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Scissors Congruence Group and the Third Homology of SL_2

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Tese de Doutorado do Programa de Pós-Graduação em Matemática (PPG-Mat)

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Grupo de Congruência de Tesoura e a Terceira Homología de SL_2

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

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To my family, who have been supporting me since I decided to do what I love. I hope that someday we can all study without pressure and in what we love, not just to survive.

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The main thanks go to my advisor who introduced me to the world of algebraic K-theory, a theory that, I believe, will be talked about for many years to come.

My special thanks goes to ICMC in which I was able to mature my knowledge of mathematics, and to CAPES for the scholarship that allowed me to give all my time to the subject that I love.

Both of us were thinking about that. Does Suzumiya-san have any opinion?
'Isn't that the Euler formula?'
Haruhi says that without even thinking, what a bummer.
Koizumi responds: 'You mean Leonhard Euler? The mathematician?'
'Yes, the mathematician, but I don't know his name.'
Koizumi re-examines the strange interface panel again, and stares for several seconds.
'Yes'
He snapped his fingers as if he were acting before someone
'This is the Euler's Planar Graph Formula, or rather a variation. As expected from Suzumiya-san.'
'It might to be it. That D thought must mean the dimensional factor. I guess.'

(The Rampage of Haruhi Suzumiya, Nagaru Tanigawa)

RESUMO

PÉREZ, E. T. **Grupo de Congruência de Tesoura e a Terceira Homologia de** SL_2. 2023. 83 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

O objetivo principal deste trabalho é estudar a terceira homología inteira do grupo especial linear $\mathrm{SL}_2(A)$ para um anel comutativo A e a sua relação com o grupo de congruência de tesoura $\mathscr{R}\mathscr{P}_1(A)$ (BLOCH, 2000), (HUTCHINSON, 2013a), (CORONADO; HUTCHINSON,).

Uma ferramenta importante para estudar a terceira homología de SL_2 é a existência de uma sequência exata refinada de Bloch-Wigner. Nesta tese mostramos que existe uma sequência exata refinada de Bloch-Wigner sobre domínios locais de característica 2. Na verdade, mostramos que se $\mathrm{char}(A) = 2$, então existe uma sequencia exata

$$0 \to \mathrm{Tor}(\mu(A), \mu(A)) \to H_3(\mathrm{SL}_2(A), \mathbb{Z}) \to \mathscr{R}\mathscr{B}(A) \to 0,$$

onde $\mathscr{R}\mathscr{B}(A) \subseteq \mathscr{R}\mathscr{P}_1(A)$ é o grupo refinado de Bloch de A. Além disso mostramos que se A é um dominio local tal que -1 é um quadrado, então existe uma sequencia exata da forma

$$H_3(\mathrm{SM}_2(A), \mathbb{Z}) \to H_3(\mathrm{SL}_2(A), \mathbb{Z}) \to \mathscr{R}\mathscr{B}(A) \to 0,$$

onde $\mathrm{SM}_2(A)$ é o grupo de matrizes monomiais em $\mathrm{SL}_2(A)$. O resultados da tese podem-se encontrar nos artigos (MIRZAII; PÉREZ, a), (MIRZAII; PÉREZ, b).

Palavras-chave: K-Teoria Algébrica, Homologia de grupos, refined Bloch group, refined Scissors-congruence group.

ABSTRACT

PÉREZ, E. T. **Scissors Congruence Group and the Third Homology of $\mathbf{SL}_2$**. 2023. 83 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

The main goal of this work is to study the third integer homology of the special linear group $H_3(\mathrm{SL}_2(A),\mathbb{Z})$ for a commutative ring A and its relationship with the refined scissors congruence group $\mathscr{R}\mathscr{P}_1(A)$ (BLOCH, 2000), (HUTCHINSON, 2013a), (CORONADO; HUTCHINSON,).

An important tool to study the third homology of SL_2 is the existence of a refined Bloch-Wigner exact sequence. In this thesis we show that there exist a refined Bloch-Wigner exact sequence over local domains of characteristic 2. In fact, we show that if $\mathrm{char}(A) = 2$, then there exists an exact sequence

$$0 \to \mathrm{Tor}(\mu(A),\mu(A)) \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathscr{R}\mathscr{B}(A) \to 0,$$

where $\mathscr{R}\mathscr{B}(A) \subseteq \mathscr{R}\mathscr{P}_1(A)$ is the refined Bloch group of A. Moreover, we show that if A is a local domain such that -1 is an square, then there exists an exact sequence

$$H_3(\mathrm{SM}_2(A),\mathbb{Z}) \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathscr{R}\mathscr{B}(A) \to 0,$$

where $\mathrm{SM}_2(A)$ is the group of monomial matrices in $\mathrm{SL}_2(A)$. The results of this thesis can be found in (MIRZAII; PÉREZ, a), (MIRZAII; PÉREZ, b).

Keywords: Algebraic K-theory, Group homology, refined Bloch group, refined scissors-congruence group.

CONTENTS

CHAPTER

1

INTRODUCTION

The study of the third homology of the group $\mathrm{SL}_2(A)$ is important because of its close connection to the third K-group of A (SUSLIN, 1991), (HUTCHINSON; MIRZAII; MOKARI, 2022), its appearance in the scissors congruence problem in 3-dimensional hyperbolic geometry (DUPONT; SAH, 1982), (SAH, 1989), etc.

The classical Bloch-Wigner exact sequence studies the indecomposable part of the third K-group of a field (DUPONT; SAH, 1982). The general Bloch-Wigner exact sequence for fields, claims that for any field F we have the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F),\mu(F))^{\sim} \to K_3^{\mathrm{ind}}(F) \to \mathscr{B}(F) \to 0.$$

Here $\mathscr{B}(F)$, called the Bloch group of F, is a certain subgroup of the classical scissors congruence group $\mathscr{P}(A)$ (see Section 3.2) and $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F),\mu(F))^{\sim}$ is the unique nontrivial extension of $\mu_2(F)$ by $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F),\mu(F))$ (SUSLIN, 1991), (HUTCHINSON, 2013b). This exact sequence can be extended to any local domain, where its residue field has more than 9 elements (MIRZAII, 2017).

When F is quadratically closed, we have a natural isomorphism $H_3(\mathrm{SL}_2(F),\mathbb{Z}) \simeq K_3^{\mathrm{ind}}(F)$ (MIRZAII, 2008),(SAH, 1989). In general we have a natural surjective map

$$H_3(\mathrm{SL}_2(F),\mathbb{Z}) \twoheadrightarrow K_3^{\mathrm{ind}}(F)$$

(see (HUTCHINSON; TAO, 2009)). The indecomposable group $K_3^{\mathrm{ind}}(F)$ has been studied extensively in the literature (see for example (MERKUR'EV; SUSLIN, 1990) or (LEVINE, 1989)). In many applications in algebraic K-theory and number theory it is important to understand the structure of the group $H_3(\mathrm{SL}_2(F),\mathbb{Z})$. When F is not quadratically closed, the above map has a nontrivial, and often quite large, kernel (see (HUTCHINSON, 2013b), (HUTCHINSON, 2021)).

The homology groups $H_\bullet(\mathrm{SL}_2(A),\mathbb{Z})$ are naturally $\mathscr{R}_A := \mathbb{Z}[A^\times/(A^\times)^2]$-modules and this module structure plays a central role in the study of the homology group $H_3(\mathrm{SL}_2(A),\mathbb{Z})$.

The refined Bloch group of a ring A, introduced by Hutchinson, is a certain subgroup of the refined scissors congruence group of A. The refined scissors congruence group $\mathcal{RP}_1(A)$ of A is defined by a presentation analogous to $\mathcal{P}(A)$ but as a module over the group ring $\mathcal{R}_A$ rather than as an abelian group.

In a series of papers (HUTCHINSON, 2013b), (HUTCHINSON, 2013a), (HUTCHINSON, 2017a), (HUTCHINSON, 2017b), (HUTCHINSON, 2021), Hutchinson extensively studied the homology group $H_3(\mathrm{SL}_2(A),\mathbb{Z})$ when A is a field or a local ring with sufficiently large residue field. Recently it has been proved that the third homology of SL_2 over discrete valuation rings satisfies a localization property (HUTCHINSON; MIRZAII; MOKARI, 2022).

Let $\mathcal{RB}(A) \subseteq \mathcal{RP}_1(A)$ be the refined Bloch group of A. Usually there is a natural map from the third homology of $\mathrm{SL}_2(A)$ to $\mathcal{RB}(A)$. In this thesis we study this map assuming minimum conditions on A. Let $T(A)$ and $B(A)$ be the group of diagonal and upper triangular matrices in $\mathrm{SL}_2(A)$, respectively. Assume that

(i) A is a universal GE_2-ring,

(ii) $\mu_2(A) = \{\pm 1\}$ and $-1 \in A^{\times 2}$,

(iii) $H_n(T(A),\mathbb{Z}) \simeq H_n(B(A),\mathbb{Z})$ for $n = 2,3$.

As the first main result of this thesis we prove a refined version of the Bloch-Wigner exact sequence when $2 = 0$ (3.4.5). In particular, we show that for any local domain of characteristic 2, where its residue field has more than 64 elements, we have the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)) \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathcal{RB}(A) \to 0.$$

This gives a positive answer to a question raised by Coronado and Hutchinson in (CORONADO; HUTCHINSON,) over such rings. Moreover it improves similar results of Hutchinson (see (HUTCHINSON, 2013a), (HUTCHINSON, 2017a)), as it leaves no ambiguity on 2-torsion elements.

As the second main result of this thesis we show that the sequence

$$H_3(\mathrm{SM}_2(A),\mathbb{Z}) \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathcal{RB}(A) \to 0 \qquad (1.0.1)$$

is exact, where $\mathrm{SM}_2(A)$ is the group of monomial matrices in $\mathrm{SL}_2(A)$ (Theorem 4.2.6). Moreover, if A satisfies in conditions (i) and (iii), we show that there is an exact sequence

$$I(A) \otimes_{\mathbb{Z}} \mu_2(A) \to H_3(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to \frac{\mathcal{RP}_1(A)}{\langle \psi_1(a^2) : a \in A^\times \rangle} \to 0, \qquad (1.0.2)$$

where $I(A)$ is the fundamental ideal of A (Theorem 4.4.2). As a particular case, we show that if $-1 \in A^{\times 2}$, then we have the isomorphism

$$H_3(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \simeq \mathcal{RP}_1(A).$$

The homology groups of $\mathrm{SL}_2(A)$ relative to its subgroups $T(A)$ and $\mathrm{SM}_2(A)$ seems to be important. As the third main result of this thesis we show that for any ring A satisfying conditions (i) and (iii), we have the isomorphisms

$$H_2(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}) \simeq W(A), \qquad H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \simeq K_1^{\mathrm{MW}}(A),$$

where $W(A)$ is the Witt ring of A and $K_1^{\mathrm{MW}}(A)$ is the first Milnor-Witt K-group of A. Moreover we show that

$$H_3(\mathrm{SL}_2(A), T(A), \mathbb{Z}\left[\tfrac{1}{2}\right]) \simeq \mathscr{RP}_1(A)\left[\tfrac{1}{2}\right]$$

(for the last two isomorphism we need to assume that $\mathrm{SL}_2(A)$ is perfect.)

It seems that $K_1^{\mathrm{MW}}(A)\left[\frac{1}{2}\right]$ and $\mathscr{RP}_1(A)\left[\frac{1}{2}\right]$ should be part of a chain of groups (WENDT, 2018, App. A) with certain properties similar to K-groups. These two groups appear in the unstable analogues of the fundamental theorem of K-theory for the second and third homology of SL_2 over an infinite field (HUTCHINSON, 2015), which can be used to calculate the low-dimensional homology of SL_2 of Laurent polynomials over certain fields. Moreover they have certain interesting localization property (GILLE; SCULLY; ZHONG, 2016, Theorem 6.3), (HUTCHINSON; MIRZAII; MOKARI, 2022, Theorem A).

Our main results follows from a careful analysis of a spectral sequence which converge to the homology of SL_2. Our spectral sequence is a variant of a spectral sequence which is studied by Hutchinson in his series of papers and is similar to the one studies for GL_2 in (MIRZAII, 2011). As we will see in this thesis this variant has certain advantage when it comes to calculation of some differentials.

In this thesis all rings are commutative, except probably group rings, and have the unit element 1.

CHAPTER

2

HOMOLOGY

In this chapter we give a short account on the homology of groups. The study of the third homology of SL_2 is te main topic of this thesis.

2.1 Chain Complexes

In this section A is a ring with the unit element 1 and all modules are left A-modules. The constructions can be applied to right A-modules, almost with no change.

Definition 2.1.1. A *chain complex* of A-modules is a family $C_\bullet = \{C_i, \partial_i\}_{i\in\mathbb{Z}}$ of A-modules and A-homomorphisms $\partial_i : C_i \to C_{i-1}$ called *differentials*, such that $\partial_{i-1} \circ \partial_i = 0$ for any $i \in \mathbb{Z}$.

For any chain complex $C_\bullet$, let $Z_i = Z_i(C_\bullet) := \ker(\partial_i)$ and $B_i = B_i(C_\bullet) := \mathrm{im}(\partial_{i+1})$. The elements of Z_i and B_i are called *i-cycles* and *i-boundaries* of $C_\bullet$, respectively. It is easy to see that $0 \subseteq B_i \subseteq Z_i \subseteq C_i$. The i-th homology of $C_\bullet$ is the quotient group $H_i(C_\bullet) = Z_i(C_\bullet)/B_i(C_\bullet)$. When $H_i(C_\bullet) = 0$ we say that $C_\bullet$ is *exact* in dimension i (or at C_i). If that property is satisfied for any $i \in \mathbb{Z}$, then we say that $C_\bullet$ is *exact*.

A *morphism* $f_\bullet : C_\bullet \to D_\bullet$ of chain complexes is a family $\{f_i\}_{i\in\mathbb{Z}}$ of A-homomorphisms $f_i : C_i \to D_i$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{i+2}} & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \xrightarrow{\partial_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \xrightarrow{\partial_{i+2}} & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \xrightarrow{\partial_{i-1}} & \cdots
\end{array}$$

is commutative, i.e. for any $i \in \mathbb{Z}$, $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ (note that here we use the same notation for the differentials of $C_\bullet$ and $D_\bullet$. When necessary we denote them with different notation). Thus we have the category $\mathbf{Ch}(A - \mathbf{mod})$ of chain complexes of A-modules. This is an abelian category.

A morphism $f_\bullet : C_\bullet \to D_\bullet$ between chain complexes induces a natural A-morphism $f_* : H_i(C_\bullet) \to H_i(D_\bullet)$, given by $x + B_i(C_\bullet) \mapsto f_i(x) + B_i(D_\bullet)$. If $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to E_\bullet$ are morphisms of chain complexes, then we have $(g_\bullet \circ f_\bullet)_* = g_* \circ f_*$. Moreover, the identity morphism $\mathrm{id}_{C_\bullet}$ induces the identity map $\mathrm{id}_{H_i(C_\bullet)}$.

The chain complexes have arbitrary and direct sums, if $\{C_{\bullet,j}\}_{j\in J}$ is a family of chain complexes, it is not difficult to show the commutativity of homology with direct sum, i.e.

$$H_i\left(\bigoplus_{j\in J} C_{\bullet,j}\right) \simeq \bigoplus_{j\in J} H_i(C_{\bullet,j})$$

An important particular of complexes is the concept of short exact sequence. We say that the complex

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$$

(we can complete the complex with zeros on the left and right side) is a short exact sequence if it is exact everywhere. This condition implies that f is an injection, g is a surjection and obviously $\mathrm{im}(f) = \ker(g)$.

The sequence of chain complexes

$$0 \longrightarrow C_\bullet \xrightarrow{f} D_\bullet \xrightarrow{g} E_\bullet \longrightarrow 0$$

is called exact if for any $i \in \mathbb{Z}$ the sequence

$$0 \longrightarrow C_i \xrightarrow{f_i} D_i \xrightarrow{g_i} E_i \longrightarrow 0$$

is exact.

For the proof of the next theorem we need the following useful lemma.

Lemma 2.1.2 (Snake lemma)**.** *For any commutative diagram with exact rows of A-modules*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & & \end{array}$$

there is a connecting homomorphism $\delta : \ker(h) \to coker(f)$ *such that the sequence*

$$\ker(f) \xrightarrow{\alpha} \ker(g) \xrightarrow{\beta} \ker(h) \xrightarrow{\delta} coker(f) \xrightarrow{\alpha'} coker(g) \xrightarrow{\beta'} coker(h)$$

is exact. Moreover, if α *is injective, then the induced map* $\ker(f) \overset{\alpha}{\to} \ker(g)$ *is injective. If* β' *is surjective then the induced map* $coker(g) \overset{\beta'}{\to} coker(h)$ *is surjective.*

Proof. The maps on the left and right sides of δ are induced by α, β, α' and β' respectively. So, the exactness in $\ker(g)$ and $\mathrm{coker}(g)$ are easy to verify.

The important part is the construction of the connecting homomorphism δ. For this consider $c \in \ker(h)$. By the surjectivity of β there exists an element $b \in B$ such that $\beta(b) = c$. By the commutativity of the diagram we have that $\beta'(g(b)) = h(\beta(b)) = h(c) = 0$. Thus $g(b) \in \ker(\beta') = \operatorname{im}(\alpha')$, so there is $a' \in A'$ such that $g(b) = \alpha'(a')$. We define $\delta(c) = a'$. A routine "diagram chasing" shows that this definition does not depend to the choice of b and is in fact an homomorphism. Moreover, it is straightforward to check the exactness of the sequence in $\ker(h)$ and $\operatorname{coker}(f)$. □

Proposition 2.1.3 (Long exact sequence)**.** *For any a short exact sequence of chain complexes of A-modules*

$$0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet \longrightarrow 0.$$

and for any $i \in \mathbb{Z}$ there is a connecting homomorphism $\delta_i : H_i(E_\bullet) \to H_{i-1}(C_\bullet)$ such that the sequence

$$\cdots \longrightarrow H_i(C_\bullet) \xrightarrow{f_*} H_i(D_\bullet) \xrightarrow{g_*} H_i(E_\bullet) \xrightarrow{\delta_i} H_{i-1}(C_\bullet) \xrightarrow{f_*} H_{i-1}(D_\bullet) \xrightarrow{g_*} \cdots.$$

is exact.

Proof. Here we only construct the map δ_i. For the rest of proof we refer the reader to (WEIBEL, 1994, Theorem 1.3.1). Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_i & \xrightarrow{f_i} & D_i & \xrightarrow{g_i} & E_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} & & \\
0 & \longrightarrow & C_{i-1} & \xrightarrow{f_{i-1}} & D_{i-1} & \xrightarrow{g_{i-1}} & E_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{i-1}} & & \downarrow{\scriptstyle \partial_{i-1}} & & \downarrow{\scriptstyle \partial_{i-1}} & & \\
0 & \longrightarrow & C_{i-2} & \xrightarrow{f_{i-2}} & D_{i-2} & \xrightarrow{g_{i-2}} & E_{i-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Let $e \in Z_i(E_\bullet) = \ker(\partial_i)$ represents the element $\overline{e} \in H_i(E_\bullet)$. Take $d \in D_i$ such that $e = g_i(d)$. Then (as in the proof of the snake lemma) we know that $\partial_i(d) \in \ker(g_{i-1})$. Hence $\partial_i(d) = f_{i-1}(c)$ for some $c \in C_{i-1}$. We show that $c \in Z_{i-1}(C_\bullet)$. Since $f_{i-2}(\partial_{i-1}(c)) = \partial_{i-1}(f_{i-1}(c)) = \partial_{i-1}(\partial_i(d)) = 0$, by the injectivity of f_{i-2} we have $\partial_{i-1}(c) = 0$. Thus $c \in Z_{i-1}(C_\bullet)$ represents an element of $H_{i-1}(C_\bullet)$. Now, we define $\delta(\overline{e}) = \overline{c}$. □

The connecting homomorphisms have the following *naturality* property.

Proposition 2.1.4. *From the commutative diagram of chain complexes with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet & \xrightarrow{f_\bullet} & D_\bullet & \xrightarrow{g_\bullet} & E_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_\bullet} & & \downarrow{\scriptstyle \varphi_\bullet} & & \downarrow{\scriptstyle \psi_\bullet} & & \\ 0 & \longrightarrow & C'_\bullet & \xrightarrow{f'_\bullet} & D'_\bullet & \xrightarrow{g'_\bullet} & E'_\bullet & \longrightarrow & 0 \end{array}$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_i(C_\bullet) & \xrightarrow{f_*} & H_i(D_\bullet) & \xrightarrow{g_*} & H_i(E_\bullet) & \xrightarrow{\delta} & H_{i-1}(C_\bullet) & \xrightarrow{f_*} & H_{i-1}(D_\bullet) & \xrightarrow{g_*} & \cdots \\ & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \varphi_*} & & \\ \cdots & \longrightarrow & H_i(C'_\bullet) & \xrightarrow{f'_*} & H_i(D'_\bullet) & \xrightarrow{g'_*} & H_i(E'_\bullet) & \xrightarrow{\delta'} & H_{i-1}(C'_\bullet) & \xrightarrow{f'_*} & H_{i-1}(D'_\bullet) & \xrightarrow{g'_*} & \cdots \end{array}$$

Proof. See (WEIBEL, 1994, Proposition 1.3.4). □

The following definition has its origin in algebraic topology.

Definition 2.1.5. We say that two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are *chain homotopic* if there exists a family of homomorphisms $\{s_i : C_i \to D_{i+1}\}_{i\in\mathbb{Z}}$ such that

$$f_i - g_i = \partial_i \circ s_i + s_{i-1} \circ \partial_i.$$

The family $\{s_i\}_{i\in\mathbb{Z}}$ is called a *chain homotopy* from $f_\bullet$ to $g_\bullet$. We say that $f_\bullet : C_\bullet \to D_\bullet$ is a *chain homotopy equivalence* if there is a morphism $g_\bullet : D_\bullet \to C_\bullet$ such that $g_\bullet \circ f_\bullet$ and $f_\bullet \circ g_\bullet$ are chain homotopic to the identity morphisms of $C_\bullet$ and $D_\bullet$, respectively.

Two chain homotopic maps induce equal maps on homology of complexes. More precisely:

Lemma 2.1.6. *If $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic, then the maps $f_*, g_* : H_i(C_\bullet) \to H_i(D_\bullet)$ are equal for all $i \in \mathbb{Z}$.*

Proof. It is sufficient to proof that if $f_\bullet$ and the zero morphism $0_\bullet : C_\bullet \to D_\bullet$ are chain homotopic, then $f_* : H_i(C_\bullet) \to H_i(D_\bullet)$ is the zero map. Let the element $\overline{x} \in H_i(C_\bullet)$ is represented by $x \in Z_i(C_\bullet)$. Then $f_i(x) = \partial_i(s_i(x))$. Thus $f_i(x) \in B_i(D_\bullet)$ which represents the zero element of $H_i(D_\bullet)$. □

For the definition of homology of groups, we need to define projective modules. These modules can be considered as a generalization of vector spaces over rings.

Definition 2.1.7. An A-module P is called projective if for any given, diagram with exact row ($f : M \to N$ is surjective)

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \rho} & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0, \end{array}$$

there is a lifting of ρ, i.e. there is a map $h: P \to M$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle \rho} & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

is commutative.

It is a well-known fact that for any A-module M there exists a surjective map $F \twoheadrightarrow M$ where F is free. The module F can be taken as the free A-module generated by a set of indexed by elements of M and the map $F \twoheadrightarrow M$ can be defined by taking a basis element $u_m \in F$ to the element $m \in M$. When M is projective, then the map $F \twoheadrightarrow M$ has a splitting map $M \to F$.

Proposition 2.1.8. *1. An A-module M is projective if and only if there exists an A-module N such that $M \oplus N$ is free.*

2. If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of left A-modules and P a right projective A-module, then

$$0 \longrightarrow P \otimes_A M' \xrightarrow{id_P \otimes f} P \otimes_A M \xrightarrow{id_P \otimes g} P \otimes_A M'' \longrightarrow 0$$

is exact.

Proof. 1. Let M be projective. Take a free A-module F with a surjective map $f: F \to M$. Let N be the kernel of this map. Then by the projectivity of M we have a map $k: M \to F$ such that $f \circ k = \mathrm{id}_M$. Thus the exact sequence $0 \to N \to F \xrightarrow{f} M \to 0$ splits and we have $F \simeq N \oplus M$. Now let there is a module N such that $M \oplus N$ is free. Consider the following diagram with exact row

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle \rho} & & \\ Q & \xrightarrow{f} & Q' & \longrightarrow & 0. \end{array}$$

Composing ρ with the projection $M \oplus N \xrightarrow{\pi_M} M$, and using the projectivity of $M \oplus N$, there is a morphism $k: M \oplus N \to Q$ such tat the diagram

$$\begin{array}{ccccc} & & M \oplus N & & \\ & \overset{k}{\swarrow} & \downarrow{\scriptstyle \pi_M} \uparrow{\scriptstyle i_M} & & \\ & & M & & \\ & \overset{k \circ i_M}{\swarrow} & \downarrow{\scriptstyle \rho} & & \\ Q & \xrightarrow{f} & Q' & \longrightarrow & 0. \end{array}$$

Commutes. Now if $i_M: M \to M \oplus N$ is give by $m \mapsto (m, 0)$, then the map $k \circ i_M: M \to Q$ makes the first diagram commutative and we're done.

2. If F is a free right A-module, then $F \otimes_A M'$, $F \otimes_A M$ and $F \otimes_A M''$ are direct sum of copies of the respective A-modules M', M and M''. This gives the exactness of the sequence

$$0 \longrightarrow F \otimes_A M' \xrightarrow{\mathrm{id}_F \otimes f} F \otimes_A M \xrightarrow{\mathrm{id}_F \otimes g} F \otimes_A M'' \longrightarrow 0.$$

If Q is a module such that $P \oplus Q$ is free, then the distributivity of the direct sum with respect to the tensor product gives the exactness of the sequence

$$0 \longrightarrow P \otimes_A M' \xrightarrow{\mathrm{id}_P \otimes f} P \otimes_A M \xrightarrow{\mathrm{id}_P \otimes g} P \otimes_A M'' \longrightarrow 0.$$

□

Definition 2.1.9. Let M be an A-module. A *resolution* of M is a family of modules $\{M_i\}_{i \geq 0}$, together with a family of morphisms $\{d_i : M_i \to M_{i-1}\}_{i \geq 1}$ and a map $\varepsilon : M_0 \to M$ such that the sequence

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

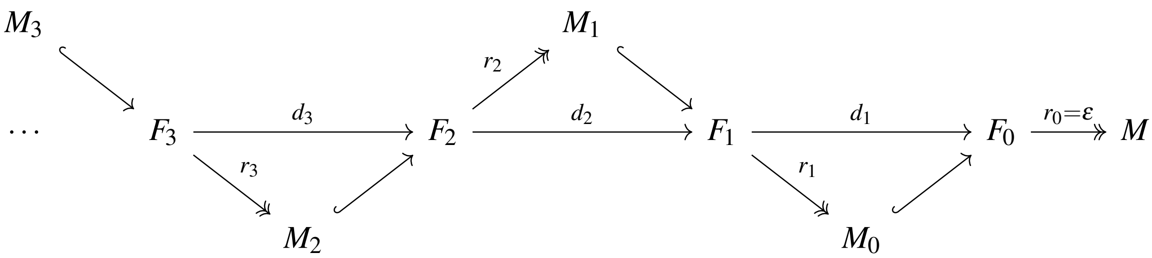
is an exact complex. This resolution of M is called *free* or *projective* if the A-modules M_i are free or projective, respectively. We denote this resolution by $M_\bullet \xrightarrow{\varepsilon} M$.

Proposition 2.1.10. *Any A-module M has a free resolution. In particular, any module M has a projective resolution.*

Proof. Let $F_0 \overset{d_0=\varepsilon}{\twoheadrightarrow} M$ be a surjective map, where F_0 is free. Let $M_0 = \ker(\varepsilon)$. Let $F_1 \overset{r_1}{\twoheadrightarrow} M_0$ be a surjective map where F_1 is free. Let d_1 be the composite $F_1 \overset{r_1}{\twoheadrightarrow} M_0 \hookrightarrow F_0$, clearly the sequence $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M$ is exact. Suppose by induction that we have constructed $F_{n-1} \overset{d_{n-1}}{\twoheadrightarrow} F_{n-2}$. Let $M_{n-1} = \ker(d_{n-1})$ and take $F_n \overset{r_n}{\twoheadrightarrow} M_{n-1}$ a surjective map such that F_n is free. Take d_n as the composition of r_n and the inclusion $M_{n-1} \to F_{n-1}$. It is easy to see that the sequence

$$F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact. For clarity see the following diagram:



□

Theorem 2.1.11 (Comparison Theorem)**.** *Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of an A-module M and $f' : M \to N$ a homomorphism of A-modules. Then for any resolution $Q_\bullet \xrightarrow{\eta} N$ there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ such that $\eta \circ f_0 = f' \circ \varepsilon$. The chain map $f_\bullet$ is unique up homotopy.*

Proof. Let $Z_{-1}(P_\bullet) := M$, $Z_{-1}(Q_\bullet) = N$, $f_{-1} := f' : Z_{-1}(P_\bullet) \to Z_{-1}(Q_\bullet)$, $\partial_0 = \varepsilon : P_0 \to M$ and $\partial_0 = \eta : Q_0 \to N$. By the projectivity of P_0 there is $f_0 : P_0 \to Q_0$ such that $f' \circ \varepsilon = f_{-1} \circ \partial_0$.

Inductively suppose that we have constructed f_i for $i \leq n$. Thus we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z_n(P_\bullet) & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} \\
 & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\
0 & \longrightarrow & Z_n(Q_\bullet) & \longrightarrow & Q_n & \xrightarrow{\partial_n} & Q_{n-1},
\end{array}$$

where f'_n is the map induced by the commutativity of the right square. By the projectivity of P_{n+1} and the diagram

$$\begin{array}{ccccc}
P_{n+1} & \xrightarrow{\partial_{n+1}} & Z_n(P_\bullet) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f'_n} & & \\
Q_{n+1} & \xrightarrow{\partial_{n+1}} & Z_n(Q_\bullet) & \longrightarrow & 0
\end{array}$$

$f'_n \partial_{n+1}$ lifts to a map $f_{n+1} : P_{n+1} \to Q_{n+1}$. Thus we can construct a chain map $f_\bullet : P_\bullet \to Q_\bullet$ with the required properties. For the uniqueness of $f_\bullet$ up to homotopy we refer the reader to (WEIBEL, 1994, Comparison Theorem 2.2.6). □

2.2 The functors Tor

In this section we introduce and study the functor Tor. For this we need the following theorem.

Theorem 2.2.1. *Let M be a right A-module and $P_\bullet \to M$ and $P'_\bullet \to M$ two projective resolutions of M. Then for any left A-module N we have $H_n(P_\bullet \otimes_A N) \simeq H_n(P'_\bullet \otimes_A N)$ for all $n \geq 0$.*

Proof. By Comparison Theorem 2.1.11 we have morphisms $f_\bullet : P_\bullet \to P'_\bullet$ and $g_\bullet : P'_\bullet \to P_\bullet$. Again by Comparison Theorem $g_\bullet \circ f_\bullet : P_\bullet \to P_\bullet$ is chain homotopic to $\mathrm{id}_{P_\bullet}$ and $f_\bullet \circ g_\bullet : P'_\bullet \to P'_\bullet$ is chain homotopic to $\mathrm{id}_{P'_\bullet}$.

Now, consider the morphisms $f_\bullet \otimes \mathrm{id}_N$ and $g_\bullet \otimes \mathrm{id}_N$. The compositions $g_\bullet \circ f_\bullet \otimes \mathrm{id}_N$ and $f_\bullet \circ g_\bullet \otimes \mathrm{id}_N$ are chain homotopic to the respective identity maps. Thus the homology of $f_\bullet \otimes \mathrm{id}_N$ gives an isomorphism. □

Theorem 2.2.2. *Let M be a right A-module and N a left A-module. If $P_\bullet \to M$ and $Q_\bullet \to N$ are projective resolutions of M and N respectively, then for any $n \geq 0$,*

$$H_n(P_\bullet \otimes_A N) \simeq H_n(M \otimes_A Q_\bullet).$$

Proof. See (ROTMAN, 1979, Theorem 7.9). □

Definition 2.2.3. Let M be a right A-module and $P_\bullet \to M$ a projective resolution of M. For a left A-module N we define

$$\mathrm{Tor}_n^A(M,N) := H_n(P_\bullet \otimes_A N).$$

By Theorem 2.2.1, this definition does not depends to the chosen projective resolution of M.

The commutativity of the direct sums and direct limits with tensor product gives the following important result

Lemma 2.2.4. *If $\{M_j\}_{j\in J}$ is a family of A-modules, then*

$$Tor_n^A\left(M, \bigoplus_{j\in J} N_j\right) \simeq \bigoplus_{j\in J} Tor_n^A(M,N_j), \qquad Tor_n^A\left(\bigoplus_{j\in J} M_j, N\right) \simeq \bigoplus_{j\in J} Tor_n^A(M_j,N),$$

if $\{M_j\}_{j\in J}$ is a direct system of A-modules, then

$$Tor_n^A\left(M, \varinjlim_{j\in J} N_j\right) \simeq \varinjlim_{j\in J} Tor_n^A(M,N_j), \qquad Tor_n^A\left(\varinjlim_{j\in J} M_j, N\right) \simeq \varinjlim_{j\in J} Tor_n^A(M_j,N).$$

Proof. The Theorem 2.2.2 gives the commutativity of Tor_n^A with direct sums and direct limits at the first component. □

In the examples below, we present some properties of the Tor-functor and its connection with the torsion subgroup of abelian groups.

Example 2.2.5. Let M and N be A-modules. Take a projective resolution $F_\bullet \to M$ of M. Since the tensor product is a right exact functor, we have the exact sequence

$$F_1 \otimes_A N \xrightarrow{d_1 \otimes \mathrm{id}} F_0 \otimes_A N \xrightarrow{\varepsilon \otimes \mathrm{id}} M \otimes_A N \longrightarrow 0.$$

This gives the isomorphism

$$\mathrm{Tor}_0^A(M,N) = H_0(F_\bullet \otimes_A M) = \mathrm{coker}(\varepsilon \otimes \mathrm{id}) \simeq M \otimes N.$$

Example 2.2.6. Let A, B abelian groups. Let $\pi : F \to A$ be a surjective map where F is a free abelian group (a free $\mathbb{Z}$-module). Since subgroups of free abelian groups are free, the complex

$$0 \longrightarrow \ker(\pi) \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$$

is free resolution of A. So, the complex

$$0 \longrightarrow \ker(\pi) \otimes_{\mathbb{Z}} B \longrightarrow F \otimes_{\mathbb{Z}} B \longrightarrow 0$$

can be used to calculate $\mathrm{Tor}_n^{\mathbb{Z}}(A,B)$. Thus

$$\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0, \qquad \text{for } n \geq 2.$$

Example 2.2.7. Let M and N be A-modules. If M is projective, for the calculation of $\mathrm{Tor}_n^A(M,N)$ we can take the projective resolution $F_\bullet \to M$ of M with $F_0 = M$, $F_i = 0$ for $i \geq 1$ and $\varepsilon = \mathrm{id}_M$. Thus

$$\mathrm{Tor}_n^A(M,N) = 0$$

for any $n \geq 1$. Now, let N be projective. If $F_\bullet \to M$ is a projective resolution of M, then $\cdots \to F_1 \otimes N \to F_0 \otimes N \to M \otimes N \to 0$ is exact. Thus $\mathrm{Tor}_n^A(M,N) = 0$ for any $n \geq 1$.

Example 2.2.8. We know that any finitely generated abelian group is direct sum of its free and torsion parts. Thus if A and B are finitely generated abelian groups we have that

$$\mathrm{Tor}_1^{\mathbb{Z}}(A,B) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(T_A,T_B) \simeq T_A \otimes_{\mathbb{Z}} T_B$$

where T_A, T_B are the torsion parts of A and B respectively and the right isomorphism can be found in (VERMANI, 2003, Corollary 6.3.16).

Now, we know that every abelian group is a direct limit of its finitely generated subgroups. Thus by lemma 2.2.4, we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(A,B) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(T_A,T_B).$$

which is a torsion group.

In the next theorem, we study the long exact sequence for the Tor-functor.

Theorem 2.2.9. *1. Let $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \to 0$ be a short exact sequence of A-modules. Then for any A-module M we have the long exact sequence*

$$\cdots \to Tor_n^A(M,N') \xrightarrow{\alpha_*} Tor_n^A(M,N) \xrightarrow{\beta_*} Tor_n^A(M,N'') \xrightarrow{\delta} Tor_{n-1}^A(M,N') \to \cdots$$

2. Let $0 \to M' \xrightarrow{\alpha'} M \xrightarrow{\beta'} M'' \to 0$ be a short exact sequence of right A-modules, then for any A-module N we have the long exact sequence

$$\cdots \to Tor_n^A(M',N) \xrightarrow{\alpha'_*} Tor_n^A(M,N) \xrightarrow{\beta'_*} Tor_n^A(M'',N) \xrightarrow{\delta} Tor_{n-1}^A(M',N) \to \cdots$$

Proof. The first claim follows from the definition of Tor and the long exact sequence for the homology of chain complexes (Proposition 2.1.3). For the second claim, take any projective resolution of N, tensorize it with the exact sequence $0 \to M' \to M \to M'' \to 0$ and then apply the Theorem 2.2.2 and proposition 2.1.3. □

2.3 Homology of groups

Now we are ready to define the homology of groups. This can be done using the Tor functor. Thus it inherits most of properties of the Tor functor.

Let G be a group and let $\mathbb{Z}G$ be the group ring of G. In this thesis we will work with left $\mathbb{Z}G$-modules. Any left $\mathbb{Z}G$-module has a natural structure of a right $\mathbb{Z}G$-module with the right action $mg := g^{-1}m$. Thus it is natural just talk about $\mathbb{Z}G$-modules. In the following we will use the notation $M \otimes_G N$ for the tensor product of M and N as left and right $\mathbb{Z}G$-modules discussed in above. We reserve the notation $M \otimes_{\mathbb{Z}G} N$ to the case when M has a natural left and right action by G (for example, when $M = \mathbb{Z}H$, where $H \leq G$).

Remark 2.3.1. Observe that the definition of $M \otimes_G N$ comes from $M \otimes_{\mathbb{Z}} N$ by adding the relations $mg \otimes n = g^{-1}m \otimes n = m \otimes gn$. If we change m by gm, the last equality turns to $m \otimes n = gm \otimes gn$. Now we define a diagonal action of G on $M \otimes_{\mathbb{Z}} N$ as $g(m \otimes n) := gm \otimes gn$, this last equality turns to $m \otimes n = g(m \otimes n)$. Thus

$$M \otimes_G N \simeq M \otimes_{\mathbb{Z}} N / \langle g(m \otimes n) - (m \otimes n) | m \in M, n \in N, g \in G \rangle.$$

Definition 2.3.2. Let G be a group and M a G-module i.e. a $\mathbb{Z}G$-module. The n-th homology of G with coefficients in M is defined as follows:

$$H_n(G,M) := \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},M),$$

where $\mathbb{Z}$ is considered as G-module with the trivial action of G, i.e. $g \cdot n = n$. When $M = \mathbb{Z}$ has the trivial action of G, then $H_n(G,\mathbb{Z})$ is called the n-th integral homology of G.

Observe that, the homology group $H_n(G,M)$ is a functor on the coefficient module M and inherits most of the properties of the Tor-functor. One important property is the long exact sequence.

Theorem 2.3.3. *Let G be a group and $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \to 0$ be a short exact sequence of left G-modules ($\mathbb{Z}G$-modules). Then there exists the following long exact sequence*

$$\cdots \longrightarrow H_n(G,N') \xrightarrow{\alpha_*} H_n(G,N) \xrightarrow{\beta_*} H_n(G,N'') \xrightarrow{\delta} H_{n-1}(G,N') \longrightarrow \cdots$$

of homology groups.

Proof. Just apply Theorem 2.2.9 to the homology functor $H_n(G,\cdot) = \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\cdot)$. □

By Lemma 2.2.4, $H_n(G,\cdot)$ commutes with direct sums and direct limits. That is for any family $\{M_j\}_{j \in J}$ of G-modules and any $n \geq 0$, we have

$$H_n(G, \bigoplus_{j \in J} M_j) \simeq \bigoplus_{j \in J} H_n(G,M_j)$$

and for any direct system $\{M_j\}_{j \in J}$ of G-modules, we have

$$H_n(G, \varinjlim_{j \in J} M_j) \simeq \varinjlim_{j \in J} H_n(G,M_j).$$

By the following theorem, direct limit can be taken on the group G.

Theorem 2.3.4. *Let $\{G_j\}_{j\in J}$ be a direct system where J is a directed set. If $G = \varinjlim_{j\in J} G_j$, then for any G-module M*

$$\varinjlim_{j\in J} H_n(G_j, M) \simeq H_n(G, M).$$

Proof. (BROWN, 2012, Page 121, Exercise 3(a)). □

For any group G, the *augmentation map* $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is defined by $\sum n_i g_i \mapsto \sum n_i$. The kernel of this map will be denoted by $I_G = \ker(\varepsilon)$ and is called the *augmentation ideal* if G. It is easy to see that I_G is generated by $\{g-1 | g \in G\}$ as free $\mathbb{Z}$-module.

For any G-module M, the $\mathbb{Z}G$-submodule $I_G M$ is generated by the elements $gm - m$ with $g \in G$ and $m \in M$. The quotient

$$M_G := M/I_G M$$

is called the group of *coinvariants* of M. Take the exact sequence of $\mathbb{Z}G$-modules

$$0 \to I_G \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

(G acts trivially on $\mathbb{Z}$). Tensoring this sequence by M we obtain the exact sequence

$$I_G \otimes_{\mathbb{Z}G} M \longrightarrow M \xrightarrow{\varepsilon\otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}G} M \longrightarrow 0.$$

This gives us the isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}G} M \simeq \mathrm{coker}(I_G \otimes_{\mathbb{Z}G} M \to M) = M/I_G M = M_G.$$

Therefore

$$H_0(G, M) = \mathrm{Tor}_0^{\mathbb{Z}G}(\mathbb{Z}, M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M \simeq M_G.$$

In particular, for any group G we have $H_0(G, \mathbb{Z}) = \mathbb{Z}$.

Example 2.3.5. In this example we study the homology group $H_1(G, \mathbb{Z})$. From the short exact sequence $0 \to I_G \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ of G-modules, we get the long exact sequence

$$H_1(G, \mathbb{Z}G) \xrightarrow{\varepsilon_*} H_1(G, \mathbb{Z}) \xrightarrow{\delta} H_0(G, I_G) \longrightarrow H_0(G, \mathbb{Z}G) \xrightarrow{\varepsilon_*} H_0(G, \mathbb{Z}) \longrightarrow 0.$$

Since $\mathbb{Z}G$ is free as $\mathbb{Z}G$-module, we have $H_1(G, \mathbb{Z}G) = \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$ (by Example 2.2.7). Moreover

$$H_0(G, \mathbb{Z}G) = \mathbb{Z} \otimes_G \mathbb{Z}G \simeq \mathbb{Z} \simeq H_0(G, \mathbb{Z}).$$

But the map

$$\varepsilon_* : \mathbb{Z} \simeq \mathbb{Z} \otimes \mathbb{Z}G \to \mathbb{Z} \otimes_G \mathbb{Z} \simeq \mathbb{Z}$$

is given by

$$n \mapsto 1 \otimes n \mapsto 1 \otimes n \mapsto n.$$

Thus

$$H_1(G,\mathbb{Z}) \simeq H_0(G,I_G) = (I_G)_G = I_G/I_G^2.$$

As we will see below I_G/I_G^2 is isomorphic to the abelianization of G. We will show this using the standard resolution of G.

Example 2.3.6. Let G be a finite cyclic group of order n with generator t. Consider the element $N = 1+t+\cdots+t^{n-1} \in \mathbb{Z}G$. It is not difficult to verify that the sequence

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Note that the maps are defined by multiplication by N or $t-1$. Tensoring this resolution with a G-module M and dropping the last term we obtain the complex

$$\cdots \longrightarrow M \xrightarrow{N} M \xrightarrow{(t-1)} M \xrightarrow{N} M \xrightarrow{(t-1)} M \longrightarrow 0.$$

Thus

$$H_n(G,M) \simeq \begin{cases} M_G & n=0 \\ \frac{M^G}{\mathrm{im} N} & \text{if } n \text{ is odd} \\ \ker(N : M_G \to M^G) & \text{if } n \text{ is even} \end{cases}$$

where $M^G := \{m \in M : gm = m \text{ for all } g \in G\}$.

Now, we will present the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, which will be very useful in future calculations. Let $\mathbf{C}'_n(G)$ be the free abelian group generated by G^{n+1}. Let $\mathbf{C}_n(G)$ be the quotient of $\mathbf{C}'_n(G)$ by the subgroup generated by the elements $(g_0,\ldots,g_n)$ where $g_i = g_{i+1}$ for some i. We denote the element of $\mathbf{C}_n(G)$ represented by $(g_0,\ldots,g_n)$ again by $(g_0,\ldots,g_n)$.

The group $\mathbf{C}_n(G) = G^{n+1}$ is a left G-module with the action:

$$g \cdot (g_0,g_1,\ldots,g_n) = (gg_0,gg_1,\ldots,gg_n)$$

We convert this left action of G to a right action by $m \cdot g := g^{-1} \cdot m$. It is not difficult to see that the elements $(1,g_1,g_2,\ldots,g_n)$, $g_i \in G$ generate $\mathbf{C}_n(G)$ as free $\mathbb{Z}G$-module. The maps

$$d_n(g_0,g_1,\ldots,g_n) := \sum_{k=0}^{n} (-1)^k (g_0,\ldots,g_{k-1},g_{k+1},\ldots,g_n), \qquad n \geq 1$$

$$\varepsilon : \mathbf{C}_0(G) \to \mathbb{Z}, \qquad (1) \mapsto 1,$$

turns $\mathbf{C}_\bullet(G) \to \mathbb{Z}$ to a free resolution of $\mathbb{Z}$ on $\mathbb{Z}G$ called the *standard resolution* of G.

Let $\mathbf{B}_n(G)$ the free $\mathbb{Z}G$-modules generated by the symbols $[g_1|g_2|\cdots|g_n]$, $g_i \neq 1$. Let the map $d_n : \mathbf{B}_n(G) \to \mathbf{B}_{n-1}(G)$ be given by

$$d_n([g_1|g_2|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n [g_1|\cdots|g_{n-1}].$$

In above if $g_ig_{i+1} = 1$, we remove the element $[g_1|\cdots|g_ig_{i+1}|g_n]$ from the above map. Moreover, let $\varepsilon : \mathbf{B}_0(G) \to \mathbb{Z}$ be given by $[] \mapsto 1$. Then $B_\bullet(G) \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, called the *bar resolution* of G.

We have the chain isomorphism $\theta_\bullet : \mathbf{C}_\bullet(G) \to \mathbf{B}_\bullet(G)$ defined by

$$(1, g_1, \ldots, g_n) \mapsto [g_1|g_1^{-1}g_2|\cdots|g_{n-1}^{-1}g_n] \tag{2.3.1}$$

with the inverse morphism $\eta_\bullet : \mathbf{B}_\bullet(G) \to \mathbf{C}_\bullet(G)$ given by

$$[g_1|g_2|\cdots|g_n] \mapsto (1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n). \tag{2.3.2}$$

Lemma 2.3.7. *Let G be a group. Then $H_1(G,\mathbb{Z}) \simeq \frac{G}{[G,G]}$, where $[G,G]$ is the commutator subgroup of G. In other words, $H_1(G,\mathbb{Z})$ coincides with the abelianization of G.*

Proof. We calculate $H_1(G,\mathbb{Z})$ using the bar resolution $\mathbf{B}_\bullet(G)$. Consider the chain complex

$$\cdots \xrightarrow{d_3\otimes \mathrm{id}_{\mathbb{Z}}} \mathbf{B}_2(G)\otimes_G \mathbb{Z} \xrightarrow{d_2\otimes \mathrm{id}_{\mathbb{Z}}} \mathbf{B}_1(G)\otimes_G \mathbb{Z} \xrightarrow{d_1\otimes \mathrm{id}_{\mathbb{Z}}} \mathbf{B}_0(G)\otimes_G \mathbb{Z} \longrightarrow 0.$$

Clearly $\mathbf{B}_1(G) = \ker(d_1 \otimes \mathrm{id}_{\mathbb{Z}})$. Thus $H_1(G) = \mathbf{B}_1(G)/\mathrm{im}(d_2 \otimes \mathrm{id}_{\mathbb{Z}})$. In this group we have $\overline{(d_2 \otimes \mathrm{id}_{\mathbb{Z}})\{([g_1|g_2]) \otimes 1\}} = 0$. and thus

$$\overline{[g_1g_2]\otimes 1} = \overline{[g_1]\otimes 1} + \overline{[g_2]\otimes 1}.$$

This implies that the maps $G/[G,G] \to H_1(G,\mathbb{Z})$, $\overline{g} \mapsto \overline{[g]\otimes 1}$, and $H_1(G,\mathbb{Z}) \to G/[G,G]$, $\overline{[g]\otimes 1} \mapsto \overline{g}$ are well defined and one is the inverse of the other. □

Remark 2.3.8. Note that by the example 2.3.5 and the above lemma we have

$$H_1(G,\mathbb{Z}) \simeq G/[G,G] \simeq I_G/I_G^2.$$

A direct map $G/[G,G] \to I_G/I_G^2$ can be given by $\overline{g} \mapsto \overline{g-1}$.

Example 2.3.9. Let G be an abelian group, and consider $\mathbf{B}_\bullet(G) \to \mathbb{Z}$ the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Consider the map

$$\begin{array}{ccc} G\times G & \longrightarrow & \mathbf{B}_2(G)\otimes_G \mathbb{Z} \\ (g_1,g_2) & \longmapsto & ([g_1|g_2]-[g_2|g_1])\otimes 1. \end{array}$$

Since $(d_2 \otimes \mathrm{id}_{\mathbb{Z}})(([g_1|g_2]-[g_2|g_1])\otimes 1) = 0$, we have the map

$$\begin{array}{ccc} G\times G & \longrightarrow & H_2(G,\mathbb{Z}) \\ (g_1,g_2) & \longmapsto & c(g_1,g_2) := \overline{([g_1|g_2]-[g_2|g_1])\otimes 1}. \end{array}$$

Now, for any $g_1, g_2, g_3 \in G$ we have the following identities

$$\begin{aligned} c(g_1g_2,g_3) &= c(g_1,g_3)+c(g_2,g_3), \\ c(g_1,g_2g_3) &= c(g_1,g_2)+c(g_1,g_3), \\ c(g_1,g_2) &= -c(g_2,g_1). \end{aligned}$$

These identities, induce the homomorphism $\bigwedge^2 G \to H_2(G)$ given by $g\wedge h \mapsto c(g,h)$. By (BROWN, 2012, Chapter V, Theorem 6.4), this map is an isomorphism.

Let H be a subgroup of G. Let $P_\bullet \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then this complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Thus for any H-module M and any projective resolution $Q_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}H$, we have

$$H_n(H,M) \simeq H_n(P_\bullet \otimes_H M) \simeq H_n(Q_\bullet \otimes_H M).$$

Example 2.3.10. Let H be a subgroup of a group G. Consider the standard resolutions $C_\bullet(G) \to \mathbb{Z}$ and $C_\bullet(H) \to \mathbb{Z}$. Let $G\backslash H$ be the set of right cosets and $s : G\backslash H \to G$ any section of the canonical projection $\pi : G \to G\backslash H$. Define the map

$$\Theta_\bullet(s) : C_\bullet(G) \longrightarrow C_\bullet(H)$$

as

$$(g_0, g_1, \dots, g_n) \longmapsto (\overline{g_0}, \overline{g_1}, \dots, \overline{g_n}),$$

where $\overline{g} := g \cdot s(\pi(g))^{-1} \in H$ for any $g \in G$. It is easy to show that this is a morphism of chain complexes. Note that if $\mathrm{inc}_\bullet : C_\bullet(H) \to C_\bullet(G)$ is the chain map induced by the inclusion, then as in the proof of Theorem 2.2.1, we can show that $\Theta_\bullet(s) \circ \mathrm{inc}_\bullet$ is chain homotopic to $\mathrm{id}_{C_\bullet(H)}$. Tensoring this by a left H-module M, we get

$$(\Theta_\bullet(s) \otimes \mathrm{id}_M)_* \circ (\mathrm{inc}_\bullet \otimes \mathrm{id}_M)_* = (\Theta_\bullet(s) \otimes \mathrm{id}_M \circ \mathrm{inc}_\bullet \otimes \mathrm{id}_M)_* = ((\Theta_\bullet(s) \circ \mathrm{inc}_\bullet) \otimes \mathrm{id}_M)_* = \mathrm{id}_{H_n(H,M)}.$$

2.4 H_n as a functor of two variables

In this section we study the homology of groups as a functor on the category of pairs (G,M), where G is a group and M is a left G-module. A morphism between (G,M) and (G',M') is a pair of maps (α, f), where $\alpha : G \to G'$ is a group homomorphism and $f : M \to M'$ is a homomorphism of G-modules where take M' as a G-module with the action $g \cdot m' := \alpha(g)m'$. More precisely we have

$$f(gm) = \alpha(g)f(m). \tag{2.4.1}$$

Let $P_\bullet \to \mathbb{Z}$ and $P'_\bullet \to \mathbb{Z}$ be two projective resolutions of $\mathbb{Z}$ over G and G', respectively. Observe that $P'_\bullet \to \mathbb{Z}$ is a resolution of $\mathbb{Z}$ over G. By Comparison Theorem 2.1.11, the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ extends to a chain map $\tau_\bullet : P_\bullet \to P'_\bullet$ which is compatible with α, i.e. $\tau_n(gm) = \alpha(g)\tau_n(m)$. Thus we have the morphism

$$\tau_\bullet \otimes f : P_\bullet \otimes_G M \to P'_\bullet \otimes_{G'} M'.$$

For any $n \geq 0$ this morphism induces the homomorphism $(\alpha, f)_* : H_n(G,M) \to H_n(G',M')$, which is given by $\overline{z_n} \mapsto \overline{(\tau \otimes f)(z_n)}$.

Example 2.4.1. Let H be a subgroup of G and consider the morphism of pairs $(\mathrm{inc}, \mathrm{id}_M) : (H,M) \to (G,M)$. The map $\mathrm{cor}_H^G : H_n(H,M) \to H_n(G,M)$ is called the *corestriction map*.

Let H be a subgroup of G and M a H-module. The induced G-module is defined as follows

$$\mathrm{Ind}_H^G(M) := \mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

With this notation we have:

Lemma 2.4.2 (Shapiro's Lemma)**.** *Let H be a subgroup of a group G and M a H-module. Then the map $(inc, \alpha) : (H, M) \to (G, \mathbb{Z}G \otimes_H M)$, where $\alpha(m) = 1 \otimes m$, induces the isomorphism*

$$H_n(H, M) \simeq H_n(G, Ind_H^G(M)).$$

Proof. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over G. Then

$$\begin{aligned} H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) &\simeq H_n(P_\bullet \otimes_G (\mathbb{Z}G \otimes_H M)) \simeq H_n((P_\bullet \otimes_G \mathbb{Z}G) \otimes_H M) \\ &\simeq H_n(P_\bullet \otimes_H M) = H_n(H, M). \end{aligned}$$

□

Example 2.4.3. Let H be a subgroup of G and M a H-module. Then $\mathrm{Hom}_H(\mathbb{Z}G, M)$ is a G-module with the G-action $(g \cdot f)(x) := f(xg)$. Let $\varphi : M \to \mathrm{Hom}_H(\mathbb{Z}G, M)$ be given by $m \to \varphi_m$, where

$$\varphi_m(x) = \begin{cases} xm, & x \in H \\ 0, & x \notin H. \end{cases}$$

This map can be extended to a G-homomorphism $\overline{\varphi} : \mathbb{Z}G \otimes_H M \to \mathrm{Hom}_H(\mathbb{Z}G, M)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & \mathrm{Hom}_H(\mathbb{Z}G, M) \\ \downarrow & \nearrow_{\overline{\varphi}} & \\ \mathbb{Z}G \otimes_H M & & \end{array}$$

is commutative, where the morphism $M \to \mathbb{Z}G \otimes_H M$ is defined by $m \mapsto 1 \otimes m$. Explicitly we define

$$\overline{\varphi}\left(\sum_{i=1}^{s} g_i \otimes m_i\right) := \sum_{i=1}^{s} g_i \cdot \varphi_{m_i}.$$

Let $(G : H) < \infty$ and consider E as a set of representatives of left cosets. Let

$$\overline{\psi} : \mathrm{Hom}_H(\mathbb{Z}G, M) \longrightarrow \mathbb{Z}G \otimes_H M,$$

$$f \longmapsto \sum_{s \in E} s \otimes f(s^{-1}).$$

It is straightforward to show that this is an inverse of $\overline{\varphi}$. So we have

$$\mathbb{Z}G \otimes_H M \simeq \mathrm{Hom}_H(\mathbb{Z}G, M).$$

For a G-module M consider the composition

$$M \xrightarrow{\varphi} \mathrm{Hom}_H(\mathbb{Z}G, M) \xrightarrow{\overline{\psi}} \mathbb{Z}G \otimes_H M.$$

By applying the homology functor $H_n(G, \cdot)$, we have the composite

$$H_n(G,M) \xrightarrow{\varphi_*} H_n(G, \mathrm{Hom}_H(\mathbb{Z}G,M)) \xrightarrow{\overline{\psi}_*} H_n(G, \mathbb{Z}G \otimes_H M) \simeq H_n(H,M).$$

Note that $\overline{\psi}_*$ is an isomorphism and the last isomorphism is given by Shapiro's lemma. This composition is called the *restriction map* or the *transfer map* and is denoted by res^G_H or tr^G_H.

Another classic and very useful case for the homology of groups is the morphism that arises from the conjugation isomorphism.

Example 2.4.4. Let G be a group, H a normal subgroup of G and M a G-module. Fix $g \in G$ and consider the morphism of pairs $(\alpha_g, f_g) : (H,M) \to (H,M)$, where

$$\alpha_g : H \to H, \qquad h \mapsto ghg^{-1}$$

and

$$f_g : M \to M, \qquad m \mapsto gm.$$

Take a projective resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Note that this is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. Consider the morphism $\tau_\bullet : P_\bullet \to P_\bullet$ defined by $x \mapsto gx$. Note that $\tau_\bullet$ satisfy 2.4.1 because

$$\tau(hx) = ghx = ghg^{-1}gx = \alpha_g(h)\tau(x).$$

By tensoring $\tau_\bullet$ with f_g, we have $\tau_\bullet \otimes f_g$ which induces

$$(\alpha_g, f_g)_* : H_*(H,M) \to H_*(H,M)$$

Note that this is given by $x \otimes m \mapsto gx \otimes gm$. Thus we have the following

1. If $g \in H$, then the action in the chain level is given by $x \otimes m \mapsto gx \otimes gm = x \otimes m$. Thus $(\alpha, f)_* = \mathrm{id}_{H_*(H,M)}$.
2. The map $(\alpha_g, f_g)_* : H_n(H,M) \to H_n(H,M)$ induces an action of G on $H_n(H,M)$. Thus by item (1), we have an action of G/M on $H_n(H,M)$.

The next two lemmas will be very useful in future calculations. The first lemma is useful for the application of Shapiro's lemma while the second lemma involves the maps cor^G_H and res^G_H.

Lemma 2.4.5. *Let G be a group and X a G-set. Let T be a set of representatives of the orbits of X. Then*

$$\mathbb{Z}X \simeq \bigoplus_{x \in T} (\mathbb{Z}G \otimes_{Stab_G(x)} \mathbb{Z})$$

where $\mathbb{Z}G$ is the free $\mathbb{Z}$-module generated by G. In particular,

$$H_n(G,\mathbb{Z}X) \simeq \bigoplus_{x\in T} H_n(Stab_G(x),\mathbb{Z}).$$

Proof. Let $x \in T$ and consider the G-homomorphism

$$\phi_x : \mathbb{Z}G \otimes_{\mathrm{Stab}_G(x)} \mathbb{Z} \longrightarrow \mathbb{Z}X$$

defined by

$$\sum_{i=1}^{N} g_i \otimes m_i \longmapsto \sum_{i=1}^{N} m_i(g_i x).$$

Taking the direct sum of ϕ_x's we have

$$\Phi := \bigoplus_{x\in T} \phi_x : \bigoplus_{x\in T} (\mathbb{Z}G \otimes_{\mathrm{Stab}_G(x)} \mathbb{Z}) \longrightarrow \mathbb{Z}X$$

which is given by

$$\left(\sum_{i=0}^{N} g_{i,x} \otimes m_{i,x}\right)_{x\in T} \longmapsto \sum_{x\in T}\left(\sum_{i=0}^{N} m_{i,x}(g_{i,x}x)\right).$$

The inverse of Φ is the following G-map

$$\Psi : \mathbb{Z}X \longrightarrow \bigoplus_{x\in T} \mathbb{Z}G \otimes_{\mathrm{Stab}_G(x)} \mathbb{Z}$$

defined by

$$\sum_{i=1}^{N} n_i y_i \longmapsto \sum_{i=1}^{N} (g_i \otimes n_i)_{x_i},$$

where y_i belongs to its orbit of x_i and $g_i x_i = y_i$. Moreover $(g_i \otimes n_i)_{x_i}$ is the element of $\bigoplus_{x\in T} \mathbb{Z}G \otimes_{\mathrm{Stab}_G(x)} \mathbb{Z}$ with $g_i \otimes n_i$ in the x_i-component and 0 in other places. It is not difficult to see that this map is well-defined and is the inverse of Φ. The second part follows from Shapiro's lemma. □

Lemma 2.4.6. *Let G be a group and X_1 and X_2 two transitive G-sets. Let $x_i \in X_i$ ($i = 1,2$) and $H_i = Stab_G(x_i)$. Let $\varphi : \mathbb{Z}[X_1] \to \mathbb{Z}[X_2]$ be a map of G-modules with*

$$\varphi(x_1) = \sum_{g\in G/H_2} n_g g x_2,$$

where $n_g \in \mathbb{Z}$. Then

1. *n_g depends only on the class of $g \in E$, where $E = H_1\backslash G/H_2$ (the set of double cosets).*
2. *If $n_g \neq 0$, $[H_1 : H_1 \cap gH_2g^{-1}] < \infty$.*
3. *The map induced by φ from $H_n(H_1,\mathbb{Z}) \to H_2(H_2,\mathbb{Z})$ is given by the formula*

$$\varphi_n(z) = \sum_{g\in E} n_g cor^{H_2}_{g^{-1}H_1g\cap H_2} \circ res^{g^{-1}H_1g}_{g^{-1}H_1g\cap H_2} g^{-1}z.$$

Proof. See (HUTCHINSON, 1989, Proof of Lemma 3, pp. 183-184). □

2.5 Relative group homology

This short section presents the relative homology of groups.

Let G be a group and M a G-module. Let G' be a subgroup of G and M' a G'-submodule of M. The inclusions define a morphism of pairs $(i,j):(G',M')\to(G,M)$. Then we have a morphism of complexes

$$\mathrm{inc}:=i_\bullet\otimes j: C_\bullet(G')\otimes_{G'}M'\to C_\bullet(G)\otimes_G M$$

where $C_\bullet(H)\to\mathbb{Z}$ denotes the standard resolution of a group H. This morphism is injective. To see this consider the composite $C_\bullet(G')\otimes_{G'}M'\to C_\bullet(G')\otimes_{G'}M\to C_\bullet(G)\otimes_G M$. The injectivity of the first morphism is obvious (because $C_n(G')$ is a free G-module). For the injectivity of the second morphism see (KNUDSON, 2001, page 153). Now we can define the relative homology groups of the pair (G,G') as follows:

$$H_n(G,G';M,M')=H_n\left(\frac{C_\bullet(G)\otimes_G M}{C_\bullet(G')\otimes_{G'}M'}\right).$$

When $M=M'$, we will denote $H_n(G,G',M,M')$ by $H_n(G,G';M)$. From the short exact sequence

$$0\longrightarrow C_\bullet(G')\otimes_{G'}M'\xrightarrow{\mathrm{inc}}C_\bullet(G)\otimes_G M\longrightarrow\frac{C_\bullet(G)\otimes_G M}{C_\bullet(G')\otimes_{G'}M'}\longrightarrow 0$$

of complexes we obtain the long exact sequence

$$\cdots\to H_n(G',M')\xrightarrow{\mathrm{inc}_*}H_n(G,M)\longrightarrow H_n(G,G';M,M')\xrightarrow{\delta}H_{n-1}(G',M)\to\cdots. \quad (2.5.1)$$

Moreover, for a chain of subgroups $G''\le G'\le G$ and a G-module M, the natural morphism

$$\overline{\mathrm{inc}}:=\overline{i_\bullet\otimes j}:\frac{C_\bullet(G')\otimes_{G'}M}{C_\bullet(G'')\otimes_{G''}M}\to\frac{C_\bullet(G)\otimes_G M}{C_\bullet(G'')\otimes_{G''}M}$$

is injective.Thus from the short exact sequence of complexes

$$0\longrightarrow\frac{C_\bullet(G')\otimes_{G'}M}{C_\bullet(G'')\otimes_{G''}M}\xrightarrow{\overline{\mathrm{inc}}}\frac{C_\bullet(G)\otimes_G M}{C_\bullet(G'')\otimes_{G''}M}\longrightarrow\frac{C_\bullet(G)\otimes_G M}{C_\bullet(G')\otimes_{G'}M}\longrightarrow 0$$

we obtain the long exact sequence of relative homology groups

$$\cdots\to H_n(G',G'',M)\xrightarrow{\overline{\mathrm{inc}}_*}H_n(G,G'',M)\longrightarrow H_n(G,G';M)\xrightarrow{\delta}H_{n-1}(G',G'',M)\to\cdots. \quad (2.5.2)$$

2.6 Spectral sequences

In next two sections, we will explore our main tool for the study of homology of SL_2, spectral sequences. Spectral sequences are very powerful computational tools.

Definition 2.6.1. A *spectral sequence* E, starting in $a\ge 0$, in an abelian category $\mathscr{A}$ is consist of the following ingredients:

1. A family $\{E^r_{p,q}\}$ of objects in $\mathscr{A}$, $p,q \in \mathbb{Z}$, $r \geq a$.

2. A family of morphisms $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$, called differentials, such that for every $p,q \in \mathbb{Z}$, $r \geq a$:
$$d^r_{p+r,q-r+1} \circ d^r_{p,q} = 0.$$

3. For every $p,q \in \mathbb{Z}$, $r \geq a$, we have the isomorphism:
$$E^{r+1}_{p,q} \simeq \frac{\ker(d^r_{p,q})}{\operatorname{im}(d^r_{p+r,q-r+1})}.$$

The upper indexes $r \geq a$ denotes the "page" of the spectral sequence. If we fix $r \geq a$, the family $\{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ is called the *r-th page* (or the *E^r-page*) of the spectral sequence. Note that, if we fix $p,q \in \mathbb{Z}$ and $r \geq a$, then the objects $\{E^r_{p-kr,q+k(r-1)}\}_{k\in\mathbb{Z}}$ with the differentials $d^r_{p-kr,q+k(r-1)}$, $k \in \mathbb{Z}$ form a complex. If we arrange the objects $E^r_{p,q}$ in a pq-plane, then the complex $\{E^r_{p-kr,q+k(r-1)}\}_{k\in\mathbb{Z}}$ lies in the line with slope $-(r-1)/r$ if $r \neq 0$, and in a vertical line if $r = 0$. With this last observation we understand that the elements in the $(r+1)$-th page are homologies of the complexes on the r-th page.

Definition 2.6.2. A morphism $f : E \to E'$ of spectral sequences is a family of maps $f^r_{p,q} : E^r_{p,q} \to E'^r_{p,q}$ in $\mathscr{A}$ with the conditions $d^r_{p,q} \circ f^r_{p,q} = f^r_{p-r,q+r-1} \circ d^r_{p,q}$, such that each $f^{r+1}_{p,q} : E^{r+1}_{p,q} \to E'^{r+1}_{p,q}$ is the induced map by $f^r_{p,q}$ on homology.

Note that the family $\{f^r_{p-kr,q+k(r-1)}\}_{k\in\mathbb{Z}}$ with p, q and r fixed form a chain map from the complex $\{E^r_{p-kr,q+k(r-1)}\}_{k\in\mathbb{Z}}$ to $\{E'^r_{p-kr,q+k(r-1)}\}_{k\in\mathbb{Z}}$. The spectral sequences in $\mathscr{A}$ with these morphisms forms a category.

The total degree of an object $E^r_{p,q}$ is the number $n := p+q$. If we fix n, we see that the objects $E^r_{p,q}$ with total degree n lie in the line with slope -1 (on the pq-plane). We will work with spectral sequences which have finite non-null objects on the lines of slope -1.

Definition 2.6.3. A spectral sequence is *bounded* if for each n have only finite number of non-null objects with total degree n.

Note that the differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ decrease the total degree by -1. This implies that if a spectral sequence is bounded, the objects $E^r_{p,q}$ (with p, q fixed) will be eventually constant as long as r grows, this means that by passing the pages we will eventually have $E^r_{p,q} = E^{r+1}_{p,q}$, because the outgoing and incoming differentials will be zero for sufficiently large r. We write $E^\infty_{p,q}$ for this stable object of $E^r_{p,q}$.

Example 2.6.4. A spectral sequence E such that $E^r_{p,q} = 0$ for $p < 0$ and $q < 0$ is called a *first quadrant* spectral sequence. In a such spectral sequence, if $r > \max\{p, q+1\}$, then $E^r_{p,q} = E^{r+1}_{p,q}$.

Definition 2.6.5. We say that a bounded spectral sequence starting in $a \geq 0$ *converges to* a family $\{H_n\}$, if for any n we have a finite filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

such that for any $p, q \in \mathbb{Z}$,

$$E^{\infty}_{p,n-p} \simeq \frac{F_p H_n}{F_{p-1} H_n}.$$

In this case we write:

$$E^a_{p,q} \Rightarrow H_{p+q}$$

Definition 2.6.6. Let $E^a_{p,q} \Rightarrow H_{p+q}$ and $E'^a_{p,q} \Rightarrow H'_{p+q}$ be two spectral sequences. We say that a family of morphisms $h_n : H_n \to H'_n$ are *compatible* with a morphism $f : E \to E'$ of spectral sequences if h_n maps $F_p H_n$ to $F_p H'_n$ such the diagram

$$\begin{array}{ccc} F_p H_n / F_{p-1} H_n & \xrightarrow{\overline{h_n}} & F_p H'_n / F_{p-1} H'_n \\ \downarrow \wr & & \downarrow \wr \\ E^{\infty}_{p,q} & \xrightarrow{f^{\infty}_{p,q}} & E'^{\infty}_{p,q} \end{array}$$

is commutative.

The spectral sequences will allow us to approximate the homology of a chain complex $C_\bullet$ by filtrations.

Definition 2.6.7. A filtration F of a chain complex $C_\bullet$ is an ordered family of subcomplexes of $C_\bullet$ as follows:

$$\cdots \subseteq F_{p-1} C_\bullet \subseteq F_p C_\bullet \subseteq F_{p+1} C_\bullet \subseteq \cdots$$

A filtration F of $C_\bullet$ is called *bounded* if for each n there are integers $s < t$ such that $F_s C_n = 0$ and $F_t C_n = C_n$.

Theorem 2.6.8 (Classical Convergence Theorem)**.** *Let $C_\bullet$ be a chain complex and F a filtration of $C_\bullet$. If F is bounded, then we have the following spectral sequence*

$$E^1_{p,q} = H_{p+q}\left(\frac{F_p C_\bullet}{F_{p-1} C_\bullet}\right) \Rightarrow H_{p+q}(C_\bullet)$$

Moreover, if $f_\bullet : C_\bullet \to C'_\bullet$ is a map of filtered complexes, then the map $f_* : H_*(C_\bullet) \to H_*(C'_\bullet)$ is compatible with the corresponding morphism of spectral sequences induced by $f_\bullet$.

Proof. See (WEIBEL, 1994, Theorem 5.5.1). □

The spectral sequences that we will study in this thesis, mostly arises from double complexes, we will apply the theory above to construct these type of spectral sequences.

Definition 2.6.9. A double complex in $\mathscr{A}$ is a family $C_{\bullet,\bullet} = \{C_{p,q}\}$ of objects in $\mathscr{A}$, together with maps $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v : C_{p,q} \to C_{p,q-1}$ such that

$$d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0.$$

The *Total complex* of $C_{\bullet,\bullet}$ is the complex $\mathrm{Tot}(C)_\bullet$ defined by

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q},$$

with the differential maps $d_n : \mathrm{Tot}(C)_n \to \mathrm{Tot}_{n-1}(C)$ that makes $(c_{p,q})_{p+q=n} \mapsto (c'_{p',q'})_{p'+q'=n-1}$ where:

$$c'_{p',q'} = d^h(c_{p'+1,q'}) + d^v(c_{p',q'+1})$$

We can represent a double complex $C_{\bullet,\bullet}$ as an anti-commutative diagram, i.e. we have a lattice

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q+1} & \xleftarrow{d^h} & C_{p,q+1} & \xleftarrow{d^h} & C_{p+1,q+1} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \longleftarrow & C_{p-1,q} & \xleftarrow{d^h} & C_{p,q} & \xleftarrow{d^h} & C_{p+1,q} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \longleftarrow & C_{p-1,q-1} & \xleftarrow{d^h} & C_{p,q-1} & \xleftarrow{d^h} & C_{p+1,q-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

where $d^v \circ d^h + d^h \circ d^v = 0$.

We have two filtrations of $\mathrm{Tot}(C)_\bullet$

$${}^{I}F_p\mathrm{Tot}(C)_n = \bigoplus_{\substack{i+j=n \\ i\leq p}} C_{i,j}$$

$${}^{II}F_p\mathrm{Tot}(C)_n = \bigoplus_{\substack{i+j=n \\ j\leq p}} C_{i,j}.$$

Let $C_{p,q} = 0$ for $p < 0$ and $q < 0$. In this case we say that $C_{\bullet,\bullet}$ is a *first quadrant* double complex, and the filtrations above are bounded. By the theorem 2.6.8, we have two first quadrant spectral sequences

$${}^{I}E^1_{p,q} = H_{p+q}\left(\frac{{}^{I}F_p\mathrm{Tot}(C)_\bullet}{{}^{I}F_{p-1}\mathrm{Tot}(C)_\bullet}\right) \Rightarrow H_{p+q}(\mathrm{Tot}(C)_\bullet)$$

and

$$^{II}E^1_{p,q} = H_{p+q}\left(\frac{^{II}F_p\mathrm{Tot}(C)_\bullet}{^{II}F_{p-1}\mathrm{Tot}(C)_\bullet}\right) \Rightarrow H_{p+q}(\mathrm{Tot}(C)_\bullet).$$

Observe that in the first spectral sequence we have

$$\frac{^{I}F_p\mathrm{Tot}(C)_\bullet}{^{I}F_{p-1}\mathrm{Tot}(C)_\bullet} = C_{p,\bullet}$$

with differential d^v. Then the $(p+q)$-homology of this complex is in fact the homology group $H_q(C_{p,\bullet})$. Thus

$$^{I}E^1_{p,q} = H_q(C_{p,\bullet}) \Rightarrow H_{p+q}(\mathrm{Tot}(C)_\bullet) \tag{2.6.1}$$

with the differential $d^1_{p,q} = d^h_* : H_q(C_{p,\bullet}) \to H_q(C_{p-1,\bullet})$. Similarly we have

$$\frac{^{II}F_p\mathrm{Tot}(C)_\bullet}{^{II}F_{p-1}\mathrm{Tot}(C)_\bullet} = C_{\bullet,p}$$

with differential d^h. Hence

$$^{II}E^1_{p,q} = H_q(C_{\bullet,p}) \Rightarrow H_{p+q}(\mathrm{Tot}(C)_\bullet) \tag{2.6.2}$$

with the differential $d^1_{p,q} = d^v_* : H_q(C_{\bullet,p}) \to H_q(C_{\bullet,p-1})$.

Remark 2.6.10. The differentials $d^r_{p,q}$ usually are very difficult to calculate. Fortunately there is an algorithm that will be very helpful in this case. For example, suppose that we need to calculate the differential $d^3_{p,q}(u)$ with $u \in E^3_{p,q} \simeq \ker(d^2_{p,q})/\mathrm{im}(d^2_{p+2,q-1})$. Let x represents u: $u = \bar{x}$. Consider the diagram

$$\begin{array}{ccccccc}
C_{p-3,q+2} & \xleftarrow{d^h_{p-2,q+2}} & C_{p-2,q+2} & & & & \\
 & & \downarrow{\scriptstyle d^v_{p-2,q+2}} & & & & \\
 & & C_{p-2,q+1} & \xleftarrow{d^h_{p-1,q+1}} & C_{p-1,q+1} & & \\
 & & & & \downarrow{\scriptstyle d^v_{p-1,q+1}} & & \\
 & & & & C_{p+1,q} & \xleftarrow{d^h_{p,q}} & C_{p,q}.
\end{array}$$

In here, we can take $x \in C_{p,q}$ (because in general $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for every r). First we apply d^h. Thus $d^h_{p,q}(x) = d^v_{p-1,q+1}(y)$ for some $y \in C_{p-1,q+1}$. Now we apply d^h and assume that $d^h_{p-1,q+1}(y) = d^v_{p-2,q+2}(z)$ for some $z \in C_{p-2,q+2}$ then $d^h_{p-2,q+2}(z) \in C_{p-3,q+2}$ represents the element $d^3_{p,q}(\bar{x})$.

Note that the images of the maps d^h of x and y are images of the maps d^v, because $u \in E^3_{p,q}$. For a justification of this algorithm see (MAC LANE, 1994, Theorem 6.1).

2.7 The spectral sequences in group homology

In this section we study the spectral sequence that are useful for the study of homology of groups. Let G be a group and $C_\bullet$ a complex of G-modules. Take a projective resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ over G and consider the double complex $D_{p,q} = P_p \otimes_G C_q$. For any $n \geq 0$ we define the n-th homology of G with coefficients in $C_\bullet$ as follows:

$$H_n(G, C_\bullet) := H_n(\mathrm{Tot}(D)_\bullet).$$

For example, if M is a G-module and $M_\bullet$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

with $M_0 = M$, then $H_n(G, M_\bullet) = H_n(G, M)$.

Theorem 2.7.1. *Let G be a group and $C_\bullet$ a complex of G-modules such that $C_n = 0$ for $n < 0$. Then there is a first quadrant spectral sequence as follows*

$$E^1_{p,q} = H_q(G, C_p) \Rightarrow H_{p+q}(G, C_\bullet).$$

If $C_\bullet$ is exact in dimension $i \geq 1$, then $H_n(G, C_\bullet) \simeq H_n(G, H_0(C_\bullet))$ for any $n \geq 0$.

Proof. Take a projective resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ over G and, consider the double complex $D_{\bullet,\bullet}$ with $D_{p,q} = P_p \otimes_G C_q$. By 2.6.1 and 2.6.2 we have two first quadrant spectral sequences

$${}^{I}E^1_{p,q} = H_q(D_{p,\bullet}) \Rightarrow H_{p+q}(\mathrm{Tot}(D)_\bullet),$$

$${}^{II}E^1_{p,q} = H_q(D_{\bullet,p}) \Rightarrow H_{p+q}(\mathrm{Tot}(D)_\bullet).$$

Since ${}^{II}E^1_{p,q} = H_q(D_{\bullet,p}) = H_q(P_\bullet \otimes_G C_p) = H_q(G, C_p)$, the second spectral sequence find the following form:

$${}^{II}E^1_{p,q} = H_q(G, C_p) \Rightarrow H_{p+q}(G, C_\bullet).$$

Now let $M = H_0(C_\bullet)$ and let $C_\bullet \to M$ be exact. Then for any p the complex $P_p \otimes_G C_\bullet \to P_p \otimes_G M$ is exact (P_p is projective), and thus

$${}^{I}E^1_{p,q} = \begin{cases} P_p \otimes_G M, & q = 0 \\ 0, & q > 0. \end{cases}$$

Passing to the E^2-page we have

$${}^{I}E^2_{p,q} = \begin{cases} H_p(P_\bullet \otimes_G M) = H_p(G, M), & q = 0 \\ 0, & q > 0 \end{cases}$$

and by the convergence ${}^{I}E^1_{p,q} \Rightarrow H_{p+q}(G, C_\bullet)$, we have that $H_n(G, C_\bullet) \simeq H_n(G, M)$ for any n. $\square$

Another useful spectral sequence for groups is the Lyndon/Hochschild-Serre spectral sequence, which relates the homology of a group and a normal subgroup.

Theorem 2.7.2 (Lyndon/Hochschild-Serre Spectral Sequence)**.** *Let G be a group and H a normal subgroup of G. If M is a G-module, then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(G/H, H_q(H,M)) \Rightarrow H_{p+q}(G,M).$$

Moreover, this convergence is natural.

Proof. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over G. Let $C_\bullet := P_\bullet \otimes_H M$. Note that $C_\bullet$ is a complex of G/H-modules and $C_p = P_p \otimes_H M = (P_p \otimes_{\mathbb{Z}} M)_H$. Let $P'_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over G/H. As the proof of Theorem 2.7.1 the double complex $D_{\bullet,\bullet}$ with $D_{p,q} = P'_p \otimes_{G/H} C_q$ gives us the spectral sequences

$${}^{I}E^1_{p,q} = H_q(P'_p \otimes_{G/H} C_\bullet) \Rightarrow H_{p+q}(G/H, C_\bullet)$$

and

$${}^{II}E^1_{p,q} = H_q(P'_\bullet \otimes_{G/H} C_p) \Rightarrow H_{p+q}(G/H, C_\bullet)$$

Since P'_p is projective, the functor $P'_p \otimes_{G/H} -$ is exact. Thus

$${}^{I}E^1_{p,q} = H_q(P'_p \otimes_{G/H} C_\bullet) \simeq P'_p \otimes_{G/H} H_q(C_\bullet) = P'_p \otimes_{G/H} H_q(P_\bullet \otimes_H M) = P'_p \otimes_{G/H} H_q(H,M).$$

Passing to the E^2-page we have

$${}^{I}E^2_{p,q} = H_p(G/H, H_q(H,M)) \Rightarrow H_{p+q}(G/H, C_\bullet)$$

where the action of G/H over $H_q(H,M)$ is defined by conjugation as the example 2.4.4. Note that the differential is horizontal and takes homology over p.

On the other hand, ${}^{II}E^1_{p,q} = H_q(G/H, C_p) = H_q(G/H, P_p \otimes_H M) = 0$ when $q > 0$. In fact, since P_p is projective, it is direct summand of a free module $F \simeq \bigoplus \mathbb{Z}G$. Thus $H_q(G/H, \mathbb{Z}G \otimes_H M) \simeq H_q(G/H, \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} M) \simeq H_q(\{1\}, M) = 0$ for $q > 0$. This last isomorphism is given by Shapiro's lemma 2.4.2). Hence we have

$${}^{II}E^1_{p,q} = \begin{cases} (C_p)_{G/H}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

But $(C_p)_{G/H} = ((P_p \otimes M)_H)_{G/H} \simeq (P_p \otimes M)_G \simeq P_p \otimes_G M$. Now passing to the E^2-page we have

$${}^{II}E^2_{p,q} = \begin{cases} H_p(P_\bullet \otimes_G M) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} = \begin{cases} H_p(G,M) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

Therefore analysing the convergence of the spectral sequence ${}^{II}E^2_{p,q} \Rightarrow H_{p+q}(G/H, C_\bullet)$ (taking the filtrations of $H_n(G/H, C_\bullet)$) we have that $H_n(G/H, C_\bullet) \simeq H_n(G, M)$ for any $n \geq 0$. Finally if we take $E^2_{p,q} := {}^{I}E^2_{p,q}$ then we have the spectral sequence

$$E^2_{p,q} = H_p(G/H, H_q(H, M)) \Rightarrow H_{p+q}(G, M).$$

□

From the previous theorem we can obtain the famous five term exact sequence

$$H_2(G,M) \to H_2(G/H, M_H) \to H_1(H,M)_{G/H} \to H_1(G,M) \to H_1(G/H, M_H) \to 0.$$

But we have this five term exact sequence in more general context.

Theorem 2.7.3 (Five-term exact sequence). *If $E^2_{p,q} \Rightarrow H_{p+q}$ is a first quadrant spectral sequence, then we have the five term exact sequence*

$$H_2 \longrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0.$$

Proof. From the spectral sequence we have a filtration $0 \subseteq F_0H_1 \subseteq F_1H_1 = H_1$, such that $F_0H_1 \simeq E^\infty_{0,1} = E^3_{0,1} = E^2_{0,1}/\mathrm{im}(d^2_{2,0})$ and $E^2_{1,0} = E^\infty_{1,0} \simeq F_1H_1/F_0/H_1$. Thus we have the exact sequences

$$0 \longrightarrow \mathrm{im}(d^2_{2,0}) \longrightarrow E^2_{0,1} \longrightarrow F_0H_1 \longrightarrow 0,$$

$$0 \longrightarrow F_0H_1 \longrightarrow F_1H_1 = H_1 \longrightarrow E^2_{1,0} \longrightarrow 0$$

from these two we obtain the exact sequence

$$E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0$$

Again from the spectral sequence we obtain a filtration $0 \subseteq F_0H_2 \subseteq F_1H_2 \subseteq F_2H_2 = H_2$, such that $F_2H_2/F_1H_2 \simeq E^\infty_{2,0} = \ker(d^2_{2,0})$. This gives the exact sequence

$$0 \longrightarrow F_1H_2 \longrightarrow F_2H_2 = H_2 \longrightarrow \ker(d^2_{2,0}) \longrightarrow 0$$

combining this with the above exact sequence, we obtain the desired exact sequence

$$H_2 \longrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0$$

□

Corollary 2.7.4. *If H is a normal subgroup of a group G, then we have the Five-term exact sequence*

$$H_2(G,M) \to H_2(G/H, M_H) \to H_1(H,M)_{G/H} \to H_1(G,M) \to H_1(G/H, M_H) \to 0.$$

Proof. This is obtained from the Theorems 2.7.2 and 2.7.3. □

The following result will be used in the next chapters.

Proposition 2.7.5. *Let G be a group and H a normal subgroup of G such that the extension*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\ j\ } G/H \longrightarrow 1$$

splits. Then for any $r \geq 2$ and $p \geq 0$, the differentials $d^r_{p,0}$ are trivial.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & G/H & \xrightarrow{\mathrm{id}_{G/H}} & G/H & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \mathrm{id}_{G/H}} & & \\
1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\ j\ } & G/H & \longrightarrow & 1
\end{array}$$

where $\alpha : G/H \to G$ is a split map of $j : G \to G/H$. Then by Theorem 2.7.2 we have a morphism of spectral sequences

$$\begin{array}{ccc}
\hat{E}^2_{p,q} = H_p(G/H, H_q(1,\mathbb{Z})) & \Longrightarrow & H_{p+q}(G/H,\mathbb{Z}) \\
\big\downarrow{\scriptstyle (\mathrm{id},i)_*} & & \big\downarrow{\scriptstyle \alpha_*} \\
E_{p,q} = H_p(G/H, H_q(H,\mathbb{Z})) & \Longrightarrow & H_{p+q}(G,\mathbb{Z}).
\end{array}$$

Since $H_q(1,\mathbb{Z}) = 0$ for $q > 0$ and $H_0(1,\mathbb{Z}) = \mathbb{Z}$, we have $E^2_{p,q} = 0$ for $q > 0$ and $E^2_{p,0} = H_p(G/H)$. Clearly the map $i_* : H_0(1,\mathbb{Z}) \to H_0(H,\mathbb{Z})$ is an isomorphism, and thus for $r \geq 2$, $(i,\mathrm{id})_* : \hat{E}^2_{p,0} \to E^2_{p,0}$ is an isomorphism. Now from the commutative diagram

$$\begin{array}{ccc}
\hat{E}^2_{p,q} = H_p(G/H, H_0(1,\mathbb{Z})) & \xrightarrow{\hat{d}^2_{p,0}} & \hat{E}^2_{p-2,1} = 0 \\
\big\downarrow{\scriptstyle \wr} & & \big\downarrow \\
E^2_{p,q} = H_p(G/H, H_q(H,\mathbb{Z})) & \xrightarrow{d^2_{p,0}} & E^r_{p-2,1}
\end{array}$$

it follows that $d^2_{p,0} = 0$ for any $p \geq 0$.

Since $d^2_{p,0} = 0$ from the above diagram follows that $\hat{E}^3_{p,0} \simeq E^3_{p,0}$. With similar argument as in above we have $d^3_{p,0}$. By continuing this process we have $d^r_{p,0} = 0$ for any $r \geq 2$ and $p \geq 0$. □

CHAPTER

3

SCISSORS CONGRUENCE GROUPS

3.1 The GE_2-rings and the complex of unimodular vectors

Let A be a commutative ring. Let $\mathrm{E}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by the elementary matrices $E_{12}(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $E_{21}(a) := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in A$. The group $\mathrm{E}_2(A)$ is generated by the matrices

$$E(a) := \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \quad a \in A.$$

In fact we have the following formulas

$$E_{12}(a) = E(-a)E(0)^{-1}, \quad E_{21}(a) = E(0)^{-1}E(a), \quad E(0) = E_{12}(1)E_{21}(-1)E_{12}(1).$$

Let $D_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by diagonal matrices. Let $\mathrm{GE}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by $D_2(A)$ and $\mathrm{E}_2(A)$. A ring A is called a *GE_2-ring* if

$$\mathrm{GE}_2(A) = \mathrm{GL}_2(A).$$

Since $\mathrm{E}_2(A) = \mathrm{SL}_2(A) \cap \mathrm{GE}_2(A)$ and $\mathrm{GL}_2(A) = \mathrm{SL}_2(A)D_2(A)$, this condition is equivalent to $\mathrm{E}_2(A) = \mathrm{SL}_2(A)$.

For any $a \in A^\times$, let $D(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Observe that $D(-a) = E(a)E(a^{-1})E(a)$. Thus $D(a) \in \mathrm{E}_2(A)$. For any $x, y \in A$ and $a \in A^\times$, we have the following relations between matrices $E(x)$ and $D(a)$:

(1) $E(x)E(0)E(y) = D(-1)E(x+y)$,

(2) $E(x)D(a) = D(a^{-1})E(a^2x)$,

(3) $D(a)D(b) = D(ab)$.

A ring A is called *universal for* GE_2 if the relations (1), (2) and (3) form a complete set of defining relations for $\mathrm{E}_2(A)$. A GE_2-ring which is universal for GE_2 is called a *universal GE_2-ring*. Thus a universal GE_2-ring is characterized by the property that $\mathrm{SL}_2(A)$ is generated by the matrices $E(x)$ and $D(a)$, with (1)-(3) as a complete set of defining relations.

Any local ring is a universal GE_2-rings (COHN, 1966, Theorem 4.1). Moreover Euclidean domains are GE_2-rings (COHN, 1966, §2). For more example of GE_2-rings and rings universal for GE_2 see (COHN, 1966) and (HUTCHINSON, 2022).

A (column) vector $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in A^2$ is said to be unimodular if there exists a vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that the matrix $(\boldsymbol{u}, \boldsymbol{v}) := \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ is an invertible matrix.

For any non-negative integer n, let $X_n(A^2)$ be the free abelian group generated by the set of all $(n+1)$-tuples $(\langle \boldsymbol{v}_0 \rangle, \ldots, \langle \boldsymbol{v}_n \rangle)$, where every $\boldsymbol{v}_i \in A^2$ is unimodular and for any two distinct vectors $\boldsymbol{v}_i, \boldsymbol{v}_j$, the matrix $\boldsymbol{v}_i, \boldsymbol{v}_j$ is invertible. Observe that $\langle \boldsymbol{v} \rangle \subseteq A^2$ is the line $\{\boldsymbol{v}a : a \in A\}$.

We consider $X_l(A^2)$ as a left $\mathrm{GL}_2(A)$-module (resp. left $\mathrm{SL}_2(A)$-module) in a natural way. If necessary, we convert this action to a right action by the definition $m.g := g^{-1}m$. Let us define the l-th differential operator

$$\partial_l : X_l(A^2) \to X_{l-1}(A^2), \quad l \geq 1,$$

as an alternating sum of face operators which throws away the i-th component of generators. Let $\partial_{-1} = \varepsilon : X_0(A^2) \to \mathbb{Z}$ be defined by $\sum_i n_i(\langle v_{0,i} \rangle) \mapsto \sum_i n_i$. Hence we have the complex

$$X_\bullet(A^2) \to \mathbb{Z} : \ \cdots \longrightarrow X_2(A^2) \xrightarrow{\partial_2} X_1(A^2) \xrightarrow{\partial_1} X_0(A^2) \to \mathbb{Z} \to 0.$$

We say that the above complex is exact in dimension $< k$ if the complex

$$X_k(A^2) \xrightarrow{\partial_k} X_{k-1}(A^2) \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} X_1(A^2) \xrightarrow{\partial_1} X_0(A^2) \to \mathbb{Z} \to 0$$

is exact.

Proposition 3.1.1 (Hutchinson). *Let A be a commutative ring.*

(i) The complex $X_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 1 if and only if A is a GE_2-ring.

(ii) If A is universal for GE_2, then $X_\bullet(A^2)$ is exact in dimension 1, i.e. $H_1(X_\bullet(A^2)) = 0$.

Proof. See (HUTCHINSON, 2022, Theorem 3.3, Theorem 7.2 and Corollary 7.3). □

Remark 3.1.2. In (HUTCHINSON, 2022, Theorem 3.3, Theorem 7.2) Hutchinson calculated H_0 and H_1 of the complex $X_\bullet(A^2)$ for any commutative ring A.

Let the complex $X_\bullet(A^2) \to \mathbb{Z}$ be exact in dimension < 1, (i.e. A is a GE$_2$-ring by Proposition 3.1.1) and let $Z_1(A^2) := \ker(\partial_1)$. From the complex

$$0 \to Z_1(A^2) \overset{\text{inc}}{\to} X_1(A^2) \overset{\partial_1}{\to} X_0(A^2) \to 0, \tag{3.1.1}$$

we obtain the double complex

$$D_{\bullet,\bullet}: 0 \to F_\bullet \otimes_{\mathrm{SL}_2(A)} Z_1(A^2) \xrightarrow{\mathrm{id}_{F_\bullet}\otimes \mathrm{inc}} F_\bullet \otimes_{\mathrm{SL}_2(A)} X_1(A^2) \xrightarrow{\mathrm{id}_{F_\bullet}\otimes \partial_1} F_\bullet \otimes_{\mathrm{SL}_2(A)} X_0(A^2) \to 0,$$

where $F_\bullet \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ over $\mathrm{SL}_2(A)$. This gives us the first quadrant spectral sequence

$$E^1_{p.q} = \begin{cases} H_q(\mathrm{SL}_2(A), X_p(A^2)) & p = 0,1 \\ H_q(\mathrm{SL}_2(A), Z_1(A^2)) & p = 2 \\ 0 & p > 2 \end{cases} \implies H_{p+q}(\mathrm{SL}_2(A), \mathbb{Z}).$$

In our calculations we usually use the bar resolution $B_\bullet(\mathrm{SL}_2(A)) \to \mathbb{Z}$ (BROWN, 2012, Chap.I, §5).

The group $\mathrm{SL}_2(A)$ acts transitively on the sets of generators of $X_i(A^2)$ for $i = 0,1$. Let

$$\boldsymbol{\infty} := \langle \boldsymbol{e}_1 \rangle, \quad \mathbf{0} := \langle \boldsymbol{e}_2 \rangle, \quad \boldsymbol{a} := \langle \boldsymbol{e}_1 + a\boldsymbol{e}_2 \rangle, \quad a \in A^\times,$$

where $\boldsymbol{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We choose $(\boldsymbol{\infty})$ and $(\boldsymbol{\infty}, \mathbf{0})$ as representatives of the orbit of the generators of $X_0(A^2)$ and $X_1(A^2)$, respectively. Therefore

$$X_0(A^2) \simeq \mathrm{Ind}_{B(A)}^{\mathrm{SL}_2(A)} \mathbb{Z}, \qquad X_1(A^2) \simeq \mathrm{Ind}_{T(A)}^{\mathrm{SL}_2(A)} \mathbb{Z},$$

where

$$B(A) := \mathrm{Stab}_{\mathrm{SL}_2(A)}(\infty) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in A^\times, b \in A \right\},$$

$$T(A) := \mathrm{Stab}_{\mathrm{SL}_2(A)}(\boldsymbol{\infty}, \mathbf{0}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in A^\times \right\}.$$

Note that $T(A) \simeq A^\times$. In our calculations usually we identify $T(A)$ with $A^\times$. Thus by Shapiro's lemma we have

$$E^1_{0,q} \simeq H_q(B(A), \mathbb{Z}), \qquad E^1_{1,q} \simeq H_q(T(A), \mathbb{Z}).$$

In particular, $E^1_{0,0} \simeq \mathbb{Z} \simeq E^1_{1,0}$. Moreover $d^1_{1,q} = H_q(\sigma) - H_q(\mathrm{inc})$, where $\sigma: T(A) \to B(A)$ is given by $\sigma(X) = wXw^{-1} = X^{-1}$ for $w := E(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This easily implies that $d^1_{1,0}$ is trivial, $d^1_{1,1}$ is induced by the map $T(A) \to B(A)$, $X \mapsto X^{-2}$, and $d^1_{1,2}$ is trivial. Thus $\ker(d^1_{1,1}) = \mu_2(A) = \{a \in A^\times : a^2 = 1\}$. It is straightforward to check that $d^1_{2,0}: H_0(\mathrm{SL}_2(A), Z_1(A^2)) \to \mathbb{Z}$ is surjective and for any $b \in \mu_2(A)$, $d^1_{2,1}([b] \otimes \partial_2(\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a})) = b$. Hence $E^2_{1,0} = 0$ and $E^2_{1,1} = 0$.

3.2 The refined scissors congruence group

For a ring A, let $\mathcal{W}_A$ be the set of $a \in A^\times$ such that $1-a \in A^\times$. Thus

$$\mathcal{W}_A := \{a \in A : a(1-a) \in A^\times\}.$$

Let $\mathcal{G}_A := A^\times/(A^\times)^2$ and set $\mathcal{R}_A := \mathbb{Z}[\mathcal{G}_A]$. The element of $\mathcal{G}_A$ represented by $a \in A^\times$ is denoted by $\langle a \rangle$. We set $\langle\langle a \rangle\rangle := \langle a \rangle - 1 \in \mathcal{R}_A$.

Let $Z_2(A^2) := \ker(\partial_2)$. Following Coronado and Hutchinson (CORONADO; HUTCHINSON, , § 3) we define

$$\mathcal{RP}(A) := H_0(\mathrm{SL}_2(A), Z_2(A^2)) = Z_2(A^2)_{\mathrm{SL}_2(A)}.$$

Note that $\mathcal{RP}(A)$ is a $\mathcal{G}_A$-module. The inclusion $\mathrm{inc} : Z_2(A^2) \to X_2(A^2)$ induces the map

$$\lambda : \mathcal{RP}(A) = Z_2(A^2)_{\mathrm{SL}_2(A)} \xrightarrow{\overline{\mathrm{inc}}} X_2(A^2)_{\mathrm{SL}_2(A)}.$$

The orbits of the action of $\mathrm{SL}_2(A)$ on $X_2(A)$ is represented by $\langle a \rangle[\ ] := (\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a})$, $\langle a \rangle \in \mathcal{G}_A$. Therefore $X_2(A^2)_{\mathrm{SL}_2(A)} \simeq \mathbb{Z}[\mathcal{G}_A]$. The $\mathcal{G}_A$-module

$$\mathcal{RP}_1(A) := \ker\left(\lambda : \mathcal{RP}(A) \to \mathbb{Z}[\mathcal{G}_A]\right)$$

is called the *refined scissors congruence group* of A. We call

$$\mathrm{GW}(A) := H_0(\mathrm{SL}_2(A), Z_1(A^2))$$

the *Grothendieck-Witt group* of A. Let $\varepsilon := d^1_{2,0} : \mathrm{GW}(A) \to \mathbb{Z}$. The kernel of ε is called the *fundamental ideal* of A and is denoted by $I(A)$.

Consider the sequence

$$X_4(A^2)_{\mathrm{SL}_2(A)} \xrightarrow{\overline{\partial_4}} X_3(A^2)_{\mathrm{SL}_2(A)} \xrightarrow{\overline{\partial_3}} \mathcal{RP}(A) \to 0$$

of $\mathcal{G}_A$-modules. The orbits of the action of $\mathrm{SL}_2(A)$ on $X_3(A)$ and $X_4(A)$ are represented by

$$\langle a \rangle[x] := (\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}, \boldsymbol{ax}), \ \text{ and } \ \langle a \rangle[x,y] := (\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}, \boldsymbol{ax}, \boldsymbol{ay}), \ \ \langle a \rangle \in \mathcal{G}_A, x, y, x/y \in \mathcal{W}_A,$$

respectively. Thus $X_3(A^2)_{\mathrm{SL}_2(A)}$ is the free $\mathbb{Z}[\mathcal{G}_A]$-module generated by the symbols $[x]$, $x \in \mathcal{W}_A$ and $X_4(A^2)_{\mathrm{SL}_2(A)}$ is the free $\mathbb{Z}[\mathcal{G}_A]$-module generated by the symbols $[x,y]$, $x, y, x/y \in \mathcal{W}_A$. It is straightforward to check that

$$\overline{\partial_4}([x,y]) = [x] - [y] + \langle x \rangle \left[\frac{y}{x}\right] - \langle x^{-1} - 1 \rangle \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \langle 1-x \rangle \left[\frac{1-x}{1-y}\right].$$

Let $\overline{\mathcal{RP}}(A)$ be the quotient of the free $\mathcal{G}_A$-module generated by the symbols $[x]$, $x \in \mathcal{W}_A$ over the subgroup generated by the elements

$$[x] - [y] + \langle x \rangle \left[\frac{y}{x}\right] - \langle x^{-1} - 1 \rangle \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \langle 1-x \rangle \left[\frac{1-x}{1-y}\right], \qquad x, y, x/y \in \mathcal{W}_A.$$

Thus we have the natural map $\overline{\mathscr{RP}}(A) \to \mathscr{RP}(A)$. It is straightforward to check that the composite

$$\overline{\mathscr{RP}}(A) \to \mathscr{RP}(A) \xrightarrow{\lambda} \mathbb{Z}[\mathscr{G}_A]$$

is given by

$$[x] \mapsto -\langle\langle x\rangle\rangle\langle\langle 1-x\rangle\rangle.$$

Let $\overline{\mathscr{RP}}_1(A)$ be the kernel of this composite. Thus we have a natural map

$$\overline{\mathscr{RP}}_1(A) \to \mathscr{RP}_1(A).$$

The sequence

$$X_3(A^2)_{\mathrm{SL}_2(A)} \overset{\overline{\partial_3}}{\to} X_2(A^2)_{\mathrm{SL}_2(A)} \overset{\overline{\partial_2}}{\to} \mathrm{GW}(A) \to 0$$

induces the natural map

$$\overline{\mathrm{GW}}(A) := \mathbb{Z}[\mathscr{G}_A]/\big\langle \langle\langle a\rangle\rangle\langle\langle 1-a\rangle\rangle : a \in \mathscr{W}_A \big\rangle \to \mathrm{GW}(A).$$

Let $\mathscr{I}_A$ be the kernel of the augmentation map $\mathbb{Z}[\mathscr{G}_A] \to \mathbb{Z}$ and set

$$\bar{I}(A) := \mathscr{I}_A/\big\langle \langle\langle a\rangle\rangle\langle\langle 1-a\rangle\rangle : a \in \mathscr{W}_A \big\rangle.$$

Thus we have a natural map $\bar{I}(A) \to I(A)$.

If the complex $X_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 2, then $\bar{I}(A) \to I(A)$ is surjective. If the complex is exact in dimension < 3, then the maps

$$\overline{\mathscr{RP}}(A) \to \mathscr{RP}(A) \text{ and } \overline{\mathscr{RP}}_1(A) \to \mathscr{RP}_1(A)$$

are surjective and $\bar{I}(A) \simeq I(A)$. Moreover, if the complex is exact in dimension < 4, then $\overline{\mathscr{RP}}(A) \simeq \mathscr{RP}(A)$ and $\overline{\mathscr{RP}}_1(A) \simeq \mathscr{RP}_1(A)$.

Remark 3.2.1. Let $X_\bullet(A^2) \to \mathbb{Z}$ be exact in dimension < 2. From the exact sequence

$$0 \to Z_2(A^2) \to X_2(A^2) \to Z_1(A^2) \to 0$$

we obtain the exact sequence $\mathscr{RP}(A) \xrightarrow{\lambda} \mathbb{Z}[\mathscr{G}_A] \to \mathrm{GW}(A) \to 0$. This induces the exact sequence

$$\mathscr{RP}(A) \xrightarrow{\lambda} \mathscr{I}_A \to I(A) \to 0.$$

If we set

$$[a]' = (\boldsymbol{\infty},\mathbf{0},\boldsymbol{a}) + (\mathbf{0},\boldsymbol{\infty},\boldsymbol{a}) - (\boldsymbol{\infty},\mathbf{0},\mathbf{1}) - (\mathbf{0},\boldsymbol{\infty},\mathbf{1}) \in \mathscr{RP}(A),$$

then $\lambda([a]') = p_{-1}^{+}\langle\langle a\rangle\rangle$, where $p_{-1}^{+} := \langle -1\rangle + 1 \in \mathbb{Z}[\mathscr{G}_A]$. This induces a natural surjection

$$\mathscr{I}_A/p_{-1}^{+}\mathscr{I}_A \twoheadrightarrow I(A).$$

3.3 The map $H_n(T(A),\mathbb{Z}) \to H_n(B(A),\mathbb{Z})$

The groups $B(A)$ and T(A) sit in the extension $1 \to N(A) \to B(A) \to T(A) \to 1$, where

$$N(A) := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in A \right\} \simeq A.$$

This extension splits canonically and $T(A)$ acts as follow on N:

$$a.\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix}.$$

So if we assume that $T(A) = A^\times$ and $N(A) = A$, then the action of $A^\times$ on A is given by $a.b := a^2b$. Thus

$$H_n(B(A),\mathbb{Z}) \simeq H_n(T(A),\mathbb{Z}) \oplus H_n(B(A),T(A),\mathbb{Z}).$$

We denote the relative homology group $H_n(B(A),T(A),\mathbb{Z})$ by $\mathscr{S}_n$. (See Section 4.4 for an exact sequence involving this relative homology group).

By studying the Lyndon/Hochschild-Serre spectral sequence of the above extension, it follows that

$$\mathscr{S}_1 \simeq H_0(A^\times,A) = A_{A^\times} = A/\langle a^2-1 | a \in A^\times \rangle$$

and $\mathscr{S}_2$ sits in the exact sequence

$$H_2(A^\times,A) \to H_2(A,\mathbb{Z})_{A^\times} \to \mathscr{S}_2 \to H_1(A^\times,A) \to 0.$$

Lemma 3.3.1. *Let G be an abelian group, A a commutative ring, M an A-module and $\varphi : G \to A^\times$ a homomorphism of groups which turns A and M into G-modules. If $H_0(G,A) = 0$, then for any $n \geq 0$, $H_n(G,M) = 0$.*

Proof. See (SUSLIN; NESTERENKO, 1989, Lemma 1.8). □

Corollary 3.3.2. *Let A be a ring and let $A^\times$ acts on A as $a.x := a^2x$. If $H_0(A^\times,A) = 0$, then $H_n(A^\times,A) = 0$ for any $n \geq 0$.*

Proof. Use the above lemma by considering $\varphi : A^\times \to A^\times$, $a \mapsto a^2$. □

Example 3.3.3. (i) If A is a local ring such that $|A/\mathfrak{m}_A| > 3$, then always we can find $a \in A^\times$ such that $a^2 - 1 \in A^\times$. Thus $H_0(A^\times,A) = 0$.

(ii) Let A be a ring such that $6 \in A^\times$. Then

$$1 = 3(2^2-1) + (-1)(3^2-1) \in \langle a^2-1 : a \in A^\times \rangle.$$

Hence $H_0(A^\times,A) = 0$.

Example 3.3.4. If $H_0(A^\times,A) = 0$, then by the above corollary $H_n(A^\times,A) = 0$ for $n \geq 0$. Thus $\mathscr{S}_1 = 0$ and $\mathscr{S}_2 \simeq H_2(A,\mathbb{Z})_{A^\times}$. Therefore $H_1(T(A),\mathbb{Z}) \simeq H_1(B(A),\mathbb{Z})$ and we have the exact sequence

$$0 \to H_2(A,\mathbb{Z})_{A^\times} \to H_2(B(A),\mathbb{Z}) \to H_2(T(A),\mathbb{Z}) \to 0.$$

Moreover we have the exact sequence

$$H_3(A,\mathbb{Z})_{A^\times} \to \mathscr{S}_3 \to H_1(A^\times, A \wedge A) \to 0.$$

Lemma 3.3.5. *If A is a subring of $\mathbb{Q}$, then for any $n \geq 0$,*

$$H_n(B(A),\mathbb{Z}) \simeq H_n(T(A),\mathbb{Z}) \oplus H_{n-1}(A^\times, A).$$

In particular if $6 \in A^\times$, then $H_n(T(A),\mathbb{Z}) \simeq H_n(B(A),\mathbb{Z})$.

Proof. It is well known that any finitely generated subgroup of $\mathbb{Q}$ is cyclic. Thus A is a direct limit of infinite cyclic groups. Since $H_n(\mathbb{Z},\mathbb{Z}) = 0$ for any $n \geq 2$ (BROWN, 2012, page 58) and since homology commutes with direct limit (BROWN, 2012, Exer. 6, § 5, Chap. V), we have $H_n(A,\mathbb{Z}) = 0$ for $n \geq 2$. Now the claim follows from an easy analysis of the Lyndon/Hochschild-Serre spectral sequence associated to the split extension $1 \to N(A) \to B(A) \to T(A) \to 1$.

If $6 \in A^\times$, then by Example 3.3.3(ii) we have $H_0(A^\times,A) = 0$. So by Corollary 3.3.2, $H_n(A^\times,A) = 0$ for any n. Therefore the claim follows from the first part of the lemma. □

Example 3.3.6. (i) Let $A = \mathbb{Z}$. Since $\mathbb{Z}^\times = \{\pm 1\}$, the action of $\mathbb{Z}^\times$ on $A = \mathbb{Z}$ is trivial. Thus $H_n(\mathbb{Z}^\times,\mathbb{Z})$ is $\mathbb{Z}$ if $n = 0$, is trivial if n is even and is $\mathbb{Z}/2$ if n is odd. Now by the previous lemma we have

$$H_1(B(\mathbb{Z}),\mathbb{Z}) \simeq H_1(T(\mathbb{Z}),\mathbb{Z}) \oplus \mathbb{Z},$$

and for any positive integer m,

$$H_{2m}(B(\mathbb{Z}),\mathbb{Z}) \simeq H_{2m}(T(\mathbb{Z}),\mathbb{Z}) \oplus \mathbb{Z}/2 \simeq \mathbb{Z}/2,$$

$$H_{2m+1}(B(\mathbb{Z}),\mathbb{Z}) \simeq H_{2m+1}(T(\mathbb{Z}),\mathbb{Z}) \simeq \mathbb{Z}/2.$$

(ii) Let p be a prime and let $A := \mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} | a,b \in \mathbb{Z}, p \nmid b\}$. Then $\mathbb{Z}_{(p)}$ is local and its residue field is isomorphic to $\mathbb{F}_p$. If $p \neq 2,3$, then the residue field of A has more than 3 elements. Thus

$$H_n(T(\mathbb{Z}_{(p)}),\mathbb{Z}) \simeq H_n(B(\mathbb{Z}_{(p)}),\mathbb{Z})$$

for any $n \geq 0$ (Example 3.3.3).

Let $B = \mathbb{Z}_{(2)}$. Consider the action of $B^\times$ on $\mathbb{Q}$ as usual: $b.x := b^2x$. It is straightforward to check that $H_0(B^\times,\mathbb{Q}) = 0$. Thus by Lemma 3.3.1, $H_n(B^\times,\mathbb{Q}) = 0$ for any $n \geq 0$. Consider the exact sequence $0 \to B \to \mathbb{Q} \to \mathbb{Q}/B \to 0$. Note that $\mathbb{Q}/B \simeq \mathbb{Z}_{2^\infty} := \mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$. From the long exact sequence associated to this short exact sequence, we obtain

$$H_{n-1}(B^\times,B) \simeq H_n(B^\times,\mathbb{Z}_{2^\infty}), \qquad n \geq 1.$$

We have a similar result for $B = \mathbb{Z}_{(3)}$. Therefore for $p = 2,3$, we have

$$H_n(B(\mathbb{Z}_{(p)}),\mathbb{Z}) \simeq H_n(T(\mathbb{Z}_{(p)}),\mathbb{Z}) \oplus H_n(\mathbb{Z}_{(p)}^{\times},\mathbb{Z}_{p^\infty}).$$

Note that $H_n(\mathbb{Z}_{(2)}^{\times},\mathbb{Z}_{2^\infty})$ and $H_n(\mathbb{Z}_{(3)}^{\times},\mathbb{Z}_{3^\infty})$ are 2-power and 3-power torsion groups, respectively. One easily can show that $H_0(\mathbb{Z}_{(2)}^{\times},\mathbb{Z}_{(2)}) \simeq \mathbb{Z}/8$ and $H_0(\mathbb{Z}_{(3)}^{\times},\mathbb{Z}_{(3)}) \simeq \mathbb{Z}/3$.

Lemma 3.3.7. *Let p be a prime number and let $A_p = \mathbb{Z}[\frac{1}{p}]$. Then*

(i) $H_1(B(A_p),\mathbb{Z}) \simeq H_1(T(A_p),\mathbb{Z}) \oplus \mathbb{Z}/(p^2-1)$,

(ii) for any $n \geq 2$, $H_n(T(A_2),\mathbb{Z}) \simeq H_n(B(A_2),\mathbb{Z})$,

(iii) for $p \neq 2$ and $n \geq 2$, we have $H_n(B(A_p),\mathbb{Z}) \simeq H_n(T(A_p),\mathbb{Z}) \oplus \mathbb{Z}/2$.

Proof. We need to calculate $H_n(A_p^{\times},A_p)$. The rest follows from Lemma 3.3.5. In the following we will use the calculation of the homology groups of cyclic groups (BROWN, 2012, page 58).

From the extension $1 \to \mu_2(A_p) \to A_p^{\times} \to \langle p\rangle \to 1$ we obtain the Lyndon/Hochschild-Serre spectral sequence

$$E'^2_{r,s} = H_r(\langle p\rangle, H_s(\mu_2(A_p),A_p)) \Rightarrow H_{r+s}(A_p^{\times},A_p).$$

Since $\langle p\rangle$ is an infinite cyclic group, we have $E'^2_{r,s} = 0$ for $r \geq 2$. Moreover

$$H_s(\mu_2(A_p),A_p) \simeq \begin{cases} A_p & \text{if } s = 0 \\ A_p/2 & \text{if } s \text{ is odd} \\ 0 & \text{if } s \text{ is even.} \end{cases}$$

(i) This follows from the isomorphism $H_0(A_p^{\times},A_p) = A_p/\langle p^2-1\rangle \simeq \mathbb{Z}/(p^2-1)$.

(ii) Since $2 \in A_2^{\times}$, $A_2/2 = 0$. This implies that $E'^2_{r,s} = 0$ for any $s \geq 1$. Now from the above spectral sequence we obtain $H_n(A_2^{\times},A_2) = 0$ for any $n \geq 1$.

(iii) We need to calculate $E'^2_{0,s}$ and $E'^2_{1,s}$ for any $s \geq 1$. Note that $A_p/2 \simeq \mathbb{Z}/2$. Now it is easy to see that $H_0(\langle p\rangle,A_p/2) \simeq \mathbb{Z}/2$ and $H_1(\langle p\rangle,A_p/2) \simeq \mathbb{Z}/2$. Thus for any $s \geq 1$,

$$E'^2_{0,s} \simeq E'^2_{1,s} \simeq \begin{cases} 0 & \text{if } s \text{ is even} \\ \mathbb{Z}/2 & \text{if } s \text{ is odd.} \end{cases}$$

Now from the above spectral sequence it follows that $H_n(A_p^{\times},A_p) \simeq \mathbb{Z}/2$ for any $n \geq 1$. □

Proposition 3.3.8. *(i) Let A be a local domain such that either $A/\mathfrak{m}_A$ is infinite or if $|A/\mathfrak{m}_A| = p^d$, we have $(p-1)d > 2n$. Then $H_n(T(A),\mathbb{Z}) \simeq H_n(B(A),\mathbb{Z})$.*

(ii) Let A be a local ring such that either $A/\mathfrak{m}_A$ is infinite or if $|A/\mathfrak{m}_A| = p^d$, we have $(p-1)d > 2(n+1)$. Then $H_n(T(A),\mathbb{Z}) \simeq H_n(B(A),\mathbb{Z})$.

Proof. (i) For this see (HUTCHINSON, 2017a, Proposition 3.19).

(ii) Similar to the proof of part (i) presented in (HUTCHINSON, 2017a, Proposition 3.19), we can show that $H_n(T(A),k) \simeq H_n(B(A),k)$, where k is a prime field and $(p-1)d > 2n$. Now the claim follows from (MIRZAII, 2017, Lemma 2.3). $\square$

3.4 The refined Bloch group

Let the complex $X_\bullet(A^2) \to \mathbb{Z}$ be exact in dimension < 2. Then from the exact sequence

$$0 \to Z_2(A^2) \to X_2(A^2) \to Z_1(A^2) \to 0$$

we obtain the long exact sequence

$$H_1(\mathrm{SL}_2(A), Z_2(A^2)) \to H_1(\mathrm{SL}_2(A), X_2(A^2)) \to H_1(\mathrm{SL}_2(A), Z_1(A^2)) \xrightarrow{\delta} H_0(\mathrm{SL}_2(A), Z_2(A^2))$$

$$\to H_0(\mathrm{SL}_2(A), X_2(A^2)) \to H_0(\mathrm{SL}_2(A), Z_1(A^2)) \to 0.$$

Choose $(\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a})$, $\langle a \rangle \in \mathscr{G}_A$, as representatives of the orbits of the generators of $X_2(A^2)$. Then

$$X_2 \simeq \bigoplus_{\langle a \rangle \in \mathscr{G}_A} \mathrm{Ind}_{\mu_2(A)}^{\mathrm{SL}_2(A)} \mathbb{Z}\langle a \rangle,$$

where $\mu_2(A) \simeq \mathrm{Stab}_{\mathrm{SL}_2(A)}(\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a})$. Thus

$$H_1(\mathrm{SL}_2(A), X_2(A^2)) \simeq \bigoplus_{\langle a \rangle \in \mathscr{G}_A} H_1(\mu_2(A), \mathbb{Z}) \simeq \mathbb{Z}[\mathscr{G}_A] \otimes \mu_2(A).$$

From the above exact sequence we obtain the exact sequence

$$H_1(\mathrm{SL}_2(A), Z_2(A^2)) \to \mathbb{Z}[\mathscr{G}_A] \otimes \mu_2(A) \to H_1(\mathrm{SL}_2(A), Z_1(A^2)) \to \mathscr{RP}_1(A) \to 0.$$

The exact sequence $0 \to Z_1(A^2) \xrightarrow{\mathrm{inc}} X_1(A^2) \xrightarrow{\partial_1} X_0(A^2)$ induces the commutative diagram

$$\begin{array}{ccccccccc}
H_1(\mathrm{SL}_2(A), Z_2(A^2)) & \longrightarrow & \mathbb{Z}[\mathscr{G}_A] \otimes \mu_2(A) & \xrightarrow{\gamma} & H_1(\mathrm{SL}_2(A), Z_1(A^2)) & \xrightarrow{\delta} & \mathscr{RP}_1(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}_{\mu_2(A)}} & & \downarrow{\scriptstyle d^1_{2,1}} & & \downarrow & & \\
0 & \longrightarrow & \mu_2(A) & = & \mu_2(A) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

By the Snake lemma we have the exact sequence

$$H_1(\mathrm{SL}_2(A), Z_2(A^2)) \to \mathscr{I}_A \otimes \mu_2(A) \xrightarrow{\gamma} E^2_{2,1} \to \mathscr{RP}_1(A) \to 0.$$

Let G be a group and let g, g' be two commuting elements of G. Set

$$\boldsymbol{c}(g, g') := ([g|g'] - [g'|g]) \otimes 1 \in H_2(G, \mathbb{Z}) = H_2(B_\bullet(G) \otimes_G \mathbb{Z}).$$

Lemma 3.4.1. *The composite*

$$\mathscr{I}_A \otimes \mu_2(A) \xrightarrow{\gamma} E^2_{2,1} \xrightarrow{d^2_{2,1}} H_2(B(A),\mathbb{Z}) \simeq (A^\times \wedge A^\times) \oplus \mathscr{S}_2$$

sends $\langle\langle a \rangle\rangle \otimes b$ *to* $\big(a \wedge b, \boldsymbol{c}(\begin{pmatrix} 1 & a+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix})\big)$.

Proof. The element $\langle\langle a \rangle\rangle \otimes b \in \mathscr{I}_A \otimes \mu_2(A)$ is represented by $[b] \otimes ((\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{a}) - (\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{1}))$. Now we want to apply γ (that is induced by ∂_2). We see that $\gamma(\langle\langle a \rangle\rangle \otimes (b))$ is represented by $[b] \otimes \partial_2((\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{a}) - (\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{1})) \in B_1(\mathrm{SL}_2(A)) \otimes Z_1(A^2)$. Consider the diagram

$$\begin{array}{ccc} B_2(\mathrm{SL}_2(A)) \otimes X_0(A^2) & \xleftarrow{\mathrm{id}_{B_2} \otimes \partial_1} & B_2(\mathrm{SL}_2(A)) \otimes X_1(A^2) & & \\ & & \big\downarrow {\scriptstyle d_2 \otimes \mathrm{id}_{X_1}} & & \\ & & B_1(\mathrm{SL}_2(A)) \otimes X_1(A^2) & \xleftarrow{\mathrm{id}_{B_1} \otimes \mathrm{inc}} & B_1(\mathrm{SL}_2(A)) \otimes Z_1(A^2). \end{array}$$

If $X_{a,b} := [b] \otimes \partial_2((\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{a}) - (\boldsymbol{\infty},\boldsymbol{0},\boldsymbol{1}))$, then

$$\begin{aligned} (\mathrm{id}_{B_1} \otimes \mathrm{inc})(X_{a,b}) =& [b] \otimes ((\boldsymbol{0},\boldsymbol{a}) - (\boldsymbol{\infty},\boldsymbol{a}) - (\boldsymbol{0},\boldsymbol{1}) + (\boldsymbol{\infty},\boldsymbol{1})) \\ =& (g_a^{-1} - h_a^{-1} - g_1^{-1} + h_1^{-1})[b] \otimes (\boldsymbol{\infty},\boldsymbol{0}) \\ =& (d_2 \otimes \mathrm{id}_{X_1})(Z_{a,b} \otimes (\boldsymbol{\infty},\boldsymbol{0})) \end{aligned}$$

where

$$Z_{a,b} := [g_a^{-1}|b] - [b|g_a^{-1}] - [g_1^{-1}|b] + [b|g_1^{-1}] - [h_a^{-1}|b] + [b|h_a^{-1}] + [h_1^{-1}|b] - [b|h_1^{-1}],$$

with $g_z = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$ and $h_z = \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}$ for $z \in A^\times$. Applying $\mathrm{id}_{B_2} \otimes \partial_1$ we have

$$(\mathrm{id}_{B_2} \otimes \partial_1)(Z_{a,b} \otimes (\boldsymbol{\infty},\boldsymbol{0})) = (wZ_{a,b} - Z_{a,b}) \otimes (\boldsymbol{\infty}).$$

Now $(wZ_{a,b} - Z_{a,b}) \otimes 1$ is a representative of $(d^2_{2,1} \circ \gamma)(\langle\langle a \rangle\rangle \otimes b)$. We have the following facts:

1. For any $g \in \mathrm{SL}_2(A)$, $h \in B(A)$ and $b, b' \in \mu_2(A)$,

$$\boldsymbol{c}(hg,b) = \boldsymbol{c}(h,b) + \boldsymbol{c}(g,b), \quad \boldsymbol{c}(h,bb') = \boldsymbol{c}(h,b) + \boldsymbol{c}(h,b').$$

2. For any $g \in \mathrm{SL}_2(A)$, $w([g|b] - [b|g]) \otimes 1$ is a representative of $\boldsymbol{c}(wg,b) - \boldsymbol{c}(w,b)$, i.e.

$$\boldsymbol{c}(wg,b) - \boldsymbol{c}(w,b) = \overline{w([g|b] - [b|g]) \otimes 1}.$$

3. For any $h \in B(A)$ and $b \in \mu_2(A)$, we have

$$\boldsymbol{c}(h^{-1},b) = -\boldsymbol{c}(h,b) = \boldsymbol{c}(h,b^{-1}) = \boldsymbol{c}(h,b).$$

Now, for any $z \in A^\times$, from the identity $g_z^{-1} = -h_{z^{-1}}w$, we obtain

$$\boldsymbol{c}(g_z^{-1},b) = \boldsymbol{c}(h_{z^{-1}},b) + \boldsymbol{c}(w,b) + \boldsymbol{c}(-1,b)$$

(by just adding the null element $(d_3 \otimes \mathrm{id})([-h_{a^{-1}}|w|b] + [b|-h_{a^{-1}}|w] - [-h_{a^{-1}}|b|w])$ and using the first fact above). On the other hand, the second fact above gives, for any $z \in A^\times$, the equality

$$\overline{w([g_z^{-1}|b] - [b|g_z^{-1}]) \otimes 1} = \boldsymbol{c}(wg_z^{-1},b) - \boldsymbol{c}(w,b).$$

Moreover the formula $wg_z^{-1} = z^{-1}h_{z^{-1}}^{-1}wh_z^{-1}$ and (1) in above gives the equality

$$\overline{w([g_z^{-1}|b] - [b|g_z^{-1}]) \otimes 1} = \boldsymbol{c}(z^{-1},b) + \boldsymbol{c}(h_{z^{-1}}^{-1},b) + \boldsymbol{c}(wh_z^{-1},b) - \boldsymbol{c}(w,b).$$

Also using (2) we have

$$\overline{w([h_z^{-1}|b] - [b|h_z^{-1}]) \otimes 1} = \boldsymbol{c}(wh_z^{-1},b) - \boldsymbol{c}(w,b).$$

Now joining all the formulas above we have:

$$\begin{aligned}\overline{(wZ_{a,b} - Z_{a,b}) \otimes 1} &= \boldsymbol{c}(a^{-1},b) + \boldsymbol{c}(h_{a^{-1}}^{-1},b) - \boldsymbol{c}(h_1^{-1},b) - \boldsymbol{c}(h_{a^{-1}},b) + \boldsymbol{c}(h_a,b) \\ &= \boldsymbol{c}(a,b) + \boldsymbol{c}(h_a h_1,b) = \boldsymbol{c}(a,b) + \boldsymbol{c}\Big(\begin{pmatrix}1 & a^{-1}+1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}\Big)\end{aligned}$$

(in the last equality, we use (1) and (3)). Substituting a with a^{-1} we see that

$$(d_{2,1}^2 \circ \gamma)(\langle\langle a \rangle\rangle \otimes b) = \boldsymbol{c}(a,b) + \boldsymbol{c}\Big(\begin{pmatrix}1 & a+1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}\Big).$$

□

We believe that the element $\boldsymbol{c}\Big(\begin{pmatrix}1 & a+1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}\Big)$, appearing in the previous lemma, is trivial for many interesting rings.

For $a \in A$ and $b \in \mu_2(A)$, let $x_a := \boldsymbol{c}\Big(\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}\Big) \in H_2(B(A),\mathbb{Z})$. This element has order 2 and $x_a = x_{-a}$. Since $\begin{pmatrix}c & 0 \\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix}c & 0 \\ 0 & c^{-1}\end{pmatrix}^{-1} = \begin{pmatrix}1 & ac^2 \\ 0 & 1\end{pmatrix}$, for any $c \in A^\times$ we have $x_a = x_{ac^2}$. (In particular $x_{c^2} = x_1$.) Thus

$$x_{a(c^2-1)} = 0, \quad x_c = x_{c^{-1}}.$$

For example if $a \in \mathscr{W}_A$, then $a+1 := \frac{1}{(a-1)}(a^2-1)$ and hence

$$x_{a+1} = x_{(a-1)^{-1}(a^2-1)} = 0.$$

Example 3.4.2. (i) If $H_0(A^\times, A) = 0$, then $A = \langle c^2 - 1 | c \in A^\times \rangle$. Thus any $a \in A$ is of the form $a = \sum d(c^2 - 1)$. This implies that $x_a = 0$ for any $a \in A$.

(ii) If $2 \in A^\times$, then for any $a \in A^\times$ we have $x_a = x_{2(a/2)} = 2x_{(a/2)} = 0$.

(iii) If $F = A$ is a field, then $x_a = 0$: If $\mathrm{char}(F) = 2$, then $b = 1$ and thus $x_a = 0$. If $\mathrm{char}(F) \neq 2$, then $2 \in F^\times$, and the claim follows from (ii).

(iv) If A is a local ring such that $A/\mathfrak{m}_A$ has at least 3 elements, then $x_a = 0$: If $|A/\mathfrak{m}_A| = 3$, then $2 \in A^\times$, and thus the claim follows from (ii). If $|A/\mathfrak{m}_A| > 3$, then there is $c \in A^\times$, such that $c^2 - 1 \in A^\times$. Thus $H_0(A^\times, A) = 0$ and the claim follows from (i).

(v) Let $A = \mathbb{Z}_{(p)}$, where p is a prime. Then $x_{a+1} = 0$ for any $a \in A^\times$: For $p > 2$ the claim follows from (iv). Let $p = 2$ and let $a = a'/b' \in \mathbb{Z}_{(2)}$. Then a', b' are odd and so $a + 1 = (a' + b')/b' = 2c'$, $c' \in \mathbb{Z}_{(2)}$. Now $x_{a+1} = x_{2c'} = 2x_{c'} = 0$.

(vi) Let $A = \mathbb{Z}\left[\frac{1}{p}\right]$, where p is a prime. Then $x_{a+1} = 0$ for any $a \in A^\times$: If $p = 2$, then by (ii), $x_{a+1} = x_a = 0$. If $p \neq 2$, then $a = \pm p^n$, $n \in \mathbb{Z}$. Now we have $a + 1 = 2c$, where $c \in A$. Thus $x_{a+1} = 0$.

(vii) If $\mu_2(A) = 1$, then $x_a = 0$: Since $\mu_2(A) = 1$, we have $b = 1$ and thus $x_a = 0$.

In the rest of this article we will mostly assume that $x_{a+1} = 0$ for any $a \in A^\times$, i.e.

$$\mathrm{im}(d^2_{2,1} \circ \gamma) = A^\times \wedge \mu_2(A).$$

For example in our important results, for technical reasons, we will assume that

$$H_2(B(A), \mathbb{Z}) \simeq H_2(T(A), \mathbb{Z}),$$

i.e. $\mathscr{S}_2 = 0$. So the above condition will be satisfied.

Now by the above lemma we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathscr{I}_A \otimes \mu_2(A) & \xrightarrow{\gamma} & E^2_{2,1} & \xrightarrow{\delta} & \mathscr{RP}_1(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^\times \wedge \mu_2(A) & \longrightarrow & (A^\times \wedge A^\times) \oplus \mathscr{S}_2 & \longrightarrow & \dfrac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)} \oplus \mathscr{S}_2 & \longrightarrow & 0. \end{array} \tag{3.4.1}$$

Let $\psi_1(a) := [a] + \langle -1 \rangle [a^{-1}] \in \overline{\mathscr{RP}}(A)$. It is easy to check that

$$g(a) := p^+_{-1}[a] + \langle\langle 1 - a \rangle\rangle \psi_1(a) \in \overline{\mathscr{RP}}_1(A),$$

where $p^+_{-1} = \langle -1 \rangle + 1 \in \mathbb{Z}[\mathscr{G}_A]$. We denote the image of this elements in $\mathscr{RP}_1(A)$ by $g(a)$ again.

Proposition 3.4.3. *Then under the composite*

$$\mathscr{RP}_1(A) \to \frac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)} \oplus \mathscr{S}_2 \to \frac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)}$$

we have

$$g(a) \mapsto a \wedge (1-a).$$

Proof. From the complex $0 \to Z_1(A^2) \overset{\text{inc}}{\to} X_1(A^2) \overset{\partial_1}{\to} X_0(A^2) \to 0$ we obtain the first quadrant spectral sequence

$$\mathscr{E}^1_{p.q} = \begin{cases} H_q(\mathrm{GL}_2(A), X_p(A^2)) & p = 0,1 \\ H_q(\mathrm{GL}_2(A), Z_1(A^2)) & p = 2 \\ 0 & p > 2 \end{cases} \Longrightarrow H_{p+q}(\mathrm{GL}_2(A), \mathbb{Z}).$$

This spectral sequence have been studied in (MIRZAII, 2011, §3). Let $\mathscr{P}(A) := H_0(\mathrm{GL}_2(A), Z_2(A^2))$. We have a $\mathscr{R}_A$-map $\mathscr{RP}(A) \to \mathscr{P}(A)$, where $\mathscr{P}(A)$ has the trivial action of $\mathscr{G}_A$. Under this map $g(a) \mapsto 2[a]$. This induces a map $\mathscr{RP}_1(A) \to \mathscr{P}(A)$. One can show that $\mathscr{E}^2_{2,1} \simeq \mathscr{P}(A)$ (see (MIRZAII, 2011, Lemma 3.2)). The map $\mathrm{SL}_2(A) \to \mathrm{GL}_2(A)$ induces the morphism of spectral sequences

$$\begin{array}{ccc} E^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{SL}_2(A), \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathscr{E}^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{GL}_2(A), \mathbb{Z}). \end{array}$$

This induces the commutative diagram

$$\begin{array}{ccccc} E^2_{2,1} & \xrightarrow{d^2_{2,1}} & H_2(B(A), \mathbb{Z}) & \longrightarrow & H_2(T(A), \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{P}(A) & \xrightarrow{d^2_{2,1}} & \dfrac{H_2(B_2(A), \mathbb{Z})}{d^2_{1,2}(H_2(T_2(A), \mathbb{Z}))} & \longrightarrow & \dfrac{H_2(T_2(A), \mathbb{Z})}{d^2_{1,2}(H_2(T_2(A), \mathbb{Z}))}, \end{array}$$

where

$$B_2(A) := \mathrm{Stab}_{\mathrm{GL}_2(A)}(\boldsymbol{\infty}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in A^\times, b \in A \right\},$$

$$T_2(A) := \mathrm{Stab}_{\mathrm{GL}_2(A)}(\boldsymbol{\infty}, \mathbf{0}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in A^\times \right\}.$$

This together with diagram (3.4.1) induce the commutative diagram

$$\begin{array}{ccc} \mathscr{RP}_1(A) & \longrightarrow & \dfrac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)} \\ \downarrow & & \downarrow \\ \mathscr{P}(A) & \longrightarrow & (A^\times \wedge A^\times) \oplus S^2_{\mathbb{Z}}(A), \end{array}$$

where $S^2_{\mathbb{Z}}(A) := (A^\times \otimes A^\times)/\langle a \otimes b + b \otimes a : a, b \in A^\times \rangle$. Moreover the vertical map on the right is given by $a \wedge b \to (2a \wedge b, 2(a \otimes b))$ and the bottom horizontal map is given by $[a] \mapsto (a \wedge (1-a), -a \otimes (1-a))$. Now the claim follows from the fact that the composite

$$\mathscr{RP}_1(A) \to \mathscr{P}(A) \to (A^\times \wedge A^\times) \oplus S^2_{\mathbb{Z}}(A)$$

maps $g(a)$ to $2(a\wedge(1-a), -a\otimes(1-a))$. □

We denote the differential $d^2_{2,1}$ by λ_1:

$$\lambda_1 : \mathscr{RP}_1(A) \to H_2(B(A),\mathbb{Z}) \simeq \frac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)} \oplus \mathscr{S}_2.$$

The kernel of λ_1 is called the *refined Bloch group* of A and is denoted by $\mathscr{RB}(A)$:

$$\mathscr{RB}(A) := \ker(\lambda_1).$$

From the spectral sequence we obtain a natural surjective map

$$H_3(\mathrm{SL}_2(A),\mathbb{Z}) \twoheadrightarrow \mathscr{RB}(A).$$

Let B be an abelian group. Let $\sigma_1 : \mathrm{Tor}_1^{\mathbb{Z}}(B,B) \to \mathrm{Tor}_1^{\mathbb{Z}}(B,B)$ be obtained by interchanging the group B. It is not difficult to show that σ_1 is induced by the involution $B\otimes B \to B\otimes B$, $a\otimes b \mapsto -b\otimes a$.

Let $\Sigma'_2 = \{1,\sigma'\}$ be the symmetric group of order 2. Consider the following action of Σ'_2 on $\mathrm{Tor}_1^{\mathbb{Z}}(B,B)$:

$$(\sigma', x) \mapsto -\sigma_1(x).$$

Proposition 3.4.4. *For any abelian group B we have the exact sequence*

$$0 \to \textstyle\bigwedge^3_{\mathbb{Z}} B \to H_3(B,\mathbb{Z}) \to Tor_1^{\mathbb{Z}}(B,B)^{\Sigma'_2} \to 0,$$

where the right side homomorphism is obtained from the composition

$$H_3(B,\mathbb{Z}) \xrightarrow{\Delta_{B*}} H_3(B\oplus B,\mathbb{Z}) \to Tor_1^{\mathbb{Z}}(B,B),$$

Δ_B being the diagonal map $B \to B\oplus B$, $b \mapsto (b,b)$.

Proof. See (SUSLIN, 1991, Lemma 5.5), (BREEN, 1999, Section 6). □

Theorem 3.4.5 (Refined Bloch-Wigner in char $= 2$)**.** *Let A be a ring such that*

(i) $\mu_2(A) = 1$,

(ii) $X_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 2

(iii) $H_3(T(A),\mathbb{Z}) \simeq H_3(B(A),\mathbb{Z})$.

Then we have the exact sequence

$$Tor_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma'_2} \to H_3(SL_2(A),\mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

If A is a domain then we have the exact sequence

$$0 \to Tor_1^{\mathbb{Z}}(\mu(A),\mu(A)) \to H_3(SL_2(A),\mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

Proof. By definition we have $E^{\infty}_{2,1} \simeq E^3_{2,1} \simeq \mathscr{RB}(A)$. We show that the differential

$$d^1_{2,2} : H_2(\mathrm{SL}_2(A), Z_1(A^2)) \to A^\times \wedge A^\times$$

is surjective. For $a \in A^\times$, denote $(\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}) \in X_2(A^2)$ by X_a. Let $Y = (\boldsymbol{\infty}, \mathbf{0}) + (\mathbf{0}, \boldsymbol{\infty}) \in Z_1(A^2)$. For $a, b \in A^\times$, let

$$\lambda(a,b) \in H_2(\mathrm{SL}_2(A), Z_1(A)) = H_2(B_\bullet(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} Z_1(A))$$

be the element

$$\begin{aligned}\lambda(a,b) :=& ([a|b] + [w|ab] - [w|a] - [w|b]) \otimes Y + [wab|wab] \otimes \partial_2(X_{ab}) \\ &- [wa|wa] \otimes \partial_2(X_a) - [wb|wb] \otimes \partial_2(X_b) + [w|w] \otimes \partial_2(X_1).\end{aligned}$$

Recall that $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$\begin{aligned}d^1_{2,2}(\lambda(a,b)) =& (w+1)([a|b] + [w|ab] - [w|a] - [w|b]) \otimes (\boldsymbol{\infty}, \mathbf{0}) \\ &+ (g_{ab}^{-1} - h_{ab}^{-1} + 1)([wab|wab]) \otimes (\boldsymbol{\infty}, \mathbf{0}) \\ &- (g_a^{-1} - h_a^{-1} + 1)([wa|wa]) \otimes (\boldsymbol{\infty}, \mathbf{0}) \\ &- (g_b^{-1} - h_b^{-1} + 1)([wb|wb]) \otimes (\boldsymbol{\infty}, \mathbf{0}) \\ &+ (g_1^{-1} - h_1^{-1} + 1)([w|w]) \otimes (\boldsymbol{\infty}, \mathbf{0}),\end{aligned}$$

where $g_x := \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ and $h_x := \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$.

This element is in $H_2(\mathrm{SL}_2(A), X_1(A^2)) = H_2(B_\bullet(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_1(A))$. The morphisms

$$B_\bullet(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_1(A) \to B_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z} \to C_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z},$$

$$[g_1|\cdots|g_n] \otimes (\boldsymbol{\infty}, \mathbf{0}) \mapsto [g_1|\cdots|g_n] \otimes 1 \mapsto \otimes(1, g_1, \ldots, g_1 \cdots g_n) \otimes 1,$$

induce the isomorphisms

$$H_2(B_\bullet(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_1(A)) \simeq H_2(B_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z}) \simeq H_2(C_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z}).$$

Following these maps we see that $d^1_{2,2}(\lambda(a,b))$ as an element of $H_2(C_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z})$ find the following form

$$\begin{aligned}d^1_{2,2}(\lambda(a,b)) =& \Big((w, wa, wab) + (w, 1, ab) - (w, 1, a) - (w, 1, b) \\ &+ (1, a, ab) + (1, w, wab) + (1, w, wa) + (1, w, wb)) \otimes 1 \\ &+ ((g_{ab}^{-1}, g_{ab}^{-1} wab, g_{ab}^{-1}) - (h_{ab}^{-1}, h_{ab}^{-1} wab, h_{ab}^{-1}) + (1, wab, 1)) \otimes 1 \\ &- ((g_a^{-1}, g_a^{-1} wa, g_a^{-1}) - (h_a^{-1}, h_a^{-1} wa, h_a^{-1}) + (1, wa, 1)) \otimes 1 \\ &- ((g_b^{-1}, g_b^{-1} wb, g_b^{-1})) - (h_b^{-1}, h_b^{-1} wb, h_b^{-1}) + (1, wb, 1) \otimes 1 \\ &+ ((g_1^{-1}, g_1^{-1} w, g_1^{-1}) - (h_1^{-1}, h_1^{-1} w, h_1^{-1}) + (1, w, 1)\Big) \otimes 1.\end{aligned}$$

Now we want to find a representative of this element in $C_\bullet(T(A)) \otimes_{T(A)} \mathbb{Z}$ by the isomorphism

$$H_2(C_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z})) \simeq H_2(C_\bullet(A^\times)) \otimes_{A^\times} \mathbb{Z}).$$

Let $s : \mathrm{SL}_2(A)\backslash T(A) \to \mathrm{SL}_2(A)$ be any (set-theoretic) section of the canonical projection π : $\mathrm{SL}_2(A) \to \mathrm{SL}_2(A)\backslash T(A)$. For $g \in \mathrm{SL}_2(A)$, set $\overline{g} = (g)(s \circ \pi(g))^{-1}$. Then the morphism

$$C_\bullet(\mathrm{SL}_2(A)) \otimes_{T(A)} \mathbb{Z} \xrightarrow{s_\bullet} C_\bullet(T(A)) \otimes_{T(A)} \mathbb{Z}, \quad [g_1|\dots|g_n] \otimes 1 \mapsto [\overline{g_1}|\dots|\overline{g_n}] \otimes 1$$

induces the desired isomorphism. Choose a section $s : \mathrm{SL}_2(A)\backslash T(A) \to \mathrm{SL}_2(A)$ such that

$$s(T(A)\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 1 & a^{-1}b \\ ac & ad \end{pmatrix}, \qquad \text{if } a \in A^\times,$$

and

$$s(T(A)\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ 1 & bd \end{pmatrix}, \qquad \text{if } a = 0.$$

We see that $d^1_{2,2}(\lambda(a,b))$ in $H_2(C_\bullet(T(A)) \otimes_{T(A)} \mathbb{Z}) = H_2(C_\bullet(A^\times) \otimes_{A^\times} \mathbb{Z})$ is of the following form

$$d^1_{2,2}(\lambda(a,b)) = \Big((1,a^{-1},(ab)^{-1}) + (1,a,ab) + (1,(ab)^{-1},1) - (1,a^{-1},1) - (1,b^{-1},1)\Big) \otimes 1.$$

In $H_2(B_\bullet(A^\times) \otimes_{A^\times} \mathbb{Z})$ this elements corresponds to

$$d^1_{2,2}(\lambda(a,b)) = \Big([a^{-1}|b^{-1}] + [a|b] + [a^{-1}b^{-1}|ab] - [a^{-1}|a] - [b^{-1}|b]\Big) \otimes 1.$$

Now by adding the following null element in $H_2(A^\times,\mathbb{Z}) = H_2(B_\bullet(A^\times) \otimes_{A^\times} \mathbb{Z})$

$$d_3([a^{-1}|b^{-1}|ab] - [b^{-1}|b|a]) \otimes 1,$$

it is easy to see that

$$d^1_{2,2}(\lambda(a,b)) = \big([a|b] - [b|a]\big) \otimes 1 \in H_2(B_\bullet(A^\times) \otimes_{A^\times} \mathbb{Z}).$$

This shows that $d^1_{2,2}$ is surjective. Therefore $E^2_{1,2} = 0$.

Now we need to study $E^\infty_{0,3} = E^3_{0,3}$. To do this, first consider the differential

$$d^1_{1,3} = H_3(\sigma) - H_3(\mathrm{inc}) : H_3(T(A),\mathbb{Z}) \to H_3(B(A),\mathbb{Z}) \simeq H_3(T(A),\mathbb{Z}).$$

By Proposition 3.4.4, we have the exact sequence

$$0 \to \textstyle\bigwedge^3_{\mathbb{Z}} A^\times \to H_3(T(A),\mathbb{Z}) \to \mathrm{Tor}^{\mathbb{Z}}_1(\mu(A),\mu(A))^{\Sigma'_2} \to 0.$$

It is straightforward to check that $d^1_{1,3}\mid_{\bigwedge^3_{\mathbb{Z}} A^\times}$ coincides with multiplication by 2. Thus we have the exact sequence

$$\textstyle\bigwedge^3_{\mathbb{Z}} A^\times/2 \to E^2_{0,3} \to \mathrm{Tor}^{\mathbb{Z}}_1(\mu(A),\mu(A))^{\Sigma'_2}/\mathscr{T} \to 0,$$

for some subgroup $\mathscr{T}$ of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma_2'}$. By an easy analysis of the spectral sequence we have the exact sequence

$$E^3_{0,3} \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

We denote the image of $a\wedge b\wedge c \in \bigwedge^3_{\mathbb{Z}} A^\times/2$ in $E^2_{0,3}$ again by $a\wedge b\wedge c$. Since $d^1_{2,2}(\lambda(ab,c)-\lambda(a,c)-\lambda(b,c))=0$, we have $\lambda(ab,c)-\lambda(a,c)-\lambda(b,c)\in E^2_{2,2}$. We show that

$$d^2_{2,2}(\lambda(ab,c)-\lambda(a,c)-\lambda(b,c)) = -a\wedge b\wedge c \in E^2_{0,3}. \tag{3.4.2}$$

This would imply that there is a surjective map $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma_2'} \twoheadrightarrow E^3_{0,3}$ and therefore we obtain the exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma_2'} \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathscr{RB}(A) \to 0,$$

which proof the first claim of the theorem. Now we prove the equality (3.4.2).

Consider the diagram

$$\begin{array}{ccc} B_3(\mathrm{SL}_2(A))\otimes X_0(A^2) & \xleftarrow{\mathrm{id}_{B_3}\otimes\partial_1} & B_3(\mathrm{SL}_2(A))\otimes X_1(A^2) \\ & & \big\downarrow {\scriptstyle d_3\otimes \mathrm{id}_{X_1}} \\ & & B_2(\mathrm{SL}_2(A))\otimes X_0(A^2) \xleftarrow{\mathrm{id}_{B_2}\otimes\mathrm{inc}} B_2(\mathrm{SL}_2(A))\otimes Z_1(A^2). \end{array}$$

The element $\Lambda(a,b,c) := \lambda(ab,c)-\lambda(a,c)-\lambda(b,c)$ is

$$\begin{aligned}\Lambda(a,b,c) := \big([ab|c]-[a|c]-[b|c]+[w|abc]-[w|ac]-[w|bc]-[w|ab]+[w|a]+[w|b]+[w|c]\big)\otimes Y \\ +[wabc|wabc]\otimes\partial_2(X_{abc})-[wab|wab]\otimes\partial_2(X_{ab})-[wbc|wbc]\otimes\partial_2(X_{bc}) \\ -[wac|wac]\otimes\partial_2(X_{ac})+[wa|wa]\otimes\partial_2(X_a)+[wb|wb]\otimes\partial_2(X_b) \\ +[wc|wc]\otimes\partial_2(X_c)-[w|w]\otimes\partial_2(X_1).\end{aligned}$$

For an element $z\in A^\times$, consider $[wz|wz]\otimes\partial_2(X_z)\in B_2(\mathrm{SL}_2(A))\otimes_{\mathrm{SL}_2(A)} Z_1(A^2)$. For the matrices g_z and h_z we have the identities

$$z^{-1}g_{z^{-1}} = g_z z, \quad zh_z = h_{z^{-1}}z, \quad g_z^{-1}w = h_{z^{-1}}, \quad h_z^{-1} = h_z.$$

Using these identities we obtain

$$\begin{aligned}[wz|wz]\otimes\partial_2(X_z) = [wz|wz]\otimes(\boldsymbol{\infty},0) \\ +(d_3\otimes\mathrm{id}_{X_1})((([g_z^{-1}|wz|wz]-[h_z|wz|wz]+[z^{-1}|g_z^{-1}|wz]-[z|h_z|wz])\otimes(\boldsymbol{\infty},\mathbf{0})) \\ +(d_3\otimes\mathrm{id}_{X_1})((([z|z^{-1}|g_z]-[z^{-1}|z|h_z]+[z^{-1}|z|z^{-1}])\otimes(\boldsymbol{\infty},0)).\end{aligned}$$

If

$$\begin{aligned}\theta_z := [g_z^{-1}|wz|wz]-[h_z|wz|wz]+[z^{-1}|g_z^{-1}|wz]-[z|h_z|wz] \\ +[z|z^{-1}|g_z]-[z^{-1}|z|h_z]+[z^{-1}|z|z^{-1}],\end{aligned}$$

then by a direct calculation we have

$$\Lambda(a,b,c) = (d_3 \otimes \mathrm{id}_{X_1})(([w|ab|c] - [w|a|c] - [w|b|c]) \otimes Y + (\Phi_{a,b,c} + \Psi_{a,b,c}) \otimes (\boldsymbol{\infty}, \mathbf{0})),$$

where

$$\begin{aligned}
\Phi_{a,b,c} &= \theta_{abc} - \theta_{ab} - \theta_{bc} - \theta_{ac} + \theta_a + \theta_b + \theta_c - \theta_1 \\
\Psi_{a,b,c} &= [wab|wab|c] - [wa|wa|c] - [wb|wb|c] + [w|w|c] \\
&\quad + [c|wabc|wabc] - [c|wac|wac] - [c|wbc|wbc] + [c|wc|wc] \\
&\quad + [ab|wab|c] - [a|wa|c] - [a|c|wac] + [ab|c|wabc] - [c|ab|wabc] + [c|a|wac] \\
&\quad - [b|wb|c] - [b|c|wbc] + [c|b|wbc] - [b|a|c] + [b|c|a] - [c|b|a].
\end{aligned}$$

Since $\partial_1(Y) = 0$, we need only to study

$$(\mathrm{id}_{B_3} \otimes \partial_1)((\Phi_{a,b,c} + \Psi_{a,b,c}) \otimes (\boldsymbol{\infty}, \mathbf{0})).$$

Through the maps

$$B_3(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_1(A^2) \xrightarrow{\mathrm{id}_{B_3} \otimes \partial_1} B_3(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_0(A^2)$$

the above element maps to

$$(w\Phi_{a,b,c} - \Phi_{a,b,c}) \otimes (\boldsymbol{\infty}) + (w\Psi_{a,b,c} - \Psi_{a,b,c}) \otimes (\boldsymbol{\infty}).$$

Now consider the composite

$$B_3(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_0(A^2) \to C_3(\mathrm{SL}_2(A)) \otimes_{\mathrm{SL}_2(A)} X_0(A^2) \to C_3(\mathrm{SL}_2(A)) \otimes_{B(T)} \mathbb{Z}.$$

Then

$$\begin{aligned}
(w\theta_z - \theta_z) \otimes (\boldsymbol{\infty}) \mapsto \Big\{ & (w, wg_z^{-1}, wg_z^{-1}wz, wg_z^{-1}) - (w, wh_z, wh_zwz, wh_z) \\
& + (w, wz^{-1}, wz^{-1}g_z^{-1}, wz^{-1}g_z^{-1}wz) - (w, wz, wzh_z, wzh_zwz) \\
& + (w, wz, w, wg_z^{-1}) - (w, wz^{-1}, w, wh_z) \\
& + (w, wz^{-1}, w, wz^{-1}) - (1, g_z^{-1}, g_z^{-1}wz, g_z^{-1}) \\
& + (1, h_z, h_zwz, h_z) - (1, z^{-1}, z^{-1}g_z^{-1}, z^{-1}g_z^{-1}wz) \\
& + (1, z, zh_z, zh_zwz) - (1, z, 1, g_z^{-1}) \\
& + (1, z^{-1}, 1, h_z) - (1, z^{-1}, 1, z^{-1}) \Big\} \otimes 1
\end{aligned}$$

and

$$\begin{aligned}(w\Psi_{a,b,c}-\Psi_{a,b,c})\otimes(\infty)\mapsto\Big\{&(w,ab,w,wc)-(w,a,w,wc)-(w,b,w,wc)+(w,1,w,wc)\\&+(w,wc,ab,wc)-(w,wc,a,wc)-(w,wc,b,wc)+(w,wc,1,wc)\\&+(w,wab,1,c)-(w,wa,1,c)-(w,wa,wac,1)+(w,wab,wabc,1)\\&-(w,wc,wabc,1)+(w,wc,wac,1)-(w,wb,1,c)-(w,wb,wbc,1)\\&+(w,wc,wbc,1)-(w,wb,wab,wabc)+(w,wb,wbc,wabc)\\&-(w,wc,wbc,wabc)-(1,wab,1,c)+(1,wa,1,c)+(1,wb,1,c)\\&-(1,w,1,c)-(1,c,wab,c)+(1,c,wa,c)+(1,c,wb,c)-(1,c,w,c)\\&-(1,ab,w,wc)+(1,a,w,wc)+(1,a,ac,w)-(1,ab,abc,w)\\&+(1,c,abc,w)-(1,c,ac,w)+(1,b,w,wc)+(1,b,bc,w)\\&-(1,c,bc,w)+(1,b,ab,abc)-(1,b,bc,abc)+(1,c,bc,abc)\Big\}\otimes 1.\end{aligned}$$

Now we want to follow these elements through the maps

$$C_3(\mathrm{SL}_2(A))\otimes_{B(A)}\mathbb{Z}\xrightarrow{s}C_3(B(A))\otimes_{B(A)}\mathbb{Z}\to C_3(T(A))\otimes_{T(A)}\mathbb{Z}\to B_3(T(A))\otimes_{T(A)}\mathbb{Z}$$

where $s:\mathrm{SL}_2(A)\backslash T(A)\to\mathrm{SL}_2(A)$ is the section discussed in above. It is straightforward to check that modulo $\mathrm{im}(d_4)$ we have

$$(w\theta_z-\theta_z)\otimes(\infty)\mapsto-[z^{-1}|z|z^{-1}]\otimes 1=[z|z^{-1}|z]\otimes 1$$

Moreover

$$\begin{aligned}(w\Psi_{a,b,c}-\Psi_{a,b,c})\otimes(\infty)\mapsto\Big\{&[c^{-1}|abc|(abc)^{-1}]-[c^{-1}|ac|(ac)^{-1}]-[c^{-1}|bc|(bc)^{-1}]+[c^{-1}|c|c^{-1}]\\&-[a^{-1}|c^{-1}|ac]+[(ab)^{-1}|c^{-1}|abc]-[c^{-1}|(ab)^{-1}|abc]+[c^{-1}|a^{-1}|ac]\\&-[b^{-1}|c^{-1}|bc]+[c^{-1}|b^{-1}|bc]-[b^{-1}|a^{-1}|c^{-1}]+[b^{-1}|c^{-1}|a^{-1}]\\&-[c^{-1}|b^{-1}|a^{-1}]-[c|(abc)^{-1}|abc]+[c|(ac)^{-1}|ac]+[c|(bc)^{-1}|bc]\\&-[c|c^{-1}|c]+[a|c|(ac)^{-1}]-[ab|c|(abc)^{-1}]+[c|ab|(abc)^{-1}]\\&-[c|a|(ac)^{-1}]+[b|c|(bc)^{-1}]-[c|b|(bc)^{-1}]+[b|a|c]-[b|c|a]\\&+[c|b|a]\Big\}\otimes 1.\end{aligned}$$

Combining all these we see that $d^2_{2,2}(\Lambda(a,b,c))$ is the following element of $E^2_{0,3}$

$$\begin{aligned}
d^2_{2,2}(\Lambda(a,b,c)) =\Big\{ & [c^{-1}|abc|(abc)^{-1}] - [c^{-1}|ac|(ac)^{-1}] - [c^{-1}|bc|(bc)^{-1}] + [c^{-1}|c|c^{-1}] \\
& - [a^{-1}|c^{-1}|ac] + [(ab)^{-1}|c^{-1}|abc] - [c^{-1}|(ab)^{-1}|abc] + [c^{-1}|a^{-1}|ac] \\
& - [b^{-1}|c^{-1}|bc] + [c^{-1}|b^{-1}|bc] - [b^{-1}|a^{-1}|c^{-1}] + [b^{-1}|c^{-1}|a^{-1}] \\
& - [c^{-1}|b^{-1}|a^{-1}] - [c|(abc)^{-1}|abc] + [c|(ac)^{-1}|ac] + [c|(bc)^{-1}|bc] \\
& + [a|c|(ac)^{-1}] - [ab|c|(abc)^{-1}] + [c|ab|(abc)^{-1}] - [c|a|(ac)^{-1}] \\
& + [b|c|(bc)^{-1}] - [c|b|(bc)^{-1}] + [b|a|c] - [b|c|a] + [c|b|a] \\
& + [abc|(abc)^{-1}|abc] - [ab|(ab)^{-1}|ab] - [bc|(bc)^{-1}|bc] \\
& - [ac|(ac)^{-1}|ac] + [a|a^{-1}|a] + [b|b^{-1}|b]\Big\} \otimes 1
\end{aligned}$$

By adding the null element

$$\begin{aligned}
& d_4\Big(\Big\{ - [c|c^{-1}|abc|(abc)^{-1}] + [c|c^{-1}|ac|(ac)^{-1}] + [c|c^{-1}|bc|(bc)^{-1}] - [c|c^{-1}|a^{-1}|ac] \\
& - [c|c^{-1}|c|c^{-1}] - [c|c^{-1}|b^{-1}|bc] + [c^{-1}|c|(abc)^{-1}|abc] + [ac|a^{-1}|c^{-1}|ac] \\
& - [abc|(ab)^{-1}|c^{-1}|abc] + [bc|b^{-1}|c^{-1}|bc] + [c|ab|(ab)^{-1}|ab] - [c|b|b^{-1}|b] - [c|a|a^{-1}|a] \\
& - [c|a|b|(ab)^{-1}] - [a|b|c|(abc)^{-1}] + [abc|b^{-1}|a^{-1}|c^{-1}] - [abc|b^{-1}|c^{-1}|a^{-1}] \\
& + [a|c|c^{-1}|a^{-1}] - [a|bc|b^{-1}|c^{-1}] + [a|c|b|b^{-1}] + [ac|b|b^{-1}|a^{-1}] - [a|bc|(bc)^{-1}|a^{-1}] \\
& - [b|c|(bc)^{-1}|a^{-1}] + [c|c^{-1}|b^{-1}|a^{-1}] - [c|c^{-1}|c|(abc)^{-1}]\Big\} \otimes 1\Big)
\end{aligned}$$

we see that, modulo $\mathrm{im}(d_4)$,

$$\begin{aligned}
d^2_{2,2}(\Lambda(a,b,c)) &= -([a|b|c] + [c|a|b] + [b|c|a] - [b|a|c] - [c|b|a] - [a|c|b]) \otimes 1 \\
&= -a \wedge b \wedge c.
\end{aligned}$$

Thus we obtain the desired exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma_2'} \to H_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

Now let A be a domain. Since $\mu(A)$ is direct limit of finite cyclic groups, then

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A))^{\Sigma_2'} = \mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)).$$

Let F be the quotient field of A and $\overline{F}$ the algebraic closure of F. It is very easy to see that $\mathscr{RB}(\overline{F}) = \mathscr{B}(\overline{F})$. The classical Bloch-Wigner exact sequence claims that the sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu(\overline{F}),\mu(\overline{F})) \to H_3(\mathrm{SL}_2(\overline{F}),\mathbb{Z}) \to \mathscr{B}(\overline{F}) \to 0$$

is exact. Now the final claim follows from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)) & \longrightarrow & H_3(\mathrm{SL}_2(A),\mathbb{Z}) & \longrightarrow & \mathscr{RB}(A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mu(\overline{F}),\mu(\overline{F})) & \longrightarrow & H_3(\mathrm{SL}_2(\overline{F}),\mathbb{Z}) & \longrightarrow & \mathscr{B}(\overline{F}) & \longrightarrow & 0
\end{array}$$

and the fact that the natural map $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu(\overline{F}),\mu(\overline{F}))$ is injective. □

Corollary 3.4.6. *Let A be a local domain of characteristic 2, where its residue field has more than 2^6 elements. Then we have the refined Bloch-Wigner exact sequence*

$$0 \to Tor_1^{\mathbb{Z}}(\mu(A), \mu(A)) \to H_3(SL_2(A), \mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

Proof. This follows from Proposition 3.1.1, Proposition 3.3.8 and Theorem 3.4.5. □

Let study the map $\mathscr{I}_A \otimes \mu_2(A) \to A^\times \wedge \mu_2(A) \subseteq A^\times \wedge A^\times$ given by $\langle\langle a \rangle\rangle \otimes b \mapsto a \wedge b$ (when A is a domain). Clearly $\mathscr{I}_A^2 \otimes \mu_2(A)$ is in the kernel of this map. This induces the map

$$\mathscr{G}_A \otimes \mu_2(A) \simeq (\mathscr{I}_A/\mathscr{I}_A^2) \otimes \mu_2(A) \to A^\times \wedge \mu_2(A),$$

$$\langle a \rangle \otimes b \mapsto \langle\langle a \rangle\rangle \otimes b \mapsto a \wedge b.$$

Lemma 3.4.7. *Let A be a domain. Then the kernel of the map $\mathscr{G}_A \otimes \mu_2(A) \to A^\times \wedge A^\times$, given by $\langle a \rangle \otimes (-1) \mapsto a \wedge (-1)$, has at most two elements.*

Proof. We may assume that $\mathrm{char}(A) \neq 2$. In this case $\mathscr{G}_A \otimes \mu_2(A) \simeq \mathscr{G}_A$. Let $a \wedge (-1) = 0$ in $A^\times \wedge A^\times$. We know that $A^\times = \varinjlim H$, where H runs through all finitely generated subgroups of $A^\times$. As the direct limit commutes with wedge product, we have $A^\times \wedge A^\times = \varinjlim H \wedge H$. We may take a finitely generated subgroup H such $a, -1 \in H$ and $a \wedge (-1) = 0 \in H \wedge H$.

Let $H \simeq F \times T$, where F is torsion free and T is a finite cyclic group. Thus $-1 \in T$ and we have

$$H \wedge H \simeq (F \wedge F) \oplus (F \otimes T) \oplus (T \wedge T).$$

Clearly $T \wedge T = 0$. Let $a = p\omega$ with $p \in F$ and $T = \langle \omega \rangle$. From $a \wedge (-1) = 0 \in H \wedge H$, it follows that $p \otimes (-1) = 0$ and $\omega \wedge (-1) = 0$. As $-1 \in T$, T has even order. Thus $p \otimes (-1) = 0$ implies that p is a square. Therefore $\langle a \rangle = \langle \omega \rangle$. This completes the proof. □

Now let A be a domain. Then from the commutative diagram (3.4.1), we obtain the exact sequence

$$H_1(\mathrm{SL}_2(A), Z_2(A^2)) \to J \xrightarrow{\gamma} E^3_{2,1} \to \mathscr{RB}(A) \to 0,$$

where J sits in the exact sequence $\mathscr{I}_A^2 \otimes \mu_2(A) \to J \to (\mathbb{Z}/2)' \to 0$ with $(\mathbb{Z}/2)'$ a subgroup of $\mathbb{Z}/2$ (Lemma 3.4.7).

CHAPTER

4

THIRD HOMOLOGY OF SL_2

4.1 The low dimensional homology of SM_2

Let $\mathrm{SM}_2(A)$ denotes the group of monomial matrices in $\mathrm{SL}_2(A)$. Then $\mathrm{SM}_2(A)$ consists of matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}$, where $a \in A^\times$. Let $\hat{X}_0(A^2)$ and $\hat{X}_1(A^2)$ be the free $\mathbb{Z}$-modules generated by the sets

$$\mathrm{SM}_2(A)(\boldsymbol{\infty}) := \{g.(\boldsymbol{\infty}) : g \in \mathrm{SM}_2(A)\}, \quad \mathrm{SM}_2(A)(\boldsymbol{\infty},\mathbf{0}) := \{g.(\boldsymbol{\infty},\mathbf{0}) : g \in \mathrm{SM}_2(A)\},$$

respectively. It is easy to see that the sequence of $\mathrm{SM}_2(A)$-modules

$$\hat{X}_1(A^2) \overset{\hat{\partial}_1}{\to} \hat{X}_0(A^2) \overset{\hat{\varepsilon}}{\to} \mathbb{Z} \to 0$$

is exact and

$$\ker(\hat{\partial}_1) = \mathbb{Z}\{(\boldsymbol{\infty},\mathbf{0}) + (\mathbf{0},\boldsymbol{\infty})\}.$$

We denote this kernel by $\hat{Z}_1(A^2)$. Observe that $\hat{Z}_1(A^2) \simeq \mathbb{Z}$ and $\mathrm{SM}_2(A)$ acts trivially on it. From the complex

$$0 \to \hat{Z}_1(A^2) \overset{\widehat{\mathrm{inc}}}{\to} \hat{X}_1(A^2) \overset{\hat{\partial}_1}{\to} \hat{X}_0(A^2) \to 0, \tag{4.1.1}$$

we obtain the first quadrant spectral sequence

$$\hat{E}^1_{p,q} = \begin{cases} H_q(\mathrm{SM}_2(A), \hat{X}_p(A^2)) & p = 0,1 \\ H_q(\mathrm{SM}_2(A), \hat{Z}_1(A^2)) & p = 2 \\ 0 & p > 2 \end{cases} \Rightarrow H_{p+q}(\mathrm{SM}_2(A), \mathbb{Z}).$$

Since the complex (4.1.1) is a $\mathrm{SM}_2(A)$-subcomplex of (3.1.1), we have a natural morphism of spectral sequences

$$\begin{array}{ccc} \hat{E}^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{SM}_2(A), \mathbb{Z}) \\ \downarrow & & \downarrow \\ E^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{SL}_2(A), \mathbb{Z}). \end{array} \tag{4.1.2}$$

As in case of $\mathrm{SL}_2(A)$, we have $\hat{X}_0 \simeq \mathrm{Ind}_{T(A)}^{\mathrm{SM}_2(A)}\mathbb{Z}$ and $\hat{X}_1 \simeq \mathrm{Ind}_{T(A)}^{\mathrm{SL}_2(A)}\mathbb{Z}$. Thus by Shapiro's lemma we have

$$\hat{E}^1_{0,q} \simeq H_q(T(A),\mathbb{Z}), \quad \hat{E}^1_{1,q} \simeq H_q(T(A),\mathbb{Z}).$$

Therefore

$$\hat{E}^1_{p.q} = \begin{cases} H_q(T(A),\mathbb{Z}) & p=0,1 \\ H_q(\mathrm{SM}_2(A),\mathbb{Z}) & p=2 \\ 0 & p>2 \end{cases} \Rightarrow H_{p+q}(\mathrm{SM}_2(A),\mathbb{Z}).$$

Moreover, $\hat{d}^1_{1,q} = H_q(\hat{\sigma}) - H_q(\hat{\mathrm{inc}}) = \hat{\sigma}_* - \hat{\mathrm{inc}}_*$, where $\hat{\sigma} : T(A) \to T(A)$ is given by $X \to wXw^{-1} = X^{-1}$. Thus $\hat{d}^1_{1,0}$ is trivial, $\hat{d}^1_{1,1}$ is induced by the map $X \mapsto X^{-2}$ and $\hat{d}^1_{1,2}$ is trivial.

A direct calculation shows that the map $\hat{d}_{2,q} : H_q(\mathrm{SM}_2(A),\mathbb{Z}) \to H_q(T(A),\mathbb{Z})$ is the transfer map (BROWN, 2012, §9, Chap. III). Hence the composite

$$H_q(\mathrm{SM}_2(A),\mathbb{Z}) \overset{\hat{d}_{2,q}}{\to} H_q(T(A),\mathbb{Z}) \overset{\mathrm{inc}_*}{\to} H_q(\mathrm{SM}_2(A),\mathbb{Z})$$

coincides with multiplication by 2 (BROWN, 2012, Proposition 9.5, Chap. III). In particular, $\hat{d}_{2,0} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2. From these we obtain the exact sequence

$$1 \to \mathscr{G}_A \to H_1(\mathrm{SM}_2(A),\mathbb{Z}) \to \mathbb{Z}/2 \to 0.$$

If fact this can be obtain directly from the extension $1 \to T(A) \to \mathrm{SM}_2(A) \to \langle \overline{w} \rangle \to 1$:

$$1 \to \mathscr{G}_A \to H_1(\mathrm{SM}_2(A),\mathbb{Z}) \to \langle \overline{w} \rangle \to 1.$$

Observe that $w^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in T(A)$. A direct calculation shows that $\hat{d}^1_{2,1}(\overline{w}) = -1$ and $\hat{d}^1_{2,1} |_{\mathscr{G}_A} = 0$. Thus

$$\hat{E}^2_{1,1} = \mu_2(A)/\{\pm 1\}, \quad \hat{E}^2_{2,1} = \mathscr{G}_A.$$

Again a direct calculation shows that

$$\hat{d}^2_{2,1} : \mathscr{G}_A \to H_2(T(A),\mathbb{Z}) \simeq A^\times \wedge A^\times$$

is given by $\langle a \rangle \mapsto a \wedge (-1)$. Therefore from the spectral sequence $\hat{E}^1_{p,q} \Rightarrow H_{p+q}(\mathrm{SM}_2(A),\mathbb{Z})$ we obtain the exact sequence

$$0 \to \frac{A^\times \wedge A^\times}{A^\times \wedge \{\pm 1\}} \to H_2(\mathrm{SM}_2(A),\mathbb{Z}) \to \mu_2(A)/\{\pm 1\} \to 1.$$

Thus we have:

Lemma 4.1.1. *If $\mu_2(A) = \{\pm 1\}$, then $H_2(SM_2(A),\mathbb{Z}) \simeq \dfrac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)}$.*

Now if $\mu_2(A) = \{\pm 1\}$, then it follows from this lemma that the image of the map $\hat{d}^1_{2,2} : H_2(\mathrm{SM}_2(A),\mathbb{Z}) \to A^\times \wedge A^\times$ is $2(A^\times \wedge A^\times)$. Thus $\hat{E}^2_{1,2} \simeq \dfrac{A^\times \wedge A^\times}{2(A^\times \wedge A^\times)}$. Moreover one can show that $\hat{E}^2_{2,2} \simeq \dfrac{2(A^\times \wedge A^\times)}{A^\times \wedge \mu_2(A))}$.

4.2 The third homology of SL_2

Let the complex $X_\bullet(A^2) \to \mathbb{Z}$ be exact in dimension < 2. Then the natural map $\alpha : \mathscr{G}_A = \hat{E}^2_{2,1} \to E^2_{2,1}$ sits in the diagram

$$\begin{array}{ccccccc} & & \mathscr{G}_A & & & & \\ & & \Big\downarrow \alpha & & & & \\ \mathscr{I}_A \otimes \mu_2(A) & \xrightarrow{\gamma} & E^2_{2,1} & \xrightarrow{\delta} & \mathscr{RP}_1(A) & \longrightarrow & 0. \end{array}$$

Recall that for any $a \in A^\times$, we defined $\psi_1(a) := [a] + \langle -1 \rangle [a^{-1}] \in \mathscr{RP}(A)$.

Lemma 4.2.1. *The composite map $\delta \circ \alpha : \mathscr{G}_A \to \mathscr{RP}_1(A)$ is given by $\langle a \rangle \mapsto \psi_1(a^2)$.*

Proof. The element $\langle a \rangle \in \mathscr{G}_A$ is represented by

$$[a] \otimes \{(\boldsymbol{\infty}, \mathbf{0}) + (\mathbf{0}, \boldsymbol{\infty})\} \in H_1(\mathrm{SM}_2(A), \hat{Z}_1(A^2)).$$

Its image in $H_1(\mathrm{SL}_2(A), Z_1(A^2))$, through α, is represented by the element

$$S := [a] \otimes \partial_2((\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}^2) + (\mathbf{0}, \boldsymbol{\infty}, \boldsymbol{a}^2)).$$

We have

$$\begin{aligned} \delta(S) &= (d_1 \otimes \mathrm{id}_{Z_2(X^2)}) \Big([a] \otimes \partial_2((\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}^2) + (\mathbf{0}, \boldsymbol{\infty}, \boldsymbol{a}^2)) \Big) \\ &= [\,] \otimes \Big((\boldsymbol{\infty}, \mathbf{0}, \mathbf{1}) + (\mathbf{0}, \boldsymbol{\infty}, \mathbf{1}) - (\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a}^2) - (\mathbf{0}, \boldsymbol{\infty}, \boldsymbol{a}^2) \Big) \\ &= [\,] \otimes \partial_3 \Big((\boldsymbol{\infty}, \mathbf{0}, \mathbf{1}, \boldsymbol{a}^2) + (\mathbf{0}, \boldsymbol{\infty}, \boldsymbol{a}^2, \mathbf{1}) \Big). \end{aligned}$$

It is straightforward to check that this element represent $-\psi_1(a^2)$. Thus

$$\delta(S) = -\psi_1(a^2) = \psi_1(a^2).$$

□

For any $a \in A^\times$, let X_a and X'_a denote the elements $(\boldsymbol{\infty}, \mathbf{0}, \boldsymbol{a})$ and $(\mathbf{0}, \boldsymbol{\infty}, \boldsymbol{a})$ of $X_2(A^2)$ respectively. Let $\chi_a \in H_1(\mathrm{SL}_2(A), Z_1(A^2))$ be represented by $[wa] \otimes \partial_2(X_{-a} - X_a)$. We usually write

$$\chi_a := [wa] \otimes \partial_2(X_{-a} - X_a).$$

We remind that usually $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is denoted by a.

Lemma 4.2.2. *For any $a \in A^\times$, $\gamma(\langle\langle a \rangle\rangle \otimes (-1)) - \alpha(\langle a \rangle) = \langle -1 \rangle \langle\langle a \rangle\rangle . \chi_1$.*

Proof. Let $Y := (\boldsymbol{\infty}, \mathbf{0}) + (\mathbf{0}, \boldsymbol{\infty}) \in Z_1(A^2)$. For any $a \in A^\times$, we have

(a) $d_2([wa|wa]) = wa[wa] - [-1] + [wa]$,

(b) $d_2([w|a]) = w[a] - [wa] + [w]$.

Thus modulo $\mathrm{im}(d_2 \otimes \mathrm{id}_{Z_1(A^2)})$, we have

1. $[-1] \otimes \partial_2(X_{-a}) = [wa] \otimes \partial_2(X'_a) + [wa] \otimes \partial_2(X_{-a})$,
2. $[wa] \otimes Y = [a] \otimes Y + [w] \otimes Y$.

Hence

$$\begin{aligned}
[wa] \otimes \partial_2(X_{-a} - X_a) =& [wa] \otimes \partial_2(X_{-a}) - [wa] \otimes \partial_2(X_a) \\
=& [-1] \otimes \partial_2(X_{-a}) - [wa] \otimes \partial_2(X'_a) - [wa] \otimes \partial_2(X_a) \\
=& [-1] \otimes \partial_2(X_{-a}) - [wa] \otimes Y \\
=& [-1] \otimes \partial_2(X_{-a}) - ([a] \otimes Y + [w] \otimes Y) \\
=& [-1] \otimes \partial_2(X_{-a}) - [w] \otimes Y - \alpha(\langle a \rangle) \\
=& [-1] \otimes \partial_2(X_{-a}) - [w] \otimes \partial_2(X_1 + X'_1) - \alpha(\langle a \rangle)
\end{aligned}$$

Now, using the identity (1) in above for $a = 1$, we get

$$\begin{aligned}
[wa] \otimes \partial_2(X_{-a} - X_a) - [w] \otimes \partial_2(X_{-1} - X_1) &= [-1] \otimes \partial_2(X_{-a} - X_{-1}) - \alpha(\langle a \rangle) \\
&= \langle -1 \rangle \gamma(\langle\langle a \rangle\rangle \otimes (-1)) - \alpha(\langle a \rangle).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[wa] \otimes \partial_2(X_{-a} - X_a) - [w] \otimes \partial_2(X_{-1} - X_1) &= \langle a \rangle([w] \otimes \partial_2(X_{-1} - X_1)) - [w] \otimes \partial_2(X_{-1} - X_1) \\
&= \langle\langle a \rangle\rangle([w] \otimes \partial_2(X_{-1} - X_1)) \\
&= \langle\langle a \rangle\rangle \chi_1.
\end{aligned}$$

Therefore $\langle\langle a \rangle\rangle \cdot \chi_1 = \langle -1 \rangle \gamma(\langle\langle a \rangle\rangle \otimes (-1)) - \alpha(\langle a \rangle)$. □

Remark 4.2.3. It is straightforward to show that $\delta(\chi_1) = \psi_1(-1) \in \mathcal{RP}_1(A)$.

Corollary 4.2.4. *If $-1 \in (A^\times)^2$, then for any $a \in A^\times$, $\gamma(\langle\langle a \rangle\rangle \otimes (-1)) = \alpha(\langle a \rangle)$.*

Proof. First observe that for any $s \in A^\times$ and $X \in X_2(A^2)$, we have

$$[w] \otimes (sX - X) = [s] \otimes (wX + sX).$$

Now if $i^2 = -1$, then by the above relation we have

$$\begin{aligned}
[w] \otimes \partial_2(X_{-1} - X_1) &= [w] \otimes \partial_2(iX_1 - X_1) \\
&= [i] \otimes \partial_2(wX_1 + iX_1) \\
&= [i] \otimes \partial_2(X'_1 + X_1) \\
&= [i] \otimes Y = \alpha(\langle i \rangle).
\end{aligned}$$

Now the claim follows from Lemma 4.2.2. □

Corollary 4.2.5. *Let $\mu_2(A) = \{\pm 1\}$ and $-1 \in (A^\times)^2$. Then $\gamma(\mathscr{I}_A^2 \otimes \mu_2(A)) = 0$. In particular, we have the exact sequence*

$$\mathscr{G}_A \xrightarrow{\alpha} E_{2,1}^2 \xrightarrow{\delta} \mathscr{RP}_1(A) \to 0.$$

Proof. The ideal $\mathscr{I}_A^2$ is generated by the elements $\langle\langle a\rangle\rangle\langle\langle b\rangle\rangle = \langle\langle ab\rangle\rangle - \langle\langle a\rangle\rangle - \langle\langle b\rangle\rangle$. Thus by the above corollary

$$\gamma(\langle\langle a\rangle\rangle\langle\langle b\rangle\rangle \otimes (-1)) = \alpha(\langle ab\rangle) - \alpha(\langle a\rangle) - \alpha(\langle b\rangle) = \alpha(\langle aba^{-1}b^{-1}\rangle) = \alpha(\langle 1\rangle) = 0.$$

The second part follows from the first part and the fact that $\mathscr{I}_A/\mathscr{I}_A^2 \simeq \mathscr{G}_A$ and $\mathrm{im}(\gamma) = \mathrm{im}(\alpha)$. □

Theorem 4.2.6. *Let A be a commutative ring such that*

(i) $\mu_2(A) = \{\pm 1\}$ and $-1 \in (A^\times)^2$,

(ii) $X_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 2.

(iii) $H_i(T(A), \mathbb{Z}) \simeq H_i(B(A), \mathbb{Z})$ for $i = 2, 3$.

Then we have the exact sequence

$$H_3(SM_2(A), \mathbb{Z}) \to H_3(SL_2(A), \mathbb{Z}) \to \mathscr{RB}(A) \to 0.$$

Proof. The morphism of spectral sequences (4.1.2) induces a map of filtration

$$\begin{array}{ccccccccc}
0 \subseteq & \hat{F}_0 & \subseteq & \hat{F}_1 & \subseteq & \hat{F}_2 & \subseteq & \hat{F}_3 = H_3(\mathrm{SM}_2(A), \mathbb{Z}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \subseteq & F_0 & \subseteq & F_1 & \subseteq & F_2 & \subseteq & F_3 = H_3(\mathrm{SL}_2(A), \mathbb{Z})
\end{array}$$

where $E_{p,3-p}^\infty = F_p/F_{p-1}$ and $\hat{E}_{p,3-p}^\infty = \hat{F}_p/\hat{F}_{p-1}$. Clearly $F_2 = F_3$ and $\hat{F}_2 = \hat{F}_3$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{F}_1 & \longrightarrow & H_3(\mathrm{SM}_2(A), \mathbb{Z}) & \longrightarrow & \hat{E}_{2,1}^\infty & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{inc}_* & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & H_3(\mathrm{SL}_2(A), \mathbb{Z}) & \longrightarrow & E_{2,1}^\infty & \longrightarrow & 0.
\end{array} \tag{4.2.1}$$

By Corollary 4.2.5, we have the exact sequence $\hat{E}_{2,1}^2 \to E_{2,1}^2 \to \mathscr{RP}_1(A) \to 0$. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \hat{E}_{2,1}^2 & \longrightarrow & E_{2,1}^2 & \longrightarrow & \mathscr{RP}_1(A) & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow \hat{d}_{2,1}^2 & & \downarrow d_{2,1}^2 & & \downarrow & & \\
0 & \longrightarrow & A^\times \wedge \mu_2(A) & \longrightarrow & (A^\times \wedge A^\times) & \longrightarrow & \dfrac{A^\times \wedge A^\times}{A^\times \wedge \mu_2(A)} & \longrightarrow & 0
\end{array}$$

we obtain the exact sequence

$$\hat{E}_{2,1}^\infty \to E_{2,1}^\infty \to \mathscr{RB}(A) \to 0.$$

Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{F}_0 & \longrightarrow & \hat{F}_1 & \longrightarrow & \hat{E}^{\infty}_{1,2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & E^{\infty}_{1,2} & \longrightarrow & 0. \end{array}$$

Since $\hat{E}^1_{0,3} \simeq E^1_{0,3}$, the natural map $\hat{F}_0 \to F_0$ is surjective. Moreover, since $\hat{E}^1_{1,2} \simeq E^1_{1,2}$, the map $\hat{E}^{\infty}_{1,2} \to E^{\infty}_{1,2}$ is surjective. These imply that the map $\hat{F}_1 \to F_1$ is surjective. Now the claim follows by applying the snake lemma to the diagram (4.2.1). □

Remark 4.2.7. We think that the condition $-1 \in A^{\times 2}$ in Theorem 4.2.6 is not essential (at least when A is a domain). To remove this condition we need to prove that under the map $\gamma : \mathscr{I}_A \otimes \mu_2(A) \to E^2_{2,1}$, $\mathscr{I}_A^2 \otimes \mu_2(A)$ maps to zero. Having this, then

$$\mathscr{G}_A \simeq \mathscr{G}_A \otimes \mu_2(A) \xrightarrow{\bar{\gamma}} E^2_{2,1} \to A^{\times} \wedge \mu_2(A) \;\text{ and }\; \mathscr{G}_A \xrightarrow{\alpha} E^2_{2,1} \to A^{\times} \wedge \mu_2(A)$$

have the same kernel by Lemma 3.4.7. Then we can proceed as in the above proof.

Example 4.2.8. Here we give examples of rings that satisfy the conditions of Theorem 4.2.6:

(1) Any local domain of characteristic 2 such that its residue field has more than 64 elements satisfies in the conditions of the theorem (Proposition 3.1.1, Theorem 3.3.8).

(2) Let B be a domain such that -1 is square. Let $\mathfrak{p}$ be a prime ideal of B such that either $B/\mathfrak{p}$ is infinite or if $|B/\mathfrak{p}| = p^d$, then $(p-1)d > 6$. Then $A := B_{\mathfrak{p}}$ satisfies in the conditions of Theorem 4.2.6 (Proposition 3.1.1, Theorem 3.3.8).

(3) Any domain with many units such that -1 is an square (e.g F-algebras which are domains and F is an algebraically closed) (MIRZAII, 2011, §2).

(4) Let $A = \mathbb{Z}[\frac{1}{m}]$, where m can be expressed as a product of primes $m = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ ($\alpha_i \geq 1$) with property that $(\mathbb{Z}/p_i)^{\times}$ is generated by the residue classes $\{-1, p_1, \ldots, p_{i-1}\}$ for all $i \leq t$. In particular, $p_1 \in \{2, 3\}$. Then A satisfies in the above conditions of Theorem 4.2.6 (Lemma 3.3.5, (HUTCHINSON, 2022, Example 6.14)).

4.3 A spectral sequence for relative homology

Let G be a group and M a G-module. We denote these by a pair (G, M). A morphism of pairs $(f, \sigma) : (G', M') \to (G, M)$ is a pair of group homomorphisms $f : G' \to G$ and $\sigma : M' \to M$ such that

$$\sigma(g'm') = f(g')\sigma(m').$$

This means that σ is a map of G'-modules.

For a group H let $C_\bullet(H) \to \mathbb{Z}$ be the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}[H]$ (BROWN, 2012, Chap.I, §5). The map $f : G' \to G$, induces in a natural way a morphism of complexes $f_\bullet : C_\bullet(G') \to C_\bullet(G)$.

The morphism of the pairs $(f,\sigma) : (G',M') \to (G,M)$, induces a morphism of complexes

$$f_\bullet \otimes \sigma : C_\bullet(G') \otimes_{G'} M' \to C_\bullet(G) \otimes_G M.$$

Let G' be a subgroup of G and M' be a G'-submodule of M. We take $(i,\sigma) : (G',M') \hookrightarrow (G,M)$ as the natural pair of inclusion maps. Then the morphism

$$i_\bullet \otimes \sigma : C_\bullet(G') \otimes_{G'} M' \to C_\bullet(G) \otimes_G M$$

is injective. We denote the n-homology of the quotient complex $C_\bullet(G) \otimes_G M / C_\bullet(G') \otimes_{G'} M'$ by $H_n(G,G',M',M)$:

$$H_n(G,G',M,M') := H_n(C_\bullet(G) \otimes_G M / C_\bullet(G') \otimes_{G'} M').$$

If $M' = M$, then $H_n(G,G',M,M')$ is the usual relative homology group $H_n(G,G',M)$.

From the exact sequence of complexes

$$0 \to C_\bullet(G') \otimes_{G'} M' \to C_\bullet(G) \otimes_G M \to C_\bullet(G) \otimes_G M / C_\bullet(G') \otimes_{G'} M' \to 0$$

we obtain the long exact sequence

$$\cdots \to H_n(G',M') \to H_n(G,M) \to H_n(G,G',M,M') \to H_{n-1}(G',M')$$

$$\to H_{n-1}(G,M) \to H_{n-1}(G,G',M,M') \to \cdots$$

Proposition 4.3.1. *Let G' be a subgroup of G. Let $L'_\bullet \to M'$ be an exact G'-subcomplex of an exact G-complex $L_\bullet \to M$. Then we have the first quadrant spectral sequence*

$$\mathbb{E}^1_{p,q} = H_q(G,G',L_p,L'_p) \Rightarrow H_{p+q}(G,G',M,M').$$

Proof. Let $i : G' \hookrightarrow G$ and $\sigma_\bullet : L'_\bullet \hookrightarrow L_\bullet$ be the usual inclusions. The morphism of double complexes

$$i_\bullet \otimes \sigma_\bullet : C_\bullet(G') \otimes_{G'} L'_\bullet \to C_\bullet(G) \otimes_G L_\bullet$$

is injective. We denote its quotient by $D_{\bullet,\bullet}$: $D_{\bullet,\bullet} = \mathrm{coker}(i_\bullet \otimes \sigma_\bullet)$. This double complexes induces two spectral sequences

$$\mathscr{E}^1_{p,q}(I) = H_q(D_{p,\bullet}) \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})), \quad \mathscr{E}^1_{p,q}(II) = H_q(D_{\bullet,p}) \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})).$$

These are the spectral sequences

$$\mathscr{E}^1_{p,q}(I) = H_q\left(\frac{C_p(G) \otimes_G L_\bullet}{C_p(G') \otimes_{G'} L'_\bullet}\right) \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})),$$

and

$$\mathscr{E}^1_{p,q}(II) = H_q\left(\frac{C_\bullet \otimes_G L_p}{C'_\bullet \otimes_{G'} L'_p}\right) \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})).$$

By definition $\mathscr{E}^1_{p,q}(II) = H_q(G,G',L_p,L'_p)$. Moreover since $L_\bullet$ and $L'_\bullet$ are exact in dimension > 0, we have $\mathscr{E}^1_{p,q}(I) = 0$ for any $q > 0$. For $q = 0$, we have $\mathscr{E}^1_{p,0}(I) \simeq \dfrac{C_p(G) \otimes_G M}{C_p(G') \otimes_{G'} M'}$. Now the homology of the sequence $\mathscr{E}^1_{p+1,0}(I) \to \mathscr{E}^1_{p,0}(I) \to \mathscr{E}^1_{p,0}(I)$ is

$$\mathscr{E}^2_{p,0}(I) \simeq H_q(G,G',M,M').$$

Now by an easy analysis of the spectral sequence $\mathscr{E}^1_{p,q}(I)$, for any $n \geq 0$ we obtain the isomorphism

$$H_n(\mathrm{Tot}(D_{\bullet,\bullet})) \simeq H_n(G,G',M,M').$$

Thus if we take $\mathbb{E}^1_{p,q} := \mathscr{E}^1_{p,q}(II)$, then we obtain the spectral sequence

$$\mathbb{E}^1_{p,q} = H_q(G,G',L_p,L'_p) \Rightarrow H_{p+q}(G,G',M,M').$$

□

4.4 The groups $\mathscr{R}\mathscr{P}_1(A)$ and $H_3(\mathbf{SL}_2(A),\mathbf{SM}_2(A),\mathbb{Z})$

Let $\mu_2(A) = \{\pm 1\}$ and the complex $X_\bullet(A^2) \to \mathbb{Z}$ be exact in dimension < 1. The complex

$$0 \to \hat{Z}_1(A^2) \to \hat{X}_1(A^2) \to \hat{X}_0(A^2) \to 0$$

is a $\mathrm{SM}_2(A)$-subcomplex of the $\mathrm{SL}_2(A)$-complex

$$0 \to Z_1(A^2) \to X_1(A^2) \to X_0(A^2) \to 0.$$

By Proposition 4.3.1, from the morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{Z}_1(A^2) & \longrightarrow & \hat{X}_1(A^2) & \longrightarrow & \hat{X}_0(A^2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_1(A^2) & \longrightarrow & X_1(A^2) & \longrightarrow & X_0(A^2) & \longrightarrow & 0, \end{array}$$

we obtain the first quadrant spectral sequence

$$\mathbb{E}^1_{p,q} = \begin{cases} H_q(\mathrm{SL}_2(A),\mathrm{SM}_2(A),X_p(A^2),\hat{X}_p(A^2)) & \text{if } p = 0,1 \\ H_q(\mathrm{SL}_2(A),\mathrm{SM}_2(A),Z_1(A^2),\hat{Z}_1(A^2)) & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases} \Rightarrow H_{p+q}(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}).$$

Consider the long exact sequence

$$\cdots \to H_q(\mathrm{SM}_2(A),\hat{X}_p(A^2)) \to H_q(\mathrm{SL}_2(A),X_p(A^2)) \to \mathbb{E}^1_{p,q} \to H_{q-1}(\mathrm{SM}_2(A),\hat{X}_p(A^2))$$

$$\to H_{q-1}(\mathrm{SL}_2(A),X_p(A^2)) \to \cdots.$$

Since

$$H_q(\mathrm{SL}_2(A),X_0(A^2)) \simeq H_q(B(A),\mathbb{Z}), \quad H_q(\mathrm{SL}_2(A),X_1(A^2)) \simeq H_q(T(A),\mathbb{Z}),$$

and

$$H_q(\mathrm{SM}_2(A),\hat{X}_0(A^2)) \simeq H_q(T(A),\mathbb{Z}), \quad H_q(\mathrm{SM}_2(A),\hat{X}_1(A^2)) \simeq H_q(T(A),\mathbb{Z}),$$

from the above exact sequence, for any q, we get

$$\mathbb{E}^1_{0,q} \simeq \mathscr{S}_q \simeq H_q(B(A),T(A),\mathbb{Z}), \quad \mathbb{E}^1_{1,q} = 0.$$

Therefore

$$\mathbb{E}^2_{0,q} \simeq \mathbb{E}^1_{0,q}, \quad \mathbb{E}^2_{1,q} = 0, \quad \mathbb{E}^2_{2,q} \simeq \mathbb{E}^1_{2,q}.$$

Now by easy analysis of the spectral sequence we get the exact sequence

$$\cdots \to H_{n+2}(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to \mathbb{E}^2_{2,n} \to H_{n+1}(B(A),T(A),\mathbb{Z}) \to H_{n+1}(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z})$$
$$\to \mathbb{E}^2_{2,n-1} \to H_n(B(A),T(A),\mathbb{Z}) \to \cdots$$

where the maps $H_n(B(A),T(A),\mathbb{Z}) \to H_n(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z})$ is induced by the natural inclusion of pairs $(B(A),T(A)) \hookrightarrow (\mathrm{SL}_2(A),\mathrm{SM}_2(A))$.

It is easy to see that $\mathbb{E}^2_{0,0} = 0 = \mathbb{E}^2_{1,0}$. Moreover we have the exact sequence

$$H_0(\mathrm{SM}_2(A),\mathbb{Z}) \to H_0(\mathrm{SL}_2(A),Z_1(A^2)) \to \mathbb{E}^2_{2,0} \to 0.$$

Note that $H_0(\mathrm{SM}_2(A),\mathbb{Z}) \simeq \mathbb{Z}$ and $H_0(\mathrm{SL}_2(A),Z_1(A^2)) = \mathrm{GW}(A)$. Moreover the map $\mathbb{Z} \to \mathrm{GW}(A)$ is injective and sends 1 to $p^+_{-1} = \langle -1 \rangle + 1$. Thus

$$\mathbb{E}^2_{2,0} \simeq \mathrm{GW}(A)/\langle\langle -1 \rangle + 1\rangle \simeq W(A).$$

where $W(A)$ is the Witt group of A. Furthermore we have the exact sequence

$$H_1(\mathrm{SM}_2(A),\mathbb{Z}) \to H_1(\mathrm{SL}_2(A),Z_1(A^2)) \to \mathbb{E}^2_{2,1} \to 0.$$

From the commutative diagram

$$\begin{array}{ccccc}
H_1(\mathrm{SM}_2(A),\mathbb{Z}) & \longrightarrow & H_1(\mathrm{SL}_2(A),Z_1(A^2)) & \longrightarrow & \mathbb{E}^2_{2,1} \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
H_1(\mathrm{SM}_2(A),\hat{X}_1(A^2) & \longrightarrow & H_1(\mathrm{SL}_2(A),X_1(A^2) & \longrightarrow & 0
\end{array}$$

we obtain the exact sequence

$$\mathscr{G}_A \xrightarrow{\alpha} E^2_{2,1} \to \mathbb{E}^2_{2,1} \to 0.$$

On the other hand we have the exact sequence

$$H_3(B(A),T(A),\mathbb{Z}) \to H_3(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to \mathbb{E}^2_{2,1} \to H_2(B(A),T(A),\mathbb{Z}) \to$$
$$H_2(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to W(A) \to A_{A^\times} \to H_1(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to 0.$$

Proposition 4.4.1. *Let A be a GE_2-ring such that $H_i(T(A),\mathbb{Z}) \simeq H_i(B(A),\mathbb{Z})$ for $i \leq 3$. Then*

(i) $H_2(SL_2(A),SM_2(A),\mathbb{Z}) \simeq W(A) \simeq GW(A)/\langle\langle -1\rangle + 1\rangle$

(ii) $H_3(SL_2(A),SM_2(A),\mathbb{Z}) \simeq \mathbb{E}^2_{2,1}$. In particular we have the exact sequence

$$\mathscr{G}_A \xrightarrow{\alpha} E^2_{2,1} \to H_3(SL_2(A),SM_2(A),\mathbb{Z}) \to 0.$$

Proof. It follows from our hypothesis that $H_i(B(A),T(A),\mathbb{Z}) = 0$ for $0 \leq i \leq 3$. Now the claims follows from the above discussions. □

Theorem 4.4.2. *Let A be a universal GE_2-ring such that $H_i(T(A),\mathbb{Z}) \simeq H_i(B(A),\mathbb{Z})$ for $i \leq 3$. Then we have an exact sequence*

$$I(A) \otimes \mu_2(A) \to H_3(SL_2(A),SM_2(A),\mathbb{Z}) \to \frac{\mathscr{RP}_1(A)}{\langle \psi_1(a^2) : a \in A^\times\rangle} \to 0.$$

In particular, if $-1 \in (A^\times)^2$, then $H_3(SL_2(A),SM_2(A),\mathbb{Z}) \simeq \mathscr{RP}_1(A)$.

Proof. The first claim follows from the above Proposition, Lemma 4.2.1 and the following diagram with exact row and column:

$$\begin{array}{ccccc} & & \mathscr{I}_A \otimes \mu_2(A) & & \\ & & \downarrow & & \\ \mathscr{G}_A & \xrightarrow{\alpha} & E^2_{2,1} & \longrightarrow & H_3(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to 0 \\ & & \downarrow & & \\ & & \mathscr{RP}_1(A) & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

(Note that in above diagram we may replace $\mathscr{I}_A$ with $I(A)$.) The second claim follows from the first claim, Lemma 4.2.4 and the fact that $\psi_1(a^2) = 0$. □

Theorem 4.4.3. *Let A be ring such that $H_i(T(A),\mathbb{Z}) \simeq H_i(B(A),\mathbb{Z})$ for $i \leq 3$. Let $H_1(SL_2(A),\mathbb{Z}) = 0$.*

(i) If A is a GE_2-ring, then $H_2(SL_2(A),T(A),\mathbb{Z}) \simeq K_1^{MW}(A)$.

(ii) If A is a universal GE_2-ring, then $H_3(SL_2(A),T(A),\mathbb{Z}[\frac{1}{2}]) \simeq \mathscr{RP}_1(A)[\frac{1}{2}]$.

Proof. (i) From the inclusions $T(A) \subseteq \mathrm{SM}_2(A) \subseteq \mathrm{SL}_2(A)$, we obtain the long exact sequence

$$\cdots \to H_n(\mathrm{SM}_2(A),T(A),\mathbb{Z}) \to H_n(\mathrm{SL}_2(A),T(A),\mathbb{Z}) \to H_n(\mathrm{SL}_2(A),\mathrm{SM}_2(A),\mathbb{Z}) \to$$

$$H_{n-1}(\mathrm{SM}_2(A),T(A),\mathbb{Z}) \to \cdots$$

Since $H_1(\mathrm{SL}_2(A), \mathbb{Z}) = 0$, we have

$$H_1(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}) = 0 = H_1(\mathrm{SL}_2(A), T(A), \mathbb{Z}).$$

It is easy to see that

$$H_1(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \simeq \mathbb{Z}/2.$$

We already have seen that $H_2(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}) \simeq W(A)$ (Proposition 4.4.1). Form the exact sequences

$$H_2(T(A), \mathbb{Z}) \to H_2(\mathrm{SL}_2(A), \mathbb{Z}) \to H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \to H_1(T(A), \mathbb{Z}) \to H_1(\mathrm{SL}_2(A), \mathbb{Z}) = 0$$

and

$$H_2(T(A), \mathbb{Z}) \to H_2(\mathrm{SL}_2(A), \mathbb{Z}) \to I^2(A) \to 0$$

we obtain the exact sequence

$$0 \to I^2(A) \to H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \to K_1^M(A) \to 0. \tag{4.4.1}$$

Now consider the exact sequence

$$H_2(T(A), \mathbb{Z}) \to H_2(\mathrm{SM}_2(A), \mathbb{Z}) \to H_2(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \to H_1(T(A), \mathbb{Z})$$

$$\to H_1(\mathrm{SM}_2(A), \mathbb{Z}) \to H_1(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \to 0.$$

Since $H_2(T(A), \mathbb{Z}) \to H_2(\mathrm{SM}_2(A), \mathbb{Z})$ is surjective (by Lemma 4.1.1) and $H_1(\mathrm{SM}_2(A), \mathbb{Z})$ sites in the exact sequence $1 \to \mathscr{G}_A \to H_1(\mathrm{SM}_2(A), \mathbb{Z}) \to \mathbb{Z}/2 \to 0$, we have

$$H_2(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \simeq A^{\times 2} \simeq 2K_1^M(A).$$

Thus we have the exact sequence

$$0 \to 2K_1^M(A) \to H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \to I(A) \to 0. \tag{4.4.2}$$

It is known that the first Milnor-Witt K-group of A, $K_1^{\mathrm{MW}}(A)$, satisfies in the exact sequences (4.4.1) and (4.4.2) ((HUTCHINSON; TAO, 2010, §2)). From the exact sequences (4.4.1) and (4.4.2) we obtain the commutative diagram

$$\begin{array}{ccc} H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) & \longrightarrow & K_1^M(A) \\ \downarrow & & \downarrow \\ I(A) & \longrightarrow & I(A)/I^2(A) \end{array}$$

Since $I(A)/I^2(A) \simeq \mathscr{G}_A \simeq K_1^M(A)/2K_1^M(A)$, the above diagram is Cartesian. Thus

$$H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \simeq K_1^M(A) \times_{I(A)/I^2(A)} I(A).$$

But it is well-known that $K_1^{\mathrm{MW}}(A)$ is the Cartesian product of the maps $K_1^M(A) \to I(A)/I^2(A)$ and $I(A) \to I(A)/I^2(A)$ (or we can take this as definition). Thus

$$H_2(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \simeq K_1^M(A) \times_{I(A)/I^2(A)} I(A) \simeq K_1^{\mathrm{MW}}(A).$$

(ii) Consider the long exact sequence

$$H_3(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \to H_3(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \to H_3(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z})$$
$$\to 2K_1^M(A) \to K_1^{\mathrm{MW}}(A) \to W(A).$$

This gives us the exact sequence

$$H_3(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \to H_3(\mathrm{SL}_2(A), T(A), \mathbb{Z}) \to H_3(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}) \to 0.$$

Consider the exact sequence

$$H_3(T(A), \mathbb{Z}) \to H_3(\mathrm{SM}_2(A), \mathbb{Z}) \to H_3(\mathrm{SM}_2(A), T(A), \mathbb{Z}) \to H_2(T(A), \mathbb{Z}) \to H_2(\mathrm{SM}_2(A), \mathbb{Z})$$

We have seen that the kernel of the right hand side map is isomorphic to $A^\times \wedge \mu_2(A)$. Moreover using the spectral sequence $\hat{E}_{p,q} \Rightarrow H_{p+q}(\mathrm{SM}_2(A), \mathbb{Z})$ we obtain the exact sequence

$$0 \to (A^\times \wedge A^\times)/2 \to H_3(\mathrm{SM}_2(A), \mathbb{Z})/H_3(T(A), \mathbb{Z}) \to \mathscr{G}_A \to A^\times \wedge A^\times.$$

These show that $H_3(\mathrm{SM}_2(A), T(A), \mathbb{Z}[\frac{1}{2}]) = 0$ Thus

$$H_3(\mathrm{SL}_2(A), T(A), \mathbb{Z}[\tfrac{1}{2}]) \simeq H_3(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}[\tfrac{1}{2}]) \simeq \mathscr{RP}_1(A)[\tfrac{1}{2}].$$

□

Remark 4.4.4. It is known that $K_1^{\mathrm{MW}}(A)$ and $\mathscr{RP}_1(A)$ have certain localization property (GILLE; SCULLY; ZHONG, 2016, Theorem 6.3), (HUTCHINSON; MIRZAII; MOKARI, 2022, Theorem A). Wendt in (WENDT, 2018, App. A) have introduced a higher version of these groups. It would be interesting to see what is the connection of these groups to the relative homology groups $H_n(\mathrm{SL}_2(A), \mathrm{SM}_2(A), \mathbb{Z}[\frac{1}{2}])$.

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