7 (Corollary 43, (BOTÓS, 2020))**.** *If a complex hyperbolic disc orbibundle admits a holomorphic section, then $3\tau=2(e+\chi)$.*

Finally, we believe that for complex hyperbolic orbibundles, the inequality $3\tau\geq 2(e+\chi)$ always holds, a claim supported by all examples found so far. Additionally, we conjecture that equality happens if, and only if, there exists a holomorphic section.

CHAPTER 5

GEOMETRY OVER ALGEBRAS

5.1 Real algebras and linear spaces

Here we discuss (BOTÓS, 2022). So far, classic geometries have been mainly developed over real, complex, and quaternionic numbers. We now develop the same theory for some other unital associative finite-dimensional real algebras. Similar work, for commutative algebras, can be found in (TRETTEL, 2019). The major difference is that, in (TRETTEL, 2019), the spaces are constructed as quotients of Lie groups; we try to tackle the theory by constructing projective models directly, thus mimicking the approach in Chapter 2. For that to work, we need some linear theory over real algebras, a non-trivial problem because linear spaces over non-division algebras are ill-behaved.

The set of units $\mathbb{F}^\times$ of a finite-dimensional $\mathbb{R}$-algebra $\mathbb{F}$ is a dense open subset of $\mathbb{F}$, and its complement $\mathbb{F}\setminus\mathbb{F}^\times$ is formed by the zero-divisors of the algebra, which is a real algebraic variety.

We explore the following involutive real algebras:

- Split-complex: $\mathbb{R}+j\mathbb{R}$, $j^2=1$;
- Dual numbers: $\mathbb{R}+\varepsilon\mathbb{R}$, $\varepsilon^2=0$;
- Split-quaternions: $\mathbb{R}+i\mathbb{R}+j\mathbb{R}+k\mathbb{R}$, where

$$i^2=-1,\quad j^2=1,\quad k^2=1$$

$$ij=-ji,\quad ki=-ik,\quad jk=-kj,\quad ij=k.$$

Observe that the split-quaternions are defined by the same relations as the quaternions, with the difference being the fact that $j^2=k^2=1$.

The involutions of these algebras are analogous to the one of the complex numbers. For instance, $(a+jb)^* := a-jb$ for the split-complex numbers. In all these examples we have the norm $N(z) := zz^* = z^*z$, and z is a real number exactly when $z = z^*$. Furthermore, z is a unit if and only if its norm does not vanish.

Let $\mathbb{F}$ be one of these algebras. A Hermitian form $\langle \cdot, \cdot \rangle$ for a finitely generated left $\mathbb{F}$-module V is defined as usual. For instance, in $\mathbb{F}^n$,

$$\langle u, v \rangle = \sum_i u_i v_i^*.$$

A Hermitian form is **non-degenerate** if the only vector orthogonal to V is the zero vector.

Proposition 5.1. *Let V be a finitely generated free $\mathbb{F}$-module. Every non-degenerate Hermitian form on V admits an orthonormal basis. Additionally, every v satisfying $\langle v, v \rangle \in \mathbb{F}^\times$ belongs to an orthonormal basis.*

Consider a free module V and a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$. A point $v \in V$ is **good** if there exists u such that $\langle v, u \rangle \in \mathbb{F}^\times$. The concept of a good point does not depend on the choice of the non-degenerate Hermitian form; thus, it is a property only related to the fact that V is free. The set $V^\bullet$ of all good points is a dense open subset of V. In the case where $\mathbb{F}$ is a division algebra, $V^\bullet = V \setminus 0$. For $V = \mathbb{F}^n$, a vector v is good if, and only if, the right ideal generated by the coordinates of v is $\mathbb{F}$.

5.2 Classic geometries

Naturally, the projective space $\mathbb{P}_\mathbb{F}(V)$ is the quotient $V^\bullet / \mathbb{F}^\times$.

Proposition 5.2. *If V is free, then the projective space $\mathbb{P}_\mathbb{F}(V)$ is a manifold and the quotient map $V^\bullet \to \mathbb{P}_\mathbb{F}(V)$ is an $\mathbb{F}^\times$-principal bundle.*

From now on we assume V to be a free left $\mathbb{F}$-module endowed with a Hermitian form $\langle \cdot, \cdot \rangle$. The projective space is separated in two regions: the **regular region**

$$R(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p \rangle \neq 0\}$$

and the **singular region**

$$S(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p \rangle = 0\}.$$

For $\boldsymbol{p}$ regular we can naturally identify $T_{\boldsymbol{p}} \mathbb{P}_\mathbb{F}(V)$ with $\mathrm{lin}_\mathbb{F}(\mathbb{F}p, p^\perp)$. Here we must be careful with the use of $\mathrm{lin}_\mathbb{F}$. If V_1, V_2 are $\mathbb{F}$-modules, then we define $\mathrm{lin}_\mathbb{F}(V_1, V_2)$ as the space of all $\mathbb{R}$-linear transformations $T : V_1 \to V_2$ such that $T(kv_1) = kT(v_1)$ for every $k \in \mathbb{F}$ and $v_1 \in V_1$. Nevertheless, this space (as in the cases of quaternions and split-quaternions) is not an $\mathbb{F}$-linear module when $\mathbb{F}$ is non-commutative, but it is always a real vector space.

As in Chapter 2, in the regular region $R(V)$, we have the Hermitian metric

$$\langle t_1, t_2 \rangle_{\boldsymbol{p}} := \pm \frac{\langle t_1(p), t_2(p) \rangle}{\langle p, p \rangle},$$

where $t_1, t_2 \in T_{\boldsymbol{p}} \mathbb{P}_{\mathbb{F}}(V)$, as well as the pseudo-Riemannian metric $g_{\boldsymbol{p}} = \operatorname{Re}\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$. Here, the real part of $z \in \mathbb{F}$ is the real number $\operatorname{Re} z := (z + z^*)/2$. For all hyperbolic examples, we choose the sign $-$ for the Hermitian metric. For the other examples, we chose the sign $+$.

Like in the real and complex cases we discussed previously, we also have linear algebraic formulas for the Levi-Civita connection, Riemann curvature tensor, distances, etc in the context of classic geometries over real algebras. For example, geodesics are components of $\mathbb{P}_{\mathbb{F}}(W)$, where W is a two-dimensional real vector space such that the form $\langle \cdot, \cdot \rangle|_{W \times W}$ is real and non-zero.

The formulas for the Riemann curvature tensor and sectional curvature are exactly the ones discussed in Section 2.2. If we consider in $\mathbb{F}^n$ the Hermitian form

$$\langle u, v \rangle = \sum_i u_i v_i^*.$$

and choose the plus sign when defining the Hermitian metric, then the sectional curvatures are as follow: for the dual numbers, the curvature is 1 wherever it is defined (for the dual projective line, for instance, such curvature can not be calculated, since the tangent plane is degenerate with respect to the pseudo-Riemannian metric). For the split-complex numbers and split-quaternions, the regular region of the projective line has curvature 4 and the regular region for higher-dimensional projective lines can have any real number as curvature. For the real division algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, the corresponding projective spaces are endowed with the Fubini-study metric. The curvature in the real case is always 1 and, for the complex and quaternionic cases, the curvature is 4 for projective lines and varies in the interval $[1, 4]$ for the higher dimensional projective spaces.

5.3 Weird projective lines

Let us analyze some examples. The split-complex algebra is isomorphic to $\mathbb{F} := \mathbb{R} \times \mathbb{R}$ with involution $(a, b)^* = (b, a)$, and the numbers of the form (a, a) are the real numbers in $\mathbb{F}$. The projective space $\mathbb{P}^n_{\mathbb{F}} := \mathbb{P}_{\mathbb{F}}(\mathbb{F}^{n+1})$ is diffeomorphic to $\mathbb{P}^n_{\mathbb{R}} \times \mathbb{P}^n_{\mathbb{R}}$ via the map

$$[(a_0, b_0) : \cdots : (a_n : b_n)] \to [a_0 : \cdots : a_n] \times [b_0 : \cdots : b_n].$$

This space is also called **point-hyperplane geometry**. The reason for this name is simple: if we have a real vector space W, then $V := (1,0)W \oplus (0,1)W^*$ is an $\mathbb{F}$-linear space endowed with a natural Hermitian form

$$\langle (1,0)w_1 + (0,1)f_1, (1,0)w_2 + (0,1)f_2 \rangle := (1,0)f_1(w_2) + (0,1)f_2(w_1).$$

Given a point $p := (1,0)w + (0,1)f$, the product $\langle p, p\rangle$ equals $f(w)$; hence, the geometry of $\mathbb{P}_{\mathbb{F}}(V) = \mathbb{P}(W) \times \mathbb{P}(W^*)$ describes the relative position of the hyperplane $f = 0$ and the point w. The region $S(V)$ is described by $f(w) = 0$. The pseudo-Riemannian metric of $R(V)$ is split, i.e., it is non-degenerate with the same number of pluses and minuses on the signature.

Consider now $W = \mathbb{R}^2$ endowed with the canonical inner product. Identifying W and its dual W^*, we obtain that V is $\mathbb{F}^2$ with the Hermitian form

$$\langle (z_1, w_1), (z_2, w_2)\rangle = z_1 w_1^* + z_2 w_2^*.$$

Since $\mathbb{P}^1_{\mathbb{F}} \simeq \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, the split-complex projective line is a torus. In Figure 10 we illustrate the three types of geodesics passing through a common point in this torus. The purple dashed line represents the singular circle $S(\mathbb{F}^2)$. The blue circle does not intersect the singular circle and it is a geodesic in a positive direction. The black line contains a geodesic in a null direction, and the red curve contains a geodesic in a negative direction. Note that, in the negative and null cases, the geodesic is indeed open, because it includes only the curve before it crosses the singular circle.

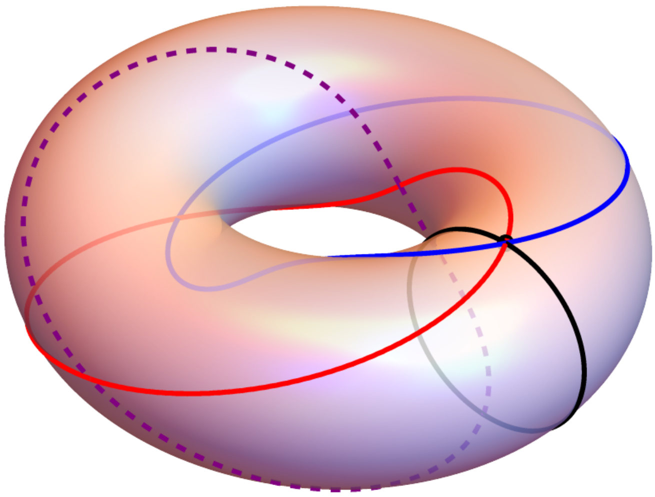


Figure 10 – Split-complex projective line

The split-complex projective line is the space of oriented geodesics of the real hyperbolic plane. Indeed, as described in Figure 1 (b), the space of all non-oriented geodesics of the hyperbolic plane is the de Sitter plane (a Möbius strip) in the Beltrami-Klein projective model. The regular region of the split-complex projective line is an isometric double covering of the de Sitter Möbius strip and each of its points represents an oriented geodesic in the real hyperbolic plane (see (BOTÓS, 2022, Subsection 5.3)). Thus, following Figure 10, we obtain that the positive geodesics (in blue) represent families of oriented geodesics all crossing in a same point of $\mathbb{H}^2_{\mathbb{R}}$. Negative geodesics (in red) represent families of pairwise ultraparallel oriented geodesics, and null geodesics correspond to families of oriented asymptotic geodesics meeting at a particular point in the absolute. Furthermore, in the cases of positive and negative geodesics, the distance between points corresponds to the angle and distance between the respective geodesics, respectively.

regular points of the split-complex projective line = oriented geodesics in $\mathbb{H}^2_{\mathbb{R}}$

Now we discuss the dual numbers $\mathbb{F} = \mathbb{R} + \varepsilon\mathbb{R}$, $\varepsilon^2 = 0$. The projective space $\mathbb{P}^n_{\mathbb{F}}$ is the tangent bundle of the real n-projective space. Indeed, consider the canonical inner product on $\mathbb{R}^{n+1}$ and write $\mathbb{F}^{n+1}$ as $\mathbb{R}^{n+1} + \varepsilon\mathbb{R}^{n+1}$. The diffeomorphism $\mathbb{P}^n_{\mathbb{F}} \to T\mathbb{P}^n_{\mathbb{R}}$ is given by

$$[a + \varepsilon b] \mapsto \frac{\langle \cdot, a\rangle}{\langle a, a\rangle}\left(b - \frac{\langle b, a\rangle}{\langle a, a\rangle} a\right),$$

where we use that $T_{\mathbf{a}}\mathbb{P}^n_{\mathbb{R}} = \mathrm{lin}_{\mathbb{R}}(\mathbb{R}a, a^{\perp})$. The pseudo-Riemannian metric of the dual numbers projective space has signature with n pluses and n zeros.

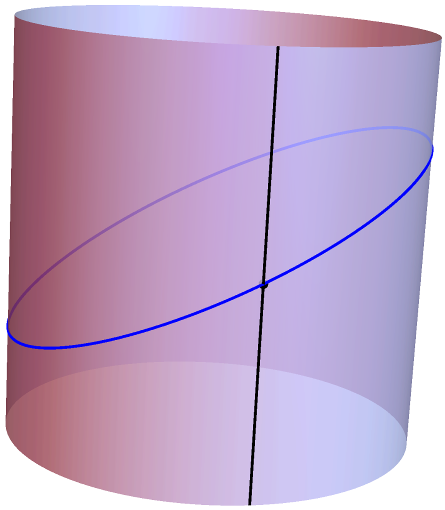


Figure 11 – Dual projective line

The projective line $\mathbb{P}^1_{\mathbb{F}}$ is the tangent space of the circle, i.e., a cylinder. The space of oriented geodesics of the Euclidean plane is also naturally identified with the tangent space of the circle. Indeed, writing

$$T\mathbb{S}^1 = \{(a,b) \in \mathbb{R}^2 \times \mathbb{R}^2 : a \in \mathbb{S}^1 \text{ and } \langle a,b \rangle = 0\},$$

each point $(a,b) \in T\mathbb{S}^2$ uniquely determines the oriented line $t \mapsto b + ta$. Furthermore, $\mathbb{P}^1_{\mathbb{F}}$ provides a natural geometry on the space of Euclidean oriented lines (see Figure 11). By that, we mean that positive geodesics (in blue) correspond to families of oriented lines passing through a common point, and null geodesics (in black) correspond to pairwise oriented parallel lines.

points of the dual number projective line = oriented geodesics in $\mathbb{E}^2$

For the sphere, the space of all oriented geodesics is the sphere itself with its natural metric. We will think of this sphere as the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.

points of the complex projective line = oriented geodesics in $\mathbb{S}^2$

As described in (TRETTEL, 2019), there is a transition between these three projective lines:

$$\mathbb{P}^1_{\mathbb{R}+i\mathbb{R}} \rightsquigarrow \mathbb{P}^1_{\mathbb{R}+\varepsilon\mathbb{R}} \rightsquigarrow \mathbb{P}^1_{\mathbb{R}+j\mathbb{R}}$$

where $j^2 = 1, \varepsilon^2 = 0, i^2 = -1$.

Here, we perform this transition by deforming these geometries inside the split-quaternionic projective line. Indeed, consider the split-quaternions $\mathbb{F} := \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ and the subalgebra $\mathbb{K}_t := \mathbb{R} + \sigma(t)\mathbb{R}$, where $\sigma(t) := (1-t)i + tj$.

$$\mathbb{K}_t \simeq \begin{cases} \mathbb{R} + i\mathbb{R} & \text{for} \quad 0 \leq t < 1/2 \\ \mathbb{R} + \varepsilon\mathbb{R} & \text{for} \quad t = 1/2 \\ \mathbb{R} + j\mathbb{R} & \text{for} \quad 1/2 < t \leq 1 \end{cases}$$

Hence we have the transition between the three projective lines inside the split-quaternionic projective line (see Figure 12) via the map

$$\begin{aligned} \mathbb{P}^1_{\mathbb{K}_t} &\hookrightarrow \mathbb{P}^1_{\mathbb{F}} \\ [z:w] &\mapsto [z:w] \end{aligned}$$

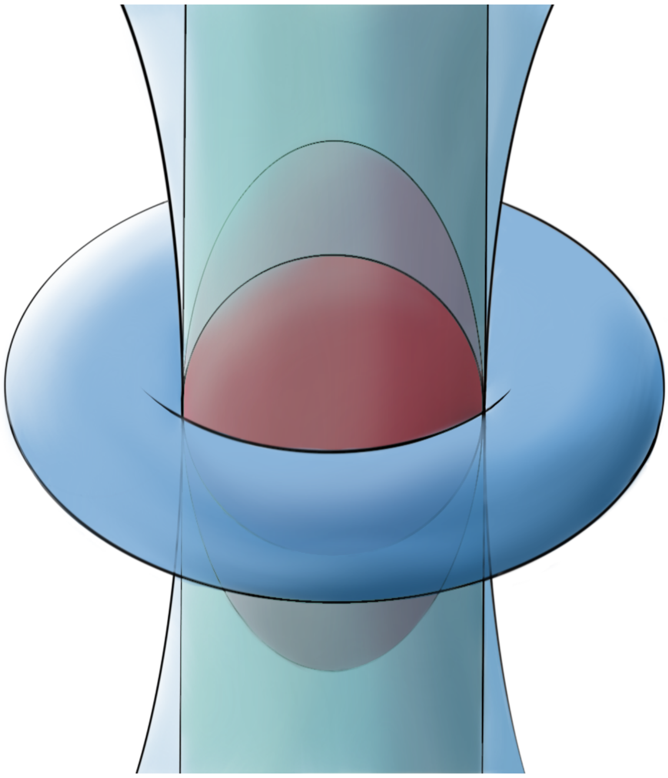


Figure 12 – Transition of geometries

With respect to the canonical Hermitian form $\langle (z_1,w_1),(z_2,w_2)\rangle = z_1 z_2^* + w_1 w_2^*$ on the split-quaternionic module $\mathbb{F}\times\mathbb{F}$, the regular region of $\mathbb{P}^1_{\mathbb{F}}$ has split metric. Inducing the Hermtian form on the modules $\mathbb{K}_t\times\mathbb{K}_t$, the above embeddings are isometric. Furthermore, these embedded spaces all share a common geodesic $\mathbb{P}^1_{\mathbb{F}}(\mathbb{R}\times\mathbb{R})\simeq\mathbb{P}^1_{\mathbb{R}}$. As previously discussed, the regular regions of the $\mathbb{P}^1_{\mathbb{K}_t}$ spaces describe the spaces of oriented geodesics of the sphere, Euclidean plane and hyperbolic plane. Thus, along side the well-known classical deformation

$$\mathbb{S}^2 \leftrightsquigarrow \mathbb{E}^2 \leftrightsquigarrow \mathbb{H}^2_{\mathbb{R}}$$

we have

$$\text{Oriented lines of } \mathbb{S}^2 \leftrightsquigarrow \text{Oriented lines of } \mathbb{E}^2 \leftrightsquigarrow \text{Oriented lines of } \mathbb{H}^2_{\mathbb{R}}.$$

5.4 Bidisc geometry

The last example we discuss is the bidisc $\mathbb{H}^1_{\mathbb{C}}\times\mathbb{H}^1_{\mathbb{C}}$ as a projective space. The algebra we consider here is $\mathbb{F}=\mathbb{C}\times\mathbb{C}$ with the involution $(z,w)^* := (\overline{z},\overline{w})$. Observe that the subalgebra fixed under conjugation is $\mathbb{R}\times\mathbb{R}$ (and not $\mathbb{R}$, as in the previous examples). This means that the Hermitian form here is $\mathbb{C}\times\mathbb{C}$-valued and $\langle u,u\rangle$ is an element of $\mathbb{R}\times\mathbb{R}$. In summary, for this algebra, $\mathbb{R}\times\mathbb{R}$ takes the place of $\mathbb{R}$. With that in mind, we have $\mathrm{Re}(z,w) = (\mathrm{Re}\,z,\mathrm{Re}\,w)$.

The space $\mathbb{P}^1_{\mathbb{F}}$ can be identified with $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ via the map $\Lambda:\mathbb{P}^1_{\mathbb{F}}\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ given by

$$[(z_0,w_0):(z_1,w_1)]\mapsto([z_0,z_1],[w_0,w_1]).$$

Considering on $\mathbb{F}^2$ the Hermitian form $\langle(u_1,v_1),(u_2,v_2)\rangle = -u_1u_2^* + v_1v_2^*$, we break $\mathbb{P}_{\mathbb{F}}^1$ in the regular and the singular regions R and S, where $[u:v]$ belongs to R when $\langle(u,v),(u,v)\rangle$ is a unit, and to S otherwise. The regular region R is composed of four connected components. We single out the component

$$B := \{[u:v] \in \mathbb{P}_{\mathbb{F}}^1 | \langle(u,v),(u,v)\rangle \in \mathbb{R}_{<0} \times \mathbb{R}_{<0}\}.$$

Consider $\mathbb{C}^2$ with the canonical Hermitian form of signature $-+$. Thus we obtain the Poincaré disc $\mathbb{H}_{\mathbb{C}}^1 = \{[z:w] \in \mathbb{P}_{\mathbb{C}}^1 | \langle(z,w),(z,w)\rangle < 0\}$ with the Riemannian structure described in Section 2.1. The diffeomorphism Λ maps B to $\mathbb{H}_{\mathbb{C}}^1 \times \mathbb{H}_{\mathbb{C}}^1$.

The $\mathbb{C}\times\mathbb{C}$-valued metric in B is defined by

$$\langle t,s\rangle := -\frac{\langle t(p),s(p)\rangle}{\langle p,p\rangle}$$

where $\boldsymbol{p} \in B$ and $t,s \in T_{\boldsymbol{p}}B$. Viewing B as $\mathbb{H}_{\mathbb{C}}^1 \times \mathbb{H}_{\mathbb{C}}^1$ via the diffeomorphism Λ, this $\mathbb{C}\times\mathbb{C}$-valued metric can be written as

$$\langle(t_1,t_2),(s_1,s_2)\rangle = (\langle t_1,s_1\rangle, \langle t_2,s_2\rangle),$$

where $(t_1,t_2),(s_1,s_2)$ are tangent vectors in $\mathbb{H}_{\mathbb{C}}^1 \times \mathbb{H}_{\mathbb{C}}^1$. The real part of the $\mathbb{C}\times\mathbb{C}$-valued metric provides an $\mathbb{R}\times\mathbb{R}$-valued metric; the usual Riemannian metric of the bidisc $\mathbb{H}_{\mathbb{C}}^1 \times \mathbb{H}_{\mathbb{C}}^1$ is then given by

$$g((t_1,t_2),(s_1,s_2)) = \mathrm{Re}\langle t_1,s_1\rangle + \mathrm{Re}\langle t_2,s_2\rangle.$$

So, the Riemannian metric of B obtained from the bidisc is the sum of the two coordinates of the $\mathbb{R}\times\mathbb{R}$-valued metric. In other words, the Riemannian geometry of the bidisc can be obtained directly from $\mathbb{P}_{\mathbb{F}}^1$.

Viewed as a Riemannian manifold, the orientation-preserving isometry group of the bidisc $\mathbb{H}_{\mathbb{C}}^1 \times \mathbb{H}_{\mathbb{C}}^1$ is generated by $\mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$ and the involution $(x,y)\mapsto(y,x)$. The group $\mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$ naturally appears as the projectivization of the group of unitary maps of $(\mathbb{F}^2,\langle\cdot,\cdot\rangle)$. Note that the disc swap involution is not an isometry of the $\mathbb{R}\times\mathbb{R}$-valued metric, but it is an isometry of the Riemannian one.

Therefore, $\mathbb{P}_{\mathbb{F}}^1$ allows one to construct a projective model for the bidisc from a Klein geometry viewpoint. The Riemannian geometry of this model is natural and its group of isometries is essentially linear.

CHAPTER

6

ARTICLE: QUOTIENTS OF THE HOLOMORPHIC 2-BALL AND THE TURNOVER

Quotients of the holomorphic 2-ball and the turnover

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Abstract

We construct two-dimensional families of complex hyperbolic structures on disc orbibundles over the sphere with three cone points. This contrasts with the previously known examples of the same type, which are locally rigid. In particular, we obtain examples of complex hyperbolic structures on trivial and cotangent disc bundles over closed Riemann surfaces.

1 Introduction

In this paper, we deal with complex hyperbolic Kleinian groups in complex dimension 2, that is, discrete holomorphic isometry groups of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. There are not so many known examples of such groups and a comprehensive survey can be found in [Kap2].

The complex hyperbolic Kleinian groups we construct here resemble those in [AGG] as they arise from discrete faithful representations of the turnover group

$$G(n_1, n_2, n_3) := \langle g_1, g_2, g_3 \mid g_1^{n_1} = g_2^{n_2} = g_3^{n_3} = 1 \text{ and } g_3 g_2 g_1 = 1\rangle$$

in the group $\mathrm{PU}(2,1)$ of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$. These discrete faithful representations lead to orbibundles over hyperbolic spheres with three cone points or, up to finite cover, to disc bundles over closed Riemann surfaces.

The examples in [AGG] come from representations with $n_1 = n_3 = n$ and $n_2 = 2$ such that $\rho(g_1), \rho(g_3)$ are regular elliptic isometries and $\rho(g_2)$ is a reflection in a complex geodesic (see Subsection 2.4 for the corresponding definitions). Here, we drop these requirements and analyze the remaining cases (except those where at least two of the $\rho(g_j)$'s are not regular, since such representations are $\mathbb{C}$-plane, see Lemma 6).

The generic representations where the $\rho(g_j)$'s are all regular elliptic are the most interesting ones because the corresponding character variety has dimension 2 (see Proposition 8). This allows us to find 2-dimensional families of pairwise non-isometric complex hyperbolic structures over a same disc orbibundle (in contrast, all the representations $G(n,2,n) \to \mathrm{PU}(2,1)$, as those in [AGG], are locally rigid). We highlight two such families of examples.

The first satisfies $e = 0$, where e stands for the Euler number of the disc orbibundle. Therefore, it gives rise to trivial disc bundles over closed Riemann surfaces. Determining whether or not a trivial bundle over a Riemann surface admits a complex hyperbolic structure was a long-standing problem; see, for instance, [Eli, Open Question 8.1], [Gol2, p. 583], and [Sch, p. 14]. It has been first solved in [AGu] using a discrete faithful representation in the isometry group of $\mathbb{H}^2_{\mathbb{C}}$ of a group generated by two reflections in points and a reflection in an $\mathbb{R}$-plane.

The second family satisfies $e/\chi = -1$; here, χ denotes the Euler characteristic of the sphere with three cone points. At the manifold level, we obtain complex hyperbolic structures on cotangent bundles of Riemann surfaces. To the best of our knowledge, the fact that the cotangent bundle of a Riemann surface has a complex hyperbolic structure was previously unknown.

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Besides e and χ, there is a third discrete invariant attached to each of our examples, the Toledo invariant (see, for instance, [Bot, Definition 35], [Krebs], [Tol]). As in [AGG], the formula $2(e+\chi) = 3\tau$ holds in all the examples we found. This formula expresses a necessary condition for the existence of a holomorphic section of the orbibundle [Bot, Corollary 43]. For the [AGG] examples, such a section does indeed exist [Kap2]; however, the proof relies on the local rigidity of representations $\rho : G(n,2,n) \to \mathrm{PU}(2,1)$ and, therefore, does not extend to the examples constructed here. Moreover, all the disc (orbi)bundles we found endorse the complex hyperbolic variant of the Gromov-Lawson-Thurston conjecture (see [AGG], [GLT]) which states that an oriented disc bundle over a closed Riemann surface admits a complex hyperbolic structure if and only if $|e/\chi| \leq 1$. Indeed, $-1 \leq e/\chi \leq 1/2$ in all the examples we constructed. (It is worthwhile mentioning that not all complex hyperbolic disc bundles over closed surfaces satisfy $2(e+\chi) = 3\tau$. Indeed, for the examples in [GKL], one has $e = \chi + |\tau/2|$.)

As in [AGG], the fundamental domains we deal with are bounded by a quadrangle of bisectors, i.e., of segments of hypersurfaces which are equidistant from a pair of points. Nevertheless, we found it necessary to develop some new tools in order to calculate the Euler number because, in the general case, there is not an explicit way to obtain a surface group (whose existence is guaranteed by the Selberg Lemma) as a finite index subgroup of the turnover. Among these tools we have the deformation Lemma 20, a central piece in calculating the Euler number.

At some point we believed that all faithful representations of the turnover in $\mathrm{PU}(2,1)$ with regular $\rho(g_j)$'s were discrete. This naive point of view turned out to be false (see the reasoning above Figure 7) but it seemed to be supported by the following observation. In order to prove discreteness, we essentially need to verify a list of inequalities involving some geometric invariants related to the fundamental domain. However, even when these inequalities are invalid (and, furthermore, even when we are able to show that the corresponding representation is not discrete) we can still apply the formulas that calculate the invariants χ, e, τ. Surprisingly, $2(e+\chi) = 3\tau$ still holds. This is in favor of studying the complex hyperbolic geometry underlying quotients of $\mathbb{H}^2_{\mathbb{C}}$ which are more singular than orbifolds and has been a central motivation for the diffeological approach started in [Bot].

2 Preliminaries

2.1. Complex hyperbolic generalities. Let V be a three-dimensional complex vector space endowed with a Hermitian form $\langle -,- \rangle : V \times V \to \mathbb{C}$ of signature $-++$. Let

$$\mathrm{B}(V) := \big\{p \in \mathbb{P}_{\mathbb{C}}(V) \mid \langle p,p \rangle < 0\big\}, \quad \mathrm{S}(V) := \big\{p \in \mathbb{P}_{\mathbb{C}}(V) \mid \langle p,p \rangle = 0\big\},$$

$$\mathrm{E}(V) := \big\{p \in \mathbb{P}_{\mathbb{C}}(V) \mid \langle p,p \rangle > 0\big\}$$

stand respectively for the subspaces of the complex projective plane $\mathbb{P}_{\mathbb{C}}(V)$ consisting of *negative, isotropic,* and *positive* points. We use the same letter to denote both a point $p \in \mathbb{P}_{\mathbb{C}}(V)$ and a representative of it in $V \setminus \{0\}$. This is harmless as long as we are referring to formulas that are independent of the choice of representatives.

The tangent space $\mathrm{T}_p\mathbb{P}_{\mathbb{C}}(V)$ to a nonisotropic point $p \in \mathbb{P}_{\mathbb{C}}(V)$ can be naturally identified with the space $\mathrm{Lin}(\mathbb{C}p, p^\perp)$ of $\mathbb{C}$-linear maps from the complex line $\mathbb{C}p$ to its orthogonal complement with respect to the Hermitian form. The *complex hyperbolic plane* $\mathbb{H}^2_{\mathbb{C}}$ is the holomorphic 2-ball $\mathrm{B}(V)$ of negative points equipped with the positive-definite *Hermitian metric*

$$\langle t_1, t_2 \rangle := -\frac{\big\langle t_1(p), t_2(p) \big\rangle}{\langle p,p \rangle}, \quad t_1, t_2 \in \mathrm{T}_p\mathrm{B}(V). \tag{1}$$

The ideal boundary of the complex hyperbolic plane in $\mathbb{P}_{\mathbb{C}}(V)$ is the 3-sphere $\mathrm{S}(V)$ called the *absolute* and denoted by $\partial\mathbb{H}^2_{\mathbb{C}}$. We write $\overline{\mathbb{H}}^2_{\mathbb{C}} := \mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$.

The real part of the Hermitian metric (1) is a Riemannian metric in $\mathbb{H}^2_{\mathbb{C}}$ whose distance function is given by $d(p,q) = \operatorname{arccosh}\sqrt{\mathrm{ta}(p,q)}$, where

$$\mathrm{ta}(p,q) := \frac{\langle p,q \rangle \langle q,p \rangle}{\langle p,p \rangle \langle q,q \rangle}$$

is the *tance* between $p, q \in \mathbb{H}^2_{\mathbb{C}}$. The imaginary part of (1) is the *Kähler 2-form* ω. For each $c \in \mathbb{H}^2_{\mathbb{C}}$,

$$P_c(t) := -\frac{1}{2}\mathrm{Im}\frac{\langle t(p), c\rangle}{\langle p, c\rangle}, \quad t \in \mathrm{T}_p\mathbb{H}^2_{\mathbb{C}}, \tag{2}$$

is a potential for ω, that is, $dP_c = \omega$. Potentials P_{c_1}, P_{c_2} based at possibly distinct points $c_1, c_2 \in \mathbb{H}^2_{\mathbb{C}}$ are related by

$$P_{c_1} = P_{c_2} + df_{c_1,c_2}, \text{ where } f_{c_1,c_2}(p) := \frac{1}{2}\mathrm{Arg}\frac{\langle c_1, p\rangle\langle p, c_2\rangle}{\langle c_1, c_2\rangle} \text{ for every } p \in \mathbb{H}^2_{\mathbb{C}} \tag{3}$$

(due to the signature of the Hermitian form, $\langle c_1, c_2\rangle \neq 0$ for all $c_1, c_2 \in \mathbb{H}^2_{\mathbb{C}}$). The above explicit relation between potentials with distinct basepoints lies at the core of the calculation of the Toledo invariant of the discrete faithful $\mathrm{PU}(2,1)$-representations that we construct (see Proposition 34).

2.2. Totally geodesic subspaces. The geodesics of the Riemannian metric are given by the nonempty intersections with $\mathbb{H}^2_{\mathbb{C}}$ of projectivizations $\mathbb{P}_{\mathbb{C}}(W) = \mathbb{P}_{\mathbb{R}}(W)$ of two-dimensional real subspaces W of V such that the Hermitian form restricted to W is real and does not vanish. A geodesic $\mathbb{P}_{\mathbb{C}}(W) \cap \mathbb{H}^2_{\mathbb{C}}$ has two distinct *vertices* $\mathbb{P}_{\mathbb{C}}(W) \cap \partial\mathbb{H}^2_{\mathbb{C}} = \{v_1, v_2\}$, $v_1 \neq v_2$. The unique geodesic determined by a pair of distinct points $c_1, c_2 \in \overline{\mathbb{H}}^2_{\mathbb{C}}$ will be denoted by $G(c_1, c_2)$ and the segment of geodesic connecting c_1, c_2, by $G[c_1, c_2]$. Note that, explicitly, $G(c_1, c_2) = \mathbb{P}_{\mathbb{C}}\big(\mathbb{R}c_1 + \mathbb{R}\langle c_1, c_2\rangle c_2\big)$.

There are two types of totally geodesic (real) surfaces in $\mathbb{H}^2_{\mathbb{C}}$: the complex geodesics and the $\mathbb{R}$-planes. The complex geodesics are the nonempty intersections of projective lines with $\mathbb{H}^2_{\mathbb{C}}$; they are nothing but copies of a Poincaré disc (of constant curvature -4) inside $\mathbb{H}^2_{\mathbb{C}}$. The $\mathbb{R}$-planes are the nonempty intersections of $\mathbb{H}^2_{\mathbb{C}}$ with projectivizations $\mathbb{P}_{\mathbb{C}}(W) = \mathbb{P}_{\mathbb{R}}(W)$ of three-dimensional real subspaces W of V such that the Hermitian form restricted to W is real of signature $-++$. They correspond to copies of a Beltrami-Klein disc (of constant curvature -1) inside $\mathbb{H}^2_{\mathbb{C}}$.

We will sometimes consider that geodesics, complex geodesics, and $\mathbb{R}$-planes are extended to the absolute $\partial\mathbb{H}^2_{\mathbb{C}}$.

Let U be a two-dimensional complex subspace of V such that the signature of the Hermitian form restricted to U is $-+$. The positive point $\mathbb{P}_{\mathbb{C}}(U^\perp) \in \mathrm{E}(V)$ is the *polar* point of the complex geodesic $\mathbb{P}_{\mathbb{C}}(U) \cap \mathbb{H}^2_{\mathbb{C}}$. So, $\mathrm{E}(V)$ is the space of all complex geodesics in $\mathbb{H}^2_{\mathbb{C}}$. Note that the geodesic $\mathbb{P}_{\mathbb{C}}(W) \cap \mathbb{H}^2_{\mathbb{C}}$ is contained in a unique complex geodesic given by $\mathbb{P}_{\mathbb{C}}(W + iW) \cap \mathbb{H}^2_{\mathbb{C}}$.

A pair of complex geodesics is called *ultraparallel, asymptotic,* or *concurrent* when the complex geodesics do not intersect in $\overline{\mathbb{H}}^2_{\mathbb{C}}$, have a single common point in $\partial\mathbb{H}^2_{\mathbb{C}}$, or have a single common point in $\mathbb{H}^2_{\mathbb{C}}$. We write $C_1||C_2$ for ultraparallel complex geodesics C_1, C_2.

Remark 4. **1.** Let L_1, L_2 be complex geodesics with polar points p_1, p_2. Then L_1, L_2 are respectively ultraparallel, asymptotic, concurrent iff $\mathrm{ta}(p_1, p_2) > 1$, $\mathrm{ta}(p_1, p_2) = 1$, $\mathrm{ta}(p_1, p_2) < 1$.

2. Let $L = \mathbb{P}_{\mathbb{C}}(U)$ be a projective line such that the Hermitian form on U is nondegenrate. Given $p \in L$, there exists a unique $q \in L$ such that $\langle p, q\rangle = 0$.

3. The tance between a complex geodesic L and a point $p \in \mathbb{H}^2_{\mathbb{C}}$ is given by

$$\mathrm{ta}(L, p) := \min\big\{\mathrm{ta}(x, p) \mid x \in L\big\} = 1 - \mathrm{ta}(p, q),$$

where q is the polar point of L.

2.3. Bisectors. There are no totally geodesic hypersurfaces in $\mathbb{H}^2_{\mathbb{C}}$. In our construction of fundamental polyhedra we use hypersurfaces known as *bisectors.* A bisector can be characterized as the equidistant locus from two distinct points in $\mathbb{H}^2_{\mathbb{C}}$. Alternatively, it is also determined by a (real) geodesic in $\mathbb{H}^2_{\mathbb{C}}$ and this is the viewpoint that we adopt and briefly describe in what follows.

Let $G = \mathbb{P}_{\mathbb{C}}(W)$ be a geodesic in $\mathbb{H}^2_{\mathbb{C}}$, let L be its complex geodesic, that is, $L = \mathbb{P}_{\mathbb{C}}(W + iW)$, and let p be the polar point of L. The bisector B with *real spine* G and *complex spine* L is given by

$$B := \mathbb{P}_{\mathbb{C}}(W + \mathbb{C}p) \cap \mathbb{H}^2_{\mathbb{C}}.$$

As before, we will sometimes consider bisectors as being extended to $\overline{\mathbb{H}}^2_{\mathbb{C}}$.

The bisector B with real spine G is foliated by complex geodesics,

$$B = \bigsqcup_{x \in G} L_x, \quad \text{where} \quad L_x := \mathbb{P}_{\mathbb{C}}(\mathbb{C}x + \mathbb{C}p) \cap \mathbb{H}^2_{\mathbb{C}}.$$

For each $x \in G$, the complex geodesic L_x is the unique complex geodesic through x orthogonal to the complex spine L in the sense of the Hermitian metric (1). The complex geodesic L_x is called the *slice* of B through x. Each point in B belongs to a unique slice of B.

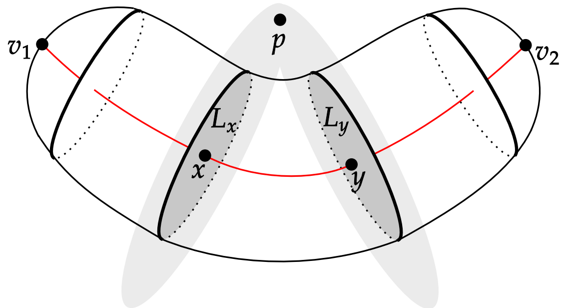


Figure 1: Bisector foliated by complex geodesics.

The bisector B with real spine $G = \mathbb{P}_{\mathbb{C}}(W)$ also admits the *meridional decomposition*

$$B = \bigcup_{\varepsilon \in \mathbb{S}^1} \mathbb{P}_{\mathbb{C}}\big(W + \mathbb{R}\varepsilon p\big) \cap \mathbb{H}^2_{\mathbb{C}},$$

where $p \in V \setminus \{0\}$ is a fixed representative of the polar point p of the complex spine L and $\varepsilon \in \mathbb{S}^1$ is a unit complex number. Given $\varepsilon \in \mathbb{S}^1$, the $\mathbb{R}$-plane $\mathbb{P}_{\mathbb{C}}(W + \mathbb{R}\varepsilon p) \cap \mathbb{H}^2_{\mathbb{C}}$ is called a *meridian* of the bisector. Every meridian of B contains the real spine G. Each point $p \in B \setminus G$ is contained in a unique meridian M of B and determines a *meridional curve* which is the curve in M through p equidistant from G (in other words, a *hypercycle* in the Beltrami-Klein disc M). We also define a meridional curve when $p \in B$ is isotropic. In this case, the intersection $M \cap \partial\mathbb{H}^2_{\mathbb{C}}$ is a circle divided by the vertices of G into two semicircles; we take the one containing p.

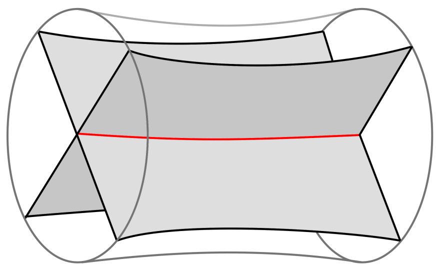


Figure 2: Meridional decomposition.

A pair of ultraparallel complex geodesics L_1, L_2 determines a unique bisector $B(L_1, L_2)$ whose real spine is the unique geodesic G that is simultaneously orthogonal to L_1 and L_2. Explicitly, this geodesic can be constructed as follows. The projective lines containing L_1, L_2 intersect at a positive point $p \in \mathrm{E}(V)$. The complex geodesic $\mathbb{P}_{\mathbb{C}}(p^{\perp})$ intersects L_i at c_i, $i = 1, 2$, and $G = G(c_1, c_2)$. The *segment of bisector* $B[L_1, L_2]$ is defined by

$$B[L_1, L_2] := \bigsqcup_{x \in G[c_1, c_2]} L_x,$$

where L_x stands for the slice of $B[L_1, L_2]$ through x. The slice of $B[L_1, L_2]$ through the middle point of $G[c_1, c_2]$ is called the *middle slice* of $B[L_1, L_2]$.

2.4. Holomorphic isometries. The group of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$ is the projective unitary group $\mathrm{PU}(2,1)$. The special unitary group $\mathrm{SU}(2,1)$ is a triple cover of $\mathrm{PU}(2,1)$ (lifts differ by a cube root of unity) and we refer to elements in $\mathrm{SU}(2,1)$ also as isometries.

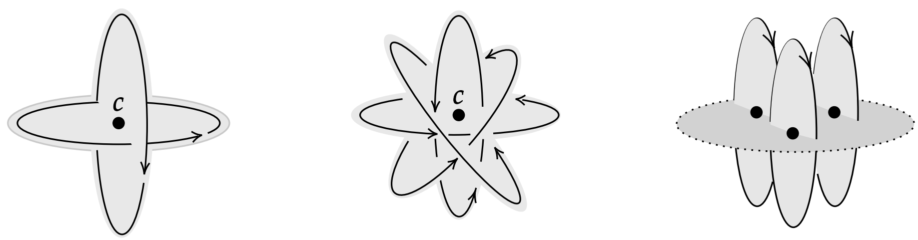
In our construction of discrete group we essentially use elliptic isometries. An isometry $I \in \mathrm{SU}(2,1)$ is said to be *elliptic* when it has a negative fixed point $c \in \mathbb{H}^2_{\mathbb{C}}$. In this case, the projective line $\mathbb{P}_{\mathbb{C}}(c^\perp)$ is I-stable. So, the isometry has a fixed point $p \in \mathbb{P}_{\mathbb{C}}(c^\perp)$. The point $q \in \mathbb{P}_{\mathbb{C}}(c^\perp)$ which is orthogonal to p (see Remark 4) must also be fixed by I. In other words, there is an orthogonal basis in V formed by eigenvectors of I. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}$ with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ be the eigenvalues corresponding respectively to c, p, q. Since none of c, p, q is isotropic, we have $|\varepsilon_i| = 1$ for $i = 1, 2, 3$. It is straightforward to see that I is given by the rule

$$I : x \mapsto (\varepsilon_1 - \varepsilon_3)\frac{\langle x, c\rangle}{\langle c, c\rangle}c + (\varepsilon_2 - \varepsilon_3)\frac{\langle x, p\rangle}{\langle p, p\rangle}p + \varepsilon_3 x. \tag{5}$$

The isometry I is called *regular* elliptic if its eigenvalues are pairwise distinct and *special* elliptic otherwise. We may describe the geometry of regular and special elliptic isometries as follows.

The regular elliptic case. The points c, p, q are the only fixed points of I. We call c the *center* of the isometry. The complex geodesics $\mathbb{P}_{\mathbb{C}}(p^\perp)$ and $\mathbb{P}_{\mathbb{C}}(q^\perp)$ intersect orthogonally at c and both are I-stable. There are no other I-stable complex geodesics. Moreover, I acts on $\mathbb{P}_{\mathbb{C}}(p^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$ as the rotation about c by the angle $\mathrm{Arg}(\varepsilon_1^{-1}\varepsilon_3)$ and on $\mathbb{P}_{\mathbb{C}}(q^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$ as the rotation about c by the angle $\mathrm{Arg}(\varepsilon_1^{-1}\varepsilon_2)$.

The special elliptic case. We can assume that not all eigenvalues of I are equal (for otherwise, I acts identically on $\mathbb{P}_{\mathbb{C}}(V)$). Hence, exactly one of the projective lines $\mathbb{P}_{\mathbb{C}}(c^\perp)$, $\mathbb{P}_{\mathbb{C}}(p^\perp)$, or $\mathbb{P}_{\mathbb{C}}(q^\perp)$ is pointwise fixed by I. If the pointwise fixed line is $\mathbb{P}_{\mathbb{C}}(c^\perp)$, that is, if $\varepsilon_2 = \varepsilon_3$, then every complex geodesic that passes through c is I-stable, the isometry acts on such complex geodesic as the rotation about c by the angle $\mathrm{Arg}(\varepsilon_1^{-1}\varepsilon_2)$, and there are no other I-stable complex geodesics. In this case, we call c the *center* of I as well. When $\mathbb{P}_{\mathbb{C}}(p^\perp)$ is pointwise fixed ($\varepsilon_1 = \varepsilon_3$), every complex geodesic intersecting $\mathbb{P}_{\mathbb{C}}(p^\perp)$ orthogonally in a negative point is stable under I, the isometry acts on such complex geodesics as the rotation about the intersection point by the angle $\mathrm{Arg}(\varepsilon_1^{-1}\varepsilon_2)$, and there are no other I-stable complex geodesics. In other words, I is a rotation with the *axis* $\mathbb{P}_{\mathbb{C}}(p^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$. The same is true for a rotation with the axis $\mathbb{P}_{\mathbb{C}}(q^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$. An important particular case of rotation about an axis is the *reflection* in a complex geodesic $L = \mathbb{P}_{\mathbb{C}}(p^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$ given by the involution $x \mapsto -x + 2\frac{\langle x,p\rangle}{\langle p,p\rangle}p$ (taking $\varepsilon_1 = \varepsilon_3 = -\varepsilon_2 = -1$ in expression (5)).



(a) (b) (c)

Figure 3: **(a)** Rotations about c on two orthogonal complex geodesics by distinct angles, **(b)** Rotation about point, and **(c)** Rotation about complex line.

3 The turnover and its $\mathrm{PU}(2,1)$-character variety

3.1. The turnover. The group

$$G(n_1, n_2, n_3) := \langle g_1, g_2, g_3 \mid g_1^{n_1} = g_2^{n_2} = g_3^{n_3} = 1 \text{ and } g_3g_2g_1 = 1\rangle$$

is called the (hyperbolic) *turnover*, where n_1, n_2, n_3 are positive integers satisfying $\sum\limits_{j=1}^{3} \frac{1}{n_j} < 1$.

We typically write simply G in place of $G(n_1, n_2, n_3)$. It is well-known that G has a discrete cocompact action on the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. Indeed, take a geodesic triangle $\Delta \subset \mathbb{H}^2_{\mathbb{R}}$ with interior angles $\pi/n_1, \pi/n_2, \pi/n_3$ and let $H(n_1, n_2, n_3)$ denote the *triangle* group generated by the reflections r_1, r_2, r_3 in the sides of Δ. The turnover G appears as the index 2 subgroup in H generated by the rotations $g_1 := r_1r_2$, $g_2 := r_3r_1$, $g_3 := r_2r_3$. By the Poincaré Polyhedron Theorem, the quadrilateral $P := \Delta \cup r_2\Delta$ with the vertices, sides, and side-pairings indicated in Picture 4, is a fundamental domain for the action of G on $\mathbb{H}^2_{\mathbb{R}}$.

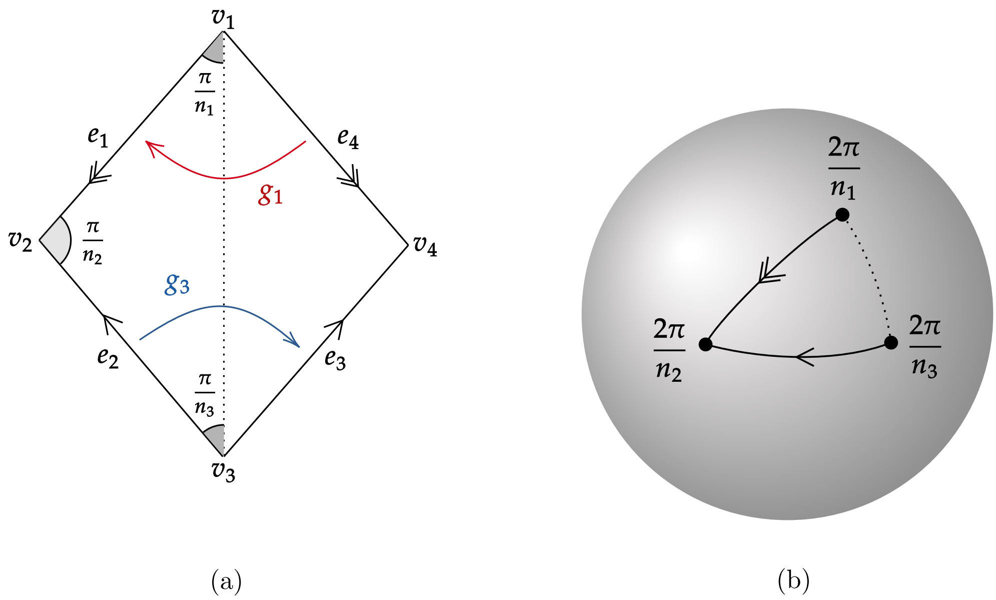


Figure 4: **(a)** Fundamental domain for G and **(b)** Orbifold $\mathbb{S}^2(n_1, n_2, n_3)$

The orbifold $\mathbb{H}^2_{\mathbb{R}}/G$ is the 2-sphere $\mathbb{S}^2(n_1, n_2, n_3)$ with 3 cone points of angles $2\pi/n_1, 2\pi/n_2, 2\pi/n_3$ and orbifold Euler characteristic (see [Sco])

$$\chi = -1 + \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

3.2. Character variety. In this subsection, we deal with the space $\mathcal{R}$ of conjugacy classes of representations $\rho : G \to \mathrm{PU}(2,1)$, where G is the turnover group defined in Subsection 3.1. More precisely, the turnover group $G := G(n_1, n_2, n_3)$ acts on the space $\mathrm{Hom}\big(G, \mathrm{PU}(2,1)\big)$ of all group homomorphisms from G to $\mathrm{PU}(2,1)$ by conjugation, i.e., $g\rho : h \mapsto \rho(g)\rho(h)\rho(g)^{-1}$. The $\mathrm{PU}(2,1)$-character variety of G is the quotient

$$\mathcal{R}(n_1, n_2, n_3) := \mathrm{Hom}\big(G, \mathrm{PU}(2,1)\big)/G.$$

Usually, we denote $\mathcal{R}(n_1, n_2, n_3)$ by $\mathcal{R}$.

Let $\rho : G \to \mathrm{PU}(2,1)$ be a faithful $\mathrm{PU}(2,1)$-representation of the turnover group. Then each isometry $I_j := \rho(g_j)$ is elliptic because a non-identical finite-order isometry in $\mathrm{PU}(2,1)$ is necessarily elliptic. Let us see how the $\mathrm{PU}(2,1)$-representations of the turnover depend on the nature of the elliptic isometries I_1, I_2, I_3.

A representation $\rho : G \to \mathrm{PU}(2,1)$ is called $\mathbb{C}$-*plane* if it stabilizes a projective line in $\mathbb{P}_{\mathbb{C}}(V)$ or, equivalently, if it possesses a fixed point in $\mathbb{P}_{\mathbb{C}}(V)$. Some components of $\mathcal{R}$ are $\mathbb{C}$-plane:

Lemma 6. *If at least two of the I_j's are special elliptic isometries, then ρ is $\mathbb{C}$-plane.*

Proof. A special elliptic isometry has a pointwise fixed projective line (see Section 2). Hence, we can find a point that is simultaneously fixed by two special elliptic isometries among I_j, $j = 1, 2, 3$. This point must also be fixed by the remaining isometry due to the relation $I_3I_2I_1 = 1$. ∎

A $\mathbb{C}$-plane representation is induced from a representation of the turnover in the isometry group of a stable complex geodesic. The well-known $\mathbb{C}$-Fuchsian representations (see [Kap2]) are constructed in this way and they lead to the complex hyperbolic $\mathbb{C}$-Fuchsian disc bundles. We will not deal with $\mathbb{C}$-plane representations here as we focus on the generic case.

We now consider the case where at least two of the I_j''s are regular elliptic. So, assume that I_1 and I_3 are regular elliptic.

We can choose representatives for the isometries I_j in $\mathrm{SU}(2,1)$ with respective eigenvalues $\alpha_j, \beta_j, \gamma_j^{-1}$, $j = 1, 2, 3$, satisfying $I_3 I_2 I_1 = 1$. The eigenvalues denoted with index 1 correspond to a negative eigenvector. Since I_1, I_3 are regular elliptic, we have $\alpha_i \neq \alpha_j$ and $\gamma_i \neq \gamma_j$ for $i \neq j$. For I_2, there are three possibilities. It can be regular elliptic ($\beta_i \neq \beta_j$ for $i \neq j$), a rotation about a point in $\mathbb{H}^2_{\mathbb{C}}$ ($\beta_2 = \beta_3$), or a rotation around a complex geodesic ($\beta_1 = \beta_2$ or $\beta_1 = \beta_3$).

So, in order to find all possible faithful representations of the turnover $G(n_1, n_2, n_3)$ in $\mathrm{PU}(2,1)$ we fix

$$(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3), (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

of order $3n_1, 3n_2, 3n_3$, respectively, satisfying

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3 = 1,$$

$$\alpha_i \neq \alpha_j, \quad \gamma_i \neq \gamma_j \quad for \quad i \neq j,$$

$$\alpha_i^{n_1} = \alpha_j^{n_1}, \quad \beta_i^{n_2} = \beta_j^{n_2}, \quad \gamma_i^{n_3} = \gamma_j^{n_3} \quad for \quad i \neq j,$$

and a regular elliptic isometry I_1 with eigenvalues α_j. We look for all I_2 such that

$$\mathrm{tr}(I_2 I_1) = \sum_{j=1}^{3} \gamma_j.$$

It follows from [Gol1, p. 204, Theorem 6.2.4] that this *trace equation* holds iff $I_3 := (I_2 I_1)^{-1}$ is a regular elliptic isometry with eigenvalues γ_j^{-1}. This strategy allows us to prove the Proposition 8.

Definition 7. Let ρ be a faithful representation where none of the ρg_j's is special elliptic. We call the representation *generic* if there exists $i \neq j$ such that the fixed points of $\rho(g_i)$ and $\rho(g_j)$ are pairwise non-orthogonal (see also 16).

Proposition 8. *Let $\rho : G \to \mathrm{PU}(2,1)$ be a faithful representation. If exactly one of the ρg_j's is special elliptic, then ρ is rigid. Assume that none of the ρg_j's is special elliptic. If ρ is generic, the corresponding component of $\mathcal{R}$ has dimension 2; otherwise, the dimension is bounded by 1.*

Section 4 is devoted to the proof of the above proposition.

Every disc bundle constructed in [AGG] corresponds to a rigid representation $\rho : G \to \mathrm{PU}(2,1)$ for $G = G(n, 2, n)$. Most of the examples highlighted in Section 4 correspond to representations lying in the two-dimensional component of $\mathcal{R}$.

Remark 9. Whenever we deal with elliptic isometries I_1, I_2, I_3 we will assume that either:

- $I_1, I_2, I_3 \in \mathrm{PU}(2,1)$ with $I_j^{n_j} = I_3 I_2 I_1 = 1$ and $\sum\limits_{j=1}^{3} \frac{1}{n_j} < 1$;
- $I_1, I_2, I_3 \in \mathrm{SU}(2,1)$ with $I_j^{n_j} = \delta_j$ and $I_3 I_2 I_1 = 1$, where $\delta_j \in \mathbb{C}$ is a cubic root of unity and $\sum\limits_{j=1}^{3} \frac{1}{n_j} < 1$.

Whether we take the isometries in $\mathrm{PU}(2,1)$ or in $\mathrm{SU}(2,1)$ will be explicitly indicated or should be clear from the context.

4 Computational results

Let us first consider the faithful representation $G(n_1, n_2, n_3) \to \mathrm{PU}(2,1)$ where each of the isometries I_1, I_2, I_3 is regular elliptic. In this case, we found 533 triples (n_1, n_2, n_3), with $3 \le n_i \le 12$ corresponding to faithful discrete representations $G(n_1, n_2, n_3) \to \mathrm{PU}(2,1)$ that lead to disc orbibundles over spheres with three conic points. These are examples of disc orbibundles $\mathbb{H}^2_{\mathbb{C}}/G \to \mathbb{H}^2_{\mathbb{R}}/G$ as discussed in Subsection 7.1. Passing to finite index, each of these disc orbibundles over an orbifold gives rise to a disc bundle over a surface with the same relative Euler number e/χ and same relative Toledo invariant τ/χ (for details about these invariants see [Bot]).

By Proposition 8 each of these orbibundles belong to a 2-dimensional family of pairwise non-isometric (nevertheless, diffeomorphic) bundles. Clearly, examples in a same family share the same discrete invariants. The corresponding relative Euler numbers vary in the interval $\{-1\} \cup (-0.65, 0.5)$ and all examples satisfy the relation $3\tau = 2(e+\chi)$, which is a necessary condition for the existence of a holomorphic section (see [Bot, Corollary 43]).

A couple of examples deserve to be highlighted: the cotangent bundle ($e/\chi = -1$) and the trivial bundle ($e/\chi = 0$). This seems to be the first instance of a complex hyperbolic structure on the cotangent bundle of a compact Riemann surface. As for the trivial bundle, an example has been constructed in [AGu] thus solving a long-standing conjecture [Eli, Open Question 8.1], [Gol2, p. 583], and [Sch, p. 14]. Our construction is quite different from the one in [AGu]; while the latter produces, at the orbiundle level, a single rigid example, the former leads to several two-dimensional families of such trivial orbibundles. Finally, we also find non-rigid discrete representations corresponding to disc orbiblundles whose relative Toledo invariant vanishes and which are not $\mathbb{R}$-Fuchsian because, for such examples, $\mathrm{tr}[I_1, I_2] \notin \mathbb{R}$ (it is well-known that $\mathbb{R}$-Fuchsian representations have vanishing Toledo invariant; in this regard, see also [CuG]).

We illustrate below a prototypical connected component of the $\mathrm{PU}(2,1)$-character variety $\mathcal{R}(n_1, n_2, n_3)$ in the coordinates $(s,t) \in \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ introduced in Section 5; here, $(n_1, n_2, n_3) = (3, 3, 4)$:

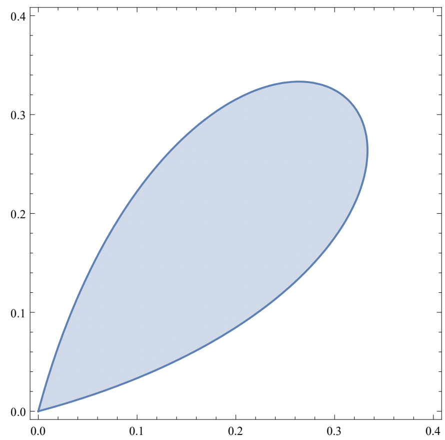


Figure 5: $\mathcal{R}(3,3,4)$ with regular I_1, I_2, I_3

In all the cases we considered, $\mathcal{R}(n_1, n_2, n_3)$ is a disjoint union of topological discs. The shaded region in Figure 6 corresponds to a family of disc orbibundles (see Subsection 7.1), i.e, a pair of distinct points in the shaded region correpond to a pair of diffeomorphic but non-isometric disc orbibundles:

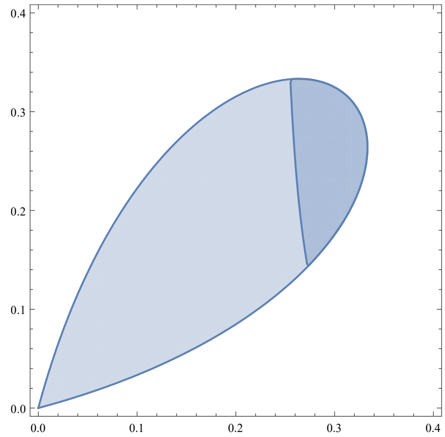


Figure 6: Some disc orbibundles over $\mathbb{S}^2(3,3,4)$

In principle, it could be that all faithful representations $G(n_1,n_2,n_3) \to \mathrm{PU}(2,1)$ with regular I_1, I_2, I_3 were discrete; for a point in the above shaded region, discreteness is guaranteed because a particular fundamental domain is shown to exist (see Section 6), but there is nothing preventing the points outside such region to also correspond to discrete representations. However, this is not the case. Indeed, consider the function $G(s,t) = f(\mathrm{tr}[I_1, I_2])$, where

$$f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27$$

is Goldman's discriminant (see [Gol1, p. 204, Theorem 6.2.4]). The region in $\mathcal{R}(3,3,4)$ described by $G(s,t) < 0$ is the shaded area in the figure

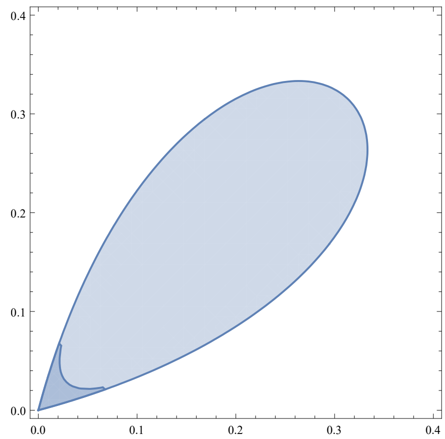


Figure 7: $\mathcal{R}(3,3,4)$ and $\mathrm{G}(s,t) < 0$

Not all examples with $G(s,t) < 0$ can be discrete because, by [Gol1, Theorem 6.2.4], $G(s,t) < 0$ means that $[I_1, I_2]$ is regular elliptic. Since in $\mathcal{R}(3,3,4)$ there are also points where $G(s,t) > 0$, we obtain uncountable many distinct negative values for $G(s,t)$. If all representations in $\mathcal{R}(3,3,4)$ were discrete, all the elliptic isometries in the group $\langle I_1, I_2, I_3 \rangle$ would be of finite order, since discrete groups of isometries have finite stabilizers leading to a countable amount of possibles values for $G(s,t) < 0$.

Now we discuss the representations where I_1, I_3 are regular elliptic and I_2 is special elliptic with $3 \leq n_1, n_3 \leq 20$ and $2 \leq n_2 \leq 20$. When I_2 is a rotation around a point, we found 6351 examples, with $e/\chi \in [-1, 0.5]$. The values $e/\chi = -1$, 0, 0.5 occur here. On the other hand, when I_2 is a rotation around a complex geodesic, we found 11017 examples, with $e/\chi \in (0, 0.5]$, including the right extreme (thus neither $e/\chi = -1$ nor $e/\chi = 0$ were observed here). As in all examples we found, the identity $3\tau = 2(e + \chi)$ is satisfied. Note that $e/\chi = 0.5$ is the maximal relative Euler number allowed by this formula (because $|\tau/\chi| \leq 1$ by Toledo Rigidity) and that this particular relative Euler number only happens for $\tau/\chi = 1$ (thus, the corresponding representation is $\mathbb{C}$-Fuchsian).

When we drop the transversalities conditions Q2 and Q3 stated in the subsection 6.2 for the quadrangles, the formula defining e still makes sense, since it only depends on information coming from the eigenvalues of I_1, I_2, I_3 and on the integer f defined in Subsection 7.6. Curiously, the formula $3\tau = 2(e+\chi)$ still holds in the majority of the cases where the transversalities conditions were dropped. In the few cases where it fails, $3/2\tau - \chi$ differs from e by an integer. This suggests that the formula for f needs a correction with respect to the topology of the "quadrangle" corresponding to such degenerate cases. We believe that this corrected Euler number $e_{\text{cor}} := 3/2\tau - \chi$ have a geometrical meaning that is related to the object obtained by gluing the sides of the "quadrangle" respectively to the relations defined by I_1, I_2, I_3. Thus, in some sense, for all points in the character variety, it seems that there exists a geometric object that behaves as a bundle over $\mathbb{S}(n_1, n_2, n_3)$ with Euler number e_{cor}.

5 Proof of proposition 8

In this section, we deal with the problem of finding elliptic isometries I_1, I_2, I_3 in $\mathrm{SU}(2,1)$ that belong to prescribed conjugacy classes and satisfy $I_3 I_2 I_1 = 1$. The results lead to the parameterization of the representation space of the turnover group discussed in Section 3. We generalize the methods used in [AGG, Section 3].

Let $I_1, I_2, I_3 \in \mathrm{SU}(2,1)$ denote elliptic isometries in given conjugacy classes: α_i, β_i, and γ_i^{-1}, $i = 1,2,3$, stand respectively for the eigenvalues of I_1, I_2, and I_3. We will assume that the first eigenvalue of each I_i corresponds to a negative eigenvector.

In view of Lemma 6 we assume that I_1, I_3 are regular. In order to determine the I_i's such that $I_3 I_2 I_1 = 1$ we fix the isometry I_1 and look for those I_2's in $\mathrm{SU}(2,1)$ satisfying the *trace equation*

$$\operatorname{tr}(I_2 I_1) = \sum_{i=1}^{3} \gamma_i. \tag{10}$$

By [Gol1, p. 204, Theorem 6.2.4], the trace equation holds if and only if $I_2 I_1$ is a regular elliptic isometry with eigenvalues $\gamma_1, \gamma_2, \gamma_3$.

We consider separately the case where I_2 is regular elliptic and the case where I_2 special elliptic (the latter is broken into the rotation about a point in $\mathbb{H}^2_{\mathbb{C}}$ and rotation about a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ subcases).

Regular case: I_2 is regular elliptic. Let $u, v \in V$ denote eigenvectors of I_2 corresponding to the eigenvalues β_1 and β_2. In particular, u is negative and v is positive. We fix a basis $\mathcal{B}$ in V of signature $-++$ consisting of eigenvectors of I_1. The corresponding eigenvalues are $\alpha_1, \alpha_2, \alpha_3$. In this basis, we write

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

We can assume that

$$u_1 > 0, \quad u_2, u_3 \geq 0, \quad \langle u, u \rangle = -1, \quad v_1, |v_2|, |v_3| \geq 0, \quad \langle v, v \rangle = 1.$$

In other words,

$$-u_1^2 + u_2^2 + u_3^2 = -1, \quad -v_1^2 + |v_2|^2 + |v_3|^2 = 1, \quad -u_1 v_1 + u_2 v_2 + u_3 v_3 = 0. \tag{11}$$

(The last equality means that $\langle u, v \rangle = 0$.)

In what follows, we show that u_2, u_3 provide parameters that describe the component of the character variety $\mathcal{R}$ (see Subsection 3.2 for the definition) corresponding to the given conjugacy classes of I_1, I_2, I_3. Roughly speaking, the isometry I_2 is essentially determined, under certain conditions, by $u_2, u_3 > 0$ and the trace equation. Note that the basis $\mathcal{B}$ defines the $\mathbb{R}$-plane $\mathbb{P}(W) \subset \mathbb{H}^2_{\mathbb{C}}$, where $W \subset V$ is spanned over $\mathbb{R}$ by $\mathcal{B}$. This $\mathbb{R}$-plane is exactly the one containing the fixed points of I_1 and the center u of I_2. Varying $u_2, u_3 > 0$ is the same as moving u inside $\mathbb{P}(W) \cap \mathbb{B}$.

First, let us write down the trace equation (10) in the basis $\mathcal{B}$. We define

$$\sqrt{s} := u_2, \quad \sqrt{t} := u_3, \quad \beta_{ij} := \beta_i - \beta_j, \quad \alpha_{ij} := \alpha_i - \alpha_j, \tag{12}$$

for $i, j = 1, 2, 3$. Hence, $u_1 = \sqrt{1 + s + t}$ and $\alpha_{ij}, \beta_{ij} \neq 0$ if $i \neq j$. Using (5), we can write I_1, I_2 in the basis $\mathcal{B}$:

$$I_1 = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} -v_1^2\beta_{23}+u_1^2\beta_{13}+\beta_3 & v_1\overline{v}_2\beta_{23}-u_2u_1\beta_{13} & v_1\overline{v}_3\beta_{23}-u_3u_1\beta_{13} \\ -v_1v_2\beta_{23}+u_1u_2\beta_{13} & |v_2|^2\beta_{23}-u_2^2\beta_{13}+\beta_3 & v_2\overline{v}_3\beta_{23}-u_3u_2\beta_{13} \\ -v_1v_3\beta_{23}+u_1u_3\beta_{13} & \overline{v}_2v_3\beta_{23}-u_2u_3\beta_{13} & |v_3|^2\beta_{23}-u_3^2\beta_{13}+\beta_3 \end{bmatrix}. \tag{13}$$

The trace equation (10) takes the form

$$\alpha_1(-v_1^2\beta_{23} + u_1^2\beta_{13} + \beta_3) + \alpha_2\big(|v_2|^2\beta_{23} - u_2^2\beta_{13} + \beta_3\big) + \alpha_3\big(|v_3|^2\beta_{23} - u_3^2\beta_{13} + \beta_3\big) = \sum_{i=1}^{3} \gamma_i$$

which is equivalent to

$$|v_2|^2\alpha_{21} + |v_3|^2\alpha_{31} = \frac{\beta_{13}}{\beta_{23}}\big(\alpha_{21}s + \alpha_{31}t\big) + k \tag{14}$$

in view of the first two equalities in (11) and in (12), where

$$k := \frac{1}{\beta_{23}}\left(\sum_{i=1}^{3}\gamma_i - \alpha_1(\beta_1 + \beta_2 - \beta_3) - \beta_3(\alpha_2 + \alpha_3)\right).$$

We rewrite equation (14) so that $|v_2|^2$ and $|v_3|^2$ are explicitly given in terms of s and t :

Lemma 15. *The determinant of* $M := \left[\begin{smallmatrix} \mathrm{Re}\alpha_{21} & \mathrm{Re}\alpha_{31} \\ \mathrm{Im}\alpha_{21} & \mathrm{Im}\alpha_{31} \end{smallmatrix}\right]$ *does not vanish. The trace equation is equivalent to the equations*

$$|v_2|^2 \det M = s\mathrm{Im}\frac{\alpha_{31}\overline{\alpha}_{21}\overline{\beta}_{13}}{\overline{\beta}_{23}} + t|\alpha_{31}|^2\mathrm{Im}\frac{\overline{\beta}_{13}}{\overline{\beta}_{23}} + \mathrm{Im}(\alpha_{31}\overline{k}),$$

$$|v_3|^2 \det M = s|\alpha_{21}|^2\mathrm{Im}\frac{\beta_{13}}{\beta_{23}} + t\mathrm{Im}\frac{\overline{\alpha}_{21}\alpha_{31}\beta_{13}}{\beta_{23}} + \mathrm{Im}(\overline{\alpha}_{21}k).$$

The coefficient of t in the first equation and that of s in the second equation do not vanish.

Proof. Note that $|\det M|$ is twice the area of the triangle with vertices $\alpha_1, \alpha_2, \alpha_3$ in the unit circle. Since $\alpha_1, \alpha_2, \alpha_3$ are pairwise distinct, this triangle has non-vanishing area. Similarly, $\mathrm{Im}\frac{\beta_{13}}{\beta_{23}} \neq 0$ because $\frac{\beta_{13}}{\beta_{23}}$ determines an internal angle of the triangle with vertices $\beta_1, \beta_2, \beta_3$.

The trace equation (10) is equivalent to

$$|v_2|^2\mathrm{Re}\alpha_{21} + |v_3|^2\mathrm{Re}\alpha_{31} = \mathrm{Re}z, \qquad |v_2|^2\mathrm{Im}\alpha_{21} + |v_3|^2\mathrm{Im}\alpha_{31} = \mathrm{Im}z,$$

where $z := \frac{\beta_{13}}{\beta_{23}}\big(\alpha_{21}s + \alpha_{31}t\big) + k$. Hence,

$$|v_2|^2 \det M = \mathrm{Im}\alpha_{31}\mathrm{Re}z - \mathrm{Re}\alpha_{31}\mathrm{Im}z = \mathrm{Im}(\alpha_{31}\overline{z}),$$

$$|v_3|^2 \det M = \mathrm{Re}\alpha_{21}\mathrm{Im}z - \mathrm{Im}\alpha_{21}\mathrm{Re}z = -\mathrm{Im}(\alpha_{21}\overline{z}).$$

■

Remark 16. Let us deal with the case where the representation of the turnover providing the isometries I_1, I_2, I_3 is not generic (see Definition 7). If $s = 0$ or $t = 0$, rechoosing the basis $\mathcal{B}$ and using the third equation in (11), we can assume that $v_2, v_3 \geq 0$. Now, the values of v_2, v_3 are determined by Lemma 15. So, we assume $s, t > 0$. If $v_1 = 0$, then $u_2v_3 = -u_3v_3$ by the third equation in (11) and $|v_2|, |v_3|$ are determined by Lemma 15. This implies that the representation space (modulo conjugation) constrained by $v_1 = 0$ have dimension at most 1.

In view of the previous remark, from now on, we assume that the representation of the turnover providing the isometries I_1, I_2, I_3 is generic.

By Lemma 15, the trace equation and the second equation in (11) imply the following inequalities:

$$\begin{aligned}
&\frac{1}{\det M}\big(s\mathrm{Im}\frac{\alpha_{31}\overline{\alpha}_{21}\overline{\beta}_{13}}{\overline{\beta}_{23}} + t|\alpha_{31}|^2\mathrm{Im}\frac{\overline{\beta}_{13}}{\overline{\beta}_{23}} + \mathrm{Im}(\alpha_{31}\overline{k})\big) > 0,\\
&\frac{1}{\det M}\big(s|\alpha_{21}|^2\mathrm{Im}\frac{\beta_{13}}{\beta_{23}} + t\mathrm{Im}\frac{\overline{\alpha}_{21}\alpha_{31}\beta_{13}}{\beta_{23}} + \mathrm{Im}(\overline{\alpha}_{21}k)\big) > 0,\\
&\frac{1}{\det M}\Big(s\big(|\alpha_{21}|^2\mathrm{Im}\frac{\beta_{13}}{\beta_{23}} + \mathrm{Im}\frac{\alpha_{31}\overline{\alpha}_{21}\overline{\beta}_{13}}{\overline{\beta}_{23}}\big)\\
&\qquad + t\big(|\alpha_{31}|^2\mathrm{Im}\frac{\overline{\beta}_{13}}{\overline{\beta}_{23}} + \mathrm{Im}\frac{\overline{\alpha}_{21}\alpha_{31}\beta_{13}}{\beta_{23}}\big) + \mathrm{Im}(\alpha_{31}\overline{k} + \overline{\alpha}_{21}k)\Big) - 1 > 0.
\end{aligned} \tag{C1}$$

Conversely, if Condition **C1** holds for a pair $(s,t) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ of positive real numbers, then the equations in Lemma 15 and the second equation in (11) provide the positive real numbers $v_1, |v_2|, |v_3|$.

In the lemma below, we state a condition, referred to as Condition **C2**, that characterizes the possibility of expressing v_2 and v_3 in terms of $s, t, v_1, |v_2|, |v_3|$.

Lemma 17. *We have*

$$v_2 = \frac{1}{2v_1\sqrt{s(1+s+t)}}\big(- t|v_3|^2 + (1+s+t)v_1^2 + s|v_2|^2 \pm i\sqrt{\Delta}\big),$$

$$v_3 = \frac{1}{2v_1\sqrt{t(1+s+t)}}\big(- s|v_2|^2 + (1+s+t)v_1^2 + t|v_3|^2 \mp i\sqrt{\Delta}\big),$$

where

$$\Delta := 4v_1^2|v_2|^2 s(1+s+t) - \big(- t|v_3|^2 + (1+s+t)v_1^2 + s|v_2|^2\big)^2 \geq 0. \tag{C2}$$

Reciprocally, let $s, t, v_1, |v_2|, |v_3|$ be given positive real numbers such that $-v_1^2 + |v_2|^2 + |v_3|^2 = 1$ and $\Delta \geq 0$. Then v_2, v_3 are well defined in terms of $s, t, v_1, |v_2|, |v_3|$ as above and satisfy $-u_1v_1 + u_2v_2 + u_3v_3 = 0$.

Proof. The third equality in 11 implies that $\mathrm{Re}\, v_3 = \dfrac{u_1v_1 - u_2\mathrm{Re}\, v_2}{u_3}$ and $\mathrm{Im}\, v_3 = -\dfrac{u_2\mathrm{Im}\, v_2}{u_3}$. So,

$$|v_3|^2 = \frac{(u_1v_1 - u_2\mathrm{Re}\, v_2)^2 + u_2^2(\mathrm{Im}\, v_2)^2}{u_3^2} = \frac{u_1^2v_1^2 - 2v_1u_1u_2\mathrm{Re}\, v_2 + u_2^2|v_2|^2}{u_3^2},$$

that is, $\mathrm{Re}\, v_2 = \dfrac{-u_3^2|v_3|^2 + u_1^2v_1^2 + u_2^2|v_2|^2}{2v_1u_1u_2}$. It follows that

$$\mathrm{Im}\, v_2 = \frac{\sigma_1}{2v_1u_1u_2}\sqrt{4v_1^2|v_2|^2u_1^2u_2^2 - \big(- u_3^2|v_3|^2 + u_1^2v_1^2 + u_2^2|v_2|^2\big)^2},$$

where $\sigma_1 \in \{-1, 1\}$. By symmetry, $\mathrm{Re}\, v_3 = \dfrac{-u_2^2|v_2|^2 + u_1^2v_1^2 + u_3^2|v_3|^2}{2v_1u_1u_3}$ and

$$\mathrm{Im}\, v_3 = \frac{\sigma_2}{2v_1u_1u_3}\sqrt{4v_1^2|v_3|^2u_1^2u_3^2 - \big(- u_2^2|v_2|^2 + u_1^2v_1^2 + u_3^2|v_3|^2\big)^2},$$

where $\sigma_2 \in \{-1, 1\}$. Taking $r := u_2^2|v_2|^2 - u_3^2|v_3|^2$ in the tautological equality

$$4v_1^2u_1^2r - (u_1^2v_1^2 + r)^2 + (u_1^2v_1^2 - r)^2 = 0,$$

we obtain

$$4v_1^2|v_2|^2u_1^2u_2^2-\big(-u_3^2|v_3|^2+u_1^2v_1^2+u_2^2|v_2|^2\big)^2=4v_1^2|v_3|^2u_1^2u_3^2-\big(-u_2^2|v_2|^2+u_1^2v_1^2+u_3^2|v_3|^2\big)^2.$$

It follows from $u_2\mathrm{Im}v_2+u_3\mathrm{Im}v_3=0$ that $\sigma_2=-\sigma_1$.

A straightforward computation implies the converse. ∎

Summarizing: Lemmas 15 and 17 imply that Conditions **C1** and **C2** are valid for an isometry I_2 satisfying the trace equation. Reciprocally, given $(s,t)\in\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ such that **C1** holds, we take the point u with coordinates $u_1:=\sqrt{1+s+t}$, $u_2:=\sqrt{s}$, and $u_3:=\sqrt{t}$. Clearly, $\langle u,u\rangle=-1$. The equations in Lemma 15 as well as the second equation in (11) provide the positive numbers $v_1,|v_2|,|v_3|$. Suppose that **C2** holds. Choosing a sign in the formulae for v_2 and v_3 in Lemma 17, we get the point v with coordinates v_1,v_2,v_3 such that $\langle v,v\rangle=1$. By Lemma 17, $\langle u,v\rangle=0$. We have just constructed an isometry I_2 with the fixed points u,v (and the third fixed point uniquely determined by u,v) satisfying the trace equation. The coordinates s,t are geometrical invariants of the representation $\rho:G\to\mathrm{PU}(2,1)$, $\rho:g_i\mapsto I_i$ (G is the turnover group defined in Subsection 3.1). Indeed, $\mathrm{ta}(u,L_1)=1+s$ and $\mathrm{ta}(u,L_2)=1+t$, where L_1,L_2 stand for the I_1-stable complex geodesics. In other words, we parameterized the generic part of the representation space in question. Let us briefly discuss the role of the sign in the formulae for v_2,v_3.

The isometries I_2 and I_2' determined by the different choices of sign in the formulae for v_2,v_3 in Lemma 17 are related as follows. Let u,v,w and u,v',w' stand respectively for the fixed points of I_2 and I_2' (w,w' are the points in $\mathbb{P}(u^\perp)$ orthogonal respectively to v,v'). In the basis $\mathcal{B}$, the reflection R in the $\mathbb{R}$-plane $\mathbb{P}(W)$ ($W\subset V$ is spanned over $\mathbb{R}$ by $\mathcal{B}$) corresponds to the complex conjugation of coordinates. Obviously, $u\in\mathbb{P}(W)$, $Ru=u$, and $Rv=v'$. This implies that $\langle Rw,u\rangle=\overline{\langle w,Ru\rangle}=\overline{\langle w,u\rangle}=0$, i.e., $Rw\in\mathbb{P}(u^\perp)$. Analogously, $\langle Rw,v'\rangle=0$. We obtain $Rq=q'$. In other words, the fixed points of I_2' are those of I_2 reflected in $\mathbb{P}(W)$. Since the eigenvalues of $I^R:=RIR^{-1}$ are complex conjugate to those of I, we obtain $I_1^R=I_1^{-1}$ and $I_2^R={I_2'}^{-1}$. So, the representation given by I_1,I_2',I_3' comes from the one given by $I_1^{-1},I_2^{-1},I_2^{-1}I_3^{-1}I_2$.

Special case: I_2 is a rotation about a point in $\mathbb{H}^2_{\mathbb{C}}$. Let $u\in\mathbb{H}^2_{\mathbb{C}}$ denote the center of I_2 with corresponding eigenvalue β_1. We fix a basis $\mathcal{B}$ in V of signature $-++$ consisting of eigenvectors of I_1 with corresponding eigenvalues $\alpha_1,\alpha_2,\alpha_3$. In this basis, we write $u=\left[\begin{smallmatrix}u_1\\u_2\\u_3\end{smallmatrix}\right]$. We can assume that $u_1,u_2,u_3\geq 0$ and $\langle u,u\rangle=-1$. In other words, $-u_1^2+u_2^2+u_3^2=-1$.

Let us write down the trace equation (10) in the basis $\mathcal{B}$. We define $\beta_{ij}:=\beta_i-\beta_j$ and $\alpha_{ij}:=\alpha_i-\alpha_j$. In particular, $\beta_{23}=0$. It follows from (5) that

$$I_1=\left[\begin{smallmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{smallmatrix}\right],\qquad I_2=\left[\begin{smallmatrix}u_1^2\beta_{13}+\beta_3&-u_2u_1\beta_{13}&-u_3u_1\beta_{13}\\u_1u_2\beta_{13}&-u_2^2\beta_{13}+\beta_3&-u_3u_2\beta_{13}\\u_1u_3\beta_{13}&-u_2u_3\beta_{13}&-u_3^2\beta_{13}+\beta_3\end{smallmatrix}\right].$$

The trace equation takes the form

$$\alpha_1(u_1^2\beta_{13}+\beta_3)+\alpha_2(-u_2^2\beta_{13}+\beta_3)+\alpha_3(-u_3^2\beta_{13}+\beta_3)=\sum_{i=1}^3\gamma_i$$

which is equivalent to

$$u_2^2\alpha_{12}+u_3^2\alpha_{13}=k,\quad k:=\frac{1}{\beta_{12}}\Big(\sum_{i=1}^3\gamma_i-\alpha_1\beta_1-\beta_2(\alpha_2+\alpha_3)\Big).$$

Lemma 18. *The determinant of $M:=\left[\begin{smallmatrix}\mathrm{Re}\alpha_{21}&\mathrm{Re}\alpha_{31}\\\mathrm{Im}\alpha_{21}&\mathrm{Im}\alpha_{31}\end{smallmatrix}\right]$ does not vanish. The trace equation is equivalent to the equations*

$$u_2^2\det M=\mathrm{Im}(\alpha_{13}\overline{k}),\qquad u_3^2\det M=\mathrm{Im}(\alpha_{21}\overline{k}).$$

Proof. The fact $\det M\neq 0$ is proven exactly as in the beginning of the proof of Lemma 15. The trace equation is equivalent to $u_2^2\mathrm{Re}\alpha_{12}+u_3^2\mathrm{Re}\alpha_{13}=\mathrm{Re}k$ and $u_2^2\mathrm{Im}\alpha_{12}+u_3^2\mathrm{Im}\alpha_{13}=\mathrm{Im}k$. Hence,

$$u_2^2\det M=\mathrm{Im}\alpha_{13}\mathrm{Re}k-\mathrm{Re}\alpha_{13}\mathrm{Im}k=\mathrm{Im}(\alpha_{13}\overline{k})$$

$$u_3^2 \det M = \mathrm{Re}\alpha_{12}\mathrm{Im}k - \mathrm{Im}\alpha_{12}\mathrm{Re}k = -\mathrm{Im}(\alpha_{12}\overline{k}).$$ ■

By Lemma 18, the trace equation implies

$$\det M\mathrm{Im}(\alpha_{13}\overline{k}) \geqslant 0, \qquad \det M\mathrm{Im}(\alpha_{21}\overline{k}) \geqslant 0.$$

Conversely, if the above inequalities hold, we obtain from Lemma 18 and from $-u_1^2+u_2^2+u_3^2=-1$ the negative point u with coordinates u_1, u_2, u_3. The corresponding isometry I_2 satisfies the trace equation. Hence, the component of the space $\mathcal{R}$ of conjugacy classes of representations $\rho : G \to \mathrm{PU}(2,1)$ (see Section 3 for the definitions) corresponding to the given conjugacy classes of I_1, I_2, I_3 is either empty or a point. We have a similar result in the case of rotation about a complex geodesic:

Special case: I_2 is a rotation about a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$. Let I_2 be a rotation about the complex geodesic $\mathbb{P}(v^\perp)$ (the eigenvalue corresponding to v is β_2). We fix an orthogonal basis of eigenvectors of I_1 (the eigenvalues are $\alpha_1, \alpha_2, \alpha_3$). In this basis, we write $v = \left[\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right]$ and assume that $v_1, v_2, v_3 \geq 0$ and that $-v_1^2 + v_2^2 + v_3^2 = 1$. The determinant of $M := \left[\begin{smallmatrix} \mathrm{Re}\alpha_{21} & \mathrm{Re}\alpha_{31} \\ \mathrm{Im}\alpha_{21} & \mathrm{Im}\alpha_{31} \end{smallmatrix}\right]$ does not vanish (see Lemma 15) and the trace equation (10) is equivalent to the equations

$$v_2^2 \det M = \mathrm{Im}(\alpha_{13}\overline{k}), \qquad v_3^2 \det M = \mathrm{Im}(\alpha_{21}\overline{k}),$$

where

$$k := \frac{1}{\beta_{12}}\Big(\sum_{i=1}^{3}\gamma_i - \alpha_1\beta_2 - \beta_1(\alpha_2+\alpha_3)\Big).$$

The trace equation and the equation $-v_1^2 + v_2^2 + v_3^2 = 1$ imply

$$\det M\mathrm{Im}(\alpha_{13}\overline{k}) \geq 0, \qquad \det M\mathrm{Im}(\alpha_{21}\overline{k}) \geq 0, \qquad \frac{1}{\det M}\big(\mathrm{Im}(\alpha_{13}\overline{k}) + \mathrm{Im}(\alpha_{21}\overline{k})\big) \geq 1.$$

Conversely, if the above inequalities hold, we obtain from the trace equation and from the equation $-v_1^2 + v_2^2 + v_3^2 = 1$ the positive point v with coordinates v_1, v_2, v_3.

6 Discreteness: fundamental quadrangle of bisectors

6.1. Quadrangle of bisectors. Following [AGG], we introduce the *quadrangle* of bisectors associated to some of the faithful representations $\rho : G \to \mathrm{PU}(2,1)$ discussed in Subsection 3.2. We expect quadrangles of bisectors to bound fundamental polyhedra for discrete actions of G on $\mathbb{H}^2_{\mathbb{C}}$ and the quotient $\mathbb{H}^2_{\mathbb{C}}/G$ to be a disc orbibundle over an orbifold (in our case, a sphere with three cone points). Passing to finite index, one arrives at a complex hyperbolic disc bundle over a closed orientable surface (this comes from the fact that a finitely generated Fuchsian group always has a finite index torsion-free subgroup).

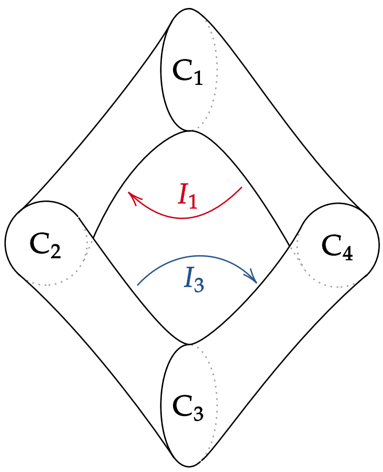


Figure 8: Quadrangle $\mathcal{Q}$

We remind here a few definitions from [AGG].

In order to orient a bisector B we only need to orient its real spine (since the fibers are complex, hence, naturally oriented). An oriented bisector B divides $\overline{\mathbb{H}}^2_{\mathbb{C}}$ into two *half-spaces* (closed 4-balls) K^+ and K^-, where K^+ stands for the half-space lying on the side of the normal vector to B.

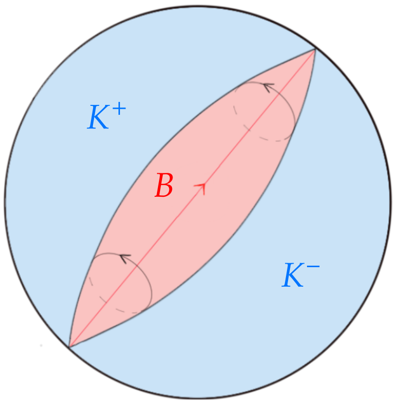


Figure 9: Orienting the real spine fixes a unique orientation of the bisector since the slices are naturally oriented. The half-space K^+ is the one on the side of the normal vector.

Let $B_1 = B_1[C_1, C_2]$ and $B_2 = B_2[C_1, C_3]$ be two oriented segments of bisectors with a common slice C_1 such that the corresponding full bisectors are transversal along that slice. The *sector* from B_1 to B_2 is defined to be either $K_1^+ \cap K_2^-$ (when the oriented angle from $B[C_1, C_2]$ to $B[C_1, C_3]$ at a point $c \in C_1$ is smaller than π) or $K_1^+ \cup K_2^-$ (when the oriented angle from $B[C_1, C_2]$ to $B[C_1, C_3]$ at a point $c \in C_1$ is greater than π). Note that, while such oriented angle does depend on the point c, it cannot equal π due to transversality.

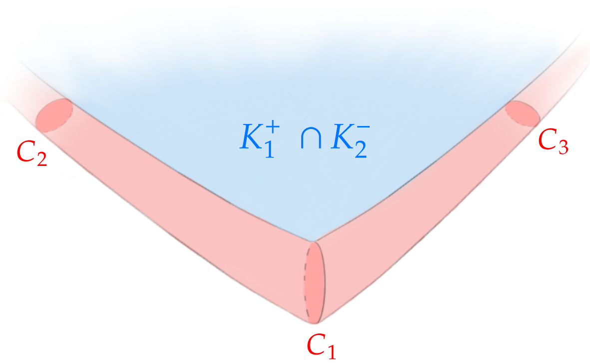


Figure 10: Sector between given by B_1 and B_2 when the angle between then is smaller than π.

Given pairwise ultraparallel complex geodesics C_1, C_2, C_3, the oriented triangle of bisectors $\Delta(C_1, C_2, C_3)$ is simply the union $B[C_1, C_2] \cup B[C_2, C_3] \cup B[C_3, C_1]$ of oriented segments of bisectors. Each such segment is a side of the oriented triangle and each of the complex geodesics C_1, C_2, C_3 is a vertex of the triangle. The triangle is *transversal* if the full bisectors containing its sides intersect transversally along the common slices.

Given three ultraparallel complex geodesics C_1, C_2, C_3 there are two possible orientations for a triangle of bisectors with vertices C_1, C_2, C_3. Assuming that such a triangle is transversal, its *counterclockwise orientation* is the one providing an acute oriented angle from $B[C_3, C_1]$ and $B[C_1, C_2]$. By [AGG, Lemma 2.13], this implies that the oriented angles from $B[C_1, C_2]$ to $B[C_2, C_3]$ and from $B[C_2, C_3]$ to $B[C_3, C_1]$ are both acute as well; moreover, in this case, each side of the triangle is contained in the sector determined by the other two.

Let $\Delta(C_1, C_2, C_4)$ and $\Delta(C_3, C_4, C_2)$ be counterclockwise oriented transversal triangles of bisectors sharing a common side. We say these triangles are *transversally adjacent* if the sector at C_1 contains a point of C_3 and the full bisectors containing the segments $B[C_1, C_2]$ and $B[C_2, C_3]$ (respectively, $B[C_3, C_4]$ and $B[C_4, C_1]$) are transversal at C_2 (respectively, at C_4). In particular (see [AGG, Lemma 2.14]), this implies that $\Delta(C_3, C_4, C_2)$ is contained in the sector at C_1; furthermore, $\Delta(C_3, C_4, C_2)$ and $\Delta(C_1, C_2, C_4)$ lie in opposite sides of the full bisector containing $B[C_2, C_4]$.

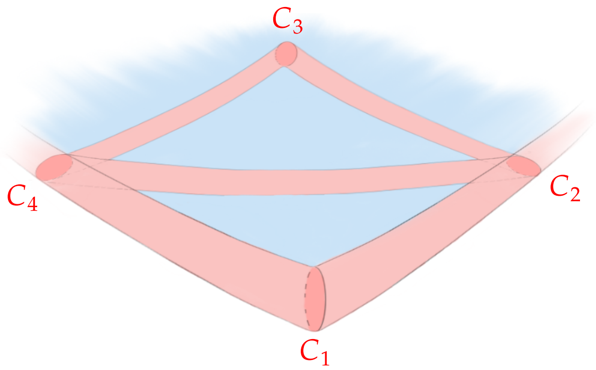


Figure 11: Transversally adjacent triangles.

Let I_1, I_2, I_3 be elliptic isometries as in Subsection 3.2. Let $c_j, p_j \in \mathbb{P}_{\mathbb{C}}(V)$ denote pairwise orthogonal distinct fixed points of I_j with $c_j \in \mathbb{H}^2_{\mathbb{C}}$. In particular, the p_j's are positive. We also define the points $c_4 := I_1^{-1}c_2$ and $p_4 := I_1^{-1}p_2$ and the complex geodesics

$$C_1 := \mathbb{P}(p_1)^{\perp} \cap \overline{\mathbb{H}}^2_{\mathbb{C}}, \qquad C_2 := \mathbb{P}(p_2)^{\perp} \cap \overline{\mathbb{H}}^2_{\mathbb{C}},$$

$$C_3 := \mathbb{P}(p_3)^{\perp} \cap \overline{\mathbb{H}}^2_{\mathbb{C}}, \qquad C_4 := \mathbb{P}(p_4)^{\perp} \cap \overline{\mathbb{H}}^2_{\mathbb{C}}.$$

Using the relation $I_3I_2I_1 = 1$ we obtain $p_4 = I_1^{-1}p_2 = I_3p_2$. Therefore $C_4 = I_1^{-1}C_2 = I_3C_2$. Note that, by Remark 4, if C_1 and C_2 are ultraparallel, $C_1 \| C_2$, then $C_1 \| C_4$ since $\operatorname{ta}(p_1, p_2) = \operatorname{ta}(I_1^{-1}p_1, I_1^{-1}p_2) = \operatorname{ta}(p_1, p_4) > 1$. Similarly, $C_3 \| C_2$ implies $C_3 \| C_4$. So, if $C_1 \| C_2$, $C_3 \| C_2$, and $C_2 \| C_4$, we get the oriented triangles of bisectors $\Delta(C_1, C_2, C_4)$ and $\Delta(C_3, C_4, C_2)$.

6.2. Quadrangle conditions. A representation $\rho : G \to \mathrm{PU}(2,1)$, $g_j \mapsto I_j$, satisfies the quadrangle conditions if

(Q1) $C_1 \| C_2$, $C_3 \| C_2$, and $C_2 \| C_4$.

(Q2) The triangles $\Delta(C_1, C_2, C_4), \Delta(C_3, C_4, C_2)$ are transversal and counterclockwise-oriented.

(Q3) The triangles $\Delta(C_1, C_2, C_4), \Delta(C_3, C_4, C_2)$ are transversally adjacent;

(Q4) The oriented angle from $\mathrm{B}[C_1, C_4]$ to $\mathrm{B}[C_1, C_2]$ at c_1 equals $\dfrac{2\pi}{n_1}$; the oriented angle from $\mathrm{B}[C_3, C_2]$ to $\mathrm{B}[C_3, C_4]$ at c_3 equals $\dfrac{2\pi}{n_3}$; the sum of the oriented angle from $\mathrm{B}[C_2, C_1]$ to $\mathrm{B}[C_2, C_3]$ at c_2 with the oriented angle from $\mathrm{B}[C_4, C_3]$ to $\mathrm{B}[C_4, C_1]$ at c_4 equals $\dfrac{2\pi}{n_2}$.

A representation satisfying the quadrangle conditions give rise to the *quadrangle of bisectors*

$$\mathcal{Q} := B[C_1, C_2] \cup B[C_2, C_3] \cup B[C_3, C_4] \cup B[C_4, C_1].$$

The quadrangle $\mathcal{Q}$ bounds polyhedron Q which is on the side of the normal vectors of the oriented segments of bisectors. Indeed, by [AGG, Lemma 2.13], there are no intersections between those segments of bisectors besides the common slices.

6.3. Discreteness. Let $\rho : G \to \mathrm{PU}(2,1)$, $g_j \mapsto I_j$, be a faithful representation satisfying the quadrangle conditions and let $\mathcal{Q}$ be the quadrangle of ρ described in the previous subsection.

Applying [AGr2, Theorem 3.2] we will show that $Q \cap \mathbb{H}^2_{\mathbb{C}}$ is a fundamental region for the action of the group K_n generated by I_1, I_3 with the defining relations $I_1^{n_1} = I_3^{n_3} = (I_3^{-1}I_1^{-1})^{n_2} = 1$ in $\mathrm{PU}(2,1)$. The main idea is to prove that, given a point x in the polyhedron P, there are corresponding copies of the polyhedron Q that tessellate a (small) ball centered at x. When x belongs to the interior of Q, the fact is immediate; when it lies in the interior of a side, it follows from the fact that the elliptic isometries I_1 and I_3 send the interior of Q to its exterior. Finally, when x is in a vertex, it is enough to understand the case $x = c_j$. Here, the tessellation follows from an infinitesimal conditional: the local tessellation of the complex geodesic normal to the vertex at c_j. For more details, see [AGr2]. This leads to a tessellation of a neighborhood of Q and, by [AGG, Lemma 2.10], such a tessellation provides a tessellation of a metric neighborhood of Q. By [AGr2, Theorem 3.2], K_n is discrete.

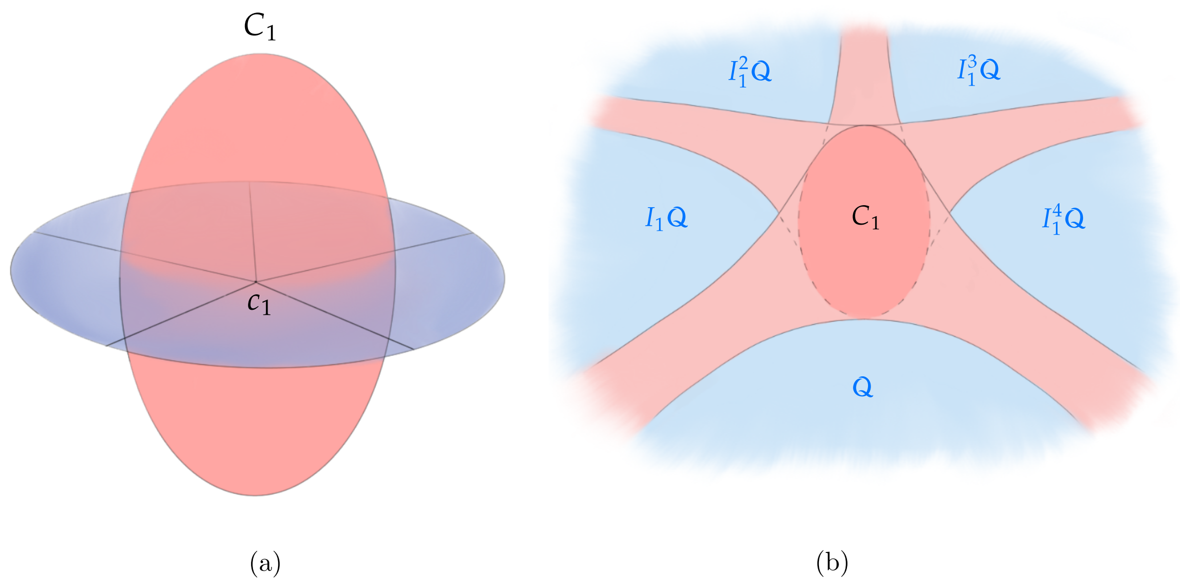


Figure 12: **(a)** Tesselation of the complex geodesic normal to C_1 at c_1, and **(b)** tesselation around the vertice C_1. In both cases, I_1 has order 5.

Theorem 19. *The group K_n is discrete and Q is a fundamental domain for its action on $\mathbb{H}^2_{\mathbb{C}}$.*

Proof. By [AGG, Lemma 2.10], we only need to verify Conditions (i) and (ii) of [AGr2, Theorem 3.2]. Since I_1 maps $\mathrm{B}[C_1, C_4]$ onto $\mathrm{B}[C_1, C_2]$ and I_3 maps $\mathrm{B}[C_3, C_1]$ onto $\mathrm{B}[C_3, C_4]$, the Condition (i) of [AGr2, Theorem 3.2] follows from the definition of counterclockwise-oriented transversal triangles. There are three (geometrical) cycles of vertices. The cycle of C_1 have total angle 2π at $c_1 \in C_1$ by [AGG, Lemma 3.4]. The same concerns the cycle of C_3 at $c_3 \in C_3$.

The geometric cycle of C_2 has length $2n_2$ due to the relation $I_2^{n_2} = 1$. Let us verify that the total angle at $c_2 \in C_2$ is 2π. Note that I_3^{-1} sends $\mathrm{B}[C_3, C_4]$ onto $\mathrm{B}[C_3, C_2]$ and sends $\mathrm{B}[C_4, C_1]$ onto $\mathrm{B}[C_2, I_2C_1]$ (indeed, $I_3^{-1}C_1 = I_3^{-1}I_1^{-1}C_1 = I_2C_1$). Therefore, by the definition of counterclockwise-orientation, the sum of the interior angle from $\mathrm{B}[C_2, C_3]$ to $\mathrm{B}[C_2, C_1]$ at c_2 with the interior angle from $\mathrm{B}[C_4, C_1]$ to $\mathrm{B}[C_4, C_3]$ at c_4 equals the angle from $\mathrm{B}[C_2, C_3]$ to $\mathrm{B}[C_2, I_2C_3]$ at c_2. By [AGG, Lemma 3.4], this angle equals $\mathrm{Arg}(\beta_1^{-1}\beta_2) = 2\pi/n_2$. □

7 Orbifold bundles and Euler number

7.1. The quadrangle conditions revisited. As in subsection 3.2, let $\rho : G(n_1, n_2, n_3) \to \mathrm{PU}(2,1)$ be a faithful representation of the turnover group and define $I_k := \rho(g_k)$. Assume that I_1, I_3 are regular elliptic. We choose $I_k \in \mathrm{SU}(2,1)$ as in Remark 9 and fix a negative eigenvector c_k of I_k as well as a positive one, p_k. Let α_i, β_i, γ_i^{-1}, $i = 1, 2, 3$, stand respectively for the eigenvalues of I_1, I_2, I_3 such that

$$I_1(c_1) = \alpha_1 c_1,\ I_2(c_2) = \beta_1 c_2,\ I_3(c_3) = \gamma_1^{-1} c_3$$

and

$$I_1(p_1) = \alpha_2 p_1,\ I_2(p_2) = \beta_2 p_2,\ I_3(p_3) = \gamma_2^{-1} p_3.$$

Since $\det I_k = 1$, we must have

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3 = 1.$$

Let us revisit the quadrangle conditions 6.2. This time, we will also formulate such conditions in terms of algebraic formulas that are used both in this section and in section 4.

Define

$$C_1 = \mathbb{P}(p_1^\perp) \cap \mathbb{H}^2_\mathbb{C}, \quad C_2 = \mathbb{P}(p_2^\perp) \cap \mathbb{H}^2_\mathbb{C}, \quad C_3 = \mathbb{P}(p_3^\perp) \cap \mathbb{H}^2_\mathbb{C} \quad \text{and} \quad C_4 = I_1^{-1} C_1 \cap \mathbb{H}^2_\mathbb{C}.$$

Note that, putting $c_4 := I_1^{-1} c_2$ and $p_4 := I_1^{-1} p_2$, we have $C_4 = \mathbb{P}(p_4^\perp) \cap \mathbb{H}^2_\mathbb{C}$. Quadrangle condition (Q1) asks that the complex geodesics C_1, C_2, C_3, C_4 are pairwise ultraparallel:

$$\operatorname{ta}(p_1, p_2) > 1, \quad \operatorname{ta}(p_2, p_3) > 1, \quad \operatorname{ta}(p_1, p_3) > 1 \quad \text{and} \quad \operatorname{ta}(p_1, p_4) > 1 \tag{Q1}$$

(see 4).

Assuming (Q1) we can define the bisectors segments $\mathrm{B}[C_1, C_2]$, $\mathrm{B}[C_2, C_3]$, $\mathrm{B}[C_2, C_3]$, and $\mathrm{B}[C_3, C_4]$.

Condition (Q2) says that the triangles of bisectors $\Delta(C_1 C_2 C_4)$ and $\Delta(C_2 C_3 C_4)$ are transversal and counterclockwise oriented; this is equivalent to the inequalities

$$\begin{gathered}\epsilon_0^2 t^2 + s^2 + t^2 < 1 + 2t^2 s \epsilon_0, \quad \epsilon_0^2 s^2 + 2t^2 < 1 + 2t^2 s \epsilon_0, \quad \epsilon_1 < 0, \\ \epsilon_0'^2 t'^2 + s^2 + t'^2 < 1 + 2t'^2 s \epsilon_0', \quad \epsilon_0'^2 s^2 + 2t'^2 < 1 + 2t'^2 s \epsilon_0', \quad \epsilon_1' < 0\end{gathered} \tag{Q2}$$

where

$$\begin{gathered} t_{12} := \sqrt{\operatorname{ta}(p_1, p_2)}, \quad t_{23} := \sqrt{\operatorname{ta}(p_2, p_4)}, \quad t_{31} := \sqrt{\operatorname{ta}(p_4, p_1)}, \\ t'_{12} := \sqrt{\operatorname{ta}(p_2, p_3)}, \quad t'_{23} := \sqrt{\operatorname{ta}(p_3, p_4)}, \quad t_{31} := \sqrt{\operatorname{ta}(p_4, p_2)}, \\ \epsilon_0 + \epsilon_1 i := \frac{\sigma}{|\sigma|}, \quad \text{where} \quad \sigma := \langle p_1, p_2 \rangle \langle p_2, p_4 \rangle \langle p_4, p_1 \rangle, \\ \epsilon_0' + \epsilon_1' i := \frac{\sigma'}{|\sigma'|}, \quad \text{where} \quad \sigma' := \langle p_2, p_3 \rangle \langle p_3, p_4 \rangle \langle p_4, p_2 \rangle \end{gathered}$$

(see [AGG, Criterion 2.27]). Note that $t_{12} = t_{31}$, $t'_{12} = t'_{23}$, and $t_{23} = t'_{31}$. We write $t := t_{12}$, $t' = t'_{12}$ and $s = t_{23}$.

The quadrangle condition (Q3) asserts that the triangles $\Delta(C_1, C_2, C_4)$ and $\Delta(C_3, C_4, C_2)$ are transversally adjacent. It is guaranteed by the conditions (Q3.1), (Q3.2), (Q3.3) below. Condition (Q3.1) concerns the transversality of the bisectors $B[C_1, C_2]$ and $B[C_2, C_3]$,

$$\left| \operatorname{Re} \left(\frac{\langle p_3, p_1 \rangle \langle p_2, p_2 \rangle}{\langle p_3, p_2 \rangle \langle p_2, p_1 \rangle} \right) - 1 \right| < \sqrt{1 - \frac{1}{\operatorname{ta}(p_2, p_3)}} \sqrt{1 - \frac{1}{\operatorname{ta}(p_2, p_1)}}. \tag{Q3.1}$$

(see [AGG, Criterion 3.3]). Similarly, (Q3.2) states the transversality of the bisectors $B[C_1, C_4]$ and $B[C_4, C_3]$,

$$\left| \operatorname{Re} \left(\frac{\langle p_1, p_3 \rangle \langle p_4, p_4 \rangle}{\langle p_1, p_4 \rangle \langle p_4, p_3 \rangle} \right) - 1 \right| < \sqrt{1 - \frac{1}{\operatorname{ta}(p_4, p_1)}} \sqrt{1 - \frac{1}{\operatorname{ta}(p_4, p_3)}}. \tag{Q3.2}$$

Finally, (Q3.3) implies that c_3 belongs to the interior of the sector at C_1 of the triangle $\Delta(C_1, C_2, C_4)$:

$$\operatorname{Im} \left(\frac{\langle p_1, c_3 \rangle \langle c_3, p_2 \rangle}{\langle p_1, p_2 \rangle} \right) \geq 0 \quad \text{and} \quad \operatorname{Im} \left(\frac{\langle p_4, c_3 \rangle \langle c_3, p_1 \rangle}{\langle p_4, p_1 \rangle} \right) \geq 0 \tag{Q3.3}$$

(see [AGG, Lemma 3.5]).

Consider the polyhedron Q bounded in $\mathbb{H}^2_\mathbb{C}$ by the quadrangle

$$\mathcal{Q} := \big(B[C_1, C_2], B[C_2, C_3], B[C_3, C_4], B[C_4, C_1] \big).$$

It follows from [AGG, Lemma 3.4] that condition (Q4) translates, in terms of the eigenvalues $\alpha_i, \beta_i, \gamma_i$, into

$$\alpha_2/\alpha_1 = \exp(-2\pi i/n_1), \quad \beta_2/\beta_1 = \exp(-2\pi i/n_2), \quad (\gamma_2/\gamma_1)^{-1} = \exp(-2\pi i/n_3) \tag{Q4}$$

7.2. Deformation lemma. Given a complex geodesic C with a chosen $c \in C$, we identify C with the unit open disc in $\mathbb{C}$ as follows. Let p be the point orthogonal to c in the complex projective line extending C. Take representatives such that $-\langle c, c\rangle = \langle p, p\rangle = 1$. Then every point in C has the form $c + \gamma p$, $|\gamma| \leq 1$. For obvious reasons, we call c the center of C.

Consider the action of $\mathbb{S}^1$ on the circle $\partial_\infty C$ by rotations centered at c. More precisely, given a unit complex number $\theta \in \mathbb{S}^1$, we define

$$\gamma[c + \theta p] = [c + \gamma\theta p].$$

In particular, we have an $\mathbb{S}^1$-action on the vertices C_i of the quadrangle $\mathcal{Q}$, where each C_i has center c_i.

Lemma 20. *Consider an orientation on V and let q_i be a vector such that c_i, p_i, q_i is a positively oriented orthonormal basis of V. There is a family of curves $c_i(\delta), p_i(\delta), q_i(\delta)$, with $\delta \in [0, 1]$, such that*

- *$c_i(0) = c_i$, $p_i(0) = p_i$ and $q_i(0) = q_i$;*
- *for each δ the vectors $c_i(\delta), p_i(\delta), q_i(\delta)$ form a positively oriented orthonormal basis of V;*
- *for all i, j, with $i \neq j$, we have $\mathrm{ta}(p_i(\delta), p_j(\delta)) > 1$;*
- *If $C_i(\delta) := \mathbb{P}(p_i(\delta)^\perp)$ and we consider the triangles of bisectors $\triangle_1(\delta)$, with vertices $C_1(\delta)$, $C_2(\delta)$ and $C_4(\delta)$, and $\triangle_2(\delta)$, with vertices $C_2(\delta)$, $C_3(\delta)$ and $C_4(\delta)$, then for each δ the triangles $\triangle_1(\delta)$ and $\triangle_2(\delta)$ are transversal and counter-clockwise oriented.*
- *$q_1(1) = q_2(1) = q_3(1) = q_4(1)$.*

The last item means that if $\mathcal{Q}(\delta)$ is the quadrangle with vertices $C_i(\delta)$, then the bisectors forming the boundary of $\mathcal{Q}(1)$ all have the same focus.

Proof: Consider $s, t, t', \epsilon_0, \epsilon_0'$ as consider in the inequalities (Q2). It is know from lemma **A.31** in [AGG] that the parameters s, t, ϵ_0, with $t, s > 1$ and $0 < \epsilon_0 < 1$, determine up to isometry the transversal counter-clockwise oriented triangle of bisectors $\triangle_1 := \triangle[C_1, C_2, C_4]$.

Suppose $t > s$. We will show that choosing a convenient $\epsilon_0 > 0$ we can reduce t until $t = s$. The inequalities which determine that $\triangle_1$ is transversal are the

$$\epsilon_0^2 s^2 + 2t^2 < 1 + 2\epsilon_0 st^2 \leq 2t^2 + s^2.$$

Consider the quadratic polynomial $f(x) = x^2s^2 - 2xst^2 + 2t^2 - 1$. The roots of $f(x) = 0$ are

$$x = \frac{t^2 \pm (t^2 - 1)}{s}$$

and, therefore, $f(x) < 0$ when $1/s < x < (2t^2 - 1)/s$. Notice $x_0 = (2s^2 - 1)/s$ is between $1/s$ and $(2t^2-1)/s$. We can reduce ϵ_0 until $1/s < \epsilon_0 < x_0$. Now, since the inequality $\epsilon_0^2 s^2 + 2t^2 < 1 + 2\epsilon_0 st^2$ is equivalent

$$t^2 > \frac{s\epsilon_0 + 1}{2},$$

and, by our choice of ϵ_0,

$$s^2 > \frac{s\epsilon_0 + 1}{2},$$

we can reduce t until $t = s$.

Note that the inequality $1 + 2\epsilon_0 st^2 \leq s^2 + 2t^2$ is kept during the above procedure.

Applying the same reasoning to t' we may suppose $1 < t, t' \leq s$.

Now, we will show that we can deform t and ϵ_0 until $t = s$ and $1 + 2\epsilon_0 s^3 = 3s^2$ always keeping $1 < t \leq s$, $0 < \epsilon_0 < 1$ and $\epsilon_0^2 t^2 + s^2 + t^2 < 1 + 2\epsilon_0 st^2$.

Indeed, if $t < s$, then increase t until one of the two following possibilities happens:

$$t = s \quad \text{or} \quad 1 + 2\epsilon_0 st^2 = 2t^2 + s^2.$$

If $t = s$, then we have $1 + 2\epsilon_0 s^3 \leq 3s^2$, which is equivalent to

$$\epsilon_0 \leq \frac{3s^2 - 1}{2s^3}.$$

Now, the function $g(x) := \frac{3x^2-1}{2x^3}$ is strictly decreasing for $x > 1$ and, therefore, $g(x) < g(1) = 1$ for $x > 1$. Therefore, we can increase ϵ_0 until

$$\epsilon_0 = \frac{3s^2 - 1}{2s^3},$$

or equivalently $1 + 2\epsilon_0 s^3 = 3s^2$.

If $1 < t \leq s$ and $1 + 2\epsilon_0 st^2 = 2t^2 + s^2$, with $0 < \epsilon_0 < 1$, then we have the inequality

$$2t^2 + s^2 < 2st^2 + 1,$$

or equivalently

$$t^2 > \frac{s^2 - 1}{2(s-1)} = \frac{s+1}{2}.$$

Therefore, we can increase t until $t = s$ and have $\epsilon_0 \in (0, 1)$ satisfying $1 + 2\epsilon_0 s^3 = 3s^2$.

So, we can deform t and ϵ_0, always keeping $t > 1$ and $0 < \epsilon_0 < 1$ during the process, and in the end we obtain $t = s$ and $1 + 2\epsilon_0 s^3 = 3s^2$.

By the same reasoning we can deform t' and ϵ_0' such that we always have $t' > 1$ and $0 < \epsilon_0' < 1$ during the deformation and in the end we obtain $t' = s$ and $1 + 2\epsilon_0' s^3 = 3s^2$.

So we reduced the problem to the case where $t = t' = s$ and $\epsilon_0 = \epsilon_0'$. Geometrically we deformed the quadrangle P inside $\mathbb{H}^2_{\mathbb{C}}$ always keeping the vertices C_2 and C_4 fixed and moving C_1 and C_3 around such that the two triangles of bisectors are kept transversal and counter-clockwise oriented. In the case we are now all sides of the quadrangle have the same length.

Now, the quadrangle P depends only on the parameters s and ϵ_0, which means it depends only on the triangle $\triangle_1$. Let q be the focus of the bisector $\mathrm{B}[C_2, C_4]$. Using that the space of transversal and counter-clockwise oriented triangles of bisectors is path-connected [AGG, Lemma 2.28] we can deform $\triangle_1$ until $q_1 = q_2 = q_3 = q$. The same deformation will be done to the triangle $\triangle_2 = \triangle[C_2, C_3, C_4]$ simultaneously using the same parameters of $\triangle_1$. Therefore, in the end of the deformation we have the desired quadrangle. $\square$

Now we apply lemma 20 to the quadrangle $\mathcal{Q}$. Deform the vertices C_1, C_2, C_3, C_4 until the focuses of the four bisectors coincide in one point $q \in \mathrm{E}(V)$. Let C_1', C_2', C_3', C_4' stand for the vertices at the end of the deformation. We can assume that the deformation is such that the centers c_1, c_2, c_3, c_4 belong to $\mathbb{P}(q^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$ at the end and are the vertices c_1', c_2', c_3', c_4' of a convex quadrilateral P. Also, we can suppose that the angles at c_1', c_2', c_3', c_4' are respectively $2\pi/n_1, \pi/n_2, 2\pi/n_3, \pi/n_2$, that is, this quadrilateral constitutes a fundamental polygon for the turnover group $\langle R_1, R_2, R_3\rangle$ action on the hyperbolic plane $\mathbb{P}(q^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$; here, R_1, R_2, R_3 are rotations in $\mathbb{P}(q^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$ with respective centers c_1', c_2', c_3' and angles $-2\pi/n_1, -2\pi/n_2, -2\pi/n_3$ and satisfying the relation $R_3 R_2 R_1 = 1$.

We have a polyhedron Q' bounded in $\mathbb{H}^2_{\mathbb{C}}$ by the quadrangle

$$\mathcal{Q}' := (\mathrm{B}[C_1', C_2'], \mathrm{B}[C_2', C_3'], \mathrm{B}[C_3', C_4'], \mathrm{B}[C_4', C_1']).$$

The deformation gives rise to a diffeomorphism

$$F : Q' \to Q$$

such that the restriction $F|_{\mathcal{Q}'} : \mathcal{Q}' \to \mathcal{Q}$ maps slices to slices isometrically. Furthermore, we can assume that the geodesic curves $G[c_i', c_{i+1}']$ are mapped by this diffeomorphism to curves g_i with end points c_i and c_{i+1} such that $I_1 g_4 = g_1$ and $I_3 g_2 = g_3$. The curve $g_1 \cup g_2 \cup g_3 \cup g_4$ intersects each slice of $\mathcal{Q}$ in one point, which we will take as a center. Given these centers, we introduce an $\mathbb{S}^1$-action on each slice of $\mathcal{Q}$ such that the map F restricted to $\mathcal{Q}'$ is $\mathbb{S}^1$-equivariant.

Note that there are two ways of mapping $B[C_1', C_2']$ to $B[C_4, C_1]$. The first one is by the map

$$[x + \gamma q] \mapsto I_1^{-1} F(x + \gamma q)$$

and the second one is

$$[x+\gamma q] \mapsto F(R^{-1}x + (\alpha_3/\alpha_1)\gamma q),$$

and since the diffeomorphism F maps slices to slices isometrically we have this two maps coincide in C_1'. Nevertheless, we want these two maps to be equal near C_1' in order to calculate the Euler number of the disc orbibundle over $\mathbb{S}^2(n_1,n_2,n_3)$ to be constructed, and the lemma bellow tell us that this is possible. The proof of this lemma is based on the idea of "twisting the tube". More visually, we have the diffeomorphism $G : B[C_1', C_2'] \to B[C_1', C_2']$ given by $[x+\gamma q] \mapsto F^{-1}I_1F(R^{-1}x + (\alpha_3/\alpha_1)\gamma q)$ and it gives us the behavior described in the figure 13.

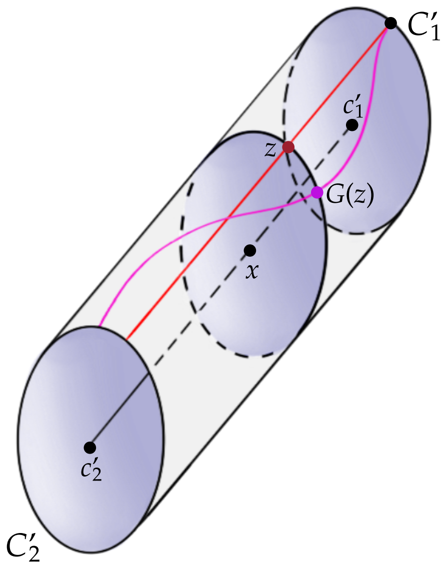


Figure 13: Consider the red curve intersecting each slice of the bisector $B[C_1', C_2']$ once. If z is in this red curve, then it can be writen as $z = x + \gamma q$, where x is a representative of the center of the disc containing z and satisfying $\langle x, x\rangle = -1$, and $G(z)$ will be a rotation depending on the center x. Therefore, for x near c_1' we have $G(x+\gamma q) = \big[x + \exp(i\theta(x))\gamma q\big]$, with $\theta(c_1') = 0$.

We want the red and the pink curve to coincide near C_1'.

Lemma 21. *We can modify the diffeomorphism $F : \mathcal{Q}' \to \mathcal{Q}$ in such a way it still maps slices to slices isometrically and*

$$F(x+\gamma q) = I_1F(R_1^{-1}x + (\alpha_3/\alpha_1)^{-1}\gamma q) \quad \textit{for} \quad x \in N_1 \cap \mathrm{G}[c_1', c_2'], \quad \langle x, x\rangle = -1,$$

for some neighborhood N_1 of c_1' in $\mathbb{P}(q^\perp)$.

Proof: On the vertices we have

$$\gamma F(c_i' + \theta q) = F(c_i' + \gamma\theta q),$$

and therefore the desired identity holds on c_1'.

By continuity, for a small neighborhood V of c_1' on the geodesic $\mathrm{G}[c_1', c_2']$ we have a smooth function $\theta : V \to \mathbb{R}$ satisfying

$$F^{-1}I_1F\big(R_1^{-1}x + (\alpha_3/\alpha_1)^{-1}\gamma q\big) = \big[x + \exp(i\theta(x))\gamma q\big] \quad \text{for} \quad x \in V \cap \mathrm{G}[c_1', c_2'], \quad \langle x, x\rangle = -1.$$

In particular, we may suppose $\theta(c_1') = 0$.

There is $\widetilde{\theta}$ in $\mathrm{G}[c_1, c_2]$ such that $\theta(x) = \widetilde{\theta}(x)$ in a small compact neighborhood $N_1 \subset V$ of c_1' such that $\operatorname{supp}(\widetilde{\theta}) \subset V$. Furthermore, we can extend $\widetilde{\theta}$ to all the quadrilateral P in such a way that $\widetilde{\theta}$ is zero over the geodesics $\mathrm{G}[c_2', c_3']$, $\mathrm{G}[c_3', c_4']$, and $\mathrm{G}[c_4', c_1']$. Therefore, we can consider $\widehat{F}(x+\gamma q) = F\Big(x + \exp\big(i\widetilde{\theta}(x)\big)\gamma q\Big)$, with $\langle x, x\rangle = -1$. With this new map we have

$$\widehat{F}(x+\gamma q) = I_1\widehat{F}\big(R_1^{-1}x + (\alpha_3/\alpha_1)^{-1}\gamma q\big) \quad \text{for} \quad x \in N_1 \cap \mathrm{G}[c_1', c_2'], \quad \langle x, x\rangle = -1,$$

where we are using $\widetilde{\theta}(R^{-1}x) = 0$ because $R_1^{-1}x \in G[c_4', c_1']$. $\square$

We may also suppose the same kind of property for the other vertices: For $i = 2, 3, 4$ we have a small neighborhood N_i of the point c_i' in $\mathbb{P}(q^\perp)$ such that

$$\begin{aligned}
F(x+\gamma q) &= I_3^{-1}F\big(R_3x + (\beta_3/\beta_1)\gamma q\big) && \text{for} \quad x \in N_2 \cap \mathrm{G}[c_2', c_3'], \quad \langle x, x\rangle = -1,\\
F(x+\gamma q) &= I_3F\big(R_3^{-1}x + (\gamma_3/\gamma_1)\gamma q\big) && \text{for} \quad x \in N_3 \cap \mathrm{G}[c_3', c_4'], \quad \langle x, x\rangle = -1,\\
F(x+\gamma q) &= I_1^{-1}F\big(R_1x + \gamma q\big) && \text{for} \quad x \in N_4 \cap \mathrm{G}[c_4', c_1'], \quad \langle x, x\rangle = -1.
\end{aligned}$$

7.3. Constructing complex hyperbolic disc orbigoodles over $\mathbb{S}^2(n_1, n_2, n_3)$. An **$n$-orbifold** B is a space locally modeled by quotients of the form $\mathbb{D}^n/\Gamma$, where Γ is a finite subgroup of $\mathrm{O}(n)$. All orbifolds considered in this paper are locally oriented, which means that we are only considering trivializations with $\Gamma \subset \mathrm{SO}(n)$. More technically by space we mean diffeological space (for details see [Bot]). A diffeomorphism $\phi : \mathbb{D}^n/\Gamma \to D$, where D stands for an open subset of B, is called **orbifold chart**. Furtheremore, if $\phi([0]) = p$ we say that the orbifold chart is centered at p. We say that p is a regular point if the finite group Γ corresponding to a chart centered at p is trivial, that is, the orbifold is locally Euclidian around p; the point is called singular otherwise and the order of the singular point is the cardinality of the group γ. Since we are interested in orbibundles over 2-orbifolds, the groups $\Gamma's$ are generated by $\exp(2\pi i/n)$, where we think of $\mathbb{D}^2$ as the unit open ball on the complex plane.

Definition 22. (see [Bot, 3.1. Orbibundles]) Consider a smooth map between orbifolds $\zeta : L \to B$. We say ζ is a **disc orbibundle** for every point $p \in B$ there is an orbifold chart $\phi : \mathbb{D}^n/\Gamma \to D$ centered at p satisfying the following properties:

- there is a smooth action of Γ on $\mathbb{D}^n \times \mathbb{D}^2$ of the form $h(x, f) = (hx, a(h, x)f)$, where $a : \Gamma \times \mathbb{D}^n \to \mathrm{Diff}(\mathbb{D}^2)$ is smooth and $\mathrm{Diff}(\mathbb{D}^2)$ stands for the group of diffeomorphisms of $\mathbb{D}^2$;
- there is a diffeomorphism $\Phi : (\mathbb{D}^n \times \mathbb{D}^2)/\Gamma \to \zeta^{-1}(D)$ such that the diagram

$$\begin{tikzcd}
(\mathbb{D}^n \times \mathbb{D}^2)/\Gamma \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_1"'] & \zeta^{-1}(D) \arrow[d, "\zeta"] \\
\mathbb{D}^n/\Gamma \arrow[r, "\phi"] & D
\end{tikzcd} \tag{23}$$

  commutes, where $\mathrm{pr}_1([x, f]) = [x]$.

A **disc orbigoodle** (see [Bot, Definition 23]) is a special case of disc orbibundle. Consider a simply-connected manifold $\mathbb{H}$ on which acts a group G properly discontinuously. If we have an action of G on $\mathbb{H} \times \mathbb{D}^2$ by diffeomorphisms of the form $g(p, v) = (gp, a(g, p)v)$ then the quotient $(\mathbb{H} \times \mathbb{D}^2)/G \to \mathbb{H}/G$ is a disc orbibundle. Such orbibundles are called disc orbigoodles. All disc orbibundles of this paper are disc orbigoodles where $\mathbb{H}$ is the hyperbolic plane.

A natural $\mathbb{S}^1$-action is defined on Q' (the polyhedron Q' is defined right after Lemma 20) because

$$Q' = \bigcup_{x \in P} \big(\mathrm{L}[q, x] \cap \mathbb{H}^2_{\mathbb{C}}\big),$$

where $\mathrm{L}[q, x]$ is the complex projective line connecting q and x, and each disc $\overline{\mathbb{H}^2_{\mathbb{C}}} \cap \mathrm{L}[q, x]$ has the point x as center. The action we define is simply given by rotation around x,

$$\gamma[x + \theta q] = [x + \gamma\theta q],$$

where $\langle x, x\rangle = -1$, $\gamma \in \mathbb{S}^1$, and $|\theta| \le 1$. Therefore, we can define an $\mathbb{S}^1$-action on Q using the diffeomorphism F. Since F is an isometry at the level of the discs foliating the quadrangles $\mathcal{Q}', \mathcal{Q}$, $I_1g_4 = g_1$ and $I_3g_2 = g_3$ (remind that the curves g_i's are image under F of the curves defining the boundary of the quadrilateral P), we conclude

$$\gamma I_1F(x) = I_1\gamma F(x) \quad \text{for} \quad x \in B[C_1', C_4'],$$

$$\gamma I_3 F(x) = I_3 \gamma F(x) \quad \text{for} \quad x \in B[C_2', C_3'],$$

$\gamma \in \mathbb{S}^1$. In particular, since $F(c_i') = c_i$ for each vertex $C_i = F(C_i')$, we obtain that $F(\gamma x)$ is the rotation of $F(x)$ with respect to the center c_i of C_i and angle given by the unitary complex number γ.

Note that the image of the quadrilateral P under F in addition to the action of G on $\mathbb{H}^2_{\mathbb{C}}$ provides an embedded disc D transversal to all discs foliating $\mathbb{H}^2_{\mathbb{C}}$ and stable under action of $\mathbb{S}^1$. Hence, the quotient $L := \mathbb{H}^2_{\mathbb{C}}/G \to D/G$ is a disc orbigoodle and by construction $D/G = \mathbb{S}^2(n_1, n_2, n_3)$.

Furthermore, from $\partial_\infty Q$ we can build the $\mathbb{S}^1$-orbibundle $\mathbb{S}^1(L) \to D/G$, from which we will deduce the formula for the Euler number of the disc bundle $L \to D/G$. Let $\pi : \partial_\infty Q \to \mathbb{S}^1(L)$ be the quotient map. It is interesting to note that the action on ∂L is not necessarily principal, i.e., there are points $x \in \partial_\infty Q$ such that the map $\mathbb{S}^1 \ni \gamma \mapsto \pi(\gamma x) \in \mathbb{S}^1(L)$ is non-injective. More precisely, the action fails to be principal on the circles $\pi(\partial C_j)$'s.

Take a small ball V_i of radius r and center c_i' on P for $i = 1, 2, 3, 4$. Let's see what happens nearby these non-principal circles. Without loss of generality, we will work with $i = 1$. We have the open set

$$U := F\left[\bigcup_{x \in V_1} \mathrm{L}(x, q) \cap \partial \mathbb{H}^2_{\mathbb{C}}\right]$$

of $\partial_\infty Q$ and the open set $W := \pi(U)$ in $\mathbb{S}^1(L)$. Let p_1' be the orthogonal point c_1' on the projective line $\mathbb{P}(q^\perp)$ such that $\langle p_1', p_1'\rangle = 1$, $c_2' \in \mathbb{R}c_1' + \mathbb{R}p_1'$ and the geodesic curve $t \mapsto [\cosh(t)c_1' + \sinh(t)p_1']$ reaches c_2' for some $t > 0$, that is, this curve goes from c_1' to c_2'.

Consider the map $\Lambda : \mathbb{S}^1 \times S \to W$ given by

$$\Lambda(\gamma, z) = \pi \circ F\left[\frac{c_1' + z p_1'}{\sqrt{1 - |z|^2}} + \gamma q\right],$$

where S is the intersection of $\overline{\mathbb{D}^2_\epsilon} \subset \mathbb{C}$, the disc of center 0 and radius ϵ such that $\cosh(r) = 1/\sqrt{1-\epsilon^2}$, and the sector given by the inequality $0 \le \arg(z) \le 2\pi/n_1$. The sides of S can be glued because, if z is real, $\Lambda(\gamma, z) = \Lambda(\gamma, \xi z)$, where $\xi = \exp(2\pi/n_1)$. Therefore, we have the smooth map

$$\Lambda : \mathbb{S}^1 \times \left(\overline{\mathbb{D}^2_\epsilon}/\langle \xi \rangle\right) \to W,$$

and using the natural projection $\overline{\mathbb{D}^2_\epsilon} \to \overline{\mathbb{D}^2_\epsilon}/\langle \xi \rangle$, we have the smooth map

$$\widetilde{\Lambda} : \mathbb{S}^1 \times \overline{\mathbb{D}^2_\epsilon} \to W.$$

Remember the eigenvalues of I_1 are α_1, α_2 and α_3. Let $e^{2\pi i l_1/n_1} = \alpha_3/\alpha_1$ and $e^{-2\pi i/n_1} = \alpha_2/\alpha_1$, with $0 \le l_1 < n_1$.

Taking the diffeomorphism $\eta(\gamma, z) := (\xi^{l_1}\gamma, \xi^{-1}z)$ on $\mathbb{S}^1 \times \overline{\mathbb{D}^2_\epsilon}$, we have the equivariant diffeomorphism

$$\widehat{\Lambda} : \left(\mathbb{S}^1 \times \overline{\mathbb{D}^2_\epsilon}\right)/\langle \eta \rangle \to W$$

as a consequence of lemma 21. So W is a solid torus (see [Bot, Lemma 20]) with an $\mathbb{S}^1$ action which is principal except for the circle $\left(\mathbb{S}^1 \times 0\right)/\langle \eta \rangle$. Hence we have a trivialization of the $\mathbb{S}^1$-orbibundle around the fiber $\pi(\partial_\infty C_1)$.

If we write $\beta_3/\beta_1 = e^{2\pi i l_2/n_2}$ and $\gamma_3/\gamma_1 = e^{2\pi i l_3/n_3}$, with $1 \le l_1 < n_i$, we obtain the same kind of trivialization of the $\mathbb{S}^1$-orbibundle as described above for $\pi(\partial_\infty C_2)$ and $\pi(\partial_\infty C_3)$.

7.4. An integer contribution to the Euler number. We now tackle the problem of calculating the Euler number of the constructed orbibundles. First, we need to introduce a particular curve d for the quadrangle $\mathcal{Q}$.

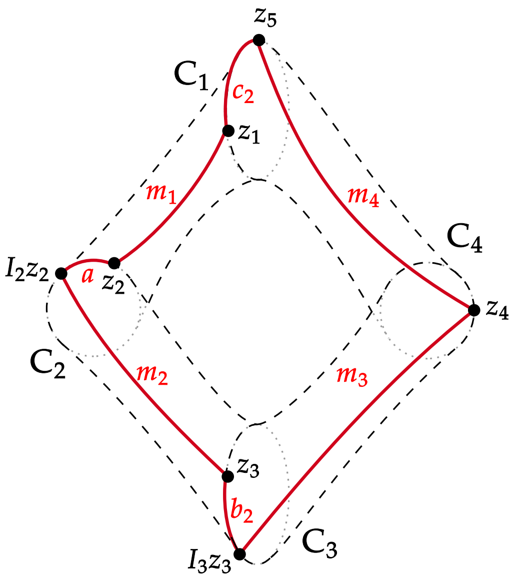


Figure 14: Meridional curve.

Take a point z_1 on $\partial_\infty C_1$ and define the following curves:

- the meridional curve $m_1 \subset \partial_\infty \mathrm{B}[C_1, C_2]$ that begins at $z_1 \in \partial_\infty C_1$ and ends at $z_2 \in \partial_\infty C_2$;
- the naturally oriented simple arc $a \subset \partial_\infty C_2$ that begins at z_2 and ends at $I_2 z_2$;
- the meridional curve $m_2 \subset \partial_\infty \mathrm{B}[C_2, C_3]$ that begins at $I_2 z_2$ and ends at $z_3 \in \partial_\infty C_3$;
- the naturally oriented simple arc $b_2 \subset \partial_\infty S$ that begins at z_3 and ends at $I_3 z_3$;
- the meridional curve $m_3 \subset \partial_\infty \mathrm{B}[C_3, C_4]$ that begins at $I_3 z_3$ and ends at $z_4 \in \partial_\infty C_4$;
- the meridional curve $m_4 \subset \partial_\infty \mathrm{B}[C_4, C_1]$ that begins at z_4 and ends at $z_5 \in \partial_\infty C_1$;
- the naturally oriented simple arc $c_2 \subset \partial_\infty C_1$ that begins at z_5 and ends at z_1.

Note that $z_4 = I_3 I_2 z_2$, because $I_3 m_2 = m_3$. Therefore, $z_4 = I_1^{-1} z_2$ and consequently $z_5 = I_1^{-1} z_1$.

Let

$$d := m_1 \cup a \cup m_2 \cup b_2 \cup m_3 \cup m_4 \cup c_2 \tag{24}$$

and let s stand for a generator of $H_1(\partial_\infty Q, \mathbb{Z})$. Then there exists $f \in \mathbb{Z}$ such that $d = fs$ in $H_1(\partial_\infty Q, \mathbb{Z})$. This integer f is an important component of the Euler number of the orbibundles we will encounter in subsection 7.5. It will be expressed in a more computational friendly manner in subsection 7.6.

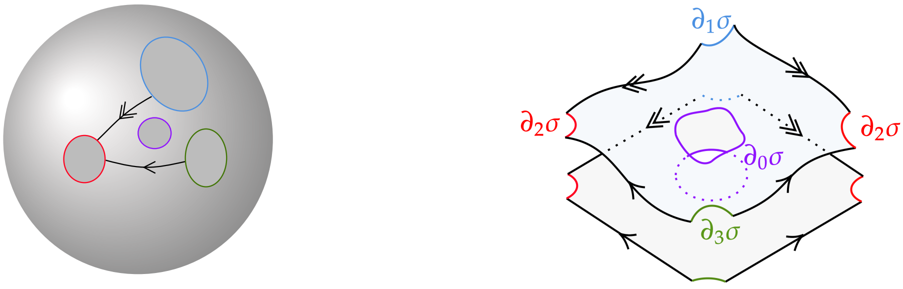
7.5. Euler Number of the constructed disc bundles. Following [Bot, 3.2. Euler number of $\mathbb{S}^1$-orbibundles over 2-orbifolds] the Euler number of the disc orbibundle $L \to D/G$ described in the subsection 7.3 is the Euler number of the $\mathbb{S}^1$-orbibundle $\mathbb{S}^1(L) \to D/G$.

In general, if $M \to B$ is an $\mathbb{S}^1$-orbibundle over a oriented compact connected 2-orbifold with singular points $x_1, \dots, x_n$ then the Euler number is calculated as follows: Take a regular point x_0 and for each $i = 0, \cdots, n$ consider a small smooth closed disc D_i centered at x_i trivializing the $\mathbb{S}^1$-orbibundle $M \to B$. The $\mathbb{S}^1$-orbibundle restricted over the surface with boundary $B' = B \setminus \sqcup_i D_i$ is trivial, since $\mathbb{S}^1$-bundles over graphs are trivial and B' is homotopically equivalent to a graph. Consider a section σ for $M|_B' \to B'$ and a fiber s over a regular point, oriented accordingly to action of $\mathbb{S}^1$ on M. The Euler number of the $\mathbb{S}^1$-orbibundle $M \to B$ is defined by the identity

$$\sigma|_{\partial B'} = -e(M)s$$

in $H_1(M, \mathbb{Q})$ (See [Bot, Definition 16]).

Now we apply the above definition of Euler number to the particular bundle $\mathbb{S}^1(L) \to D/G$. Let us also denote $\mathbb{S}^1(L)$ by M and D/G by B. Remember that B is the quotient of the hyperbolic plane by the turnover group. Here we think of B as the quotient of P by the gluing relations described by the turnover group (the quadrilateral P is the fundamental domain for the turnover group as described in Subsection 3.1). Hence we denote the point under the fiber $\pi(\partial C_i)$ by $[c_i']$. The points $[c_i']$ are the only singular points of B.



(a) (b)

Figure 15: **(a)** Surface B', and **(b)** Section $\sigma : B' \to M'$

Removing small open discs D_1, D_2 and D_3 on B around the three singular points $[c_1'], [c_3'], [c_2'] = [c_4']$ and one small disc D_0 around a regular point $[x_0]$ in B, with $x_0 \in \mathring{P}$, we have the surface with boundary $B' := B \setminus \sqcup_i D_i$. The 3-manifold $M' = \zeta^{-1}(B')$ is a principal $\mathbb{S}^1$-bundle over B'. Notice that $M \setminus M'$ is made of four solid tori W_0, W_1, W_2, W_3, where $W_i = \zeta^{-1} D_i$.

For any section $\sigma : D' \to M'$, lets denote $\sigma|_{\partial D_i}$ by $\partial_i \sigma$. Remember the curve d defined in Subsection 25. Shrinking $F^{-1}(d)$ inside the torus $\partial_\infty Q'$ we can build a section $\sigma : D' \to M'$ satisfying the identities (See figure 14)

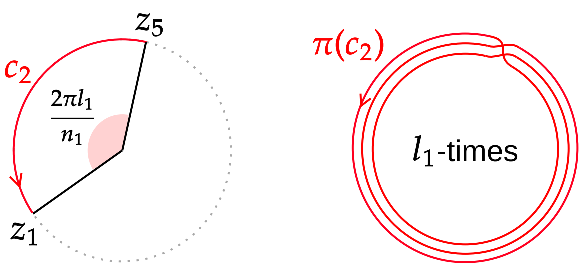
$$\partial_0\sigma = \pi(d), \quad \partial_1\sigma = -\pi(c_2), \quad \partial_2\sigma = -\pi(b_2), \quad \partial_3\sigma = -\pi(a)$$

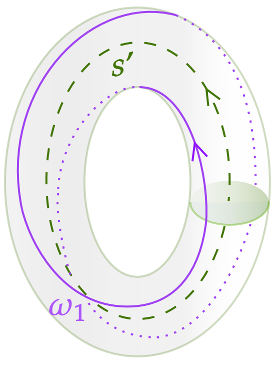
in $H_1(M, \mathbb{Q})$.

The identity $n_i \partial_i \sigma = -l_i \omega_i$ in $H_1(M, \mathbb{Q})$ holds for $i = 1, 2, 3$, where ω_i is the orbit of a point in ∂W_i. Furthermore, $\omega_0 = \omega_1 = \omega_2 = \omega_3 = s$ in $H_1(M, \mathbb{Q})$.

Let us prove the identity $n_i \partial_i \sigma = -l_i \omega_i$ for $i = 1$.

Consider a generator s' of the fundamental group of $\pi(\partial C_1)$.





(a) (b) (c)

Figure 16: **(a)** Curve c_2 in ∂C_1, **(b)** loop $\pi(c_2)$ in $\pi(\partial C_1)$, and **(c)** loop ω_1 on the solid torus W_1.

We can think of ω_1 as $\mathbb{S}^1 \to M$ given by

$$\gamma \mapsto \pi \circ F \left[\frac{(c_1' + z p_1')}{\sqrt{1 - |z|^2}} + \gamma q \right]$$

for a fixed z.

Notice $\omega_1 = n_1 s'$, because ω_1 is homotopic to the curve $\gamma \mapsto [c_1' + \gamma q]$ in M, which is a curve that goes n_1 times around the circle $\pi(\partial C_1)$, and $\partial_1\sigma = -l_1 s'$, because $\partial_1\sigma = -\pi(c_2)$ in M and $\pi(c_2)$ goes l_1 times around the circle $\pi(\partial C_1)$, since in ∂C_1 the curve c_2 is constructed as the curve going from z_5 to z_1 following the natural orientation of the circle and $I_1 z_1 = z_5$.

Therefore, we have

$$n_1\partial_1\sigma = -l_1\omega_1 \quad \text{in} \quad H_1(M,\mathbb{Q}).$$

In the case of $i = 0$, we have $\partial_0\sigma = \pi(d)$ and, therefore, we have $\partial_0\sigma = f\omega_0$ in $H_1(M',\mathbb{Z})$, because $d = fs$.

Note ∂D_i is oriented in opposite direction of $\partial B'$. Therefore, in $H_1(M,\mathbb{Q})$ we can write

$$\partial\sigma = \sum_{i=0}^{3} -\partial_i\sigma = \left(-f + \frac{l_1}{n_1} + \frac{l_2}{n_2} + \frac{l_3}{n_3}\right) s$$

and, therefore,

$$e(M) = f - \frac{l_1}{n_1} - \frac{l_2}{n_2} - \frac{l_3}{n_3}.$$

7.6. Holonomy of the quadrangle. In Subsection 25 we define the curve d, shown in Figure 14, and the integer f, necessary to calculate the Euler number. In order to express this integer explicitly we use the concept of holomony of a transversal triangle of bisectors.

Given a counterclockwise oriented transversal triangle of bisectors $\Delta(L_1, L_2, L_3)$, let M_1, M_2, M_3 be the middle slices (see Subsection 2.3) of the segments of bisectors $\mathrm{B}[L_1, L_2]$, $\mathrm{B}[L_2, L_3]$, $\mathrm{B}[L_3, L_1]$. The product I of the reflections in the middle slices M_1, M_2, M_3 (in that order) is called the *holonomy* of the triangle $\Delta(L_1, L_2, L_3)$ [AGG, Subsection 2.5.1]. Note that I stabilizes L_1.

The triangle $\Delta(L_1, L_2, L_3)$ is respectively called *elliptic, parabolic,* or *hyperbolic* when the holonomy I restricted to L_1 is an elliptic, parabolic, or hyperbolic isometry of the Poincaré disc L_1. The holonomy of a counterclockwise oriented transversal triangle cannot be trivial, that is, I restricted to L_1 is never the identical isometry; moreover, parabolic triangles are always L-parabolic, that is, the holonomy restricted to L_1 moves its non-fixed points in the clockwise sense [AGG, Theorem 2.24]. In the case of a hyperbolic triangle, the action of I on L_1 divides $\partial_\infty L_1$ into the L and R-parts: the L-part (respectively, the R-part) consists of those points that are moved by I in the clockwise sense (respectively, counterclockwise sense). In the elliptic and parabolic cases, all (non-fixed) points belong to the L-part.

A simple closed curve in the torus $\partial_\infty\Delta(L_1, L_2, L_3)$ is called a *trivialing curve* of the triangle if it generates the fundamental group of the solid torus $\Delta(L_1, L_2, L_3)$ and is contractible in the ideal boundary of the polyhedron bounded by $\Delta(L_1, L_2, L_3)$ (see Section 6).

As introduced in Subsection 7.4, let

$$d := m_1 \cup a \cup m_2 \cup b_2 \cup m_3 \cup m_4 \cup c_2 \tag{25}$$

be the oriented closed curve in the boundary of the solid torus $\partial_\infty Q$, where Q stands for the polyhedron of the quadrangle $\mathcal{Q}$. Remind that the group $H_1(\partial_\infty Q, \mathbb{Z})$ is generated by $[s]$, where $[s]$ stands for the naturally oriented boundary of C_1. Hence, $[d] = f[s]$ for some $f \in \mathbb{Z}$. In order to express f in terms of the holonomies of the triangles $\Delta(C_1, C_2, C_4)$ and $\Delta(C_3, C_4, C_2)$, we introduce more points and curves:

- the meridional curve $m_2' \subset \partial_\infty \mathrm{B}[C_2, C_3]$ that begins at z_2 and ends at $z_3' \in \partial_\infty C_3$;
- the meridional curve $m \subset \partial_\infty \mathrm{B}[C_2, C_4]$ that begins at z_2 and ends at $z_4' \in \partial_\infty C_4$;
- the meridional curve $m_3' \subset \partial_\infty \mathrm{B}[C_4, C_3]$ that begins at z_4' and ends at $z_3'' \in \partial_\infty C_3$;
- the naturally oriented arc $b \subset \partial_\infty C_3$ that begins at z_3' and ends at z_3'';
- the meridional curve $m_4' \subset \partial_\infty \mathrm{B}[C_4, C_1]$ that begins at z_4' and ends at $z_5' \in \partial_\infty C_1$;
- the naturally oriented arc $c \in \partial_\infty C_1$ that begins at z_5' and ends at z_1.

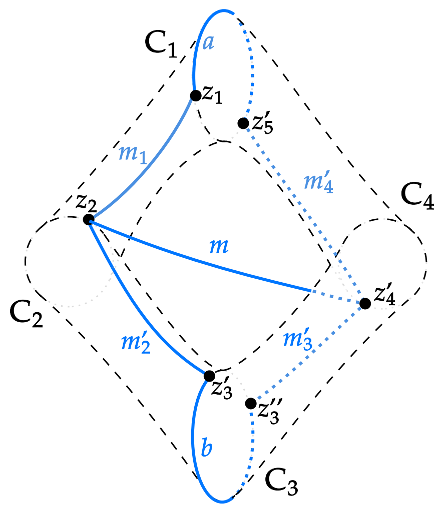


Figure 17: Auxiliary curves

Denote by I the holonomy of the triangle $\Delta(C_1, C_2, C_4)$ and by J the holonomy of the triangle $\Delta(C_3, C_4, C_2)$. By the definition of holonomy of a triangle, we have $z_3'' = J^{-1} z_3'$ and $z_5' = I z_1$.

Let us assume that z_1 belongs to the L-part of $\Delta(C_1, C_2, C_4)$ and that z_3' belongs to the L-part of $\Delta(C_3, C_4, C_2)$ (this is harmless because all the triangles that appear in the constructed orbibundles are elliptic). In this case, by [AGG, Theorem 2.24], the closed oriented curve $m_1 \cup m \cup m_4' \cup c$ is a trivializing curve of $\Delta(C_1, C_2, C_4)$. Similarly, ${m_3'}^{-1} \cup m^{-1} \cup m_2' \cup b$ is a trivializing curve of the triangle $\Delta(C_3, C_4, C_2)$. (We denote by x^{-1} the (not necessarily closed) curve x taken with the opposite orientation.) By [AGG, Remark 2.21], $m_1 \cup m_2' \cup b \cup {m_3'}^{-1} \cup m_4' \cup c$ is a trivializing curve of the quadrangle $\mathcal{Q}$, that is, it generates the fundamental group of $\mathcal{Q}$ and is contractible in $\partial_\infty Q$. In terms of 1-chains modulo boundaries, this means that

$$[m_1] + [m_2'] + [b] - [m_3'] + [m_4'] + [c] = 0. \tag{26}$$

Finally, we introduce the following arcs:

- the naturally oriented simple arc $b_1 \subset \partial_\infty C_3$ that begins at z_3' and ends at z_3;
- the naturally oriented simple arc $b_3 \subset \partial_\infty C_3$ that begins at $I_3 z_3$ and ends at z_3'';
- the naturally oriented simple arc $c_1 \subset \partial_\infty C_1$ that begins at z_5 and ends at z_5'.

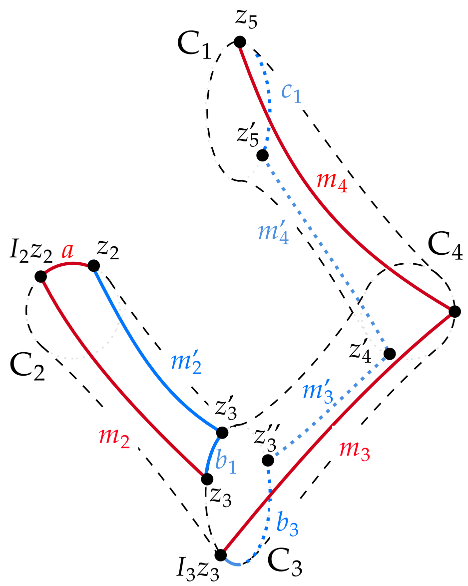


Figure 18: Cylinders $\partial_\infty \mathrm{B}[C_2, C_3]$ and $\partial_\infty \mathrm{B}[C_3, C_4] \cup \partial_\infty \mathrm{B}[C_4, C_1]$.

By looking at the cylinder $\partial_\infty \mathrm{B}[C_2, C_3]$, it is easy to see that

$$[a] + [m_2] - [b_1] - [m_2'] = 0. \tag{27}$$

Similarly, by considering the cylinder $\partial_\infty \mathrm{B}[C_3, C_4] \cup \partial_\infty \mathrm{B}[C_4, C_1]$, one obtains that

$$[m_3] + [m_4] + [c_1] - [m_4'] + [m_3'] - [b_3] = 0. \tag{28}$$

It follows from equations (25), (27), and (28) that

$$[d] = [m_1] + [a] + [m_2] + [b_2] + [m_3] + [m_4] + [c_2] = \\ [m_1] + [m_2'] + [b_1] + [b_2] + [b_3] - [m_3'] + [m_4'] - [c_1] + [c_2]. \tag{29}$$

We introduce the following notation. Let C be an oriented circle and let $t_1, t_2, t_3 \in C$. We define $o(t_1, t_2, t_3) = 1$ if t_1, t_2, t_3 are pairwise distinct and not in cyclic order. Otherwise, we put $o(t_1, t_2, t_3) = 0$.

Lemma 30. *Let C be an oriented circle and let $t_1, t_2, t_3, t_4 \in C$ be such that $t_3 \neq t_1 \neq t_4$. Following the orientation of C, we define four simple arcs (some of them may consist of a single point): $a_i \subset C$ joining t_i and t_{i+1} for $i = 1, 2, 3$ and $a \subset C$ joining t_1 and t_4. Then we have*

$$[a_1] + [a_2] + [a_3] - [a] = o(t_1, t_2, t_3)[C] + o(t_3, t_4, t_1)[C]$$

in $C_1(C, \mathbb{Z})/\partial C_0(C, \mathbb{Z})$. (Of course, we take $[C]$ as a generator of $H_1(C, \mathbb{Z})$.)

Proof. Define the following oriented simple arcs: a_4 joining t_4 and t_1; m_1 joining t_1 and t_3; and m_2 joining t_3 and t_1. It follows from $t_1 \neq t_4$ that $[a] + [a_4] = [C]$. Analogously, $t_1 \neq t_3$ implies $[m_1] + [m_2] = [C]$. By drawing the corresponding arcs in C, it is easy to see that $[a_1] + [a_2] - [m_1] = o(t_1, t_2, t_3)[C]$ and $[a_3] + [a_4] - [m_2] = o(t_3, t_4, t_1)[C]$. So,

$$[a_1] + [a_2] + [a_3] - [a] = [a_1] + [a_2] + [a_3] + [a_4] - [C] =$$

$$= o(t_1, t_2, t_3)[C] + [m_1] + o(t_3, t_4, t_1)[C] + [m_2] - [C] = o(t_1, t_2, t_3)[C] + o(t_3, t_4, t_1)[C]. \qquad \blacksquare$$

Applying Lemma 30 to the naturally oriented circle $\partial_\infty C_3$ and the points $z_3', z_3, I_3 z_3, J^{-1} z_3' \in \partial_\infty C_3$ (note that $z_3' \neq J^{-1} z_3'$ always hold and one can assume that $z_3' \neq I_3 z_3$) we obtain

$$[b_1] + [b_2] + [b_3] = [b] + o(z_3', z_3, I_3 z_3)[s] + o(I_3 z_3, J^{-1} z_3', z_3')[s]. \tag{31}$$

In the naturally oriented circle $\partial_\infty C_1$ we have

$$[c_1] + [c] = [c_2] + o(I_1^{-1} z_1, I z_1, z_1)[s] \tag{32}$$

since $[\partial_\infty C_1] = [s]$. Therefore, it follows from (29), (31), and (32) that

$$[d] = [m_1] + [m_2'] + [b] - [m_3'] + [m_4'] + [c] + o(z_3', z_3, I_3 z_3)[s] + o(I_3 z_3, J^{-1} z_3', z_3')[s] - o(I_1^{-1} z_1, I z_1, z_1)[s].$$

Hence, by (26),

$$[d] = o(z_3', z_3, I_3 z_3)[s] + o(I_3 z_3, J^{-1} z_3', z_3')[s] - o(I_1^{-1} z_1, I z_1, z_1)[s],$$

that is,

$$f = o(z_3', z_3, I_3 z_3) + o(z_3', I_3 z_3, J^{-1} z_3') - o(z_1, I_1^{-1} z_1, I z_1).$$

8 Toledo invariant

Let $\rho : G \to \mathrm{PU}(2,1)$ be a faithful $\mathrm{PU}(2,1)$-representation of the turnover group G and let $\phi : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{C}}$ be a G-equivariant map. The *Toledo invariant* of ρ is defined by the formula

$$\tau(\rho) = 4 \cdot \frac{1}{2\pi} \int_P \phi^* \omega,$$

where P is a fundamental domain in $\mathbb{H}^2_{\mathbb{R}}$ for the action of G (see subsection 3.1 and figure 4). The number τ does not depend on the choice of the G-equivariant map ϕ. For details about the Toledo invariant in the context of orbifolds, see [Bot, Definition 35] and [Krebs]). The factor 4 in our formula for the Toledo invariant comes from the fact that our metric is four times the usual one.

Let $\mathcal{Q}$ be the quadrangle associated to the representation ρ. We assume that it satisfies the quadrangle conditions in subsection 6.2. In order to calculate the Toledo invariant of ρ, we introduce in $\mathcal{Q}$ several curves as illustrated in figure 19. First, we define the oriented meridional curves

$$m_1 := [c_1', c_2] \subset \mathrm{B}[C_1, C_2], \quad m_2 := [c_2, c_3'] \subset \mathrm{B}[C_2, C_3], \quad m_3^{-1} := I_3 m_2 = [c_4, I_3 c_3'] \subset \mathrm{B}[C_3, C_4],$$

$$m_4^{-1} := I_1^{-1} m_1 = [I_1^{-1} c_1', c_4] \subset \mathrm{B}[C_4, C_1],$$

with $c_1' \in C_1$ and $c_3' \in C_3$ (note that $c_4 = I_1^{-1} c_2 = I_3 c_2$). We also introduce the oriented geodesics

$$h_1 := \mathrm{G}[c_1, c_1'] \subset C_1, \quad h_2 := \mathrm{G}[c_3', c_3] \subset C_3, \quad h_3^{-1} := I_3 h_2 = \mathrm{G}[I_3 c_3', c_3] \subset C_3,$$

$$h_4^{-1} := I_1^{-1} h_1 = \mathrm{G}[c_1, I_1^{-1} c_1'] \subset C_1$$

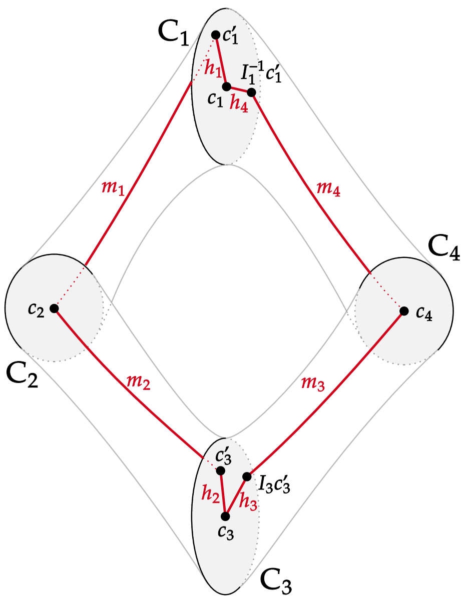


Figure 19: Curve c.

thus obtaining the closed oriented curve

$$\zeta := h_1 \cup m_1 \cup m_2 \cup h_2 \cup h_3 \cup m_3 \cup m_4 \cup h_4. \tag{33}$$

Following the notation in Subsection 7.1, let $\alpha_1, \beta_1, \gamma_1^{-1}$ be the eigenvalues of I_1, I_2, I_3 corresponding to negative eigenvectors. The proof below is similar to that of [AGG, Proposition 2.7]. The strategy of the proof is the following. By Stokes theorem, the Toledo invariant of ρ can be obtained by integrating a Kähler potential along ζ because the quadrangle conditions allow one to build a G-equivariant map $\mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{C}}$ sending ∂P to ζ (note that ζ is the boundary of a smooth disc inside the real 4-ball Q). A potential for the Kähler is obtained by choosing a basepoint $c \in \mathbb{H}^2_{\mathbb{C}}$ as in formula (2). The boundary of ζ is made of meridional curves and geodesics. Since each of these curves is contained in a real plane, it follows from formula (2) that the integral of a Kähler potential along the curve vanishes when we choose the basepoint c in the curve (say, we can take c as the starting point of the curve). So, the contributions to the Toledo invariant come from the changes of basepoints which are explicitly given in (3).

Proposition 34. *Let $\rho : G \to \mathrm{PU}(2,1)$, $g_j \mapsto I_j$, be a representation satisfying the quadrangle conditions 6.2. Then $\tau \equiv \dfrac{\operatorname{Arg}(\alpha_1\beta_1\gamma_1^{-1})}{\pi} \mod 2$, where τ stands for the Toledo invariant of ρ.*

Proof. Note that $\alpha_1\beta_1\gamma_1^{-1}$ is well-defined for ρ because we assume the equality $I_3I_2I_1 = 1$ in $\mathrm{SU}(2,1)$. We take the quadrangle of bisectors $\mathcal{Q}$ of ρ described in subsection 6.2, the closed oriented curve $\zeta \subset \mathcal{Q}$ defined in (33), and the geodesic polygon $P \subset \mathbb{H}^2_{\mathbb{R}}$ defined in subsection 3.1. Let $D \subset \mathbb{H}^2_{\mathbb{C}}$ be a disc with $\partial D = \zeta$ and let $\varphi : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{C}}$ be a ρ-equivariant map such that $\varphi P = D$, $\varphi v_j = c_j$, and

$$\varphi e_1 = h_1 \cup m_1, \quad \varphi e_2 = m_2 \cup h_2, \quad \varphi e_3 = h_3 \cup m_3, \quad \varphi e_4 = m_4 \cup h_4$$

(see Figure 4). Then $\tau = \frac{4}{2\pi}\int_P \varphi^*\omega$, that is,

$$\tau = \frac{2}{\pi}\int_D \omega = \frac{2}{\pi}\int_{\partial D} P_{c_2} = \frac{2}{\pi}\sum_{j=1}^{4}\left(\int_{m_j} P_{c_2} + \int_{h_j} P_{c_2}\right),$$

where P_{c_2} is a Kähler primitive with basepoint $c_2 \in \mathbb{H}^2_{\mathbb{C}}$. The choice of the basepoint implies $\int_{m_1} P_{c_2} = \int_{m_2} P_{c_2} = 0$. The remaining integrals can be evaluated with the aid of the formula relating primitives based on distinct points:

$$J_1 := \int_{h_2} P_{c_2} = \int_{h_2} (P_{c_2} - P_{c_3'}) = \int_{h_2} \mathrm{d}f_{c_2,c_3'} =$$

$$= \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,c_3\rangle\langle c_3,c_3'\rangle}{\langle c_2,c_3'\rangle} - \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,c_3'\rangle\langle c_3',c_3'\rangle}{\langle c_2,c_3'\rangle} = \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,c_3\rangle\langle c_3,c_3'\rangle}{\langle c_2,c_3'\rangle} - \frac{\pi}{2};$$

$$J_2 := \int_{h_3} P_{c_2} = \int_{h_3} (P_{c_2} - P_{c_3}) = \int_{h_3} \mathrm{d}f_{c_2,c_3} = \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,I_3c_3'\rangle\langle I_3c_3',c_3\rangle}{\langle c_2,c_3\rangle} - \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,c_3\rangle\langle c_3,c_3\rangle}{\langle c_2,c_3\rangle} =$$

$$= \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,I_3c_3'\rangle\langle I_3c_3',c_3\rangle}{\langle c_2,c_3\rangle} - \frac{\pi}{2} = \frac{1}{2}\operatorname{Arg}\left(\gamma_1^{-1}\frac{\langle c_2,I_3c_3'\rangle\langle c_3',c_3\rangle}{\langle c_2,c_3\rangle}\right) - \frac{\pi}{2}.$$

Similarly, one obtains

$$J_3 := \int_{m_3} P_{c_2} = \frac{1}{2}\operatorname{Arg}\left(\beta_1\frac{\langle c_2,I_1^{-1}c_2\rangle\langle c_2,c_3'\rangle}{\langle c_2,I_3c_3'\rangle}\right) - \frac{\pi}{2}; \quad J_4 := \int_{m_4} P_{c_2} = \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,I_1^{-1}c_1'\rangle\langle c_1',c_2\rangle}{\langle c_2,I_1^{-1}c_2\rangle} - \frac{\pi}{2};$$

$$J_5 := \int_{h_4} P_{c_2} = \frac{1}{2}\operatorname{Arg}\left(\alpha_1\frac{\langle c_2,c_1\rangle\langle c_1,c_1'\rangle}{\langle c_2,I_1^{-1}c_1'\rangle}\right) - \frac{\pi}{2}; \quad J_6 := \int_{h_1} P_{c_2} = \frac{1}{2}\operatorname{Arg}\frac{\langle c_2,c_1'\rangle\langle c_1',c_1\rangle}{\langle c_2,c_1\rangle} - \frac{\pi}{2}.$$

We have $\tau = \dfrac{2}{\pi}\sum_j J_j$. Calculating mod 2, we multiply the arguments of every Arg function participating in the previous sum thus obtaining the result. ∎

The Toledo invariant is in fact an invariant of the faithful representation $\rho : G \to \mathrm{PU}(2,1)$; it does not depend on the quadrangle conditions. However, we only consider here representations satisfying such condition since discreteness is our main concern (see Section 6).

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CHAPTER 7

ARTICLE: ORBIFOLDS AND ORBIBUNDLES IN COMPLEX HYPERBOLIC GEOMETRY

Orbifolds and orbibundles in complex hyperbolic geometry

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Abstract

We develop the theory of orbibundles from a geometrical viewpoint using diffeology. One of our goals is to present new tools allowing to calculate invariants of complex hyperbolic disc orbibundles over 2-orbifolds appearing in the geometry of 4-manifolds. These invariants are the Euler number of disc orbibundles and the Toledo invariant of $\mathrm{PU}(2,1)$-representations of 2-orbifold groups.

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1 Introduction

We study orbibundles via diffeology (following the suggestion in [Igl2, pag. 94]). Using this framework, we describe essential invariants appearing in complex hyperbolic geometry and prove orbifold generalizations of some classical results in the area. These new tools were developed while investigating the complex variant of the Gromov-Lawson-Thurston conjecture.

A Riemannian manifold N is uniformized by a simply connected complete Riemannian manifold M if there is a Riemannian covering of N by M. Uniformization plays an important role in classifying manifolds in dimensions 2 and 3 respectively via uniformization of Riemann surfaces and Thurston's geometrization conjecture (proved by Perelman in 2006).

Uniformization in dimension 4 is far from being well understood. In this regard, it is natural to investigate disc bundles over surfaces, one of the simplest types of 4-manifolds. Along these lines stands the **Gromov-Lawson-Thurston conjecture:** An oriented disc bundle $M \to B$ over an oriented, connected, compact surface B with negative Euler characteristic is uniformized by the real hyperbolic space $\mathbb{H}^4_{\mathbb{R}}$ (i.e., the usual hyperbolic 4-space) if, and only if, $|e_R(M)| \leq 1$, where $e_R(M)$ is the relative Euler number of the bundle, defined as the quotient of the Euler number $e(M)$ of the disc bundle by the Euler characteristic $\chi(B)$ of the surface (see [GLT]). Observe that the relative Euler number is preserved under pullbacks of M by finite covers of B. Both directions of the conjecture are open (see [ACh], [GLT], [Kap1], and [Kui] for details).

The Euler number of an oriented disc bundle $M \to B$ is the oriented intersection number (see [GPo, Chapter 3]) of two transversal sections and the Euler characteristic is the Euler number of the tangent bundle of the surface. Since an oriented disc bundle $M \to B$ is determined, up to isomorphism, by its Euler number, the relative Euler number measures how different from the tangent bundle $\mathrm{T}B \to B$ the disc bundle $M \to B$ is. Up to bundle isomorphism, there are three distinguished cases: the tangent, the cotangent, and the trivial disc bundles over B. The relative Euler numbers are respectively $1, -1, 0$.

In 2011, new examples of complex hyperbolic disc bundles $M \to B$ were discovered [AGG]. Complex hyperbolic means that the total space M is uniformized by the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ (see Subsection 2.2). They satisfy $|e_R| \leq 1$ and, therefore, support the **complex GLT-conjecture** (same statement with $\mathbb{H}^2_{\mathbb{C}}$ in place of $\mathbb{H}^4_{\mathbb{R}}$). A large number of examples backing the conjecture can also be found in [GKL], [AGu], and [BGr], including the above distinguished cases. The fact that the statement of the conjecture is the same in both cases may be seen as evidence that the conjecture is more about negative curvature than about constant negative curvature.

In this paper, we develop a theory of $\mathbb{S}^1$-orbibundles over compact oriented 2-orbifolds via diffeology and use it to introduce the concept of Euler number for oriented vector orbibundles of rank 2. The tools obtained here are essential for calculating the Euler number of the disc orbibundles constructed in [BGr] and provide a general framework that may also be applied to the examples obtained in [AGG] and [AGu]. At the core of these calculations and of the technology developed here lies Lemma 20 linking the geometry and the topology of $\mathbb{S}^1$-orbibundles.

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At first glance, creating a framework for calculating the Euler number at the level of orbibundles may seem unnecessary since in the previous works [AGG] and [AGu] the authors were able to avoid such an approach. However, in their examples, the involved orbifolds B were particularly simple and it was therefore possible to explicitly describe compact oriented surfaces covering them; through the relations between the fundamental domains of the orbifold and of such surfaces, the authors were able to calculate the Euler number by reducing the problem to the case of a disc bundle over surface. In [BGr], on the other hand, this method does not work because there are no special explicit surfaces to rescue us (clearly, by Selberg's lemma, there exist finite covers by surfaces of the orbifolds used in [BGr], but there is no practical way of determining them). Hence, we found necessary to develop the technology presented in this article.

Since the fibers of a complex hyperbolic disc bundle $M \to B$ over a surface are contractible, the fundamental groups of B and M are equal and $\pi_1(M)$ (viewed as a deck group) is a subgroup of the orientation preserving isometry group $\mathrm{PU}(2,1)$ of $\mathbb{H}^2_{\mathbb{C}}$ because M is uniformized by $\mathbb{H}^2_{\mathbb{C}}$. Therefore, we have a discrete faithful representation $\rho : \pi_1(B) \to \mathrm{PU}(2,1)$ and M is isometric to $\mathbb{H}^2_{\mathbb{C}}/\pi_1(B)$. Isometric complex hyperbolic disc bundles correspond to conjugated representations. With that in mind, we can view complex hyperbolic disc bundles (up to isometry of bundles) over the surface B as points in the $\mathrm{PU}(2,1)$-character variety of B (that is, the space of all $\mathrm{PU}(2,1)$-representations of $\pi_1(B)$ modulo conjugation). It is worthwhile mentioning that character varieties are the essential geometrical objects in Higher Teichmüller theory and we believe that the methods developed here can be of use in some settings of that science as well.

It is also important to point out that this work led us to a perspective shift. Originally, our objective in [BGr] was to produce disc bundles over surfaces with complex hyperbolic structures as in [AGG] and [AGu]. Nevertheless, it is actually not desirable to reduce the examples found at the orbifold level to the surface level by pulling back the disc orbibundle to a disc bundle over a surface. By doing that, we lose track of the $\mathrm{PU}(2,1)$-character variety we are dealing with, thus losing information (for instance, rigid representations may become flexible). The techniques developed here allow us to deal with orbibundles on their own.

To work with complex hyperbolic geometry on orbibundles, we slightly generalize the usual definition of orbibundle (see Subsection 3.1) and introduce the concept of an **orbigoodle** (see Definition 23), a natural class of orbibundles over good orbifolds. The latter concept is well-behaved under pullbacks and enables a Chern-Weil theory for vector orbigoodles (see Theorem 25 and Section 5). Analytical expressions for the Euler number of rank 2 vector orbigoodles via Chern-Weil theory are given in Theorems 31 and 32.

Besides the Euler number, there is another important invariant associated to a complex hyperbolic disc orbigoodle $M \to B$, called the **Toledo invariant** (see Definition 35). It is a real-valued function τ defined on the $\mathrm{PU}(2,1)$-character variety of B (see Definition 33). As the relative Euler number, the **relative Toledo invariant** given by $\tau_R := \tau/\chi(B)$ is also preserved under finite covers of B. When B is a surface, D. Toledo proved that $|\tau_R| \le 1$ as well as the famous Toledo rigidity: a representation $\rho : \pi_1(B) \to \mathrm{PU}(2,1)$ is maximal, i.e., the identity $|\tau_R(\rho)| = 1$ holds, if, and only if, there is a complex geodesic (see Subsection 2.2) in $\mathbb{H}^2_{\mathbb{C}}$ stable under action of ρ. We give a proof of the Toledo rigidity for 2-orbifolds (see Theorem 44). It is interesting to point out that the maximality of the Toledo invariant of a representation $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$ implies that the representation ρ is discrete [BIW, Theorem 4.1].

We also show that the Toledo invariant of a representation $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$ belongs to $\frac{2}{3}\big(\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \cdots + \frac{1}{m_n}\mathbb{Z}\big)$ for a 2-orbifold B whose singularities are conic points of angles $2\pi/m_1, \dots, 2\pi/m_n$ (see Corollary 41). When B is a surface we recover the integrality property proved in [GKL]. Note that this result implies the discreteness of the Toledo invariant. For surfaces, the Toledo invariant indexes the connected components of the character variety (see [Xia, Theorem 1.1]): different connected components have different Toledo invariants. We conjecture the same for 2-orbifolds.

All examples found in [AGG], [AGu], and [BGr] satisfy

$$\frac{3}{2}\tau_R(\rho) = e_R(M) + 1, \tag{1}$$

where ρ is the representation associated to a complex hyperbolic disc bundle $M \to B$. The other non-trivial known examples of complex hyperbolic disc bundles are the ones found in [GKL], which satisfy $\frac{3}{2}\tau_R(\rho) \le e_R(M) + 1$.

It is known (see [AGG] or Corollary 43) that the existence of a holomorphic section of a complex hyperbolic disc bundle implies the identity (1). By holomorphic section (when B is a surface) we mean a section $\sigma : B \to M$ that, viewed as a submanifold of M (which is a complex manifold), is a Riemann surface. The existence of holomorphic sections for the examples in [AGG] was proved by Misha Kapovich (see [Kap3, Example 8.9]). Nevertheless, his technique does not seem to work for the examples in [BGr] because the complex hyperbolic disc orbigoodles we found are not rigid (they form 2-dimensional regions in the corresponding character variety which means these orbigoodles, while equal as smooth orbigoodles, are geometrically distinct). Inspired by Toledo rigidity and based on the above observations, Carlos Grossi suggests the following conjecture: Given a complex hyperbolic disc bundle $M \to B$ over a compact oriented surface with negative Euler characteristic, we have

$$\frac{3}{2}\tau_R(\rho) \leq e_R(M) + 1$$

and the equality holds if, and only if, there is a holomorphic disc embedded in $\mathbb{H}^2_{\mathbb{C}}$ stable under action of ρ.

The theory of orbibundles is well-known in geometry and is usually approached from the Lie groupoid perspective (see [ALR] and [Ame]). Nevertheless, we won't follow this path for two reasons: 1) the way we empirically discovered the formula for the Euler number of the orbibundles in [BGr] does not have an algebraic flavor; it is actually low-tech and very geometric in nature, not fitting the language of Lie groupoids and 2) we plan to generalize the technology developed here for spaces much more singular than orbibundles because we believe there is a correspondence between the faithful part of the $\mathrm{PU}(2,1)$-character variety of hyperbolic spheres with 3 conic points B (see Figure 1) and some very singular "bundles". Roughly speaking, the (wild) speculations are the following: for each faithful representation ρ in the $\mathrm{PU}(2,1)$-character variety of B, we can find a "polyhedron" $Q \subset \mathbb{H}^2_{\mathbb{C}}$ (analogous to the actual polyhedra appearing as fundamental domains for disc orbigoodles in [BGr]). The representation ρ provides gluing relations between the sides of Q and by taking the quotient of Q by these relations we obtain a diffeological space M. The space Q can be seen as an immersion of $D \times P$ in $\mathbb{H}^2_{\mathbb{C}}$, where D is a disc and P is a fundamental domain for the hyperbolic sphere with three conic points. By gluing the sides of Q we also glue the sides of the immersed fundamental domain P, obtaining a "pinched orbifold" (see Figure 1). We believe the quotient M to be, in some sense, a "disc bundle" over this pinched orbifold. Our suspicions come from computational observations: the formula for the Euler number in [BGr] is well-defined and the identity (1) holds even when the representation ρ is no longer discrete. With these strange bundles in mind, it seemed appropriate to approach the subject through diffeology, a differential-geometry like science that deals with very singular spaces.

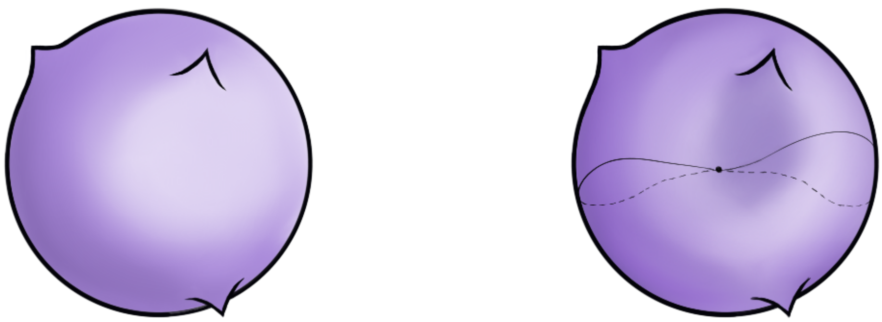


Figure 1: Orbifold and its pinched version.

Acknowledgments: Thank you Carlos H. Grossi and Sasha Anan'in for converting me to geometry.

2 Preliminaries

2.1. Orbifolds as diffeological spaces. Orbifolds are the simplest type of singular spaces. They are almost like smooth manifolds, but allow singularities. In the case we are interested in (compact and oriented two-dimensional orbifold with only isolated singularities), orbifolds are, vaguely speaking, topological surfaces that from the smooth perspective are locally Euclidean except around the singular points, where the smooth structure is modeled by cones. The classical approach to orbifolds can be found, for instance, in [Sco], [Kap2, Chapter 6] and [Thu, Chapter 13].

In order to develop the theory of "bundles" over orbifolds we are interested in, the language of diffeology will come in handy. Since this language is not commonly used by hyperbolic geometers, we state some basic definitions. For a more complete treatment, see [Igl1] and [Igl2].

We will call an open set $U \subset \mathbb{R}^n$ an Euclidean open set (note that $\mathbb{R}^0 = 0$).

Definition 2. Let M be a set. A **diffeology** on M is a family $\mathcal{F}$ of functions from Euclidean open sets to M such that:

- every map $0 \to M$ belongs to $\mathcal{F}$;
- if $\phi : U \to M$ belongs to $\mathcal{F}$ and $g : V \to U$ is smooth, where V is an Euclidean open set, then $\phi \circ g \in \mathcal{F}$;
- if we have a function $\phi : U \to M$ and every point $x \in U$ has a neighborhood $U_x \subset U$ such that $\phi|_{U_x} \in \mathcal{F}$, then $\phi \in \mathcal{F}$.

The pair $(M, \mathcal{F})$ is a **diffeological space**. The elements of $\mathcal{F}$ are called **plots** of M.

Structurally, the definition of a diffeology resembles that of a topology. In particular, every set M admits two extreme diffeologies:

- The **discrete diffeology**: the set of all functions from Euclidean open sets to M that are locally constant.
- The **indiscrete diffeology**: the set of all functions from Euclidean open sets to M.

A map $f : M_1 \to M_2$ is **smooth** if, for every plot $\phi : U \to M_1$ of M_1, the map $f \circ \phi : U \to M_2$ is a plot of M_2. In particular, plots are smooth. Therefore, we have the category of diffeological spaces where the objects are diffeological spaces and the morphisms are the smooth maps. A diffeomorphism is an isomorphism in this category.

The basic constructions we will need are motivated by the ideas of **initial and final diffeologies** (analogous to the concepts of initial and final topologies). Given a set M and a family of diffeological spaces M_α with maps $\pi_\alpha : M \to M_\alpha$, the initial diffeology on M is the largest one that turns each π_α into a smooth map. The dual definition is the final diffeology, i.e., given a set M and a family of diffeological spaces M_α with maps $i_\alpha : M_\alpha \to M$, the final diffeology of M is the smallest one such that each i_α is smooth.

Some basic constructions arising from initial diffeologies include the subspace diffeology and the product diffeology, given respectively by the inclusion and the projection maps. Final diffeologies provide, for example, the quotient and the coproduct diffeologies, given respectively by the quotient map and the natural inclusions in the disjoint union.

If M is a diffeological space, then we consider on M the largest topology that makes all plots of M continuous (a final topology).

Remark 3. The category of (finite dimensional) smooth manifolds does not have arbitrary pullbacks (transversality is required) and quotients. Furthermore, the space of smooth maps between manifolds as well as diffeomorphism groups are not smooth manifolds. On the other hand, the category of diffeological spaces is bicomplete and it is Cartesian closed; in particular, the space of smooth maps between diffeological spaces is a diffeological space and diffeomorphism groups are diffeological groups. Moreover, the theories of bundles and differential forms are very simple and practical in this language.

With these basics definitions, we can define an n-dimensional smooth manifold as a Hausdorff and second countable diffeological space locally diffeomorphic to open subsets of $\mathbb{R}^n$, i.e., given $x \in M$, there is a neighborhood D of x in the diffeological topology of M, such that D is diffeomorphic to an open subset of $\mathbb{R}^n$. Similarly, we have orbifolds:

Definition 4. An **n-dimensional orbifold** B is a Hausdorff and second countable diffeological space locally diffeomorphic to $\mathbb{B}^n/\Gamma$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is the unit open n-ball centered at 0 and Γ is a finite subgroup of the orthogonal group $\mathrm{O}(n)$. We call such diffeomorphism $\phi : \mathbb{B}^n/\Gamma \to D$, where D is an open subset of M, an orbifold chart and say the chart ϕ is centered at x if $\phi\big([0]\big) = x$. An orbifold is **locally orientable** if it is locally modeled on $\mathbb{B}^n/\Gamma$ with $\Gamma \subset \mathrm{SO}(n)$.

For a detailed treatment of orbifolds as diffeological spaces see [IKZ].

All orbifolds in this article are supposed to be locally orientable because we do not allow non-isolated singularities in two-dimensional orbifolds.

Definition 5. Consider an orbifold B. We say $x \in B$ is a **singular point** if there is a chart $\phi : \mathbb{B}^n/\Gamma \to D$ centered at x and $\Gamma \neq \{1\}$. If a point is non-singular, we call it regular. Since we are dealing with locally orientable orbifolds, B is connected if, and only if, $B \setminus S$ is connected, where S is the set of all singular points of B. We say B is an **oriented orbifold** if $B\setminus S$ is oriented.

Remark 6. We will need the concepts of orbifold fundamental group and orbifold universal cover. For details, the reader may take a look at [Kap2, Chapter 6] and [Thu, Chapter 13]. We will also use the concept of a good orbifold, which is an orbifold covered (in the sense of orbifolds) by manifolds. In this case, an orbifold B can be seen as the quotient of a simply connected manifold $\mathbb{H}$ by a group G acting properly discontinuously on $\mathbb{H}$, and $\pi_1^{\mathrm{orb}}(B) = G$.

2.2. Complex hyperbolic geometry. In this subsection we present a few basic facts about complex hyperbolic geometry that will be needed later. The reader is referred to [AGr] and [Gol1] for details.

Consider an $(n+1)$-dimensional complex vector space V endowed with a Hermitian form $\langle\cdot,\cdot\rangle$ of signature $-+\cdots+$. The projective space $\mathbb{P}(V)$ is divided into three parts,

$$\mathbb{H}^n_{\mathbb{C}} = \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p,p\rangle < 0\}, \quad \mathrm{S}(V) = \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p,p\rangle = 0\},$$
$$\mathrm{E}(V) = \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p,p\rangle > 0\},$$

where we denote by $\boldsymbol{p}$ a point in the projective space and by p a representative of $\boldsymbol{p}$ in V. The n-dimensional ball $\mathbb{H}^n_{\mathbb{C}}$ is called **complex hyperbolic space**. We say the points of $\mathbb{H}^n_{\mathbb{C}}$, $\mathrm{S}(V)$ and $\mathrm{E}(V)$ are negative, isotropic, and positive, respectively.

The tangent space of $\mathrm{T}_{\boldsymbol{p}}\mathbb{P}(V)$ is naturally identified with the space of linear transformations from $\mathbb{C}p$ to $p^\perp := \big\{v \in V : \langle p,v\rangle = 0\big\}$ whenever $\boldsymbol{p}$ is non-isotropic. Furthermore, over the non-isotropic region we define the **Hermitian metric**

$$h(s,t) := -\frac{\langle s(p),t(p)\rangle}{\langle p,p\rangle}$$

for $t,s \in \mathrm{T}_{\boldsymbol{p}}\mathbb{P}(V)$ (the definition does not depend on the representatives for $\boldsymbol{p}$ because t,s are linear). The **pseudo-Riemannian metric** g and the **Kähler form** ω are defined by $g := \mathrm{Re}\,h$ and $\omega := \mathrm{Im}\,h$, respectively. Furthermore, the metric g is complete and it is Riemannian on $\mathbb{H}^n_{\mathbb{C}}$. Hence, $\mathbb{H}^n_{\mathbb{C}}$ is a Kähler manifold.

The **Riemann curvature tensor** R of the Levi-Civita connection is

$$R(t_1,t_2)s = -st_1^*t_2 - t_2t_1^*s + st_2^*t_1 + t_1t_2^*s \tag{7}$$

where the adjoint of a tangent vector $t : \mathbb{C}p \to p^\perp$ at $\boldsymbol{p}$ is the map $t^* : p^\perp \to \mathbb{C}p$ given by $t^*(v) := \frac{\langle v,t(p)\rangle}{\langle p,p\rangle}p$.

Remark 8. We use the definition $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, which differs by a sign from the one in [AGG].

From the above formula, it is easy to deduce that the Gaussian curvature of $\mathbb{H}^1_{\mathbb{C}}$ is -4, hence it is a Poincaré disc. Additionally, $E(V)$ is also a Poincaré disc with the same curvature. So the projective line $\mathbb{P}^1_{\mathbb{C}}$ is formed by two Poincaré discs glued along the boundary, and we call it a **Riemann-Poincaré sphere**.

The **complex hyperbolic plane** $\mathbb{H}^2_{\mathbb{C}}$ has non-constant sectional curvature varying on the interval $[-4,-1]$. Furthermore, given a positive point $\boldsymbol{p}$, the projective line $\mathbb{P}(p^{\perp})$ is a Riemann-Poincaré sphere, and its intersection with $\mathbb{H}^2_{\mathbb{C}}$ is a **complex geodesic**. Note that two distinct points in the complex hyperbolic plane determine a unique complex geodesic.

It may seem strange that we do not scale the metric in order to obtain complex geodesics of curvature -1. The reason is that there exist Beltrami-Klein hyperbolic discs inside $\mathbb{H}^2_{\mathbb{C}}$ as well. They are given by $\mathbb{H}^2_{\mathbb{C}}\cap\mathbb{P}(\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3)$, where e_1,e_2,e_3 is an orthonormal basis of V. These discs have curvature -1, are called **$\mathbb{R}$-planes**, and are very different from complex geodesics: they are not uniquely determined by two points, they are not embedded Riemann surfaces, and are Lagrangian submanifolds (whereas complex geodesics are symplectic).

The group of holomorphic isometries of $\mathbb{H}^n_{\mathbb{C}}$ is the group $\mathrm{PU}(n,1)$, the projectivization of the group $\mathrm{SU}(n,1)$ formed by the linear isomorphisms of V preserving the Hermitian form.

A non-identical isometry that fixes a point in $\mathbb{H}^n_{\mathbb{C}}$ is called an **elliptic isometry**. An elliptic isometry can have a unique fixed point in $\mathbb{H}^2_{\mathbb{C}}$ or a fixed complex geodesic: if $\widehat{I}\in\mathrm{SU}(2,1)$ is a representative of an elliptic isometry I (the representatives differ by a cube root of the unit), then $\widehat{I}$ has an orthonormal basis c,p,q consisting of eigenvectors, where c is negative. The projective lines $\mathbb{P}(\mathbb{C}c\oplus\mathbb{C}p)$, $\mathbb{P}(\mathbb{C}c\oplus\mathbb{C}q)$, and $\mathbb{P}(\mathbb{C}p\oplus\mathbb{C}q)$ are stable under the action of I, where the first two are Riemann-Poincaré spheres and the third is a round sphere (of positive points). If two of the eigenvectors have the same eigenvalue, then the projective line determined by them is fixed. So, an elliptic isometry has either a unique fixed point in $\mathbb{H}^2_{\mathbb{C}}$ or a unique fixed complex geodesic.

3 Orbibundles and the Euler number

3.1. Orbibundles. Let M,F be diffeological spaces and let B be an orbifold. A smooth map $\zeta:M\to B$ is an **orbibundle** with fiber F if for every point $p\in B$ there is an orbifold chart $\phi:\mathbb{B}^n/\Gamma\to D$ centered at p satisfying the following properties:

- there is a smooth action of Γ on $\mathbb{B}^n\times F$ of the form $h(x,f)=(hx,a(h,x)f)$, where $a:\Gamma\times\mathbb{B}^n\to\mathrm{Diff}(F)$ is smooth and $\mathrm{Diff}(F)$ stands for the group of diffeomorphisms of F endowed with its natural diffeology (see [Igl2, Section 1.61]);
- there is a diffeomorphism $\Phi:(\mathbb{B}^n\times F)/\Gamma\to\zeta^{-1}(D)$ such that the diagram

$$\begin{tikzcd} (\mathbb{B}^n\times F)/\Gamma \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_1"'] & \zeta^{-1}(D) \arrow[d, "\zeta"] \\ \mathbb{B}^n/\Gamma \arrow[r, "\phi"] & D \end{tikzcd} \tag{9}$$

commutes, where $\mathrm{pr}_1([x,f])=[x]$.

Note that a fiber bundle in the sense of [Igl2, Chapter 8] (when the base space is an orbifold) is the particular case of an orbibundle where $a(h,x)f$ always equals f. If $F=\mathbb{B}^2$, then we say that ζ is a **disc orbibundle**.

Remark 10. Our definition was originally discovered studying examples in complex hyperbolic geometry and is heavily inspired by Audin's approach to Seifert manifolds [Aud]. We later found out a similar definition in [Car, 3.3 Orbibundles and Frobenius' Theorem] which does not use diffeology and assumes the total space to be an orbifold. Not requiring the total space to be an orbifold makes the description of a G-orbibundle (see Definition 13) more natural. Moreover, the language of diffeological spaces simplifies the theory due to its good categorical properties (a highlight being the fact that the category has quotients).

If F is a discrete (and countable) diffeological space, we say that $\zeta:M\to B$ is an **orbifold covering map** and if F is finite with d elements we say the orbifold cover has degree d. Let us analyze more precisely what this object is. Following the above diagram, $\zeta^{-1}(D)$ is modeled by $(\mathbb{B}^n\times F)/\Gamma$. Writing $F/\Gamma=\{\Gamma f_1,\dots,\Gamma f_l\}$, the set of the disjoint orbits of F, we have the diffeomorphism

$$\coprod_{i=1}^{l} \mathbb{B}^n/\mathrm{stab}_\Gamma(f_i) \to (\mathbb{B}^n \times F)/\Gamma$$
$$([x], f_i) \mapsto [x, f_i]$$

Therefore, M is an orbifold.

Definition 11. An orbibundle $\zeta : M \to B$ with fiber $\mathbb{R}^k$ is a **real vector orbibundle of rank** k if:

- every fiber of ζ is a real vector space,
- for each orbifold chart $\phi : \mathbb{B}^n/\Gamma \to D$, following the notation in diagram (9), the map $a(h,x)$ is a linear isomorphism;
- for each $u \in \mathbb{B}^n/\Gamma$, the map $\Psi : \mathrm{pr}_1^{-1}(u) \to \zeta^{-1}(\phi(u))$ is an $\mathbb{R}$-linear isomorphism.

Complex vector orbibundles are defined analogously. Furthermore, we say $M \to B$ is an **oriented vector orbibundle** if removing the singular points of B we obtain an oriented vector bundle.

Remark 12. Each fiber of $\mathrm{pr}_1 : (\mathbb{B}^n \times \mathbb{R}^k)/\Gamma \to \mathbb{B}^n/\Gamma$ is naturally a vector space: $\zeta^{-1}[0] = \mathbb{R}^k/\Gamma$, and on the fiber of $[z] \neq [0]$ we induce the linear structure using the bijections $l_{z,h} : \zeta^{-1}[z] \to \mathbb{R}^k$ given by $[z,v] \mapsto a(h,z)v$, with $h \in \Gamma$ (this is possible because the map $l_{z,h} \circ l_{z,h'}^{-1}$ is always linear and $l_{gz,h} = l_{z,hg} l_{z,g}^{-1}$).

Definition 13. Consider a diffeological group G. We say an orbibundle $\zeta : M \to B$ with fiber G is a G-**orbibundle** if

- G acts smoothly on M;
- for each orbifold chart $\phi : \mathbb{B}^n/\Gamma \to D$, according to the notation in diagram (9), we consider $a : \Gamma \times \mathbb{B}^n \to G$ and the action given by $h(x,g) := (hx, a(h,x)g)$;
- the map $\Phi : (\mathbb{B}^n \times G)/\Gamma \to \zeta^{-1}(D)$ is G-equivariant, where G acts on $(\mathbb{B}^n \times G)/\Gamma$ on the right.

In this paper, $\mathbb{S}^1$ stands for the group of unit complex numbers.

Proposition 14. *For $\mathbb{S}^1$-orbibundles, up to an $\mathbb{S}^1$-equivariant diffeomorphism, we can assume that the map $a : \Gamma \times \mathbb{B}^n \to \mathbb{S}^1$ does not depend on x.*

Proof. Given a small ball $U \subset \mathbb{B}^n$ centered at 0, define $f : U \times \mathbb{S}^1 \to U \times \mathbb{S}^1$,

$$f(x,s) := \Big(x, \mathrm{norm}\big(\sum_{h\in\Gamma} a(h^{-1},0)a(h,x)\big)s\Big),$$

where $\mathrm{norm}(z) := z/|z|$, $z \in \mathbb{C}$. Define actions of Γ on the domain by $h(x,s) := (hx, a(h,x)s)$ and on the codomain by $h(x,s) := (hx, a(h,0)s)$. Since $a(hg,x) = a(h,gx)a(g,x)$, we have

$$\begin{aligned} f(g(x,s)) &= \Big(gx, \mathrm{norm}\big(\sum_{h\in\Gamma} a(h^{-1},0)a(h,gx)\big)a(g,x)s\Big) \\ &= \Big(gx, \mathrm{norm}\big(\sum_{h\in\Gamma} a(h^{-1},0)a(hg,x)\big)s\Big) \\ &= \Big(gx, \mathrm{norm}\big(\sum_{h\in\Gamma} a(gh^{-1},0)a(h,x)\big)s\Big) \\ &= \Big(gx, a(g,0)\mathrm{norm}\big(\sum_{h\in\Gamma} a(h^{-1},0)a(h,x)\big)s\Big) \\ &= gf(x,s). \end{aligned}$$

Hence the map f is a Γ-equivariant diffeomorphism and

$$f : (U \times \mathbb{S}^1)/\Gamma \to (U \times \mathbb{S}^1)/\Gamma$$

is a diffeomorphism. Since the action of $\mathbb{S}^1$ on both quotients is on the right, we conclude it is an $\mathbb{S}^1$-equivariant diffeomorphism. □

Remark 15. The above argument can be easily adapted to the case of GL_n and SL_n-orbibundles.

3.2. Euler number of $\mathbb{S}^1$-orbibundles over 2-orbifolds. In this subsection we establish some tools to calculate Euler numbers and to characterize $\mathbb{S}^1$-orbibundles by such invariant.

Consider a 2-orbifold B, a point $x \in B$, and an orbifold chart $\phi : \mathbb{B}^2/\Gamma \to B$ centered at x. Note that Γ is a finite subgroup of $\mathrm{SO}(2)$ and, therefore, our orbifolds have only isolated singularities since Γ acts freely in $\mathbb{B}^2 \setminus \{0\}$.

Thinking of $\mathbb{B}^2$ as the unit disc in $\mathbb{C}$ centered at 0, we have $\Gamma = \langle \xi \rangle$, where $\xi = \exp(2\pi i/n)$. The number n is the **order** of the point x. If $n = 1$, then $\Gamma = 1$ and the point is regular. In the case $n \geq 2$, the quotient $\mathbb{B}^2/\Gamma$ is a cone with cone point of angle $2\pi/n$.

The orbifolds we are interested in are compact and, therefore, have a finite number of singular points. In this section, when we consider a 2-orbifold B, we assume that it is connected, compact, and oriented. Moreover, $x_1, \dots, x_n$ denote its singular points.

Definition 16 (Euler number of $\mathbb{S}^1$-orbibundles)**.** Let $\zeta : M \to B$ be an $\mathbb{S}^1$-orbibundle and let $x_0 \in B$ be a regular point. For each x_i, $i = 0, 1, \dots, n$, consider a small smooth disc $D_i \subset B$ centered at x_i. We have the surface with boundary $B' = B \setminus \sqcup_k D_k$. Note that $\zeta : M' \to B'$, where $M' := \zeta^{-1}(B')$, is an ordinary $\mathbb{S}^1$-bundle and B' is homotopically equivalent to a graph. Therefore, $\zeta : M' \to B'$ admits a global section σ because $\mathbb{S}^1$-bundles over graphs are trivial. Fix an arbitrary fiber s of $\zeta : M \to B$ over a regular point, that is, a generator of $H_1(M, \mathbb{Q})$ (every time we talk about an $\mathbb{S}^1$-fiber we will assume that it is parameterized and that the curve goes around the fiber just once in the same direction of the $\mathbb{S}^1$-orbits). We define the **Euler number** $e(M)$ by the formula

$$\sigma|_{\partial B'} = -e(M)s$$

in $H_1(M, \mathbb{Q})$.

Let us prove that $e(M)$ does not depend on the choice of σ. Given sections σ_1 and σ_2 of $\zeta : M' \to B'$ we define $f : B' \to \mathbb{S}^1$ such that $\sigma_1(x) = f(x)\sigma_2(x)$, $x \in B'$. Note that in homology $\sigma_1|_{\partial D_k} = \deg(f|_{\partial D_k})s + \sigma_2|_{\partial D_k}$ and, therefore,

$$\sum_k \sigma_1|_{\partial D_k} - \sum_k \sigma_2|_{\partial D_k} = \sum_k \deg(f|_{\partial D_k})s.$$

On the other hand, if $d\theta := -idz/z$ is the angle 1-form of $\mathbb{S}^1$, then, using Stokes theorem and the fact that the orientation of ∂D_k is opposite to that of $\partial B'$, we obtain

$$\sum_k \deg(f|_{\partial D_k}) = \sum_k \frac{1}{2\pi} \int_{\partial D_k} f^* d\theta = -\frac{1}{2\pi} \int_{B'} d(f^* d\theta) = 0,$$

$$\sigma_1|_{\partial B'} = -\sum_k \sigma_1|_{\partial D_k} = -\sum_k \sigma_2|_{\partial D_k} = \sigma_2|_{\partial B'}.$$

Hence, the Euler number is well defined. It is easy to verify that the definition of the Euler number does not depend on the choice of the regular point x_0.

Next, we define the Euler number of oriented rank 2 real vector orbibundles. Recall that by oriented here we mean that, removing the singular points of the base orbifold, one obtains an oriented vector bundle over a surface.

As a motivation, consider an oriented real vector bundle $\zeta : L \to B$ of rank 2 over a connected, compact, and oriented surface. If X is a section transversal to the 0 section in L, then these sections intersect in a finite number of points whose projections onto B are denoted by $x'_1, \dots, x'_n$. On L we consider local trivializations $\xi_x : L|_{\overline{D}_x} \to \overline{D}_x \times \mathbb{R}^2$ such that D_x is a small open disc centered in x with smooth boundary and this atlas of L is compatible with the orientation of the bundle. Consider as well the surface with boundary $B' := B \setminus \sqcup_i D_{x_i}$.

By the Poincaré-Hopf theorem, the Euler number of the vector bundle ζ is given by the formula

$$e(M) = \sum_{k=1}^{n} \operatorname{index}(X, x'_k). \tag{17}$$

Remark 18. In a finite-dimensional real vector space V consider the sphere $\mathbb{S}(V) := V^\times/\mathbb{R}_{>0}$, where $V^\times := V \setminus 0$. The topology and smooth structure of $\mathbb{S}(V)$ can be induced from a sphere obtained from an inner product in V. Furthermore, these two structures do not depend on the choice of the inner product. Analogously, given a real vector orbibundle $L \to B$ we can define the natural sphere orbibundle $\mathbb{S}(L) \to B$. On the case of a real vector orbibundle of rank 2, it is always possible to introduce an $\mathbb{S}^1$-action compatible with the natural orientation of the fibers (just consider a metric on the vector orbibundle); the Euler number will not depend on the choice of the action.

The term $\operatorname{index}(X, x'_k)$ in equation (17) is the degree of the map $\partial D_{x'_k} \to \mathbb{S}(\mathbb{R}^2)$ between circles given by $x \mapsto \operatorname{pr}_2 \xi_{x'_k}\big(X(x)\big)$, where $\operatorname{pr}_2 : D_{x'_k} \times (\mathbb{R}^2)^\times \to \mathbb{S}(\mathbb{R}^2)$ is the projection on the second coordinate composed with the quotient map $(\mathbb{R}^2)^\times \to \mathbb{S}(\mathbb{R}^2)$.

Note that $\sigma(x) := [X(x)]$ is a section of $\mathbb{S}(L)|_{B'} \to B'$ and the group $H_1(\mathbb{S}(L), \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$, is generated by any fiber s of the projection $\mathbb{S}(L) \to B$. It is easy to see that $\sigma|_{\partial D_{x_k}} = \operatorname{index}(X, x'_k)s$ and

$$\sigma|_{\partial B'} = -e\big(\mathbb{S}(L)\big)s \text{ in } H_1(\mathbb{S}(L), \mathbb{Z})$$

because $\sigma|_{\partial B'} = -\sum_k \sigma|_{\partial D_k}$ since the orientation of ∂D_k is opposite to that of $\partial B'$. So, we can reduce our calculations to ordinary $\mathbb{S}^1$-bundles.

Definition 19. Consider an oriented real vector orbibundle $\zeta : L \to B$ of rank 2, where B is a connected, compact, and oriented 2-orbifold. We define the Euler number of L to be the Euler number of $\mathbb{S}(L) \to B$.

An example of oriented 2-dimensional real vector orbibundle is the tangent bundle TB of B. For each $x \in B$ we have the stalk C_x^∞ of the sheaf $C^\infty(-, \mathbb{R})$ and the space $\mathrm{Der_x}$ of derivations at x, which is a real vector space. A tangent vector at a point $x \in B$ is a derivation v that comes from a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$, i.e., for every $f \in C_x^\infty$ we have $v(f) := (f \circ \gamma)'(0)$. The real vector space $\mathrm{T}_x B$ is defined as $\operatorname{Span}_{\mathbb{R}}\{v \in \mathrm{Der_x} : v \text{ is a tangent vector at } x\}$. It is easy to verify that $\mathrm{T}B$ is a rank 2 real vector orbibundle.

Lemma 20. *Let Γ_1 and Γ_2 be finite subgroups of $\mathbb{S}^1$ such that Γ_2 is a subgroup of Γ_1. Also consider a continuous action $\Gamma_1 \times \mathbb{S}^1 \to \mathbb{S}^1$ and the solid torus $\mathrm{T} = \overline{\mathbb{B}}^2 \times \mathbb{S}^1$. We have the following commutative diagram*

$$\begin{array}{ccc} \mathrm{T}/\Gamma_2 & \xrightarrow{G} & \mathrm{T}/\Gamma_1 \\ {\scriptstyle\zeta_2}\downarrow & & \downarrow{\scriptstyle\zeta_1} \\ \overline{\mathbb{B}}^2/\Gamma_2 & \xrightarrow{g} & \overline{\mathbb{B}}^2/\Gamma_1 \end{array}$$

of natural continuous maps.

With respect to the map $\zeta_i : \mathrm{T}/\Gamma_i \to \overline{\mathbb{B}}^2/\Gamma_i$, consider the fiber $s_i : \mathbb{S}^1 \to \zeta_i^{-1}([1])$ over $[1] \in \overline{\mathbb{B}}^2/\Gamma_i$ given by $s_i(z) := [1, z]$, and the fiber $s'_i : \mathbb{S}^1/\Gamma_i \to \zeta_i^{-1}([0])$ over $[0]$ given by $s'_i([z]) := [0, z]$.

Additionally, let σ_i be a continuous section of the $\mathbb{S}^1$-principal bundle $\zeta_i : (\mathbb{S}^1 \times \mathbb{S}^1)/\Gamma_i \to \mathbb{S}^1/\Gamma_i$, with $i = 1, 2$, such that σ_2 is on top of σ_1, i.e, $G\sigma_2 = \sigma_1 g$. Then we have:

- *T/Γ_i is homeomorphic to T and any fiber of ζ_i is a generator of $H_1\big(\mathrm{T}/\Gamma_i, \mathbb{Q}\big)$,*
- *$s_i = |\Gamma_i| s'_i$ in $H_1\big(\mathrm{T}/\Gamma_i, \mathbb{Q}\big)$,*
- *$\sigma_1 = q s'_1$ in $H_1\big(\mathrm{T}/\Gamma_1, \mathbb{Q}\big)$ if, and only if, $\sigma_2 = q s'_2$ in $H_1\big(\mathrm{T}/\Gamma_2, \mathbb{Q}\big)$.*

Proof. Let us prove the first item. Without loss of generality we can suppose that the action of Γ_i on $\mathbb{S}^1$ is $\Gamma_i \times \mathbb{S}^1 \to \mathbb{S}^1$, $(\xi, \gamma) \mapsto \xi^k \gamma$, where $\xi := \exp(2\pi i/n)$ and $n := |\Gamma_i|$. The result is obvious for $k = 0$. We assume $1 \leq k \leq n-1$. If $d := \gcd(n,k)$, then there is an integer l such that $kl = d \mod n$ and $1 \leq l \leq n-1$. The continuous map $\lambda : \mathrm{T} \to \mathrm{T}$ given by $\lambda(z,\gamma) = (\gamma^{-l} z^d, \gamma^{n/d})$ is Γ_i-invariant and surjective. Besides, if $\lambda(z',\gamma') = \lambda(z,\gamma)$, then there exists $\omega \in \Gamma_i$ satisfying $(\omega z, \omega^k \gamma) = (z', \gamma')$. Indeed, there exists an integer s such that $\gamma' = \xi^{sd}\gamma$, because $\gamma'^{n/d} = \gamma^{n/d}$ and, consequently, $z'^d = \xi^{lsd} z^d$. By the same reasoning, there is an integer t satisfying $z' = \xi^{ls+(n/d)t} z$. Taking $\omega := \xi^{ls+(n/d)t}$ we have $\omega^k = \xi^{sd}$ and $(z',\gamma') = (\omega z, \omega^k \gamma)$. Therefore, we have the continuous bijection $\mathrm{T}/\Gamma_i \to \mathrm{T}$, $[z,\gamma] \mapsto \lambda(z,\gamma)$, which is a homeomorphism because T/Γ_i is compact.

The second item follows from the homotopy $R : [0,1] \times \mathbb{S}^1 \to \mathrm{T}/\Gamma_i$, $R(t,z) := [t,z]$, because $R(0,-) = |\Gamma_i| s_i'$ and $R(1,-) = s_i$ in $H_1\big(\mathrm{T}/\Gamma_i, \mathbb{Q}\big)$.

Let us prove the third item. At the homology level, $\sigma_1 \circ g = [\Gamma_1 : \Gamma_2]\sigma_1$, since $g : \mathbb{S}^1/\Gamma_2 \to \mathbb{S}^1/\Gamma_1$ is a covering map of degree $[\Gamma_1 : \Gamma_2]$. Similarly, $s_1' \circ g = [\Gamma_1 : \Gamma_2] s_1'$ at the homology level. Therefore, if $\sigma_2 = d s_2'$ then

$$\begin{aligned} G_*(\sigma_2) &= d\, G_*(s_2'), \\ [\Gamma_1 : \Gamma_2]\sigma_1 &= d[\Gamma_1 : \Gamma_2] s_1', \\ \sigma_1 &= d s_1' \end{aligned}$$

in homology, due to $G\sigma_2 = \sigma_1 g$ and $G s_2' = s_1' g$. □

The Euler number of $\mathrm{T}B$ is the **Euler characteristic** of B, denoted by $\chi(B)$.

We prove the next theorem, which is a standard result in the theory of orbifolds, in order to illustrate a practical application of Lemma 20. This may also be useful as the proofs of Theorems 22 and 25 follow similar lines of thought.

Theorem 21. *The Euler characteristic of B is given by*

$$\chi(B) = \chi(\tilde{B}) + \sum_{k=1}^{n} \Big(-1 + \frac{1}{m_k} \Big),$$

where the numbers $m_1, \ldots, m_n$ are the orders of the singular points $x_1, \ldots, x_n$ of B and $\tilde{B}$ stands for the smooth surface obtained by removing the singular discs and gluing regular ones.

Proof. Consider a regular point x_0 and remove from B small open discs D_k centered at x_k, where $k = 0, \ldots, n$, thus obtaining a surface with boundary B'. For each $1 \leq k \leq n$ consider a vector field V_k on ∂D_k pointing outwards B'. By [Aud, Lemma I.3.6], there is a global section σ of the $\mathbb{S}^1$-bundle $\mathbb{S}(\mathrm{T}B')$ such that $\sigma|_{\partial D_k}(x) = [V_k(x)] \in \mathbb{S}(\mathrm{T}_x B)$ for $k = 1, \ldots, n$. Let s be a generic fiber of $\zeta : \mathbb{S}(\mathrm{T}B) \to B$ over a regular point, and let s_k be the fiber over x_k. By Lemma 20 we have $s = m_k s_k$ in $H_1(M, \mathbb{Q})$.

We can assume that for each k there is a chart $\phi_k : 2\mathbb{B}^2/\Gamma_k \to 2D_k$ centered at x_k satisfying $\phi_k\big(\overline{\mathbb{B}}^2/\Gamma_k\big) = \overline{D}_k$, where $\Gamma_k = \langle \exp(2\pi/m_k) \rangle$, $2\mathbb{B}^2$ is the disc of radius 2 in the complex plane, and $2D_k$ is an open subset of B containing D_k. In particular, we have the diffeomorphism $\phi_k : \mathbb{S}^1/\Gamma_k \to \partial D_k$. Furthermore, we have a trivialization of the vector bundle $\mathrm{T}B$ given by the chart ϕ_k as described in the commutative diagram

$$\begin{tikzcd} (2\mathbb{B}^2 \times \mathbb{R}^2)/\Gamma_k \arrow[r, "\Phi_k"] \arrow[d] & TB|_{2D_k} \arrow[d, "\zeta"] \\ 2\mathbb{B}^2/\Gamma_k \arrow[r, "\phi_k"] & 2D_k \end{tikzcd}$$

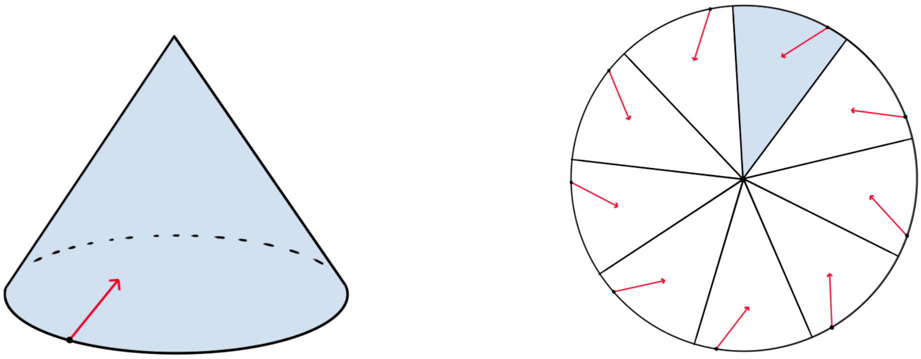
In particular, we have the commutative diagram

$$\begin{tikzcd} \overline{\mathbb{B}}^2 \times \mathbb{S}^1 \arrow[r] \arrow[d] & (\overline{\mathbb{B}}^2 \times \mathbb{S}^1)/\Gamma_k \arrow[r, "\Phi_k"] \arrow[d] & \mathbb{S}(\mathrm{T}B)|_{\overline{D}_k} \arrow[d, "\zeta"] \\ \overline{\mathbb{B}}^2 \arrow[r] & \overline{\mathbb{B}}^2/\Gamma_k \arrow[r, "\phi_k"] & \overline{D}_k \end{tikzcd}$$

Now the section $\sigma|_{\partial D_k} : \partial D_k \to \mathbb{S}(\mathrm{T}B)|_{\partial D_k}$ can be lifted to a section $\tilde{\sigma}_k : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ satisfying $\Phi_k([\tilde{\sigma}_k(z)]) = \sigma|_{\partial D_k} \circ \phi_k([z])$, as in Figure 2. Since the $\mathbb{S}^1$-bundle $\overline{\mathbb{B}}^2 \times \mathbb{S}^1 \to \overline{\mathbb{B}}^2$ is trivial, $\tilde{\sigma}_k$ equals the fiber $0 \times \mathbb{S}^1$ in $H_1(\overline{\mathbb{B}}^2 \times \mathbb{S}^1, \mathbb{Q})$. Then, by Lemma 20, we have

$$\sigma|_{\partial D_k} = s'_k$$

in $H_1\big(\mathbb{S}(TB)|_{\overline{B}_k}, \mathbb{Q}\big)$.



(a) (b)

Figure 2: **(a)** Section of the $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 \times \mathbb{S}^1)/H_k \to \mathbb{S}^1/H_k$. **(b)** The copy of this section on the bundle $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

Since $s = m_k s_k$ in $H_1\big(\mathbb{S}(TB), \mathbb{Q}\big)$,

$$\sigma|_{\partial B'} = \Big(-f - \sum_{k=1}^{n} \frac{1}{m_k} \Big) s$$

and $\chi(B) = f + \sum_{k=1}^{n} \frac{1}{m_k}$, where $\sigma|_{\partial D_0} = fs$.

By the same argument, in $H_1(\mathbb{S}(T\tilde{B}), \mathbb{Q})$ we have

$$\sigma|_{\partial B'} = \Big(-f - \sum_{k=1}^{n} 1 \Big) s$$

and $\chi(\tilde{B}) = f + n$. It remains to compare the formulas for $\chi(B)$ and $\chi(\tilde{B})$. □

The following theorem states that Euler numbers of $\mathbb{S}^1$-orbibundles are not arbitrary rational numbers but form a particular discrete subset of $\mathbb{R}$.

Theorem 22. *Consider the connected, compact, and oriented 2-orbifold B with isolated singularities $x_1, \dots, x_n$, where m_k is the order of x_k. The Euler number of an $\mathbb{S}^1$-orbibundle $\zeta : M \to B$ belongs to $\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \cdots + \frac{1}{m_n}\mathbb{Z}$.*

Proof. Let x_0 be a regular point in B, which has order $m_0 = 1$. As in the proof of Theorem 21, extract from B small open discs D_k, $k = 0, 1, \dots, n$, corresponding to the image of $\mathbb{B}^2/\Gamma_k$ under the orbifold chart $\phi_k : 2\mathbb{B}^2/\Gamma_k \to 2D_k$ centered at x_k, obtaining the surface B' with boundary. The bundle $M|_{B'} \to B'$ is trivial (because B' has the homotopy type of a graph) and, consequently, there exists a global section $\sigma : B' \to M|_{B'}$. If s_k is the fiber over x_k, then $\sigma|_{\partial D_k} = q_k s_k$ in homology for some $q_k \in \mathbb{Z}$. Considering an arbitrary regular fiber s and the identity $s = m_k s_k$, which follows from Lemma 20, we conclude that

$$\sigma|_{\partial B} = -\sum_{k=0} \frac{q_k}{m_k} s$$

in $H_1(M, \mathbb{Q})$. The result follows from Definition 16. □

4 Euler number for orbigoodles

An orbifold B_1 is a **good orbifold** if it is orbifold covered by a simply connected manifold $\mathbb{H}$. If G_1 is the orbifold fundamental group of B_1, i.e., the deck group of a fixed orbifold covering map $\mathbb{H} \to B_1$, then $B_1 = \mathbb{H}/G_1$, where G_1 acts properly discontinuously on $\mathbb{H}$. If $p : B_2 \to B_1$ is an orbifold covering, then there exists $G_2 \subset G_1$ such that $B_2 = \mathbb{H}/G_2$ and p is the quotient map.

Definition 23. Let V be a diffeological space. Consider an action of G_1 on $\mathbb{H} \times V$ by diffeomorphisms such that $g(x, v) = (gx, a(g, x)v)$, where $g \in G_1$ and $(x, v) \in \mathbb{H} \times V$. An **orbigoodle (good orbibundle)** with fiber V is the natural projection $\zeta_1 : L_1 \to B_1$ where $L_1 := (\mathbb{H} \times V)/G_1$. If G_2 is a subgroup of G_1, then **the pullback of the orbigoodle** ζ_1 by the orbifold covering $p : B_2 \to B_1$ is the orbigoodle $\zeta_2 : L_2 \to B_1$ given by restricting the action of G_1 to G_2 with the natural map P according to the diagram

$$\begin{array}{ccc} L_2 & \xrightarrow{P} & L_1 \\ {\scriptstyle\zeta_2}\downarrow & & \downarrow{\scriptstyle\zeta_1} \\ B_2 & \xrightarrow{p} & B_1 \end{array}$$

Sometimes, we denote L_2 by p^*L_1.

In the case $G_1' = hG_1h^{-1}$, where $h : \mathbb{H} \to \mathbb{H}'$ is a diffeomorphism (or, equivalently, G_1' is the deck group of another universal orbifold covering map $\mathbb{H}' \to B_1$), there is a natural action of G_1' on $\mathbb{H}' \times V$ which arises from the action of G_1 on $\mathbb{H} \times V$ and is defined by

$$g_1'(x, v) := \big(g_1'x, a(h^{-1}g_1'h, h^{-1}g_1'hx)v\big).$$

The map $H : (\mathbb{H} \times V)/G_1 \to (\mathbb{H}' \times V)/G_1'$, given by $H[x, v] := [hx, v]$, is a bundle isomorphism on the top of $h : \mathbb{H}/G_1 \to \mathbb{H}/G_1'$. Therefore, an orbigoodle brings forth natural orbigoodles for each universal orbifold covering of B_1, and all these orbigoodles are isomorphic.

Remark 24. The orbigoodle structure depends on how G_1 acts on $\mathbb{H}\times V$. If V is an n-dimensional vector space and the action is by (orientation preserving) linear isomorphisms, that is, each map $a(g, x) : V \to V$ is a linear isomorphism, then we obtain a (oriented) **vector orbigoodle** of rank n. If V is the group $\mathbb{S}^1$ and the action is by multiplication, then we obtain an **$\mathbb{S}^1$-orbigoodle**. If $V = \mathbb{B}^2$ and the action is by diffeomorphisms, then we obtain a **disc orbigoodle**.

Theorem 25. *If B_1 is a connected, compact, oriented good 2-orbifold, the orbifold covering map $p : B_2 \to B_1$ has degree d, and $\zeta : L \to B_1$ is an oriented vector orbigoodle of rank 2, then*

$$e(p^*L) = de(L).$$

Proof. Postponed to the end of the section. □

A direct consequence of this result is the identity $\chi\big(\mathbb{H}/G_1\big) = [G_1 : G_2]\chi\big(\mathbb{H}/G_2\big)$.

Definition 26. If B is a connected, compact, oriented good 2-orbifold and $L \to B$ is an oriented real vector orbigoodle of rank 2, then the number $e(L)/\chi(B)$ is called the **relative Euler number** of L. It is invariant under pullbacks by orbifold coverings of finite degree.

Now we examine the behavior of orbigoodles under pullbacks, according to Definition 23. Since $B_1 = \mathbb{H}/G_1$ admits a Riemannian metric, we assume $\mathbb{H}$ is a Riemannian manifold and G_1 is a group of isometries acting properly discontinuously on $\mathbb{H}$, i.e., the orbits are discrete and the stabilizer of points are finite. Let us locally trivialize the orbigoodle $\zeta_1 : L_1 \to B_1$.

Given $x \in \mathbb{H}$, there exists $r > 0$ such that $B(x, r) \cap gB(x, r) = \varnothing$ for every $g \in G_1 \setminus \operatorname{stab}_{G_1}(x)$, where $B(x, r)$ stands for the ball in $\mathbb{H}$ of radius r centered at x. We have the trivializations

$$\begin{array}{ccc} (B(x,r) \times V)/\operatorname{stab}_{G_1}(x) & \longrightarrow & \zeta_1^{-1}(D) \\ \downarrow & & \downarrow{\scriptstyle\zeta_1} \\ B(x,r)/\operatorname{stab}_{G_1}(x) & \longrightarrow & D \end{array}$$

where $D = B(x,r)/\operatorname{stab}_{G_1}(x)$, $\zeta_1^{-1}(D) = (B(x,r) \times V)/\operatorname{stab}_{G_1}(x)$, and the horizontal maps are identities.

Associated to the orbifold covering $p : B_2 \to B_1$, where $B_2 = \mathbb{H}/G_2$ for a subgroup G_2 of G_1 and p is the quotient map, we have the pullback L_2 of L_1 by p. The following lemma proves that p is indeed an orbifold cover and lay down the reasoning for why $P : L_2 \to L_1$ behaves as a covering map as well. Writing $F := G_1/G_2$, the disjoint orbits of the action of G_2 on G_1 on the left, we obtain the model fiber for the covering map p with discrete diffeology. The group G_1 acts on F on the right.

Lemma 27. *The map $p : \mathbb{H}/G_2 \to \mathbb{H}/G_1$ is an orbifold cover with fiber F.*

Proof. Consider the group $\Gamma := \operatorname{stab}_{G_1}(x)$ acting on F on the right and the quotient $F/\Gamma = \{f_i\Gamma\}$. Also fix a representative $s_i \in \mathrm{G}_1$ of $f_i \in F$ and denote by π_2 the quotient map $\mathbb{H} \to \mathbb{H}/G_2$.

Following the notation used in the previous diagram, the balls $D_i := \pi_2(B(s_i x, r))$ are pairwise disjoint, because $G_2 s_i = f_i$. Additionally, they do not depend on the representative s_i of f_i.

If $g_1 \in G_1$, then $G_2 g_1 \Gamma$ is one of the $f_i\Gamma$'s. Hence, there are $g_2 \in G_2$ and $h_1 \in \Gamma$ such that $g_2 g_1 h_1 = s_i$, and

$$\pi_2(B(s_i x, r)) = \pi_2(B(g_2 g_1 h_1 x, r)) = \pi_2(B(g_1 x, r)).$$

Therefore,

$$p^{-1}(D) = \coprod_i \pi_2(B(s_i x, r)).$$

Furthermore, $B(s_i x, r)/\big(s_i \operatorname{stab}_\Gamma(f_i) s_i^{-1}\big) = \pi_2(B(s_i x, r))$ and we conclude that

$$p^{-1}(D) = \coprod_i B(s_i x, r)/\big(s_i \operatorname{stab}_\Gamma(f_i) s_i^{-1}\big).$$

The diffeomorphism

$$\psi : \coprod_i B(x,r)/(\operatorname{stab}_\Gamma(f_i)) \to \coprod_i B(s_i x, r)/(s_i \operatorname{stab}_\Gamma(f_i) s_i^{-1}),$$

defined by $\psi([z], i) = ([s_i z], i)$ and the natural diffeomorphism

$$\coprod_i B(x,r)/(\operatorname{stab}_\Gamma(f_i)) \simeq (B(x,r) \times F)/\Gamma$$

produce a local trivialization of p

$$\begin{tikzcd}
(B(x,r) \times F)/\Gamma \arrow[r] \arrow[d, "\psi"] & B(x,r)/\Gamma \arrow[d, "\mathrm{id}"] \\
p^{-1}(D) \arrow[r, "p"'] & D
\end{tikzcd}$$

proving that p is an orbifold covering map. $\square$

Applying to the map P the same arguments as those in the proof of Lemma 27 we obtain the commutative diagram

$$\begin{tikzcd}
(B(x,r) \times V \times F)/\Gamma \arrow[rrr] \arrow[dr, "\Phi"] \arrow[ddd] & & & (B(x,r) \times V)/\Gamma \arrow[dl, "\mathrm{id}"'] \arrow[ddd] \\
& \zeta_2^{-1}(p^{-1}(D)) \arrow[r, "P"] \arrow[d, "\zeta_2"'] & \zeta_1^{-1}(D) \arrow[d, "\zeta_1"] & \\
& p^{-1}(D) \arrow[r, "p"] & D & \\
(B(x,r) \times F)/\Gamma \arrow[rrr] \arrow[ur, "\phi"] & & & B(x,r)/\Gamma \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

If ζ_1 is a vector orbigoodle, then P is a legit orbifold cover, and P and p have the same degree $|F| = [G_1 : G_2]$.

Proof of Theorem 25. Let $x_1, \ldots, x_n$ be the singular points of B_1 and x_0 be a regular point. For each $x_k \in B_1$ consider an orbifold chart $\phi_1^k : \mathbb{B}^2/\Gamma_k \to D_k$ centered at x_k on B_1 which trivializes the orbifold covering map p and the real vector orbigoodle ζ_1 of rank 2 as described in the diagram

$$\begin{tikzcd}
\coprod_{i=1}^{l_k} (\mathbb{B}^2 \times \mathbb{R}^2)/\mathrm{stab}_{\Gamma_k}(f_i^k) \arrow[r, "\Phi_2^k"] \arrow[dd, "\mathrm{pr}_1"'] & \zeta_2^{-1}(p^{-1}(D_k)) \arrow[r, "P"] \arrow[d, "\zeta_2"'] & \zeta_1^{-1}(D_k) \arrow[d, "\zeta_1"] & (\mathbb{B}^2 \times \mathbb{R}^2)/\Gamma_k \arrow[l, "\Phi_1^k"'] \arrow[dd, "\mathrm{pr}_1"] \\
 & p^{-1}(D_k) \arrow[r, "p"] & D_k & \\
\coprod_{i=1}^{l_k} \mathbb{B}^2/\mathrm{stab}_{\Gamma_k}(f_i^k) \arrow[ur, "\phi_2^k"] \arrow[rrr] & & & \mathbb{B}^2/\Gamma_k \arrow[ul, "\phi_1^k"']
\end{tikzcd}$$

where $F/\Gamma_k = \{f_i^k \Gamma_k\}_{i=1}^{l_k}$ is the set of disjoint orbits of F given by the action of Γ_k.

Consider the surfaces with boundary $B_1' := B_1 \setminus \sqcup_i D_i$ and $B_2' := p^{-1}(B_1')$, and a non-vanishing section $\xi_1 : B_1' \to L_1|_{B_1'}$. The fundamental groups identities

$$\pi_1(L_1|_{B_1'}) = \pi_1(B_1'), \quad \pi_1(L_2|_{B_2'}) = \pi_1(B_2')$$

along with

$$(\xi_1 \circ p)_*(\pi_1(B_2')) \subset P_*(\pi_1(L_2|_{B_2'})) \subset \pi_1(L_1|_{B_1'})$$

guarantees the existence of a section $\xi_2 : B_2' \to L_2|_{B_2'}$ on the top of ξ_1. Indeed, for a point $x \in B_2'$ and a vector $v \in P^{-1}(\xi_1 \circ p(x))$, the map $\xi_1 \circ p : B_2' \to L_1|_{B_1'}$ can be lifted to $\xi_2 : B_2' \to L_2|_{B_2'}$ by the covering map P such that $\xi_2(x) = v$:

$$\begin{tikzcd}
 & L_2|_{B_2'} \arrow[d, "P"] \\
B_2' \arrow[ur, "\xi_2"] \arrow[r, "\xi_1 \circ p"'] & L_1|_{B_1'}
\end{tikzcd}$$

The smooth map $\xi_2 : B_2' \to L_2|_{B_2'}$ is a section and satisfy $P \circ \xi_2 = \xi_1 \circ p$. Define the sections $\sigma_1 = [\xi_1]$ and $\sigma_2 = [\xi_2]$ on the $\mathbb{S}^1$-bundle $\mathbb{S}(L_1|_{B_1'})$ and $\mathbb{S}(L_2|_{B_2'})$.

Let us prove that if s_1 and s_2 are fibers of $\mathbb{S}(L_1)$ and $\mathbb{S}(L_2)$ over regular points, then whenever

$$\sigma_1|_{\partial D_k} = \frac{q_k}{|\Gamma_k|} s_1$$

in $H_1(\mathbb{S}(L_1), \mathbb{Q})$,

$$\sigma_2|_{p^{-1}(\partial D_k)} = \frac{d q_k}{|\Gamma_k|} s_2$$

in $H_1(\mathbb{S}(L_2), \mathbb{Q})$ and, consequently, we obtain the relation

$$e(L_2) = d e(L_1).$$

Indeed,

$$p^{-1}(D_k) = \coprod_{i=1}^{l_k} D(f_i^k),$$

where $D(f_i^k) := \phi_2^k\big(\mathbb{B}^2/\mathrm{stab}_{\Gamma_k}(f_i^k)\big)$ is a disc with center $y_i^k := \phi_2^k([0])$.

If s_i^k is the fiber above y_i^k on $\mathbb{S}(L_2)$, then by Lemma 20 we have

$$s_i^k = \frac{1}{|\mathrm{stab}_{\Gamma_k}(f_i^k)|} s_2, \quad \sigma_2|_{\partial D(f_i^k)} = q_k s_i^k$$

and, therefore,

$$\sigma_2|_{\partial D(f_i^k)} = \frac{q_k}{|\mathrm{stab}_{\Gamma_k}(f_i^k)|} s_2$$

in $H_1(\mathbb{S}(L_2),\mathbb{Q})$. So,

$$\sigma_2|_{\partial p^{-1}(D_k)} = \sum_{i=1}^{l_k} \frac{q_k}{|\mathrm{stab}_{\Gamma_k}(f_i^k)|} s_2.$$

Since

$$\sum_{i=1}^{l_k} |\Gamma_k|/|\mathrm{stab}_{\Gamma_k}(f_i^k)| = |F| = d,$$

because F is the disjoint union of the orbits $\Gamma_k f_i^k$, we have

$$\sigma_2|_{\partial p^{-1}(D_k)} = \frac{dq_k}{|\Gamma_k|} s_2.$$

□

5 Euler number via Chern-Weil theory

A differential k-form on a diffeological space M is a function ω that maps each plot $\phi : U \to M$ to a differential k-form $\phi^*\omega$ on U satisfying the following property: if $\phi : U \to M$ is a plot and $g : V \to U$ is a smooth function between Euclidean open sets, then $g^*(\phi^*\omega) = (\phi g)^*\omega$. The space $\Omega^k(M)$ of all differential k-forms is a real vector space. Moreover, the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ can be defined by $\phi^*(\mathrm{d}\omega) := \mathrm{d}(\phi^*\omega)$. Similarly, the wedge product $\omega_1 \wedge \omega_2$ of two differential forms ω_1 and ω_2 is defined by $\phi^*(\omega_1 \wedge \omega_2) := (\phi^*\omega_1) \wedge (\phi^*\omega_2)$. Following the same reasoning, the pullback $f^*\omega$ of a differential form ω on N under a smooth map $f : M \to N$ is given by $\phi^*(f^*\omega) := (f\phi)^*\omega$.

Now, consider an n-orbifold B and an orbifold chart $\phi : \mathbb{B}^n/\Gamma \to D$. Denote by $\widetilde{\phi} : \mathbb{B}^n \to D$ the map $\widetilde{\phi}(x) := \phi([x])$. The integral of an n-form ω over D is defined by

$$\int_D \omega := \frac{1}{|\Gamma|} \int_{\mathbb{B}^n} \widetilde{\phi}^*\omega.$$

The intuition behind this definition is very simple. For the sake of argument, we assume that the orbifold is two-dimensional and $\Gamma = \langle \exp(2\pi i/m)\rangle$. The action of Γ in $\mathbb{B}^2$ has the sector $S := \{z \in \mathbb{B}^2 \mid z = 0 \text{ or } 0 \le \arg(z) \le 2\pi/m\}$ as a fundamental domain. Therefore, the quotient $\mathbb{B}^2/\Gamma$ is the cone obtained by gluing the sides of S. Taking a region R in the cone, the inverse image $\widetilde{R}$ of R by the projection $\mathbb{B}^2 \to \mathbb{B}^2/\Gamma$ is made of $|\Gamma|$ copies of R. Hence, it is natural to define the area of R as the area of $\widetilde{R}$ divided by $|\Gamma|$.

Furthermore, we want to integrate over all the orbifold and, in order to do that, we imitate the definition of integral over smooth manifolds. If B is a compact orbifold, then consider a finite open cover $D_1, \dots, D_k$ of B such that each D_i comes from an orbifold chart $\phi_i : \mathbb{B}^n/\Gamma_i \to D_i$. If ρ_i is a smooth partition of unit subordinate to the cover D_i, then the integral of an n-form ω over B is defined by

$$\int_B \omega := \sum_{i=1}^{k} \int_{D_i} \rho_i\omega$$

(see [ALR, pag. 34-35] and [Car, pag. 36-37]).

Theorem 28. *If $p : B_2 \to B_1$ is an orbifold convering map of degree d, where B_1, B_2 are compact n-orbifolds, and ω is a differential n-form on B_1, then*

$$\int_{B_2} p^*\omega = d \int_{B_1} \omega.$$

Proof. The fact follows immediately from the definitions of integral and orbifold cover. □

Remark 29. By [Thu, Theorem 13.3.6], every connected, compact, oriented 2-orbifold B with negative Euler characteristic is diffeomorphic to $\mathbb{H}^1_{\mathbb{C}}/G$, where G is a cocompact Fuchsian group and, in particular, $\pi_1^{\mathrm{orb}}(B) = G$. Moreover, since every cocompact group is finitely generated, B is finitely orbifold covered by a compact surface, because G admits a normal torsion-free finite index subgroup by Selberg's lemma [Ratc, pag. 331, Corollary 5].

Consider a connected, compact, oriented good 2-orbifold B with negative Euler number. By Remark 29 above, we can assume $B = \mathbb{H}^1_{\mathbb{C}}/G$ for some Fuchsian group G. Consider as well an action of G on $\mathbb{H}^1_{\mathbb{C}} \times V$ that gives rise, as in Definition 23, to the vector orbigoodle $L := (\mathbb{H}^1_{\mathbb{C}} \times V)/G$ over B, where V is an n-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In the real case we assume the vector orbigoodle to be oriented.

A connection on the orbigoodle $L \to B$ is a connection ∇ on the trivial bundle $\mathbb{H}^1_{\mathbb{C}} \times V \to \mathbb{H}^1_{\mathbb{C}}$ which is invariant under the action of G, i.e., $g^{-1}\nabla_{gu}gs = \nabla_u s$, where s is a section of the trivial bundle, u is a tangent vector of the hyperbolic plane, and the action of G on sections is given by the formula $gs(x) := gs(g^{-1}x)$.

Lemma 30. *Every vector orbigoodle admits a connection.*

Proof. Consider an open cover by orbifold charts $B(x_i, r_i)/\Gamma_i$ of B, where $1 \leq i \leq k$ and Γ_i is the stabilizer of x_i on G. Let f_i be a partition of unity subordinate to this cover. Over the open sets $U_i := \bigsqcup_{x \in Gx_i} B(x, r_i)$, the smooth maps $\rho_i : U_i \to [0, 1]$ given by $\rho_i(x) := f_i[x]$ form a partition of unity of $\mathbb{H}^1_{\mathbb{C}}$ subordinate to U_i.

Fix an arbitrary connection $\widetilde{\nabla}$ on the trivial bundle $\mathbb{H}^1_{\mathbb{C}} \times V \to \mathbb{H}^1_{\mathbb{C}}$ and define the connection ∇^i on $B(x_i, r_i) \times V \to B(x_i, r_i)$ by the formula

$$\nabla^i_v s := \frac{1}{|\Gamma_i|} \sum_{h \in \Gamma_i} h^{-1}\widetilde{\nabla}_{hv} hs.$$

For $B(g^{-1}x_i, r_i) \times V \to B(g^{-1}x_i, r_i)$ define $\nabla^i_v s := g^{-1}\nabla^i_{gv} gs$. The connection ∇^i is G-invariant on U_i. Hence, the connection $\nabla := \sum_{i=1}^k \rho_i \nabla^i$ is G-invariant on $\mathbb{H}^1_{\mathbb{C}}$. $\square$

In what follows, R is the Riemann curvature tensor of a G-invariant connection ∇. In the real case, we have the G-invariant $2n$-form $\mathrm{pf}(R)$, the Pfaffian of the curvature tensor. In the complex case, we have the first Chern number $e(L) := \frac{1}{2\pi i}\int_B \mathrm{tr}(R)$.

Theorem 31. *If $\zeta : L \to B$ is an oriented real vector orbigoodle of rank 2 and ∇ is a connection on L, then*

$$e(L) = \frac{1}{2\pi}\int_B \mathrm{pf}(R).$$

Proof. The surface case is well-knwon (Gauss-Bonnet-Chern theorem). Otherwise, there exists an orbifold covering $p : B' \to B$ of degree d, where B' is a compact, oriented surface, and we consider the pullback p^*L of L by p. From Theorems 25 and 28 we conclude that

$$e(L) = \frac{1}{d}e(p^*L) = \frac{1}{d}\frac{1}{2\pi}\int_{B'} \mathrm{pf}(R) = \frac{1}{2\pi}\int_B \mathrm{pf}(R).$$

$\square$

Theorem 32. *If $\zeta : L \to B$ is an oriented complex line orbigoodle and ∇ is a connection on L, then*

$$e(L) = c_1(L).$$

Proof. Analogous to the proof of Theorem 31. $\square$

6 Applications to complex hyperbolic geometry

In this section, the 2-orbifolds are connected, compact, oriented, and have negative Euler characteristic.

6.1. Complex hyperbolic disc orbigoodles and $\mathrm{PU}(2,1)$-character varieties. Consider a disc orbigoodle $\zeta : M \to B$ over a 2-orbifold B. The orbifold M is complex hyperbolic if it is diffeomorphic to a 4-orbifold $\mathbb{H}^2_{\mathbb{C}}/G$, where G is a discrete subgroup of $\mathrm{PU}(2,1)$. Let $p : \mathbb{H}^1_{\mathbb{C}} \to B$ be an universal covering map of B. The pullback of ζ by p provides the universal cover of M as $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{B}^2$ with deck group $\pi_1^{\mathrm{orb}}(B)$. Therefore, the orbifold fundamental group of M is isomorphic to $\pi_1^{\mathrm{orb}}(B)$ and, consequently, we obtain a discrete faithful representation $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$ such that $\mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B) = M$. Hence, complex hyperbolic disc orbigoodles arise from particular discrete faithful representations of $\pi_1^{\mathrm{orb}}(B)$ in $\mathrm{PU}(2,1)$.

The orbifold $\mathbb{H}^2_{\mathbb{C}}/G$ has a natural geometric structure. The Riemannian metric g of $\mathbb{H}^2_{\mathbb{C}}$ can be induced in the 4-orbifold because g is $\mathrm{PU}(2,1)$-invariant. Furthermore, we say that $\mathbb{H}^2_{\mathbb{C}}/G_1$ and $\mathbb{H}^2_{\mathbb{C}}/G_2$ are isometric if there is an orientation preserving diffeomorphism $f : \mathbb{H}^2_{\mathbb{C}}/G_1 \to \mathbb{H}^2_{\mathbb{C}}/G_2$ such that $f^*g = g$. The quotient maps $P_i : \mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}/G_i$ (which are universal covers) provide two universal covering maps P_2 and $P_2' := f \circ P_1$ for $\mathbb{H}_2/G_2$. Hence, there exists a diffeomorphism $F : \mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{H}^2_{\mathbb{C}} & \xrightarrow{\quad F \quad} & \mathbb{H}^2_{\mathbb{C}} \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ \mathbb{H}^2_{\mathbb{C}}/G_1 & \xrightarrow{\quad f \quad} & \mathbb{H}^2_{\mathbb{C}}/G_2 \end{array}$$

commutes and the map F is clearly an isometry. If we have a diffeomorphism $\mathbb{H}^2_{\mathbb{C}}/G_1 \simeq M$ then, by composing with the isometry f, we obtain $\mathbb{H}^2_{\mathbb{C}}/G_2 \simeq M$. The corresponding discrete faithful representations $\rho_i : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$, $i = 1,2$, have images G_i and satisfy $\rho_2(\cdot) = F\rho_1(\cdot)F^{-1}$. Therefore, isometric complex hyperbolic structures on M correspond to the same representation up to conjugation in $\mathrm{PU}(2,1)$.

Definition 33. The $\mathrm{PU}(2,1)$**-character variety** of the connected, compact, oriented 2-orbifold B with negative Euler characteristic is the space

$$\hom\big(\pi_1^{\mathrm{orb}}(B), \mathrm{PU}(2,1)\big)/\mathrm{PU}(2,1),$$

where $\hom\big(\pi_1^{\mathrm{orb}}(B), \mathrm{PU}(2,1)\big)$ is the space of all group homomorphisms $\pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$ on which $\mathrm{PU}(2,1)$ acts by conjugation.

It is clear from the above discussion that complex hyperbolic disc orbigoodles over B, considered up to isometry, can be seen as points in the $\mathrm{PU}(2,1)$-character variety of B. Nevertheless, not all representations correspond to complex hyperbolic orbigoodles (say, there are faithful non-discrete representations).

6.2. Discreteness of the Toledo invariant, holomorphic section identity and Toledo rigidity.

Lemma 34. *Consider a 2-orbifold $\mathbb{H}^1_{\mathbb{C}}/G$ and a representation $\rho : G \to \mathrm{PU}(2,1)$, where G is a Fuchsian group. There exists a smooth G-equivariant map $\mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$. Furthermore, if we have two such G-equivariant maps f_0, f_1, then there exists a smooth homotopy f_t, $t \in \mathbb{R}$, such that f_t is G-equivariant for every t.*

Proof. Let $[x_1], \dots, [x_n]$ be the singular points of $\mathbb{H}^1_{\mathbb{C}}/G$, consider the orbigoodle

$$\zeta : (\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^2_{\mathbb{C}})/G \to \mathbb{H}^1_{\mathbb{C}}/G$$

and observe that each point x_i of $\mathbb{H}^1_{\mathbb{C}}$ satisfies $\mathrm{stab}_G(x_i) \neq 1$. Take a small geodesic disc $B(x_i, r_i)$ centered at x_i and let D_i be its image under the quotient map $\mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^1_{\mathbb{C}}/G$. The trivialization of

ζ is given by the commutative diagram

$$\begin{tikzcd}
\big(B(x_i,r_i)\times \mathbb{H}^2_{\mathbb{C}}\big)/\mathrm{stab}_H(x_i) \arrow[r, "\Phi_i"] \arrow[d] & \zeta^{-1}(D_i) \arrow[d, "\zeta"] \\
B(x_i,r_i)/\mathrm{stab}_H(x_i) \arrow[r, "\phi_i"] & D_i
\end{tikzcd}$$

Since G is Fuchsian, the subgroup $\mathrm{stab}_G(x_i)$ is finite and, therefore, it is generated by a rotation R_i. Then $I_i := \rho(R_i)$ has finite order and, consequently, is an elliptic isometry or the identity. Let p_i be a fixed point of I_i and define the map

$$F_{x_i} : B(x_i,r_i)/\mathrm{stab}_G(x_i) \to (B(x_i,r_i)\times \mathbb{H}^2_{\mathbb{C}})/\mathrm{stab}_G(x_i)$$

by the formula $F_{x_i}([x]) = ([x,p_i])$, producing a section on the neighborhood of each $[x_i]\in \mathbb{H}^1_{\mathbb{C}}/G$ as a consequence of the trivialization described in the above diagram. Now, we extend these sections constructed around each singular point to the entire orbifold (a standard argument for fiber bundles over manifolds). Hence, we have a global section $F : \mathbb{H}^1_{\mathbb{C}}/G \to (\mathbb{H}^1_{\mathbb{C}}\times\mathbb{H}^2_{\mathbb{C}})/G$. Considering $\widetilde{F} : \mathbb{H}^1_{\mathbb{C}} \to (\mathbb{H}^1_{\mathbb{C}}\times\mathbb{H}^2_{\mathbb{C}})/G$ given by $\widetilde{F}(x) = F([x])$ and writing $\widetilde{F}(x) = [x, f(x)]$ we obtain a G-equivariant smooth function $f : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$.

Now, we consider two G-equivariant maps f_0, f_1 which define sections

$$F_k : \mathbb{H}^1_{\mathbb{C}}/G \to (\mathbb{H}^1_{\mathbb{C}}\times\mathbb{H}^2_{\mathbb{C}})/G$$

by the formula $F_k([x]) = ([x, f_k(x)])$. From the trivialization described above, F_k is given, around the singular point $[x_i]$, by the map $\widetilde{F}_{k,i} = \Phi_i^{-1}\circ F_k\circ\phi_i$. Note that $f_0(x_i)$ and $f_1(x_i)$ are fixed points of I_i. Furthermore, deforming the sections we can assume that $\tilde{F}_{k,i}([x]) = [x, f_k(x_i)]$.

The set of fixed points of I_i is always connected (see Subsection 2.2): it can be a point or a complex geodesic when I_i is elliptic, or the whole space when $I_i = 1$. Hence we can, in a small neighborhood of $[x_i]$, deform F_0 to F_1 smoothly and therefore assume $F_1 = F_0$ near the singular points. The rest of the deformation is made in a smooth manifold, which is possible since $\mathbb{H}^2_{\mathbb{C}}$ is a ball. So, we have a homotopy F_t between F_0 and F_1 and, thus, a homotopy f_t between f_0 and f_1 such that f_t is G-equivariant for all $t\in\mathbb{R}$. □

Note that $\mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$ is a diffeological space (but not necessarily an orbifold!) and the Kähler form ω is well defined on it because it is invariant under the action of $\pi_1^{\mathrm{orb}}(B)$.

By Lemma 34, there exist smooth maps $f : B \to \mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$ satisfying the following property: for each universal orbifold covering map $\mathbb{H}^1_{\mathbb{C}} \to B$ with deck group G, the map f can be lifted to a smooth G-equivariant map $\tilde{f} : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$ such that the diagram

$$\begin{tikzcd}
\mathbb{H}^1_{\mathbb{C}} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{H}^2_{\mathbb{C}} \arrow[d] \\
\mathbb{H}^1_{\mathbb{C}}/G \arrow[r, "f"'] & \mathbb{H}^2_{\mathbb{C}}/G
\end{tikzcd}$$

commutes. We call these functions **good smooth maps** from B to $\mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$.

Question: Are all smooth maps $B \to \mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$ good?

Definition 35. Let B be a connected, compact, oriented 2-orbifold with negative Euler characteristic. The **Toledo invariant** of a representation $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$ is given by the integral

$$\tau(\rho) := \frac{4}{2\pi}\int_B f^*\omega,$$

where $f : B \to \mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$ is a good smooth map and ω is the Kähler form on $\mathbb{H}^2_{\mathbb{C}}/\pi_1^{\mathrm{orb}}(B)$.

Regarding the definition of the Toledo invariant, see also [Krebs].

Lemma 36. *The definition of the Toledo invariant does not depend on the choice of f.*

Proof. We take $B = \mathbb{H}^1_{\mathbb{C}}/G$, where G is a Fuchsian group.

Consider two G-equivariant maps $f_0, f_1 : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$. By Lemma 34 there exists a homotopy f_t between these maps such that f_t is G-equivariant for all t.

For each singular point $x_i \in B$ consider the chart $\phi_i : 2\mathbb{B}^2/H_i \to 2D_i$ centered at x_i, where $2D_i$ is sufficiently small. Removing the discs $D_i := \phi_i\big(\mathbb{B}^2/H_i\big)$ we obtain the open surface B'. Applying Stokes theorem

$$0 = \int_{B'\times[0,1]} \mathrm{d}(f_t^*\omega) = \int_{B'} f_1^*\omega - \int_{B'} f_0^*\omega + \int_{\partial B'\times[0,1]} f_t^*\omega \tag{37}$$

because $\mathrm{d}(f_t^*\omega) = f_t^*\mathrm{d}\omega = 0$ (the Kähler form ω is closed).

For each chart ϕ_i we have

$$\int_{D_i} f_t^*\omega = \frac{1}{|H_i|}\int_{\mathbb{B}^2} (f_t \circ \widetilde{\phi}_i)^*\omega,$$

where $\widetilde{\phi}_i(x) := \phi_i([x])$. By Stokes theorem,

$$0 = \frac{1}{|H_i|}\int_{\mathbb{B}^2\times[0,1]} \mathrm{d}(f_t \circ \widetilde{\phi}_i)^*\omega = \int_{D_i} f_1^*\omega - \int_{D_i} f_0^*\omega + \int_{\partial D_i\times[0,1]} f_t^*\omega. \tag{38}$$

By equations (37) and (38) we conclude

$$\int_B f_0^*\omega = \int_B f_1^*\omega.$$

$\square$

Definition 39. The relative Toledo invariant of a representation $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$, denoted by $\tau_R(\rho)$, is the number $\tau(\rho)/\chi(B)$. This number is unchanged under a finite cover $\widetilde{B}$ of B by Theorems 28 and 25, i.e.,

$$\frac{\tau(\rho)}{\chi(B)} = \frac{\tau\left(\rho|_{\pi_1^{\mathrm{orb}}(\widetilde{B})}\right)}{\chi(\widetilde{B})}.$$

The following Theorem 40 and Corollary 41 are generalizations for orbifolds of results proved in [GKL]. They establish the integrality property of the Toledo invariant.

Let $B = \mathbb{H}^1_{\mathbb{C}}/G$, where G is a Fuchsian group. Consider a representation $\rho : G \to \mathrm{PU}(2,1)$ and a smooth G-equivariant map $f : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$. Let $f^*\mathrm{T}\mathbb{H}^2_{\mathbb{C}} \to \mathbb{B}^2$ be the pullback of the tangent bundle $\mathrm{T}\mathbb{H}^2_{\mathbb{C}}$ by f. We have the vector orbigoodle $\zeta : E \to B$, where $E := (f^*\mathrm{T}\mathbb{H}^2_{\mathbb{C}})/G$.

Theorem 40. *The following formula holds*

$$c_1(E) = \frac{3}{2}\tau(\rho).$$

Proof. A little modification of the argument in the proof of Lemma 34 allows us to assume that f is an immersion out of the singular points. We can pullback the metric, connection, and Kähler form of $\mathrm{T}\mathbb{H}^2_{\mathbb{C}}$ to $f^*\mathrm{T}\mathbb{H}^2_{\mathbb{C}}$, and since these objects are invariant under action of $\mathrm{PU}(2,1)$, we conclude that they are well defined in E. The first Chern number of E is given by

$$c_1(E) := \frac{1}{2\pi i}\int_B \eta$$

where $\eta = \mathrm{tr}(R)$ and R is curvature tensor of $(f^*\mathrm{T}\mathbb{H}^2_{\mathbb{C}}, \nabla)$.

In order to calculate the curvature tensor at a regular point we can assume that we are dealing with an embedded surface $S \subset \mathbb{H}^2_{\mathbb{C}}$ and are calculating the curvature tensor of $\mathrm{T}\mathbb{H}^2_{\mathbb{C}}|_S$, because f is a local embedding around regular points. So, take $\boldsymbol{p} \in S$. Let $t_1, t_2 \in \mathrm{T}_{\boldsymbol{p}}S$ be unit tangent vectors to S at $\boldsymbol{p}$ orthogonal with respect to the Riemannian metric $g_{\boldsymbol{p}}$. Consider a unit tangent vector $n \in \mathrm{T}_{\boldsymbol{p}}\mathbb{H}^2_{\mathbb{C}}$ orthogonal to t_1 with respect to the Hermitian form $h_{\boldsymbol{p}}$. Since $t := t_1$ and n span $\mathrm{T}_{\boldsymbol{p}}\mathbb{H}^2_{\mathbb{C}}$

as a complex vector space, we have $t_2 = at + bn$, where $a, b \in \mathbb{C}$. Furthermore, $\operatorname{Re} a = 0$ because $g_{\boldsymbol{p}}(t_1, t_2) = 0$. The curvature tensor R on $\mathrm{T}_{\boldsymbol{p}}\mathbb{H}^2_{\mathbb{C}}$ is given by the formula (7). Since

$$tt^*t_2 = at, \quad t_2t^*t = at + bn, \quad tt_2^*t = \overline{a}t, \quad nt^*t_2 = an, \quad t_2t^*n = 0, \quad tt_2^*n = \overline{b}t, \quad nt_2^*t = \overline{a}n,$$

we obtain

$$R(t_1, t_2)t = -4at - bn,$$
$$R(t_1, t_2)n = +\overline{b}t - 2an.$$

Hence, $\operatorname{tr}(R)(t_1, t_2) = -6a$. On the other hand, $\omega_{\boldsymbol{p}}(t_1, t_2) = \operatorname{Im} h_{\boldsymbol{p}}(t_1, t_2) = ia$ and we conclude that $\operatorname{tr}(R) = 6i\omega$ on the surface S. Therefore, $c_1(E) = \frac{6}{2\pi}\int_B f^*\omega$ and the result follows. ☐

Corollary 41. *If the orbifold B has singularities $x_1, \dots, x_n$ of order $m_1, \dots, m_n$, then*

$$\tau(\rho) \in \frac{2}{3}\Big(\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \dots + \frac{1}{m_n}\mathbb{Z}\Big).$$

In particular, the Toledo invariant is a discrete invariant.

Proof. By Theorem 40 we have $\tau(\rho) = \frac{2}{3}c_1(E)$. Observe that E is a complex vector orbigoodle of rank 2 and, thus, $c_1(\wedge^2 E) = c_1(E)$. Since $\wedge^2 E$ is a complex line orbibundle, the number $c_1(E)$ belongs to $\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \dots + \frac{1}{m_n}\mathbb{Z}$ by Theorem 32. ☐

Remark 42. In the above proof, we used $\wedge^2 E$. One way of building this line bundle is by pulling back E to the universal cover of B, taking the second wedge power and then quotient it back to B.

It is worth noting that the above result holds for $\mathbb{H}^n_{\mathbb{C}}$ as well, where the Toledo invariant belongs to

$$\frac{2}{n+1}\Big(\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \dots + \frac{1}{m_n}\mathbb{Z}\Big),$$

and the proof is analogous.

Corollary 43. *If a complex hyperbolic disc orbigoodle $M \to B$ has a holomorphic section, i.e., a section $B \to M$ originated from a $\pi_1^{\mathrm{orb}}(B)$-equivariant holomorphic embedding $\mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}$, then*

$$\frac{3}{2}\tau_R(M) = e_R(M) + 1.$$

Proof. We can suppose that B is a surface and it is embedded in M as a Riemann surface because the identity is between relative invariants. Take a point $x_0 \in B$, remove a small disc D_0 centered at x_0, and consider $B' = B \setminus D_0$. Let t and s be sections of $\mathbb{S}(\mathrm{T}B')$ and $\mathbb{S}(L|_{B'})$, where L is the kernel of the vector bundle morphism $TM|_B \to TB$. Note that $L \to B$ is the vector bundle whose fibers are the tangent planes of the fibers of the disc bundle $M \to B$ at the points where they intersect B. Since M is a Riemannian manifold, we suppose these circle bundles are unit bundles. As B is a Riemann surface, $t \wedge s$ is non-zero in $\wedge^2\mathrm{T}M|_{B'}$, because t and s are never $\mathbb{C}$-linear dependent. Take local sections t_0 and s_0 of $\mathrm{T}B$ and L in a neighborhood of D_0 (that we shrink if necessary) such that $t_0 \wedge s_0$ is non-zero at every point, i.e., the vectors t_0 and s_0 are never in the same complex line. Writing $t = ft_0$ and $s = gs_0$ on ∂D_0, where f, g are maps from ∂D_0 to $\mathbb{S}^1$, we obtain $t \wedge s = fgt_0 \wedge s_0$ and, therefore, $e(\wedge^2\mathrm{T}M|_B) = e(\mathrm{T}B) + e(L)$ because $\deg(fg) = \deg(f) + \deg(g)$. By Theorem 40, we have $\frac{3}{2}\tau_R(\rho) = e_R(M) + 1$.

☐

The following is the generalization for 2-orbifolds of the classical Toledo rigidity theorem [Tol].

Theorem 44. (Toledo Rigidity) *The inequality*

$$|\tau_R(\rho)| \leq 1$$

always holds for representations $\rho : \pi_1^{\mathrm{orb}}(B) \to \mathrm{PU}(2,1)$. Furthermore, the relative Toledo number is 1 in absolute value if, and only if, there is a stable complex geodesic.

Proof. Since the inequality holds for surfaces and the relative Toledo number is unchanged under finite orbifold covers, the inequality is true for orbifolds as well.

The only thing we have to prove is that if $|\tau_R(\rho)| = 1$, then there exists a stable complex geodesic. We can assume $B = \mathbb{H}^1_{\mathbb{C}}/G$, where G is a Fuchsian group.

By Remark 29 there is a normal torsion-free finite index subgroup H of G and by Toledo's rigidity theorem for surfaces there exists a Riemann-Poincaré sphere L stable under the action of H. We write $G/H = \{Hg_1, \dots, Hg_n\}$, where H acts on G on the left. The projective lines L and g_iL intersect at a point p_i. The line L is broken into two Poincaré discs and, therefore, p_i is in one of the Poincaré discs or in their common boundary. The action of H on this Poincaré disc is faithful and discrete by Goldman's theorem (see [Gol2, Theorem A]).

For each $h \in H$ we have $g_ihg_i^{-1}p_i \in g_iL$, because $p_i \in g_iL$. On the other hand, $g_ihg_i^{-1}p_i \in L$ because H is normal. So, assuming $L \neq g_iL$, we obtain that the group H is in the stabilizer of p_i which is impossible because H is the fundamental group of a surface of genus ≥ 2. So, $L = g_iL$ and the complex geodesic L is stable under G. □

E. Xia has shown in [Xia, Theorem 1.1] that, given a compact, connected, and oriented surface B with negative Euler characteristic, the number of connected components of the $\mathrm{PU}(2,1)$-character variety of B is the number of τ's satisfying

$$\tau \in \frac{2}{3}\mathbb{Z} \quad \text{and} \quad \left|\frac{\tau}{\chi(B)}\right| \leq 1.$$

Analogously, it is interesting to ask if the same holds in the case of orbifolds. More precisely, given a compact, connected, and oriented 2-orbifold B with negative Euler characteristic, we conjecture that the number of connected components of the $\mathrm{PU}(2,1)$-character variety of B equals the number of τ's satisfying

$$\tau \in \frac{2}{3}\left(\mathbb{Z} + \frac{1}{m_1}\mathbb{Z} + \cdots + \frac{1}{m_n}\mathbb{Z}\right) \quad \text{and} \quad \left|\frac{\tau}{\chi(B)}\right| \leq 1.$$

This conjecture is supported by the following sketchy argument. The map associating each component of the $\mathrm{PU}(2,1)$-character variety to $\tau(\rho)$, where ρ is a representation in that component, is well-defined because whenever we deform ρ along a curve, we are at some level continuously moving the fixed points of some elliptic isometries. For each instant of this deformation, we can construct an equivariant map (with respect to the current representation) accordingly to Lemma 34 and we may assume that, as we deform the representation, we are simultaneously deforming the corresponding equivariant map. Hence, the Toledo invariant varies continuously along such curve; nevertheless, the Toledo invariant is discrete by Corollary 41 and so it must be constant along the curve.

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CHAPTER

8

ARTICLE: GEOMETRY OVER ALGEBRAS

Geometry over algebras

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Abstract

We study geometric structures arising from Hermitian forms on linear spaces over real algebras beyond the division ones. Our focus is on the dual numbers, the split-complex numbers, and the split-quaternions. The corresponding geometric structures are employed to describe the spaces of oriented geodesics in the hyperbolic plane, the Euclidean plane, and the round 2-sphere. We also introduce a simple and natural geometric transition between these spaces. Finally, we present a projective model for the hyperbolic bidisc, that is, the Riemannian product of two hyperbolic discs.

1 Introduction

Following [AGr], classic geometries emerge from a linear space endowed with a Hermitian form. Typical examples of such geometries are the real/complex/quaternionic projective spaces with Fubini-Study metric, the real/complex/quaternionic hyperbolic spaces, the de Sitter spaces, and anti-de Sitter spaces, among others. Here, we extend the framework of classic geometries to the case of linear structures over real algebras other than the real numbers, the complex numbers, and the quaternions. This is necessary if one wants to, for example, describe natural geometric structures on the spaces of geodesics in usual classic geometries (for the spaces of geodesics in spherical, Euclidean, and hyperbolic geometries, see Section 5). As in [AGr], we take the coordinate-free route and describe (pseudo)-Riemannian concepts and formulas in a simple algebraic form which is well suited, say, for scientific computation.

The algebras we consider here, besides the associative real division algebras, are the simplest associative unital finite-dimensional involutive real algebras: split-complex numbers $\mathbb{R}[x]/(x^2-1)$ (also known as hyperbolic numbers), dual numbers $\mathbb{R}[x]/(x^2)$, and split-quaternions. The reason why we cling to these algebras is that the linear algebra over them is not too ill-behaved (see Section 2). Moreover, they are enough to describe the above mentioned spaces of geodesics.

Geometries over split-complex and dual numbers were previously studied by S. Trettel [Tre] from a homogeneous spaces approach. In contrast, we work with projective spaces (whose definition is, essentially, the usual one) where geometric structures are induced from a Hermitian form. So, our work is to [Tre] as [AGr] is to the usual symmetric/homogeneous approach to projective geometries. For instance, S. Trettel develops a theory of transition of geometries showing how the one-parameter family of algebras $\mathbb{R}[x]/(x^2+\delta)$ provides a transition between geometries over complex ($\delta>0$), dual numbers ($\delta=0$), and split-complex ($\delta<0$) hyperbolic geometries. We describe this transition inside the split-quaternionic projective spaces. It is important to mention that geometries constructed from this one-parameter family of algebras appear in the work of J. Danciger [Dan1], [Dan2], [Dan3]. Among several other results, J. Danciger presents a transition between the three dimensional hyperbolic and anti-de Sitter spaces passing through a pipe geometry.

Curiously, the hyperbolic bidisc (product of two Poincaré discs) appears as a projective classic geometry in the present context. The bidisc is an important space when it comes to uniformization questions in dimension 4 [CGr]. For instance, the known examples of disc bundles uniformized by the bidisc support a bidisc variant of the Gromov-Lawson-Thurston conjecture [GLT]. The original GLT conjecture says that an oriented disc bundle over a closed oriented surface of genus ≥ 2 admits a complete metric of constant negative curvature (a real hyperbolic structure) if, and only if, the Euler number e of the bundle satisfies $|e|\leq|\chi|$, where χ is the Euler characteristic of the surface. However, all known examples of disc bundles uniformized by the complex hyperbolic space (see [AGG], [BGr], [GKL]) and by the bidisc (see [CGr]) support that the GLT conjecture might hold also for these geometries. Describing the bidisc as a projective classic geometry may be a step

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towards understanding the relationships between such versions of the GLT conjecture; for example, the existence of a transition between the real, the complex and the bidisc hyperbolic geometries inside of a bigger classic geometry might connect the different versions of the GLT conjecture.

2 Linear algebra over real algebras

The goal of this section is establishing basic linear algebra tools to properly develop projective geometry over some non-division algebras.

2.1. Finite dimensional real algebras. Consider a real finite-dimensional unital associative algebra $\mathbb{F}$. There is a natural $\mathbb{R}$-algebra embedding $T : \mathbb{F} \to \mathrm{Lin}_{\mathbb{R}}(\mathbb{F}, \mathbb{F})$ given by $a \mapsto T_a$, where $T_a(x) := ax$. So, left and right zero divisors coincide and a left inverse is also a right inverse. Denote by $\mathbb{F}_z$ the set of zero-divisors and by $\mathbb{F}^\times$ the set of units.

Proposition 1. $\mathbb{F} = \mathbb{F}_z \sqcup \mathbb{F}^\times$

Proof. Clearly, $\mathbb{F}_z \cap \mathbb{F}^\times = \emptyset$. Take $a \in \mathbb{F} \setminus \mathbb{F}^\times$. The map $T_a : \mathbb{F} \to \mathbb{F}$, $x \mapsto ax$, is $\mathbb{R}$-linear. It cannot be surjective because that would imply $a \in \mathbb{F}^\times$. Hence, its kernel is non-trivial. So, $a \in \mathbb{F}_z$. □

Proposition 2. *The subset $\mathbb{F}_z$ of $\mathbb{F}$ is a non-trivial real algebraic set.*

Proof. Consider the map $p : \mathbb{F} \to \mathbb{R}$, $a \mapsto \det(T_a)$. Then $a \in \mathbb{F}_z$ if, and only if, $p(a) = 0$. Since p is a (non-zero, several variables) real polynomial, $\mathbb{F}_z = \{x \in \mathbb{F} : p(x) = 0\}$ is a non-trivial real algebraic set. □

A real algebraic set can be written as a union of finite smooth manifolds [Sh] and we define its dimension as the largest dimension among such manifolds.

Corollary 3. *The subset $\mathbb{F}_z$ of $\mathbb{F}$ is a finite union of manifolds of dimension smaller than* $\dim_{\mathbb{R}} \mathbb{F}$. *In particular, $\mathbb{F}^\times$ is open and dense in $\mathbb{F}$.*

From now on we assume that $\mathbb{F}$ has an involutive structure: there is an algebra antiautomorphism $x \mapsto x^*$ of $\mathbb{F}$ such that $x^{**} = x$. By antiautomorphism we mean that this map is an $\mathbb{R}$-linear isomorphism, $1^* = 1$, and $(xy)^* = y^* x^*$. An element x is called self-adjoint when $x^* = x$. With a single exception (see Section 6), we will restrict ourselves to involutive real algebras where the self-adjoint elements of $\mathbb{F}$ are exactly the real numbers.

In the case when the quadratic form $N(x) := xx^*$ is non-degenerate, these algebras are called (associative) composition algebras. When N is definite, then $\mathbb{F}$ is one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, where $\mathbb{H}$ stands for the quaternions. Otherwise, the algebra is either the split-complex numbers $\mathbb{C}_s$ or the split-quaternions $\mathbb{H}_s$:

- **Split-complex numbers** $\mathbb{C}_s := \mathbb{R} + j\mathbb{R}$, with $j^2 = 1$ and involution $(x + jy)^* = x - jy$;
- **Split-quaternions** $\mathbb{H}_s := \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$, with

$$i^2 = -1, \quad j^2 = 1, \quad k^2 = 1,$$

$$ij = k, \quad ij = -ji, \quad ik = -ki, \quad jk = -kj,$$

and involution $(x + iy + zj + wk)^* = x - iy - zj - wk$.

There are also cases where N is degenerate. For example, we have the

- **Dual numbers** $\mathbb{D} := \mathbb{R} + \varepsilon\mathbb{R}$, with $\varepsilon^2 = 0$ and involution $(x + \varepsilon y)^* = x - \varepsilon y$.

Note that the split-quaternions contain copies of the complex, split-complex, and dual numbers (the last one happens, for example, taking $\varepsilon := i + j$).

The split-complex numbers $\mathbb{C}_s$ can be naturally identified with the algebra $\mathbb{R} \times \mathbb{R}$ endowed with the involution $(a, b)^\times = (b, a)$. The isomorphism is given by the map

$$\begin{aligned} \mathbb{R} + j\mathbb{R} &\to \mathbb{R} \times \mathbb{R} \\ x + jy &\mapsto (x + y, x - y) \end{aligned}$$

From this identification, we obtain that the set units $\mathbb{C}_s^\times$ of the split-complex numbers is $\mathbb{R}^\times \times \mathbb{R}^\times$. The units of the dual numbers $\mathbb{D}$ are of the form $a + \varepsilon b$, where $a \in \mathbb{R}^\times$.

2.2. Hermitian form. All modules in this paper are finite-dimensional free left-modules. Observe that, when V is also free, the concept of dimension of V as an $\mathbb{F}$-module is well defined. Indeed, let V be a free $\mathbb{F}$-module with basis $e_1, \dots, e_n$. Since each $\mathbb{F}e_i$ is a real vector space with dimension $\dim_{\mathbb{R}} \mathbb{F}$, the number $n = \dim_{\mathbb{R}} V / \dim_{\mathbb{R}} \mathbb{F}$ does not depend on the choice of basis.

Definition 4. A Hermitian form on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ satisfying the following properties:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \quad u, v, w \in V;$
- $\langle zu, w \rangle = z \langle u, v \rangle, \quad u, v \in V, z \in \mathbb{F};$
- $\langle u, v \rangle^* = \langle v, u \rangle, \quad u, v \in V.$

In particular $\langle u, u \rangle \in \mathbb{R}$ for all $u \in V$.

From now on, V is a finite-dimensional left $\mathbb{F}$-module equipped with a Hermitian form. An orthonormal basis consists of $b_1, \dots, b_n \in V$ such that $\langle b_i, b_i \rangle = \pm 1$, $\langle b_i, b_j \rangle = 0$ for $i \neq j$, and $V = \mathbb{F}b_1 \oplus \cdots \oplus \mathbb{F}b_n$.

Definition 5. A Hermitian form is non-degenerate if zero is the only vector perpendicular to all vectors.

Lemma 6. *Consider a finite-dimensional free $\mathbb{F}$-module V equiped with a non-degenerate Hermitian form. If W is a proper subspace of V that admits a orthonormal basis, then there exists $u \in V$ such that $\langle u, u \rangle \neq 0$ and $\langle W, u \rangle = 0$. (In particular, there always exists $u \in V$ such $\langle u, u \rangle \neq 0$.)*

Proof. We assume $\mathbb{F} \neq \mathbb{R}$ (otherwise, the fact is trivial).

Fix an orthonormal basis $b_1, \dots, b_m$ for W.

Let $W^\perp := \{u \in V \mid \langle u, W \rangle = 0\}$. Note that $W \cap W^\perp = 0$ and that for each $u \in V$, the vector

$$u' := u - \sum_i \frac{\langle u, b_i \rangle}{\langle b_i, b_i \rangle} b_i$$

belongs to $W^\perp$. Therefore, $V = W \oplus W^\perp$.

Suppose that for all $v \in W^\perp$ we have $\langle v, v \rangle = 0$. Let us show that such assumption leads to a contradiction, thus proving the result.

Fix $u \in W^\perp$. Note that $\langle u + h, u + h \rangle = 0$ for all $h \in W^\perp$. So, $\langle u, h \rangle + \langle h, u \rangle = 0$ for all $h \in W^\perp$. Clearly, $\langle u, h \rangle + \langle h, u \rangle = 0$ for all $h \in V$.

If $\mathbb{F}$ is the split-complex or the complex numbers, there is $j \in \mathbb{F}^\times$ such that $j^* = -j$. So, $\langle u, jh \rangle + \langle jh, u \rangle = 0$ which implies $\langle u, h \rangle - \langle h, u \rangle = 0$ for all $h \in V$, that is, $\langle u, h \rangle = 0$ for all $h \in V$. Therefore, $u = 0$ (the Hermitian form is non-degenerate), contradicting $W^\perp \neq 0$.

For the quaternions and split-quaternions, we have the numbers $i, j, k \in \mathbb{F}^\times$ that anti-commute among themselves and satisfy

$$i^* = -i, \quad j^* = -j, \quad k^* = -k.$$

For each of this numbers, we obtain

$$i\langle u, h \rangle = \langle u, h \rangle i,$$
$$j\langle u, h \rangle = \langle u, h \rangle j,$$
$$k\langle u, h \rangle = \langle u, h \rangle k.$$

Since only real numbers commute with i, j, k, we conclude that $\langle u, h \rangle = 0$ for all h. Hence $u = 0$, contradicting $W^\perp \neq 0$.

For the dual numbers, we have $\mathbb{F} = \mathbb{R} \oplus \mathbb{R}\epsilon$, $\epsilon^2 = 0$. Proceeding as above, we obtain the identity $\epsilon\langle u, h \rangle - \epsilon\langle h, u \rangle = 0$ for all $h \in V$ which implies $\langle \epsilon u, h \rangle = 0$ and, therefore, $\epsilon u = 0$. Since V is free, this implies $u \in \epsilon V$. Thus, $W^\perp$ is a subspace of ϵV implying $\epsilon W \oplus W^\perp \subset \epsilon V$. In particular,

$$\dim_{\mathbb{R}} \epsilon W + \dim_{\mathbb{R}} W^\perp \leq \dim_{\mathbb{R}} \epsilon V.$$

Since W is free, $2\dim_{\mathbb{R}} \epsilon W = \dim_{\mathbb{R}} W$. Thus, it follows from $V = W \oplus W^{\perp}$ that

$$\frac{\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W^{\perp}}{2} + \dim_{\mathbb{R}} W^{\perp} \leq \dim_{\mathbb{R}} \epsilon V$$

and, therefore,

$$\dim_{\mathbb{R}} W^{\perp} \leq 2\dim_{\mathbb{R}} \epsilon V - \dim_{\mathbb{R}} V.$$

We reached a contradiction: $2\dim_{\mathbb{R}} \epsilon V - \dim_{\mathbb{R}} V = 0$ because V is free, but $\dim_{\mathbb{R}} W^{\perp} > 0$.

Finally, we conclude that there exists $u \in W^{\perp}$ such that $\langle u, u\rangle \neq 0$. □

We have the following corollaries:

Corollary 7. *If the Hermitian form is non-degenerate then the finite-dimensional free $\mathbb{F}$-module V has an orthonormal basis.*

Corollary 8. *Let V be a finite dimensional free $\mathbb{F}$-module endowed with a non-degenerate Hermitian form. If W is a free submodule of V where the Hermitian form is non-degenerate, then any orthonormal basis of W can be completed to an orthogonal basis of V.*

Corollary 9. *If V has an orthonormal basis then the Hermitian form is non-degenerate.*

2.3. Good points.

Definition 10. We say that $u \in V$ is a **good point** if there exists a basis for V such that $\sum_i u_i\mathbb{F} = \mathbb{F}$, where $u_1, \ldots, u_n$ are the coordinates of u on such basis. We denote the set of all good points by $V^{\bullet}$.

(Clearly, for the concept of good point to be well defined, we need V to be free.) Alternatively, a point u is good if for every non-degenerate Hermitian form $\langle\cdot,\cdot\rangle$ on V there exists $v \in V$ such that $\langle u, v\rangle = 1$ (see Proposition 13).

Proposition 11. *If $u \in V$ is a good point, then for every basis of V we have $\sum_i u_i\mathbb{F} = \mathbb{F}$, where $u_1, \ldots, u_n$ are the coordinates of u on such basis.*

Proof. Take a basis e_i such that

$$u = \sum_i u_i e_i \quad \text{and} \quad \sum_i u_i\mathbb{F} = \mathbb{F}.$$

The second condition means that there are $v_1, \ldots, v_n \in \mathbb{F}$ such that $\sum_i u_i v_i = 1$.

Consider another basis f_j. We have $f_j = \sum_i \alpha_{ij} e_i$ and $e_j = \sum_i \beta_{ij} f_i$. Therefore, for each i, j, we have

$$\sum_k \alpha_{ki}\beta_{jk} = \sum_k \beta_{ki}\alpha_{jk} = \delta_{ij}.$$

The coordinates of u on the basis f_k are given by $\widetilde{u_k} = \sum_i u_i\beta_{ki}$. For $\widetilde{v_k} := \sum_j \alpha_{jk} v_j$ we obtain

$$\sum_k \widetilde{u_k}\widetilde{v_k} = \sum_{i,j,k} u_i\beta_{ki}\alpha_{jk} v_j = \sum_{i,j} \delta_{ij} u_i v_j = 1,$$

thus proving that

$$\sum_i \tilde{u}_i\mathbb{F} = \mathbb{F}.$$

□

Proposition 12. *The set of good points $V^{\bullet}$ is an open dense subset of V.*

Proof. We may suppose $V{:=}\mathbb{F}^n$ because V is free. For $u \in V^{\bullet}$ there is $v \in \mathbb{F}^n$ such that $\sum_i u_i v_i = 1$. Note that

$$U = \{x \in V : \sum_i x_i v_i \in \mathbb{F}^{\times}\} \subset V^{\bullet}$$

is an open neighborhood of u, since $\mathbb{F}^{\times}$ is open in $\mathbb{F}$. Thus, $V^{\bullet}$ is open. To see that it is also dense, just note that $V^{\bullet}$ contains the dense subset $(\mathbb{F}^{\times})^n$, where we are using that $\mathbb{F}^{\times}$ is dense in $\mathbb{F}$. □

Proposition 13. *Assume that V is equipped with a non-degenerate Hermitian form. A point $u \in V$ is good if, and only if, the map $h \mapsto \langle u, h\rangle$ from V to $\mathbb{F}$ is surjective.*

Proof. Consider an orthogonal basis $b_1, \ldots, b_n$ and define $c_i = \langle b_i, b_i\rangle$. Take $u \in V^\bullet$. Writing $u = \sum_i u_i b_i$, let $\alpha_i \in \mathbb{F}$ be such that

$$\sum_i u_i \alpha_i = 1.$$

The vector

$$h = \sum_i c_i^{-1} \alpha_i^* b_i$$

satisfies

$$\langle u, h\rangle = 1.$$

Thus, the map $\langle u, -\rangle$ is surjective. The converse is analogous. □

2.4. Orthogonal complement of a good point.

Proposition 14. *Let $p \in V^\bullet$. There exist $b_2, \ldots, b_n \in V$ such that $p, b_2, \ldots, b_n$ is a basis.*

Proof. We assume $V = \mathbb{F}^n$. Consider the Hermitian form

$$\langle x, y\rangle = \sum_i x_i y_i^*.$$

If $\langle p, p\rangle \neq 0$, then the result follows from Lemma 6. Thus, we may assume $\langle p, p\rangle = 0$.

Since $p \in (\mathbb{F}^n)^\bullet$, there exists $q \in \mathbb{F}^n$ such that $\langle p, q\rangle = 1$.

Consider the $\mathbb{F}$-linear space $W = \mathbb{F}p + \mathbb{F}q$. Note that $\mathbb{F}p \cap \mathbb{F}q = 0$. Indeed, if $xp + yq = 0$ then $y = 0$ since $x\langle p, p\rangle + y\langle q, p\rangle = y$. We obtain $xp = 0$ which implies $\langle xp, q\rangle = x = 0$. Thus, $W = \mathbb{F}p \oplus \mathbb{F}q$. We will show that W admits an orthonormal basis and the result will follow from Lemma 6.

First we assume $\langle q, q\rangle \neq 0$. We may suppose $\langle q, q\rangle = \pm 1$. For $p' := p - \frac{\langle p,q\rangle}{\langle q,q\rangle} q$, we obtain $W = \mathbb{F}p' + \mathbb{F}q$, $\langle p', p'\rangle = \mp 1$, $\langle p', q\rangle = 0$. Thus, p', q form an orthonormal basis of W.

Finally, we take $\langle q, q\rangle = 0$. Take $p' = p + q$ and $q' = p - q$. We have $W = \mathbb{F}p' \oplus \mathbb{F}q'$, $\langle p', p'\rangle = -\langle q', q'\rangle = 2$ and $\langle p', q'\rangle = 0$. □

Proposition 15. *Assume that V is equipped with a Hermitian form. Let $p \in V$ be such that $\langle p, p\rangle \neq 0$. Then $V = \mathbb{F}p \oplus p^\perp$ and $p^\perp$ is free, where*

$$p^\perp := \{v \in V : \langle v, p\rangle = 0\}$$

stands for the orthogonal complement of p.

Proof. It is easy to see that $V = \mathbb{F}p \oplus p^\perp$. Indeed, $\mathbb{F}p \cap p^\perp = 0$ and for every $v \in V$ we have

$$v = \frac{\langle v, p\rangle}{\langle p, p\rangle} p + \left(v - \frac{\langle v, p\rangle}{\langle p, p\rangle} p\right).$$

By Proposition 14 there exist $b_2, \ldots, b_n \in V$ such that $p, b_2, \ldots, b_n$ is a basis of V. The vectors

$$b_k - \frac{\langle b_k, p\rangle}{\langle p, p\rangle} p$$

form a basis for $p^\perp$. □

3 Projective geometry

In this section we study the smooth structure of projective spaces over algebras. Their tangent spaces and vector fields are easily described via linear tools, analogous to what is done in [AGr].

Let V be a finite dimensional free $\mathbb{F}$-module. The projective space $\mathbb{P}_{\mathbb{F}}(V)$ is defined as

$$\mathbb{P}_{\mathbb{F}}(V) := V^\bullet/\mathbb{F}^\times.$$

Whenever there is no possible confusion we omit the subscript $\mathbb{F}$ and just write $\mathbb{P}(V)$; given $p \in V^\bullet$, we denote by $\boldsymbol{p} \in \mathbb{P}(V)$ the corresponding point on the projective space.

Proposition 16. *The space* $\mathbb{P}(V)$ *is a smooth manifold of dimension* $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \mathbb{F}$. *Furthermore, the quotient map* $V^\bullet \to \mathbb{P}(V)$ *is a principal* $\mathbb{F}^\times$*-bundle.*

Proof. Note that $\mathbb{F}^\times$ is a Lie group and $V^\bullet$ is an open subset of V. Consider the smooth injective map $\lambda : \mathbb{F}^\times \times V^\bullet \to V^\bullet \times V^\bullet$ defined by $\lambda(\alpha, v) := (\alpha v, v)$. Let us prove that such map is proper.

Consider a compact $K \subset V^\bullet \times V^\bullet$ and a sequence $(\alpha_n, v_n) \in \lambda^{-1}(K)$. Since K is compact we may assume, without loss of generality, that $(\alpha_n v_n, v_n)$ converges in K to a limit (w, v).

Since V is free, it admits a non-degenerate Hermitian form $\langle -, - \rangle$. Take $h, h' \in V$ such that $\langle v, h \rangle = \langle w, h' \rangle = 1$. Consider the scalars $\alpha := \langle w, h \rangle$ and $\beta := \langle v, h' \rangle$.

Note that

$$\alpha_n = \frac{\langle \alpha_n v_n, h \rangle}{\langle v_n, h \rangle} \to \alpha \quad \text{and} \quad \alpha\beta = \lim \alpha_n \langle v_n, h' \rangle = \langle w, h' \rangle = 1.$$

Thus, every sequence on $\lambda^{-1}(K)$ admits a convergent subsequence and, therefore, the map λ is proper.

Since the action of $\mathbb{F}^\times$ on $V^\bullet$ is free and proper, $\mathbb{P}(V) := V^\bullet/\mathbb{F}^\times$ is a smooth manifold and the quotient map $V^\bullet \to \mathbb{P}(V)$ is a principal $\mathbb{F}^\times$-bundle. □

Corollary 17. *We have the natural isomorphism*

$$C^\infty(\mathbb{P}(V)) \simeq \{f \in C^\infty(V^\bullet) : f \textit{ is } \mathbb{F}^\times\textit{-invariant}\}.$$

Based on this corollary we will always think of smooth functions on the projective space as $\mathbb{F}^\times$-invariant smooth functions on $V^\bullet$.

Example 18. If $\mathbb{F}$ stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, then we obtain the usual real, complex and quaternionic projective spaces $\mathbb{P}^n_{\mathbb{F}} := (\mathbb{F}^{n+1})^\bullet/\mathbb{F}^\times$. Observe that the projective lines are spheres in this case: $\mathbb{P}^1_{\mathbb{R}} \simeq \mathbb{S}^1$, $\mathbb{P}^1_{\mathbb{C}} \simeq \mathbb{S}^2$, $\mathbb{P}^1_{\mathbb{H}} \simeq \mathbb{S}^4$.

Example 19. If $\mathbb{C}_s = \mathbb{R} \times \mathbb{R}$ stands for the split-complex numbers, then the projective space $\mathbb{P}^n_{\mathbb{C}_s}$ is diffeomorphic to $\mathbb{P}^n_{\mathbb{R}} \times \mathbb{P}^n_{\mathbb{R}}$. Indeed, the diffeomophism is

$$\mathbb{P}^n_{\mathbb{C}_s} \to \mathbb{P}^n_{\mathbb{R}} \times \mathbb{P}^n_{\mathbb{R}}$$
$$[(x_0, y_0) : \cdots : (x_n, y_n)] \mapsto ([x_0 : \cdots : x_n], [y_0 : \cdots : y_n])$$

In particular, the corresponding projective line is the torus $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$.

The split-complex projective spaces model the point-hyperplane geometry. Indeed, consider a real vector space W and let W^* be its dual space. We define the $\mathbb{C}_s$-module

$$V := (1, 0)W \oplus (0, 1)W^*.$$

The region $V^\bullet$ of good points is $(1, 0)(W \setminus 0) \oplus (0, 1)(W^* \setminus 0)$ and its projectivization give us $\mathbb{P}_{\mathbb{C}_s}(V) = \mathbb{P}(W) \times \mathbb{P}(W^*)$. Note that a point $(1, 0)p + (0, 1)\phi$ in this space represents a point and a hyperplane on $\mathbb{P}_{\mathbb{R}}(W)$.

Example 20. If we consider the dual numbers $\mathbb{D} := \mathbb{R} + \varepsilon\mathbb{R}$, then $\mathbb{P}^n_{\mathbb{D}}$ is the tangent bundle of $\mathbb{P}^n_{\mathbb{R}}$. Indeed, consider a real vector space W and the $\mathbb{D}$-module $V := W \oplus \varepsilon W$. As shown in Proposition 23, the tangent space $T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{R}}(W)$ of $\mathbb{P}_{\mathbb{R}}(W)$ at the point $\boldsymbol{p}$ can be identified with $\mathrm{Lin}_{\mathbb{R}}(\mathbb{R}p, W/\mathbb{R}p)$. Thus we obtain the diffeomorphism

$$\begin{aligned} \mathbb{P}_{\mathbb{D}}(V) &\to T\mathbb{P}_{\mathbb{R}}(W) \\ p+\varepsilon v &\mapsto \varphi_{p+\varepsilon v} \end{aligned}$$

where $\varphi_{p+\varepsilon v} : \mathbb{R}p \to V/\mathbb{R}p$ is the tangent vector at $\boldsymbol{p}$ defined by $rp \mapsto rv + \mathbb{R}p$. Observe that the map is well-defined: if $\zeta := \alpha + \varepsilon\beta \in \mathbb{D}^{\times}$ then $\alpha \neq 0$ and $\varphi_{\zeta(p+\varepsilon v)} = \varphi_{\alpha p+(\alpha v+\beta p)\varepsilon} = \varphi_{\alpha p+\alpha v\varepsilon}$.

This map can be better visualized if we endow W with an inner product $\langle\cdot,\cdot\rangle$. In this case, $T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{R}}(W) = \mathrm{Lin}_{\mathbb{R}}(\mathbb{R}p, p^{\perp})$ and we have the diffeomorphism

$$\begin{aligned} \mathbb{P}_{\mathbb{F}}(V) &\to T\mathbb{P}_{\mathbb{R}}(W) \\ p+\varepsilon v &\mapsto \frac{\langle -,p\rangle}{\langle p,p\rangle}\left(v - \frac{\langle v,p\rangle}{\langle p,p\rangle}p\right) \end{aligned}$$

The dual number projective line is the tangent bundle of a circle, i.e., a cylinder.

In the same way that $\mathbb{R}$ and $\mathbb{C}$ projective spaces can be embedded in quaternionic projective spaces ($\mathbb{R}$ and $\mathbb{C}$ are subalgebras of $\mathbb{H}$), the projective spaces over $\mathbb{R}, \mathbb{C}, \mathbb{D}, \mathbb{C}_s$ can be embedded in the split-quaternionic projective space:

Example 21. The projective space $\mathbb{P}^n_{\mathbb{H}_s}$ over the split-quaternions $\mathbb{H}_s$ is an ambient space for the previously described geometries. Indeed, taking $t \in [0,1]$ and defining $\sigma(t) := (1-t)i + tj \in \mathbb{H}_s$, we obtain the one parameter family of subalgebras $\mathbb{K}_t := \mathbb{R} + \sigma(t)\mathbb{R}$ of the split-quaternions. Note that

$$\mathbb{K}_t \simeq \begin{cases} \mathbb{R} + i\mathbb{R} & \text{for} \quad 0 \le t < 1/2 \\ \mathbb{R} + \varepsilon\mathbb{R} & \text{for} \quad t = 1/2 \\ \mathbb{R} + j\mathbb{R} & \text{for} \quad 1/2 < t \le 1 \end{cases}$$

because $\sigma(t)^2 = -(1-t)^2 + t^2$. Observe that these algebras are $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{C}_s$, respectively. Thus, we have the following one parameter family of embeddings

$$\begin{aligned} \mathbb{P}^n_{\mathbb{K}_t} &\hookrightarrow \mathbb{P}^n_{\mathbb{H}_s} \\ [z_0 : \cdots : z_n] &\mapsto [z_0 : \cdots : z_n] \end{aligned}$$

Therefore, there is a natural transition between the $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{C}_s$ projective geometries (see Figure 1).

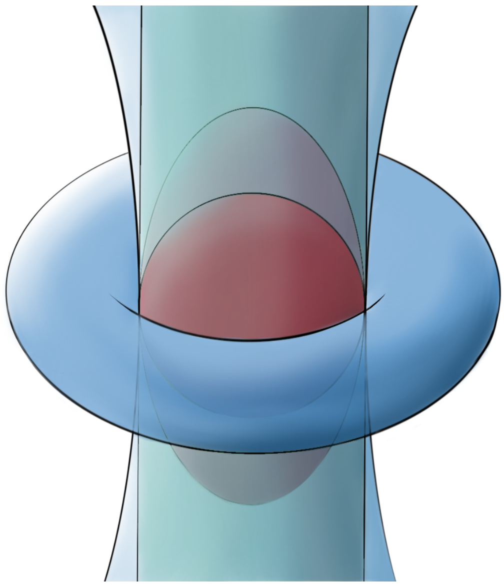


Figure 1: Transition between the split-complex (blue torus), dual number (green cylinder) and complex (red sphere) projective lines

This transition of geometries is described in a different fashion in [Tre].

Definition 22. Given finite dimension free $\mathbb{F}$-modules V_1 and V_2, we define $\mathrm{Lin}_{\mathbb{F}}(V_1, V_2)$ as the space of all real linear transformations $\phi : V_1 \to V_2$ satisfying $\phi(\alpha v) = \alpha\phi(v)$ for every $\alpha \in \mathbb{F}$, $v \in V_1$. This space is $\mathbb{R}$-linear in general and $\mathbb{F}$-linear when $\mathbb{F}$ is commutative.

The quotient $V/\mathbb{F}p$ is free by Proposition 14 and its dimension with respect to $\mathbb{F}$ is $\dim_{\mathbb{F}} V - 1$. An element $\phi \in \mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, V/\mathbb{F}p)$ is uniquely determined by $\phi(p)$, and thus the real dimension of $\mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, V/\mathbb{F}p)$ is $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \mathbb{F}$.

Given a map $\phi \in \mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, V/\mathbb{F}p)$ we define a tangent vector $t_\phi \in T_{\boldsymbol{p}}\mathbb{P}(V)$ by the formula:

$$t_\phi(f) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(p+\varepsilon v),$$

where p is a representative of $\boldsymbol{p}$ and v is a representative of $\phi(p)$. Note that the above definition does not depend on the choice of v: if $v, v' \in V$ satisfy $[v] = [v'] = \phi(p)$, then $v - v' = \alpha p$ for some $\alpha \in \mathbb{F}$ and

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon v) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon v') + \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon\alpha p).$$

Since for suficiently small $\varepsilon \neq 0$ the element $(1+\varepsilon\alpha)$ is a unit, we conclude that $f(p+\varepsilon\alpha p) = f(p)$. Therefore,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon v) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon v').$$

Finally, the definition does not depend on the choice of a representative of $\boldsymbol{p}$. So, $t_\phi \in T_{\boldsymbol{p}}\mathbb{P}(V)$.

Proposition 23. *Let $\boldsymbol{p} \in \mathbb{P}(V)$. The map $t : \mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, V/\mathbb{F}p) \to T_{\boldsymbol{p}}\mathbb{P}(V)$ mapping ϕ to t_ϕ is an $\mathbb{R}$-isomorphism.*

Proof. Note that both spaces have the same real dimension $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \mathbb{F}$. We just have to prove that t is surjective. Let $\gamma : \mathbb{R} \to \mathbb{P}_{\mathbb{F}}(V)$ be a smooth curve, $\gamma(0) = \boldsymbol{p}$. We lift γ around 0 to a map $\tilde{\gamma}$ with codomain $V^\bullet$ such that $\tilde{\gamma}(0) = p$. Hence,

$$\gamma'(0)f = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(\tilde{\gamma}(\varepsilon)).$$

Expanding in Taylor series, we have $\tilde{\gamma}(\varepsilon) = p + \varepsilon\tilde{\gamma}'(0) + o(\varepsilon)$ and, consequently,

$$\gamma'(0)f = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(p+\varepsilon\tilde{\gamma}'(0)).$$

Thus, defining $\phi : \mathbb{F}p \to V/\mathbb{F}p$ by the formula $\phi(\alpha p) = [\alpha\tilde{\gamma}'(0)]$ we conclude that $t_\phi = \gamma'(0)$. □

With the above proposition in mind, every time we write $T_{\boldsymbol{p}}\mathbb{P}(V)$ we mean $\mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, V/\mathbb{F}p)$.

Now consider a Hermitian form $\langle\cdot,\cdot\rangle$ on V. The projective space $\mathbb{P}(V)$ has two distinguished regions

$$S(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p\rangle = 0\},$$
$$R(V) := \{\boldsymbol{p} \in \mathbb{P}(V) : \langle p, p\rangle \neq 0\}.$$

The points of $S(V)$ are called **singular** and, those of $R(V)$, **regular**.

Proposition 24. *If $\boldsymbol{p} \in R(V)$, then*

$$T_{\boldsymbol{p}}\mathbb{P}(V) \simeq \mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, p^\perp).$$

Proof. Follows directly from Proposition 15. □

Whenever working with a tangent space at a regular point $\boldsymbol{p}$ we will think of $T_{\boldsymbol{p}}\mathbb{P}(V)$ as $\mathrm{Lin}_{\mathbb{F}}(\mathbb{F}p, p^\perp)$.

Definition 25. The Hermitian form $\langle\cdot,\cdot\rangle$ induces a Hermitian metric on $R(V)$, defined by

$$\langle\phi,\psi\rangle_{\boldsymbol{p}} = \pm\frac{\langle\phi(p),\psi(p)\rangle}{\langle p,p\rangle}$$

for $\phi,\psi \in T_{\boldsymbol{p}}\mathbb{P}(V)$. The sign is to be fixed conveniently. Associated to this Hermitian metric is the pseudo-Riemannian metric

$$g_{\boldsymbol{p}}(\phi,\psi) = \mathrm{Re}\,\langle\phi,\psi\rangle_{\boldsymbol{p}},$$

where $\mathrm{Re}\,u := (u+u^*)/2$.

Example 26. Let $V := \mathbb{F}^{n+1}$ be endowed with the Hermitian form

$$\langle u, v\rangle := \sum_i u_i v_i^*.$$

Consider the pseudo-Riemannian metric g on the regular region $R(V)$ obtained from the Hermitian metric in Definition 25 with positive sign.

For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the metric g is Riemannian, the usual Fubini-Study metric.

For split-algebras $\mathbb{F} = \mathbb{C}_s, \mathbb{H}_s$, the metric g is split, i.e., its signature has the same number of pluses and minuses.

The regular region over dual numbers $\mathbb{D}$ is the whole projective space, and the signature has n pluses and n zeros. The vectors ϕ parallel to the fibers of $\mathbb{P}^n_{\mathbb{D}} \to \mathbb{P}^n_{\mathbb{R}}$, $[u + \varepsilon v] \mapsto [u]$, are the ones with null norm, i.e., $g(\phi, \phi) = 0$.

For projective lines, the metrics have the following signatures:

$\mathbb{P}^1_{\mathbb{R}}$ has signature $+$	$\mathbb{P}^1_{\mathbb{D}}$ has signature $+0$
$\mathbb{P}^1_{\mathbb{C}}$ has signature $++$	$\mathbb{P}^1_{\mathbb{C}_s}$ has signature $+-$
$\mathbb{P}^1_{\mathbb{H}}$ has signature $++++$	$\mathbb{P}^1_{\mathbb{H}_s}$ has signature $++--$

Example 27. The split-complex projective space (point-hyperplane geometry) arising from $V := (1,0)W \oplus (0,1)W$, as described in the Example 19, has a natural geometry.

Given two vectors $v_1 := (1,0)w_1 + (0,1)\varphi_1$ and $v_2 = (1,0)w_2 + (0,1)\varphi_2$, we have the natural Hermitian form

$$\langle v_1, v_2 \rangle := (1,0)\phi_1(v_2) + (0,1)\phi_2(v_1).$$

In particular, if $v := (1,0)w + (0,1)\varphi$, then $\langle v, v \rangle = (1,1)\phi(v)$. Thus, the regular region describes the pair of points $[w]$ and hyperplanes $\varphi(x) = 0$ such that the point is not in the hyperplane. Taking $W = \mathbb{R}^{n+1}$ and identifying $W^* = W$ via the standard Euclidean metric, the metric associated to the above Hermitian form coincides with the one described in the Example 26. That is, the signature of $\mathbb{P}_{\mathbb{C}_s}(V)$ is split.

Example 28. The n-dimensional real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ is the ball $\{\boldsymbol{p} \in \mathbb{P}^n_{\mathbb{R}} : \langle p, p \rangle < 0\}$, where $\langle \cdot, \cdot \rangle$ is the canonical real Hermitian form on $\mathbb{R}^{n+1}$ with signature $- + \cdots +$. The hyperbolic Hermitian metric is obtained from Definition 25 using the minus sign. The complex and quaternionic hyperbolic spaces are defined likewise.

The region $\mathrm{dS}^n = \{\boldsymbol{p} \in \mathbb{P}^n_{\mathbb{R}} : \langle p, p \rangle > 0\}$ with the metric defined above is the projectivization of the de Sitter space, and it is a Lorentz manifold.

Now, let us discuss vector fields. For a regular point $\boldsymbol{p}$ we can think of $T_{\boldsymbol{p}}\mathbb{P}(V)$ as a subset of $\operatorname{Lin}(V, V)$, because $V = \mathbb{F}p \oplus p^{\perp}$. More precisely, $T_{\boldsymbol{p}}\mathbb{P}(V)$ can be seen as the linear maps $\phi \in \operatorname{Lin}(V, V)$ such that $\phi(p) \in p^{\perp}$ and $\phi(p^{\perp}) = 0$.

Definition 29. A vector field on an open subset U of $R(V)$ is a smooth map $X : U \to \operatorname{Lin}(V, V)$ satisfying $X(\boldsymbol{p}) \in T_{\boldsymbol{p}}V$ for all $\boldsymbol{p} \in U$. We denote the space of all vector fields by $\mathfrak{X}(U)$.

Among the vector fields there are special ones called called **spread vector fields**. Given $\boldsymbol{p} \in R(V)$ we define the two projections $\pi'[\boldsymbol{p}] : V \to \mathbb{F}p$ and $\pi[\boldsymbol{p}] : V \to p^{\perp}$ by

$$\pi'[\boldsymbol{p}]v = \frac{\langle v, p \rangle}{\langle p, p \rangle} p \quad \text{and} \quad \pi[\boldsymbol{p}]v = v - \pi'[\boldsymbol{p}]v.$$

Both formulas are well defined because they do not depend on the choice of a representative p of $\boldsymbol{p}$.

Definition 30. A spread vector field T is a vector field defined by

$$T_{\boldsymbol{q}} := \pi[\boldsymbol{q}] \circ t \circ \pi'[\boldsymbol{q}]$$

for a given $t \in \operatorname{Lin}(V, V)$. The vector field T is said to be spread from t.

The importance of spread vector fields lies on the fact that if we have $\phi \in T_{\boldsymbol{p}}\mathbb{P}(V)$ for a regular $\boldsymbol{p}$, then the spread Φ from ϕ is a vector field satisfying $\phi = \Phi_{\boldsymbol{p}}$ (in other words, we have a natural way to extend vectors to vector fields). Furthermore, calculating tensors is largely simplified by the use of spread vector fields; this is analogous to what happens in Lie groups when working with left-invariant vector fields.

4 Connection and geodesics

Following [AGr], we give an algebraic description of the (pseudo-)Riemannian geometry on the previously discussed projective spaces.

4.1. Levi-Civita connection. A vector field X on the open set $U \subset R(V)$ is, in particular, a smooth map $X : U \to \mathrm{Lin}(V,V)$. We remind that the quotient map $\mathrm{proj} : V^\bullet \to \mathbb{P}(V)$ defines a principal $\mathbb{F}^\times$-bundle. For $\tilde{U} := \mathrm{proj}^{-1} U$ there is a smooth map $\tilde{X} : \tilde{U} \to \mathrm{Lin}(V,V)$ which is $\mathbb{F}^\times$-invariant and satisfies $X(\boldsymbol{p}) = \tilde{X}(p)$. Hence, we can always think of vector fields as $\mathbb{F}^\times$-invariants smooth functions defined on $\mathbb{F}^\times$-stable open subsets of $V^\bullet$.

If $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$, with $\boldsymbol{p} \in U$, then

$$dX(t) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X\big(p + \varepsilon t(p)\big).$$

Note that this derivative does not depend on the choice of a representative p for $\boldsymbol{p}$.

Definition 31. The connection ∇ defined on the vector fields of $R(V)$ is given by

$$\nabla_t X(\boldsymbol{p}) = \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X\big(p + \varepsilon t(p)\big)\right)_{\boldsymbol{p}},$$

where X is a vector field, $\boldsymbol{p}$ is a point on the domain of X, and $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$.

Here we are using the notation $M_{\boldsymbol{p}} := \pi[\boldsymbol{p}] \circ M \circ \pi'[\boldsymbol{p}]$, where $M \in \mathrm{Lin}(V,V)$. So, the connection is defined as the derivative of vector fields up to the projections necessary to ensure that $\nabla_t X(\boldsymbol{p})$ is in the tangent space at $\boldsymbol{p}$. If X, Y are vector fields, then $\nabla_Y X$ is the vector field $\boldsymbol{p} \mapsto \nabla_{Y(\boldsymbol{p})} X$. It is easy to see that ∇ is a connection.

The facts/expressions in [AGr] involving the connection hold in the case of $\mathbb{F}$-modules as well:

Definition 32. Given $t : V \to V$ and $\boldsymbol{p} \in R(V)$ define $t^* : V \to V$ by the formula

$$t^* v := \frac{\langle v, tp\rangle}{\langle p, p\rangle} p.$$

We call this function the **adjoint** of t.

Lemma 33. *Given $\boldsymbol{p} \in R(V)$ and $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$ we have*

$$\langle tu, v\rangle = \langle u, t^* v\rangle.$$

Proof. We can write $t = t \circ \pi'[\boldsymbol{p}]$ because $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$. Just note that

$$\langle tu, v\rangle = \frac{\langle u, p\rangle}{\langle p, p\rangle}\langle tp, v\rangle, \quad \langle u, t^* v\rangle = \langle u, p\rangle \frac{\langle v, tp\rangle^*}{\langle p, p\rangle}.$$

So, $\langle tu, v\rangle = \langle u, t^* v\rangle$. ∎

Lemma 34 (see Lemma 4.2 [AGr])**.** *Let $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$ with $\boldsymbol{p} \in R(V)$. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi'[p + \varepsilon tp] = -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi[p + \varepsilon tp] = t + t^*.$$

Proof. The relation between the derivatives follows from $\pi[\boldsymbol{x}] + \pi'[\boldsymbol{x}] = \mathrm{id}$ for $\boldsymbol{x} \in R(V)$. Now, note that

$$\pi'[p + \varepsilon tp]v = \frac{\langle v, p + \varepsilon tp\rangle}{\langle p + \varepsilon tp, p + \varepsilon tp\rangle}(p + \varepsilon tp) = \frac{\langle v, p + \varepsilon tp\rangle}{\langle p, p\rangle + \varepsilon^2\langle tp, tp\rangle}(p + \varepsilon tp).$$

Since

$$\frac{1}{\langle p, p\rangle + \varepsilon^2\langle tp, tp\rangle} = \frac{1}{\langle p, p\rangle} + o(\varepsilon),$$

$$\langle v, p + \varepsilon tp\rangle(p + \varepsilon tp) = \langle v, p\rangle p + \varepsilon\Big(\langle v, tp\rangle p + \langle v, p\rangle tp\Big) + o(\varepsilon),$$

and

$$tv = \frac{\langle v, p\rangle}{\langle p, p\rangle} tp$$

we conclude that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi'[p + \varepsilon tp] = t + t^*.$$

∎

The derivatives of a spread vector field with respect to a spread vector field is particularly simple:

Proposition 35 (see Lemma 4.3 [AGr]). *Consider $t, s \in T_{\boldsymbol{p}}\mathbb{P}(V)$ with $\boldsymbol{p} \in R(V)$. Let T and S be the vector fields spread from t and s, respectively. Then*

$$\nabla_T S(\boldsymbol{x}) = [s\pi[\boldsymbol{x}]t - t\pi'[\boldsymbol{x}]s]_{\boldsymbol{x}}\,.$$

In particular, $\nabla_T S(\boldsymbol{p}) = 0$.

Proof. Since $S(x + \varepsilon T_{\boldsymbol{x}} x) = \pi[x + \varepsilon T_{\boldsymbol{x}} x] \circ s \circ \pi'[x + \varepsilon T_{\boldsymbol{x}} x]$, we have, by Lemma 34,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(x + \varepsilon T_{\boldsymbol{x}} x) = -(T_{\boldsymbol{x}} + T^*_{\boldsymbol{x}}) \circ s \circ \pi'[\boldsymbol{x}] + \pi[\boldsymbol{x}] \circ s \circ (T_{\boldsymbol{x}} + T^*_{\boldsymbol{x}}).$$

From $\pi[x] \circ T^*_{\boldsymbol{x}} = T^*_{\boldsymbol{x}} \circ \pi'[x] = 0$ we obtain

$$\left[\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(x + \varepsilon T_{\boldsymbol{x}} x)\right]_{\boldsymbol{x}} = -T_{\boldsymbol{x}} \circ s \circ \pi'[\boldsymbol{x}] + \pi[\boldsymbol{x}] \circ s \circ T_{\boldsymbol{x}}$$

and $T_{\boldsymbol{x}} = \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}]$ implies

$$\nabla_T S(\boldsymbol{x}) = -\pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}] \circ s \circ \pi'[\boldsymbol{x}] + \pi[\boldsymbol{x}] \circ s \circ \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}] = [s\pi[\boldsymbol{x}]t - t\pi'[\boldsymbol{x}]s]_{\boldsymbol{x}}\,.$$

□

Proposition 36. *Consider $t, s \in T_{\boldsymbol{p}}\mathbb{P}(V)$ with $\boldsymbol{p} \in R(V)$. Let T and S be the vector fields spread from t and s respectively. Then $[T, S](\boldsymbol{p}) = 0$.*

Proof. Let $f \in C^\infty(\mathbb{P}(V))$. We have

$$S_{\boldsymbol{x}}(f) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(x + \varepsilon S_{\boldsymbol{x}}(x)\big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(x + \varepsilon\pi[\boldsymbol{x}]s(x)\big).$$

So,

$$T_{\boldsymbol{p}}(Sf) = t(Sf) = \frac{d}{d\delta}\Big|_{\delta=0}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(p + \delta tp + \varepsilon\pi[p + \delta tp]s(p + \delta tp)\big).$$

It follows from Lemma 34 that

$$\pi[p + \delta tp]s(p + \delta tp) = \pi[\boldsymbol{p}]sp + \delta(-(t + t^*)sp + \pi[\boldsymbol{p}]stp) + o(\delta) = sp - \delta t^* sp + o(\delta)$$

where we use that $ts = st = 0$ and $\pi[\boldsymbol{p}]sp = sp$ since $s, t \in T_{\boldsymbol{p}}\mathbb{P}(V)$. Hence,

$$T_{\boldsymbol{p}}(Sf) = \frac{d}{d\delta}\Big|_{\delta=0}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(p + \delta tp + \varepsilon sp),$$

which implies $T_{\boldsymbol{p}}(Sf) = S_{\boldsymbol{p}}(Tf)$. □

Corollary 37. *The connection ∇ is torsion free.*

Proof. Follows immediately from Propositions 35 and 36. □

Corollary 38 (see Proposition 4.4 [AGr]). *The connection ∇ is compatible with the Hermitian metric.*

Proof. Consider the tensor $B := \nabla\langle -, - \rangle$, i.e., $B(S, T_1, T_2) := S\langle T_1, T_2\rangle - \langle \nabla_S T_1, T_2\rangle - \langle T_1, \nabla_S T_2\rangle$. Let $\boldsymbol{p} \in R(V)$, let $t_1, t_2, s \in T_{\boldsymbol{p}}\mathbb{P}(V)$, and let S, T_1, T_2 be the vector fields spread respectively from s, t_1, t_2. By Proposition 35 we have

$$B(s, t_1, t_2) = s\langle T_1, T_2\rangle.$$

Fix a representative p for $\boldsymbol{p}$ and define $u := sp$, $v_1 := t_1 p$, $v_2 := t_2 p$.

$$s\langle T_1, T_2\rangle = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \langle (T_1)_{p+\varepsilon u}, (T_2)_{p+\varepsilon u}\rangle$$
$$= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\langle (T_1)_{p+\varepsilon u}(p+\varepsilon u), (T_2)_{p+\varepsilon u}(p+\varepsilon u)\rangle}{\langle p+\varepsilon u, p+\varepsilon u\rangle}.$$

Since $\langle p, u\rangle = 0$, we have $\langle p+\varepsilon u, p+\varepsilon u\rangle^{-1} = \langle p, p\rangle^{-1} + o(\varepsilon)$, and therefore

$$s\langle T_1, T_2\rangle = \frac{1}{\langle p, p\rangle} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \langle (T_1)_{p+\varepsilon u}(p+\varepsilon u), (T_2)_{p+\varepsilon u}(p+\varepsilon u)\rangle.$$

Now,

$$\begin{aligned}(T_i)_{p+\varepsilon u}(p+\varepsilon u) &= \pi[p+\varepsilon u]t_i\pi'[p+\varepsilon u](p+\varepsilon u)\\ &= \pi[p+\varepsilon u]v_i\\ &= v_i - \frac{\langle v_i, p+\varepsilon u\rangle}{\langle p+\varepsilon u, p+\varepsilon u\rangle}(p+\varepsilon u)\\ &= v_i - \varepsilon\frac{\langle v_i, u\rangle}{\langle p, p\rangle}p + o(\varepsilon),\end{aligned}$$

where we use that $tu = 0$ and $\langle v_i, p\rangle = 0$. Hence,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (T_i)_{p+\varepsilon u}(p+\varepsilon u) = -\frac{\langle v_i, u\rangle}{\langle p, p\rangle}p$$

and we obtain

$$s\langle T_1, T_2\rangle = -\frac{1}{\langle p, p\rangle^2}\left(\langle v_1, u\rangle\langle p, v_2\rangle + \langle v_2, u\rangle^*\langle v_1, p\rangle\right) = 0,$$

because $\langle v_i, p\rangle = 0$. Thus, $B(s, t_1, t_2) = 0$. □

Corollaries 37 and 38 say that the Hermitian and the pseudo-Riemannian metric are the Levi-Civita ones.

4.2. Geodesics. The geodesics in $\mathbb{P}(V)$, as we will see in Proposition 42, are of linear nature (analogous to the geodesics on a sphere).

Definition 39. Consider a 2-dimensional real subspace W of V such that the restriction to W of the Hermitian form $\langle\cdot,\cdot\rangle$ is $\mathbb{R}$-valued and non-null. We call the projectivization $\mathbb{P}_{\mathbb{F}}(W)$ a **geodesic**, where by $\mathbb{P}_{\mathbb{F}}(W)$ we mean the image of $W \cap V^\bullet$ under the quotient map $V^\bullet \to \mathbb{P}_{\mathbb{F}}(V)$.

Proposition 40. *The natural map $\phi : \mathbb{P}_{\mathbb{R}}(W \cap V^\bullet) \to \mathbb{P}_{\mathbb{F}}(V)$ is an immersion and its image is $\mathbb{P}_{\mathbb{F}}(W)$. Furthermore, if $\boldsymbol{p} \in \mathbb{P}_{\mathbb{F}}(W)$ is regular, then*

$$T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{F}}(W) = \{t \in T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{F}}(V) : t(p) \in W\},$$

where $p \in W$ is a representative of $\boldsymbol{p}$.

Proof. Since the form restricted to W is real and non-null, there exists $p \in W$ with $\langle p, p\rangle \neq 0$. Consider an orthonormal basis p, q for W. Observe that $\mathbb{R}q$ contains the only points in W which can be non-good since, for every $x \in W \setminus \mathbb{R}q$ we have $\langle p, x\rangle \neq 0$. So, $W \cap V^\bullet$ is either $W \setminus \{0\}$ or $W \setminus \mathbb{R}q$. Clearly, the image of ϕ is $\mathbb{P}_{\mathbb{F}}(W)$.

We now prove that ϕ is injective. If $[\alpha p + \beta q] = [\alpha' p + \beta' q]$ in $\mathbb{P}_{\mathbb{F}}(W)$, where the coefficients $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$, then there exists $\gamma \in \mathbb{F}^\times$ such that $\alpha p + \beta q = \gamma(\alpha' p + \beta' q)$. If $W \cap V^\bullet = W \setminus \{0\}$, then p, q are $\mathbb{F}$-linearly independent and it follows that $\alpha = \gamma\alpha'$ and $\beta = \gamma\beta'$. If $W \cap V^\bullet = W \setminus \mathbb{R}q$, then $\alpha \neq 0$ and, consequently, $\alpha\langle p, p\rangle = \gamma\alpha'\langle p, p\rangle$, implying that $\gamma \in \mathbb{R}$. Thus $[\alpha p + \beta q] = [\alpha' p + \beta' q]$ in $\mathbb{P}_{\mathbb{R}}(W \cap V^\bullet)$.

The smoothness of ϕ follows from the commutative diagram bellow

$$\begin{tikzcd} W \cap V^\bullet \arrow[r, hook, "i"] \arrow[d, "\mathrm{proj}_W"'] & V^\bullet \arrow[d, "\mathrm{proj}_V"] \\ \mathbb{P}_\mathbb{R}(W \cap V^\bullet) \arrow[r, "\phi"] & \mathbb{P}_\mathbb{F}(V) \end{tikzcd}$$

because the natural map $\mathrm{proj}_V \circ i : W \cap V^\bullet \to \mathbb{P}_\mathbb{F}(V)$ is smooth and $\mathbb{R}^\times$-invariant. Let us show that ϕ is an immersion. If v is a tangent vector at $\boldsymbol{x} \in \mathbb{P}_\mathbb{R}(W \cap V^\bullet)$, then there is a vector $\tilde{v} \in W$ tangent to W at x such that $\mathrm{d}\mathrm{proj}_W(\tilde{v}) = v$. The image of $\tilde{v}$ by $\mathrm{d}i$ is $\tilde{v} \in V$ itself, and the image of this vector by $\mathrm{d}\mathrm{proj}_V$ is $\mathrm{d}\phi(v)$. If $\mathrm{d}\phi(v) = 0$, then $\tilde{v}$ is tangent to the fiber of proj_V at x, which means $\tilde{v} = kx$ for some $k \in \mathbb{F}$. It remains to show that $k \in \mathbb{R}$ since this implies that $v = \mathrm{d}\mathrm{proj}_W(\tilde{v}) = 0$. If $\langle x, x\rangle \neq 0$, it follows from $\langle \tilde{v}, x\rangle = k\langle x, x\rangle$ that $k \in \mathbb{R}$. Otherwise, assume $\langle x, x\rangle = 0$ and $k \notin \mathbb{R}$. Then $x, y := \tilde{v} - x$ is a basis for W satisfying $\langle x, x\rangle = \langle y, y\rangle = \langle x, y\rangle = 0$, a contradiction. □

In order to prove that (the regular parts of) the geodesics introduced in Definition 39 coincide with the geodesics of the Levi-Civita connection we will use a distinguished vector field introduced bellow.

The **tance** between two regular points $\boldsymbol{p}, \boldsymbol{q}$ is defined by

$$\mathrm{ta}(\boldsymbol{p}, \boldsymbol{q}) := \frac{\langle p, q\rangle\langle q, p\rangle}{\langle p, p\rangle\langle q, q\rangle}.$$

Clearly, the tance between two points is always a real number.

Let $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$, where $\boldsymbol{p} \in R(V)$. We define the vector field $\mathrm{Tn}(t)$ by the formula

$$\mathrm{Tn}(t)(\boldsymbol{x}) := \frac{T_{\boldsymbol{x}}}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})},$$

where T is the spread vector field from t. Note that $\mathrm{Tn}(t)$ is a smooth vector field defined on the region described by $\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x}) \neq 0$. The tance and the vector field $\mathrm{Tn}(t)$ are extensions of the corresponding concepts introduced in [AGG] and [AGr], respectively.

Let us show that the integral curve of $\mathrm{Tn}(t)$ starting at $\boldsymbol{p}$ is the geodesic passing through $\boldsymbol{p}$ with velocity t. We need the following lemma:

Lemma 41 (see Lemma 5.3 [AGr])**.** *Let $\boldsymbol{p} \in R(V)$ and $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$. If T is the spread vector field from t, then*

$$T_{\boldsymbol{x}}\mathrm{ta}(\boldsymbol{p}, \cdot) = -2\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})\mathrm{Re}\,\frac{\langle tx, x\rangle}{\langle x, x\rangle}$$

for all $\boldsymbol{x}$ satisfying $\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x}) \neq 0$.

Proof. Let $\xi = T_{\boldsymbol{x}}x$ for some representative x. By definition

$$T_{\boldsymbol{x}}\mathrm{ta}(\boldsymbol{p}, \cdot) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{\langle p, x + \varepsilon\xi\rangle\langle x + \varepsilon\xi, p\rangle}{\langle p, p\rangle\langle x + \varepsilon\xi, x + \varepsilon\xi\rangle} = 2\mathrm{Re}\,\frac{\langle p, x\rangle\langle \xi, p\rangle}{\langle p, p\rangle\langle x, x\rangle}.$$

Since

$$\xi = T_{\boldsymbol{x}}x = \pi[\boldsymbol{x}]tx = tx - \frac{\langle tx, x\rangle}{\langle x, x\rangle}x$$

and $tx \in p^\perp$ we obtain

$$\langle \xi, p\rangle = -\langle tx, x\rangle\frac{\langle x, p\rangle}{\langle x, x\rangle},$$

concluding the proof. □

Proposition 42 (see Thm 5.4 [AGr])**.** *Let $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$, $t \neq 0$, with $\boldsymbol{p} \in R(V)$. Consider the geodesic $\mathbb{P}_\mathbb{F}(W)$, where $W = \mathbb{R}p + \mathbb{R}tp$ (note that $\mathbb{P}_\mathbb{F}(W)$ does not depend on the choice of representative p for $\boldsymbol{p}$). Let c be a curve on $R(V) \cap \mathbb{P}_\mathbb{F}(W)$ satisfying*

$$c'(\theta) = \mathrm{Tn}(t)_{c(\theta)}, \quad c(0) = \boldsymbol{p} \quad \textit{and} \quad c'(0) = t.$$

The curve c is the geodesic of the Levi-Civita connection passing through $\boldsymbol{p}$ with velocity t.

Proof. Fix a representative p of $\boldsymbol{p}$ and a lift $\tilde{c}$ of c such that $\tilde{c}(0) = p$. Since

$$\frac{D}{d\theta} c'(\theta) = \nabla_{\mathrm{Tn}(t)(c(\theta))} \mathrm{Tn}(t)(c(\theta)),$$

it is enough to show that $\nabla_T \mathrm{Tn}(t) = 0$. By definition,

$$\nabla_{T_{\boldsymbol{x}}} \mathrm{Tn}(t) = [\mathrm{dTn}(t)(T_{\boldsymbol{x}})]_{\boldsymbol{x}}.$$

From $\mathrm{Tn}(t) = \mathrm{ta}(\boldsymbol{p}, \cdot)^{-1} T$ we obtain

$$\begin{aligned} \mathrm{dTn}(t)(T_{\boldsymbol{x}}) &= \frac{-1}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})^2} (T_{\boldsymbol{x}} \mathrm{ta}(\boldsymbol{p}, \cdot)) T_{\boldsymbol{x}} + \frac{1}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \mathrm{d}T(T_{\boldsymbol{x}}) \\ &= \frac{2}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \mathrm{Re} \frac{\langle tx, x\rangle}{\langle x, x\rangle} T_{\boldsymbol{x}} + \frac{1}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \mathrm{d}T(T_{\boldsymbol{x}}). \end{aligned}$$

Taking into account that $\boldsymbol{x} \in \mathbb{P}_{\mathbb{F}}(\mathbb{R}p + \mathbb{R}tp)$, we can take $x \in \mathbb{R}p + \mathbb{R}tp$. So,

$$\nabla_{T_{\boldsymbol{x}}} \mathrm{Tn}(t) = \frac{2}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \frac{\langle tx, x\rangle}{\langle x, x\rangle} T_{\boldsymbol{x}} + \frac{1}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \nabla_{T_{\boldsymbol{x}}}(T).$$

By proposition 35,

$$\nabla_{T_{\boldsymbol{x}}} T = [t\pi[\boldsymbol{x}]t - t \circ \pi'[\boldsymbol{x}]t]_{\boldsymbol{x}} = \pi[\boldsymbol{x}]t\pi[\boldsymbol{x}]t\pi'[\boldsymbol{x}] - \pi[\boldsymbol{x}]t\pi'[\boldsymbol{x}]t\pi'[\boldsymbol{x}].$$

Using that $t = \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} tp$, we have

$$\begin{aligned} \pi[\boldsymbol{x}]t\pi[\boldsymbol{x}]t\pi'[\boldsymbol{x}] &= \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \pi[\boldsymbol{x}]t\pi[\boldsymbol{x}]tp = -\frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \frac{\langle tp, x\rangle}{\langle x, x\rangle} \pi[\boldsymbol{x}]tx, \\ \pi[\boldsymbol{x}]t\pi'[\boldsymbol{x}]t\pi'[\boldsymbol{x}] &= \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \pi[\boldsymbol{x}]t\pi'[\boldsymbol{x}]tp = \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \frac{\langle tp, x\rangle}{\langle x, x\rangle} \pi[\boldsymbol{x}]tx, \end{aligned}$$

which implies

$$\nabla_{T_{\boldsymbol{x}}} T = -2 \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \frac{\langle tp, x\rangle}{\langle x, x\rangle} \pi[\boldsymbol{x}]tx.$$

On the other hand,

$$T_{\boldsymbol{x}} = \pi[\boldsymbol{x}] \circ t \circ \pi'[\boldsymbol{x}] = \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \pi[\boldsymbol{x}]tp \quad \text{and} \quad \langle tx, x\rangle = \frac{\langle x, p\rangle}{\langle p, p\rangle} \langle tp, x\rangle.$$

So,

$$\nabla_{T_{\boldsymbol{x}}} \mathrm{Tn}(t) = \frac{2}{\mathrm{ta}(\boldsymbol{p}, \boldsymbol{x})} \frac{\langle \cdot\,, p\rangle}{\langle p, p\rangle} \left(\frac{\langle tp, x\rangle}{\langle x, x\rangle} - \frac{\langle tp, x\rangle}{\langle x, x\rangle} \right) \pi[\boldsymbol{x}]tx = 0.$$

$\square$

The following lemma allows us to explicitly find tangent vectors to geodesics.

Lemma 43 (see Lemma A.1 [AGG]). *Let c be a smooth curve passing through $\boldsymbol{p} \in R(V)$ at the instant 0 and let p be a representative of $\boldsymbol{p}$. For any lift $\tilde{c}$ to $V^\bullet$ of the curve c passing through p at the instant 0 we have*

$$c'(0) = \frac{\langle \cdot\,, \tilde{c}(0)\rangle}{\langle \tilde{c}(0), \tilde{c}(0)\rangle} \pi[\boldsymbol{p}]\tilde{c}'(0).$$

Proof. Consider a smooth funcion $f : \mathbb{P}(V) \to \mathbb{R}$. We have

$$c'(0)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(\tilde{c}(\varepsilon)\big).$$

Expanding $\tilde{c}$ in Taylor series we obtain $\tilde{c}(\varepsilon) = \tilde{c}(0) + \tilde{c}'(0)\varepsilon + o(\varepsilon)$ and, therefore,

$$c'(0)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(\tilde{c}(0) + \varepsilon\tilde{c}'(0)\big).$$

Since the component of $\tilde{c}'(0)$ parallel to p does not contribute to the derivative, we have

$$c'(0)f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(\tilde{c}(0) + \varepsilon\pi[\boldsymbol{p}]\tilde{c}'(0)\big),$$

implying the result. $\square$

Geodesics appear in three types. Consider $\boldsymbol{p} \in R(V)$ and $t \in T_{\boldsymbol{p}}\mathbb{P}(V)$, $t \neq 0$ with $\langle p, p\rangle \pm 1$ and $\langle t, t\rangle \in \{-1, 0, 1\}$.

Assume that the form on $W := \mathbb{R}p + \mathbb{R}tp$ is nondegenerate definite. This means that $\langle tp, tp\rangle$ and $\langle p, p\rangle$ have the same sign and, therefore, $\langle tp, tp\rangle = \langle p, p\rangle$. We parametrize the geodesic $\mathbb{P}_{\mathbb{F}}(W)$ starting at $\boldsymbol{p}$ with velocity t by

$$\theta \mapsto [\cos(\theta)p + \sin(\theta)tp].$$

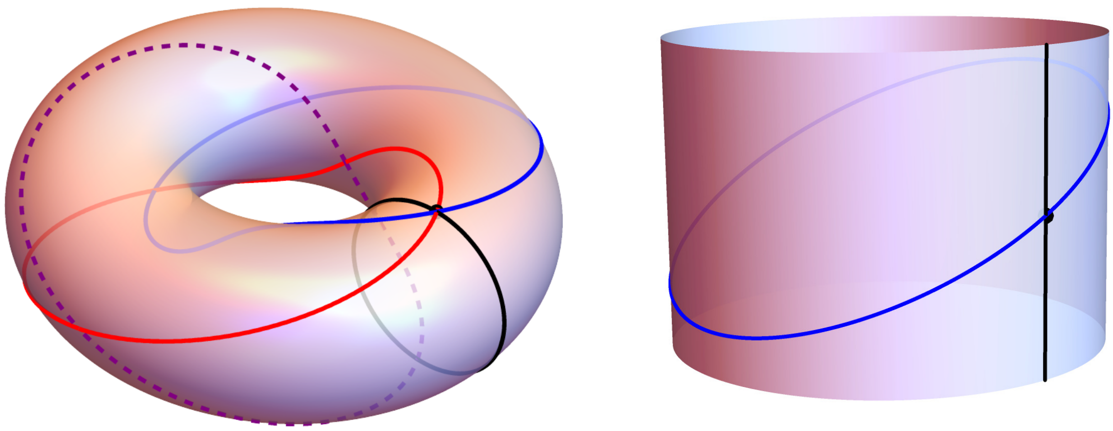
When the form on W is nondegenerate indefinite, $\langle p, p\rangle$ and $\langle tp, tp\rangle$ have opposite signs and, therefore, $\langle p, p\rangle = -\langle tp, tp\rangle$. The geodesic $\mathbb{P}_{\mathbb{F}}(W)$ is now parametrized by

$$\theta \mapsto [\cosh(\theta)p + \sinh(\theta)tp].$$

Finally, when the form on W is degenerate, that is, $\langle tp, W\rangle = 0$, then the parametrization in question is

$$\theta \mapsto [p + \theta tp].$$

The verification that these curves are indeed geodesics follows from Lemma 43 and Proposition 42.



(a) (b)

Figure 2: **(a)** Geodesics of the $\mathbb{C}_s$-projective line and **(b)** geodesics of the $\mathbb{D}$-projective line.

Example 44. Consider the settings of the Example 26 and fix a regular point $\boldsymbol{p}$. Without loss of generality, we assume $\langle p, p\rangle = 1$. Take a tangent vector t at p and consider a geodesic curve starting at $\boldsymbol{p}$ with velocity t. We will say that this geodesic is positive, negative or null respectively when $\langle t, t\rangle$ is positive, negative or null.

Let us focus on geodesics in the projective lines. For the division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the only geodesics appearing are the positive ones (the geodesics of the Fubini Study geometry). On the other hand, for $\mathbb{D}$ we have two types of geodesics: the positive and the null ones (tangent to the fibers). For the split-algebras $\mathbb{C}_s$ and $\mathbb{H}_s$, we have three types of geodesics: positive, negative and null. In Figure 2, we represent the positive, negative and null geodesics by blue, red and black colors, respectively.

Example 45. Consider $\mathbb{R}^3$ with a Hermitian form of signature $-++$. As described in Example 28, the regular region of $\mathbb{P}^2_{\mathbb{R}}$ is formed by two connected components, the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ and the projective de Sitter space dS^2. The space of all non-oriented geodesics in the hyperbolic plane is the space dS^2, a result obtained via point-plane duality. Indeed, for each $\boldsymbol{p} \in \mathrm{dS}^2$, we have the geodesic $\mathbb{P}(\boldsymbol{p}^\perp)$, and all geodesics of $\mathbb{H}^2_{\mathbb{R}}$ are of this type (see Figure 3). A positive geodesic of dS^2 correspond to a one parameter family of geodesics in the hyperbolic plane sharing a common point in $\mathbb{H}^2_{\mathbb{R}}$; a null geodesic correspond to a family of geodesics meeting at a common point in the absolute $\partial\mathbb{H}^2_{\mathbb{R}}$; and a negative geodesic corresponds to a one parameter family of ultra-parallel geodesics on the hyperbolic plane perpendicular to it.

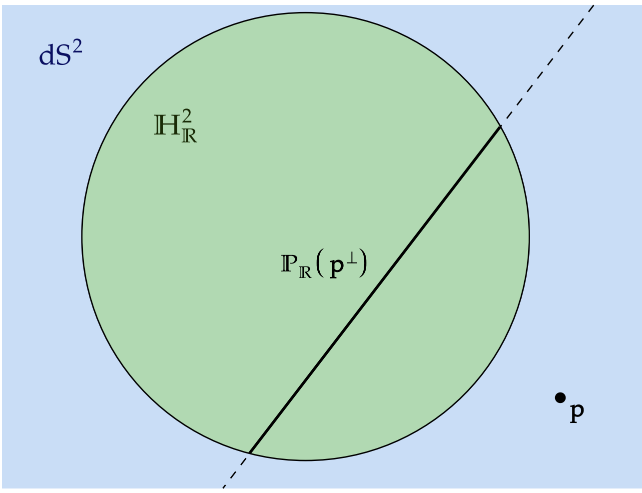


Figure 3: Real hyperbolic plane, projective de Sitter space, and space of geodesics.

4.3. Curvature tensor. Let the Hermitian metric be given by

$$\langle t_1, t_2 \rangle := \frac{\langle t_1(p), t_2(p) \rangle}{\langle p, p \rangle},$$

where t_1, t_2 are tangent vectors at a regular point $\boldsymbol{p}$. The pseudo-Riemannian metric g is the real part of the Hermitian metric.

The Riemann curvature tensor

$$R(U, V)W := \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W,$$

defined for vector fields U, V, W, has a very simple expression in the settings of classical geometries [AGr]:

$$R(t_1, t_2)s = -s(t_1^* t_2 - t_2^* t_1) + (t_1 t_2^* - t_2 t_1^*)s,$$

where $t_1, t_2, s \in T_{\boldsymbol{p}} \mathbb{P}_{\mathbb{F}}(V)$, where s^* is the adjoint of s (see Definition 32) and the same goes for t_1^*, t_2^*. The proof of this fact follows from using spread vector fields (Definition 30) and applying Propositions 35 and 36.

For a real subspace $W = \mathbb{R}t_1 \oplus \mathbb{R}t_2$ of $T_{\boldsymbol{p}} \mathbb{P}_{\mathbb{F}}(V)$, where t_1, t_2 are tangent vectors at $\boldsymbol{p}$ orthonormal with respect to g, the sectional curvature

$$K(W) := \frac{g(R(t_1, t_2)t_2, t_1)}{g(t_1, t_1)g(t_2, t_2) - g(t_1, t_2)^2}$$

is given by

$$K(W) = \frac{\langle t_1, t_1 \rangle \langle t_2, t_2 \rangle - 2\mathrm{Re}\,\langle t_1, t_2 \rangle^2 + \langle t_1, t_2 \rangle \langle t_2, t_1 \rangle}{\langle t_1, t_1 \rangle \langle t_2, t_2 \rangle}.$$

Writting $a_i := \langle t_i, t_i \rangle$ and $b := \langle t_1, t_2 \rangle$, we have $a_i \in \{1, -1\}$ and $\mathrm{Re}\, b = 0$, because t_1, t_2 is an orthonormal basis for W with respect to g. Thus,

$$K(W) = 1 + \frac{bb^* - 2\mathrm{Re}\, b^2}{a_1 a_2} = 1 - \frac{3b^2}{a_1 a_2}$$

where the last equality holds for the algebras under consideration.

When $\mathbb{F} = \mathbb{R}$, the curvature is constant and equals 1. For the dual numbers, the same happens because $b^2 = 0$. Note that the tangent planes to points in the dual numbers projective line do not possess a non-degenerate real two-dimensional subspace plane W. Thus, sectional curvatures is not defined in this case (that is not what happens in higher dimensions). The cases where $\mathbb{F} = \mathbb{C}, \mathbb{H}$ are detailed at [AGr, Section 4.6].

Now we analyse the case where $\mathbb{F}$ is $\mathbb{C}_s$ or $\mathbb{H}_s$. If V is two-dimensional, then $T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{F}}(V)$ has dimension one as an $\mathbb{F}$-module. Therefore, $t_1 = kt_2$ for some $k \in \mathbb{F}^{\times}$. Note that $b = ka_2$ and $a_1 = -k^2 a_2$. Therefore,

$$K(W) = 1 - \frac{3b^2}{a_1 a_2} = 4.$$

If V has $\mathbb{F}$-dimension higher than 2, then $K(W)$ can be any real number. Indeed, consider the tangent vectors e_1, e_2 at $\boldsymbol{p}$ such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$ and $\langle e_1, e_2\rangle = 0$, which exist because $T_{\boldsymbol{p}}\mathbb{P}_{\mathbb{F}}(V)$ is at least two dimensional as an $\mathbb{F}$-module. For $t_1 = e_1$ and $t_2 = \sinh(\theta) j e_1 + \cosh(\theta) e_2$, we obtain $K(W) = 1 - 3\sinh(\theta)^2$. For $t_1 = e_1$ and $t_2 = \cosh(\theta) j e_1 + \sinh(\theta) e_2$, we obtain $K(W) = 1 + 3\cosh(\theta)^2$. Finally, for $t_1 = e_1$ and $t_2 = j(\cos(\theta) e_1 + \sin(\theta) e_2)$, we have $K(W) = 1 + 3\cos(\theta)^2$.

Summarizing, for the algebras other than the division ones, we have: for dual numbers, the curvature is always one; for split algebras, the curvature equals 4 in the projective line case and can be any number otherwise.

Note that, had we taken the Hermitian metric with a negative sign, then the curvature formula would have its sign changed as well. For the hyperbolic models, for instance, we take the negative sign (Example 28). Thus, the real hyperbolic spaces have curvature -1. For the complex and quaternionic hyperbolic spaces the obtained curvature is -4 for one dimensional spaces and, for higher dimension, the curvature lies in $[-4, -1]$.

5 Spaces of oriented geodesics on Euclidean, elliptical and hyperbolic two-dimensional geometries.

In the following examples, we consider the algebras $\mathbb{F} = \mathbb{C}$, $\mathbb{D}$, $\mathbb{C}_s$ and the $\mathbb{F}$-module $\mathbb{F}^2$ endowed with the Hermitian form $\langle u, v\rangle = u_1 v_1^* + u_2 v_2^*$. We will see that the regular components of the spaces $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{D}}$ and $\mathbb{P}^1_{\mathbb{C}_s}$ are the spaces of geodesics of the round sphere, Euclidean plane and hyperbolic plane, respectively.

5.1. Points in the complex projective line = oriented geodesics in $\mathbb{S}^2$. In these settings, the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ is a constant curvature sphere. So, a given point $\boldsymbol{p} \in \mathbb{P}^1_{\mathbb{C}}$ determines a unique equator (the geodesic equidistant from $\boldsymbol{p}$ and its antipodal point). This geodesic is oriented in the counterclockwise direction as seen from $\boldsymbol{p}$. The Hermitian metric measures the oriented angle between two oriented geodesics and at an intersection point.

5.2. Points in the dual number projective line = oriented geodesics in $\mathbb{E}^2$. Identify $\mathbb{E}^2$ with the complex plane $\mathbb{C}$. The cylinder $\mathbb{S}^1 \times \mathbb{R}$ can be identified with $T\mathbb{S}^1$ via the map $(e, s) \mapsto (e, sie)$, where $\mathbb{S}^1$ is taken as the circle of unit complex numbers. An oriented line in the Euclidean plane $\mathbb{E}^2$ is given by $c_{e,s}(t) := sie + et$, where e is a unit vector and s is a real number. Thus, we have a one-to-one correspondence between points in the tangent bundle $T\mathbb{S}^1$ and the space of oriented lines in the plane which is given by $\mathbb{S}^1 \times \mathbb{R} \ni (e, s) \mapsto c_{e,s}$.

Fix a point $p = a + ib$ in the plane. The lines passing through p are $c(e(\theta), s(\theta))$, where $e(\theta) = \exp(i\theta)$ and $s(\theta) := -a\sin(\theta) + b\cos(\theta)$. Thus, taking the coordinates $(x + iy, s) \in \mathbb{C} \times \mathbb{R}$, where $\mathbb{C} \times \mathbb{R}$ is the ambient space of the cylinder $\mathbb{S}^1 \times \mathbb{R}$, the previously described family of oriented lines is obtained by intersecting the linear subspace $s = bx - ay$ with the cylinder. Thus, families of oriented lines sharing a fixed point p correspond to planes that cut the cylinder in an ellipse. Each of the remaining planes cut the cylinder in two components (lines); one of them corresponds to a family of oriented geodesics in $\mathbb{E}^2$ and, the other, to the same family of geodesics in $\mathbb{E}^2$ with the opposite orientation.

The described curves in $T\mathbb{S}^1$ are geodesics of the following metric on the cylinder:

$$g_{T\mathbb{S}^1}((u_1, s_1), (u_2, s_2)) := \operatorname{Re} u_1 \overline{u_2}.$$

Thus, the distance between two points on $T\mathbb{S}^1$ is the angle between the corresponding oriented lines.

Now, we just have to identify the described cylinder with the dual number projective line. Take $V = \mathbb{C} + \varepsilon\,\mathbb{C}$ as a $\mathbb{D}$-module. The diffeomorphism $f : \mathbb{P}^1_{\mathbb{D}} \to \mathbb{S}^1 \times \mathbb{R}$ given by $[e + k\varepsilon ie] \mapsto (e^2, 2k)$ is the desired isometry (up to rescaling the metrics), where e is a unit complex number. Indeed,

$df^* g_{T\mathbb{S}^1} = 4g_{\mathbb{P}^1_{\mathbb{D}}}$. Therefore, $\mathbb{P}^1_{\mathbb{D}}$ is the space of all oriented Euclidean lines. On the dual numbers projective line, a positive geodesic represents a family of oriented lines rotating around a common point in the Euclidean plane while a null geodesic represents a one parameter family of parallels lines (see Figure 2(b)).

5.3. Regular points of the split-complex projective line = oriented geodesics in $\mathbb{H}^2_{\mathbb{R}}$. As we discussed in Example 45, the projective de Sitter space dS^2 is the space of all non-oriented hyperbolic geodesics. Topologically, dS^2 is an open Möbius strip. The regular part of the split-complex projective line is a cylinder (see Figure 2(a); the regular region is the cylinder obtained from removing the singular circle, the dashed curve in purple, from the torus) and it is an isometric double cover of dS^2 (up to rescaling the metrics). Furthermore, it constitutes the space of oriented geodesics of the hyperbolic plane.

The cross product on $\mathbb{R}^3$ endowed with the canonical Minkowski metric $-++$ is given by $e_1 \times e_2 = e_3$, $e_2 \times e_3 = -e_1$, $e_3 \times e_1 = e_2$, where e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$. Equivalently, the cross product is defined by the formula $\langle u \times v, w\rangle e_1 \wedge e_2 \wedge e_3 = u \wedge v \wedge w$. Given two points $\boldsymbol{p}, \boldsymbol{q} \in \partial\mathbb{H}^2_{\mathbb{R}}$, the vector $p \times q$ represents the point in dS^2 corresponding to the geodesic G of the hyperbolic plane connecting $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e., $G = \mathbb{P}_{\mathbb{R}}\big((p \times q)^{\perp}\big)$.

Taker $V = \mathbb{C}^2_s$ with the Hermitian form defined in Example 26. For each $[(a,a') : (b,b')] \in R(V)$ we have the points $A(a,b) := [a^2+b^2 : a^2-b^2 : 2ab]$ and $B(a',b') := [a'^2+b'^2 : -a'^2+b'^2 : -2a'b']$ on $\partial\mathbb{H}^2_{\mathbb{R}}$ and, thus, the oriented geodesic of the hyperbolic plane connecting $B(a',b')$ to $A(a,b)$. Observe that the condition for $[(a,a') : (b,b')]$ to be in $R(V)$ is $aa'+bb' \neq 0$, and the same condition guarantees $A(a,b) \neq B(a',b')$ in $\mathbb{P}^2_{\mathbb{R}}$. Therefore, we obtain a correspondence between $R(V)$ and oriented geodesics in $\mathbb{H}^2_{\mathbb{R}}$. The point $A(a,b) \times B(a',b')$ in dS^2 corresponds to the non-oriented geodesic containing $A(a,b)$ and $B(a',b')$ (see Example 45).

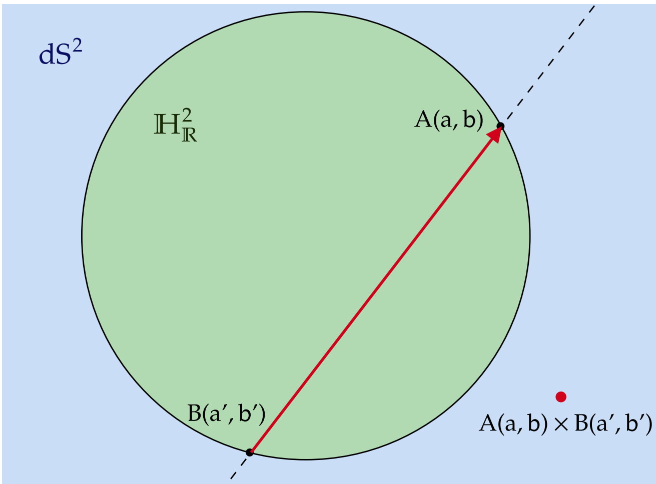


Figure 4: Points $A(a,b)$, $B(a',b')$ and $A(a,b) \times B(a',b')$.

The double cover $f : R(V) \to \mathrm{dS}^2$ of interest is given by $[(a,a') : (b,b')] \mapsto A(a,b) \times B(a',b')$, that is,

$$f([(a,a') : (b,b')]) = [ab' - a'b : ab' + a'b : -aa' + bb'].$$

A direct computation, sketched bellow, shows that $df^* g_{\mathrm{dS}^2} = -4g_{\mathbb{P}^1_{\mathbb{C}_s}}$. Thus, up to rescaling the metrics, f is an isometric 2 to 1 cover map.

In order to verify that $df^* g_{\mathrm{dS}^2} = -4g_{\mathbb{P}^1_{\mathbb{C}_s}}$, consider a point $\boldsymbol{p} \in R(V)$ with representative $p = ((a,a'),(b,b'))$. We assume that $aa' + bb' = 1$. The tangent vectors $t_1 = \langle \cdot, p\rangle v$ and $t_2 = \langle \cdot, p\rangle(1,-1)v$ at $\boldsymbol{p}$, where $v = ((b',b),(-a',-a))$, satisfy $g_{\mathbb{P}^1_{\mathbb{C}_s}}(t_1,t_1) = 1$, $g_{\mathbb{P}^1_{\mathbb{C}_s}}(t_2,t_2) = -1$, and $g_{\mathbb{P}^1_{\mathbb{C}_s}}(t_1,t_2) = 0$. The curves $\gamma_1(\theta) = [p + \theta v]$ and $\gamma_2(\theta) = [p + \theta(1,-1)v]$ have respective velocities t_1 and t_2 at $\theta = 0$.

Now we compute $df(t_i)$ using the curve γ_i. We have $f \circ \gamma_1(\theta) = [q + \theta w_1 + o(\theta)]$, where

$$q = (ab' - a'b, ab' + a'b, -aa' + bb'), \quad \text{and} \quad w_1 = (-a^2 - b^2 + a'^2 + b'^2, -a^2 + b^2 - a'^2 + b'^2),$$

and, by Lemma 43, we obtain $df(t_1) = \langle \cdot, q\rangle w_1$. Similarly, $f \circ \gamma_2(\theta) = [q + \theta w_2 + o(\theta)]$ and $df(t_2) = \langle \cdot, q\rangle w_2$, where

$$w_2 = (a^2 + b^2 + a'^2 + b'^2, a^2 - b^2 - a'^2 + b'^2, 2ab - 2a'b').$$

The identity $df^* g_{\mathrm{dS}^2} = -4g_{\mathbb{P}^1_{\mathbb{C}_s}}$ follows from the fact that

$$df^* g^2_{\mathrm{dS}^2}(t_1, t_1) = -4, \quad df^* g_{\mathrm{dS}^2}(t_2, t_2) = 4, \quad \text{and} \quad df^* g_{\mathrm{dS}^2}(t_1, t_2) = 0.$$

Considering the split-quaternions $\mathbb{H}_s$ and the module $\mathbb{H}_s \times \mathbb{H}_s$, we obtain an ambient space for a transition between the three described geometries. Indeed, with the Hermitian form $\langle u, v\rangle = u_1 v_1^* + u_2 v_2^*$ on $\mathbb{H}_s \times \mathbb{H}_s$, the maps in the Example 21 are isometric embeddings (when restricted to the regular region). Thus, there exists a natural transition between the regular regions of the three discussed 2-dimensional geometries inside the split-quaternionic projective line.

Remark 46. Taking $\mathbb{H}_s \times \mathbb{H}_s$ with the Hermitian form $\langle u, v\rangle = -u_1 v_1^* + u_2 v_2^*$, we obtain the hyperbolic spaces $\mathbb{H}^1_{\mathbb{C}}$, $\mathbb{H}^1_{\mathbb{D}}$ and $\mathbb{H}^1_{\mathbb{C}_s}$, where $\mathbb{H}^1_{\mathbb{F}}$ is formed by the regular points $\boldsymbol{p}$ admiting a representative p satisfying $\langle p, p\rangle < 0$. As above, we can geometrically transition between this geometries inside the split-quaternionic projective line. A transition between hyperbolic geometries is also studied in [Tre] via a more abstract route. In contrast, here we use that these geometries share a common ambient space.

6 Bidisc geometry

The bidisc is the Riemannian manifold $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$, the product of two Poincaré discs, with the canonical Riemannian product metric. The metric we take in the Poincaré disc $\mathbb{H}^1_{\mathbb{C}}$ is the one defined in Example 28. In this section, we want to show how the bidisc appears as part of a projective line. For that purpose, we use an algebra not previously considered.

Let $\mathbb{F}$ be the real algebra $\mathbb{C} \times \mathbb{C}$ with the involution $(a, b)^* = (\overline{a}, \overline{b})$. The algebra of self-adjoint elements of $\mathbb{F}$ in this case is $\mathbb{R} \times \mathbb{R}$. The projective line $\mathbb{P}^1_{\mathbb{F}}$ is diffeomorphic to the product of two Riemann spheres $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Indeed, the diffeomorphism is given by the map $\Lambda : \mathbb{P}^1_{\mathbb{F}} \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $[(a_1, b_1), (a_2, b_2)] \mapsto ([a_1, a_2], [b_1, b_2])$.

Consider in $\mathbb{F}^2$ the $\mathbb{F}$-valued Hermitian form

$$\langle u, v\rangle = -u_1 v_1^* + u_2 v_2^*.$$

For $u = ((a_1, b_1), (a_2, b_2))$,

$$\langle u, u\rangle = (-|a_1|^2 + |a_2|^2, -|b_1|^2 + |b_2|^2) \in \mathbb{R} \times \mathbb{R}.$$

We define the regular region R of $\mathbb{P}^1_{\mathbb{F}}$ as the set of all $\boldsymbol{u}$ such that $\langle u, u\rangle$ is a unit, which means that both coordinates of $\langle u, u\rangle$ are non-zero real numbers. Observe that R is the union of four disjoint 4-balls. Indeed, these balls are

$$B_{++} = \{\boldsymbol{u} \in \mathbb{P}^1_{\mathbb{F}} \mid \langle u, u\rangle \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}\},$$

$$B_{-+} = \{\boldsymbol{u} \in \mathbb{P}^1_{\mathbb{F}} \mid \langle u, u\rangle \in \mathbb{R}_{<0} \times \mathbb{R}_{>0}\},$$
$$B_{+-} = \{\boldsymbol{u} \in \mathbb{P}^1_{\mathbb{F}} \mid \langle u, u\rangle \in \mathbb{R}_{>0} \times \mathbb{R}_{<0}\},$$
$$B_{--} = \{\boldsymbol{u} \in \mathbb{P}^1_{\mathbb{F}} \mid \langle u, u\rangle \in \mathbb{R}_{<0} \times \mathbb{R}_{<0}\},$$

and $R = B_{++} \sqcup B_{+-} \sqcup B_{-+} \sqcup B_{--}$. We denote B_{--} by $\mathbb{B}^4$.

Let $\lambda : \mathbb{F}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ be given by the formula $((a_1, b_1), (a_2, b_2)) \mapsto ((a_1, a_2), (b_1, b_2))$. If π_1, π_2 stand respectively for the projections $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ in the first and second coordinates, we define $\lambda_1 = \pi_1 \circ \lambda$ and $\lambda_2 = \pi_2 \circ \lambda$.

Consider on $\mathbb{C}^2$ the $\mathbb{C}$-valued Hermitian form

$$\langle (a_1, a_2), (a'_1, a'_2)\rangle := -a_1 \overline{a'_1} + a_2 \overline{a'_2}.$$

For $u, u' \in \mathbb{F}^2$ we have
$$\langle u, u'\rangle = (\langle \lambda_1(u), \lambda_1(u')\rangle, \langle \lambda_2(u), \lambda_2(u')\rangle).$$

The complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$ is formed by $\boldsymbol{p} \in \mathbb{P}^1_{\mathbb{C}}$ such that $\langle p, p\rangle < 0$. Therefore, the map $\Lambda : \mathbb{P}^1_{\mathbb{F}} \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ provides a diffeomorphism between $\mathbb{B}^4$ and $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$. The ball $\mathbb{B}^4$ will be our projective model for the bidisc.

A unitary operator $T : \mathbb{F}^2 \to \mathbb{F}^2$ is a $\mathbb{F}$-linear map satisfying $\langle T(u), T(v)\rangle = \langle u, v\rangle$. Writting T as the matrix
$$\begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{pmatrix}$$
we obtain that T is unitary if, and only if, the matrices (a_{ij}) and (b_{ij}) are unitary as well. Thus we have the map $\mathrm{U}(\mathbb{F}^2, \langle\cdot,\cdot\rangle) \to \mathrm{U}(1,1)\times\mathrm{U}(1,1)$, $(a_{ij}, b_{ij}) \mapsto \big((a_{ij}), (b_{ij})\big)$ which is a group isomorphism. The action of unitary transformations on $\mathbb{B}^4$ correspond to the action of $\mathrm{U}(1,1)\times\mathrm{U}(1,1)$ on $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$. If we restrict ourselves to determinant 1 matrices, the above isomorphism holds for SU matrices as well: $\mathrm{SU}(\mathbb{F}^2, \langle\cdot,\cdot\rangle) \simeq \mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$. The same goes for projective unitary group: $\mathrm{PU}(\mathbb{F}^2, \langle\cdot,\cdot\rangle) \simeq \mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$.

Now we consider the Hermitian metric on $\mathbb{H}^1_{\mathbb{C}}$ introduced in Example 28. In a similar fashion, we consider the $\mathbb{F}$-valued Hermitian metric
$$\langle t, t'\rangle = -\frac{\langle t(p), t'(p)\rangle}{\langle p, p\rangle}$$
for $t, t' \in T_{\boldsymbol{p}}\mathbb{P}^1_{\mathbb{F}}$, where $\boldsymbol{p}$ is a regular point. Writing the tangent vector $t \in T_{\boldsymbol{p}}\mathbb{P}^1_{\mathbb{F}}$ as
$$t = \frac{\langle\cdot, p\rangle}{\langle p, p\rangle} t(p)$$
we obtain that its image in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is (s_1, s_2), where
$$s_j := \frac{\langle \cdot, \lambda_j(p)\rangle}{\langle \lambda_j(p), \lambda_j(p)\rangle}\lambda_j(t(p)).$$
Therefore, the Hermitian metric on the $\mathbb{F}$ projective line corresponds to the pair of Hermitian metrics arising from the two Riemann spheres. More precisely
$$\langle t, t'\rangle = (\langle s_1, s_2\rangle, \langle s_1', s_2'\rangle).$$

From this $\mathbb{F}$-Hermitian metric, we obtain a Riemannian metric by taking the real part of the $\mathbb{F}$-value Hermitian metric, which gives an element of $\mathbb{R}\times\mathbb{R}$, and then summing the obtained coordinates. Let us denote this metric by $g_{\mathbb{B}^4}$. Therefore, the map $\Lambda : \mathbb{B}^4 \to \mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$ is an isometry:
$$g_{\mathbb{B}^4}(t, t') = g_{\mathbb{H}^1_{\mathbb{C}}}(s_1, s_2) + g_{\mathbb{H}^1_{\mathbb{C}}}(s_1', s_2').$$

Finally, the group of orientation preserving isometries of the bidisc $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^1_{\mathbb{C}}$ is generated by $\mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$ and the map that swaps the coordinates of the two hyperbolic discs $\tau : (\boldsymbol{p}, \boldsymbol{q}) \mapsto (\boldsymbol{q}, \boldsymbol{p})$. In $\mathbb{B}^4$, this map τ is given by $\tau : [(a_1, b_1) : (a_2, b_2)] \mapsto [(b_1, a_1) : (b_2, a_2)]$. Hence, $\mathbb{B}^4$ is a projective model for the bidisc and the unitary group of $(\mathbb{F}^2, \langle\cdot,\cdot\rangle)$ together with τ provides the orientation preserving isometries. Furthermore, there exists in $\mathbb{B}^4$ an orientation preserving isometry sending the pair of points $\boldsymbol{u}, \boldsymbol{u}'$ to the pair of points $\boldsymbol{v}, \boldsymbol{v}'$ if, and only if, either $\mathrm{ta}(\boldsymbol{u}, \boldsymbol{u}') = \mathrm{ta}(\boldsymbol{v}, \boldsymbol{v}')$ or $\mathrm{ta}(\boldsymbol{u}, \boldsymbol{u}') = \mathrm{ta}(\tau\boldsymbol{v}, \tau\boldsymbol{v}')$, where the tance here is $\mathbb{R}\times\mathbb{R}$-valued, obtained from the $\mathbb{F}$-valued Hermitian form defined on $\mathbb{F}^2$.

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