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Instituto de Ciências Matemáticas e de Computação

Asymptotic dynamics of wave equations on compact Riemannian manifolds: sharp localized damping and supercritical forcing

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Paulo Nicanor Seminario Huertas

Dinâmica assintótica de equações de onda sobre variedades Riemannianas compactas: dissipação localizada ótima e forças supercríticas

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Ma To Fu

USP – São Carlos
Janeiro de 2019

PAULO NICANOR SEMINARIO HUERTAS

ASYMPTOTIC DYNAMICS OF WAVE EQUATIONS ON COMPACT

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação - ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências: Matemática.

Aprovado em 07 de fevereiro de 2019.

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Dedico este trabajo a los que siempre estuvieron conmigo, día a día, codo a codo.

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"Sé una gota en el jardín
sigue el curso de agua
que nos lleve donde nunca fuimos.
Por senderos que se bifurcan
por mundos paralelos."

(Gustavo Cerati)

RESUMO

SEMINARIO-HUERTAS, P. N. **Dinâmica assintótica de equações de onda sobre variedades Riemannianas compactas: dissipação localizada ótima e forças supercríticas**. 2019. 107 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

A presente tese é dedicada ao estudo da dinâmica a longo prazo de equações de ondas definidas sobre variedades Riemannianas compactas, com bordo, que possuam dissipação localizada e forças com crescimento Sobolev supercrítico. O objetivo principal é construir regiões de dissipação com medida total (interior e fronteira) arbitrariamente pequena, de forma a garantir a existência de atratores globais regulares de dimensão finita. Entre outros resultados, provaremos uma versão supercrítica de um teorema de continuação única de Triggiani and Yao (2002).

Palavras-chave: Equações da onda Riemannianas, amortecimento não linear localizado, atratores globais.

ABSTRACT

SEMINARIO-HUERTAS, P. N. **Asymptotic dynamics of wave equations on compact Riemannian manifolds: sharp localized damping and supercritical forcing**. 2019. 107 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

The present thesis is concerned with long-time dynamics of wave equations, defined on compact Riemannian manifolds, with boundary, and featuring localized damping and nonlinear forcing terms with supercritical Sobolev growth. The main objective is to construct optimal damping regions with arbitrarily small summed interior/boundary measure that imply the existence of a regular finite-dimensional global attractor. To this end, among other results, we prove a supercritical extension of a unique continuation theorem of Triggiani and Yao (2002).

Keywords: Riemannian wave equations, nonlinear localized damping, global attractors.

CONTENTS

CHAPTER

1

INTRODUCTION

In the field of partial differential equations, the asymptotic analysis of the dynamics of the evolution system is always a constant subject of study that generates a great interest within the mathematical community, in particular for systems with localized dissipation, since they allow us to model more realistic problems

Studying equations with localized dissipation involves a geometric study of the spatial domain, which often leads to a deeper understanding on the geometry of these domains.

Therefore, this work has both a geometric approach and an functional approach, complemented with an applications in Chapter 5 and 6.

1.1 Framework

An important question in differential equations has always been regarding the study of energy decay, stabilization and controllability, where damping plays an important role. In the case of wave equations, for example, Ralston (RALSTON, 1969) and Russell (RUSSELL, 1971a; RUSSELL, 1971b) are pioneers in studying the stability and controllability of these equations. Rauch (RAUCH, 1976) studied the action of damping on limited domains based on what is proposed in (RAUCH; TAYLOR; PHILLIPS, 1974) by showing the exponential decay of energy.

In 1975, Rauch and Taylor (RAUCH; TAYLOR, 1975a; RAUCH; TAYLOR, 1975b) developed an idea in relation to the decay of energy with respect to the location of the damping on a certain region ω of the domain in such a way that it suffices to locate the dissipation on said region in order to achieve the exponential decay of the energy. Finding minor region of dissipation has several of applications in science and technology, from constructions or analysis of seismic waves, among others. In order to construct such a the region ω they studied the effects of the geometrical optics on the whole domain as proposed in (RALSTON, 1969), defining a *G*eometric Control Condition (GCC for short) that was later generalized in the context of Riemannian manifolds by Bardos et al. (BARDOS; LEBEAU; RAUCH, 1992) from the study of the geodesics.

This idea of being able to locate the dissipation had repercussions in different areas of the study of the wave equations, for example, applications are obtained in the exact controllability on the boundary of the region (e.g. (BURQ; GÉRARD, 1997; ALABAU-BOUSSOUIRA, 2005; MARTINEZ, 1999)), in the localized dissipations in exterior domains (e.g. (NAKAO, 1996b; NAKAO, 1996a; NAKAO, 2005; BAE; NAKAO, 2005)) or in the study of attractors and stability of waves with localized damping exposed to subcritical, critical or supercritical forces (e.g. (DEHMAN; LEBEAU; ZUAZUA, 2003; FEIREISL; ZUAZUA, 1993; JOLY; LAURENT, 2013; CHUESHOV; LASIECKA; TOUNDYKOV, 2008; CHUESHOV; LASIECKA; TOUNDYKOV, 2009; CAVALCANTI *et al.*, 2010; CAVALCANTI *et al.*, 2009)).

In the Euclidean case, an example of the construction of ω satisfying the condition (GCC), used in several of the aforementioned references (see Figure 1) for a wave equation with damping located on a domain $M \subset \mathbb{R}^3$, given by

$$\partial_t^2 u - \Delta u + \chi_\omega \partial_t u = 0 \text{ in } M \times (0,\infty),$$

is to set a $x_0 \in \mathbb{R}^3 \setminus M$, i.e. an *o*bserver outside the M such that ω is a neighborhood of the closure is $\mathbb{R}^3$ of the set

$$\{x \in \partial M \;:\; (x - x_0)\cdot n(x) \geq 0\},$$

where $n(x)$ represents the outward normal unit vector in $x \in M$.

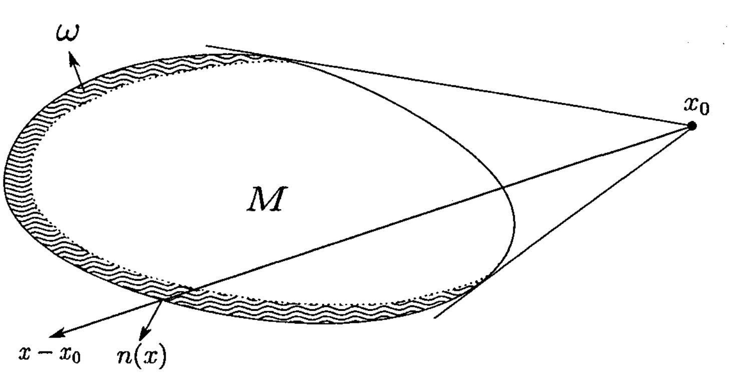

Figure 1 – Because the distribution of ω over the boundary of M, the measure of ω with respect to M can be as small as desired.

The concept of the *o*bserver point leads to the study of different properties generated by ω. Different authors (cf. (RAUCH; TAYLOR, 1975a; RAUCH; TAYLOR, 1975b; BAR-

DOS; LEBEAU; RAUCH, 1992; ENRIKE, 1990; ZUAZUA, 1991; TRIGGIANI *et al.*, 2002; LASIECKA; TRIGGIANI; ZHANG, 2000) among others) study the effects of said location, such as the properties of *o*bservability related to the boundary conditions on the equation, or the *u*nique continuation property, that is, if the solution is nukk on ω then it is null on the whole. These properties are fundamental in the study of the decay of energy, as in the theory of attractors (see, e.g. (CAVALCANTI *et al.*, 2010; CAVALCANTI *et al.*, 2009; JOLY; LAURENT, 2013; CHUESHOV; LASIECKA; TOUNDYKOV, 2008; CHUESHOV; LASIECKA; TOUNDYKOV, 2009)), which are mainly based on the Carleman estimates (see, e.g. (LASIECKA; TRIGGIANI; ZHANG, 2000; TRIGGIANI *et al.*, 2002; YAO, 2011)) or on Hörmander's results (see, e.g. (HÖRMANDER, 1997; ??; TATARU, 1999; JOLY; LAURENT, 2013; ROBBIANO; ZUILY, 1998)).

On the other hand, regarding the *o*ptimization of the location ω, Cavalcanti et al. (CAVALCANTI *et al.*, 2010; CAVALCANTI *et al.*, 2009) study a *s*harp localization, in the sense of the control of the measure ω, not only with respect to the domain measure, also with respect to the measure of the boundary, where the construction of the dissipation region involves the boundary and the interior of the set, always taking into account the property of being (GCC) (see Figure 2).

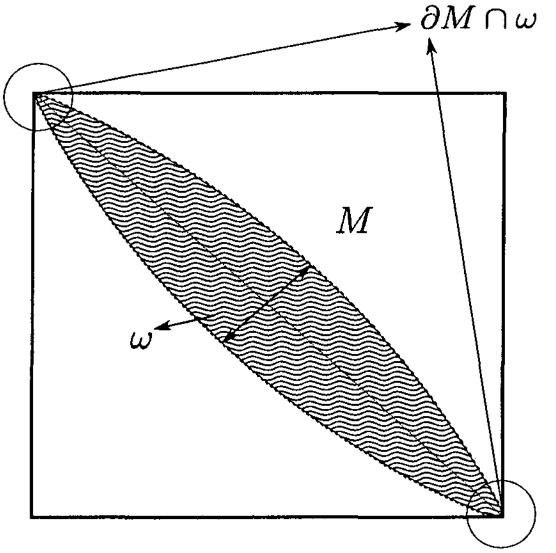

Figure 2 – M represents the whole square. It is easy to see that there is full control in the area that occupies ω in relation to M as well as, the measure of $\omega \cap \partial M$.

It is important to note that the construction of the sharp dissipation regions presented in (CAVALCANTI *et al.*, 2010; CAVALCANTI *et al.*, 2009) not only depends on the geometry of the manifold, but also on the equation to be studied. This is a disadvantage when we want to use the construction for other systems.

Regarding the study of asymptotic dynamics for the Riemannian wave equation with nonlinear localized damping and nonlinear forces, Chueshov et al. (CHUESHOV; LASIECKA; TOUNDYKOV, 2009) prove the existence of a smooth global attractor with finite fractal dimension, being characterized by the unstable manifolds of stationary points, from the method of contractive functions and proof a new observability inequality from the use of Carleman estimates. It should be noted that in this case the nonlinearity has Sobolev's critical growth.

For the supercritical case with linear localized damping, Joly and Laurent (JOLY; LAURENT, 2013) show the gradient structure of the system and the existence of a global attractor for it, by means of the Strichartz estimates (cf. (STRICHARTZ *et al.*, 1977; KAPITANSKI, 1995; KAPITANSKI, 1989; IVANOVICI, ; GINIBRE; VELO, 1985; GINIBRE; VELO, 1989)) and the works of Hörmander. It is important to observe that the optimal regularity and the finite fractal dimension are not possible to prove mainly by the observability inequality, since the initial energy is controlled from the kinetic energy of the system, this does not allow to possess the sufficient regularity in the control, which makes the authors use the decomposition method of the semigroup in an exponentially decaying part and a compact part.

In (CHUESHOV; LASIECKA; TOUNDYKOV, 2009; JOLY; LAURENT, 2013), the location of the damping is in the classical sense, that is, it is not sharp.

Thus, after a review in the literature, the main objective is to study the effects of a new observability inequality and unique continuity property in the existence of a smooth global attractor with a finite fractal dimension and caracaterized by the unstable manifolds of stationary points, for two waves equations: a Riemannian wave equation with nonlinear localized damping and critical forces and a Riemannian wave equation with linear localized damping and supercritical forces; where the damping region is sharp in the sense of (CAVALCANTI *et al.*, 2010; CAVALCANTI *et al.*, 2009).

To this end, we will divide the present work into six Chapters, the first of which is intended to describe the previous notions for the subsequent analysis of the results, highlighting the observability inequality and unique continuation property for finite collection overlapping subdomains from the study of Carleman estimates as shown in (TRIGGIANI *et al.*, 2002).

Chapter 2 is intended to proof the construction of the sharp admissible damping regions, which is distinctly geometric and independent of the equation. While in Chapter 3 the geometric consequences of this construction are shown, highlighting the decomposition in overlapping sets of the spatial domain.

One of the important results shown in this work is described in Chapter 4, showing new observability inequality and unique continuity property, that when applied in two wave equations: a Riemannian wave equation with nonlinear localized damping and critical forces and a Riemannian wave equation with linear localized damping and supercritical forces; is achieved prove the quasi-stability of the systems. This is detailed in the Chapter 5 and 6, respectively.

1.2 Setting

Given that the most important results of this work can be divided into three parts, we will divide the hypotheses and considerations for each of the objectives into three sections: the first of these being the proof of an inequality of observability and unique continuity property on a $N-$dimensional Riemannian manifold; the second is with respect to the prove of a global attractor for a *three*−dimensional Riemannian wave equation with nonlinear localized damping and critical forces; while the third one is with respect to the study of the properties of a global attractor for a *three*−dimensional Riemannian wave equation with linear localized damping and supercritical forces. Note that our results and their proofs should easily extend to any space $N > 3$.

1.2.1 About the observability inequality and unique continuation property

Let $(M, \mathbf{g})$ be a $N-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Let us consider the wave problem with $T > 0$ sufficiently large, given by

$$\begin{cases} \partial_t^2 w - \Delta w = p_0 w + p_1 \partial_t w \text{ in } M \times (0,T], \\ w = 0 \text{ on } \partial M \times (0,T], \end{cases} \tag{1.1}$$

where Δ represents the Laplace Beltrami operator on M and $p_0, p_1 : M \times (0,T] \to \mathbb{R}$ such that

$$p_1 \in L^\infty(0,T;L^\infty(M)), \;\; p_0 \in L^2(0,T;L^2(M)). \tag{1.2}$$

We assume that:

1. For all $z \in H^{2,2}(M \times (0,T])$

$$p_0 z \in H^{1,1}(M \times (0,T]), \;\; \partial_t(p_0 z) \in L^2(0,T;L^2(M)). \tag{1.3}$$

2. There exists $C_{p_0T} > 0$ such that

$$\|p_0 z\|_{L^2(0,T;L^2(M))} \leq C_{p_0T} \|z\|_{L^2(0,T;H^1(M))}, \;\; \forall z \in H^{2,2}(M \times (0,T]). \tag{1.4}$$

Taking into account that

$$H^{1,1}(M \times (0,T]) := L^2(0,T;H^1(M)) \cap H^1(0,T;L^2(M)),$$

$$H^{2,2}(M \times (0,T]) := L^2(0,T;H^2(M)) \cap H^2(0,T;L^2(M)).$$

1.2.2 About the applications to wave equations with critical forces

Let $(M,\mathbf{g})$ be a *three*−dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Let us consider the damped wave problem with localized damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)g(\partial_t u) + f(u) = h(x) \text{ in } M \times (0,\infty), \\ u = 0 \text{ on } \partial M \times (0,\infty), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ x \in M, \end{cases} \tag{1.5}$$

where Δ represents the Laplace Beltrami operator on M.

We assume that:

1. **Respect to damping:**

   Existen constantes $m_1, m_2 > 0$ such that

$$g \in C^1(\mathbb{R}), \quad g(0) = 0 \quad \text{and} \quad m_1 \leq g'(z) \leq m_2, \quad \forall z \in \mathbb{R}, \tag{1.6}$$

$$a \in L^\infty(M) \text{ such that } a(x) \geq 0 \text{ for all } x \in M. \tag{1.7}$$

2. **Respect to forces:**

   There exists a constants $C_f > 0$ such that

$$f \in C^1(\mathbb{R}),\ f(0) = 0, \tag{1.8}$$

$$|f(z)| \leq C_f(1+|z|^3),\ |f'(z)| \leq C_f(1+|z|^2),\ |f''(z)| \leq C_f(1+|z|), \tag{1.9}$$

$$f(z)z \geq F(z) - \frac{\lambda_1}{2}(1-\nu)|u|^2 - m_f,\ F(u) \geq -\frac{\lambda_1}{2}(1-\mu)|u|^2 - m_f, \tag{1.10}$$

   for some $\nu > 0$ and $m_f \geq 0$. Here $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet operator $-\Delta$ and $F(z) = \int_0^z f(y)dy$.

   The external force h is time-independent and

$$h \in L^2(M). \tag{1.11}$$

Later we will add a hypothesis of location with respect to damping, that is to say that there will be a constant $a_0 > 0$ such that for a certain open subset $\omega \subset M$ is fulfilled

$$a \geq a_0 > 0, \text{ a.e. in } \omega, \tag{1.12}$$

This hypothesis is fundamental for the proof of the existence of a global attractor in Chapter 5.

1.2.3 About the applications to wave equations with supercritical forces

Let $(M,\mathbf{g})$ be a *three*−dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Let us consider the damped wave problem with localized damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u + f(x,u) = 0 \text{ in } M\times(0,\infty), \\ u = 0 \text{ on } \partial M\times(0,\infty), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ x\in M, \end{cases} \tag{1.13}$$

where Δ represents the Laplace Beltrami operator on M.

We assume that:

1. **Respect to damping:**

$$a\in L^\infty(M) \text{ such that } a(x)\geq 0 \text{ for all } x\in M. \tag{1.14}$$

There exist and open set $\omega\subset M$, $a_0>0$, $x_0\in M$ and $R\geq 0$ such that

$$\forall x\in\omega,\ \ a(x)\geq a_0>0,\ \ M\setminus B(x_0,R)\subset\omega. \tag{1.15}$$

$$\omega \text{ satisfies the geometric control condition.} \tag{1.16}$$

The definition of ω *satisfying the geometric control condition* is discussed in detail in Section 3.1.

2. **Respect to forces:**

There exists a constants $C_f>0, p\in[3,5)$ and $R>0$ such that all $(x,u)\in M\times\mathbb{R}$

$$f\in C^\infty(M\times\mathbb{R},\mathbb{R}),\ \ f(0)=0, \tag{1.17}$$

$$|f(x,u)|\leq C_f(1+|u|^p),\ \ |\partial_x f(x,u)|\leq C_f(1+|u|^p),\ \ |\partial_u f(x,u)|\leq C(1+|u|)^{p-1} \tag{1.18}$$

$$(x\notin B(x_0,R) \text{ or } |u|\geq R)\Longrightarrow f(x,u)u\geq 0. \tag{1.19}$$

where x_0 denotes a fixed point of the manifold.

1.3 Goals and new results

As commented when reviewing the previous literature that involves the work, the main objective is to prove two great results:

The first main result, the Theorem 1.1, shows a new observability inequality and unique continuation property for a class of ε-controllable. The definition of ε-controllable is shown later in the work, in the Definition 3.1:

Theorem 1.1. Let $(M,\mathbf{g})$ be a $N-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Assume that statements (1.2)-(1.4) hold and that $\varepsilon > 0$ and $T > 0$ sufficiently large is given. Then for all ε-controllable set $\omega \subset M$ sharp admissible damping region and assuming that $T > 0$ is , and

$$w \in L^2(0,T;H_0^1(M)) \cap H^1(0,T;L^2(M))$$

is a solution of the linear wave equation (1.1) where $p_0, p_1 : M \times [0,T] \to \mathbb{R}$ satisfy the hypothesis 3 in Theorem 2.25. Then there exists $K_T > 0$ depending on $\varepsilon, T, \mathbf{g}$ and k_T such that

$$\int_0^T \int_\omega |\nabla w|^2 d\omega\, dt \geq K_T \big(\|(w,\partial_t w)(0)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} + \|(w,\partial_t w)(T)\|^2_{H^1(\Omega)\times L^2(\Omega)} \big). \quad (1.20)$$

If in addition $w = 0$ in $\omega \times [0,T]$, it follows that $w = 0$ in $M \times [0,T]$.

The proof of this Theorem is found in Chapter 4.

The second main result, is a direct application of Theorem 1.1, getting to proof a smooth global attractor with finite fractal dimension characterized by unstable manifolds of stationary points of the dynamical system associated with the wave equation with nonlinear localized damping and critical forces (1.5). This result is presented below:

Theorem 1.2. Let $(M,\mathbf{g})$ be a *three*$-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Assume that assumption (1.6)-(1.11) hold and that $\varepsilon > 0$ is given. Then for some ε-controllable set $\omega \subset M$ sharp admissible damping region, such that (1.12) is fulfilled, the dynamical system associated to the problem (1.5) possesses a global attractor $\mathscr{A}$ of finite fractal dimension characterized by unstable manifolds of stationary points of the dynamical system. Moreover, the attractor is smooth in the sense of $\mathscr{A} \subset \big(H^2(M) \cap H_0^1(M)\big) \times H_0^1(M)$, i.e., any full trajectory $\{(u(t), \partial_t u(t)) \mid t \in \mathbb{R}\} \subset \mathscr{A}$ has the property that

$$\partial_t u \in L^\infty(\mathbb{R}; H_0^1(M)) \cap C(\mathbb{R}; L^2(M)), \quad (1.21)$$

with bound

$$\|\nabla \partial_t u(t)\|_2^2 + \|\partial_t^2 u(t)\|_2^2 \leq C, \quad (1.22)$$

where the constant C independent of t.

The proof of this Theorem is found in Chapter 5.

Finally, the third main result, is an application of the observability inequality (1.20) for the wave equation with localized linear damping and supercritical forces (1.13), getting to show the fractal dimension and optimal regularity for a global attractor of the system.

Theorem 1.3. Let $(M,\mathbf{g})$ be a *three*−dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M. Assume that assumption (1.17)-(1.19) hold and that $\varepsilon > 0$ is given. Then for some ε-controllable set $\omega \subset M$ sharp admissible damping region, such that (1.14)-(1.16) is fulfilled, the dynamical system associated to the problem (1.13) possesses a global attractor $\mathscr{A}$ of finite fractal dimension characterized by unstable manifolds of stationary points of the dynamical system. Moreover, the attractor is smooth in the sense of $\mathscr{A} \subset \left(H^2(M)\cap H_0^1(M)\right) \times H_0^1(M)$, i.e., any full trajectory $\{(u(t),\partial_t u(t)) \mid t \in \mathbb{R}\} \subset \mathscr{A}$ has the property that

$$\partial_t u \in L^\infty(\mathbb{R};H_0^1(M))\cap C(\mathbb{R};L^2(M)), \tag{1.23}$$

with bound

$$\|\nabla \partial_t u(t)\|_2^2 + \|\partial_t^2 u(t)\|_2^2 \leq C, \tag{1.24}$$

where the constant C independent of t.

The proof of this Theorem is found in Chapter 6.

In addition to the previous theorems, the work presents a series of new results in terms of literature. These are listed below:

1. Recover Carleman estimates for supercritical wave equations (see Chapter 1 and 4).
2. From a $\varepsilon > 0$, construct a class of sets sharp admissible damping region from the new definition of ε-controllable sets (see Chapter 2). In addition, this construction is distinctly geometric (independent of any PDE).
3. Prove a series of geometric consequences for the class of sharp admissible damping region sets, highlighting a result of decomposition in overlapping sets and a coarea formula (see Chapter 3).
4. Prove the existence of a smooth global attractor for the dynamic systems associated with (1.5) and (1.13), through the study of quasi-stable dynamical systems (see Chapter 5 and 6).

CHAPTER

2

MATHEMATICAL BACKGROUND

Now we recall the different notations and results regarding the differential geometry that will be used throughout the present work. For more details see (ABRAHAM; MARSDEN; RATIU, 2012; O'NEILL, 1983; SAKAI, 1996; BURKE; BURKE, 1985; CARMO, 1992; TAYLOR, 2013).

Let $(M, \mathbf{g})$ be a compact Riemannian manifold, $N-$dimensional, with smooth boundary, with the Riemannian metric $\mathbf{g}(\cdot,\cdot) = \langle \cdot,\cdot \rangle$.

The interior of M will be denoted by $int(M)$ and the boundary of M by ∂M. On the other hand, we will denote the tangent space of M in the point $p \in M$ by T_pM. Let's set a coordinate system $(U, \psi = (x_1, \dots, x_N))$ of M. The canonical basis of T_pM associated with this neighborhood is $B^{\psi} = (e_1, \dots, e_N)$ such that $e_i(p) = \partial_{x_i}(p)$.

Given $d \in C^{\infty}(M)$ and $v_p = [\gamma]$ a vector tangent to M in p, we define the derivative of d in the direction of v_p with $v_p(d) = \frac{d}{dt}|_{t=0}(d \circ \gamma) \in \mathbb{R}$. We will denote the differential of d in p by $Dd(p)$ such that

$$\begin{aligned} Dd(p): \ T_pM &\longrightarrow \mathbb{R} \\ v_p &\mapsto v_p(d). \end{aligned}$$

Let $\widetilde{M}$ be another differentiable manifold and $f : M \to \widetilde{M}$ a differentiable application among them. The differential of f in p will be denoted by $Df(p)$ and is defined as

$$\begin{aligned} Df(p): \ T_pM &\longrightarrow T_{f(p)}\widetilde{M} \\ [\gamma] &\mapsto [f \circ \gamma]. \end{aligned}$$

A vector field X on M is an application that assigns to each point $p \in M$ a vector tangent to M at that point, $X(p) \in T_pM$. That is, an application $X : M \to TM$ such that $\pi \circ X = Id_M$ where TM is the tangent manifold of M and $\pi : TM \to M$ the canonical projection.

An X field is said to be differentiable if X is C^∞ differentiable as an application between manifolds.

We will denote by $\mathfrak{X}(M)$ the set of all vector fields on M. Note with the internal operator the sum and product by real scalars, $(\mathfrak{X}(M),+,\cdot\mathbb{R})$ is a real vector space of infinite dimension, and with the product defined by $(d\cdot X)(p)=d(p)X(p)$ for all $p\in M$ and $d\in C^\infty(M)$, $(\mathfrak{X}(M),+,\cdot)$ is a module on the ring $C^\infty(M)$.

Fix $X\in\mathfrak{X}(M)$, for each function $d\in C^\infty(M)$ we can define the application

$$\begin{aligned} X(d):\ & M\longrightarrow\mathbb{R}\\ & p\mapsto X(p)(d), \end{aligned}$$

that satisfies Leibniz's rule, this is

$$X(f\cdot g)=X(f)\cdot g+f\cdot X(g).$$

Given $X\in\mathfrak{X}(M)$ a vector field and $\gamma: I\to M$, with $I\subset\mathbb{R}$ a differentiable curve. We will say that γ is an integral curve of X if $\gamma'(t)=X(\gamma(t))$ for all $t\in I$. In the case that $I=\mathbb{R}$ we will say that γ is complete and that X is complete along γ. If X is complete throughout all of its integral curves, it is said that X is complete.

Remark 2.1: An important result about complete vector fields is that if M is a compact manifold then any vector field $X\in\mathfrak{X}(M)$ is complete (cf. eg. (O'NEILL, 1983)).

Let $(\mathcal{V},\mathbb{R})$ be a vector space. A tensor r times covariant and s times contravariant (or type (r,s)) on $\mathcal{V}$ is a multilinear application

$$\begin{aligned} T:\ & \mathcal{V}^r\times(\mathcal{V}^*)^s\longrightarrow\mathbb{R}\\ & (u_1,\ldots,u_r,\psi_1,\ldots,\psi_s)\mapsto T(u_1,\ldots,u_r,\psi_1,\ldots,\psi_s). \end{aligned}$$

We will denote by $\mathcal{T}_{r,s}(\mathcal{V})$ the set of tensors type (r,s) over $(\mathcal{V},\mathbb{R})$. It is easily verified that $(\mathcal{T}_{r,s}(\mathcal{V}),+,\cdot\mathbb{R})$ has vector space structure.

Let $T\in\mathcal{T}_{r,s}(\mathcal{V})$ and $T'\in\mathcal{T}_{r',s'}(\mathcal{V})$. The tensor product of T by T' is defined as

$$\begin{aligned} T\otimes T':\ & \mathcal{V}^{r+r'}\times(\mathcal{V}^*)^{s+s'}\longrightarrow\mathbb{R}\\ & ((u_1,\ldots,u_{r+r'},\psi_1,\ldots,\psi_{s+s'})\mapsto T\otimes T'(u_1,\ldots,u_{r+r'},\psi_1,\ldots,\psi_{s+s'}), \end{aligned}$$

where

$$T\otimes T'(u_1,\ldots,u_{r+r'},\psi_1,\ldots,\psi_{s+s'})=T(u_1,\ldots,u_r,\psi_1,\ldots,\psi_s)\cdot T'(u_{r+1},\ldots u_{r+r'},\psi_{s+1},\ldots,\psi_{s+s'}).$$

In the particular case that $\mathscr{V} = TM$ and $\mathscr{V}^* = (T_pM)^*$, we will call the flat application $\flat$ (*l*owering an index) between TM and $(TM)^*$ associated to the metric **g**,

$$\begin{aligned} \flat: \ TM &\longrightarrow (TM)^* \\ v &\mapsto v^\flat := \langle v, \cdot \rangle, \end{aligned}$$

with

$$\begin{aligned} \langle v, \cdot \rangle : \ TM &\longrightarrow \mathbb{R} \\ w &\mapsto v^\flat := \langle v, w \rangle. \end{aligned}$$

We will denote by $\sharp$ to the inverse application of $\flat$, which will be called sharp application (*r*aising an index), and is characterized by the relation $\langle \psi^\sharp, w\rangle = \psi(w)$ for all $\psi \in (TM)^*$ and $w \in TM$.

A tensor field T type (r,s) over M is an assignment of a tensor $T_p \in \mathscr{T}_{r,s}(T_pM)$ at each point $p \in M$. Given T and a coordinate system (U, ψ), there exist functions $t^{j_1,\dots,j_s}_{i_1,\dots,i_r}$, $i_1,\dots,j_s \in \{1,2,3\}$ defined in U such that

$$T_p = \sum_{i_1,\dots,j_s=1}^{3} t^{j_1,\dots,j_s}_{i_1,\dots,i_r}(p) \cdot Dx_{i_1}(p) \otimes \dots \otimes Dx_{i_r}(p) \otimes e_{j_1}(p) \otimes \dots \otimes e_{j_s}(p), \ \ \forall p \in U, \qquad (2.1)$$

where $B_p = \{e_1(p), e_2(p), e_3(p)\}$ and $B_p^* = \{Dx_1(p), Dx_2(p), Dx_3(p)\}$ they are basis of T_pM and $(T_pM)^*$ respectively.

We say that T is continuous (resp. differentiable $C^r(M)$) in p if for all functions $t^{j_1,\dots,j_s}_{i_1,\dots,i_r}$ are continuous (resp. differentiable $C^r(M)$) in p, in particular if T is differentiable $C^\infty(M)$ we say that T is a tensor field.

We denote $\mathscr{T}_{r,s}(M) := \bigcup_{p\in M} \mathscr{T}_{r,s}(T_pM)$. A tensor field type (r,s) is an application $T : M \to \mathscr{T}_{r,s}(M)$ such that $\pi_{r,s} \circ T = Id_M$, where $\pi_{r,s} : \mathscr{T}_{r,s}(M) \to M$ is the canonical projection, that is, $\pi_{r,s}(T_p) = p$ for all $T_p \in \mathscr{T}_{r,s}(M)$. We denote by $\mathfrak{X}_{r,s}(M)$ the set of all tensor fields (r,s) over M.

A differential form type $(1,0)$ or 1-differential form α is a field of linear forms, that is, tensors type $(1,0)$. We will denote by $\Lambda^1(M)$ the set of all 1-differential forms, endowed with their natural operations.

A differential form type $(2,0)$ or 2-differential form β on a manifold M is a tensor field $(2,0)$ that is antisymmetric, this is, $\beta_p(v_p, w_p) = -\beta(w_p, v_p)$ for all $v_p, w_p \in T_pM$ and for all $p \in M$. We will denote by $\Lambda^2(M)$ the set of all 2-differential forms, endowed with their natural operations.

We say that 1-differential form $\alpha \in \Lambda^1(M)$ is exact if there exists a function $d \in C^\infty(M)$ such that $\alpha = Dd$. Let $\alpha = \sum_{i=1}^{N} \alpha_i Dx_i$ be the expression of a differential form in a coordinate system (U,ψ) of M, and define the external differential of α as

$$D\alpha = \sum_{i=1}^{N}\sum_{j=1}^{N} \partial_{x_j}\alpha_i \cdot Dx_j \wedge Dx_i,$$

with $\psi \wedge \phi = \psi \otimes \phi - \phi \otimes \psi$. Moreover, if $\gamma\colon [a,b] \to M$ is a differentiable curve, we denote the circulation of α along γ as

$$\int_\gamma \alpha := \int_a^b \alpha(\gamma'(t))dt.$$

On $(M,\mathbf{g})$, we denote by $X^\flat \in \Lambda^1(M)$ to the flat differential form of a field $X \in \mathfrak{X}(M)$, and $\alpha^\sharp \in \mathfrak{X}(M)$ to the field of a sharp differential form $\alpha \in \Lambda^1(M)$. In coordinates (U,ψ) these fields have the expressious

$$X = \sum_{i=1}^{N} X_i e_i, \quad X^\flat = \sum_{i,j=1}^{N} \mathbf{g}_{ij} X_i Dx_j,$$
$$\alpha = \sum_{i=1}^{N} \alpha_i Dx_i, \quad \alpha^\sharp = \sum_{i,j=1}^{N} \mathbf{g}^{ij}\alpha_i e_j,$$

where $\mathbf{g}_{ij}$ and $\mathbf{g}^{ij}$ as in (2.1) with $T = \mathbf{g}$, so for $d \in C^\infty(M)$ we will denote the gradient vector field of d by

$$\nabla d(p) = (Dd(p))^\sharp, \quad \forall p \in M,$$

using the above coordinates, we can call the ∇f as

$$\nabla d = \sum_{i,j=N}^{3} \mathbf{g}^{ij}\partial_{x_i} d \cdot e_j.$$

Additionally, for all $X \in \mathfrak{X}$ we have that

$$\nabla_X d := \langle \nabla d, X\rangle = X(d).$$

In what follows, we will denote the application $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $\nabla(X,Y) = \nabla_X Y$ as the Levi-Civita connection of $(M,\mathbf{g})$ such that $\nabla_X Y$ satisfies the Koszul formula and for the coordinates (U,ψ) it has to

$$\nabla_{e_i} e_j = \sum_{k=1}^{N} \Gamma_{ij}^k e_k,$$

where the coefficients Γ_{ij}^k are the Christoffel symbols of $\mathbf{g}$ and are given by

$$\Gamma_{ij}^k = \frac{1}{N}\sum_{m=1}^{3} \left(\partial_{x_i}\mathbf{g}_{jm} + \partial_{x_j}\mathbf{g}_{im} - \partial_{x_m}\mathbf{g}_{ij}\right)\mathbf{g}^{hk}.$$

Given $\gamma \colon [a,b] \to M$ a differentiable curve, we define the length of γ as

$$l(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{\mathbf{g}_{\gamma(t)}(\gamma'(t),\gamma'(t))}dt.$$

Moreover, if $(M,\mathbf{g})$ is connected, we will denote the distance between two points $p,q \in M$ as

$$d(p,q) = \inf_{\gamma}\{l(\gamma) \mid \gamma \colon [a,b] \to M, \gamma(a) = p,\ \gamma(b) = q\}, \tag{2.2}$$

Let $\gamma \colon [a,b] \to M$ be a differentiable curve and $t_0 \in (a,b)$ such that $\gamma'(t_0) \neq 0$. We say that $X \in \mathfrak{X}(\gamma)$, if there exists $\varepsilon > 0$ and $\overline{X} \in \mathfrak{X}(M)$ such that $\overline{X}_{\gamma(t)} = X(t)$ provided that $|t - t_0| < \varepsilon$.

We denote the covariant derivative of $X \in \mathfrak{X}(\gamma)$ in t_0, by

$$\frac{DX}{dt}(t_0) := \nabla_{\gamma'(t_0)}\overline{X}.$$

In particular, about the coordinates (U,ψ) will be given by

$$\frac{DX}{dt} = \sum_{k=1}^{N}\left(X_k' + \sum_{i,j=1}^{3}(x_i \circ \gamma)'X_j(\Gamma_{ij}^k \circ \gamma)\right)e_k,$$

where $X = \sum_{i=1}^N X_i e_i$ and Γ_{ij}^k are the Christoffel symbols in that chart.

A regular curve $\gamma \colon [a,b] \to M$ is said to be a geodesic of M if $\frac{D\gamma'}{dt} = 0$ in (a,b).

Let be $X \in \mathfrak{X}(M)$. The divergence of X is the function $div(X) : M \to \mathbb{R}$ given by

$$(divX)(p) = Trace\left\{\begin{array}{c} T_pM \longrightarrow T_pM \\ Y \longrightarrow \nabla_Y X \end{array}\right\} = \langle \nabla_{e_i(p)}X, e_i(p)\rangle.$$

Note that if $d \in C^\infty(M)$ is a function defined on M, then

$$div(d \cdot X) = d \cdot divX + \mathbf{g}(\nabla d, X).$$

On the other hand, if $p \in M$ is a critical point of d, the Hessian of d in the critical point p, $(\nabla^2 d)_p : T_pM \times T_pM \to \mathbb{R}$, is defined through

$$(\nabla^2 d)_p(X,Y) = X(W(d)),$$

where $W \in \mathfrak{X}(M)$ satisfies $W(p) = Y$.

In particular,

$$(\nabla^2 d)_p(X,X) = (d \circ \gamma)''(0),$$

with $\gamma \in C^\infty((-\varepsilon,\varepsilon),M)$ holds that $\gamma(0) = p$, $\gamma'(0) = X$.

Given $d \in C^\infty(M)$, the Hessian of d is defined as $\nabla^2 d : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ where

$$(\nabla^2 d)(X,Y) = X(Y(d)) - (\nabla_X Y)(d) = \langle \nabla_X \nabla d, Y \rangle, \;\; \forall X,Y \in \mathfrak{X}(M).$$

The Laplace-Beltrami operator of a given function d on a manifold, it is given by

$$\Delta d = div(\nabla d) = Trace(\nabla^2 d) \in C^\infty(M).$$

For the L^p spaces on Riemannian manifolds, the notations and definitions presented in (TAYLOR, 2013) will be followed. Thus, over the space $\mathfrak{X}_{0,k}(M)$ of all tensor fields on type $(0,k)$ the internal product is defined by

$$\langle T_1, T_2 \rangle_{\mathfrak{X}_{0,k}(M)} = \int_M \langle T_1, T_2 \rangle_{\mathscr{T}_{0,k}(M)} dV_{\mathbf{g}}, \;\; T_1, T_2 \in \mathfrak{X}_{0,k}(M),$$

where,

$$\langle T_1, T_2 \rangle_{\mathscr{T}_{0,k}(M)} = \sum_{i_1,\dots,i_k}^{N} T_1(e_{i_1},\dots,e_{i_k}) T_2(e_{i_1},\dots,e_{i_k}),$$

and $dV_{\mathbf{g}}$ is the volumen form of M for the metric $\mathbf{g}$.

Remark 2.2: Note that $\mathscr{T}_{0,k}(M) = \bigcup_{p\in M} T_p^k M$.

We denote

$$L^2(M, \mathfrak{X}_{0,k}(M)) = \{T \in \mathfrak{X}_{0,k}(M) \;\Big|\; \int_M \langle T, T \rangle_{\mathscr{T}_{0,k}(M)} dV_{\mathbf{g}} < \infty\}.$$

Analogously, $L^2(M)$ is the completion of $C^\infty(M)$ with the inner product

$$(u,v)_{L^2(M)} = \int_M u(x)v(x) dV_{\mathbf{g}}, \;\; u,v \in C^\infty(M).$$

2.1 Sobolev spaces on Riemannian manifolds

This section is intended to show the main definitions and results about Sobolev spaces on Riemannian manifolds. For this we will continue as the main reference (HEBEY, 2000).

Let $(M,\mathbf{g})$ be a Riemannian manifold. For k an integer and $u \in C^\infty(M)$, $\nabla^k u$ denotes the $k-th$ covariant derivative of u (with the convention $\nabla^0 u = u$). As an example, the components of ∇u in local coordinates are given by $(\nabla u)_i = \partial_i u$, while the components of $\nabla^2 u$ in local coordinates are given by

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma^k_{ij} \partial_k u.$$

By definition one has that

$$|\nabla^k u|^2 = \mathbf{g}^{i_1,j_1} \dots \mathbf{g}^{i_k,j_k} (\nabla^k u)_{i_1 \dots i_k} (\nabla^k u)_{j_1 \cdots j_k}.$$

For k an integer and $p \geq 1$ real, we denote by $C^p_k(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j u| \in L^p(M)$ for any $j = 0, \dots, k$. Hence,

$$C^p_k(M) = \{u \in C^\infty(M) \mid \forall j = 0, \dots, k, \int_M |\nabla^j u|^p dV_{\mathbf{g}} < \infty\},$$

where, in local coordinates, $dV_{\mathbf{g}} = \sqrt{\det(\mathbf{g}_{ij})} dx$, and where dx stands for the Lebesgue's volume element of $\mathbb{R}^N$, $N = dimM$. If M is compact, one has that $C^p_k(M) = C^\infty(M)$ for all k and $p \geq 1$.

Remark 2.3: When the dependency of $\mathbf{g}$ on the volume form $dV_{\mathbf{g}}$ is clear, the classic dx or dM notations will be used instead.

Definition 2.4: The Sobolev space $W^{k,p}(M)$ is the completion of $C^p_k(M)$ with respecto to the norm

$$\|u\|_{W^{k,m}(M)} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p dV_{\mathbf{g}} \right)^{1/p}.$$

Proposition 2.5. If $p = 2$, $H^k(M) := W^{k,2}(M)$ is a Hilbert space when equipped with the equivalent norm

$$\|u\|_{H^k(M)} = \sqrt{\sum_{j=0}^{k} \int_M |\nabla^j u|^2 dV_{\mathbf{g}}}.$$

The scalar product $\langle \cdot, \cdot \rangle_{H^k(M)}$ associated to $\| \cdot \|$ is defined by

$$\langle u, v \rangle_{H^k(M)} = \sum_{m=0}^{k} \int_M \left(\mathbf{g}^{i_1,j_1} \dots \mathbf{g}^{i_m,j_m} (\nabla^m u)_{i_1 \dots i_m} (\nabla^m v)_{j_1 \cdots j_m} \right) dV_{\mathbf{g}}.$$

Proposition 2.6. If M is compact, $W^{k,p}(M)$ does not depend on the Riemannian metric.

Proposition 2.7. If $p > 1$, $W^{k,p}(M)$ id reflexive.

Definition 2.8: The Sobolev space $W^{k,p}_0(M)$ is the closure of $D(M)$ in $W^{k,p}(M)$ where

$$D(M) = \{\varphi \in C^\infty(M) \mid \varphi \text{ have compact support in } M\}.$$

Theorem 2.9. If $(M, \mathbf{g})$ is complete, then, for any $p \geq 1$, $W^{1,p}_0(M) = W^{1,p}(M)$.

Theorem 2.10. Let (M, g) be a complete Riemannian manifold with positive injectivity radius and let $k \geq 2$ be an integer. Suppose that there exists a positive constant C such that for any $j = 0, \ldots, k-2$, $|\nabla^j Rc_{M,g}| \leq C$. Then for any $p \geq 1$, $W_0^{k,p}(M) = W^{k,p}(M)$.

Theorem 2.11. Let (M, g) be a compact Riemannian $N-$manifold. For any real numbers $1 \leq q < p$ and any integers $0 \leq m < k$ satisfying $\frac{1}{p} = \frac{1}{q} - \frac{k-m}{N}$, it is true that the embedding

$$W^{k,q}(M) \subset W^{m,p}(M) \text{ is continuous.}$$

Theorem 2.12. Let (M, g) be a compact Riemannian $N-$manifold. For any integers $j \geq 0$ and $m \geq 1$, any real number $q \geq 1$, and any real number p such that $1 \leq p < \frac{Nq}{N-mq}$, the embedding of $W^{j+m,q}(M)$ in $W^{j,p}(M)$ is compact.

Corollary 2.13. Let (M, g) be a compact Riemannian *three*-manifold. Then:

(*i*) for any $1 \leq p \leq 6$, the embedding of $H^1(M)$ in $L^p(M)$ is continuous,

(*ii*) for any $1 \leq p < 6$, the embedding of $H^1(M)$ in $L^p(M)$ is compact.

2.2 Carleman estimates for wave equations

For this section, we will follow the works of Triggini and Yao (TRIGGIANI *et al.*, 2002) in order to show a result of observability inequality and unique continuation property. With this objective, we will study these results for two different cases: for a subdomain of an compact Riemannian manifold and for an finite collection overlapping subdomains for an compact Riemannian manifold.

We consider (M, g) a $N-$dimensional connected compact Riemannian manifold of class C^∞ with metric $\mathrm{g}(\cdot,\cdot) = \langle \cdot,\cdot \rangle$, squared norm $|X|^2 = \mathrm{g}(X,X)$, and with smooth boundary ∂M. Moreover, for the temporary space, we will consider $T > 0$ large enough.

2.2.1 *Case I: Observability inequality and unique continuation property for subdomains of M*

Let us consider Ω such that is an open bounded, conected, compact set of M with smooth boundary $\partial\Omega \subset \partial M$. We let n denote the outward unit normal field along the boundary ∂M.

In this section we will study a observability and unique continuation result for the next system on Ω,

$$\begin{cases} \partial_t^2 w - \Delta w = p_0 w + p_1 \partial_t w \text{ in } Q_{\Omega,T} := \Omega \times (0,T], \\ w = 0 \text{ on } \Sigma_{\Omega,T} := \Omega \times (0,T], \end{cases} \tag{2.3}$$

where $p_0, p_1 : Q_{\Omega,T} \to \mathbb{R}$ such that

$$p_1 \in L^\infty(0,T;L^\infty(\Omega)), \;\; p_0 \in L^2(0,T;L^2(\Omega)). \tag{2.4}$$

We assume that:

1. For all $z \in H^{2,2}(M \times (0,T])$

$$p_0 w \in H^{1,1}(Q_{\Omega,T}), \;\; \partial_t(p_0 w) \in L^2(0,T;L^2(\Omega)), \tag{2.5}$$

2. There exists $C_{p_0T} > 0$ such that

$$\|p_0 w\|_{L^2(0,T;L^2(\Omega))} \leq C_{p_0T} \|w\|_{L^2(0,T;H^1(\Omega))}. \tag{2.6}$$

Taking into account that

$$H^{1,1}(Q_{\Omega,T}) := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

$$H^{2,2}(Q_{\Omega,T}) := L^2(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega)).$$

Moreover, we will denote by $\mathscr{E}_\Omega : [0,\infty) \to \mathbb{R}$ to the energy of the system with respect to the norm in $H^1(\Omega) \times L^2(\Omega)$, i.e.

$$\mathscr{E}_\Omega(t) = \int_\Omega \left(|\partial_t w(t)|^2 + |\nabla w(t)|^2 + |w(t)|^2\right) dM. \tag{2.7}$$

Functional Approaches

We will start by defining an generator of the local escape vector fields, denoted by the functional d. This functional and its local escape vector fields ∇d are fundamental for the application of Carleman estimates.

Definition 2.14: Let $d : \overline{\Omega} \to \mathbb{R}$ a function such that

$(d.1)_\Omega \;\; d \in C^3(\overline{\Omega}), \;\; \min_{q \in \overline{\Omega}} d(q) > 0,$

$(d.2)_\Omega \;\; \nabla^2 d(X,X) > |X|^2_{\mathbf{g}}, \quad \forall X \in T_qM, \; \forall q \in \Omega,$

$(d.3)_\Omega \;\; \inf_\Omega |\nabla d| > 0\,,$

Consider the following definitions from the definition of d:

For T or sufficiently large, note that there is a $\delta > 0$ such that

$$T^2 > 4\max_{x\in\overline{\Omega}} d(x) + 4\delta.$$

For this $\delta > 0$, there is a constant c such that $0 < c < 1$, where

$$cT^2 > 4\max_{x\in\overline{\Omega}} d(x) + 4\delta$$

Thus, we define the pseudo-convex function $\phi : M \times \mathbb{R} \to \mathbb{R}$ of class C^3 given by

$$\phi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2, \quad 0 \le t \le T, \ \ x \in M.$$

Note that ϕ satisfies the following properties

(ϕ.1.)

$$\phi(x,0) = \phi(x,T) = d(x) - c\frac{T^2}{4} \le \max_{x\in\overline{\Omega}} d(x) - c\frac{T^2}{4} \le -\delta, \ \text{uniformemente em } x \in \overline{\Omega}.$$

(ϕ.2.) Let t_0, t_1 such that $0 < t_0 < T/2 < t_1 < T$, then

$$\min_{(x,t)\in\overline{\Omega}\times[t_0,t_1]} \phi(x,t) \ge \sigma, \ \ 0 < \sigma < m,$$

where $m := \min_{x\in\overline{\Omega}} d(x) > 0$.

Now, repeating the done in (TRIGGIANI *et al.*, 2002; LASIECKA; TRIGGIANI; ZHANG, 2000), there is a rescaling for d such that we can define the function

$$\alpha(x) := \Delta d(x) - c - 1,$$

satisfying the following properties

(α.1.)

$$\Delta d(x) - 2c - \alpha(x) = 1 - c > 0, \ \ \forall x \in \overline{\Omega},$$

(α.2.)

$$[2c+\Delta d-\alpha]|\nabla d|^2+2\nabla^2 d(\nabla d,\nabla d)-4c^2(\Delta d+6c-\alpha)\left(t-\frac{T}{2}\right)^2 \geq 4(1+7c)\phi^*(x$$

for all $(x,t) \in Q_{\Omega,T}$, where

$$\phi^*(x,t) = d(x) - c^2\left(t-\frac{T}{2}\right)^2, \quad \forall (x,t) \in Q_{\Omega,T}.$$

Remark 2.15: Note that

$$\phi^*(x,t) \geq \phi(x,t), \quad \forall (x,t) \in Q_{\Omega,T}.$$

We now define the following sets in $Q_{\Omega,T}$,

$$Q_{\Omega,T}(\sigma) = \{(x,t) \in Q_{\Omega,T} \mid \phi(x,t) \geq \sigma > 0\},$$
$$Q^*_{\Omega,T}(\sigma^*) = \{(x,t) \in Q_{\Omega,T} \mid \phi^*(x,t) \geq \sigma^* > 0\}, \quad 0 < \sigma^* < \sigma,$$

for some constant σ^* such that $0 < \sigma^* < \sigma < m$. Observe that

$$\Omega_{\Omega,T} \times [t_0,t_1] \subset Q_{\Omega,T}(\sigma) \subset Q^*_{\Omega,T}(\sigma^*) \subset Q_{\Omega,T}.$$

In addition, an important property regarding the function $\phi*$ is the control estimate of the terms it l.o.t.. This is

$$4(1+7c)\phi^*(x,t) \geq 4(1+7c)\sigma* > 0, \quad \forall (x,t) \in Q^*_{\Omega,T}(\sigma^*).$$

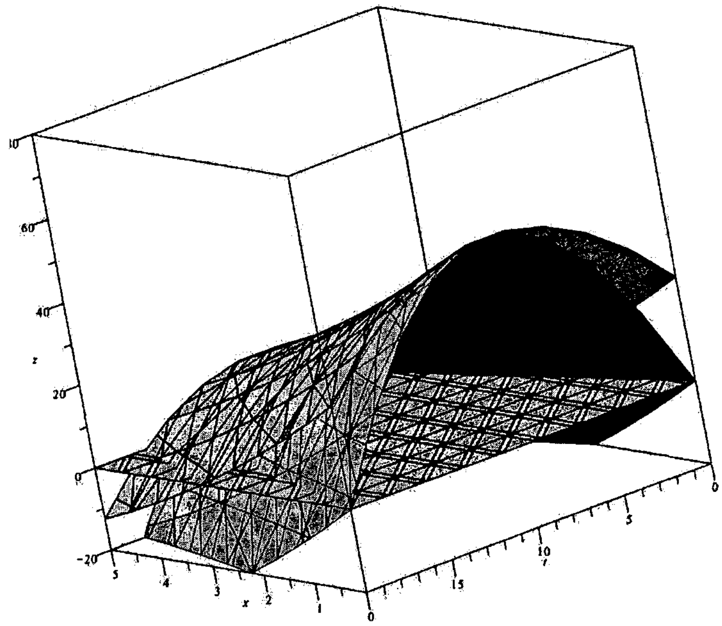

Figure 3 – In this example we consider $(M,g) = ([0,L],|\cdot|)$, $\Omega = (0,L)$, $d = \frac{1}{2}|x-10|^2$ with $c = \frac{1}{2}$ and $T = 20$. The plane denotes the region Q, on the other hand the surface generated by ϕ^* is represented by the top surface and the surface generated by ϕ by the bottom surface. Note that both functions are pseudo-convex functions.

Operational notations

Let $f(x,t), h(x,t) \in C^1(M \times \mathbb{R})$ e $X \in \mathscr{X}(M)$, then, we will use the following notations for the this section

(*i*) $\widehat{\nabla} f := (f_t, -\nabla f)$,

(*ii*) $\widehat{div}(h,X) := h_t + divX$,

(*iii*) $(h,X)(f) := hf_t + X(f)$,

(*iv*) $\mathscr{A}w := \widehat{div}\widehat{\nabla}w$.

The following lemma, found in (TRIGGIANI *et al.*, 2002, Lemma 3.0.), shows some important equivalences referring to the new notations

Lemma 2.16. Considering $(i) - (iv)$, we have the following identities:

(a) $\mathscr{A}w = w_{tt} - \Delta w$,

(b) $\widehat{div}(f(h,X)) = f\widehat{div}(h,X) + (h,X)(f)$,

(c) $\widehat{\nabla} f(h) = f_t h_t - \nabla f(h)$,

(d) $\widehat{div} f(X) = f\widehat{div} X + X(f)$.

Auxiliary functions

Since the objective is to approximate the solution $w \in C^2(Q_{\Omega,\infty})$ from the pseudo-convex function ϕ, it is necessary to construct different auxiliary functions that have goods properties over $Q_{\Omega,T}$ and $Q^*_{\Omega,T}$. Therefore, given $w \in C^2(Q_{\Omega,\infty})$, $d \in C^3(\Omega)$ and $\alpha(x) \in C^1(\Omega)$ the functions defined above, and let $\tau > 0$ be an arbitrary parameter, we define

$$l(x,t) = \tau\left(d(x) - c\left(t - \frac{T}{2}\right)^2\right) = \tau\phi(x,t), \tag{2.8}$$

$$\Psi(x) = \tau\alpha(x), \quad \theta(x,t) = e^{l(x,t)} = e^{\tau\phi(x,t)}, \tag{2.9}$$

$$a(x,t) = \tau^2\left(|\phi_t|^2 - |\nabla d|^2\right) + (c-1)\tau = \tag{2.10}$$

$$\tau^2\left(4c^2\left(t - \frac{T}{2}\right)^2 - |\nabla d|^2\right) + 2c\tau + \tau\Delta d - \Psi.$$

Note that next to the operational notations is clearly the following Lemma

Lemma 2.17. In the context of the functions defined above with $c \in (0,1)$. The following identities are satisfied

(i) $\Psi_t = 0$,

(ii) $l_t = -2\tau c\left(t - \frac{T}{2}\right)$,

(iii) $l_{tt} = -2c\tau$,

(iv) $\nabla l = \tau\nabla d$,

(*v*) $\nabla\Psi = \tau\nabla\alpha$,

(*vi*) $\nabla l_t = 0$,

(*vii*) $\Delta l = \tau\Delta d$,

(*viii*) $a = \tau^2\left(4c^2\left(t-\frac{T}{2}\right)^2 - |\nabla d|^2\right) + \mathscr{O}(\tau)$, where $\mathscr{O}(\tau)$ represents a positive linear error dependent on tau, in particular $\tau(c-1)$.

On the other hand, given that one of the objectives is to be able to prove an observability result, we need estimates on $\partial\Omega$. Note that Carleman estimates do not require a particular condition on the boundary of the manifold. So, in order to facilitate the mathematical calculations, we will consider that the system has Dirichlet conditions. We will denote the term that contains the terms about $\Sigma_{\Omega,T}$ as $BT|_{\Omega,T}$ such that

$$BT|_{\Omega,T} = 2\tau\int_{\Sigma_{\Omega,T}} \theta^2\left(\frac{\partial w}{\partial n}\right)^2 \langle\nabla d, n\rangle d\Sigma_{\Omega,T}. \tag{2.11}$$

Also, we define

$$\Gamma_0^\Omega = \{x\in\partial\Omega\,|\langle\nabla d(x), n(x)\rangle \le 0\},\;\; \Gamma_1^\Omega = \partial\Omega\setminus\Gamma_0^\Omega, \tag{2.12}$$

well, we have to

$$BT|_{\Omega,T} \le 2\tau\int_0^T\int_{\Gamma_1^\Omega} \theta^2\left(\frac{\partial w}{\partial n}\right)^2 \langle\nabla d, n\rangle d\Sigma_{\Omega,T}.$$

Carleman estimates

As seen in (LASIECKA; TRIGGIANI; ZHANG, 2000; TRIGGIANI *et al.*, 2002), Carleman estimates are fundamental to the demonstration of the observability inequality and unique continuation property result. Next, we state the result found in (TRIGGIANI *et al.*, 2002) for the case of classical solutions and strong solutions in $H^{2,2}(Q_{\Omega,T})$.

Theorem 2.18. Let d as in the definition 2.14 and in the context of the functional approaches and auxiliary functions defined above. Given $w \in C^2(Q_{\Omega,\infty})$ solution of the system (2.3), with p_0, p_1 satisfying (2.4)-(2.6), then for $\rho = 1 - c$ and $\varepsilon > 0$ small enough, can be defined $\beta := 4(1+7c)\sigma^* - \varepsilon\rho \max_{\overline{Q_{\Omega,T}}}(|\partial_t \phi|^2 + |\nabla\phi|^2) > 0$ for all $\tau > 0$ large enough, such that

$$\begin{aligned} BT|_{\Omega,T} + C_{1,T}e^{2\tau\sigma}\int_0^T \mathscr{E}_\Omega(t)dt \geq & [\tau\varepsilon\rho - 2C_{p_0p_1T}]\int_0^T\int_\Omega e^{2\tau\phi}[|\partial_t w|^2 + |\nabla w|^2]d\Omega dt \\ & + (2\tau^3\beta + \mathscr{O}(\tau^2) - 2C_{p_0p_1T})\int_{Q_{\Omega,T}(\sigma)} e^{2\tau\phi}|w|^2 dxdt \\ & - C_{2,T}\tau^3 e^{-2\tau\delta}[\mathscr{E}_\Omega(0) + \mathscr{E}_\Omega(T)], \end{aligned}$$

where $C_{1,T}, C_{2,T} > 0$ they are constant depending on $T > 0$ e d.

Remark 2.19: The previous Theorem is also valid for solutions in $H^{2,2}(Q_{\Omega,T})$.

Observability inequality and unique continuation property

One of the most important results found in (TRIGGIANI *et al.*, 2002), are the result of observability and unique continuation. Next we present the version for subdomains of M.

Theorem 2.20. Let d as in the Definition 2.14 and in the context of the functional approaches and auxiliary functions defined above. Given $w \in H^{1,1}_{Q_{\Omega,T}}$ solution of the system (2.3), with p_0, p_1 satisfying (2.4)-(2.6). Then, considering (2.12) and $T > 0$ sufficiently large, there exists $k_T > 0$ depending on d, T and C_T such that

$$\int_0^T\int_{\Gamma_1^\Omega}\left(\frac{\partial w}{\partial n}\right)^2 d\Gamma_1^\Omega dt \geq k_T\big(\mathscr{E}_\Omega(0) + \mathscr{E}_\Omega(T)\big). \tag{2.13}$$

If in addition

$$\left.\frac{\partial w}{\partial n}\right|_{\Gamma_1^\Omega\times(0,T)} = 0, \tag{2.14}$$

then $w = 0$ in $Q_{\Omega,T}$.

Remark 2.21: The above Theorem is a modification of Theorem 8.1. presented in (TRIGGIANI *et al.*, 2002), which takes advantage of the key estimates about the system solutions (2.3) that generate the inequality (2.13), which they are at first in the strong space $H^{2,2}(Q_{\Omega,T})$ analogously to that shown in (LASIECKA; TRIGGIANI; ZHANG, 2000). The inequality (2.13) is satisfied for the weak solutions in $H^{1,1}(Q_{\Omega,T})$ by a passage of density, continuity and convergence, as in (LASIECKA; TRIGGIANI; ZHANG, 2000, Theorem 8.2.). When decreasing the regularity of the function p_0 it is necessary to have the same approximations with respect to the space $H^{2.2}(Q_{\Omega,T})$ and to be able to pass through a limit to the space $H^{1,1}(Q_{\Omega,T})$, is so considering $p_0 \in L^1(0,T;L^2(\Omega))$ instead of $L^2(0,T;L^2(\Omega))$, we need the estimates of p_0 with respect to the space $H^{2,2}(Q_{\Omega,T})$ (hypothesis (2.4)-(2.6)) ensuring that (2.13) is met over $H^{2,2}(Q_{\Omega,T})$, in addition to satisfying the elliptical regularity as it is shown in (LIONS; MAGENES, 1968). For the passage to the limit on the boundary, we use the continuation property for a trace function of the space $L^2(0,T;L^2(\partial\Omega))$ into space $L^1(0,T;L^2(\Omega))$ (cf. (??LASIECKA; TRIGGIANI, 1987; ??)). It is important to keep in mind that the hypothesis (2.6) allows an approximation of the energy of the system in space $H^{1,1}(Q_{\Omega,T})$, where the term $p_0 w$ will be absorbed by the energy at the initial moment.

2.2.2 Case II: Observability inequality and unique continuation property for finite collection overlapping subdomains of M

Another important result about Carleman estimates found in (TRIGGIANI *et al.*, 2002; LASIECKA; TRIGGIANI; ZHANG, 2000) is respect to manifold decomposed in a finite collection overlapping subdomains $\{\Omega_j\}_{j\in I}$ of M, such that

(Ω.1) $\bigcup_{j\in I}\Omega_j = M$,

(Ω.2) for all $\Omega_j \in \{\Omega_j\}_{j\in I}$, there is at least one Ω_k in the $\{\Omega_j\}_{j\in I}$ family such that $\Omega_j \cap \Omega_k \neq \emptyset$.

For this case we will focus on studying the system (1.1) with $p_0, p_1 : Q_{M,T} \to \mathbb{R}$ satisfying (1.2)-(1.4), where

$$Q_{M,T} := M \times (0,T], \; \Sigma_{M,T} := \partial M \times (0,T].$$

Also, we will denote by $\mathscr{E}_M : [0,\infty) \to \mathbb{R}$ to the energy of the system with respect to the norm in $H^1(M) \times L^2(M)$, this is

$$\mathscr{E}_M(t) = \int_M \left(|\partial_t w(t)|^2 + |\nabla w(t)|^2 + |w(t)|^2 \right) dM. \tag{2.15}$$

Functional environment

We will proceed analogously to what was developed for Case I. So, for this case, consider that for each $\Omega_j \in \{\Omega_j\}_{j\in I}$, there is a function $d_j : M \to \mathbb{R}$ such that it is satisfied

$(d_j.1)$ $d_j \in C^\infty(M), \;\; \min_{q\in\overline{\Omega}} d_j(q) > 0,$

$(d_j.2)$ $\nabla^2 d_j(X,X) > |X|_{\mathbf{g}}, \quad \forall X \in T_qM, \; \forall q \in \Omega_j,$

$(d_j.3)$ $\inf_{\Omega_j} |\nabla d_j| > 0.$

In addition, it will be considered:

(i) $Q_{j,T} := Q_{\Omega_j,T}, \; \Sigma_{j,T} := \Sigma_{\Omega_j,T}, \; Q_{j,T}(\sigma) := Q_{\Omega_j,T}(\sigma), \; Q^*_{j,T}(\sigma^*) := Q^*_{\Omega_j,T}(\sigma^*),$ for all $j \in I$,

(ii) repeating the same as in Case I, the functions related to d_j, will be denoted with the sub-index j,

(iii) Let $w \in C^2(Q_{M,\infty})$ solution of the system (1.1), we will define $\omega_j(x,t)$ by

$$\omega_j(x,t) = \chi_j \omega(x,t), \;\; j \in I, \tag{2.16}$$

such that it satisfies the system

$$\mathscr{A} w_j = \partial_t^2 w_j - \Delta w_j = p_0 w_j + p_1 w_j + [\partial_t^2 - \Delta - p_0 - p_1 \partial_t, \chi_j] w, \;\; j \in I, \tag{2.17}$$

$$w_j(\cdot,0) = \chi_j(\cdot,0)w(\cdot,0), \ \partial_t w_j(\cdot,0) = \chi_j(\cdot,0)w(\cdot,0) + \chi_j(\cdot,0)\partial w(\cdot,0), \tag{2.18}$$

where, $\chi_j(x,t)$ be a smooth cutt-off function such that

$$|\chi_j| < \infty, \ \chi_j(x,t) = 1 \ \text{on}\ Q_{j,T}(\sigma),$$

and $[\partial_t^2 - \Delta - p_0 - p_1\partial_t, \chi_j]$ is the commutator active only on $supp\chi_j$ for all $j \in I$.

Remark 2.22: Given the above definitions, we will have to:

(i) Boundary condition:

$$w_j|_{\partial M} = 0, \ \ j \in I. \tag{2.19}$$

(ii) Estimate for $w \in H^{2,2}(M)$ solution of system (1.1):

$$\int_0^T \int_M |\mathscr{A} w_j|^2 \leq C_{p_0 p_1 T}\left(\|\partial_t w_j\|^2_{L^2(0,T;L^2(M)} + \|w_j\|^2_{L^2(0,T;H^1(M))} + \int_0^T \mathscr{E}_M(t)\right), \ j \in I.$$

Carleman estimates: observability inequality and unique continuation property

Proceeding as in Case I, the following result will be obtained with respect to Carleman estimates:

Theorem 2.23. In the context of the above definitions, let $w \in C^2(Q_{M,\infty})$ solution of the system (1.1) with $w_j \in C^2(Q_{j,\infty})$ like in (2.16), with p_0, p_1 fulfilling (1.2)-(1.4), then for $\rho = 1 - c$ and $\varepsilon > 0$ small enough, can be defined $\beta := 4(1+7c)\sigma^* - \varepsilon\rho \max_{\overline{Q_{j,T}}}(|\partial_t \phi_j|^2 + |\nabla \phi_j|^2) > 0$ for all $\tau > 0$ large enough, such that

$$\begin{aligned} BT|_{\Omega_j,T} + C_{1,T}e^{2\tau\sigma}\int_0^T \mathscr{E}_M(t)dt \geq & [\tau\varepsilon\rho - 2C_{p_0p_1T}]\int_0^T\int_M e^{2\tau\phi_j}[|\partial_t w_j|^2 + |\nabla w_j|^2]dMdt \\ & + (2\tau^3\beta + \mathscr{O}(\tau^2) - 2C_{p_0p_1T})\int_{Q(\sigma)} e^{2\tau\phi_j}|w|^2 dxdt \\ & - C_{2,T}\tau^3 e^{-2\tau\delta}[\mathscr{E}_M(0) + \mathscr{E}_M(T)], \end{aligned}$$

where $C_{1,T}, C_{2,T} > 0$ they are constant depending on $T > 0$ e d_j.

Remark 2.24: (i) The previous Theorem is also valid for solutions in $H^{2,2}(M)$.

(ii) In (TRIGGIANI *et al.*, 2002) the Theorem 2.23 is proof for the case $I = \{1,2\}$. The extension for a finite number of domains is immediate, provided it is considered that if $\Omega_j \subset \{\Omega_j\}_{j\in I}$ has no geometric boundary, then

$$BT|_{\Omega_j,T} = 0.$$

(iii) We note that

$$\begin{aligned}\sum_{j\in I} BT|_{\Omega_j,T} =& 2\tau \sum_{j\in I} \int_{\Sigma_{j,T}} \theta^2 \left(\frac{\partial w_j}{\partial n}\right) \langle \nabla d_j, n\rangle d\Sigma_{j,T} \\ &\leq 2\tau \sum_{j\in I} \int_0^T \int_{\Gamma_1 \cap \partial\Omega_j} \theta^2 \left(\frac{\partial w_j}{\partial n}\right) \langle \nabla d_j, n\rangle dxdt,\end{aligned}$$

where

$$\Gamma_0 = \bigcup_{j\in I} \left\{ x \in \partial M \mid \langle \nabla d_j(x), n(x)\rangle = \frac{\partial d_j(x)}{\partial n} \leq 0 \right\}, \quad \Gamma_1 = \partial M \setminus \Gamma_0. \tag{2.20}$$

Regarding the observability inequality and the unique continuation property, we will have an analogous version to that of case I, which will be fundamental for the proof of the existence of a global attractor presented in Chapter 5.

Theorem 2.25. Let $(M, \mathbf{g})$ be a $N-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M such that there is a finite collection overlapping subdomains $\{\Omega_j\}_{j\in I}$. We assume that the following statements are holds

1. The collection $\{\Omega_j\}_{j\in I}$ satisfies $(\Omega.1) - (\Omega.2)$.
2. The collection $\{d_j\}_{j\in I}$ with $d_j : M \to \mathbb{R}$ satisfies $(d_j.1) - (d_j.3)$ for all $j \in I$.

3. $p_0, p_1 : Q_{M,T} \to \mathbb{R}$ satisfies (1.2)-(1.4) where $T > 0$ is a temporary constant sufficiently large.

Then, there exists $k_T > 0$ depending on d_j, T and C_T such that

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial n}\right)^2 d\Gamma_1 \, dt \geq k_T \left(\|(w, \partial_t w)(0)\|^2_{H^1(\Omega)\times L^2(\Omega)} + \|(w, \partial_t w)(T)\|^2_{H^1(\Omega)\times L^2(\Omega)} \right). \tag{2.21}$$

where $w \in H^{1,1}(Q_{M,T})$ is solution of the linear problem (1.1) and Γ_1 is given in (2.20). If in addition $\frac{\partial w}{\partial n}\Big|_{\Gamma_1 \times [0,T]} = 0$, then $w = 0$ in $M \times [0,T]$.

2.3 Dynamical systems and global attractors

In this section, we will review some topics related to the theory of dynamic systems in a Banach space $\mathscr{H}$. We will review the results presented in Temam (TEMAM, 2012), Hale (HALE, 2010), Babin and Vishik (BABIN,), and we will make a more detailed study of the (CHUESHOV; LASIECKA, 2008; CHUESHOV; LASIECKA, 2010).

Definition 2.26: A *d*ynamical system is a pair of objects $(S(t), \mathscr{H})$ consisting of a Banach space $\mathscr{H}$ and a family of continuous mappings $\{S(t) \mid t \geq 0\}$ of $\mathscr{H}$ into itself with the semigroup properties:

$$S(0) = I_{\mathscr{H}}, \;\; S(t+s) = S(t)S(s) \;\; \forall t, s \geq 0.$$

We also assume that the map $[0, \infty) \times \mathscr{H} \ni (t, x) \longmapsto S(t)(x) \in X$ is continuous for any $x \in \mathscr{H}$. Moreover, the liner operator A defined by

$$D(A) = \left\{ z \in \mathscr{H} \mid \lim_{t \to 0^+} \frac{S(t)z - z}{t} \text{ exists} \right\},$$

and

$$Az = \lim_{t \to 0^+} \frac{S(t)z - z}{t} = \frac{dS(t)z}{dt}\Big|_{t=0} \quad \text{for } z \in D(A)$$

is the *i*nfinitesimal generator of the dynamical system $(S(t), \mathscr{H})$, with domain $D(A)$.

Remark 2.27: Therewith $\mathscr{H}$ is called a *p*hase space and $S(t)$ is called an *e*volution semigroup (or evolution operator).

Definition 2.28: Let $(S(t),\mathscr{H})$ be a dynamical system.

(i) A closed set $B \subset \mathscr{H}$ is said to be *a*bsorbing for $(S(t),\mathscr{H})$ iff for any bounded set $D \subset \mathscr{H}$ there exists $t_0(D)$ such that $S(t)D \subset B$ for all $t \geq t_0(D)$.

(ii) $(S(t),\mathscr{H})$ is said to be (bounded, or ultimately) *d*issipative iff it possesses a bounded absorbing set B.

(iii) $(S(t),\mathscr{H})$ is said to be *a*symptotically compact iff there exists an attracting compact set K; that is, for any bounded set D we have

$$\lim_{t\to\infty} d_{\mathscr{H}}(S(t)D,K) = 0, \tag{2.22}$$

where $d_{\mathscr{H}}(A,B) = \sup_{x\in A} dist_{\mathscr{H}}(x,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_{\mathscr{H}}$.

(iv) $(S(t),\mathscr{H})$ is said to be *a*symptotically smooth iff for any bounded set D such that $S(t)D \subset D$ for $t > 0$ there exists a compact set K in the closure $\overline{D}$ of D, such that (2.22) holds.

Proposition 2.29. Assume that $(S(t),\mathscr{H})$ is a dissipative dynamical system. Then the following assertions are equivalent

- $(S(t),\mathscr{H})$ is asymptotically compact.
- $(S(t),\mathscr{H})$ is asymptotically smooth.

Definition 2.30: A bounded closed set $\mathscr{A} \subset \mathscr{H}$ is said to be *g*lobal attractor of the dynamical system $(S(t),\mathscr{H})$ iff the following properties hold

(i) $\mathscr{A}$ is an invariant set; that is, $S(t)\mathscr{A} = \mathscr{A}$ for $t \geq 0$,

(ii) $\mathscr{A}$ is uniformly attracting; that is, for all bounded set $D \subset \mathscr{H}$

$$\lim_{t\to\infty} d_{\mathscr{H}}(S(t)D,\mathscr{A}) = 0.$$

Definition 2.31: Let K be a compact set in a Banach space $\mathscr{H}$, the *f*ractal dimension $dim_f K$ of K is defined by

$$dim_f K = \limsup_{\varepsilon\to 0} \frac{\ln n(K,\varepsilon)}{\ln(1/\varepsilon)},$$

where $n(K,\varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set K.

Definition 2.32: Let $\mathcal{N}$ be the set of stationary points of the dynamical system $(S(t),\mathcal{H})$

$$\mathcal{N} = \{z \in \mathcal{H} \mid S(t)z = z \text{ for all } t \geq 0\}.$$

Definition 2.33: Given a set $B \subset \mathcal{H}$, its unstable manifold $\mathbb{M}^u(B)$ is the set of points $z \in \mathcal{H}$ that belong to some complete trajectory $\{y(t)\}_{t\in\mathbb{R}}$ and satisfy

$$y(0) = z \quad \text{and} \quad \lim_{t\to-\infty} d_{\mathcal{H}}(y(t),B) = 0.$$

Definition 2.34: A function $\Psi \in C(\mathcal{H},\mathbb{R})$ is called a Lyapunov functional if

(*i*) $t \mapsto \Psi(S(t)z)$ is decreasing for all $z \in \mathcal{H}$;

(*ii*) If $\Phi(S(t)z) = \Psi(z)$ for all $t \geq 0$, then z is a stationary point of $S(\cdot)$.

A dynamical system $(S(t),\mathcal{H})$ is called gradient if there exists a Lyapunov functional Ψ.

Now we will present a known result about the existence of global attractors for semigroups that will be the cornerstone for the proof of the existence of the global attractor in our work, the interested reader can consult Hale (HALE, 2010).

Theorem 2.35. Let $(S(t),\mathcal{H})$ be a dynamical system. We assume that:

$(H.1.)$ $(S(t),\mathcal{H})$ is gradient with Lyapunov functional $\Psi \in C(\mathcal{H},\mathbb{R})$,

$(H.2.)$ the set of stationary points $\mathcal{N}$ of the dynamical system $(S(t),\mathcal{H})$ is bounded in $\mathcal{H}$,

$(H.3.)$ $(S(t),\mathcal{H})$ is asymptotically compact,

$(H.4.)$ $\Psi(z) \to \infty$ if and only if $\|z\|_{\mathcal{H}} \to \infty$,

then, $(S(t),\mathcal{H})$ possesses a global attractor $\mathcal{A}$ such that $\mathcal{A} = \mathbb{M}^u(\mathcal{N})$.

Remark 2.36: Thanks to hypothesis $(H.1.)$ and $(H.4.)$ It is possible to consider that the dynamic system $(S(t),\mathscr{H})$ is asymptotically smooth instead of being asymptotically compact.

Since the system presented in Chapter 5 presents too many difficulties in the verification of asymptotic compactness, we introduce the concept of $q$uasi-stable, which will be used together with the results studied by Chueshov and Lasiecka (CHUESHOV; LASIECKA, 2008; CHUESHOV; LASIECKA, 2010). In addition, this result allows a study regarding the optimal regularity of the global attractor and the fractal dimension.

Definition 2.37: We say that the dynamical system $(S(t),\mathscr{H})$ is $q$uasi-stable on the set $B\subset\mathscr{H}$, if there exist a compact semi-norm $n_{\mathscr{H}}$ on $\mathscr{H}$ and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0,\infty)$, and $b(t)\in L^1(0,\infty)$ with $\lim_{t\to\infty}b(t)=0$, such that,

$$\|S(t)z^1-S(t)z^2\|^2_{\mathscr{H}}\leq a(t)\|z^1-z^2\|^2_{\mathscr{H}},$$

and

$$\|S(t)z^1-S(t)z^2\|^2_{\mathscr{H}}\leq b(t)\|z^1-z^2\|^2_{\mathscr{H}}+c(t)\sup_{0<s<t}|n_{\mathscr{H}}(u^1(s)-u^2(s))|^2, \tag{2.23}$$

for any $z^1,z^2\in B$, where $S(t)z^i=(u^i(t),\partial_t u^i(t)), i=1,2$.

Theorem 2.38. Let X and Y be reflexive Banach spaces, X is compactly embedded in Y. We endow the space $\mathscr{H}=X\times Y$ with the norm

$$\|z\|^2_{\mathscr{H}}=\|u_0\|^2_X+\|u_1\|^2_Y,\ \ z=(u_0,u_1).$$

We assume that $(S(t),\mathscr{H})$ is a quasi-stable dynamical system on every bounded forward invariant set $B\in\mathscr{H}$ (i.e. $S(t)B\subset B$) with the evolution operator of the form

$$S(t)z=(u(t),\partial_t u(t)),\ \ z=(u_0,u_1)\in\mathscr{H},$$

where the function $u(t)$ possess the propertie

$$u\in C([0,\infty),X)\cap C^1([0,\infty),Y).$$

Then, $(S(t),\mathscr{H})$ is asymptotically smooth. In addition, if the dynammical system $(S(t),\mathscr{H})$ possesses a global attractor $\mathscr{A}$ and is quasi-stable on $\mathscr{A}$ then the attractor $\mathscr{A}$ of has a finite fractal dimension. Moreover, we assume that (2.23) holds with the function $c(t)$ possessing the property $c_\infty = \sup_{t\geq 0} c(t) < \infty$, then any full trajectory $\{(u(t),\partial_t u(t)) \mid t \in \mathbb{R}\}$ that belongs to the global attractor enjoys the following regularity properties,

$$\partial_t u \in L^\infty(\mathbb{R},X) \cap C(\mathbb{R},Y), \quad \partial_t^2 u \in L^\infty(\mathbb{R},Y),$$

and there exists $R > 0$ such that

$$\|\partial_t u(t)\|_X^2 + \|\partial_t^2 u(t)\|_Y^2 \leq R^2, \quad t \in \mathbb{R},$$

where R depends on the constant c_∞, on the seminorm $n_{\mathscr{H}}$ in Definition 2.37, and also on the embedding properties of X into Y.

2.4 Preliminaries for the wave equation with critical forces

In this section we will show the basic results to prove the well-posedness of the wave equation described in Chapter 5. Thus, the basic theory will be followed regarding about the Cauchy abstract problem (cf. (BARBU, 1976; PAZY, 2012)). It is important to note that for the case with super critical forces, this theory does not apply, because the forces are not locally Lipschitz.

2.4.1 *Abstract Cauchy Problem*

Let $\mathbb{A} : D(\mathbb{A}) \subset \mathscr{H} \to \mathscr{H}$ be the infinitesimal generator of a dynamical system $(S(t),\mathscr{H})$ a and $\mathbb{F} : [0,T) \times \mathscr{H} \to \mathscr{H}$ an arbitrary function. Let us consider the following inhomogeneous initial value problem

$$\partial_t U = \mathbb{A}U + \mathbb{F}(t,U(t)), \quad t > 0 \tag{2.24}$$

$$U(0) = U_0 \in \mathscr{H}. \tag{2.25}$$

Definition 2.39:

(*i*) A classical solution of the system (2.24)-(2.25) in the interval $[0,T)$ it is a function $U:[0,T)\to\mathscr{H}$ if U is continuous on $(0,T]$, continuously differentiable on $(0,T)$, $U(t)\in D(\mathbb{A})$ for $0<t<T$ and (2.24)-(2.25) is satisfied on $[0,T)$,

(*ii*) A strong solution of the system (2.24)-(2.25) in the interval $[0,T)$ it is a function $U:[0,T)\to\mathscr{H}$ if U is differentiable almost everywhere on $[0,T]$ such that $\partial_t U\in L^1(0,T;\mathscr{H})$ and (2.24)-(2.25) is satisfied almost everywhere on $[0,T]$.

(*iii*) A weak solution of the system (2.24)-(2.25) in the interval $[0,T)$ it is a function $U:[0,T)\to\mathscr{H}$ if $U\in C([0,T];\mathscr{H})$ such that satisfy

$$U(t)=S(t)U_0+\int_0^t S(t-s)\mathbb{F}(t,U(s))ds,\ \ t\in[0,T].$$

Theorem 2.40. Let $\mathbb{F}:[0,\infty)\times\mathscr{H}\to\mathscr{H}$ be continuous in t for $t\geq 0$ and locally Lipschitz continuous in $\mathscr{H}$, uniformly in $[0,\infty)$ on bounded intervals. If $\mathbb{A}:D(\mathbb{A})\subset\mathscr{H}\to\mathscr{H}$ is the infinitesimal generator of the dynamical system $(S(t),\mathscr{H})$ then for every initial date $U_0\in\mathscr{H}$ there is a $T_{\max}\leq\infty$ such that the initial value problem (2.24)-(2.25) has a unique weak solution U on $[0,T_{\max})$. Moreover, if $T_{\max}<\infty$ then $\lim_{t\to T_{\max}^+}\|U(t)\|_{\mathscr{H}}=\infty$. In addition, if $\mathbb{F}$ is also continuously differentiable, there are strong solutions $U_n:[0,T_{\max})\to\mathscr{H}$, with $n\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\sup_{0\leq t<T_{\max}}\|U_n(t)-U(t)\|_{\mathscr{H}}=0.$$

Theorem 2.41. In the context of the previous Theorem with $\mathscr{H}$ a reflexive Banach space. If $U_0\in D(\mathbb{A})$ and $U\in C([0,T_{\max};\mathscr{H})$ is the weak solution of the problem (2.24)-(2.25), then U is the strong solution of this problem.

2.5 Preliminaries for the wave equation with supercritical forces

In this section we present some known results for the wave equations with supercritical forces, mainly with respect to the well-posedness and the existence of a global attractor. Therfore, we will state the most important results shown in Joly and Laurent (JOLY; LAURENT, 2013). It is important to note that the Strichartz estimaties, the result of Hörmander (HÖRMANDER, 1997) and Robbiano-Zuily (ROBBIANO; ZUILY, 1998) play a fundamental role in the study of the asymptotic dynamics of these equations (cf. (GINIBRE; VELO, 1985; GINIBRE; VELO, 1989; STRICHARTZ *et al.*, 1977; KAPITANSKI, 1989; BURQ; LEBEAU; PLANCHON, 2008; BLAIR; SMITH; SOGGE, 2009)).

One of the main difficulties with respect to proof of well-posedness for supercritical *three*−dimensional wave equations is that the forces are not locally Lipschitz for $p \in (3,5)$, this prevents the method from being semigroups, shown in the previous section. Ginibre and Velo (GINIBRE; VELO, 1989; GINIBRE; VELO, 1985), Kapitanski (KAPITANSKI, 1995) and Joly and Laurent (JOLY; LAURENT, 2013) manage to show the well-posedness of the problem from the study of certain intermediary spaces, initially proving the existence on Besov spaces and then via density get well-posed in the expected phase space. Kalantarov et al. (KALANTAROV; SAVOSTIANOV; ZELIK, 2016) get the well-posedness from a Galerkin scheme and studying the solutions in the sense of Shatah-Struwe. The prove in detail of the well-posedness, as well as the existence of a global attractor for the problem (1.13) is not a subject of study in the present work, given that the objective is to study the properties on said attractor.

Theorem 2.42 (Strichartz estimates). Let $T > 0$ and (q,r) satisfying

$$\frac{1}{q}+\frac{3}{r}=\frac{1}{2}, \quad q \in \left[\tfrac{7}{2},\infty\right]. \tag{2.26}$$

There exists $C = C(T,q) > 0$ such that for every $H \in L^1(0,T;L^2(M))$ and every

$(u_0, u_1) \in \mathcal{H}$, the solution u of

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = H & \text{in } M \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } M, \end{cases} \tag{2.27}$$

satisfies the estimate

$$\|u\|_{L^q(0,T;L^r(M))} \leq C\left(\|u_0\|_{H_0^1(M)} + \|u_1\|_{L^2(M)} + \|H\|_{L^1(0,T;L^2(M))}\right). \tag{2.28}$$

Remark 2.43: For the case $p = 3$, the previous Theorem is valid, considering the equation

$$\begin{cases} \partial_t^2 u - \Delta u = p_1 \partial_t u + H & \text{in } M \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } M, \end{cases} \tag{2.29}$$

provided that $p_1 \in L^\infty(0,T;L^\infty(M))$, and $p_1(x,t) < 0$ a.e. $(x,t) \in M \times [0,T]$.

Theorem 2.44 (Well-possedness). Assume that (1.14)-(1.19) hold. Then

(*i*) For any initial data $(u_0, u_1) \in \mathcal{H}$, problem (1.5) possesses a unique weak solution

$$u \in C(\mathbb{R}^+; H_0^1(M)) \cap C^1(\mathbb{R}^+; L^2(M)), \tag{2.30}$$

(*ii*) Given $T > 0$ and (q,r) satisfying (2.26), there exists a constant $C = C(T,q) > 0$ such that

$$\|u\|_{L^q(0,T;L^r(M))} \leq C\left(\|u_0\|_{H_0^1(M)} + \|u_1\|_{L^2(M)}\right) \tag{2.31}$$

(*iii*) Given $T > 0$ and two solutions $z^i = (u^i, \partial_t u^i)$ with initial value $z_0^i \in B$, where B is a bounded set of $\mathcal{H}$, $i = 1, 2$, one has

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \leq C_{BT}\|z_0^1 - z_0^2\|_{\mathcal{H}}^2, \quad \forall t \in [0,T], \tag{2.32}$$

where $C_{BT} > 0$ is constant.

Theorem 2.45. Let the assumptions of Theorem 2.44 and the initial data be more smooth, i.e.,

$$(u_0, u_1) \in \mathcal{H}^1 := [H^2(M) \cap H_0^1(M)] \times H_0^1(M). \tag{2.33}$$

Then, the corresponding weak solution (en el sentido de (6.3)) is more regular as well:

$$(u(t), \partial_t u(t)) \in \mathscr{H}^1, \tag{2.34}$$

for all $t \geq 0$.

Corollary 2.46. Under the assumptions of Theorem 2.44, then the solution operator of problem (1.5) generates a strongly continuous semigroup defined by

$$S(t) : \mathscr{H} \to \mathscr{H}, \quad (u_0, u_1) \mapsto (u(t), \partial_t u(t)), \quad t \geq 0, \tag{2.35}$$

where $(u, \partial_t u)$ is the weak solution corresponding to initial data (u_0, u_1).

Theorem 2.47 (Global attractor). Under the assumptions of Theorem 2.44. Then, the dynamical system generated by (1.13) in $\mathscr{H}$ is gradient and admits a compact global attractor $\mathscr{A} := \mathbb{M}^u(\mathscr{N})$, where $\mathscr{N}$ is the set of stationary points in the system.

CHAPTER

3

SHARP TYPE GEOMETRIC CONSTRUCTION

One of the most important results in the present work, is to construct an admissible damping region where the dissipation of different systems can be located such that this location in a sense is *s*harp. This has a huge physical sense when you want to optimize the region where the damping will be placed for a certain modeling. For example, when you want to study the vibrations on a plate, and you want to place a certain material that allows these vibrations to be damped, the most convenient thing is for that material to occupy minus possible space inside the plate in order to optimize expenses. This idea of *o*ccupy less is translated within the domain, that the admissible damping region possesses the smallest possible measure.

In this sense, the literature (cf. (BARBU, 1993; RAUCH; TAYLOR, 1975a; RAUCH; TAYLOR, 1975b; ENRIKE, 1990; ZUAZUA, 1991) among others) shows that, regarding the measure of the set, we can always build a damping region as small as we want.

More recently in (CAVALCANTI *et al.*, 2010; **??**) show that the interest to optimize the measurement of the region, now not only lies in optimizing the measurement with respect to the domain, but also in relation to the boundary of it, so in this meaning, these new admissible damping regions that optimize both measures are *sharp*.

A major problem that arises with respect to these constructions it is not possible to separate the construction of the admissible damping region with the equation to be studied. This does not allow the same construction to be used for a different group of systems, but it has to be repeated and adapted for each one of them, even though the intrinsic idea of the dissipation of the systems from the damping is a concept clearly geometric, from the study of geodesics in the manifold.

Thus, in this Chapter a clearly geometric construction of admissible damping regions in a *sharp* sense will be presented, allowing to study later uses in different systems (see Chapter 5), as well as certain geometric consequences (see Chapter 3) and consequences with Carleman estimates (see Chapter 4). To this end, we will divide the chapter into two sections, the first of which shows the basic definitions on the subject and the idea of the optimization of the measure with respect to the domain and its boundary. While the second section will be responsible for showing the distinctly geometric construction of the admissible damping regions in the sharp sense.

3.1 ε-controllable sets

The objective of this section is to study the construction of ε-controllable sets with the purpose of obtaining *sharp* admissible damping regions. Throughout the section, we will consider $(M, \mathbf{g})$ be a $N-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M.

Definition 3.1: We say that measurable subset ω of M, with the Lebesgue measure, is ε-controllable in measure if given $\varepsilon > 0$,

$$\mathrm{m}eas_M(\omega) + \mathrm{m}eas_{\partial M}(\omega \cap \partial M) < \varepsilon, \tag{3.1}$$

where $\mathrm{m}eas_A(B)$ represents the measure of B with respect to the Lebesgue measure defined in A. Moreover, the class of ε-controllable set of M is denoted by $\chi_\varepsilon(M)$.

Note that thanks to the properties of the Lebesgue measure (cf. eg. (??)), the following result is satisfied.

Proposition 3.2. Let $\varepsilon, \varepsilon_i > 0$ for all $i = 1, ..., N$ with $N \in \mathbb{N}$, then:

(i) If $\omega_j \in \chi_{\varepsilon_j}(M)$ then $\bigcup_{j=1}^{N} \omega_j \in \chi_{\varepsilon_1+\varepsilon_2+...+\varepsilon_N}(M)$,

(ii) The (arbitrary) intersection of elements of $\chi_\varepsilon(M)$ is an element of $\chi_\varepsilon(M)$,

(iii) Any set with null measures with respect to the measure of ∂M and M, belongs to $\chi_\varepsilon(M)$,

(iv) Given $\varepsilon' > 0$ such that $\varepsilon < \varepsilon'$ then $\chi_\varepsilon(M) \subset \chi_{\varepsilon'}(M)$,

(v) Given $M \subset \widetilde{M}$, then $\chi_\varepsilon(M) \subset \chi_\varepsilon(\widetilde{M})$,

(vi) Given $r \in \mathbb{R}$, $\omega \in \chi_\varepsilon(M)$ and $p \in M$ such that $r\omega + p := \{rx + p \ : \ x \in \omega\} \subset M$, then $r\omega + p \in \chi_{|r|\varepsilon}(M)$.

3.2 Construction to the sharp admissible damping regions on compact Riemannian manifolds

We will take advantage of the definition and properties of the sets ε-controllable to establish one of the main results of (CAVALCANTI *et al.*, 2010), which is summarized in the following

Theorem 3.3. Let $(M, \mathbf{g})$ be a connected compact Riemannian $N-$manifold of class C^∞ with smooth boundary ∂M. Then, given $\varepsilon > 0$ and $\varepsilon_0 \in (0, \varepsilon)$, the following holds:

1. There exists an open set $V \subset M$, with smooth boundary $\overline{\partial V \cap int(M)}$, that intercepts ∂M transversally and satisfies

$$\overline{M \setminus V} \in \chi_{\varepsilon_0}(M).$$

2. There exists a function $d : M \to \mathbb{R}$ satisfying:

(d.1.) $d \in C^3(\overline{V})$,

(d.2.) $\nabla^2 d(X,X) > 0, \ \forall X \in T_x M, \ x \in \overline{V}$,

(d.3.) $\inf_V |\nabla d| > 0$,

(d.4.) $\langle \nabla d, n \rangle < 0$ on $\partial M \cap \overline{V}$.

3. There exists an open set $\omega \in \chi_\varepsilon(M)$, such that

$$\overline{M \backslash V} \subset \omega, \quad \omega \cap V \in \chi_{\varepsilon-\varepsilon_0}(M).$$

Proof. We follow the ideas from (CAVALCANTI *et al.*, 2010; ??). We start by proving the existence of d and V locally, both in the case that a neighborhood of an interior point of M and a a neighborhood of boundary point of M. Then taking advantage of the compactness of M, we build globally both d and V. We split the proof in three parts: local analysis for points in the interior, local analysis for points on the boundary and global construction.

Claim 1: *For* any $p \in int(M)$ there exist a neigborhood V_p of p and a function $d : V_p \to \mathbb{R}$ such that satisfying $(d.1) - (d.3)$ with $V = V_p$.

Let $p \in int(M)$, so there is an orthonormal basis $(e_1, \ldots, e_N)$ of T_pM and a coordinate system $(x_1, \ldots, x_N)$ over a neighborhood V_p of p contained in some (U, ψ) chart of the atlas of M such that $\partial x_i(p) = e_i(p)$ for $i = 1, \ldots, N$. Note that the Chistoffel symbols respect to $(x_1, \ldots, x_N)$ satisfy that $\Gamma_{ij}^k(p) = 0$ (see, for instance, (CARMO, 1992) for details).

We define the function $d : V_p \to \mathbb{R}$ by

$$d(q) = \frac{1}{2} \sum_{j=1}^{N} x_j^2(q) + m.$$

for some $m > 0$. It is clear that $(d.1)$ is fulfilled and also

$$|\nabla d(p)| > 0, \quad \Delta d(p) = N, \quad \inf_{q \in V_p} d(q) \geq m > 0, \quad \text{for some } m > 0,$$

therefore $\nabla^2 d(p)(X,Y) = \mathbf{g}(X,Y)$ for all $X, Y \in T_pM$, which implies that $\nabla^2 d(p)(X,X) = |X|^2 > 0$ for all $X \in T_pM$.

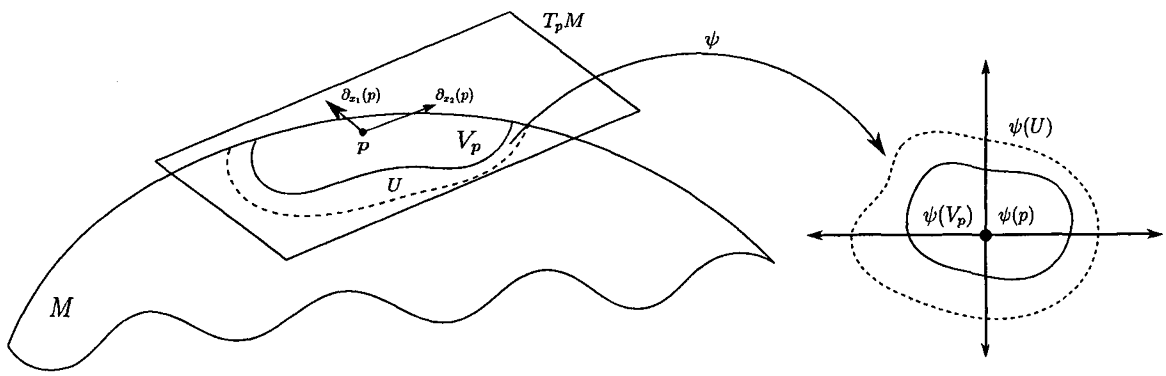

Figure 4 – The figure shows the previous constructions in the *two*-dimensional case. In this case (U, ψ) represents the chart with respect to an atlas. Note that the coordinate system can be considered the same for all elements in V_p.

Note that taking $V_p \subset\subset U$ small enough, $(d.2)$ is satisfied with $V = V_p$, and because the coordinate system is the same for any element in V_p, we can define the same function d on V_p such that

$$\nabla^2 d(q)(X,X) = |X|^2, \quad X \in T_qM,$$

for some other point $q \in V_p$, which proves $(d.3)$. Then the Claim 1 is fulfilled.

Claim 2: *Let* $p \in \partial M$. Then there exists a neigborhood V_p of p with smooth boundary $\overline{\partial V_p \cap int(M)}$ which intercepts ∂M transversally and a functión $d : V_p \to \mathbb{R}$ satisfying $(d.1) - (d.4)$ with $V = V_p$.

Fix $p \in \partial M$. Due to a Riemannian geometry result (CAVALCANTI *et al.*, 2010, Lemma 6.4.), there exist a Riemannian manifold $\widetilde{M}$ and an isometric immersion $f : M \to \widetilde{M}$ such that $\overline{f(M)} \subset int(\widetilde{M})$.

Taking the orthonormal basis $(e_1, \ldots, e_N)$ of $T_p\widetilde{M}$ such that $n(p) = -e_1$ be the outward normal vector field in the point p respect to ∂M. Proceeding as in the previous case taking $\widetilde{M}$ instead of M we have that there exists a neighborhood $\widetilde{V}'_p \subset \widetilde{M}$ of p. Due to the regularity of $\partial \widetilde{V}'_p \cap \partial M$ there is an open set $\widetilde{V}_p \subset\subset \widetilde{V}'_p$ with $p \in \widetilde{V}_p$ such that $n(q) = -e_1$ for all $q \in \overline{\widetilde{V}_p} \cap \partial M$. Moreover, we define $d : \widetilde{V}_p \to \mathbb{R}$ such that

$$d(q) = x_1(q) + \frac{1}{2}\sum_{j=1}^{N} x_j^2(q) + m,$$

for some $m > 0$. It is evident that $\inf_{q\in V_p} |\nabla d(q)| > 0$, $\inf_{q\in V_p} d(q) \geq m$, $\Delta d(p) = N$, and $\nabla^2 d(p)(X,Y) = \mathbf{g}(X,Y)$ for all $X,Y \in T_pM$ and proceeding analogously that Claim 1, it holds that $(d.1)-(d.3)$ for $V = \widetilde{V}_p$. Additionally,

$$\langle \nabla d(q), n(q)\rangle < 0, \quad q \in \widetilde{V}_p. \tag{3.2}$$

Finally, as shown in Figure 5, we can find a neighborhood $V_p \subset \widetilde{V}_p \cap M$ such that $\overline{\partial V_p \cap int(M)}$ intercepts ∂M transversally, completing the proof of claim 2.

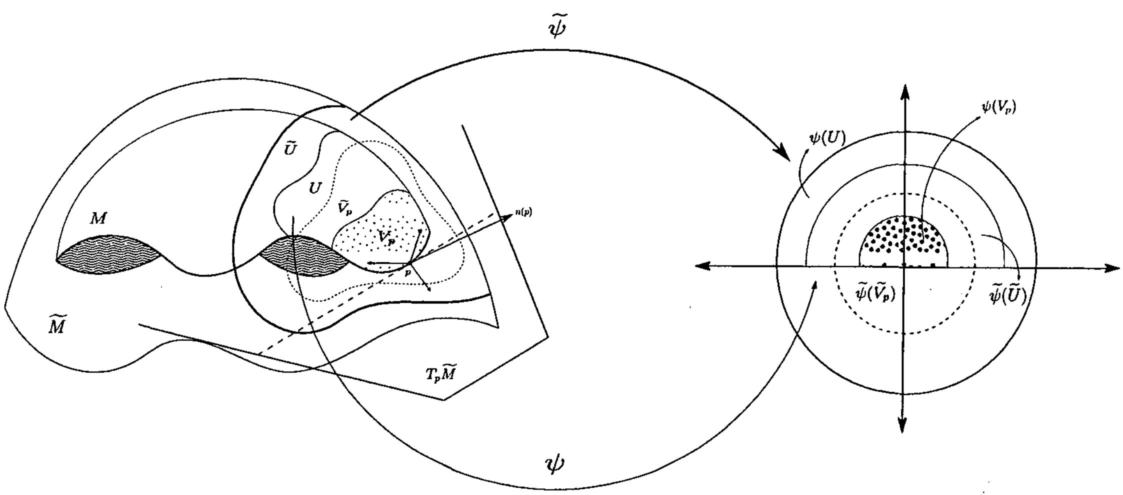

Figure 5 – In the figure (U,ψ) and $(\widetilde{U},\widetilde{\psi})$ represent the charps M and $\widetilde{M}$ respectively containing p. Note that $\psi(V_p) \subset \psi(U) \subset \widetilde{\psi}(\widetilde{U})$ so we can use the same coordinate system for every point in $V_p \subset M$.

Claim 3: Conclusion of Proposition 3.3.

In order to prove this claim we borrow the next auxiliary Lemma from (CAVALCANTI *et al.*, 2010).

Lemma 3.4. *((CAVALCANTI et al., 2010, Lemma 6.9))* Consider two subsets A and B such that $d(A,B) := \inf_{(x,y)\in A\times B} d(x,y) > 0$. Suppose that $\overline{A}$ and $\overline{B}$ are compact. Then there exist open subsets $O_A \supset\supset A$ and $O_B \supset\supset B$ with smooth boundaries such that $d(O_A, O_B) > 0$. Moreover, there exists a smooth (cut-off) function $\rho : M \to \mathbb{R}$ such that $\rho|_{O_A} = 1, \rho|_{O_B} = 0$ and $\rho(M) \subset [0,1]$.

Remark 3.5: The sets O_A and O_B in the above lemma can be constructed, for any $\varepsilon \in (0, d(A,B)/3)$, such that $A \subset\subset O_A \subset\subset A_\varepsilon$ and $B \subset\subset O_B \subset\subset B_\varepsilon$, where

$$A_\varepsilon = \{x \in M \mid d(x,A) < \varepsilon\}, \quad B_\varepsilon = \{x \in M \mid d(x,B) < \varepsilon\},$$

and $d(x,Y)$ is the usual point-set distance defined by $d(x,Y) = \inf_{y\in Y} d(x,y)$ with $d(x,y) = |x-y|_{\mathbf{g}}$, since M is compact.

Repeating the strategy applied in the Claim 2, we can extend M to a Riemannian manifold $\widetilde{M}$ such that, for each $p \in M$, one can choose a neighborhood $\widetilde{W_p}$ of p, and a function $d_p \in C^\infty(\widetilde{W_p})$ such that

- If $p \in int(M)$, then choose $\widetilde{W_p} = V_p$ as in the Claim 1.
- If $p \in \partial M$, then choose $\widetilde{W_p} = \widetilde{V_p} \subset int(\widetilde{M})$ as in the Claim 2.

Then, due to the compactness of M, we can choose a finite sub-cover $\{\widetilde{W_j}\}_{j=1}^k$ of M such that if $p \in W_j$ for some $j = 1,...,k$ denote by $\widetilde{d_j} = d_p|_{W_j}$. Let $B = \bigcup_{j=1}^k \partial\widetilde{W_j} \cap M$ where clearly $M \setminus B$ is an open subset of M. As seen in Figure 6, denoting $(M\setminus B)\cap\widetilde{W_1}$ for W_1 and $(M\setminus B)\cap\left(\widetilde{W_j}\setminus\bigcup_{i=1}^{j-1}\widetilde{W_i}\right)$ for W_j for $j = 2,...,k$, it is show that $M \setminus B = \bigcup_{j=1}^k W_j$.

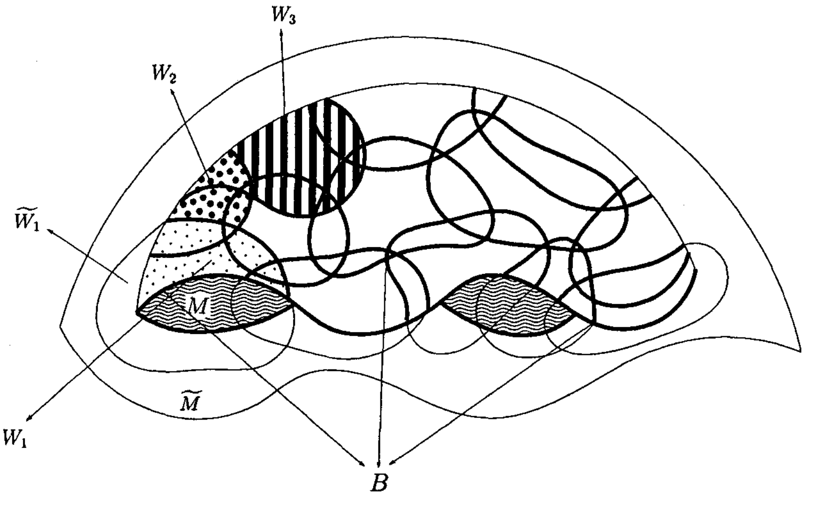

Figure 6 – Note that each $W_j \subset \widetilde{W_j}$ for $j = 1,..,k$ is an open set of M being the union of connected components of $M \setminus B$ where it is well defined $\widetilde{d_j}$.

On the other hand, fixed $\varepsilon > 0$, for each $\varepsilon_0 \in (0,\varepsilon)$ and W_j with $j = 1,...,k$, it is possible to build an open U_j of M such that $\overline{U_j} \subset W_j$ and $meas_M(W_j \setminus$

$U_j) < \frac{\varepsilon_0}{2k}$ (see Figure 7). In addition, if W_j is a neighborhood of a boundary point of M, then we can take U_j such that $meas_{\partial M}(\partial M \cap (W_j \setminus U_j)) < \frac{\varepsilon_0}{2k}$ (see (CAVALCANTI *et al.*, 2010, Lemma 6.7) for more details).

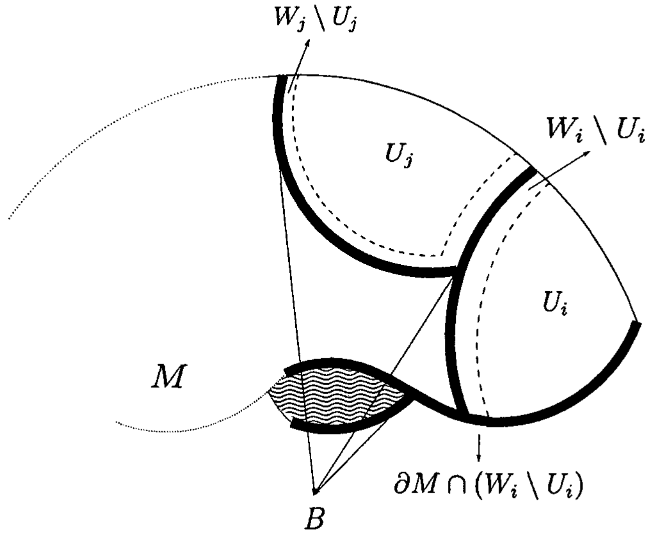

Figure 7 – Given $j, i = 1, ..., k$, we have total control over the measure of $W_i \setminus U_i$ and $\partial M \cap (W_i \setminus U_i)$ for M and ∂M provided they are positive. Note that if $\frac{\varepsilon_0}{2k} \geq \min\{meas_M(W_i), meas_{\partial M}(\partial M \cap W_i)\} > 0$ it is possible to choose some $0 < \frac{\varepsilon'}{2k} < \{meas_M(W_i), meas_{\partial M}(\partial M \cap W_i)\}$ such that the measure of the aforementioned sets are less than $\frac{\varepsilon_0}{2k'}$ where $k' = \frac{k\varepsilon_0}{\varepsilon'}$.

Because $\overline{U_j} \subset W_j$, we can define $d_j = \widetilde{d_j}|_{U_j}$. Also, from the compactness of B and $\overline{U_j}$, there are numbers $\delta_j > 0$, $j = 1, ..., k$, such that $d(B, \overline{U_j}) = \delta_j$. Then by Lemma 3.4, exist open sets $V_j \supset\supset U_j$ and $O_j \supset\supset M \setminus W_j$ of M with smooth boundaries, and a function $\rho_j : M \to \mathbb{R}$ such that $\rho_j|_{V_j} = 1, \rho_j|_{O_j} = 0$ and $\rho_j(M) \subset [0,1]$. Note that in view of the Remark 3.5, we can construct V_j such that $V_j \subset W_j$, so that $\{V_j\}_{j=1}^k$ is a disjoint family of open and $\widetilde{d_j}$ is defined on each V_j.

Note that if V_j is a neighborhood intersecting ∂M, then it is possible assume that V_j has smooth boundary $\overline{\partial V_j \cap int(M)}$ that intercepts ∂M transversally. Thus, we define $d_j = \widetilde{d_j}|_{\overline{V_j}}$, $\rho = \sum_{j=1}^k \rho_j$ and

$$V = \bigcup_{j=1}^{k} V_j, \tag{3.3}$$

so that $\rho|_V = 1$ and (3.2) it is satisfied.

For the construction of d, it is enough to define

$$d(x)=\begin{cases} d_j(x)\rho(x) & \text{if } x\in W_j,\\ 0 & \text{otherwise,}\end{cases} \tag{3.4}$$

which clearly satisfy $(d.1)$-$(d.4)$.

Finally, from the construction of V, there is an open set $\omega\supset\overline{M\setminus V}$ such that $\omega\cap V\in\chi_{\varepsilon-\varepsilon_0}(M)$. From (3.2) we see that ω is ε-controllable. This ends the proof to Theorem. □

Remark 3.6: The choice of $\varepsilon_0\in(0,\varepsilon)$ is independent of any other condition, that is, the result is valid for any $\varepsilon_0\in(0,\varepsilon)$ that is chosen. This value represents the measure that is to be granted to the set $\overline{M\setminus V}$ where the damping will be effective that will allow to prove a unique continuation property (more details see Chapter 4).

We note that in the Theorem 3.3, once taken ε_0, V and the function d, the choice of ω involves mainly three properties:

(a) ω is an open subset to M,

(b) $\left(\overline{M\setminus V}\right)\dot{\cup}(\omega\cap V)=\omega$,

(c) $\overline{M\setminus V}\in\chi_{\varepsilon_0}(M)$ and $\omega\cap V\in\chi_{\varepsilon-\varepsilon_0}(M)$.

Therefore, let $\overline{M\setminus V}\subset M$, it is possible to build different sets ω such that $\omega\cap V\in\chi_{\varepsilon-\varepsilon_0}(M)$. This motivates the definition of the class of admissible ε-controllable sets

$$[\omega_\varepsilon]=\left\{\omega\in\chi_\varepsilon(M)\,|\,\omega \text{ is given by Theorem 3.3 for some } \varepsilon_0\in(0,\varepsilon)\right\}, \tag{3.5}$$

which is sharp in the sense of (CAVALCANTI *et al.*, 2010). This class be called the class of sharp admissible damping regions associated to ε.

CHAPTER

4

GEOMETRICAL CONSEQUENCES

The construction presented in 3.3 allows a series of consequences applied both in the geometry and in the equations, in this Chapter it will show some relevant applications and facts in the geometry regarding said result. Throughout the Chapter, we will consider $(M,\mathbf{g})$ be a $N-$dimensional connected compact Riemannian manifold of class C^∞ with smooth boundary ∂M.

4.1 Geometric control condition and the sharp admissible damping regions

The objective of this section is to study the consequence of sets ε-controllable built in the Theorem 3.3 regarding the geometric control condition (GCC for short), this is:

(GCC) The set ω satisfies $(GCC) \iff$ There exists $T_0 > 0$ such that every geodesic traveling at speed 1 and issued at $t = 0$ enter the open set ω before the time T_0.

Which implies having a observability inequality (cf. (BARDOS; LEBEAU; RAUCH, 1992; BURQ; GÉRARD, 1997)) of type

$$\|(u(0),\partial_t u(0)\|^2_{\mathscr{H}} \leq C_T \int_0^T \int_\omega |\partial_t u|^2 dxdt, \ T > T_0, \tag{4.1}$$

for $(u,\partial_t u) \in H_0^1(M) \times L^2(M)$ solution of the problem (1.5) with $f = h = a(x) = 0$ and some constant $C_T > 0$ that depends on T.

Note that each element of the class $[\omega_\varepsilon]$ contains $\overline{M \setminus V}$, and this property will play a fundamental role in the proof the unique continuation theorem in the next Chapter. The part $\omega \cap V$ guarantees that the set ω is open in M and because we have a control $\varepsilon - \varepsilon_0$ in its measure, it is possible to build all the elements of this class, which will allow to prove the existence of a $\omega \in [\omega_\varepsilon]$ that satisfies the (GCC), that is, the part $\omega \cap V$ has a close relationship with the (GCC) and the observability inequality (4.1).

Another detail to take into account is regarding the *o*bservability is seen in (CAVALCANTI *et al.*, 2010), where the authors prove the exponential decay of energy for the system

$$\begin{cases} \partial_t^2 u - \Delta u + a(x) g(\partial_t u) = 0 \text{ in } M \times (0,\infty), \\ u = 0 \text{ on } \partial M \times (0,\infty), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ x \in M, \end{cases} \tag{4.2}$$

where $a(x) > 0$ on a open set $\omega \in (V,d)$ and g is a monotonically increasing function such that $k|s| \leq |g(s)| \leq K|s|$ for all $|s| \geq 1$. This result, together with the fact that ω is ε-controllable, shows that the system (4.2) has sharp localized damping, which inspires the name for $[\omega_\varepsilon]$. Additionally, it is proved in this same work, that the solution of the system (4.2) for $a = 0$ satisfies an inequality of the type (4.1), which makes us think that the construction of V and d directly proof the property (GCC) for ω, but, for exemple, for the typical case of M as the hemisphere, the observability inequality is satisfied but not the (GCC) for ω.

Motivated by this fact, we will prove that it is always possible to choose a $\omega \in [\omega_\varepsilon]$ satisfying Theorem 3.3 that complies with the (GCC).

Proposition 4.1. Fix $\varepsilon > 0$. In the context of Theorem 3.3, given $\varepsilon_0 \in (0,\varepsilon)$, $V \subset M$ as in (3.3) and $d : M \to \mathbb{R}$ as in (3.4), then it is possible to choose the open set $\widetilde{\omega} \in [\omega_\varepsilon]$ satisfying the (GCC).

Proof. Because for every $p \in V_j$ with $j = 1, 2, ..., k$ of the definition of V in Theorem 3.3, there exists a N_j totally normal neighborhood of p, such that $N_j \subset V_j$. Thus, for all $q \in N_j$ and $X \in T_qM$, the only geodesic such that $\gamma(0) = q, \gamma_t(0) = X$, satisfies that $\gamma(0) \subset \exp(B_\delta(0))$, where $\delta > 0$ depends on N_j, but for the compactness of M, all geodesic is defined above $\mathbb{R}$, then all geodesic associated with q and X intersects N_j, that is, the whole geodesic associated with every point of N_j and each field of this point intersects N_j. Again by the compactness of M, we can choose the finite family of totally normal neighborhoods $\{N_j^i\}_{i=1,...,r}$ that cover V_j, making them sufficiently small and repeating the process of constructing V on these families, there will be a $\widetilde{\omega} \in [\omega_\varepsilon]$ such that $\omega \subset \widetilde{\omega}$ and $\tilde{\omega}$ satisfies (GCC). □

4.2 Smooth boundary of the sharp admissible damping regions

On the other hand, note that the Theorem 3.3, allows the construction of the set V with smooth boundary, in this sense, you have to understand the smoothness of ∂V in the following sense (see (CARMO, 1992; CAVALCANTI *et al.*, 2010; ABRAHAM; MARSDEN; RATIU, 2012) among others):

Definition 4.2: An open set $V \subset M$ is an open set of the topological space M. Therefore it can intercept the boundary. We say that an open subset $V \subset M$ has smooth boundary $\overline{\partial V \cap int(M)}$ if $\overline{\partial V \cap int(M)}$ is a smooth hypersurface of M with smooth boundary $\overline{\partial V \cap int(M)} \cap \partial M$. Therfore the term smooth ignores $\partial V \cap \partial M$.

Thus, we will try to gain regularity at the boundary of ω, given as in Theorem 3.3, from the regularity of V. For this, the following classical result of Riemannian Geometry shows the existence of an intermediate set between $\overline{M \setminus V}$ and ω with the same regularity of V.

Proposition 4.3. Let M be a differentiable manifold with boundary. Suppose that $V \subset M$ is an open subset with smooth boundary $\overline{\partial V \cap int(M)}$ which intercepts

∂M transversally. Let ω be an arbitrary open subset of M such that $\overline{M\setminus V}\subset\omega$. Then there exists an open subset $W\subset M$ with smooth boundary $\overline{\partial W\cap int(M)}$ which intercepts ∂M transversally such that $\overline{M\setminus V}\subset W$ and $\overline{W}\subset\omega$.

Theorem 4.4. Fix $\varepsilon>0$. In the context of the Theorem 3.3, given $\varepsilon_0\in(0,\varepsilon)$, $V\subset M$ as in (3.3) and $d:M\to\mathbb{R}$ as in (3.4), then exists a set (GCC) $W\in[\omega_\varepsilon]$ with smooth boundary $\overline{\partial W\cap int(M)}$ that intersects transversally to ∂M.

Proof. Fix $\varepsilon>0$, for some $\varepsilon_0\in(0,\varepsilon)$, we have that exist $\omega\in\chi_\varepsilon(M)$, V and d as in Theorem 3.3, and by Proposition 4.1, we can consider ω satisfying (GCC). Therefore, by Theorem 3.3 and by Proposition 4.3, there is a $\widetilde{\delta}\in(0,\varepsilon-\varepsilon_0)$ small enough such that there is a set $\widetilde{\omega}\in[\omega_\varepsilon]$ satisfying (GCC) with $V\cap\widetilde{\omega}\in\chi_{\widetilde{\delta}}(M)$, and exists an open subset $W\subset M$ with smooth boundary $\overline{\partial W\cap int(M)}$ which intercepts ∂M transversally such that $M\setminus V\subset\subset\widetilde{\omega}\subset\subset W\subset\subset\omega$. The chain of inclusions shows that $\left(\overline{M\setminus V}\right)\dot{\cup}(W\cap V)=W$ and that there is a $\delta_0\in(\widetilde{\delta},\varepsilon-\varepsilon_0)$ such that $V\cap W\in\chi_{\delta_0}(M)$. Then, given that $W\in\chi_\varepsilon(M)$ we have to $W\in[\omega_\varepsilon]$ and as $\widetilde{\omega}$ satisfying (GCC), then W is also (GCC). This ensures that it is always possible to choose a set (GCC) $W\in[\omega_\varepsilon]$ with smooth boundary $\overline{\partial W\cap int(M)}$ that intersects transversally to ∂M. □

4.3 Construction of the local escape vector fields

An important hypothesis in the analysis of observability, unique continuation theorem and control theory in several PDE's, is a geometric condition on the domain. A classical condition (see (YAO, 2011)) is given by the existence of an escape vector field over an open set in the manifold, in this sense we have the following definition

Definition 4.5: Given $(M,\mathbf{g})$ a Riemannian manifold. Let $\Omega\subset M$ be an open set of M, and let H be a vector field on Ω. The vector field H is said to be an escape vector field for Ω of there is a function v on $\overline{\Omega}$ such that

$$\nabla H(X,X)\le v(x)|X|_{\mathbf{g}},\quad \text{for } X\in T_pM,\quad p\in\Omega,$$

and

$$\inf_{x\in\overline{\Omega}} |H| > 0, \ \inf_{x\in\Omega} \nu(x) > 0.$$

This field H is commonly used to divide the boundary of $\overline{\Omega}$ into two regions depends on the boundary conditions of the equations to be studied. This division is given by a signal condition on $\langle H, n\rangle$ for all $x \in \partial\Omega$, where n represents the outward unit normal field along the boundary ∂M. For example, in Theorem ref prop-marcelo1, $H := \nabla d$ is a escape vector field for V, with the signal condition

$$\langle \nabla d, n\rangle < 0, \ \partial M \cap \overline{V}.$$

In Cavalcanti et al. (CAVALCANTI *et al.*, 2010), we observe the need for a vector field escape for an open small enough that intersects $M \setminus V$ and is included in ω, in order to prove an inequality of observability (4.1) and unique continuation theorem.

The Theorem 3.3, allows us to construct locally within ω an escape vector field in the context of sets ε-controllable, independent of the equation. So we have the following result.

Theorem 4.6. Fix $\varepsilon > 0$. In the context of the Proposition 3.3, given $\varepsilon_0 \in (0, \varepsilon)$, $V \subset M$ as in (3.3) and $d : M \to \mathbb{R}$ as in (3.4), then for all $\omega \subset [\omega_\varepsilon]$ exists a smooth escape vector field $H \in \mathfrak{X}(M)$ over an open small enough $U \subset M$, such that

$$\langle H, n\rangle = 1, \ \text{ on } \partial M \cap (M \setminus V).$$

Proof. It is enough to make the geometric construction made in Cavalcanti et al. (CAVALCANTI *et al.*, 2010, Theorem 5.1.), considering the existence of a $W \in [\omega_\varepsilon]$ as in Theorem 4.4. □

Note that the H construction in the previous Theorem is a natural extension of Lion's vector field defined in (LIONS, 1988, Lemma 3.1.) for the Euclidean case.

4.4 Decomposition in overlapping sets

In order to prove a unique observability and continuation result from Carleman estimates (see 2.25), it is often not possible to hide a single escape vector field over the entire domain of the equation. For example, for the Euclidean case, as can be seen in (LASIECKA; TRIGGIANI; ZHANG, 2000) for domains with a part of the boundary of the *f*lat type, it is usually necessary to have more than one vector field escape allowing to divide the total boundary from different conditions of sign over $\langle H,n\rangle$ (see Figure 8)

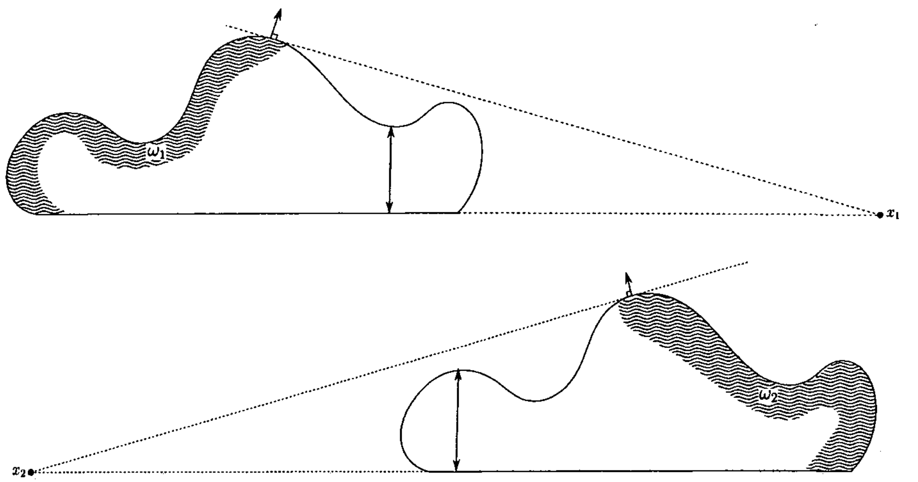

Figure 8 – Let us consider $(\mathbb{R}^2, |\cdot|_{\mathbb{R}^2})$. In this case the region $\omega = \omega_1 \cup \omega_2$ with $\omega_1 \cap \omega_2 \neq \emptyset$, where the functions associated with each one ω_i are given by $d_i(x) = \|x - x_i\|^2$, with the escapes vector fields $h_i(x) = 2\nabla d_i(x)$ for all $i = 1,2$. This generates a sign condition $\langle h_i, n\rangle$ for every point on the boundary of the domain.

This is due to the convexity of the functional, since it can not always be guaranteed that, for a *o*ne escape vector field, it is fulfilled that does not have critical points over the whole domain, or that the Hessian does not cancel; but locally each of the functional ones if they possess these properties.

As seen in the Theorem 2.25, we present a generalization of the applications of the Carleman estimates for domains decomposed in a finite collection overlapping subdomains, such that an escape vector field is associated to each of the elements of the decomposition. With this in mind, we proof a decomposition by overlapping domains over ω.

Theorem 4.7. Fix $\varepsilon > 0$. In the context of Theorem 3.3, given $\varepsilon_0 \in (0,\varepsilon)$, $V \subset M$ as in (3.3) and $d : M \to \mathbb{R}$ as in (3.4), then for all $\omega \subset [\omega_\varepsilon]$ exits a finite collection overlapping subdomains $\{\omega_j\}_{j\in\Lambda}$ in M such that

(ω_j.1) $\bigcup_{j\in\Lambda} \omega_j = M$,

(ω_j.2) for all $\omega_j \in \{\omega_j\}_{j\in\Lambda}$, there is at least one ω_k in the $\{\omega_j\}_{j\in\Lambda}$ family such that $\omega_j \cap \omega_k \neq \emptyset$.

Moreover, for each $\omega_j \in \{\omega_j\}_{j\in\Lambda}$, there is a function $d_j : M \to \mathbb{R}$ such that is fulfilled

$(d_j.1)_\omega$ $d_j \in C^\infty(M)$, $\min_{q\in\overline{\omega_j}} d_j(q) > 0$,

$(d_j.2)_\omega$ $\nabla^2 d_j(X,X) > |X|_g$, $\forall X \in T_qM$, $\forall q \in \omega_j$,

$(d_j.3)_\omega$ $\inf_{\omega_j} |\nabla d_j| > 0$,

$(d_j.4)_\omega$ there is a set of indexes $J \subset \Lambda$ such that $\overline{\omega_j} \cap \partial M \neq \emptyset$ for all $j \in J$. Moreover:

1. $\left(\bigcup_{j\in\Lambda\setminus J} \omega_j\right) \cap \partial M = \emptyset$,
2. $\langle \nabla d_j, n\rangle \geq 0$ on $\partial M \cap \overline{\omega_j}$, $\forall j \in J$.

Proof. We note that by constructing ω, $\omega \cap \partial M$ can be divided into a finite number of connected components denoted by $\{\Gamma^j_\omega\}_{j=1}^l$ for some $l \in \mathbb{N}$.

Observe that, setting $j = 1,2,...,l$ and proceeding analogously to the Theorem 3.3, given $p \in \Gamma^j_\omega$, we can take an orthonormal basis $(\partial_{x_1},\dots,\partial_{x_N})$ of $T_p\widetilde{M}$, where $\widetilde{M}$ is the Riemannian manifold given in Claim 2 in proof of Theorem 3.3. Thus, we can assume that $n(p) = -\partial_{x_1}(p)$ outward unit normal field at the point p in ∂M. Then, for a neighborhood $V_p \subset \widetilde{M}$ of p small enough that $n(x) = -\partial_{x_1}(x)$ for all $x \in \overline{V_p} \cap \partial M$, we define the function $d_j : V_p \to \mathbb{R}$ given by

$$d_j(q) = -\alpha x_1(q) + \beta + \frac{1}{2} x_j^2(q), \quad q \in V_p,$$

for an $\alpha > 0$ large enough, and $\beta > \frac{C^2}{2}$, such that

$$\inf_{q\in V_p} d_j(q) \geq m > 0, \quad \text{for some } m > 0,$$

$$\langle \nabla d_j(p), n(p)\rangle < 0, \ \ \langle \nabla d_j(q), n(q)\rangle \leq 0, \ \ q \in V_p \cap \partial M.$$

Also, we have to $d_j \in C^\infty(M)$, $\Delta d_j(q) > 0$ and $\nabla^2 d_j(q)(X,Y) = \mathbf{g}(X,Y)$ for all $X,Y \in T_qM$ with $q \in V_p$.

On the other hand, without loss of generality, we can assume that $meas_{\partial M}(\Gamma^j_\omega)$ is small enough, that there is an open set $V_j \subset V_p$ neighborhood of p and exist a $\delta_j > 0$, such that $\Gamma^j_\omega \subset V_j$ and $\mathcal{O}_{\delta_j}(V_j) \cap \Gamma^k_\omega = \emptyset$ for all $k \in \{1,2,...,l\} \setminus \{j\}$.

Considering the function of the partition of the unit $\rho_j : M \to \mathbb{R}$ such that $\rho_j|_{V_j} = 1, \rho_j|_{M\setminus\left(\bigcup_{j=1}^{l} \mathcal{O}_{\delta_j}(V_j)\right)} = 0, \rho(M) \subset [0,1]$, we have that there is an open set $\omega_1 = \bigcup_{i=1}^{l} V_j \cap \omega \subset \omega$ and a function $d_1 : M \to \mathbb{R}$ in $C^\infty(M)$ such that $d_1 = \Sigma_{i=j}^{l} \rho_j d_j$ satisfies

(*i*) $\nabla^2 d_1(X,X) > |X|_{\mathbf{g}}, \ \forall X \in T_pM, \ \forall p \in \omega_1,$

(*ii*) $\inf_{q\in\omega_1} |\nabla d_1(q)| > 0, \ \ \min_{q\in\overline{\omega}_1} d(q) > 0,$

(*iii*) $\langle \nabla d_1, n\rangle \geq 0,$ on $\partial M \cap \overline{\omega}_1,$

(*iv*) $(\omega \setminus \omega_1) \cap \partial M = \emptyset.$

Now, on the open set $\omega \setminus \overline{\omega_1} \subset int(M)$, proceeding similarly to the previous one, and given the compactness of M, there exists a finite index $\widehat{\Lambda}$, such that ω can be decomposed into $\{\omega_j\}_j \in \Lambda$ satisfying $(\omega_j.1) - (\omega_j.2)$, to which are associated functionals $d_j : M \to \mathbb{R}$ fulfilling $(d_j.1)_\omega - (d_j.3)_\omega$. Finally, we note that not necessarily ω_1 is connected. Then, decomposing the set into its connected components, there is a finite set of indexes J such that $(d_j.4)_\omega$ is fulfilled. In addition, it is important to note that, as ω is connected, then for everything ω_j with $j \in I$, there will be a $i \in \Lambda$ such that $\omega_i \cap \omega_j \neq \emptyset$. Thus, taking $\Lambda = \widehat{\Lambda} \cup I$ the Theorem is proved.

□

Remark 4.8: It is possible to construct a function $d_\omega \in C^3(M)$ that satisfies $(d_j.1)_\omega - (d_j.4)_\omega$ mostly ω under certain conditions on the sectional curvature using inf-convolutions, provided that you can construct convex functions, strictly convex or lower semicontinuous functions on $(M,\mathbf{g})$. (Cf. (GREENE; WU; CHERN, 1973; GREENE; WU; CHERN, 1976; GREENE; WU, 1979; AZAGRA; FERRERA, 2005) among others).

4.5 Coarea formula for the sharp admissible damping regions

One of the main objectives of the construction of the ω is to proofs the observability inequality and unique continuation property analogous to the fact in Theorem 2.25. For this, a geometric technical lemma is needed in a context of Riemannian manifolds, which will allow carrying area integrals over volume integrals. To this end, it is necessary a result of coareas for subdomains of small volume, this will allow the application on sets ε-controllable. Thus we enunciate a known result about the coareas and Sard's theorem, the interested reader can consult for example Chavel (CHAVEL, 2006) among others.

Proposition 4.9. Let (W,g) be a C^∞ $N-$dimensional Riemannian manifold, and let $\phi : W \to \mathbb{R} \in C^\infty(W)$. Then, for any measurable function $f : M \to \mathbb{R}$, which is everywhere nonnegative or is in $L^1(W)$, one has

$$\int_W f dV_g = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{f}{|\nabla \phi|} dV_{\tilde{g},t} dt,$$

where $\Gamma(t) := \phi^{-1}(t) = \{p \in W \mid \phi(p) = t\}$ and $dV_{\tilde{g},t}$ is the induced measure on $\Gamma(t)$.

Using the previous Proposition and Theorem 3.3 it is possible to have the following result for it cubes small enough

Theorem 4.10. Fix $\varepsilon > 0$. In the context of Theorems 3.3 and 4.7, exist $0 < \varepsilon_1 < \varepsilon$ such that for all $\varepsilon_0 \in (0,\varepsilon_1)$, there is a constant $C > 0$ that depends of $\varepsilon_1 > 0$ and the metric $\mathbf{g}$, such that

$$\int_{\widehat{\Gamma}_1} f dV_{g|_{\partial M}} \leq C \int_\omega f dV_g, \tag{4.3}$$

for all $\omega \in [\omega_\varepsilon]$ and

$$f \geq 0 \text{ a.e. in } M, \ \ f \in L^1(M), \tag{4.4}$$

where Γ_1 is defined in (2.20), this is

$$\Gamma_1 = \bigcup_{j\in\Lambda}\{x \in \partial M \mid \langle \nabla d_j, n\rangle > 0\},$$

and $dV_{g|_{\partial M}}$ is the volume form on ∂M.

Proof. Step 1. Let $\varepsilon > 0$, by Theorem 3.3, for all $0 < \varepsilon_0 < \varepsilon$ it is possible to build the class $[\omega_\varepsilon]$ so that if $\omega \in [\omega_\varepsilon]$, then

$$\overline{M\setminus V} \in \chi_{\varepsilon_0}(M), \quad \omega \cap V \in \chi_{\varepsilon-\varepsilon_0}(M).$$

Moreover, by constructing $\omega \in M$ and by the compactness of M, we have a finite number of connected components of ω that intersect ∂M, that is, there exists a number $k \in \mathbb{N}$ such that

$$\omega \cap \partial M = \bigcup_{j=1}^{l} \Gamma_\omega^j,$$

where Γ_ω^j is the j-th connected component of $\omega \cap \partial M$. Note that $\Gamma_1 \subset \bigcup_{j\in I}\omega_j \subset \omega$ (see Theorem 4.7), that is

$$meas_{\partial M}\Gamma_1 < \varepsilon_0.$$

Given the $\Gamma_1 \subset \omega \cap \partial M$, then we will denote by Γ_1^j to the connected components of Γ_1 for all $j = 1,2,...l$, given by

$$\Gamma_1^j = \Gamma_1 \cap \Gamma_\omega^j,$$

such that $meas_{\partial M}\Gamma_1^j < \varepsilon_0$ for each $j = 1,2,...,l$.

Step 2. We define about $\mathbb{R}^N$ the set $\mathrm{P}_h(A)$ associated with the constant $h > 0$ and the set $A \subset \mathbb{R}^{N-1}$, given by

$$\mathrm{P}_h(A) := \{(x_1,\dots,x_N) \in \mathbb{R}^N \mid (x_1,\dots,x_{N-1}) \in int(A)\ \forall i = 1,\dots,N-1 \text{ and } 0 \le x_N < h\},$$

which we call *o*pen prism of base A and height h.

$\mathrm{P}_h(A)$ will play a fundamental role in the proof of Theorem.

We will fix $j = 1,2,...,k$. Be $\varepsilon_1^j \in (0,\varepsilon)$ small enough, such that there is a $p \in \Gamma_1^j$ and a chart related to that point $(U_j, \phi_j = (x_1,\dots,x_N))$ with $\phi_j(p) = (0,\dots,0)$ such that the connected component Γ_ω^j be totally within that chart. Thus, there is a constant $h > 0$ small enough, such that $P_h(\Gamma_1^j) \subset \Gamma_\omega^j \subset\subset \phi_j(U_j)$ of M and $\Gamma_1^j = \partial\phi_j^{-1}(P_h(\Gamma_1^j))$ (see Figure 9).

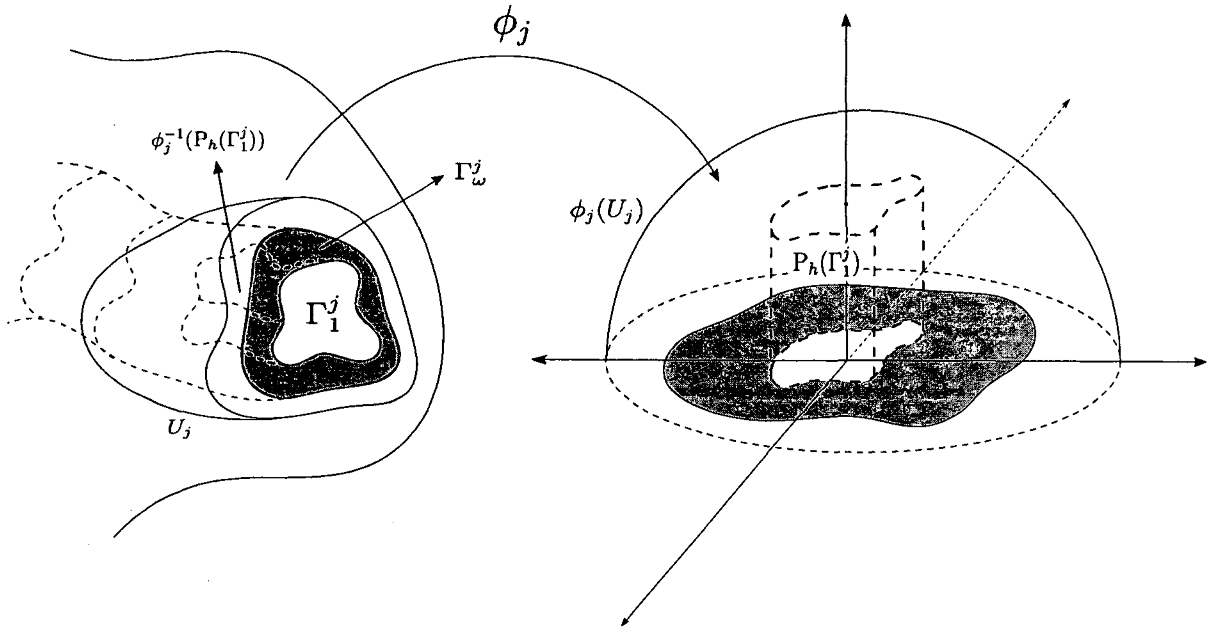

Figure 9 – Note that $h > 0$ depends directly on $\varepsilon_0 > 0$, thus, the open set $\phi_j^{-1}(P_h(\Gamma_1^j)) \subset U_j$ depends on the class $[\omega_\varepsilon]$.

Observe that $(\phi_j^{-1}(P_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(P_h(\Gamma_1^j))})$ is a smooth Riemannian submanifold of M with boundary and dimension N, with the induced metric of M, in particular, we have to

(*i*) The Lebesgue σ-algebra associated with $(\phi_j^{-1}(P_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(P_h(\Gamma_1^j))})$ is the Lebesgue σ-algebra associated to $(M,\mathbf{g})$ intersected with $\phi_j^{-1}(P_h(\Gamma_1^j))$,

(*ii*) If $B \subset \phi_j^{-1}(P_h(\Gamma_1^j)) \subset \omega$ is a measurable set in $(M,\mathbf{g})$, then in particular it is measurable in $(\omega,\mathbf{g}|_\omega)$ and $(\phi_j^{-1}(P_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(P_h(\Gamma_1^j))})$. Moreover,

$$meas_M(B) = meas_\omega(B) = meas_{\phi_j^{-1}(P_h(\Gamma_1^j))}(B).$$

This is because the three manifolds have the same dimension. Also $(\omega,\mathbf{g}|_\omega)$ and $(\phi_j^{-1}(P_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(P_h(\Gamma_1^j))})$ they have the M metric restricted to each of the manifolds.

Step 3. Consider the Riemannian $N-$manifold $(\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))})$ with boundary and the map in $C^\infty(\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j)))$ given by $x_N : \phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j)) \to \mathbb{R}$, where

$$\Gamma_1^j = x_N^{-1}(0) = \Gamma(0), \;\; Img(x_N) = [0,h), \;\; 0 < |\nabla x_N| < C_{\mathbf{g}}, \tag{4.5}$$

for a constant $C_{\mathbf{g}} > 0$ that depends on the metric $\mathbf{g}$.

Thus, by Proposition 4.9, with $\psi = x_N$ and $(W,g) = (\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j)), \mathbf{g}|_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))})$, we have to

$$\int_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))} f dV_{\mathbf{g}|_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))}} = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{f}{|\nabla x_N|} dV_{\mathbf{g}|_{\partial\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))}} dt,$$

for all $f : M \to \mathbb{R}$ which is everywhere nonnegative or is in $L^1(\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j)))$.

Then, taking f fulfilling (4.4), $(i)-(ii)$ and (4.5), we obtain

$$\begin{aligned}
\int_{\Gamma_1^j} f dV_{\mathbf{g}|\partial M} &\leq \int_{\Gamma(0)} f dV_{\mathbf{g}|_{\partial\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))}} \\
&\leq \int_{\mathbb{R}} \int_{\Gamma(t)} f dV_{\mathbf{g}|_{\partial\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))}} dt \\
&\leq C_{\mathbf{g}} \int_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))} f dV_{\mathbf{g}|_{\phi_j^{-1}(\mathrm{P}_h(\Gamma_1^j))}} \\
&\leq C_{\mathbf{g}} \int_{\omega} f dV_{\mathbf{g}}.
\end{aligned}$$

Finally, repeating the process for each $j = 1,2,...,l$ the proof is completed.

□

CHAPTER

5

CONSEQUENCES WITH CARLEMAN ESTIMATES

One of the main consequences of the class of sharp admissible damping regions is the proof of a it new inequality of observability and a unique continuation property.

As explained in Chapter 3, because we can always choose a representative in the class that is (GCC), for a linear wave equation (for example (4.2)), we have an observability inequality as (4.1). Thanks to this inequality in the literature, several results are found for these equations, but it has a certain *l*imitation when we want to study these wave equations with supercritical external forces.

Regarding the critical case of Sobolev, it can be observed in (CHUESHOV; LASIECKA; TOUNDYKOV, 2008), that it is possible to prove the existence of a global attractor for the problem, showing a geometric characterization from the study of unstable manifolds of stationary points for dynamic systems of the gradient type and the finitude of the fractal dimension on the attractor, as well as the optimal regularity of this. It is important to remember that in this work, the authors prove a new inequality of observability based on a new Carleman estimates for the system. Another important detail is regarding the strategy used for the global attractor proof, since the authors use the method of contractive functions (e.g. (CHUESHOV; LASIECKA, 2010; CHUESHOV; LASIECKA, 2008)) and not the quasi-stability (see Definition 2.37).

In the recent work of Joly and Laurent cite joly-laurent, the authors studied the supercritical case for the nonlinearity in the wave equation, from the Strichartz Estimates (see (2.26)-(2.28)). In this case the authors do a first study for nonlinearity belonging to the class of analytic functions (cf. (??TATARU, 1999; HÖRMANDER, 1997; ROBBIANO; ZUILY, 1998)) among others), and by density they get the expected result, being able to demonstrate the existence of a global attractor characterized by unstable varieties of stationary points. The study of the fractal dimension and optimal regularity of the attractor are not proven, mainly because of the limitations in the observability inequality.

For the proof of existence of a global attractor, the literature shows us that the inequality (4.1) is compatible with the method of contractive functions and the decomposition of the semigroup into a compact part and a part that decays exponentially. Regarding quasi-stability method, both for the critical case of Sobolev or subcritical Strichartz, (4.1) is not compatible.

The advantage of studying quasi-stable systems is the extra properties that the attractor gains (see Theorem 2.38). The lack of compatibility with the inequality (4.1) is due to the regularity of the propagation speed of the waves.

Thus, in this section, a new observability inequality is proof for the class of sharp admissible damping regions, where the control of the initial energy is given by the gravitational potential energy and not the kinetic energy. This allows to have a greater regularity with the control term, which makes this new inequality compatible with the quasi-stable systems for supercritical nonlinearity of Sobolev or subcritical Strichartz (see Theorem 6.7). In addition, the unique continuation, proven as a consequence of this new observability, allows us to prove characterizations of the global attractor from the unstable manifolds of the stationary points.

5.1 New observability inequality and unique continuation property

The goal of this section is to establish a new observability inequality and unique continuation property for the system (1.1) where $p_0, p_1 : M \times [0,T] \to \mathbb{R}$ satisfy (1.2)-(1.4). That is, we will proof Theorem 1.1.

As explained above, the objective of the new observability inequality is to be able to control the initial energy from the gravitational potential energy. So the methodology to be able to achieve this inequality will be via Carleman Estimates using the results shown by Triggiani and Yao (TRIGGIANI *et al.*, 2002), more exactly, will be used Theorem 2.25.

Since Theorem 2.25 needs a finite collection overlapping subdomains of M associated with a collection of functions satisfying $(d_j.1)_\omega - (d_j.4)_\omega$, the construction to the sharp admissible damping regions $[\omega_\varepsilon]$ plays a fundamental role, since from a $\varepsilon > 0$, the Theorem 3.3 allows a decomposition of M in two overlapping open sets, this is

$$M = \omega \cup V.$$

Note that $\omega \subset M$ is connected, while V is divided into k connected components as in (3.3), where, by construction, each of these components intersects ω . In addition, V is associated with a functional $d : M \to \mathbb{R}$ satisfying $(d.1.) - (d.4.)$.

On the other hand, the Theorem 4.7 shows a decomposition in overlapping subdomains of ω such that each element of the collection is associated with a functional $d_j : M \to \mathbb{R}$ satisfying $(d_j.1)_\omega - (d_j.4)_\omega$. Note that this will allow you to use Theorem 2.25 over M.

It is important to note that the inequality (2.21) in Theorem 2.25, shows the control of the initial energy from the gravitational potential energy on $\Gamma_1 \subset \partial M$ defined as in (2.20), that is, the form volume is defined on ∂M and not on M, so a geometric result of coarea, in particular Theorem 4.10, will be necessary.

Proof of Theorem 1.1:

Proof. Fix $\varepsilon > 0$, for Theorem 2.20 and Theorem 4.7, there is a finite index $\Lambda = \{1, \dots, \lambda_0\}$ and an integer $k > 0$ such that

I.

$$M = \bigcup_{j\in\Lambda} \omega_j \cup V, \;\; V = \bigcup_{j=1}^{k} V_j, \;\; V_j \cap \omega \neq \emptyset \;\; \forall j = 1, \dots, k.$$

II. There is a functional $d : M \to \mathbb{R}$ satisfying

- $d \in C^3(\overline{V})$,
- $\nabla^2 d(X,X) > 0, \;\; \forall X \in T_xM, \;\; x \in \overline{V}$,
- $\inf_V |\nabla d| > 0$,
- $\langle \nabla d, n\rangle < 0$ on $\partial M \cap \overline{V}$.

III. For each $\omega_j \in \{\omega_j\}_{j\in\Lambda}$, there is a function $d_j : M \to \mathbb{R}$ such that is fulfilled

- $d_j \in C^\infty(M), \;\; \min_{q\in\overline{\omega_j}} d_j(q) > 0$,
- $\nabla^2 d_j(X,X) > |X|_{\mathbf{g}}, \quad \forall X \in T_qM, \; \forall q \in \omega_j$,
- $\inf_{\omega_j} |\nabla d_j| > 0$,
- there is a set of indexes $J \subset \Lambda$ such that $\overline{\omega_j} \cap \partial M \neq \emptyset$ for all $j \in J$. Moreover:
  a) $\left(\bigcup_{j\in\Lambda\setminus J} \omega_j\right) \cap \partial M = \emptyset$,
  b) $\langle \nabla d_j, n\rangle \geq 0$ on $\partial M \cap \overline{\omega_j}, \;\; \forall j \in J$.

Then, considering $\Lambda \cap \{1, \dots, k\} \neq \emptyset$, we can define finite collection overlapping subdomains of M, denoted by $\{\Omega_j\}_{j\in I}$ for a finite index $I = 1, \dots, k \cup \Lambda$, such that

$$\Omega_j = \begin{cases} V_j & \text{if } j = \{1, \dots, k\}, \\ \omega_j & \text{if } j \in \Lambda. \end{cases} \tag{5.1}$$

Analogously, we define a collection of functional associates, denoted by $\{\widehat{d_j}\}_{j\in I}$ such that

$$\widehat{d_j} = \begin{cases} d & \text{if } j = \{1, \dots, k\}, \\ d_j & \text{if } j \in \Lambda. \end{cases} \tag{5.2}$$

Note that $\{\Omega_j\}_{j\in I}$ and $\{\widehat{d_j}\}_{j\in I}$ fulfil hypotheses 1 and 2 of Theorem 2.25, since the collection $\{V_j\}_{j=1,\dots,k}$ is connected. Then, we have all the hypotheses of the Theorem 2.25, therefore there exists $k_T > 0$ depending on ε, T and C_T such that

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial n}\right)^2 d\Gamma_1\, dt \geq k_T \left(\|(w(0), \partial_t w(0))\|^2_{H^1(\Omega)\times L^2(\Omega)} + \|(w(T), \partial_t w(T))\|^2_{H^1(\Omega)\times L^2(\Omega)}\right)$$

where

$$\Gamma_1 = \bigcup_{j\in I} \left\{x \in \partial M \mid \langle \nabla d_j(x), n(x)\rangle > 0\right\}.$$

Again, by Theorem 4.7 and Theorem 3.3 we have to $\Gamma_1 \subset \bigcup_{j\in J} \omega_j \subset \omega$ and

$$\Gamma_1 = \bigcup_{j\in J} \{x \in \partial M \mid \langle \nabla d_j(x), n(x)\rangle > 0\}.$$

Finally, using the Theorem 4.10 with $f = |\nabla w|^2$, (1.20) is fulfilled, since

$$|\langle \nabla w, n\rangle|^2 \leq |\nabla w|^2, \quad \widehat{\Gamma_1} = \Gamma_1.$$

Additionally, if $w = 0$ in $\omega \times [0,T]$ is clear that $w = 0$ on $M \times [0,T]$, proving Theorem. □

CHAPTER

6

APPLICATIONS TO WAVE EQUATIONS: CRITICAL CASE

The objective of this Chapter is to proof the Theorem 1.2. For this purpose, the new inequality of observability and unique continuation property shown in Theorem 1.1 will play a fundamental role.

The strategy to be followed will be using gradient and quasi-stable dynamic systems, that is, we will use the Theorem 2.38 (cf. (CHUESHOV; LASIECKA, 2010; CHUESHOV; LASIECKA, 2008)). This is how we will divide the Theorem 1.2 proof into four sections: the first is intended to show the well-posedness of the problem (1.5) and some notable inequalities that will serve as the basis for the subsequent proofs, the second section shows that the dynamical system associated with (1.5) has a gradient structure, while the third section proves that this dynamic system is also quasi-stable. Finally, the Theorem 1.2 proof will be concluded in the fourth section.

Let us write (with $A = -\Delta$)

$$U = \begin{bmatrix} u \\ \partial_t u \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} 0 & -I \\ A & a(x)g(\cdot) \end{bmatrix}, \quad \mathbb{F} = \begin{bmatrix} 0 & 0 \\ f(\cdot) & 0 \end{bmatrix}, \quad \mathbb{H} = \begin{bmatrix} 0 \\ h \end{bmatrix}. \tag{6.1}$$

Then problem (1.5) is equivalent to the Cauchy problem

$$\partial_t U + \mathbb{A}U + \mathbb{F}U = \mathbb{H}, \quad U(0) = (u_0, u_1) \tag{6.2}$$

defined in $\mathscr{H} := H_0^1(M) \times L^2(M)$ with domain

$$D(\mathbb{A}) = \{(u,v) \in (H_0^1(M))^2 \,|\, Au + a(x)g(v) \in L^2(M)\} =: \mathscr{H}^1,$$

In order to solve Cauchy problem 6.2, we can use Sobolev embedding to obtain estimates of the solutions. See for instance (BARBU, 1993; CHUESHOV; ELLER; LASIECKA, 2002; CHUESHOV; LASIECKA; TOUNDYKOV, 2008) for details.

Theorem 6.1 (Well-possedness). Assume that (1.6)-(1.11) hold. Then

(*i*) For any initial data $(u_0, u_1) \in \mathscr{H}$, problem (1.5) possesses a unique weak solution

$$u \in C(\mathbb{R}^+; H_0^1(M)) \cap C^1(\mathbb{R}^+; L^2(M)), \tag{6.3}$$

(*ii*) Given $T > 0$ and two solutions $z^i = (u^i, \partial_t u^i)$ with initial value $z_0^i \in B$, where B is a bounded set of $\mathscr{H}$, $i = 1,2$, one has

$$\|z^1(t) - z^2(t)\|_{\mathscr{H}}^2 \le C_{BT} \|z_0^1 - z_0^2\|_{\mathscr{H}}^2, \quad \forall t \in [0,T], \tag{6.4}$$

where $C_{BT} > 0$ is constant.

The total energy of the system defined by

$$\mathscr{E}_u(t) = \frac{1}{2}\|(u, \partial_t u)\|_{\mathscr{H}}^2 + \int_M F(u(t))\,dx - \int_M h(x)u(t)dx, \tag{6.5}$$

with $F(s) = \int_0^s f(r)dr$.

Proposition 6.2 (Energy identity). Under the assumptions of Theorem 6.1, then the corresponding solution of (1.5) satisfies

$$\mathscr{E}_u(t_2) + \int_{t_1}^{t_2} \int_M a(x)g(\partial_t u)\partial_t u dx dt = \mathscr{E}_u(t_1), \ \forall t_2 \ge t_1 \ge 0, \tag{6.6}$$

where (the subscript is included just for the sake of clarity, indicating the solution to problem (1.5))

Proof. Once g is assumed linearly bounded at infinity the argument is standard and readily follows for strong solutions from multiplication by $\partial_t u$ and integration by parts. After that, the identity can be extended by density to all weak solutions, since the functionals in (6.6) are continuous with respect to the topology of $\mathscr{H}$. □

$$\frac{d}{dt}\mathscr{E}_u(t) = \int_M a(x)g(\partial_t u)\partial_t u\,dx, \quad t \geq 0. \tag{6.7}$$

There exist positive constants $\beta, C_{f,h,M}$ and $C_{F,h,M}$, the two last ones depending on f, h and M such that

$$\beta\|(u(t),\partial_t u(t))\|^2_{\mathscr{H}} - C_{f,h,M} \leq \mathscr{E}_u(t) \leq C_{F,h,M}\left(1+\|(u(t),\partial_t u(t))\|^4\right), \tag{6.8}$$

An important consequence of the energy identity is that the forward-in-time evolution of the linear energy is controlled by the linear energy at the initial time. We state it rigorously now.

Proposition 6.3. Under the assumptions of Theorem 6.1, there exists $C_0 > 0$, depending on f, h and M, such that for any $(u_0, u_1) \in \mathscr{H}$ the solution associated to problem (1.5) with initial data (u_0, u_1) fulfils

$$\|(u(t),\partial_t u(t))\|^2_{\mathscr{H}} \leq C_0(1+\|(u_0,u_1)\|^4_{\mathscr{H}}), \quad \forall t \geq 0. \tag{6.9}$$

Proof. Since $\mathscr{E}_u$ is non-increasing, we have that $\mathscr{E}_u(t) \leq \mathscr{E}_u(0)$ for any $t \geq 0$. Combining this with (6.8) at time $t = 0$ in the right hand side, the result follows. □

Lemma 6.4. Let us consider $(u, \partial_t u) \in C([0,T];\mathscr{H}^1)$ strong solutions for the problem (1.1) and $f \in C^1(\mathbb{R})$ satisfying (1.9)-(1.10). Then, considering $p_0 = f'(u)$ and $p_1 \in L^\infty(0,T;L^\infty(M))$, (1.3)-(1.4) they are fulfilled.

Proof. It is clear starting from the Sobolev embedding. □

6.1 Gradient structure

Theorem 6.5 (Gradient structure). Assume that hypotheses (1.6)-(1.11) are satisfied and $\varepsilon > 0$ be given.Then, for some ε-controllable measure set $\omega \subset M$

sharp admissible damping region, such that (1.12) is fulfilled where $a_0 > 0$ is a constant that depend on ω, the dynamical system $(S(t), \mathscr{H})$ corresponding to the problem (1.5) is gradient and if it possesses a global attractor, it will be characterized as the unstable manifold $\mathbb{M}^u(\mathscr{N})$ of the set $\mathscr{N}$ of stationary solutions to (1.5).

Proof. For $z = (u, \partial_t u)$, let us define

$$\Psi(z) = \frac{1}{2}\|z\|^2_{\mathscr{H}} + \int_M (F(u) - h(x)u)dx.$$

Thus, we need to prove $(i) - (ii)$ according to the definition of the Lyapunov functional (see Definition 2.34)

From the definition the energy of the system (1.5), is fulfilled

$$\frac{d}{dt}\Psi(S(t)z) = -\int_M a(x)g(\partial_t u)\partial_t u\, dx \tag{6.10}$$

and using (1.6), the right-hand-side is negative, this yields (i).

For to prove (ii), suppose that z_0 satisfies

$$\Psi(S(t)z_0) = z_0, \quad \forall t \geq 0. \tag{6.11}$$

Then (6.10) implies that $S(t)z_0 = (u, \partial_t u)$

$$\int_M a(x)g(\partial_t u)\partial_t u\, dx = 0,$$

and from assumption (1.6) we infer that

$$\int_\omega |\partial_t u|^2\, dx = 0 \quad \text{and} \quad \int_M a(x)|g(\partial_t u)|^2\, dx = 0.$$

Therefore $S(t)z_0 = (u(t), \partial_t u(t))$ is a $C^0([0,T]; \mathscr{H})$ solution of the undamped system

$$\begin{cases} \partial_t^2 u - \Delta u + f(u) = h & \text{in } M \times (0,T], \\ u = 0 \quad \text{on } \partial M \times (0,T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } M, \end{cases} \tag{6.12}$$

with supplementary condition

$$\partial_t u = 0 \quad \text{a.e. in } \omega \times (0,T]. \tag{6.13}$$

To conclude that u is stationary, we shall apply unique continuation property (Theorem 1.1) with $w = \partial_t u$, in order to show that $\partial_t u = 0$ in M. However, we need $(u, \partial_t u)$ regular.

We first observe that (6.13) implies $\partial_t u(0) = 0$ in ω. Then, it turns out that any point $z_0 = (u_0, u_1)$ satisfying (6.11) must satisfy compatibility condition

$$u_1 = 0 \text{ a.e. in } \omega. \tag{6.14}$$

Thus, the system (6.12)-(6.13) is equivalent to the Cauchy problem

$$\partial_t U + \mathbb{B}U + \mathbb{F}(U) = \mathbb{H}, \;\; U(0) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \tag{6.15}$$

where

$$U = \begin{bmatrix} u \\ \partial_t u \end{bmatrix}, \;\; \mathbb{F} = \begin{bmatrix} 0 \\ f(\cdot) \end{bmatrix}, \;\; \mathbb{B} = \begin{bmatrix} 0 & -I|_V \\ A & 0 \end{bmatrix}, \;\; \mathbb{H} = \begin{bmatrix} 0 \\ h \end{bmatrix}, \tag{6.16}$$

with $A = -\Delta$ and

$$V = \{v \in D(A^{1/2}) \mid v = 0 \text{ in } \omega\}, \tag{6.17}$$

then $D(\mathbb{B}) = D(A) \times V$. We will denote by

$$\mathscr{V} = \{(z, v) \in \mathscr{H} \mid v = 0 \text{ in } \omega\} \subset \mathscr{H}, \tag{6.18}$$

to the phase space of the problem (6.15). It is clear that $(\mathscr{V}, \|\cdot\|_{\mathscr{H}})$ is a Hilbert space with the internal product of $\mathscr{H}$ and $D(A) \times V \subset \mathscr{V}$ with compact imbedding $D(A) \times V \rightarrow \mathscr{V}$.

It is not difficult to prove that by Theorem 6.1, in this new contex, the solution operator of problem (6.15) generates a strongly continuous semigroup defined by

$$T(t) : \mathscr{V} \rightarrow \mathscr{V}, \;\; (v_0, v_1) \mapsto (v(t), \partial_t v(t)), \;\; t \geq 0,$$

where $(v, \partial_t v)$ is the weak solution corresponding to initial data (v_0, v_1). In addition, $T(t)$ is also strongly continuous semigroup on $\mathscr{V}$ associated with the system (6.12) satisfying the compatibility condition (6.13).

Then, $(u, \partial_t u)$ is also a solution to the problem (6.15), that is

$$(u, \partial_t u) \in C^0([0, T]; \mathscr{V}). \tag{6.19}$$

Remark 6.6: It is not difficult to show that a solution $(u, \partial_t u) \in C^0([0,T];\mathscr{W})$ of the system (6.15), is also a solution for (6.12). In particular

$$(u, \partial_t u) \in C^0([0,T];\mathscr{H}) \text{ and } \partial_t u = 0 \text{ in } \omega \times [0,T]. \tag{6.20}$$

Then, thanks to (6.19), exists a sequence $(u_0^k, u_1^k)_{k\in\mathbb{N}} \subset H^2(M) \cap H_0^1(M) \times V$ such that $(u_0^k, u_1^k) \to (u_0, u_1)$ in $\mathscr{V}$ and $(u^k, \partial_t u^k) := T(t)(u_0^k, u_1^k) \to (u, \partial_t u)$ in $C^0([0,T];\mathscr{V})$ where $(u^k, \partial_t u^k)_{k\in\mathbb{N}} \subset C^0([0,T]; H^2(M) \cap H_0^1(M) \times V)$ is a sequence of strong solutions of the system (6.15). Additionally, by Remark 6.6, for each $k \in \mathbb{N}$ we have to $(u^k, \partial_t u^k)$ satisfy

$$\begin{cases} \partial_t^2 u^k - \Delta u^k + f(u^k) = h, \text{ in } M \times (0,T] \\ u^k = 0 \quad \text{on } \partial M \times (0,T], \\ \partial_t u^k = 0 \quad \text{in } \omega \times [0,T], \\ u^k(x,0) = u_0^k(x), \ \partial_t u^k(x,0) = u_1^k(x), \ x \in M, \end{cases} \tag{6.21}$$

with $(u^k, \partial_t u^k) \in C^0([0,T];\mathscr{H}^1)$.

On the other hand, setting $v^k = \partial_t u^k$ and differentiating (6.21) with respect to time, we obtain, in the distributional sense

$$\begin{cases} \partial_t^2 v^k - \Delta v^k + f'(u^k) v^k = 0, \text{ in } M \times (0,T], \\ v^k = 0 \quad \text{in } \omega \times [0,T], \\ v^k = 0 \quad \text{on } \partial M \times (0,T]. \end{cases} \tag{6.22}$$

Note that the previous system fits the system (1.1), where $p_0 = f'(u^k)$ and $p_1 = 0$. Now, fix $k \in \mathbb{N}$ and using the Lemma 6.4 it is possible to apply the Theorem 1.1. So, we get $v^k = 0$ in $M \times [0,T]$ for each $k \in \mathbb{N}$, from which it follows that $v^k(t) \to 0$ in $H_0^1(M)$ for all $t \in [0,T]$, in particular $v^k(t) \to 0$ in $L^2(M)$. Thus, of the uniqueness of the limit, (6.12) and (6.13), we have $\partial_t u(t) = 0$ a.e. in M, for all $t \in [0,T]$. Thus, we conclude that

$$S(t)z_0 = z_0 = (u_0, 0),$$

which means that z_0 is a stationary point. This proves that solution operator of (1.5) is a gradient system on $\mathscr{H}$.

Additionally, note that

$$\Psi(z) \to \infty \iff \|z\|_{\mathscr{H}} \to \infty, \tag{6.23}$$

owing to (6.7)-(6.8) this immediately establishs.

For the final statement, suppose that then exists a global attractor for the dynamical system $(S(t), \mathscr{H})$. Then due a well-known abstract result for gradient systems e.g., cf. (HALE, 2010, Theorem 2.4.6) or (MIRANVILLE; ZELIK, 2008, Theorem 2.26). It asserts that an asymptotically compact gradient system $(S(t), \mathscr{H})$ the set $\mathscr{N}$ of its stationary points is bounded, then it has a compact global attractor characterized by $\mathscr{A} = \mathbb{M}^u(\mathscr{N})$. Then it suffices to prove that the set of stationary points is bounded. To this aim, note that any stationary solution $(u, 0)$ of $S(\cdot)$ satisfy

$$-\Delta u + f(u) = h,$$

whence $\|u\|_{H_0^1(M)} \leq c$ for some $c > 0$, completing the proof. □

6.2 Quasi-stability inequality

Theorem 6.7 **(Quasi-stability).** Assume the assumptions (1.6)-(1.11) are satisfied and $\varepsilon > 0$ be given. Then, for some ε-controllable set $\omega \subset M$ sharp admissible damping region, and given a bounded set $B \subset \mathscr{H}$, let $z^1 = (u^1, \partial_t u^1)$ and $z^2 = (u^2, \partial_t u^2)$ be two solutions to problem (1.5) such that $z^1(0), z^2(0) \in B$, there are time independent constants $C_1, \varkappa > 0$ such that

$$\|z^1(t) - z^2(t)\|_{\mathscr{H}}^2 \leq C_1 e^{-\varkappa t} \|z^1(0) - z^2(0)\|_{\mathscr{H}}^2 + C_1 \sup_{s \in [0,t]} \|u^1(s) - u^2(s)\|_{L^3(M)}^2. \tag{6.24}$$

Remark 6.8: The quasi-stability system show to decomposition of the flow into exponentially stable and compact part. An immediate consequence of the quasi-stability inequality presented in (6.24) is the existence of a regular global attractor with finite fractal dimension. For more details the interested reader can consult (CHUESHOV; LASIECKA, 2008; CHUESHOV; LASIECKA, 2010; CHUESHOV, 2015).

Proof. Let us denote $w = u^1 - u^2$. Then $(w, \partial_t w)$ satisfies the equation

$$\begin{cases} \partial_t^2 w - \Delta w = p_0 w + p_1 \partial_t w \text{ in } M \times (0,\infty), \\ w = 0 \text{ on } \partial M \times (0,\infty), \\ w(x,0) = u^1(0) - u^2(0) =: w_0,\ \partial_t w(x,0) = v^1(0) - v^2(0) =: w_1,\ x \in M, \end{cases} \tag{6.25}$$

where

$$p_0(x,t) = f'(\lambda_0 u^1 + (1-\lambda_0)u^2), \tag{6.26}$$

and

$$p_1(x,t) = -a(x)g'(\lambda_1 \partial_t u^1 + (1-\lambda_1)\partial_t u^2), \tag{6.27}$$

for some $\lambda_0, \lambda_1 \in [0,1]$.

Remark 6.9: Of the hypotheses about g and $a(\cdot)$, we have to

$$-a(x)m_2 \leq p_1(x,t) \leq -a(x)m_1.$$

In particular,

$$p_1(x,t) \leq -a_0 m_1,\ \ x \in \omega, \forall t \geq 0.$$

It is clear to note that (1.6)-(1.12) $p_1 \in L^\infty(0,T;L^\infty(M))$. Also, $p_0 w \in L^1(0,T;L^2(M))$ for $(w,\partial_t w) \in C^0([0,T];\mathscr{H})$. On the other hand, if $(u^i, \partial_t u^i) \in C^0([0,T];\mathscr{H}^1)$ with $i = 1,2$, then

$$\begin{aligned}\int_M |f'(\lambda_0 u^1 + (1-\lambda_0)u^2)|^2 |w|^2 &\leq C_0 \|w\|^2_{H^1(M)} + \int_M |u^1|^4 |w|^2 dx + \int_M |u^2|^4 |w|^2 dx \\ &\leq C_0 \|w\|^2_{H^1(M)} + \left(\|u^1\|^4_{L^{16}(M)} + \|u^2\|^6_{L^{16}(M)}\right) \|w\|^2_{L^4(M)} \\ &\leq C_0 \|w\|^2_{H^1(M)}.\end{aligned}$$

Moreover, we need to use the Strichartz estimates with $(4,12)$, (see Theorem 2.42) for the system (6.25) with $H = p_0 w \in L^1(0,T;L^2(M))$. Then, it is clear that (for detail, see (7.6))

$$\|p_0 w\|_{L^1(0,T;L^2(M))} \leq C_{BT} \sup_{t\in[0,T]} \|w\|_{L^3(M)}. \tag{6.28}$$

From what was seen before and from the Theorem 1.1 and Lemma 6.4, we have that observability inequality (1.20) is valid for the system (6.25), then

$$\|(w(0),\partial_t w(0))\|^2_{\mathscr{H}} \leq k_T \int_0^T \int_\omega |\nabla w|^2 = k_T \int_0^T \|\nabla w\|^2_{L^2(\omega)}. \tag{6.29}$$

Nuevamente, por las Strichartz estimates with $(4,12)$ se tiene que (for detail, see Theorem 7.2):

$$\int_0^t \int_M p_0|w|^2 \leq C_{BT} \sup_{t\in[0,T]} \|w\|^2_{L^3(M)}, \tag{6.30}$$

$$\int_0^t \int_M p_0 w\partial_t w \leq C_{BT} \sup_{t\in[0,T]} \|w\|^2_{L^3(M)} + \rho \int_0^t \|\nabla w\|^2_{L^2(\omega)}, \tag{6.31}$$

for some $\rho > 0$.

These estimates will be the cornerstone in the proof of (6.24).

Now, for each $U \in C^0([0,T];H^1_0(M)) \cap C^1([0,T];L^2(M))$, we define the functional $E_U : [0,T] \to \mathbb{R}$, such that

$$E_U(t) = \frac{1}{2}\|(U(t),\partial_t U(t))\|^2_{\mathscr{H}_W}, \tag{6.32}$$

in particular, $E_w(t)$ is the linear energy of the equation (6.25).

Besides that, we define the functional $\Upsilon : [0,T] \to \mathbb{R}$ by

$$\Upsilon(t) = \mu E_w(t) + \eta\chi(t) + \vartheta\kappa(t)$$

Where $\mu > 0, \eta, \vartheta > 0$ will be fixed later, and

$$\mathscr{X}(t) = \int_M w(t)\partial_t w(t)dx, \quad \kappa(t) = \langle\chi_\omega w, \partial_t w\rangle,$$

Lemma 6.10. There exist constants $\beta_1, \beta_2 > 0$ such that

$$\beta_1 E_w(t) \leq \Upsilon(t) \leq \beta_2 E_w(t), \quad t \geq 0, \tag{6.33}$$

with $\lambda_1\mu > \eta + \vartheta$.

Proof. From the definition of $\mathscr{X}$ e κ

$$|\eta\mathscr{X}(t)+\vartheta\kappa(t)|\leq\frac{\eta+\vartheta}{2}\|\partial_t w(t)\|^2_{L^2(M)}+\frac{\eta+\vartheta}{2}\|w(t)\|^2_{L^2(M)}\leq\frac{\eta+\vartheta}{\lambda_1}E_w(t),$$

then

$$\beta_1 E_w(t)\leq\Upsilon(t)\leq\beta_2 E_w(t),$$

for $\beta_1=\mu-\frac{\eta+\vartheta}{\lambda_1}$ and $\beta_2=\mu+\frac{\eta+\vartheta}{\lambda_1}$. Thus, we have the result for all $\eta,\vartheta,\mu>0$ such that $\lambda_1\mu>\eta+\vartheta$. □

Lemma 6.11. Given $a_0,m_1>0$ in (1.6)-(1.12), then

$$E'_w(t)\leq -a_0m_1\|\partial_t w\|_{L^2(\omega)}-\int_M p_0 w\partial_t w dx.$$

Proof. We note that by the equation (6.25) and using (1.6)-(1.12), be get that

$$\begin{aligned}E'_w(t)&=\int_M p_1|\partial_t w|^2dx-\int_M p_0 w\partial_t w dx\\&\leq -a_0m_1\int_\omega|\partial_t w|^2dx-\int_M p_0 w\partial_t w dx,\end{aligned}$$

this completes the proof. □

Lemma 6.12. There exists a constant $C_{f,B,a,g}>0$ such that

$$\mathscr{X}'(t)\leq -E_w(t)-\frac{1}{2}\|\nabla w\|^2_{L^2(M)}+2\|\partial_t w\|^2_{L^2(M)}+C_{f,B,a,g}\|w\|^2_{L^3(M)}+\int_M p_0|w|^2dx,$$

for any $\varepsilon>0$.

Proof. Since

$$\mathscr{X}'(t)=\int_M\partial_t^2 w(t)w(t)dx+\|\partial_t w(t)\|^2_{L^2(M)},$$

by applying w in the equation (6.25), comparing with $\chi'(t)$ and using (??)-(1.6) we have that

$$\begin{aligned}\mathscr{X}'(t) &\leq -\|\nabla w\|^2_{L^2(M)} + \int_M p_1 \partial_t w w dx + \int_M p_0 |w|^2 dx + \|\partial_t w\|^2_{L^2(M)} \\ &\leq -\|\nabla w\|^2_{L^2(M)} + \left(\frac{1}{2}+1\right)\|\partial_t w\|^2_{L^2(M)} + C_{f,B,a,g}\|w\|^2_{L^3(M)} + \int_M p_0|w|^2 dx \\ &\leq -E_w(t) - \frac{1}{2}\|\nabla w\|^2_{L^2(M)} + 2\|\partial_t w\|^2_{L^2(M)} + C_{f,B,a,g}\|w\|^2_{L^3(M)} + \int_M p_0|w|^2 dx,\end{aligned}$$

thus proving the result. □

Lemma 6.13. There exists a constant $C_{f,B,a,g} > 0$ such that

$$\kappa'(t) \leq -\|\nabla w\|^2_{L^2(\omega)} + 2\|\partial_t w\|^2_{L^2(\omega)} + C_{f,B,a,g}\|w\|^2_{L^3(M)} + \int_\omega p_0|w|^2 dx.$$

Proof. Since

$$\kappa'(t) = \langle \chi_\omega \partial_t w, \partial_t w\rangle + \langle \chi_\omega w, \partial_t^2 w\rangle = \|\partial_t w(t)\|^2_{L^2(\omega)} + \langle \chi_\omega w, \partial_t^2 w\rangle$$

by applying w in the equation (6.25), comparing with $\kappa'(t)$, then

$$\kappa'(t) = -\|\nabla w\|_{L^2(\omega)} + \int_\omega p_1 \partial_t w w dx + \int_\omega p_0|w|^2 dx + \|\partial_t w\|^2_{L^2(\omega)}.$$

So, proceeding the same as the previous Lemma, we have to

$$\begin{aligned}\kappa'(t) &\leq -\|\nabla w\|_{L^2(\omega)} + \int_\omega p_1 \partial_t w w dx + \int_\omega p_0|w|^2 dx + \|\partial_t w\|^2_{L^2(\omega)} \\ &\leq -\|\nabla w\|^2_{L^2(\omega)} + 2\|\partial_t w\|^2_{L^2(\omega)} + C_{f,B,a,g}\|w\|^2_{L^3(M)} + \int_\omega p_0|w|^2 dx.\end{aligned}$$

□

Then, thanks to Lemmas 6.11-6.13, we obtain that

$$\begin{aligned}\Upsilon'(t) &= \mu E_w'(t) + \eta \mathscr{X}'(t) + \vartheta \kappa'(t) \\ &\leq -\eta E_w(t) + M(t)\end{aligned}$$

where

$$M(t) = -\vartheta\|\nabla w\|^2_{L^2(\omega)} + 2\eta\|w_t\|^2_{L^2(M)} - \mu\int_M p_0 w w_t + (\vartheta+\eta)\int_M |p_0 w||w| dx$$

$$+(\eta+\vartheta)C_{f,B,a,g}\|w\|^2_{L^3(M)}, \tag{6.34}$$

provided that

$$\mu > \frac{2\vartheta}{a_0 m_1}. \tag{6.35}$$

Considering

$$\lambda_1\mu > \eta + \vartheta, \tag{6.36}$$

and using Lemma 6.10 and applying the inequality of Gronwall in the differential form, one has to

$$\Upsilon(t) \le e^{-\frac{\eta t}{\beta_2}}\Upsilon(0) + \int_0^t e^{-\frac{\eta(t-\tau)}{\beta_2}} M(t)d\tau.$$

For the properties of the function $J(\tau) = e^{-\frac{\eta(t-\tau)}{\beta}}$ with $\tau \in [0,T]$, for the estimates (6.4), (6.29) and (6.28), and inequalities (6.28), (6.30) e (6.31) there has to be a constant $C_0 > 0$ depending on f,B,a,g,T such that

$$\Upsilon(t) \le e^{-\frac{\eta t}{\beta_2}}\Upsilon(0) - \frac{\vartheta e^{-\frac{\eta T}{\beta_2}}}{2}\int_0^t \|\nabla w\|^2_{L^2(\omega)} + C_0 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)},$$

for $\vartheta > 0$ large enough, $\rho, \eta > 0$ small enough and fulfilling (6.35), (6.36). Also, note that for the estimate (6.29) we have to

$$\Upsilon(t) \le e^{-\frac{\eta t}{\beta_2}}\Upsilon(0) - \frac{\vartheta k_T e^{-\frac{\eta T}{\beta_2}}}{2} E_w(0) + C_0 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)}.$$

Now, for the Lemma 6.10,by inequality 6.4 and considering $\vartheta > 2\beta_2 C_{BT}$ with $C_{BT} > 0$ given by inequality (6.4), we have

(*i*)

$$E_w(t) \le C_0 e^{-\varkappa} t E_w(0) + C_0 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)},$$

(*ii*)

$$E_w(T) \le C_T(E_w(0) - E_w(T)) + C_0 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)},$$

For a certain $T > 0$ large enough and $\varkappa > 0$ a positive constant.

Note that (i) already shows the property of quasi-stability, but (ii) has to exist $\gamma \in (0, 1/2)$ independent of time $T > 0$, such that

$$E_w(T) \leq \gamma E_w(0) + C_0 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)}.$$

Then proceeding analogously to what is shown by Chueshov and Lasiecka (CHUESHOV; LASIECKA, 2010, Lemma 8.5.5.), there is a constant time independent $C_1 > 0$ such that

$$E_w(t) \leq C_1 e^{-\varkappa t} E_w(0) + C_1 \sup_{t\in[0,T]} \|w\|^2_{L^3(M)},$$

proving the Theorem 6.7. □

6.3 Proof of Theorem 1.2

We will divide the proof in three steps:

Step 1. Note that due to Theorem 6.5, the dynamic system $(S(t), \mathscr{H})$ associated with (1.5) is gradient, where the Lyapounov function satisfies (6.23). In addition, it was shown that the set of stationary points $\mathscr{N}$ is bounded in $\mathscr{H}$.

Step 2. From the Theorem 6.7 the dynamical system $(S(t), \mathscr{H})$ is quasi-stable on every bounded forward invariant set $B \subset \mathscr{H}$ (see Definition 2.37), then by the Theorema 2.38, the dynamic system $(S(t), \mathscr{H})$ is asymptotically smooth.

Step 3. By the Theorem 2.35 and the Remark 2.36, there is a global attractor $\mathscr{A} = \mathbb{M}^u(\mathscr{N})$. In particular, again by the Theorem 6.7, $(S(t), \mathscr{H})$ is quasi-stable on $\mathscr{A}$, and for the indepence at the time of the constant $C_1 > 0$, $\mathscr{A}$ has finite fractal dimension and optimal regularity (see Theorem 2.38), which proves the Theorem.

CHAPTER

7

APPLICATIONS TO WAVE EQUATIONS: SUPERCRITICAL CASE

In this Chapter we present a proof of Theorem 1.3. The methodology will be the same as that used for the critical case (see Chapter 5).

So it is necessary to use the Theorem 1.1. To this end, we will show that the Theorem 2.47 is valid for ω a sharp admissible damping region.

Lemma 7.1. Let $\varepsilon > 0$, the Theorems 2.44 and 2.47 they are valid for some ε-controllable set $\omega \subset M$ sharp admissible damping region, such that (1.15)-(1.16) is fulfilled for the system (1.13).

Proof. For Theorem 3.3, there is a $\omega \subset M$ that defines the class of sharp admissible damping regions associated to ε: $[\omega_\varepsilon]$.

Note that associated with this class, there is an open set V such that $M \setminus \omega \cap V \neq \emptyset$. So, there is a $x_0 \in M$ and $R > 0$ such that $B(x_0, R) \subset V$ and $\omega \cap B(x_0, R) = \emptyset$. Now, by Theorem 4.1 we can always choose within the class $[\omega_\varepsilon]$ a representative that satisfies (GCC). So, without loss of generality, we can consider ω satisfying (GCC).

That is, there is $\omega \in [\omega_\varepsilon]$ such that fulfills (1.15)-(1.16).

This shows that we can change the hypothesis (1.15)-(1.16) for the following statement:

(Sharp H.) There exist and open set $\omega \subset M$ ε-controllable such that

$$a(x) \geq a_0 > 0, \ \ \forall x \in \omega.$$

What proves the result. □

7.1 Quasi-stability inequality

Theorem 7.2 (Quasi-stability). Assume the assumptions (1.14)-(1.19) are satisfied and $\varepsilon > 0$ be given. Then, for some ε-controllable set $\omega \subset M$ sharp admissible damping region, and given a bounded set $B \subset \mathscr{H}$, let $z^1 = (u^1, \partial_t u^1)$ and $z^2 = (u^2, \partial_t u^2)$ be two solutions to problem (1.13) such that $z^1(0), z^2(0) \in B$, there are time independent constants $C_1, \varkappa > 0$ such that

$$\|z^1(t) - z^2(t)\|^2_{\mathscr{H}_W} \leq C_1 e^{-\varkappa t}\|z^1(0) - z^2(0)\|^2_{\mathscr{H}} + C_1 \sup_{s\in[0,t]} \|u^1(s) - u^2(s)\|^2_{L^{\frac{12}{7-p}}(M)}. \tag{7.1}$$

Proof. Let us denote $w = u^1 - u^2$. Then $(w, \partial_t w)$ satisfies the equation

$$\begin{cases} \partial_t^2 w - \Delta w = p_0 w + p_1 \partial_t w \text{ in } M \times (0,\infty), \\ w = 0 \text{ on } \partial M \times (0,\infty), \\ w(x,0) = u^1(0) - u^2(0) =: w_0, \ \partial_t w(x,0) = v^1(0) - v^2(0) =: w_1, \ x \in M, \end{cases} \tag{7.2}$$

where

$$p_0(x,t) = f'(\lambda_0 u^1 + (1-\lambda_0)u^2), \tag{7.3}$$

and

$$p_1(x,t) = -a(x), \tag{7.4}$$

for some $\lambda_0 \in [0,1]$.

The test is analogous to that performed for the critical case, provided that the estimates are available (6.28)-(6.31).

Thus, for $p \in [3,5)$ given that

$$\frac{p-1}{6}+\frac{7-p}{6}=1,\ \frac{p-1}{4}+\frac{5-p}{4}=1,$$

we have to

$$\begin{aligned}\int_0^T\left(\int_M |u^i|^{2p-2}|w|^2dx\right)^{\frac{1}{2}}dt &\leq \int_0^T \|u^i\|^{p-1}_{L^{12}(M)}\|w\|_{L^{\frac{7}{7-p}}(M)}dt\\ &\leq \|u^i\|^{p-1}_{L^4(0,T;L^{12}(M))}\|w\|_{L^{\frac{4}{5-p}}(0,T;L^{\frac{12}{7-p}}(M))}\\ &\leq C_{BT}\|w\|_{L^\infty(0,T;L^{\frac{12}{7-p}}(M))},\end{aligned}$$

for $i=1,2$ e $(q,r)=(4,12)$ in (2.31).

Then, there is a constant $C_{BT}>0$, depending on the initial data, such that

$$\begin{aligned}\|p_0 w\|_{L^1(0,T;L^2(M))} &\leq CT\|w\|_{L^\infty(0,T;L^2(M))}+C_{BT}\|w\|_{L^\infty(0,T;L^{\frac{12}{7-p}}(M))}\\ &\leq C_{BT}\|w\|_{L^\infty(0,T;L^{\frac{12}{7-p}}(M))}\\ &\leq C_{BT}\|(w(0),\partial_t w(0))\|_{\mathscr{H}}.\end{aligned}$$

From what was seen before and from the Theorem 1.1, we have that observability inequality (1.20) is valid for the system (7.2), then

$$\|(w(0),\partial_t w(0))\|^2_{\mathscr{H}_W}\leq k_T\int_0^T\int_\omega |\nabla w|^2 = k_T\int_0^T \|\nabla w\|^2_{L^2(\omega)}. \tag{7.5}$$

Furthermore, is fulfilled

$$\|p_0 w\|_{L^1(0,T;L^2(M))}\leq C_{BT}\sup_{t\in[0,T]}\|w\|_{L^{\frac{12}{7-p}}(M)}. \tag{7.6}$$

and

$$\begin{aligned}\int_0^T\int_M p_0|w|^2 &\leq \int_0^t \|p_0 w\|_{L^2(M)}\|w\|_{L^2(M)}\\ &\leq \|p_0 w\|_{L^1(0,T;L^2(M))}\|w\|_{L^\infty(0,T;L^2(M))}\\ &\leq C_{BT}\|w\|^2_{L^\infty(0,T;L^{\frac{12}{7-p}}(M))},\end{aligned}$$

this is

$$\int_0^T \int_M p_0 |w|^2 \leq C_{BT} \sup_{t\in[0,T]} \|w\|^2_{L^{\frac{12}{7-p}}(M)}. \tag{7.7}$$

On the other hand, notice that of the observability inequality (1.20), we have to

$$\|\partial_t w(t)\|^2_{L^2(M)} \leq k_T \int_0^t \|\nabla w\|^2_{L^2(\omega)}.$$

Thus, it is true that

$$\begin{aligned}\int_0^T \int_M p_0 w \partial_t w &\leq \|p_0 w\|_{L^1(0,T;L^2(M)} \|\partial_t w\|_{L^\infty(0,t;L^2(M))} \\ &\leq \|p_0 w\|_{L^1(0,T;L^2(M)} \sup_{s\in[0,t]} \|\partial_t w\|_{L^2(M)} \\ &\leq C_{BT} \sup_{t\in[0,T]} \|w\|^2_{L^{\frac{12}{7-p}}(M)} + \rho \sup_{s\in[0,t]} \left(\int_0^s \|\nabla w\|^2_{L^2(\omega)}\right),\end{aligned}$$

this is

$$\int_0^T \int_M p_0 w \partial_t w \leq C_{BT} \sup_{t\in[0,T]} \|w\|^2_{L^{\frac{12}{7-p}}(M)} + \rho \int_0^t \|\nabla w\|^2_{L^2(\omega)}, \tag{7.8}$$

for some $\rho > 0$.

Proceeding analogously to the critical case has the result. □

7.2 Proof of Theorem 1.3

We will divide the proof in two steps:

*S*tep 1. Thank to Lemma 7.1, it is enough to assume the hypothesis (Sharp H.) instead of (1.15)-(1.16). Moreover, Theorems 2.44 and 2.47 are satisfied. Therefore, the dynamic system associated with (1.13) is a gradient and has a global attractor characterized by the unstable manifolds of the stationary points.

*S*tep 2. From the Theorems 6.7 and 2.38, the dynamic system associated to (1.13) has a smooth global attractor with finite fractal dimension, proving the Theorem.

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