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Integrability and geometry of quadratic differential systems with invariant hyperbolas

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Integrabilidade e geometria de sistemas diferenciais quadráticos com hipérboles invariantes

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RESUMO

TRAVAGLINI, A. M. **Integrabilidade e geometria de sistemas diferenciais quadráticos com hipérboles invariantes**. 2021. 395 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Os sistemas diferenciais polinômiais planares ocorrem com muita frequência em vários ramos da matemática aplicada, na modelagem de fenômenos naturais, na astrofísica, nas equações de continuidade que descrevem as interações de íons, elétrons e espécies neutras na física de plasma, entre outras situações. Tais sistemas diferenciais também têm importância teórica. Vários problemas expostos a mais de cem anos atrás em sistemas diferenciais polinomiais ainda estão em aberto, por exemplo, a segunda parte do 16º problema de Hilbert relatado por Hilbert em (HILBERT, 1902), o problema de integrabilidade algébrica relatado por Poincaré (POINCARÉ, 1891a), (POINCARÉ, 1891b), problemas de integrabilidade resultantes do trabalho de Darboux (DARBOUX, 1878) e o problema do centro também relatado por Poincaré (POINCARÉ, 1885). Estes problemas ainda estão em aberto, exceto pelo problema do centro que foi resolvido no caso quadrático. Nesta tese, denotamos por **QSH** toda a classe de sistemas diferenciais quadráticos planares não degenerados que possuem pelo menos uma hipérbole invariante. **QSH** é uma rica família de sistemas que exibem vários tipos de integrabilidade: polinomial, algébrica (racional), Darboux, Darboux generalizado e Liouvilliana. O objetivo desta investigação é estudar esta classe do ponto de vista da teoria de Darboux: Separar os sistemas integráveis em **QSH**, classificá-los de acordo com o tipo de integral primeira que eles possuem e estudar sua geometria. Nossa principal motivação e objetivo, além de coletar dados, é estudar a relação entre a integrabilidade e a geometria dos sistemas expressa em suas configurações das curvas algébricas invariantes, estudar as bifurcações de suas configurações, bem como suas relações com as bifurcações dos retratos de fase.

Palavras-chave: Sistema diferencial quadrático, curva algébrica invariante, hipérbole invariante, integrabilidade de Darboux, integrabilidade Liouvilianna, configuração das curvas algébricas invariantes, bifurcação de configurações, singularidade, bifurcação de singularidades.

ABSTRACT

TRAVAGLINI, A. M. **Integrability and geometry of quadratic differential systems with invariant hyperbolas**. 2021. 395 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Planar polynomial differential systems occur very often in various branches of applied mathematics, in modeling natural phenomena, in astrophysics, in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics, among other situations. Such differential systems have also theoretical importance. Several problems stated more than one hundred years ago on polynomial differential systems are still open, for instance, the second part of Hilbert's 16th problem stated by Hilbert in (HILBERT, 1902), the problem of algebraic integrability stated by Poincaré in (POINCARÉ, 1891a), (POINCARÉ, 1891b), problems on integrability resulting from the work of Darboux (DARBOUX, 1878) and the problem of the center also stated by Poincaré (POINCARÉ, 1885). They are still unsolved, except for the problem of the center solved only in the quadratic case. In this thesis we denote by **QSH** be the whole class of non-degenerate planar quadratic differential systems possessing at least one invariant hyperbola. **QSH** is a rich family of systems displaying various kinds of integrability: polynomial, algebraic (rational), Darboux, generalized Darboux, Liouvillian. The goal of this investigation is to study this class from the viewpoint of the theory of Darboux: To separate the integrable system in **QSH**, to classify them according to the kind of first integral they possess and study their geometry. Our main motivation and goal, apart from gathering data, is to study the relationship between integrability and the geometry of the systems as expressed in their configurations of invariant algebraic curves, to study the bifurcations of their configurations as well as their relations with the bifurcations of the phase portraits.

Keywords: Quadratic differential system, Invariant algebraic curve, Invariant hyperbola, Darboux integrability, Liouvillian integrability, configuration of invariant algebraic curves, bifurcation of configurations, singularity, bifurcation of singularities.

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1

INTRODUTION

Let $\mathbb{F}[x,y]$ be the set of all polynomials with coefficients in $\mathbb{F}$ and in the variables x and y where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Consider the planar system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$ and P, $Q \in \mathbb{R}[x,y]$. We define the degree of a system (1.1) as $\max\{\deg P, \deg Q\}$. In the case where the polynomials P and Q are relatively prime i. e. they do not have a non-constant common factor, we say that (1.1) is *non-degenerate.*

Consider

$$\chi = P(x,y)\,\frac{\partial}{\partial x} + Q(x,y)\,\frac{\partial}{\partial y} \tag{1.2}$$

the polynomial vector field associated to (1.1).

A real *quadratic differential system* is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a},x,y) + p_2(\tilde{a},x,y) \equiv p(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(\tilde{a},x,y) + q_2(\tilde{a},x,y) \equiv q(\tilde{a},x,y) \end{aligned} \tag{1.3}$$

with $\max\{\deg p, \deg q\} = 2$ and

$$\begin{aligned} p_0 &= a, \quad p_1(\tilde{a},x,y) = cx + dy, \quad p_2(\tilde{a},x,y) = gx^2 + 2hxy + ky^2, \\ q_0 &= b, \quad q_1(\tilde{a},x,y) = ex + fy, \quad q_2(\tilde{a},x,y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a,c,d,g,h,k,b,e,f,l,m,n)$ the 12-tuple of the coefficients of system (1.3). Thus a quadratic system can be identified with a point $\tilde{a}$ in $\mathbb{R}^{12}$.

The class of all quadratic differential systems is denoted by **QS**.

Planar polynomial differential systems appear in various branches of applied mathematics, in modeling natural phenomena, such as, modeling the time evolution of conflicting species, in biology, in chemical reactions, in economics, in astrophysics, in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (see, for example (LOTKA, 1920), (VOLTERRA, 1931), (CHANDRASEKHAR, 1939) and (ROTH, 1969)). Polynomial systems occur also in shock waves, in neural networks etc. This kind of differential systems have also theoretical importance since several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900, the problem of algebraic integrability stated by Poincaré in 1891 (POINCARÉ, 1891a), (POINCARÉ, 1891b), the problem of the center stated by Poincaré in 1885 (POINCARÉ, 1885), and problems on integrability resulting from the work of Darboux (DARBOUX, 1878) published in 1878. Except for the problem of the center for quadratic differential systems, which was solved, all the other problems mentioned above, are still unsolved even in the quadratic case.

Although the theory of Darboux is for complex differential systems, we can use it also for real systems. Every system (1.1) with real coefficients yields also a complex system by considering $x, y \in \mathbb{C}$. The integrability theory of Darboux is based on the notion of invariant algebraic curve.

Definition 1. (DARBOUX, 1878) An algebraic curve $f(x,y) = 0$ with $f(x,y) \in \mathbb{C}[x,y]$ is called an *invariant algebraic curve* of system (1.1) if it satisfies the following identity:

$$f_x P + f_y Q = K f, \tag{1.4}$$

for some $K \in \mathbb{C}[x,y]$ where f_x and f_y are the derivative of f with respect to x and y. K is called the *cofactor* of the curve $f = 0$.

For simplicity we write the curve f instead of the curve $f = 0$ in $\mathbb{C}[x,y]$. Note that if system (1.1) has degree m then the cofactor of an invariant algebraic curve f of the system has degree $m-1$.

Definition 2. (DARBOUX, 1878) Consider a planar polynomial system (1.1). An algebraic solution of (1.1) is an algebraic invariant curve f which is irreducible over $\mathbb{C}$.

Definition 3. Let U be an open subset of $\mathbb{R}^2$. A real function $H \colon U \to \mathbb{R}$ is a *first integral* of system (1.1) if it is constant on all solution curves $(x(t), y(t))$ of system (1.1), i.e., $H(x(t), y(t)) = k$, where k is a real constant, for all values of t for which the solution $(x(t), y(t))$ is defined on U. If H is differentiable in U then H is a first integral on U if and only if

$$H_x P + H_y Q = 0. \tag{1.5}$$

Observation 4. Any system has a constant first integral.

Definition 5. Let U be an open subset of $\mathbb{R}^2$ and a consider differential system (not necessarily polynomial). We say the system is *integrable* on U if there exists a first integral which is nonconstant in any open subset of U.

The problem of integrating a polynomial system by using its algebraic invariant curves over $\mathbb{C}$ was considered for the first time by Darboux in (DARBOUX, 1878).

Theorem 6. (DARBOUX, 1878) Suppose that a polynomial system (1.1) of degree n has m invariant algebraic curves $f_i(x,y)=0$, $i\leq m$, with $f_i\in\mathbb{C}[x,y]$ and with $m>n(n+1)/2$ where n is the degree of the system. Then we can compute complex numbers $\lambda_1,\ldots,\lambda_m$ not all zero such that $f_1^{\lambda_1}\ldots f_m^{\lambda_m}$ is a first integral of the system.

Definition 7. If a system (1.1) has a first integral of the form

$$H(x,y)=f_1{}^{\lambda_1}\ldots f_p{}^{\lambda_p} \tag{1.6}$$

where f_i are the invariant algebraic curves of system (1.1) and $\lambda_i\in\mathbb{C}$ not all zero then we say that system (1.1) is *Darboux integrable* and we call the function H a *Darboux function*.

Consider the divergence of system (1.1) defined as $\operatorname{div}(P,Q)=P_x+Q_y$. The next definition leads to a generalization of the notion of Darboux integrability.

Definition 8. Let U be an open subset of $\mathbb{R}^2$ and let $R:U\to\mathbb{R}$ be a differentiable function which is not identically zero on U. The function R is an integrating factor of a polynomial system (1.1) on U if one of the following two equivalent conditions holds:

$$\operatorname{div}(RP,RQ)=0,\quad R_xP+R_yQ=-R\operatorname{div}(P,Q),$$

on U.

A first integral H of

$$\dot{x}=RP,\quad \dot{y}=RQ$$

associated to the integrating factor R is given by

$$H(x,y)=\int R(x,y)P(x,y)dy+h(x),$$

where $H(x,y)$ is a function satisfying $H_x=-RQ$. Then,

$$\dot{x}=H_y,\quad \dot{y}=-H_x.$$

In order that this function H be well defined the open set U must be simply connected.

The condition in Darboux' theorem is only sufficient for Darboux integrability and it is not always necessary. For instance, consider the system

$$\begin{cases}\dot{x}=3+2x^2+xy\\ \dot{y}=3+xy+2y^2.\end{cases} \tag{1.7}$$

This system admits the invariant line $x-y=0$ and the invariant hyperbola $2+xy=0$. Then, $m=2<3=n(n+1)/2$. However we still have here a Darboux first integral $H(x,y)=(x-$

$y)^{-3/2}(2+xy)$. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability stated in the theorem of Darboux is in general greater than necessary. The following question arises then naturally: *Could we find a necessary and sufficient condition for Darboux integrability?*

The theory of Darboux has been improved by including for example independent singular points, exponential factor and the multiplicity of invariant algebraic curves (see (LLIBRE; ZHANG, 2009a)). But a deeper understanding about Darboux integrability is still lacking.

The simplest integrable systems (1.1) are the Hamiltonian ones having a polynomial first integral. Next we have the systems (1.1) which admit a rational first integral. These were called by Poincaré algebraically integrable systems. In (POINCARÉ, 1891a) and (POINCARÉ, 1891b), Poincaré stated the following problem (which remains open): *Can we recognize when a system* (1.1) *admits a rational first integral?*

To advance knowledge on algebraic or more general Darboux integrability it is useful to have a large number of examples to analyze. In the literature, scattered isolated examples were analyzed but a more systematic approach is still needed.

This more systematic approach was initiated in the papers of Schlomiuk and Vulpe (SCHLOMIUK; VULPE, 2008b), (SCHLOMIUK; VULPE, 2008c), (SCHLOMIUK; VULPE, 2008a), (SCHLOMIUK; VULPE, 2008d) and (SCHLOMIUK; VULPE, 2004) where they classified topologically the phase portraits of quadratic systems with invariant lines of at least four total multiplicity as well as the quadratic systems with the line at infinity filled up with singularities and proved their Liouvillian integrability.

In (OLIVEIRA *et al.*, 2017) the authors classified the family **QSH** of non-degenerate quadratic differential systems possessing an invariant hyperbola according to “configurations of invariant hyperbolas and lines". They proved that the family **QSH** is geometrically rich as it has 205 distinct configurations of invariant hyperbolas (see Chapter 3). The authors did not study the integrability of the systems in this family. We do this in this work. This family is very interesting since it displays a considerable amount of systems of various kinds of integrability as we see in the next chapters. This thesis is motivated by the desire to explore the relationship between the integrability according to the theory of Darboux and the geometric properties of the configurations of invariant curves of a system. We believe that the data and results collected in our work will be pertinent for the deeper exploration of the Darboux theory of integrability.

The work is organized as follows:

In Chapter 2 we give an overview of Darboux theory, including all essential new notions not used in Darboux’ work, as well as new results, extensions of his theory.

In Chapter 3 we discuss the class **QSH**, that is, the class of non-degenerate planar quadratic systems possessing at least one invariant hyperbola. We list all the normal forms of **QSH** (given in (OLIVEIRA *et al.*, 2017)) and we explain briefly how they were split. In

(OLIVEIRA *et al.*, 2017) the authors also calculated the invariant algebraic curves (lines and hyperbolas) of each normal form in **QSH**.

In Chapter 4 we introduce a number of geometrical concepts which are very helpful in understanding the relation between the geometry of the configuration of invariant algebraic curves and the integrability of the systems.

In Chapter 5 we present in section 5.1 tables containing the invariant algebraic curves, exponential factors, cofactors and integrating factors or first integrals (whenever they can be calculated) for each one of the normal forms in **QSH**. We also display in these tables the corresponding normal forms in (OLIVEIRA *et al.*, 2017) as well as the configurations of invariant algebraic curves and we give information about their integrability. In section 5.2 we prove the non-integrability for the cases where the number of invariant curves and exponential factors were not enough to find a first integral or integrating factor for normal forms in **QSH**.

In Chapter 6 we present a detailed geometric analysis for 22 normal forms in **QSH**. We exhibit the bifurcation diagrams of the configurations of invariant algebraic curves as well as the bifurcation diagrams of the systems and study the interactions between these two kinds of bifurcations. Phase portraits for quadratic system with an invariant hyperbola and an invariant straight line were also constructed in (LLIBRE; YU, 2018). However, we point out that the authors of (LLIBRE; YU, 2018) did not get all of the phase portraits. This is due to the fact that their normal form for this family misses some of the systems in the family. In this chapter we point out 16 missing phase portraits. We also point out 1 missing phase portraits in (CAIRÓ; FEIX; LLIBRE, 1999), 3 missing configurations in (OLIVEIRA *et al.*, 2017) and 1 missing configuration in (SCHLOMIUK; VULPE, 2008c). Furthermore, we solve the Poincaré problem of algebraic integrability for 6 of the families studied.

In Chapter 7 we highlight some significant points raised in this work, explain the relation between the bifurcations of configurations of invariant curves and topological bifurcations, raise a number of questions and state some problems.

The main results of our work are given in section 3.2 and in Chapter 6.

Interested in studying the integrable systems in **QSH** from the topological, dynamical and algebraic geometric viewpoints, we perform the study of three normal forms (H), (J) and (O) (see section 3.1) of **QSH**. We construct their topological bifurcation diagrams as well as the bifurcation diagrams of their configurations of invariant hyperbolas and lines and point out the relationship between them. We give a global answer to the problem of Poincaré for the normal form (H) by producing a Diophantine geometric necessary and sufficient condition for a system in this family to have a rational first integral. This is content of the following paper

OLIVEIRA, R.; SCHLOMIUK, D.; TRAVAGLINI, A.M. Geometry and integrability of quadratic systems with invariant hyperbolas. ***Electronic Journal of Qualitative Theory of Differential Equations****, v. 2021, n.6, p. 1-56, 2021.*

The theory of integrability of Darboux has been very much extended and it is now an active area of research. In surveys it is not usually presented following its conceptual historical evolution and its significant connections to Poincaré's problem. Motived by this fact we give, in a concise way, following the history of the subject, its conceptual development. This is the first main goal of the paper

OLIVEIRA, R.; SCHLOMIUK, D.; TRAVAGLINI, A.M.; VALLS, C. Geometry, integrability and bifurcation diagrams of a family of quadratic differential systems as application of the Darboux theory of integrability. ***Submitted****, 2021.*

Our second goal in this paper is first to display the many aspects of the theory of Darboux as we have today, by using it for studying the special family of planar quadratic differential systems possessing an invariant hyperbola, and having either two singular points at infinity or the infinity filled up with singularities. We investigate the integrability for the normal forms of this class and we perfom the geometric analysis for the families (P), (Q) (see section 3.1). Furthermore, we study the interaction between bifurcation of configurations of invariant algebraic curves and the bifurcation of phase portraits for each one of the normal forms considered. Finally, we solve the problem of Poincaré of algebraic integrability for some of the normal forms we studied.

We have a third work, in progress,

OLIVEIRA, R.; SCHLOMIUK, D.; TRAVAGLINI, A.M.; VALLS, C. The interplay between the geometry and the Darboux integrability of a family of quadratic differential systems with invariant hyperbolas. ***Preprint****, 2021.*

Our objective in this paper is to present the investigation of the integrability of the class of planar quadratic differential systems possessing an invariant hyperbola, and having three singular points at infinity and its geometric analysis. The interplay between the geometry of the systems in this class and their integrability, as well as investigate the solutions for Poincaré's problem for this family is the main target of this work.

CHAPTER

2

THE EXTENSIONS OF DARBOUX THEORY

Poincaré was enthustiastic about the work of Darboux (DARBOUX, 1878) which he called “oeuvre magistrale" in (POINCARÉ, 1891b). He stated the problem of algebraic integrability which asks to recognize when a polynomial vector field has a rational first integral. In further exploring the evolution of ideas and development of the theory of Darboux it is important to mention the connections between this theory and the problem of the center also stated by Poincaré in (POINCARÉ, 1885). These connections have done much to draw attention to the theory of Darboux and its unifying power in proving integrability of families of polynomial systems. We indicate here some of these connections as well as the story of the solution of the problem of the center for quadratic systems and in proving their integrability in a unified way by the method of Darboux.

For quadratic systems the problem of the center was solved by Dulac. But unlike Poincaré, Dulac considered differential systems defined over $\mathbb{C}$. In (DULAC, 1904) he defined the following notion of center: *A singular point of a planar holomorphic differential system with non-zero eigenvalues is a center if and only if the quotient of its eigenvalues is negative and rational and the system has a local analytic first integral.* In his paper (DULAC, 1908), Dulac mentions that the general case is more difficult to treat, he supposes that the quotient of the eigenvalues is -1. Placing the singular point at the origin, he used the following normal form for quadratic systems:

$$\begin{aligned} \dot{x} &= x + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \dot{y} &= -y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned}$$

To solve the problem of the center for quadratic systems means to find necessary and sufficient conditions in terms the coefficients a_{ij} and b_{ij} so that the origin be a center. He solved this problem in (DULAC, 1908) by using the method of integration of Darboux.

This work of Dulac could not be readily applied for real systems. Indeed, in the normal form considered by Dulac, if we assume that the coefficients of the equations are real than this real system has a saddle at the origin and we cannot pass from this normal form to the normal form used by Poincaré (where the linear terms of the two equations are respectively $-y$, x) by a real linear transformation. Thus the conditions for the center obtained by Dulac cannot be readily used in the case of real systems for centers as defined by Poincaré.

The first major result after the publication of the work of Darboux followed up by the work of Poincaré and Dulac was a result of Jouanolou published a century after Darboux' work. In (JOUANOLOU, 1979) Jouanolou gave a sufficient condition for algebraic integrability.

Theorem 9. (JOUANOLOU, 1979) Consider a polynomial system (1.1) of degree n and suppose that it admits m invariant algebraic curves $f_i(x,y) = 0$ where $f_i \in \mathbb{C}[x,y]$ and $1 \leq i \leq m$, then if $m \geq 2 + \frac{n(n+1)}{2}$, there exists integers $N_1, N_2, \ldots, N_m$ not all zero such that $I(x,y) = \prod_{i=1}^{m} f_i^{N_i}$ is a first integral of (1.1).

In the previous chapter we mentioned only three types of first integrals: polynomial, rational and Darboux first integrals (which could be rational or transcendental). But we have other types of first integrals in this hierarchy, for instance, the *elementary first integrals*. Roughly speaking these are functions which are constructed by using addition, multiplication, composition of finitely many rational functions, trigonometric and exponential functions and their inverses. In general, elementary first integrals are defined in the context of differential algebra (for more details see (CHRISTOPHER; LLIBRE, 1999)).

Definition 10. 1) A *derivation* over a ring A is an operation $\delta : A \to A$ such that, for all $x, y \in A$ we have:

$$\delta(x+y) = \delta(x) + \delta(y), \quad \delta(xy) = \delta(x)y + x\delta(y).$$

2) A *differential field* is a pair (F, δ) where F is a field and δ is a derivation $\delta : F \to F$.

3) A set of differential fields (F_i, δ_i) where $i \in \{0, 1, \ldots, n\}$ is called a *tower of differential fields* if

$$F_0 \subset F_1 \subset \ldots \subset F_n$$

and $\delta_i : F_i \to F_i$ where $\delta_{i-1} = \delta_i\,|_{F_{i-1}}$ for all $i \in \{0, 1, \ldots, n\}$.

This next tower of fields, arise by adding exponentials, logarithms or the solutions of algebraic equations based on the previous set of functions.

Definition 11. Consider the tower of fields $F_i = F_0(\theta_1, \ldots, \theta_i)$, where one of the following holds:

(i) $\delta\theta_i = \theta_i \delta g$, for some $g \in F_{i-1}$ and for each derivation δ.

(ii) $\delta\theta_i = g^{-1}\delta g$, for some $g \in F_{i-1}$ and for each derivation δ.

(iii) θ_i is algebraic over F_{i-1}.

We say that F is an elementary extension of F_0 if there exists a tower of differential fields (F_i, δ_i) where $i \in \{0, 1, ..., n\}$ such that $F = F_n$.

Definition 12. The set of all elements of a differential field which are annihilated by all the derivations of the field is called *field of constants*.

We shall always assume that the field of constants is algebraically closed.

Definition 13. We say that system (1.1) has an elementary first integral if there is an element u in an elementary extension field of the field of rational functions $\mathbb{C}(x,y)$ with the same field of constants such that $\delta u = 0$.

The derivations on $\mathbb{C}(x,y)$ are of course d/dx and d/dy.

The next result was obtained by Prelle and Singer in 1983 and it involves elementary first integrals. The original result was stated for more general vector fields in $\mathbb{C}^n$ in differential algebra language. Here we consider only the case of planar differential systems (1.1).

Theorem 14. (PRELLE; SINGER, 1983) If a polynomial differential system (1.1) has an elementary first integral, then the system has a first integral of the following form:

$$f(x,y) + c_1 log(f_1(x,y)) + c_2 log(f_2(x,y)) + ... + c_k log(f_k(x,y))$$

where f and f_i, are algebraic functions over $\mathbb{C}(x,y)$ and $c_i \in \mathbb{C}$, $i = 1, 2, \ldots k$.

Taking the exponential of the above expression we obtain the following Corollary:

Corolário 1. If a polynomial differential system (1.1) possesses an elementary first integral then it also admits a first integral of the form:

$$e^{f(x,y)} f_1(x,y)^{c_1} f_2(x,y)^{c_2} \ldots f_k^{c_k}.$$

where f and f_i, are algebraic functions over $\mathbb{C}(x,y)$ and $c_i \in \mathbb{C}$, $i = 1, 2, \ldots k$.

The expression for the first integral in this result differs from a Darboux first integral by the exponential factor $e^{f(x,y)}$ which appears in the first integral for the first time, though not explicitly, in Prelle-Singer's paper and also f_i's are here algebraic and not just polynomials over $\mathbb{C}$.

In (PRELLE; SINGER, 1983) Prelle and Singer talk about "Algorithmic considerations" and they say:

The preceding work was motivated by our desire to develop a decision procedure for finding elementary first integrals. These results show that we need only look for elementary integrals of a prescribed form. In this section we shall discuss the problem of finding an elementary *first integral for a two-dimensional autonomous system of differential equations and reduce this problem to that of bounding the degrees of algebraic solutions of this system.*

Their algorithm was based on the following two propositions:

Proposition 15. (PRELLE; SINGER, 1983) If the planar system (1.1) has an elementary first integral, then there exists an integer n and an invariant algebraic curve f such that

$$Pf_x + Qf_y = -n(P_x + Q_y) f.$$

Proposition 16. If the equations of (1.1) have an elementary first integral, then there exists an element R algebraic over $\mathbb{C}(x,y)$ such that $R_xP + R_yQ = -(P_x + Q_y)R$.

The next result is a version of the Prelle-Singer algorithm provided in (GORIELY, 2001).

Theorem 17. (PRELLE; SINGER, 1983), (GORIELY, 2001)

(1) Let $N = 1$.

(2) Find all the invariant algebraic curves $C : f(x,y) = 0$ with

$$Pf_x + Qf_y = Kf$$

such that $K(x,y) \in \mathbb{C}[x,y]$ and $deg(f) \leq N$.

(3) Decide if there exist constants $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{C}$, not all zero, such as

$$\sum_{i=0}^{m} \lambda_i K_i = 0,$$

where K_i is cofactor of a curve f_i found in (2). If such λ_i's exist, then $I = \prod_{i=0}^{m} f_i^{\lambda_i}$ is a first integral. Otherwise, go to (4).

(4) Decide if there exist constants $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{C}$, not all zero, such as

$$\sum_{i=0}^{m} \lambda_i K_i = -(P_x + Q_y),$$

where K_i is cofactor of a curve f_i found in (2).

If such λ_i's exist, then $R = \prod_{i=0}^{m} f_i^{\lambda_i}$ is an integrating factor and a first integral can be obtained by integrating the equations:

$$I_x = RQ$$
$$I_y = -RP.$$

If such λ_i's do not exist, return to (1) increasing N by 1 and continue the process.

There is still another type of first integrals we need to mention, the *Liouvillian first integrals*. In (SINGER, 1992) Singer describes Liouvillian functions as follows: Liouvillian functions are functions that are built up from rational functions using exponentiation, integration, and algebraic functions.

In general, Liouvillian functions are defined in the context of differential algebra (for more details see (CHRISTOPHER; LLIBRE, 1999)).

Definition 18. We say that an extension F_n is a *Liouvillian* extension of F_0 if there is a tower of differential fields as in Definition 11 which satisfies conditions (i), (iii) or

(ii)$'$ $\delta_k\theta_i = h_k$ for some elements $h_k \in F_{i-1}$ such that $\delta_k h_j = \delta_j h_k$.

This last condition, mimics the introduction of line integrals into the class of functions. Clearly (ii) is included in (ii)$'$.

This class of functions represents those functions which are obtainable “by quadratures”.

Definition 19. An element u of a Liouvillian extension field of $\mathbb{C}(x,y)$ with the same field of constants is said to be a *Liouvillian* first integral.

The following result was proved by Singer in 1992.

Theorem 20. (SINGER, 1992) If the system (1.1) has a Liouvillian first integral, then it has an integrating factor of the form

$$e^{\int Udx+Vdy}, \quad U_y = V_x,$$

where U and V are rational functions over $\mathbb{C}[x,y]$.

A consequence of Singer’s theorem is the following.

Corolário 2. (SINGER, 1992) A system of differential equations (1.1) has a Liouvillian first integral if and only if it has an integrating factor of the form

$$R(x,y) = e^{\int Udx+Vdy}, \;\; U_y = V_x \;\; (U,V \text{ are rational function over } \mathbb{C}[x,y])$$

in which case

$$F(x,y) = \int R(x,y)Q(x,y)dx - R(x,y)P(x,y)dy$$

is a Liouvillian first integral.

It is important to mention that a Liouvillian integrable system does not necessarily have an affine (finite) invariant algebraic curve. An example of such a polynomial differential system is presented in (GINÉ; LLIBRE, 2012).

The following notion was defined by Christopher in 1994 (see (CHRISTOPHER, 1994)) where he called it “degenerate invariant algebraic curve".

Definition 21. Let $F(x,y) = \exp\left(\dfrac{G(x,y)}{H(x,y)}\right)$ with G, $H \in \mathbb{C}[x,y]$ coprime. We say that F is an *exponential factor* of system (1.1) if it satisfies the equality

$$F_x P + F_y Q = LF, \tag{2.1}$$

for some $L \in \mathbb{C}[x,y]$. The polynomial L is called the *cofactor* of the exponential factor F.

Proposition 22. (CHRISTOPHER, 1994) If $F = \exp(G/H)$ is an exponential factor of system (1.1) with cofactor L then $H = 0$ is an invariant algebraic curve of the system (1.1) with cofactor K_H and G satisfies the equation

$$PG_x + QG_y = K_H G + LH, \text{ where } G, H, L, K_H \in \mathbb{C}[x,y]. \tag{2.2}$$

See (CHRISTOPHER; LLIBRE, 2000) for a detailed proof.

We need the following concept for the next result.

If $S(x,y) = \sum_{i+j=0}^{m-1} a_{ij} x^i y^j$ is a polynomial of degree at most $m-1$ with $m(m+1)/2$ coefficients in $\mathbb{C}$, then we write $S \in \mathbb{C}_{m-1}[x,y]$. We identify the linear space $\mathbb{C}_{m-1}[x,y]$ with $\mathbb{C}^{m(m+1)/2}$ through the isomorphism

$$S \to (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1}).$$

Definition 23. (CHAVARRIGA; LLIBRE; SOTOMAYOR, 1997) We say that r singular points $(x_k, y_k) \in \mathbb{C}^2$, $k = 1, \ldots, r$ of a differential system (1.1) of degree m are independent with respect to $\mathbb{C}_{m-1}[x,y]$ if the intersection of the r hyperplanes

$$\sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, \; k = 1, \ldots, r,$$

in $\mathbb{C}^{m(m+1)/2}$ is a linear subspace of dimension $[m(m+1)/2] - r$.

We remark that the maximum number of isolated singular points of the polynomial system (1.1) of degree m is m^2 (by Bézout's Theorem), that the maximum number of independent isolated singular points of the system is $m(m+1)/2$, and that $m(m+1)/2 < m^2$ for $m \geq 2$.

The next result we present is an extension of the theory of Darboux involving the notion of independent singular points. This result is due to Chavarriga, Llibre and Sotomayor (see (CHAVARRIGA; LLIBRE; SOTOMAYOR, 1997)).

Theorem 24. (CHAVARRIGA; LLIBRE; SOTOMAYOR, 1997) Assume that a real (complex) polynomial system of degree m admits $q = m(m+1)/2 + 1 - p$ algebraic solutions $f_i = 0$, $i = 1, 2, \ldots q$, not passing through p real (complex) independent singular points (x_k, y_k), $k = 1, 2, \ldots p$, then the system has a first integral of the form $f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_q^{\lambda_q}$ with $\lambda_i \in \mathbb{R}$ $(\mathbb{C})$.

Observation 25. (i) This result is interesting because it reduces the number of invariant algebraic curves we need to have when compared with Darboux' Theorem.

(ii) The theory of Darboux was formulated by Darboux over the complex projective space. In (CHAVARRIGA; LLIBRE; SOTOMAYOR, 1997) as in the other extensions of Darboux theory we mention in this thesis, we deal with the affine Darboux theory done over $\mathbb{C}$. As every system (1.1) with real coefficients generates a complex system we can apply Darboux theory to systems with real coefficients and even obtain integrability results leading to a real first integral.

Example 26.

$$\begin{cases} \dot{x} = 3 + 2x^2 + xy \\ \dot{y} = 3 + xy + 2y^2. \end{cases}$$

The line $f_1(x,y) = x - y = 0$ and the hyperbola $f_2(x,y) = 2 + xy = 0$ are invariant for this system with cofactors $K_1(x,y) = 2x + 2y$ and $K_2(x,y) = 3x + 3y$. Here $m = 2 = n$ and hence $m < n(n+1)/2$. Still, the number of curves suffices to compute the first integral $H(x,y) = (x - y)^{-3/2}(2 + xy)$ although the condition in the theorem of Darboux is not satisfied by this number. But here we have that the singular points $P_{1,2} = \pm(-i\sqrt{3}, i\sqrt{3})$ of the system are independent. Indeed, solving the system $H_1 = a_{00} - i\sqrt{3}a_{10} + i\sqrt{3}a_{01} = 0$, $H_2 = a_{00} + i\sqrt{3}a_{10} - i\sqrt{3}a_{01} = 0$, we get $a_{00} = 0$ and $a_{10} = a_{01}$ and hence $\dim(H_1 \cap H_2) = 1$. Also $f_1(P_i) \neq 0$ and $f_2(P_i) \neq 0$. So the points P_i's are independent. Applying the above theorem we have $q = 2$, $p = 2$, $n = 2$ and we have $q = n(n+1)/2 + 1 - p$.

Definition 27. A singular point (x_0, y_0) of system (1.1) is called weak if the divergence of system (1.1) at (x_0, y_0) is zero.

The next result is also a generalization of Darboux's theorem, now taking into account exponential factors, independent points and invariants. This result was stated and proved by Christopher and Llibre in 2000 (CHRISTOPHER; LLIBRE, 2000). An earlier version appeared in (CAIRÓ; FEIX; LLIBRE, 1999).

Theorem 28. (CHRISTOPHER; LLIBRE, 2000) Suppose that a $\mathbb{C}-$polynomial system (1.1) of degree m admits p algebraic solutions $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$, q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$, and r independent singular points $(x_k, y_k) \in \mathbb{C}^2$ such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \dots, p$ and for $k = 1, \dots, r$.

(i) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

if and only if the (multi-valued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} \tag{2.3}$$

is a first integral of system (1.1).

(ii) If $p+q+r \geq [m(m+1)/2]+1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=1}^{q} \mu_j L_j = 0.$$

(iii) If $p+q+r \geq [m(m+1)/2]+2$, then system (1.1) has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.

(iv) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P,Q),$$

if and only if function (2.3) is an integrating factor of system (1.1).

(v) If $p+q+r = m(m+1)/2$ and the r independent singular points are weak, then function (2.3) is a first integral if

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=1}^{q} \mu_j L_j = 0,$$

or an integrating factor if

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P,Q),$$

under the condition that not all $\lambda_i, \mu_j \in \mathbb{C}$ are zero.

(vi) If there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$$

for some $s \in \mathbb{C} \backslash \{0\}$, then the (multi-valued) function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q} \exp(st) \tag{2.4}$$

is an invariant of system (1.1).

Of course, each irreducible factors of each h_j is one of the f_i's.

Definition 29. If system (1.1) has a first integral of the form

$$H(x,y) = f_1{}^{\lambda_1} \ldots f_p{}^{\lambda_p} F_1{}^{\mu_1} \ldots F_q{}^{\mu_q} \tag{2.5}$$

where f_i and F_j are respectively the invariant algebraic curves and exponential factors of a system (1.1) and $\lambda_i, \mu_j \in \mathbb{C}$, then we say that the system is *generalized Darboux integrable*. We call the function H a *generalized Darboux function*.

Observation 30. In (DARBOUX, 1878) Darboux considered functions of the type (1.6), not of type (2.5). In recent works functions of type (2.5) were called Darboux functions. Since in this work we need to pay attention to the distinctions among the various kinds of first integral we call (1.6) a Darboux and (2.5) a generalized Darboux first integral.

Proposition 31. (DUMORTIER; LLIBRE; ARTÉS, 2006) For a real polynomial system (1.1) the function $\exp(G/H)$ is an exponential factor with cofactor K if and only if the function $\exp(\overline{G}/\overline{H})$ is an exponential factor with cofactor $\overline{K}$.

Observation 32. (DUMORTIER; LLIBRE; ARTÉS, 2006) If among exponential factors of the real system (1.1) a complex pair $F = \exp(G/H)$ and $\overline{F} = \exp(\overline{G}/\overline{H})$ occurs, then the first integral (2.5) has a real factor of the form

$$(\exp(G/H))^{\mu}\left(\exp(\overline{G}/\overline{H})\right)^{\overline{\mu}} = \exp\left(2\,Re(\mu(G/H))\right),$$

where $\mu \in \mathbb{C}$ and $Im(\mu)Im(F) \neq 0$. This means that function (2.5) is real when system (1.1) is real.

Considering the definition of generalized Darboux function we can rewrite Corollary 2 as follows.

Theorem 33. (SINGER, 1992), (CHRISTOPHER, 1994) A planar polynomial differential system (1.1) has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.

For a proof see also (ZHANG, 2017), page 134.

We can also state easily the following result of Preller-Singer.

Theorem 34. (PRELLE; SINGER, 1983), (CHAVARRIGA *et al.*, 2003) If a planar polynomial vector field (1.2) has a generalized Darboux first integral, then it has a rational integrating factor.

In 2019, a converse of the previous result was proved in (CHRISTOPHER *et al.*, 2019) as a consequence of (ROSENLICHT, 1976).

Theorem 35. (CHRISTOPHER *et al.*, 2019) If a planar polynomial vector field (1.2) has a rational integrating factor, then it has a generalized Darboux first integral.

We have the following table summing up these results.

First integral		Integrating factor
Generalized Darboux	$\Leftrightarrow$	Rational
Liouvillian	$\Leftrightarrow$	Generalized Darboux

To study how the integrability change within families of polynomial differential systems we must consider perturbations in a system of such a family. For instance, if we have a system possessing two invariant algebraic curve we could have, after a perturbation, that one of this invariant curves splits in several others. Or we could also have the coalescence of these two invariant curves. So there arises the necessity of the concept that today is known as *multiplicity of an invariant algebraic curve*.

Suppose that a polynomial differential system has an algebraic solution $f(x,y)=0$ where $f(x,y)\in\mathbb{C}[x,y]$ is of degree n given by

$$f(x,y)=c_0+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2+\ldots+c_{n0}x^n+c_{n-1,1}x^{n-1}y+\ldots+c_{0n}y^n,$$

with $\hat{c}=(c_0,c_{10},\ldots,c_{0n})\in\mathbb{C}^N$ where $N=(n+1)(n+2)/2$. Consider $P_{N-1}(\mathbb{C})$ the complex projective space of degree $N-1$. We note that the equation

$$\lambda f(x,y)=0,\ \lambda\in\mathbb{C}^*=\mathbb{C}-\{0\}$$

yields the same locus of complex points in the plane as the locus induced by $f(x,y)=0$. Therefore, a curve of degree n is defined by $\hat{c}$ where

$$[\hat{c}]=[c_0:c_{10}:\ldots:c_{0n}]\in P_{N-1}(\mathbb{C}).$$

We say that a sequence of curves $f_i(x,y)=0$, each one of degree n, *converges* to a curve $f(x,y)=0$ if and only if the sequence of points $[c_i]=[c_{i0}:c_{i10}:\ldots:c_{i0n}]$ converges to $[\hat{c}]=[c_0:c_{10}:\ldots:c_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t'=\lambda t$ by a positive constant λ the geometry of the systems (1.1) (phase curves) does not change. So for our purposes we can identify a system (1.1) of degree n with a point

$$[a_0:a_{10}:\ldots:a_{0n}:b_0:b_{10}:\ldots:b_{0n}]\in\mathbb{S}^{N-1}(\mathbb{R})$$

where $N=(n+1)(n+2)$. We compactify the space of all the polynomial differential systems of degree n on $\mathbb{S}^{N-1}$ by multiplying the coefficients of each system with $1/(\sum(a_{ij}^2+b_{ij}^2))^{1/2}$.

Definition 36. (SCHLOMIUK; VULPE, 2004)

(1) We say that an invariant curve

$$\mathscr{L}:f(x,y)=0,\ f\in\mathbb{C}[x,y]$$

for a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N=(n+1)(n+2)$ such that each (S_k) has m distinct invariant curves

$$\mathscr{L}_{1,k}:f_{1,k}(x,y)=0,\ldots,\mathscr{L}_{m,k}:f_{m,k}(x,y)=0$$

over $\mathbb{C}$, $deg(f)=deg(f_{i,k})=r$, converging to $\mathscr{L}$ as $k\to\infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R=(r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity

$$\mathscr{L}_\infty : Z = 0$$

of a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N = (n+1)(n+2)$ such that each (S_k) has $m-1$ distinct invariant lines

$$\mathscr{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathscr{L}_{m-1,k} : f_{m-1,k}(x,y) = 0$$

over $\mathbb{C}$, converging to the line at infinity $\mathscr{L}_\infty$ as $k \to \infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In 2007 the authors of (CHRISTOPHER; LLIBRE; PEREIRA, 2007) introduced the following notion of geometric multiplicity:

Definition 37. (CHRISTOPHER; LLIBRE; PEREIRA, 2007) Consider χ a polynomial vector field of degree d. An invariant algebraic curve $f = 0$ of degree n of the vector field χ has geometric multiplicity m if m is the largest integer for which there exists a sequence of vector fields $(\chi_i)_{i>0}$ of bounded degree, converging to $h\chi$, for some polynomial h, not divisible by f, such that each χ_r has m distinct invariant algebraic curves, $f_{r,1} = 0, f_{r,2} = 0, \ldots, f_{r,m} = 0$, of degree at most n, which converge to $f = 0$ as r goes to infinity. If $h = 1$, then we say that the curve has strong geometric multiplicity m.

Definition 38. (PEREIRA, 2001), (CHRISTOPHER; LLIBRE; PEREIRA, 2007) Let $\mathbb{C}_m[x,y]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x,y]$ of degree at most m and of dimension $R = \binom{2+m}{2}$. Let $\{v_1, v_2, \ldots, v_R\}$ be a base of $\mathbb{C}_m[x,y]$. We denote by $M_R(m)$ the $R \times R$ matrix

$$M_R(m) = \begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \chi(v_1) & \chi(v_2) & \ldots & \chi(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \chi^{R-1}(v_1) & \chi^{R-1}(v_2) & \ldots & \chi^{R-1}(v_R) \end{pmatrix}, \tag{2.6}$$

where $\chi^{k+1}(v_i) = \chi(\chi^k(v_i))$. The mth extactic curve of χ, $\mathscr{E}_m(\chi)$, is given by the equation $\det M_R(m) = 0$. We also call $\mathscr{E}_m(\chi)$ the mth extactic polynomial.

From the properties of the determinant we note that the extactic curve is independent of the choice of the base of $\mathbb{C}_m[x,y]$.

Theorem 39. (PEREIRA, 2001) Consider a planar vector field (1.2). We have $\mathscr{E}_m(\chi) = 0$ and $\mathscr{E}_{m-1}(\chi) \neq 0$ if and only if χ admits a rational first integral of exact degree m.

Observe that if $f = 0$ is an invariant algebraic curve of degree m of χ, then f divides $\mathscr{E}_m(\chi)$. This is due to the fact that if f is a member of a base of $\mathbb{C}_m[x,y]$, then f divides the whole column in which f is located.

Definition 40. (CHRISTOPHER; LLIBRE; PEREIRA, 2007) We say that an invariant algebraic curve $f=0$ of degree $m\geq 1$ has algebraic multiplicity k if $\det M_R(m)\neq 0$ and k is the maximum positive integer such that f^k divides $\det M_R(m)$; and it has no defined algebraic multiplicity if $\det M_R(m)\equiv 0$.

Definition 41. (CHRISTOPHER; LLIBRE; PEREIRA, 2007) We say that an invariant algebraic curve $f=0$ of degree $m\geq 1$ has integrable multiplicity k with respect to χ if k is the largest integer for which the following is true: there are $k-1$ exponential factors $\exp(g_j/f^j)$, $j=1,\ldots,k-1$, with $\deg g_j\leq jm$, such that each g_j is not a multiple of f.

In the next result we see that the algebraic and integrable multiplicity coincide if $f=0$ is an irreducible invariant algebraic curve.

Theorem 42. (CHRISTOPHER; LLIBRE; PEREIRA, 2007) Consider an algebraic solution $f=0$ of degree $m\geq 1$ of χ. Then f has algebraic multiplicity k if and only if the vector field (1.2) has $k-1$ exponential factors $\exp(g_j/f^j)$, where $(g_j,f)=1$ and g_j is a polynomial of degree at most jm, for $j=1,\ldots,k-1$.

In (CHRISTOPHER; LLIBRE; PEREIRA, 2007) the authors show that the definitions of geometric, algebraic and integrable multiplicity are equivalent when $f=0$ is an algebraic solution of the vector field (1.2). The algebraic multiplicity has the advantage that we have the possibility of calculating it via the extactic curve and if the curve is irreducible then this coincides with either the integrable (reflected in the exponential factors) or the geometric one. Christopher, Llibre and Pereira also stated and proved the following theorem about Darboux theory of integrability that takes into account the multiplicity of the invariant algebraic curves.

Theorem 43. (CHRISTOPHER; LLIBRE; PEREIRA, 2007) Consider a planar vector field (1.2). Assume that (1.2) has p distinct irreducible invariant algebraic curves $f_i=0$, $i=1,\ldots,p$ of multiplicity m_i, and let $N=\sum_{i=1}^{p} m_i$. Suppose, furthermore, that there are q critical points $p_1,\ldots,p_q$ which are independent with respect to $\mathbb{C}_{m-1}[x,y]$, and $f_j(p_k)\neq 0$ for $j=1,\ldots,p$ and $k=1,\ldots,q$. We have:

(a) If $N+q\geq[m(m+1)/2]+2$, then χ has a rational first integral.

(b) If $N+q\geq[m(m+1)/2]+1$, then χ has a Darboux first integral.

(c) If $N+q\geq[m(m+1)/2]$ and p_1 are weak, then χ has either a Darboux first integral or a Darboux integrating factor.

This theorem was generalized by Llibre and Zhang in (LLIBRE; ZHANG, 2009a) for invariant hypersurfaces in $\mathbb{C}^n$. In the same paper they also generalized the theorem of Jouanolou and gave a simplified, elementary proof.

We consider now the result of Llibre and Zhang in (LLIBRE; ZHANG, 2009b). To state it the authors generalized the Poincaré compactification on the sphere for planar differential systems to the Poincaré compactification of polynomial differential systems in $\mathbb{R}^n$ which they constructed in the Appendix of (LLIBRE; ZHANG, 2009b).

To talk about multiplicity of the hyperplane at infinity they only needed to pass by central projection from the systems in $\mathbb{R}^n$, considered as the hyperplane $Z = 1$ in $\mathbb{R}^{n+1}$ tangent to the n-sphere with radius 1 centered at the origin of $\mathbb{R}^{n+1}$, and then further into the chart $x_1 = 1$ and obtain $(x_1,\dots,x_n,1) = \lambda(1,y_2,\dots,y_n,Z)$ for some non-zero real λ. Hence we must have $\lambda = x_1$ and therefore $y_2 = x_2/x_1,\dots,\ y_n = x_n/x_1$, $Z = 1/x_1$ and $x_1 = 1/Z$, $x_2 = y_2/Z,\dots,\ x_n = y_n/Z$. Transferring the vector field in this chart we obtain that it has a pole on $Z = 0$. In complete analogy with the compactification of the plane we can obtain an analytic vector field on the n-sphere which is conjugate to the vector field thus obtained. In this way our initial hyper-surface at infinity, becomes just an affine hypersurface in the chart $x_1 = 1$ and hence we can apply to it our notions of multiplicity. Let $\overline{\chi} = (P_1(x),P_2(x),...,P_n(x))$ be the expression of the compactified vector field χ. We say that the infinity of χ has algebraic multiplicity k if $Z = 0$ has algebraic multiplicity k for the vector field $\overline{\chi}$; and that it has no defined algebraic multiplicity if $Z = 0$ has no defined algebraic multiplicity for χ. One thing the authors did not say is that this definition of the multiplicity of the infinite hypersurface does not depend on the chart x_1 we chose, and that it leads to the same value if we replace this chart by any other chart $x_i = 1$ with $i \neq 1$.

Theorem 44. (LLIBRE; ZHANG, 2009b) Let $\overline{\chi}$ be the expression of the compactified vector field χ. Assume that χ restricted to $Z = 0$ has no rational first integral. Then $Z = 0$ has algebraic multiplicity k for $\overline{\chi}$ if and only if $\overline{\chi}$ has $k-1$ exponential factors $\exp(\overline{g_j}/Z^j)$ where $j = 1,...,k-1$ with $\overline{g_j} \in \mathbb{C}_j[Z,y_2,...,y_n]$ having no factor Z.

The next result provides a relation between the exponential factors of χ and those of $\overline{\chi}$ associated with $Z = 0$.

Proposition 45. (LLIBRE; ZHANG, 2009b) For the exponential factors associated with the hyperplane at infinity the following statements hold.

(a) If $E = \exp(g(x))$ with g a polynomial of degree k is an exponential factor of χ with cofactor $L_E(x)$, then $\overline{E} = \exp\left(\frac{\overline{g}}{Z^k}\right)$ with $\overline{g} = Z^k g\left(\frac{1}{Z},\frac{y_2}{Z},...,\frac{y_n}{Z}\right)$ is an exponential factor of $\overline{\chi}$ with cofactor $L_{\overline{E}} = Z^{d-1} L_E\left(\frac{1}{Z},\frac{y_2}{Z},...,\frac{y_n}{Z}\right)$.

(b) Conversely if $\overline{F} = \exp\left(\frac{\overline{h}}{Z^k}\right)$ with $\overline{h} \in \mathbb{R}_k[Z,y_2,...,y_n]$ is an exponential factor of $\overline{\chi}$ with cofactor $L_{\overline{F}}$, then $F = \exp(h(x))$ with $h(x) = x^k \overline{h}\left(\frac{1}{x_1},\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right)$ is an exponential factor of χ with cofactor $L_F = x^{d-1} L_{\overline{F}}\left(\frac{1}{x_1},\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right)$.

The following result was proved in 2009 by Llibre and Zhang.

Theorem 46. (LLIBRE; ZHANG, 2009b) Assume that the polynomial vector field χ in $\mathbb{R}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1,...,p$ and the invariant hyperplane at infinity.

(i) If one of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field χ has a rational first integral.

(ii) Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have algebraic multiplicity m_i for $i = 1,...,p$ and that the invariant hyperplane at infinity has algebraic multiplicity k. If the vector field restricted to the hyperplane at infinity or any invariant hypersurface with multiplicity larger than 1 has no rational first integrals, then the following hold

(a) If $\sum_{i=1}^{p} m_i + k = N + 2$, then the vector field χ has a real Darboux first integral, where $N = \binom{n+d-1}{n}$.

(b) If $\sum_{i=1}^{p} m_i + k = N + n + 1$, then (1.2) has a real rational first integral.

For two-dimensional polynomial vector fields, the additional condition in Theorem 46 on the nonexistence of rational first integrals of the vector field restricted to the invariant algebraic curves including the line at infinity is not necessary.

Darboux constructed his theory over the complex projective space which we think is the natural field and natural space for this theory. Firstly the complex numbers form an algebraically closed field. So an essential ingredient in his theory, the theory of algebraic curves, can be properly done (Bézout theorem cannot be proved over the reals). Secondly the complex projective plane is a compact space and in particular "the line at infinity" of the affine plane completely looses its special status in the projective plane. It is like any other line.

On the other hand it is important to observe that when we consider the theory of Darboux for real systems, we can go to their complexification and these systems could have complex invariant algebraic curves $f(x,y) = 0$ with $f \in \mathbb{C}[x,y]$. We can therefore end up with more invariant curves than those with real coefficients. Let us consider an example.

Example 47.

$$\begin{cases} \dot{x} = x^2 + 1 \\ \dot{y} = x + y. \end{cases}$$

This system clearly has two invariant lines which are complex $x \pm i = 0$ with respective co-factors $x \mp i$. It can easily be checked that the line at infinity has the multiplicity two. So the total multiplicity of invariant lines over $\mathbb{C}$ is four. This system was proved to be integrable in (SCHLOMIUK; VULPE, 2008c) having the inverse Darboux integrating factor

$$(x+i)^{1+i/2}(x-i)^{1-i/2}.$$

Let us now consider this real system without taking into consideration its complexification. Suppose now that we want to prove just by using real curves that the system is integrable. The lines $x \pm i = 0$ are defined over $\mathbb{C}$ and it is only their union, the conic $x^2+1=0$ which is defined over $\mathbb{R}$. This is an invariant curve of the real system with the cofactor $\alpha_1 = 2x$. We also have an exponential factor $e^{g_0+g_2y}$ with cofactor $\alpha_2 = g_2(x+y)$ where $g_0, g_2 \in \mathbb{C}$. However this is insufficient for proving integrability as we can check by trying to apply the usual algorithm for computing an integrating factor. Indeed, considering $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\begin{aligned} \lambda_1\ \alpha_1 + \lambda_2\ \alpha_2 &= -div(P_1, Q_1) \\ (2\lambda_1 + g_2\lambda_2)x + g_2\lambda_2 y &= -2x - 1, \end{aligned}$$

this equality does not have a solution. So although the real system is integrable and has a real first integral, we cannot compute this real first integral without considering the two invariant lines. So this supports the idea that the full extension of the Darboux theory that also covers the line at infinity with its own multiplicity cannot produce all the real integrable systems.

In conclusion we really need an extension of the Darboux theory over $\mathbb{C}$ that includes the multiplicity of the line at infinity.

CHAPTER

3

THE FAMILY QSH OF QUADRATIC DIFFERENTIAL SYSTEMS WITH AN INVARIANT HYPERBOLA

The notion of *configuration of invariant curves* of a polynomial differential system appears in several works, see for instance (SCHLOMIUK; VULPE, 2004).

Definition 48. (SCHLOMIUK; VULPE, 2004) Consider a real planar polynomial system (1.1) with a finite number of singular points. By *configuration of algebraic solutions* of the system we mean a set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.

Definition 49. (OLIVEIRA *et al.*, 2017) Suppose we have two systems (S_1), (S_2) in **QSH** with a finite number of singularities, finite or infinite, a finite set of invariant hyperbolas $\mathscr{H}_i^1 : h_i^1(x,y) = 0,\ i = 1,\dots,k$ of (S_1) (respectively $\mathscr{H}_i^2 : h_i^2(x,y) = 0,\ i = 1,\dots,k$ of (S_2)) and a finite set (which could also be empty) of invariant straight lines $\mathsf{L}_j^1 : g_j^1(x,y) = 0,\ j = 1,\dots,k'$ of (S_1) (respectively $\mathsf{L}_j^2 : g_j^2(x,y) = 0,\ j = 1,\dots,k'$ of (S_2)). We say that the two configurations C_1, C_2 of hyperbolas and lines of these systems are *equivalent* if there is a one-to-one correspondence Φ_h between the hyperbolas of C_1 and C_2 and a one-to-one correspondence Φ_l between the lines of C_1 and C_2 such that:

(i) the correspondences conserve the multiplicities of the hyperbolas and lines (in case there are any) and also send a real invariant curve to a real invariant curve and a complex invariant curve to a complex invariant curve;

(ii) for each hyperbola $\mathscr{H} : h(x,y) = 0$ of C_1 (respectively each line $\mathsf{L} : g(x,y) = 0$) we have a one-to-one correspondence between the real singular points on $\mathscr{H}$ (respectively on L) and the real singular points on $\Phi_h(\mathscr{H})$ (respectively $\Phi_l(\mathsf{L})$) conserving their multiplicities, their location on branches of hyperbolas and their order on these branches (respectively on the lines);

(iii) Furthermore, consider the total curves $\mathscr{F}^1 : \prod H_i^1(X,Y,Z) \prod G_j^1(X,Y,Z)Z = 0$ (respectively $\mathscr{F}^2 : \prod H_i^2(X,Y,Z) \prod G_j^2(X,Y,Z)Z = 0$ where $H_i^1(X,Y,Z) = 0,\ G_j^1(X,Y,Z) = 0$ (respectively $H_i^2(X,Y,Z) = 0,\ G_j^2(X,Y,Z) = 0$)) are the projective completions of $\mathscr{H}_i^1$, L_j^1 (respectively $\mathscr{H}_i^2$, L_j^2). Then, there is a one-to-one correspondence ψ between the singularities of the curves $\mathscr{F}^1$ and $\mathscr{F}^2$ conserving their multiplicities as singular points of the total curves.

This notion was used in (SCHLOMIUK; VULPE, 2008b), (SCHLOMIUK; VULPE, 2008c), (SCHLOMIUK; VULPE, 2008a), (SCHLOMIUK; VULPE, 2008d), (SCHLOMIUK; VULPE, 2004) and (OLIVEIRA *et al.*, 2017) also to classify systems in **QS** possessing invariant algebraic curves according to the kind of configurations these systems could possess. In particular, in the last one (OLIVEIRA *et al.*, 2017), **QSH** was classified according to the configuration of invariant hyperbolas and lines the systems possess. This classification led to 205 distinct configurations. The family **QSH** is rich, displaying all kinds of Liouvillian integrability and as well as non-integrable systems.

It is important to assume that the systems (1.3) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSH** have a finite number of finite singular points.

In the family **QSH** we also have cases where we have an infinite number of hyperbolas. In these cases, by a Jouanolou result (see Theorem 9 on Chapter 2), we have a rational first integral.

In (OLIVEIRA *et al.*, 2017) the authors study the class **QSH** of non-degenerate quadratic differential systems having an invariant hyperbola. They classified the family **QSH**, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. If a quadratic system has an infinite number of hyperbolas then the system has a finite number of invariant affine straight lines (see (ARTÉS; GRÜNBAUM; LLIBRE, 1998)). Therefore, we can talk about *equivalence of configurations of the invariant affine lines associated to the system*. Given two such configurations C_{1l} and C_{2l} associated to systems (S_1) and (S_2) of (1.1), we say they are *equivalent* if and only if there is a one-to-one correspondence Φ between the lines of C_{1l} and C_{2l} such that:

(i) the correspondence preserve the multiplicities of the lines and also sends a real (respectively complex) invariant line to a real (respectively complex) invariant line;

(ii) for each line $\mathsf{L} : g(x,y) = 0$ we have a one-to-one correspondence between the real singularities on L and the real singularities on Φ preserving their multiplicities and their order on the lines.

Definition 50. (OLIVEIRA *et al.*, 2017) Consider two systems (S_1) and (S_2) in **QSH** each one with an infinite number of invariant hyperbolas. Consider the configurations C_{1l} and C_{2l} of invariant affine straight lines $\mathsf{L}_j^1 : g_j^1(x,y) = 0$ where $j = 1,2,...,k$ of system (S_1) and respectively $\mathsf{L}_j^2 : g_j^2(x,y) = 0$ where $j = 1,2,...,k$ of system (S_2). We say that the two configurations C_{1l} and C_{2l} are equivalent with respect to the hyperbolas of the systems if and only if:

(i) they are equivalent as configurations of invariant lines, and

(ii) taking any hyperbola $\mathscr{H}_1 : h_1(x,y) = 0$ of (S_1) and any hyperbola $\mathscr{H}_2 : h_2(x,y) = 0$ of (S_2), then we must have a one-to-one correspondence between the real singularities of system (S_1) located on $\mathscr{H}_1$ and of real singularities of system (S_2) located on $\mathscr{H}_2$, preserving their multiplicities, their location and order on branches.

Furthermore, consider the curves $\mathscr{F}_1 : \prod h_1(x,y) \prod g_j^1 = 0$ and $\mathscr{F}_2 : \prod h_2(x,y) \prod g_j^2 = 0$. Then, we have a one-to-one correspondence between the singularities of the curve $\mathscr{F}_1$ with those in the curve $\mathscr{F}_2$ preserving their multiplicities as singularities of these curves.

The definition above is independent of the choice of the two hyperbolas $\mathscr{H}_1 : h_1(x,y) = 0$ of (S_1) and $\mathscr{H}_2 : h_2(x,y) = 0$ of (S_2).

Here we introduce some invariant polynomials that play an important role in the study of polynomial vector fields. Considering $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2,\xi], \quad M(\tilde{a},x,y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. It is known that the singular points at infinity of quadratic systems are given by the solutions in x and y of $C_2(\tilde{a},x,y) = 0$. If $\eta < 0$ then this means we have one real singular point at infinity and two complex ones.

Observation 51. We note that since a system in **QSH** always has an invariant hyperbola then clearly we always have at least 2 real singular points at infinity. So we must have $\eta \geq 0$.

The family **QSH** can be splitted as: $\mathbf{QSH}_{(\eta=0)}$ of systems which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities and $\mathbf{QSH}_{(\eta>0)}$ of systems which possess three distinct real singularities at infinity in $P_2(\mathbb{C})$.

In (OLIVEIRA *et al.*, 2017) the authors give necessary and sufficient conditions for a quadratic system to have an invariant hyperbola. These conditions were given in terms of 59 affine invariant polynomial so they are independent of the normal forms in which the systems may be presented. For the sake of completeness, we give below in the following tables these conditions.

In the next two tables (i) and (ii) we present in the first column the number associated with the equations in (OLIVEIRA *et al.*, 2017), which are the normal forms for the systems in **QSH**. In the second column are the necessary and sufficient conditions.

For a proof see (OLIVEIRA *et al.*, 2017).

(i) $\eta > 0$

Equations	Invariants
(3.4)	$[\eta>0,\theta\neq0],\beta_1\neq0,\beta_2^2+\beta_3^2\neq0\ (\mathfrak{C}_1),B_1\neq0$
(3.25)	$[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_2\ [(\mathfrak{C}_2)],\chi_A^{(2)}<0$ $[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_2\ [(\mathfrak{C}_2)],\chi_A^{(2)}>0$
(3.49)	$[\eta>0,\theta\neq0],[\beta_1=0],\beta_6\neq0,\mathscr{C}_7\ [\beta_2=0,\gamma_5=0,\mathscr{R}_4\neq0]$
(3.13)	$[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_1\ [(\mathfrak{C}_1),B_1=0],\chi_E^{(1)}\neq0$
(3.23)	$[\eta>0,\theta\neq0],\beta_1\neq0,\beta_2^2+\beta_3^2\neq0,B_1\neq0$
(3.66)	$[\eta>0,\theta=0],N\neq0,\beta_6\neq0,\beta_{10}\neq0,\gamma_7=0,\mathscr{R}_6\neq0,B_1\neq0$
(3.66) $h=1/3$	$[\eta>0,\theta=0],N\neq0,\beta_6\neq0,\beta_{10}=0,\gamma_4=0,\beta_2\mathscr{R}_3\neq0$
(3.66) $h=1/2$	$[\eta>0,\theta=0],N\neq0,\beta_6\neq0,\beta_{10}\neq0,\gamma_7=0,\mathscr{R}_6\neq0,B_1=0$
(3.15) $a=1/4h^2$	$[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_2\ [(\mathfrak{C}_2)],\chi_A^{(2)}=0$
(3.20)	$[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_4\ [(\mathfrak{C}_1),B_1=0,\chi_E^{(1)}=0,B_2=0]$
(3.15)	$[\eta>0,\theta\neq0],\beta_1\neq0,\mathscr{C}_1\ [(\mathfrak{C}_1),B_1=0],\chi_E^{(1)}=0,B_2\neq0$
(3.29)	$[\eta>0,\theta\neq0],\mathscr{C}_3\ [\beta_1=0],\beta_6\neq0,\beta_2\neq0,\gamma_4=0,\mathscr{R}_5\neq0,\delta_1\neq0,B_1\neq0$
(3.33)	$[\eta>0,\theta\neq0],[\beta_1=0],\beta_6\neq0,\beta_2\neq0,\mathscr{C}_5\ [\delta_1\neq0,B_1=0],\chi_E^{(3)}\neq0$
(3.35)	$[\eta>0,\theta\neq0],[\beta_1=0],\beta_6\neq0,\beta_2\neq0,\mathscr{C}_5\ [\delta_1\neq0,B_1=0],\chi_E^{(3)}\neq0,B_2\neq0$
(3.41)	$[\eta>0,\theta\neq0],[\beta_1=0],\beta_6\neq0,\beta_2\neq0,\mathscr{C}_5\ [\delta_1\neq0,B_1=0],\chi_E^{(3)}\neq0,B_2=0$
(3.43)	$[\eta>0,\theta\neq0],[\beta_1=0],\beta_6\neq0,\beta_2\neq0,\mathscr{C}_6\ [\delta_1=0]$
(3.50)	$[\eta>0,\theta\neq0],[\beta_1=0],\mathscr{C}_8\ [\beta_6=0],\beta_7\neq0,\gamma_5=0,\mathscr{R}_5\neq0,\beta_8^2+\delta_2^2\neq0$
(3.50) $h=1/4$	$[\eta>0,\theta\neq0],[\beta_1=0,\beta_6=0],\beta_7\neq0,\mathscr{C}_{11}\ [\beta_8=\delta_2=0]$
(3.55)	$[\eta>0,\theta\neq0],[\beta_1=0,\beta_6=0],\mathscr{C}_{12}\ [\beta_7=0],\beta_9\neq0,\gamma_5=0,\mathscr{R}_5\neq0,\delta_3\neq0$
(3.58)	$[\eta>0,\theta\neq0],[\beta_1=0,\beta_6=0],\mathscr{C}_{12}\ [\beta_7=0],\beta_9\neq0,\gamma_5=0,\mathscr{R}_5\neq0,\delta_3=0,\beta_8\neq0$
(3.58) $h=1/4$	$[\eta>0,\theta\neq0],[\beta_1=0,\beta_6=0],\mathscr{C}_{12}\ [\beta_7=0],\beta_9\neq0,\gamma_5=0,\mathscr{R}_5\neq0,\delta_3=0,\beta_8=0$
(3.62)	$[\eta>0,\theta\neq0],[\beta_1=0,\beta_6=0],\mathscr{C}_{12}\ [\beta_7=0],\beta_9=0,\gamma_6=0,\mathscr{R}_5\neq0$
(3.73)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\beta_2\neq0,\beta_7\neq0,\delta_4\neq0,B_2\neq0,\mathscr{C}_{13}\ [\mu_0\neq0]$
(3.73) $h=0$	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\beta_2\neq0,\beta_7\neq0,\delta_4\neq0,B_2\neq0,\mathscr{C}_{14}\ [\mu_0=0]$
(3.76)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\beta_2\neq0,\beta_7\neq0,\delta_4\neq0,\mathscr{C}_{15}\ [\beta_2=0]$
(3.79)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\beta_2\neq0,\beta_7\neq0,\mathscr{C}_{16}\ [\delta_4=0]$
(3.82)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\mathscr{C}_{17}\ [\beta_2\neq0,\beta_7=0,\gamma_9=0,\mathscr{R}_8\neq0]$
(3.85)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\mathscr{C}_{18}\ [\beta_2=0],\beta_7\neq0,\beta_{10}\neq0,\gamma_7\gamma_8=0,\mathscr{R}_5\neq0,\beta_8^2+\delta_2^2\neq0,\gamma_7=0$
(3.88)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\mathscr{C}_{18}\ [\beta_2=0],\beta_7\neq0,\beta_{10}\neq0,\gamma_7\gamma_8=0,\mathscr{R}_5\neq0,\beta_8^2+\delta_2^2\neq0,\gamma_8=0$
(3.91)	$[\eta>0,\theta=0],N\neq0\ [\beta_6=0],\mathscr{C}_{18}\ [\beta_2=0],\beta_7\neq0,\beta_{10}\neq0,\gamma_7\gamma_8=0,\mathscr{R}_5\neq0,\beta_8=\delta_2=0$

(3.95)	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7\neq 0, \gamma_{10}<0, B_2\neq 0$
(3.95) $4a+v^2=0$	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7\neq 0, \gamma_{10}<0, B_2=0$
(3.97)	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7\neq 0, \gamma_{10}>0, B_2\neq 0$
(3.97) $4a-v^2=0$	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7\neq 0, \gamma_{10}>0, B_2=0$
(3.97) $v=0$	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7\neq 0, \gamma_{10}=0$
(3.99)	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7\neq 0, \beta_{10}=0, \mathscr{R}_3\neq 0, \gamma_7=0$
(3.102)	$[\eta>0,\theta=0], N\neq 0\ [\beta_6=0], \mathscr{C}_{18}\ [\beta_2=0], \beta_7=0, \gamma_7=0, \mathscr{R}_3\neq 0$
(3.105)	$\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2\neq 0, \beta_1=\gamma_{11}=0, \mathscr{R}_9\neq 0, \chi_A^{(8)}<0$ $\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2\neq 0, \beta_1=\gamma_{11}=0, \mathscr{R}_9\neq 0, \chi_A^{(8)}>0$
(3.105) $b=0$	$\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2\neq 0, \beta_1=\gamma_{11}=0, \mathscr{R}_9\neq 0, \chi_A^{(8)}=0$
(3.107)	$\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2=0\ (\mathfrak{C}_3), \gamma_{12}=0$
(3.109)	$\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2=0\ (\mathfrak{C}_3), \gamma_{13}=0, \mathscr{R}_9<0$ $\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2=0\ (\mathfrak{C}_3), \gamma_{13}=0, \mathscr{R}_9>0$
(3.109) $a=0$	$\mathscr{C}_{19}\ [\eta>0,\theta=0,N=0], \beta_2=0\ (\mathfrak{C}_3), \gamma_{13}=0, \mathscr{R}_9=0$

(ii) $\eta=0$

Equations	Invariants
(4.4)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1\neq 0, \mathscr{R}_1\neq 0, B_1\neq 0$
(4.10)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1\neq 0, \mathscr{R}_1\neq 0, B_1=0$
(4.11)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1=0, \gamma_1=0, \mathscr{R}_3\neq 0$
(4.13)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathscr{R}_{10}\neq 0$
(4.13) $g=1/4$	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathscr{R}_{10}\neq 0, \beta_7\beta_8=0, \mathscr{R}_{10}<0, \beta_8=0$ $\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathscr{R}_{10}\neq 0, \beta_7\beta_8=0, \mathscr{R}_{10}>0, \beta_8=0$
(4.13) $g=1/2$	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathscr{R}_{10}\neq 0, \beta_7\beta_8=0, \mathscr{R}_{10}<0, \beta_7=0$ $\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathscr{R}_{10}\neq 0, \beta_7\beta_8=0, \mathscr{R}_{10}>0, \beta_7=0$
(4.16)	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathscr{R}_7\neq 0, \chi_A^{(7)}<0$ $\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathscr{R}_7\neq 0, \chi_A^{(7)}>0$
(4.16) $c^2=a$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathscr{R}_7\neq 0, \chi_A^{(7)}=0$
(4.18)	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2\neq 0$
(4.18) $g=0$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2=0, \gamma_{16}\neq 0$
(4.18) $c=0$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2=0, \gamma_{16}=0$
(4.22)	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}<0$ $\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}>0$
(4.22) $\varepsilon=0$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}=0$
(4.25) $c\neq 0$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}\neq 0, \gamma_{10}=\gamma_{17}=0, \mathscr{R}_{11}\neq 0, \gamma_{16}\neq 0$
(4.25) $c=0$	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}\neq 0, \gamma_{10}=\gamma_{17}=0, \mathscr{R}_{11}\neq 0, \gamma_{16}=0$
(4.27)	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_2\neq 0$

(4.28)	$\eta=0, \mathscr{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_2=0$
(4.30)	$\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9<0$ $\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9>0$
(4.31)	$\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9=0$
(4.34)	$\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}\neq 0, H_2\neq 0$
(4.36)	$\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}\neq 0, H_2=0$
(4.38)	$\eta=0, \mathscr{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}=0$

In this table we denote by (4.25) the following system that appears in (OLIVEIRA *et al.*, 2017) without number

$$\dot{x}=-\frac{3c^2}{16}+cx-x^2, \quad \dot{y}=1-2xy.$$

If $c\neq 0$ we may assume $c=4$ by the rescaling $(x,y,t)\mapsto(cx/4, 4y/c, 4t/c)$. So we obtain the system denoted by (4.25) in (OLIVEIRA *et al.*, 2017) which we denote here by (4.25) $c\neq 0$.

3.1 Normal forms in QSH

The normal forms numbered in the tables indicated by the number (3.4) up to (4.38) and obtained in (OLIVEIRA *et al.*, 2017) appear below in two more condensed tables in the following two propositions.

Proposition 52. (OLIVEIRA *et al.*, 2017) Every system in $\mathbf{QSH}_{(\eta>0)}$ can be brought by an affine transformation and time rescaling to one of the following 13 normal forms, where a, g, h, b, v are real parameters. Next to each normal form we present the respective invariant hyperbola.

$$\begin{cases} \dot{x}=a(2h-1)+x+gx^2+(h-1)xy \\ \dot{y}=a(2g-1)-y+(g-1)xy+hy^2, \end{cases} \qquad \Phi(x,y)=a+xy \qquad \text{(A)}$$
where $a(g-1)(h-1)(3g-1)(3h-1)\neq 0$.

$$\begin{cases} \dot{x}=a(2h-1)+gx^2+(h-1)xy \\ \dot{y}=a(2g-1)+(g-1)xy+hy^2, \end{cases} \qquad \Phi(x,y)=a+xy \qquad \text{(B)}$$
where $a(g-1)(h-1)(2g-1)(2h-1)\neq 0$.

$$\begin{cases} \dot{x}=a+\left(\frac{1-2h}{2}\right)x^2+(h-1)xy \\ \dot{y}=a-\left(\frac{2h+1}{2}\right)xy+hy, \end{cases} \qquad \begin{aligned}\Phi_1(x,y)&=-\frac{a}{2h-1}+x(x-y)\\ \Phi_2(x,y)&=\frac{a}{h}+2y(x-y)\end{aligned} \qquad \text{(C)}$$
where $ah(h-1)(2h\pm 1)\neq 0$.

$$\begin{cases} \dot{x}=(a-1)(2h-1)+(3h-1)x-hx^2+(h-1)xy \\ \dot{y}=2a(h-1)+(3h-1)y-(h+1)xy+hy^2, \end{cases} \qquad \Phi(x,y)=1-a-2x+x(x-y) \qquad \text{(D)}$$
where $(a-1)(h\pm 1)(2h\pm 1)(3h\pm 1)\neq 0$.

$$\begin{cases} \dot{x} = x - \frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b + y - \frac{3xy}{2} + \frac{y^2}{2}, \end{cases} \qquad \Phi(x,y) = 4 + b - 4x + x(x-y) \qquad \text{(E)}$$
where $(b+4) \neq 0$.

$$\begin{cases} \dot{x} = a(2h-1) - hx^2 + (h-1)xy \\ \dot{y} = 2a(h-1) - (h+1)xy + hy^2, \end{cases} \qquad \Phi(x,y) = a - x(x-y) \qquad \text{(F)}$$
where $a(h\pm 1)(2h\pm 1)(3h\pm 1) \neq 0$.

$$\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a + 3v^2 - \frac{4xy}{3} + \frac{y^2}{3} \end{cases} \qquad \Phi(x,y) = 3a \pm 3ivx + x(x-y) \qquad \text{(G)}$$
where $av \neq 0$.

$$\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \frac{4xy}{3} + \frac{y^2}{3}, \end{cases} \qquad \Phi_{1,2}(x,y) = 3a \pm 3vx + x(x-y) \qquad \text{(H)}$$
where $a \neq 0$.

$$\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 5a - \frac{4xy}{3} + \frac{y^2}{3} \end{cases} \qquad \begin{aligned} &\Phi_{1,2}(x,y) = 3a \pm \sqrt{-3a}x + x(x-y) \\ &\Phi_3(x,y) = -3a + xy \end{aligned} \qquad \text{(I)}$$
where $a \neq 0$.

$$\begin{cases} \dot{x} = -\frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b - \frac{3xy}{2} + \frac{y^2}{2} \end{cases} \qquad \begin{aligned} &\Phi_1(x,y) = b - 2xy \\ &\Phi_2(x,y) = b + x(x-y) \end{aligned} \qquad \text{(J)}$$
where $b \neq 0$.

$$\begin{cases} \dot{x} = 4b - 1 + 4y + x^2 \\ \dot{y} = b + y^2 \end{cases} \qquad \Phi(x,y) = b - 1 - x + 3y + y(x-y) \qquad \text{(K)}$$
where $(b+1) \neq 0$.

$$\begin{cases} \dot{x} = a + x^2 \\ \dot{y} = 4a + y^2, \end{cases} \qquad \Phi(x,y) = a - x(x-y) \qquad \text{(L)}$$
where $a \neq 0$.

$$\begin{cases} \dot{x} = a + x^2 \\ \dot{y} = a + y^2. \end{cases} \qquad \Phi(x,y) = 2a - r(x-y) + 2xy \qquad \text{(M)}$$

Observation 53. Consider the class of all non-degenerate quadratic systems possessing at least one invariant hyperbola and three distinct real singularities at infinity. Using the invariants described earlier in this chapter (see pages 52,53 and 54) which guarantees the existence of an invariant hyperbola for systems (1.3), in (OLIVEIRA *et al.*, 2017) the authors arrived at the system:

$$(3.1)\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + (h-1)xy \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2. \end{cases}$$

with invariants $\eta = 1$ and $\theta = -(g-1)(h-1)(g+h)/2$.

(i) The case $\theta \neq 0$ gives the condition $(g-1)(h-1)(g+h) \neq 0$ for (3.1) and we arrive at the normal forms

- (A) where $a(g-1)(h-1)(g+h) \neq 0$,
- (B) where $a(g-1)(h-1)(g+h) \neq 0$,
- (C) where $ah(h-1)(2h \pm 1) \neq 0$.

(ii) For the case $\theta = 0$ we consider $g = -h$ in (3.1) and we arrive at another invariant $N = 9(h^2-1)(x-y)^2$.

(ii.1) For $N \neq 0$ we arrive at the normal forms:

- (A) where $g = -h$,
- (B) where $g = -h$,
- (D) where $(a-1)(h \pm 1)(2h \pm 1)(3h \pm 1) \neq 0$,
- (E) where $b \neq -4$,
- (F) where $a(h \pm 1)(2h \pm 1)(3h \pm 1) \neq 0$,
- (G) where $av \neq 0$,
- (H) where $a \neq 0$,
- (I) where $a \neq 0$,
- (J) where $b \neq 0$.

(ii.2) The case $N = 0$ we arrive at the normal forms:

- (K) where $b \neq -1$,
- (L) where $a \neq 0$,
- (M).

Proposition 54. (OLIVEIRA *et al.*, 2017) Every system in $\mathbf{QSH}_{(\eta=0)}$ can be brought by an affine transformation and time rescaling to one of the following 13 normal forms, where a, g, c, ε are real parameters. Next to each normal form we present the respective invariant hyperbola.

$$\begin{cases} \dot{x} = 2a + x + gx^2 + xy \\ \dot{y} = a(2g-1) - y + (g-1)xy + y^2, \end{cases} \qquad \Phi(x,y) = a + xy \qquad \text{(N)}$$

where $a(g-1) \neq 0$

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases} \qquad \Phi(x,y) = a + xy \tag{O}$$
where $a(g-1) \neq 0$

$$\begin{cases} \dot{x} = 2a + 3cx + x^2 + xy \\ \dot{y} = a - c^2 + y^2, \end{cases} \qquad \Phi(x,y) = a + cx + xy \tag{P}$$
where $a \neq 0$

$$\begin{cases} \dot{x} = (c+x)(c(2g-1) + gx) \\ \dot{y} = 1 + (g-1)xy, \end{cases} \qquad \Phi(x,y) = \frac{1}{(-1+2g)} + cy + xy \tag{Q}$$
where $(g \pm 1)(3g-1)(2g-1) \neq 0$

$$\begin{cases} \dot{x} = x^2 + \varepsilon \\ \dot{y} = 1 - 2xy \end{cases} \qquad \Phi_{1,2}(x,y) = -1 \pm i\sqrt{\varepsilon}y + xy \tag{R}$$

$$\begin{cases} \dot{x} = (x-1)(3-x) \\ \dot{y} = 1 - 2xy. \end{cases} \qquad \Phi(x,y) = \frac{1}{3} + y - xy \tag{S}$$

$$\begin{cases} \dot{x} = -x^2 \\ \dot{y} = 1 - 2xy. \end{cases} \qquad \Phi(x,y) = -1 + 3xy \tag{T}$$

$$\begin{cases} \dot{x} = (2x-1)(2x+1)/4 \\ \dot{y} = y. \end{cases} \qquad \Phi(x,y) = -\frac{q}{2} + qx + \frac{y}{2} + 2xy, \ q \neq 0 \tag{U}$$

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1. \end{cases} \qquad \Phi(x,y) = 1 + rx + xy \tag{V}$$

$$\begin{cases} \dot{x} = a + y + x^2 \\ \dot{y} = xy. \end{cases} \qquad \Phi(x,y) = a + 2y + x^2 - m^2y^2 \tag{W}$$

$$\begin{cases} \dot{x} = (1+3x)(2+3x)/9 \\ \dot{y} = xy. \end{cases} \qquad \Phi(x,y) = 4 + 12x + 9x^2 + my + 3mxy \tag{X}$$

$$\begin{cases} \dot{x} = a + x^2 \\ \dot{y} = xy, \end{cases} \qquad \Phi(x,y) = a + x^2 - m^2xy \tag{Y}$$
where $a \neq 0$.

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1 + xy. \end{cases} \qquad \Phi(x,y) = 1 + mx^2 + 2xy \tag{Z}$$

Observation 55. Consider the class of all non-degenerate quadratic systems possessing at least one invariant hyperbola. Using the invariants described previously in order to guarantee the existence of an invariant hyperbola for systems (1.3), in (OLIVEIRA *et al.*, 2017) the authors arrived at two possibilities: $M(\tilde{a},x,y) \neq 0$ (i.e. at infinity we have two distinct real singularities) and $M = 0 = C_2$ (when we have an infinite number of singularities at infinity).

(i) $M(\tilde{a},x,y) \neq 0$: This brings systems (1.3) to the systems

$$(4.1)\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2. \end{cases}$$

with invariants $C_2(x,y) = x^2y$ and $\theta = -h^2(g-1)/2$.

(i.1) The case $\theta \neq 0$ gives the condition $h(g-1) \neq 0$ for (4.1) and we arrive at the normal forms

- (N) where $a(g-1) \neq 0$,
- (O) where $a(g-1) \neq 0$.

(i.2) The case $\theta = 0$ gives the condition $h(g-1) = 0$ for (4.1) and we arrive at another invariant $\mu_0 = gh^2$.

(i.2.1) For $\mu_0 \neq 0$ we have the normal form

- (P) where $a \neq 0$.

(i.2.2) For $\mu_0 = 0$ they calculated the invariant $N = 9(g-1)(g+1)x^2$ and we need to consider two possibilities.

(i.2.2.1) For the case $N \neq 0$ we have the normal forms

- (Q) where $(g-1)(g+1)(2g-1)(3g-1) \neq 0$,
- (R) where $\varepsilon \neq 0$.

(i.2.2.2) The case $N = 0$ gives the condition $(g-1)(g+1) = 0$ and we have the normal forms

- (S),
- (T),
- (U),
- (V).

(ii) $M(\tilde{a},x,y) = 0 = C_2$: We arrive at the normal forms

- (W),
- (X),
- (Y) where $a \neq 0$,
- (Z).

Observation 56. The invariant hyperbolas involve:

(i) sometimes all the parameters of the system (such as (C));

(ii) sometimes only some parameters (such as (N)) and

(iii) sometimes additional parameters (such as (W)).

Observation 57. We studied the integrability and the geometric aspects of the normal forms with their respective conditions given in Propositions 52 and 54. When one of these conditions is not satisfied then either the system does not belong to the family, even though we may have an invariant hyperbola such as for the family (F) when $h=-1/2$, or the system does not belong to **QSH**. It is important to mention that the conditions for the normal forms of **QSH** were given in (OLIVEIRA *et al.*, 2017).

3.2 Main Theorems

The next two resuls are part of the main results of this thesis, they treat about the integrability in the classe **QSH**. The proofs of this theorems are given in Chapter 5.

Consider the following sets:

$$\mathrm{E}_1=\{(a,g,h): h=1/2 \text{ and } a\neq 0\},\ \ \mathrm{E}_2=\{(a,g,h): g=0 \text{ and } a\neq 0\},$$

$$\mathrm{E}_3=\{(a,g,h): g=1/2 \text{ and } a\neq 0\},\ \ \mathrm{E}_4=\{(a,g,h): h=0 \text{ and } a\neq 0\},$$

$$\mathrm{E}_5=\{(a,g,h): g=h \text{ and } a\neq 0\},$$

$$\mathrm{E}_6=\cup_{k\in\mathbb{N}}\mathrm{E}_{6,k},\ \text{where } \mathrm{E}_{6,k}=\{(a,g,h): g+h=-k \text{ and } a\neq 0\},\ k\in\mathbb{N},$$

$$\mathrm{E}_7=\cup_{k\in\mathbb{N}}\mathrm{E}_{7,k},\ \text{where } \mathrm{E}_{7,k}=\{(a,g,h): g+h=-\tfrac{k}{2} \text{ and } a\neq 0\},\ k\in\mathbb{N},$$

$$\mathrm{E}_8=\{(a,g,h): g+h=1 \text{ and } a\neq 0\},\ \ \mathrm{E}_9=\{(a,g,h): 4agh=1\},$$

$$\mathrm{E}_{10}=\left\{(a,g,h): a(g+h)^2=1\right\},\ \ \mathrm{E}_{11}=\{(a,g,h): g=1/4 \text{ and } a\neq 0\},$$

$$\mathrm{E}_{12}=\{(a,g,h): h=1/4 \text{ and } a\neq 0\},\ \ \mathrm{E}_{13}=\{(a,g,h): g=-h \text{ and } a\neq 0\},$$

$$\mathrm{E}_{14}=\{(a,g,h): g=2,\ h=-2/5 \text{ and } a\neq 0\},\ \ \mathrm{E}_{15}=\{(a,g,h): h=2,\ g=-2/5 \text{ and } a\neq 0\}.$$

Main Theorem 1. Consider the polynomial systems in $\mathbf{QSH}_{(\eta>0)}$.

(a) The 11 normal forms (C) – (M) are all Liouvillian integrable. The following table sums up the results regarding the types of integrability:

Systems	Parameters	Type of first integral
(C)	$h=1/4$ and $a\neq 0$	Polynomial
(C)	$ah(h-1)(2h\pm 1)(4h-1)\neq 0$	Darboux
(D)	$a=0$	Darboux
(D)	$a=-(h-1)^2(2h+1)/(3h+1)^2$	Darboux
(D)	$a=h=0$	Generalized Darboux
(D)	$h=0$ and $a=-1$	Generalized Darboux

(D)	$(a-1)(h\pm1)(2h\pm1)(3h\pm1)\neq0$	Liouvillian
(E)	$b=0$ or $b=8/25$	Rational
(E)	$b\neq-4$	Darboux
(F)	$h=1/4$, $\bar{a}=-a/2$ and $a\neq0$	Rational
(F)	$a(h\pm1)(2h\pm1)(3h\pm1)(4h\pm1)\neq0$	Liouvillian
(G)	$a=3v^2$ or $a=-3v^2/4$ or $a=-8v^2/9$ and $a\neq0$	Rational
(G)	$av(a+v^2)(a-3v^2)(a+3v^2/4)(a+8v^2/9)\neq0$	Darboux
(G)	$a=-v^2$ and $a\neq0$	Generalized Darboux
(H)	$a=-3v^2$ or $a=3v^2/4$ or $a=8v^2/9$ and $a\neq0$	Rational
(H)	$av(a-v^2)(a+3v^2)(a-3v^2/4)(a-8v^2/9)\neq0$	Darboux
(H)	$a=v^2$ or $v=0$ and $a\neq0$	Generalized Darboux
(I)	$a\neq0$	Rational
(J)	$b\neq0$	Rational
(K)	$b\neq-1,-1/4$	Liouvillian
(K)	$b=-1/4$	Generalized Darboux
(L)	$a\neq0$	Rational
(M)	$a\in\mathbb{R}$	Rational

(b) For the normal forms (A) and (B) we have the following:

(i) If $(a,g,h)\in\mathbb{R}^3-\mathrm{E}$, where $\mathrm{E}=\cup_{i=1}^{10}\mathrm{E}_i$ then systems (A) are not Liouvillian integrable.

(i.1) If $(a,g,h)\in\mathrm{E}_9\cup\mathrm{E}_{10}$ then systems (A) are not Liouvillian integrable.

(i.2) If $(a,g,h)\in\mathrm{E}_5\cup(\mathrm{E}_{10}\cap\mathrm{E}_3)$ then systems (A) are Liouvillian integrable.

(ii) If $(a,g,h)\in\mathbb{R}^3-\mathrm{F}$, where $\mathrm{F}=\mathrm{E}_2\cup\mathrm{E}_4\cup\left(\cup_{i=5}^{8}\mathrm{E}_i\right)\cup\left(\cup_{i=11}^{15}E_i\right)$ then systems (B) are not Liouvillian integrable.

(ii.1) If $(a,g,h)\in\cup_{i=11}^{15}\mathrm{E}_i\cup\mathrm{E}_5$ then systems (B) are Liouvillian integrable.

The following table sums up the results regarding the types of integrability:

Systems	Parameters	Type of first integral
(A)	$g=h$ and $a\neq0$	Darboux
(A)	$a=1/(g+h)^2$ and $g=1/2$	Liouvillian
(A)	$\bar{a}=-36/289$, $h=1/3$ and $g=-4/3$	Liouvillian
(A)	$\bar{a}=-3/(3g+1)^2$, $h=1/3$ and $g=1/2$	Liouvillian
(A)	$(a,g,h)\in\mathbb{R}^3-\tilde{\mathrm{E}}$, where $\tilde{\mathrm{E}}=\cup_{i=1}^{8}\mathrm{E}_i$	Not Liouvillian integrable

(B)	$g = h$ and $a \neq 0$	Darboux
(B)	$g = 1/2$, $h = 0$, $\bar{a} = -a$ and $a \neq 0$	Generalized Darboux
(B)	$g = 1/4$ or $h = 1/4$ or $g = -h$ and $a \neq 0$	Liouvillian
(B)	$g = 2$, $h = -2/5$ and $a \neq 0$	Liouvillian
(B)	$h = 2$, $g = -2/5$ and $a \neq 0$	Liouvillian
(B)	$(a,g,h) \in \mathbb{R}^3 - \tilde{\mathrm{F}}$, where $\tilde{\mathrm{F}} = \mathrm{E}_2 \cup \mathrm{E}_4 \cup_{i=6}^{8} \mathrm{E}_i$	Not Liouvillian integrable

Observation 58. The Liouvillian integrability of any system in class (A) with $(a,g,h) \in \cup_{i=1}^{4} \mathrm{E}_i \cup \left(\cup_{i=6}^{8} \mathrm{E}_i\right)$ or in class (B) with $(a,g,h) \in \mathrm{E}_2 \cup \mathrm{E}_4 \cup \left(\cup_{i=6}^{8} \mathrm{E}_i\right)$ is still open. The reason is that the methods applied in this thesis for proving the existence or non-existence of a Liouvillian first integral do not work in these cases and so new ideas are needed for proving or disproving their Liouvillian integrability.

Consider the following sets:

$$\mathrm{L}_1 = \cup_{k\in\mathbb{N}} \mathrm{L}_{1,k}, \text{ where } \mathrm{L}_{1,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/2 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N},$$

$$\mathrm{L}_2 = \cup_{k\in\mathbb{N}} \mathrm{L}_{2,k}, \text{ where } \mathrm{L}_{2,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/3 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N},$$

$$\mathrm{L}_3 = \left\{(a,g) \in \mathbb{R}^2 : g = 1/4 \text{ and } a \neq 0\right\},$$

$$\mathrm{C}' = \cup_{k\in\mathbb{N}} \mathrm{C}_k, \text{ where } \mathrm{C}_k = \left\{(a,g) \in \mathbb{R}^2 : g = (2+a-2ak)/4a \text{ and } a \neq 0\right\}, \ k \in \mathbb{N}.$$

Main Theorem 2. Consider the polynomial systems in $\mathbf{QSH}_{(\eta=0)}$.

(a) The 11 normal forms (P) – (Z) are all Liouvillian integrable. The following table sums up the results regarding the types of integrability:

Systems	Parameters	Type of first integral
(P)	$a = 8c^2/9$ and $a \neq 0$	Generalized Darboux
(P)	$a(a - 8c^2/9) \neq 0$	Liouvillian
(Q)	$g(g \pm 1)(2g-1)(3g-1) \neq 0$	Darboux
(Q)	$g = 0$ and $c \neq 0$	Generalized Darboux
(R)	$\varepsilon \in \mathbb{R}$	Polynomial (hamiltonian)
(S)	-	Rational
(T)	-	Rational
(U)	-	Rational
(V)	-	Rational
(W)	$a \in \mathbb{R}$	Rational
(X)	-	Rational
(Y)	$a \neq 0$	Rational
(Z)	-	Rational

(b) For the normal forms (N) and (O) we have the following:

(i) If $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$, then systems (N) are not Liouvillian integrable.

(i.1) If $(a,g) \in \mathrm{L}_{1,1}$ then systems (N) are not Liouvillian integrable.

(i.2) If $(a,g) \in \mathrm{L}_{2,1}$ then systems (N) are not Liouvillian integrable.

(ii) If $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$, then systems (O) are not Liouvillian integrable.

(ii.1) If $(a,g) \in \mathrm{L}_{1,1}$ then systems (O) are generalized Darboux integrable.

(ii.2) If $(a,g) \in \mathrm{L}_3$ then systems (O) are Liouvillian integrable.

The following table sums up the results regarding the types of integrability:

Systems	Parameters	Type of first integral
(N)	$(a,g) \in \mathbb{R} - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$	Not Liouvillian integrable
(O)	$g = 1/2$ and $a \neq 0$	Generalized Darboux
(O)	$g = 1/4$ and $a \neq 0$	Liouvillian
(O)	$(a,g) \in \mathbb{R} - (\mathrm{L}_1 \cup \mathrm{L}_3)$	Not Liouvillian integrable

Observation 59. The Liouvillian integrability of any system in class (N) with $(a,g) \in (\mathrm{L}_1 - \mathrm{L}_{1,1}) \cup \mathrm{L}_2 \cup \mathrm{C}'$ or in class (O) with $(a,g) \in (\mathrm{L}_1 - \mathrm{L}_{1,1})$ is still open. The reason is that the methods applied in this thesis for proving the existence or non-existence of a Liouvillian first integral do not work in these cases and so new ideas are needed for proving or disproving their Liouvillian integrability.

Observation 60. Note that we have very interesting cases where arithmetic or algebraic conditions change the type of integrability (such as systems (P)).

CHAPTER

4

GEOMETRICAL CONCEPTS AND RESULTS

In the theory of Darboux presented in the preceding chapter the main tools used to prove the integrability are the number of invariant curves, their multiplicities and the number of independent points. In 1993 Christopher and Kooij stated a theorem in (KOOIJ; CHRISTOPHER, 1993) that can be reformulate in geometric terms. This result shows a relation between the geometry of the "configuration of invariant curves" and their Darboux integrability. This theorem was proved in (CHRISTOPHER *et al.*, 2002).

Theorem 61. (KOOIJ; CHRISTOPHER, 1993) Consider a polynomial system (1.1) that has k algebraic solutions $C_i = 0$ such that

(a) all curves $C_i = 0$ are non-singular and have no repeated factor in their highest order terms,

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves have a common factor in their highest order terms,

(d) the sum of the degrees of the curves is $n+1$, where n is the degree of system (1.1).

Then system (1.1) has an integrating factor

$$\mu(x,y) = 1/(C_1C_2...C_k).$$

This theorem has a geometric content which is not completely explicit in the algebraic way they stated the result. We rewrite the theorem above in geometric terms as follows:

Theorem 62. Consider a polynomial system (1.1) that has k algebraic solutions $C_i = 0$ such that

(a) all curves $C_i = 0$ are non-singular and they intersect transversally the line at infinity $Z = 0$,

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves intersect at a point on the line at infinity $Z = 0$,

(d) the sum of the degrees of the curves is $n+1$, where n is the degree of system (1.1).

Then system (1.1) has an integrating factor

$$\mu(x,y) = 1/(C_1C_2...C_k).$$

In the hypotheses of this theorem the way the curves are placed with respect to one another in the totality of the curves, in other words the "geometry of the configuration of invariant algebraic curves" has an impact of the kind of integrating factor we could have.

One of our motivations is relating the geometry of the invariant algebraic curves curves taken in their totality with the various kinds of integrability. To begin doing this we need to recall some concepts.

Poincaré stated a number of definitions and among them we have the following.

Let $H = f/g$ be a rational first integral of the polynomial vector field (1.2). We say that H has degree n if n is the maximum of the degrees of f and g. We say that the degree of H is *minimal* among all the degrees of the rational first integrals of χ if any other rational first integral of χ has a degree greater than or equal to n. Let $H = f/g$ be a rational first integral of χ. According to Poincaré (POINCARÉ, 1891b) we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable value* of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x,y]$. Here, if $c = \infty$, then $f + cg$ denotes g. Note that for all $c \in \mathbb{C}$ the algebraic curve $f + cg = 0$ is invariant. The curves in the factorization of $f + cg$, when c is a remarkable value, are called *remarkable curves*.

Now suppose that c is a remarkable value of a rational first integral H and that $u_1^{\alpha_1}...u_r^{\alpha}$ is the factorization of the polynomial $f + cg$ into reducible factors in $\mathbb{C}[x,y]$. If at least one of the α_i is larger than 1 then we say, following again Poincaré (see for instance (FERRAGUT; LLIBRE, 2007)), that c is a *critical remarkable value* of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable curve* of the vector field (1.2) with exponent α_i.

Since we can think of $\mathbb{C} \cup \{\infty\}$ as the projective line $P_1(\mathbb{R})$ we can also use the following definition.

Definition 63. Consider $\mathscr{F}_{(c_1,c_2)} : c_1 f - c_2 g = 0$ where f/g is a rational first integral of (1.2). We say that $[c_1 : c_2]$ is a remarkable value of the curve $\mathscr{F}_{(c_1,c_2)}$ if $\mathscr{F}_{(c_1,c_2)}$ is reducible over $\mathbb{C}$.

It is proved in (CHAVARRIGA *et al.*, 2003) that there are finitely many remarkable values for a given rational first integral H and if (1.2) has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.

Given $H = f/g$ a rational first integral, consider $F_{(c_1,c_2)} = c_1 f - c_2 g$ where $f_1, f_2 \in \mathbb{C}[x,y]$ and $\deg F_{(c_1,c_2)} = n$. If $F_{(c_1,c_2)} = f_1 f_2$ where $\deg f_i = n_i < n$ then necessarily the points on the intersection of $f_1 = 0$ and $f_2 = 0$ must be singular points of the curve $F_{(c_1,c_2)}$.

Lemma 64. (CHRISTOPHER, 1994) Assume that system (1.1) with degree m has an invariant algebraic curve f of degree n. Let f_n, P_m and Q_m be the homogeneous parts of f with degree n, P and Q with degree m. Then each one of the irreducible factors of f_n divides $yP_m - xQ_m$.

In geometric terms, this lemma means that the points at infinity of any algebraic invariant curve $f = 0$ of a system (1.1) are singularities of this system.

Let us recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 65. Let V be an irreducible algebraic variety of dimension n over a field $\mathbb{K}$. A cycle of dimension r or r-cycle on V is a formal sum

$$\sum_W n_W W$$

where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_W's are non-zero. We call degree of an r-cycle the sum

$$\sum_W n_W.$$

An $(n-1)$-cycle is called a divisor.

Definition 66. For a non-degenerate polynomial differential system (S) possessing a finite number of algebraic solutions

$$\mathscr{F} = \{f_i\}_{i=1}^m, \ f_i(x,y) = 0, \ f_i(x,y) \in \mathbb{C}[x,y],$$

each with multiplicity n_i and a finite number of singularities at infinity, we define the algebraic solutions divisor (also called the invariant curves divisor) on the projective plane,

$$ICD_{\mathscr{F}} = \sum_{n_i} n_i \mathscr{C}_i + n_\infty \mathscr{L}_\infty$$

where $C_i : F_i(X,Y,Z) = 0$ are the projective completions of $f_i(x,y) = 0$, n_i is the multiplicity of the curve $C_i = 0$ and n_∞ is the multiplicity of the line at infinity $\mathscr{L}_\infty : Z = 0$.

Proposition 67. (ARTÉS; GRÜNBAUM; LLIBRE, 1998) Every polynomial differential system of degree n and with a finite number of invariant lines has at most $3n$ invariant straight lines, including the line at infinity.

In particular the maximum number of invariant lines for a quadratic system with a finite number of invariant lines is six. In the case we consider here, we have a particular instance of the divisor *ICD* because the invariant curves will be invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we can construct the divisor of the invariant straight lines which are always in finite number.

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system, located on the algebraic solutions of the system.

Definition 68. 1. Suppose a real quadratic system (1.3) has a non-empty finite set of invariant hyperbolas

$$\mathscr{H}_i : h_i(x,y) = 0,\ i = 1,2,...k$$

and a finite number of affine invariant lines

$$\mathscr{L}_j : f_j(x,y) = 0,\ j = 1,2,...,l$$

where $h_i, f_j \in \mathbb{C}[x,y]$. We denote the line at infinity $\mathscr{L}_\infty : Z = 0$. Let us assume that on the line at infinity we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the system is the following

$$ICD = n_1\mathscr{H}_1 + ... + n_k\mathscr{H}_k + m_1\mathscr{L}_1 + ... + m_l\mathscr{L}_l + m_\infty\mathscr{L}_\infty$$

where n_i (respectively m_j) is the multiplicity of the hyperbola $\mathscr{H}_i$ (respectively m_j of the line $\mathscr{L}_j$), and m_∞ is the multiplicity of $\mathscr{L}_\infty$. We also mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by $\mathscr{H}_i^C$ (respectively $\mathscr{L}_i^C$). We define the total multiplicity TM of the divisor as the sum $\sum_i n_i + \sum_j m_j + m_\infty$.

2. The zero-cycle on the real projective plane, of singularities of a quadratic system (1.3) located on a configuration of invariant lines and invariant hyperbolas, is given by

$$M_{0CS} = r_1P_1 + ... + r_lP_l + v_1P_1^\infty + ... + v_nP_n^\infty$$

where P_i (respectively P_j^∞) are all the finite (respectively infinite) real singularities of the system and r_i (respectively v_j) are their corresponding multiplicities. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of zero-cycles as the sum $\sum_i r_i + \sum_j v_j$.

Definition 69. (1) In case we have an infinite number of hyperbolas and just two or three singular points at infinity but we have a finite number of invariant straight lines we define the invariant lines divisor as

$$ILD = m_1\mathscr{L}_1 + ... + m_l\mathscr{L}_l + m_\infty\mathscr{L}_\infty.$$

(2) In case we have an infinite number of hyperbolas, the line at infinity is filled up with singularities and we have a finite number of affine lines, we define the invariant lines divisor as

$$ILD = m_1\mathscr{L}_1 + \dots + m_l\mathscr{L}_l.$$

Definition 70. (1) Suppose we have a finite number of invariant hyperbolas and invariant straight lines of a system (S) and that they are given by equations

$$f_i(x,y) = 0,\ i \in \{1,2,\dots,k\},\ f_i \in \mathbb{C}[x,y].$$

Set $F_i(X,Y,Z) = 0$ the projection completion of the invariant curves $f_i = 0$ in $P_2(\mathbb{C})$. The total invariant algebraic curve of the system (S) in QSH, on $P_2(\mathbb{R})$, is the curve

$$T(S) = \prod_i F_i(X,Y,Z)^{m_i} Z^{m_\infty} = 0,$$

where m_i is the multiplicity of $f_i = 0$, $i = 1,\dots,k$ and m_∞ is the multiplicity of the line at infinity.

(2) Suppose that a system (S) has an infinite number of invariant hyperbola. Then the system (S) has a finite number of invariant affine straight lines (see (OLIVEIRA *et al.*, 2017)). In this case, the total invariant curve is formed only by the invariant lines of system (S). Set $L_i(X,Y,Z) = 0$ the projective completions of the invariant lines $l_i(x,y) = 0$, $i \in \{1,2,\dots,k\}$ in $P_2(\mathbb{C})$.

(i) If there are just two or three singular points at infinity, the total invariant curve of system (S) is

$$T(S) = \prod_i L_i(X,Y,Z)^{m_i} Z^{m_\infty} = 0,$$

where m_i is the multiplicity of the line $l_i = 0$, $i = 1,\dots,k$ and m_∞ is the multiplicity of the line at infinity.

(ii) If the line at infinity is filled up with singularities, the total invariant curve of system (S) is

$$T(S) = \prod_i L_i(X,Y,Z)^{m_i} = 0,$$

where m_i is the multiplicity of the line $l_i = 0$, $i = 1,\dots,k$.

For example, if a system (S) admits a invariant hyperbola $h(x,y)$ with multiplicity two and the line at infinity $Z = 0$ has multiplicity one, then the total invariant curve of this system is

$$T(S) = H(X,Y,Z)^2 Z = 0$$

where $H(X,Y,Z)$ is the projection completion of $h = 0$. The degree of $T(S)$ is 5.

The singular points of the system (S) situated on $T(S)$ are of two kinds: those which are simple (or smooth) points of $T(S)$ and those which are multiple points of $T(S)$.

Observation 71. To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, when these singular points are situated on the total curve, we also have the multiplicity of these points as points on the total curve $T(S)$. Through a singular point of the systems there may pass several of the curves $F_i = 0$ and $Z = 0$. Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve $T(S)$. The simple points of the curve $T(S)$ are those of multiplicity one. They are also the smooth points of this curve.

Definition 72. (i) Suppose a system (S) has a finite number of singularities finite or infinite. The zero-cycle of the total curve $T(S)$ of system (S) is given by

$$M_{0CT} = r_1 P_1 + ... + r_l P_l + v_1 P_1^{\infty} + ... + v_n P_n^{\infty}$$

where P_i (respectively P_j^{∞}) are all the finite (respectively infinite) singularities situated on $T(S)$ and r_i (respectively v_j) are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of zero-cycles of the total invariant curve as the sum $\sum_i r_i + \sum_j v_j$.

(ii) Suppose a system (S) possesses the line at infinity filled up with singularities. The zero-cycle of the total curve $T(S)$ of system (S) is given by

$$M_{0CT} = r_1 P_1 + ... + r_l P_l$$

where P_i are all the finite singularities situated on $T(S)$ and r_i are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of zero-cycles of the total invariant curve as the sum $\sum_i r_i$.

Observation 73. If two curves intersects transversally, this point will be a simple point of intersection. If they are tangent, we would have an intersection multiplicity higher than or equal to two.

Definition 74. Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2 and preserving the orientation.

To cut the number of non equivalent phase portraits in half we use here another equivalence relation.

Definition 75. Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2, preserving or reversing the orientation.

Notation: $\cong_{top}$.

In (ARTÉS *et al.*, 2020) the authors provide a complete classification of **QS** according to the equivalence relation of topological configurations of singularities, finite or infinite. In this thesis we choose to use a terminology and notation for singularities introduced in (ARTÉS *et al.*, 2020) except by the way to describe the parabolic sectors and the multiplicities of the finite singularities. We give more details in what follows.

We say that a singular point is *elemental* if it possess two non-zero eigenvalues; *semi-elemental* if it possess exactly one eigenvalue equal to zero and *nilpotent* it possesses two zero eigenvalues and the linear part is not zero. We call *intricate* a singular point with its Jacobian matrix identically zero.

We will place first the finite singular points which will be denoted with lower case letters and secondly we will place the infinite singular points which will be denoted by capital letters, separating them by a semicolon ';'.

In our study we will have real and complex finite singular points for real systems and from the topological viewpoint only the real ones are interesting. When we have a simple (respectively double) complex finite singular point we use the notation © (respectively $©_{(2)}$).

For the elemental singular points we use the notation 's', 'S' for saddles, 'n', 'N' for nodes, 'f' for foci and 'c' for centers.

Non-elemental singular points are multiple points. We denote by $\binom{a}{b}$ the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point at infinity. For example, $\binom{1}{1}SN$ and $\binom{0}{2}SN$ correspond to two saddle-nodes at infinity which are locally topologically distinct since the first arise from the coalescence of a finite with an infinite singularity and the second from the coalescence of two infinity singularities.

The semi-elemental singular points can either be nodes, saddles or saddle-nodes (finite or infinite). If they are finite singular points we will denote them by '$n_{(3)}$', '$s_{(3)}$' and '$sn_{(2)}$', respectively and if they are infinite singular points by '$\binom{a}{b}N$', '$\binom{a}{b}S$' and '$\binom{a}{b}SN$', where $\binom{a}{b}$ indicates their multiplicity. We note that semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles.

The nilpotent singular points can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by '$es_{(4)}$' and '$cp_{(4)}$'.

In the case of nilpotent infinite points, we use the similar notation described below for the intricate singular points.

The intricate singular points are degenerate singular points. The neighbourhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic (p), hyperbolic (h) and elliptic

(e) (see (DUMORTIER; LLIBRE; ARTÉS, 2006)). In this work we have the following finite intricate singular points of multiplicity four described according their sectoral decomposition:

- $hpphpp_{(4)}$, $pphpph_{(4)}$, $phph_{(4)}$
- $epep_{(4)}$
- $hhhhhh_{(4)}$

where (4) indicates that the points have multiplicity four.

For intricate and nilpotent singular points at infinity, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}P-HHP$ and $\binom{2}{2}PH-HP$, for instance. When describing a single finite nilpotent or intricate singular point, one can always apply an affine change of coordinates to the system, so it does not really matter which sector starts the sequence, or the direction(clockwise or counter-clockwise) we choose. If it is an infinite nilpotent or intricate singular point, then we will always start with a sector bordering the infinity (to avoid using two dashes). The lack of finite singular points after the removal of degeneracies, will be encapsulated in the notation $\emptyset$ (i.e. small size $\emptyset$). In similar cases when we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset$. Finally there is also the possibility that we have an infinite number of finite or of infinite singular points. In the first case, this means that the quadratic polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points.

The line at infinity is filled up with singularities, then it is known that any such system has in a sufficiently small neighbourhood of infinity one of 7 topological distinct phase portraits (see (SCHLOMIUK; VULPE, 2008a)). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. Following (ARTÉS *et al.*, 2021) we use the notation $[\infty;\emptyset]$, $[\infty;N]$, $[\infty;N^d]$ (one-direction node, that is, a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal), $[\infty;S]$, $[\infty;C]$, $[\infty;\binom{0}{2}SN]$, $[\infty;\binom{0}{3}ES]$ indicating the singularities obtained after removing the line filled with singularities.

The degenerate systems are systems with a common factor in the polynomials defining the system. We will denote this case with the symbol $\ominus$. The degeneracy can be produced by a common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. Following (ARTÉS *et al.*, 2021) we will indicated each case by the following symbols:

- $\ominus[|]$ for a real straight line;
- $\ominus[)(]$ for an hyperbola;

- $\ominus[\times]$ for two real straight lines intersecting at a finite point.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If some singular points remain on this curve we will use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remain after the removal of the degeneracy, may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we will need to keep the geometrical information associated to that singularity. In this study we use the following notations:

- $(\ominus[|]; n^*)$ denotes the presence of a real straight line filled up with singular points in the system such that the reduced system has a node n^* on this line where n^* is a star node, that is, a node with two identical eigenvalues whose Jacobian matrix is diagonal;
- $(\ominus[|]; n^d)$ denotes the presence of a real straight line filled up with singular points in the system such that the reduced system has a node n^d on this line where n^d is a one-direction node, that is, a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal;
- $(\ominus[|]; s)$ denotes the presence of a real straight line filled up with singular points in the system such that the reduced system has a saddle on this line;
- $(\ominus[\times]; \emptyset)$ denotes the presence of two real straight lines filled up with singular points in the system such that the reduced system has no singularity on these lines;
- $(\ominus[)(]; \emptyset)$ denotes the presence of a hyperbola filled up with singular points in the system such that the reduced system has no singularity on this hyperbola.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity. There is a detailed description of this notation in (ARTÉS *et al.*, 2021). In case that after the removal of the finite degeneracy, a singular point at infinity remains at the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. In this study we use the following notations:

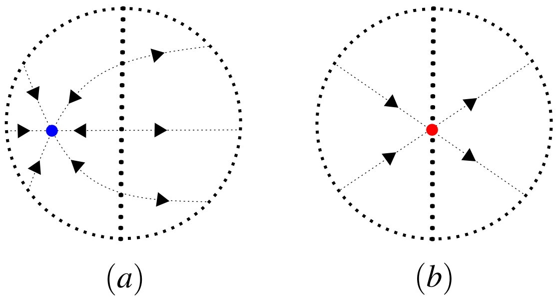
- $N, N, (\ominus[|]; \emptyset)$ means that the system has at infinity two nodes and one non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has no infinite singular point in that position;
- $N, (\ominus[\times]; \emptyset, \emptyset)$ means that the system has at infinity one node and two non-isolated singular point which are part of two real intersecting straight lines filled up with singularities, and that the reduced constant system has no singularity in those positions;

- $\binom{0}{2}SN, (\ominus[|];\emptyset)$ that means that the system has at infinity a saddle-node, and one non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has no infinite singular point in that position;

- $(\ominus[)(];N,\emptyset)$ that means that the system has at infinity two non-isolated singular points which are part of a hyperbola filled up with singularities, and that the reduced constant system has a node in one of those positions and no singularity in the other.

Degenerate systems with the line at infinity filled up with singularities: According to (ARTÉS *et al.*, 2021) there are only two geometrical configurations of this class which are also topologically distinct, and which produce just the two phase portraits given in Figure 1. The notations of configurations of infinite singularities in (ARTÉS *et al.*, 2021) are $[\infty;(\ominus[|];\emptyset_3)]$ for picture (a) and $[\infty;(\ominus[|];\emptyset_2)]$, for picture (b).

Figure 1 – Phase portraits of quadratic degenerate systems with infinite line filled up with singularities.



Source: Elaborated by the author.

See (ARTÉS *et al.*, 2021) and (ARTÉS *et al.*, 2020) for more details.

In order to distinguish topologically the phase portraits of the systems we obtained, we also use some invariants introduced in (SCHLOMIUK; VULPE, 2008c). Let SC be the total number of separatrix connections, i.e. of phase curves connecting two singularities which are local separatrices of the two singular points. We denote by

- SC_f^f the total number of SC connecting two finite singularities,
- SC_f^∞ the total number of SC connecting a finite with an infinite singularity,
- SC_∞^∞ the total number of SC connecting two infinite.

A *graphic* as defined in (DUMORTIER; ROUSSARIE; ROUSSEAU, 1994) is formed by a finite sequence of singular points $p_1, p_2, \ldots, p_n$, $p_{n+1} = p_1$ and oriented regular orbits $s_1, \ldots, s_n$ connecting them such that s_j has p_j as α-limit set and p_{j+1} as ω-limit set for $j < n$ and s_n has p_n as α-limit set and p_1 as ω-limit set. Graphics may or may not have a return map. Particular graphics are given special names. A *loop* is a graphic through a unique singular point and with a return map. A *polycycle* is a graphic through several singular points and with a return map. A *degenerate graphic* as defined in (DUMORTIER; ROUSSARIE; ROUSSEAU, 1994) is formed by singular points $p_1, p_2, ..., p_n, p_{n+1} = p_1$, oriented regular orbits and segments $s_1, \ldots s_n$ of curves of singular points (which are also oriented) such that either s_j is a orbit that has p_j as α-limit and p_{j+1} as ω-limit for $j < n$ and s_n has p_n as α-limit set and p_1 as ω-limit set or an open segment of a curve of singular points with end points p_j and p_{j+1}, for each $j < n$. Moreover, the regular orbits and the curves of singular points have coherent corientations in the sense that if s_{j-1} has left hand orientation then so does s_j. For more details, see (DUMORTIER; ROUSSARIE; ROUSSEAU, 1994).

Now let us give some details on the notations used in the drawings of the configurations and phase portraits.

(i) In the **bifurcation diagrams of configurations**:

(i.i) The dashed lines in the parameter space are limit cases of the family studied,

(i.ii) the multiple invariant curves of the configurations are emphasized and we indicated the multiplicities next to the drawing of each curve,

(i.iii) the complex curves in the configurations are drawn as dashed,

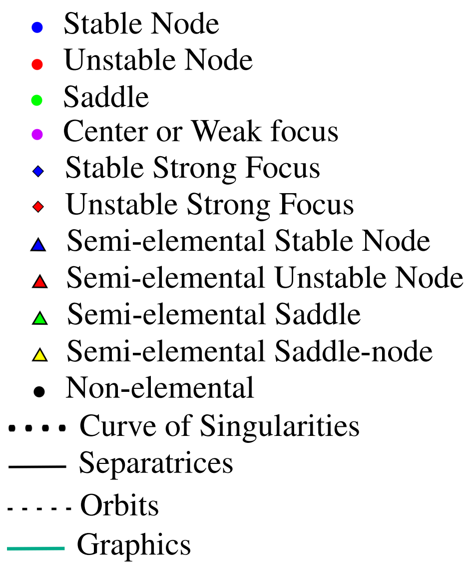
(i.iv) the dots represents the real singular points that are located on the invariant curves,

(i.v) the numbers indicated inside the parenthesis are the multiplicities of each singular points,

(i.vi) the curves appearing drawn as dotted represents a curve filled up with singularities.

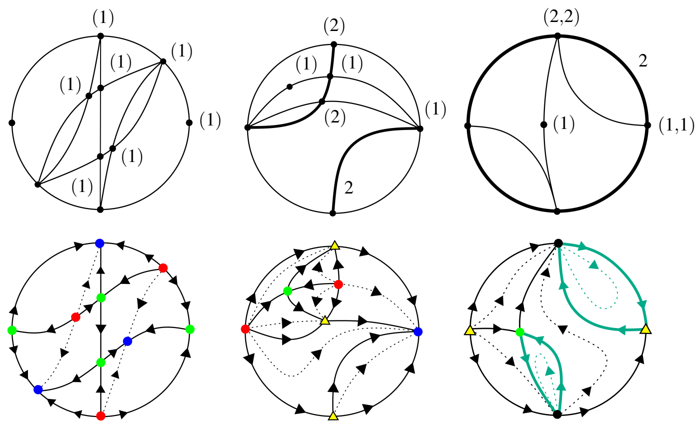
(ii) In the **topological bifurcation diagrams** we use the following notations.

Figure 2 – Notations used on the phase portraits.



In what follows we present an example of the notation used in this thesis to describe the global configuration of singularities of **QSH**.

Figure 3 – Some examples of configurations and phase portraits.



The notation used to describe the topological type of the singularities is

$$(n,s,n,s;\binom{0}{2}SN,N)$$
$$(s,sn_{(2)},n;\binom{0}{2}SN,N)$$
$$(s;\binom{2}{2}PPEP-PEPP,\binom{1}{1}SN)$$

for each phase portrait apearing in the respective order. The first letters appearing with lower case represents the topological type of the finite singularities. Here '$sn_{(2)}$' denotes a saddle-node which arises from the coalescence of a finite saddle with a finite node so this is a singularity of multiplicity two. When we do not indicate the multiplicity this means the singularity is simple, which is the case of '*n*' (elemental node) and '*s*' (elemental saddle). The capital letters give the topological type of the singularities at infinity, starting from north pole following clockwise: '$\binom{0}{2}SN$' denotes a saddle-node which arises from the coalescence of two infinite singularities (saddle and node) so this is a double singularity, '$\binom{1}{1}SN$' also denotes a saddle-node but here this multiplicity arises from the coalescence of a finite with an infinite singularity, '$\binom{2}{2}PPEP-PEPP$' denotes an intricate singularity arising from the coalesce of two finite singularities with two infinite singularities and the neighbourhood of this singularity is formed by two parabolic sectors (PP), one elipctic sector (E) and other parabolic sector (P) and the dash (hyphen) between the sectors split those which appear on one side or the other of the equator of the sphere. The cases where we do no indicate the multiplicity means the singularity is simple, which is the case of '*S*' (elemental saddle) and '*N*' (elemental node).

CHAPTER

5

INTEGRABILITY IN QSH

5.1 Proof of Main Theorem 1 and Main Theorem 2 for the integrable cases

The family **QSH** displays all the features of the theory of Darboux as we know it today. We apply here this theory and the Prelle-Singer algorithm (including the exponential factors, when they exist) in order to prove the integrability as stated in Main theorem 1 and Main theorem 2. The proof for the cases which do not have a Liouvillian first integral will be done in section 5.2.

The result of our calculations are given in the tables presented in subsection 5.1.1 where we have the invariant algebraic curves, exponential factors and their cofactors, first integrals or integrating factors for each normal form of Proposition 52 and Proposition 54, obtained using the software *Mathematica*.

In the first column are the normal forms for **QSH**. In the second column are the invariant algebraic curves, the exponential factors and the respective cofactors. In the third column are the expressions of the first integrals or the expressions of the integrating factors. If we give the expression for the first integral then it is not necessary to give the integrating factor to guarantee the integrability. When we give the expression for the integrating factor instead of the first integral this means that we could not compute the expression for the first integral using *Mathematica* and we use the notation “—". When “—" appears in both the first integral and integrating factor this means that we could find neither of them applying the Prelle-Singer algorithm. In the fourth and fifth columns are the normal forms and their possible configurations as in (OLIVEIRA *et al.*, 2017). The notation “—" appears when we do not have them appearing in (OLIVEIRA *et al.*, 2017). In the sixth column are indicated the types of integrability of each normal form using the following notations.

Notation
N-I : Systems admit neither a Darboux nor a Liouvillian first integral;
D: Systems are Darboux integrable;
GD: Systems are generalized Darboux integrable;
L: Systems are Liouvillian integrable;
P: Systems admit a polynomial first integral;
R: Systems admit a rational first integral;
HAM: Systems are Hamiltonian;
open case: We could prove neither the integrability nor the non-integrability;
$\mathscr{R}$: Represents an integrating factor;
$\mathscr{F}$: Represents a first integral;
J : Represents invariant algebraic curves;
α : Represents cofactors;
[*] : Equation in (OLIVEIRA *et al.*, 2017).

The precise integrating factor, first integral, invariant algebraic curve and cofactor that did not fit in the table will be given subsequently using the notations above.

5.1.1 Table.

i) $\eta > 0$

Orbit representative $a,g,h,b,v \in \mathbb{R} : a \neq 0$	Invariant curves/ ExpFac	Integrating Factor $\mathscr{R}_i$	[*]	Config.H	Integ.
	Respective cofactors	First integral $\mathscr{F}_i$			
(A) $\begin{cases} \dot{x} = a(2h-1)+x+gx^2+(h-1)xy \\ \dot{y} = a(2g-1)-y+(g-1)xy+hy^2 \end{cases}$	$a+xy$	—	(3.4)	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34	N-I
where $(g-1)(h-1)(3g-1)(3h-1)(2g-1)(2h-1)\neq 0$, $a\left(a-\frac{1}{(g+h)^2}\right)(g\pm h)(3g+3h-4)\neq 0$	$(-1+2g)x+(-1+2h)y$	—			
(A) where $g=1/2$	$y,\ a+xy$	—	(3.13)	38, 39, 41, 42, 43, 44, 46, 47, 48, 49, 55, 64, 65, 66, 67, 70, 71, 72, 73, 74, 75	open case
	$-1-\frac{x}{2}+hy,\ (-1+2h)y$	—			
(A) where $h=1/2$	$x,\ a+xy$	—	—	—	open case
	$gx-\frac{y}{2}+1,\ (2g-1)x$	—			

(A) where $g=h$	$h(x-y)+1,\ a+xy$		(3.25)	37, 52, 53, 45	D
	$h(x+y),\ (2h-1)(x+y)$	$\mathscr{F}_{A,3}$			
(A) where $a=1/(g+h)^2$	$1+(g+h)^2xy,\ 1+\frac{(g+h)}{2}(x-y)$	—	(3.15)	50, 51, 54, 56, 57, 59, 60, 61, 62, 63, 68, 69	N-I
	$(-1+2g)x+(-1+2h)y,\ \left(\frac{-g+h}{g+h}\right)+gx+hy$	—			
(A) where $a=1/(g+h)^2$ and $g=1/2$	$\frac{1}{4}((2h+1)(x-y)+4),\ y,\ 1+\frac{(1+2h)^2}{4}xy$	$\mathscr{R}_{A,4}$	(3.20)	81, 82, 83, 84, 85, 86	L
	$\frac{2h-1}{2h+1}+\frac{x}{2}+hy,\ -1-\frac{x}{2}+hy,\ (-1+2h)y$	—			
(A) where $g=4/3-h$	$a+xy$	—	(3.23)	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34	open case
	$\frac{1}{3}(5-6h)x+(-1+2h)y$	—			
(A) where $g=-h$	$a+xy$	—	(3.66)	2, 8, 17, 18, 21, 22, 25, 38, 42, 46, 65, 72, 75, 76, 77	N-I
	$(-1-2h)x+(-1+2h)y$	—			

(A) where $g=-h$ and $h=1/3$	$a+xy$	—	(3.70)	2, 8, 17, 18, 21	N-I
	$(-1-2/3)x+(-1+2/3)y$	—			
(A) where $g=-h$ and $h=1/2$	$a+xy,\ x$	—	—	38, 46, 65, 72, 75	open case
	$-2x,\ 1-\frac{x}{2}-\frac{y}{2}$	—			
(A) where $a=1/4h^2$ and $g=h$	$\frac{1}{h}+x-y,\ \frac{1}{4h^2}+xy$		—	45	D
	$h(x+y),\ (-1+2h)x+(-1+2h)y$	$\mathscr{F}_{A,5}$			
(A) where $g=h=1/3$	$3+x-y,\ a+xy$		(3.49)	37, 52, 53, 45	P
	$\frac{x}{3}+\frac{y}{3},\ -\frac{x}{3}-\frac{y}{3}$	$\mathscr{F}_{A,7}=(3+x-y)(a+xy)$			
(A) where $h=1/3$ and $a=-3\overline{a}$	$3\overline{a}-xy$	—	(3.29)	1, 2, 3, 7, 8, 9,10, 11, 14, 16, 17, 18, 19, 20, 21, 23, 26, 32, 34, 29	open case
	$(-1+2g)x-y/3$	—			
(A) where $h=1/3$, $g=1/2$ and $a=-3\overline{a}$	$3\overline{a}-xy,\ y$	—	(3.33)	47, 49, 66, 67, 73, 74	open case
	$-\frac{y}{3},\ -1-\frac{x}{2}+\frac{y}{3}$	—			

(A) where $a=-\frac{3}{(3g+1)^2}$ and $h=1/3$	$\frac{-9}{(3g+1)^2}-xy,\quad 1+\frac{1}{6}(1+3g)(x-y)$	—	(3.33)	50, 57, 59, 60, 61, 62, 68, 69	open case
	$(-1+2g)x-\frac{y}{3},\quad \frac{1-3g}{3g+1}+gx+\frac{y}{3}$	—			
(A) where $a=\frac{6(3g-1)}{(3g+1)^2}$ and $h=1/3$	$1-\frac{(3g+1)^2xy}{54g-18},\ J_{A,5}$	—	(3.43)	124, 125, 127, 128, 129, 130, 135	open case
	$(2g-1)x-\frac{y}{3},\ \alpha_{A,5}$	—			
(A) where $a=-\frac{36}{289}$, $h=1/3$ and $g=-4/3$	$xy-\frac{36}{289},\ J_{A,6},\ J_{A,7}$	$\mathscr{R}_{A,9}$	—	—	L
	$-\frac{11x}{3}-\frac{y}{3},\ \alpha_{A,6},\ \alpha_{A,7}$	—			
(A) where $a=-\frac{3}{(3g+1)^2}$, $h=1/3$ and $g=1/2$	$xy+\frac{36}{25},\ x-y+\frac{12}{5},\ y$	$\mathscr{R}_{A,10}$	(3.41)	86	L
	$-\frac{y}{3},\ \frac{x}{2}+\frac{y}{3}-\frac{1}{5},\ -\frac{x}{2}+\frac{y}{3}-1$	—			
(B) $\begin{cases}\dot{x}=a(2h-1)+gx^2+(h-1)xy\\ \dot{y}=a(2g-1)+(g-1)xy+hy^2\end{cases}$	$a+xy$	—	(3.50)	1, 2, 5, 6, 17, 19, 27, 28, 35, 36	N-I
where $\begin{matrix}a(g-1)(h-1)(g\pm h)(2g-1)(2h-1)(4g-1)(4h-1)\neq 0\\ (2g+2h-1)(g-2)(h-2)(5g+2)(5h+2)\neq 0\end{matrix}$	$(2g-1)x+(2h-1)y$	—			
(B) where $g=1/4$	$a+xy,\ a+xy-y^2$	$\mathscr{R}_{B,3}=(a+xy-y^2)^{\frac{1}{2}-2h}(a+xy)^{2h-1}$	—	—	L
	$(2h-1)y-\frac{x}{2},\ 2hy-\frac{x}{2}$	—			

(B) where $h = 1/4$	$a+xy,\ a-x^2+xy$	$\mathscr{R}_{B,6} = \left(a-x^2+xy\right)^{\frac{1}{2}-2g}(a+xy)^{2g-1}$	—	121, 122, 123, 126, 131	L
	$(2g-1)x-\frac{y}{2},\ 2gx-\frac{y}{2}$	—			
(B) where $g = h$	$x-y,\ a+xy$		—	37, 53	D
	$hx+hy,\ (2h-1)(x+y)$	$\mathscr{F}_{B,7} = (x-y)^{\frac{1-2h}{h}}(a+xy)$			
(B) where $g = -h$	$-2\sqrt{a}+x-y,\ 2\sqrt{a}+x-y,\ a+xy$	$\mathscr{R}_{B,8}$	(3.85)	80, 91	L
	$-2\sqrt{a}h-hx+hy,\ 2\sqrt{a}h-hx+hy,\ (-2h-1)x+(2h-1)y$	$\mathscr{F}_{B,8}$			
(B) where $g = 2$ and $h = -2/5$	$\frac{y^2}{5\sqrt{5}\sqrt{a}}+\frac{3\sqrt{a}}{\sqrt{5}}+x-\frac{3y}{5},\ -\frac{y^2}{5\sqrt{5}\sqrt{a}}-\frac{3\sqrt{a}}{\sqrt{5}}+x-\frac{3y}{5},\ a+xy$	$\mathscr{R}_{B,9}$	—	—	L
	$-\frac{6\sqrt{a}}{\sqrt{5}}+2x-\frac{4y}{5},\ \frac{6\sqrt{a}}{\sqrt{5}}+2x-\frac{4y}{5},\ 3x-\frac{9y}{5}$	—			
(B) where $h = 2$ and $g = -2/5$	$-\frac{x^2}{3\sqrt{5}\sqrt{a}}-\sqrt{5}\sqrt{a}+x-\frac{5y}{3},\ \frac{x^2}{3\sqrt{5}\sqrt{a}}+\sqrt{5}\sqrt{a}+x-\frac{5y}{3},\ a+xy$	$\mathscr{R}_{B,10}$	—	—	L
	$-\frac{6\sqrt{a}}{\sqrt{5}}-\frac{4x}{5}+2y,\ \frac{6\sqrt{a}}{\sqrt{5}}-\frac{4x}{5}+2y,\ 3y-\frac{9x}{5}$	—			

(B) where $g = 1/2 - h$	$a+xy$	—	(3.55)	1, 2, 17, 19	open case
	$(2h-1)y-2hx$	—			
(B) where $g = 1/2,\ h = 0$ and $a = -\overline{a}$	$y,\ -\overline{a}+xy,\ e^{xy-y^2+1}$		(3.62)	40, 58	GD
	$-\frac{x}{2},\ -y,\ \overline{a}y$	$\mathscr{F}_{B,11}$			
(C) $\begin{cases} \dot{x} = a + \left(\frac{1-2h}{2}\right)x^2 + (h-1)xy \\ \dot{y} = a - \left(\frac{2h+1}{2}\right)xy + hy^2 \end{cases}$ where $ah(h-1)(2h\pm1)(4h-1)\neq0$	$x-y,\ \frac{a}{2h-1}-x^2+xy,\ -\frac{a}{2h}+xy-y^2$		(3.58)	132, 133, 134, 136	D
	$\frac{1}{2}(1-2h)x+hy,\ (1-2h)x+(-1+2h)y,\ -2hx+2hy$	$\mathscr{F}_C$			
(C) where $h = 1/4$	$x-y,\ \ -2a+xy,\ \ -2a-x^2+xy,\ \ -2a+xy-y^2$		—	156, 157	P
	$\frac{x}{4}+\frac{y}{4},\ \ -\frac{x}{2}-\frac{y}{2},\ \ \frac{x}{2}-\frac{y}{2},\ \ \frac{y}{2}-\frac{x}{2}$	$\mathscr{F}_{C,1}$			
(D) $\begin{cases} \dot{x} = (a-1)(2h-1)+(3h-1)x-hx^2+(h-1)xy \\ \dot{y} = 2a(h-1)+(3h-1)y-(h+1)xy+hy^2 \end{cases}$ where $ah(a-1)(h\pm1)(2h\pm1)(3h\pm1)\left(a+\frac{(2h+1)(h-1)^2}{(3h+1)^2}\right)\neq0$	$J_{D,1},\ J_{D,2},\ J_{D,3}$	$\mathscr{R}_D$	(3.73)	78, 79,87, 88, 89, 90, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103	L
	$\alpha_{D,1},\ \alpha_{D,2},\ \alpha_{D,3}$	—			
(D) where $h = 0$	$-1-a+x-y,\ \ -1+a+x(2-x+y),\ \ e^{x-y+1}$	$\mathscr{R}_{D,1}$	—	104, 105, 106, 107	L
	$-1,\ -2-y,\ a-x+y+1$	—			

(D) where $a=0$	$y,\ -1+x-y,\ 1-2h+hx-hy,\ -1+2x-x^2+xy$		(3.76)	110, 111, 112	D
	$(3h-1)+(-1-h)x+hy,\ (2h-1)-hx+hy,\ h-hx+hy,\ 2(-1+2h)-2hx+(-1+2h)y$	$\mathscr{F}_{D,2}$			
(D) where $a=h=0$	$y,\ -1+x-y,\ -1+2x-x^2+xy,\ e^{x-y+1}$		—	116	GD
	$-1-x,\ -1,\ 2-y,\ -x+y+1$	$\mathscr{F}_{D,3}$			
(D) where $a=-(h-1)^2(2h+1)/(3h+1)^2$	$J_{D,5},\ J_{D,6},\ J_{D,7},\ J_{D,8}$		(3.79)	139, 140	D
	$\alpha_{D,5},\ \alpha_{D,6},\ \alpha_{D,7},\ \alpha_{D,8}$	$\mathscr{F}_{D,4}$			
(D) where $h=0$ and $a=-1$	$x-y,\ -x^2+xy+2x-2,\ xy-y^2+2y-2,\ e^{x-y+1}$		—	146	GD
	$-1,\ -2-y,\ -2-x,\ -x+y$	$\mathscr{F}_{D,5}$			
(E) $\begin{cases} \dot{x}=x-\dfrac{x^2}{2}-\dfrac{xy}{2} \\ \dot{y}=b+y-\dfrac{3xy}{2}+\dfrac{y^2}{2} \end{cases}$ where $b(b+4)(25b-8)\neq 0$	$J_{E,1},\ J_{E,2},\ J_{E,3},\ J_{E,4}$		(3.82)	113, 114, 115, 117	D
	$\alpha_{E,1},\ \alpha_{E,2},\ \alpha_{E,3},\ \alpha_{E,4}$	$\mathscr{F}_{E}$			

(E) where $b=0$	$y,\ x,\ x-y,\ -2+x-y,\ x(-x+y+4)-4$		—	119	R
	$-\frac{3x}{2}+\frac{y}{2}+1,\ -\frac{x}{2}-\frac{y}{2}+1,\ -\frac{x}{2}+\frac{y}{2}+1,\ \frac{y}{2}-\frac{x}{2},\ -x$	$\mathscr{F}_{E,1}$			
(E) where $b=8/25$	$-5x+5y+2,\ 8-5x+5y,\ x,\ x(-x+y+4)-\frac{108}{25},\ \left(x-\frac{4}{5}\right)y-y^2-\frac{4}{25}$		—	147	R
	$\frac{1}{10}(-5x+5y+8),\ \frac{1}{10}(-5x+5y+2),\ -\frac{x}{2}-\frac{y}{2}+1,\ -x,\ -2x+y+\frac{8}{5}$	$\mathscr{F}_{E,2}$			
(F) $\begin{cases}\dot{x}=a(2h-1)-hx^2+(h-1)xy\\ \dot{y}=2a(h-1)-(h+1)xy+hy^2\end{cases}$ where $ah(h\pm1)(2h\pm1)(3h\pm1)(4h\pm1)\neq0$	$-\frac{\sqrt{a}}{\sqrt{h}}+x-y,\ \frac{\sqrt{a}}{\sqrt{h}}+x-y,\ a-x(x-y)$	$\mathscr{R}_F$	(3.88)	78, 79, 95, 96	L
	$-\sqrt{a}\sqrt{h}-hx+hy,\ \sqrt{a}\sqrt{h}-hx+hy,\ (2h-1)y-2hx$	—			
(F) where $h=0$	$a-x(x-y),\ e^{-\frac{x^2}{2}+xy-\frac{y^2}{2}+1}$	$\mathscr{R}_{F,1}$	—	108, 109	L
	$-y,\ ay-ax$	—			

(F) where $h=1/4$ and $a=-2\overline{a}$	$-2i\sqrt{2\overline{a}}+x-y,\ 2i\sqrt{2\overline{a}}+x-y,\ -2\overline{a}+xy,\ -2\overline{a}-x(x-y)$		(3.91)	141, 143	R
	$-i\sqrt{\frac{\overline{a}}{2}}-\frac{x}{4}+\frac{y}{4},\ i\sqrt{\frac{\overline{a}}{2}}-\frac{x}{4}+\frac{y}{4},\ -\frac{3}{2}x-\frac{y}{2},\ -\frac{x}{2}-\frac{y}{2}$	$\mathscr{F}_{F,2}$			
(G) $\begin{cases}\dot{x}=a-\dfrac{x^2}{3}-\dfrac{2xy}{3}\\ \dot{y}=4a+3v^2-\dfrac{4xy}{3}+\dfrac{y^2}{3}\end{cases}$ where $av(a+v^2)(a-3v^2)(a+3v^2/4)(a+8v^2/9)\neq 0$	$-3i\sqrt{a+v^2}-x+y,\ 3i\sqrt{a+v^2}-x+y,\ -3a+x(3iv-x+y),\ -3a+x(-3iv-x+y)$		(3.95)	144, 145	D
	$i\sqrt{a+v^2}-\frac{x}{3}+\frac{y}{3},\ -i\sqrt{a+v^2}-\frac{x}{3}+\frac{y}{3},\ -iv-\frac{2x}{3}-\frac{y}{3},\ iv-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{G}$			
(G) where $a=-v^2$	$x-y,\ 3v^2+3ivx-x^2+xy,\ 3v^2-3ivx-x^2+xy,\ e^{\frac{1}{x-y}}$		—	153	GD
	$-\frac{x}{3}+\frac{y}{3},\ -iv-\frac{2x}{3}-\frac{y}{3},\ iv-\frac{2x}{3}-\frac{y}{3},\ \frac{1}{3}$	$\mathscr{F}_{G,1}$			

(G) where $a = 3v^2$	$-6iv+x-y,\ 6iv+x-y,\ -9v^2+xy,\ -9v^2+3ivx-x^2+xy,\ -9v^2-3ivx-x^2+xy$		—	—	R
	$-2iv-\frac{x}{3}+\frac{y}{3},\ 2iv-\frac{x}{3}+\frac{y}{3},\ -\frac{5x}{3}-\frac{y}{3},\ -iv-\frac{2x}{3}-\frac{y}{3},\ iv-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{G,2}$			
(G) where $a = -3v^2/4$	$-\frac{3iv}{2}+x-y,\ \frac{3iv}{2}+x-y,\ y,\ \frac{ix^2}{3v}-\frac{ixy}{3v}-\frac{3iv}{4}+x,\ -\frac{ix^2}{3v}+\frac{ixy}{3v}+\frac{3iv}{4}+x$		—	151	R
	$\frac{1}{6}(-3iv-2x+2y),\ \frac{1}{6}(3iv-2x+2y),\ \frac{y}{3}-\frac{4x}{3},\ -iv-\frac{2x}{3}-t\frac{y}{3},\ iv-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{G,3}$			
(G) where $a = -8v^2/9$	$-iv+x-y,\ iv+x-y,\ y(x-y)+\frac{v^2}{3},\ \frac{8v^2}{3}+3ivx-x^2+xy,\ \frac{8v^2}{3}-3ivx-x^2+xy$		—	—	R
	$\frac{1}{3}(-iv-x+y),\ \frac{1}{3}(iv-x+y),\ \frac{2y}{3}-\frac{5x}{3},\ -iv-\frac{2x}{3}-\frac{y}{3},\ iv-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{G,4}$			

(H) $\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \frac{4xy}{3} + \frac{y^2}{3} \end{cases}$ where $av(a-v^2)(a+3v^2)(a-3v^2/4)(a-8v^2/9)\neq 0$	$-3\sqrt{-a+v^2} - x + y,\ 3\sqrt{-a+v^2} - x + y,\ -3a + x(3v - x + y),\ -3a+x(-3v-x+y)$		(3.97)	137, 138, 142	D
	$\sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3},\ -\sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3},\ -v-\frac{2x}{3}-\frac{y}{3},\ v-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_H$			
(H) where $a = v^2$	$x-y,\ -3v^2+x(3v-x+y),\ -3v^2+x(-3v-x+y),\ e^{\frac{1}{x-y}}$		—	152	GD
	$-\frac{x}{3}+\frac{y}{3},\ -v-\frac{2x}{3}-\frac{y}{3},\ v-\frac{2x}{3}-\frac{y}{3},\ \frac{1}{3}$	$\mathscr{F}_{H,1}$			
(H) where $a = -3v^2$	$-6v+x-y,\ 6v+x-y,\ 9v^2+xy,\ 9v^2+3vx-x^2+xy,\ 9v^2-3vx-x^2+xy$		—	—	R
	$-2v-\frac{x}{3}+\frac{y}{3},\ 2v-\frac{x}{3}+\frac{y}{3},\ -\frac{5x}{3}-\frac{y}{3},\ -v-\frac{2x}{3}-\frac{y}{3},\ v-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{H,2}$			

(H) where $a = 3v^2/4$	$-\frac{3v}{2}+x-y,\ \frac{3v}{2}+x-y,\ y,\ -\frac{x^2}{3v}+\frac{xy}{3v}-\frac{3v}{4}+x,\ \frac{x^2}{3v}-\frac{xy}{3v}+\frac{3v}{4}+x$		—	149	R
	$\frac{1}{6}(-3v-2x+2y),\ \frac{1}{6}(3v-2x+2y),\ \frac{y}{3}-\frac{4x}{3},\ \frac{1}{3}(-3v-2x-y),\ v-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{H,3}$			
(H) where $v = 0$	$-3i\sqrt{a}+x-y,\ 3i\sqrt{a}+x-y,\ -3a-x^2+xy,\ e^{\frac{x}{x(y-x)-3a}}$		—	154, 155	GD
	$-i\sqrt{a}-\frac{x}{3}+\frac{y}{3},\ i\sqrt{a}-\frac{x}{3}+\frac{y}{3},\ -\frac{2x}{3}-\frac{y}{3},\ -\frac{1}{3}$	$\mathscr{F}_{H,4}$			
(H) where $a = 8v^2/9$	$-v+x-y,\ v+x-y,\ y(x-y)-\frac{v^2}{3},\ -\frac{8v^2}{3}+3vx+x(y-x),\ -\frac{8v^2}{3}-3vx+x(y-x)$		—	—	R
	$\frac{1}{3}(-v-x+y),\ \frac{1}{3}(v-x+y),\ \frac{2y}{3}-\frac{5x}{3},\ -v-\frac{2x}{3}-\frac{y}{3},\ v-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_{H,5}$			

(I) $\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 5a - \frac{4xy}{3} + \frac{y^2}{3} \end{cases}$ where $a \neq 0$	$-2i\sqrt{3a}+x-y,\ 2i\sqrt{3a}+x-y,\ -3a+xy,\ -3a+i\sqrt{3a}x+x(x-y),\ -3a-i\sqrt{3a}x+x(x-y)$		(3.99)	158, 159	R
	$-2i\sqrt{\frac{a}{3}}-\frac{x}{3}+\frac{y}{3},\ 2i\sqrt{\frac{a}{3}}-\frac{x}{3}+\frac{y}{3},\ -\frac{5x}{3}-\frac{y}{3},\ -i\sqrt{\frac{a}{3}}-\frac{2x}{3}-\frac{y}{3},\ i\sqrt{\frac{a}{3}}-\frac{2x}{3}-\frac{y}{3}$	$\mathscr{F}_I$			
(J) $\begin{cases} \dot{x} = -\frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b - \frac{3xy}{2} + \frac{y^2}{2} \end{cases}$ where $b \neq 0$	$-i\sqrt{2b}-x+y,\ i\sqrt{2b}-x+y,\ x,\ -b-x^2+xy,\ -\frac{b}{2}+xy$		(3.102)	148, 150	R
	$i\sqrt{\frac{b}{2}}-\frac{x}{2}+\frac{y}{2},\ -i\sqrt{\frac{b}{2}}-\frac{x}{2}+\frac{y}{2},\ -\frac{x}{2}-\frac{y}{2},\ -x,\ -2x$	$\mathscr{F}_J$			
(K) $\begin{cases} \dot{x} = 4b-1+4y+x^2 \\ \dot{y} = b+y^2 \end{cases}$ where $b(b+1/4)(b+1) \neq 0$	$1-i\frac{y}{\sqrt{b}},\ 1+i\frac{y}{\sqrt{b}},\ -1+b-x+3y+xy-y^2$	$\mathscr{R}_K$	(3.105)	79, 92, 93, 96	L
	$-i\sqrt{b}+y,\ i\sqrt{b}+y,\ -1+x+2y$	—			
(K) where $b=0$	$y,\ -1-x+3y+xy-y^2,\ e^{1/y}$	$\mathscr{R}_{K,1}$	—	101	L
	$y,\ -1+x+2y,\ -1$	—			

(K) where $b=-1/4$	$1-2y,\ 1+2y,\ -\frac{5}{4}-x+3y+xy-y^2,\ e^{\frac{4\left(4x(y-8)y+x+52y^2-44y+1\right)}{4x(y-1)-4(y-3)y-5}}$		—	87	GD
	$y+\frac{1}{2},\ y-\frac{1}{2},\ -1+x+2y,\ -8(1+2y)$	$\mathscr{F}_{K,2}$			
(L) $\begin{cases}\dot{x}=a+x^2\\ \dot{y}=4a+y^2\end{cases}$ where $a\neq 0$	$1-i\frac{y}{2\sqrt{a}},\ 1+i\frac{y}{2\sqrt{a}},\ 1-i\frac{x}{\sqrt{a}},\ 1+i\frac{x}{\sqrt{a}},\ a-x^2+xy$		(3.107)	118, 120	R
	$-2i\sqrt{a}+y,\ 2i\sqrt{a}+y,\ -i\sqrt{a}+x,\ i\sqrt{a}+x,\ 2x+y$	$\mathscr{F}_L$			
(M) $\begin{cases}\dot{x}=a+x^2\\ \dot{y}=a+y^2\end{cases}$	$1-i\frac{y}{\sqrt{a}},\ 1+i\frac{y}{\sqrt{a}},\ 1-i\frac{x}{\sqrt{a}},\ 1+i\frac{x}{\sqrt{a}},\ x-y,\ 2a-r(x-y)+2xy$		(3.109)	160, 161	R
	$-2i\sqrt{a}+y,\ 2i\sqrt{a}+y,\ -i\sqrt{a}+x,\ i\sqrt{a}+x,\ x+y,\ x+y$	$\mathscr{F}_M$			
(M) where $a=0$	$x,\ y,\ x-y,\ r(x-y)+xy,\ e^{\frac{x+1}{x}},\ e^{\frac{y+1}{y}}$		—	162	R
	$x,\ y,\ x+y,\ x+y,\ -1,\ -1$	$\mathscr{F}_{M,1}=\frac{xy}{x-y}$			

ii) $\eta = 0$

Orbit representative $a, g, c, \varepsilon \in \mathbb{R} : a \neq 0$	Invariant curves/ ExpFac / Respective cofactors	Integrating Factor $\mathscr{R}_i$ / First integral $\mathscr{F}_i$	[*]	Config. $\tilde{H}$	Integ.
(N) $\begin{cases} \dot{x} = 2a + x + gx^2 + xy \\ \dot{y} = a(2g-1) - y + (g-1)xy + y^2, \end{cases}$	$a + xy$	—	(4.4)	1, 3, 4, 5, 6, 7, 8, 9, 10, 11	N-I
where $a(g-1)(2g-1)(3g-1)(g-2)(25a-3) \neq 0$	$(-1+2g)x + 2y$	—			
(N) where $g = 1/2$	$y,\ a + xy$	—	(4.10)	12, 13, 14, 15, 16, 17	N-I
	$-1 - \frac{x}{2} + y,\ 2y$	—			
(N) where $g = 2$ and $a = 3/25$	$xy + \frac{3}{25},\ x + \frac{5y^2}{9} - y + \frac{3}{5}$	—	—	—	open case
	$3x + 2y,\ 2x + 2y - \frac{1}{5}$	—			
(N) where $g = 1/3$ and $a = \overline{a}/2$	$\overline{a} + 2xy$	—	(4.11)	1, 4, 5, 6, 10	open case
	$2y - \frac{x}{3}$	—			
(O) $\begin{cases} \dot{x} = 2a + gx^2 + xy \\ \dot{y} = a(2g-1) + (g-1)xy + y^2 \end{cases}$	$a + xy$	—	(4.13)	1, 2, 6	N-I
where $a(g-1)(2g-1)(4g-1) \neq 0$	$(-1+2g)x + 2y$	—			
(O) where $g = 1/2$	$y,\ a + xy,\ e^{-\frac{a+2y^2}{2(a+xy)}}$		—	31, 32	GD
	$-\frac{x}{2} + y,\ 2y,\ y$	$\mathscr{F}_{O,1} = e^{\frac{a+2y^2}{a+xy}}(a + xy)$			
(O) where $g = 1/4$	$a + xy,\ e^{\frac{y^2}{a+xy}}$	$\mathscr{R}_{O,3} = \dfrac{e^{\frac{2y^2}{a+xy}}}{\sqrt{a+xy}}$	—	29, 30	L
	$-\frac{x}{2} + 2y,\ -y$	—			

(P) $\begin{cases} \dot{x} = 2a + 3cx + x^2 + xy \\ \dot{y} = a - c^2 + y^2, \end{cases}$ where $a(c^2-a)(9a-8c^2) \neq 0$	$y + \sqrt{c^2-a}$, $y - \sqrt{c^2-a}$, $a + cx + xy$	$\mathscr{R}_P$	(4.16)	18, 20, 21, 22, 33	L
	$y - \sqrt{c^2-a}$, $y + \sqrt{c^2-a}$, $2c + x + 2y$	—			
(P) where $a = c^2$	y, $c^2 + cx + xy$, $e^{\frac{1}{y}}$	$\mathscr{R}_{P,1}$	—	23	L
	y, $2c + x + 2y$, -1	—			
(P) where $a = 8c^2/9$	$3y - c$, $3y + c$, $8c^2 + 9cx + 9xy$, $e^{\frac{-cx+48cy+63xy-24y^2}{48c(8c^2+9cx+9xy)}}$		—	33	GD
	$\frac{c}{3} + y$, $y - \frac{c}{3}$, $2c + x + 2y$, $\frac{y}{18c} - \frac{1}{54}$	$\mathscr{F}_{P,2}$			
(Q) $\begin{cases} \dot{x} = (c+x)(c(2g-1) + gx) \\ \dot{y} = 1 + (g-1)xy \end{cases}$ where $cg(g\pm1)(2g-1)(3g-1) \neq 0$	$x + c$, $c(2g-1) + gx$, $\frac{1}{(-1+2g)} + cy + xy$		(4.18)	19	D
	$c(-1+2g) + gx$, $cg + gx$, $c(-1+2g) + (-1+2g)x$	$\mathscr{F}_Q$			
(Q) where $g = 0$ and $c \neq 0$	$c + x$, $-1 + cy + xy$, e^{x+1}		—	24	GD
	$-c$, $-c - x$, $-c^2 - cx$	$\mathscr{F}_{Q,1} = e^{x+1}(y(c+x)-1)^{-c}$			
(Q) where $c = 0$ and $g \neq 0, 1/2$	x, $\frac{1}{-1+2g} + xy$, $e^{\frac{1}{x}}$, $e^{\frac{2gxy+1}{x^2}}$		—	25, 34	D
	gx, $(-1+2g)x$, $-g$, $-2gy$	$\mathscr{F}_{Q,2} = \frac{x^{\frac{1}{g}-2}(2gxy - xy + 1)}{2g-1}$			

(R) $\begin{cases} \dot{x} = x^2+\varepsilon \\ \dot{y} = 1-2xy, \end{cases}$	$x+i\sqrt{\varepsilon},\ x-i\sqrt{\varepsilon},$ $-1+i\sqrt{\varepsilon}y+xy,$ $-1-i\sqrt{\varepsilon}y+xy$		(4.22)	27, 28	P/HAM
where $\varepsilon \neq 0$	$x-i\sqrt{\varepsilon},\ x+i\sqrt{\varepsilon},$ $-x-i\sqrt{\varepsilon},\ -x+i\sqrt{\varepsilon}$	$\mathscr{F}_R = (x^2+\varepsilon)\left((xy-1)^2+y^2\varepsilon\right)$			
(R) where $\varepsilon = 0$	$x,\ -1+xy,\ e^{\frac{1}{x}},$ $e^{\frac{2xy+1}{x^2}}, e^{\frac{y}{xy-1}},\ e^{\frac{y^2(2xy-3)}{(xy-1)^2}}$		—	34	P/HAM
	$x,\ -x,\ -1,$ $-6y,\ -1,\ -6y$	$\mathscr{F}_{R,1} = x(-1+xy)$			
(S) $\begin{cases} \dot{x} = (x-1)(3-x) \\ \dot{y} = 1-2xy \end{cases}$	$1-x,\ 3-x,$ $\frac{1}{3}-y+xy,$ $-\frac{19}{8}+x+3y-\frac{x^2}{8}$		(4.25)	19	R
	$3-x,\ 1-x,\ 3-3x,$ $-2x$	$\mathscr{F}_S = \dfrac{(3-x)^2}{\frac{1}{3}-y+xy}$			
(T) $\begin{cases} \dot{x} = -x^2 \\ \dot{y} = 1-2xy \end{cases}$	$x,\ -1+3xy,\ e^{\frac{1}{x}}$ $e^{\frac{1-2xy}{x^2}},\ e^{\frac{1-3xy-2x^2y+x}{x^3}}$		—	26	R
	$-x,\ -3x,\ 1,$ $2y,\ 2y$	$\mathscr{F}_T = \dfrac{x^3}{-1+3xy}$			
(U) $\begin{cases} \dot{x} = (2x-1)(2x+1)/4 \\ \dot{y} = y \end{cases}$	$1+2x,\ 1-2x,\ y,$ $-\frac{q}{2}+qx+y+2xy,$ $e^y,\ e^{\frac{-2x+y+1}{1-2x}}$		(4.27)	35	R
	$-\frac{1}{2}+x,\ \frac{1}{2}+x,\ 1,$ $\frac{1}{2}+x,\ y,\ \frac{y}{2}$	$\mathscr{F}_U = \dfrac{(2x+1)y}{q\left(x-\frac{1}{2}\right)+2xy+y}$			

(V) $\begin{cases}\dot{x}=x^2\\ \dot{y}=1\end{cases}$	$x,\ 1+rx+xy,\ e^{\frac{x+1}{x}}$ $e^{\frac{x^2+2xy+x+1}{x^2}},\ e^{y^2+y+1}$		(4.28)	36	R
	$x,\ x,\ -1,$ $-1-2y,\ 2y+1$	$\mathscr{F}_V=\dfrac{x}{1+rx+xy}$			
(W) $\begin{cases}\dot{x}=a+y+x^2\\ \dot{y}=xy\end{cases}$	$y,\ -i\sqrt{a}+x-\frac{iy}{\sqrt{a}},$ $i\sqrt{a}+x+\frac{iy}{\sqrt{a}},$ $a+2y+x^2-m^2y^2$		(4.30)	39, 41	R
where $a\neq 0$	$x,\ i\sqrt{a}+x,$ $-i\sqrt{a}+x,\ 2x$	$\mathscr{F}_W=\dfrac{y^2}{a+2y+x^2-m^2y^2}$			
(W) where $a=0$	$y,\ 2y+x^2-m^2y^2,$ $e^{\frac{x}{y}},\ e^{\frac{x^2+2xy+2y^2}{2y^2}}$		—	43	R
	$x,\ 2x,\ 1,\ 1$	$\mathscr{F}_{W,1}=\dfrac{y^2}{2y+x^2-m^2y^2}$			
(X) $\begin{cases}\dot{x}=(1+3x)(2+3x)/9\\ \dot{y}=xy\end{cases}$	$y,\ \ 2+3x,\ \ 1+3x,\ \ 4+$ $12x+9x^2+my+3mxy$		(4.34)	37	R
	$x,\ \frac{1}{3}+x,\ \frac{2}{3}+x, \frac{2}{3}+2x$	$\mathscr{F}_X=\dfrac{(3x+1)y}{(3x+2)^2}$			
(Y) $\begin{cases}\dot{x}=a+x^2\\ \dot{y}=xy\end{cases}$	$y,\ 1-\frac{ix}{\sqrt{a}},\ 1+\frac{ix}{\sqrt{a}},$ $a+x^2-m^2y^2$		(4.36)	38,40	R
where $a\neq 0$	$x,\ -i\sqrt{a}+x,\ i\sqrt{a}+x, 2x$	$\mathscr{F}_Y=\dfrac{x^2+a}{ay^2}$			

(Z) $\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1 + xy \end{cases}$	$x, \ 1 + mx^2 + 2xy,$ $e^{1/x}, \ e^{\frac{x^2+2xy+x+1}{x^2}}$		(4.38)	42	R
	$x, \ 2x, \ -1, \ -1$	$\mathscr{F}_Z = \dfrac{x^2}{1 + mx^2 + 2xy}$			

$$\mathscr{R}_{A,1} = \frac{\left(\frac{-\sqrt{1-4ah}-1}{2h}+y\right)^{\frac{2a(h-1)}{4ah+\sqrt{1-4ah}-1}} \left(\frac{\sqrt{1-4ah}-1}{2h}+y\right)^{\frac{(h-1)\left(2ah+\sqrt{1-4ah}-1\right)}{h\left(4ah+\sqrt{1-4ah}-1\right)}}}{(a+xy)^2};$$

$$\mathscr{F}_{A,3} = (a+xy)(h(x-y)+1)^{\frac{1}{h}-2};$$

$$\mathscr{R}_{A,4} = 8y^{\frac{1}{2}-h}(2hx-(2h+1)y+x+4)^{-h-\frac{1}{2}}\left((2h+1)^2xy+4\right)^{h-1};$$

$$\mathscr{F}_{A,5} = \left(\frac{1}{h}+x-y\right)^{\frac{1-2h}{h}}\left(\frac{1}{4h^2}+xy\right);$$

$$J_{A,5} = \frac{(3g+1)^2x^2}{54g-18} - \frac{(3g+1)^2xy}{18(3g-1)} + \frac{(3g+1)x}{3(3g-1)} + 1;$$

$$\alpha_{A,5} = 2gx + \frac{2}{3g+1} - \frac{y}{3};$$

$$J_{A,6} = -\frac{68x^2}{147} + \frac{136xy}{147} + x - \frac{68y^2}{147} + \frac{76y}{49} - \frac{708}{833};$$

$$J_{A,7} = -\frac{x^3}{2} + x^2y + \frac{18x^2}{17} - \frac{xy^2}{2} + \frac{30xy}{17} - \frac{252x}{289} + \frac{18y}{289} - \frac{216}{4913};$$

$$\alpha_{A,6} = -\frac{8x}{3} + \frac{2y}{3} - \frac{15}{17};$$

$$\alpha_{A,7} = \frac{3}{17} - 4x;$$

$$\mathscr{R}_{A,9} = \frac{1}{\sqrt[3]{xy-\frac{36}{289}}\left(-\frac{x^3}{2}+x^2\left(y+\frac{18}{17}\right)-\frac{1}{578}x(17y(17y-60)+504)+\frac{18(17y-12)}{4913}\right)^{5/6}\sqrt[6]{-\frac{68x^2}{147}+\frac{136xy}{147}+x-\frac{4(17y(17y-57)+531)}{2499}}};$$

$$\mathscr{R}_{A,10} = \frac{\sqrt[6]{y}}{\left(x-y+\frac{12}{5}\right)^{5/6}\left(xy+\frac{36}{25}\right)^{2/3}};$$

$$\mathscr{R}_{B,8} = \frac{1}{\sqrt{(x-y)^2-4a}\,(a+xy)};$$

$$\mathscr{F}_{B,8} = \left(\sqrt{(x-y)^2-4a}+x-y\right)^{-h-1}(a+xy)^h\left(y^2\left(\sqrt{(x-y)^2-4a}-x+y\right)-a\left(\sqrt{(x-y)^2-4a}+x+3y\right)\right)^{-h};$$

$$\mathscr{R}_{B,9} = -\frac{125a}{\sqrt[3]{a+xy}\,(225a^2-5a\,(25x^2-30xy+3y^2)+y^4)};$$

$$\mathscr{R}_{B,10} = -\frac{45a}{\sqrt[3]{a+xy}\,(225a^2-5a\,(3x^2-30xy+25y^2)+x^4)};$$

$$\mathscr{F}_{B,11} = e^{y(x-y)+1}(xy-\overline{a})^{\overline{a}};$$

$$\mathscr{F}_{C} = \left(\frac{a}{2h-1}-x^2+xy\right)\left(-\frac{a}{2h}+xy-y^2\right)^{-\frac{2h-1}{2h}};$$

$$\mathscr{F}_{C,1} = (x-y)^2(-(2a+x(x-y)))(2a-xy)(2a+y(y-x));$$

$$J_{D,1} = -\frac{3}{2} + \frac{1}{2h} - \frac{\sqrt{1+h(-2+4a+h)}}{2h} + x - y;$$

$$J_{D,2}=-\frac{3}{2}+\frac{1}{2h}+\frac{\sqrt{1+h(-2+4a+h)}}{2h}+x-y;$$

$$J_{D,3}=-1+a+x(2-x+y);$$

$$\alpha_{D,1}=\frac{1}{2}(-1-\sqrt{1+h(-2+4a+h)}+h(3-2x+2y));$$

$$\alpha_{D,2}=\frac{1}{2}(-1+\sqrt{1+h(-2+4a+h)}+h(3-2x+2y));$$

$$\alpha_{D,3}=2(-1+2h)-2hx+(-1+2h)y;$$

$$\mathscr{R}_D=\frac{\left(\frac{\sqrt{h(4a+h-2)+1}+1}{2h}+x-y-\frac{3}{2}\right)^{\frac{(h-1)\left(\sqrt{h(4a+h-2)+1}+h+1\right)}{2h\sqrt{h(4a+h-2)+1}}}\left(-\frac{\sqrt{h(4a+h-2)+1}+3h-1}{2h}+x-y\right)^{\frac{(h-1)\left(\sqrt{h(4a+h-2)+1}-h-1\right)}{2h\sqrt{h(4a+h-2)+1}}}}{(a+x(-x+y+2)-1)^2};$$

$$\mathscr{R}_{D,1}=\frac{e^{-x+y-1}(-a+x-y-1)^{1-a}}{(a+x(-x+y+2)-1)^2};$$

$$\mathscr{F}_{D,2}=y^h(h(x-y-2)+1)^{h-1}(x(-x+y+2)-1)^{-h};$$

$$\mathscr{F}_{D,3}=\frac{ye^{-x+y-1}}{x(-x+y+2)-1};$$

$$J_{D,5}=x-y-\frac{4h}{3h+1};$$

$$J_{D,6}=x-y+\frac{1-5h^2}{h(3h+1)};$$

$$J_{D,7}=\frac{2(-1+h)^3}{(3h+1)^2}+\frac{(2-6h)y}{(3h+1)}+xy-y^2;$$

$$J_{D,8}=-\frac{2(1+h)^3}{(3h+1)^2}+2x-x^2+xy;$$

$$\alpha_{D,5}=\frac{5h^2-1}{3h+1}-hx+hy;$$

$$\alpha_{D,6}=\frac{4h^2}{3h+1}-hx+hy;$$

$$\alpha_{D,7}=\frac{2(-1+h+6h^2)}{3h+1}+(-1-2h)x+2hy;$$

$$\alpha_{D,8}=2(-1+2h)-2hx+(-1+2h)y;$$

$$\mathscr{F}_{D,4}=\frac{\left(x-y+\frac{1-5h^2}{h(3h+1)}\right)\left(-\frac{2(1+h)^3}{(3h+1)^2}+2x-x^2+xy\right)^h}{\left(\frac{2(-1+h)^3}{(3h+1)^2}+\frac{(2-6h)y}{(3h+1)}+xy-y^2\right)^h};$$

$$\mathscr{F}_{D,5}=\frac{e^{-x+y-1}\left(-(x+2)y+y^2+2\right)}{x^2-x(y+2)+2};$$

$$J_{E,1}=1-\sqrt{1-2b}-x+y;$$

$$J_{E,2} = 1+\sqrt{1-2b}-x+y;$$

$$J_{E,3} = x;$$

$$J_{E,4} = -4-b+4x-x^2+xy;$$

$$\alpha_{E,1} = \frac{1}{2}(1+\sqrt{1-2b}-x+y);$$

$$\alpha_{E,2} = \frac{1}{2}(1-\sqrt{1-2b}-x+y);$$

$$\alpha_{E,3} = 1-\frac{x}{2}-\frac{y}{2};$$

$$\alpha_{E,4} = -x;$$

$$\mathscr{F}_E = -x^{\frac{2b-3\sqrt{1-2b}-1}{b+4}}\left(\sqrt{1-2b}+x-y-1\right)\left(\sqrt{1-2b}-x+y+1\right)^{\frac{b-3\sqrt{1-2b}-5}{b+4}}(-b+x(-x+y+4)-4)^{\frac{-2b+3\sqrt{1-2b}+1}{b+4}};$$

$$\mathscr{F}_{E,1} = \frac{y}{x(-x+y+2)^2};$$

$$\mathscr{F}_{E,2} = \frac{25x(25x(x-y-4)+108)(5x-5y-8)^3(-5x+5y+2)^2}{(5y(-5x+5y+4)+4)^2};$$

$$\mathscr{R}_F = \frac{\left((x-y)^2-\frac{a}{h}\right)^{\frac{h-1}{2h}}}{(a+x(y-x))^2};$$

$$\mathscr{R}_{F,1} = \frac{\left(e^{-\frac{x^2}{2}+xy-\frac{y^2}{2}+1}\right)^{-1/a}}{\left(a-x^2+xy\right)^2};$$

$$\mathscr{F}_{F,2} = -\frac{(2\overline{a}+x(x-y))^2\left(8\overline{a}+(x-y)^2\right)}{2\overline{a}-xy};$$

$$\mathscr{F}_G = \frac{(-3i\sqrt{a+v^2}-x+y)(-3a+x(3iv-x+y))^{\frac{\sqrt{a+v^2}}{v}}}{(3i\sqrt{a+v^2}-x+y)(-3a+x(-3iv-x+y))^{\frac{\sqrt{a+v^2}}{v}}};$$

$$\mathscr{F}_{G,1} = \frac{\left(e^{\frac{1}{x-y}}\right)^{6iv}\left(3v^2+3ivx-x^2+xy\right)}{3v^2-3ivx-x^2+xy};$$

$$\mathscr{F}_{G,2} = \frac{\left(9v^2-3ivx+x(x-y)\right)^2(6iv+x-y)}{xy-9v^2};$$

$$\mathscr{F}_{G,3} = \frac{\left(9v^2+4(x-y)^2\right)^2\left(4xy+(3v-2ix)^2\right)\left(4xy+(3v+2ix)^2\right)}{2304v^2y^2};$$

$$\mathscr{F}_{G,4} = \frac{(-iv+x-y)^3\left(8v^2-9ivx+3x(y-x)\right)}{v^2+3y(x-y)};$$

$$\mathscr{F}_H = \frac{(-3\sqrt{-a+v^2}-x+y)(-3a+x(3v-x+y))^{\frac{\sqrt{-a+v^2}}{v}}}{(3\sqrt{-a+v^2}-x+y)(-3a+x(-3v-x+y))^{\frac{\sqrt{-a+v^2}}{v}}};$$

$$\mathscr{F}_{H,1} = \frac{\left(e^{\frac{1}{x-y}}\right)^{6v}\left(3v^2-3vx+x(x-y)\right)}{3v^2+3vx+x(x-y)};$$

$$\mathscr{F}_{H,2} = \frac{(-6v+x-y)\left(9v^2-3vx+x(y-x)\right)^2}{9v^2+xy};$$

$$\mathscr{F}_{H,3} = \frac{(3v-2x+2y)^2\left((3v+2x)^2-4xy\right)}{48vy};$$

$$\mathscr{F}_{H,4} = \frac{\left(e^{\frac{x}{x(y-x)-3a}}\right)^{-6i\sqrt{a}}(-3i\sqrt{a}+x-y)}{3i\sqrt{a}+x-y};$$

$$\mathscr{F}_{H,5} = -\frac{\left(8v^2+9vx+3x(x-y)\right)(v-x+y)^3}{v^2+3y(y-x)};$$

$$\mathscr{F}_{I} = \frac{(12a+(x-y)^2)(9a^2+3ax(3x-2y)+x^2(x-y)^2)^2}{(-3a+xy)^2};$$

$$\mathscr{F}_{J} = -\frac{2(b+x(x-y))^2}{b-2xy};$$

$$\mathscr{R}_{K} = \frac{\left(1-i\frac{y}{\sqrt{b}}\right)^{\frac{-i+\sqrt{b}}{\sqrt{b}}}\left(1+i\frac{y}{\sqrt{b}}\right)^{\frac{i-\sqrt{b}}{\sqrt{b}}}}{(-1+b-x+3y+xy-y^2)^2};$$

$$\mathscr{R}_{K,1} = \frac{y^2\exp^{2/y}}{(-1-x+3y+xy-y^2)^2};$$

$$\mathscr{F}_{K,2} = (1-2y)^{16}e^{\frac{4\left(4x(y-8)y+x+52y^2-44y+1\right)}{4x(y-1)-4(y-3)y-5}}$$

$$\mathscr{F}_{L} = \frac{\left(1-i\frac{y}{2\sqrt{a}}\right)\left(1+i\frac{x}{\sqrt{a}}\right)^2}{\left(1+i\frac{y}{2\sqrt{a}}\right)\left(1-i\frac{x}{\sqrt{a}}\right)^2};$$

$$\mathscr{F}_{M} = \frac{(\sqrt{a}+ix)(\sqrt{a}-iy)}{a(x-y)};$$

$$\mathscr{R}_{P} = (y+\sqrt{c^2-a})^{\frac{1}{2}\left(1+\frac{c}{\sqrt{c^2-a}}\right)}(y-\sqrt{c^2-a})^{\frac{1}{2}\left(1-\frac{c}{\sqrt{c^2-a}}\right)}(a+cx+xy)^{-2};$$

$$\mathscr{R}_{P,1} = y\left(c+x+\frac{xy}{c}\right)^{-2}e^{\frac{-c}{y}};$$

$$\mathscr{F}_{P,2} = (c+3y)\left(e^{\frac{-cx+48cy+63xy-24y^2}{48c\left(8c^2+9cx+9xy\right)}}\right)^{-18c};$$

$$\mathscr{F}_{Q} = (c(2g-1)+gx)\left(y(c+x)+\frac{1}{2g-1}\right)^{-\frac{g}{2g-1}}.$$

5.2 Proof of Main Theorem 1 and Main Theorem 2 for the non-integrable cases

5.2.1 Systems with $\eta > 0$

Consider the following sets:

$E_1 = \{(a,g,h) : h = 1/2 \text{ and } a \neq 0\}$, $E_2 = \{(a,g,h) : g = 0 \text{ and } a \neq 0\}$,

$E_3 = \{(a,g,h) : g = 1/2 \text{ and } a \neq 0\}$, $E_4 = \{(a,g,h) : h = 0 \text{ and } a \neq 0\}$,

$E_5 = \{(a,g,h) : g = h \text{ and } a \neq 0\}$,

$E_6 = \cup_{k\in\mathbb{N}} E_{6,k}$ where $E_{6,k} = \{(a,g,h) : g+h = -k \text{ and } a \neq 0\}$, $k \in \mathbb{N}$,

$E_7 = \cup_{k\in\mathbb{N}} E_{7,k}$ where $E_{7,k} = \{(a,g,h) : g+h = -\frac{k}{2} \text{ and } a \neq 0\}$, $k \in \mathbb{N}$,

$E_8 = \{(a,g,h) : g+h = 1 \text{ and } a \neq 0\}$, $E_9 = \{(a,g,h) : 4agh = 1\}$,

$E_{10} = \left\{(a,g,h) : a(g+h)^2 = 1\right\}$, $E_{11} = \{(a,g,h) : g = 1/4 \text{ and } a \neq 0\}$,

$E_{12} = \{(a,g,h) : h = 1/4 \text{ and } a \neq 0\}$, $E_{13} = \{(a,g,h) : g = -h \text{ and } a \neq 0\}$,

$E_{14} = \{(a,g,h) : g = 2,\ h = -2/5 \text{ and } a \neq 0\}$, $E_{15} = \{(a,g,h) : h = 2,\ g = -2/5 \text{ and } a \neq 0\}$.

5.2.1.1 *The systems* (A)

$$\begin{cases} \dot{x} = a(2h-1) + x + gx^2 + (h-1)xy \\ \dot{y} = a(2g-1) - y + (g-1)xy + hy^2, \end{cases}$$

where $a(g-1)(h-1)(3g-1)(3h-1) \neq 0$.

Theorem 76. (a) If $(a,g,h) \notin E := \bigcup_{i=1}^{7} E_i$ then the only affine invariant algebraic curves of a system in the family (A) are of the form $J_1^m = 0$ where $J_1(x,y) = a + xy$ and m is a positive integer.

(b) If $(a,g,h) \notin \tilde{E} := E \cup E_8$ then any system in the family (A) has no exponential factors.

(c) If $(a,g,h) \notin \tilde{E}$ then any system in the family (A) is not Liouvillian integrable.

Proof.

(a) By a simple calculation can be verified that $J_1(x,y) = a + xy$ is an invariant algebraic curve with cofactor $\alpha_1(x,y) = (-1+2g)x + (-1+2h)y$.

STEP 1: Set $C = \sum_{i=0}^{n} C_i(x,y) = 0$ another invariant algebraic curve of the systems (A) with cofactor $K = K_0 + K_1x + K_2y$, where C_i are homogeneous polynomial of degree i where

$0 \leq i \leq n$. From the definition of the invariant algebraic curve, we have:

$$\begin{aligned} &\left[a(2h-1)+x+gx^2+(h-1)xy\right] \sum_{i=0}^{n} C_{i,x}+ \\ &+\left[a(2g-1)-y+(g-1)xy+hy^2\right] \sum_{i=0}^{n} C_{i,y}= \\ &=(K_0+K_1x+K_2y)\sum_{i=0}^{n} C_i \end{aligned} \tag{5.1}$$

Separating from (5.1) the terms of degree $n+1$:

$$\left[gx^2+(h-1)xy\right] C_{n,x}+\left[(g-1)xy+hy^2\right] C_{n,y}=(K_1x+K_2y)\,C_n. \tag{5.2}$$

For the systems (A) we have:

$$yP_2-xQ_2=xy(x-y).$$

Then, from Lemma 64 we can assume that:

$$C_n=x^m\, y^l\, (x-y)^p, \text{ where } n=m+l+p.$$

Substituting C_n into (5.2) and doing some computations we obtain:

$$\begin{aligned} K_1 &= gm+(g-1)l+gp, \\ K_2 &= (h-1)m+hl+hp. \end{aligned}$$

STEP 2: We rename by J the hyperbola J_1 and by α its cofactor α_1. Consider the change using a birational transformation:

$$(x,y)\longrightarrow \left(x,\tfrac{J-a}{x}\right).$$

Considering the new variables x and J, set

$$\begin{array}{rccc} \Phi: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,J) & \longmapsto & \left(x,\tfrac{J-a}{x}\right). \end{array}$$

We have that the transformed invariant algebraic curve

$$\hat{f}=J\circ\Phi(x,J)=a+x\left(\tfrac{J-a}{x}\right)=J$$

with transformed cofactor

$$\hat{K}=\alpha\circ\Phi(x,J)=(-1+2g)x+(-1+2h)\tfrac{J-a}{x}$$

is associated to the new system obtained by this change

$$\begin{cases} \dot{x}=ah+x+(h-1)J+gx^2=:\tilde{P}(x,J) \\ \\ \dot{J}=\left((-1+2g)x+(-1+2h)\tfrac{J-a}{x}\right)J=:\tilde{Q}(x,J). \end{cases} \tag{5.3}$$

(For a reference, see (FERRAGUT; GASULL, 2015)). If before we wanted to find an invariant algebraic curve

$$C(x,y)=\sum_{i+j=0}^{n}a_{i,j}x^iy^j$$

now the transformed curve is

$$\tilde{h}(x,J)=C\circ\Phi(x,J)=C\left(x,\tfrac{J-a}{x}\right)=\sum_{i+j=0}^{n}a_{i,j}x^i\left(\tfrac{J-a}{x}\right)^j=\sum_{j=0}^{n}\tilde{h}_j(x)J^j,$$

where $\tilde{h}_j(x)=\dfrac{D_j(x)}{x^{\tilde{n}}}$ with $D_j(x)$ a polynomial in x and $\tilde{n}\in\mathbb{N}$. If before the cofactor was

$$K(x,y)=K_0+K_1x+K_2y,$$

now after the transformation K becomes

$$\tilde{K}(x,J)=K\circ\Phi(x,J)=K\left(x,\tfrac{J-a}{x}\right)=K_0+K_1x+K_2\left(\tfrac{J-a}{x}\right).$$

Doing $y=\frac{J-a}{x}$ in (5.1) and simplifying the result we observe that in fact we can put the result in this form:

$$\tilde{h}_x\tilde{P}+\tilde{h}_J\tilde{Q}=\tilde{K}\tilde{h}.$$

Therefore,

$$\left(\frac{\partial\tilde{h}_0}{\partial x}+\frac{\partial\tilde{h}_1}{\partial x}J+\frac{\partial\tilde{h}_2}{\partial x}J^2+\dots\right)\tilde{P}+\left(\tilde{h}_1+2\tilde{h}_2J+3\tilde{h}_3J^2+\dots\right)\tilde{Q}=$$
$$=\left(K_0+K_1x+K_2\left(\tfrac{J-a}{x}\right)\right)\left(\tilde{h}_0+\tilde{h}_1J+\tilde{h}_2J^2+\dots\right).$$

By doing $J=0$ we have:

$$\frac{\partial\tilde{h}_0}{\partial x}(ah+x+gx^2)=\left(K_0+K_1x-K_2\left(\tfrac{a}{x}\right)\right)\tilde{h}_0$$

$$\int\frac{d\tilde{h}_0}{\tilde{h}_0}=\int\frac{\left(K_0+K_1x-K_2\left(\frac{a}{x}\right)\right)}{ah+x+gx^2}dx$$

Therefore,

$$\tilde{h}_0(x)=\left(\tfrac{-1+4agh}{4g}\right)^{\beta}x^{-\frac{k2}{h}}\left(1-\frac{i(2gx+1)}{\sqrt{4agh-1}}\right)^{\beta+\alpha}\left(1+\frac{i(2gx+1)}{\sqrt{4agh-1}}\right)^{\beta-\alpha}$$

where

$$\beta=\frac{1}{2}\left(\frac{K_1}{g}+\frac{K_2}{h}\right),\ \alpha=\frac{i(g(2hK_0+K_2)-hK_1)}{2gh\sqrt{4agh-1}},\ (a,g,h)\notin \mathrm{E}_2,\mathrm{E}_4,\mathrm{E}_9.$$

We know that

$$\tilde{h}_0(x)=\frac{D_0(x)}{x^{\tilde{n}}},$$

where D_0 is a polynomial. Then,

$$\frac{K_2}{h}=\tilde{n},\ \tilde{n}\in\mathbb{N}.$$

Consider the linear map

$$\sigma : x \to -y,\ \ y \to -x,\ \ g \to h,\ \ h \to g. \tag{5.4}$$

Note that the system

$$W(X) := \begin{cases} \dot{x} = a(2g-1) + x + gx^2 + (h-1)xy \\ \dot{y} = a(2h-1) - y + (g-1)xy + hy^2 \\ \dot{g} = 0 \\ \dot{h} = 0 \end{cases}$$

is invariant by the transformation $((x,y,g,h),t) \to (\sigma(x,y,g,h),-t)$ since

$$\sigma(W(X)) = (-1)W(\sigma(X)), \text{ for } (a,g,h) \notin \cup_{i=1}^{4} \mathrm{E}_i.$$

Then, by lemma 2.2 of (FERRAGUT; GASULL, 2015) we obtain the change in the cofactor:

$$K_0 + K_1 x + K_2 y \ \to -\overline{K}_0 + \overline{K}_2 x + \overline{K}_1 y, \tag{5.5}$$

where $\overline{K}_i = K_i\,|_{\{g\to h,\ h\to g\}}$, $i = 0,1,2$. Therefore,

$$K_1 \to \overline{K}_2 = K_2\,|_{\{g\to h,\ h\to g\}}\,.$$

Then, as

$$K_2 = h\tilde{n},\ \tilde{n} \in \mathbb{N}$$

it follows that

$$K_1 = g\tilde{n},\ \tilde{n} \in \mathbb{N}.$$

Therefore, we have:

$$\beta = \tilde{n},\ \alpha = \frac{iK_0}{\sqrt{-1+4agh}},\ \tilde{n} \in \mathbb{N}.$$

On the other hand, for

$$\left(1 - \frac{i(2gx+1)}{\sqrt{4agh-1}}\right)^{\beta+\alpha} \left(1 + \frac{i(2gx+1)}{\sqrt{4agh-1}}\right)^{\beta-\alpha}$$

to be a polynomial we must have

$$\begin{cases} \beta + \alpha = n_1,\ n_1 \in \mathbb{N} \\ \beta - \alpha = n_2,\ n_2 \in \mathbb{N} \end{cases}$$

and $-1+4agh = -p^2$ or $n_1 = n_2$.

<u>**Case 1**</u>: Suppose $-1+4agh = -p^2$. Then,

$$\begin{cases} K_1 = \left(\dfrac{n_1+n_2}{2}\right)g,\ \ K_2 = \left(\dfrac{n_1+n_2}{2}\right)h,\ \ K_0 = \left(\dfrac{n_1-n_2}{2}\right)p, \\ \\ \tilde{h}_0(x) = \tilde{C}\,x^{-\left(\frac{n_1+n_2}{2}\right)}(p-(1+2gx))^{n_1}(p+(1+2gx))^{n_2}, \end{cases}$$

where $(a,g,h) \notin \bigcup_{i=1}^{4} \mathrm{E}_i \cup \mathrm{E}_9$.

STEP 3:

By step 1, we have:

$$C_n = x^m y^l (x-y)^p.$$

If $(a,g,h) \notin \cup_{i=1}^{4} \mathrm{E}_i$, then $W(X)$ is invariant by the transformation

$$((x,y,g,h),t) \to (\sigma(x,y,g,h),-t)$$

and by lemma 2.2 of (FERRAGUT; GASULL, 2015), we have:

$$C_n = x^m y^l (x-y)^p \to (-1)^{m+l} x^l y^m (x-y)^p.$$

Therefore, $m = l$ and $C_n = (xy)^l (x-y)^{n-2l}$. By step 2, we have:

$$K_1 = \tilde{n}g, \quad K_2 = \tilde{n}h, \text{ where } \tilde{n} = \frac{n_1+n_2}{2}.$$

Now, let's use C_n, K_1 and K_2 to find conditions over $\tilde{n}$ or l. Separating from (5.1) the terms of degree $n+1$, we have:

$$\left[gx^2 + (h-1)xy\right] C_{n,x} + \left[(g-1)xy + hy^2\right] C_{n,y} = (K_1 x + K_2 y) C_n.$$

That is,

$$-(x-y)^{n-2l}(xy)^l((g\tilde{n} + l - gn)x + (h\tilde{n} + l - hn)y) = 0.$$

Then,

$$\begin{cases} h = g \text{ and } l = g(n-\tilde{n}) \text{ or} \\ l = 0 \text{ and } \tilde{n} = n. \end{cases}$$

If $(a,g,h) \notin \mathrm{E}_5$ then $l = 0$ and $\tilde{n} = n$. Therefore,

$$\begin{cases} C_n = (x-y)^n, \\ K_1 = ng, \\ K_2 = nh. \end{cases} \tag{5.6}$$

Using (5.6), let's obtain C_{n-1}.

STEP 4: Separating from (5.1) the terms of degree n :

$$\begin{aligned} &x\, C_{n,x} + [gx + (h-1)y]\, x\, C_{n-1,x} - y\, C_{n,y} + [(g-1)x + hy]\, y\, C_{n-1,y} = \\ &= K_0 C_n + (K_1 x + K_2 y)\, C_{n-1}. \end{aligned} \tag{5.7}$$

It is a know result that: $x\, C_{N,x} + y\, C_{N,y} = N\, C_N$, for all $N \in \mathbb{N}$ (for a reference see (WEISSTEIN, 2011)). Then,

$$x\, C_{N,x} = NC_N - y\, C_{N,y}, \text{ for all } N \in \mathbb{N}. \tag{5.8}$$

Replacing (5.8) and (5.6) in (5.7), we obtain:

$$(x-y)\left[-y\, C_{n-1,y} + (x-y)^{n-2}\left(n(x+y) - K_0(x-y)\right)\right] =$$

$$= (gx + (h+n-1)y)\, C_{n-1}.$$

Then,

$$C_{n-1} = (x-y)\, T_{n-2},$$

where $g \neq -h-n+1$ and T_{n-2} is a polynomial of degree $n-2$. Using that

$$C_{n-1,y} = -T_{n-2} + (x-y)T_{n-2,y}$$

and replacing above, we obtain:

$$(x-y)\left[-y\, T_{n-2,y} + (x-y)^{n-3}\left(n(x+y) - K_0(x-y)\right)\right] =$$

$$= (gx + (h+n-2)y)\, T_{n-2}.$$

Then,

$$T_{n-2} = (x-y)T_{n-3},$$

where $g \neq -h-n+2$ and T_{n-3} is a polynomial of degree $n-3$. Using that

$$T_{n-2,y} = -T_{n-3} + (x-y)T_{n-3,y}$$

and continuing this recursively, we concluded that:

$$(x-y)\left[-y\, T_{1,y} + (x-y)^0\left(n(x+y) - K_0(x-y)\right)\right] = (gx + (h+1)y)\, T_1.$$

Then,

$$T_1 = (x-y)T_0,$$

where $g \neq -h-1$ and T_0 is a constant. Using that

$$T_{1,y} = -T_0 + (x-y)T_{0,y} = -T_0,$$

we have:

$$\begin{aligned} \left[y\, T_0 + \left(n(x+y) - K_0(x-y)\right)\right] &= (gx + (h+1)y)\, T_0, \\ (n - K_0 - gT_0)x + (n + K_0 - hT_0)y &= 0. \end{aligned}$$

Therefore,

$$K_0 = \frac{(-g+h)n}{g+h}, \quad T_0 = \frac{2n}{g+h}, \quad C_{n-1} = \frac{2n}{g+h}(x-y)^{n-1}, \tag{5.9}$$

where $(a,g,h) \notin \mathrm{E}_6$.

Separating from (5.1) the terms of degree $n-1$:

$$a(2h-1)\, C_{n,x} + x\, C_{n-1,x} + [gx+(h-1)y]\, x\, C_{n-2,x} + a(2g-1)\, C_{n,y} - y\, C_{n-1,y} +$$

$$+[(g-1)x+hy]\, y\, C_{n-2,y} = K_0 C_{n-1} + (K_1 x + K_2 y)\, C_{n-2}. \tag{5.10}$$

Replacing (5.8), (5.6) and (5.9) in (5.10) we have:

$$(x-y)\Big[\left(a(2h-1)n - a(2g-1)n - \frac{(-g+h)}{(g+h)^2} 2n^2\right)(x-y)^{n-2} +$$

$$+\frac{2n(n-1)}{g+h}(x+y)(x-y)^{n-3} - y\, C_{n-2,y}\Big] = (2gx+(2h+n-2)y)\, C_{n-2}.$$

Then,

$$C_{n-2} = (x-y)\ \tilde{T}_{n-3},$$

where $2g \neq -2h-n+2$ and $\tilde{T}_{n-3}$ is a polynomial of degree $n-3$. Using that

$$C_{n-2,y} = -\tilde{T}_{n-3} + (x-y)\tilde{T}_{n-3,y}$$

and replacing above, we have:

$$(x-y)\Big[\left(a(2h-1)n - a(2g-1)n - \frac{(-g+h)}{(g+h)^2} 2n^2\right)(x-y)^{n-3} +$$

$$+\frac{2n(n-1)}{g+h}(x+y)(x-y)^{n-4} - y\, \tilde{T}_{n-3,y}\Big] = (2gx+(2h+n-3)y)\, \tilde{T}_{n-3}.$$

Then,

$$\tilde{T}_{n-3} = (x-y)\tilde{T}_{n-4},$$

where $2g \neq -2h-n+3$ and $\tilde{T}_{n-4}$ is a polynomial of degree $n-4$. Using that

$$\tilde{T}_{n-3,y} = -\tilde{T}_{n-4} + (x-y)\tilde{T}_{n-4,y}$$

and continuing this recursively, we concluded that:

$$(x-y)\Big[\left(a(2h-1)n - a(2g-1)n - \frac{(-g+h)}{(g+h)^2} 2n^2\right)(x-y) +$$

$$+\frac{2n(n-1)}{g+h}(x+y)(x-y)^{0} - y\, \tilde{T}_{1,y}\Big] = (2gx+(2h+1)y)\, \tilde{T}_{1}.$$

Then,

$$\tilde{T}_1 = (x-y)\tilde{T}_0,$$

where $2g \neq -2h-1$ and $\tilde{T}_0$ is a constant. Using that

$$\tilde{T}_{1,y} = -\tilde{T}_0 + (x-y)\tilde{T}_{0,y} = -\tilde{T}_0$$

we have:

$$(x-y)\Big[\left(a(2h-1)n - a(2g-1)n - \frac{(-g+h)}{(g+h)^2}2n^2\right)(x-y)+$$

$$+\frac{2n(n-1)}{g+h}(x+y)\Big] = (2gx+2hy)\,\tilde{T}_0.$$

Therefore,

$$\begin{cases} n=0 \text{ and } \tilde{T}_0 = 0 \text{ or} \\ a = \dfrac{1}{(g+h)^2},\ n=1 \text{ and } \tilde{T}_0 = 0 \text{ or} \\ g=h,\ n=1 \text{ and } \tilde{T}_0 = 0,\ \text{for } (a,g,h) \notin \mathrm{E}_7. \end{cases}$$

We conclude that for $(a,g,h) \notin \bigcup_{i=1}^{7} \mathrm{E}_i \cup \mathrm{E}_9 \cup \mathrm{E}_{10}$ the systems (A) will not have an algebraic solution different from the hyperbolas $a+xy=0$.

Note that if $(a,g,h) \in \mathrm{E}_{10}$, this is $a = \frac{1}{(g+h)^2}$, then we also do not have additional algebraic solutions according to the above proof. This concludes the case 1.

Case 2: Suppose $n_1 = n_2$. Then,

$$\begin{cases} K_1 = n_1 g,\ \ K_2 = n_1 h,\ \ K_0 = 0, \\ \tilde{h}_0(x) = \tilde{C}\,x^{-n_1}(4g(ah+x+gx^2))^{n_1}. \end{cases}$$

By STEP 3 done above, we have exactly the same calculations. Therefore,

$$C_n = (x-y)^n,\ \ K_1 = ng,\ \ K_2 = nh.$$

By STEP 4, following the same calculations above and doing $K_0 = 0$ we arrive at the conditions:

$$\begin{cases} g = h \text{ or} \\ n = 0 \text{ and } T_0 = 0. \end{cases}$$

We conclude that for $(a,g,h) \notin \bigcup_{i=1}^{4} \mathrm{E}_i \cup \mathrm{E}_9$ the systems (A) will not have an algebraic solution different from the hyperbolas $a+xy=0$. This concludes case 2.

Now, if $(a,g,h) \in \mathrm{E}_9$ then $a = \frac{1}{4gh}$ and returning to STEP 2 we have:

$$\tilde{h}_0(x) = \left(-\frac{1}{2gx}\right)^{\frac{(-2gK_0+K_1)}{g}} (1+2gx)^{\frac{K_1}{g}+\frac{(-2gK_0+K_1)}{h}}.$$

Using the transformation (5.4) we obtain:

$$K_0 = 0, \;\; K_1 = g\tilde{n}, \;\; K_2 = h\tilde{n}, \text{ where } \tilde{n} \in \mathbb{N}.$$

Then,

$$\tilde{h}_0(x) = \left(-\frac{1}{2gx}\right)^{\frac{K_1}{g}} (1+2gx)^{\frac{K_1}{g}+\frac{K_1}{h}}.$$

Following STEP 3 done previously, we have:

$$C_n = (x-y)^n, \;\; K_1 = ng, \;\; K_2 = nh.$$

Following STEP 4, as we have that $K_0 = 0$, we arrive at the conditions:

$$\begin{cases} \tilde{T}_0 = 0 \text{ and } n = 0 \text{ or} \\ \\ g = h. \end{cases}$$

Then, if $(a,g,h) \notin \mathrm{E}_5$ it follows that the system (A) for $(a,g,h) \in \mathrm{E}_9$ does not admit an algebraic solution different from the hyperbolas $J_1(x,y) = a + xy$.

(b) Considering $(a,g,h) \notin E := \bigcup_{i=1}^{7} \mathrm{E}_i$, the only algebraic solution of systems (A) are the hyperbolas $J_1(a,y) = a + xy$. Calculating the 2th extactic polynomial, we obtain that the multiplicity of J_1 is 1. Then, if systems (A) have an exponential factor, we can assume that it has the form

$$F = \exp(G)$$

with a cofactor $L = L_0 + L_1x + L_2y$, where l is non-negative integers and

$$G(x,y) = \sum_{i=0}^{n} G_i(x,y),$$

where G_i is a homogeneous polynomial of degree i. From (2.2) we have that G satisfy:

$$\begin{aligned} &\left[a(2h-1)+x+gx^2+(h-1)xy\right] G_x + \left[a(2g-1)-y+(g-1)xy+hy^2\right] G_y = \\ &= L_0 + L_1x + L_2y \end{aligned} \tag{5.11}$$

Note that the degree from the first part of the above equality is $n+1$. We have the following cases to consider:

(i) If $n=0$ then $G(x,y)$ is a constant. Therefore, $L\equiv 0$ what can not happen.

(ii) If $n=1$ then

$$G(x,y)=g_0+g_1x+g_2y.$$

Replacing $G(x,y)$ in (5.11) we obtain that for $(a,g,h)\notin E$,

$$L_0=L_1=L_2=0.$$

Therefore, $L\equiv 0$ what can not happen.

(iii) If $n+1>1$ consider $G_n=\sum_{i=0}^{n} c_{n-i}\, x^{n-i}y^i$, where c_{n-i} are constants.

Equating of (5.11) the terms of degree $n+1$ we have:

$$[gx^2+(h-1)xy]\sum_{i=0}^{n}(n-i)c_{n-i}\, x^{n-i-1}y^i+$$
$$+[(g-1)xy+hy^2]\sum_{i=0}^{n} ic_{n-i}\, x^{n-i}y^{i-1}=0$$

Doing some calculations we can write this last equation as

$$\sum_{i=0}^{n+1}[(gn-i)c_{n-i}+((h-1)n+i-l)c_{n-i+1}]x^{n-i+1}y^i=0,$$

where $c_i=0$ for $i<0$ and $i>n$. The last equation is equivalent to

$$(gn+(h-1)n)\ c_n+(gn+(h-1)n)\ c_{n-1}+(gn+(h-1)n)\ c_{n-2}+\ldots$$

$$\ldots+(gn+(h-1)n)\ c_1+(gn+(h-1)n)\ c_0=0.$$

Therefore, if $(a,g,h)\notin \mathrm{E}_8$, where $\mathrm{E}_8=\{(a,g,h):g+h=1\}$ it follows that:

$$G(x,y)\equiv 0,$$

what can not happen. Then, for $(a,g,h)\notin \tilde{E}=\bigcup_{i=1}^{8}\mathrm{E}_i\supset E$, systems (A) do not admit exponential factors.

(c) If $(a,g,h)\notin \tilde{E}$ according to (a) and (b) systems (A) have only the algebraic solution

$$J_1(x,y)=a+xy$$

with cofactors $\alpha_1(x,y)=(-1+2g)x+(-1+2h)y$ and they have no exponential factor. Under this assumptions

$$\begin{cases}\lambda_1\alpha_1=0\Leftrightarrow\lambda_1=0,\\[2ex] \lambda_1\alpha_1=-div(P,Q)\Leftrightarrow\lambda_1=0.\end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (A) are not Liouvillian integrable.

□

5.2.1.2 The systems (B)

$$\begin{cases} \dot{x} = a(2h-1) + gx^2 + (h-1)xy \\ \dot{y} = a(2g-1) + (g-1)xy + hy^2, \end{cases}$$

where $a(g-1)(h-1)(2g-1)(2h-1) \neq 0$.

Theorem 77. (a) If $(a,g,h) \notin E := \mathrm{E}_2 \cup \mathrm{E}_4 \bigcup_{i=5}^{7} \mathrm{E}_i \cup \mathrm{E}_{13}$ then the only affine invariant algebraic curves of a system in the family (B) are of the form $J_1^m = 0$ where $J_1(x,y) = a + xy$ and m is a positive integer.

(b) If $(a,g,h) \notin \tilde{E} := E \cup \mathrm{E}_8$ then any system in the family (B) has no exponential factors.

(c) If $(a,g,h) \notin \tilde{E}$ then any system in the family (B) is not Liouvillian integrable.

Proof.

(a) By a simple calculation can be verified that $J_1(x,y) = a + xy$ is an invariant algebraic curve with cofactor $\alpha_1(x,y) = (-1+2g)x + (-1+2h)y$.

STEP 1: Set $C = \sum_{i=0}^{n} C_i(x,y) = 0$ another invariant algebraic curve of the systems (A) with cofactor $K = K_0 + K_1 x + K_2 y$, where C_i are homogeneous polynomial of degree i where $0 \leq i \leq n$. From the definition of the invariant algebraic curve, we have:

$$\left[a(2h-1) + gx^2 + (h-1)xy\right] \sum_{i=0}^{n} C_{i,x} + \left[a(2g-1) + (g-1)xy + hy^2\right] \sum_{i=0}^{n} C_{i,y} = \\ = (K_0 + K_1 x + K_2 y) \sum_{i=0}^{n} C_i. \tag{5.12}$$

Separating from (5.12) the terms of degree $n+1$:

$$\left[gx^2 + (h-1)xy\right] C_{n,x} + \left[(g-1)xy + hy^2\right] C_{n,y} = (K_1 x + K_2 y) C_n. \tag{5.13}$$

For the systems (B) we have:

$$yP_2 - xQ_2 = xy(x-y).$$

Then, from Lemma 64 we can assume that:

$$C_n = x^m\, y^l\, (x-y)^p, \text{ where } n = m + l + p.$$

Substituting C_n into (5.13) and doing some computations we obtain:

$$K_1 = gm + (g-1)l + gp,$$
$$K_2 = (h-1)m + hl + hp.$$

STEP 2: We rename by J the hyperbola J_1 and by α its cofactor α_1. Consider the change using a birational transformation:

$$(x,y) \longrightarrow \left(x, \tfrac{J-a}{x}\right).$$

Considering the new variables x and J, set

$$\begin{array}{rccc} \Phi: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,J) & \longmapsto & \left(x, \tfrac{J-a}{x}\right). \end{array}$$

We have that the transformed invariant algebraic curve

$$\hat{f} = J \circ \Phi(x,J) = a + x\left(\tfrac{J-a}{x}\right) = J$$

with transformed cofactor

$$\hat{K} = \alpha \circ \Phi(x,J) = (-1+2g)x + (-1+2h)\tfrac{J-a}{x}$$

is associated to the new system obtained by this change

$$\begin{cases} \dot{x} = ah + (h-1)J + gx^2 =: \tilde{P}(x,J) \\ \\ \dot{J} = \left((-1+2g)x + (-1+2h)\tfrac{J-a}{x}\right)J =: \tilde{Q}(x,J). \end{cases} \tag{5.14}$$

(For a reference, see (FERRAGUT; GASULL, 2015)). If before we wanted to find an invariant algebraic curve

$$C(x,y) = \sum_{i+j=0}^{n} a_{i,j}x^i y^j$$

now the transformed curve is

$$\tilde{h}(x,J) = C \circ \Phi(x,J) = C\left(x, \tfrac{J-a}{x}\right) = \sum_{i+j=0}^{n} a_{i,j}x^i \left(\tfrac{J-a}{x}\right)^j = \sum_{j=0}^{n} \tilde{h}_j(x)J^j,$$

where $\tilde{h}_j(x) = \dfrac{D_j(x)}{x^{\tilde{n}}}$ with $D_j(x)$ a polynomial in x and $\tilde{n} \in \mathbb{N}$. If before the cofactor was

$$K(x,y) = K_0 + K_1 x + K_2 y,$$

now after the transformation K becomes

$$\tilde{K}(x,J) = K \circ \Phi(x,J) = K\left(x, \tfrac{J-a}{x}\right) = K_0 + K_1 x + K_2\left(\tfrac{J-a}{x}\right).$$

Doing $y=\frac{J-a}{x}$ in (5.12) and simplifying the result we observe that in fact we can put the result in this form:

$$\tilde{h}_x\tilde{P}+\tilde{h}_J\tilde{Q}=\tilde{K}\tilde{h}.$$

Therefore,

$$\left(\frac{\partial\tilde{h}_0}{\partial x}+\frac{\partial\tilde{h}_1}{\partial x}J+\frac{\partial\tilde{h}_2}{\partial x}J^2+\dots\right)\tilde{P}+\left(\tilde{h}_1+2\tilde{h}_2J+3\tilde{h}_3J^2+\dots\right)\tilde{Q}=$$
$$=\left(K_0+K_1x+K_2\left(\tfrac{J-a}{x}\right)\right)\left(\tilde{h}_0+\tilde{h}_1J+\tilde{h}_2J^2+\dots\right).$$

By doing $J=0$ we have:

$$\frac{\partial\tilde{h}_0}{\partial x}(ah+gx^2)=\left(K_0+K_1x-K_2\left(\tfrac{a}{x}\right)\right)\tilde{h}_0$$

$$\int\frac{d\tilde{h}_0}{\tilde{h}_0}=\int\frac{\left(K_0+K_1x-K_2\left(\frac{a}{x}\right)\right)}{ah+gx^2}dx$$

Therefore,

$$\tilde{h}_0(x)=\left(\tfrac{1}{ah}\right)^{\beta}x^{-\frac{K2}{h}}\left(1-\frac{i\sqrt{g}x}{\sqrt{ah}}\right)^{\beta+\alpha}\left(1+\frac{i\sqrt{g}x}{\sqrt{ah}}\right)^{\beta-\alpha}$$

where

$$\beta=\frac{1}{2}\left(\frac{K_1}{g}+\frac{K_2}{h}\right),\quad \alpha=\frac{iK_0}{2\sqrt{agh}},\quad (a,g,h)\notin \mathrm{E}_2,\mathrm{E}_4.$$

We know that

$$\tilde{h}_0(x)=\frac{D_0(x)}{x^{\tilde{n}}},$$

where D_0 is a polynomial. Then,

$$\frac{K_2}{h}=\tilde{n},\ \tilde{n}\in\mathbb{N}.$$

Consider the linear map

$$\sigma: x\to -y,\ \ y\to -x,\ \ g\to h,\ \ h\to g. \tag{5.15}$$

Note that the system

$$\tilde{W}(X):=\begin{cases}\dot{x}=a(2g-1)+gx^2+(h-1)xy\\ \dot{y}=a(2h-1)+(g-1)xy+hy^2\\ \dot{g}=0\\ \dot{h}=0\end{cases}$$

is invariant by the transformation $((x,y,g,h),t)\to(\sigma(x,y,g,h),-t)$ since

$$\sigma(\tilde{W}(X))=(-1)\tilde{W}(\sigma(X)),\ \text{for}\ (a,g,h)\notin \mathrm{E}_2\cup\mathrm{E}_4.$$

Then, by lemma 2.2 of (FERRAGUT; GASULL, 2015) we obtain the change in the cofactor:

$$K_0+K_1x+K_2y\ \to -\overline{K}_0+\overline{K}_2x+\overline{K}_1y, \tag{5.16}$$

where $\overline{K}_i = K_i \mid_{\{g\to h,\ h\to g\}}$, $i = 0,1,2$. Therefore,

$$K_1 \to \overline{K}_2 = K_2 \mid_{\{g\to h,\ h\to g\}}.$$

Then, as

$$K_2 = h\tilde{n},\ \tilde{n} \in \mathbb{N}$$

it follows that

$$K_1 = g\tilde{n},\ \tilde{n} \in \mathbb{N}.$$

Therefore, we have:

$$\beta = \tilde{n},\ \alpha = \frac{iK_0}{2\sqrt{agh}},\ \tilde{n} \in \mathbb{N}.$$

On the other hand, for

$$\left(1 - \frac{i\sqrt{g}x}{\sqrt{ah}}\right)^{\beta+\alpha}\left(1 + \frac{i\sqrt{g}x}{\sqrt{ah}}\right)^{\beta-\alpha}$$

to be a polynomial we must have

$$\begin{cases} \beta + \alpha = n_1,\ n_1 \in \mathbb{N} \\ \beta - \alpha = n_2,\ n_2 \in \mathbb{N}. \end{cases}$$

and $\dfrac{g}{ah} = -p^2$ or $n_1 = n_2$.

<u>**Case 1**</u>: Suppose that $\dfrac{g}{ah} = -p^2$. Then,

$$\begin{cases} K_1 = \left(\dfrac{n_1+n_2}{2}\right)g,\ \ K_2 = \left(\dfrac{n_1+n_2}{2}\right)h,\ \ K_0 = \left(\dfrac{n_1-n_2}{2}\right)ahp, \\ \tilde{h}_0(x) = \tilde{C}\, x^{-\frac{n_1+n_2}{2}}(1+px)^{n_1}(1-px)^{n_2}, \end{cases}$$

where $(a,g,h) \notin \mathrm{E}_2 \cup \mathrm{E}_4$.

STEP 3: By step 1, we have:

$$C_n = x^m y^l (x-y)^p.$$

If $(a,g,h) \notin \mathrm{E}_2 \cup \mathrm{E}_4$, then $\tilde{W}(X)$ is invariant by the transformation

$$((x,y,g,h),t) \to (\sigma(x,y,g,h),-t)$$

and by lemma 2.2 of (FERRAGUT; GASULL, 2015), we have:

$$C_n = x^m y^l (x-y)^p \to (-1)^{m+l} x^l y^m (x-y)^p.$$

Therefore, $m = l$ and $C_n = (xy)^l (x-y)^{n-2l}$. By step 2, we have:

$$K_1 = \tilde{n}g,\ \ K_2 = \tilde{n}h,\ \text{where } \tilde{n} \in \mathbb{N}.$$

Now, let's use C_n, K_1 and K_2 to find conditions over $\tilde{n}$ or l. Separating from (5.12) the terms of degree $n+1$ we have:

$$\left[gx^2+(h-1)xy\right]C_{n,x}+\left[(g-1)xy+hy^2\right]C_{n,y}=(K_1x+K_2y)\,C_n.$$

That is,

$$-(x-y)^{n-2l}(xy)^l((g\tilde{n}+l-gn)x+(h\tilde{n}+l-hn)y)=0$$

Then,

$$\begin{cases} h=g \text{ and } l=g(n-\tilde{n}) \text{ or} \\ l=0 \text{ and } \tilde{n}=n. \end{cases}$$

If $(a,g,h)\notin \mathrm{E}_5$ then $l=0$ and $\tilde{n}=n$. Therefore,

$$\begin{cases} C_n=(x-y)^n, \\ K_1=ng, \\ K_2=nh. \end{cases} \tag{5.17}$$

Using (5.17), let's obtain C_{n-1}.

STEP 4: Separating from (5.12) the terms of degree n :

$$\begin{aligned} &[gx+(h-1)y]\,x\,C_{n-1,x}+[(g-1)x+hy]\,y\,C_{n-1,y}= \\ &=K_0C_n+(K_1x+K_2y)\,C_{n-1}. \end{aligned} \tag{5.18}$$

Replacing (5.8) and (5.17) in (5.18), we obtain:

$$(x-y)\left[-y\,C_{n-1,y}-K_0(x-y)^{n-1}\right]=(gx+(h+n-1)y)\,C_{n-1}.$$

Then,

$$C_{n-1}=(x-y)\;T_{n-2},$$

where $g\neq -h-n+1$ and T_{n-2} is a polynomial of degree $n-2$. Using that

$$C_{n-1,y}=-T_{n-2}+(x-y)T_{n-2,y}$$

and replacing above, we have:

$$(x-y)\left[-y\,T_{n-2,y}-K_0(x-y)^{n-2}\right]=(gx+(h+n-2)y)\,T_{n-2}.$$

Then,

$$T_{n-2}=(x-y)T_{n-3},$$

where $g\neq -h-n+2$ and T_{n-3} is a polynomial of degree $n-3$. Using that

$$T_{n-2,y}=-T_{n-3}+(x-y)T_{n-3,y}$$

and continuing this recursively, we concluded that:

$$(x-y)\left[-y\,T_{1,y}-K_0(x-y)\right]=(gx+(h+1)y)\,T_1.$$

Then,

$$T_1=(x-y)T_0,$$

where $g\neq -h-1$ and T_0 is a constant. Using that

$$T_{1,y}=-T_0$$

we have:

$$\begin{aligned}[y\,T_0-K_0(x-y)] &= (gx+(h+1)y)\,T_0,\\ (K_0+gT_0)x+(-K_0+hT_0)y &= 0.\end{aligned}$$

Therefore,

$$\begin{cases} K_0=-gT_0 \text{ and } h=-g \text{ or}\\ \\ K_0=0 \text{ and } T_0=0, \text{ for } (a,g,h)\notin \mathrm{E}_6.\end{cases} \tag{5.19}$$

Suppose that $(a,g,h)\notin \mathrm{E}_{13}$. Then, $K_0=0$ and $C_{n-1}\equiv 0$.

Separating from (5.12) the terms of degree $n-1$:

$$\begin{aligned}&a(2h-1)\,C_{n,x}+[gx+(h-1)y]\,x\,C_{n-2,x}+a(2g-1)\,C_{n,y}+\\ &+[(g-1)x+hy]\,y\,C_{n-2,y}=(gnx+hny)\,C_{n-2}.\end{aligned} \tag{5.20}$$

Replacing (5.8), (5.17) and (5.19) in (5.20), we have:

$$(x-y)\left[(2h-2g)an(x-y)^{n-2}-y\,C_{n-2,y}\right]=(2gx+(2h+n-2)y)\,C_{n-2}.$$

Then,

$$C_{n-2}=(x-y)\,\tilde{T}_{n-3},$$

where $2g\neq -2h-n+2$ and $\tilde{T}_{n-3}$ is a polynomial of degree $n-3$. Using that

$$C_{n-2,y}=-\tilde{T}_{n-3}+(x-y)\tilde{T}_{n-3,y}$$

and replacing above, we have:

$$(x-y)\left[(2h-2g)an(x-y)^{n-3}-y\,\tilde{T}_{n-3,y}\right]=(2gx+(2h+n-3)y)\,\tilde{T}_{n-3}.$$

Then,

$$\tilde{T}_{n-3}=(x-y)\tilde{T}_{n-4},$$

where $2g\neq -2h-n+3$ and $\tilde{T}_{n-4}$ is a polynomial of degree $n-4$. Using that

$$\tilde{T}_{n-3,y}=-\tilde{T}_{n-4}+(x-y)\tilde{T}_{n-4,y}$$

and continuing this recursively, we concluded that:

$$(x-y)\left[(2h-2g)an(x-y)-y\,\tilde{T}_{1,y}\right]=(2gx+(2h+1)y)\,\tilde{T}_1.$$

Then,

$$\tilde{T}_1=(x-y)\tilde{T}_0,$$

where $2g\neq -2h-1$ and $\tilde{T}_0$ is a constant. Using that

$$\tilde{T}_{1,y}=-\tilde{T}_0$$

we have:

$$(2h-2g)an(x-y)=(2gx+2hy)\,\tilde{T}_0$$

$$(2g\tilde{T}_0-(2h-2g)an)x+(2h\tilde{T}_0+(2h-2g)an)y=0.$$

Therefore,

$$\begin{cases} g=h=0 \text{ or} \\ \\ h=-g \text{ and } \tilde{T}_0=-2an \text{ or} \\ \\ h=g \text{ and } \tilde{T}_0=0 \text{ or} \\ \\ \tilde{T}_0=0 \text{ and } n=0 \text{ or} \\ \\ g=0,\; n=0 \text{ and } \tilde{T}_0=0, \text{ for } (a,g,h)\notin \mathrm{E}_7. \end{cases}$$

We conclude that for $(a,g,h)\notin \mathrm{E}_2\cup\mathrm{E}_4\bigcup_{i=5}^{7}\mathrm{E}_i\cup\mathrm{E}_{13}=:E$ the systems (B) will not have an algebraic solution different from the hyperbolas $a+xy=0$.

Case 2: Suppose that $n_1=n_2$. Then,

$$\begin{cases} K_1=n_1g,\;\; K_2=n_1h,\;\; K_0=0, \\ \\ \tilde{h}_0(x)=\tilde{C}\,x^{-n_1}(ah+gx^2))^{n_1}. \end{cases}$$

By STEP 3 done above, we have exactly the same calculations. Therefore,

$$C_n=(x-y)^n,\;\; K_1=ng,\;\; K_2=nh.$$

By STEP 4, following the same calculations above and doing $K_0=0$ we arrive at the conditions:

$$\begin{cases} g=h=0 \text{ or} \\ \\ T_0=0, \text{ for } (a,g,h)\notin \mathrm{E}_6. \end{cases}$$

Then, for $(a,g,h) \notin \mathrm{E}_2 \cup \mathrm{E}_4 \cup \mathrm{E}_6$ we have $C_{n-1} \equiv 0$. Separating the terms of degree $n-1$ to obtain C_{n-2} we obtain again that:

$$\begin{cases} g = h = 0 \text{ or} \\ h = -g \text{ and } \tilde{T}_0 = -2an \text{ or} \\ h = g \text{ and } \tilde{T}_0 = 0 \text{ or} \\ \tilde{T}_0 = 0 \text{ and } n = 0 \text{ or} \\ g = 0,\ n = 0 \text{ and } \tilde{T}_0 = 0, \text{ for } (a,g,h) \notin \mathrm{E}_7. \end{cases}$$

We conclude that for $(a,g,h) \notin \mathrm{E}_2 \cup \mathrm{E}_4 \bigcup_{i=5}^{7} \mathrm{E}_i \cup \mathrm{E}_{13} =: E$ the systems (B) will not have an algebraic solution different from the hyperbolas $a + xy = 0$. This concludes case 2.

(b) Considering $(a,g,h) \notin E$, the only algebraic solution of systems (B) are the hyperbolas $J_1(a,y) = a + xy$. Calculating the 2th extactic polynomial, we obtain that the multiplicity of J_1 is 1. Then, if systems (B) have an exponential factor, we can assume that it has the form

$$F = \exp(G)$$

with a cofactor $L = L_0 + L_1 x + L_2 y$, where l is non-negative integers and

$$G(x,y) = \sum_{i=0}^{n} G_i(x,y),$$

where G_i is a homogeneous polynomial of degree i. From (2.2) we have that G satisfy:

$$\begin{aligned} &\left[a(2h-1) + gx^2 + (h-1)xy\right] G_x + \left[a(2g-1) + (g-1)xy + hy^2\right] G_y = \\ &= L_0 + L_1 x + L_2 y \end{aligned} \tag{5.21}$$

Note that the degree from the first part of the above equality is $n+1$. We have the following cases to consider:

(i) If $n = 0$ then $G(x,y)$ is a constant. Therefore, $L \equiv 0$ what can not happen.

(ii) If $n = 1$ then

$$G(x,y) = g_0 + g_1 x + g_2 y.$$

Replacing $G(x,y)$ in (5.21) we obtain that for $(a,g,h) \notin E$,

$$L_0 = L_1 = L_2 = 0.$$

Therefore, $L \equiv 0$ what can not happen.

(iii) If $n+1>1$ consider $G_n=\sum_{i=0}^{n} c_{n-i}\, x^{n-i}y^i$, where c_{n-i} are constants.

Equating the terms of (5.21) with degree $n+1$ we have:

$$[gx^2+(h-1)xy]\sum_{i=0}^{n}(n-i)c_{n-i}\, x^{n-i-1}y^i+$$
$$+[(g-1)xy+hy^2]\sum_{i=0}^{n} ic_{n-i}\, x^{n-i}y^{i-1}=0$$

Doing some calculations we can write this last equation as

$$\sum_{i=0}^{n+1}[(gn-i)c_{n-i}+((h-1)n+i-l)c_{n-i+1}]x^{n-i+1}y^i=0,$$

where $c_i=0$ for $i<0$ and $i>n$. The last equation is equivalent to

$$(gn+(h-1)n)\; c_n+(gn+(h-1)n)\; c_{n-1}+(gn+(h-1)n)\; c_{n-2}+\ldots$$

$$\ldots+(gn+(h-1)n)\; c_1+(gn+(h-1)n)\; c_0=0.$$

Therefore, if $(a,g,h)\notin \mathrm{E}_8$, where $\mathrm{E}_8=\{(a,g,h): g+h=1 \text{ and } a\neq 0\}$ it follows that:

$$G(x,y)\equiv 0,$$

what can not happen. Then, for $(a,g,h)\notin \tilde{E}:=E\cup \mathrm{E}_8$, systems (A) do not admit exponential factors.

(c) If $(a,g,h)\notin \tilde{E}$ according to (a) and (b) systems (B) have only the algebraic solution

$$J_1(x,y)=a+xy$$

with cofactors $\alpha_1(x,y)=(-1+2g)x+(-1+2h)y$ and they have no exponential factor. Under this assumptions

$$\begin{cases} \lambda_1\alpha_1=0 \Leftrightarrow \lambda_1=0 \text{ or } g=h=\dfrac{1}{2}. \\ \\ \lambda_1\alpha_1=-div(P,Q) \Leftrightarrow \lambda_1=0 \text{ or } g=h. \end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (B) are not Liouvillian integrable.

□

5.2.2 Systems with $\eta = 0$

Consider the sets:

$$\mathrm{L}_1 = \cup_{k\in\mathbb{N}} \mathrm{L}_{1,k}, \text{ where } \mathrm{L}_{1,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/2 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N},$$

$$\mathrm{L}_2 = \cup_{k\in\mathbb{N}} \mathrm{L}_{2,k}, \text{ where } \mathrm{L}_{2,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/3 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N},$$

$$\mathrm{L}_3 = \left\{(a,g) \in \mathbb{R}^2 : g = 1/4 \text{ and } a \neq 0\right\},$$

$$\mathrm{C}' = \cup_{k\in\mathbb{N}} \mathrm{C}_k, \text{ where } \mathrm{C}_k = \left\{(a,g) \in \mathbb{R}^2 : g = (2+a-2ak)/4a \text{ and } a \neq 0\right\}, \ k \in \mathbb{N}.$$

5.2.2.1 The systems (N)

$$\begin{cases} \dot{x} = 2a + x + gx^2 + xy, \\ \dot{y} = a(2g-1) - y + (g-1)xy + y^2, \end{cases}$$

where $a(g-1) \neq 0$.

Theorem 78. (a) If $(a,g) \notin \mathrm{L}_1$ then the only affine invariant algebraic curves of a system in the family (N) are of the form $J_1^m = 0$ where $J_1(x,y) = a + xy$ and m is a positive integer.

(b) If $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$ then any system in the family (N) has no exponential factors.

(c) If $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$ then any system in the family (N) is not Liouvillian integrable.

Observation 79. When $(a,g) \in \mathrm{L}_{1,1}$ the systems (N) also posses the invariant line $y = 0$ but this additional invariant curve is still not enough to prove the existence of a first integral. The proof for the non existence of a Liouvillian first integral in this case can be done just by adapting $g = 1/2$ in the proof below. The only invariant algebraic curves will be of the form $y^l(a+xy)^m$ where l and m are positive integers.

Proof.

(a) By a straightforward computation, we can verify that $J_1(x,y) = a + xy$ is an invariant hyperbola with cofactor $\alpha_1(x,y) = (-1+2g)x + 2y$. Assume that

$$C = \sum_{i=0}^{n} C_i(x,y) = 0$$

is an invariant algebraic curve of the system (N) with cofactor $K = K_0 + K_1x + K_2y$, where C_i are homogeneous polynomial of degree i where $0 \leq i \leq n$. From the definition of invariant algebraic curve (1.4), we have:

$$(2a + x + gx^2 + xy)\sum_{i=0}^{n} C_{i,x} + (a(2g-1) - y + (g-1)xy + y^2)\sum_{i=0}^{n} C_{i,y} = \\ = (K_0 + K_1x + K_2y)\sum_{i=0}^{n} C_i. \tag{5.22}$$

Taking from (5.22) the terms of degree $n+1$ we have:

$$\left(gx^2+xy\right)\,C_{n,x}+\left((g-1)xy+y^2\right)\,C_{n,y}=(K_1x+K_2y)\,C_n. \tag{5.23}$$

For this system we have

$$yP_2-xQ_2=x^2y.$$

Then, from Lemma 64 we can assume that

$$C_n=x^my^l \text{ where } n=m+l.$$

Substituting C_n in (5.23) and doing some computations we terminate that

$$K_1=gm+(g-1)l; \qquad K_2=m+l.$$

Now, taking from (5.22) the terms of degree n we have:

$$\begin{aligned} &x\,C_{n,x}+(gx^2+xy)\,C_{n-1,x}-y\,C_{n,y}+((g-1)xy+y^2)\,C_{n-1,y}\\ &=K_0\,C_n+[(gm+(g-1)l)\,x+(m+l)\,y]\,C_{n-1}. \end{aligned} \tag{5.24}$$

Set $C_{n-1}=\sum_{i=0}^{n-1}c_{n-1-i}x^{n-1-i}y^i$. Replacing C_n, C_{n-1} in (5.24) and doing some calculations, we obtain

$$\begin{aligned} &\sum_{i=0}^{m+l-1}(l-i-g)\,c_{m+l-1-i}\,x^{m+l-i}\,y^i-\sum_{i=0}^{m+l-1}c_{m+l-1-i}\,x^{m+l-1-i}\,y^{i+1}\\ &=(K_0-m+l)\,x^my^l. \end{aligned}$$

Note that this equation can be written as

$$\sum_{i=0}^{m+l}[(l-i-g)\,c_{m+l-1-i}-c_{m+l-i}]\,x^{m+l-i}\,y^i=(K_0-m+l)\,x^my^l,$$

where $c_i=0$ for $i<0$ and $i>m+l-1$. Equating the coefficients of x^iy^j in the above equation, we get:

$$\begin{cases} (l-i-g)\,c_{m+l-1-i}-\,c_{m+l-i}=0, \text{ where } i=0,1,\dots,l-1,l+1,\dots,m+l\\ \\ (-g)\,c_{m-1}-\,c_m=K_0-m+l. \end{cases} \tag{5.25}$$

For $i=m+l, m+l-1,\dots,l+1$ we have

$$c_0=c_1=\dots=c_{m-1}=0.$$

Then $c_m=-K_0+m-l$. Working recursively we have

$$\begin{aligned} &c_{m+1}=(-g+1)(-K_0+m-l),\\ &c_{m+2}=(-g+2)(-g+1)(-K_0+m-l),\dots\\ &c_{m+l-1}=(-g+l-1)\dots(-g+2)(-g+1)(-K_0+m-l). \end{aligned}$$

Replacing c_{m+l-1} in (5.25) where $i=0$, we get

$$(-g+l)(-g+l-1)...(-g+2)(-g+1)(-K_0+m-l)=0.$$

Note that

$$(-g+l)(-g+l-1)...(-g+2)(-g+1)$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_1^l the set of roots. If $g \notin S_1^l$ then $K_0 = m-l$. Therefore, we can conclude that

$$K = (m-l)+(gm+(g-1)l)x+(m+l)y,$$

since $g \notin S_1^l$. This is

$$C_{n-1} \equiv 0.$$

Now, taking from (5.22) the terms of degree $n-1$ we have:

$$\begin{aligned} 2a\, C_{n,x} + (gx^2+xy)\, C_{n-2,x} + a(2g-1)\, C_{n,y} + ((g-1)xy+y^2)\, C_{n-2,y} \\ = [(gm+(g-1)l)\, x + (m+l)\, y]\, C_{n-2}. \end{aligned} \tag{5.26}$$

Setting $C_{n-2} = \sum_{i=0}^{n-2} c_{n-2-i} x^{n-2-i} y^i$ and replacing C_n, C_{n-2} in (5.26) we obtain

$$\sum_{i=0}^{m+l-2} (l-i-2g)\, c_{m+l-2-i}\, x^{m+l-1-i}\, y^i + \sum_{i=0}^{m+l-2} (-2)\, c_{m+l-2-i}\, x^{m+l-2-i}\, y^{i+1} =$$

$$= -a(2g-1)l\, x^m y^{l-1} - 2am\, x^{m-1} y^l.$$

This equation can be written as

$$\sum_{i=0}^{m+l-2} [(l-i-2g)\, c_{m+l-2-i} + (-2)\, c_{m+l-1-i}]\, x^{m+l-1-i}\, y^i =$$

$$= -a(2g-1)l\, x^m y^{l-1} - 2am\, x^{m-1} y^l$$

where $c_i = 0$ for $i<0$ and $i > m+l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we get

$$\begin{cases} (l-i-2g)\, c_{m+l-2-i} - 2\, c_{m+l-1-i} = 0, \text{ where } i = 0,1,...,l-2, l+1,...,m+l-1, \\ (-2g+1)c_{m-1} - 2c_m = -a(2g-1)l, \\ (-2g)c_{m-2} - 2c_{m-1} = -2am. \end{cases} \tag{5.27}$$

For $i = m+l-1, m+l-2, ..., l+1$ we have

$$c_0 = c_1 = ... = c_{m-2} = 0.$$

Then $c_{m-1} = am$. Working recursively we have

$$c_m = \frac{a(-2g+1)(m-l)}{2},$$

$$c_{m+1} = \frac{a(-2g+1)(-2g+2)(m-l)}{4}, ...$$

$$c_{m+l-2} = \frac{a(-2g+1)(-2g+2)...(-2g+l-1)(m-l)}{2^{l-1}}.$$

Replacing c_{m+l-2} in (5.27) where $i = 0$, we get

$$\frac{a(-2g+1)(-2g+2)...(-2g+l-1)(-2g+l)(m-l)}{2^{l-1}} = 0.$$

Note that

$$(-2g+1)(-2g+2)...(-2g+l-1)(-2g+l)$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_2^l the set of roots. If $g \notin S_1^l \cup S_2^l$ then $a = 0$ or $m = l$.

The hyperbola is not an invariant algebraic curve when $a = 0$ and this cases does not matter for us. We assume $m = l$. Then, we have the following:

$$\begin{aligned} &K = (2g-1)m\,x + 2m\,y, \\ &C_n = x^m y^m, \;\; C_{n-1} \equiv 0, \;\; C_{n-2} = amx^{m-1}y^{m-1}, \end{aligned} \tag{5.28}$$

for $g \notin S_1^m \cup S_2^m$ which is a numerable set.

Following similar arguments for terms of degree $n-2, n-3, ...$ in (5.22) we conjecture that $C = (a+xy)^m$. Now we prove this statement by induction:

Suppose that for $k = 1, 2, ..., L$ we have

$$C_{n-(2k-1)} \equiv 0, \quad C_{n-2k} = \frac{a^k(m-(k-1))!}{k!}x^{m-k}y^{m-k}. \tag{5.29}$$

We shall prove that:

$$C_{n-2L-1} \equiv 0, \quad C_{n-2L-2} = \frac{a^{L+1}(m-L)!}{(L+1)!}x^{m-L-1}y^{m-L-1}.$$

Considering in (5.22) the terms of degree $n-2L$ we have:

$$2a\,C_{n-2L+1,x}+x\,C_{n-2L,x}+(gx^2+xy)\,C_{n-2L-1,x}+a(2g-1)\,C_{n-2L+1,y}+$$

$$-y\,C_{n-2L,y}+\left((g-1)xy+y^2\right)\,C_{n-2L-1,y}=((2g-1)m\,x+2m\,y)\,C_{n-2L-1}.$$

By the induction hypothesis $C_{n-2L+1}\equiv 0$ then:

$$x\,C_{n-2L,x}+(gx^2+xy)\,C_{n-2L-1,x}-y\,C_{n-2L,y}+((g-1)xy+y^2)\,C_{n-2L-1,y}= \\ =[(2g-1)m\,x+2m\,y]\,C_{n-2L-1}. \tag{5.30}$$

Setting $C_{n-2L-1}=\sum_{i=0}^{2m-2L-1} c_{2m-2L-1-i}x^{2m-2L-1-i}y^i$ and replacing C_{n-2L},C_{n-2L-1} in (5.30) we obtain

$$\sum_{i=0}^{2m-2L-1}(m-i-g(2L+1))\,c_{2m-2L-1-i}\,x^{2m-2L-i}\,y^i\;+$$

$$+\sum_{i=0}^{2m-2L-1}(-2L-1)\,c_{2m-2L-1-i}\,x^{2m-2L-1-i}\,y^{i+1}=0.$$

This equation can be written as

$$\sum_{i=0}^{2m-2L}\left[(m-i-g(2L+1))\,c_{2m-2L-1-i}+(-2L-1)\,c_{2m-2L-i}\right]\,x^{2m-2L-i}\,y^i=0,$$

where $c_i=0$ for $i<0$ and $i>2m-2L-1$. Equating the coefficients of x^iy^j in the above equation, we have:

$$(m-i-g(2L+1))\,c_{2m-2L-1-i}+(-2L-1)\,c_{2m-2L-i}=0,$$

for $i=0,1,...,2m-2L$. As $L\in\mathbb{N}$ then $L\neq -1/2$ and:

$$c_{2m-2L-1}=c_{2m-2L-2}=...=c_1=c_0=0.$$

Therefore,

$$C_{n-2L-1}\equiv 0.$$

Now, considering in (5.22) the terms of degree $n-2L-1$ we have:

$$2a\,C_{n-2L,x}+x\,C_{n-2L-1,x}+(gx^2+xy)\,C_{n-2L-2,x}+a(2g-1)\,C_{n-2L,y}+$$

$$-y\,C_{n-2L-1,y}+((g-1)xy+y^2)\,C_{n-2L-2,y}=[(2g-1)m\,x+2m\,y]\,C_{n-2L-2}.$$

We just proved that $C_{n-2L-1}\equiv 0$, then we have:

$$2a\,C_{n-2L,x}+(gx^2+xy)\,C_{n-2L-2,x}+a(2g-1)\,C_{n-2L,y}+ \\ +((g-1)xy+y^2)\,C_{n-2L-2,y}=[(2g-1)m\,x+2m\,y]\,C_{n-2L-2} \tag{5.31}$$

By the induction hypothesis follows that $C_{n-2L} = \dfrac{a^L(m-(L-1))!}{L!}x^{m-L}y^{m-L}$. Setting $C_{n-2L-2} = \sum_{i=0}^{2m-2L-2} c_{2m-2L-2-i}x^{2m-2L-2-i}y^i$ and replacing C_{n-2L}, C_{n-2L-2} in (5.31) we have:

$$\sum_{i=0}^{2m-2L-2} (m-i-g(2(L+1))\, c_{2m-2L-2-i}\, x^{2m-2L-1-i}\, y^i +$$

$$+\sum_{i=0}^{2m-2L-2} (-2L-2)\, c_{2m-2L-2-i}\, x^{2m-2L-2-i}\, y^{i+1} =$$

$$= -(2g-1)\frac{a^{L+1}(m-L)!}{L!}x^{m-L}y^{m-L-1} - \frac{2a^{L+1}(m-L)!}{L!}x^{m-L-1}y^{m-L}.$$

This equation can be rewritten as

$$\sum_{i=0}^{2m-2L-1} [(m-i-g(2(L+1))\, c_{2m-2L-2-i} + (-2L-2)\, c_{2m-2L-1-i}]\, x^{2m-2L-1-i}\, y^i =$$

$$= -(2g-1)\frac{a^{L+1}(m-L)!}{L!}x^{m-L}y^{m-L-1} - \frac{2a^{L+1}(m-L)!}{L!}x^{m-L-1}y^{m-L}$$

where $c_i = 0$ for $i < 0$ and $i > 2m-2L-2$. Equating the coefficients of x^iy^j in the above equation, we get the following equations

$$\begin{cases} (m-i-g(2(L+1)))\, c_{2m-2L-2-i} + (-2L-2)\, c_{2m-2L-1-i} = 0, \\ \\ (L+1-g(2(L+1)))c_{m-L-1} + (-2L-2)c_{m-L} = -(2g-1)\dfrac{a^{L+1}(m-L)!}{L!}, \\ \\ (L-g(2(L+1)))c_{m-L-2} + (-2L-2)c_{m-L-1} = -\dfrac{2a^{L+1}(m-L)!}{L!}, \end{cases}$$

for $i = 0, 1, ..., m-L-2, m-L+1, ..., 2m-2L-1$. As $L \in \mathbb{N}$ then $L \neq -1$ and

$$c_{m-L-2} = ... = c_1 = c_0 = 0.$$

Then,

$$c_{m-L-1} = \frac{a^{L+1}(m-L)!}{(L+1)!}, \quad c_{m-L} = 0.$$

When $i = m-L-2, ..., 0$, we obtain

$$c_{m-L+1} = c_{m-L+2} = ... = c_{2m-2L-2} = 0.$$

Therefore,

$$C_{n-2L-2} = \frac{a^{L+1}(m-L)!}{(L+1)!}x^{m-L-1}y^{m-L-1}.$$

This finishes the induction proof. It follows that

$$C = J_1^m, \;\; m \in \mathbb{N}$$

for all $(a,g) \notin \mathrm{L}_1$, where $\mathrm{L}_1 = \bigcup_{k\in\mathbb{N}} \mathrm{L}_{1,k} = \bigcup_{k\in\mathbb{N}} \left\{(a,g) \in \mathbb{R}^2 : g = \frac{k}{2} \text{ and } a \neq 0\right\}$.

(b) From (a) systems (N) have only the algebraic solution $J_1(x,y) = a + xy$ for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$. Then by Proposition 22, if systems (N) have an exponential factor, it must have the form:

$$F = \exp\left(G/J_1^l\right)$$

with cofactor $L = \overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y$ and where l is non-negative integer. Since the invariant algebraic curve $J_1^l = 0$ has the cofactor

$$K = l\alpha_1 = l(-1+2g)x + 2ly,$$

it follows by (2.2) that G satisfies the following equation:

$$\begin{aligned} &\left[2a + x + gx^2 + xy\right] G_x + \left[a(2g-1) - y + (g-1)xy + y^2\right] G_y + \\ &+ \left[l(1-2g)x - 2ly\right] G = \left[\overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y\right] \sum_{k=0}^{l} \binom{l}{k} a^k x^{l-k} y^{l-k} \end{aligned} \tag{5.32}$$

From now on we assume $G(x,y) = \sum_{i=0}^{n} G_i(x,y)$, where G_i is a homogeneous polynomial of degree i and split the study in cases.

Case 1: $n+1 < 2l$.

By equating the homogeneous terms of highest degree in (5.32) we obtain that

$$\overline{L}_1 = \overline{L}_2 = 0 \text{ and } \overline{L}_0 = 0.$$

Thus, G is an invariant algebraic curve. Then, $G = c\, J_1^l$ where c is a constant. Therefore, F is constant and it cannot be an exponential factor of systems (N).

Case 2: $n+1 = 2l$.

By equating the homogeneous terms of highest degree in (5.32) we obtain that

$$\overline{L}_1 = \overline{L}_2 = 0.$$

Set $G_n = \sum_{i=0}^{n} c_{n-i} x^{n-i} y^i$ where c_{n-i} are constants. Equating the terms of degree $n+1$ in (5.32) and using that $n+1 = 2l$, we have:

$$\sum_{i=0}^{2l} \left[(-g-i+l)c_{2l-i-1} + (-1)c_{2l-i}\right] x^{2l-i} y^i = \overline{L}_0 x^l y^l,$$

where $c_i = 0$ for $i < 0$ and $i > n$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)\; c_{2l-i-1} - \; c_{2l-i} = 0, \text{ where } i = 0,1,...,l-1,l+1,...,2l \\ \\ (-g)\; c_{l-1} - \; c_l = \overline{L}_0. \end{cases}$$

For $i = 2l, 2l-1, ..., l+1$ we obtain:

$$c_0 = c_1 = ... = c_{l-1} = 0.$$

Then $c_l = -\overline{L}_0$. Working recursively,

$$\begin{aligned} c_{l+1} &= (-g+1)(-\overline{L}_0), \\ c_{l+2} &= (-g+2)(-g+1)(-\overline{L}_0), ... \\ c_{2l-1} &= (-g+l)(-g+l-1)...(-g+1)(-\overline{L}_0). \end{aligned}$$

Therefore, if $(a,g) \notin \mathrm{L}_1$ we have $\overline{L}_0 = 0$ then

$$L = 0.$$

Consequently, systems (N) have no exponential factors for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$.

Case 3: $n = 2l$.

Consider the notation for G_n introduced in the study of Case 2. Equating the terms of degree $n+1$ in (5.32) we have

$$\sum_{i=0}^{2l+1} (l-i)\; c_{2l-i} x^{2l-i+1} y^i = \overline{L}_1\; x^{l+1} y^l + \overline{L}_2\; x^l y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > n$. These equations are equivalent to

$$\begin{cases} (l-i)\; c_{2l-i} = 0, \text{ where } i = 0,1,...,l-1,l+2,l+3,...,2l+1 \\ \\ 0\; c_l = \overline{L}_1, \\ \\ (-1)\; c_{l-1} = \overline{L}_2, \end{cases} \tag{5.33}$$

For $i = 2l, 2l-1, ..., l+2$ we obtain:

$$c_0 = \; c_1 = ... = \; c_{l-2} = 0.$$

Then,

$$c_{l-1} = -\overline{L}_2, \;\; c_l \text{ is free}, \;\; \overline{L}_1 = 0 \text{ and } c_{l+1} = c_{l+2} = ... = c_{2l} = 0.$$

Therefore,

$$G_n = c_l\; x^l y^l - \overline{L}_2 x^{l-1} y^{l+1}.$$

Since $c_l \neq 0$, without loss of generality, we assume that $c_l = 1$.

Equating the terms of degree n in (5.32) we have

$$x\,G_{n,x} + \left[gx^2 + xy\right] G_{n-1,x} - y\,G_{n,y} + \left[(g-1)xy + y^2\right] G_{n-1,y} +$$

$$+ \left[l(1-2g)x - 2ly\right] G_{n-1} = \overline{L}_0 x^l y^l.$$

Set $G_{n-1} = \sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^i$. Replacing G_{n-1} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l} \left[(-g-i+l)c_{2l-i-1} - c_{2l-i}\right] x^{2l-i} y^i = \overline{L}_0\, x^l y^l + 2\overline{L}_2\, x^{l-1} y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > 2l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)c_{2l-i-1} - c_{2l-i} = 0, \text{ where } i = 0,1,\ldots,l-1,l+2,\ldots,2l \\ \\ (-g)c_{l-1} - c_l = \overline{L}_0, \\ \\ (-g-1)c_{l-2} - c_{l-1} = 2\overline{L}_2. \end{cases} \tag{5.34}$$

For $i = 2l,\ldots,l+2$ we have:

$$c_0 = c_1 = \ldots = c_{l-2} = 0.$$

Then, $c_{l-1} = -2\overline{L}_2$, $c_l = 2g\overline{L}_2 - \overline{L}_0$. Working recursively,

$$\begin{aligned} c_{l+1} &= (-g+1)(2g\overline{L}_2 - \overline{L}_0), \\ c_{l+2} &= (-g+1)(-g+2)(2g\overline{L}_2 - \overline{L}_0), \ldots \\ c_{2l-1} &= (-g+1)(-g+2)\ldots(-g+l-1)(2g\overline{L}_2 - \overline{L}_0). \end{aligned}$$

Replacing c_{2l-1} in (5.34) where $i = 0$, we have:

$$(-g+l)(-g+l-1)\ldots(-g+1)(2g\overline{L}_2 - \overline{L}_0) = 0.$$

Then, if $g \notin \{l, l-1, \ldots, 1\}$ we must have $\overline{L}_0 = 2g\overline{L}_2$ and

$$G_{n-1} = -2\overline{L}_2 x^{l-1} y^l.$$

Therefore, $L = 2g\overline{L}_2 + \overline{L}_2 y$.

Equating the terms of degree $n-1$ in (5.32) we have:

$$2aG_{n,x} + x\,G_{n-1,x} + \left[gx^2 + xy\right] G_{n-2,x} + a(2g-1)G_{n,y} - y\,G_{n-1,y} +$$

$$+ \left[(g-1)xy + y^2\right] G_{n-2,y} + \left[l(1-2g)x - 2ly\right] G_{n-2} = al\overline{L}_2 x^{l-1} y^l.$$

Set $G_{n-2} = \sum_{i=0}^{n-2} c_{n-i-2}x^{n-i-2}y^i$. Replacing G_{n-2} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l}[(-2g-i+l)c_{2l-i-2} - 2c_{2l-i-1}]x^{2l-i-1}y^i = al(-2g+1)\overline{L}_2\, x^l y^{l-1} +$$

$$+\left(\overline{L}_2(al-2) - 2al + a(2g-1)(l+1)\overline{L}_2\right)x^{l-1}y^l + 2a(l-1)\overline{L}_2\, x^{l-2}y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > 2l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-2g-i+l)c_{2l-i-1} - 2c_{2l-i-1} = 0, \text{ where } i = 0,1,...,l-2,l+2,...,2l-1 \\ (-2g+1)c_{l-1} - 2c_l = al(-2g+1)\overline{L}_2, \\ (-2g)c_{l-2} - 2c_{l-1} = L_2(al-2) - 2al + a(2g-1)(l+1)\overline{L}_2, \\ (-2g-1)c_{l-3} - 2c_{l-2} = 2a(l-1)\overline{L}_2. \end{cases} \tag{5.35}$$

For $i = 2l-1,...,l+2$, we obtain:

$$c_0 = c_1 = ... = c_{l-3} = 0.$$

Then,

$$c_{l-2} = -a(l-1)L_2, \quad c_{l-1} = al + \overline{L}_2\left(1 + \frac{a}{2} - 2ag\right),$$

$$c_l = \frac{al}{2}(2g-1)\overline{L}_2 + \frac{(-2g+1)}{2}\left(L_2 + al + \frac{a\overline{L}_2}{2} - 2ag\overline{L}_2\right) \doteq A$$

Working recursively, we have:

$$c_{l+1} = \frac{(-2g+2)A}{2},$$

$$c_{l+2} = \frac{(-2g+2)(-2g+3)A}{2^2},...$$

$$c_{2l-2} = \frac{(-2g+l-1)(-2g+l-2)...(-2g+2)A}{2^{l-2}}.$$

Replacing c_{2l-2} in (5.35) where $i = 0$ we obtain:

$$\frac{(-2g+l)(-2g+l-1)...(-2g+2)A}{2^{l-2}} = 0, \text{ or}$$

$$\frac{(-2g+l)(-2g+l-1)...(-2g+2)(-2g+1)}{2^{l-1}}\left(-al\overline{L}_2 + \overline{L}_2 + al + \frac{a\overline{L}_2}{2} - 2ag\overline{L}_2\right) = 0$$

If $g \notin \left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, ..., \frac{l}{2}\right\}$ this is $(a,g) \notin \mathrm{L}_1$ then we must have:

$$-al\overline{L}_2 + \overline{L}_2 + al + \frac{a\overline{L}_2}{2} - 2ag\overline{L}_2 = 0,$$

that happens if, and only if,

$$\overline{L}_2 = \frac{-2al}{-2al+2+a-4ag}, \text{ for } -2al+2+a-4ag \neq 0 \text{ or}$$

$$g = \frac{2+a-2al}{4a}.$$

Suppose that $(a,g) \notin \mathrm{L}_1 \cup \mathrm{C}'$, where $\mathrm{C}' = \bigcup_{k\in\mathbb{N}} \mathrm{C}_k = \bigcup_{k\in\mathbb{N}} \left\{(a,g) : g = \frac{2+a-2ak}{4a} \text{ and } a \neq 0\right\}$. Therefore, we have:

$$G_n = x^l y^l + \frac{2al}{(-2al+2+a-4ag)} x^{l-1} y^{l+1}, \quad G_{n-1} = \frac{4al}{(-2al+2+a-4ag)} x^{l-1} y^l,$$

$$G_{n-2} = \frac{2a^2 l(l-1)}{(-2al+2+a-4ag)} x^{l-2} y^l - \frac{2a^2 l^2}{(-2al+2+a-4ag)} x^{l-1} y^{l-1},$$

$$L = \frac{-4agl}{(-2al+2+a-4ag)} - \frac{2al}{(-2al+2+a-4ag)} y.$$

Equating the terms of degree $n-2$ in (5.32) we have:

$$2aG_{n-1,x} + x\,G_{n-2,x} + \left[gx^2 + xy\right] G_{n-3,x} + a(2g-1)G_{n-1,y} - y\,G_{n-2,y} +$$

$$+\left[(g-1)xy + y^2\right] G_{n-3,y} + \left[l(1-2g)x - 2ly\right] G_{n-3} = \frac{-4a^2 g l^2}{(-2al+2+a-4ag)} x^{l-1} y^{l-1}.$$

Set $G_{n-3} = \sum_{i=0}^{n-3} c_{n-i-3} x^{n-i-3} y^i$. Replacing G_{n-3} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l-2} \left[(-3g-i+l)c_{2l-i-3} - 3c_{2l-i-2}\right] x^{2l-i-2} y^i = \frac{4a^2 l^2 (1-3g)}{(-2al+2+a-4ag)} x^{l-1} y^{l-1} +$$

$$-\frac{4a^2 l(l-1)}{(-2al+2+a-4ag)} x^{l-2} y^l,$$

where $c_i = 0$ for $i < 0$ and $i > 2l-3$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-3g-i+l)c_{2l-i-3} - 3c_{2l-i-2} = 0, \text{ where } i = 0,1,...,l-2,l+1,...,2l-2 \\ (-3g+1)c_{l-2} - 3c_{l-1} = \dfrac{4a^2 l^2 (1-3g)}{(-2al+2+a-4ag)}, \\ (-3g)c_{l-3} - 3c_{l-2} = -\dfrac{4a^2 l(l-1)}{(-2al+2+a-4ag)}. \end{cases} \tag{5.36}$$

For $i = 2l-2, \ldots, l+1:$

$$c_0 = c_1 = \ldots = c_{l-3} = 0.$$

Then,

$$c_{l-2} = \frac{4a^2 l(l-1)}{3(-2al+2+a-4ag)}, \quad c_{l-1} = \frac{(4a^2 l(2l-1))(1-3g)}{(-9)(-2al+2+a-4ag)} \doteq B.$$

Working recursively, we obtain:

$$c_l = \frac{(-3g+2)B}{3},$$

$$c_{l+1} = \frac{(-3g+2)(-3g+3)B}{3^2}, \ldots$$

$$c_{2l-3} = \frac{(-3g+2)(-3g+3)\ldots(-3g+l-1)B}{3^{l-3}}.$$

Replacing c_{2l-3} in (5.36) where $i = 0:$

$$\frac{(-3g+l)(-3g+l-1)\ldots(-3g+2)B}{3^{l-2}} = 0, \text{ or}$$

$$\frac{(-3g+l)(-3g+l-1)\ldots(-3g+2)(-3g+1)}{3^l}\left(\frac{4a^2 l(2l-1)}{2al-2-a+4ag}\right) = 0.$$

Consider $\mathrm{L}_2 = \bigcup_{k\in\mathbb{N}} \mathrm{L}_{2,k} = \bigcup_{k\in\mathbb{N}} \left\{(a,g) : g = \frac{k}{3} \text{ and } a \neq 0\right\}$. Then, if $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$ we must have

$$4a^2 l(2l-1) = 0.$$

What happens if, and only if

$$a = 0 \text{ or } l = 0 \text{ or } l = 1/2.$$

Therefore, systems (N) have no exponential factors for $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$.

Case 4: $n > 2l$.

Consider the notation for G_n introduced in the study of Case 2. Equating the terms of degree $n+1$ in (5.32) we have:

$$\left[gx^2 + xy\right] \sum_{i=0}^{n} (n-i) c_{n-i} x^{n-i-1} y^i + \left[(g-1)xy + y^2\right] \sum_{i=0}^{n} i c_{n-i} x^{n-i} y^{i-1} +$$

$$+ \left[l(1-2g)x - 2ly\right] \sum_{i=0}^{n} c_{n-i} x^{n-i} y^i = 0.$$

Working in a similar way to the previous cases, we obtain:

$$\sum_{i=0}^{n+1} \left[(gn - i + l(1-2g)) c_{n-i} + (n-2l) c_{n-i+1}\right] x^{n-i+1} y^i = 0,$$

when $c_i = 0$ for $i < 0$ and $i > n$. Therefore,

$$(gn - i + l(1-2g))c_{n-i} + (n-2l)c_{n-i+1} = 0,$$

for $i = 0, 1, ..., n+1$. As $n \neq 2l$ we have

$$c_0 = c_1 = ... = c_n = 0.$$

Then, $G_n = 0$.

Summing up this four cases the proof follows.

(c) Suppose $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$. Then, by (a) and (b) we get that the systems (N) have only the algebraic solution

$$J_1(x,y) = a + xy$$

with cofactor $\alpha_1 = (-1+2g)x + 2y$ and they have no exponential factor. Under this assumptions

$$\begin{cases} \lambda_1 \alpha_1 = 0 \Leftrightarrow \lambda_1 = 0 \text{ and} \\ \\ \lambda_1 \alpha_1 = -div(P,Q) = -(1+2gx+y) - (-1+(g-1)x+2y) \text{ has no solution.} \end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (N) are not Liouvillian integrable.

□

5.2.2.2 The systems (O)

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases}$$

where $a(g-1) \neq 0$.

Theorem 80. (a) If $(a,g) \notin \mathrm{L}_1$ then the only affine invariant algebraic curves of a system in the family (O) are of the form $J_1^m = 0$ where $J_1(x,y) = a + xy$ and m is a positive integer.

(b) If $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ then any system in the family (O) has no exponential factors.

(c) If $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ then any system in the family (O) is not Liouvillian integrable.

Proof.

(a) By a straightforward computation, we can verify that $J_1(x,y) = a + xy$ is an invariant hyperbola with cofactor $\alpha_1(x,y) = (-1+2g)x + 2y$. Assume that

$$C = \sum_{i=0}^{n} C_i(x,y) = 0$$

is an invariant algebraic curve of the systems (O) with cofactor $K = K_0 + K_1 x + K_2 y$, where C_i are homogeneous polynomial of degree i where $0 \le i \le n$. From the definition of the invariant algebraic curve (1.4), we have:

$$\begin{aligned}\left(2a + gx^2 + xy\right) \sum_{i=0}^{n} C_{i,x} + \left(a(2g-1) + (g-1)xy + y^2\right) \sum_{i=0}^{n} C_{i,y} = \\ = (K_0 + K_1 x + K_2 y) \sum_{i=0}^{n} C_i.\end{aligned} \tag{5.37}$$

The step where we take the terms of degree $n+1$ in (5.37) is exactly the same made for system (N). Then we have:

$$C_n = x^m\, y^l \quad \text{where } n = m + l,$$

$$K_1 = gm + (g-1)l; \quad K_2 = m + l.$$

Now, taking from (5.37) the terms of degree n we have:

$$\begin{aligned}\left(gx^2 + xy\right)\, C_{n-1,x} + \left((g-1)xy + y^2\right)\, C_{n-1,y} = K_0\, C_n + \\ + \left[(gm + (g-1)l)\, x + (m+l)\, y\right]\, C_{n-1}.\end{aligned} \tag{5.38}$$

Set $C_{n-1} = \sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^i$. Replacing C_n, C_{n-1} in (5.38) and doing some calculations, we obtain:

$$\sum_{i=0}^{m+l-1} (l-i-g)\, c_{m+l-1-i}\, x^{m+l-i}\, y^i - \sum_{i=0}^{m+l-1} c_{m+l-1-i}\, x^{m+l-1-i}\, y^{i+1} = K_0\, x^m y^l.$$

This equation can be written as

$$\sum_{i=0}^{m+l} \left[(l-i-g)\, c_{m+l-1-i} - c_{m+l-i}\right] x^{m+l-i}\, y^i = K_0\, x^m y^l,$$

where $c_i = 0$ for $i < 0$ and $i > m+l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we get:

$$\begin{cases} (l-i-g)\, c_{m+l-1-i} - c_{m+l-i} = 0, \text{ where } i = 0, 1, \ldots, l-1, l+1, \ldots, m+l \\ (-g)\, c_{m-1} - c_m = K_0. \end{cases} \tag{5.39}$$

For $i = m+l, m+l-1, ..., l+1$ we have:

$$c_0 = c_1 = ... = c_{m-1} = 0.$$

Then $c_m = -K_0$. Working recursively we have

$$\begin{aligned}c_{m+1} &= (-g+1)(-K_0),\\ c_{m+2} &= (-g+1)(-g+2)(-K_0), ...\\ c_{m+l-1} &= (-g+1)(-g+2)...(-g+l-1)(-K_0).\end{aligned}$$

Replacing c_{m+l-1} in (5.39) where $i = 0$, we get

$$(-g+l)(-g+l-1)...(-g+2)(-g+1)(-K_0) = 0.$$

Note that

$$(-g+l)(-g+l-1)...(-g+2)(-g+1)$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_1^l the set of roots. If $g \notin S_1^l$ we have that $K_0 = 0$. Therefore, we can conclude that

$$K = (gm + (g-1)l)x + (m+l)y,$$

since $g \notin S_1^l$. This is

$$C_{n-1} \equiv 0.$$

The step where we take the terms of degree $n-1$ in (5.37) is exactly the same made for system (N). Then we have $m = l$ which leads us to

$$\begin{aligned}&K = (2g-1)mx + 2my,\\ &C_n = x^m y^m, \quad C_{n-1} \equiv 0, \quad C_{n-2} = amx^{m-1}y^{m-1},\end{aligned} \tag{5.40}$$

for $g \notin S_1^m \cup S_2^m$ numerable set, where S_2^m is the set of roots of the polynomial

$$(-2g+1)(-2g+2)...(-2g+m-1)(-2g+m).$$

Following similar arguments for terms of degree $n-2, n-3, ...$ in (5.37) we conjecture that $C = (a+xy)^m$. Now we prove this statement by induction:

Suppose that for $k = 1, 2, ..., L$ we have

$$C_{n-(2k-1)} \equiv 0, \quad C_{n-2k} = \frac{a^k(m-(k-1))!}{k!} x^{m-k} y^{m-k}. \tag{5.41}$$

We shall prove that:

$$C_{n-2L-1} \equiv 0, \quad C_{n-2L-2} = \frac{a^{L+1}(m-L)!}{(L+1)!} x^{m-L-1} y^{m-L-1}.$$

Considering in (5.37) the terms of degree $n-2L$ we have:

$$2a\, C_{n-2L+1,x}+\left(gx^2+xy\right)\, C_{n-2L-1,x}+a(2g-1)\, C_{n-2L+1,y}+$$

$$+\left((g-1)xy+y^2\right)\, C_{n-2L-1,y}=((2g-1)m\,x+2m\,y)\, C_{n-2L-1}.$$

By the induction hypothesis $C_{n-2L+1}\equiv 0$ then:

$$\begin{aligned}\left(gx^2+xy\right)\, C_{n-2L-1,x}+\left((g-1)xy+y^2\right)\, C_{n-2L-1,y}= \\ =((2g-1)m\,x+2m\,y)\, C_{n-2L-1}.\end{aligned} \tag{5.42}$$

Setting $C_{n-2L-1}=\sum_{i=0}^{2m-2L-1} c_{2m-2L-1-i}x^{2m-2L-1-i}y^i$ and replacing C_{n-2L},C_{n-2L-1} in (5.42) we obtain:

$$\sum_{i=0}^{2m-2L-1}(m-i-g(2L+1))\, c_{2m-2L-1-i}\, x^{2m-2L-i}\, y^i+$$

$$+\sum_{i=0}^{2m-2L-1}(-2L-1)\, c_{2m-2L-1-i}\, x^{2m-2L-1-i}\, y^{i+1}=0.$$

This is exactly the same equation solved for system (N) in the induction proof. Therefore, we have:

$$C_{n-2L-1}\equiv 0.$$

Now, considering in (5.37) the terms of degree $n-2L-1$ we have:

$$\begin{aligned}2a\, C_{n-2L,x}+\left(gx^2+xy\right)\, C_{n-2L-2,x}+a(2g-1)\, C_{n-2L,y}+ \\ +\left((g-1)xy+y^2\right)\, C_{n-2L-2,y}=((2g-1)m\,x+2m\,y)\, C_{n-2L-2}\end{aligned} \tag{5.43}$$

Note that (5.43) is exactly the same as (5.31), solved in system (N). Therefore,

$$C_{n-2L-2}=\frac{a^{L+1}(m-L)!}{(L+1)!}x^{m-L-1}y^{m-L-1}.$$

This finishes the induction proof. It follows that

$$C=J_1^m,\;\; m\in\mathbb{N}$$

for all $(a,g)\notin \mathrm{L}_1$, where $\mathrm{L}_1=\bigcup_{k\in\mathbb{N}}\mathrm{L}_{1,k}=\bigcup_{k\in\mathbb{N}}\left\{(a,g)\in\mathbb{R}^2: g=\frac{k}{2}\right\}$.

(b) From (a) systems (O) have only the algebraic solution $J_1(x,y)=a+xy$ for $(a,g)\in\mathbb{R}^2-\mathrm{L}_1$. Then, by Proposition 22, if systems (O) have an exponential factor, it must have the form

$$F=\exp\left(G/J_1^l\right)$$

with a cofactor $L = \overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y$ and where l is non-negative integers. Since the invariant algebraic curve $J_1^l = 0$ has the cofactor

$$K = l\alpha_1 = l(-1+2g)x + 2ly,$$

it follows by (2.2) that G satisfies the following equation:

$$\begin{aligned}\left[2a + gx^2 + xy\right] G_x + \left[a(2g-1) + (g-1)xy + y^2\right] G_y + [l(1-2g)x - 2ly]\, G \\ = \left[\overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y\right] \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} a^k x^{l-k} y^{l-k}\end{aligned} \tag{5.44}$$

From now on we assume $G(x,y) = \sum_{i=0}^{n} G_i(x,y)$, where G_i is a homogeneous polynomial of degree i and split the study in cases.

Case 1: $n+1 < 2l$.

This case is exactly the same proved for systems (N). We have that F is constant, what can not happen.

Case 2: $n+1 = 2l$.

This case is also the same proved for systems (N). We have that if $g \notin \{l, l-1, ..., 1\}$ then $L = 0$. Consequently, systems (O) have no exponential factors for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$.

Case 3: $n = 2l$.

Set $G_n = \sum_{i=0}^{n} c_{n-i} x^{n-i} y^i$, where c_{n-i} are constants. Equating the terms of (5.44) with degree $n+1$ we have

$$\sum_{i=0}^{2l+1} (l-i) c_{2l-i} x^{2l-i+1} y^i = \overline{L}_1 x^{l+1} y^l + \overline{L}_2 x^l y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > n$. This is the same equation solved for systems (N) in case 3. Then we have:

$$G_n = \; x^l y^l - \overline{L}_2 x^{l-1} y^{l+1} \text{ and } \overline{L}_1 = 0.$$

Equating the terms of degree n in (5.44) we have

$$\left[gx^2 + xy\right] G_{n-1,x} + \left[(g-1)xy + y^2\right] G_{n-1,y} + [l(1-2g)x - 2ly]\, G_{n-1} = \overline{L}_0 \, x^l y^l.$$

Set $G_{n-1} = \sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^i$. Replacing G_{n-1} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l} [(-g-i+l) c_{2l-i-1} - c_{2l-i}] x^{2l-i} y^i = \overline{L}_0 \, x^l y^l,$$

where $c_i = 0$ for $i < 0$ and $i > 2l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)c_{2l-i-1} - c_{2l-i} = 0, \text{ where } i = 0,1,...,l-1,l+1,...,2l \\ (-g)c_{l-1} - c_l = \overline{L}_0. \end{cases} \tag{5.45}$$

For $i = 2l, 2l-1, ..., l+1$:

$$c_0 = c_1 = ... = c_{l-1} = 0.$$

Then, $c_l = -\overline{L}_0$. Working recursively,

$$\begin{aligned} c_{l+1} &= (-g+1)(-L_0), \\ c_{l+2} &= (-g+1)(-g+2)(-\overline{L}_0), ... \\ c_{2l-1} &= (-g+1)(-g+2)...(-g+l-1)(-\overline{L}_0). \end{aligned}$$

Replacing c_{2l-1} in (5.45) where $i = 0$, we have:

$$(g+l)(-g+l-1)...(-g+1)(-\overline{L}_0) = 0.$$

Then, if $g \notin \{l, l-1, ..., 1\}$ we must have $\overline{L}_0 = 0$. Therefore,

$$G_{n-1} \equiv 0 \text{ and } L = \overline{L}_2 y.$$

Equating the terms of degree $n-1$ in (5.44) we have:

$$2aG_{n,x} + \left[gx^2 + xy\right] G_{n-2,x} + a(2g-1)G_{n,y} + \left[(g-1)xy + y^2\right] G_{n-2,y} +$$
$$+ \left[l(1-2g)x - 2ly\right] G_{n-2} = al\overline{L}_2 x^{l-1} y^l.$$

Set $G_{n-2} = \sum_{i=0}^{n-2} c_{n-i-2} x^{n-i-2} y^i$. Replacing G_{n-2} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l} \left[(-2g-i+l)c_{2l-i-2} - 2c_{2l-i-1}\right] x^{2l-i-1} y^i = (al(1-2g))x^l y^{l-1} +$$
$$+(\overline{L}_2 al - 2al + a(2g-1)(l+1)\overline{L}_2)x^{l-1}y^l + 2a(l-1)\overline{L}_2 x^{l-2} y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > 2l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-2g-i+l)c_{2l-i-2} + (-2)c_{2l-i-1} = 0, \text{ where } i = 0,...,l-2,l+2,...,2l-1 \\ (-2g+1)c_{l-1} + (-2)c_l = al(1-2g), \\ (-2g)c_{l-2} + (-2)c_{l-1} = \overline{L}_2 al - 2al + a(2g-1)(l+1)\overline{L}_2, \\ (-2g-1)c_{l-3} + (-2)c_{l-2} = 2a(l-1)\overline{L}_2. \end{cases} \tag{5.46}$$

For $i = 2l-1, 2l-2, ..., l+1$ we obtain

$$c_0 = c_1 = ... = c_{l-3} = 0.$$

Then,

$$c_{l-2} = -a(l-1)\overline{L}_2, \quad c_{l-1} = \frac{a}{2}(2l+\overline{L}_2) - 2ag\overline{L}_2,$$

$$c_l = \frac{\overline{L}_2 a}{2}\left(-2g+\frac{1}{2}\right)(-2g+1) \doteq A$$

Working recursively,

$$c_{l+1} = \frac{(-2g+2)A}{2},$$

$$c_{l+2} = \frac{(-2g+2)(-2g+3)A}{4}, \ldots$$

$$c_{2l-2} = \frac{(-2g+l-1)(-2g+l-2)...(-2g+2)}{2^{l-2}}A.$$

Replacing c_{2l-2} in (5.46) where $i = 0$ we have:

$$\frac{(-2g+l)(-2g+l-1)...(-2g+2)}{2^{l-2}}A = 0, \text{ or}$$

$$\frac{(-2g+l)(-2g+l-1)...(-2g+2)(-2g+1)}{2^{l-2}}\left[\frac{\overline{L}_2 a}{2}\left(-2g+\frac{1}{2}\right)\right] = 0.$$

Then, if $g \notin \{1,2,...,1\} \cup \left\{\frac{1}{2},\frac{2}{2},...,\frac{l}{2}\right\} \cup \{1/4\}$, this is, $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ we must have

$$\overline{L}_2 = 0.$$

Therefore, $L = 0$ and systems (O) have no exponential factors for $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$.

Case 4: $n > 2l$.

This case is the same proved for systems (N). Then, $G_n = 0$ that can not happen.

Summing up this four cases the proof follows.

(c) Suppose $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$. Then, by (a) and (b) we get that the systems (O) have only the algebraic solution

$$J_1(x,y) = a + xy$$

with cofactor $\alpha_1 = (-1+2g)x + 2y$ and they have no exponential factor. Under this assumptions

$$\begin{cases} \lambda_1\alpha_1 = 0 \Leftrightarrow \lambda_1 = 0 \text{ and} \\ \\ \lambda_1\alpha_1 = -div(P,Q) = -(2gx+y) - ((g-1)x+2y) \text{ has no solution.} \end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (O) are not Liouvillian integrable.

□

CHAPTER

6

GEOMETRIC ANALYSIS

In this chapter we present a detailed study of 22 of the normal forms for the family **QSH**. Our goal is gathering data on the geometry of polynomial systems as expressed in the configurations of invariant algebraic curves and their impact on integrability. This data is useful in gaining more insight about the Darboux theory of integrability. We also present here the bifurcation diagrams of the normal forms from two viewpoints: their topological bifurcations as well as the bifurcations diagrams of their configurations.

We first present the results of our calculations of the geometric features of the configurations as well as the information on the singularities. Afterwards we sum up these in a proposition and in pictures of the two bifurcation diagrams. We also give a proof for Poincaré's problem for 6 of the cases studied. It is important to mention that we restricted our study to the case where the first integral is obtained using invariant algebraic curves of degree at most two.

Observation 81. In the following study we have some cases where the total multiplicity of the invariant lines, including the line at infinity, is at least four or the line at infinity is filled up with singularities. These cases were studied in the papers: (SCHLOMIUK; VULPE, 2008c) (for the cases where the total multiplicity of invariant lines is four), (SCHLOMIUK; VULPE, 2008b) (for the cases where the total multiplicity of invariant lines is at least five) and (SCHLOMIUK; VULPE, 2008a) (for the cases where the line at infinity is filled up with singularities). For the sake of completeness we include these cases in our study too but we also bring some new information about them, such as, about the way the curves intersect, remarkable curves, bifurcation diagrams of configurations of invariant curves and bifurcations of phase portraits of the normal forms.

Observation 82. The exponents λ_i appearing in the expressions of some first integrals and integrating factor cannot be all zero since we want to get nonconstant expressions.

Observation 83. In the following study we present first integrals and integrating factors for many families of systems in **QSH**. In each case they are obtained only using invariant algebraic curves of degree at most two. This means that we could have others first integrals and integrating factors expressed by invariant algebraic curves of higher degrees.

6.1 Geometric study for systems with $\eta > 0$

In this section we present a detailed study of 10 normal forms for the class $\mathbf{QSH}_{(\eta>0)}$, namely the families (C), (E), (F), (G), (H), (I), (J), (K), (L) and (M). Following the geometric study of this families we give an anwser to Poincaré's problem, but only for the cases when its solution does not follows directly from the expressions of the first integral.

The normal forms (A) and (B) are 3-parameters families which makes this study more complicated and these cases will be studied in further works. Here we present only a particular case for one of these two normal forms, the case (B) with $h = 1/4$. We also did not present in this thesis the geometric analysis of family (D) due to complicated expressions for its invariant algebraic curves and singularities. The study for this case is long and in it arises more complicated bifurcation diagrams than the cases studied here. The study of this case is in progress and it will be published in further works.

6.1.1 Geometric Analysis of Family (B) *with* $h = 1/4$

Consider the family

$$(\mathrm{B})\ h=\tfrac{1}{4}:\quad \begin{cases} \dot{x} = -\frac{a}{2} + x^2 - \frac{3xy}{4} \\ \dot{y} = a(-1+2g) + (g-1)xy + \frac{y^2}{4}, \end{cases}$$

where $a(g-1)(2g-1) \neq 0$.

This is a two parameters family depending on a and g such that $a(g-1)(2g-1) \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (B) with $h = 1/4$ we study here also the limit cases $a(g-1)(2g-1) = 0$ where the equations are still defined.

We display below the full geometric analysis of this family, which is endowed with at least two invariant hyperbolas. In the **generic case**

$$ag(g-1)(2g-1)(3g-4)(4g-1)(4g+1) \neq 0$$

the systems have two invariant hyperbolas J_1, J_2 with cofactors α_1, α_2 given by

$$\begin{aligned} J_1 &= a + xy, & \alpha_1 &= (-1+2g)x - \tfrac{y}{2} \\ J_2 &= a + xy - x^2, & \alpha_2 &= 2gx - \tfrac{y}{2}. \end{aligned}$$

We note that when $g = -1/4$ we have two additional invariant lines. When $g = 1/4$ we have an additional invariant hyperbola and one additional invariant line. The multiplicities of each invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 2nd extactic polynomial.

(i) The generic case: $ag(g-1)(2g-1)(3g-4)(4g-1)(4g+1) \neq 0$.

Table 10 – Invariant curves, cofactors, singularities and intersection points of family (B) $h = 1/4$ for the generic case.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $J_2 = a + xy - x^2$ $\alpha_1 = (-1+2g)x - \frac{y}{2}$ $\alpha_2 = 2gx - \frac{y}{2}$	$P_1 = \left(-\frac{\sqrt{a}}{\sqrt{3-4g}}, \frac{2\sqrt{a}(1-2g)}{\sqrt{3-4g}}\right)$ $P_2 = \left(\frac{\sqrt{a}}{\sqrt{3-4g}}, \frac{2\sqrt{a}(2g-1)}{\sqrt{3-4g}}\right)$ $P_3 = \left(-\frac{i\sqrt{a}}{2\sqrt{g}}, -2i\sqrt{a}\sqrt{g}\right)$ $P_4 = \left(\frac{i\sqrt{a}}{2\sqrt{g}}, 2i\sqrt{a}\sqrt{g}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; N, N, S if $g < 0$ ©, ©, $s, s; N, N, N$ if $0 < g < \frac{3}{4}$ $n, n, s, s; N, S, N$ if $g > \frac{3}{4}$ For $a > 0$ we have $s, s, n, n; N, N, S$ if $g < 0$ s, s, ©, ©; N, N, N if $0 < g < \frac{3}{4}$ ©, ©, ©, ©; N, S, N if $g > \frac{3}{4}$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ quadruple $\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 11 – Divisor and zero-cycles of family (B) $h=1/4$ for the generic case.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = \begin{cases} P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } \begin{cases} a>0 \text{ and } g>3/4 \text{ or} \\ a<0 \text{ and } g<0 \end{cases} \\ P_1^C+P_2^C+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a<0 \text{ and } 0<g<3/4 \\ P_1+P_2+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a>0 \text{ and } 0<g<3/4 \\ P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } \begin{cases} a>0 \text{ and } g<0 \text{ or} \\ a<0 \text{ and } g>3/4 \end{cases} \end{cases}$	7 7 7 7 7 7
$T = Z\bar{J}_1\bar{J}_2 = 0$	5
$M_{0CT} = \begin{cases} P_1^C+P_2^C+P_3^C+P_4^C+3P_1^\infty+2P_2^\infty+2P_3^\infty \text{ if } \begin{cases} a>0 \text{ and } g>3/4 \text{ or} \\ a<0 \text{ and } g<0 \end{cases} \\ P_1^C+P_2^C+P_3+P_4+3P_1^\infty+2P_2^\infty+2P_3^\infty \text{ if } a<0 \text{ and } 0<g<3/4 \\ P_1+P_2+P_3^C+P_4^C+3P_1^\infty+2P_2^\infty+2P_3^\infty \text{ if } a>0 \text{ and } 0<g<3/4 \\ P_1+P_2+P_3+P_4+3P_1^\infty+2P_2^\infty+2P_3^\infty \text{ if } \begin{cases} a>0 \text{ and } g<0 \text{ or} \\ a<0 \text{ and } g>3/4 \end{cases} \end{cases}$	11 11 11 11 11 11

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

Table 12 – First integral and integrating factor of family (B) $h=1/4$ for the generic case.

	First integral	Integrating Factor
General	I	$R = J_1^{-1+2g} J_2^{\frac{1}{2}(1-4g)}$
Simple example	$\mathscr{I}$	$\mathscr{R} = J_1^{-1+2g} J_2^{\frac{1}{2}(1-4g)}$

Source: Elaborated by the author.

$$I = \mathscr{I} = \frac{(a+xy)^{2g}(a+x(y-x))^{\frac{3}{2}-2g}\left(3a(a+xy)\,{}_2F_1\left(1,\frac{5}{2};\frac{5}{2}-2g;-\frac{-x^2+yx+a}{x^2}\right)+(4g-3)x^2\left(a+4gx^2\right)\right)}{8g(4g-3)x^5},$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function which has a branch cut discontinuity in the complex z plane running from 1 to ∞ and has series expansion ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{a_k b_k z^k}{c_k k!}$.

(ii) The non-generic cases: $ag(g-1)(2g-1)(3g-4)(4g-1)(4g+1)=0$.

(ii.1) $g=-\frac{1}{4}$ and $a\neq 0$.

Here we have, apart from the two invariant hyperbolas, two additional invariant lines.

Table 13 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = -1/4$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = -2\sqrt{a} + x - y$ $J_2 = 2\sqrt{a} + x - y$ $J_3 = a + xy$ $J_4 = a + xy - x^2$ $\alpha_1 = -\frac{\sqrt{a}}{2} - \frac{x}{4} + \frac{y}{4}$ $\alpha_2 = \frac{\sqrt{a}}{2} - \frac{x}{4} + \frac{y}{4}$ $\alpha_3 = -\frac{3x}{2} - \frac{y}{2}$ $\alpha_4 = -\frac{x}{2} - \frac{y}{2}$	$P_1 = \left(-\sqrt{a}, \sqrt{a}\right)$ $P_2 = \left(\sqrt{a}, -\sqrt{a}\right)$ $P_3 = \left(-\frac{\sqrt{a}}{2}, \frac{3\sqrt{a}}{2}\right)$ $P_4 = \left(\frac{\sqrt{a}}{2}, -\frac{3\sqrt{a}}{2}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; N, N, S For $a > 0$ we have $n, n, s, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_2$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_4 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ double $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_3 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ quadruple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 14 – Divisor and zero-cycles of family (B) when $h = 1/4$, $g = -1/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	17 17

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double,

2) four distinct tangents at P_2^∞.

Table 15 – First integral and integrating factor of family (B) when $h = 1/4$, $g = -1/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{-\lambda_1} J_4^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{-\lambda_1 - \frac{3}{2}} J_4^{1+2\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_2 J_4^2}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3^{\frac{1}{2}} J_4}$

Source: Elaborated by the author.

Observation 84. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_2 J_4^2 - c_2 J_3 = 0$, $\deg \mathscr{F}_1 = 6$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -4a^2]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-4a^2)} = \left(a(y-3x) + x(x-y)^2\right)^2, \quad \mathscr{F}^1_{(1,0)} = J_1 J_2 J_4^2.$$

Therefore, J_1, J_2, J_4 and $J_5 := a(y-3x) + x(x-y)^2$ are remarkable curves, $[1 : -4a^2]$ and $[1 : 0]$ are all critical remarkable values and J_4, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_2 for $\mathscr{F}^1_{(1,-4a^2)}$ and P_3, P_4 for $\mathscr{F}^1_{(1,0)}$.

Observation 85. $J_5 := a(y-3x) + x(x-y)^2$ is an invariant algebraic curve of degree 3 of family (B) when $h = 1/4$, $g = -1/4$ and $a \neq 0$, with cofactor given by $\alpha_5 = -\frac{3x}{4} - \frac{y}{4}$.

Observation 86. Note that the rational first integral $\mathscr{I}_1$ in Table 15 has the rational integrating factor $R = \frac{J_3 J_5}{J_4^2}$ expressed by an invariant curve of degree higher than two.

(ii.2) $g = 0$ and $a \neq 0$.

Table 16 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = 0$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $J_2 = a + xy - x^2$ $\alpha_1 = -x - \frac{y}{2}$ $\alpha_2 = -\frac{y}{2}$	$P_1 = \left(\frac{\sqrt{a}}{\sqrt{3}}, -\frac{2\sqrt{a}}{\sqrt{3}}\right)$ $P_2 = \left(-\frac{\sqrt{a}}{\sqrt{3}}, \frac{2\sqrt{a}}{\sqrt{3}}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $©, ©; N, N, \binom{2}{1} S$ For $a > 0$ we have $s, s; N, N, \binom{2}{1} N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ quadruple $\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 17 – Divisor and zero-cycles of family (B) when $h = 1/4$, $g = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_1^\infty + P_2^\infty + 3P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_1^\infty + P_2^\infty + 3P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2 = 0$	5
$M_{0CT} = \begin{cases} P_1^C + P_2^C + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	9 9

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

Table 18 – First integral and integrating factor of family (B) when $h = 1/4$, $g = 0$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = \frac{1}{2}\left((y-x)\sqrt{a+x(y-x)} + a\tan^{-1}\left(\frac{\sqrt{a+x(y-x)}}{x}\right)\right)$	$R = J_1^{-1}J_2^{\frac{1}{2}}$
Simple example	$\mathscr{I} = \frac{1}{2}\left((y-x)\sqrt{a+x(y-x)} + a\tan^{-1}\left(\frac{\sqrt{a+x(y-x)}}{x}\right)\right)$	$\mathscr{R} = J_1^{-1}J_2^{\frac{1}{2}}$

Source: Elaborated by the author.

(ii.3) $g = \frac{1}{4}$ and $a \neq 0$.

Here we have, apart from the two invariant hyperbolas, a third invariant hyperbola and one additional invariant line.

Table 19 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = 1/4$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = a + xy$ $J_3 = a + xy - x^2$ $J_4 = a + xy - y^2$ $\alpha_1 = \frac{x}{4} + \frac{y}{4}$ $\alpha_2 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_3 = \frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -\frac{x}{2} + \frac{y}{2}$	$P_1 = \left(-i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(i\sqrt{a}, i\sqrt{a}\right)$ $P_3 = \left(-\frac{\sqrt{a}}{\sqrt{2}}, \frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_4 = \left(\frac{\sqrt{a}}{\sqrt{2}}, -\frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $s, s, ©, ©; N, N, N$ For $a > 0$ we have $©, ©, s, s; N, N, N$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_3 = P_2^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ quadruple $\bar{J}_2 \cap \bar{J}_4 = P_3^\infty$ quadruple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_3 \text{ simple} \\ P_4 \text{ simple} \\ P_2^\infty \text{ double} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 87. 1) We see here that taking J_1 and J_2, the conditions of the theorem of C-K are satisfied and hence we can also have an inverse integrating factor as $J_1 J_2$.

2) Note that there is a Darboux first integral for this case since we have $4 = \frac{n(n+1)}{2}$ invariant algebraic curves.

Table 20 – Divisor and zero-cycles of family (B) when $h = 1/4$, $g = 1/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	5
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	8
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty \text{ if } a > 0 \end{cases}$	18 18

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞ (and at P_3^∞), but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is triple.

Table 21 – First integral and integrating factor of family (B) when $h = 1/4$, $g = 1/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{2}} J_3^{\lambda_3} J_4^{\lambda_3}$	$R = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{2}-\frac{1}{2}} J_3^{\lambda_3} J_4^{\lambda_3}$
Simple example	$\mathscr{I}_1 = J_1^2 J_2 \quad \mathscr{I}_2 = J_3 J_4$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

(ii.4) $g = \frac{3}{4}$ and $a \neq 0$.

Table 22 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = 3/4$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $J_2 = a + xy - x^2$ $\alpha_1 = \frac{x}{2} - \frac{y}{2}$ $\alpha_2 = \frac{3x}{2} - \frac{y}{2}$	$P_1 = \left(-\frac{i\sqrt{a}}{\sqrt{3}}, -i\sqrt{3}\sqrt{a}\right)$ $P_2 = \left(\frac{i\sqrt{a}}{\sqrt{3}}, i\sqrt{3}\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $s, s; N, \binom{2}{1} N, N$ For $a > 0$ we have $©, ©; N, \binom{2}{1} S, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$quadruple $\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 23 – Divisor and zero-cycles of family (B) for $h = 1/4$, $g = 3/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = \begin{cases} P_1 + P_2 + P_1^\infty + 3P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_1^\infty + 3P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2 = 0$	5
$M_{0CT} = \begin{cases} P_1 + P_2 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	9 9

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

Table 24 – First integral and integrating factor of family (B) for $h = 1/4$, $g = 3/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = \frac{1}{4}\left(2y\sqrt{a+xy} + 2a\tanh^{-1}\left(\frac{\sqrt{a+xy}}{x}\right)\right)$	$R = J_1^{\frac{1}{2}} J_2^{-1}$
Simple example	$\mathscr{I} = \frac{1}{4}\left(2y\sqrt{a+xy} + 2a\tanh^{-1}\left(\frac{\sqrt{a+xy}}{x}\right)\right)$	$\mathscr{R} = J_1^{\frac{1}{2}} J_2^{-1}$

Source: Elaborated by the author.

(ii.5) $g = \frac{1}{2}$ and $a \neq 0$.

Under this condition the systems do not belong to family (B) $h = 1/4$. Here we have, apart from the two invariant hyperbolas, a third invariant hyperbola and one additional invariant line.

Table 25 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = 1/2$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = a + xy$ $J_3 = a + xy - x^2$ $J_4 = -a + xy - y^2$ $\alpha_1 = -\frac{x}{2} + \frac{y}{4}$ $\alpha_2 = -\frac{y}{2}$ $\alpha_3 = x - \frac{y}{2}$ $\alpha_4 = \frac{y}{2}$	$P_1 = \left(-\sqrt{a}, 0\right)$ $P_2 = \left(\sqrt{a}, 0\right)$ $P_3 = \left(-\frac{i\sqrt{a}}{\sqrt{2}}, -i\sqrt{2}\sqrt{a}\right)$ $P_4 = \left(\frac{i\sqrt{a}}{\sqrt{2}}, i\sqrt{2}\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $©, ©, s, s; N, N, N$ For $a > 0$ we have $s, s, ©, ©; N, N, N$	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ double $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = P_3^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ quadruple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_3 \text{ simple} \\ P_4 \text{ simple} \\ P_3^\infty \text{ double} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_4 = P_2^\infty$ quadruple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 88. We see here that taking J_1 and J_3, the conditions of the theorem of C-K are satisfied and hence we can have an inverse integrating factor as $J_1 J_3$.

Table 26 – Divisor and zero-cycles of family (B) when $h = 1/4$, $g = 1/2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	5
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4 = 0$	8
$M_{0CT} = \begin{cases} 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty \text{ if } a < 0 \\ 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty \text{ if } a > 0 \end{cases}$	18 18

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞ (and at P_2^∞), but one of them is double,

2) only two distinct tangents at P_3^∞, but one of them is triple.

Table 27 – First integral and integrating factor of family (B) when $h = 1/4$, $g = 1/2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\frac{\lambda_1}{2}} J_4^{\lambda_2}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\frac{\lambda_1}{2}-\frac{1}{2}} J_4^{\lambda_2}$
Simple example	$\mathscr{I}_1 = J_1^2 J_3 \quad \mathscr{I}_2 = J_2 J_4$	$\mathscr{R} = \dfrac{1}{J_1 J_3}$

Source: Elaborated by the author.

(ii.6) $g = 1$ and $a \neq 0$.

Under this condition the systems do not belong to family (B) $h = 1/4$. Here we have, apart from the two invariant hyperbolas, two additional invariant lines.

Table 28 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $g = 1$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{2\sqrt{a}}$ $J_2 = 1 + \frac{iy}{2\sqrt{a}}$ $J_3 = a + xy$ $J_4 = a + xy - x^2$ $\alpha_1 = \frac{y}{4} - \frac{i\sqrt{a}}{2}$ $\alpha_2 = \frac{y}{4} + \frac{i\sqrt{a}}{2}$ $\alpha_3 = x - \frac{y}{2}$ $\alpha_4 = 2x - \frac{y}{2}$	$P_1 = \left(-\frac{i\sqrt{a}}{2}, -2i\sqrt{a}\right)$ $P_2 = \left(\frac{i\sqrt{a}}{2}, 2i\sqrt{a}\right)$ $P_3 = \left(-i\sqrt{a}, -2i\sqrt{a}\right)$ $P_4 = \left(i\sqrt{a}, 2i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $s, s, n, n; N, S, N$ For $a > 0$ we have ©, ©, ©, ©; N, S, N	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = P_3$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = P_4$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ quadruple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 89. Note that there is a Darboux first integral for this case since we have $4 = \frac{n(n+1)}{2}$ invariant algebraic curves.

Table 29 – Divisor and zero-cycles of family (B) when $h = 1/4$, $g = 1$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 2P_2^\infty + 4P_3^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 2P_2^\infty + 4P_3^\infty \text{ if } a > 0 \end{cases}$	17 17

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double,

2) four distinct tangents at P_3^∞.

Table 30 – First integral and integrating factor of family (B) when $h = 1/4$, $g = 1$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{-\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{-\frac{3}{2}-\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_2 J_3^2}{J_4}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4^{\frac{1}{2}}}$

Source: Elaborated by the author.

Observation 90. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_2 J_3^2 - c_2 J_4 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 6$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:a]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,a)} = \frac{\left(2ax+ay+xy^2\right)^2}{4a}, \quad \mathscr{F}^1_{(1,0)} = J_1 J_2 J_3^2.$$

Therefore, J_1, J_2, J_3 and $J_5 := 2ax+ay+xy^2$ are remarkable curves, $[1:a]$ and $[1:0]$ are all critical remarkable values and J_3, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_2 for $\mathscr{F}^1_{(1,0)}$ and P_3, P_4 for $\mathscr{F}^1_{(1,a)}$.

Observation 91. $J_5 := 2ax+ay+xy^2$ is an invariant algebraic curve of family (B) when $h = 1/4$, $g = 1$ and $a \neq 0$, with cofactor given by $\alpha_5 = x - \frac{y}{4}$.

Observation 92. Note that the rational first integral $\mathscr{I}_1$ in Table 30 has the rational integrating factor $R = \frac{J_4}{J_1 J_2 J_3 J_5^3}$ expressed by an invariant curve of degree higher than two.

(ii.7) $a = 0$ and $g \neq 0, \frac{3}{4}$.

Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

(ii.7.1) $a = 0$ and $g \neq -\frac{1}{4}, 0, \frac{3}{4}, 1$.

Under this condition, systems (B) $h = 1/4$ do not belong to **QSH**. The affine invariant lines are $y = 0$, $-x+y = 0$ and $x = 0$ that are all simple. By perturbing the reducible conics $xy = 0$ and $x(x-y) = 0$ we can produce the two distinct hyperbolas $a+xy = 0$ and $a-x^2+xy = 0$ respectively.

Table 31 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $a = 0$ and $g \neq -1/4, 0, 3/4, 1$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = -x+y$ $J_3 = x$ $\alpha_1 = (-1+g)x + \frac{y}{4}$ $\alpha_2 = gx + \frac{y}{4}$ $\alpha_3 = gx - \frac{3y}{4}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N,N,S$ if $g < 0$ $hhhhhh_{(4)}; N,N,N$ if $0 < g < 3/4$ $pphpph_{(4)}; N,S,N$ if $g > 3/4$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Observation 93. Here we have one of the C-K conditions not satisfied ($(0,0)$ is point of intersection of the 3 curves) but we still have the conclusion i.e. $J_1J_2J_3$ is an inverse integrating factor.

Table 32 – Divisor and zero-cycles of family (B) when $h = 1/4$, $a = 0$ and $g \neq -1/4, 0, 3/4, 1$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	4
$M_{0CT} = 3P_1 + 2P_1^\infty + 2P_2^\infty + 2P_3^\infty$	9

Source: Elaborated by the author.

Table 33 – First integral and integrating factor of family (B) when $h = 1/4$, $a = 0$ and $g \neq -1/4, 0, 3/4, 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{(4g-3)\lambda_1}{4g}} J_3^{\frac{\lambda_1}{4g}}$	$R = J_1^{\lambda_1} J_2^{\frac{3(\lambda_1+1)}{4g} - \lambda_1 - 2} J_3^{\frac{1-4g+\lambda_1}{4g}}$
Simple example	$\mathscr{I} = J_1^{4g} J_2^{3-4g} J_3$	$\mathscr{R} = \frac{1}{J_1J_2J_3}$

Source: Elaborated by the author.

(ii.7.2) $a = 0$ and $g = -1/4$.

Under this condition, systems (B) $h = 1/4$ do not belong to **QSH**. The affine invariant lines are $y = 0$ and $x = 0$ that are both simple and $x - y = 0$ that is double. By perturbing the reducible conics $xy = 0$ and $x(x-y) = 0$ we can produce the two distinct hyperbolas $a + xy = 0$ and $a - x^2 + xy = 0$ respectively.

Table 34 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $a = 0$ and $g = -1/4$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x - y$ $J_3 = x$ $E_4 = e^{\frac{g_0+g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{5x}{4} + \frac{y}{4}$ $\alpha_2 = -\frac{x}{4} + \frac{y}{4}$ $\alpha_3 = -\frac{x}{4} - \frac{3y}{4}$ $\alpha_4 = \frac{g_0}{4}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 35 – Divisor and zero-cycles of family (B) when $h = 1/4$, $a = 0$ and $g = -1/4$.

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2^2\bar{J}_3 = 0$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	9

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is double.

Table 36 – First integral and integrating factor of family (B) when $h = 1/4$, $a = 0$ and $g = -1/4$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-4\lambda_1} J_3^{-\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-5-4\lambda_1} J_3^{-2-\lambda_1} E_4^0$
Simple example	$\mathscr{I} = J_1 J_2^{-4} J_3^{-1}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

(ii.7.3) $a = 0$ and $g = 1$.

Under this condition, systems (B) $h = 1/4$ do not belong to **QSH**. The affine invariant lines are $-x+y=0$ and $x=0$ that are both simple and $y=0$ that is double. By perturbing the reducible conics $xy=0$ and $x(x-y)=0$ we can produce the two distinct hyperbolas $a+xy=0$ and $a-x^2+xy=0$ respectively.

Table 37 – Invariant curves, cofactors, singularities and intersection points of family (B) when $h = 1/4$, $a = 0$ and $g = 1$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = -x+y$ $J_3 = x$ $E_4 = e^{\frac{g_0+g_1 y}{y}}$ $\alpha_1 = \frac{y}{4}$ $\alpha_2 = x+\frac{y}{4}$ $\alpha_3 = x-\frac{3y}{4}$ $\alpha_4 = -\frac{g_0}{4}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $pphpph_{(4)};N,S,N$	$\overline{J}_1 \cap \overline{J}_2 = P_1$ simple $\overline{J}_1 \cap \overline{J}_3 = P_1$ simple $\overline{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = P_1$ simple $\overline{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 38 – Divisor and zero-cycles of family (B) when $h = 1/4$, $a = 0$ and $g = 1$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2\overline{J}_3 = 0$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 2P_2^\infty + 3P_3^\infty$	9

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double,

2) only two distinct tangents at P_3^∞, but one of them is double.

Table 39 – First integral and integrating factor of family (B) when $h = 1/4$, $a = 0$ and $g = 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{\lambda_1}{4}} J_3^{\frac{\lambda_1}{4}} E_4^0$	$R = J_1^{\lambda_1} J_2^{-\frac{\lambda_1}{4}-\frac{5}{4}} J_3^{\frac{\lambda_1}{4}-\frac{3}{4}} E_4^0$
Simple example	$\mathscr{I} = J_1^4 J_2^{-1} J_3$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 94. Here again we have one condition of C-K broken but we still have the conclusion i.e. $J_1J_2J_3$ is an inverse integrating factor.

(ii.8) $a = g = 0$.

Under this condition, systems (B) when $h = 1/4$ do not belong to **QSH**. The system here is $\dot{x} = -\frac{3xy}{4}$, $\dot{y} = \frac{y(-4x+y)}{4}$. This is a degenerate system where the line $y = 0$ is filled up with singularities. The affine invariant lines are $x - y = 0$ and $x = 0$ that are both simple. We get a polynomial first integral.

Table 40 – Invariant curves, cofactors, singularities and intersection points for the reduced of family (B) when $h = 1/4$ and $a = g = 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = x$ $\alpha_1 = \frac{1}{4}$ $\alpha_2 = -\frac{3}{4}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $(\ominus[|];s);N,N,(\ominus[|];\emptyset)$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 41 – Divisor and zero-cycles for the reduced of family (B) when $h = 1/4$ and $a = g = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = P_1 + P_1^\infty + P_2^\infty$	3
$T = Z\bar{J}_1\bar{J}_2 = 0$	3
$M_{0CT} = 2P_1 + 2P_1^\infty + 2P_2^\infty$	6

Source: Elaborated by the author.

Table 42 – First integral and integrating factor for the reduced system of family (B) when $h = 1/4$ and $a = g = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{3}}$	$R = J_1^{\lambda_1} J_2^{\frac{\lambda_1 - 2}{3}}$
Simple example	$\mathscr{I} = J_1^3 J_2$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}$ are also first integrals for family (B) when $h = 1/4$ and $a = g = 0$.

(ii.9) $a=0$ and $g=\dfrac{3}{4}$.

Under this condition, systems (B) when $h=1/4$ do not belong to **QSH**. The system here is $\dot{x}=\frac{3x(x-y)}{4}$, $\dot{y}=-\frac{y(x-y)}{4}$. This is a degenerate system where the line $x-y=0$ is filled up with singularities. The affine invariant lines are $y=0$ and $x=0$ that are both simple. We get a polynomial first integral.

Table 43 – Invariant curves, cofactors, singularities and intersection points for the reduced system of family (B) when $h=1/4$, $g=3/4$ and $a=0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1=y$ $J_2=x$ $\alpha_1=-\frac{1}{4}$ $\alpha_2=\frac{3}{4}$	$P_1=(0,0)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:0:0]$ $(\ominus[\vert];s);N,(\ominus[\vert];\emptyset),N$	$\bar{J}_1\cap\bar{J}_2=P_1$ simple $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\mathscr{L}_\infty=P_1^\infty$ simple

Source: Elaborated by the author.

Table 44 – Divisor and zero-cycles for the reduced system of family (B) when $h=1/4$, $g=3/4$ and $a=0$.

Divisor and zero-cycles	Degree
$ICD=J_1+J_2+\mathscr{L}_\infty$	3
$M_{0C}=P_1+P_1^\infty+P_2^\infty$	3
$T=Z\bar{J}_1\bar{J}_2=0$	3
$M_{0CT}=2P_1+2P_1^\infty+2P_2^\infty$	6

Source: Elaborated by the author.

Table 45 – First integral and integrating factor for the reduced system of family (B) when $h=1/4$, $g=3/4$ and $a=0$.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_2^{\frac{\lambda_1}{3}}$	$R=J_1^{\lambda_1}J_1^{\frac{\lambda_1-2}{3}}$
Simple example	$\mathscr{I}=J_1^3J_2$	$\mathscr{R}=\dfrac{1}{J_1J_2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}$ are also first integrals for family (B) when $h=1/4$, $g=3/4$ and $a=0$.

We sum up the topological, dynamical and algebraic geometric features of family (B) $h = 1/4$ in the following proposition. We also confront our results with previous results in literature in the following proposition.

Proposition 95. (a) For the family (B) when $h = 1/4$ we obtained nine distinct configurations $C_1^{(B)}$ up to $C_9^{(B)}$ of invariant hyperbolas and lines (see Figure 4 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is $ag(g-1)(g\pm 1/4)(g-1/2)(g-3/4) = 0$. Its complement is a union of 12 disjoint sets. In this parameter set the bifurcation of configurations of the systems is formed by the lines $g = -1/4$, $g = 0$, $g = 1/4$ and $g = 3/4$ with $a \neq 0$. On $g = 0$ or $g = \frac{3}{4}$ and $a \neq 0$ we have just the two invariant hyperbolas. On $g = -1/4$ we have two additional invariant lines. On $g = 1/4$ we have one additional invariant hyperbola and one invariant line. For the limiting set of the parameter space of the considered family we have the following: On $g = 1/2$ and $a \neq 0$ we have one additional invariant line and one additional invariant hyperbola. On $g = 1$ and $a \neq 0$ we have two additional invariant lines. On $a = 0$ the invariant hyperbolas become reducible. On $a = g = 0$ the line $y = 0$ is filled up with singularities. On $a = 0$ and $g = 3/4$ the line $x - y = 0$ is filled up with singularities.

(b) The family (B) when $h = 1/4$ is Liouvillian integrable if $a(g-1)(2g-1)(4g-1)(4g+1) \neq 0$. When $g = -1/4$ the family (B) with $h = 1/4$ admits a rational first integral and the plane is foliated into invariant algebraic curves of degree six. The remarkable curves are J_1, J_2, J_3 and J_5 corresponding to this case. When $h = g = 1/4$ the family (B) admits a polynomial first integral.

(c) For the family (B) when $h = 1/4$ we have four topologically distinct phase portraits $P_1^{(B)} - P_4^{(B)}$. The topological bifurcation diagram of family (B) is done in Figure 5. The bifurcation set are the lines $g = 0$, $g = 3/4$, $a = 0$ and the half lines $g = 1/2$ with $a > 0$ and $g = 1/4$ with $a < 0$. The lines $g = 0$, $g = 3/4$ and $a = 0$ are bifurcation sets of singularities and the half lines $g = 1/2$ with $a > 0$ and $g = 1/4$ with $a < 0$ are bifurcations of separatrix from saddle to saddle connection. The phase portraits $P_1^{(B)}$ and $P_3^{(B)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 95.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (B) when $h = 1/4$:

Table 46 – Configurations for family (B) when $h = 1/4$.

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(B)}$	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + P_3 + P_4 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty$
$C_2^{(B)}$	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CS} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \\ P_1^C + P_2^C + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \end{cases}$ $M_{0CT} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \\ P_1^C + P_2^C + P_3 + P_4 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty \end{cases}$
$C_3^{(B)}$	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$ $M_{0CT} = P_1^C + P_2^C + P_3^C + P_4^C + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty$
$C_4^{(B)}$	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CS} = \begin{cases} P_1^C + P_2^C + P_1^\infty + 3P_2^\infty + P_3^\infty \\ P_1^C + P_2^C + P_1^\infty + P_2^\infty + 3P_3^\infty \end{cases}$ $M_{0CT} = P_1^C + P_2^C + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty$
$C_5^{(B)}$	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CS} = \begin{cases} P_1 + P_2 + P_1^\infty + 3P_2^\infty + P_3^\infty \\ P_1 + P_2 + P_1^\infty + P_2^\infty + 3P_3^\infty \end{cases}$ $M_{0CT} = P_1 + P_2 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty$
$C_6^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty$
$C_7^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty$
$C_8^{(B)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_9^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(B)}$ up to $C_9^{(B)}$ are all distinct. For the limit case of family (B) when $h = 1/4$ we have the following configuration:

Table 47 – Configurations for the limit cases of family (B) when $h = 1/4$.

Configuration	Divisors and zero-cycles of the total inv. curve T
$C_6^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty$
$C_7^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 4P_3^\infty$
$C_8^{(B)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_9^{(B)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$
c_1	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_1^\infty + 2P_2^\infty + 2P_3^\infty$
c_2	$ICD = J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$
c_3	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_1^\infty + 2P_2^\infty$
c_4	$ICD = 2J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 2P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves of family (B) when $h = 1/4$ and $g = -1/4$ were done in Remark 84. When $h = g = 1/4$ and $a \neq 0$ the family (B) has an inverse integrating factor which is polynomial (see Table 21).

(c) We have:

Table 48 – Phase portraits for family (B) when $h = 1/4$.

Phase Portraits	Sing. at ∞	Sing. at $< \infty$	Separatrix connections
$P_1^{(B)}$	(N,N,S)	(s,s,n,n)	$2SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(B)}$	(N,N,N) $(N,\binom{2}{1}N,N)$ $(N,N,\binom{2}{1}N)$	$(s,s,©,©)$ (s,s) (s,s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(B)}$	(N,S,N) $(N,\binom{2}{1}S,N)$ $(N,N,\binom{2}{1}S)$	$(©,©,©,©)$ $(©,©)$ $(©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(B)}$	(N,N,N)	$(s,s,©,©)$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have four distinct phase portraits for systems (B) where $h = 1/4$. For the limit cases of family (B) when $h = 1/4$ we have the following phase portraits:

Table 49 – Phase portraits for the limit cases of family (B) when $h = 1/4$.

Phase Portraits	Sing. at ∞	Sing. at $< \infty$	Separatrix connections
p_1	(N,N,S)	$hpphpp_{(4)}$ $pphpph_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_2	(N,N,N)	$hhhhhh_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_3	$(N,N,(\ominus[|];\emptyset))$ $(N,(\ominus[|];\emptyset),N)$	$(\ominus[|];s)$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

On the table below we list all the phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,N,S) and with 0 or 4 real singular points in the finite region.

Table 50 – Phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,N,S) and with 0 or 4 real singular points in the finite region.

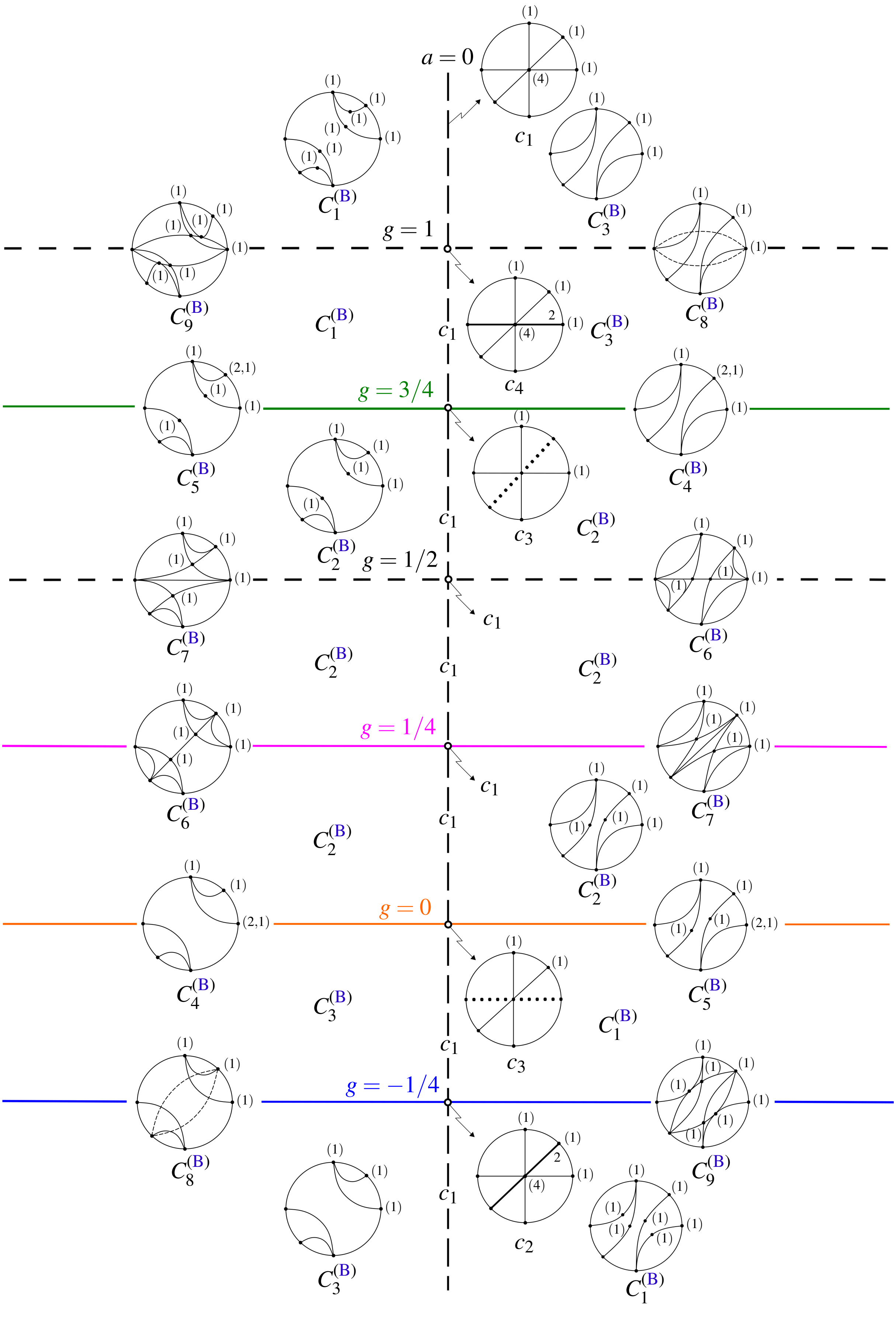
Phase Portraits	Sing. at ∞	Sing. at $< \infty$	Separatrix connections
R_{01,Ω_6}	(N,S,N)	0	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_{8,Ω_1}	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, the phase portraits $P_1^{(B)}$ and $P_3^{(B)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

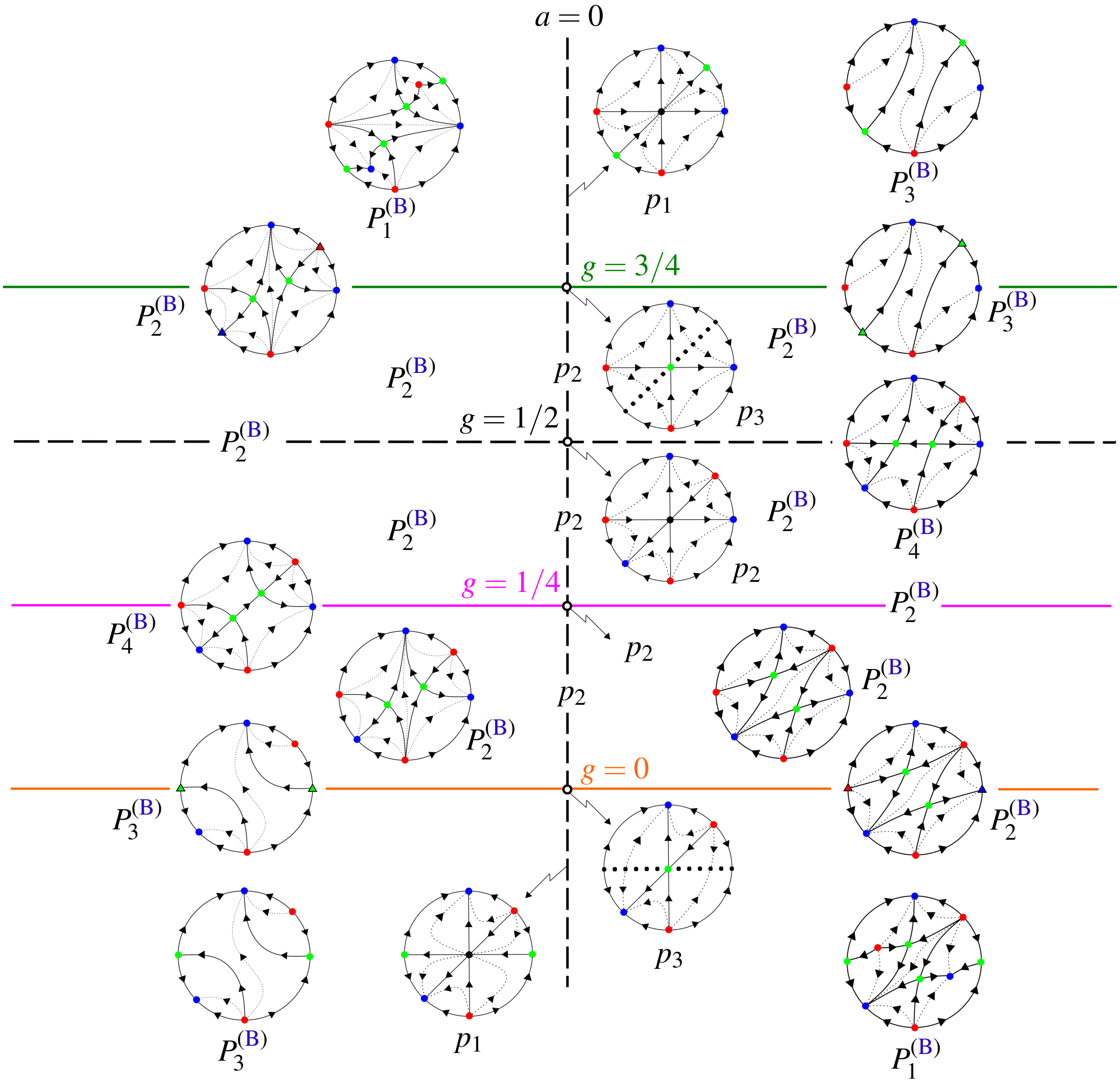
□

Figure 4 – Bifurcation diagram of configurations for family (B) when $h = 1/4$.



Source: Elaborated by the author.

Figure 5 – Topological bifurcation diagram for family (B) when $h = 1/4$.



Source: Elaborated by the author.

6.1.1.1 *The solution of the Poincaré problem for the family* (B) *when* $h = 1/4$.

The following theorem solves the problem of Poincaré for the family defined by the equations (B) when $h = 1/4$ and $(a, g) \in \mathbb{R}^2$.

Theorem 96. A necessary and sufficient condition for a system (S) defined by the equations (B) with $h = 1/4$ and $(a, g) \in \mathbb{R}^2$ to have a rational first integral given by invariant algebraic curves of degree at most two is that: i) $g = \pm\frac{1}{4}, \frac{1}{2}, 1$ and $a \neq 0$, or ii) $a = 0$ and there exist integers m_1, m_2 such that $g = \frac{m_1}{4m_2}$ where $m_1, m_2 \neq 0$ and $m_2 \neq \frac{m_1}{3}$.

Proof. The proof of this result is based on the formulas obtained for the first integrals for the family (B). We remember that these first integrals are obtained using the invariant algebraic curves of degree at most two.

In the generic case $ag(g-1)(2g-1)(3g-4)(4g\pm 1)\neq 0$ we have a Liouvillian first integral I given by a hypergeometric function (see Table 12). In this case, the integrating factor is given by

$$R = J_1^{-1+2g} J_2^{\frac{1}{2}(1-4g)}.$$

Then, R is rational if and only if

$$\begin{cases} -1+2g = m_1, \ m_1 \in \mathbb{Z} \\ \frac{1}{2}(1-4g) = m_2, \ m_2 \in \mathbb{Z}. \end{cases}$$

Solving the first equation we obtain $g = \frac{m_1}{2} + \frac{1}{2}$. Replacing this expression in the second equation we get

$$\tfrac{1}{2} - 2\left(\tfrac{m_1}{2} + \tfrac{1}{2}\right) = m_2 \Rightarrow \tfrac{1}{2} = m_1 + 1 + m_2 \in \mathbb{Z}.$$

As this cannot happen, R is not rational and it follows from Theorem 34 that this system does not have a generalized Darboux first integral, accordingly, a rational first integral.

In the non-generic case $ag(g-1)(2g-1)(3g-4)(4g\pm 1) = 0$ we also have a Liouvillian first integral in the cases $g(3g-4) = 0$ and $a \neq 0$ (see Table 18 and 24). In both cases, the expressions of the first integral is given by square roots and $\tan^{-1}$ or $\tanh^{-1}$. Therefore, the systems do not have a rational first integral.

When $g = \pm\frac{1}{4}, \frac{1}{2}, 1$ and $a \neq 0$ we obtain rational first integrals (see Tables 21, 15, 27 and 30). When $a = 0$ and $g \neq -\frac{1}{4}, 0, \frac{3}{4}, 1$ then the first integral of family (B) with $h = \frac{1}{4}$ is given by

$$I = J_1^{\lambda_1} J_2^{-\frac{(4g-3)\lambda_1}{4g}} J_3^{\frac{\lambda_1}{4g}} \tag{6.1}$$

where $\lambda_1 \neq 0$ and J_1, J_2, J_3, J_4 are given in table 31. This is a rational first integral if and only if

$$\begin{cases} \lambda_1 &= m_1, \ m_1 \in \mathbb{Z}\backslash\{0\} \\ \frac{\lambda_1}{4g} &= m_2, \ m_2 \in \mathbb{Z}\backslash\{0\} \\ -\frac{(4g-3)}{4g}\lambda_1 &= m_3, \ m_3 \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{6.2}$$

Replacing $\lambda_1 = m_1$ in the second equation of (6.2) we obtain

$$\tfrac{m_1}{4g} = m_2 \Rightarrow g = \tfrac{m_1}{4m_2}, \ m_1, m_2 \in \mathbb{Z}\backslash\{0\}, \ m_2 \neq -m_1, \ m_2 \neq \tfrac{m_1}{3} \text{ and } m_2 \neq \tfrac{m_1}{4}.$$

Note that as $g = \frac{m_1}{4m_2}$ with $m_1, \ m_2 \neq 0$ then

$$m_3 = -\frac{(4g-3)}{4g} m_1 = \frac{-4\left(\frac{m_1}{4m_2}\right)+3}{4\left(\frac{m_1}{4m_2}\right)} m_1 = -m_1 + 3m_2 \in \mathbb{Z}\backslash\{0\}.$$

Therefore, if I is rational then $g = \frac{m_1}{4m_2}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq \frac{m_1}{3}$ and $m_2 \neq \frac{m_1}{4}$. Conversely, replacing $g = \frac{m_1}{4m_2}$ and $\lambda_1 = m_1$ in (6.1) where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq \frac{m_1}{3}$ and $m_2 \neq \frac{m_1}{4}$ we obtain that

$$I = J_1^{m_1} J_2^{-m_1+3m_2} J_3^{m_2}$$

which is rational. Therefore, when $a=0$ and $g \neq -\frac{1}{4}, 0, \frac{3}{4}, 1$, the systems are algebraically integrable if and only if $g=\frac{m_1}{4m_2}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq \frac{m_1}{3}$ and $m_2 \neq \frac{m_1}{4}$.

When $a=0$ and $(g+1/4)(g-1)=0$ we obtain a rational first integrals (see Tables 36 and 39). Note that

$$\begin{cases} \frac{m_1}{4m_2} = -\frac{1}{4} \Leftrightarrow m_2 = -m_1 \\ \frac{m_1}{4m_2} = 1 \Leftrightarrow m_2 = \frac{m_1}{4} \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$. When $a=g=0$ or $a=0$ and $g=\frac{3}{4}$ the systems (B) with $h=\frac{1}{4}$ are degenerate.

□

6.1.2 Geometric Analysis of family (C)

Consider the family

$$(C) \quad \begin{cases} \dot{x} = a + \left(\frac{1-2h}{2}\right)x^2 + (h-1)xy \\ \dot{y} = a - \left(\frac{2h+1}{2}\right)xy + hy^2, \end{cases}$$

where $ah(h-1)(2h\pm 1) \neq 0$.

This is a two parameter family depending on a and h such that $ah(h-1)(2h\pm1)\neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (C) we study here also the limit cases $ah(h-1)(2h\pm1)=0$ where the equations are still defined.

We display below the full geometric analysis of the systems in this family, which is endowed with at least three invariant algebraic curves. In the **generic case**

$$ah(h-1)(h-1/2)(h+1/2)(h-1/4) \neq 0$$

the systems have only one invariant line J_1 and two invariant hyperbolas J_2 and J_3 with cofactors α_i, $1 \leq i \leq 3$ given by

$$\begin{array}{ll} J_1 = x - y, & \alpha_1 = \frac{1}{2}(1-2h)x + hy, \\ J_2 = \frac{a}{2h-1} - x^2 + xy, & \alpha_2 = (1-2h)x + (-1+2h)y, \\ J_3 = -\frac{a}{2h} + xy - y^2, & \alpha_3 = -2hx + 2hy. \end{array}$$

We note that when $h=1/4$ we have one additional invariant hyperbola. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $ah(h-1)(h-1/2)(h+1/2)(h-1/4) \neq 0$.

Table 51 – Invariant curves, cofactors, singularities and intersection points of family (C) for the generic case.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = \frac{a}{2h-1} - x^2 + xy$ $J_3 = -\frac{a}{2h} + xy - y^2$ $\alpha_1 = \frac{1}{2}(1-2h)x + hy$ $\alpha_2 = (1-2h)(x-y)$ $\alpha_3 = -2hx + 2hy$	$P_1 = \left(-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a}\right)$ $P_2 = \left(\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a}\right)$ $P_3 = \left(-\frac{\sqrt{2}\sqrt{a}\sqrt{h}}{\sqrt{2h-1}}, -\frac{\sqrt{a}\sqrt{2h-1}}{\sqrt{2}\sqrt{h}}\right)$ $P_4 = \left(\frac{\sqrt{2}\sqrt{a}\sqrt{h}}{\sqrt{2h-1}}, \frac{\sqrt{a}\sqrt{2h-1}}{\sqrt{2}\sqrt{h}}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; S, N, N if $h < 0$ ©, ©, $s, s; N, N, N$ if $0 < h < 1/2$ ©, ©, ©, ©; N, N, S if $h > 1/2$ For $a > 0$ we have $s, s, n, n; S, N, N$ if $h < 0$ s, s, ©, ©; N, N, N if $0 < h < 1/2$ $s, s, n, n; N, N, S$ if $h > 1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \bar{J}_3 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ double} \\ P_3 \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 52 – Divisors and zero-cycles of family (C) for the generic case.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$ if $a < 0$ and $h < 0$ or $a < 0$ and $h > 1/2$	7 7
$M_{0CS} = P_1^C + P_2^C + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$ if $a < 0$ and $0 < h < 1/2$	7
$M_{0CS} = P_1 + P_2 + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$ if $a > 0$ and $0 < h < 1/2$	7
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$ if $a > 0$ and $h > 1/2$ or $a > 0$ and $h < 0$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	6
$M_{0CT} = P_1^C + P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$ if $a < 0$ and $h < 0$ or $a < 0$ and $h > 1/2$	14 14
$M_{0CT} = P_1^C + P_2^C + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$ if $a < 0$ and $0 < h < 1/2$	14
$M_{0CT} = P_1 + P_2 + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$ if $a > 0$ and $0 < h < 1/2$	14
$M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$ if $a > 0$ and $h > 1/2$ or $a > 0$ and $h < 0$	14 14

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_2^∞, but one of them is triple.

Table 53 – First integral and integrating factor of family (C) for the generic case.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\frac{(2h-1)\lambda_2}{2h}}$	$R = J_1^1 J_2^{\lambda_2} J_3^{\frac{-2h\lambda_2-4h+\lambda_2+1}{2h}}$
Simple example	$\mathscr{I} = J_2^{2h} J_3^{1-2h}$	$\mathscr{R} = \dfrac{J_1}{J_2 J_3}$

Source: Elaborated by the author.

(ii) The non-generic case: $ah(h-1)(h-1/2)(h+1/2)(h-1/4) = 0$.

(ii.1) $h = 1/4$ and $a \neq 0$.

Here we have, apart from one line and two hyperbolas, a third invariant hyperbola.

Table 54 – Invariant curves, cofactors, singularities and intersection points of family (C) when $h = 1/4$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -2a + xy$ $J_3 = -2a - x^2 + xy$ $J_4 = -2a + xy - y^2$ $\alpha_1 = \frac{x}{4} + \frac{y}{4}$ $\alpha_2 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_3 = \frac{x}{2} - \frac{y}{2}$ $\alpha_4 = \frac{y}{2} - \frac{x}{2}$	$P_1 = (-i\sqrt{a}, i\sqrt{a})$ $P_2 = (i\sqrt{a}, -i\sqrt{a})$ $P_3 = (-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a})$ $P_4 = (\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $s, s, ©, ©; N, N, N$ For $a > 0$ we have $©, ©, s, s; N, N, N$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_3 \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_3 = P_2^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ quadruple $\bar{J}_2 \cap \bar{J}_4 = P_3^\infty$ quadruple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ double} \\ P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 97. We see here that taking J_1 and J_2, the conditions of the theorem of C-K are satisfied and hence we can have an inverse integrating factor as $J_1 J_2$. We can also note that according to Darboux' theorem we have a Darboux first integral.

Table 55 – Divisor and zero-cycles of family (C) when $h = 1/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	5
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0.$	8
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 3P_3^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 3P_3^\infty \text{ if } a > 0 \end{cases}$	18 18

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double;

2) only two distinct tangents at P_2^∞, but one of them is triple and

3) only two distinct tangents at P_3^∞, but one of them is double.

Table 56 – First integral and integrating factor of family (C) when $h = 1/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{2}} J_3^{\lambda_3} J_4^{\lambda_3}$	$R = J_1^{\lambda_1} J_2^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_3^{\lambda_3} J_4^{\lambda_3}$
Simple example	$\mathscr{I} = J_1^2 J_2$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

(ii.2) $h = -1/2$ and $a \neq 0$.

Under this condition the systems do not belong to family (C). Here we have three invariant lines and two invariant hyperbolas. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

Table 57 – Invariant curves, cofactors, singularities and intersection points of family (C) when $h = -1/2$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = 1 - \frac{y}{\sqrt{2}\sqrt{a}}$ $J_3 = 1 + \frac{y}{\sqrt{2}\sqrt{a}}$ $J_4 = a + xy - y^2$ $J_5 = -a - 2x^2 + 2xy$ $\alpha_1 = x - \frac{y}{2}$ $\alpha_2 = -\frac{\sqrt{a}}{\sqrt{2}} - \frac{y}{2}$ $\alpha_3 = \frac{\sqrt{a}}{\sqrt{2}} - \frac{y}{2}$ $\alpha_4 = x - y$ $\alpha_5 = 2x - 2y$	$P_1 = \left(-\frac{\sqrt{a}}{\sqrt{2}}, -\sqrt{2}\sqrt{a}\right)$ $P_2 = \left(\frac{\sqrt{a}}{\sqrt{2}}, \sqrt{2}\sqrt{a}\right)$ $P_3 = (-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a})$ $P_4 = (\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; S, N, N For $a > 0$ we have $n, n, s, s; S, N, N$	$\bar{J}_1 \cap \bar{J}_2 = P_4$ simple $\bar{J}_1 \cap \bar{J}_3 = P_3$ simple $\bar{J}_1 \cap \bar{J}_4 = P_2^\infty$ double $\bar{J}_1 \cap \bar{J}_5 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_4 = \{ P_3^\infty$ simple, P_2 simple $\bar{J}_2 \cap \bar{J}_5 = P_2$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \{ P_3^\infty$ simple, P_1 simple $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \{ P_1$ simple, P_2 simple, P_2^∞ double $\bar{J}_4 \cap \mathscr{L}_\infty = \{ P_2^\infty$ simple, P_3^∞ simple $\bar{J}_5 \cap \mathscr{L}_\infty = \{ P_1^\infty$ simple, P_2^∞ simple

Source: Elaborated by the author.

Observation 98. According to Jouanolou's theorem we must have a rational first integral.

Table 58 – Divisor and zero-cycles of family (C) when $h = -1/2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 4P_3^\infty \text{ if } a < 0 \\ 3P_1 + 3P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 4P_3^\infty \text{ if } a > 0 \end{cases}$	20 20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is triple and

3) four distinct tangents at P_3^∞.

Table 59 – First integral and integrating factor of family (C) when $h = -1/2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{2}} J_3^{\frac{\lambda_1}{2}} J_4^{\lambda_4} J_5^{-\frac{\lambda_1}{2}-\frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_3^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_4^{\lambda_4} J_5^{-1-\frac{\lambda_1}{2}-\frac{\lambda_4}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2 J_2 J_3}{J_5} \quad \mathscr{I}_2 = \dfrac{J_4^2}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 99. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_2 J_3 - c_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:-1/2]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,-1/2)} = -\frac{J_4^2}{2a}, \ \ \mathscr{F}^1_{(1,0)} = J_1^2 J_2 J_3.$$

Therefore, J_1, J_2, J_3, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1:-1/2]$ and $[1:0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_4 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_2 for $\mathscr{F}^1_{(1,-1/2)}$ and P_3, P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_4^2 - c_2 J_5$ we have the remarkable values $[1:-a]$ and $[1:0]$ and the same remarkable curves J_1, J_2, J_3, J_4, J_5. However, the singular points are P_1, P_2 for $\mathscr{F}^2_{(1,0)}$ and P_3, P_4 for $\mathscr{F}^2_{(1,-a)}$.

(ii.3) $h = 0$ and $a \neq 0$.

Under this condition the systems do not belong to family (C). Here we have one invariant line and one invariant hyperbola. We also could find an exponential factor.

Table 60 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (C) when $h = 0$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -a - x^2 + xy$ $E_3 = e^{g_0 + g_1 y(x-y)}$ $\alpha_1 = \frac{x}{2}$ $\alpha_2 = x - y$ $\alpha_3 = a g_1 (x - y)$	$P_1 = (-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a})$ $P_2 = (\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©; $\binom{2}{1}S, N, N$ For $a > 0$ we have $s, s; \binom{2}{1}N, N, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 61 – Divisor and zero-cycles of family (C) when $h = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = \begin{cases} P_1^C + P_2^C + 3P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + 3P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2 = 0$	4
$M_{0CT} = \begin{cases} P_1^C + P_2^C + 2P_1^\infty + 3P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + 2P_1^\infty + 3P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	8 8

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_2^∞, but one of them is double.

Table 62 – First integral and integrating factor of family (C) when $h = 0$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{-ag_1\lambda_3} E_3^{\lambda_3}$	$R = J_1^1 J_2^{-1-ag_1\lambda_3} E_3^{\lambda_3}$
Simple example	$\mathscr{I} = J_2^{-a} E_3$	$\mathscr{R} = \dfrac{J_1}{J_2}$

Source: Elaborated by the author.

(ii.4) $h = 1/2$ and $a \neq 0$.

Under this condition the systems do not belong to family (C). Here we have one invariant line and one invariant hyperbola. We also could find an exponential factor.

Table 63 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (C) when $h = 1/2$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -a + xy - y^2$ $E_3 = e^{g_0 - g_1 x(x-y)}$ $\alpha_1 = \frac{y}{2}$ $\alpha_2 = -x + y$ $\alpha_3 = -ag_1(x-y)$	$P_1 = (-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a})$ $P_2 = (\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©; $N, N, \binom{2}{1} S$ For $a > 0$ we have $s, s; N, N, \binom{2}{1} N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 64 – Divisor and zero-cycles of family (C) when $h = 1/2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathcal{L}_\infty$	3
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_1^\infty + P_2^\infty + 3P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_1^\infty + P_2^\infty + 3P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2 = 0$	4
$M_{0CT} = \begin{cases} P_1^C + P_2^C + P_1^\infty + 3P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_1^\infty + 3P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	8 8

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_2^∞, but one of them is double.

Table 65 – First integral and integrating factor of family (C) when $h = 1/2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{-ag_1\lambda_3} E_3^{\lambda_3}$	$R = J_1^1 J_2^{-1-ag_1\lambda_3} E_3^{\lambda_3}$
Simple example	$\mathscr{I} = J_2^{-a} E_3$	$\mathscr{R} = \dfrac{J_1}{J_2}$

Source: Elaborated by the author.

(ii.5) $h = 1$ and $a \neq 0$.

Under this condition the systems do not belong to family (C). Here we have three invariant lines and two invariant hyperbolas. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

Table 66 – Invariant curves, cofactors, singularities and intersection points of family (C) when $h = 1$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = 1 - \frac{x}{\sqrt{2}\sqrt{a}}$ $J_3 = 1 + \frac{x}{\sqrt{2}\sqrt{a}}$ $J_4 = a - x^2 + xy$ $J_5 = -a + 2xy - 2y^2$ $\alpha_1 = -\frac{x}{2} + y$ $\alpha_2 = -\frac{\sqrt{a}}{\sqrt{2}} - \frac{x}{2}$ $\alpha_3 = \frac{\sqrt{a}}{\sqrt{2}} - \frac{x}{2}$ $\alpha_4 = -x + y$ $\alpha_5 = -2x + 2y$	$P_1 = \left(-\sqrt{2}\sqrt{a}, -\frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_2 = (-\sqrt{2}\sqrt{a}, -\sqrt{2}\sqrt{a})$ $P_3 = \left(\sqrt{2}\sqrt{a}, \frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_4 = (\sqrt{2}\sqrt{a}, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; N, N, S For $a > 0$ we have $n, s, n, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_4$ simple $\bar{J}_1 \cap \bar{J}_3 = P_2$ simple $\bar{J}_1 \cap \bar{J}_4 = P_2^\infty$ double $\bar{J}_1 \cap \bar{J}_5 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = P_3$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1 \text{ simple} \\ P_3 \text{ simple} \\ P_2^\infty \text{ double} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 100. According to Jouanolou's theorem we must have a rational first integral.

Table 67 – Divisor and zero-cycles of family (C) when $h = 1$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1^C + 2P_2^C + 3P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ 3P_1 + 2P_2 + 3P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	20 20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_3), but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is triple and

3) four distinct tangents at P_1^∞.

Table 68 – First integral and integrating factor of family (C) when $h = 1$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\frac{\lambda_1}{2}} J_3^{\frac{\lambda_1}{2}} J_4^{\lambda_4} J_5^{-\frac{\lambda_1}{2}-\frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_3^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_4^{\lambda_4} J_5^{-1-\frac{\lambda_1}{2}-\frac{\lambda_4}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2 J_2 J_3}{J_5}$ $\quad\mathscr{I}_2 = \dfrac{J_4^2}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 101. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_2 J_3 - c_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:-1/2]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,-1/2)} = -\frac{J_4^2}{2a}, \quad \mathscr{F}^1_{(1,0)} = J_1^2 J_2 J_3.$$

Therefore, J_1, J_2, J_3, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1:-1/2]$ and $[1:0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_4 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_3 for $\mathscr{F}^1_{(1,-1/2)}$ and P_2, P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_4^2 - c_2 J_5$ we have the remarkable values $[1:-a]$ and $[1:0]$ and the same remarkable curves J_1, J_2, J_3, J_4, J_5. However, the singular point are P_1, P_3 for $\mathscr{F}^2_{(1,0)}$ and P_2, P_4 for $\mathscr{F}^2_{(1,-a)}$.

(ii.6) $a = 0$ and $h \neq -1/2, 0, 1/2, 1$.

Under this condition, systems (C) do not belong to **QSH**. The affine invariant lines are $x - y = 0$, $x = 0$ and $y = 0$ that are all simple. Considering the line at infinity

$Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

Table 69 – Invariant curves, cofactors, singularities and intersection points of family (C) when $a = 0$ and $h \neq -1/2, 0, 1/2, 1$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = x$ $J_3 = y$ $\alpha_1 = \frac{1}{2}(1-2h)x + hy$ $\alpha_2 = \frac{1}{2}(1-2h)x + hy$ $\alpha_3 = \frac{1}{2}(-1-2h)x + hy$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $phpphp_{(4)}; S, N, N$ if $h < 0$ $hhhhhh_{(4)}; N, N, N$ if $0 < h < 1/2$ $hpphpp_{(4)}; N, N, S$ if $h > 1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple

Source: Elaborated by the author.

Table 70 – Divisor and zero-cycles of family (C) when $a = 0$ and $h \neq -1/2, 0, 1/2, 1$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0.$	4
$M_{0CT} = 3P_1 + 2P_1^\infty + 2P_2^\infty + 2P_3^\infty$	9

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_1.

Table 71 – First integral and integrating factor of family (C) when $a = 0$ and $h \neq -1/2, 0, 1/2, 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{2h\lambda_1} J_3^{-(-1+2h)\lambda_1}$	$R = J_1^{\frac{-2h+\lambda_2+1}{2h}} J_2^{\lambda_2} J_3^{\frac{-2h\lambda_2-4h+\lambda_2+1}{2h}}$
Simple example	$\mathscr{I} = J_1 J_2^{2h} J_3^{1-2h}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

(ii.7) $a = 0$ and $h = -1/2$.

Under this condition, system (C) does not belong to **QSH**. The affine invariant lines are $x - y = 0$, $x = 0$ that are simple and $y = 0$ that is double. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 72 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (C) when $a = 0$ and $h = -1/2$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = x$ $J_3 = y$ $E_4 = e^{\frac{g_0}{y} + g_1}$ $\alpha_1 = x - \frac{y}{2}$ $\alpha_2 = x - \frac{3y}{2}$ $\alpha_3 = -\frac{y}{2}$ $\alpha_4 = \frac{g_0}{2}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $phpphp_{(4)}; S, N, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple

Source: Elaborated by the author.

Table 73 – Divisor and zero-cycles of family (C) when $a = 0$ and $h = -1/2$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 2P_2^\infty + 3P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, one of them double;

2) only two distinct tangents at P_3^∞, but one of them is double.

Table 74 – First integral and integrating factor of family (C) when $a = 0$ and $h = -1/2$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{2\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{1+2\lambda_1} E_4^0$
Simple example	$\mathscr{I} = J_1 J_2^{-1} J_3^2$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 102. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_3^2 - c_2 J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1 J_3^2.$$

Therefore, J_1, J_3 are remarkable curves of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(1,0)}$.

(ii.8) $a = h = 0$.

Under this condition, system (C) does not belong to **QSH**. The system here is $\dot{x} = \frac{x(x-2y)}{2}$, $\dot{y} = -\frac{xy}{2}$. This is a degenerate system where the line $x = 0$ is filled up with singularities. The affine invariant lines are $x - y = 0$ and $y = 0$ that are both simple. We get a polynomial first integral.

Table 75 – Invariant curves, cofactors, singularities and intersection points for the reduced system of family (C) when $a = h = 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = y$ $\alpha_1 = \frac{1}{2}$ $\alpha_2 = -\frac{1}{2}$	$P_1 = (0,0)$ $P_1^\infty = [1:1:0]$ $P_2^\infty = [1:0:0]$ $(\ominus[\|];s);(\ominus[\|];\emptyset),N,N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple

Source: Elaborated by the author.

Table 76 – Divisor and zero-cycles for the reduced system of family (C) when $a = h = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = P_1 + P_1^\infty + P_2^\infty$	3
$T = Z\bar{J}_1\bar{J}_2 = 0.$	3
$M_{0CT} = 2P_1 + 2P_1^\infty + 2P_2^\infty$	6

Source: Elaborated by the author.

Table 77 – First integral and integrating factor for the reduced system of family (C) when $a = h = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_1}$
Simple example	$\mathscr{I}_1 = J_1 J_2$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}_1$ are also first integrals for family (C) when $a = h = 0$.

(ii.9) $a = 0$ and $h = 1/2$.

Under this condition, system (C) does not belong to **QSH**. The system here is $\dot{x} = -\frac{xy}{2}$, $\dot{y} = \frac{y(-2x+y)}{2}$. This is a degenerate system where the line $y = 0$ is filled up with singularities. The affine invariant lines are $x - y = 0$ and $x = 0$ that are both simple. We get a polynomial first integral.

Table 78 – Invariant curves, cofactors, singularities and intersection points for the reduced system of family (C) when $a = 0$ and $h = 1/2$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = x$ $\alpha_1 = \frac{1}{2}$ $\alpha_2 = -\frac{1}{2}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $(\ominus[\|];s);N,N,(\ominus[\|];\emptyset)$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple

Source: Elaborated by the author.

Table 79 – Divisor and zero-cycles for the reduced system of family (C) when $a = 0$ and $h = 1/2$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = P_1 + P_1^\infty + P_2^\infty$	3
$T = Z\bar{J}_1\bar{J}_2 = 0.$	3
$M_{0CT} = 2P_1 + 2P_1^\infty + 2P_2^\infty$	6

Source: Elaborated by the author.

Table 80 – First integral and integrating factor for the reduced system of family (C) when $a = 0$ and $h = 1/2$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_1}$
Simple example	$\mathscr{I} = J_1 J_2$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}$ are also first integrals for family (C) when $a = 0$ and $h = 1/2$.

(ii.10) $a = 0$ and $h = 1$.

Under this condition, system (C) does not belong to **QSH**. The affine invariant lines are $x-y=0$, $y=0$ that are simple and $x=0$ that is double. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 81 – Invariant curves,exponential factors, cofactors, singularities and intersection points of family (C) when $a=0$ and $h=1$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x-y$ $J_2 = x$ $J_3 = y$ $E_4 = e^{\frac{g_0}{x}+g_1}$ $\alpha_1 = -\frac{x}{2}+y$ $\alpha_2 = -\frac{x}{2}$ $\alpha_3 = -\frac{3x}{2}+y$ $\alpha_4 = \frac{g_0}{2}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)};N,N,S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple

Source: Elaborated by the author.

Table 82 – Divisor and zero-cycles of family (C) when $a=0$ and $h=1$.

Divisor and zero-cycles	Degree
$ICD = J_1+2J_2+J_3+\mathscr{L}_\infty$	5
$M_{0CS} = 4P_1+P_1^\infty+P_2^\infty+P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2^2\bar{J}_3 = 0.$	5
$M_{0CT} = 4P_1+3P_1^\infty+2P_2^\infty+2P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, one of them double;

2) only two distinct tangents at P_1^∞, but one of them is double.

Table 83 – First integral and integrating factor of family (C) when $a=0$ and $h=1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1}J_2^{2\lambda_1}J_3^{-\lambda_1}E_4^0$	$R = J_1^{\lambda_1}J_2^{1+2\lambda_1}J_3^{-2-\lambda_1}E_4^0$
Simple example	$\mathscr{I} = J_1J_2^2J_3^{-1}$	$\mathscr{R} = \dfrac{1}{J_1J_2J_3}$

Source: Elaborated by the author.

Observation 103. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_2^2 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1 J_2^2.$$

Therefore, J_1, J_2 are remarkable curves of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (C) and we also confront our results with previous results in the literature in the following proposition.

Proposition 104. (a) For the family (C) we have six distinct configurations $C_1^{(C)} - C_6^{(C)}$ of invariant hyperbolas and lines (see Figure 6 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $ah(h-1)(h-1/2)(h+1/2)(h-1/4) = 0$. Its complement is a union of 12 disjoint sets. On $h = 1/4$ and $a \neq 0$ we have one additional invariant hyperbola. For the limiting set of the parameter space of the considered family we have the following: On $h = -1/2$ and $a \neq 0$ or $h = 1$ and $a \neq 0$ we have three invariant lines and two invariant hyperbola. On $h = 0$ and $a \neq 0$ or $h = 1/2$ and $a \neq 0$ we have one invariant line and one invariant hyperbola. On $a = 0$ the invariant hyperbolas become reducible. On $a = h = 0$ the line $x = 0$ is filled up with singularities. On $a = 0$ and $h = 1/2$ the line $y = 0$ is filled up with singularities.

(b) The family (C) is Darboux integrable if $ah(h-1)(h-1/2)(h+1/2)(h-1/4) \neq 0$. When $h = 1/4$ the family (C) admits a polynomial first integral.

(c) For the family (C) we have four topologically distinct phase portraits $P_1^{(C)} - P_4^{(C)}$. The topological bifurcation diagram of family (C) is done in Figure 7. The bifurcation set is $ah(h-1/2) = 0$ and it is a bifurcation of singularities. The phase portrait $P_4^{(C)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 104:

(a) We have the following types of divisors and zero-cycles of the total invariant curve T for the configurations of family (C) :

Table 84 – Configurations for family (C).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_2^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + P_2^C + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_3^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_4^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_5^{(C)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 3P_3^\infty$
$C_6^{(C)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(C)}$ up to $C_6^{(C)}$ are all distinct. For the limit cases of family (C) we have the following configurations:

Table 85 – Configurations for the limit cases of family (C).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + P_2^C + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
c_2	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
c_3	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
c_4	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
c_5	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_1^\infty + 2P_2^\infty + 2P_3^\infty$
c_6	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 2P_2^\infty + 3P_3^\infty$
c_7	$ICD = J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_1^\infty + 2P_2^\infty$
c_8	$ICD = J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 2P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

The other statement in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. When $h = 1/4$ and $a \neq 0$ the family (C) has an inverse integrating factor which is polynomial (see Table 56).

(c) We have:

Table 86 – Phase portraits for family (C).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(C)}$	(S,N,N)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_2^{(C)}$	(N,N,N)	$(©,©,s,s)$	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(C)}$	(N,N,N)	$(s,s,©,©)$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_4^{(C)}$	(S,N,N)	(s,s,n,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have four distinct phase portraits for systems (C). For the limit cases of family (C) we have the following phase portraits:

Table 87 – Phase portraits for the limit cases of family (C).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(C)}$	$(\binom{2}{1}S,N,N)$	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_3^{(C)}$	$(\binom{2}{1}N,N,N)$	$(s,s,©,©)$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_1	(S,N,N) (N,N,S)	$phpphp_{(4)}$ $hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_2	(N,N,N)	$hhhhhh_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_3	$((\ominus[\|];\emptyset),N,N)$ $(N,N,(\ominus[\|];\emptyset))$	$(\ominus[\|],s)$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Table 88 – Phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,N,S), and it has either 0 or 4 real singular points in the finite region and phase portraits that admit 3 singular points at infinity with the type (N,N,N), and it has two real singular points in the finite region.

Phase Portraits	Sing. at ∞	Real finite sing.	Separatrix connections
R_{01},Ω_6	(N,S,N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
L_{34}	(N,N,N)	(s,s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
R_4,Ω_4	(N,N,N)	(s,s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
R_{03},Ω_3	(N,N,N)	(s,s)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, the phase portraits $P_4^{(C)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018). Note that $P_1^{(C)} \cong_{top} P_1^{(B)}$ is also missing and it was listed in the geometric study of family (B).

□

Figure 6 – Bifurcation diagram of configurations for family (C).

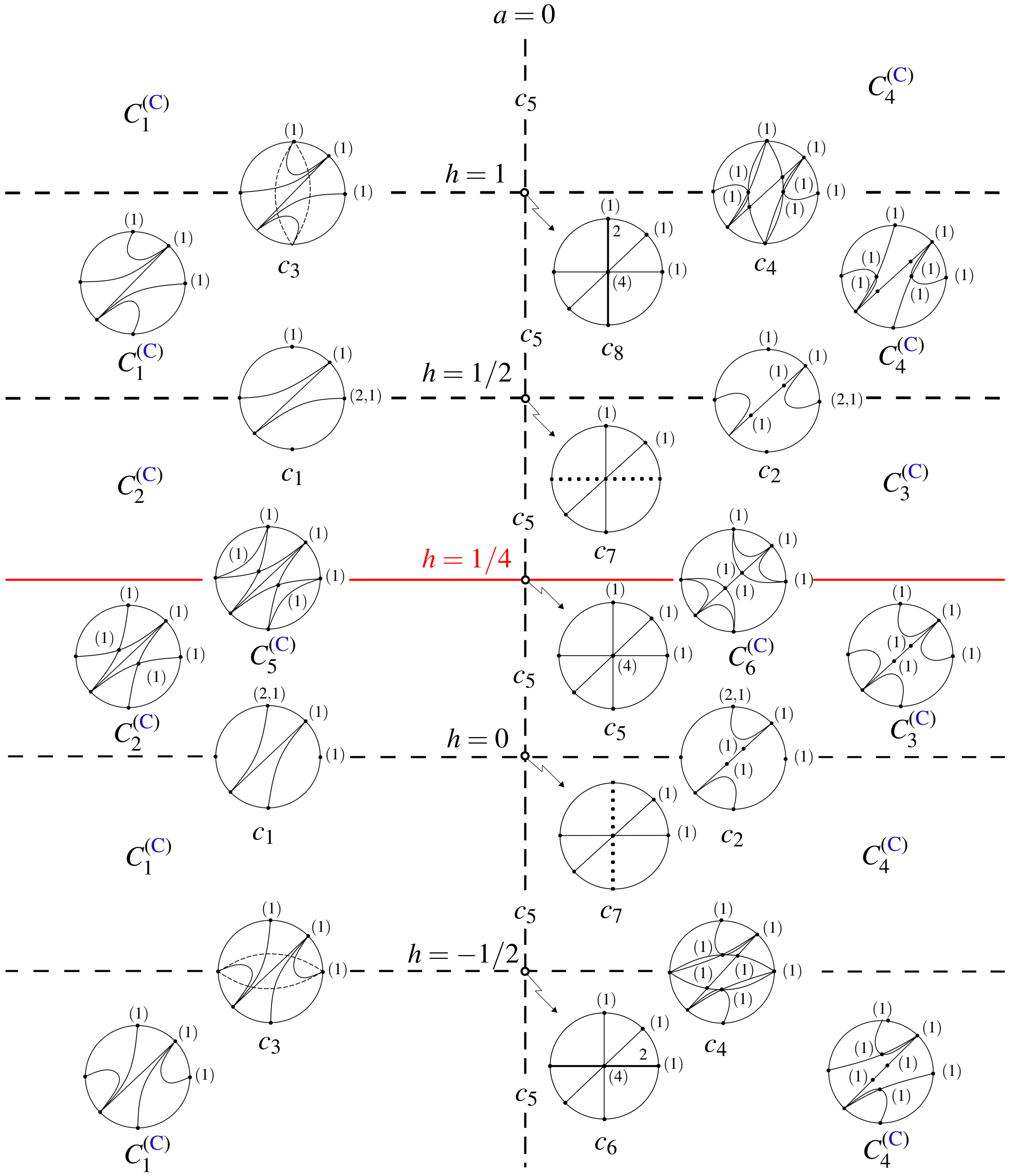
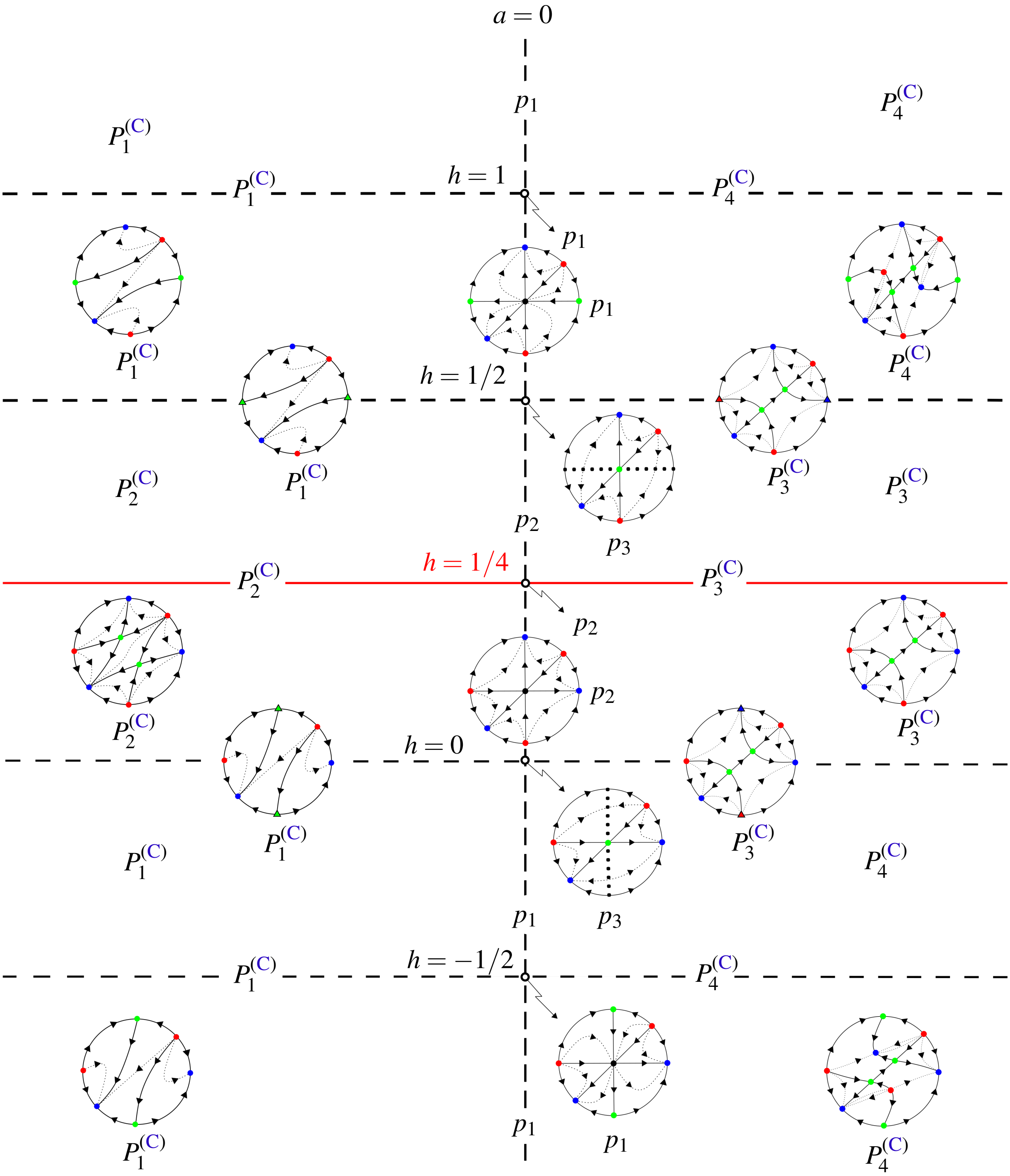


Figure 7 – Topological bifurcation diagram for family (C).



6.1.2.1 The solution of the Poincaré problem for the family (C).

The following theorem solves the problem of Poincaré for the family defined by the equations (C).

Theorem 105. A necessary and sufficient condition for a system (S) defined by the equations (C) with $(a,h) \in \mathbb{R}^2$ to have a rational first integral given by invariant algebraic curves of degree at most two is that: i) $ah(2h-1) \neq 0$ and there exist integers m_1, m_2 such that $h = \frac{m_1}{2(m_1+m_2)}$ where

$m_1, m_2 \in \mathbb{Z}\backslash\{0\}$ and $m_1 \neq -m_2$, or ii) $a = 0$ and there exist integers m_1, m_2 such that $h = \frac{m_2}{2m_1}$ where $m_1 \neq 0$.

Proof. The proof of this result is based on the formulas obtained for the first integrals for the family (C).

Suppose we are in the generic case $ah(h-1)(2h \pm 1)(4h-1) \neq 0$. A first integral of family (C) is of the form

$$I = J_2^{\lambda_2} J_3^{-\frac{(2h-1)\lambda_2}{2h}}, \tag{6.3}$$

where $\lambda_2 \neq 0$ and J_2, J_3 are given in table 51. This is a rational first integral if and only if

$$\begin{cases} \lambda_2 = m_1 \in \mathbb{Z}\backslash\{0\} \\ -\frac{(2h-1)}{2h}\lambda_2 = m_2 \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{6.4}$$

Replacing $\lambda_2 = m_1$ in the second equation of (6.4) we obtain

$$-\frac{(2h-1)}{2h} m_1 = m_2 \Rightarrow -(2h-1)m_1 = 2hm_2 \Rightarrow 2h(m_1+m_2) = m_1 \Rightarrow$$

$$h = \frac{m_1}{2(m_1+m_2)},\ m_1, m_2 \in \mathbb{Z}\backslash\{0\},\ m_2 \neq -2m_1,\ m_2 \neq m_1,\ m_2 \neq -\frac{m_1}{2} \text{ and } m_1 \neq -m_2.$$

Therefore, if I is rational then $h = \frac{m_1}{2(m_1+m_2)}$, $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -2m_1$, $m_2 \neq m_1$, $m_2 \neq -\frac{m_1}{2}$ and $m_1 \neq -m_2$. Conversely, replacing $h = \frac{m_1}{2(m_1+m_2)}$ and $\lambda_2 = m_1$ in (6.3) where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -2m_1$, $m_2 \neq m_1$, $m_2 \neq -\frac{m_1}{2}$ and $m_1 \neq -m_2$ we obtain that

$$I = J_2^{m_1} J_3^{m_2}$$

which is rational. Therefore, in the generic case, the systems are algebraically integrable if and only if $h = \frac{m_1}{2(m_1+m_2)}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -2m_1$, $m_2 \neq m_1$, $m_2 \neq -\frac{m_1}{2}$ and $m_1 \neq -m_2$.

Now suppose we are in non-generic case $ah(h-1)(2h \pm 1)(4h-1) = 0$. When $(h + 1/2)(h-1/4)(h-1) = 0$ and $a \neq 0$ we obtain rational first integrals (see Tables 54, 68 and 59). Note that

$$\begin{cases} \frac{m_1}{2(m_1+m_2)} = -\frac{1}{2} \Leftrightarrow m_2 = -2m_1 \\ \frac{m_1}{2(m_1+m_2)} = \frac{1}{4} \Leftrightarrow m_2 = m_1 \\ \frac{m_1}{2(m_1+m_2)} = 1 \Leftrightarrow m_2 = -\frac{m_1}{2} \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$.

When $h = 0$ and $a \neq 0$ then the first integral of family (C) is of the form

$$I = J_1^0 J_2^{-ag_1\lambda_3} E_3^{\lambda_3}$$

where J_1, J_2, E_3 are given in table 60. Therefore, I cannot be rational. When $h = 1/2$ and $a \neq 0$ then the first integral of family (C) is of the form

$$I = J_1^0 J_2^{-ag_1\lambda_3} E_3^{\lambda_3}$$

where J_1, J_2, E_3 are given in table 63. Therefore, I cannot be rational. Note that

$$\begin{cases} \frac{m_1}{2(m_1+m_2)} = 0 \Leftrightarrow m_1 = 0 \\ \frac{m_1}{2(m_1+m_2)} = \frac{1}{2} \Leftrightarrow m_2 = 0. \end{cases}$$

When $a = 0$ and $h \neq -1/2, 0, 1/2, 1$ a first integral of family (C) is of the form

$$I = J_1^{\lambda_1} J_2^{2h\lambda_1} J_3^{-(-1+2h)\lambda_1} \tag{6.5}$$

where $\lambda_1 \neq 0$ and J_1, J_2, J_3 are given in table 69. This is a rational first integral if and only if

$$\begin{cases} \lambda_1 &= m_1,\ m_1 \in \mathbb{Z}\backslash\{0\} \\ 2h\lambda_1 &= m_2,\ m_2 \in \mathbb{Z}\backslash\{0\} \\ -(-1+2h)\lambda_1 &= m_3,\ m_3 \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{6.6}$$

Replacing $\lambda_1 = m_1$ in the second equation of (6.6) we obtain $h = \frac{m_2}{2m_1}$, $m_1,\ m_2 \in \backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq m_1$ and $m_2 \neq 2m_1$. Then,

$$m_3 = (1-2h)m_1 = \left(1 - 2\left(\frac{m_2}{2m_1}\right)\right) m_1 = m_1 - m_2 \in \mathbb{Z}\backslash\{0\}$$

where $m_1,\ m_2 \neq 0$ and $m_1 \neq m_2$. Therefore, if I is rational then $h = \frac{m_2}{2m_1}$ where $m_1,\ m_2 \in \backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq m_1$ and $m_2 \neq 2m_1$. Conversely, replacing $h = \frac{m_2}{2m_1}$ and $\lambda_1 = m_1$ in (6.5) where $m_1,\ m_2 \in \backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq m_1$ and $m_2 \neq 2m_1$ we obtain that

$$I = J_1^{m_1} J_2^{m_2} J_3^{m_1 - m_2}$$

which is rational. Therefore, the systems are algebraically integrable if and only if $h = \frac{m_2}{2m_1}$ where $m_1,\ m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -m_1$, $m_2 \neq m_1$ and $m_2 \neq 2m_1$.

When $a = 0$ and $(h+1/2)h(h-1/2)(h-1) = 0$ we obtain rational first integrals (see Tables 74, 77, 80 and 83). Note that

$$\begin{cases} \frac{m_2}{2m_1} = -\frac{1}{2} \Leftrightarrow m_2 = -m_1 \\ \frac{m_2}{2m_1} = 0 \Leftrightarrow m_2 = 0 \\ \frac{m_2}{2m_1} = \frac{1}{2} \Leftrightarrow m_2 = m_1 \\ \frac{m_2}{2m_1} = 1 \Leftrightarrow m_2 = 2m_1. \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}$ and $m_1 \neq 0$. When $a = h = 0$ or $a = 0$ and $h = \frac{1}{2}$ the system (C) is degenerate.

In conclusion, we get a rational first integral for a system (S) defined by the equations (C) if and only if i) $ah(2h-1) \neq 0$ and there exist integers m_1, m_2 such that $h = \frac{m_1}{2(m_1+m_2)}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$ and $m_1 \neq -m_2$, or ii) $a = 0$ and there exist integers m_1, m_2 such that $h = \frac{m_2}{2m_1}$ where $m_1 \neq 0$.

□

6.1.3 Geometric Analysis of Family (E)

Consider the family

$$(E)\quad \begin{cases} \dot{x} = x - \frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b + y - \frac{3xy}{2} + \frac{y^2}{2}, \end{cases}$$

where $b \neq -4$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (E) we study here also the limit case $b = -4$ where the equations are still defined. We display below the full geometric analysis of the systems in this family which is endowed with at least four invariant algebraic curves: three lines and one hyperbola. Considering the line at infinity $Z = 0$ the total multiplicity of invariant lines is four so this family was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. In the **generic case**

$$b(b-8/25)(b-1/2)(b+4) \neq 0$$

the systems have three invariant lines J_1, J_2, J_3 and with one invariant hyperbola J_4 with cofactors α_i, $1 \leq i \leq 4$ given by

$$\begin{aligned} &J_1 = 1 - \sqrt{1-2b} - x + y, \quad \alpha_1 = \tfrac{1}{2}\left(1 + \sqrt{1-2b} - x + y\right), \\ &J_2 = 1 + \sqrt{1-2b} - x + y, \quad \alpha_2 = \tfrac{1}{2}\left(1 - \sqrt{1-2b} - x + y\right), \\ &J_3 = x, \quad \alpha_3 = -\tfrac{x}{2} - \tfrac{y}{2} + 1, \\ &J_4 = -4 - b + x(4 - x + y), \quad \alpha_4 = -x. \end{aligned}$$

Then according to Darboux' theorem we must have a Darboux first integral. We note that when $b = 1/2$ the lines J_1 and J_2 coalesce yielding a double line. When $b = 0$ we have obtain an additional invariant line. When $b = 8/25$ we obtain an additional invariant hyperbola. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola

(i) The generic case: $b(b-8/25)(b-1/2)(b+4) \neq 0$.

Table 89 – Invariant curves, cofactors, singularities and intersection points of family (E) when $b(b+4)(b-8/25)(b-1/2) \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1-\sqrt{1-2b}-x+y$ $J_2 = 1+\sqrt{1-2b}-x+y$ $J_3 = x$ $J_4 = -4-b+x(4-x+y)$ $\alpha_1 = \frac{1}{2}\left(1+\sqrt{1-2b}-x+y\right)$ $\alpha_2 = \frac{1}{2}\left(1-\sqrt{1-2b}-x+y\right)$ $\alpha_3 = 1-\frac{x}{2}-\frac{y}{2}$ $\alpha_4 = -x$	$P_1 = \left(0,-\sqrt{1-2b}-1\right)$ $P_2 = \left(\frac{1}{2}\left(\sqrt{1-2b}+3\right),\frac{1}{2}\left(1-\sqrt{1-2b}\right)\right)$ $P_3 = \left(0,\sqrt{1-2b}-1\right)$ $P_4 = \left(\frac{1}{2}\left(3-\sqrt{1-2b}\right),\frac{1}{2}\left(\sqrt{1-2b}+1\right)\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $s,n,s,n;N,N,S$ if $b<-4$ $s,n,n,s;N,N,S$ if $-4<b<1/2$ ©,©,©,©$;N,N,S$ if $b>1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_3$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_4 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 90 – Divisor and zero-cycles of family (E) when $b(b+4)(b-8/25)(b-1/2) \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1+J_2+J_3+J_4+\mathscr{L}_\infty \text{ if } b<1/2 \\ J_1^C+J_2^C+J_3+J_4+\mathscr{L}_\infty \text{ if } b>1/2 \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } b<1/2 \\ P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } b>1/2 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	6
$M_{0CT} = \begin{cases} 2P_1+2P_2+2P_3+2P_4+3P_1^\infty+4P_2^\infty+P_3^\infty \text{ if } b<1/2 \\ 2P_1^C+2P_2^C+2P_3^C+2P_4^C+3P_1^\infty+4P_2^\infty+P_3^\infty \text{ if } b>1/2 \end{cases}$	16 16

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them double and

2) four distinct tangents at P_2^∞.

Table 91 – First integral and integrating factor of family (E) when $b(b+4)(b-8/25)(b-1/2) \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{(-\sqrt{1-2b}-3)\lambda_1}{\sqrt{1-2b}-3}} J_3^{\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3}} J_4^{-\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3}}$	R
Simple example	$\mathscr{I} = J_1^{\sqrt{1-2b}-3} J_2^{\sqrt{1-2b}+3} J_3^{2\sqrt{1-2b}} J_4^{-2\sqrt{1-2b}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

where $R = J_1^{\lambda_1} J_2^{\frac{6}{\sqrt{1-2b}-3} - \frac{(-\sqrt{1-2b}-3)\lambda_1}{\sqrt{1-2b}-3}} J_3^{\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3} - \frac{-\sqrt{1-2b}-3}{\sqrt{1-2b}-3}} J_4^{-\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3} - \frac{3(\sqrt{1-2b}-1)}{\sqrt{1-2b}-3}}$

(ii) $b = 0$.

Here we have, apart from the three lines and one hyperbola, an additional invariant line. Considering the line at infinity $Z = 0$ the total multiplicity of invariant lines is five so this system was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 92 – Invariant curves, cofactors, singularities and intersection points of family (E) when $b = 0$.

Inv.cur./Exp.Fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $J_4 = -2 + x - y$ $J_5 = -4 + 4x - x^2 + xy$ $\alpha_1 = 1 - \frac{3x}{2} + \frac{y}{2}$ $\alpha_2 = 1 - \frac{x}{2} - \frac{y}{2}$ $\alpha_3 = 1 - \frac{x}{2} + \frac{y}{2}$ $\alpha_4 = -\frac{x}{2} + \frac{y}{2}$ $\alpha_5 = -x$	$P_1 = (0, -2)$ $P_2 = (1, 1)$ $P_3 = (2, 0)$ $P_4 = (0, 0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $s, s, n, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_4$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = P_3$ simple $\bar{J}_1 \cap \bar{J}_5 = P_3$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_4$ simple $\bar{J}_2 \cap \bar{J}_4 = P_1$ simple $\bar{J}_2 \cap \bar{J}_5 = P_1^\infty$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_3 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 106. According to Jouanolou's theorem the corresponding system has a rational first integral.

Table 93 – Divisor and zero-cycles of family (E) when $b = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	7
$M_{0CT} = 2P_1 + 2P_2 + 3P_3 + 3P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$	19

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_3, but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) four distinct tangents at P_2^∞.

Table 94 – First integral and integrating factor of family (E) when $b = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{-\lambda_1-\lambda_2}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{-2-\lambda_1-\lambda_2}$
Simple example	$\mathscr{I} = \dfrac{J_1}{J_3 J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4}$

Source: Elaborated by the author.

Observation 107. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_3 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[4:1]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(4,1)} = -J_2 J_4, \quad \mathscr{F}^1_{(0,1)} = -J_3 J_5.$$

Therefore, J_2, J_3, J_4, J_5 are remarkable curves of $\mathscr{I}_1$ and $[4:1]$ and $[0:1]$ are remarkable values of $\mathscr{I}_1$. The singular points are P_1, P_3, P_4 for $\mathscr{F}^1_{(4,1)}$ and P_2 for $\mathscr{F}^1_{(0,1)}$.

(iii) $b = \frac{8}{25}$.

Here we have, apart from the three lines and one hyperbola, an additional invariant hyperbola. According to Jouanolou's theorem the corresponding system has a rational first integral.

Table 95 – Invariant curves, cofactors, singularities and intersection points of family (E) when $b = \frac{8}{25}$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 2-5x+5y$ $J_2 = 8-5x+5y$ $J_3 = x$ $J_4 = -\frac{108}{25}+4x+x(-x+y)$ $J_5 = -\frac{4}{25}-\frac{4y}{5}-y(-x+y)$ $\alpha_1 = \frac{1}{10}(-5x+5y+8)$ $\alpha_2 = \frac{1}{10}(-5x+5y+2)$ $\alpha_3 = 1-\frac{x}{2}-\frac{y}{2}$ $\alpha_4 = -x$ $\alpha_5 = \frac{8}{5}-2x+y$	$P_1 = \left(0, -\frac{8}{5}\right)$ $P_2 = \left(0, -\frac{2}{5}\right)$ $P_3 = \left(\frac{9}{5}, \frac{1}{5}\right)$ $P_4 = \left(\frac{6}{5}, \frac{4}{5}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $s, n, n, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_2$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_4 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_3 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_3 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_3 \cap \bar{J}_5 = P_2$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_3 \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 108. According to Jouanolou's theorem the corresponding system has a rational first integral.

Table 96 – Divisor and zero-cycles of family (E) when $b = \frac{8}{25}$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = 2P_1 + 3P_2 + 3P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$	20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_2 (and P_3), but one of them is double and

2) only two distinct tangents at P_1^∞, but one of them is double.

Table 97 – First integral and integrating factor of family (E) when $b = \frac{8}{25}$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\frac{\lambda_2}{3}} J_4^{\frac{\lambda_1}{2}} J_5^{-\frac{\lambda_1}{2}-\frac{\lambda_2}{3}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{2}{3}+\frac{\lambda_2}{3}} J_4^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_5^{-\frac{5}{6}-\frac{\lambda_1}{2}-\frac{\lambda_2}{3}}$
Simple example	$\mathscr{I} = \dfrac{J_1^2 J_4}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 109. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_4 - c_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:108]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,108)} = \tfrac{1}{5} J_2^3 J_3, \quad \mathscr{F}^1_{(1,0)} = J_1^2 J_4.$$

Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathscr{I}_1$, $[1:108]$ and $[1:0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_3 for $\mathscr{F}^1_{(1,108)}$ and P_2, P_4 for $\mathscr{F}^1_{(1,0)}$.

(iv) $b = \frac{1}{2}$.

Here two lines coalesce yielding a double line so we compute the exponential factor E_4.

Table 98 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (E) when $b = \frac{1}{2}$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = 1 - x + y$ $J_2 = x$ $J_3 = -\frac{9}{8} + x - \frac{x^2}{4} + \frac{xy}{4}$ $E_4 = e^{\frac{g_0 + g_1 x - g_1 y}{1 - x + y}}$ $\alpha_1 = \frac{1}{2} - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = 1 - \frac{x}{2} - \frac{y}{2}$ $\alpha_3 = -x$ $\alpha_4 = -\frac{g_0}{2} - \frac{g_1}{2}$	$P_1 = (0, -1)$ $P_2 = \left(\frac{3}{2}, \frac{1}{2}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 99 – Divisor and zero-cycles of family (E) when $b = \frac{1}{2}$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2\overline{J}_3 = 0$	6
$M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 4P_2^\infty + P_3^\infty$	14

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and P_2), but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) only three distinct tangents at P_2^∞, but one of them is double.

Table 100 – First integral and integrating factor of family (E) when $b = \frac{1}{2}$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{-\lambda_1} e^{g_0\lambda_4 + \frac{3\lambda_1(x-y)}{(-x+y+1)}}$	$R = J_1^{-2} J_2^{-1} J_3^{-1} E_4^0$
Simple example	$\mathscr{I} = \dfrac{J_1 J_2 e^{g_0 + \frac{3(x-y)}{-x+y+1}}}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1^2 J_2 J_3}$

Source: Elaborated by the author.

(v) $b = -4$.

Under this condition, systems (E) do not belong to **QSH**. The affine invariant lines are $-2 - x + y = 0$, $4 - x + y = 0$ and $x = 0$ that are all simple. Considering the line at infinity $Z = 0$ the total multiplicity of invariant lines is four so this family was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. Furthermore, the conic $x(4 - x + y) = 0$ has integrable multiplicity two so we get an exponetial factor that allow us to find a generalized Darboux first integral. By perturbing the reducible conic $x(4 - x + y) = 0$ we can produce the hyperbola $-4 - b + 4x - x^2 + xy = 0$.

Table 101 – Invariant curves, cofactors, singularities and intersection points of family (E) when $b = -4$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -2 - x + y$ $J_2 = 4 - x + y$ $J_3 = x$ $E_4 = e^{\frac{x(g_0-6g_1)(-x+y+4)+8g_0(x-3)}{48x(-x+y+4)}}$ $\alpha_1 = 2 - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = -1 - \frac{x}{2} + \frac{y}{2}$ $\alpha_3 = 1 - \frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -\frac{g_0}{12}$	$P_1 = (0, -4)$ $P_2 = (3, -1)$ $P_3 = (0, 2)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $s, n, sn_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_3$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 102 – Divisor and zero-cycles of family (E) when $b = -4$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + 2P_3 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0.$	4
$M_{0CT} = 2P_1 + P_2 + 2P_3 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_2^∞.

Table 103 – First integral and integrating factor of family (E) when $b = -4$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{\frac{36\lambda_1}{g_0}}$	$R = J_1^{\lambda_1} J_2^{-3-\lambda_1} J_3^{-2} E_4^{\frac{36(\lambda_1+1)}{g_0}}$
Simple example	$\mathscr{I} = \dfrac{J_1 E_4}{J_2}$	$\mathscr{R} = \dfrac{1}{J_1 J_2^2 J_3^2}$

Source: Elaborated by the author.

We sum up the topological, dynamical and algebraic geometric features of family (E) in the following proposition and we also confront our results with previous results in the literature.

Proposition 110. (a) For the family (E) we have six distinct configurations $C_1^{(E)} - C_6^{(E)}$ of invariant hyperbolas and lines (see Figure 8 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $b(b+4)(b-8/25)(b-1/2) = 0$. Its complement is a union of 5 disjoint sets. On

$b=0$ we have one additional invariant line. On $b=8/25$ we have one additional invariant hyperbola. On $b=1/2$ two lines of the generic case coalesce yielding to a double line and two double finite singularities that are located in this line. For the limiting set of the parameter space, i.e. on $b=-4$ the invariant hyperbola becomes reducible producing the lines $x=0$ and $4-x+y=0$.

(b) The family (E) is Darboux integrable if $b(b-8/25)(b-1/2)(b+4)\neq 0$. When $b=0$ the family (E) admits a rational first integral and the plane is foliated into cubic invariant algebraic curves. The remarkable curves corresponding to this case are J_2, J_3, J_4 and J_5. When $b=8/25$ the family (E) admits a rational first integral and the plane is foliated into quartic invariant algebraic curves. The remarkable curves corresponding to this case are J_1, J_2, J_3 and J_4. When $b=1/2$ the family (E) is generalized Darboux integrable. All systems in family (E) have an integrating factor which is polynomial.

(c) For the family (E) we have four topologically distinct phase portraits $P_1^{(E)} - P_4^{(E)}$. The topological bifurcation diagram in the full parameter space is done in Figure 9 . The bifurcation is $(b+4)(b-1/2)=0$ and it is a bifurcation set of singularities. The phase portrait $P_4^{(E)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of proposition 110:

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (E):

Table 104 – Configurations for family (E).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(E)}$	$ICD = J_1+J_2+J_3+J_4+\mathscr{L}_\infty$ $M_{0CT} = 2P_1+2P_2+2P_3+2P_4+3P_1^\infty+4P_2^\infty+P_3^\infty$
$C_2^{(E)}$	$ICD = J_1+J_2+J_3+J_4+\mathscr{L}_\infty$ $M_{0CT} = 2P_1+2P_2+2P_3+2P_4+3P_1^\infty+4P_2^\infty+P_3^\infty$
$C_3^{(E)}$	$ICD = J_1^C+J_2^C+J_3+J_4+\mathscr{L}_\infty$ $M_{0CT} = 2P_1^C+2P_2^C+2P_3^C+2P_4^C+3P_1^\infty+4P_2^\infty+P_3^\infty$
$C_4^{(E)}$	$ICD = J_1+J_2+J_3+J_4+J_5+\mathscr{L}_\infty$ $M_{0CT} = 2P_1+2P_2+3P_3+3P_4+3P_1^\infty+4P_2^\infty+2P_3^\infty$
$C_5^{(E)}$	$ICD = J_1+J_2+J_3+J_4+J_5+\mathscr{L}_\infty$ $M_{0CT} = 2P_1+3P_2+3P_3+2P_4+3P_1^\infty+5P_2^\infty+2P_3^\infty$
$C_6^{(E)}$	$ICD = 2J_1+J_2+J_3+\mathscr{L}_\infty$ $M_{0CT} = 3P_1+3P_2+3P_1^\infty+4P_2^\infty+P_3^\infty$

Source: Elaborated by the author.

Although $C_1^{(E)}$ and $C_2^{(E)}$ admit the same type of divisors and zero-cycles we can see that the configurations are different because in $C_1^{(E)}$ each branch of the hyperbola intersects one of the parallel lines while $C_2^{(E)}$ have one branch intersecting both parallel lines and the other branch does not intersect any line. The configuration $C_4^{(E)}$ and $C_5^{(E)}$ are also clearly distinct since $C_4^{(E)}$ possess three affine lines and one hyperbola while $C_5^{(E)}$ possess two affine lines and two hyperbolas.

Therefore, the configurations $C_1^{(E)}$ up to $C_6^{(E)}$ are distinct. For the limit case of family (E) we have the following configuration:

Table 105 – Configuration for the limit case of family (E).

Configuration	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 107 and 109.

(c) We have:

Table 106 – Phase portraits for family (E).

Phase Portraits	Sing. at ∞	Sing. at $< \infty$	Separatrix connections
$P_1^{(E)}$	(N,N,S)	(s,n,s,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(E)}$	(N,N,S)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(E)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(E)}$	(N,N,S)	$(sn_{(2)}, sn_{(2)})$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have four distinct phase portraits for systems (E). For the limit case of family (E) we have the following phase portrait:

Table 107 – Phase portrait for the limit case of family (E).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	$(s,n,sn_{(2)})$	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Table 108 – Phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,N,S) that possess 0, 2 or 4 real singular points in the finite region.

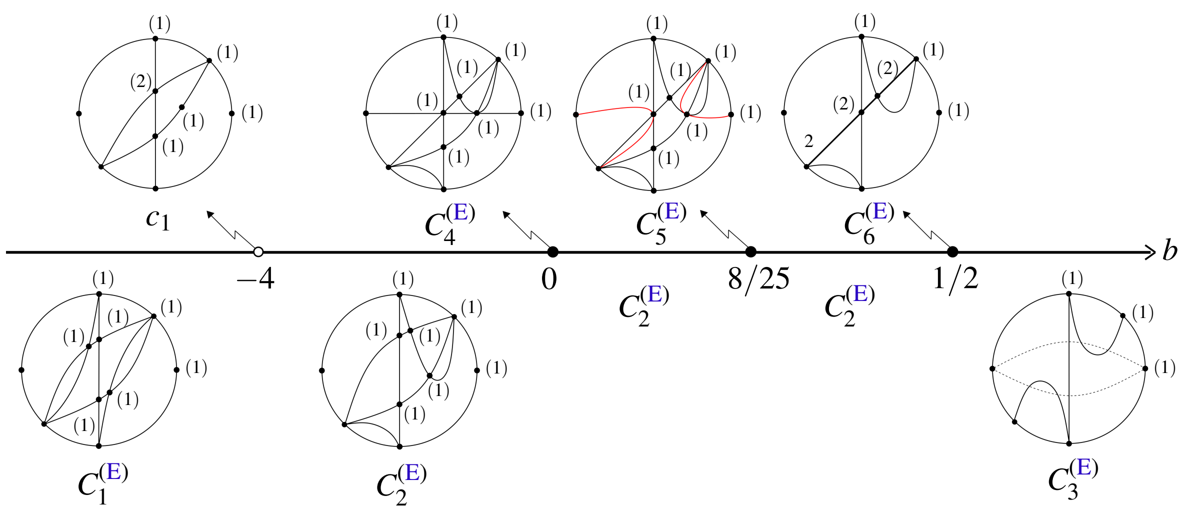
Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
L_{31}, L_{32}	(N,S,N)	(s,es)	$2C_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_1, R_2	(N,S,N)	(s,c)	$1C_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
R_{01,Ω_6}	(N,S,N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_{8,Ω_1}	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, the phase portrait $P_4^{(E)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018). Note that $P_1^{(E)} \cong_{top} P_4^{(C)}$ and $P_3^{(E)} \cong_{top} P_3^{(B)}$ are also missing and they were listed in the geometric study of families (C) and (B), respectively.

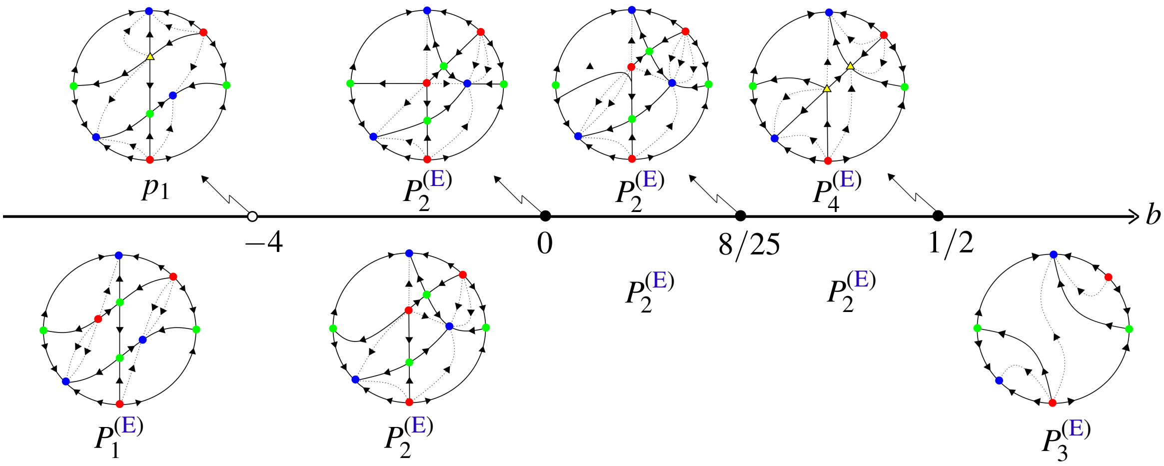
$\square$

Figure 8 – Bifurcation diagram of configurations for family (E).



Source: Elaborated by the author.

Figure 9 – Topological bifurcation diagram for family (E).



Source: Elaborated by the author.

6.1.3.1 The solution of the Poincaré problem for the family (E).

The following theorem solves the problem of Poincaré for the family defined by the equations (E) when $b \in \mathbb{R}$.

Theorem 111. A necessary and sufficient condition for a system (S) defined by the equations (E) with $b \in \mathbb{R}$ to have a rational first integral given by invariant algebraic curves of degree at most two is that there exist integers m_1, m_2 such that $b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2}$ where $m_1 \neq 0$, $m_2 \neq 0$, $m_1 \neq m_2$ and $m_1 \neq -m_2$.

Proof. The proof of this result is based on the formulas obtained for the first integrals for the family (E).

In the generic case $b(b-8/25)(b-1/2)(b+4) \neq 0$ we have a Darboux first integral given by

$$I = J_1^{\lambda_1} J_2^{-\frac{(-\sqrt{1-2b}-3)\lambda_1}{\sqrt{1-2b}-3}} J_3^{\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3}} J_4^{-\frac{2\sqrt{1-2b}\lambda_1}{\sqrt{1-2b}-3}}, \tag{6.7}$$

where $\lambda_1 \neq 0$, $\sqrt{1-2b}-3 \neq 0$ and J_1, J_2, J_3, J_4 are given in table 89. This is a rational first integral if and only if

$$\begin{cases} \lambda_1 = m_1, \; m_1 \in \mathbb{Z}\backslash\{0\} \\ -\frac{(-\sqrt{1-2b}-3)}{\sqrt{1-2b}-3}\lambda_1 = m_2, \; m_2 \in \mathbb{Z}\backslash\{0\} \\ \frac{2\sqrt{1-2b}}{\sqrt{1-2b}-3}\lambda_1 = m_3, \; m_3 \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{6.8}$$

Replacing $\lambda_1 = m_1$ in the second equation of (6.8) we obtain

$$-\frac{\left(-\sqrt{1-2b}-3\right)}{\sqrt{1-2b}-3}m_1 = m_2 \Rightarrow \frac{\left(\sqrt{1-2b}+3\right)}{\sqrt{1-2b}-3}m_1 = m_2 \Rightarrow \left(\sqrt{1-2b}+3\right)m_1 = \left(\sqrt{1-2b}-3\right)m_2 \Rightarrow$$
$$\sqrt{1-2b}(m_1-m_2)+3(m_1+m_2) = 0 \overset{m_1-m_2\neq 0}{\Rightarrow} \sqrt{1-2b} = -\frac{3(m_1+m_2)}{(m_1-m_2)} \Rightarrow 1-2b = \frac{9(m_1+m_2)^2}{(m_1-m_2)^2} \Rightarrow$$
$$b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2},\ m_1, m_2 \in \mathbb{Z}\backslash\{0\},\ m_1 \neq m_2 \text{ and } m_1 \neq -m_2.$$

Note that as $\sqrt{1-2b} = -\frac{3(m_1+m_2)}{(m_1-m_2)}$ is rational, then

$$m_3 = \frac{2\sqrt{1-2b}}{\sqrt{1-2b}-3}m_1 = \frac{2\left(\frac{-3(m_1+m_2)}{m1-m_2}\right)}{\left(\frac{-3(m_1+m_2)}{m1-m_2}\right)-3}m_1 = m_1+m_2 \in \mathbb{Z}.$$

Therefore, if I is rational then $b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_1 \neq m_2$ and $m_1 \neq -m_2$. Conversely, replacing $b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2}$ and $\lambda_1 = m_1$ in (6.7) where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_1 \neq m_2$ and $m_1 \neq -m_2$ we obtain that

$$I = J_1^{m_1} J_2^{m_2} J_3^{m_1+m_2} J_4^{-m_1-m_2},$$

which is rational. Therefore, in the generic case, the systems are algebraically integrable if and only if $b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_1 \neq m_2$ and $m_1 \neq -m_2$.

In the non-generic case $b(b-8/25)(b-1/2)(b+4) = 0$ we have rational first integrals when $b(b-8/25) = 0$ (see Table 94 and 97). Note that

$$\begin{cases} \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2} = 0 \Leftrightarrow m_1 = -2m_2 \text{ or } m_2 = -2m_1 \\ \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2} = \frac{8}{25} \Leftrightarrow 2m_1 = -3m_2 \text{ or } 2m_2 = -3m_1 \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$. When $b = 1/2$ the first integral is of the form

$$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{-\lambda_1} e^{g_0\lambda_4 + \frac{3\lambda_1(x-y)}{(-x+y+1)}}$$

where J_1, J_2, J_3 are given in table 98. Therefore, I cannot be rational. When $b = -4$ the first integral is of the form

$$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{0} E_4^{\frac{36\lambda_1}{g_0}}$$

where J_1, J_2, J_3 are given in table 101. Therefore, I cannot be rational. Note that

$$\begin{cases} \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2} = -4 \Leftrightarrow m_1 = 0 \text{ or } m_2 = 0 \\ \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2} = \frac{1}{2} \Leftrightarrow m_1 = -m_2 \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}$.

In conclusion, we get a rational first integral I for a system (S) defined by the equations (E) if and only if $b = \frac{1}{2} - \frac{9(m_1+m_2)^2}{2(m_1-m_2)^2}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_1 \neq m_2$ and $m_1 \neq -m_2$.

□

We see that the Poincaré problem for families involves in the final instance questions of a number theoretic nature which need to be treated with meticulous care.

6.1.4 Geometric Analysis of family (F)

Consider the family

$$(F)\quad \begin{cases} \dot{x} = a(2h-1) - hx^2 + (h-1)xy \\ \dot{y} = 2a(h-1) - (h+1)xy + hy^2, \end{cases}$$

where $a(h\pm 1)(2h\pm 1)(3h\pm 1) \neq 0$.

This is a two parameter family depending on a and h such that $a(h\pm 1)(2h\pm 1)(3h\pm 1) \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (F) we study here also the limit cases $a(h\pm 1)(2h\pm 1)(3h\pm 1) = 0$ where the equations are still defined.

We display below the full geometric analysis of the systems in this family, which is endowed with at least one invariant hyperbola. In the **generic case**

$$ah(h\pm 1)(2h\pm 1)(3h\pm 1)(4h-1) \neq 0$$

the systems have two invariant lines J_1 and J_2 and one invariant hyperbola J_3 with cofactors α_i, $1 \leq i \leq 3$ given by

$$\begin{array}{ll} J_1 = -\frac{\sqrt{a}}{\sqrt{h}} + x - y, & \alpha_1 = -\sqrt{a}\sqrt{h} - hx + hy, \\ J_2 = \frac{\sqrt{a}}{\sqrt{h}} + x - y, & \alpha_2 = \sqrt{a}\sqrt{h} - hx + hyy, \\ J_3 = a - x^2 + xy, & \alpha_3 = (2h-1)y - 2hx. \end{array}$$

We note that when $h = 0$ we do not have any affine invariant line. The four finite singularities have gone to infinity and two of them coalesced with a node at infinity producing a triple node and the other two coalesced with a saddle at infinity producing a triple saddle. Also in this case, the line of infinity $Z = 0$ is triple. We inquire when we could have an additional hyperbola and calculations yield that this happens when $h = 1/4$. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbolas.

(i) The generic case: $ah(h\pm 1)(2h\pm 1)(3h\pm 1)(4h-1) \neq 0$.

Table 109 – Invariant curves, cofactors, singularities and intersection points of family (F) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\frac{\sqrt{a}}{\sqrt{h}} + x - y$ $J_2 = \frac{\sqrt{a}}{\sqrt{h}} + x - y$ $J_3 = a - x^2 + xy$ $\alpha_1 = -\sqrt{a}\sqrt{h} - hx + hy$ $\alpha_2 = \sqrt{a}\sqrt{h} - hx + hy$ $\alpha_3 = (2h-1)y - 2hx$	$P_1 = \left(-\sqrt{a}\sqrt{h}, -\frac{\sqrt{a}(h-1)}{\sqrt{h}}\right)$ $P_2 = \left(\sqrt{a}\sqrt{h}, \frac{\sqrt{a}(h-1)}{\sqrt{h}}\right)$ $P_3 = \left(\frac{\sqrt{a}(1-2h)}{\sqrt{h}}, -2\sqrt{a}\sqrt{h}\right)$ $P_4 = \left(\frac{\sqrt{a}(2h-1)}{\sqrt{h}}, 2\sqrt{a}\sqrt{h}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a<0$ we have $n,n,s,s;S,N,N$ if $h<0$ ©,©,©,©$;N,N,S$ if $h>0$ For $a>0$ we have ©,©,©,©$;S,N,N$ if $h<0$ $s,s,n,n;N,N,S$ if $0<h<1/3$ $n,n,s,s;N,N,S$ if $h>1/3$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 110 – Divisors and zero-cycles of family (F) for the generic case.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathscr{L}_\infty \text{ if } ah>0 \\ J_1^C + J_2^C + J_3 + \mathscr{L}_\infty \text{ if } ah<0 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } ah>0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } ah<0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty + P_3^\infty \text{ if } ah>0 \\ 2P_1^C + 2P_2^C + P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty + P_3^\infty \text{ if } ah<0 \end{cases}$	13 13

Source: Elaborated by the author.

where the total curve T has four distinct tangents at P_2^∞.

Observation 112. Mathematica could not give a response for the computation of the first integral of family (F) in the generic case.

Table 111 – Integrating factor of family (F) for the generic case.

	Integrating Factor
General	$R = J_1^{\frac{h-1}{2h}} J_2^{\frac{h-1}{2h}} J_3^{-2}$
Simple example	$\mathscr{R} = J_1^{\frac{h-1}{2h}} J_2^{\frac{h-1}{2h}} J_3^{-2}$

Source: Elaborated by the author.

(ii) The non-generic case: $ah(h\pm 1)(2h\pm 1)(3h\pm 1)(4h-1) = 0$.

(ii.1) $h = 0$ and $a \neq 0$.

Here we have just one invariant hyperbola. The line at infinity $Z = 0$ is triple and we could find two exponential factors.

Table 112 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (F) when $h = 0$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = a - x^2 + xy$ $E_2 = e^{g_0+g_1(x-y)}$ $E_3 = e^{l_0+\frac{1}{2}(x-y)(2l_1+l_2(y-x))}$ $\alpha_1 = -y$ $\alpha_2 = ag_1$ $\alpha_3 = al_1 - al_2x + al_2y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $\binom{2}{1}N, N, \binom{2}{1}S$ For $a > 0$ we have $\binom{2}{1}S, N, \binom{2}{1}N$	$\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 113 – Divisor and zero-cycles of family (F) when $h = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + 3\mathscr{L}_\infty$	4
$M_{0CS} = 3P_1^\infty + P_2^\infty + 3P_3^\infty$	7
$T = Z^3\bar{J}_1 = 0.$	5
$M_{0CT} = 2P_1^\infty + 2P_2^\infty + P_3^\infty$	5

Source: Elaborated by the author.

Table 114 – First integral and integrating factor of family (F) when $h = 0$ and $a \neq 0$.

	First integral	Integrating Factor
General	I	$R = J_1^{-2} E_2^{-\frac{l_1 \lambda_3}{g_1}} \left(e^{\frac{(x-y)(2a\lambda_3 l_1 + x - y)}{2a\lambda_3}} + l_0 \right)^{\lambda_3}$
Simple example	$\mathscr{I}$	$\mathscr{R} = J_1^{-2} e^{\frac{(x-y)^2}{2a}}$

Source: Elaborated by the author.

where

$$I = -\frac{\left(\sqrt{2}(a+x(y-x))F\left(\frac{x-y}{\sqrt{2}\sqrt{a}}\right) + \sqrt{a}x\right)\left(e^{\frac{(x-y)(2a\lambda_3 l_1 + x-y)}{2a\lambda_3}} + l_0\right)^{\lambda_3} \left(e^{g_0 + g_1 x - g_1 y}\right)^{-\frac{\lambda_3 l_1}{g_1}}}{\sqrt{a}(a+x(y-x))} \tag{6.9}$$

$$\mathscr{I} = e^{\frac{(x-y)^2}{2a}} \left(-\frac{\sqrt{2}F\left(\frac{x-y}{\sqrt{2}\sqrt{a}}\right)}{\sqrt{a}} - \frac{x}{a+x(y-x)} \right)$$

and $F(z)$ is the Dawson integral defined by $F(z) = e^{-z^2} \int_0^z e^{y^2}\, dy$.

(ii.2) $h = 1/4$ and $a \neq 0$.

Here we have, apart from the two lines and one hyperbola, an additional invariant hyperbola so we know that we have a Darboux first integral.

Table 115 – Invariant curves, cofactors, singularities and intersection points of family (F) when $h = 1/4$ and $a \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = -2\sqrt{a} + x - y$ $J_2 = 2\sqrt{a} + x - y$ $J_3 = a - x^2 + xy$ $J_4 = a + xy$ $\alpha_1 = -\frac{\sqrt{a}}{2} - \frac{x}{4} + \frac{y}{4}$ $\alpha_2 = \frac{\sqrt{a}}{2} - \frac{x}{4} + \frac{y}{4}$ $\alpha_3 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -\frac{3x}{2} - \frac{y}{2}$	$P_1 = \left(-\sqrt{a}, \sqrt{a}\right)$ $P_2 = \left(-\frac{\sqrt{a}}{2}, \frac{3\sqrt{a}}{2}\right)$ $P_3 = \left(\frac{\sqrt{a}}{2}, -\frac{3\sqrt{a}}{2}\right)$ $P_4 = \left(\sqrt{a}, -\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; N, N, S For $a > 0$ we have $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = P_4$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = P_1$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ quadruple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 116 – Divisor and zero-cycles of family (F) when $h = 1/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	17 17

Source: Elaborated by the author.

where the total curve T has

1) only one distinct tangents at P_1 (and P_4), but they are double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) four distinct tangents at P_2^∞.

Table 117 – First integral and integrating factor of family (F) when $h = 1/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{-\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{-\frac{3}{2}-\lambda_1}$
Simple example	$\mathscr{I} = \dfrac{J_1 J_2 J_3^2}{J_4}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4^{\frac{1}{2}}}$

Source: Elaborated by the author.

Observation 113. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_2 J_3^2 - c_2 J_4 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 6$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -4a^2]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-4a^2)} = -\left(a(y-3x) + x(x-y)^2\right)^2, \quad \mathscr{F}^1_{(1,0)} = J_1 J_2 J_3^2.$$

Therefore, J_1, J_2, J_3 and $J_5 := a(y-3x) + x(x-y)^2$ are remarkable curves of $\mathscr{I}_1$, $[1 : -4a^2]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_3, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_4 for $\mathscr{F}^1_{(1,-4a^2)}$ and P_2, P_3 for $\mathscr{F}^1_{(1,0)}$.

Observation 114. $J_5 := a(y-3x) + x(x-y)^2$ is an invariant algebraic curve of degree 3 of family (F) when $h = 1/4$ and $a \neq 0$, with cofactor given by $\alpha_5 = -\frac{3x}{4} - \frac{y}{4}$.

Observation 115. Note that the rational first integral $\mathscr{I}$ in Table 117 has the rational integrating factor $R = \frac{J_3 J_5}{J_4^2}$ expressed by an invariant curve of degree higher than two.

(ii.3) $h = -1$ and $a \neq 0$.

Under this condition the systems do not belong to family (F). Here we have four invariant lines and one invariant hyperbola. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 118 – Invariant curves, cofactors, singularities and intersection points of family (F) when $h = -1$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -i\sqrt{a} + x - y$ $J_2 = i\sqrt{a} + x - y$ $J_3 = 1 - \frac{iy}{2\sqrt{a}}$ $J_4 = 1 + \frac{iy}{2\sqrt{a}}$ $J_5 = a - x^2 + xy$ $\alpha_1 = i\sqrt{a} + x - y$ $\alpha_2 = -i\sqrt{a} + x - y$ $\alpha_3 = 2i\sqrt{a} - y$ $\alpha_4 = -2i\sqrt{a} - y$ $\alpha_5 = 2x - 3y$	$P_1 = (-i\sqrt{a}, -2i\sqrt{a})$ $P_2 = (i\sqrt{a}, 2i\sqrt{a})$ $P_3 = (-3i\sqrt{a}, -2i\sqrt{a})$ $P_4 = (3i\sqrt{a}, 2i\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $n, n, s, s; S, N, N$ For $a > 0$ we have ©, ©, ©, ©; S, N, N	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \bar{J}_4 = P_4$ simple $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \bar{J}_4 = P_2$ simple $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = P_2$ double $\bar{J}_4 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 119 – Divisor and zero-cycles of family (F) when $h = -1$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + J_3^C + J_4^C + J_5 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	7
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 3P_3^\infty \text{ if } a < 0 \\ 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 3P_3^\infty \text{ if } a > 0 \end{cases}$	19 19

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and P_2);

2) four distinct tangents at P_2^∞

3) three distinct tangents at P_3^∞.

Table 120 – First integral and integrating factor of family (F) when $h = -1$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\frac{\lambda_2}{2}} J_4^{\frac{\lambda_1}{2}} J_5^{-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{1}{2}+\frac{\lambda_2}{2}} J_4^{-\frac{1}{2}+\frac{\lambda_1}{2}} J_5^{-1-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}}$
Simple example	$\mathscr{I} = \dfrac{J_1^2 J_4}{J_2^2 J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 116. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_4 - c_2 J_2^2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:0]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^2 J_4, \quad \mathscr{F}^1_{(0,1)} = J_2^2 J_3.$$

Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathscr{I}$, $[1:0]$ and $[0:1]$ are the only two critical remarkable values of $\mathscr{I}$ and J_1, J_2 are critical remarkable curves. The singular points are P_1, P_4 for $\mathscr{F}^1_{(1,0)}$ and P_2, P_3 for $\mathscr{F}^1_{(0,1)}$.

(ii.4) $h = -1/2$ and $a \neq 0$.

Under this condition the systems do not belong to family (F).

Here we have exactly the same results appearing in the generic case replacing $h = -1/2$ however we can calculated the expression of the first integral, it is given by

$$I = \frac{\sqrt{-i\sqrt{2}\sqrt{a}+x-y}\sqrt{i\sqrt{2}\sqrt{a}+x-y}\left(a(5y-9x)+y(x-y)^2\right)}{2(a+x(y-x))} - 3a\log\left(\sqrt{-i\sqrt{2}\sqrt{a}+x-y}\sqrt{i\sqrt{2}\sqrt{a}+x-y}+x-y\right). \tag{6.10}$$

(ii.5) $h=-1/3$ and $a\neq 0$.

Under this condition the systems do not belong to family (F). Here we have exactly the same results from the generic case replacing $h=-1/3$ however we can calculated the expression of the first integral, it is given by

$$I = \frac{9a^2(5y-8x)+2a(4x+5y)(x-y)^2+y(x-y)^4}{3(a+x(y-x))}. \tag{6.11}$$

(ii.6) $h=1/3$ and $a\neq 0$.

Under this condition the systems do not belong to family (F). Here we have two invariant lines and one double hyperbola, which allow us to find an exponential factor.

Table 121 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (F) when $h=1/3$ and $a\neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1=-\sqrt{3}\sqrt{a}+x-y$ $J_2=\sqrt{3}\sqrt{a}+x-y$ $J_3=a-x^2+xy$ $E_4=e^{\frac{ag_1+x(g_0+g_1(y-x))}{a+x(y-x)}}$ $\alpha_1=-\frac{\sqrt{a}}{\sqrt{3}}-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=\frac{\sqrt{a}}{\sqrt{3}}-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-\frac{2x}{3}-\frac{y}{3}$ $\alpha_4=-\frac{g_0}{3}$	$P_1=\left(-\frac{\sqrt{a}}{\sqrt{3}},\frac{2\sqrt{a}}{\sqrt{3}}\right)$ $P_2=\left(\frac{\sqrt{a}}{\sqrt{3}},-\frac{2\sqrt{a}}{\sqrt{3}}\right)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ For $a<0$ we have ©$_{(2)}$,©$_{(2)}$;N,N,S For $a>0$ we have $sn_{(2)},sn_{(2)};N,N,S$	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_2 \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_1 \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 122 – Divisor and zero-cycles of family (F) when $h = 1/3$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1^C + J_2^C + 2J_3 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1 + J_2 + 2J_3 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} 2P_1^C + 2P_2^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0$	7
$M_{0CT} = \begin{cases} 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	15 15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double,

2) only four distinct tangents at P_2^∞, but one of them is double and

3) only two distinct tangents at P_1^∞, but one of them is double.

Table 123 – First integral and integrating factor of family (F) when $h = 1/3$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{2\sqrt{3}\sqrt{a}\lambda_1}{g_0}}$	$R = J_1^{-1} J_2^{-1} J_3^{-2}$
Simple example	$\mathscr{I}_1 = J_1 J_2^{-1} \left(e^{\frac{x}{a+x(y-x)}} + 1 \right)^{-2\sqrt{3}\sqrt{a}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3^2}$

Source: Elaborated by the author.

(ii.7) $h = 1/2$ and $a \neq 0$.

Under this condition the systems do not belong to family (F). Here we have three invariant lines and two invariant hyperbolas. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

Table 124 – Invariant curves, cofactors, singularities and intersection points of family (F) when $h = 1/2$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\sqrt{2}\sqrt{a} - x + y$ $J_2 = \sqrt{2}\sqrt{a} - x + y$ $J_3 = x$ $J_4 = a - x^2 + xy$ $J_5 = a + 2xy$ $\alpha_1 = \frac{\sqrt{a}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = -\frac{\sqrt{a}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_3 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -x$ $\alpha_5 = -2x$	$P_1 = \left(\frac{\sqrt{a}}{\sqrt{2}}, -\frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_2 = \left(-\frac{\sqrt{a}}{\sqrt{2}}, \frac{\sqrt{a}}{\sqrt{2}}\right)$ $P_3 = (0, -\sqrt{2}\sqrt{a})$ $P_4 = (0, \sqrt{2}\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©,©,©,©; N, N, S For $a > 0$ we have $n, n, s, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = P_2$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = P_1$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_3 \cap \bar{J}_5 = P_1^\infty$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ double} \\ P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 125 – Divisor and zero-cycles of family (F) when $h = 1/2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	20 20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double,

2) only two distinct tangents at P_1^∞, but one of them is triple and

3) four distinct tangents at P_2^∞.

Table 126 – First integral and integrating factor of family (F) when $h = 1/2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1 - \frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{-\frac{3}{2} - \lambda_1 - \frac{\lambda_4}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_2 J_3^2}{J_5} \quad \mathscr{I}_2 = \dfrac{J_4^2}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 117. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_2 J_3^2 - c_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -a]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-a)} = J_4^2, \ \ \mathscr{F}^1_{(1,0)} = J_1 J_2 J_3^3.$$

Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathscr{I}_1$, $[1 : -a]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_3, J_4 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_2 for $\mathscr{F}^1_{(1,-a)}$ and P_3, P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_4^2 - c_2 J_5$ with $\deg \mathscr{F}^2_{(c_1,c_2)} = 4$. We have the remarkable values $[1 : a]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_3, J_4. However, the singular point are P_1, P_2 for $\mathscr{F}^2_{(1,0)}$ and P_3, P_4 for $\mathscr{F}^2_{(1,a)}$.

(ii.8) $h = 1$ and $a \neq 0$.

Under this condition the systems do not belong to family (F). Here we have five invariant lines and a family of invariant hyperbolas $a + ry - x^2 + xy$, where $r \neq -\sqrt{a}$. Therefore, these systems are algebraically integrable. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 127 – Invariant curves, cofactors, singularities and intersection points of family (F) when $h=1$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = 1 - \frac{x}{\sqrt{a}}$ $J_3 = 1 + \frac{x}{\sqrt{a}}$ $J_4 = 1 - \frac{x}{\sqrt{a}} + \frac{y}{\sqrt{a}}$ $J_5 = 1 + \frac{x}{\sqrt{a}} - \frac{y}{\sqrt{a}}$ $J_{6,r} = a + ry - x^2 + xy$ $\alpha_1 = -2x + y$ $\alpha_2 = -\sqrt{a} - x$ $\alpha_3 = \sqrt{a} - x$ $\alpha_4 = -\sqrt{a} - x + y$ $\alpha_5 = \sqrt{a} - x + y$ $\alpha_6 = -2x + y$	$P_1 = (-\sqrt{a}, 0)$ $P_2 = (-\sqrt{a}, -2\sqrt{a})$ $P_3 = (\sqrt{a}, 0)$ $P_4 = (\sqrt{a}, 2\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have ©, ©, ©, ©; N, N, S For $a > 0$ we have $n, s, n, s; N, N, S$	$\overline{J}_1 \cap \overline{J}_2 = P_3$ simple $\overline{J}_1 \cap \overline{J}_3 = P_1$ simple $\overline{J}_1 \cap \overline{J}_4 = P_3$ simple $\overline{J}_1 \cap \overline{J}_5 = P_1$ simple $\overline{J}_1 \cap \overline{J}_{6,r} = \begin{cases} P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = P_1^\infty$ simple $\overline{J}_2 \cap \overline{J}_4 = P_3$ simple $\overline{J}_2 \cap \overline{J}_5 = P_4$ simple $\overline{J}_2 \cap \overline{J}_{6,r} = \begin{cases} P_1^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\overline{J}_3 \cap \overline{J}_4 = P_2$ simple $\overline{J}_3 \cap \overline{J}_5 = P_1$ simple $\overline{J}_3 \cap \overline{J}_{6,r} = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\overline{J}_4 \cap \overline{J}_5 = P_2^\infty$ simple $\overline{J}_4 \cap \overline{J}_{6,r} = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_4 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_5 \cap \overline{J}_{6,r} = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_5 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_{6,r} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 128 – Divisor and zero-cycles of family (F) when $h=1$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ILD = \begin{cases} J_1 + J_2^C + J_3^C + J_4^C + J_5^C + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4\overline{J}_5 = 0$	6
$M_{0CT} = \begin{cases} 3P_1^C + 2P_2^C + 3P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty \text{ if } a < 0 \\ 3P_1 + 2P_2 + 3P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty \text{ if } a > 0 \end{cases}$	18 18

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_1, P_3, P_1^∞ and P_2^∞.

Table 129 – First integral and integrating factor of family (F) when $h = 1$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_3} J_4^{\lambda_3} J_5^{\lambda_2} J_{6,r}^{-\lambda_1-\lambda_2-\lambda_3}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_3} J_4^{\lambda_3} J_5^{\lambda_2} J_{6,r}^{-2-\lambda_1-\lambda_2-\lambda_3}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_5}$ $\mathscr{I}_2 = \dfrac{J_1}{J_3 J_4}$	$\mathscr{R} = \dfrac{1}{J_2 J_3 J_4 J_5}$

Source: Elaborated by the author.

Observation 118. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $\left[-\sqrt{a}/2 : 1\right]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(-\sqrt{a}/2,1)} = \frac{\sqrt{a}}{2} J_3 J_4, \quad \mathscr{F}^1_{(0,1)} = J_2 J_5.$$

Therefore, J_2, J_3, J_4, J_5 are remarkable curves of $\mathscr{I}_1$. The singular points are P_2 for $\mathscr{F}^1_{(-\sqrt{a}/2,1)}$ and P_4 for $\mathscr{F}^1_{(0,1)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_1 - c_2 J_3 J_4$ we have the remarkable values $\left[\sqrt{a}/2 : 1\right]$, $[0:1]$ and the remarkable curves J_2, J_3, J_4, J_5. The singular point are P_2 for $\mathscr{F}^2_{(0,1)}$ and P_4 for $\mathscr{F}^2_{(\sqrt{a}/2,1)}$.

(ii.9) $a = 0$ and $h \neq 0, \pm 1$.

Under this condition, systems (F) do not belong to **QSH**. The affine invariant lines are $x = 0$, $y = 0$ that are both simple and $x - y = 0$ which is double so we get an exponential factor. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 130 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (F) when $a = 0$ and $h \neq 0, \pm 1$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1 x - g_1 y}{x - y}}$ $\alpha_1 = (-1-h)x + hy$ $\alpha_2 = -hx + (-1+h)y$ $\alpha_3 = -hx + hy$ $\alpha_4 = g_0 h$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $phpphp_{(4)}; S, N, N$ if $h < 0$ $hpphpp_{(4)}; N, N, S$ if $h > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 131 – Divisor and zero-cycles of family (F) when $a = 0$ and $h \neq 0, \pm 1$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double and

2) only two distinct tangents at P_2^∞, but one of them is double.

Table 132 – First integral and integrating factor of family (F) when $a = 0$ and $h \neq 0, \pm 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-\frac{\lambda_1}{h}} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{\frac{-1-h-\lambda_1}{h}} E_4^0$
Simple example	$\mathscr{I} = J_1^h J_2^{-h} J_3^{-1}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

(ii.10) $a = 0$ and $h = -1$.

Under this condition the system does not belong to family (F). The affine invariant lines are $x = 0$ that is simple and $y = 0$, $x - y = 0$ that are both double so we get two exponential factors associated to them. Here we also have a family of invariant hyperbolas $rx + xy - y^2 = 0$, where $r \in \mathbb{R} \backslash \{0\}$. Therefore, this system is algebraically integrable. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 133 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (F) when $a = 0$ and $h = -1$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $J_{4,m} = rx + xy - y^2$ $E_5 = e^{\frac{g_0 + g_1 y}{y}}$ $E_6 = e^{\frac{h_0 + h_1 (x-y)}{x-y}}$ $\alpha_1 = -y$ $\alpha_2 = x - 2y$ $\alpha_3 = x - y$ $\alpha_4 = x - 2y$ $\alpha_5 = g_0$ $\alpha_6 = -h_0$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $phpphp_{(4)}; S, N, N$	$\overline{J}_1 \cap \overline{J}_2 = P_1$ simple $\overline{J}_1 \cap \overline{J}_3 = P_1$ simple $\overline{J}_1 \cap \overline{J}_{4,m} = \begin{cases} P_3^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = P_1$ simple $\overline{J}_2 \cap \overline{J}_{4,m} = P_1$ double $\overline{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\overline{J}_3 \cap \overline{J}_{4,m} = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_{4,m} \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 134 – Divisor and zero-cycles of family (F) when $a = 0$ and $h = -1$.

Divisor and zero-cycles	Degree
$ILD = 2J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	6
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1^2 \overline{J}_2 \overline{J}_3^2 = 0.$	6
$M_{0CT} = 5P_1 + 2P_1^\infty + 3P_2^\infty + 3P_3^\infty$	13

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but two of them are double;

2) only two distinct tangents at P_2^∞, but one of them is double and

2) only two distinct tangents at P_3^∞, but one of them is double.

Table 135 – First integral and integrating factor of family (F) when $a = 0$ and $h = -1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_3} J_2^{\lambda_2} J_3^{\lambda_3} J_{4,m}^{-\lambda_2 - \lambda_3} E_5^{\lambda_5} E_6^{\frac{g_0 \lambda_5}{h_0}}$	$R = J_1^{\lambda_3} J_2^{\lambda_2} J_3^{\lambda_3} J_{4,m}^{-2-\lambda_2-\lambda_3} E_5^{\lambda_5} E_6^{\frac{g_0 \lambda_5}{h_0}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_2}{J_1 J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 119. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_2 - c_2 J_1 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_1 J_3.$$

Therefore, J_1, J_3 are remarkable curves of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$.

(ii.11) $a = h = 0$.

Under this condition, systems (F) do not belong to **QSH**. The system here is $\dot{x} = -xy$, $\dot{y} = -xy$. This is a degenerate system where the lines $x = 0$ and $y = 0$ are filled up with singularities.

Table 136 – Singularities of reduced system when $a = h = 0$ for family (F).

Singularities
$P_1^\infty = [1:1:0]$ $(\ominus[\times];\emptyset); N, (\ominus[\times];\emptyset,\emptyset)$

Source: Elaborated by the author.

Table 137 – First integral and integrating factor for the reduced system of family (F) when $a = h = 0$.

	First integral	Integrating Factor
General	$I = x - y$	$R = 1$
Simple example	$\mathscr{I} = x - y$	$\mathscr{R} = 1$

Source: Elaborated by the author.

Note that I and $\mathscr{I}$ are also first integrals for family (F) when $a = h = 0$.

(ii.12) $a = 0$ and $h = 1$.

Under this condition the system does not belong to family (F). The affine invariant lines are $y = 0$ that is simple and $x = 0$, $x - y = 0$ that are both double so we get two exponential factors associated to them. Here we also have a family of invariant hyperbolas $rx - x^2 + xy = 0$, where $r \in \mathbb{R} \backslash \{0\}$. Therefore, this system is algebraically integrable. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 138 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (F) when $a = 0$ and $h = 1$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $J_{4,m} = rx - x^2 + xy$ $E_5 = e^{\frac{g_0+g_1x}{x}}$ $E_6 = e^{\frac{h_0+h_1(x-y)}{x-y}}$ $\alpha_1 = -2x + y$ $\alpha_2 = -x$ $\alpha_3 = -x + y$ $\alpha_4 = -2x + y$ $\alpha_5 = g_0$ $\alpha_6 = h_0$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \bar{J}_{4,m} = P_1$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_{4,m} = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_{4,m} = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_{4,m} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 139 – Divisor and zero-cycles of family (F) when $a = 0$ and $h = 1$.

Divisor and zero-cycles	Degree
$ILD = J_1 + 2J_2 + 2J_3 + \mathscr{L}_\infty$	6
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2^2\bar{J}_3^2 = 0.$	6
$M_{0CT} = 5P_1 + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty$	13

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but two of them are double;

2) only two distinct tangents at P_1^∞, but one of them is double and

2) only two distinct tangents at P_2^∞, but one of them is double.

Table 140 – First integral and integrating factor of family (F) when $a = 0$ and $h = 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_3} J_3^{\lambda_3} J_{4,m}^{-\lambda_1-\lambda_3} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$	$R = J_1^{\lambda_1} J_2^{\lambda_3} J_3^{\lambda_3} J_{4,m}^{-2-\lambda_1-\lambda_3} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 120. Although the three lines J_1, J_2, J_3 intersect at the origin we still have $J_1J_2J_3$ as an inverse integrating factor as in C-K Theorem.

Observation 121. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_1 - c_2J_2J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_2J_3.$$

Therefore, J_2, J_3 are remarkable curves of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (F) and we also confront our results with previous results in the literature in the following proposition.

Proposition 122. (a) For the family (F) we have nine distinct configurations $C_1^{(F)} - C_9^{(F)}$ of invariant hyperbolas and lines (see Figure 10 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $ah(h\pm1)(2h-1)(3h-1)(4h-1) = 0$. Its complement is a union of 14 disjoint sets. Oh $h = 0$ and $a \neq 0$ we have just one invariant hyperbola. On $h = 1/4$ we have an additional invariant hyperbola. For the limiting set of the parameter space of the considered family we have the following: On $h = -1$ and $a \neq 0$ we have four invariant lines and one invariant hyperbola. On $h = -1/2$ or $h = -1/3$ and $a \neq 0$ we have two invariant lines and one invariant hyperbola. On $h = 1/3$ and $a \neq 0$ we have two invariant lines and one double hyperbola. On $h = 1/2$ and $a \neq 0$ we have three invariant lines and two invariant hyperbolas. On $h = 1$ and $a \neq 0$ we have five invariant lines and a family of invariant hyperbolas. On $a = 0$ and $h \neq \pm1, 0$ the invariant hyperbola becomes reducible producing the lines $x = 0$ and $x - y = 0$. On $a = 0$ and $h = \pm1$ we have three invariant lines and a family of invariant hyperbolas. On $a = h = 0$ the lines $x = 0$ and $y = 0$ are filled up with singularities.

(b) The family (F) is Liouvillian integrable if $a(h\pm1)(2h\pm1)(3h\pm1)(4h-1) \neq 0$. When $h = 1/4$ the family (F) admits a rational first integral and the plane is foliated into 6th algebraic curves. The remarkable curves are J_1, J_2, J_3 and J_5 corresponding to this case. The systems in family (F) have a polynomial inverse integrating factor when $h = \frac{1}{1+2m}$ where $m \in \mathbb{N}$.

(c) For the family (F) we have three topologically distinct phase portraits $P_1^{(F)} - P_3^{(F)}$. The topological bifurcation diagram of family (F) is done in Figure 11. The bifurcation set is $ah = 0$ and the half lines $h = 1$ with $a < 0$, $h = 1/2$ with $a > 0$ and $h = 1/3$ for $a > 0$. The bifurcation set of singularities are the lines $a = 0$, $h = 0$ and the half line $h = 1/3$ with $a > 0$. The half lines $h = 1$ with $a < 0$ and $h = 1/2$ with $a > 0$ are bifurcations of saddle to saddle connection. The phase portraits $P_3^{(F)}$ is not topologically equivalent with anyone of the phase portraits in (CAIRÓ; FEIX; LLIBRE, 1999). We also have one phase portraits in a limit case of family (F) that does not appear in (LLIBRE; YU, 2018).

Proof of Proposition 122:

(a) We have the following types of divisors and zero-cycles of the total invariant curve T for the configurations of family (F) :

Table 141 – Configurations for family (F).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(F)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty + P_3^\infty$
$C_2^{(F)}$	$ICD = J_1^C + J_2^C + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty + P_3^\infty$
$C_3^{(F)}$	$ICD = J_1^C + J_2^C + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty + P_3^\infty$
$C_4^{(F)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty + P_3^\infty$
$C_5^{(F)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty + P_3^\infty$
$C_6^{(F)}$	$ICD = J_1 + 3\mathscr{L}_\infty$ $M_{0CT} = 2P_1^\infty + 2P_2^\infty + P_3^\infty$
$C_7^{(F)}$	$ICD = J_1 + 3\mathscr{L}_\infty$ $M_{0CT} = 2P_1^\infty + 2P_2^\infty + P_3^\infty$
$C_8^{(F)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_9^{(F)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Although $C_1^{(F)}$, $C_4^{(F)}$ and $C_5^{(F)}$ admit the same type of divisors and zero-cycles we can see they are different because of the position of the branches of the hyperbola and also because of the location of the singular points on the invariant lines. We also have that $C_2^{(F)}$, $C_3^{(F)}$ and $C_6^{(F)}$, $C_7^{(F)}$ are distinct because of the position of the branches of the hyperbola.

Therefore, the configurations $C_1^{(F)}$ up to $C_9^{(F)}$ are all distinct. For the limit cases of family (F) we have the following configurations:

Table 142 – Configurations for the limit cases of family (F).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
c_2	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
c_3	$ICD = J_1^C + J_2^C + 2J_3 + + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
c_4	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
c_5	$ILD = J_1 + J_2^C + J_3^C + J_4^C + J_5^C + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 2P_2^C + 3P_3^C + 2P_4^C + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty$
c_6	$ILD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 3P_3 + 2P_4 + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty$
c_7	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 4P_2^\infty + 3P_3^\infty$
c_8	$ICD = J_1^C + J_2^C + J_3^C + J_4^C + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 4P_2^\infty + 3P_3^\infty$
c_9	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$
c_{10}	$ICD = J_1 + 2J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 5P_1 + 3P_1^\infty + 3P_2^\infty + 2P_3^\infty$
c_{11}	$ICD = 2J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 5P_1 + 2P_1^\infty + 3P_2^\infty + 3P_3^\infty$
c_{12}	$ICD = \mathscr{L}_\infty$ $M_{0CT} = P_1^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) The part about the integrability of statement (b), the expression for the integrating factor and the invariant curves used below follows from the tables previously presented. The computations for the remarkable curves when $h = 1/4$ were done in Remark 113. Let us show when the family (F) admit a polynomial inverse integrating factor using invariant algebraic curves of degree at most two.

For the **generic case** $ah(h-1/4)(h\pm1)(2h\pm1)(3h\pm1) \neq 0$ we have the integrating factor

$$R = J_1^{\frac{h-1}{2h}} J_2^{\frac{h-1}{2h}} J_3^{-2}.$$

In order to R^{-1} be polynomial we must have that $\frac{h-1}{2h} = -m,\ m \in \mathbb{N}$. Then, $h = \frac{1}{1+2m}$ where $m \in \mathbb{N}$. Therefore, R^{-1} is polynomial when $h = \frac{1}{1+2m}$ for $m \in \mathbb{N}$.

(c) We have:

Table 143 – Phase portraits for family (F).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(F)}$	(S,N,N)	(n,n,s,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(F)}$	(N,N,S) $((\binom{2}{1})N,N,(\binom{2}{1})S)$ $((\binom{2}{1})S,N,(\binom{2}{1})N)$	$(©,©,©,©)$ $\emptyset$ $\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_3^{(F)}$	(N,N,S)	(s,s,n,n)	$2SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have three distinct phase portraits for systems (F). For the limit cases of family (F) we have the following phase portraits:

Table 144 – Phase portraits for the limit cases of family (F).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(F)}$	(S,N,N)	(n,n,s,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(F)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
p_1	(N,N,S)	(n,n,s,s)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_2	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
p_3	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
p_4	(S,N,N) (N,N,S)	$phpphp_{(4)}$ $hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_5	$(N,(\ominus[\times];\emptyset,\emptyset))$	$(\ominus[\times];\emptyset)$	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Table 145 – Phase portraits in in (CAIRÓ; FEIX; LLIBRE, 1999) that admit 3 singular points at infinity with the type (N,N,S), and it has either 0, 1, 2 or 4 real singular points in the finite region.

Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
(20)	(N,N,S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
(42)	(N,N,S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
(59)	(N,N,S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
(21)	(N,N,S)	cp	$0SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
(43)	(N,S,N)	cp	$0SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
(57)	(N,N,S)	$pphpph$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(22)	(N,N,S)	(s,c)	$1SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
(23)	(N,N,S)	(s,c)	$0SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
(28)	(N,N,S)	(s,c)	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
(44)	(N,N,S)	(s,c)	$1SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
(45)	(N,N,S)	(es,s)	$2SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
(58)	(N,N,S)	(sn,sn)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(77)	(N,N,S)	(sn,sn)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
(102)	(N,N,S)	(s,es)	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(35)	(N,N,S)	(n,s,s,n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(115)	(N,N,S)	(n,s,s,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

We recall that in the generic case $ah(4h-1)(h\pm1)(2h\pm1)(3h\pm1)\neq0$ the inverse integrating factor R^{-1} is polynomial if $h=\frac{1}{1+2m}$ where $m\in\mathbb{N}$, according to the proof done in (b). Therefore, the phase portraits for these cases should appear in (CAIRÓ; FEIX; LLIBRE, 1999). However, we could not find any phase portrait topologically equivalent with $P_3^{(F)}$.

Table 146 – Phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,N,S), and it has either 0,1, 2 or 4 real singular points in the finite region.

Phase Portraits	Sing. at ∞	Real finite sing.	Separatrix connections
R_{01},Ω_6	(N,S,N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
L_{11},L_{12}	(N,S,N)	cp	$0SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
P_2	(N,S,N)	$pphpph$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
L_{31},L_{32}	(N,S,N)	(s,es)	$2SC_f^f$ $6SC_f^\infty$ $2SC_\infty^\infty$
L_{33}	(N,S,N)	(c,es)	$1SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
R_1,R_2	(N,S,N)	(s,c)	$1SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
R_3,Ω_5	(N,S,N)	(c,c)	$2SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_8,Ω_1	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Consider the phase portrait p_2 appearing in the limit case $h = 1/3$ of family (F). When $h = 1/3$ we have two invariant lines and one invariant hyperbola so the phase portraits of this case should appear in (LLIBRE; YU, 2018). However we did not find any phase portrait topologically equivalent with p_2. Note that $P_2^{(\mathrm{F})} \cong_{top} P_3^{(\mathrm{B})}$, $P_3^{(\mathrm{F})} \cong_{top} P_1^{(\mathrm{B})}$ and $p_1 \cong_{top} P_4^{(\mathrm{C})}$ are also missing and they were listed in the geometric study of families (B) and (C).

□

Figure 10 – Bifurcation diagram of configurations for family (F).

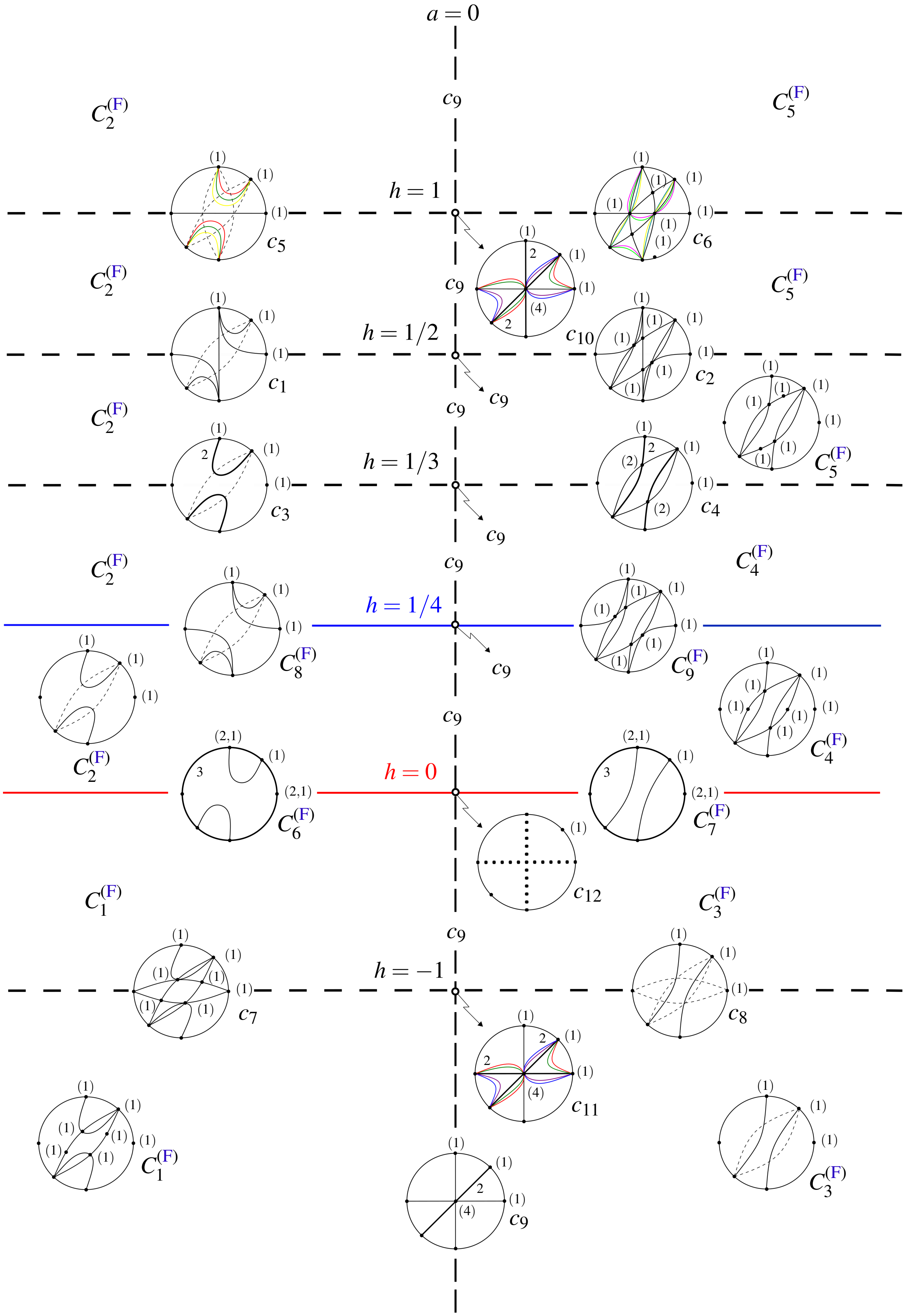
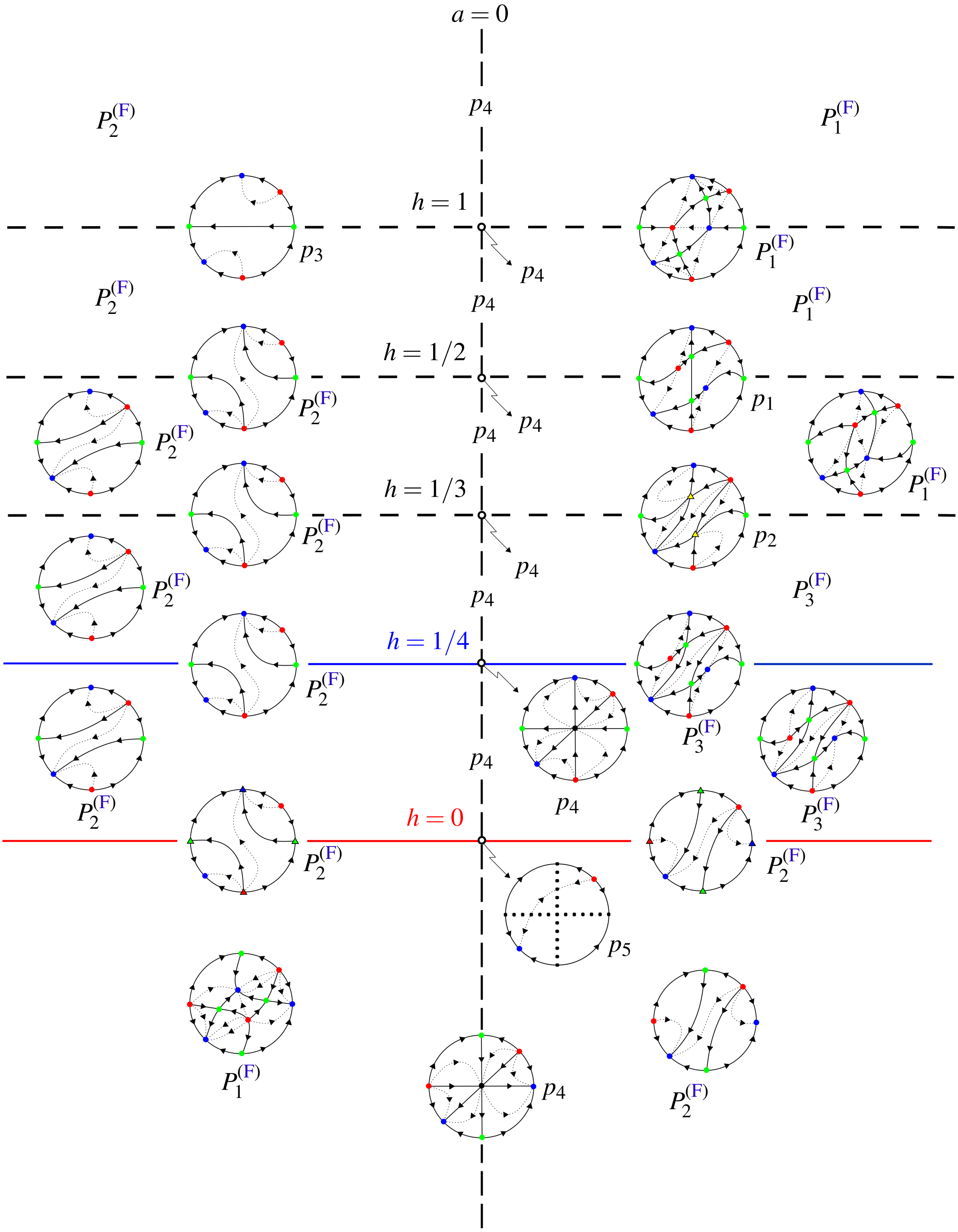


Figure 11 – Topological bifurcation diagram for family (F).



6.1.5 Geometric Analysis of Family (G)

Consider the family

$$(G)\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a + 3v^2 - \frac{4xy}{3} + \frac{y^2}{3}, \end{cases}$$

where $av \neq 0$.

This is a two parameter family depending a and v such that $av \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (G) we study here also the limit cases $av = 0$ where the equations are still defined.

We display below the full geometric analysis of the systems in this family, which is endowed with at least three invariant algebraic curves. In the **generic case**

$$av(a+v^2)(a+3v^2/4)(a-3v^2)(a+8v^2/9) \neq 0$$

the systems have two invariant lines J_1 and J_2 and two invariant hyperbolas J_3 and J_4 with cofactors α_i, $1 \leq i \leq 4$ given by

$$\begin{array}{ll} J_1 = -3i\sqrt{a+v^2} - x + y, & \alpha_1 = i\sqrt{a+v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_2 = 3i\sqrt{a+v^2} - x + y, & \alpha_2 = -i\sqrt{a+v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_3 = -3a + 3ivx - x^2 + xy, & \alpha_3 = -iv - \frac{2x}{3} - \frac{y}{3}, \\ J_4 = -3a - 3ivx - x^2 + xy, & \alpha_4 = iv - \frac{2x}{3} - \frac{y}{3}. \end{array}$$

Then according to Darboux' theorem we must have a Darboux first integral. We note that when $a = -v^2$ the two lines coincide and we get a double line. We inquire when we could have an additional line. Calculations yield that this happens when $(a + 3v^2/4) = 0$. We also inquire when we could have an additional hyperbola. Calculations yield that this happens when $(a - 3v^2)(a + 8v^2/9) = 0$. The multiplicities of each invariant line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $av(a+v^2)(a+3v^2/4)(a-3v^2)(a+8v^2/9) \neq 0$.

Table 147 – Invariant curves, cofactors, singularities and intersection points of family (G) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -3i\sqrt{a+v^2} - x + y$ $J_2 = 3i\sqrt{a+v^2} - x + y$ $J_3 = -3a + 3ivx - x^2 + xy$ $J_4 = -3a - 3ivx - x^2 + xy$ $\alpha_1 = i\sqrt{a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = i\sqrt{a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -iv - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = iv - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-i\left(\sqrt{a+v^2}-v\right), i\left(2\sqrt{a+v^2}+v\right)\right)$ $P_2 = \left(-i\left(\sqrt{a+v^2}+v\right), -i\left(v-2\sqrt{a+v^2}\right)\right)$ $P_3 = \left(i\left(\sqrt{a+v^2}+v\right), i\left(v-2\sqrt{a+v^2}\right)\right)$ $P_4 = \left(i\left(\sqrt{a+v^2}-v\right), -i\left(2\sqrt{a+v^2}+v\right)\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ ©, ©, ©, ©; N, N, S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 148 – Divisor and zero-cycles of family (G) for the generic case.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3^C + J_4^C + \mathscr{L}_\infty \text{ if } a+v^2 < 0 \\ J_1^C + J_2^C + J_3^C + J_4^C + \mathscr{L}_\infty \text{ if } a+v^2 > 0 \end{cases}$	5 5
$M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$	17

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) five distinct tangents at P_2^∞.

Table 149 – First integral and integrating factor of family (G) for the generic case.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1\sqrt{v^2+a}}{v}} J_4^{-\frac{\lambda_1\sqrt{v^2+a}}{v}}$	$R = J_1^{\lambda_1} J_2^{-\lambda_1-2} J_3^{\frac{(\lambda_1+1)\sqrt{v^2+a}}{v}-1} J_4^{-\frac{(\lambda_1+1)\sqrt{v^2+a}}{v}-1}$
Simple example	$\mathscr{I} = \frac{J_1}{J_2}\left(\frac{J_3}{J_4}\right)^{\frac{\sqrt{v^2+a}}{v}}$	$\mathscr{R} = \frac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

(ii) The non-generic case: $av(a+v^2)(a+3v^2/4)(a-3v^2)(a+8v^2/9)=0$.

(ii.1) $a=-v^2$ and $a\neq 0$.

Here the two lines coalesce yielding a double line so we compute the exponential factor E_4.

Table 150 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (G) when $a=-v^2$ and $a\neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x-y$ $J_2 = 3v^2+3ivx-x^2+xy$ $J_3 = 3v^2-3ivx-x^2+xy$ $E_4 = e^{\frac{g_0+g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{x}{3}+\frac{y}{3}$ $\alpha_2 = -iv-\frac{2x}{3}-\frac{y}{3}$ $\alpha_3 = iv-\frac{2x}{3}-\frac{y}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (-iv,-iv)$ $P_2 = (iv,iv)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $©_{(2)}, ©_{(2)}; N, N, S$	$\bar{J}_1\cap\bar{J}_2 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_1\cap\bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_3\cap\mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 151 – Divisor and zero-cycles of family (G) when $a=-v^2$ and $a\neq 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2^C + J_3^C + \mathscr{L}_\infty$	5
$M_{0CS} = 2P_1^C + 2P_2^C + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2\bar{J}_3 = 0$	7
$M_{0CT} = 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$	15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) only four distinct tangents at P_2^∞, but one of them is double.

Table 152 – First integral and integrating factor of family (G) when $a = -v^2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\lambda_2} E_4^{\frac{6iv\lambda_2}{g_0}}$	$R = J_1^{-2} J_2^{\lambda_2} J_3^{-2-\lambda_2} E_4^{\frac{6iv(1+\lambda_2)}{g_0}}$
Simple example	$\mathscr{I} = \dfrac{J_2 E_4{}^{6iv}}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1^2 J_2 J_3}$

Source: Elaborated by the author.

(ii.2) $a = -3v^2/4$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant line. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 153 – Invariant curves, cofactors, singularities and intersection points of family (G) when $a = -3v^2/4$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\frac{3iv}{2} + x - y$ $J_2 = \frac{3iv}{2} + x - y$ $J_3 = y$ $J_4 = \frac{ix^2}{3v} - \frac{ixy}{3v} - \frac{3iv}{4} + x$ $J_5 = -\frac{ix^2}{3v} + \frac{ixy}{3v} + \frac{3iv}{4} + x$ $\alpha_1 = \frac{1}{6}(-3iv - 2x + 2y)$ $\alpha_2 = \frac{1}{6}(3iv - 2x + 2y)$ $\alpha_3 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_4 = -iv - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = iv - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{iv}{2}, -2iv\right)$ $P_2 = \left(\frac{iv}{2}, 2iv\right)$ $P_3 = \left(-\frac{3iv}{2}, 0\right)$ $P_4 = \left(\frac{3iv}{2}, 0\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ ©, ©, ©, ©; N, N, S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_4$ double $\bar{J}_3 \cap \bar{J}_5 = P_3$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 154 – Divisor and zero-cycles of family (G) when $a = -3v^2/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$	6
$M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^c + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$	20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) five distinct tangents at P_2^∞.

Table 155 – First integral and integrating factor of family (G) when $a = -3v^2/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}} J_4^{\frac{\lambda_2}{2}} J_5^{\frac{\lambda_1}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-1-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}} J_4^{-\frac{1}{2}+\frac{\lambda_2}{2}} J_5^{-\frac{1}{2}+\frac{\lambda_1}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2 J_5}{J_3}$ $\quad \mathscr{I}_2 = \dfrac{J_2^2 J_4}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 123. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_5 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -9v^2/2]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-9v^2/2)} = -J_2^2 J_4, \quad \mathscr{F}^1_{(1,0)} = J_1^2 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : -9v^2/2]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_2, P_3 for $\mathscr{F}^1_{(1,-9v^2/2)}$ and P_1, P_4 for $\mathscr{F}^1_{(1,0)}$.
Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2^2 J_4 - c_2 J_3$ we have the same remarkable values $[1 : -9v^2/2]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. However, the singular point are P_1, P_4 for $\mathscr{F}^2_{(1,-9v^2/2)}$ and P_2, P_3 for $\mathscr{F}^2_{(1,0)}$.

(ii.3) $a = 3v^2$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant hyperbola. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 156 – Invariant curves, cofactors, singularities and intersection points of family (G) when $a = 3v^2$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -6iv + x - y$ $J_2 = 6iv + x - y$ $J_3 = -9v^2 + xy$ $J_4 = -9v^2 + 3ivx - x^2 + xy$ $J_5 = -9v^2 - 3ivx - x^2 + xy$ $\alpha_1 = -2iv - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = 2iv - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{5x}{3} - \frac{y}{3}$ $\alpha_4 = -iv - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = iv - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-iv, 5v)$ $P_2 = (iv, -5iv)$ $P_3 = (-3iv, 3iv)$ $P_4 = (3iv, -3iv)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ ©, ©, ©, ©; N, N, S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ double $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 157 – Divisor and zero-cycles of family (G) when $a = 3v^2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$	6
$M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$	21

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is triple,

2) five distinct tangents at P_2^∞.

Table 158 – First integral and integrating factor of family (G) when $a = 3v^2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{1+2\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_5^2}{J_3}$ $\quad \mathscr{I}_2 = \dfrac{J_2 J_4^2}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 124. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_5^2 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : 108iv^3]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,108iv^3)} = J_2 J_4^2, \qquad \mathscr{F}^1_{(1,0)} = J_1 J_5^2.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : 108iv^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_4, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_4 for $\mathscr{F}^1_{(1,108iv^3)}$ and P_2, P_3 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_4^2 - c_2 J_3$ we have the remarkable values $[1 : -108iv^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_2, P_3 for $\mathscr{F}^2_{(1,-108iv^3)}$ and P_1, P_4 for $\mathscr{F}^2_{(1,0)}$.

(ii.4) $a = -8v^2/9$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant hyperbola. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 159 – Invariant curves, cofactors, singularities and intersection points of family (G) when $a = -8v^2/9$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -iv + x - y$ $J_2 = iv + x - y$ $J_3 = y(x-y) + \frac{v^2}{3}$ $J_4 = -\frac{8v^2}{3} + 3ivx + x(y-x)$ $J_5 = -\frac{8v^2}{3} - 3ivx + x(y-x)$ $\alpha_1 = \frac{1}{3}(-iv - x + y)$ $\alpha_2 = \frac{1}{3}(iv - x + y)$ $\alpha_3 = \frac{2y}{3} - \frac{5x}{3}$ $\alpha_4 = -iv - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = iv - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{2iv}{3}, -\frac{5iv}{3}\right)$ $P_2 = \left(\frac{2iv}{3}, \frac{5iv}{3}\right)$ $P_3 = \left(-\frac{4iv}{3}, -\frac{iv}{3}\right)$ $P_4 = \left(\frac{4iv}{3}, \frac{iv}{3}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ ©, ©, ©, ©; N, N, S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 160 – Divisor and zero-cycles of family (G) when $a = -8v^2/9$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$	6
$M_{0CS} = P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$	21

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) six distinct tangents at P_2^∞.

Table 161 – First integral and integrating factor of family (G) when $a = -8v^2/9$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}} J_4^{\frac{\lambda_2}{3}} J_5^{\frac{\lambda_1}{3}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}-\frac{2}{3}} J_4^{\frac{\lambda_2}{3}-\frac{2}{3}} J_5^{\frac{\lambda_1}{3}-\frac{2}{3}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^3 J_5}{J_3}$ $\quad \mathscr{I}_2 = \dfrac{J_2^3 J_4}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 125. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^3 J_5 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : 16iv^3]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,16iv^3)} = J_2^3 J_4, \quad \mathscr{F}^1_{(1,0)} = J_1^3 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : 16iv^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_2, P_3 for $\mathscr{F}^1_{(1,16iv^3)}$ and P_1, P_4 for $\mathscr{F}^1_{(1,0)}$.
Considering the first integral $\mathscr{I}_2$ with its associated curves $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2^3 J_4 - c_2 J_3$ we have the remarkable values $[1 : -16iv^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_2, P_3 for $\mathscr{F}^2_{(1,0)}$ and P_1, P_4 for $\mathscr{F}^2_{(1,-16iv^3)}$.

(ii.5) $v = 0$ and $a \neq 0$.

Under this condition the systems do not belong to family (G). Here we have two invariant lines and one double hyperbola so we compute the exponential factor E_4.

Table 162 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (G) when $v = 0$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -3i\sqrt{a} + x - y$ $J_2 = 3i\sqrt{a} + x - y$ $J_3 = -3a + x(y - x)$ $E_4 = e^{\frac{g_1 x}{-3a + x(y-x)}}$ $\alpha_1 = -i\sqrt{a} - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = i\sqrt{a} - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = -\frac{g_1}{3}$	$P_1 = (-i\sqrt{a}, 2i\sqrt{a})$ $P_2 = (i\sqrt{a}, -2i\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $sn_{(2)}, sn_{(2)}; N, N, S$ For $a > 0$ we have $©_{(2)}, ©_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 163 – Divisor and zero-cycles of family (G) when $v = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + 2J_3 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + 2J_3 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	7
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	15 15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) only four distinct tangents at P_2^∞, but one of them is double.

Table 164 – First integral and integrating factor of family (G) when $v = 0$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{6i\sqrt{a}\lambda_1}{g_1}}$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-2} E_4^{-\frac{6i\sqrt{a}(1+\lambda_1)}{g_1}}$
Simple example	$\mathscr{I} = \dfrac{J_1}{J_2 E_4{}^{6i\sqrt{a}}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3^2}$

Source: Elaborated by the author.

(ii.6) $a = 0$ and $v \neq 0$.

Under this condition, systems (G) do not belong to **QSH**. The affine invariant lines are $x = 0$ and $\pm 3iv - x + y = 0$ that are all simple. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. Perturbing this system in the family (G) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics $x(3iv - x + y) = 0$ and $x(-3iv - x + y) = 0$ we can produce two distinct hyperbolas $-3a + 3ivx - x^2 + xy = 0$ and $-3a - 3ivx - x^2 + xy = 0$, respectively. Furthermore, the cubic $x(3iv - x + y)(-3iv - x + y) = 0$ has integrable multiplicity two.

Table 165 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (G) when $a=0$ and $v\neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1=-3iv-x+y$ $J_2=3iv-x+y$ $J_3=x$ $E_4=e^{\frac{3g_0\left(6v^2+x(x-y)\right)}{x\left(9v^2+(x-y)^2\right)}-\frac{g_1}{2}}$ $\alpha_1=iv-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=-iv-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-\frac{x}{3}-\frac{2y}{3}$ $\alpha_4=g_0$	$P_1=(2iv,-iv)$ $P_2=(-2iv,iv)$ $P_3=(0,-3iv)$ $P_4=(0,3iv)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ ©,©,©,©;N,N,S	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=P_4$ simple $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=P_3$ simple $\bar{J}_2\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\mathscr{L}_\infty=P_1^\infty$ simple

Source: Elaborated by the author.

Table 166 – Divisor and zero-cycles of family (G) when $a=0$ and $v\neq 0$.

Divisor and zero-cycles	Degree
$ICD=J_1^C+J_2^C+J_3+\mathscr{L}_\infty$	4
$M_{0CS}=P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty$	7
$T=Z\bar{J}_1\bar{J}_2\bar{J}_3=0.$	4
$M_{0CT}=P_1^C+P_2^C+2P_3^C+2P_4^C+2P_1^\infty+3P_2^\infty+P_3^\infty$	12

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_2^∞.

Table 167 – First integral and integrating factor of family (G) when $a=0$ and $v\neq 0$.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_2^{-\lambda_1}J_3^0E_4^{-\frac{2iv\lambda_1}{g_0}}$	$R=J_1^{\lambda_1}J_2^{-4-\lambda_1}J_3^{-2}E_4^{-\frac{2iv(\lambda_1+2)}{g_0}}$
Simple example	$\mathscr{I}=\dfrac{J_1}{J_2E_4^{2iv}}$	$\mathscr{R}=\dfrac{1}{J_1^2J_2^2J_3^2}$

Source: Elaborated by the author.

(ii.7) $a=v=0$.

Under this condition, systems (G) do not belong to **QSH**. The affine invariant lines are $x=0$, $y=0$ that are simple and $x-y=0$ that is double. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation

81. This system has a rational first integral that foliates the plane into quartic invariant algebraic curves. The lines $x = 0$ and $x - y = 0$ are remarkable curves. Perturbing this system in the full family (H) we can obtain up to ten distinct configurations of lines and hyperbolas. By perturbing the reducible conic $x(x-y) = 0$ we can produce 2 distinct hyperbolas $-3a + 3vx - x^2 + xy = 0$ and $-3a - 3vx - x^2 + xy = 0$. Perturbing the reducible conic $y(x-y) = 0$ we can produce a third hyperbola $y(x-y) - \frac{v^2}{3} = 0$ and by perturbing $xy = 0$ we can produce the hyperbola $9v^2 + xy = 0$. We get a double hyperbola $-3a + x(y-x) = 0$ by perturbing the double reducible conic $x^2(x-y)^2 = 0$.

Table 168 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (G) when $a = v = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_2 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_3 = \frac{y}{3} - \frac{x}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 169 – Divisor and zero-cycles of family (G) when $a = v = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	10

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double;

2) only two distinct tangents at P_2^∞, but one of them is double.

Table 170 – First integral and integrating factor of family (G) when $a = v = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-3\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-4-3\lambda_1} E_4^0$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_3^3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 126. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_2 J_3^3.$$

Therefore, J_2, J_3 are remarkable curves of $\mathscr{I}_1$, $[0:1]$ is the only critical remarkable values of $\mathscr{I}_1$ and J_3 is critical remarkable curve of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (G) and we also confront our results with previous results in the literature in the following proposition.

Proposition 127. (a) For the family (G) we have seven distinct configurations $C_1^{(G)} - C_7^{(G)}$ of invariant hyperbolas and lines (see Figure 12 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $av(a+v^2)(a-3v^2)(a+3v^2/4)(a+8v^2/9) = 0$. Its complement is a union of 12 disjoint sets. On $a = -v^2$ the invariant line is double. On $(a+3v^2/4)(a-3v^2)(a+8v^2/8) = 0$ we have an additional invariant curve. For the limiting set of the parameter space of the considered family we have the following: On $v = 0$ and $a \neq 0$ the invariant hyperbola is double. On $a = 0$ the invariant hyperbolas become reducible producing the lines $x = 0$, $-3iv - x + y = 0$ and $x = 0$, $3iv - x + y = 0$. The configurations $C_2^{(G)}$ and $C_6^{(G)}$ are not equivalent with anyone of the configurations in (OLIVEIRA *et al.*, 2017).

(b) The family (G) is Darboux integrable if $av(a+v^2)(a+3v^2/4)(a-3v^2)(a+8v^2/9) \neq 0$. When $a = -v^2$ the family (G) is generalized Darboux first integrable. In all the following three cases, we have a rational first integral. If $a = -3v^2/4$ then the systems have an additional invariant line and the plane is foliated into quartic invariant algebraic curves. If $a = 3v^2$ then the systems have an additional invariant hyperbola and the plane is foliated by quintic invariant algebraic curves. If $a = -8v^2/9$ then the systems have an additional invariant hyperbola and the plane is foliated in quintic invariant algebraic curves. The remarkable curves are J_1, J_2, J_4, J_5 for these three algebraically integrable cases of family (G) for each case correspondingly. All systems in family (G) have an inverse integrating factor which is polynomial.

(c) For the family (G) we have two topologically distinct phase portraits $P_1^{(G)}$ and $P_2^{(G)}$. The topological bifurcation diagram of family (G) is done in Figure 13. The bifurcation set is the half line $v = 0$ with $a < 0$ and the parabola $a = -3v^2/4$. The half line $v = 0$ with $a < 0$ is a bifurcation of singularities and the parabola $a = -3v^2/4$ is a bifurcation of separatrix from saddle to saddle connection.

Proof of Proposition 127:

(a) We have the following types of divisors and zero-cycles of the total invariant curve T for the configurations of family (G) :

Table 171 – Configurations for family (G).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(G)}$	$ICD = J_1^C + J_2^C + J_3^C + J_4^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_2^{(G)}$	$ICD = J_1^C + J_2^C + J_3^C + J_4^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_3^{(G)}$	$ICD = J_1 + J_2 + J_3^C + J_4^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_4^{(G)}$	$ICD = 2J_1 + J_2^C + J_3^C + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_5^{(G)}$	$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_6^{(G)}$	$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_7^{(G)}$	$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Although $C_1^{(G)}$ and $C_2^{(G)}$ admit the same type of divisors and zero-cycles we can see they are different because in $C_1^{(G)}$ each branch of the hyperbolas intersects one line while $C_2^{(G)}$ have two branches intersecting both lines and two branches intersecting no line. Therefore, the configurations $C_1^{(G)}$ up to $C_7^{(G)}$ are all distinct. For the limit cases of family (G) we have the following configurations:

Table 172 – Configurations for the limit cases of family (G).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
c_2	$ICD = J_1^C + J_2^C + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
c_3	$ICD = J_1^C + J_2^C + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + P_2^C + 2P_3^C + 2P_4^C + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
c_4	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

In (OLIVEIRA *et al.*, 2017) the authors presented all the normal forms for **QSH** and they also give all the configurations for each normal form. However we notice two configurations missing in the study of (3.95) (here family (G)). The configurations they gave in this study was H.144 (which is $C_2^{(G)}$), H.145 (which is $C_3^{(G)}$), H.151 (which is $C_5^{(G)}$), H.153 (which is $C_4^{(G)}$) and H.159 (which is $C_6^{(G)}$). Therefore, they missed the configurations $C_1^{(G)}$ and $C_7^{(G)}$.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 123, 124 and 125 .

(c) We have that:

Table 173 – Phase portraits for family (G).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(G)}$	(N,N,S)	$(©,©,©,©)$ $(©_{(2)},©_{(2)})$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_2^{(G)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have two distinct phase portraits for systems (G). For the limit cases of family (G) we have the following phase portraits:

Table 174 – Phase portraits for the limit cases of family (G).

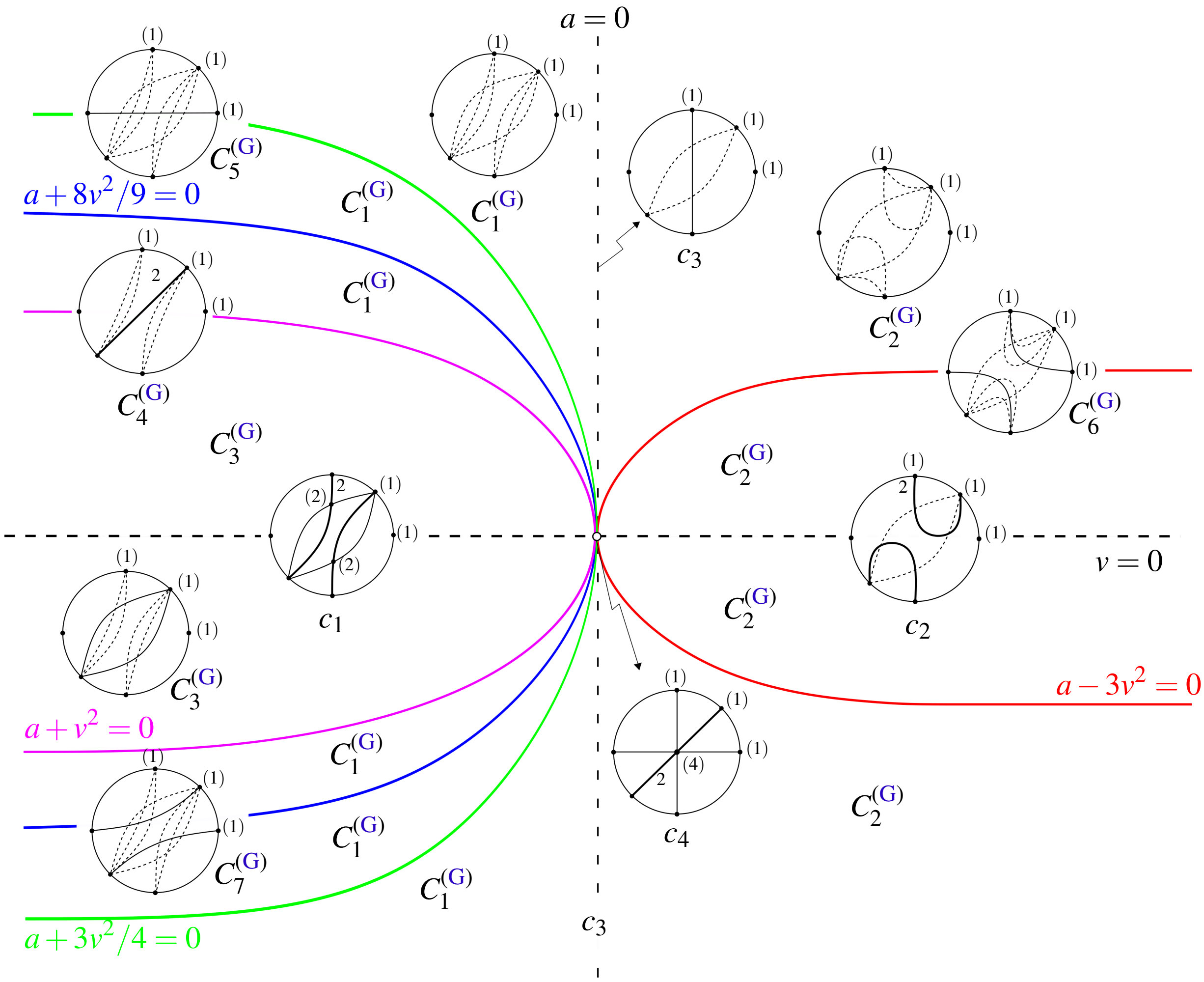
Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
p_2	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

We note that the phase portraits $P_1^{(G)} \cong_{top} P_3^{(B)}$ and $p_1 \cong_{top} p_{2\ (F)}$ (where p_2 is a phase portrait in the limit cases of family (F)) are missing in (LLIBRE; YU, 2018) and they were listed in the geometric studies of families (B) and (F).

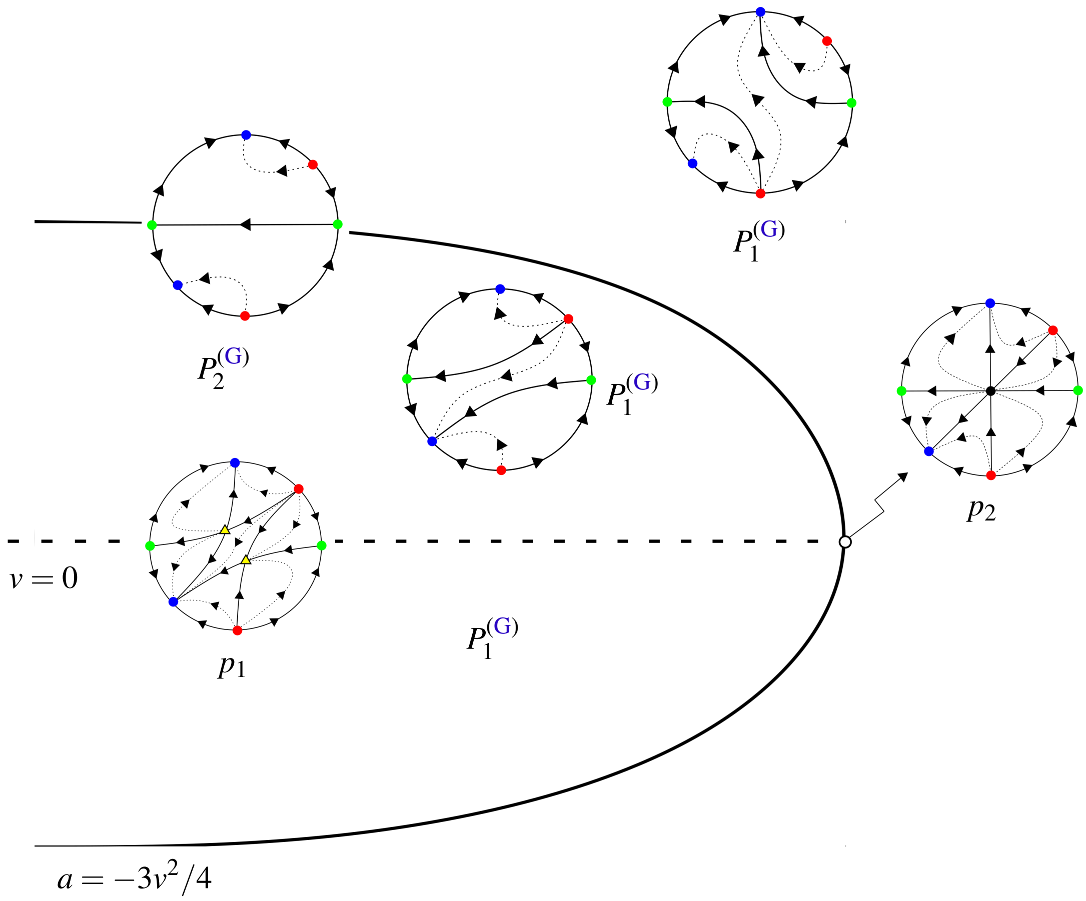
□

Figure 12 – Bifurcation diagram of configurations for family (G).



Source: Elaborated by the author.

Figure 13 – Topological bifurcation diagram for family (G).



Source: Elaborated by the author.

6.1.5.1 The solution of the Poincaré problem for the family (G)

We can recognize when a system in this family has a rational first integral. The following is the answer to Poincaré's problem for the family (G):

Theorem 128. i) A necessary and sufficient condition for a system in family (G) to have a rational first integral given by invariant algebraic curves of degree at most two, is that $v^2 + a > 0$ and that (a, v) be situated on a parabola of the form $a = (r^2 - 1)v^2$ with $r \in \mathbb{Q}$. ii) The set of all points (a, v)'s satisfying these two conditions is dense in the set $v^2 + a > 0$ with $v \neq 0$.

Proof. i) We first prove that the condition is necessary. So assume that we have a system of parameters (a, v) that has a rational first integral. Assume now that (a, v) is in the generic situations $(a + v^2)(a + 3v^2/4)(a - 3v^2)(a + 8v^2/9) \neq 0$. Any first integral of the system is then of the following general form:

$$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1 \sqrt{v^2+a}}{v}} J_4^{-\frac{\lambda_1 \sqrt{v^2+a}}{v}}.$$

This is a rational first integral if and only if $\lambda_1 \in \mathbb{Z}$ and $\frac{\lambda_1\sqrt{v^2+a}}{v} \in \mathbb{Z}$ in which case we must have that $r = \sqrt{v^2+a}/v$ must be a rational number. In view of our generic hypothesis $r \neq 0$. Since $r = \sqrt{v^2+a}/v$ is rational we have $v^2+a \geq 0$ and by hypothesis $v^2+a \neq 0$. Therefore $v^2+a > 0$. We also have $a = (r^2-1)v^2$ and therefore the condition is necessary in this case. Consider now the case when $(a+v^2)(a+3v^2/4)(a-3v^2)(a+8v^2/9) = 0$. Since on $a = -v^2$ we cannot have a rational first integral because as we see in the tables for this case, we have exponential factors in the first integrals and hence we must have $a \neq -v^2$. Therefore our previous assumption is reduced to $(a+3v^2/4)(a-3v^2)(a+8v^2/9) = 0$. Suppose first that the point (a,v) is located on the parabola $a = 3v^2$. Then this parabola can be written as $a = (r^2-1)v^2$ where $r = 2$. We then have $v^2+a = r^2v^2 = 4v^2 > 0$. If the point (a,v) is on the parabola $a+8y^2/9 = 0$ then this parabola can be written as $a = (r^2-1)v^2$ for $r = 1/3$. Here again we have that $v^2+a = r^2v^2 = v^2/9 > 0$. So the system situated on the parabola $a+8y^2/9 = 0$ satisfies $v^2+a > 0$ and for $r = 1/3$ the point is located on the parabola $a = (r^2-1)v^2$. So also in this case these conditions are necessary. There remains only the case when (a,v) is on the parabola $a+3v^2/4 = 0$. In this case we can write this parabola as $a = (r^2-1)v^2$ by taking $r = 1/2$. Also here $v^2+a = r^2v^2 = v^2/4 > 0$, i.e. $v^2+a > 0$. So the necessity of the conditions is proved in this case too.

We now prove the sufficiency of the conditions. Let us assume that $v^2+a > 0$, $v \neq 0$ and (a,v) is located on a parabola $a = (r^2-1)v^2$ with $r \in \mathbb{Q}$. Then clearly $r \neq 0$, otherwise $v^2+a = r^2v^2 = 0$ contrary to our assumption. In case $r = 2, 1/3, 1/2$ we are on one of the three parabolas obtained from the condition $(a-3v^2)(a+8v^2/9)(a+3v^2/4) = 0$ and for these parabolas the tables give us rational first integrals. If the generic condition is satisfied, i.e. $(a+v^2)(a-3v^2)(a+8v^2/9)(a+3v^2/4) \neq 0$, then we know that we have the corresponding first integral indicated in the Tables for this case where the exponents for the curves J_i are λ_1 and $\lambda_1\sqrt{v^2+a}/v$. But we know by our assumption that (a,v) is located on a parabola $a = (r^2-1)v^2$ for some rational number r. From this equation we have that $r^2 = (a+v^2)/v^2$. Hence $r = \sqrt{v^2+a}/v$ is rational. We may suppose $r = m/n$ with $m,n \in \mathbb{Z}$ and m,n coprime. Then by taking in the general expression of the first integral $\lambda_1 = n$ and $r = \sqrt{v^2+a}/v$ we obtained a rational first integral in this case.

ii) Let us denote by $\mathscr{P}_r$ the parabola corresponding to a rational number r, i.e.

$$\mathscr{P}_r := \{(a,v) \in \mathbb{R}^2 : (r^2-1)v^2 = a\}.$$

Thus a system in the family (H) has a rational first integral if and only if it corresponds to a point (a,v) such that $v^2+a > 0$ with $v \neq 0$ and the point is situated on a parabola $\mathscr{P}_r$ for some rational number r. In the parameter plane $\mathbb{R}^2$ let the $a-$axis be the horizontal line and the $v-$axis be the vertical one. The parabolas $a = (r^2-1)v^2$ are symmetric with respect to the $a-$axis. Because of this it would suffice to prove the density of points (a,v) on parabolas $\mathscr{P}_r$ and inside $v^2+a > 0$ and $v > 0$.

Claim: The set of all points in $A =: \cup_{r\in\mathbb{Q}}\mathscr{P}_r$ with $v > 0$ is dense in the set $S^+ = \{(a,v) : v^2 + a > 0,\ v > 0\}$.

Take an arbitrary point point $p_0 = (a_0, v_0) \in S^+$. So we have $v_0^2 + a_0 > 0$ and $v_0 > 0$. We only need to consider p_0 inside the first or second quadrant. Indeed the line $a = 0$ is outside the parameter space of our family. So $a_0 \neq 0$. In view of our assumption we have that $(v_0^2 + a_0)/v_0^2 > 0$. So let $r_0 = \sqrt{(v_0^2 + a_0)/v_0^2} > 0$. Hence we have $a_0 = (r_0^2 - 1)v_0^2$. Here r_0 is not necessarily a rational number. But it can be approximated with rational numbers. So take a sequence of rational numbers r_n such that $r_n \to r_0$. At this point let us assume that the point (a_0, v_0) is in the first quadrant, i.e. $a_0 > 0$. In this case $r_0 > 1$ and since $r_n \to r_0$ there exists a number N such that for $n > N$ $r_n > 1$ and hence $r_n^2 > 1$ for all $n > N$. So $\sqrt{a_0/(r_n^2 - 1)} > 0$. Denote by $v_n = \sqrt{a_0/(r_n^2 - 1)}$. Then $v_n \to v_0$ and hence $(a_0, v_n) \to (a_0, v_0)$. But each point (a_0, v_n) is located on the corresponding parabola $a_0 = (r_n^2 - 1)v_n^2$ and hence p_0 is an accumulation point of points situated on such parabolas with r_n rational. Assume now that the point p_0 is in the second quadrant. Then $a_0 < 0$ and since $(v_0^2 + a_0)/v_0^2 > 0$ we have that $0 < r_0 = \sqrt{1 + a_0/v_0^2} < 1$ which means that there exists a natural number N such that for $n > N$ we have $0 < r_n < 1$ and hence $r_n^2 < 1$ and hence we can take again $v_n = \sqrt{a_0/(r_n^2 - 1)}$. Then clearly $v_n \to v_0$ and we obtain a sequence of points (a_0, v_n) sitting on parabolas $a_0 = (r_n^2 - 1)v_n^2$ with r_n rational. And $v_0^2 + a_0 = r_n^2 > 0$. Since $v_0 > 0$ then there is a natural number M such that for all $n > M$ $v_n > 0$.

□

Considering $r = m_1/m_2$ where $m_1, m_2 \in \mathbb{N}$ we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2}\left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (G) when $a = (1 + (m_1/m_2)^2)v^2$. Consider

$$\mathscr{F}_{(c_1,c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have that $[1:0]$ and $[0:1]$ are remarkable values for $\mathscr{I}$, since

$$\mathscr{F}_{(1,0)} = J_1^{m_2} J_3^{m_1},\ \ \mathscr{F}_{(0,1)} = -J_2^{m_2} J_4^{m_1}.$$

The case $m_1 = m_2 = 1$ is when $a = 0$ and this case was done previously. Suppose $m_1 \neq 1$ or $m_2 \neq 1$. If $m_1 > 1$ then $[1:0]$ and $[0:1]$ are two critical remarkable values for $\mathscr{I}$ and J_3, J_4 are critical remarkable curves. If follows from the Main Theorem in (CHAVARRIGA *et al.*, 2003) that these are the unique critical remarkable values of $\mathscr{I}$. If we also have $m_2 > 1$ then J_1, J_2 also are critical remarkable curves.

Observation 129. Note that if $r < 0$ then we can suppose that $r = -m_1/m_2$ where $m_1, m_2 \in \mathbb{N}$ and

$$I = \left(\frac{J_1}{J_2}\right)^{m_2}\left(\frac{J_3}{J_4}\right)^{-m_1} = \left(\frac{J_1}{J_2}\right)^{m_2}\left(\frac{J_4}{J_3}\right)^{m_1}.$$

Considering $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^{m_2} J_4^{m_1} - c_2 J_2^{m_2} J_3^{m_1} = 0$ we still have the same conclusions as before.

6.1.6 Geometric Analysis of family (H)

Consider the family

$$(H)\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \frac{4xy}{3} + \frac{y^2}{3}, \end{cases}$$

where $a \neq 0$.

This is a two parameter family depending on a and v such that $a \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (H) we study here also the limit case $a = 0$ where the equations are still defined.

We display below the full geometric analysis of the systems in this family, which is endowed with at least three invariant algebraic curves. In the **generic case**

$$av(a-v^2)(a-3v^2/4)(a+3v^2)(a-8v^2/9) \neq 0$$

the systems have only two invariant lines J_1 and J_2 and only two invariant hyperbolas J_3 and J_4 with cofactors α_i, $1 \leq i \leq 4$ given by

$$\begin{array}{ll} J_1 = -3\sqrt{-a+v^2} - x + y, & \alpha_1 = \sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_2 = 3\sqrt{-a+v^2} - x + y, & \alpha_2 = -\sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_3 = -3a + 3vx - x^2 + xy, & \alpha_3 = -v - \frac{2x}{3} - \frac{y}{3}, \\ J_4 = -3a - 3vx - x^2 + xy, & \alpha_4 = v - \frac{2x}{3} - \frac{y}{3}. \end{array}$$

Then according to Darboux' theorem we must have a Darboux first integral. We note that if $v = 0$ then the two hyperbolas coincide and we get a double hyperbola. Also if $a = v^2$ the two lines coincide and we get a double line. So to have four distinct curves we need to put $v(a - v^2) \neq 0$. We inquire when we could have an additional line. Calculations yield that this happens when $(a - 3v^2/4) = 0$. We also inquire when we could have an additional hyperbola. Calculations yield that this happens when $(a + 3v^2)(a - 8v^2/9) = 0$. The multiplicities of each invariant line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $av(a-v^2)(a-3v^2/4)(a+3v^2)(a-8v^2/9) \neq 0$.

Table 175 – Invariant curves, cofactors, singularities and intersection points of family (H) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1=-3\sqrt{-a+v^2}-x+y$ $J_2=3\sqrt{-a+v^2}-x+y$ $J_3=-3a+3vx-x^2+xy$ $J_4=-3a-3vx-x^2+xy$ $\alpha_1=\sqrt{-a+v^2}-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=\sqrt{-a+v^2}-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-v-\frac{2x}{3}-\frac{y}{3}$ $\alpha_4=v-\frac{2x}{3}-\frac{y}{3}$	$P_1=(-v-\sqrt{v^2-a},-v+2\sqrt{v^2-a})$ $P_2=(v-\sqrt{v^2-a},v+2\sqrt{v^2-a})$ $P_3=(-v+\sqrt{v^2-a},-v-2\sqrt{v^2-a})$ $P_4=(v+\sqrt{v^2-a},v-2\sqrt{v^2-a})$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ For $v^2>a$ we have $n,s,s,n;N,N,S$ if $v>0$ $s,n,n,s;N,N,S$ if $v<0$ For $v^2<a$ we have ©,©,©,©;N,N,S	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_2 \text{ simple}\end{cases}$ $\bar{J}_1\cap\bar{J}_4=\begin{cases}P_2^\infty \text{ simple}\\ P_1 \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_4 \text{ simple}\end{cases}$ $\bar{J}_2\cap\bar{J}_4=\begin{cases}P_2^\infty \text{ simple}\\ P_3 \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\bar{J}_4=\begin{cases}P_1^\infty \text{ triple}\\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_3\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_4\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 176 – Divisors and zero-cycles of family (H) for the generic case.

Divisor and zero-cycles	Degree
$ICD=\begin{cases}J_1+J_2+J_3+J_4+\mathscr{L}_\infty \text{ if } v^2>a\\ J_1^C+J_2^C+J_3+J_4+\mathscr{L}_\infty \text{ if } v^2<a\end{cases}$	5 5
$M_{0CS}=\begin{cases}P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } v^2>a\\ P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } v^2<a\end{cases}$	7 7
$T=Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4=0$	7
$M_{0CT}=\begin{cases}2P_1+2P_2+2P_3+2P_4+3P_1^\infty+5P_2^\infty+P_3^\infty \text{ if } v^2>a\\ 2P_1^C+2P_2^C+2P_3^C+2P_4^C+3P_1^\infty+5P_2^\infty+P_3^\infty \text{ if } v^2<a\end{cases}$	17 17

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) five distinct tangents at P_2^∞.

Table 177 – First integral and integrating factor of family (H) for the generic case.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1\sqrt{v^2-a}}{v}} J_4^{-\frac{\lambda_1\sqrt{v^2-a}}{v}}$	$R = J_1^{\lambda_1} J_2^{-\lambda_1-2} J_3^{\frac{(\lambda_1+1)\sqrt{v^2-a}}{v}-1} J_4^{-\frac{(\lambda_1+1)\sqrt{v^2-a}}{v}-1}$
Simple example	$\mathscr{I} = \frac{J_1}{J_2}\left(\frac{J_3}{J_4}\right)^{\frac{\sqrt{v^2-a}}{v}}$	$\mathscr{R} = \frac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

(ii) The non-generic case: $av(a-v^2)(a-3v^2/4)(a+3v^2)(a-8v^2/9) = 0$.

(ii.1) $v = 0$ and $a \neq 0$.

Here the two hyperbolas coalesce yielding a double hyperbola so we compute the exponential factor E_4.

Table 178 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (H) when $v = 0$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -3i\sqrt{a}+x-y$ $J_2 = 3i\sqrt{a}+x-y$ $J_3 = -3a+x(y-x)$ $E_4 = e^{\frac{g_1 x}{-3a+x(y-x)}}$ $\alpha_1 = -i\sqrt{a}-\frac{x}{3}+\frac{y}{3}$ $\alpha_2 = i\sqrt{a}-\frac{x}{3}+\frac{y}{3}$ $\alpha_3 = -\frac{2x}{3}-\frac{y}{3}$ $\alpha_4 = -\frac{g_1}{3}$	$P_1 = (-i\sqrt{a}, 2i\sqrt{a})$ $P_2 = (i\sqrt{a}, -2i\sqrt{a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $sn_{(2)}, sn_{(2)}; N, N, S$ For $a > 0$ we have $©_{(2)}, ©_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 179 – Divisor and zero-cycles of family (H) when $v = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + 2J_3 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + 2J_3 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	7
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a < 0 \\ 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	15 15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) only four distinct tangents at P_2^∞, but one of them is double.

Table 180 – First integral and integrating factor of family (H) when $v = 0$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{6i\sqrt{a}\lambda_1}{g_1}}$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-2} E_4^{-\frac{6i\sqrt{a}(1+\lambda_1)}{g_1}}$
Simple example	$\mathscr{I} = \dfrac{J_1}{J_2 E_4{}^{6i\sqrt{a}}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3^2}$

Source: Elaborated by the author.

(ii.2) $a = v^2$ and $a \neq 0$.

Here the two lines coalesce yielding a double line so we compute the exponential factor E_4.

Table 181 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (H) when $a = v^2$ and $a \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -3v^2 + 3vx - x^2 + xy$ $J_3 = -3v^2 - 3vx - x^2 + xy$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{x}{3} + \frac{y}{3}$ $\alpha_2 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_3 = v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (-v, -v)$ $P_2 = (v, v)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ triple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 182 – Divisor and zero-cycles of family (H) when $a = v^2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2\bar{J}_3 = 0$	7
$M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$	15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 and at P_2, but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) only four distinct tangents at P_2^∞, but one of them is double.

Table 183 – First integral and integrating factor of family (H) when $a = v^2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\lambda_2} E_4^{\frac{6v\lambda_2}{g_0}}$	$R = J_1^{-2} J_2^{\lambda_2} J_3^{-2-\lambda_2} E_4^{\frac{6v(1+\lambda_2)}{g_0}}$
Simple example	$\mathscr{I} = \dfrac{J_2 E_4^{6v}}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1^2 J_2 J_3}$

Source: Elaborated by the author.

(ii.3) $a = 3v^2/4$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant line. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 184 – Invariant curves, cofactors, singularities and intersection points of family (H) when $a = 3v^2/4$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\frac{3v}{2} + x - y$ $J_2 = \frac{3v}{2} + x - y$ $J_3 = y$ $J_4 = -\frac{x^2}{3v} + \frac{xy}{3v} - \frac{3v}{4} + x$ $J_5 = \frac{x^2}{3v} - \frac{xy}{3v} + \frac{3v}{4} + x$ $\alpha_1 = \frac{1}{6}(-3v - 2x + 2y)$ $\alpha_2 = \frac{1}{6}(3v - 2x + 2y)$ $\alpha_3 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_4 = \frac{1}{3}(-3v - 2x - y)$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{3v}{2}, 0\right)$ $P_2 = \left(-\frac{v}{2}, -2v\right)$ $P_3 = \left(\frac{v}{2}, 2v\right)$ $P_4 = \left(\frac{3v}{2}, 0\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_4$ double $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 185 – Divisor and zero-cycles of family (H) when $a = 3v^2/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$	20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_4), but one of them is double,

2) only two distinct tangents at P_1^∞, but one of them is double and

3) five distinct tangents at P_2^∞.

Table 186 – First integral and integrating factor of family (H) when $a = 3v^2/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}} J_4^{\frac{\lambda_2}{2}} J_5^{\frac{\lambda_1}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-1-\frac{\lambda_1}{2}-\frac{\lambda_2}{2}} J_4^{-\frac{1}{2}+\frac{\lambda_2}{2}} J_5^{-\frac{1}{2}+\frac{\lambda_1}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2 J_5}{J_3}$ $\mathscr{I}_2 = \dfrac{J_2^2 J_4}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 130. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_5 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : 9v^2/2]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,9v^2/2)} = -J_2^2 J_4, \;\; \mathscr{F}^1_{(1,0)} = J_1^2 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : 9v^2/2]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_3 for $\mathscr{F}^1_{(1,9v^2/2)}$ and P_2, P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2^2 J_4 - c_2 J_3$ we have the same remarkable values $[1 : 9v^2/2]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. However, the singular point are P_1, P_3 for $\mathscr{F}^2_{(1,0)}$ and P_2, P_4 for $\mathscr{F}^2_{(1,9v^2/2)}$.

(ii.4) $a = -3v^2$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant hyperbola. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 187 – Invariant curves, cofactors, singularities and intersection points of family (H) when $a=-3v^2$ and $a\neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1=-6v+x-y$ $J_2=6v+x-y$ $J_3=9v^2+xy$ $J_4=9v^2+3vx-x^2+xy$ $J_5=9v^2-3vx-x^2+xy$ $\alpha_1=-2v-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=2v-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-\frac{5x}{3}-\frac{y}{3}$ $\alpha_4=-v-\frac{2x}{3}-\frac{y}{3}$ $\alpha_5=v-\frac{2x}{3}-\frac{y}{3}$	$P_1=(-3v,3v)$ $P_2=(-v,5v)$ $P_3=(v,-5v)$ $P_4=(3v,-3v)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ $n,s,s,n;N,N,S$	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=P_4$ double $\bar{J}_1\cap\bar{J}_4=\begin{cases}P_2^\infty \text{ simple}\\ P_4 \text{ simple}\end{cases}$ $\bar{J}_1\cap\bar{J}_5=\begin{cases}P_2^\infty \text{ simple}\\ P_3 \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=P_1$ double $\bar{J}_2\cap\bar{J}_4=\begin{cases}P_2^\infty \text{ simple}\\ P_2 \text{ simple}\end{cases}$ $\bar{J}_2\cap\bar{J}_5=\begin{cases}P_2^\infty \text{ simple}\\ P_1 \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\bar{J}_4=\begin{cases}P_1^\infty \text{ triple}\\ P_4 \text{ simple}\end{cases}$ $\bar{J}_3\cap\bar{J}_5=\begin{cases}P_1^\infty \text{ triple}\\ P_1 \text{ simple}\end{cases}$ $\bar{J}_3\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_3^\infty \text{ simple}\end{cases}$ $\bar{J}_4\cap\bar{J}_5=\begin{cases}P_1^\infty \text{ triple}\\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_4\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$ $\bar{J}_5\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 188 – Divisor and zero-cycles of family (H) when $a=-3v^2$ and $a\neq 0$.

Divisor and zero-cycles	Degree
$ICD=J_1+J_2+J_3+J_4+J_5+\mathscr{L}_\infty$	6
$M_{0CS}=P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty$	7
$T=Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5=0$	9
$M_{0CT}=3P_1+2P_2+2P_3+3P_4+4P_1^\infty+5P_2^\infty+2P_3^\infty$	21

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_4), but one of them is double,

2) only two distinct tangents at P_1^∞, but one of them is triple,

3) only four tangents at P_2^∞, but one of them is double.

Table 189 – First integral and integrating factor of family (H) when $a = -3v^2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{1+2\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_5^2}{J_3}$ $\mathscr{I}_2 = \dfrac{J_2 J_4^2}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 131. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_5^2 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -108v^3]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-108v^3)} = J_2 J_4^2, \quad \mathscr{F}^1_{(1,0)} = J_1 J_5^2.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : -108v^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_4, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_2, P_4 for $\mathscr{F}^1_{(1,-108v^3)}$ and P_1, P_3 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_4^2 - c_2 J_3$ we have the remarkable values $[1 : 108v^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_1, P_3 for $\mathscr{F}^2_{(1,108v^3)}$ and P_2, P_4 for $\mathscr{F}^2_{(1,0)}$.

(ii.5) $a = 8v^2/9$ and $a \neq 0$.

Here we have, apart from two lines and two hyperbolas, an additional invariant hyperbola. Then, we have five invariant algebraic curves and according to Jouanolou's theorem the corresponding system has a rational first integral.

Table 190 – Invariant curves, cofactors, singularities and intersection points of family (H) when $a = 8v^2/9$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -v + x - y$ $J_2 = v + x - y$ $J_3 = y(x-y) - \frac{v^2}{3}$ $J_4 = -\frac{8v^2}{3} + 3vx + x(y-x)$ $J_5 = -\frac{8v^2}{3} - 3vx + x(y-x)$ $\alpha_1 = \frac{1}{3}(-v - x + y)$ $\alpha_2 = \frac{1}{3}(v - x + y)$ $\alpha_3 = \frac{2y}{3} - \frac{5x}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{4v}{3}, -\frac{v}{3}\right)$ $P_2 = \left(-\frac{2v}{3}, -\frac{5v}{3}\right)$ $P_3 = \left(\frac{2v}{3}, \frac{5v}{3}\right)$ $P_4 = \left(\frac{4v}{3}, \frac{v}{3}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 191 – Divisor and zero-cycles of family (H) when $a = 8v^2/9$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$	21

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_4), but one of them is double,

2) only two distinct tangents at P_1^∞, but one of them is double and

3) six distinct tangents at P_2^∞.

Table 192 – First integral and integrating factor of family (H) when $a = 8v^2/9$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}} J_4^{\frac{\lambda_2}{3}} J_5^{\frac{\lambda_1}{3}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}-\frac{2}{3}} J_4^{\frac{\lambda_2}{3}-\frac{2}{3}} J_5^{\frac{\lambda_1}{3}-\frac{2}{3}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^3 J_5}{J_3}$ $\mathscr{I}_2 = \dfrac{J_2^3 J_4}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 132. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^3 J_5 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -16v^3]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-16v^3)} = J_2^3 J_4, \quad \mathscr{F}^1_{(1,0)} = J_1^3 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : -16v^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_3 for $\mathscr{F}^1_{(1,-16v^3)}$ and P_2, P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curves $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2^3 J_4 - c_2 J_3$ we have the remarkable values $[1 : 16v^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_1, P_3 for $\mathscr{F}^2_{(1,0)}$ and P_2, P_4 for $\mathscr{F}^2_{(1,16v^3)}$.

(ii.6) $a = 0$ and $v \neq 0$.

Under this condition, systems (H) do not belong to **QSH**. The affine invariant lines are $x = 0$ and $\pm 3v - x + y = 0$ that are all simple. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. Perturbing this system in the family (H) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics $x(-3v - x + y) = 0$ and $x(3v - x + y) = 0$ we can produce two distinct hyperbolas $-3a - 3vx - x^2 + xy = 0$ and $-3a + 3vx - x^2 + xy = 0$, respectively. Furthermore, the cubic $x(3v - x + y)(-3v - x + y) = 0$ has integrable multiplicity two.

Table 193 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (H) when $a=0$ and $v\neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1=-3v-x+y$ $J_2=3v-x+y$ $J_3=x$ $E_4=e^{-\frac{6g_0\left(6v^2+x(y-x)\right)+g_1x\left((x-y)^2-9v^2\right)}{2x(-3v+x-y)(3v+x-y)}}$ $\alpha_1=v-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=-v-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-\frac{x}{3}-\frac{2y}{3}$ $\alpha_4=g_0$	$P_1=(0,-3v)$ $P_2=(2v,-v)$ $P_3=(-2v,v)$ $P_4=(0,3v)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ For $v\neq 0$ we have $s,n,n,s;N,N,S$	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=P_4$ simple $\bar{J}_1\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=P_1$ simple $\bar{J}_2\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\mathscr{L}_\infty=P_1^\infty$ simple

Source: Elaborated by the author.

Table 194 – Divisor and zero-cycles of family (H) when $a=0$ and $v\neq 0$.

Divisor and zero-cycles	Degree
$ICD=J_1+J_2+J_3+\mathscr{L}_\infty$ if $v\neq 0$	4
$M_{0CS}=P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty$ if $v\neq 0$	7
$T=Z\bar{J}_1\bar{J}_2\bar{J}_3=0.$	4
$M_{0CT}=2P_1+P_2+P_3+2P_4+2P_1^\infty+3P_2^\infty+P_3^\infty$ if $v\neq 0$	12

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_2^∞.

Table 195 – First integral and integrating factor of family (H) when $a=0$ and $v\neq 0$.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_2^{-\lambda_1}J_3^0E_4^{-\frac{2\lambda_1 v}{g_0}}$	$R=J_1^{\lambda_1}J_2^{-4-\lambda_1}J_3^{-2}E_4^{-\frac{2v(\lambda_1+2)}{g_0}}$
Simple example	$\mathscr{I}=\dfrac{J_1}{J_2E_4^{2v}}$	$\mathscr{R}=\dfrac{1}{J_1^2J_2^2J_3^2}$

Source: Elaborated by the author.

(ii.7) $a=v=0$.

Under this condition, systems (H) do not belong to **QSH**. The affine invariant lines are $x=0$, $y=0$ that are both simple and $x-y=0$ that is double. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. This system has a rational first integral that foliates the plane into quartic invariant algebraic curves. The lines $x = 0$ and $x - y = 0$ are remarkable curves. Perturbing this system in the full family (H) we can obtain up to ten distinct configurations of lines and hyperbolas. By perturbing the reducible conic $x(x-y) = 0$ we can produce 2 distinct hyperbolas $-3a + 3vx - x^2 + xy = 0$ and $-3a - 3vx - x^2 + xy = 0$. Perturbing the reducible conic $y(x-y) = 0$ we can produce a third hyperbola $y(x-y) - \frac{v^2}{3} = 0$ and by perturbing $xy = 0$ we can produce the hyperbola $9v^2 + xy = 0$. We get a double hyperbola $-3a + x(y-x) = 0$ by perturbing the double reducible conic $x^2(x-y)^2 = 0$.

Table 196 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (H) when $a = v = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_2 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_3 = \frac{y}{3} - \frac{x}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 197 – Divisor and zero-cycles of family (H) when $a = v = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	10

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double;

2) only two distinct tangentes at P_2^∞, but one of them is double.

Table 198 – First integral and integrating factor of family (H) when $a = v = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-3\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-4-3\lambda_1} E_4^0$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_3^3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 133. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_2 J_3^3.$$

Therefore, J_2, J_3 are remarkable curves of $\mathscr{I}_1$, $[0:1]$ is the only critical remarkable values of $\mathscr{I}_1$ and J_3 is critical remarkable curve of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (H) and we also confront our results with previous results in the literature in the following proposition.

Proposition 134. (a) For the family (H) we have nine distinct configurations $C_1^{(\mathrm{H})} - C_9^{(\mathrm{H})}$ of invariant hyperbolas and lines (see Figure 14 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $av(a-v^2)(a+3v^2)(a-3v^2/4)(a-8v^2/9) = 0$. Its complement is a union of 12 disjoint sets. On $v(a-v^2) = 0$ one of the algebraic solutions is double. On $(a-3v^2/4)(a+3v^2)(a-8v^2/8) = 0$ we have an additional line or an additional hyperbola. For the limiting set of the parameter space, i.e. on $a = 0$ the invariant hyperbolas become reducible producing the lines $x = 0$, $-3v - x + y = 0$ and $x = 0$, $3v - x + y = 0$. The configuration $C_9^{(\mathrm{H})}$ is not equivalent with anyone of the configurations in (OLIVEIRA *et al.*, 2017).

(b) The family (H) is Darboux integrable if $av(a-v^2)(a-3v^2/4)(a+3v^2)(a-8v^2/9) \neq 0$. When $v(a-v^2) = 0$ the family (H) is generalized Darboux integrable. In all the following three cases, we have a rational first integral. If $a = 3v^2/4$ then the systems have an additional invariant line and the plane is foliated into quartic invariant algebraic curves. If $a = -3v^2$ the plane is foliated by quintic invariant algebraic curves. If $a = 8v^2/9$ then the systems have an additional invariant hyperbola and the plane is foliated in quintic invariant algebraic curves. The remarkable curves are J_1, J_2, J_4, J_5 for these three algebraically integrable cases of family (H) for each case correspondingly. All systems in family (H) have an inverse integrating factor which is polynomial.

(c) For the family (H) we have five topologically distinct phase portraits $P_1^{(\mathrm{H})} - P_5^{(\mathrm{H})}$. The topological bifurcation diagram of family (H) is done in Figure 15. The bifurcation set is the line $a = 0$, the parabola $a = v^2$ and the half line $v = 0$ with $a < 0$. The line $a = 0$ is a bifurcation of separatrix from saddle to saddle connection. The parabola $a = v^2$ and the half line $v = 0$ with $a < 0$ is are bifurcation sets of singularities.

Proof of Proposition 134:

(a) We have the following types of divisors and zero-cycles of the total invariant curve T for the configurations of family (H) :

Table 199 – Configurations for family (H).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(H)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_2^{(H)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_3^{(H)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_4^{(H)}$	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_5^{(H)}$	$ICD = J_1^C + J_2^C + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_6^{(H)}$	$ICD = 2J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_7^{(H)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_8^{(H)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_9^{(H)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Although $C_1^{(H)}$ and $C_2^{(H)}$ admit the same type of divisors and zero-cycles we can see they are different because in $C_1^{(H)}$ each branch of the hyperbolas intersects one line while $C_2^{(H)}$ have two branches intersecting both lines and two branches intersecting no line. Therefore, the configurations $C_1^{(H)}$ up to $C_9^{(H)}$ are all distinct. For the limit cases of family (H) we have the following configurations:

Table 200 – Configurations for the limit cases of family (H).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
c_2	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

In (OLIVEIRA *et al.*, 2017) the authors presented all the normal forms for **QSH** and they also give all the configurations for each normal form. However we notice one configuration missing in the study of (3.97) (here family (H)). The configurations they gave in this study was H.142 (which is $C_3^{(\mathrm{H})}$), H.137 (which is $C_1^{(\mathrm{H})}$), H.138 (which is $C_2^{(\mathrm{H})}$), H.152 (which is $C_6^{(\mathrm{H})}$), H.149 (which is $C_7^{(\mathrm{H})}$), H.155 (which is $C_5^{(\mathrm{H})}$), H.154 (which is $C_4^{(\mathrm{H})}$) and H.158 (which is $C_8^{(\mathrm{H})}$). Therefore, they missed in the listing of configurations for systems (3.97) the configuration $C_9^{(\mathrm{H})}$.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 130, 131 and 132 .

(c) We have that:

Table 201 – Phase portraits for family (H).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(\mathrm{H})}$	(N,N,S)	(n,s,s,n)	$2SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(\mathrm{H})}$	(N,N,S)	(n,s,s,n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(\mathrm{H})}$	(N,N,S)	$(\copyright,\copyright,\copyright,\copyright)$ $(\copyright_{(2)},\copyright_{(2)})$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(\mathrm{H})}$	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_5^{(\mathrm{H})}$	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have five distinct phase portraits for systems (H). For the limit cases of family (H) we have the following phase portraits:

Table 202 – Phase portraits for the limit cases of family (H).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	(n,s,s,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
p_2	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

We note that the phase portraits $P_1^{(H)} \cong_{top} P_1^{(B)}$, $P_3^{(H)} \cong_{top} P_3^{(B)}$, $P_4^{(H)} \cong_{top} P_4^{(E)}$, $P_5^{(H)} \cong_{top} p_{2\ (E)}$ (where p_2 is a phase portrait in the limit cases of family (E)) and $p_1 \cong_{top} P_4^{(C)}$ are missing in (LLIBRE; YU, 2018) and they were listed in the geometric studies of families (B), (C) and (E). We also have that $P_1^{(H)}$ is missing in (CAIRÓ; FEIX; LLIBRE, 1999) and it was listed in the geometric study of family (B).

□

Figure 14 – Bifurcation diagram of configurations for family (H)

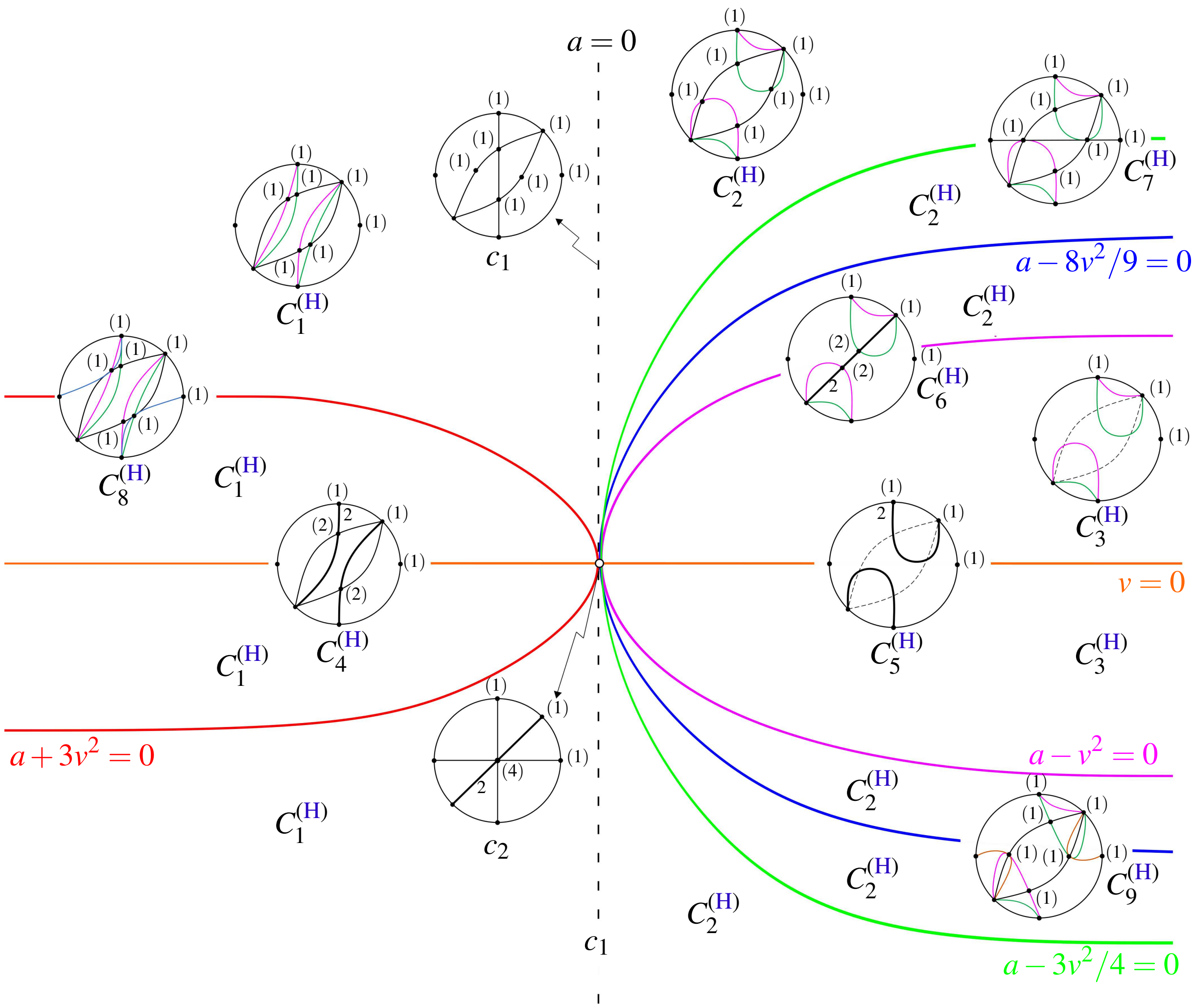
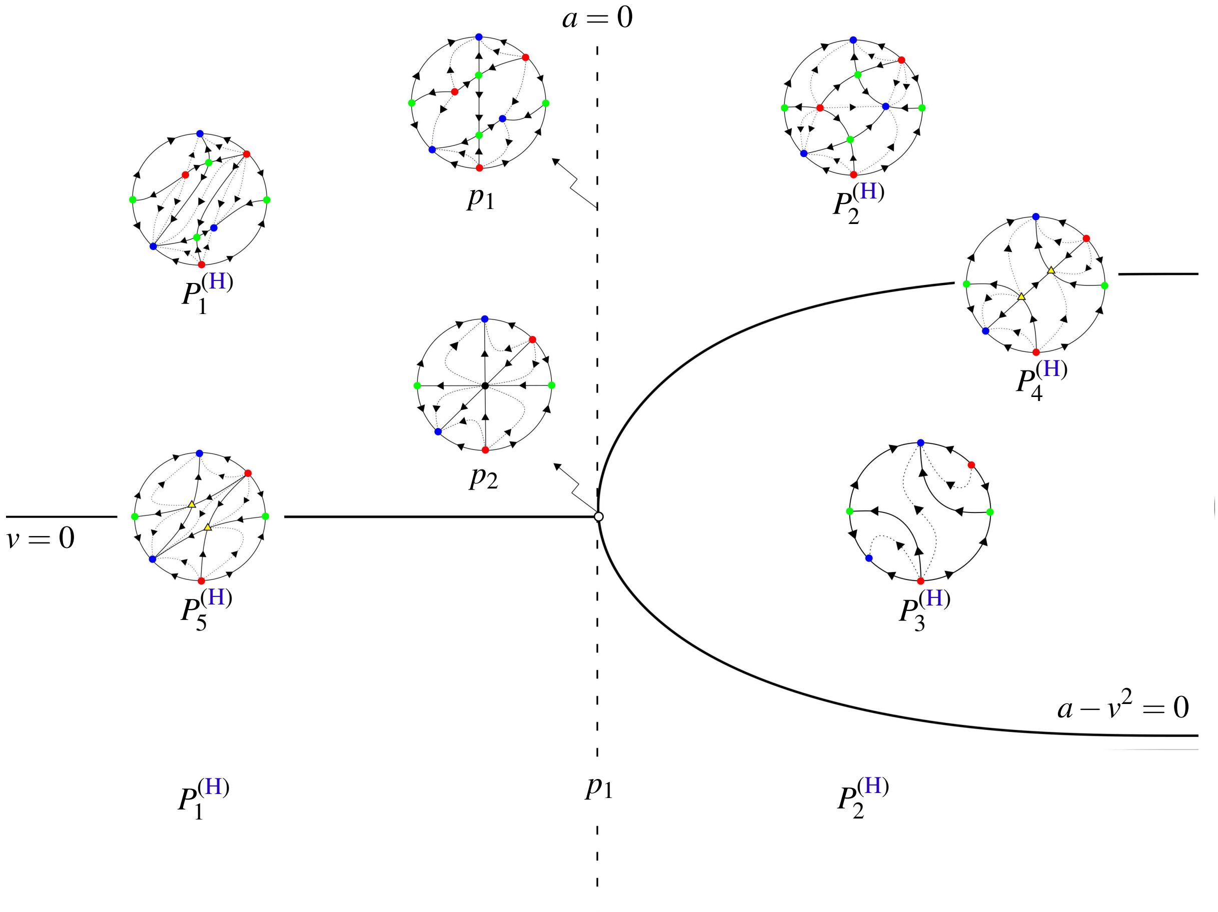


Figure 15 – Topological bifurcation diagram for family (H).



6.1.6.1 The solution of the Poincaré problem for the family (H)

We can recognize when a system in this family has a rational first integral. The following is the answer to Poincaré's problem for the family (H):

Theorem 135. i) A necessary and sufficient condition for a system in family (H) to have a rational first integral given by invariant algebraic curves of degree at most two, is that $v^2-a>0$ and that (a,v) be situated on a parabola of the form $a=(1-r^2)v^2$ with $r\in\mathbb{Q}$. ii) The set of all points (a,v)'s satisfying these two conditions is dense in the set $v^2-a>0$ with $v\neq 0$.

Proof. i) We first prove that the condition is necessary. So assume that we have a system of parameters (a,v) that has a rational first integral. Assume now that (a,v) is in the generic situations $v(a-v^2)(a+3v^2)(a-8v^2/9)(a-3v^2/4)\neq 0$. Any first integral of the system is then of the following general form:

$$I=J_1^{\lambda_1}J_2^{-\lambda_1}J_3^{\frac{\lambda_1\sqrt{v^2-a}}{v}}J_4^{-\frac{\lambda_1\sqrt{v^2-a}}{v}}.$$

This is a rational first integral if and only if $\lambda_1 \in \mathbb{Z}$ and $\frac{\lambda_1\sqrt{v^2-a}}{v} \in \mathbb{Z}$ in which case we must have that $r = \sqrt{v^2-a}/v$ must be a rational number. In view of our generic hypothesis $r \neq 0$. Since $r = \sqrt{v^2-a}/v$ is rational we have $v^2 - a \geq 0$ and by hypothesis $v^2 - a \neq 0$. Therefore $v^2 - a > 0$. We also have $a = (1-r^2)v^2$ and therefore the condition is necessary in this case. Consider now the case when $v(a-v^2)(a+3v^2)(a-8v^2/9)(a-3v^2/4) = 0$. Since on $v(a-v^2) = 0$ we cannot have a rational first integral because as we see in the tables for these two cases, we have exponential factors in the first integrals and hence we must have $v(a-v^2) \neq 0$. Therefore our previous assumption is reduced to $(a+3v^2)(a-8v^2/9)(a-3v^2/4) = 0$. Suppose first that the point (a,v) is located on the parabola $a = -3v^2$. Then this parabola can be written as $a = (1-r^2)v^2$ where $r = 2$. We then have $v^2 - a = r^2v^2 = 4v^2 > 0$. If the point (a,v) is on the parabola $a - 8y^2/9 = 0$ then this parabola can be written as $a = (1-r^2)v^2$ for $r = 1/3$. Here again we have that $v^2 - a = r^2v^2 = v^2/9 > 0$. So the system situated on the parabola $a - 8y^2/9 = 0$ satisfies $v^2 - a > 0$ and for $r = 1/3$ the point is located on the parabola $a = (1-r^2)v^2$. So also in this case these conditions are necessary. There remains only the case when (a,v) is on the parabola $a - 3v^2/4 = 0$. In this case we can write this parabola as $a = (1-r^2)v^2$ by taking $r = 1/2$. Also here $v^2 - a = r^2v^2 = v^2/4 > 0$, i.e. $v^2 - a > 0$. So the necessity of the conditions is proved in this case too.

We now prove the sufficiency of the conditions. Let us assume that $v^2 - a > 0$, $v \neq 0$ and (a,v) is located on a parabola $a = (1-r^2)v^2$ with $r \in \mathbb{Q}$. Then clearly $r \neq 0$, otherwise $v^2 - a = r^2v^2 = 0$ contrary to our assumption. In case $r = 2, 1/3, 1/2$ we are on one of the three parabolas obtained from the condition $(a+3v^2)(a-8v^2/9)(a-3v^2/4) = 0$ and for these parabolas the tables give us rational first integrals. If the generic condition is satisfied, i.e. $v(a-v^2)(a+3v^2)(a-8v^2/9)(a-3v^2/4) \neq 0$, then we know that we have the corresponding first integral indicated in the Tables for this case where the exponents for the curves J_i are λ_1 and $\lambda_1\sqrt{v^2-a}/v$. But we know by our assumption that (a,v) is located on a parabola $a = (1-r^2)v^2$ for some rational number r. From this equation we have that $r^2 = (a-v^2)/v^2$. Hence $r = \sqrt{v^2-a}/v$ is rational. We may suppose $r = m/n$ with $m,n \in \mathbb{Z}$ and m,n coprime. Then by taking in the general expression of the first integral $\lambda_1 = n$ and $r = \sqrt{v^2-a}/v$ we obtained a rational first integral in this case.

ii) Let us denote by $\mathscr{P}_r$ the parabola corresponding to a rational number r, i.e.

$$\mathscr{P}_r := \{(a,v) \in \mathbb{R}^2 : (1-r^2)v^2 = a\}.$$

Thus a system in the family (H) has a rational first integral if and only if it corresponds to a point (a,v) such that $v^2 - a > 0$ with $v \neq 0$ and the point is situated on a parabola $\mathscr{P}_r$ for some rational number r. In the parameter plane $\mathbb{R}^2$ let the $a-$axis be the horizontal line and the $v-$axis be the vertical one. The parabolas $a = (1-r^2)v^2$ are symmetric with respect to the $a-$axis. Because of this it would suffice to prove the density of points (a,v) on parabolas $\mathscr{P}_r$ and inside $v^2 - a > 0$ and $v > 0$.

Claim: The set of all points in $A =: \cup_{r\in\mathbb{Q}} \mathscr{P}_r$ with $v > 0$ is dense in the set $S^+ = \{(a,v) : v^2 - a > 0,\ v > 0\}$.

Take an arbitrary point point $p_0 = (a_0, v_0) \in S^+$. So we have $v_0^2 - a_0 > 0$ and $v_0 > 0$. We only need to consider p_0 inside the first or second quadrant. Indeed the line $a = 0$ is outside the parameter space of our family. So $a_0 \neq 0$. In view of our assumption we have that $(v_0^2 - a_0)/v_0^2 > 0$. So let $r_0 = \sqrt{(v_0^2 - a_0)/v_0^2} > 0$. Hence we have $a_0 = (1 - r_0^2)v_0^2$. Here r_0 is not necessarily a rational number. But it can be approximated with rational numbers. So take a sequence of rational numbers r_n such that $r_n \to r_0$. At this point let us assume that the point (a_0, v_0) is in the second quadrant, i.e. $a_0 < 0$. In this case $r_0 > 1$ and since $r_n \to r_0$ there exists a number N such that for $n > N$ $r_n > 1$ and hence $r_n^2 > 1$ for all $n > N$. So $\sqrt{a_0/(1 - r_n^2)} > 0$. Denote by $v_n = \sqrt{a_0/(1 - r_n^2)}$. Denote by $v_n = \sqrt{a_0/(1 - r_n^2)}$. Then $v_n \to v_0$ and hence $(a_0, v_n) \to (a_0, v_0)$. But each point (a_0, v_n) is located on the corresponding parabola $a_0 = (1 - r_n^2)v_n^2$ and hence p_0 is an accumulation point of points situated on such parabolas with r_n rational. Assume now that the point p_0 is in the first quadrant. Then $a_0 > 0$ and since $(v_0^2 - a_0)/v_0^2 > 0$ we have that $0 < r_0 = \sqrt{1 - a_0/v_0^2} < 1$ which means that there exists a natural number N such that for $n > N$ we have $0 < r_n < 1$ and hence $r_n^2 < 1$ and hence we can take again $v_n = \sqrt{a_0/(1 - r_n^2)}$. Then clearly $v_n \to v_0$ and we obtain a sequence of points (a_0, v_n) sitting on parabolas $a_0 = (1 - r_n^2)v_n^2$ with r_n rational. And $v_0^2 - a_0 = r_n^2 > 0$. Since $v_0 > 0$ then there is a natural number M such that for all $n > M$ $v_n > 0$.

□

Considering $r = m_1/m_2$ where $m_1, m_2 \in \mathbb{N}$ we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (H) when $a = (1 - (m_1/m_2)^2)v^2$. Consider

$$\mathscr{F}_{(c_1,c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have that $[1:0]$ and $[0:1]$ are remarkable values for $\mathscr{I}$, since

$$\mathscr{F}_{(1,0)} = -J_1^{m_2} J_3^{m_1}, \quad \mathscr{F}_{(0,1)} = -J_2^{m_2} J_4^{m_1}.$$

The case $m_1 = m_2 = 1$ is when $a = 0$ and this case was done previously. Suppose $m_1 \neq 1$ or $m_2 \neq 1$. If $m_1 > 1$ then $[1:0]$ and $[0:1]$ are two critical remarkable values for $\mathscr{I}$ and J_3, J_4 are critical remarkable curves. If follows from the Main Theorem in (CHAVARRIGA *et al.*, 2003) that these are the unique critical remarkable values of $\mathscr{I}$. If we also have $m_2 > 1$ then J_1, J_2 also are critical remarkable curves.

Observation 136. Note that if $r < 0$ then we can suppose that $r = -m_1/m_2$ where $m_1, m_2 \in \mathbb{N}$ and

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{-m_1} = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_4}{J_3}\right)^{m_1}.$$

Considering $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^{m_2} J_4^{m_1} - c_2 J_2^{m_2} J_3^{m_1} = 0$ we still have the same conclusions as before.

There are some additional remarkable curves when $a = (1-(m_1/m_2)^2)v^2$ for especial values of m_1 and m_2, see examples in Appendix A. We could find among these examples curves of degree $5, 6, 7, 8, 10, 12$ etc.

6.1.7 Geometric Analysis of Family (I)

Consider the family

$$\text{(I)} \quad \begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 5a - \frac{4xy}{3} + \frac{y^2}{3}, \end{cases}$$

where $a \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (I) we study here also the limit case $a = 0$ where the equations are still defined.

Every system in the family (I) is endowed with five invariant algebraic curves: two lines J_1, J_2 and three hyperbolas J_3, J_4, J_5 with cofactors α_i, $1 \leq i \leq 5$ given by

$$\begin{array}{l}
J_1 = -2i\sqrt{3}\sqrt{a} + x - y, \;\; \alpha_1 = -\frac{2i\sqrt{a}}{\sqrt{3}} - \frac{x}{3} + \frac{y}{3}, \\
J_2 = 2i\sqrt{3}\sqrt{a} + x - y, \;\; \alpha_2 = \frac{2i\sqrt{a}}{\sqrt{3}} - \frac{x}{3} + \frac{y}{3}, \\
J_3 = -3a + xy, \;\; \alpha_3 = -\frac{5x}{3} - \frac{y}{3}, \\
J_4 = i\sqrt{3}\sqrt{a}x - 3a + x(y-x), \;\; \alpha_4 = -\frac{i\sqrt{a}}{\sqrt{3}} - \frac{2x}{3} - \frac{y}{3}, \\
J_5 = -i\sqrt{3}\sqrt{a}x - 3a + x(y-x), \;\; \alpha_5 = \frac{i\sqrt{a}}{\sqrt{3}} - \frac{2x}{3} - \frac{y}{3}.
\end{array}$$

Since the number of invariant curve is five, these systems are algebraically integrable. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) $a \neq 0$.

Table 203 – Invariant curves, cofactors, singularities and intersection points of family (I) when $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -2i\sqrt{3}\sqrt{a}+x-y$ $J_2 = 2i\sqrt{3}\sqrt{a}+x-y$ $J_3 = -3a+xy$ $J_4 = i\sqrt{3}\sqrt{a}x-3a+x(y-x)$ $J_5 = -i\sqrt{3}\sqrt{a}x-3a+x(y-x)$ $\alpha_1 = -\frac{2i\sqrt{a}}{\sqrt{3}}-\frac{x}{3}+\frac{y}{3}$ $\alpha_2 = \frac{2i\sqrt{a}}{\sqrt{3}}-\frac{x}{3}+\frac{y}{3}$ $\alpha_3 = -\frac{5x}{3}-\frac{y}{3}$ $\alpha_4 = -\frac{i\sqrt{a}}{\sqrt{3}}-\frac{2x}{3}-\frac{y}{3}$ $\alpha_5 = \frac{i\sqrt{a}}{\sqrt{3}}-\frac{2x}{3}-\frac{y}{3}$	$P_1 = \left(-\frac{i\sqrt{a}}{\sqrt{3}}, \frac{5i\sqrt{a}}{\sqrt{3}}\right)$ $P_2 = \left(\frac{i\sqrt{a}}{\sqrt{3}}, -\frac{5i\sqrt{a}}{\sqrt{3}}\right)$ $P_3 = \left(-i\sqrt{3}\sqrt{a}, i\sqrt{3}\sqrt{a}\right)$ $P_4 = \left(i\sqrt{3}\sqrt{a}, -i\sqrt{3}\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a<0$ we have $s,s,n,n;N,N,S$ For $a<0$ we have ©,©,©,©;N,N,S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ double $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 204 – Divisor and zero-cycles of family (I) when $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1+J_2+J_3+J_4+J_5+\mathscr{L}_\infty \text{ if } a<0 \\ J_1^C+J_2^C+J_3+J_4^C+J_5^C+\mathscr{L}_\infty \text{ if } a>0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a<0 \\ P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a>0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = \begin{cases} 2P_1+2P_2+3P_3+3P_4+4P_1^\infty+5P_2^\infty+2P_3^\infty \text{ if } a<0 \\ 2P_1^C+2P_2^C+3P_3^C+3P_4^C+4P_1^\infty+5P_2^\infty+2P_3^\infty \text{ if } a>0 \end{cases}$	21 21

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_3 (and at P_4), but one of them is double,

2) only two distinct tangents at P_1^∞, but one of them is triple,

2) only four distinct tangents at P_2^∞, but one of them is double.

Table 205 – First integral and integrating factor of family (I) when $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{1+2\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_5^2}{J_3} \quad \mathscr{I}_2 = \dfrac{J_2 J_4^2}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4 J_5}$

Source: Elaborated by the author.

Observation 137. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_5^2 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 5$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : 12i\sqrt{3}a^{3/2}]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1:12i\sqrt{3}a^{3/2})} = J_2 J_4^2, \quad \mathscr{F}^1_{(1,0)} = J_1 J_5^2.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathscr{I}_1$, $[1 : 12i\sqrt{3}a^{3/2}]$ and $[1 : 0]$ are the only two critical remarkable values and J_4, J_5 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1, P_4 for $\mathscr{F}^1_{(1:12i\sqrt{3}a^{3/2})}$ and P_2, P_3 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_4^2 - c_2 J_3$ we have the remarkable values $[1 : -12i\sqrt{3}a^{3/2}]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_2, P_3 for $\mathscr{F}^2_{(1:-12i\sqrt{3}a^{3/2})}$ and P_1, P_4 for $\mathscr{F}^2_{(1,0)}$.

(ii) $a = 0$.

Under this condition, systems (I) do not belong to **QSH**. The affine invariant lines are $x = 0$, $y = 0$ that are both simple and $x - y = 0$ that is double. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. This system has a rational first integral that foliates the plane into quartic invariant algebraic curves. The lines $x = 0$ and $x - y = 0$ are remarkable curves.

Table 206 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (I) when $a = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{4x}{3} + \frac{y}{3}$ $\alpha_2 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_3 = -\frac{x}{3} + \frac{y}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 207 – Divisor and zero-cycles of family (I) when $a = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double;

2) only two distinct tangents at P_2^∞

Table 208 – First integral and integrating factor of family (I) when $a = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-3\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-4-3\lambda_1} E_4^0$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_3^3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 138. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^3 = 0$, $\deg \mathscr{F}_1 = 4$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_2 J_3^2.$$

Therefore, J_2, J_3 are remarkable curves of $\mathscr{I}_1$, $[0:1]$ is the only critical remarkable values of $\mathscr{I}_1$ and J_3 is critical remarkable curve of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (I) in the following proposition.

Proposition 139. (a) For the family (I) we have two distinct configurations $C_1^{(I)}$ and $C_2^{(I)}$ of invariant hyperbolas and lines (see Figure 16 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space contains only the point $a = 0$. Its complement is a union of 2 disjoint sets. For the limiting set of the parameter space, i.e. on $a = 0$ the invariant hyperbolas become reducible.

(b) The family (I) have a rational first integral and the plane is foliated into quintic invariant algebraic curves. The remarkable curves are J_1, J_2, J_4, J_5 for family (I). All systems in family (I) have an inverse integrating factor which is polynomial.

(c) For the family (I) we have two topologically distinct phase portraits $P_1^{(I)}$ and $P_2^{(I)}$. The topological bifurcation diagram of family (I) is done in Figure 17. The bifurcation set is the point $a = 0$ and it is a bifurcation of singularities.

Proof of Proposition 139.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (I):

Table 209 – Configurations for family (I).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(I)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 3P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_2^{(I)}$	$ICD = J_1^C + J_2^C + J_3 + J_4^C + J_5^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 3P_3^C + 3P_4^C + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(I)}$ and $C_2^{(I)}$ are distinct. For the limit case of family (I) we have the following configuration:

Table 210 – Configuration for the limit case of family (I).

Configuration	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

(b) It follows directly from Jouanolou's theorem that we always have a rational first integral for family (I) since we have five invariant algebraic curves. The computations for the remarkable curves were done in Remark 137. The other statement follows from the study done previously.

(c) We have that:

Table 211 – Phase portraits for family (I).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(I)}$	(N,N,S)	(n,s,s,n)	$2SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(I)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have two distinct phase portraits for systems (I). For the limit case of family (I) we have the following phase portrait:

Table 212 – Phase portrait for the limit case of family (I).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

We note that the phase portraits $P_1^{(I)} \cong_{top} P_1^{(B)}$ and $P_2^{(I)} \cong_{top} P_3^{(B)}$ are missing in (LLIBRE; YU, 2018) and they were listed in the geometric study of family (B). We also have that $P_1^{(I)}$ is missing in (CAIRÓ; FEIX; LLIBRE, 1999) and it was listed in the geometric study of family (G).

□

Figure 16 – Bifurcation diagram of configurations for family (I).

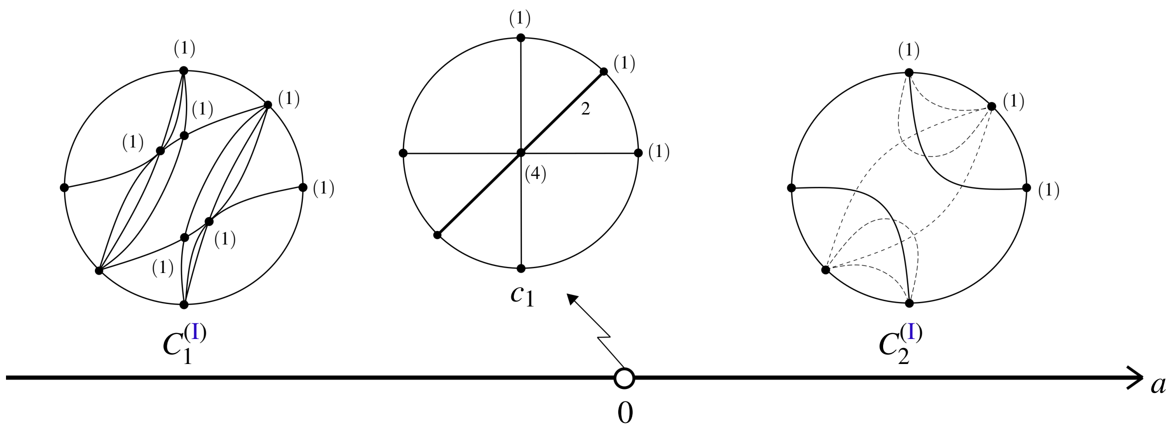
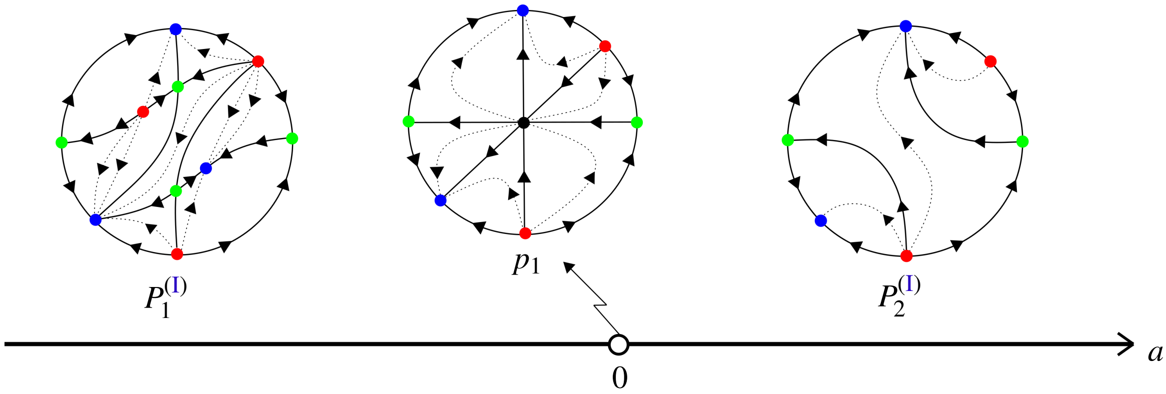


Figure 17 – Topological bifurcation diagram for family (I).



6.1.8 Geometric Analysis of Family (J)

Consider the family

$$(J)\ \begin{cases} \dot{x} = -\frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b - \frac{3xy}{2} + \frac{y^2}{2}, \end{cases}$$

where $b \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (J) we study here also the limit case $b = 0$ where the equations are still defined.

Every system in the family (J) is endowed with five invariant algebraic curves: three lines J_1, J_2, J_3 and two hyperbolas J_4, J_5 with cofactors α_i, $1 \leq i \leq 5$ given by

$$\begin{array}{ll} J_1 = -i\sqrt{2}\sqrt{b} - x + y, & \alpha_1 = \frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}, \\ J_2 = i\sqrt{2}\sqrt{b} - x + y, & \alpha_2 = -\frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}, \\ J_3 = x, & \alpha_3 = -\frac{x}{2} - \frac{y}{2}, \\ J_4 = x(y - x) - b, & \alpha_4 = -x, \\ J_5 = xy - \frac{b}{2}, & \alpha_5 = -2x. \end{array}$$

Considering the line at infinity $Z = 0$ the total multiplicity of invariant lines is four so this family was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

(i) $b \neq 0$.

Table 213 – Invariant curves, cofactors, singularities and intersection points of family (J) when $b \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -i\sqrt{2}\sqrt{b} - x + y$ $J_2 = i\sqrt{2}\sqrt{b} - x + y$ $J_3 = x$ $J_4 = x(y-x) - b$ $J_5 = xy - \frac{b}{2}$ $\alpha_1 = \frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = -\frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_3 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -x$ $\alpha_5 = -2x$	$P_1 = \left(\frac{i\sqrt{b}}{\sqrt{2}}, -\frac{i\sqrt{b}}{\sqrt{2}}\right)$ $P_2 = \left(-\frac{i\sqrt{b}}{\sqrt{2}}, \frac{i\sqrt{b}}{\sqrt{2}}\right)$ $P_3 = \left(0, -i\sqrt{2}\sqrt{b}\right)$ $P_4 = \left(0, i\sqrt{2}\sqrt{b}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $b<0$ we have $n, n, s, s; N, N, S$ For $b>0$ we have ©, ©, ©, ©; N, N, S	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = P_2$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = P_1$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_3 \cap \bar{J}_5 = P_1^\infty$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1, P_2 \text{ simple} \\ P_1^\infty \text{ double} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 214 – Divisor and zero-cycles of family (J) when $b \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } b<0 \\ J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } b>0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b<0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b>0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } b<0 \\ 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } b<0 \end{cases}$	20 20

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double,

2) only three distinct tangents at P_1^∞, but one of them is double and

3) four distinct tangents at P_2^∞.

Table 215 – First integral and integrating factor of family (J) when $b \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1-\frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1-\frac{\lambda_4}{2}-\frac{3}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_4^2}{J_5} \quad \mathscr{I}_2 = \dfrac{J_1 J_2 J_3^2}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 140. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_4^2 - c_2 J_5 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -2b]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-2b)} = J_1 J_2 J_3^2, \quad \mathscr{F}^1_{(1,0)} = J_4^2.$$

Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathscr{I}_1$, $[1 : -2b]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathscr{I}_1$ and J_3, J_4 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_3, P_4 for $\mathscr{F}^1_{(1,-2b)}$ and P_1, P_2 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curves $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_1 J_2 J_3^2 - c_2 J_5 = 0$ we have the remarkable values $[1 : 2b]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_3, J_4. The singular points are P_1, P_2 for $\mathscr{F}^2_{(1,2b)}$ and P_3, P_4 for $\mathscr{F}^2_{(1,0)}$.

(ii) $b = 0$.

Under this condition, the system (J) does not belong to **QSH**. The affine invariant lines are $x = 0$, $y = 0$ that are both simple and $x - y = 0$ that is double so we compute the exponential factor E_4. This system has a rational first integral that foliates the plane into cubic invariant algebraic curves. The lines $x = 0$ and $x - y = 0$ are remarkable curves.

Table 216 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (J) when $b = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{2} - \frac{3x}{2}$ $\alpha_2 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_3 = \frac{y}{2} - \frac{x}{2}$ $\alpha_4 = \frac{g_0}{2}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 217 – Divisor and zero-cycles of family (J) when $b = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	11

Source: Elaborated by the author.

where the total curve T has

i) only three distinct tangents at P_1, but one of them is double;

ii) only two distinct tangents at P_2^∞, but one of them is double.

Table 218 – First integral and integrating factor of family (J) when $b = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-2\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-3-2\lambda_1} E_4^0$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2 J_3^2} \quad \mathscr{I}_2 = \dfrac{J_2 J_3^2}{J_1}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 141. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_2 J_3^2.$$

Therefore, J_2, J_3 are remarkable curves of $\mathscr{I}_1$, $[1:0]$ is the only critical remarkable values of $\mathscr{I}_1$ and J_3 is critical remarkable curve of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(0,1)}$. Considering the first integral $\mathscr{I}_2$ with its associated curves $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_3^2 - c_2 J_1$ we have the remarkable value $[1:0]$ and the same remarkable curves J_2, J_3. The singular point is P_1 for $\mathscr{F}^2_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (J).

Proposition 142. (a) For the family (J) we have two distinct configurations $C_1^{(J)}$ and $C_2^{(J)}$ of invariant hyperbolas and lines (see Figure 18 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space contains only the point $b = 0$. Its complement is a union of 2 disjoint sets. For the limiting set of the parameter space, i.e. on $b = 0$ the invariant hyperbolas become reducible.

(b) The family (J) have a rational first integral and the plane is foliated into quartic invariant algebraic curves. The remarkable curves are J_1, J_2, J_3, J_4 for family (J). All systems in family (J) have an inverse integrating factor which is polynomial.

(c) For the family (J) we have two topologically distinct phase portraits $P_1^{(J)}$ and $P_2^{(J)}$. The topological bifurcation diagram in the full parameter space is done in Figure 19. The bifurcation set of singularities is the point $b = 0$ and it is a bifurcation of singularities.

Proof of Proposition 142.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (J):

Table 219 – Configurations for family (J).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(J)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_2^{(J)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 3P_2^C + 2P_3^C + 2P_4^C + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(J)}$ and $C_2^{(J)}$ are distinct. For the limit case of family (J) we have the following configuration:

Table 220 – Configuration for the limit case of family (J).

Configuration	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

Source: Elaborated by the author.

(b) It follows directly from Jouanolou's theorem that we always have a rational first integral for family (J) since we have five invariant algebraic curves. The computations for the remarkable curves were done in Remark 140. The other statement follows from the study done previously.

(c) We have:

Table 221 – Phase portraits for family (J).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(J)}$	(N,N,S)	(n,s,s,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(J)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have two distinct phase portraits for systems (J). For the limit case of family (J) we have the following phase portrait:

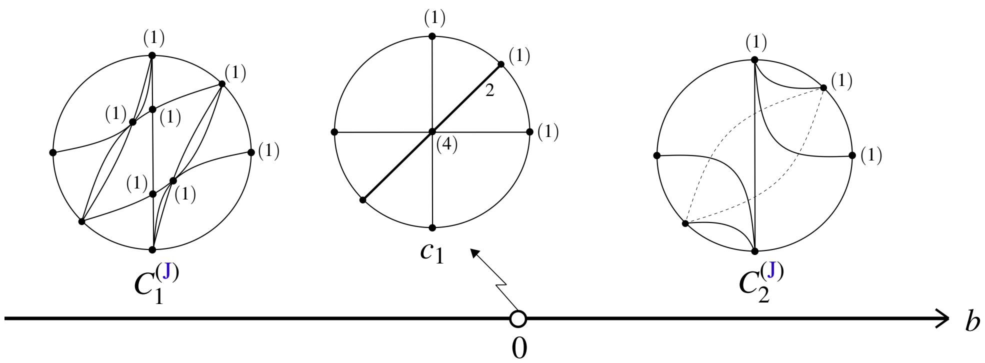
Table 222 – Phase portrait for the limit case of family (J).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

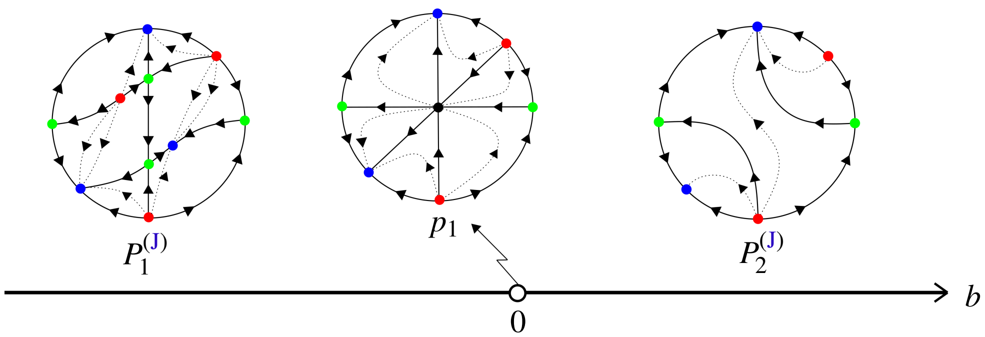
We note that the phase portraits $P_1^{(J)} \cong_{top} P_4^{(C)}$ and $P_2^{(J)} \cong_{top} P_3^{(B)}$ are missing in (LLIBRE; YU, 2018) and they were listed in the geometric study of families (B) and (C).

Figure 18 – Bifurcation diagram of configurations for family (J).



Source: Elaborated by the author.

Figure 19 – Topological bifurcation diagram for family (J).



Source: Elaborated by the author.

6.1.9 Geometric Analysis of Family (K)

Consider the family

$$(K) \begin{cases} \dot{x} = 4b - 1 + 4y + x^2 \\ \dot{y} = b + y^2, \end{cases}$$

where $b \neq -1$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (K) we study here also the limit case $b = -1$ where the equations are still defined. We display below the full geometric analysis of the systems in this family, which is endowed with at least two invariant algebraic curves. When $b(b+1) \neq 0$ the systems have two invariant lines J_1, J_2 and one invariant hyperbola J_3 with cofactors α_i, $1 \leq i \leq 3$ given by

$$\begin{aligned} &J_1 = 1 - \tfrac{iy}{\sqrt{b}}, \ \ \alpha_1 = y - i\sqrt{b}, \\ &J_2 = 1 + \tfrac{iy}{\sqrt{b}}, \ \ \alpha_2 = y + i\sqrt{b}, \\ &J_3 = (-1+b) - x + 3y + xy - y^2, \ \ \alpha_3 = -1 + x - 2y. \end{aligned}$$

We note that when $b = 0$ the two lines coalesce yielding a multiple line. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $b(b+1/4)(b+1) \neq 0$.

Table 223 – Invariant curves, cofactors, singularities and intersection points of family (K) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{\sqrt{b}}$ $J_2 = 1 + \frac{iy}{\sqrt{b}}$ $J_3 = (-1+b) - x + 3y + xy - y^2$ $\alpha_1 = y - i\sqrt{b}$ $\alpha_2 = y + i\sqrt{b}$ $\alpha_3 = -1 + x + 2y$	$P_1 = \left(-\sqrt{-4b+4i\sqrt{b}+1}, -i\sqrt{b}\right)$ $P_2 = \left(\sqrt{-4b+4i\sqrt{b}+1}, -i\sqrt{b}\right)$ $P_3 = \left(-\sqrt{-(2\sqrt{b}+i)^2}, i\sqrt{b}\right)$ $P_4 = \left(\sqrt{-(2\sqrt{b}+i)^2}, i\sqrt{b}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $b < 0$ we have $s, n, n, s; N, S, N$ For $b > 0$ we have $©, ©, ©, ©; N, S, N$	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_3 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 224 – Divisor and zero-cycles of family (K) for the generic case.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathscr{L}_\infty \text{ if } b < 0 \\ J_1^C + J_2^C + J_3 + \mathscr{L}_\infty \text{ if } b > 0 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} P_1 + 2P_2 + 2P_3 + P_4 + P_1^\infty + 2P_2^\infty + 4P_3^\infty \text{ if } b < 0 \\ P_1^C + 2P_2^C + 2P_3^C + P_4^C + P_1^\infty + 2P_2^\infty + 4P_3^\infty \text{ if } a > 0 \end{cases}$	13 13

Source: Elaborated by the author.

where the total curve T has four distinct tangents at P_3^∞.

Observation 143. Mathematica could not give a response for the computation of the first integral of family (K) in the generic case.

Table 225 – Integrating factor of family (K) for the generic case.

	Integrating Factor
General	$R = J_1^{-\frac{-\sqrt{b}+i}{\sqrt{b}}} J_2^{-\frac{-\sqrt{b}-i}{\sqrt{b}}} J_3^{-2}$
Simple example	$\mathscr{R} = J_1^{-\frac{-\sqrt{b}+i}{\sqrt{b}}} J_2^{-\frac{-\sqrt{b}-i}{\sqrt{b}}} J_3^{-2}$

Source: Elaborated by the author.

(ii) The non-generic cases: $b(b+1/4)(b+1) = 0$

(ii.1) $b = -\frac{1}{4}$.

As in the generic case, we have two invariant lines and one invariant hyperbola, but here we have the coalescence of two finite singular points yielding to a semi-hyperbolic singularity.

Table 226 – Invariant curves, cofactors, singularities and intersection points of family (K) when $b = -\frac{1}{4}$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - 2y$ $J_2 = 1 + 2y$ $J_3 = -\frac{5}{4} - x + 3y + xy - y^2$ $E_4 = e^{\frac{G(x,y)}{4x(y-1)-4(y-3)y-5}}$ $\alpha_1 = y + \frac{1}{2}$ $\alpha_2 = y - \frac{1}{2}$ $\alpha_3 = -1 + x + 2y$ $\alpha_4 = 8(2y+1)(4g_0 - 5g_1)$	$P_1 = \left(0, \frac{1}{2}\right)$ $P_2 = \left(-2, -\frac{1}{2}\right)$ $P_3 = \left(2, -\frac{1}{2}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $sn_{(2)}, n, s; N, S, N$	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

where $G(x,y) = 4\left(-4g_0(4x+51)y^2 + 4g_0(31x+41)y + g_0 + 4g_1 y(x(5y-39)+64y-52) + g_1 x\right)$ with $g_0, g_1 \in \mathbb{C}$.

Table 227 – Divisor and zero-cycles of family (K) when $b = -\frac{1}{4}$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = 2P_1 + P_2 + P_3 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = 2P_1 + 2P_2 + P_3 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$	12

Source: Elaborated by the author.

where the total curve T has four distinct tangents at P_3^∞.

Table 228 – First integral and integrating factor of family (K) when $b = -\frac{1}{4}$.

	First integral	Integrating Factor
General	$I = J_1^{-16(4g_0\lambda_4 - 5g_1\lambda_4)} J_2^0 J_3^0 E_4^{\lambda_4}$	$R = J_1^{-1-64g_0\lambda_4+80g_1\lambda_4} J_2^3 J_3^{-2} E_4^{\lambda_4}$
Simple example	$\mathscr{I} = J_1^{16} e^{\frac{4\left(4x(y-8)y+x+52y^2-44y+1\right)}{4x(y-1)-4(y-3)y-5}}$	$\mathscr{R} = J_1^{-1} J_2^3 J_3^{-2}$

Source: Elaborated by the author.

(ii.2) $b = 0$.

Here the two lines J_1 and J_2 of the generic case coalesce yielding a double line so we compute the exponential factor E_3.

Table 229 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (K) when $b=0$.

Inv.cur./Exp.Fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = -1+-x+3y+xy-y^2$ $E_3 = e^{\frac{g_0+g_1 y}{y}}$ $\alpha_1 = y$ $\alpha_2 = -1+x+2y$ $\alpha_3 = -g_0$	$P_1 = (-1,0)$ $P_2 = (1,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; N, S, N$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 230 – Divisor and zero-cycles of family (K) when $b=0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + \mathscr{L}_\infty$	4
$M_{0CS} = 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2 = 0$	5
$M_{0CT} = 3P_1 + 2P_2 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$	12

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1, but one of them is double;

2) only one double tangent at P_2 and

3) only three distinct tangents at P_3^∞, but one of them is double.

Table 231 – First integral and integrating factor of family (K) when $b=0$.

	First integral	Integrating Factor
General	I	$R = J_1^2 J_2^{-2} E_3^{\frac{2}{g_0}}$
Simple example	$\mathscr{I}$	$\mathscr{R} = J_1^2 J_2^{-2} E_3^2$

Source: Elaborated by the author.

$$I = \mathscr{I} = \left(-4e^{-2/y}E_i\left(\frac{2}{y}\right) - \frac{y(x(y-2)+5y-2)}{-xy+x+y^2-3y+1}\right)\left(-\left(e^{\frac{g_0}{y}+g_1}\right)^{2/g_0}\right)$$

where $E_i(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral function that has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.

(ii.3) $b = -1$.

Under this condition, systems (K) do not belong to **QSH**. The affine invariant lines are $1+y=0$, $1-y=0$ and $2+x-y=0$ that are simple. We also could find an exponential factor.

Table 232 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (K) when $b = -1$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = 1+y$ $J_2 = 1-y$ $J_3 = 2+x-y$ $E_4 = e^{\frac{g_0(x+10)y^2 - g_0(5x+7)y + g_0 + g_1 y(x(9-2y)-19y+11) + g_1 x}{(y-1)(-x+y-2)}}$ $\alpha_1 = -1+y$ $\alpha_2 = 1+y$ $\alpha_3 = -2+x+y$ $\alpha_4 = 4(y+1)(g_0 - 2g_1)$	$P_1 = (-3,-1)$ $P_2 = (-1,1)$ $P_3 = (1,1)$ $P_4 = (3,-1)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n,s,n,s;N,S,N$	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_2$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 233 – Divisor and zero-cycles of family (K) when $b = -1$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0.$	4
$M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + P_1^\infty + 2P_2^\infty + 3P_3^\infty$	12

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_3^∞.

Table 234 – First integral and integrating factor of family (K) when $b=-1$.

	First integral	Integrating Factor
General	$I=J_1^0 J_2^{-4(g_0\lambda_4-2g_1\lambda_4)} J_3^0 E_4^{\lambda_4}$	$R=R=J_1^2 J_2^{-2(1+2g_0\lambda_4-4g_1\lambda_4)} J_3^{-2} E_4^{\lambda_4}$
Simple example	$\mathscr{I}=(y-1)^4 e^{-\frac{-(x+9)y^2+4(x+1)y+x+1}{(y-1)(x-y+2)}}$	$\mathscr{R}=J_1^2 J_2^{-2} J_3^{-2}$

Source: Elaborated by the author.

We sum up the topological, dynamical and algebraic geometric features of family (K) and we also confront our results with previous results in the literature in the following proposition.

Proposition 144. (a) For the family (K) we obtained five distinct configurations $C_1^{(K)}-C_5^{(K)}$ of invariant hyperbolas and lines (see Figure 20 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $b(b+1)(b+1/4)=0$. Its complement is the union of 4 disjoint sets. On $b=-1/4$ we have the coalescence of two of the four finite singular points producing a saddle-node which has multiplicity two. On $b=0$ we have the coalescence of two lines yielding a double line. For the limiting set of the parameter space, i.e. on $b=-1$ the invariant hyperbola becomes reducible producing the lines $2+x-y=0$ and $1-y=0$. In this case, we also have the invariant affine line $1+y=0$ and the invariant line at infinity. The configuration c_1 appearing on the limite case $b=-1$ of systems (K) is not equivalent with anyone of the configurations in (SCHLOMIUK; VULPE, 2008c).

(b) The family (K) is Liouvillian integrable when $(b+1/4)(b+1)\neq 0$. When $b=-1/4$ the family (K) is generalized Darboux integrable.

(c) For the family (K) we have five topologically distinct phase portraits $P_1^{(K)}-P_5^{(K)}$. The topological bifurcation diagram in the full parameter space is done in Figure 21 . The bifurcation set of singularities is $b(b+1)(b+1/4)=0$. The points $b=-1/4$ and $b=0$ are bifurcation of singularities and the point $b=-1$ is a bifurcation of separatrices from saddle to saddle connection. The phase portraits $P_2^{(K)}$, $P_4^{(K)}$ and $P_5^{(K)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of proposition 144:

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (K):

Table 235 – Configurations for family (K).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(K)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + 2P_2 + 2P_3 + P_4 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$
$C_2^{(K)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + 2P_2 + 2P_3 + P_4 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$
$C_3^{(K)}$	$ICD = J_1^C + J_2^C + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + 2P_2^C + 2P_3^C + P_4^C + P_1^\infty + 2P_2^\infty + 4P_3^\infty$
$C_4^{(K)}$	$ICD = 2J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$
$C_5^{(K)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_1^\infty + 2P_2^\infty + 4P_3^\infty$

Source: Elaborated by the author.

Although $C_1^{(K)}$ and $C_2^{(K)}$ admit the same type of divisors and zero-cycles we can see they are different because in $C_1^{(K)}$ each branch of the hyperbola intersects one line while $C_2^{(K)}$ have one branch of the hyperbola intersecting both lines and the other branch does not intersect any line. Therefore, the configurations $C_1^{(K)}$ up to $C_5^{(K)}$ are distinct. For the limit case of family (K) we have the following configuration:

Table 236 – Configuration for the limit case of family (K).

Configuration	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + P_1^\infty + 2P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

In (SCHLOMIUK; VULPE, 2008c) the authors presented all the configurations and phase portraits for real quadratic differential system having invariant lines of total multiplicity four and a finite set of singulatirities at infinity. However, considering the system defined by the equations (K) when $b = -1$ we have three affine invariant lines that are simple and we also have the line at infinity as an invariant line, which is also simple. So the total multiplicity of the invariant lines is four but we could not find the configuration c_1 in their work.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables.

(c) We have:

Table 237 – Phase portraits for family (K).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(K)}$	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(K)}$	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$
$P_3^{(K)}$	(N,S,N)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(K)}$	(N,S,N)	$(sn_{(2)},sn_{(2)})$	$1SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$
$P_5^{(K)}$	(N,S,N)	$(sn_{(2)},n,s)$	$3SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have five distinct phase portraits for systems (K). For the limit case of family (K) we have the following phase portrait:

Table 238 – Phase portrait for the limit case of family (K).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,S,N)	(n,s,n,s)	$4SC_f^f$ $5SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Table 239 – Phase portraits in (LLIBRE; YU, 2018) that admit 3 singular points at infinity with the type (N,S,N) that possess 0, 2, 3 or 4 real singular points in the finite plane.

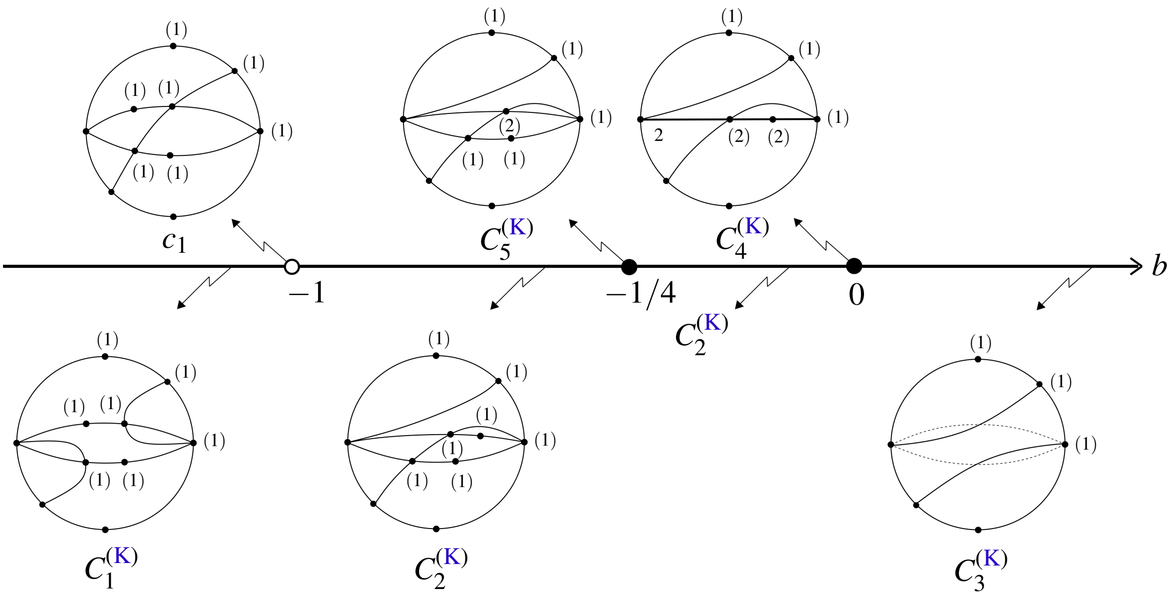
Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
L_{31},L_{32}	(N,S,N)	(s,es)	$2C_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_1,R_2	(N,S,N)	(s,c)	$1C_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
R_{01,ω_6}	(N,S,N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_{8,ω_1}	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, the phase portraits $P_2^{(K)}$, $P_4^{(K)}$ and $P_5^{(K)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018). We also note that the phase portrait $P_3^{(K)} \cong_{top} P_3^{(B)}$ is missing in (LLIBRE; YU, 2018) and it was listed in the geometric study of family (B).

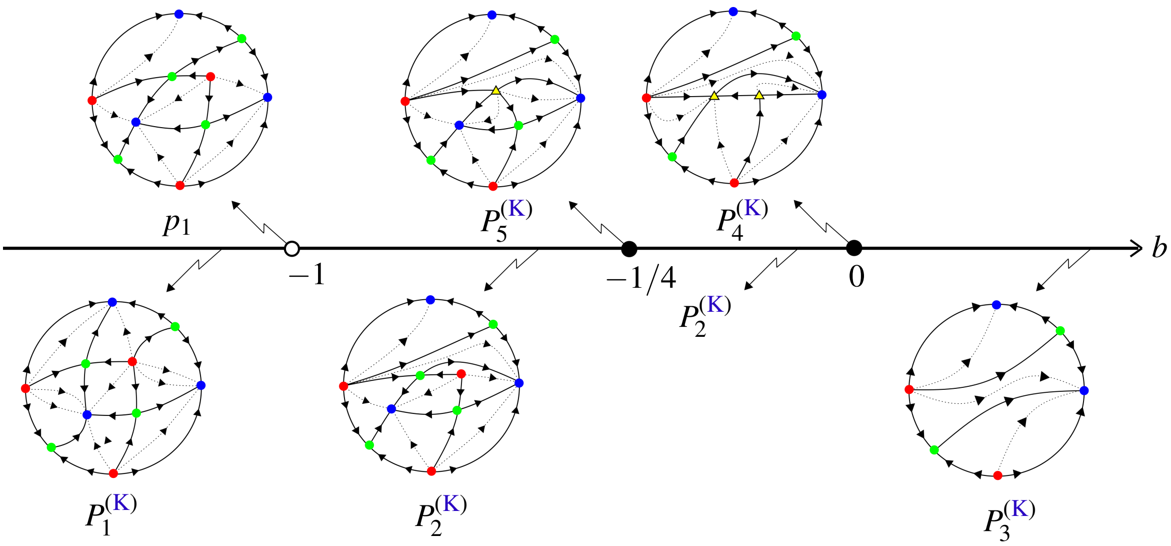
□

Figure 20 – Bifurcation diagram of configurations for family (K).



Source: Elaborated by the author.

Figure 21 – Topological bifurcation diagram for family (K).



Source: Elaborated by the author.

6.1.10 Geometric Analysis of Family (L)

Consider the family

$$(L) \begin{cases} \dot{x} = a + x^2 \\ \dot{y} = 4a + y^2, \end{cases}$$

where $a \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (L) we study here also the limit case $a = 0$ where the equations are still defined. Every system in family (L) is endowed with five invariant algebraic curves: four lines J_1, J_2, J_3, J_4 and one hyperbola J_5 with cofactors α_i, $1 \le i \le 5$ given by

$$\begin{aligned} J_1 &= 1 - \tfrac{iy}{2\sqrt{a}}, \quad \alpha_1 = y - 2i\sqrt{a},\\ J_2 &= 1 + \tfrac{iy}{2\sqrt{a}}, \quad \alpha_2 = y + 2i\sqrt{a},\\ J_3 &= 1 - \tfrac{ix}{\sqrt{a}}, \quad \alpha_3 = x - i\sqrt{a},\\ J_4 &= 1 + \tfrac{ix}{\sqrt{a}}, \quad \alpha_4 = x + i\sqrt{a},\\ J_5 &= a - x^2 + xy, \quad \alpha_5 = 2x + y. \end{aligned}$$

Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. Since the number of invariant curve is five, it follows by Jouanolou's theorem that these systems are algebraically integrable. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

(i) $a \neq 0$.

Table 240 – Invariant curves, cofactors, singularities and intersection points of family (L) when $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{2\sqrt{a}}$ $J_2 = 1 + \frac{iy}{2\sqrt{a}}$ $J_3 = 1 - \frac{ix}{\sqrt{a}}$ $J_4 = 1 + \frac{ix}{\sqrt{a}}$ $J_5 = a - x^2 + xy$ $\alpha_1 = y - 2i\sqrt{a}$ $\alpha_2 = y + 2i\sqrt{a}$ $\alpha_3 = x - i\sqrt{a}$ $\alpha_4 = x + i\sqrt{a}$ $\alpha_5 = 2x + y$	$P_1 = \left(-i\sqrt{a}, -2i\sqrt{a}\right)$ $P_2 = \left(-i\sqrt{a}, 2i\sqrt{a}\right)$ $P_3 = \left(i\sqrt{a}, -2i\sqrt{a}\right)$ $P_4 = \left(i\sqrt{a}, 2i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $n, s, s, n; N, S, N$ For $a > 0$ we have ©, ©, ©, ©; N, S, N	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \bar{J}_4 = P_3$ simple $\bar{J}_1 \cap \bar{J}_5 = P_1$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_2$ simple $\bar{J}_2 \cap \bar{J}_4 = P_4$ simple $\bar{J}_2 \cap \bar{J}_5 = P_4$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_1 \text{ simple} \\ P_1^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_4 \text{ simple} \\ P_1^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_5 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 241 – Divisor and zero-cycles of family (L) when $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1+J_2+J_3+J_4+J_5+\mathscr{L}_\infty \text{ if } a<0 \\ J_1^C+J_2^C+J_3^C+J_4^C+J_5+\mathscr{L}_\infty \text{ if } a>0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1+P_2+P_3+P_4+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a<0 \\ P_1^C+P_2^C+P_3^C+P_4^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a>0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	7
$M_{0CT} = \begin{cases} 3P_1+2P_2+2P_3+3P_4+4P_1^\infty+2P_2^\infty+3P_3^\infty \text{ if } a<0 \\ 3P_1^C+2P_2^C+2P_3^C+3P_4^C+4P_1^\infty+2P_2^\infty+3P_3^\infty \text{ if } a>0 \end{cases}$	19 19

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1 (and at P_4), but one of them is double;

2) four distinct tangents at P_1^∞;

3) three distinct tangents at P_3^∞.

Table 242 – First integral and integrating factor of family (L) when $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1}J_2^{\lambda_2}J_3^{2\lambda_2}J_4^{2\lambda_1}J_5^{-\lambda_1-\lambda_2}$	$R = J_1^{\lambda_1}J_2^{\lambda_2}J_3^{1+2\lambda_2}J_4^{1+2\lambda_1}J_5^{-2-\lambda_1-\lambda_2}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1J_4^2}{J_5}$ $\mathscr{I}_2 = \dfrac{J_2J_3^2}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1J_2J_3J_4}$

Source: Elaborated by the author.

Observation 145. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_1J_4^2 - c_2J_5$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $\left[1 : \frac{2}{a}\right]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{\left(1,\frac{2}{a}\right)} = -J_2J_3^2, \quad \mathscr{F}^1_{(1,0)} = J_1J_4^2.$$

Therefore, J_1,J_2,J_3,J_4 are remarkable curves of $\mathscr{I}_1$, $\left[1 : \frac{2}{a}\right]$ and $[1:0]$ are critical remarkable values of $\mathscr{I}_1$ and J_3,J_4 are critical remarkable curves of $\mathscr{I}_1$. The singular points are P_1,P_2 for $\mathscr{F}^1_{\left(1,\frac{2}{a}\right)}$ and P_3,P_4 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1J_2J_3^2 - c_2J_5$ we have the same remarkable values and remarkable curves as before. The singular points are P_1,P_2 for $\mathscr{F}^2_{(1,0)}$ and P_3,P_4 for $\mathscr{F}^2_{\left(1,\frac{2}{a}\right)}$.

(ii) $a = 0$.

Under this condition the system does not belong to family (L). The affine invariant lines are $x=0$, $y=0$ that are double (so we compute two exponential factors) and $x-y=0$ that is simple. We also have a family of invariant hyperbolas. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. This system has a rational first integral that foliates the plane into conic invariant algebraic curves. The lines $x=0$ and $y=0$ are remarkable curves.

Table 243 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (L) when $a=0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1=x$ $J_2=y$ $J_3=x-y$ $J_{4,r}=r(x-y)+2xy$ $E_5=e^{\frac{g_0+g_1x}{x}}$ $E_6=e^{\frac{h_0+h_1y}{y}}$ $\alpha_1=x$ $\alpha_2=y$ $\alpha_3=x+y$ $\alpha_4=x+y$ $\alpha_4=-g_0$ $\alpha_5=-h_0$	$P_1=(0,0)$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ $pphpph_{(4)};N,S,N$	$\bar{J}_1\cap\bar{J}_2=P_1$ simple $\bar{J}_1\cap\bar{J}_3=P_1$ simple $\bar{J}_1\cap\bar{J}_{4,r}=\begin{cases}P_1 \text{ simple}\\ P_3^\infty \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathscr{L}_\infty=P_1^\infty$ simple $\bar{J}_2\cap\bar{J}_3=P_1$ simple $\bar{J}_2\cap\bar{J}_{4,r}=\begin{cases}P_1 \text{ simple}\\ P_3^\infty \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty=P_3^\infty$ simple $\bar{J}_3\cap\bar{J}_{4,r}=P_1$ double $\bar{J}_3\cap\mathscr{L}_\infty=P_2^\infty$ simple $\bar{J}_{4,r}\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_3^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 244 – Divisor and zero-cycles of family (L) when $a=0$.

Divisor and zero-cycles	Degree
$ILD=2J_1+2J_2+J_3+\mathscr{L}_\infty$	6
$M_{0CS}=4P_1+P_1^\infty+P_2^\infty+P_3^\infty$	7
$T=Z\bar{J}_1^2\bar{J}_2^2\bar{J}_3=0.$	6
$M_{0CT}=4P_1+3P_1^\infty+2P_2^\infty+3P_3^\infty$	13

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but two of them are double;

2) only two distinct tangents at P_1^∞ (and P_2^∞), but one of them double.

Table 245 – First integral and integrating factor of family (L) when $a = 0$.

	First integral	Integrating Factor
General	$I = J_1^{-\lambda_3-\lambda_4} J_2^{-\lambda_3-\lambda_4} J_3^{\lambda_3} J_{4,r}^{\lambda_4} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$	$R = J_1^{-2-\lambda_3-\lambda_4} J_2^{-2-\lambda_3-\lambda_4} J_3^{\lambda_3} J_{4,r}^{\lambda_4} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_3}{J_1J_2}$	$\mathscr{R} = \dfrac{1}{J_1J_2J_3}$

Source: Elaborated by the author.

Observation 146. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_3 - c_2J_1J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_1J_2.$$

Therefore, J_1, J_2 are remarkable curves of $\mathscr{I}_1$. The singular points is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (L) in the following proposition.

Proposition 147. (a) For the family (L) we have two distinct configurations $C_1^{(L)}$ and $C_2^{(L)}$ of invariant hyperbolas and lines (see Figure 22 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space contains only the point $a = 0$. Its complement is the union of 2 disjoint sets. When $a < 0$ we have four real lines and when $a > 0$ we have four complex lines. For the limiting set of the parameter space, i.e. on $a = 0$ we have a family of invariant hyperbola and three invariant lines.

(b) The family (L) admits a rational first integral and the plane is foliated into cubic invariant algebraic curves. The remarkable curves are J_1, J_2, J_3, J_4 for family (L). All systems in family (L) have an inverse integrating factor which is polynomial.

(c) For the family (L) we have two topologically distinct phase portraits $P_1^{(L)}$ and $P_2^{(L)}$. The topological bifurcation diagram in the full parameter space is done in Figure 23 . The bifurcation set is the point $a = 0$ and it is a bifurcation of singularities.

Proof of proposition 147:

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (L):

Table 246 – Configurations for family (L).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(L)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 2P_2^\infty + 3P_3^\infty$
$C_2^{(L)}$	$ICD = J_1^C + J_2^C + J_3^C + J_4^C + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 2P_2^C + 2P_3^C + 3P_4^C + 4P_1^\infty + 2P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(L)}$ and $C_2^{(L)}$ are distinct. For the limit case of family (L) we have the following configuration:

Table 247 – Configuration for the limit case of family (L).

Configuration	Divisors and zero-cycles of the total inv. curve T
c_1	$ILD = 2J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 5P_1 + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) It follows directly from Jouanolou's theorem that we always have a rational first integral for family (L). The computations for the remarkable curves were done in Remark 145.

(c) We have:

Table 248 – Phase portraits for family (L).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(L)}$	(N,S,N)	(n,s,s,n)	$4SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$
$P_2^{(L)}$	(N,S,N)	$(©,©,©,©)$	$0SC_f^f\ 0SC_f^\infty\ 2SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have two distinct phase portraits for systems (L). For the limit case of family (L) we have the following phase portrait:

Table 249 – Phase portrait for the limit case of family (L).

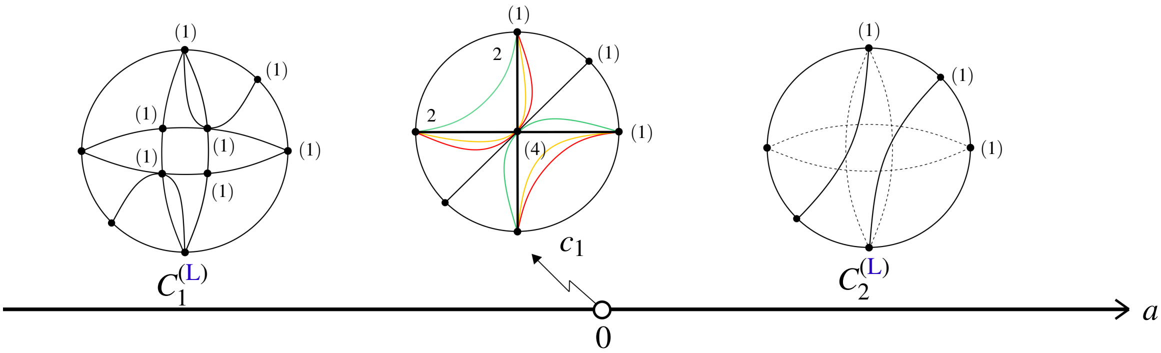
Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,S,N)	$pphpph_{(4)}$	$0SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$

Source: Elaborated by the author.

We note that $P_2^{(L)} \cong_{top} P_3^{(B)}$ is missing in (LLIBRE; YU, 2018) and it was listed in the geometric study of family (B).

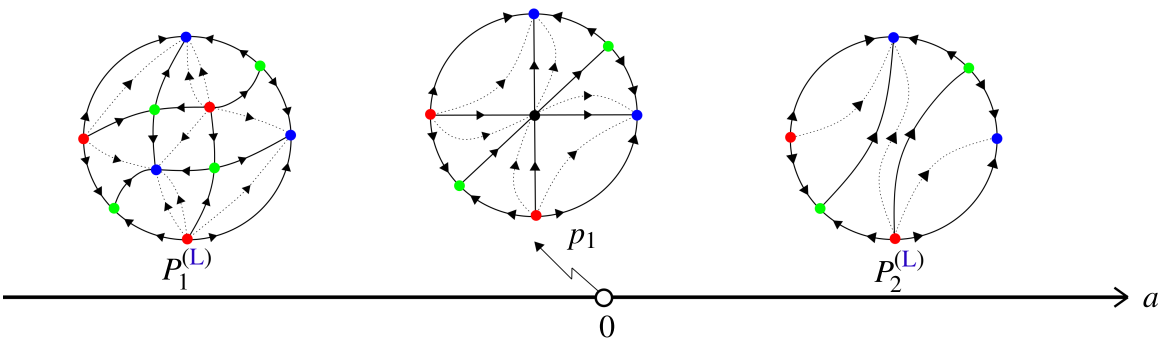
□

Figure 22 – Bifurcation diagram of configurations for family (L).



Source: Elaborated by the author.

Figure 23 – Topological bifurcation diagram for family (L).



Source: Elaborated by the author.

6.1.11 Geometric Analysis of Family (M)

Consider the family

$$(M) \begin{cases} \dot{x} = a + x^2 \\ \dot{y} = a + y^2. \end{cases}$$

This is a one parameter family depending on $a \in \mathbb{R}$. We display below the full geometric analysis of this family. When $a \neq 0$ every system in family (M) is endowed with five invariant algebraic lines J_1, J_2, J_3, J_4, J_5 and with a family of invariant hyperbolas $J_{6,r}$ with cofactors α_i,

$1 \leq i \leq 6$ given by

$$
\begin{aligned}
&J_1 = 1 - \tfrac{iy}{\sqrt{a}}, \;\; \alpha_1 = y - i\sqrt{a},\\
&J_2 = 1 + \tfrac{iy}{\sqrt{a}}, \;\; \alpha_2 = y + i\sqrt{a},\\
&J_3 = 1 - \tfrac{ix}{\sqrt{a}}, \;\; \alpha_3 = x - i\sqrt{a},\\
&J_4 = 1 + \tfrac{ix}{\sqrt{a}}, \;\; \alpha_4 = x + i\sqrt{a},\\
&J_5 = x - y, \;\; \alpha_5 = x + y,\\
&J_{6,r}(x,y) = 2a - r(x-y) + 2xy, \;\; \alpha_6 = x + y.
\end{aligned}
$$

When $a = 0$ the lines J_1 coalesce with J_2 and J_3 coalesce with J_4 yielding to two double lines. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. This family is clearly algebraically integrable. The multiplicities of each invariant invariant line appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

(i) $a \neq 0$.

Table 250 – Invariant curves, cofactors, singularities and intersection points of family (M) when $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{\sqrt{a}}$ $J_2 = 1 + \frac{iy}{\sqrt{a}}$ $J_3 = 1 - \frac{ix}{\sqrt{a}}$ $J_4 = 1 + \frac{ix}{\sqrt{a}}$ $J_5 = x - y$ $J_{6,r} = 2a - r(x-y) + 2xy$ $\alpha_1 = y - i\sqrt{a}$ $\alpha_2 = y + i\sqrt{a}$ $\alpha_3 = x - i\sqrt{a}$ $\alpha_4 = x + i\sqrt{a}$ $\alpha_5 = x + y$ $\alpha_6 = x + y$	$P_1 = \left(-i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(-i\sqrt{a}, i\sqrt{a}\right)$ $P_3 = \left(i\sqrt{a}, -i\sqrt{a}\right)$ $P_4 = \left(i\sqrt{a}, i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $a < 0$ we have $n, s, s, n; N, S, N$ For $a > 0$ we have ©, ©, ©, ©; N, S, N	$\bar{J}_1 \cap \bar{J}_2 = P_3^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \bar{J}_4 = P_3$ simple $\bar{J}_1 \cap \bar{J}_5 = P_1$ simple $\bar{J}_1 \cap \bar{J}_{6,r} = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_2$ simple $\bar{J}_2 \cap \bar{J}_4 = P_4$ simple $\bar{J}_2 \cap \bar{J}_5 = P_4$ simple $\bar{J}_2 \cap \bar{J}_{6,r} = \begin{cases} P_4 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_5 = P_1$ simple $\bar{J}_3 \cap \bar{J}_{6,r} = \begin{cases} P_1 \text{ simple} \\ P_1^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = P_4$ simple $\bar{J}_4 \cap \bar{J}_{6,r} = \begin{cases} P_4 \text{ simple} \\ P_1^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_5 \cap \bar{J}_{6,r} = \begin{cases} P_1 \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_{6,r} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 251 – Divisor and zero-cycles of family (M) when $a \neq 0$.

Divisor and zero-cycles	Degree
$ILD = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + J_3^C + J_4^C + J_5 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	6
$M_{0CT} = \begin{cases} 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty \text{ if } a < 0 \\ 3P_1^C + 2P_2^C + 2P_3^C + 3P_4^C + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty \text{ if } a > 0 \end{cases}$	18 18

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_1, P_4, P_1^∞ and P_3^∞.

Table 252 – First integral and integrating factor of family (M) when $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_4^{\lambda_1} J_5^{\lambda_5} J_{6,m}^{-\lambda_1-\lambda_2-\lambda_5}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_4^{\lambda_1} J_5^{\lambda_5} J_{6,m}^{-2-\lambda_1-\lambda_2-\lambda_5}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_4}{J_5} \quad \mathscr{I}_2 = \dfrac{J_2 J_3}{J_5}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_3 J_4}$

Source: Elaborated by the author.

Observation 148. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_4 - c_2 J_5$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable values of $\mathscr{F}^1_{(c_1,c_2)}$ are $\left[1 : \frac{2i}{\sqrt{a}}\right]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{\left(1,\frac{2i}{\sqrt{a}}\right)} = J_2 J_3, \quad \mathscr{F}^1_{(1,0)} = J_1 J_4.$$

Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathscr{I}_1$, $\left[1 : \frac{2i}{\sqrt{a}}\right]$ and $[1:0]$ are remarkable values of $\mathscr{I}_1$. The singular points are P_1, P_2, P_4 for $\mathscr{F}^1_{\left(1,\frac{2i}{\sqrt{a}}\right)}$ and P_3 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_3 - c_2 J_5$ we have the remarkable values $\left[1 : -\frac{2i}{\sqrt{a}}\right]$ and $[1:0]$ and the same remarkable curves as before. The singular points are P_1, P_3, P_4 for $\mathscr{F}^2_{\left(1,-\frac{2i}{\sqrt{a}}\right)}$ and P_2 for $\mathscr{F}^2_{(1,0)}$.

(ii) $a = 0$.

Here the line J_1 coalesce with J_2 and the line J_3 coalesce J_4 yielding to two double lines so we compute the exponential factors factors E_5 and E_6. This system has a rational first integral that foliates the plane into conic invariant algebraic curves. The remarkable curves are the double lines.

Table 253 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (M) when $a = 0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = y$ $J_3 = x - y$ $J_{4,r} = r(x-y) + 2xy$ $E_5 = e^{\frac{g_0+g_1x}{x}}$ $E_6 = e^{\frac{h_0+h_1y}{y}}$ $\alpha_1 = x$ $\alpha_2 = y$ $\alpha_3 = x + y$ $\alpha_4 = x + y$ $\alpha_4 = -g_0$ $\alpha_5 = -h_0$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $pphpph_{(4)}; N, S, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \bar{J}_{4,r} = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_{4,r} = \begin{cases} P_1 \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_3^\infty$ simple $\bar{J}_3 \cap \bar{J}_{4,r} = P_1$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_{4,r} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 254 – Divisor and zero-cycles of family (M) when $a = 0$.

Divisor and zero-cycles	Degree
$ILD = 2J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$	6
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2^2\bar{J}_3 = 0.$	6
$M_{0CT} = 4P_1 + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty$	13

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1, but two of them are double;

2) only two distinct tangents at P_1^∞ (and P_2^∞), but one of them double.

Table 255 – First integral and integrating factor of family (M) when $a = 0$.

	First integral	Integrating Factor
General	$I = J_1^{-\lambda_3-\lambda_4} J_2^{-\lambda_3-\lambda_4} J_3^{\lambda_3} J_{4,r}^{\lambda_4} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$	$R = J_1^{-2-\lambda_3-\lambda_4} J_2^{-2-\lambda_3-\lambda_4} J_3^{\lambda_3} J_{4,r}^{\lambda_4} E_5^{\lambda_5} E_6^{-\frac{g_0\lambda_5}{h_0}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_3}{J_1J_2}$	$\mathscr{R} = \dfrac{1}{J_1J_2J_3}$

Source: Elaborated by the author.

Observation 149. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_3 - c_2 J_1 J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathscr{F}^1_{(0,1)} = -J_1 J_2.$$

Therefore, J_1, J_2 are remarkable curves of $\mathscr{I}_1$. The singular points is P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (M) in the following proposition.

Proposition 150. (a) For the family (M) we have three distinct configurations $C_1^{(M)}, C_2^{(M)}$ and $C_3^{(M)}$ of invariant hyperbolas and lines (see Figure 24 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations contain only the point $a = 0$. On $a < 0$ we have five real lines and on $a > 0$ we have four complex lines and one real line. On $a = 0$ we have the coalescence of two lines with other two lines yielding to two double lines.

(b) The family (M) admits a rational first integral and the plane is foliated into cubic invariant algebraic curves. The remarkable curves for family (M) are J_1, J_2, J_3, J_4 when $a \neq 0$ and J_1, J_2 when $a = 0$. All systems in family (M) have an inverse integrating factor which is polynomial.

(c) For the family (M) we have three topologically distinct phase portraits $P_1^{(M)}, P_2^{(M)}$ and $P_3^{(M)}$. The topological bifurcation diagram is done in Figure 25. The bifurcation set is the point $a = 0$ and it is a bifurcation of singularities.

Proof of proposition 150:

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (M):

Table 256 – Configurations for family (M).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(M)}$	$ILD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty$
$C_2^{(M)}$	$ILD = J_1^C + J_2^C + J_3^C + J_4^C + J_5 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1^C + 2P_2^C + 2P_3^C + 3P_4^C + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty$
$C_3^{(M)}$	$ILD = 2J_1 + 2J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 5P_1 + 3P_1^\infty + 2P_2^\infty + 3P_3^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(M)}, C_2^{(M)}$ and $C_3^{(M)}$ are all distinct.

The other statements in (a) follows from the study done previously.

(b) It follows directly from Jouanolou's theorem that we always have a rational first integral for family (M). The computations for the remarkable curves were done in Remarks 148 and 149.

(c) We have:

Table 257 – Phase portraits for family (M).

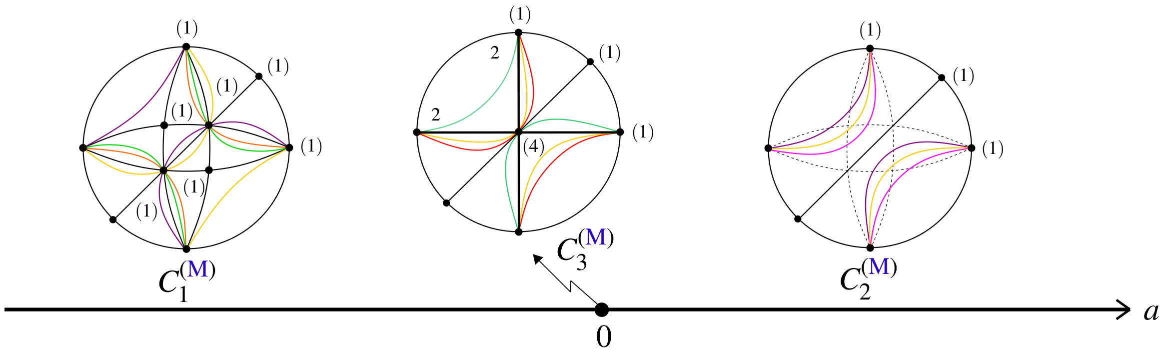
Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(M)}$	(N,S,N)	(n,s,s,n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(M)}$	(N,S,N)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
$P_3^{(M)}$	(N,S,N)	$pphpph_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have three distinct phase portraits for systems (M).

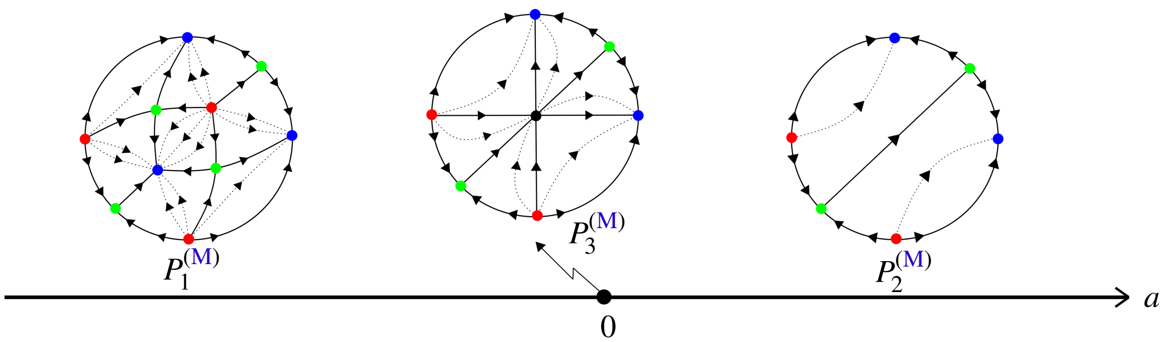
□

Figure 24 – Bifurcation diagram of configurations for family (M).



Source: Elaborated by the author.

Figure 25 – Topological bifurcation diagram for family (M).



Source: Elaborated by the author.

6.2 Systems with $\eta = 0$

In this section we present a detailed study of 12 normal forms for the class $\mathbf{QSH}_{(\eta=0)}$, namely the families (O), (P), (Q), (R), (S), (T), (U), (V), (W), (X), (Y) and (Z). Following the geometric study of these families we give an anwser to Poincaré's problem, but only for the cases when its solution does not follows directly from the expressions of the first integral.

We did not present in this thesis the geometric analysis of the normal form (N) due to its complicated expressions for the finite singularities, the study for this case is more difficult and it arises to more complicated bifurcation diagrams. This study will be done in further works using different methods.

6.2.1 Geometric Analysis of Family (O)

Consider the family

$$(\mathrm{O})\quad \begin{cases} \dot{x} = 2a + gx^2 + xy \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases}$$

where $a(g-1) \neq 0$.

This is a two parameter family depending on a and g such that $a(g-1) \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (O) we study here also the limit cases $a(g-1) = 0$ where the equations are still defined. We display below the full geometric analysis of the systems in this family, which is endowed with at least one invariant hyperbola J_1 with cofactor α_1 given by

$$J_1 = a + xy, \quad \alpha_1 = (-1+2g)x + 2y.$$

Except for a denumerable set of lines in the parameter space, i.e. except for

$$L_k : 2g - k = 0,\ k \in \mathbb{N} = \{0, 1, 2, \dots\} \text{ and } L : 4g - 1 = 0,$$

systems in (O) are not Liouvillian integrable (see (OLIVEIRA *et al.*, 2021)). It thus remains to be shown what happens on these lines and we consider here the cases L_1 and L. The multiplicities of each invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 2nd extactic polynomial.

(i) The generic case: $ag(g-1)(2g-1)(4g-1) \neq 0$.

In (OLIVEIRA *et al.*, 2021) it is proved that except for the denumerable set of lines $\cup_{k\in\mathbb{N}} L_k \cup L$,

$$L_k = \{(a, g) \in \mathbb{R}^2 \backslash \{(0, 1)\} : 2g - k = 0\},\ k \in \mathbb{N},$$
$$L = \{(a, g) \in \mathbb{R}^2 \backslash \{(0, 1)\} : 4g - 1 = 0\}$$

systems (O) are neither Darboux nor Liouvillian integrable. We prove below that when $(a,g) \in L_1$ systems (O) are generalized Darboux integrable and when $(a,g) \in L$ systems (O) are Liouvillian integrable. The cases where $(a,g) \in \cup_{k\in\mathbb{N}} L_k - L_1$ are still open. For these cases we were not able to prove the non-integrability and we also could not find other invariant algebraic curves, which we managed to search up to degree four. Although we are unable to guarantee the existence of a first integral in $\cup_{k\in\mathbb{N}} L_k - L_1$, it is still possible to obtain the complete topological bifurcation diagram of this family.

Table 258 – Invariant curve, cofactor, singularities and intersection points of family (O) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = (-1+2g)x + 2y$	$P_1 = \left(-2i\sqrt{a}, i\left(2\sqrt{a}g - \sqrt{a}\right)\right)$ $P_2 = \left(2i\sqrt{a}, -i\left(2\sqrt{a}g - \sqrt{a}\right)\right)$ $P_3 = \left(-\frac{i\sqrt{a}}{\sqrt{g}}, -i\sqrt{a}\sqrt{g}\right)$ $P_4 = \left(\frac{i\sqrt{a}}{\sqrt{g}}, i\sqrt{a}\sqrt{g}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $a < 0$ we have $f, f, ©, ©; \binom{0}{2}SN, S$ if $g < 0$ $f, f, s, s; \binom{0}{2}SN, N$ if $0 < g < \frac{7}{32}$ $n, n, s, s; \binom{0}{2}SN, N$ if $\frac{7}{32} \leq g < \frac{1}{4}$ $s, s, n, n; \binom{0}{2}SN, N$ if $g > \frac{1}{4}$ For $a > 0$ we have $©, ©, n, n; \binom{0}{2}SN, S$ if $g < 0$ $©, ©, ©, ©; \binom{0}{2}SN, N$ if $g > 0$	$\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 259 – Divisor and zero-cycles of family (O) for the generic case.

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathscr{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1+P_2+P_3^C+P_4^C+2P_1^\infty+P_2^\infty \text{ if } a<0 \text{ and } g<0 \\ P_1+P_2+P_3+P_4+2P_1^\infty+P_2^\infty \text{ if } a<0 \text{ and } g>0 \\ P_1^C+P_2^C+P_3+P_4+2P_1^\infty+P_2^\infty \text{ if } a>0 \text{ and } g<0 \\ P_1^C+P_2^C+P_3^C+P_4^C+2P_1^\infty+P_2^\infty \text{ if } a>0 \text{ and } g>0 \end{cases}$	7 7 7 7
$T = Z\bar{J}_1 = 0$	3
$M_{0CT} = \begin{cases} P_3^C+P_4^C+2P_1^\infty+2P_2^\infty \text{ if } ag>0 \\ P_3+P_4+2P_1^\infty+2P_2^\infty \text{ if } ag<0 \end{cases}$	6 6

Source: Elaborated by the author.

(ii) The non-generic case: $ag(g-1)(2g-1)(4g-1)=0$.

(ii.1) $g=0$ and $a \neq 0$.

Under this condition, $(a,g) \in L_0$ which corresponds to an open case regarding the integrability.

Table 260 – Invariant curves, cofactors, singularities and intersection points of family (O) when $g=0$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = -x + 2y$	$P_1 = \left(2i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(2i\sqrt{a}, i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $f, f; \binom{0}{2}SN, \binom{1}{2}S$ if $a<0$ $©, ©; \binom{0}{2}SN, \binom{1}{2}N$ if $a>0$	$\bar{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 261 – Divisor and zero-cycles of family (O) when $g = 0$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathscr{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1 + P_2 + 2P_1^\infty + 3P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + 2P_1^\infty + 3P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\overline{J_1} = 0$	3
$M_{0CT} = 2P_1^\infty + 2P_2^\infty$	4

Source: Elaborated by the author.

(ii.2) $g = \frac{1}{4}$ and $a \neq 0$.

Here the hyperbola becomes double so we compute the exponential factor E_2.

Table 262 – Invariant curve, exponential factor, cofactors, singularities and intersection points of family (O) when $g = 1/4$ and $a \neq 0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $E_2 = e^{\frac{ag_0 + g_0 xy + g_1 y^2}{(a+xy)}}$ $\alpha_1 = -\frac{x}{2} + 2y$ $\alpha_2 = -g_1 y$	$P_1 = \left(2i\sqrt{a}, -\frac{i\sqrt{a}}{2}\right)$ $P_2 = \left(2i\sqrt{a}, \frac{i\sqrt{a}}{2}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; \binom{0}{2}SN, N$ if $a < 0$ $©_{(2)}, ©_{(2)}; \binom{0}{2}SN, N$ if $a > 0$	$\overline{J}_1 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 263 – Divisor and zero-cycles of family (O) when $g = 1/4$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathscr{L}_\infty$	3
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\overline{J}_1^2 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 3P_1^\infty + 3P_2^\infty \text{ if } a > 0 \end{cases}$	10 10

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞ (and P_2^∞), but one of them is double.

Table 264 – First integral and integrating factor of family (O) when $g = 1/4$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = \frac{1}{2}\left(2\sqrt{2}aDawsonF\left(\frac{\sqrt{2}y}{\sqrt{a+xy}}\right)+x\sqrt{a+xy}\right)\left(e^{\frac{ag_0+g_0xy+g_1y^2}{a+xy}}\right)^{\frac{2}{g_1}}$	$R = J_1^{-\frac{1}{2}}E_2^{\frac{2}{g_1}}$
Simple example	$\mathscr{I} = \frac{1}{2}\left(2\sqrt{2}aDawsonF\left(\frac{\sqrt{2}y}{\sqrt{a+xy}}\right)+x\sqrt{a+xy}\right)\left(e^{\frac{y^2}{a+xy}}\right)^2$	$\mathscr{R} = J_1^{-\frac{1}{2}}E_2^2$

Source: Elaborated by the author.

where $DawsonF[z]$ gives the Dawson integral defined by $F(z) = e^{-z^2}\int_0^z e^{y^2}dy$.

(ii.3) $g = \frac{1}{2}$ and $a \neq 0$.

Here we have an additional invariant line which is simple and the invariant hyperbola becomes double so we compute the exponential factor E_3.

Table 265 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (O) when $g = 1/2$ and $a \neq 0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = a + xy$ $E_3 = e^{-\frac{a(2g_1-g_0)+g_1xy-g_0y^2}{2(a+xy)}}$ $\alpha_1 = \frac{x}{2} + y$ $\alpha_2 = 2y$ $\alpha_3 = g_0y$	$P_1 = \left(-2i\sqrt{a}, 0\right)$ $P_2 = \left(2i\sqrt{a}, 0\right)$ $P_3 = \left(-i\sqrt{2}\sqrt{a}, -\frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_4 = \left(i\sqrt{2}\sqrt{a}, \frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, s, n, n; \binom{0}{2}SN, N$ if $a < 0$ ©, ©, ©, ©; $\binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty =$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 266 – Divisor and zero-cycles of family (O) when $g = 1/2$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathscr{L}_\infty$	4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2^2 = 0$	6
$M_{0CT} = \begin{cases} P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + 2P_3^C + 2P_4^C + 3P_1^\infty + 4P_2^\infty \text{ if } a > 0 \end{cases}$	13 13

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is triple.

Table 267 – First integral and integrating factor of family (O) when $g = 1/2$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} E_3^{\frac{-2\lambda_2}{g_0}}$	$R = J_1^1 J_2^{\lambda_2} E_3^{\frac{-2(2+\lambda_2)}{g_0}}$
Simple example	$\mathscr{I} = \dfrac{J_2}{E_3^2}$	$\mathscr{R} = \dfrac{J_1}{J_2^2}$

Source: Elaborated by the author.

(ii.4) $g = 1$ and $a \neq 0$.

Under this condition the systems do not belong to family (O). Here we have, apart from a simple hyperbola, two additional invariant lines (real or complex, depending on the sign of the parameter a).

Table 268 – Invariant curves, cofactors, singularities and intersection points of family (O) when $g = 1$ and $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{\sqrt{a}}$ $J_2 = 1 + \frac{iy}{\sqrt{a}}$ $J_3 = a + xy$ $\alpha_1 = y - i\sqrt{a}$ $\alpha_2 = y + i\sqrt{a}$ $\alpha_3 = x + 2y$	$P_1 = \left(-i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(i\sqrt{a}, i\sqrt{a}\right)$ $P_3 = \left(-2i\sqrt{a}, i\sqrt{a}\right)$ $P_4 = \left(2i\sqrt{a}, -i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $n, n, s, s; \binom{0}{2}SN, N$ if $a < 0$ $©, ©, ©, ©; \binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 269 – Divisor and zero-cycles of family (O) when $g = 1$ and $a \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathscr{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + J_3 + \mathscr{L}_\infty \text{ if } a > 0 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty \text{ if } a > 0 \end{cases}$	12 12

Source: Elaborated by the author.

where the total curve T has four distinct tangents at P_2^∞.

Table 270 – First integral and integrating factor of family (O) when $g = 1$ and $a \neq 0$.

	First integral	Integrating Factor
General	$I = \left(\sqrt{a+y^2} + y\right)^{-\frac{\sqrt{\frac{a+y^2}{a}}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{\frac{a+y^2}{a}}(x-y)}{a+xy}}$	$R = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$
Simple example	$\mathscr{I} = \left(\sqrt{a+y^2} + y\right)^{-\frac{\sqrt{\frac{a+y^2}{a}}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{\frac{a+y^2}{a}}(x-y)}{a+xy}}$	$\mathscr{R} = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$

Source: Elaborated by the author.

(ii.5) $a = 0$ and $g \neq 0, 1$.

Under this condition, systems (O) do not belong to **QSH**. The affine invariant lines are $y = 0$ that is simple and $x = 0$ that is double so we compute the exponential factor E_3. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. By perturbing the reducible conic $xy = 0$ we produce the hyperbola $a + xy = 0$.

Table 271 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (O) when $a = 0$ and $g \neq 0, 1$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0 x + g_1 y}{x}}$ $\alpha_1 = (-1+g)x + y$ $\alpha_2 = gx + y$ $\alpha_3 = -g_1 y$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $epep_{(4)}; \binom{0}{2}SN, S$ if $g < 0$ $phph_{(4)}; \binom{0}{2}SN, N$ if $g > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 272 – Divisor and zero-cycles of family (O) when $a = 0$ and $g \neq 0, 1$.

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathscr{L}_\infty$ if $g \neq 1$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$ if $g \neq 0$	7
$T = Z\bar{J}_1\bar{J}_2^2 = 0$ if $g \neq 1$	3
$M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$ if $g \neq 0$	8

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

Table 273 – First integral and integrating factor of family (O) when $a = 0$ and $g \neq 0, 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g}} E_3^{\frac{\lambda_1}{g_1 g}}$	$R = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g} - \frac{3g-1}{g}} E_3^{\frac{1+\lambda_1}{g_1 g}}$
Simple example	$\mathscr{I} = J_1^g J_2^{(1-g)} E_3$	$\mathscr{R} = \dfrac{1}{J_1 J_2^2}$

Source: Elaborated by the author.

(ii.6) $a = g = 0$.

Under this condition, systems (O) do not belong to **QSH**. The system here is $\dot{x} = xy$, $\dot{y} = y(-x+y)$. This is a degenerate system where the line $y = 0$ is filled up with singularities. The affine line $x = 0$ is double so we compute the exponential factor E_2.

Table 274 – Invariant curves, exponential factors, cofactors, singularities and intersection points for the reduced system of family (O) when $a = g = 0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $E_2 = e^{\frac{g_0 x + g_1 y}{x}}$ $\alpha_1 = 1$ $\alpha_2 = -g_1$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $(\ominus[\vert];n^d);\binom{0}{2}SN,(\ominus[\vert];\emptyset)$	$\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 275 – Divisor and zero-cycles for the reduced system of family (O) when $a = g = 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathscr{L}_\infty$	3
$M_{0CS} = P_1 + 2P_1^\infty$	3
$T = Z\bar{J}_1^2 = 0$	3
$M_{0CT} = 2P_1 + 3P_1^\infty$	5

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

Table 276 – First integral and integrating factor for the reduced system of family (O) when $a = g = 0$.

	First integral	Integrating Factor
General	$I = J_1^{g_1\lambda_2} E_2^{\lambda_2}$	$R = J_1^{-2+g_1\lambda_2} E_2^{\lambda_2}$
Simple example	$\mathscr{I} = J_1 E_2$	$\mathscr{R} = \dfrac{1}{J_1^2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}$ are also first integrals for family (O) when $a = g = 0$.

(ii.7) $a = 0$ and $g = 1$.

Under this condition, systems (O) do not belong to **QSH**. The affine invariant lines are $y = 0$ and $x = 0$ that are both double so we compute the exponential factor E_3 and E_4. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. By perturbing the reducible conic $xy = 0$ we produce the hyperbola $a + xy = 0$.

Table 277 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (O) when $a = 0$ and $g = 1$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0x+g_1y}{x}}$ $E_4 = e^{\frac{h_0+h_1y}{y}}$ $\alpha_1 = y$ $\alpha_2 = x+y$ $\alpha_3 = -g_1y$ $\alpha_4 = -h_0$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $phph_{(4)}; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 278 – Divisor and zero-cycles of family (O) when $a = 0$ and $g = 1$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + 2J_2 + \mathscr{L}_\infty$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2^2 = 0$	5
$M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$	10

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞ (and P_2^∞), but one of them is double.

Table 279 – First integral and integrating factor of family (O) when $a = 0$ and $g = 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^0 E_3^{\frac{\lambda_1}{g_1}} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2} E_3^{\frac{1+\lambda_1}{g_1}} E_4^0$
Simple example	$\mathscr{I} = J_1E_3$	$\mathscr{R} = \dfrac{1}{J_1J_2^2}$

Source: Elaborated by the author.

We sum up the topological, dynamical and algebraic geometric features of family (O) and also confront our results with previous results in literature in the following proposition.

Proposition 151. (a) For the family (O) we obtained seven distinct configurations $C_1^{(O)} - C_7^{(O)}$ of invariant hyperbolas and lines (see Figure 26 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is $ag(g-1)(g-1/2)(g-1/4) = 0$. Its complement is a union of 10 disjoint sets. On $(g-1/2)(g-1/4) = 0$ the invariant hyperbola is double. On $g = 1/2$ we have an additional invariant line. On $g = 0$ we just have a simple invariant hyperbola. For the limiting set of the parameter space of the considered family we have the following: On $g = 1$ we have two additional invariant lines. On $a = 0$ the hyperbola becomes reducible producing two lines and when $a = g = 0$ one of the lines is filled up with singularities.

(b) The family (O) is generalized Darboux integrable when $g = 1/2$ and it is Liouvillian integrable when $g = 1/4$.

(c) For the family (O) we have seven topologically distinct phase portraits $P_1^{(O)} - P_7^{(O)}$. The topological bifurcation diagram of family (O) is done in Figure 27. The bifurcation set are the half lines $g = 1/4$ and $g = 1/2$ with $a < 0$ and the lines $g = 0$ and $a = 0$. The half line $g = 1/4$ with $a < 0$ and the lines $g = 0$, $a = 0$ are bifurcation sets of singularities and the half line $g = 1/2$ with $a < 0$ is a bifurcation of saddle to saddle connection. The phase portraits $P_2^{(O)}$, $P_4^{(O)}$ and $P_6^{(O)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 151.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (O):

Table 280 – Configurations for family (O).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(O)}$	$ICD = J_1 + \mathscr{L}_\infty$ $M_{0CT} = P_3 + P_4 + 2P_1^\infty + 2P_2^\infty$
$C_2^{(O)}$	$ICD = J_1 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^\infty + 2P_2^\infty$
$C_3^{(O)}$	$ICD = J_1 + \mathscr{L}_\infty$ $M_{0CT} = P_3^C + P_4^C + 2P_1^\infty + 2P_2^\infty$
$C_4^{(O)}$	$ICD = J_1 + 2J_2 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty$
$C_5^{(O)}$	$ICD = J_1 + 2J_2 + \mathscr{L}_\infty$ $M_{0CT} = P_1^C + P_2^C + 3P_1^\infty + 4P_2^\infty$
$C_6^{(O)}$	$ICD = 2J_1 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty$
$C_7^{(O)}$	$ICD = 2J_1 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 3P_1^\infty + 3P_2^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(O)}$ up to $C_7^{(O)}$ are all distinct. For the limit cases of family (O) we have the following configurations:

Table 281 – Configurations for the limit cases of family (O).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + 2J_2 + \mathscr{L}_\infty$ $M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$
c_2	$ICD = 2J_1 + 2J_2 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$
c_3	$ICD = 2J_1 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 3P_1^\infty$
c_4	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
c_5	$ICD = J_1^C + J_2^C + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables.

(c) We have that:

Table 282 – Phase portraits for family (O).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	(n, n, s, s)	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	(s, s, n, n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(O)}$	$\left(\binom{0}{2}SN, S\right)$ $\left(\binom{0}{2}SN, \binom{1}{2}S\right)$	$(f, f, ©, ©)$ (f, f)	$0SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(O)}$	$\left(\binom{0}{2}SN, N\right)$ $\left(\binom{0}{2}SN, N\right)$ $\left(\binom{0}{2}SN, \binom{1}{2}N\right)$	$(©, ©, ©, ©)$ $(©_{(2)}, ©_{(2)})$ $(©, ©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_5^{(O)}$	$\left(\binom{0}{2}SN, S\right)$	$(©, ©, n, n)$	$0SC_f^f$ $2SC_f^\infty$ $0SC_\infty^\infty$
$P_6^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	(s, s, n, n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_7^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	$(sn_{(2)}, sn_{(2)})$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have seven distinct phase portraits for systems (O). For the limit cases of family (O) we have the following phase portraits:

Table 283 – Phase portraits for the limit cases of family (O).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_2^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	(s,s,n,n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_4^{(O)}$	$\left(\binom{0}{2}SN, N\right)$	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
p_1	$\left(\binom{0}{2}SN, N\right)$	$phph_{(4)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
p_2	$\binom{0}{2}SN, (\ominus[\|]; \emptyset)$	$(\ominus[\|]; n^d)$	$0SC_f^f$ $2SC_f^\infty$ $0SC_\infty^\infty$
p_3	$\left(\binom{0}{2}SN, S\right)$	$epep_{(4)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Table 284 – Phase portraits in (LLIBRE; YU, 2018) that admit 2 singular points at infinity and with at most 4 real finite singular points. We describe the infinity singularities given the type of each sector in the neighbourhood of the singular point. The sectors can be of three types: P (parabolic sector), H (hyperbolic sector) and E (elliptic sector).

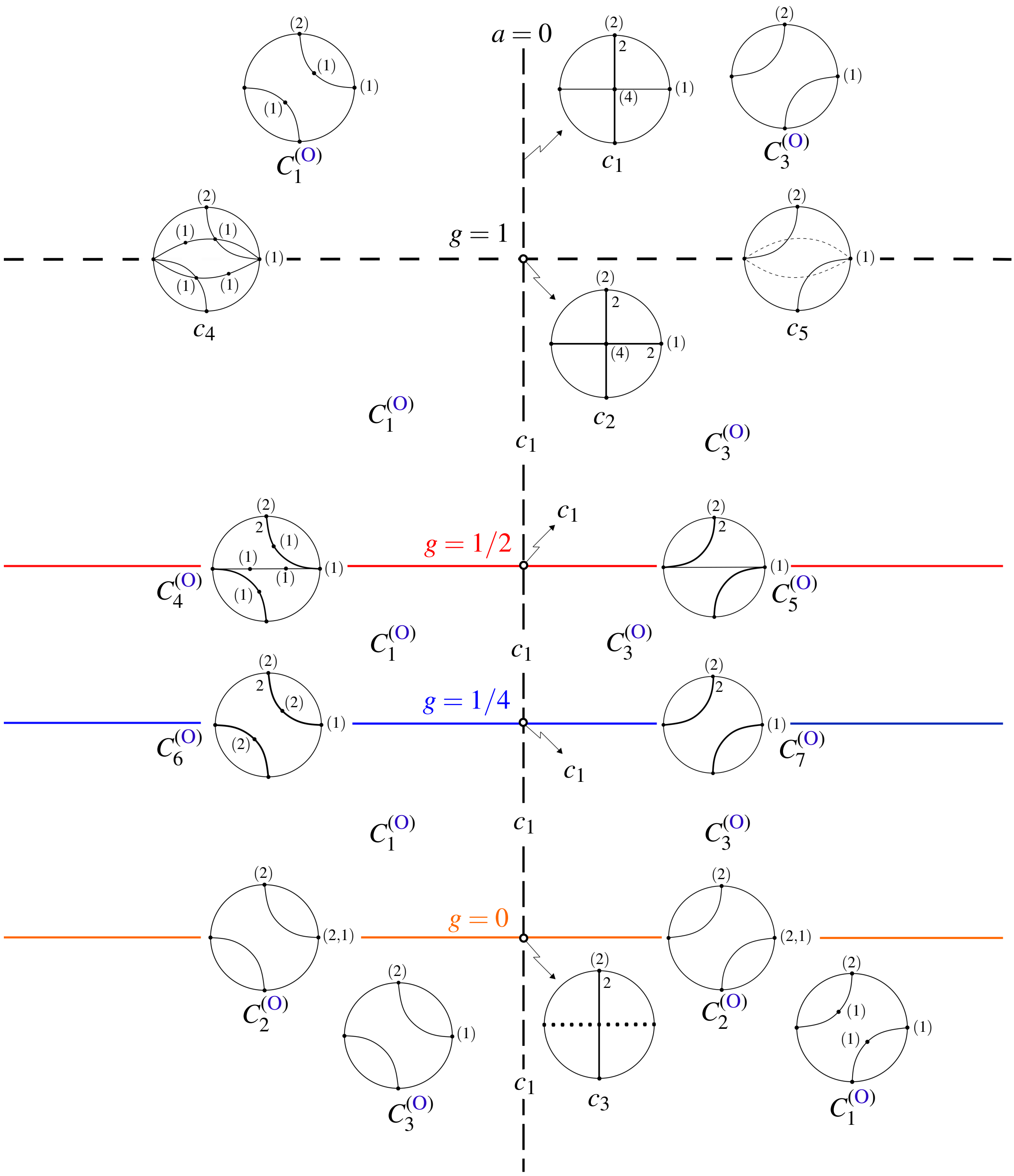
Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
L_{01}	(HP, N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
L_{02}	(PPE, S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
L_{03}	(PHP, N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
ω_1	(PPH, N)	(s,n)	$1SC_f^f$ $5SC_f^\infty$ $0SC_\infty^\infty$
ω_2	$(PEPP, S)$	(s,n)	$1SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
ω_3	$(PEPP, S)$	(s,s)	$1SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$

Source: Elaborated by the author.

The phase portraits $P_2^{(O)}$, $P_4^{(O)}$ and $P_6^{(O)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018). They are however phase portraits of systems possessing an invariant line and an invariant hyperbola (when $g = 1/2$ and $g = 1$).

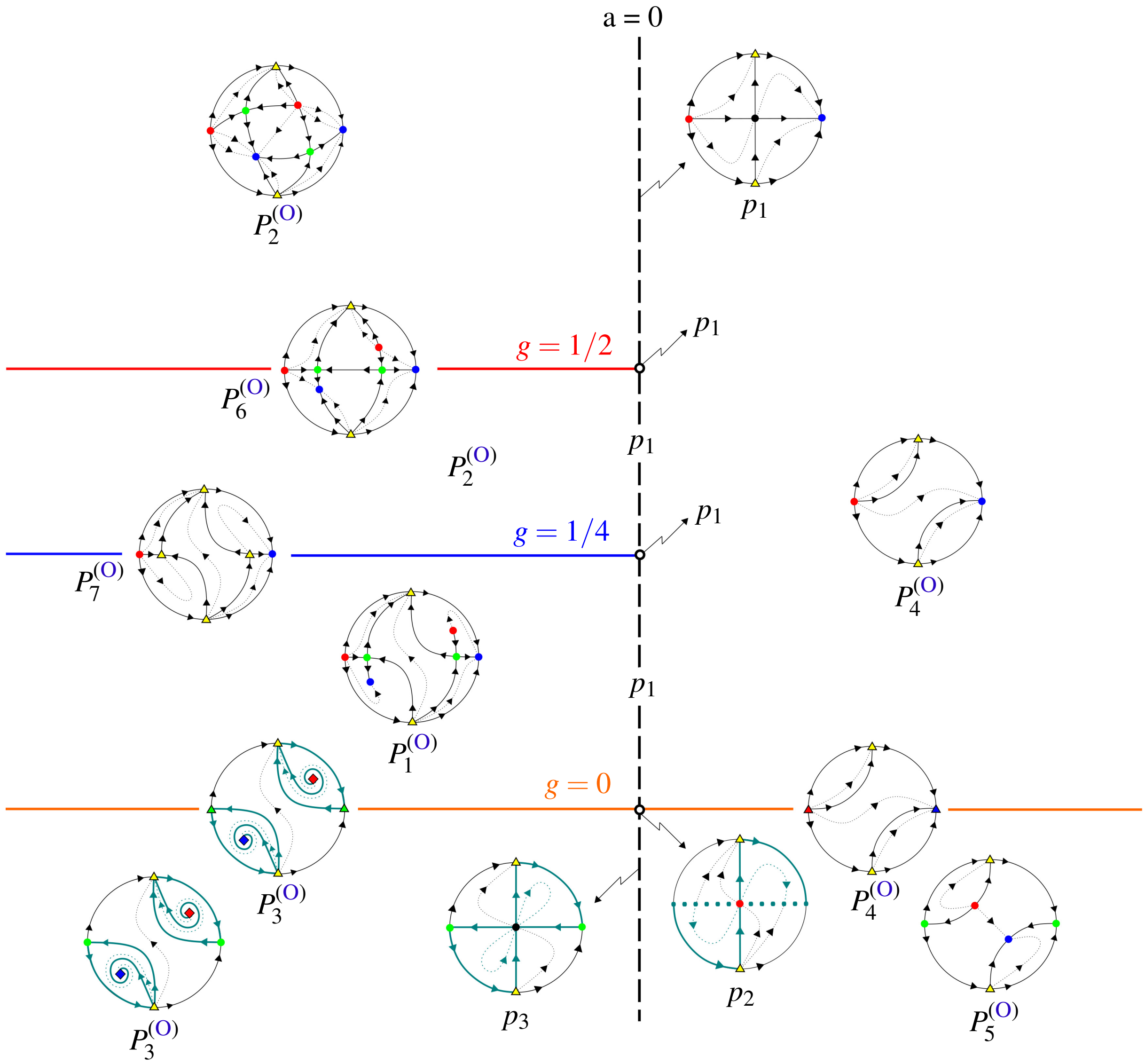
□

Figure 26 – Bifurcation diagram of configurations for family (O).



Source: Elaborated by the author.

Figure 27 – Topological bifurcation diagram for family (O).



Source: Elaborated by the author.

Note that the phase portraits p_4, p_5 and $P_3^{(O)}$ possess graphics in their first and third quadrant.

6.2.2 Geometric Analysis of Family (P)

Consider the family

$$(P) \begin{cases} \dot{x} = 2a + 3cx + x^2 + xy \\ \dot{y} = a - c^2 + y^2, \end{cases}$$

where $a \neq 0$.

This is a two parameter family depending on a and c such that $a \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations

(P) we study here also the limit case $a = 0$ where the equations are still defined. We display below the full geometric analysis of the systems in this family, which is endowed with at least two invariant algebraic curves. In the **generic case**

$$a(a-c^2)(a-8c^2/9) \neq 0$$

the systems have two invariant lines J_1 and J_2 and only one invariant hyperbolas J_3 with cofactors α_i, $1 \le i \le 3$ given by

$$\begin{aligned} J_1 &= y-\sqrt{c^2-a}, & \alpha_1 &= y+\sqrt{c^2-a},\\ J_2 &= y+\sqrt{c^2-a}, & \alpha_2 &= y-\sqrt{c^2-a},\\ J_3 &= a+cx+xy, & \alpha_3 &= 2c+x+2y. \end{aligned}$$

We note that if $a = c^2$ the two lines coalesce and we get a double line. Also if $a = 8v^2/9$ we get a double hyperbola. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $a(a-c^2)(a-8c^2/9) \neq 0$.

Table 285 – Invariant curves, cofactors, singularities and intersection points of family (P) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y-\sqrt{c^2-a}$ $J_2 = y+\sqrt{c^2-a}$ $J_3 = a+cx+xy$ $\alpha_1 = y+\sqrt{c^2-a}$ $\alpha_2 = y-\sqrt{c^2-a}$ $\alpha_3 = 2c+x+2y$	$P_1 = (-\sqrt{c^2-a}-c, -\sqrt{c^2-a})$ $P_2 = (-2(\sqrt{c^2-a}+c), \sqrt{c^2-a})$ $P_3 = (\sqrt{c^2-a}-c, \sqrt{c^2-a})$ $P_4 = (2(\sqrt{c^2-a}-c), -\sqrt{c^2-a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $a < 8c^2/9$ we have $n,s,n,s; \binom{0}{2}SN,N$ For $8c^2/9 < a < c^2$ we have $s,s,n,n; \binom{0}{2}SN,N$ if $c<0$ $n,n,s,s; \binom{0}{2}SN,N$ if $c>0$ For $c^2 < a$ we have $©,©,©,©; \binom{0}{2}SN,N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 286 – Divisor and zero-cycles of family (P) for the generic.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathscr{L}_\infty \text{ if } a < c^2 \\ J_1^C + J_2^C + J_3 + \mathscr{L}_\infty \text{ if } a > c^2 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < c^2 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > c^2 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty \text{ if } a < c^2 \\ 2P_1^C + P_2^C + 2P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty \text{ if } a < c^2 \end{cases}$	12 12

Source: Elaborated by the author.

where the total curve T has four distinct tangents at P_2^∞.

Observation 152. Mathematica could not give a response for the computation of the first integral of family (P) in the generic case.

Table 287 – Integrating factor of family (P) for the generic case.

	Integrating Factor
General	$R = J_1^{-\frac{-\sqrt{c^2-a}-c}{2\sqrt{c^2-a}}} J_2^{-\frac{c-\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}$
Simple example	$\mathscr{R} = J_1^{-\frac{-\sqrt{c^2-a}-c}{2\sqrt{c^2-a}}} J_2^{-\frac{c-\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}$

Source: Elaborated by the author.

(ii) The non-generic case: $a(a-c^2)(a-8c^2/9) = 0$.

(ii.1) $a = c^2$ and $c \neq 0$.

Table 288 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (P) when $a = c^2$ and $c \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = \frac{xy}{c} + c + x$ $E_3 = e^{\frac{g_0 + g_1 y}{y}}$ $\alpha_1 = y$ $\alpha_2 = 2c + x + 2y$ $\alpha_3 = -g_0$	$P_1 = (-2c, 0)$ $P_2 = (-c, 0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 289 – Divisor and zero-cycles of family (P) when $a = c^2$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + \mathscr{L}_\infty$	4
$M_{0CS} = 2P_1 + 2P_2 + 2P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2 = 0$	5
$M_{0CT} = 2P_1 + 3P_2 + 2P_1^\infty + 4P_2^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only one double tangent at P_1;

2) only two distinct tangents at P_2, but one of them double and

3) only three distinct tangents at P_2^∞, but one of them is double.

Table 290 – First integral and integrating factor of family (P) when $a = c^2$ and $c \neq 0$.

	First integral	Integrating Factor
General	I	$R = J_1 J_2^{-2} J_3^{-\frac{c}{g_0}}$
Simple example	$\mathscr{I}$	$\mathscr{R} = J_1 J_2^{-2} J_3^{-c}$

Source: Elaborated by the author.

$I = \mathscr{I} = \dfrac{c^2\left(e^{c/y}E_i\left(-\frac{c}{y}\right)\left(c^2+cx+xy\right)+y(c+x-y)\right)\left(e^{\frac{g_0}{y}+g_1}\right)^{-\frac{c}{g_0}}}{c^2+cx+xy}$ where $E_i(z)$ is the exponential integral function given by $E_i(z) = -\displaystyle\int_{-z}^{\infty}\frac{e^{-t}}{t}\,dt$ which has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.

(ii.2) $a = 8c^2/9$ and $c \neq 0$.

Table 291 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (P) when $a = 8c^2/9$ and $c \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -c + 3y$ $J_2 = c + 3y$ $J_3 = 8c^2 + 9cx + 9xy$ $E_4 = e^{\frac{c^2(48g_0 - g_1 x + 48 g_1 y) + 54 c g_0 x + 3 c g_1 y (21x - 8y) + 54 g_0 xy}{48c^2(8c^2 + 9cx + 9xy)}}$ $\alpha_1 = y + \frac{c}{3}$ $\alpha_2 = y - \frac{c}{3}$ $\alpha_3 = 2c + x + 2y$ $\alpha_4 = -\frac{g_1(c - 3y)}{54c}$	$P_1 = \left(-\frac{8c}{3}, \frac{c}{3}\right)$ $P_2 = \left(-\frac{4c}{3}, -\frac{c}{3}\right)$ $P_3 = \left(-\frac{2c}{3}, \frac{c}{3}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, sn_{(2)}, n; \binom{0}{2} SN, N$	$\overline{J}_1 \cap \overline{J}_2 = P_2^\infty$ simple $\overline{J}_1 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\overline{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 292 – Divisor and zero-cycles of family (P) when $a = 8c^2/9$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$	5
$M_{0CS} = P_1 + 2P_2 + P_3 + 2P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3^2 = 0$	7
$M_{0CT} = P_1 + 3P_2 + 3P_3 + 3P_1^\infty + 5P_2^\infty$	15

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_2, but one of them double;

2) only two distinct tangents at P_3, but one of them double;

3) only two distinct tangents at P_1^∞, but one of them double and

4) only four distinct tangents at P_2^∞, but one of them is double.

Table 293 – First integral and integrating factor of family (P) when $a = 8c^2/9$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^0 E_4^{-\frac{18c\lambda_2}{g_1}}$	$R = J_1^2 J_2^{\lambda_2} J_3^{-2} E_4^{-\frac{18(c\lambda_2 + c)}{g_1}}$
Simple example	$\mathscr{I} = J_2 E_4^{-18c}$	$\mathscr{R} = \dfrac{J_1^2}{J_2 J_3^2}$

Source: Elaborated by the author.

(ii.3) $a = 0$ and $c \neq 0$.

Under this condition, systems (P) do not belong to **QSH**. The affine invariant lines are $x = 0$ and $\pm c + y = 0$ that are all simple. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81. By perturbing the reducible conic $x(c+y) = 0$ we can produce the hyperbola $a + cx + xy = 0$. Furthermore, the conic $x(c+y) = 0$ has integrable multiplicity two.

Table 294 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (P) when $a = 0$ and $c \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -c+y$ $J_2 = c+y$ $J_3 = x$ $E_4 = e^{\left(-\frac{c^2(g_0-3g_1x)+2cg_0(x-y)-3cg_1xy+g_0y^2}{3cx(c+y)}\right)}$ $\alpha_1 = c+y$ $\alpha_2 = -c+y$ $\alpha_3 = 3c+x+y$ $\alpha_4 = \frac{g_0(y-c)}{3c}$	$P_1 = (-4c, c)$ $P_2 = (-2c, -c)$ $P_3 = (0, -c)$ $P_4 = (0, c)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, n, s, n; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 295 – Divisor and zero-cycles of family (P) when $a = 0$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	4
$M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty$	11

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_2^∞.

Table 296 – First integral and integrating factor of family (P) when $a = 0$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^0 E_4^{-\frac{3c\lambda_2}{g_0}}$	$R = J_1 J_2^{\lambda_2} J_3^{-2} E_4^{-\frac{3(2c+c\lambda_2)}{g_0}}$
Simple example	$\mathscr{I} = J_2 E_4^{-3c}$	$\mathscr{R} = \frac{J_1}{J_2^2 J_3^2}$

Source: Elaborated by the author.

(ii.4) $a = c = 0$.

Under this condition, systems (P) do not belong to **QSH**. The affine invariant lines are $x = 0$ and $y = 0$ that are both double. Considering the line at infinity $Z = 0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. This system has a generalized Darboux first integral. By perturbing the reducible conic $xy = 0$ we can produce the hyperbola $a + cx + xy = 0$.

Table 297 – Invariant curves, exponential factor, cofactors, singularities and intersection points of family (P) when $a = c = 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = y$ $E_3 = e^{g_0 + \frac{g_1 y}{x}}$ $E_4 = e^{h_0 + \frac{h_1}{y}}$ $\alpha_1 = c + y$ $\alpha_2 = -c + y$ $\alpha_3 = -g_1 y$ $\alpha_4 = -h_1$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $phph_{(4)}; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_2^\infty$ simple

Source: Elaborated by the author.

Table 298 – Divisor and zero-cycles of family (P) when $a = c = 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + 2J_2 + \mathscr{L}_\infty$	5
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2^2 = 0$	5
$M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$	10

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1, but two of them double;

2) only two distinct tangents at P_1^∞ (and P_2^∞), but one of them double.

Table 299 – First integral and integrating factor of family (P) when $a = c = 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{g_1\lambda_3} E_3^{\lambda_3} E_4^0$	$R = J_1^{-2} J_2^{-1+g_1\lambda_3} E_3^{\lambda_3} E_4^0$
Simple example	$\mathscr{I} = J_2 E_3$	$\mathscr{R} = \dfrac{1}{J_1^2 J_2}$

Source: Elaborated by the author.

We sum up the topological, dynamical and algebraic geometric features of family (P) and also confront our results with previous results in literature in the following proposition.

Proposition 153. (a) For the family (P) we obtained six distinct configurations $C_1^{(P)} - C_6^{(P)}$ of invariant hyperbolas and lines (see Figure 28 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is $a(a-c^2)(a-8c^2/9) = 0$ and it is made of points of bifurcation due to change in the multiplicities of the invariant algebraic invariant curves: On $a = c^2$ and $c \neq 0$ the invariant lines coalesce into a double line. On $a = 8c^2/9$ and $c \neq 0$ the hyperbola becomes double. For the limiting set of the parameter space, i.e. on $a = 0$ the invariant hyperbola becomes reducible producing the lines $x = 0$ and $c + y = 0$ and when also $c = 0$ then $x = 0$ and $y = 0$ become double lines.

(b) The family (P) is Liouvillian integrable for $a(a-8c^2/9) \neq 0$ and generalized Darboux integrable for $a = 8c^2/9$. All systems in family (P) do not have a polynomial inverse integrating factor. Outside the parameter space, i.e. on $a = 0$ we have a polynomial inverse integrating factor only when $c = 0$.

(c) For the family (P) we have five topologically distinct phase portraits $P_1^{(P)} - P_5^{(P)}$. The topological bifurcation diagram of family (P) is done in Figure 29. The parabolas $a = c^2$ and $a = 8c^2/9$ are bifurcation sets of singularities and the line $a = 0$ is a bifurcation of separatrices connection. The phase portraits $P_2^{(P)}$, $P_4^{(P)}$ and $P_5^{(P)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 153.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (P):

Table 300 – Configurations for family (P).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(P)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_2^{(P)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_3^{(P)}$	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_4^{(P)}$	$ICD = J_1^C + J_2^C + J_3 \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + P_2^C + 2P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty$
$C_5^{(P)}$	$ICD = 2J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 2P_1^\infty + 4P_2^\infty$
$C_6^{(P)}$	$ICD = J_1 + J_2 + 2J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + 3P_2 + 3P_3 + 3P_1^\infty + 5P_2^\infty$

Source: Elaborated by the author.

Note that $C_1^{(P)}$, $C_2^{(P)}$ and $C_3^{(P)}$ admit the same type of divisor and zero-cycles but the configurations are non equivalent. In fact, consider the convex quadrilateral in Figure 28 formed by the four finite singularities in these configurations. In $C_1^{(P)}$ any two consecutive or opposite points of this quadrilateral are not joined by anyone of the two branches of the hyperbola, in $C_2^{(P)}$, two opposite points are joined by a branch of the hyperbola and in $C_3^{(P)}$ two consecutive points of this quadrilateral is joined by a branch of the hyperbola.

Therefore, the configurations $C_1^{(P)}$ up to $C_6^{(P)}$ are all distinct. For the limit cases of family (P) we have the following configurations:

Table 301 – Configurations for the limit cases of family (P).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty$
c_2	$ICD = 2J_1 + 2J_2 + \mathscr{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$

Source: Elaborated by the author.

The other statements in (a) follows from the study done previously.

(b) In the generic case $a(a-c^2)(a-8c^2/9) \neq 0$ the three cofactors α_1, α_2, α_3 of J_1, J_2, J_3 are linearly independent. Hence we cannot get a Darboux first integral by using these curves. Furthermore the curves are each of multiplicity 1 and hence we cannot have exponential factors attached to them. However we obtained an integrating factor for (P) in the generic case. Using Mathematica we could not obtain an expression for the first integral of these systems but we know that it exists and it is Liouvillian. For the non-generic cases we obtained first integrals and they were given in previously exhibited tables.

Let us show that the family does not admit a polynomial inverse integrating factor.

(i) The **generic case**: $a(a-c^2)(a-8c^2/9) \neq 0$.

We have the following integrating factor

$$R = J_1^{\frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_2^{\frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}.$$

In order to R^{-1} to be polynomial we must have that

$$\begin{cases} \frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}} = \dfrac{c}{2\sqrt{c^2-a}} + \dfrac{1}{2} & = -m_1,\ m_1 \in \mathbb{N} \\ \\ \frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}} = \dfrac{-c}{2\sqrt{c^2-a}} + \dfrac{1}{2} & = -m_2,\ m_2 \in \mathbb{N}. \end{cases}$$

Adding up these two expressions we have

$$1 = -(m_1+m_2),\ m_1, m_2 \in \mathbb{N}$$

and this equation does not have a solution. Therefore, R^{-1} cannot be polynomial.

(ii) The **non-generic case**: $a(a-c^2)(a-8c^2/9) = 0$.

(ii.1) $a = c^2$: We have the integrating factor

$$R = J_1 J_2^{-2} E_3^{-c/g_0}$$

and it is clear that R^{-1} cannot be polynomial.

(ii.2) $a = 8c^2/9$: We have the integrating factor

$$R = J_1^2 J_2^{\lambda_2} J_3^{-2} E_4^{-18(c\lambda_2+c)/g_1}$$

again it is clear that R^{-1} cannot be polynomial.

(ii.3) $a = 0$ and $c \neq 0$. We have the integrating factor

$$R = J_1 J_2^{\lambda_2} J_3^{-2} E_4^{-3(2c+c\lambda_2)/g_0}.$$

again it is clear that R^{-1} cannot be polynomial.

(ii.4) $a = 0$ and $c = 0$: We have the integrating factor

$$R = J_1^{-2} J_2^{-1+g_1\lambda_3} E_3^{\lambda_3} E_4^0.$$

Taking $\lambda_3 = 0$ we have that $R^{-1} = J_1^2 J_2$ which is polynomial.

(c) We have:

Table 302 – Phase portraits for family (P).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(P)}$	$\left(\binom{0}{2}SN,N\right)$	(n,s,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(P)}$	$\left(\binom{0}{2}SN,N\right)$	(n,s,n,s)	$4SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$
$P_3^{(P)}$	$\left(\binom{0}{2}SN,N\right)$	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(P)}$	$\left(\binom{0}{2}SN,N\right)$	$(sn_{(2)},sn_{(2)})$	$1SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$
$P_5^{(P)}$	$\left(\binom{0}{2}SN,N\right)$	$(s,sn_{(2)},n)$	$3SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have five distinct phase portraits for systems (P). For the limit case of family (P) we have the following phase portraits:

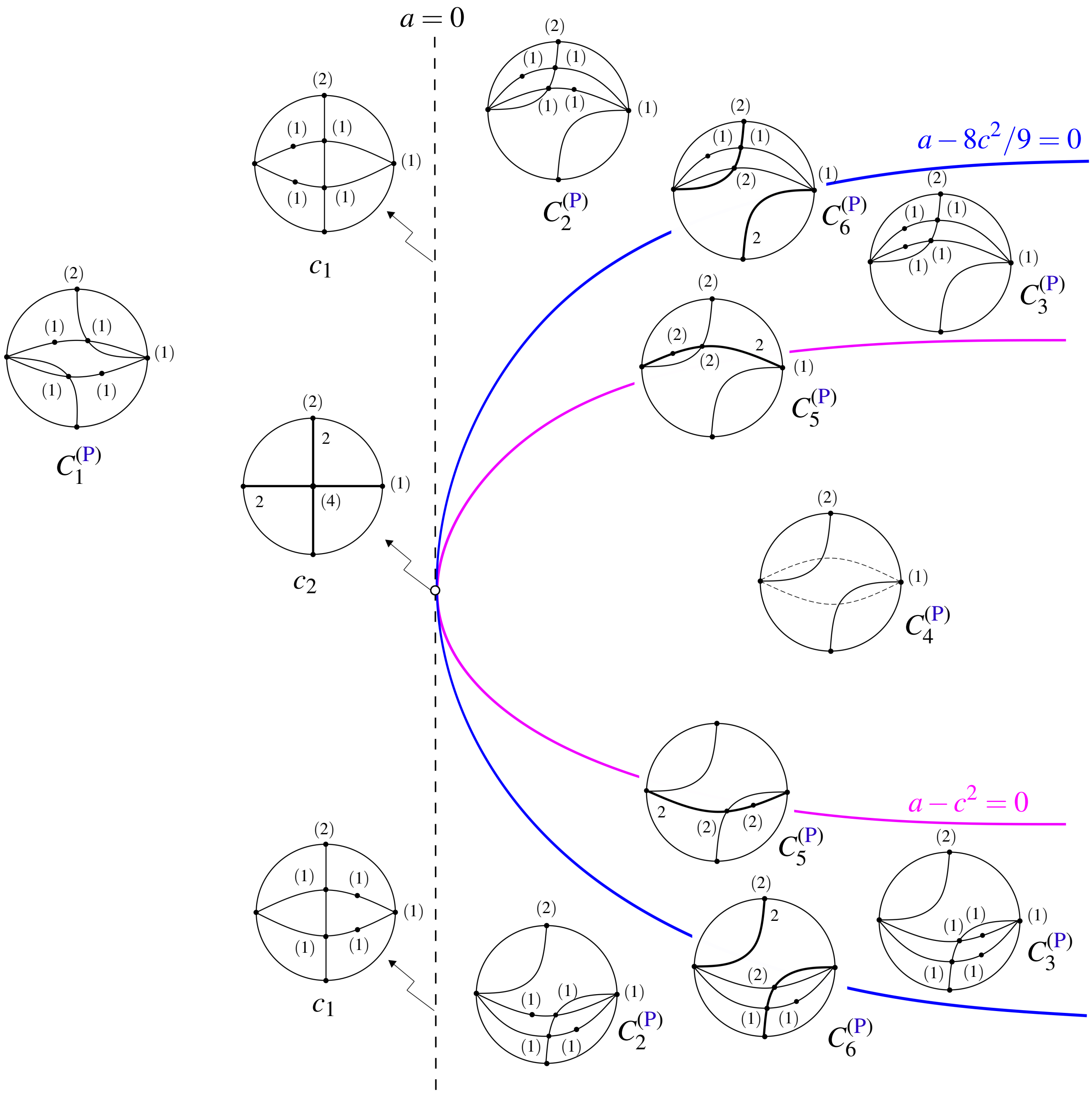
Table 303 – Phase portraits for the limit case of family (P).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_1	$\left(\binom{0}{2}SN,N\right)$	(s,n,s,n)	$4SC_f^f$ $5SC_f^\infty$ $0SC_\infty^\infty$
p_2	$\left(\binom{0}{2}SN,N\right)$	$phph_{(4)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

The phase portraits $P_2^{(P)}, P_4^{(P)}$ and $P_5^{(P)}$ are not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018) (see the table 284). Note that $P_1^{(P)} \cong_{top} P_2^{(O)}$ and $P_3^{(P)} \cong_{top} P_4^{(O)}$ are also missing in (LLIBRE; YU, 2018) and they were listed in the geometric study of family (O).

Figure 28 – Bifurcation diagram of configurations for family (P).

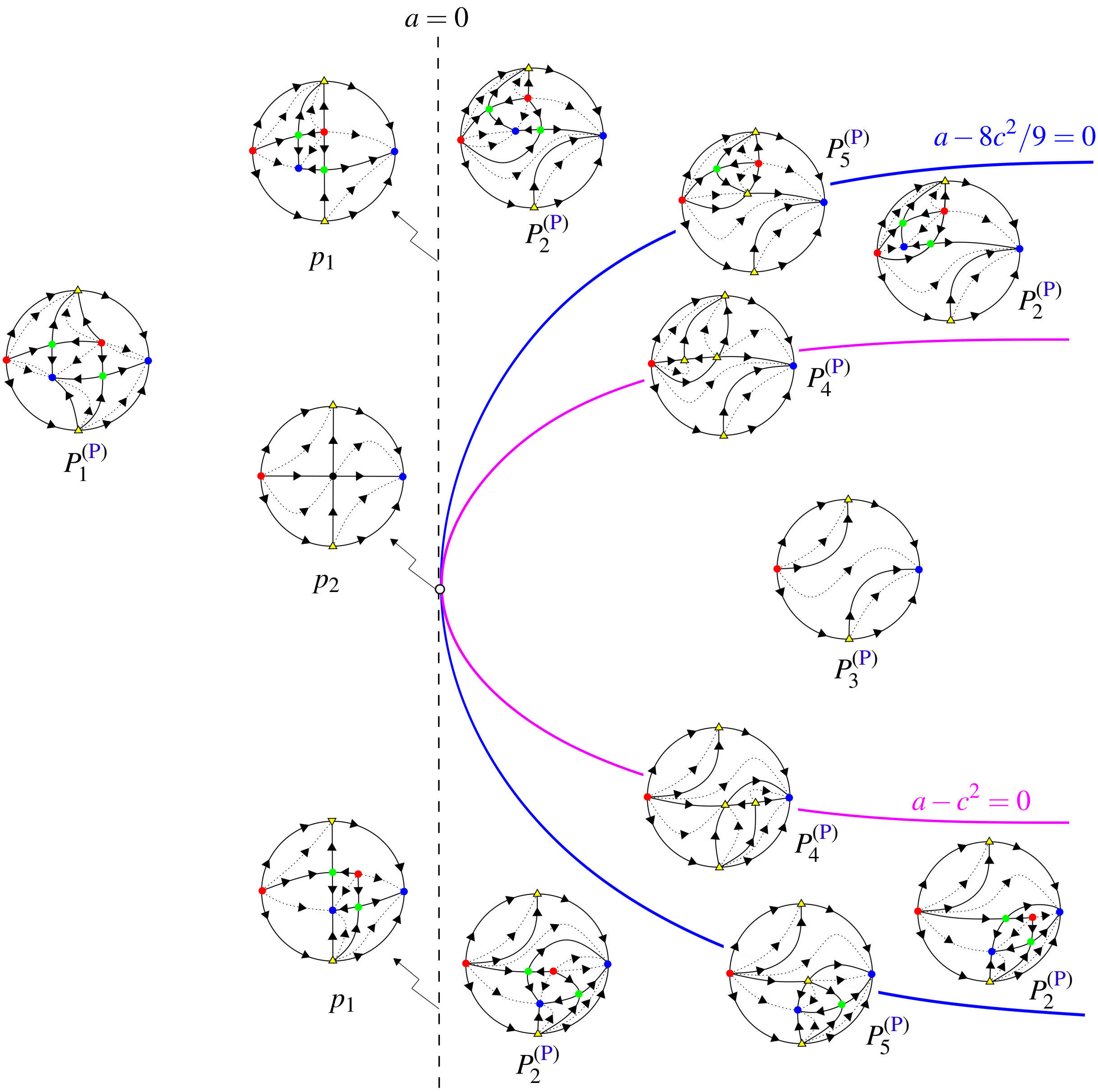


Source: Elaborated by the author.

Observation 154. Note that c_1 (for example, for $c > 0$) has three distinct lines, each line is an irreducible curve and for these lines the algebraic, integrable and geometric multiplicities coincide and this multiplicity is one. Hence in perturbations the line $y + c = 0$ can produce at most one line and in this case, it produces the line $y + \sqrt{c^2 - a} = 0$.

Observation 155. Note that the necessary and sufficient condition for systems defined by the equations (P) to have a double hyperbola or a double line is that it has two singularities of the system of multiplicity two or just one singularity of multiplicity four.

Figure 29 – Topological bifurcation diagram for family (P).



Source: Elaborated by the author.

6.2.3 Geometric Analysis of Family (Q)

Consider the family

$$(\mathrm{Q})\quad \begin{cases} \dot{x} = (c+x)(c(2g-1)+gx) \\ \dot{y} = 1+(g-1)xy, \end{cases}$$

where $(g\pm 1)(3g-1)(2g-1) \neq 0$ and $c^2+g^2 \neq 0$.

This is a two parameter family depending on c and g such that $(g\pm 1)(3g-1)(2g-1) \neq 0$ and $c^2+g^2 \neq 0$ but for a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (Q) we study here also the limit cases $(g \pm 1)(3g-1)(2g-1) = 0$ and $g = c = 0$ where the equations are still defined. We display below the full geometric analysis of the systems in this family, which is endowed with at least two invariant algebraic curves. In the **generic case**

$$cg(g \pm 1)(3g-1)(2g-1) \neq 0$$

the systems have two invariant lines J_1 and J_2 and only one invariant hyperbola J_3 with cofactors α_i, $1 \leq i \leq 3$ given by

$$\begin{array}{ll} J_1 = c + x, & \alpha_1 = c(-1+2g) + gx, \\ J_2 = c(-1+2g) + gx, & \alpha_2 = -cg + gx, \\ J_3 = \frac{1}{2g-1} + y(c+x), & \alpha_3 = c(-1+2g) + (-1+2g)x. \end{array}$$

We note that when $c = 0$ and $g \neq 0$ then the two lines coincide and we get a multiple line. The multiplicities of each invariant line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $cg(g \pm 1)(3g-1)(2g-1) \neq 0$.

Table 304 – Invariant curves, cofactors, singularities and intersection points of family (Q) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = c(-1+2g) + gx$ $J_3 = \frac{1}{2g-1} + y(c+x)$ $\alpha_1 = c(-1+2g) + gx$ $\alpha_2 = -cg + gx$ $\alpha_3 = c(-1+2g) + (-1+2g)x$	$P_1 = \left(-c, \frac{1}{c(g-1)}\right)$ $P_2 = \left(c\left(\frac{1}{g}-2\right), \frac{g}{2cg^2-3cg+c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $c < 0$ we have $s,n; \binom{2}{2}PEP-EPP,S$ if $g < 0$ $s,s; \binom{2}{2}PPEP-PEPP,N$ if $0<g<1/2$ $s,n; \binom{2}{2}HPP-HPP,N$ if $g > 1/2$ For $c > 0$ we have $s,n; \binom{2}{2}PPE-PEP,S$ if $g < 0$ $s,s; \binom{2}{2}PPEP-PEPP,N$ if $0<g<1/2$ $s,n; \binom{2}{2}PPH-PPH,N$ if $g > 1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 305 – Divisor and zero-cycles of family (Q) for the generic case.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = P_1 + 2P_2 + 4P_1^\infty + 2P_2^\infty$	9

Source: Elaborated by the author.

where the total curve T has only three distinct tangents at P_1^∞, but one of them is double.

Table 306 – First integral and integrating factor of family (Q) for the generic case.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\frac{g\lambda_2}{2g-1}}$	$R = J_1^0 J_2^{\lambda_2} J_3^{\frac{1-g(\lambda_2+3)}{2g-1}}$
Simple example	$\mathscr{I} = J_2 J_3^{-\frac{g}{2g-1}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

(ii) The non-generic case: $cg(g \pm 1)(3g-1)(2g-1) = 0$.

(ii.1) $c = 0$ and $g(g \pm 1)(3g-1)(2g-1) \neq 0$.

Here the two lines coalesce yielding a triple line so we compute the exponential factor E_3 and E_4.

Table 307 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (Q) when $c = 0$ and $g \neq 0, 1/2$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = \frac{1}{2g-1} + xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{2gh_0xy+h_0+x(h_1+h_2x)}{x^2}}$ $\alpha_1 = gx$ $\alpha_2 = (2g-1)x$ $\alpha_3 = -gg_0$ $\alpha_4 = -g(h_1 + 2h_0y)$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $g < 0$ we have $\emptyset$; $\binom{4}{2}PPE - EPP, S$ For $g > 0$ we have $\emptyset$; $\binom{4}{2}PHP - PHP, N$ if $g < 1/2$ $\emptyset$; $\binom{4}{2}PPH - HPP, N$ if $g > 1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 308 – Divisor and zero-cycles of family (Q) when $c = 0$ and $g \neq 0, 1/2$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + J_2 + \mathscr{L}_\infty$	5
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3\bar{J}_2 = 0.$	6
$M_{0CT} = 5P_1^\infty + 2P_2^\infty$	7

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is quadruple.

Table 309 – First integral and integrating factor of family (Q) when $c = 0$ and $g \neq 0, 1/2$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{g\lambda_1}{2g-1}} E_3^0 E_4^0$	$R = J_1^{\lambda_1} J_2^{\frac{1-g(\lambda_1+3)}{2g-1}} E_3^0 E_4^0$
Simple example	$\mathscr{I} = J_1 J_2^{-\frac{g}{2g-1}}$	$\mathscr{R} = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

(ii.2) $g = 0$ and $c \neq 0$.

Here we have only one affine invariant line and one invariant hyperbola both of them are simple. The line at infinity $Z = 0$ is double so we compute the exponential factor E_3.

Table 310 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (Q) when $g = 0$ and $c \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = -1 + cy + xy$ $E_3 = e^{g_0 + g_1 x}$ $\alpha_1 = -c$ $\alpha_2 = -c - x$ $\alpha_3 = -c^2 g_1 - c g_1 x$	$P_1 = \left(-c, -\frac{1}{c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $c < 0$ we have $s; \binom{2}{2} PEP - EPP, \binom{1}{1} SN$ For $c > 0$ we have $s; \binom{2}{2} PPE - PEP, \binom{1}{1} SN$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 311 – Divisor and zero-cycles of family (Q) when $g = 0$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2\mathscr{L}_\infty$	4
$M_{0CS} = P_1 + 4P_1^\infty + 2P_2^\infty$	7
$T = Z^2\bar{J}_1\bar{J}_2 = 0.$	5
$M_{0CT} = P_1 + 4P_1^\infty + 3P_2^\infty$	8

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1^∞, but one of them is double and

1) only two distinct tangents at P_2^∞, but one of them is double.

Table 312 – First integral and integrating factor of family (Q) when $g = 0$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} E_3^{-\frac{\lambda_2}{cg_1}}$	$R = J_1^0 J_2^{\lambda_2} E_3^{-\frac{1+\lambda_2}{cg_1}}$
Simple example	$\mathscr{I} = J_2^c E_3^{-1}$	$\mathscr{R} = \dfrac{1}{J_2}$

Source: Elaborated by the author.

(ii.3) $g = -1$ and $c \neq 0$.

Under this condition the systems do not belong to family (Q). Here we have two invariant lines, one invariant hyperbola and one invariant parabola. We note that in the case $c = g = -1$ this system is exactly family (S).

Table 313 – Invariant curves, cofactors, singularities and intersection points of system (Q) when $g = -1$ and $c \neq 0$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = 3c + x$ $J_3 = -1 + 3cy + 3xy$ $J_4 = 3c^2y + \frac{x^2}{8c} + \frac{19c}{8} + x$ $\alpha_1 = -3c - x$ $\alpha_2 = -c - x$ $\alpha_3 = -3c - 3x$ $\alpha_4 = -2x$	$P_1 = \left(-c, -\frac{1}{2c}\right)$ $P_2 = \left(-3c, -\frac{1}{6c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, n; \binom{2}{2} PPPE - PPEP, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = P_1^\infty$ double

Table 314 – Divisor and zero-cycles of system (Q) when $g = -1$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	5
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0.$	7
$M_{0CT} = 2P_1 + 3P_2 + 5P_1^\infty + 2P_2^\infty$	12

where the total curve T has

1) only three distinct tangents at P_1^∞, two of them double and one simple,

2) two distinct tangents at P_2, but one of them is double.

Table 315 – First integral and integrating factor of system (Q) when $g = -1$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1 - \frac{\lambda_2}{3}} J_4^{\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{4}{3} - \lambda_1 - \frac{\lambda_2}{3}} J_4^{\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_2^3}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4}$

Observation 156. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_2^3 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1 : -8c^3]$ and $[1 : 0]$ for which we have

$$\mathscr{F}^1_{(1,-8c^3)} = 8cJ_1J_4, \quad \mathscr{F}^1_{(1,0)} = J_2^3.$$

Therefore, J_1, J_2, J_4 are remarkable curves and $[1:-8c^3]$, $[1:0]$ are remarkable values of $\mathscr{I}_1$. Moreover, $[1:0]$ is a critical remarkable values and J_2 is critical remarkable curve of $\mathscr{I}_1$. The singular points are P_1 for $\mathscr{F}^1_{(1,-8c^3)}$ and P_2 for $\mathscr{F}^1_{(1,0)}$.

(ii.4) $g = 1/3$ and $c \neq 0$.

Under this condition the systems do not belong to family (Q). Here we have two invariant lines and two hyperbolas. These systems are Hamiltonian so they admit a polynomial first integral.

Table 316 – Invariant curves, cofactors, singularities and intersection points of family (Q) when $g = 1/3$ and $c \neq 0$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = -c + x$ $J_2 = c + x$ $J_3 = -3 - cy + xy$ $J_4 = -3 + cy + xy$ $\alpha_1 = \frac{c}{3} + \frac{x}{3}$ $\alpha_2 = \frac{x}{3} - \frac{c}{3}$ $\alpha_3 = \frac{c}{3} - \frac{x}{3}$ $\alpha_4 = -\frac{c}{3} - \frac{x}{3}$	$P_1 = \left(-c, -\frac{3}{2c}\right)$ $P_2 = \left(c, \frac{3}{2c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, s; \binom{2}{2} PPEP - PEPP, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 317 – Divisor and zero-cycles of family (Q) when $g = 1/3$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	5
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0.$	7
$M_{0CT} = 2P_1 + 2P_2 + 5P_1^\infty + 3P_2^\infty$	12

where the total curve T has

1) only three distinct tangents at P_1^∞, but two of them are double;

2) three distinct tangents at P_2^∞.

Table 318 – First integral and integrating factor of family (Q) when $g = 1/3$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_4^{\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_4^{\lambda_1}$
Simple example	$\mathscr{I}_1 = J_1 J_4$	$\mathscr{R}_1 = \dfrac{1}{J_1 J_4}$

(ii.5) $g = 1/2$ and $c \neq 0$.

Under this condition, systems (Q) do not belong to **QSH**. Here we have two invariant lines. We also could find an exponential factor but it did not arise from multiple curves since by calculating the 1st extactic polynomial we checked that the multiplicity of the affine invariant lines is one. We also checked the multiplicity of the line at infinity and it is also simple.

Table 319 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (Q) when $g = 1/2$ and $c \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = \frac{x}{c} + 1$ $E_3 = e^{cg_1 y + g_0 + g_1 xy}$ $\alpha_1 = \frac{c}{2} + \frac{x}{2}$ $\alpha_2 = \frac{x}{2}$ $\alpha_3 = cg_1 + g_1 x$	$P_1 = \left(-c, -\frac{2}{c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $c < 0$ we have $s; \binom{3}{2} PPEP - HPP, N$ For $c > 0$ we have $s; \binom{3}{2} PPH - PEPP, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 320 – Divisor and zero-cycles of family (Q) when $g = 1/2$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathscr{L}_\infty$	3
$M_{0CS} = P_1 + 5P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2 = 0.$	3
$M_{0CT} = P_1 + 3P_1^\infty + P_2^\infty$	5

Source: Elaborated by the author.

where the total curve T has three distinct tangents at P_1^∞.

Table 321 – First integral and integrating factor of family (Q) when $g = 1/2$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^0 E_3^{-\frac{\lambda_1}{2g_1}}$	$R = J_1^{\lambda_1} J_2^0 E_3^{-\frac{\lambda_1}{2g_1}-\frac{1}{2g_1}}$
Simple example	$\mathscr{I} = J_1^2 E_3^{-1}$	$\mathscr{R} = \dfrac{1}{J_1}$

Source: Elaborated by the author.

(ii.6) $g = 1$ and $c \neq 0$.

Under this condition, systems defined by the equations (Q) do not belong to family (Q). Here the systems possess an invariant line with multiplicity three and a family of invariant hyperbolas

$$1 + rc + rx + cy + xy,$$

where $r \in \mathbb{R}$. The line at infinity $\mathscr{L}_\infty : Z = 0$ has multiplicity 3.

Table 322 – Invariant curves, exponential factors, cofactors, singularities and intersection points of system (Q) when $g = 1$ and $c \neq 0$.

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_{2,r} = 1 + rc + rx + cy + xy$ $E_3 = e^{g_0 + g_1 y + g_2 y^2}$ $E_4 = e^{\frac{h_0 + h_1 x}{c+x}}$ $E_5 = e^{-\frac{l_2 - 2(c+x)(l_1(c+x) - l_0 - l_2 y)}{2(c+x)^2}}$ $\alpha_1 = c + x$ $\alpha_2 = c + x$ $\alpha_3 = g_1 + 2g_2 x$ $\alpha_4 = -h_0 + ch_1$ $\alpha_5 = l_0 + l_2 y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2} PH - HP, N$	$\bar{J}_1 \cap \bar{J}_{2,r} = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_{2,r} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 323 – Divisor and zero-cycles of system (Q) when $g = 1$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ILD = 3J_1 + 3\mathscr{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z^3 \bar{J}_1^3 = 0$	6
$M_{0CT} = 6P_1^\infty + 3P_2^\infty$	9

where the total curve T has

1) only two distinct tangents at P_1^∞, both of them triple and

2) only one triple tangent at P_2^∞.

Table 324 – First integral and integrating factor of system (Q) when $g = 1$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_{2,r}^{-\lambda_1} E_3^{\lambda_3} E_4^{-\frac{\lambda_3(2g_2l_0-g_1l_2)}{l_2(h_0-ch_1)}} E_5^{-\frac{2g_2\lambda_3}{l_2}}$	$R = J_1^{\lambda_1} J_{2,r}^{-2-\lambda_1} E_3^{\lambda_3} E_4^{-\frac{\lambda_3(2g_2l_0-g_1l_2)}{l_2(h_0-ch_1)}} E_5^{-\frac{2g_2\lambda_3}{l_2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_{2,1}}$	$\mathscr{R}_1 = \dfrac{1}{J_1^2}$

Observation 157. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_1 - c_2J_{2,1} = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. We do not have any remarkable values and remarkable curves for $\mathscr{I}_1$.

(ii.7) $g = -1$ and $c = 0$.

Under this condition the systems do not belong to family (Q). Here we have one invariant line which has multiplicity four (so we compute the exponential factors E_1, E_2 and E_3) and one simple invariant hyperbola. We note that this system is exactly family (T).

Table 325 – Invariant curves, cofactors, singularities and intersection points of system (Q) when $g = -1$ and $c = 0$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = -1 + 3xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{-2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $E_5 = e^{\left(\frac{-3l_0xy+l_0+x(-2l_1xy+l_1+x(l_2+l_3x))}{x^3}\right)}$ $\alpha_1 = -x$ $\alpha_2 = -3x$ $\alpha_3 = g_0$ $\alpha_4 = h_1 + 2h_0y$ $\alpha_5 = l_2 + 2l_1y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PE - EP, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 326 – Divisor and zero-cycles of system (Q) when $g = -1$ and $c = 0$.

Divisor and zero-cycles	Degree
$ICD = 4J_1 + J_2 + \mathscr{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^4\bar{J}_2 = 0.$	7
$M_{0CT} = 6P_1^\infty + 2P_2^\infty$	8

where the total curve T has only two distinct tangents at P_1^∞, but one of them is quintuple.

Table 327 – First integral and integrating factor of system(Q) when $g=-1$ and $c=0$.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_2^{-\frac{\lambda_1}{3}}E_3^{-\frac{(h_1l_1-h_0l_2)\lambda_4}{g_0l_1}}E_4^{\lambda_4}E_5^{-\frac{h_0\lambda_4}{l_1}}$	$R=J_1^{\lambda_1}J_2^{-\frac{4}{3}-\frac{\lambda_1}{3}}E_3^{-\frac{(h_1l_1-h_0l_2)\lambda_4}{g_0l_1}}E_4^{\lambda_4}E_5^{-\frac{h_0\lambda_4}{l_1}}$
Simple example	$\mathscr{I}=\dfrac{J_1^3}{J_2}$	$\mathscr{R}=\dfrac{1}{J_1J_2}$

Observation 158. Consider $\mathscr{F}^1_{(c_1,c_2)}=c_1J_1^3-c_2J_2=0$, $\deg\mathscr{F}^1_{(c_1,c_2)}=3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)}=J_1^3.$$

Therefore, J_1 is a critical remarkable curves and $[1:0]$ is a critical remarkable value of $\mathscr{I}_1$.

(ii.8) $g=1/3$ and $c=0$.

Under this condition the systems do not belong to family (Q). Here we have one affine invariant line and one invariant hyperbola, both of them are triple so we compute the exponential factors E_3, E_4, E_5 and E_6. This system is Hamiltonian so it admits a polynomial first integral.

Table 328 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (Q) when $g=1/3$ and $c=0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1=x$ $J_2=-3+xy$ $E_3=e^{\frac{g_0+g_1x}{x}}$ $E_4=e^{\frac{h_0+h_1x+h_2x^2+\frac{2h_0xy}{3}}{x^2}}$ $E_5=e^{\frac{l_0y}{-3+xy}}$ $E_6=e^{\frac{m_0}{9}+\frac{y(m_1(6-2xy)+3m_2y(2xy-9))}{6(xy-3)^2}}$ $\alpha_1=\frac{x}{3}$ $\alpha_2=-\frac{x}{3}$ $\alpha_3=-\frac{g_0}{3}$ $\alpha_4=-\frac{h_1}{3}-\frac{2h_0y}{3}$ $\alpha_5=-\frac{l_0}{3}$ $\alpha_6=\frac{m_1}{9}-m_2y$	$P_1^\infty=[0:1:0]$ $P_2^\infty=[1:0:0]$ $\emptyset;\binom{4}{2}PEP-PEP,N$	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ double $\bar{J}_1\cap\mathscr{L}_\infty=P_1^\infty$ simple $\bar{J}_2\cap\mathscr{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$

Table 329 – Divisor and zero-cycles of family (Q) when $g = 1/3$ and $c = 0$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + 3J_2 + \mathscr{L}_\infty$	7
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3\bar{J}_2^3 = 0$	10
$M_{0CT} = 7P_1^\infty + 4P_2^\infty$	11

where the total curve T has

1) only three distinct tangents at P_1^∞, but two of them are triple;

2) only two distinct tangents at P_2^∞, but one of them is triple.

Table 330 – First integral and integrating factor of family (Q) when $g = 1/3$ and $c = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} E_3^{\lambda_3} E_4^{\lambda_4} E_5^{-\frac{g_0\lambda_3}{l_0} - \frac{\lambda_4(2h_0m_1+9h_1m_1)}{9l_0m_2}} E_6^{-\frac{2h_0\lambda_4}{3m_2}}$	$R = I$
Simple example	$\mathscr{I}_1 = J_1J_2$	$\mathscr{R}_1 = \dfrac{1}{J_1J_2}$

(ii.9) $g = 1/2$ and $c = 0$.

Under this condition, systems (Q) do not belong to **QSH**. Here we have only one triple invariant line so we compute the exponential factors E_2 and E_3. We have that the line at infinity $Z = 0$ also is triple. Therefore, the total multiplicity of the invariant lines is six so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81.

Table 331 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (Q) when $g = 1/2$ and $c = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $E_2 = e^{\frac{g_0+g_1x}{x}}$ $E_3 = e^{\frac{h_0xy+h_0+x(h_1+h_2x)}{x^2}}$ $E_4 = e^{l_0+l_1xy}$ $\alpha_1 = \frac{x}{2}$ $\alpha_2 = -\frac{g_0}{2}$ $\alpha_3 = -h_0y - \frac{h_1}{2}$ $\alpha_4 = l_1x$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PH - HP, N$	$\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple

Source: Elaborated by the author.

Table 332 – Divisor and zero-cycles of family (Q) when $g = 1/2$ and $c = 0$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + \mathscr{L}_\infty$	4
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3 = 0.$	4
$M_{0CT} = 4P_1^\infty + P_2^\infty$	5

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, one of them triple.

Table 333 – First integral and integrating factor of family (Q) when $g = 1/2$ and $c = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} E_2^0 E_3^0 E_4^{-\frac{\lambda_1}{2l_1}}$	$R = J_1^{\lambda_1} E_2^0 E_3^0 E_4^{-\frac{(1+\lambda_1)}{2l_1}}$
Simple example	$\mathscr{I} = J_1^2 E_4^{-1}$	$\mathscr{R} = \dfrac{1}{J_1}$

Source: Elaborated by the author.

(ii.10) $g = c = 0$.

Under this condition, systems (Q) do not belong to **QSH**. The system here is $\dot{x} = 0,\ \dot{y} = -1 + xy$. This is a degenerate system where the hyperbola $-1 + xy = 0$ filled up with singularities.

Table 334 – Singularities for the reduced system of system of family (Q) when $g = c = 0$.

Singularities
$P_1^\infty = [0:1:0]$ $(\ominus[)(];\emptyset);(\ominus[)(];N,\emptyset)$

Table 335 – First integral and integrating factor for the reduced system of family (Q) when $g = c = 0$.

	First integral	Integrating Factor
General	$I = -x$	$R = -1$
Simple example	$\mathscr{I} = -x$	$\mathscr{R} = -1$

Note that I and $\mathscr{I}$ are also first integrals for family (Q) when $g = c = 0$.

(ii.11) $g = 1$ and $c = 0$.

Under this condition, systems defined by the equations (Q) do not belong to family (Q). The system defined by the equation (Q) when $c = 0$ and $g = 1$ is exactly the family (V). Here the systems possess an invariant line with multiplicity three and a family of invariant hyperbolas

$$1 + rx + xy,$$

where $r \in \mathbb{R}$. The line at infinity $\mathscr{L}_\infty : Z = 0$ has multiplicity 3.

Table 336 – Invariant curves, exponential factors, cofactors, singularities and intersection points of system (Q) when $c = 0$ and $g = 1$.

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = x$ $J_{2,r} = 1 + rx + xy$ $E_3 = e^{g_0 + g_1 y + g_2 y^2}$ $E_4 = e^{\frac{h_0 + h_1 x}{c + x}}$ $E_5 = e^{\frac{l_0 + l_1 x + l_2 x^2 + 2h_0 xy}{x^2}}$ $\alpha_1 = x$ $\alpha_2 = x$ $\alpha_3 = g_1 + 2g_2 x$ $\alpha_4 = -h_0$ $\alpha_5 = -l_1 - 2l_0 y$	$P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $\emptyset; \binom{4}{2} PH - HP, N$	$\bar{J}_1 \cap \bar{J}_{2,r} = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_{2,r} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 337 – Divisor and zero-cycles of system (Q) when $c = 0$ and $g = 1$.

Divisor and zero-cycles	Degree
$ILD = 3J_1 + 3\mathscr{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z^3 \bar{J}_1^3 = 0$	6
$M_{0CT} = 6P_1^\infty + 3P_2^\infty$	9

where the total curve T has

1) only two distinct tangents at P_1^∞, both of them triple and

2) only one triple tangent at P_2^∞.

Table 338 – First integral and integrating factor of system (Q) when $c=0$ and $g=1$.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_{2,r}^{-\lambda_1}E_3^{\lambda_3}E_4^{-\frac{\lambda_3(g_2l_1-g_1l_0)}{h_0l_0}}E_5^{\frac{g_2\lambda_3}{l_0}}$	$R=J_1^{\lambda_1}J_{2,r}^{-2-\lambda_1}E_3^{\lambda_3}E_4^{-\frac{\lambda_3(g_2l_1-g_1l_0)}{h_0l_0}}E_5^{\frac{g_2\lambda_3}{l_0}}$
Simple example	$\mathscr{I}_1=\dfrac{J_1}{J_{2,1}}$	$\mathscr{R}_1=\dfrac{1}{J_1^2}$

Observation 159. Consider $\mathscr{F}^1_{(c_1,c_2)}=c_1J_1-c_2J_{2,1}=0$, $\deg\mathscr{F}^1_{(c_1,c_2)}=2$. We do not have any remarkable values and remarkable curves for $\mathscr{I}_1$.

We sum up the topological, dynamical and algebraic geometric features of family (Q) and also confront our results with previous results in literature in the following proposition.

Proposition 160. (a) For the family (Q) we obtained three distinct configurations $C_1^{(Q)}-C_3^{(Q)}$ of invariant hyperbolas and lines (see Figure 30 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is $cg(g\pm1)(2g-1)(3g-1)=0$. On $c=0$ and $g\neq0$ the invariant lines coalesce and two finite singularities coalesced with an infinite singularity. On $g=0$ and $c\neq0$ the line at infinity has multiplicity two and we have just one invariant line. For the limiting set of the parameter space of the considered family we have the following: On $g=-1$ and $c\neq0$ we have an invariant parabola. On $g=1/3$ and $c\neq0$ we have an additional invariant hyperbola. On $g=1$ we have a family of invariant hyperbolas and the invariant lines coalesce. On $g=1/2$ the invariant hyperbola becomes reducible and we have two invariant lines $c+x=0$ and $x=0$ when $c\neq0$ and only one triple line $x=0$ when $c=0$. On $g=c=0$ the hyperbola is filled up with singularities.

(b) The family (Q) is Darboux integrable in the generic case $cg(g\pm1)(3g-1)(2g-1)\neq0$ and also when $c=0$ and $g\neq0,\pm1,1/3,1/2$. When $g=0$ and $c\neq0$ the family (Q) is generalized Darboux integrable. All systems in family (Q) have an inverse integrating factor which is polynomial.

(c) For the family (Q) we have seven topologically distinct phase portraits $P_1^{(Q)}-P_7^{(Q)}$. The topological bifurcation diagram of family (Q) is done in Figure 31. The bifurcation set is $cg(2g-1)(g-1)=0$ ans it is a bifurcation of singularities. The phase portraits $P_7^{(Q)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 160.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (Q):

Table 339 – Configurations for family (Q).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(Q)}$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + 2P_2 + 4P_1^\infty + 2P_2^\infty$
$C_2^{(Q)}$	$ICD = 3J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = 5P_1^\infty + 2P_2^\infty$
$C_3^{(Q)}$	$ICD = J_1 + J_2 + 2\mathcal{L}_\infty$ $M_{0CT} = P_1 + 4P_1^\infty + 3P_2^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(Q)}$ up to $C_3^{(Q)}$ are all distinct. For the limit cases of family (Q) we have the following configurations:

Table 340 – Configurations for the limit cases of family (Q).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 5P_1^\infty + 3P_2^\infty$
c_2	$ICD = 3J_1 + 3J_2 + \mathcal{L}_\infty$ $M_{0CT} = 7P_1^\infty + 4P_2^\infty$
c_3	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 6P_1^\infty + 2P_2^\infty$
c_4	$ICD = 4J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 2P_2^\infty$
c_5	$ILD = 3J_1 + 3\mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 3P_2^\infty$
c_6	$ICD = J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + 3P_1^\infty + P_2^\infty$
c_7	$ICD = 3J_1 + 3\mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 3P_2^\infty$
c_8	$ICD = \mathcal{L}_\infty$ $M_{0CT} = P_1^\infty$

Source: Elaborated by the author.

The other statements is (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables.

(c) We have:

Table 341 – Phase portraits for family (Q).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(Q)}$	$\left(\binom{2}{2}PEP-EPP,S\right)$ $\left(\binom{2}{2}PPE-PEP,S\right)$	(s,n)	$1SC_f^f \; 4SC_f^\infty \; 1SC_\infty^\infty$
$P_2^{(Q)}$	$\left(\binom{2}{2}PPEP-PEPP,N\right)$	(s,s)	$0SC_f^f \; 8SC_f^\infty \; 0SC_\infty^\infty$
$P_3^{(Q)}$	$\left(\binom{2}{2}HPP-HPP,N\right)$ $\left(\binom{2}{2}PPH-PPH,N\right)$	(s,n)	$1SC_f^f \; 5SC_f^\infty \; 0SC_\infty^\infty$
$P_4^{(Q)}$	$\left(\binom{4}{2}PPE-EPP,S\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 2SC_\infty^\infty$
$P_5^{(Q)}$	$\left(\binom{4}{2}PHP-PHP,N\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 3SC_\infty^\infty$
$P_6^{(Q)}$	$\left(\binom{4}{2}PH-HP,N\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 1SC_\infty^\infty$
$P_7^{(Q)}$	$\left(\binom{2}{2}PEP-EPP,\binom{1}{1}SN\right)$ $\left(\binom{2}{2}PPE-PEP,\binom{1}{1}SN\right)$	s	$0SC_f^f \; 4SC_f^\infty \; 1SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have seven distinct phase portraits for systems (Q). For the limit case of family (Q) we have the following phase portrait:

Table 342 – Phase portraits for the limit case of family (Q).

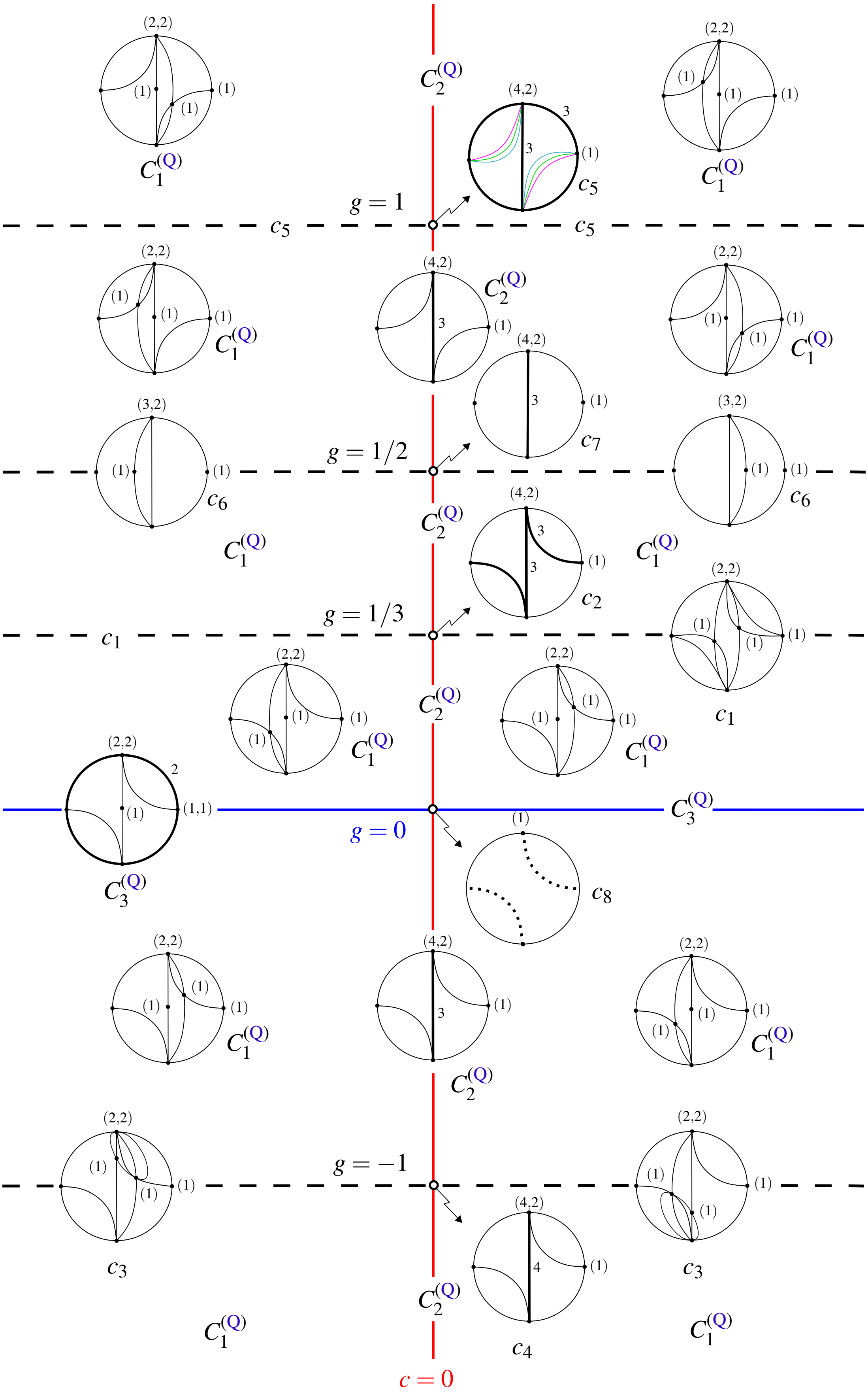
Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(Q)}$	$\left(\binom{2}{2}PEP-EPP,S\right)$ $\left(\binom{2}{2}PPE-PEP,S\right)$	(s,n)	$1SC_f^f \; 4SC_f^\infty \; 1SC_\infty^\infty$
$P_2^{(Q)}$	$\left(\binom{2}{2}PPEP-PEPP,N\right)$	(s,s)	$0SC_f^f \; 8SC_f^\infty \; 0SC_\infty^\infty$
$P_4^{(Q)}$	$\left(\binom{4}{2}PPE-EPP,S\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 2SC_\infty^\infty$
$P_5^{(Q)}$	$\left(\binom{4}{2}PHP-PHP,N\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 3SC_\infty^\infty$
$P_6^{(Q)}$	$\left(\binom{4}{2}PH-HP,N\right)$	$\emptyset$	$0SC_f^f \; 0SC_f^\infty \; 1SC_\infty^\infty$
p_1	$\left(\binom{3}{2}PPEP-HPP,N\right)$ $\left(\binom{3}{2}PPH-PEPP,N\right)$	s	$0SC_f^f \; 4SC_f^\infty \; 1SC_\infty^\infty$
p_2	$(\ominus[)(];N,\emptyset)$	$(\ominus[)(];\emptyset)$	$0SC_f^f \; 0SC_f^\infty \; 0SC_\infty^\infty$

Source: Elaborated by the author.

The phase portrait $P_7^{(Q)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018) since they do not have any phase portrait with 2 singular points at infinity and with only 1 finite singular point.

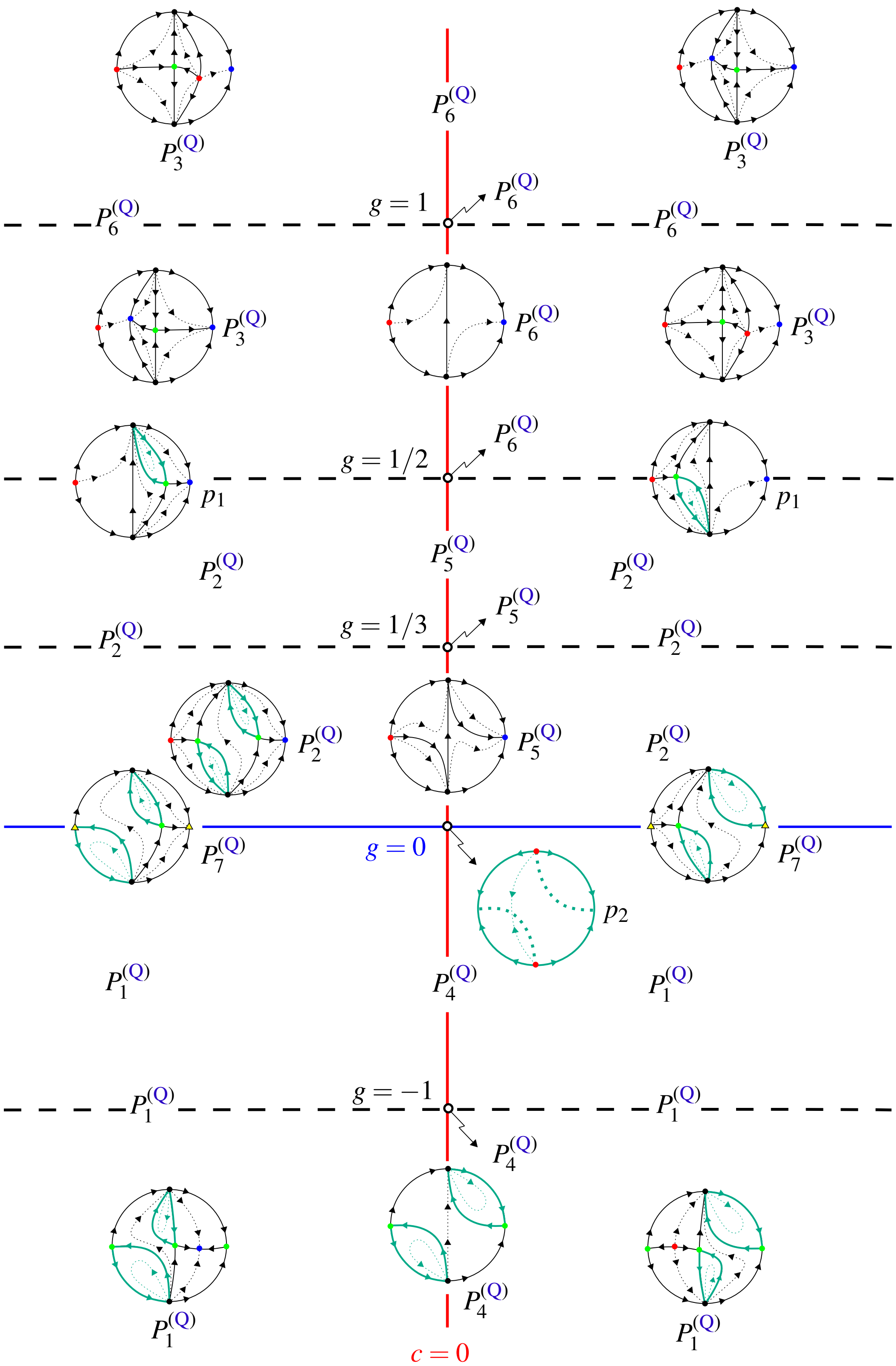
□

Figure 30 – Bifurcation diagram of configurations for family (Q).



Source: Elaborated by the author.

Figure 31 – Topological bifurcation diagram for family (Q).



Source: Elaborated by the author.

Note that the phase portraits $P_1^{(Q)}$, $P_2^{(Q)}$, $P_4^{(Q)}$, $P_7^{(Q)}$, p_1 and p_2 possess graphics.

6.2.3.1 The solution of the Poincaré problem for the family (Q).

The following theorem solves the problem of Poincaré for the family defined by the equations (Q) when $(c, g) \in \mathbb{R}^2$.

Theorem 161. A necessary and sufficient condition for a system (S) defined by the equations (Q) with $(c, g) \in \mathbb{R}^2$ to have a rational first integral given by invariant algebraic curves of degree at most two, is that there exist integers m_1, m_2 such that $g = \frac{m_2}{2m_2+m_1}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$ and $2m_2 + m_1 \neq 0$.

Proof. The proof of this result is based on the formulas obtained for the first integrals of family (Q).

In the generic case $cg(g\pm1)(2g-1)(3g-1) \neq 0$ a first integral of family (Q) is of the form

$$I = J_2^{\lambda_2} J_3^{-\frac{g\lambda_2}{2g-1}} \tag{6.12}$$

where $\lambda_2 \neq 0$ and J_2, J_3 are given in table 304. This is a rational first integral if and only if

$$\begin{cases} \lambda_2 = m_1,\ m_1 \in \mathbb{Z}\backslash\{0\} \\ -\frac{g\lambda_2}{2g-1} = m_2,\ m_2 \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{6.13}$$

Replacing $\lambda_2 = m_1$ in the second equation of (6.13) we obtain

$$-\frac{gm_1}{2g-1} = m_2 \Rightarrow -gm_1 = 2gm_2 - m_2 \Rightarrow g(2m_2+m_1) = m_2 \overset{2m_2\neq -m_1}{\Rightarrow}$$
$$g = \frac{m_2}{2m_2+m_1},\ m_1, m_2 \in \mathbb{Z}\backslash\{0\},\ m_2 \neq -\frac{m_1}{3},\ m_2 \neq \pm m_1 \text{ and } 2m_2 \neq -m_1.$$

Therefore, if I is rational then $g = \frac{m_2}{2m_2+m_1}$ with $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -\frac{m_1}{3}$, $m_2 \neq \pm m_1$ and $2m_2 \neq -m_1$. Conversely, replacing $g = \frac{m_2}{2m_2+m_1}$ and $\lambda_2 = m_1$ in (6.12) where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -\frac{m_1}{3}$, $m_2 \neq \pm m_1$ and $2m_2 \neq -m_1$ we obtain

$$I = J_2^{m_1} J_3^{m_2}$$

which is rational. Therefore, in the generic case, the systems are algebraically integral if and only if $g = \frac{m_2}{2m_2+m_1}$ where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -\frac{m_1}{3}$, $m_2 \neq \pm m_1$ and $2m_2 \neq -m_1$.

Now let us study the non-generic case $cg(g\pm1)(2g-1)(3g-1) = 0$. When $c = 0$ and $g(g\pm1)(2g-1)(3g-1) \neq 0$ then a first integral of family (Q) is of the form

$$I = J_1^{\lambda_1} J_2^{-\frac{g\lambda_1}{2g-1}}$$

where $\lambda_1 \neq 0$ and J_1, J_2 are given in table 307. The proof here is exactly the same of the generic case. Therefore, the systems are algebraically integral if and only if $g = \frac{m_2}{2m_2+m_1}$ where

$m_1, m_2 \in \mathbb{Z}\backslash\{0\}$, $m_2 \neq -\frac{m_1}{3}$, $m_2 \neq \pm m_1$ and $2m_2 \neq -m_1$. When $(3g-1)(g\pm 1)=0$ we have rational first integrals (see Tables 315, 318, 324, 327, 330 and 338). Note that

$$\begin{cases} \frac{m_2}{2m_2+m_1} = -1 \Leftrightarrow m_2 = -\frac{m_1}{3} \\ \frac{m_2}{2m_2+m_1} = \frac{1}{3} \Leftrightarrow m_2 = m_1 \\ \frac{m_2}{2m_2+m_1} = 1 \Leftrightarrow m_2 = -m_1 \end{cases}$$

where $m_1, m_2 \in \mathbb{Z}\backslash\{0\}$. When $g=0$ and $c \neq 0$ then the first integral of family (Q) is of the form

$$I = J_2^{\lambda_2} E_3^{-\frac{\lambda_2}{cg_1}}$$

where $\lambda_2 \neq 0$ and J_2, E_3 are given in table 310. As E_3 is an exponential factor this first integral is always generalized Darboux. So we can not have a rational first integral for this case. When $(2g-1)=0$ and $c \neq 0$ then the first integral of family (Q) is of the form

$$I = J_1^{\lambda_1} J_2^0 E_3^{-\frac{\lambda_1}{2g_1}}$$

where J_1, J_2 and E_3 are given in table 319. As E_3 is an exponential factor this first integral is always generalized Darboux. So we can not have a rational first integral for this case. When $(2g-1)=0$ and $c=0$ then the first integral of family (Q) is of the form

$$I = J_1^{\lambda_1} E_2^0 E_3^0 E_4^{-\frac{\lambda_1}{2l_1}}$$

where J_1, E_2, E_3 and E_3 are given in table 331. As E_4 is an exponential factor this first integral is always generalized Darboux. So we can not have a rational first integral for this case. When $c=g=0$ the system (Q) is degenerate.

□

We note that the set of systems defined by the equations (Q) for $(c,g) \in \mathbb{R}^2$ that are algebraically integrable is dense in $\mathbb{R}^2$.

6.2.4 Geometric Analysis of Family (R)

Consider the family

$$(\mathrm{R}) \begin{cases} \dot{x} = x^2 + \varepsilon \\ \dot{y} = 1 - 2xy. \end{cases}$$

This is a Hamiltonian family in one parameter $\varepsilon \in \mathbb{R}$. We display below the full geometric analysis of this family, which is endowed with at least two invariant curves. When $\varepsilon \neq 0$ every system in family (R) have two invariant lines J_1, J_2 and two invariant hyperbolas J_3 and J_4 with cofactor α_i, $1 \leq i \leq 4$ given by

$$
\begin{array}{ll}
J_1 = \sqrt{\varepsilon} - ix, & \alpha_1 = x - i\sqrt{\varepsilon}, \\
J_2 = \sqrt{\varepsilon} + ix, & \alpha_2 = x + i\sqrt{\varepsilon}, \\
J_3 = -1 + iy\sqrt{\varepsilon} + xy, & \alpha_3 = -x - i\sqrt{\varepsilon}, \\
J_4 = -1 - iy\sqrt{\varepsilon} + xy, & \alpha_3 = -x + i\sqrt{\varepsilon}.
\end{array}
$$

When $\varepsilon = 0$ the lines J_1 and J_2 coalesce as well as the hyperbolas J_3 and J_4 yielding to multiple curves. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

(i) $\varepsilon \neq 0$.

Table 343 – Invariant curves, cofactors, singularities and intersection points of family (R) when $\varepsilon \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = \sqrt{\varepsilon} - ix$ $J_2 = \sqrt{\varepsilon} + ix$ $J_3 = -1 + iy\sqrt{\varepsilon} + xy$ $J_4 = -1 - iy\sqrt{\varepsilon} + xy$ $\alpha_1 = x - i\sqrt{\varepsilon}$ $\alpha_2 = x + i\sqrt{\varepsilon}$ $\alpha_3 = -x - i\sqrt{\varepsilon}$ $\alpha_4 = -x + i\sqrt{\varepsilon}$	$P_1 = \left(-i\sqrt{\varepsilon}, \frac{i}{2\sqrt{\varepsilon}}\right)$ $P_2 = \left(i\sqrt{\varepsilon}, -\frac{i}{2\sqrt{\varepsilon}}\right)$ $P_1^{\infty} = [0:1:0]$ $P_2^{\infty} = [1:0:0]$ For $\varepsilon < 0$ we have $s, s; \binom{2}{2}PPEP - PEPP, N$ For $\varepsilon > 0$ we have $©, ©; \binom{2}{2}H - H, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^{\infty}$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^{\infty}$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_1^{\infty} \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_{\infty} = P_1^{\infty}$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^{\infty} \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = P_1^{\infty}$ double $\bar{J}_2 \cap \mathscr{L}_{\infty} = P_1^{\infty}$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^{\infty} \text{ simple} \\ P_2^{\infty} \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_{\infty} = \begin{cases} P_1^{\infty} \text{ simple} \\ P_2^{\infty} \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_{\infty} = \begin{cases} P_1^{\infty} \text{ simple} \\ P_2^{\infty} \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 344 – Divisor and zero-cycles of family (R) when $\varepsilon \neq 0$.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1+J_2+J_3+J_4+\mathscr{L}_\infty \text{ if } \varepsilon<0 \\ J_1^C+J_2^C+J_3^C+J_4^C+\mathscr{L}_\infty \text{ if } \varepsilon>0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1+P_2+4P_1^\infty+P_2^\infty \text{ if } \varepsilon<0 \\ P_1^C+P_2^C+4P_1^\infty+P_2^\infty \text{ if } \varepsilon>0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1+2P_2+5P_1^\infty+3P_2^\infty \text{ if } \varepsilon<0 \\ 2P_1^C+2P_2^C+5P_1^\infty+3P_2^\infty \text{ if } \varepsilon>0 \end{cases}$	12 12

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1^∞, but two of them are double;

2) three distinct tangents at P_2^∞.

Table 345 – First integral and integrating factor of family (R) when $\varepsilon \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1}J_2^{\lambda_2}J_3^{\lambda_2}J_4^{\lambda_1}$	$R = J_1^{\lambda_1}J_2^{\lambda_2}J_3^{\lambda_2}J_4^{\lambda_1}$
Simple example	$\mathscr{I}_1 = J_1J_4$	$\mathscr{R}_1 = \dfrac{1}{J_1J_4}$

Source: Elaborated by the author.

(ii) $\varepsilon = 0$.

Here the invariant line $x=0$ and the invariant hyperbola $-1+xy=0$ are triple so we compute the exponential factors E_3, E_4, E_5 and E_6. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is four so this case was studied in (SCHLOMIUK; VULPE, 2008c). We include this case as indicated in Observation 81.

Table 346 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (R) when $\varepsilon = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = -1 + xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $E_5 = e^{\frac{l_0y}{-1+xy}}$ $E_6 = e^{\left(m_0+\frac{y(m_1(2-2xy)+m_2y(2xy-3))}{2(-1+xy)^2}\right)}$ $\alpha_1 = x$ $\alpha_2 = -x$ $\alpha_3 = -g_0$ $\alpha_4 = -h_1 - 6h_0y$ $\alpha_5 = -l_0$ $\alpha_6 = m_1 - 3m_2y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PEP-PEP,N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 347 – Divisor and zero-cycles of family (R) when $\varepsilon = 0$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + 3J_2 + \mathscr{L}_\infty$	7
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3\bar{J}_2^3 = 0$	10
$M_{0CT} = 7P_1^\infty + 4P_2^\infty$	11

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1^∞, but two of them are triple;

2) only two distinct tangents at P_2^∞, but one of them is triple.

Table 348 – First integral and integrating factor of family (R) when $\varepsilon = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} E_3^{\lambda_3} E_4^{-\frac{\lambda_6 m_2}{2h_0}} E_5^{\frac{\lambda_6(2h_0m_1+h_1m_2)}{2h_0l_0} - \frac{g_0\lambda_3}{l_0}} E_6^{\lambda_6}$	$R = I$
Simple example	$\mathscr{I}_1 = J_1J_2$	$\mathscr{R}_1 = \dfrac{1}{J_1J_2}$

Source: Elaborated by the author.

We sum up the topological, dynamical and algebraic geometric features of family (R) and we confront our results with previous results in the literature in the following proposition.

Proposition 162. (a) For the family (R) we have three distinct configurations $C_1^{(R)}$, $C_2^{(R)}$ and $C_3^{(R)}$ of invariant hyperbolas and lines (see Figure 32 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations contains only the point $\varepsilon = 0$. Its complement is a union of 2 disjoint sets. On $\varepsilon = 0$ we have a triple line and a triple hyperbola arising from the coalescence of two lines and two hyperbolas.

(b) The family (R) admits a polynomial first integral that foliates the plane into cubic invariant algebraic curves. All systems in family (R) have an inverse integrating factor which is polynomial.

(c) For the family (R) we have three topologically distinct phase portraits $P_1^{(R)}$, $P_2^{(R)}$ and $P_3^{(R)}$. The topological bifurcation diagram is done in Figure 33. The bifurcation set is the point $\varepsilon = 0$ and it is a bifurcation of singularities. The phase portrait $P_2^{(R)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018).

Proof of Proposition 162.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (R):

Table 349 – Configurations for family (R).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(R)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 5P_1^\infty + 3P_2^\infty$
$C_2^{(R)}$	$ICD = J_1^C + J_2^C + J_3^C + J_4^C + \mathscr{L}_\infty$ $M_{0CT} = 2P_1^C + 2P_2^C + 5P_1^\infty + 3P_2^\infty$
$C_3^{(R)}$	$ICD = 3J_1 + 3J_3 + \mathscr{L}_\infty$ $M_{0CT} = 7P_1^\infty + 4P_2^\infty$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(R)}$, $C_2^{(R)}$ and $C_3^{(R)}$ are distinct.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables.

(c) We have:

Table 350 – Phase portraits for family (R).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(R)}$	$\binom{2}{2}PPEP-PPEP,\ N$	(s,s)	$0SC_f^f\ \ 8SC_f^\infty\ \ 0SC_\infty^\infty$
$P_2^{(R)}$	$\binom{2}{2}H-H,N$	$(©,©)$	$0SC_f^f\ \ 0SC_f^\infty\ \ 0SC_\infty^\infty$
$P_3^{(R)}$	$\binom{4}{2}PHP-PHP,N$	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 3SC_\infty^\infty$

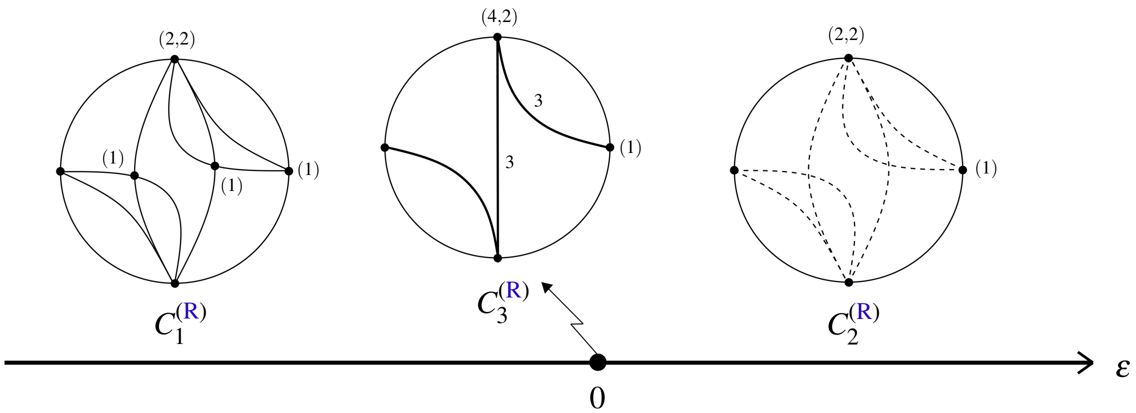
Source: Elaborated by the author.

Therefore, we have three distinct phase portraits for systems (R).

The phase portraits $P_2^{(R)}$ is not topologically equivalent with anyone of the phase portraits in (LLIBRE; YU, 2018) (see table 284).

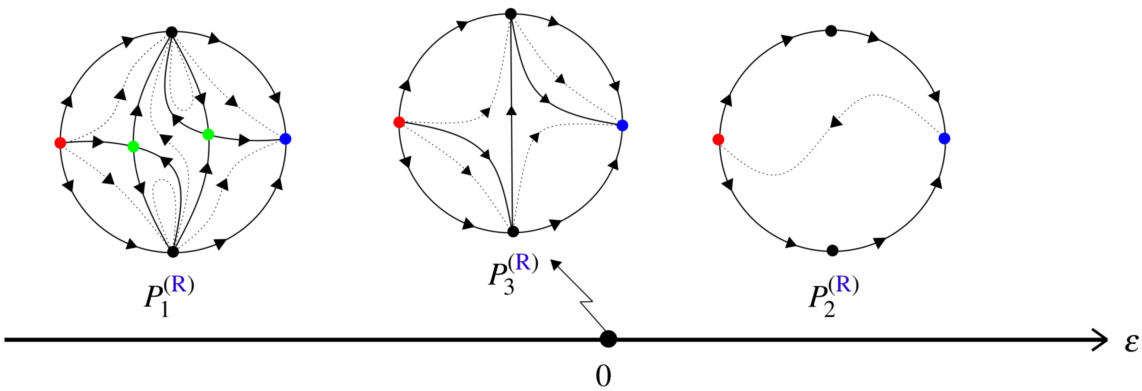
□

Figure 32 – Bifurcation diagram of configurations for family (R).



Source: Elaborated by the author.

Figure 33 – Topological bifurcation diagram for family (R).



Source: Elaborated by the author.

6.2.5 Geometric Analysis of Normal Form (S)

Consider the system

$$(S)\begin{cases} \dot{x} = (x-1)(3-x) \\ \dot{y} = 1-2xy. \end{cases}$$

System (S) is endowed with two invariant lines, one invariant hyperbola and one invariant parabola. The multiplicities of each invariant curve appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola and the parabola.

Table 351 – Invariant curves, cofactors, singularities and intersection points of system (S).

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = 1-x$ $J_2 = 3-x$ $J_3 = -\frac{1}{3} - y + xy$ $J_4 = -\frac{19}{8} + x + 3y - \frac{x^2}{8}$ $\alpha_1 = 3-x$ $\alpha_2 = 1-x$ $\alpha_3 = 3-3x$ $\alpha_4 = -2x$	$P_1 = \left(1, \frac{1}{2}\right)$ $P_2 = \left(3, \frac{1}{6}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, n; \binom{2}{2}PPPE - PPEP, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathscr{L}_\infty = P_1^\infty$ double

Source: Elaborated by the author.

Table 352 – Divisor and zero-cycles of system (S).

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0.$	7
$M_{0CT} = 2P_1 + 3P_2 + 5P_1^\infty + 2P_2^\infty$	12

Source: Elaborated by the author.

where the total curve T has

1) only three distinct tangents at P_1^∞, two of them double and one simple,

2) two distinct tangents at P_2, but one of them is double.

Table 353 – First integral and integrating factor of system (S).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\frac{\lambda_2}{3}} J_4^{\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{4}{3}-\lambda_1-\frac{\lambda_2}{3}} J_4^{\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_2^3}{J_3}$	$\mathscr{R} = \dfrac{1}{J_1 J_2 J_4}$

Source: Elaborated by the author.

Observation 163. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_2^3 - c_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:-24]$ and $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,-24)} = -8J_1J_4, \quad \mathscr{F}^1_{(1,0)} = J_2^2.$$

Therefore, J_1, J_2, J_4 are remarkable curves and $[1:-24]$, $[1:0]$ are remarkable values of $\mathscr{I}_1$. Moreover, $[1:0]$ is a critical remarkable values and J_2 is critical remarkable curve of $\mathscr{I}_1$. The singular point are P_1 for $\mathscr{F}^1_{(1,-24)}$ and P_2 for $\mathscr{F}^1_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (R) in the following proposition.

Proposition 164. (a) For system (S) we have one configuration $C_1^{(S)}$ of invariant hyperbola, parabola and lines (see Figure 34).

(b) The system (S) has a rational first integral and the plane is foliated into cubic invariant algebraic curves. The remarkable curves for system (S) are J_1, J_2 and J_4. The system (S) has an inverse integrating factor which is polynomial.

(c) The phase portrait of system (S) is $P_1^{(S)}$ in Figure 34.

Proof of Proposition 164.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (S):

Table 354 – Configuration of system (S).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(S)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 5P_1^\infty + 2P_2^\infty$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves of system (S) were done in Remark 163.

(c) We have:

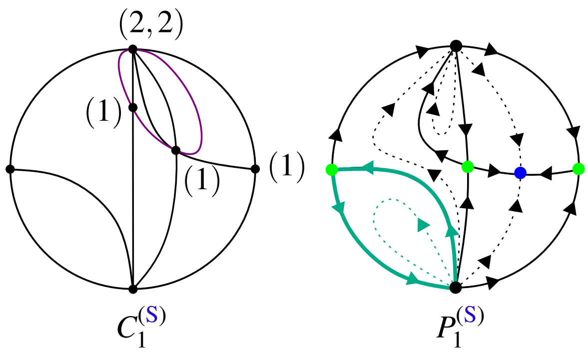
Table 355 – Phase portrait for system (S).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(S)}$	$\binom{2}{2}PEP-EPP,\ S$	(s,n)	$1SC_f^f\ \ 4SC_f^\infty\ \ 1SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 34 – Configuration and phase portrait for system(S).



Source: Elaborated by the author.

Note that the phase portrait $P_1^{(S)}$ possess a graphic in the third quadrant.

6.2.6 Geometric Analysis of Normal Form (T)

Consider the system

$$(T)\begin{cases}\dot{x}=-x^2\\ \dot{y}=1-2xy.\end{cases}$$

System (T) is endowed with one invariant line of multiplicity four and also with one invariant hyperbola. Considering the line at infinity $Z=0$ the total multiplicity of the invariant lines is five so this case was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. The multiplicities of the invariant curves appearing in the divisor ICD were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

Table 356 – Invariant curves, exponential factors, cofactors, singularities and intersection points of system (T).

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = -1+3xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{-2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $E_5 = e^{\left(\frac{-3l_0xy+l_0+x(-2l_1xy+l_1+x(l_2+l_3x))}{x^3}\right)}$ $\alpha_1 = -x$ $\alpha_2 = -3x$ $\alpha_3 = g_0$ $\alpha_4 = h_1+2h_0y$ $\alpha_5 = l_2+2l_1y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PE-EP,S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 357 – Divisor and zero-cycles of system (T).

Divisor and zero-cycles	Degree
$ICD = 4J_1+J_2+\mathscr{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^4\bar{J}_2 = 0.$	7
$M_{0CT} = 6P_1^\infty + 2P_2^\infty$	8

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1^∞, but one of them is quintuple.

Table 358 – First integral and integrating factor of system(T).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{\lambda_1}{3}} E_3^{-\frac{(h_1l_1-h_0l_2)\lambda_4}{g_0l_1}} E_4^{\lambda_4} E_5^{-\frac{h_0\lambda_4}{l_1}}$	$R = J_1^{\lambda_1} J_2^{-\frac{4}{3}-\frac{\lambda_1}{3}} E_3^{-\frac{(h_1l_1-h_0l_2)\lambda_4}{g_0l_1}} E_4^{\lambda_4} E_5^{-\frac{h_0\lambda_4}{l_1}}$
Simple example	$\mathscr{I} = \dfrac{J_1^3}{J_2}$	$\mathscr{R} = \dfrac{1}{J_1J_2}$

Source: Elaborated by the author.

Observation 165. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_1^3 - c_2J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^3.$$

Therefore, J_1 is a critical remarkable curves and $[1:0]$ is a critical remarkable value of $\mathscr{I}_1$.

We sum up the topological, dynamical and algebraic geometric features of family (T) in the following proposition.

Proposition 166. (a) For system (T) we have one configuration $C_1^{(T)}$ of invariant hyperbolas and lines (see Figure 35).

(b) The system (T) has a rational first integral and the plane is foliated into cubic invariant algebraic curves. The remarkable curve for system (T) is J_1 . The system (T) have an inverse integrating factor which is polynomial.

(c) The phase portrait of system (T) is $P_1^{(T)}$ in Figure 35.

Proof of Proposition 166.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (T):

Table 359 – Configuration of system (T).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(T)}$	$ICD = 4J_1 + J_2 + \mathscr{L}_\infty$ $M_{0CT} = 6P_1^\infty + 2P_2^\infty$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remark 165.

(c) We have:

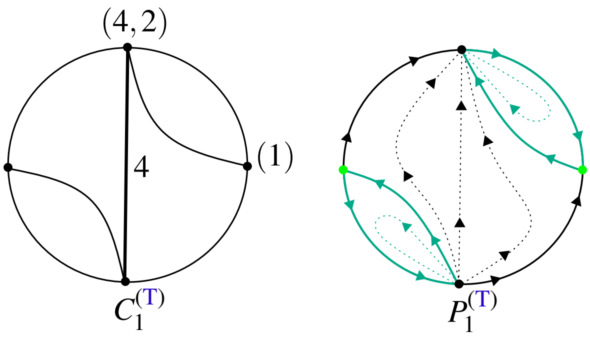
Table 360 – Phase portrait for system (T).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(T)}$	$\binom{4}{2}PE-EP,\ S$	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 2SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 35 – Configuration and phase portrait for system (T).



Source: Elaborated by the author.

Note that the phase portrait $P_1^{(T)}$ possess graphics in the first and in the third quadrant.

6.2.7 Geometric Analysis of Normal Form (U)

Consider the system

$$(U) \begin{cases} \dot{x} = (2x-1)(2x+1)/4 \\ \dot{y} = y. \end{cases}$$

System (U) is endowed with a family of invariant hyperbolas $m\left(x-\frac{1}{2}\right)+2xy+y$, where $m \neq 0$, and with three affine invariant lines, one of them double. The line at infinity $\mathscr{L}_\infty : Z = 0$ is also double. Therefore, as the total multiplicity of the invariant lines is six this system was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. The multiplicities of each invariant line appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

Table 361 – Invariant curves, exponential factors, cofactors, singularities and intersection points of system (U).

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = 1+2x$ $J_2 = 1-2x$ $J_3 = y$ $J_{4,m} = m\left(x-\frac{1}{2}\right)+2xy+y$ $E_5 = e^{\frac{-2g_0x+g_0+g_1y}{1-2x}}$ $E_6 = e^{h_0+h_1y}$ $\alpha_1 = x-\frac{1}{2}$ $\alpha_2 = x+\frac{1}{2}$ $\alpha_3 = 1$ $\alpha_4 = x+\frac{1}{2}$ $\alpha_5 = \frac{g_1y}{2}$ $\alpha_6 = h_1y$	$P_1 = \left(-\frac{1}{2},0\right)$ $P_2 = \left(\frac{1}{2},0\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s,n;\binom{2}{2}PPH-PPH,N$	$\bar{J}_1\cap\bar{J}_2 = P_1^\infty$ simple $\bar{J}_1\cap\bar{J}_3 = P_1$ simple $\bar{J}_1\cap\bar{J}_{4,m} = P_1^\infty$ double $\bar{J}_1\cap\mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_2\cap\bar{J}_3 = P_2$ simple $\bar{J}_2\cap\bar{J}_{4,m} = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3\cap\bar{J}_{4,m} = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple}\end{cases}$ $\bar{J}_3\cap\mathscr{L}_\infty = P_2^\infty$ simple $\bar{J}_{4,m}\cap\mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple}\end{cases}$

Source: Elaborated by the author.

Table 362 – Divisor and zero-cycles of system (U).

Divisor and zero-cycles	Degree
$ILD = J_1+2J_2+J_3+2\mathscr{L}_\infty$	6
$M_{0CS} = P_1+P_2+4P_1^\infty+1P_2^\infty$	7
$T = Z^2\bar{J}_1\bar{J}_2^2\bar{J}_3 = 0$	6
$M_{0CT} = 2P_1+3P_2+5P_1^\infty+3P_2^\infty$	13

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_2, but one of them is double;

2) only three distinct tangents at P_1^∞, but two of them are double and

3) only two distinct tangents at P_2^∞, but one of them is double.

Table 363 – First integral and integrating factor of system (U).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_1} J_{4,m}^{-\lambda_1-\lambda_2} E_5^{\lambda_5} E_6^{-\frac{g_1\lambda_5}{h_1}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_1} J_{4,m}^{-2-\lambda_1-\lambda_2} E_5^{\lambda_5} E_6^{-\frac{g_1\lambda_5}{h_1}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_3}{J_2}$	$\mathscr{R}_1 = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 167. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_3 - c_2 J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1 J_3.$$

Therefore, J_1 and J_3 are remarkable curves and $[1:0]$ is remarkable value of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (U) in the following proposition.

Proposition 168. (a) For system (U) we have one configuration $C_1^{(U)}$ of invariant hyperbolas and lines (see Figure 36).

(b) The system (U) has a rational first integral and the plane is foliated into quadratic invariant algebraic curves. The remarkable curves for system (U) are J_1 and J_3. The system (U) have an inverse integrating factor which is polynomial.

(c) The phase portrait of system (U) is $P_1^{(U)}$ in Figure 36.

Proof of Proposition 168.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (U):

Table 364 – Configuration for system (U).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(U)}$	$ILD = J_1 + 2J_2 + J_3 + 2\mathscr{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 5P_1^\infty + 3P_2^\infty$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remark 167.

(c) We have:

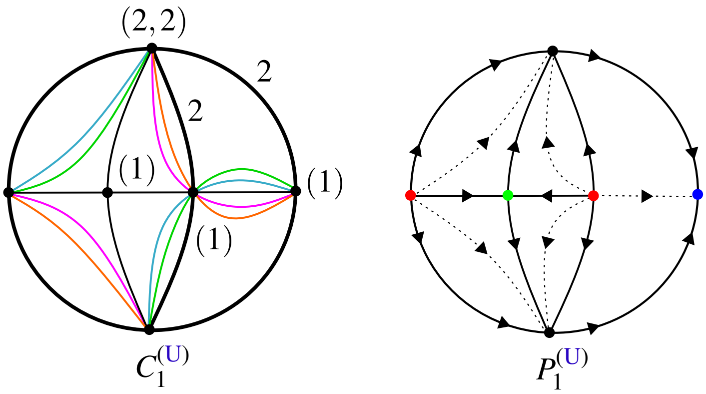
Table 365 – Phase portraits for system (U).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(U)}$	$\binom{2}{2}PPH-PPH,\ N$	s,n	$1SC_f^f\ \ 5SC_f^\infty\ \ 0SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 36 – Configuration and phase portrait for system (U).



Source: Elaborated by the author.

6.2.8 Geometric Analysis of Normal Form (V)

Consider the system

$$(V)\ \begin{cases} \dot{x} = x^2 \\ \dot{y} = 1. \end{cases}$$

System (V) is endowed with a family of invariant hyperbolas $1+mx+xy$, where $m \in \mathbb{R}$, and with one affine invariant line which is triple. The line at infinity $\mathscr{L}_\infty : Z = 0$ is also triple. Therefore, as the total multiplicity of the invariant lines is six this system was studied in (SCHLOMIUK; VULPE, 2008b). We include this case as indicated in Observation 81. The multiplicities of the invariant lines appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

Table 366 – Invariant curves, exponential factors, cofactors, singularities and intersection points of system (V).

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = x$ $J_{2,m} = 1 + mx + xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $E_5 = e^{l_0+l_1y}$ $E_6 = e^{m_0+m_1y+m_2y^2}$ $\alpha_1 = x$ $\alpha_2 = 2x$ $\alpha_3 = -g_0$ $\alpha_4 = -2h_0y - h_1$ $\alpha_5 = l_1$ $\alpha_6 = m_1 + 2m_2y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PH-HP,N$	$\bar{J}_1 \cap \bar{J}_{2,m} = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_{2,m} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 367 – Divisor and zero-cycles of system (V).

Divisor and zero-cycles	Degree
$ILD = 3J_1 + 3\mathscr{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z^3\bar{J}_1^3 = 0$	6
$M_{0CT} = 6P_1^\infty + 3P_2^\infty$	9

Source: Elaborated by the author.

where the total curve T has

1) only two distinct tangents at P_1^∞, both of them double and

2) only one distinct tangent at P_2^∞.

Table 368 – First integral and integrating factor of system (V).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_{2,m}^{-\lambda_1} E_3^{\lambda_3} E_4^{\lambda_4} E_5^{\frac{g_0\lambda_3}{l_1} - \frac{\lambda_4(h_0m_1-h_1m_2)}{l_1m_2}} E_6^{\frac{h_0\lambda_4}{m_2}}$	$R = J_1^{\lambda_1} J_{2,m}^{-2-\lambda_1} E_4^{\lambda_4} E_5^{\frac{g_0\lambda_3}{l_1} - \frac{\lambda_4(h_0m_1-h_1m_2)}{l_1m_2}} E_6^{\frac{h_0\lambda_4}{m_2}}$
Simple example	$\mathscr{I}_1 = \frac{J_1}{J_{2,1}}$	$\mathscr{R}_1 = \frac{1}{J_1^2}$

Source: Elaborated by the author.

Observation 169. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_{2,1} = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. We do not have any remarkable values and remarkable curves for $\mathscr{I}_1$.

We sum up the topological, dynamical and algebraic geometric features of family (V) in the following proposition.

Proposition 170. (a) For system (V) we have one configuration $C_1^{(V)}$ of invariant hyperbolas and lines (see Figure 37).

(b) The system (V) has a rational first integral and the plane is foliated into quadratic invariant algebraic curves. The system (V) have an inverse integrating factor which is polynomial.

(c) The phase portrait of system (V) is $P_1^{(V)}$ in Figure 37.

Proof of Proposition 170.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (V):

Table 369 – Configuration for system (V).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(V)}$	$ILD = 3J_1 + 3\mathscr{L}_\infty$ $M_{0CT} = 6P_1^\infty + 3P_2^\infty$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables.

(c) We have:

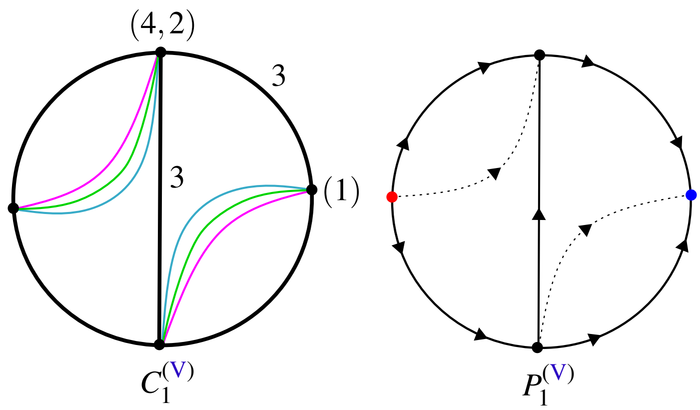
Table 370 – Phase portrait for system (V).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(V)}$	$\binom{4}{2} PH-HP,\ N$	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 1SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 37 – Configuration and phase portrait for system (V).



Source: Elaborated by the author.

6.2.9 Geometric Analysis of Family (W)

Consider the family

$$(\text{W}) \begin{cases} \dot{x} = a + y + x^2 \\ \dot{y} = xy. \end{cases}$$

This is an algebraically integrable family in one parameter $a \in \mathbb{R}$. Every system in the family (W) is endowed with at lest one invariant line and with a family of invariant hyperbolas. When $a \neq 0$ we have three invariant lines J_1, J_2, J_3 and a family of invariant hyperbolas $J_{4,m}$ with cofactors α_1, α_4 given by

$$\begin{aligned} &J_1 = y, \ \ \alpha_1 = x \\ &J_2 = x - \tfrac{i(a+y)}{\sqrt{a}}, \ \ \alpha_2 = x + i\sqrt{a} \\ &J_3 = x + \tfrac{i(a+y)}{\sqrt{a}}, \ \ \alpha_3 = x - i\sqrt{a} \\ &J_{4,m} = a + 2y + x^2 - m^2y^2, \ \ \alpha_4 = 2x \end{aligned}$$

where $m \in \mathbb{R}\backslash\{0\}$ and $a \neq -1/m^2$. When $a = 0$ the lines J_2 and J_3 coalesce with J_1 yielding a triple invariant line. The line at infinity $\mathscr{L}_\infty : Z = 0$ is filled up with singularities. Therefore, this family was studied in (SCHLOMIUK; VULPE, 2008a) but we include this case here as indicated in Observation 81. The multiplicities of the invariant lines appearing in the divisor ILD of invariant algebraic line were calculated by using the 1st extactic polynomial.

(i) $a \neq 0$.

Table 371 – Invariant curves, cofactors, singularities and intersection points of family (W) for $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x - \frac{i(a+y)}{\sqrt{a}}$ $J_3 = x + \frac{i(a+y)}{\sqrt{a}}$ $J_{4,m} = a + 2y + x^2 - m^2y^2$ $\alpha_1 = x$ $\alpha_2 = x + i\sqrt{a}$ $\alpha_3 = x - i\sqrt{a}$ $\alpha_4 = 2x$	$P_1 = (0, -a)$ $P_2 = \left(-i\sqrt{a}, 0\right)$ $P_3 = \left(i\sqrt{a}, 0\right)$ For $a < 0$ we have $s, n, n; [\infty; \emptyset]$ For $a > 0$ we have $c, ©, ©; [\infty; \emptyset]$	$\overline{J}_1 \cap \overline{J}_2 = P_3$ simple $\overline{J}_1 \cap \overline{J}_3 = P_2$ simple $\overline{J}_1 \cap \overline{J}_{4,m} = \begin{cases} P_2 \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathscr{L}_\infty = [1:0:0]$ simple $\overline{J}_2 \cap \overline{J}_3 = P_1$ simple $\overline{J}_2 \cap \overline{J}_{4,m} = P_3$ double $\overline{J}_2 \cap \mathscr{L}_\infty = [1:-i\sqrt{a}:0]$ simple if $a < 0$ $\overline{J}_3 \cap \overline{J}_{4,m} = P_2$ double $\overline{J}_3 \cap \mathscr{L}_\infty = [1:i\sqrt{a}:0]$ simple if $a < 0$ $\overline{J}_{4,m} \cap \mathscr{L}_\infty = \begin{cases} [m:1:0] \text{ simple} \\ [-m:1:0] \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 171. We see here that taking J_1 and $J_{4,\overline{m}}$ for some $\overline{m}$, the conditions of the theorem of C-K are satisfied and hence we can also have an inverse integrating factor as $J_1 J_{4,\overline{m}}$.

Table 372 – Divisor and zero-cycles of family (W) for $a \neq 0$.

Divisor and zero-cycles	Degree
$ILD = \begin{cases} J_1 + J_2 + J_3 \text{ if } a < 0 \\ J_1 + J_2^C + J_3^C \text{ if } a > 0 \end{cases}$	3 3
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 \text{ if } a < 0 \\ P_1 + P_2^C + P_3^C \text{ if } a > 0 \end{cases}$	3 3
$T = \overline{J}_1 \overline{J}_2 \overline{J}_3 = 0$	3
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3 \text{ if } a < 0 \\ 2P_1 + 2P_2^C + 2P_3^C \text{ if } a > 0 \end{cases}$	6 6

Source: Elaborated by the author.

Table 373 – First integral and integrating factor of family (W) for $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_{4,m}^{-\frac{\lambda_1}{2} - \lambda_2}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_{4,m}^{-\frac{3}{2} - \frac{\lambda_1}{2} - \lambda_2}$
Simple example	$\mathscr{I}_1 = \frac{J_1^2}{J_2 J_3}$ $\mathscr{I}_2 = \frac{J_2 J_3}{J_{4,\overline{m}}}$	$\mathscr{R}_1 = \frac{1}{J_1 J_2 J_3}$ $\mathscr{R}_2 = \frac{1}{J_1 J_{4,\overline{m}}}$

Source: Elaborated by the author.

Observation 172. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1J_1^2 - c_2J_2J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:0]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^2, \quad \mathscr{F}^1_{(0,1)} = -J_2J_3.$$

Therefore, J_1, J_2, J_3 are remarkable curves and $[1:0]$, $[0:1]$ are remarkable values of $\mathscr{I}_1$. Moreover, $[1:0]$ is a critical remarkable values and J_1 is critical remarkable curve of $\mathscr{I}_1$. The singular point are P_1 for $\mathscr{F}^1_{(0,1)}$ and P_2, P_3 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1J_2J_3 - c_2J_{4,\overline{m}}$ we have the remarkable values $[1:1]$ and $[1:0]$ and the same remarkable curves J_1, J_2, J_3. The singular point are P_2, P_3 for $\mathscr{F}^2_{(1,1)}$ and P_1 for $\mathscr{F}^2_{(1,0)}$.

(ii) $a = 0$.

Here the invariant line $y = 0$ is triple so we compute the exponential factor E_3 and E_4. This system has a rational first integral that foliates the plane into quadratic invariant algebraic curves.

Table 374 – Invariant curves, exponential factors, cofactors, singularities and intersection points of family (W) for $a = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_{2,m} = 2y + x^2 - m^2y^2$ $E_3 = e^{\frac{g_0x+g_1y}{y}}$ $E_4 = e^{\frac{2h_0y^2+2y(h_1+h_2x)+h_1x^2}{2y^2}}$ $\alpha_1 = x$ $\alpha_2 = 2x$ $\alpha_3 = g_0$ $\alpha_4 = h_2$	$P_1 = (0,0)$ $es_{(3)}; [\infty; \emptyset]$	$\overline{J}_1 \cap \overline{J}_{2,m} = P_1$ double $\overline{J}_1 \cap \mathscr{L}_\infty = [1:0:0]$ simple $\overline{J}_{2,m} \cap \mathscr{L}_\infty = \begin{cases} [m:1:0] \text{ simple} \\ [-m:1:0] \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 375 – Divisor and zero-cycles of family (W) for $a = 0$.

Divisor and zero-cycles	Degree
$ILD = 3J_1$	3
$M_{0CS} = 3P_1$	3
$T = \overline{J}_1^3 = 0$	3
$M_{0CT} = 3P_1$	3

Source: Elaborated by the author.

where the total curve T has one triple tangent at P_1.

Table 376 – First integral and integrating factor of family (W) for $a = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_{2,m}^{-\frac{\lambda_1}{2}} E_3^{\lambda_3} E_4^{-\frac{g_0\lambda_3}{h_2}}$	$R = J_1^{\lambda_1} J_{2,m}^{-\frac{3}{2}-\frac{\lambda_1}{2}} E_3^{\lambda_3} E_4^{-\frac{g_0\lambda_3}{h_2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2}{J_{2,1}}$	$\mathscr{R}_1 = \dfrac{1}{J_1^3}$

Source: Elaborated by the author.

Observation 173. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 - c_2 J_{2,1} = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^2.$$

Therefore, $[1:0]$ is a critical remarkable values and J_1 is critical remarkable curve of $\mathscr{I}_1$. The singular point is P_1 for $\mathscr{F}^1_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (W) in the following proposition.

Proposition 174. (a) For the family (W) we have three distinct configurations $C_1^{(W)}$, $C_2^{(W)}$ and $C_3^{(W)}$ of invariant hyperbolas and lines (see Figure 38 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations contains only the point $a = 0$. On $a = 0$ we have one triple invariant line arising from the coalescence of the three invariant lines.

(b) The family (W) is algebraically integrable. When $a \neq 0$ the plane is foliated into quadratic invariant algebraic curves and the remarkable curves are J_1, J_2, J_3. When $a = 0$ the plane is foliated into quadratic invariant algebraic curves and the remarkable curve is J_1. All systems in family (W) have an inverse integrating factor which is polynomial.

(c) For the family (W) we have three topologically distinct phase portraits $P_1^{(W)}$, $P_2^{(W)}$ and $P_3^{(W)}$. The topological bifurcation diagram is done in Figure 39. The bifurcation set is the singleton $a = 0$ and it is a bifurcation point of singularities.

Proof of Proposition 174.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (W):

Table 377 – Configurations for family (W).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(W)}$	$ILD = J_1 + J_2 + J_3$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3$
$C_2^{(W)}$	$ILD = J_1 + J_2^C + J_3^C$ $M_{0CT} = 2P_1 + 2P_2^C + 2P_3^C$
$C_3^{(W)}$	$ILD = 3J_1$ $M_{0CT} = 3P_1$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(W)}$, $C_2^{(W)}$ and $C_3^{(W)}$ are distinct.

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 172 and 173.

(c) We have:

Table 378 – Phase portraits for family (W).

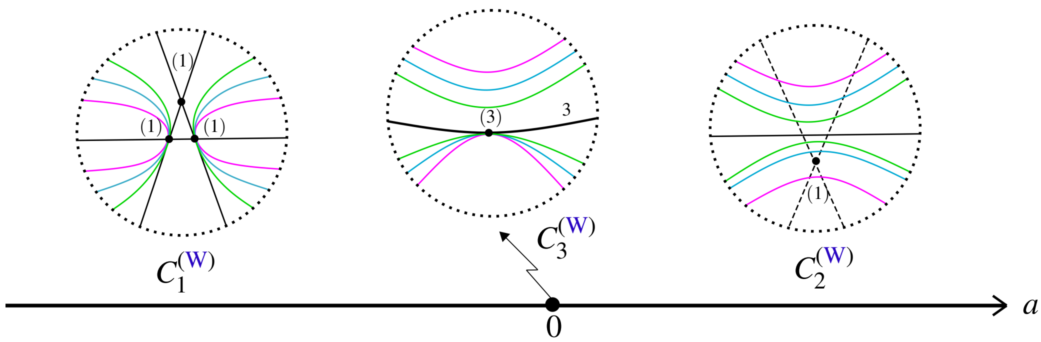
Phase Portraits	Sing. at ∞	Sing. at $< \infty$	Separatrix connections
$P_1^{(W)}$	$[\infty, \emptyset]$	(s,n,n)	$2SC_f^f$ $2SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(W)}$	$[\infty, \emptyset]$	c	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(W)}$	$[\infty, \emptyset]$	$es_{(3)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have three distinct phase portraits for systems (W).

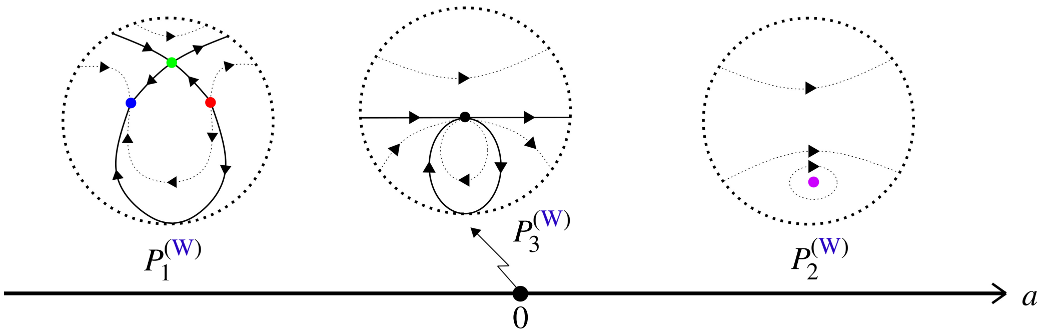
□

Figure 38 – Bifurcation diagram of configurations for family (W).



Source: Elaborated by the author.

Figure 39 – Topological bifurcation diagram for family (W).



Source: Elaborated by the author.

6.2.10 Geometric Analysis of Normal Form (X)

Consider the system

$$(X)\begin{cases} \dot{x} = (1+3x)(2+3x)/9 \\ \dot{y} = xy. \end{cases}$$

System (X) is endowed with three invariant lines and with a family of invariant hyperbolas $(2+3x)^2+m(3x+1)y$, where $m \neq 0$. The line at infinity $\mathscr{L}_\infty : Z = 0$ is filled up with singularities. Therefore, this system was studied in (SCHLOMIUK; VULPE, 2008a) but we include this case here as indicated in Observation 81. The multiplicities of each invariant line appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

Table 379 – Invariant curves, cofactors, singularities and intersection points of system (X).

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = 2+3x$ $J_3 = 1+3x$ $J_{4,m} = (2+3x)^2+m(3x+1)y$ $\alpha_1 = x$ $\alpha_2 = \frac{1}{3}+x$ $\alpha_3 = \frac{2}{3}+x$ $\alpha_4 = \frac{2}{3}+2x$	$P_1 = (-2/3, 0)$ $P_2 = (-1/3, 0)$ $P_1^\infty = [0:1:0]$ $n, s;\ [\infty; N]$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_2$ simple $\bar{J}_1 \cap \bar{J}_{4,m} = P_1$ double $\bar{J}_1 \cap \mathscr{L}_\infty = [1:0:0]$ simple $\bar{J}_2 \cap \bar{J}_3 = [0:1:0]$ simple $\bar{J}_2 \cap \bar{J}_{4,m} = \begin{cases} [0:1:0] \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathscr{L}_\infty = [0:1:0]$ simple $\bar{J}_3 \cap \bar{J}_{4,m} = [0:1:0]$ double $\bar{J}_3 \cap \mathscr{L}_\infty = [0:1:0]$ simple $\bar{J}_{4,m} \cap \mathscr{L}_\infty = \begin{cases} [0:1:0] \text{ simple} \\ [1:-\frac{3}{m}:0] \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 380 – Divisor and zero-cycles of system (X).

Divisor and zero-cycles	Degree
$ILD = J_1 + J_2 + J_3$	3
$M_{0CS} = P_1 + P_2 + P_1^{\infty}$	3
$T = \overline{J}_1 \overline{J}_2 \overline{J}_3 = 0$	3
$M_{0CT} = 2P_1 + 2P_2$	4

Source: Elaborated by the author.

where the total curve T has only two distinct tangents at P_1, but one of them is double.

Table 381 – First integral and integrating factor of system (X).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_1} J_{4,m}^{-\lambda_1 - \frac{\lambda_2}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_1} J_{4,m}^{-\frac{3}{2} - \lambda_1 - \frac{\lambda_2}{2}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1 J_3}{J_2^2}$	$\mathscr{R}_1 = \dfrac{1}{J_1 J_2 J_3}$

Source: Elaborated by the author.

Observation 175. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 J_3 - c_2 J_2^2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:0]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1 J_3, \quad \mathscr{F}^1_{(0,1)} = -J_2^2.$$

Therefore, J_1, J_2, J_3 are remarkable curves and $[1:0]$, $[0:1]$ are remarkable values of $\mathscr{I}_1$. Moreover, $[0:1]$ is a critical remarkable values and J_2 is critical remarkable curve of $\mathscr{I}_1$. The singular point are P_2 for $\mathscr{F}^1_{(1:0)}$ and P_1 for $\mathscr{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (X) in the following proposition.

Proposition 176. (a) For system (X) we have one configuration $C_1^{(X)}$ of invariant hyperbolas and lines (see Figure 40).

(b) The system (X) has a rational first integral and the plane is foliated into quadratic invariant algebraic curves. The remarkable curves are J_1, J_2, J_3. The system (X) have an inverse integrating factor which is polynomial.

(c) The phase portrait of system (X) is $P_1^{(X)}$ in Figure 40.

Proof of Proposition 176.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (X):

Table 382 – Configurations for system (X).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(X)}$	$ILD = J_1 + J_2 + J_3$ $M_{0CT} = 2P_1 + 2P_2$

Source: Elaborated by the author.

(b) It follows directly from the tables. The computations for the remarkable curves were done in Remark 175.

(c) We have:

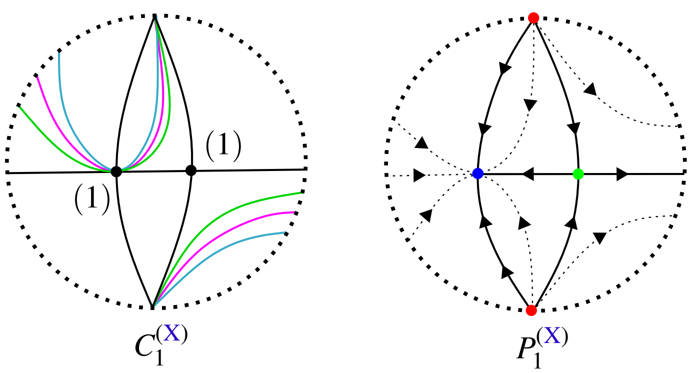
Table 383 – Phase portrait for system (X).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(X)}$	$[\infty, N]$	(n, s)	$1SC_f^f$ $5SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 40 – Configuration and phase portrait for system (X).



Source: Elaborated by the author.

6.2.11 Geometric Analysis of Family (Y)

Consider the family

$$(Y) \begin{cases} \dot{x} = a + x^2 \\ \dot{y} = xy, \end{cases}$$

where $a \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (Y) we study here also the limit case $a = 0$ where the equations are still defined. Every system in the family (Y) is endowed with three invariant lines and with a family of invariant hyperbolas $a + x^2 - m^2y^2$, where $m \neq 0$. The line at infinity $\mathscr{L}_\infty : Z = 0$ is filled up with singularities. Therefore, this family was studied in (SCHLOMIUK; VULPE, 2008a) but we include this case here as indicated in Observation 81. The multiplicities of each invariant line appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

(i) $a \neq 0$.

Table 384 – Invariant curves, cofactors, singularities and intersection points of family (Y) for $a \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = 1 - \frac{ix}{\sqrt{a}}$ $J_3 = 1 + \frac{ix}{\sqrt{a}}$ $J_{4,m} = a + x^2 - m^2y^2$ $\alpha_1 = x$ $\alpha_2 = x - i\sqrt{a}$ $\alpha_3 = x + i\sqrt{a}$ $\alpha_4 = 2x$	$P_1 = \left(-i\sqrt{a}, 0\right)$ $P_2 = \left(i\sqrt{a}, 0\right)$ $P_1^\infty = [0:1:0]$ For $a < 0$ we have $n, n; [\infty; S]$ For $a > 0$ we have ©, ©; $[\infty; C]$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_2$ simple $\bar{J}_1 \cap \bar{J}_{4,m} = \begin{cases} P_1 \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathscr{L}_\infty = [1:0:0]$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_{4,m} = P_1$ double $\bar{J}_2 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_{4,m} = P_2$ double $\bar{J}_3 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_{4,m} \cap \mathscr{L}_\infty = \begin{cases} [m:1:0] \text{ simple} \\ [-m:1:0] \text{ simple} \end{cases}$

Source: Elaborated by the author.

Observation 177. We see here that taking J_1 and $J_{4,\overline{m}}$ for some $\overline{m}$, the conditions of the theorem of C-K are satisfied and hence we can also have an inverse integrating factor as $J_1 J_{4,\overline{m}}$.

Table 385 – Divisor and zero-cycles of family (Y) for $a \neq 0$.

Divisor and zero-cycles	Degree
$ILD = \begin{cases} J_1 + J_2 + J_3 \text{ if } a < 0 \\ J_1 + J_2^C + J_3^C \text{ if } a > 0 \end{cases}$	3 3
$M_{0CS} = \begin{cases} P_1 + P_2 + P_1^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_1^\infty \text{ if } a > 0 \end{cases}$	3 3
$T = \overline{J}_1 \overline{J}_2 \overline{J}_3 = 0$	3
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 \text{ if } a < 0 \\ 2P_1^C + 2P_2^C \text{ if } a > 0 \end{cases}$	4 4

Source: Elaborated by the author.

Table 386 – First integral and integrating factor of family (Y) for $a \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_{4,m}^{-\frac{\lambda_1}{2} - \lambda_2}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_{4,m}^{-\frac{3}{2} - \frac{\lambda_1}{2} - \lambda_2}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2}{J_2 J_3} \quad \mathscr{I}_2 = \dfrac{J_2 J_3}{J_{4,\overline{m}}}$	$\mathscr{R}_1 = \dfrac{1}{J_1 J_2 J_3} \quad \mathscr{R}_2 = \dfrac{1}{J_1 J_{4,\overline{m}}}$

Source: Elaborated by the author.

Observation 178. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 - c_2 J_2 J_3 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ are $[1:0]$ and $[0:1]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^2, \quad \mathscr{F}^1_{(0,1)} = -J_2 J_3.$$

Therefore, J_1, J_2, J_3 are remarkable curves and $[1:0]$, $[0:1]$ are remarkable values of $\mathscr{I}_1$. Moreover, $[1:0]$ is a critical remarkable values and J_1 is critical remarkable curve of $\mathscr{I}_1$. The singular point are P_1 and P_2 for $\mathscr{F}^1_{(1,0)}$.

Considering the first integral $\mathscr{I}_2$ with its associated curve $\mathscr{F}^2_{(c_1,c_2)} = c_1 J_2 J_3 - c_2 J_{4,\overline{m}}$ we have the remarkable values $[a:1]$ and $[1:0]$ and have the same remarkable curves J_1, J_2, J_3. The singular point are P_1, P_2 for $\mathscr{F}^2_{(a,1)}$.

(ii) $a = 0$.

Under this condition the system does not belong to family (Y). The system here is $\dot{x} = x^2$, $\dot{y} = xy$. This is a degenerate system where the line $x = 0$ is filled up with singularities. The affine invariant lines are $y = 0$ and $x + y = 0$ that are both simple. This system has a rational first integral that foliates the plane into quadratic invariant curves.

Table 387 – Invariant curves, cofactors, singularities and intersection points for the reduced system of family (Y) when $a = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x + y$ $\alpha_1 = 1$ $\alpha_2 = 1$	$P_1 = (0,0)$ $(\ominus[\|];n^*);[\infty;(\ominus[\|];\emptyset_2)]$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathscr{L}_\infty = [1:0:0]$ simple $\bar{J}_2 \cap \mathscr{L}_\infty = [1:1:0]$ simple

Source: Elaborated by the author.

Table 388 – Divisor and zero-cycles for the reduced system of family (Y) when $a = 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2$	2
$M_{0CS} = P_1$	1
$T = \bar{J}_1 \bar{J}_2$	2
$M_{0CT} = 2P_1$	2

Source: Elaborated by the author.

Table 389 – First integral and integrating factor for the reduced system of family (Y) when $a = 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1}$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1}{J_2}$	$\mathscr{R}_1 = \dfrac{1}{J_1 J_2}$

Source: Elaborated by the author.

Note that I and $\mathscr{I}_1$ are also first integrals for family (Y) when $a = 0$.

Observation 179. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 1$. We do not have any remarkable values and remarkable curves for $\mathscr{I}_1$.

We sum up the topological, dynamical and algebraic geometric features of family (Y) in the following proposition.

Proposition 180. (a) For the family (Y) we have two distinct configurations $C_1^{(Y)}$ and $C_2^{(Y)}$ of invariant hyperbolas and lines (see Figure 41 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is the point $a = 0$. For the limiting set of the parameter space, i.e. on $a = 0$ the line $x = 0$ is filled up with singularities.

(b) The family (Y) has a rational first integral and and the plane is foliated into quadratic invariant algebraic curves. The remarkable are J_1, J_2, J_3. All systems in family (Y) have an inverse integrating factor which is polynomial.

(c) For the family (Y) we have two topologically distinct phase portraits $P_1^{(Y)}$, $P_2^{(Y)}$. The topological bifurcation diagram in the full parameter space is done in Figure 42. The bifurcation set is the point $a = 0$ and it is a bifurcation of singularities.

Proof of Proposition 180.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (Y):

Table 390 – Configurations for family (Y).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(Y)}$	$ILD = J_1 + J_2 + J_3$ $M_{0CT} = 2P_1 + 2P_2$
$C_2^{(Y)}$	$ILD = J_1 + J_2^C + J_3^C$ $M_{0CT} = 2P_1^C + 2P_2^C$

Source: Elaborated by the author.

Therefore, the configurations $C_1^{(Y)}$ and $C_2^{(Y)}$ are distinct. We have the following configuration for the limit case:

Table 391 – Configuration for the limit case of family (Y).

Configuration	Divisor and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2$ $M_{0CT} = 2P_1$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remark 178.

(c) We have:

Table 392 – Phase portraits for family (Y).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(Y)}$	$[\infty, S]$	(n, n)	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(Y)}$	$[\infty, C]$	$(©, ©)$	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

Therefore, we have two distinct phase portraits for systems (Y). We have the following phase portrait for the limit case:

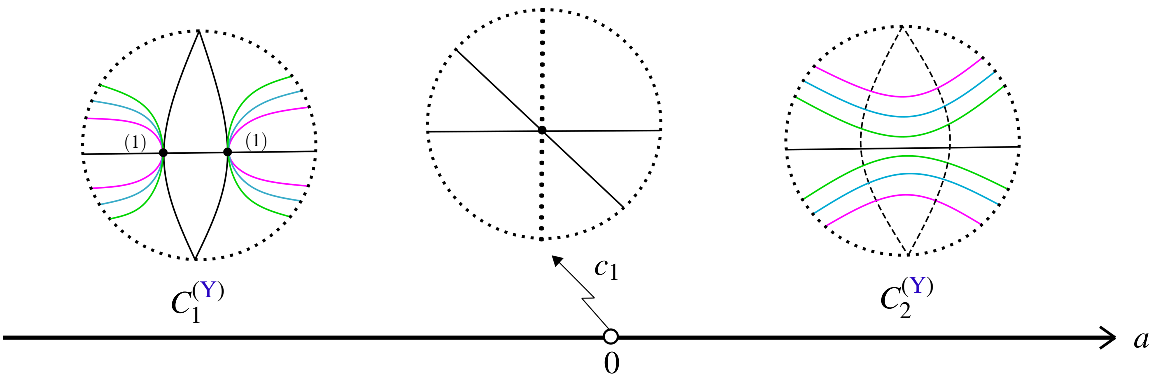
Table 393 – Phase portrait for the limit cases of family (Y).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_1	$[\infty;(\ominus[|];\emptyset_2)]$	$(\ominus[|];n^*)$	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$

Source: Elaborated by the author.

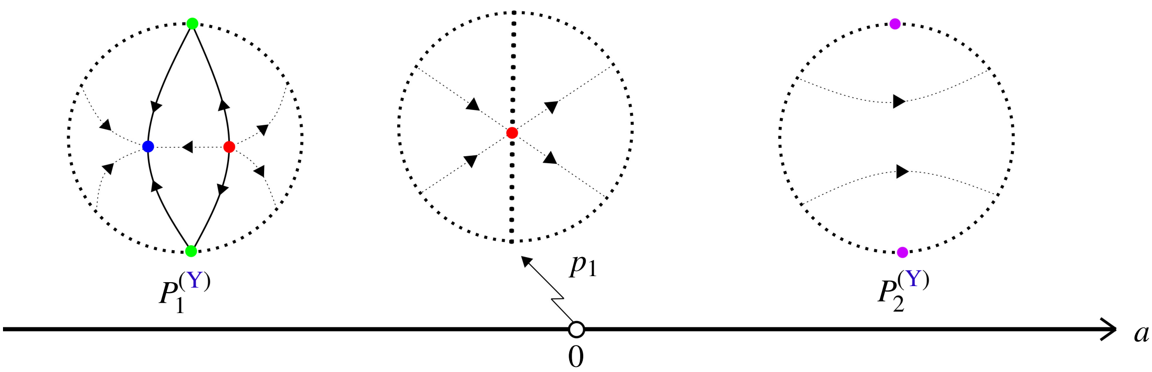
□

Figure 41 – Bifurcation diagram of configurations for family (Y).



Source: Elaborated by the author.

Figure 42 – Topological bifurcation diagram for family (Y).



Source: Elaborated by the author.

6.2.12 Geometric Analysis of Normal Form (Z)

Consider the system

$$(Z)\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1 + xy. \end{cases}$$

System (Z) is endowed with one invariant line which is triple and with a family of invariant hyperbolas $1+mx^2+2xy$, where $m \in \mathbb{R}$. The line at infinity $\mathscr{L}_\infty : Z = 0$ is filled up with singularities. Therefore, this system was studied in (SCHLOMIUK; VULPE, 2008a) but we include this case here as indicated in Observation 81. The multiplicities of each invariant line appearing in the divisor ILD of invariant algebraic lines were calculated by using the 1st extactic polynomial.

Table 394 – Invariant curves, exponential factors, cofactors, singularities and intersection points for system (Z).

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = x$ $J_{2,m} = 1+mx^2+2xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $\alpha_1 = x$ $\alpha_2 = 2x$ $\alpha_3 = -g_0$ $\alpha_4 = -h_1$	$P_1^\infty = [0:1:0]$ $\emptyset;\ [\infty; \binom{3}{0})ES]$	$\bar{J}_1 \cap \bar{J}_{2,m} = P_1^\infty$ double $\bar{J}_1 \cap \mathscr{L}_\infty = P_1^\infty$ simple $\bar{J}_{2,m} \cap \mathscr{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ [1 : -\frac{m}{2} : 0] \text{ simple} \end{cases}$

Source: Elaborated by the author.

Table 395 – Divisor and zero-cycles for system (Z).

Divisor and zero-cycles	Degree
$ILD = 3J_1$	3
$M_{0CS} = 3P_1^\infty$	3
$T = \bar{J}_1^3 = 0$	3

Source: Elaborated by the author.

Table 396 – First integral and integrating factor for system (Z).

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_{2,m}^{-\frac{\lambda_1}{2}} E_3^{\lambda_3} E_4^{-\frac{g_0\lambda_3}{h_1}}$	$R = J_1^{\lambda_1} J_{2,m}^{\frac{-3-\lambda_1}{2}} E_3^{\lambda_3} E_4^{-\frac{g_0\lambda_3}{h_1}}$
Simple example	$\mathscr{I}_1 = \dfrac{J_1^2}{J_{2,1}}$	$\mathscr{R}_1 = \dfrac{1}{J_1^3}$

Source: Elaborated by the author.

Observation 181. Consider $\mathscr{F}^1_{(c_1,c_2)} = c_1 J_1^2 - c_2 J_{2,1} = 0$, $\deg \mathscr{F}^1_{(c_1,c_2)} = 2$. The remarkable value of $\mathscr{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathscr{F}^1_{(1,0)} = J_1^2.$$

Therefore, J_1 is remarkable curve and $[1:0]$ is remarkable value of $\mathscr{I}_1$. Moreover, $[1:0]$ is a critical remarkable value and J_1 is critical remarkable curve of $\mathscr{I}_1$.

We sum up the topological, dynamical and algebraic geometric features of family (Z) in the following proposition.

Proposition 182. (a) For system (Z) we have one configuration $C_1^{(Z)}$ of invariant hyperbolas and lines (see Figure 43).

(b) The system (Z) has a rational first integral and the plane is foliated into quadratic invariant algebraic curves. The remarkable curve is J_1. The system (Z) has an inverse integrating factor which is polynomial.

(c) The phase portrait of system (Z) is $P_1^{(Z)}$ in Figure 43.

Proof of Proposition 182.

(a) We have the following type of divisor and zero-cycle of the total invariant curve T for the configuration of system (Z):

Table 397 – Configuration for system (Z).

Configuration	Divisor and zero-cycle of the total inv. curve T
$C_1^{(Z)}$	$ILD = 3J_1$

Source: Elaborated by the author.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remark 181.

(c) We have the following phase portrait for system (Z):

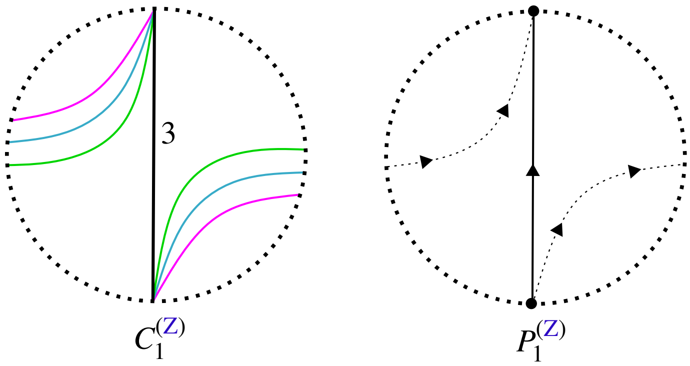
Table 398 – Phase portrait for system (Z).

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(Z)}$	$[\infty; \binom{3}{0} ES]$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$

Source: Elaborated by the author.

□

Figure 43 – Configuration and phase portrait for system (Z).



Source: Elaborated by the author.

CHAPTER 7

QUESTIONS AND CONCLUDING COMMENTS

As we have seen in the geometric analysis of the families we discussed in Chapter 6, the class **QSH** forms a rich testing ground for exploring integrability in terms of the global algebraic geometric features of the systems occurring in these normal forms. The geometric analysis of the systems we studied brings out a number of questions. We expect to find answers to some of these questions once the full study of all the normal forms of **QSH** will be completed.

7.1 The problem of generalizing the Christopher-Kooij Theorem 61

We saw that under the "generic" conditions of Christopher and Kooij (C-K), formulated algebraically on the algebraic invariant curves $f_1(x,y),\ldots,f_k(x,y)$ of a polynomial differential system, we are assured to have a polynomial inverse integrating factor of the special form

$$f_1(x,y)\ldots f_k(x,y).$$

In this article we see cases where these "generic conditions" of (C-K) are not satisfied and yet we still have an integrating factor which is polynomial. Furthermore, in some cases, this polynomial inverse integrating factor is of the same form as the one in the (C-K) theorem. Here are some examples occurring in the families we considered.

(I) For the family (H).

(1) All the systems in family (H) have an inverse integrating factor which is polynomial, they are Darboux integrable and have in the generic case only two invariant lines J_1, J_2 and two invariant hyperbolas J_3, J_4. An inverse polynomial factor is $J_1J_2J_3J_4$ just like in C-K theorem. The condition (a) of the C-K theorem 61 is satisfied since our curves are lines and hyperbolas

and they are, of course, non-singular and irreducible. The condition (b) is also satisfied since the coefficients in M_{0ST} are all equal to 2. The condition (c) is not satisfied because both of the hyperbolas J_3 and J_4 intersect the line at infinity at P_1^∞ and they are tangent at this point. The condition (d) is not satisfied because the sum of the degrees of the curves is 6 and not 3. However, the conclusion is the same as in theorem 61.

(2) In the non-generic cases $(a-3v^2/4)(a+3v^2)(a-8v^2/9)=0$ we have a similar situation and an inverse polynomial integrating factor. We have, as in the generic case, the two invariant lines J_1, J_2 and we have, apart from the two invariant hyperbolas J_4, J_5 and additional invariant curve J_3. We again have (a) and (b) satisfied but not (c) and (d) of (C-K) theorem 61. However, if we restrict our attention only to the remarkable curves J_1, J_2, J_4, J_5 then we still have an inverse integrating factor of the form $J_1J_2J_4J_5$ as in the (C-K) theorem.

(II) Consider now the family (J).

(1) The systems in the family (J) have in the generic case three invariant lines J_1, J_2, J_3 and two invariant hyperbolas J_4, J_5. Let us now consider for our discussion only the remarkable curves, the three lines J_1, J_2, J_3 and the hyperbola J_4. These of course satisfy the conditions (a) and (b). However they do not satisfy (c) because for example J_1, J_2, J_4 intersect at P_2. They also do not satisfy (d). If we limit our attention to the four curves J_1, J_2, J_3, J_4 we see that we have as an inverse integrating factor the polynomial $J_1J_2J_3J_4$ which we get by taking in the general expression of the integrating factor $\lambda_1=\lambda_2=\lambda_4=-1$.

So we can ask the following questions:

Question 1: *How should the geometry of the configuration of algebraic solutions $J_1, \ldots, J_k$ of a polynomial system be so as to have an inverse integrating factor which is polynomial? In particular, how should this geometry be in order to have an inverse integrating factor $J_1 \ldots J_k$? How could we relax, generalize, the hypotheses in the (C-K) theorem such that the same conclusion holds?*

Question 2: *If a systems has a rational first integral do we always have an inverse integrating factor involving only remarkable curves?*

Consider now the non-generic case $v(a-v^2)=0$ of the family (H). We have that one of the invariant curves becomes double. In the case $v=0$ we have two simple invariant lines J_1, J_2 and one double invariant hyperbola J_3. A polynomial inverse integrating factor in this case is $J_1J_2J_3^2$. In the case $a-v^2=0$ we have a double line J_1 and two simple hyperbolas J_2, J_3. We have a polynomial integrating factor $J_1^2J_2J_3$. In this case we still have a polynomial inverse integrating factor.

Question 3: *Can we generalize the (C-K) Theorem 61 so as to include multiplicity?*

In what cases there is a relation between the multiplicity s of an algebraic solution J_i and the exponent of J_i appearing in the polynomial inverse integrating factor?

7.2 The problem of Poincaré

For 19 of the 23 families studied here we give an answer to Poincaré's problem. For 6 of them this answer is entirely geometric (see, for instance, Theorem 96 in Subsection 6.1.1.1) and for 13 of them the solution is straightforward from the expressions of the first integral, i. e., the families are algebraically integrable (see for instance, the geometric analysis of family (S)).

For the 4 remaining normal forms for which we did not give an answer to Poincaré's problem there eighter occurs: i) not Liouvillian integrability, or ii) a general kind of Liouvillian integrability, or iii) generalized Darboux integrability but not Darboux integrability, or iv) open case regarding the integrability.

7.3 The existence of Exponential Factors

We saw in Theorems 42 and 44 that the existence of an exponential factor is associated with the existence of a multiple invariant algebraic curve (affine or the line at infinity). Let us recall some examples that appear in our work where we could find an 'special' kind of exponential factor.

- For the family (K) when $b = -1/4$.

  This is a single system that admits two invariant lines J_1 and J_2 and one invariant hyperbola J_3. By computing the 1st extactic polynomial we see that J_1 and J_2 are simple. By computing the 2nd extactic polynomial we see that J_3 is simple. By computing the 1st extactic polynomial for the compactified field we see that the line at infinity $\mathscr{L}_\infty : Z = 0$ is also simple. However we could find the exponential factor:

  $$E_4 = e^{\frac{G(x,y)}{4J_3}}$$

  where $G(x,y) = 4\big(-4g_0(4x+51)y^2+4g_0(31x+41)y+g_0+4g_1y(x(5y-39)+64y-52)+g_1x\big)$, $g_0 \in \mathbb{R}$. We emphasize that the existence of such first integral does not contradicts Theorem 42 since $\deg G(x,y) = 3$.

  Considering family (K) when $b = -1/4$ we have three finite singular points: P_1, P_2 and P_3 where P_1 is a double point which is located in the intersection of J_1 and J_3. Motivated by this fact we raise the following question:

  Question 4: *Could the existence of this exponential factor be related to the existence of this multiple singularity? Or this exponential factor comes from a multiple curve which is product of the invariant algebraic curves J_1, J_2, J_3?*

- For the family (Q) when $g = 1/2$ and $c \neq 0$.

  This is a family in one parameter $a \in \mathbb{R}\backslash\{0\}$ that admits two invariant lines J_1 and J_2. By computing the 1st extactic polynomial we see that J_1 and J_2 are simple. By computing the 1st extactic polynomial for the compactified field we see that the line at infinity $\mathscr{L}_\infty : Z = 0$ is also simple. However we could find the exponential factor:

  $$E_3 = e^{g_0 + cg_1 y + g_1 xy}.$$

  Considering the compactified field we obtained the exponential factor:

  $$\overline{E}_3 = e^{\frac{g_1 Y + cg_1 YZ + g_0 Z^2}{Z^2}}.$$

  Considering family (Q) when $g = 1/2$ and $c \neq 0$ we have two infinite singular points: P_1^∞ and P_2^∞. The singularity P_1^∞ is of multiplicity 5 (of type $\binom{3}{2}$)) and it is located in the intersection of J_1 and $Z = 0$. Motivated by this fact we raise the following question:

  Question 5: *Could the existence of this exponential factor be related to the existence of this multiple singularity at infinity?*

7.4 On the bifurcation diagrams

We have two kinds of bifurcation diagrams: topological and geometrical, i.e., of geometric configurations of algebraic solutions (lines and hyperbolas). In this Section we are interested in the relationship between these two bifurcation diagrams, more precisely we show how the dynamics of the systems expressed in their topological bifurcations impacts the bifurcations of the geometry of the configurations and the resulting bifurcations in integrability.

7.4.1 *Family* (H)

In all the families here the topological bifurcation set of the phase portraits is a subset of the bifurcation set of configurations of algebraic solutions. This inclusion is strict for some families.

The bifurcation set $Bif_{(\mathrm{H})}$ for topological phase portraits in the family (H) is formed by the half-line of $v = 0$, $a < 0$ ($(Bif_{(\mathrm{H})})^{(1)}$); the non-zero points on the parabola $a = v^2$ ($(Bif_{(\mathrm{H})})^{(2)}$).

On $(Bif_{(\mathrm{H})})^{(1)}$ and on $(Bif_{(\mathrm{H})})^{(2)}$ 4 real finite singular points coalesce into 2 real finite double points. In the first case, after crossing the half-line they split again into 4 real singular points, while in the second case they split into 4 complex finite singular points which are finite points of intersection of the complexifications of each one of two real hyperbolas with the two complex invariant lines, respectively.

It is interesting to observe that these topological bifurcation points have an impact on the bifurcation set of geometrical configurations. Indeed, first we mention that above and below the half-line $v = 0$ and $a < 0$ we have two couples of real singularities, the points in each couple are located on distinct branches of one hyperbola. When two singular points on different hyperbolas coalesce this yields the coalescence of the respective branches and also of the two hyperbolas, producing a double hyperbola.

On the non-zero points of the parabola $a = v^2$ the coalescence of the 4 real finite singular points into two couples of double real singular points yields the coalescence of the two lines into a double (even real) line which afterwards splits into two complex lines.

We note that we have a saddle to saddle connection on the parabola $a = v^2$ for $(a, v) \neq (0, 0)$.

On the bifurcation points situated on the remaining three parabolas we either have the occurrence of an additional hyperbola (on $a - 8v^2/9 = 0$ or on $a + 3v^2 = 0$) or the appearance of an additional invariant line (on $a - 3v^2/4 = 0$). The presence of these additional invariant curves does not affect in any way the bifurcation diagram of the systems.

In conclusion we have:

(i) Impact of the topological bifurcations on the bifurcations of configurations: The bifurcation points of singularities located on the algebraic solutions, when singular points become multiple, become also bifurcation points for the multiplicity of the algebraic solutions, inducing coalescence of the respective curves and hence of their geometric configuration.

(ii) The bifurcation points of configurations due to the appearance of additional invariant curves (three hyperbolas instead of two or three lines instead of two lines) have no consequence for the topological bifurcation diagram of this family.

(iii) Inside the parabola $a = v^2$ where we have complex singularities we have no bifurcation points of phase portraits but we have, on the half-line $v = 0$, $a > 0$ bifurcation points of configurations, the two hyperbolas coalescing into a double hyperbola. Here we need to stress the fact that on this half-line we have two double complex singularities and while this fact has no impact on the topological bifurcation it is important for the bifurcations of the configurations. Indeed, when the four complex singularities become two double complex singularities on this half-line, the two hyperbolas on which they are lying coalesce becoming a double hyperbola.

Limit points of the bifurcation diagram for (H) Let us discuss the bifurcation phenomena which occur at the limiting points of our parameter space for systems in the family (H), i.e. the points on $a = 0$. The topological bifurcation on this line is easy to understand. Indeed, except for the the point $(0, 0)$ where all four singularities collide, all the other points on $a = 0$ are bifurcation points of saddle to saddle connections. All the points on the line $a = 0$ are also points of bifurcation of configurations of algebraic solutions. However this bifurcation is a bit harder to understand. Indeed, at these points say on $v > 0$ we have a configuration with three *simple* affine invariant lines, the vertical line intersecting the two parallel line at two points and forming a saddle-to saddle connection. It is clear that this configuration splits into the configuration $C_1^{(H)}$ on the left which has two hyperbolas and two invariant lines. So in some sense the configuration on $a = 0$ should be considered as a *multiple configuration* since it yields new algebraic solutions. Analyzing the bifurcation phenomenon we see that each one of the two hyperbolas splits into two lines on $a = 0$ and $v > 0$. Indeed, the hyperbola J_4 splits into the line $x = 0$ and the line J_1 and the hyperbola J_3 splits into J_2 and $x = 0$. So that although for $a = 0$ each one of the lines is simple, each line contributes to the multiplicity of the configuration. Considering the composite cubic curve $xJ_1J_2 = 0$ we may say that this configuration has (geometric) multiplicity two in this family as it splits into two cubic curves J_1J_4 and J_2J_3. On the other hand we see that we have on $a = 0$ an exponential factor involving in its exponent at the denominator of the rational function, the polynomial xJ_1J_2 which turns out to be of integrable multiplicity two. The notions of integrable multiplicity and geometric multiplicity in (CHRISTOPHER; LLIBRE; PEREIRA, 2007) are not restricted to algebraic solutions. But the authors say there clearly that the equivalence between integrable and geometric multiplicities occurs only for integrable solutions. In the example above these two multiplicities coincide. So we have the following

Question 6: *Under what condition on (finite) configurations of algebraic solutions do the two multiplicities, integrable and geometric coincide?*

7.4.2 Family (P)

The parameter space for this family is $\{(a,c) \in \mathbb{R}^2 : a \neq 0\}$, its topological bifurcation set is $(a-8c^2/9)(a-c^2) = 0$ and it is formed of bifurcation points of finite singularities. On $a-8c^2/9 = 0$ we see coalescence of two finite singularities, both situated *on the same one of the two invariant lines, yielding a double singular point on this line*. On $a-c^2 = 0$ we have two coalesces, but each one of them being a *coalescence of two points situated on distinct lines yielding a double singular point situated on a double line*.

The bifurcation set for configurations of invariant lines and hyperbolas is also $(a-8c^2/9)(a-c^2) = 0$. On $a-8c^2/9 = 0$ the coalescence of the two singularities on the same line yielding a double singular point generates a distinct configuration than the one in the generic case surrounding points on this parabola where none of the singularities is double. As already mentioned above, on the parabola $a-c^2 = 0$ we get a double line due to the coalescence of the four singularities located on the two lines into two double singularities on the double line. Can we explain in a similar way the appearance of a double hyperbola on the parabola $a-8c^2/9 = 0$? Within this family this is however not possible. Indeed moving in all directions from a point on this parabola, we always get just one hyperbola, no two hyperbolas coalesce when on $a-8c^2/9 = 0$ in the parameter space of this family.

We claim however that the same kind of phenomenon occurs as on $a-c^2 = 0$, namely that a bifurcation of singularities does occur on $a-8c^2/9 = 0$ but when we unfold these systems in a larger family that includes systems with three distinct singular points at infinity and hence for these systems we have $\eta > 0$. Looking at the families of systems in the set of systems with $\eta > 0$ in (OLIVEIRA *et al.*, 2017) we find the configuration denoted by Config.H.139 (see Figure 44) with three singular points at infinity in the real projective plane and with 4 singular points in the affine plane. This configuration has two hyperbolas that coalesce when two of the three singular points at infinity collide and we also have collision of two finite singular points located on distinct hyperbolas. To prove this, consider the systems:

$$\begin{cases} \dot{x} = \frac{72c^2(1-\varepsilon)(2+\varepsilon)}{(-9+\varepsilon^2)^2} + 3cx + x^2 + (1+\varepsilon)xy \\ \dot{y} = -\frac{9c^2(1+\varepsilon^2)}{(-9+\varepsilon^2)^2} + y^2, \end{cases} \tag{7.1}$$

where ε is sufficiently small. These systems possess the configuration Config.H.139 (see Figure 44) of (OLIVEIRA *et al.*, 2017) for any value of $\varepsilon > 0$ as we can show that it satisfies the required conditions on the polynomial invariants. On the other hand, the systems (7.1) form a perturbation of the system obtained by setting $\varepsilon = 0$ which has the configuration Config.$\tilde{H}$.33 (see Figure 44) of (OLIVEIRA *et al.*, 2017) (here family (P) when $a = 8v^2/9$ with configuration $C_6^{(P)}$).

We conclude that on both parabolas $a - c^2 = 0$ and $a - 8c^2/9 = 0$ bifurcation of multiple singular points produce bifurcation points of configurations corresponding to multiple invariant curves but this time we have apart from coalescence of finite singularities, also coalescence of two infinite singularities.

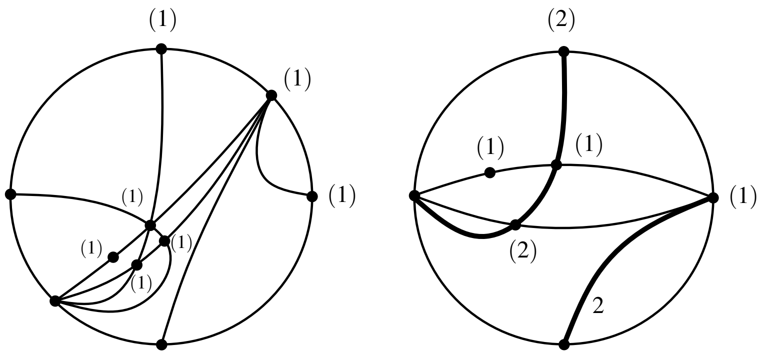
In the article (OLIVEIRA *et al.*, 2017) the classification of **QSH** according to the configurations of invariant hyperbolas and lines was done separately for the two subfamilies corresponding to $\eta > 0$ and $\eta = 0$ leading to two non-integrated bifurcation diagrams in terms of invariant polynomials.

As the above example clearly illustrates there is the need of obtaining an *integrated bifurcation diagram* of **QSH**. We thus propose the following problem:

Problem: *Obtain an integrated bifurcation diagram for the family **QSH** of the configurations of invariant hyperbolas and lines that systems in **QSH** have, by finding a common set of invariant polynomials to be applied jointly to both subfamilies $\eta > 0$ and $\eta = 0$.*

Finally, in the full (extended) parameter space we observe that on $a = 0$ the hyperbola becomes reducible. For $c \neq 0$ the hyperbola splits into the lines $x = 0$ and $c + y = 0$. On $a = 0 = c$, the two lines coincide yielding a double line $x = 0$ and in addition the hyperbola splits into the lines $x = 0$ and $y = 0$.

Figure 44 – *Config.H*.139 and *Config.$\tilde{H}$*.33 (respectively) from (OLIVEIRA *et al.*, 2017). The left configuration becomes the right one when the hyperbola with infinite points $[1:1:0]$ and $[1:0:0]$ is identified with the other hyperbola by moving the point here at $[1:0:0]$ to coincide with $[0:1:0]$ in $P_2(\mathbb{C})$.



Source: Elaborated by the author.

7.4.3 *Family* (Q)

The parameter space for this family is $\{(c,g)\in\mathbb{R}^2 : (g\pm 1)(3g-1)(2g-1)\neq 0$ and $c^2+g^2\neq 0\}$. The topological bifurcation set for this family is the set $cg=0$, with $c^2+g^2\neq 0$. On $c=0$ and $g\neq 0$ we have that all the singularities of the systems are at infinity and this occurs nowhere else. Moreover, on $c=0$ and $g\neq 0$ we have that $[0:1:0]$ is of multiplicity $(2,4)$ while $[1:0:1]$ is of multiplicity one. On $g=0$ and $c\neq 0$ the singular point $[1:0:1]$ is of multiplicity $\binom{1}{1}$ while for neighbouring parameters this point is of multiplicity 1.

The bifurcation set of the configurations is again $cg=0$, with $c^2+g^2\neq 0$. On $c=0$ the line $x=0$ is a triple line, except for the value $(c,g)=(0,-1)$ where $x=0$ is a quadruple line. This phenomenon is forced by the topological bifurcation of singularities. Indeed, on this line two of the finite singularities, one on a line and one at the intersection of the hyperbola with the line coalesced with $[0:1:0]$ producing the a line of multiplicity at least two. In fact calculation indicates that the multiplicity of $x=0$ is actually 3 for $g\neq 0$. Everywhere else in the parameter space of (Q) we either have just one simple invariant line (this occurs on $g=0$) or two simple invariant lines. This proves that $g=0$ is a bifurcation line of configurations.

Thus for both families of systems (P) and (Q) the bifurcation of configurations is produced by coalescence of singularities either finite or infinite or coalescence of a finite with an infinite singularities.

Observation 183. Finally we observe that if we take in the family of systems with equations (Q) when $c=0$ and $g=-1$ we obtain exactly the system denoted by (T) in the list of normal forms. The normal form (S) is also for just one system. This system coincides with the system in the family (Q) when $g=c=-1$. If we take $c=0$ and $g=1$ in the systems defined by equations (Q) we obtain exactly (V). Hence in the study of family (Q) we covered four of the normal forms listed in Proposition 54: (Q), (S) and (T), (V).

Apart from the fact that we are interested in producing the phase portraits of family **QSH** as well as fully understanding the integrability of this family, the questions and problems raised above are additional motivation for completing the study of this class of systems.

7.5 Phase Portraits

It is important to emphasize that the study for **QSH** is not yet complete. However, it is worth mentioning how many distinct phase portraits we found in our investigation so far. In the table below we present the phase portrait in **QSH** and also the phase portraits appearing in the limit cases. In the geometric analysis we simply used the notation "p_i" where $i = 1, 2, \ldots$ for the phase portraits on the limit cases of the normal forms but here we need to clarify in which normal form the phase portrait appeared, so we use the notation "$p_i(*)$" where $*$ is the normal form associated with the phase portrait.

Phase Portrait	Singularities at Infinity	Real Finite Singularities	Separatrix Connections	Top. equivalent Phase Portraits
$P_1^{(B)}$	(N,N,S)	(s,s,n,n)	$2SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$	$P_3^{(F)}$, $P_1^{(H)}$, $P_1^{(I)}$
$P_2^{(B)}$	(N,N,N)	(s,s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$	$P_2^{(C)}$
$P_3^{(B)}$	(N,S,N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$	$P_1^{(C)}, P_3^{(E)}, P_2^{(F)}$ $P_1^{(G)}, P_3^{(H)}, P_2^{(I)}$ $P_2^{(J)}, P_3^{(K)}, P_2^{(L)}$
$P_4^{(B)}$	(N,N,N)	(s,s)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	$P_3^{(C)}$
p_1(B)	(N,N,S)	$hpphpp$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	p_1(C), p_2(G), p_2(H) p_1(I), p_1(J) p_1(L), $P_3^{(M)}$
p_2(B)	(N,N,N)	$hhhhhh$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	p_2(C), p_4(F)
p_3(B)	$(N,N,(\ominus[\vert];\emptyset))$	$(\ominus[\vert];s)$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$	p_3(C)
$P_4^{(C)}$	(S,N,N)	(s,s,n,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	$P_1^{(E)}$, p_1(F) p_1(H), $P_1^{(J)}$
$P_2^{(E)}$	(N,N,S)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	$P_1^{(F)}, P_2^{(H)}, P_1^{(K)}$ $P_1^{(L)}, P_1^{(M)}$
$P_4^{(E)}$	(N,N,S)	(sn,sn)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	$P_4^{(H)}$
p_1(E)	(N,N,S)	(s,n,sn)	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$	–
p_2(F)	(N,N,S)	(sn,sn)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$	p_1(G), $P_5^{(H)}$
p_3(F)	(N,N,S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$	$P_2^{(G)}, P_2^{(M)}$
p_5(F)	$(N,(\ominus[\times];\emptyset,\emptyset))$	$(\ominus[\times];\emptyset)$	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$	–
$P_2^{(K)}$	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $5SC_f^\infty$ $1SC_\infty^\infty$	–

$P_4^{(K)}$	(N,S,N)	$(sn_{(2)},sn_{(2)})$	$1SC_f^f\ 5SC_f^\infty\ 1SC_\infty^\infty$	–
$P_5^{(K)}$	(N,S,N)	$(sn_{(2)},n,s)$	$3SC_f^f\ 5SC_f^\infty\ 1SC_\infty^\infty$	–
$p_{1(K)}$	(N,S,N)	(n,s,n,s)	$4SC_f^f\ 5SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(O)}$	(SN,N)	(n,n,s,s)	$2SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$	–
$P_2^{(O)}$	(SN,N)	(s,s,n,n)	$4SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$	$P_1^{(P)}$
$P_3^{(O)}$	(SN,S)	(f,f)	$0SC_f^f\ 2SC_f^\infty\ 2SC_\infty^\infty$	–
$P_4^{(O)}$	(SN,N)	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 2SC_\infty^\infty$	$P_3^{(P)}$
$P_5^{(O)}$	(SN,S)	(n,n)	$0SC_f^f\ 2SC_f^\infty\ 0SC_\infty^\infty$	–
$P_6^{(O)}$	(SN,N)	(s,s,n,n)	$3SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$	–
$P_7^{(O)}$	(SN,N)	(sn,sn)	$0SC_f^f\ 6SC_f^\infty\ 0SC_\infty^\infty$	–
$p_{1(O)}$	(SN,N)	$phph$	$0SC_f^f\ 4SC_f^\infty\ 0SC_\infty^\infty$	–
$p_{2(O)}$	$SN,(\ominus[\|];\emptyset)$	$(\ominus[\|];n^d)$	$0SC_f^f\ 2SC_f^\infty\ 0SC_\infty^\infty$	–
$p_{3(O)}$	(SN,S)	$epep$	$0SC_f^f\ 4SC_f^\infty\ 0SC_\infty^\infty$	$p_{2(P)}$
$P_2^{(P)}$	(SN,N)	(n,s,n,s)	$4SC_f^f\ 5SC_f^\infty\ 1SC_\infty^\infty$	–
$P_4^{(P)}$	(SN,N)	(sn,sn)	$1SC_f^f\ 5SC_f^\infty\ 1SC_\infty^\infty$	–
$P_5^{(P)}$	(SN,N)	$(s,sn_{(2)},n)$	$3SC_f^f\ 5SC_f^\infty\ 1SC_\infty^\infty$	–
$p_{1(P)}$	(SN,N)	(s,n,s,n)	$4SC_f^f\ 5SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(Q)}$	$(PEP-EPP,S)$	(s,n)	$1SC_f^f\ 4SC_f^\infty\ 1SC_\infty^\infty$	$P_1^{(S)}$
$P_2^{(Q)}$	$(PPEP-PEPP,N)$	(s,s)	$0SC_f^f\ 8SC_f^\infty\ 0SC_\infty^\infty$	$P_1^{(R)}$
$P_3^{(Q)}$	$(HPP-HPP,N)$	(s,n)	$1SC_f^f\ 5SC_f^\infty\ 0SC_\infty^\infty$	$P_1^{(U)}$
$P_4^{(Q)}$	$(PPE-EPP,S)$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 2SC_\infty^\infty$	$P_1^{(T)}$
$P_5^{(Q)}$	$(PHP-PHP,N)$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 3SC_\infty^\infty$	$P_3^{(R)}$
$P_6^{(Q)}$	$(PH-HP,N)$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 1SC_\infty^\infty$	$P_1^{(V)}$
$P_7^{(Q)}$	$(PEP-EPP,SN)$	s	$0SC_f^f\ 4SC_f^\infty\ 1SC_\infty^\infty$	–
$p_{1(Q)}$	$(PPEP-HPP,N)$	s	$0SC_f^f\ 4SC_f^\infty\ 1SC_\infty^\infty$	–
$p_{2(Q)}$	$(\ominus[)(];N,\emptyset)$	$(\ominus[)(];\emptyset)$	$0SC_f^f\ 0SC_f^\infty\ 0SC_\infty^\infty$	–
$P_2^{(R)}$	$(H-H,N)$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(W)}$	$[\infty,\emptyset]$	(s,n,n)	$2SC_f^f\ 2SC_f^\infty\ 0SC_\infty^\infty$	–
$P_2^{(W)}$	$[\infty,\emptyset]$	c	$0SC_f^f\ 0SC_f^\infty\ 0SC_\infty^\infty$	–
$P_3^{(W)}$	$[\infty,\emptyset]$	es	$0SC_f^f\ 4SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(X)}$	$[\infty,N]$	(n,s)	$1SC_f^f\ 5SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(Y)}$	$[\infty,S]$	(n,n)	$0SC_f^f\ 4SC_f^\infty\ 0SC_\infty^\infty$	–
$P_2^{(Y)}$	$[\infty,C]$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 0SC_\infty^\infty$	–
$p_{1(Y)}$	$[\infty;(\ominus[\|];\emptyset_2)]$	$(\ominus[\|];n^*)$	$0SC_f^f\ 0SC_f^\infty\ 0SC_\infty^\infty$	–
$P_1^{(Z)}$	$[\infty;ES]$	$\emptyset$	$0SC_f^f\ 0SC_f^\infty\ 1SC_\infty^\infty$	–

In conclusion, considering separately $\mathbf{QSH}_{(\eta>0)}$ and $\mathbf{QSH}_{(\eta=0)}$ we have:

	Phase portraits in **QSH**	Phase portraits in **QSH** + limit cases
$\eta > 0$	13	18
$\eta = 0$	25	32
Total	38	50

We found in our investigation:

(i) 16 phase portraits missing in (LLIBRE; YU, 2018): $P_1^{(B)}$, $P_3^{(B)}$, $P_4^{(C)}$, $P_4^{(E)}$, $p_{2(F)}$, $P_2^{(K)}$, $P_4^{(K)}$, $P_5^{(K)}$, $P_2^{(O)}$, $P_4^{(O)}$, $P_6^{(O)}$, $P_2^{(P)}$, $P_4^{(P)}$, $P_5^{(P)}$, $P_7^{(Q)}$ and $P_2^{(R)}$.

(ii) 1 phase portrait missing in (CAIRÓ; FEIX; LLIBRE, 1999): $P_3^{(F)}$.

(iii) 3 configurations missing in (OLIVEIRA *et al.*, 2017): $C_1^{(G)}$, $C_7^{(G)}$ and $C_9^{(H)}$.

(iv) 1 configuration missing in (SCHLOMIUK; VULPE, 2008c): $c_{1(K)}$.

7.6 Further works

In further works we intend to give results concerning the questions stated in Chapter 7 and to investigate the geometric analysis of the 3-parameters normal forms of **QSH** not studied in this thesis.

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APPENDIX

A

ON THE REMARKABLE CURVES OF FAMILY (H)

Considering $r = m_1/m_2$ where $m_1, m_2 \in \mathbb{N}$ we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (H) when $a = (1-(m_1/m_2)^2)v^2$. Consider

$$\mathscr{F}_{(c_1,c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have the following:

- Taking $m_1 = 2$ and $m_2 = 4$ (i.e. $a = 3v^2/4$) we have that

  $$\mathscr{F}_{(1,1)} = -\tfrac{27}{16} v^3 y \left(81v^4 + 36v^2\left(-2x^2 + xy + y^2\right) + 16x(x-y)^3\right).$$

  Therefore, we have a line and a quartic as remarkable curves.

- Taking $m_1 = 2$ and $m_2 = 6$ (i.e. $a = 8v^2/9$) we have that

  $$\mathscr{F}_{(1,1)} = \tfrac{32}{9} v^3 \left(v^2 + 3y(y-x)\right)\left(3v^4(5x-8y) - 2v^2(x-y)^2(5x+4y) + 3x(x-y)^4\right).$$

  Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking $m_1 = 2$ and $m_2 = 8$ (i.e. $a = 15v^2/16$) we have that

  $$\begin{aligned} \mathscr{F}_{(1,1)} = \; & \tfrac{27}{262144} v^3 (36v^2x - 45v^2y - 80x^2y + 160xy^2 - 80y^3) \\ & (3645v^6 + 19440v^4x^2 - 58320v^4xy + 38880v^4y^2 - 11520v^2x^4 + 23040v^2x^3y + \\ & -23040v^2xy^3 + 11520v^2y^4 + 4096x^6 - 20480x^5y + 40960x^4y^2 - 40960x^3y^3 + \\ & +20480x^2y^4 - 4096xy^5). \end{aligned}$$

  Therefore, we have a cubic and a polynomial of degree 6 as remarkable curves.

- Taking $m_1 = 3$ and $m_2 = 6$ (i.e. $a = 3v^2/4$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} &= \tfrac{81}{512}v^3y(6561v^8 - 1944v^6\left(6x^2-3xy-5y^2\right)+\\ &+1296v^4(x-y)^2\left(6x^2+2xy+y^2\right) - 1152v^2x(x-y)^4(2x+y)+256x^2(x-y)^6).\end{aligned}$$

Therefore, we have a line and a polynomial of degree 8 as remarkable curves.

- Taking $m_1 = 3$ and $m_2 = 9$ (i.e. $a = 8v^2/9$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} &= \tfrac{16}{27}v^3\left(v^2-3xy+3y^2\right)(64v^{10}+675v^8x^2-2544v^8xy+2112v^8y^2+\\ &-900v^6x^4+2520v^6x^3y-612v^6x^2y^2-2736v^6xy^3+1728v^6y^4+\\ &+570v^4x^6-2232v^4x^5y+3420v^4x^4y^2-2760v^4x^3y^3+1530v^4x^2y^4+\\ &-720v^4xy^5+192v^4y^6-180v^2x^8+936v^2x^7y-1836v^2x^6y^2+1440v^2x^5y^3+\\ &+180v^2x^4y^4-1080v^2x^3y^5+684v^2x^2y^6-144v^2xy^7+27x^{10}-216x^9y+\\ &+756x^8y^2-1512x^7y^3+1890x^6y^4-1512x^5y^5+756x^4y^6-216x^3y^7+27x^2y^8).\end{aligned}$$

Therefore, we have a hyperbola and a polynomial of degree 10 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 2$ (i.e. $a = -3v^2$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} &= 216v^3(9v^2+xy)(405v^4x-81v^4y-45v^2x^3+63v^2x^2y-18v^2xy^2+\\ &+x^5-3x^4y+3x^3y^2-x^2y^3).\end{aligned}$$

Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 6$ (i.e. $a = 5v^2/9$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} &= -\tfrac{8}{81}v^3(100v^4-21v^2x^2+270v^2xy+75v^2y^2-45x^3y+90x^2y^2-45xy^3)\\ &(420v^6x+300v^6y-385v^4x^3+255v^4x^2y+105v^4xy^2+25v^4y^3+105v^2x^5+\\ &-285v^2x^4y+225v^2x^3y^2-15v^2x^2y^3-30v^2xy^4-9x^7+45x^6y-90x^5y^2+\\ &+90x^4y^3-45x^3y^4+9x^2y^5).\end{aligned}$$

Therefore, we have a quartic and a polynomial of degree 7 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 8$ (i.e. $a = 3v^2/4$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} &= -\tfrac{27}{2048}v^3y(81v^4-72v^2x^2+36v^2xy+36v^2y^2+16x^4-48x^3y+\\ &+48x^2y^2-16xy^3)(6561v^8-11664v^6x^2+5832v^6xy+17496v^6y^2+\\ &+7776v^4x^4-12960v^4x^3y+3888v^4x^2y^2+1296v^4y^4-2304v^2x^6+\\ &+8064v^2x^5y-9216v^2x^4y^2+2304v^2x^3y^3+2304v^2x^2y^4-1152v^2xy^5+\\ &+256x^8-1536x^7y+3840x^6y^2-5120x^5y^3+3840x^4y^4-1536x^3y^5+256x^2y^6).\end{aligned}$$

Therefore, we have a line, a quartic and a polynomial of degree 8 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 12$ (i.e. $a = 8v^2/9$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} = {} & \tfrac{64}{81}v^3(v^2-3xy+3y^2)(15v^4x-24v^4y-10v^2x^3+12v^2x^2y+6v^2xy^2+ \\ & -8v^2y^3+3x^5-12x^4y+18x^3y^2-12x^2y^3+3xy^4)(64v^{10}+225v^8x^2+ \\ & -1104v^8xy+960v^8y^2-300v^6x^4+840v^6x^3y+180v^6x^2y^2-1680v^6xy^3+ \\ & +960v^6y^4+190v^4x^6-744v^4x^5y+1140v^4x^4y^2-920v^4x^3y^3+510v^4x^2y^4+ \\ & -240v^4xy^5+64v^4y^6-60v^2x^8+312v^2x^7y-612v^2x^6y^2+480v^2x^5y^3+ \\ & +60v^2x^4y^4-360v^2x^3y^5+228v^2x^2y^6-48v^2xy^7+9x^{10}-72x^9y+252x^8y^2+ \\ & -504x^7y^3+630x^6y^4-504x^5y^5+252x^4y^6-72x^3y^7+9x^2y^8).\end{aligned}$$

Therefore, we have a hyperbola, a quintic and a polynomial of degree 10 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 16$ (i.e. $a = 15v^2/16$) we have that

$$\begin{aligned}\mathscr{F}_{(1,1)} = {} & \tfrac{27}{2199023255552}v^3(36v^2x-45v^2y-80x^2y+160xy^2-80y^3) \\ & (3645v^6+19440v^4x^2-58320v^4xy+38880v^4y^2-11520v^2x^4+23040v^2x^3y+ \\ & -23040v^2xy^3+11520v^2y^4+4096x^6-20480x^5y+40960x^4y^2-40960x^3y^3+ \\ & +20480x^2y^4-4096xy^5)(13286025v^{12}+383582304v^{10}x^2-1029814560v^{10}xy+ \\ & +661348800v^{10}y^2+293932800v^8x^4-3174474240v^8x^3y+8406478080v^8x^2y^2+ \\ & -8465264640v^8xy^3+2939328000v^8y^4-418037760v^6x^6+2090188800v^6x^5y+ \\ & -2090188800v^6x^4y^2-4180377600v^6x^3y^3+10450944000v^6x^2y^4+ \\ & -7942717440v^6xy^5+2090188800v^6y^6+291962880v^4x^8-1804861440v^4x^7y+ \\ & +4830658560v^4x^6y^2-7431782400v^4x^5y^3+7431782400v^4x^4y^4+ \\ & -5202247680v^4x^3y^5+2601123840v^4x^2y^6-849346560v^4xy^7+132710400v^4y^8+ \\ & -94371840v^2x^{10}+660602880v^2x^9y-1887436800v^2x^8y^2+2642411520v^2x^7y^3+ \\ & -1321205760v^2x^6y^4-1321205760v^2x^5y^5+2642411520v^2x^4y^6+ \\ & -1887436800v^2x^3y^7+660602880v^2x^2y^8-94371840v^2xy^9+16777216x^{12}+ \\ & -167772160x^{11}y+754974720x^{10}y^2-2013265920x^9y^3+3523215360x^8y^4+ \\ & -4227858432x^7y^5+3523215360x^6y^6-2013265920x^5y^7+754974720x^4y^8+ \\ & -167772160x^3y^9+16777216x^2y^{10}).\end{aligned}$$

Therefore, we have a cubic, a polynomial of degree 6 and a polynomial of degree 12 as remarkable curves.

These computations suggest that the remarkable curves of algebraically integrable systems in the family (H) have an unbounded degree.

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