UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Variations in selection principles and selective topological games

Juan Francisco Camasca Fernández

Tese de Doutorado do Programa de Pós-Graduação em Matemática (PPG-Mat)

ICMC USP
SÃO CARLOS

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: ______________________

Juan Francisco Camasca Fernández

Variations in selection principles and selective topological games

Doctoral dissertation submitted to the Instituto de Ciências Matemáticas e de Computação – ICMC-USP, in partial fulfillment of the requirements for the degree of the Doctorate Program in Mathematics. *FINAL VERSION*

Concentration Area: Mathematics

Advisor: Prof. Dr. Leandro Fiorini Aurichi

USP – São Carlos
March 2023

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

C173v

Camasca Fernández, Juan Francisco
Variations in selection principles and selective topological games / Juan Francisco Camasca Fernández; orientador Leandro Fiorini Aurichi. -- São Carlos, 2023.
103 p.

Tese (Doutorado - Programa de Pós-Graduação em Matemática) -- Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2023.

1. selection principles. 2. selective topological games . 3. function spaces . 4. topology . 5. open covers. I. Aurichi, Leandro Fiorini, orient. II. Título.

Bibliotecários responsáveis pela estrutura de catalogação da publicação de acordo com a AACR2:
Gláucia Maria Saia Cristianini - CRB - 8/4938
Juliana de Souza Moraes - CRB - 8/6176

Juan Francisco Camasca Fernández

Variações em principios seletivos e jogos topologicos de seleção

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Leandro Fiorini Aurichi

USP – São Carlos
Março de 2023

Á Deus,
meus Pais
e o amor da minha vida,
Heydy.

ACKNOWLEDGEMENTS

First of all, I thank God. I thank my parents and my brothers for their support during this long process away from them. I especially thank Heydy, for being my constant support and strength to continue to go every day. I love you!

I thank my advisor Prof. Leandro Fiorini Aurichi for his infinite patience and his constant support in the face of my mistakes.

I would like to thank all my friends and family who have supported me on this long journey.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

“ The world was not prepared for this.
It was too far ahead of time,
but the same laws will prevail, and one day
make it a triumphant success.”
(Nikola Tesla.)

ABSTRACT

FERNANDEZ, J. F. C. **Variations in selection principles and selective topological games**. 2023. 103 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

In this work, we study the relation of different variations of selection principles and selective topological games. In particular, we study the case of selection principles and selective topological games when we consider the case of a class of dense subsets of a topological space, and we obtain a result of equivalence in the case of the space of continuous functions with the compact-open topology. Furthermore, we include a translation of some results with different dense families, and we include a little of selection star principles and selectively c.c.c property.

Keywords: selection principles, selective topological games, function spaces, topology, open covers.

RESUMO

FERNANDEZ, J. F. C. **Variações em principios seletivos e jogos topologicos de seleção**. 2023. 103 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

Neste trabalho, estudamos a relação entre diferentes variações de princípios de seleção e jogos topológicos de seleção. Particularmente, estudamos o caso de princípios de seleção e jogos topológicos quando consideramos o caso da classe de subconjuntos densos de um espaço topológico, e obtemos um resultado de equivalência no caso do espaço de funções contínuas com a topologia compacta-aberta. Além disso, incluímos uma tradução de alguns resultados com diferentes famílias densas, e incluímos um pouco a repeito de princípios seletivos estrela e da propriedade seletivamente c.c.c.

Palavras-chave: princípios seletivos, jogos topológicos de seleção, espaço de funções, topologia, coberturas abertas.

LIST OF SYMBOLS

ω — set of natural numbers

${}^{<\omega}X$ — Set of all finite sequences of X.

$C(X)$ — Set of all continuous real functions defined in X.

$\aleph_0$ — cardinality of the ordinal number ω.

$C_p(X)$ — C(X) equipped with the topology of pointwise convergence

$C_k(X)$ — C(X) equipped with the compact-open topology

$\wp(X)$ — Power set of X.

A^B — The set of all functions of B in A.

$[X]^{<\aleph_0}$ — Set of finite subsets of X.

$|A|$ — cardinality of the set A

$\mathbb{R}$ — set of all real numbers.

$[X]^k$ — Set of all subsets of X with cardinality k

$w(X)$ — the least cardinality of a basis of X.

$d(X)$ — the least cardinality of a dense subset of X.

CONTENTS

CHAPTER

1

INTRODUCTION

In 1996, Marion Scheepers introduced the selection principles $S_1(\mathscr{A},\mathscr{B})$ and $S_{fin}(\mathscr{A},\mathscr{B})$, with $\mathscr{A}$ and $\mathscr{B}$ classes of sets. In short, a selection principle is a property that allows us to describe a particular property in terms of a specific class ($\mathscr{B}$) by making a certain choice over a succession of elements from another family ($\mathscr{A}$).

For example, the selection principle $S_1(\mathscr{D}_X,\mathscr{D}_X)$ indicates the following property: for any sequence $\langle D_n : n \in \omega \rangle$ of dense subsets in a topological space X, there is, for any $n \in \omega$, $x_n \in D_n$ such that $\{x_n : n \in \omega\}$ is a dense subset. Many topological properties are described with these selection principles when we considere different classes of families of subsets in a topological space. The most studied case in the literature, and that was the basis for its formalization, is when we take the family of open covers of a topological space (namely, Rothberger and Menger properties). We must emphasize that in the notation, the sub-index 1 and fin indicate the number of elements selected from each element of the sequence (fin indicates that it is selected finitely many elements). Then, naturally, we can define variations of these selection principles, such as $S_2(\mathscr{A},\mathscr{B})$, $S_3(\mathscr{A},\mathscr{B})$, etc.

On the other hand, the term "topological game" was introduced for the first time in 1957 by Claude Berge. Later, in 1974, Rastislav Telgársky introduced a different meaning for it using the concept "topological properties defined by games". The first game studied was the Banach-Mazur game. This game was introduced in the famous Scottish Book, a compilation of problems and discussions by celebrated mathematicians of the time. The problem was proposed by Mazur and was answered by Banach. A selective topological game is one formulated following the same idea of a selection principle.

For example, the topological game $G_1(\mathscr{O}_X,\mathscr{O}_X)$ (which is called the Rothberger game) is played as follows: in each inning, Player I chooses an open cover of X and Player II chooses an element in the open cover chosen by Player I. Player II wins if his choices form an open cover of X. Note that we can analogously define a game by changing the number of elements chosen by Player *II*. For example, we can say that Player *II* chooses a finite subset rather than a single element in the open cover played by Player *I*. This game is denoted by $G_{fin}(\mathscr{O}_X, \mathscr{O}_X)$, and it is a different game from $G_1(\mathscr{O}_X, \mathscr{O}_X)$.

The main objective of this text is to study variations of selection principles and selective topological games when considering a distinct number of elements chosen in each element of a sequence and a distinct number of elements played in each inning for Player *II*, respectively. Furthermore, we study these variations when considering different classes of families of subsets of a topological space X, focusing mainly on dense families. This work is divided into sections as follows:

In Chapter 2 we have compiled some basic facts with respect to selection principles and selective topological games, with an emphasis on types of selection principles, about the number of choices, winning strategies of games, duality, and equivalences about selective topological games. We conclude with some results in spaces of continuous functions with bornologies.

In Chapter 3 we present some results about the equivalences between the different variations in selection principles in certain types of classes of subsets of topological spaces, and particular cases are revisited.

In Chapter 4 we present additional results on the equivalence of variations in topological games on the class of k-covers (and some on the class of bornologies covers). Using a translation between k-covers in a topological space and tightness in the space of continuous functions with the compact-open topology, we obtain an equivalence in some variations of topological games about the class of dense subsets when we work in the space of continuous functions with the compact-open topology.

In Chapter 5 we have some generalizations of translations in some selection principles with dense families. Additionally, we also present translations for a certain selective topological game.

In Chapter 6 we considered some variations on the selectively ccc property, the star selectively ccc property, and the game versions of the last properties.

CHAPTER

2

PRELIMINARIES

In this chapter, we make a compilation of selection principles and selective topological games (where we focus on winning strategies and duality).

For references in general topology, we cite (ENGELKING, 1989).

2.1 Selection principles and selective topological games

2.1.1 Families of sets

Definition 2.1. Let (X,τ) be a topological space and let $\mathscr{U}$ be an open cover of X. We say that $\mathscr{U}$ is:

- a ω-cover if $X \notin \mathscr{U}$ and for all finite subset $F \subset X$, there is $U \in \mathscr{U}$ such that $F \subset U$;
- a k-cover if $X \notin \mathscr{U}$ and for all compact subset $K \subseteq X$, there is $U \in \mathscr{U}$ such that $K \subseteq U$;
- a γ-cover if $X \notin \mathscr{U}$, $\mathscr{U}$ is infinite and for all $x \in X$, the set $\{U \in \mathscr{U} : x \notin U\}$ is finite;
- a large cover if for all $x \in X$ the set $\{U \in \mathscr{U} : x \in U\}$ is infinite.

Definition 2.2. A topological space (X,τ) is called ω-Lindelöf, if for all ω -cover $\mathscr{U}$ there is $\mathscr{U}' \subseteq \mathscr{U}$ such that $\mathscr{U}'$ is a countable ω-cover.

Definition 2.3. Let (X,τ) be a topological space. A family $\mathbf{D}$ is said to be a dense family if all its elements are open sets and $\bigcup \mathbf{D}$ is dense in X.

Definition 2.4. Let (X,τ) be a topological space. A family $\mathbf{A}$ is cellular if all its elements are pairwise disjoint open sets. A family $\mathbf{A}$ is maximal cellular if it is cellular and maximal with respect to $\subseteq$ in the family of all cellular families.

Definition 2.5. Let (X,τ) be a topological space, $x \in X$ and $A \subseteq X$. A set A converges to a point x, and it is denoted by $x := \lim A$, if A is infinite, $x \notin A$ and $A \setminus U$ is finite for any neighborhood U of x. In this case, we say that x is a limit point of A.

Denote by $[A]_{seq}$ the set of all limit points of sequences of $A \subseteq X$.

Definition 2.6. Let (X,τ) be a topological space. A subset $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$.

Definition 2.7. Let (X,τ) be a topological space. X is called strongly sequentially dense if any dense subset is sequentially dense.

Definition 2.8. Let (X,τ) be a topological space. X is said to be Fréchet if, for all $x \in X$ and $A \subseteq X$, with $x \in \overline{A}$, there is a sequence $\langle x_n : n \in \omega \rangle$ in A such that $\lim\{x_n : n \in \omega\} = x$.

Definition 2.9. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X. The set X satisfies a property $\begin{pmatrix} \mathscr{A} \\ \mathscr{B} \end{pmatrix}$ if, for all $A \in \mathscr{A}$, there is a $B \subseteq A$ such that $B \in \mathscr{B}$.

If (X,τ) is a topological space, we will use the following notation:

1. $\mathscr{O}_X$ is the class of all open covers of X;
2. Ω_X is the class of all ω-covers of X;
3. $\mathscr{K}_X$ is the class of all k-covers of X;
4. Γ_X is the class of all γ-covers of X;
5. $\mathscr{D}_X$ is the class of all dense subsets of X;
6. Λ_X is the class of all large covers;
7. $\mathscr{S}_X$ is the class of all sequentially dense subsets of X;
8. $\mathscr{D}_o$ is the class of all dense families;
9. $\mathscr{M}_c$ is the class of all maximal cellular families;
10. $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
11. $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

2.1.2 Selection principles and selective topological games

In this section, we will mention the best known results about selection principles and selective topological games. For reference we cite (AURICHI; DIAS, 2019).

The main motivation for continuing to study selective topological games and selection principles is that they can characterize some topological properties and even define new ones.

Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X. We will denote by ω the set of natural numbers.

Definition 2.10. $S_1(\mathscr{A},\mathscr{B})$ is the following selection principle: for any sequence $\langle A_n : n \in \omega\rangle$ of elements of $\mathscr{A}$, there is a sequence $\langle b_n : n \in \omega\rangle$ such that $b_n \in A_n$, for all $n \in \omega$, and $\{b_n : n \in \omega\} \in \mathscr{B}$.

Definition 2.11. $S_{fin}(\mathscr{A},\mathscr{B})$ is the following selection principle: for any sequence $\langle A_n : n \in \omega\rangle$ of elements of $\mathscr{A}$, there is a sequence $\langle B_n : n \in \omega\rangle$ such that $B_n \subseteq A_n$ is finite for all $n \in \omega$, and $\bigcup_{n\in\omega} B_n \in \mathscr{B}$.

The following are the most well-known selection principles:

- $S_1(\mathscr{O}_X,\mathscr{O}_X)$ is the Rothberger property;
- $S_{fin}(\mathscr{O}_X,\mathscr{O}_X)$ is the Menger property;
- $S_1(\Omega_X,\Omega_X)$ is the Ω-Rothberger property;
- $S_{fin}(\Omega_X,\Omega_X)$ is the Ω-Menger property;
- $S_1(\Omega_x,\Omega_x)$ is the strong countable tightness property;
- $S_{fin}(\Omega_x,\Omega_x)$ is the countable tightness property;
- $S_1(\mathscr{D}_X,\Omega_x)$ is the strong countable tightness property with respect to dense subsets;
- $S_{fin}(\mathscr{D}_X,\Omega_x)$ is the countable tightness property with respect dense subsets;
- $S_1(\mathscr{D}_X,\mathscr{D}_X)$ is the R-separable property;
- $S_{fin}(\mathscr{D}_X,\mathscr{D}_X)$ is the M-separable property or SS property;
- $S_1(\mathscr{M}_c,\mathscr{D}_o)$ is the selectively ccc property.

We can obtain the following variation in the selection principles:

Definition 2.12. Let $f : \omega \to \omega\setminus\{0\}$ be a function. $S_f(\mathscr{A},\mathscr{B})$ is the following selection principle: for any sequence $\langle A_n : n \in \omega\rangle$ of elements of $\mathscr{A}$, there is a sequence $\langle B_n : n \in \omega\rangle$ such that $B_n \in [A_n]^{\leq f(n)}$, for all $n \in \omega$, and $\bigcup_{n\in\omega} B_n \in \mathscr{B}$.

By $k \in \omega \backslash \{0\}$, we write $S_k(\mathscr{A}, \mathscr{B})$ when $f = f_k$. Clearly, for all $k \in \omega \backslash \{0\}$, if $S_k(\mathscr{A}, \mathscr{B})$ holds, then $S_{k+1}(\mathscr{A}, \mathscr{B})$ holds, for any $\mathscr{A}$ and $\mathscr{B}$ classes of families of subsets of a set X.

Observation 2.13. If $\mathscr{A}$ and $\mathscr{B}$ are classes of families of subsets of a set X, the following implications are immediate:

1. If $S_1(\mathscr{A}, \mathscr{B})$ holds, then $S_f(\mathscr{A}, \mathscr{B})$ holds for all $f : \omega \to \omega \backslash \{0\}$.
2. For all fixed $f : \omega \to \omega \backslash \{0\}$, if $S_f(\mathscr{A}, \mathscr{B})$ holds, then $S_{fin}(\mathscr{A}, \mathscr{B})$ holds.

Another way to characterize properties in topological spaces is through selective topological games.

Definition 2.14. The game $G_1(\mathscr{A}, \mathscr{B})$ is defined as follows: in each inning $n \in \omega$, Player *I* chooses $A_n \in \mathscr{A}$, then Player *II* chooses $b_n \in A_n$. The winner is Player *II* if $\{b_n : n \in \omega\} \in \mathscr{B}$. Otherwise, the winner is Player *I*.

Definition 2.15. The game $G_{fin}(\mathscr{A}, \mathscr{B})$ is defined as follows: in each inning $n \in \omega$, Player *I* chooses $A_n \in \mathscr{A}$, then Player *II* chooses $B_n \subseteq A_n$ finite. The winner is Player *II* if $\bigcup_{n\in\omega} B_n \in \mathscr{B}$. Otherwise, the winner is Player *I*.

The most popular selective topological games are the following:

- $G_1(\mathscr{O}_X, \mathscr{O}_X)$ is the Rothberger game;
- $G_{fin}(\mathscr{O}_X, \mathscr{O}_X)$ is Menger game;
- $G_1(\Omega_X, \Omega_X)$ is the Ω-Rothberger game;
- $G_{fin}(\Omega_X, \Omega_X)$ is the Ω-Menger game;
- $G_1(\Omega_x, \Omega_x)$ is the strong countable tightness game;
- $G_{fin}(\Omega_x, \Omega_x)$ is the countable tightness game;
- $G_1(\mathscr{D}_X, \Omega_x)$ is the strong countable tightness with respect to dense subsets game;
- $G_{fin}(\mathscr{D}_X, \Omega_x)$ is the countable tightness with respect to dense subsets game;

By changing the number of elements in the choice made by Player *II*, we can obtain the following version of the selective topological game:

Definition 2.16. Let $f : \omega \to \omega \backslash \{0\}$ be a function. The game $G_f(\mathscr{A}, \mathscr{B})$ is defined as follows: in each inning $n \in \omega$, Player *I* chooses $A_n \in \mathscr{A}$, then Player *II* chooses $B_n \in [A_n]^{\leq f(n)}$. The winner is Player *II* if $\bigcup_{n\in\omega} B_n \in \mathscr{B}$. Otherwise, the winner is Player *I*.

For $k \in \omega \backslash \{0\}$, we denote by $G_k(\mathscr{A},\mathscr{B})$ the case where $f = f_k$, where f_k is the constant function k.

2.1.3 Winning strategies

Informally, a strategy is a fixed way that allows a certain player to make his choices in each inning $n \in \omega$. More formally, for $\mathscr{A}$ and $\mathscr{B}$ classes of families of subsets of a set X, we define the following:

Definition 2.17. A strategy (with complete information) for Player I in $G_1(\mathscr{A},\mathscr{B})$ is a function $\sigma : {}^{<\omega}(\bigcup \mathscr{A}) \to \mathscr{A}$. A strategy σ for Player I is called a winning strategy if, for any choice $b_n \in \sigma(\langle b_m : m < n \rangle)$, for all $n \in \omega$, $\{b_n : n \in \omega\} \notin \mathscr{B}$. $I \uparrow G_1(\mathscr{A},\mathscr{B})$ denotes the existence of a winning strategy for Player I.

Definition 2.18. A strategy (with complete information) for Player II in $G_1(\mathscr{A},\mathscr{B})$ is a function $\rho : {}^{<\omega}\mathscr{A} \to \bigcup \mathscr{A}$. A strategy ρ for Player II is called a winning strategy if, for any choice $(A_n)_{n\in\omega}$, then $\{\rho(\langle A_0,...,A_n \rangle) : n \in \omega\} \in \mathscr{B}$. $II \uparrow G_1(\mathscr{A},\mathscr{B})$ denotes the existence of a winning strategy for Player II

In the same form as is defined strategies considering all of the previous selection of the both of players, is defined strategies where only is considerate partially the history of game.

Definition 2.19. A predetermined strategy for Player I in $G_1(\mathscr{A},\mathscr{B})$ is one that considers only the number of the current inning. Formally, a predetermined strategy is a function $\sigma : \omega \to \mathscr{A}$. $I \underset{pre}{\uparrow} G_1(\mathscr{A},\mathscr{B})$ denotes the existence of a predetermined winning strategy for Player I.

Similarly, we can define strategies with complete information and predetermined strategies for all variations G_f, for all $f : \omega \to \omega \backslash \{0\}$, and G_{fin}.

Observation 2.20. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X. Then the following implications are immediate:

$$II \uparrow G_1(\mathscr{A},\mathscr{B}) \Rightarrow II \uparrow G_f(\mathscr{A},\mathscr{B}), \text{ for all } f : \omega \to \omega \backslash \{0\} \Rightarrow II \uparrow G_{fin}(\mathscr{A},\mathscr{B})$$

Furthermore, it is immediate the following implication:

$$II \uparrow G_k(\mathscr{A},\mathscr{B}) \Rightarrow II \uparrow G_{k+1}(\mathscr{A},\mathscr{B}),$$

for any $\mathscr{A}$ and $\mathscr{B}$ classes of families of subsets of a set X and for all $k \in \omega \backslash \{0\}$.

On the other hand, a result that linking selection topological games and selective principles is the following:

$$I \not\uparrow G_1(\mathscr{A},\mathscr{B}) \Rightarrow S_1(\mathscr{A},\mathscr{B}) \textit{ holds.}$$

The same implications are valid for variations G_f, for all $f : \omega \to \omega\backslash\{0\}$, and G_{fin}. The reciprocal of the previous result is not necessarily true. However, we have the following result:

Proposition 2.21 (Folkrore). $I \underset{pre}{\uparrow} G_1(\mathscr{A}, \mathscr{B})$ if, and only if, $S_1(\mathscr{A}, \mathscr{B})$ holds.

Proof. If $S_1(\mathscr{A}, \mathscr{B})$ does not hold, there is a sequence $(A_n)_{n\in\omega}$, with $A_n \in \mathscr{A}$ for all $n \in \omega$, such that, for any sequence $(b_n)_{n\in\omega}$, where $b_n \in A_n$ for all $n \in \omega$, we have $\{b_n : n \in \omega\} \notin \mathscr{B}$. Define $\sigma : \omega \to \mathscr{A}$ with $\sigma(n) = A_n$, for $n \in \omega$. Then ρ is a winning predetermined strategy for Player I in $G_1(\mathscr{A}, \mathscr{B})$.

Reciprocally, suppose that $S_1(\mathscr{A}, \mathscr{B})$ holds and let $\sigma : \omega \to \mathscr{A}$ be a strategy of Player I in $G_1(\mathscr{A}, \mathscr{B})$. As $(\sigma(n))_{n\in\omega}$ is a sequence of elements in $\mathscr{A}$, it follows that, for all $n \in \omega$, there is a $b_n \in \sigma(n)$ such that $\{b_n : n \in \omega\} \in \mathscr{B}$. That is, the play

$$\sigma(0), b_0, \sigma(1), b_1, ..., \sigma(n), b_n, ...$$

is winner by Player II. Since σ was arbitrary, we conclude that $I \underset{pre}{\not\uparrow} G_1(\mathscr{A}, \mathscr{B})$. □

It is easy to see that the previous proposition is still valid for variations G_f, for all $f : \omega \to \omega\backslash\{0\}$, and G_{fin}. In the case of complete information strategy is not always true, buy it is still true for some classes of families. For example, we have the following results:

Theorem 2.22. (HUREWICZ, 1925) Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\mathscr{O}_X, \mathscr{O}_X)$ holds;
2. $I \not\uparrow G_{fin}(\mathscr{O}_X, \mathscr{O}_X)$.

Theorem 2.23. (PAWLIKOWSKI, 1994) Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_1(\mathscr{O}_X, \mathscr{O}_X)$ holds;
2. $I \not\uparrow G_1(\mathscr{O}_X, \mathscr{O}_X)$.

A recent proof of these results was given in (SZEWCZAK; TSABAN, 2020). This result is valid in the Ω_X class as well:

Theorem 2.24. (SCHEEPERS, 1997) Let (X, τ) be a topological space. Then:

1. $S_1(\Omega_X, \Omega_X)$ holds $\Leftrightarrow I \not\uparrow G_1(\Omega_X, \Omega_X)$.
2. $S_{fin}(\Omega_X, \Omega_X)$ holds $\Leftrightarrow I \not\uparrow G_{fin}(\Omega_X, \Omega_X)$.

2.2 Bornologies

Bornologies and bornological analysis has a principal motivation in the fact that bornological spaces provide an important setting for homological algebra in functional analysis, because the category contains bornological spaces is additive, complete and has a tensor product adjoint to an internal *hom*. For reference we cite (HOGBE-NLEND, 1977).

Definition 2.25. Let (X,τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. $\mathfrak{B}$ is called a bornology in X if $\mathfrak{B}$ is an ideal (that is, it is closed by finite union and if $A \subset B$ and $B \in \mathfrak{B}$ then $A \in \mathfrak{B}$) that covers X.

Definition 2.26. Let (X,τ) be a topological space. A (compact) base $\mathfrak{B}'$ for a bornology $\mathfrak{B}$ in X, is a subset of $\mathfrak{B}$ such that for all $B \in \mathfrak{B}$, there is a $B' \in \mathfrak{B}'$ such that $B \subset B'$ (such that all its elements are compact subsets).

Definition 2.27. Let $\mathfrak{B}$ be a bornology with a compact base in a Tychonoff space (X,τ). We call it a uniform convergence topology in $\mathfrak{B}$, denoted by $\tau_{\mathfrak{B}}$, the topology in $C(X)$ that has as a base of neighborhoods, in each $f \in C(X)$, sets of the form:

$$[f,B,\varepsilon] := \{g \in C(X) : |f(x)-g(x)| < \varepsilon, \text{ for all } x \in B\},$$

where $B \in \mathfrak{B}$ and $\varepsilon > 0$. We denote by $C_{\mathfrak{B}}(X)$ the topological space $(C(X),\tau_{\mathfrak{B}})$.

In the presence of the topology above, we can find that $C_{\mathfrak{B}}(X)$ has a good separation property. First, we recall the following result:

Theorem 2.28. Let (X,τ) be a Tychonoff space. If $A \subseteq X$ is compact and $B \subseteq X$ is closed such that $A \cap B = \emptyset$, then there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

Now, we can prove that $C_{\mathfrak{B}}(X)$ is a Tychonoff space. Indeed, note that Theorem 1.1.5 and Theorem 1.2.3 in (MCCOY; NTANTU, 1988) are still valid when we switch from the hypothesis of a compact network to a bornology with a compact base. As the constructions made in that work are valid by we considered a bornology with a compact base instead of a compact network the result is obtained.

There is a form to generalized the notions of ω-covers and k-covers as follows:

Definition 2.29. Let (X,τ) be a topological space, $\mathfrak{B}$ be a family of subsets of X, and $\mathscr{U}$ be a family of open subsets of X. $\mathscr{U}$ is called a $\mathfrak{B}-$cover of X if $X \notin \mathscr{U}$ and, for all $B \in \mathfrak{B}$, there is $U \in \mathscr{U}$ such that $B \subset U$.

We denote by $\mathscr{O}_{\mathfrak{B}}^X$ the set of all $\mathfrak{B}-$covers of X.

Observation 2.30. Note that $\mathscr{O}_{\mathfrak{B}}^X$ satisfies the following property: if $\mathscr{U} \in \mathscr{O}_{\mathfrak{B}}^X$ and $\mathscr{F} \in [\mathscr{U}]^{<\aleph_0}$, then $\mathscr{U} \setminus \mathscr{F} \in \mathscr{O}_{\mathfrak{B}}^X$.

Example 2.31. Let (X,τ) be a topological space. We have that the sets $\mathbf{F}=[X]^{<\aleph_0}$ and

$$\mathbf{K}=\{A\subset X:\exists\, K\subset X \textit{ compact subsets and } A\subset K\}$$

(if X is Hausdorff, note that $\mathbf{K}=\{A\subset X:\overline{A}$ is compact$\}$) are bornologies with compact base. Furthermore, $C_{\mathbf{F}}(X)=C_p(X)$ and $\mathscr{O}^X_{\mathbf{F}}=\Omega_X$. If X is a Hausdorff space, we have that $C_{\mathbf{K}}(X)=C_k(X)$ and $\mathscr{O}^X_{\mathbf{K}}=\mathscr{K}_X$. For a discuss of this kind of coincidence in a more general setting we cite (NOKHRIN; OSIPOV, 2009).

Definition 2.32. Let (X,τ) be a topological space, $\mathfrak{B}$ be a family of subsets of X, and $\mathscr{U}$ be a family of open subsets of X. $\mathscr{U}$ is called a $\mathfrak{B}-$cofinite cover of X, if it is infinite and for all $B\in\mathfrak{B}$, the set $\{U\in\mathscr{U}:B\not\subset U\}$ is finite.

Denote by $\Gamma^X_{\mathfrak{B}}$ the set of all $\mathfrak{B}$-cofinite covers of X.

Definition 2.33. Let (X,τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. The space X is called $\mathfrak{B}$-Lindelöf if for all $\mathscr{U}\in\mathscr{O}^X_{\mathfrak{B}}$ it has $\mathscr{U}'\subseteq\mathscr{U}$, with $\mathscr{U}'\in\mathscr{O}^X_{\mathfrak{B}}$ countable.

We have that the result in Theorem 2.24 was generalized to the following result:

Theorem 2.34. (MEZABARBA; AURICHI, 2019) Let (X,τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. Then:

1. $S_1(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$ holds if, and only if, $I\not\uparrow G_1(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$.
2. $S_{fin}(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$ holds if, and only if, $I\not\uparrow G_{fin}(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$.

2.3 Function spaces and selection principles

The following results show some translations of a topological space (X,τ) into the space of continuous functions $C_{\mathfrak{B}}(X)$.

Theorem 2.35. (SAKAI, 1988) Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $S_1(\Omega_X,\Omega_X)$ holds;
2. $S_1(\Omega_g,\Omega_g)$ is true for all $g\in C_p(X)$;
3. $S_1(\mathscr{D}_{Cp(X)},\Omega_g)$ holds, for all $g\in C_p(X)$.

Theorem 2.36. (ARKHANGEL'SKII, 1986)(CLONTZ, 2019) Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $S_{fin}(\Omega_X,\Omega_X)$ holds;

2. $S_{fin}(\Omega_g, \Omega_g)$ holds for all $g \in C_p(X)$;

3. $S_{fin}(\mathscr{D}_{C_p(X)}, \Omega_g)$ holds, for all $g \in C_p(X)$.

Additionally, we have the following results:

Theorem 2.37. (KOCINAC, 2003) Let (X, τ) be a Tychonoff space. The following statements are equivalent:

1. $S_1(\mathscr{K}_X, \mathscr{K}_X)$ holds;

2. $S_1(\Omega_g, \Omega_g)$ holds, for all $g \in C_k(X)$;

3. $S_1(\mathscr{D}_{C_k(X)}, \Omega_g)$ holds, for all $g \in C_k(X)$.

Theorem 2.38. (LIN; LIU; TENG, 1994)(OSIPOV, 2018c) Let (X, τ) be a Tychonoff space. The following statements are equivalent:

1. $S_{fin}(\mathscr{K}_X, \mathscr{K}_X)$ hold;

2. $S_{fin}(\Omega_g, \Omega_g)$ holds, for all $g \in C_k(X)$;

3. $S_{fin}(\mathscr{D}_{C_k(X)}, \Omega_g)$ holds, for all $g \in C_k(X)$.

The previous results were generalized to the following ones:

Theorem 2.39. (MEZABARBA; AURICHI, 2019) Let (X, τ) be a Tychonoff space and let $\mathfrak{B}$ be a bornology with a compact base. Let $f : \omega \to \omega \setminus \{0\}$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$ holds;

2. $S_f(\Omega_g, \Omega_g)$ holds for all $g \in C_{\mathfrak{B}}(X)$;

3. $S_f(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \Omega_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$.

Theorem 2.40. (MEZABARBA; AURICHI, 2019) Let (X, τ) be a Tychonoff space and let $\mathfrak{B}$ be a bornology with compact base. The following statements are equivalent:

1. $S_{fin}(\mathscr{O}_{\mathfrak{B}}, \mathscr{O}_{\mathfrak{B}})$ holds;

2. $S_{fin}(\Omega_g, \Omega_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$;

3. $S_{fin}(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \Omega_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$.

Observation 2.41. Let (X, τ) be a topological space and $\mathscr{U} \in \Gamma_X$. If $\mathscr{V} \subseteq \mathscr{U}$ is infinite and $X \notin \mathscr{V}$, then $\mathscr{V} \in \Gamma_X$ because, for any $x \in X$, we have $\{V \in \mathscr{V} : x \notin V\} \subseteq \{U \in \mathscr{U} : x \notin U\}$. In particular, any $\mathscr{U} \in \Gamma_X$ has a countable subset $\mathscr{V}$ such that $\mathscr{V} \in \Gamma_X$.

From the previous observation, we have the following translation:

Theorem 2.42. (GERLITS; NAGY, 1982) Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $\begin{pmatrix} \Omega_X \\ \Gamma_X \end{pmatrix}$ holds;
2. $S_1(\Omega_X,\Gamma_X)$ holds;
3. $S_1(\Omega_g,\Gamma_g)$ holds, for all $g \in C_p(X)$;
4. $C_p(X)$ is Fréchet.

And its more general version:

Theorem 2.43. (MCCOY; NTANTU, 1988) Let (X,τ) be a Tychonoff space and let $\mathfrak{B}$ be a bornology with compact base. The following statements are equivalent:

1. $\begin{pmatrix} \mathscr{O}^X_{\mathfrak{B}} \\ \Gamma^X_{\mathfrak{B}} \end{pmatrix}$ holds;
2. $S_1(\mathscr{O}^X_{\mathfrak{B}},\Gamma^X_{\mathfrak{B}})$ holds;
3. $S_1(\Omega_g,\Gamma_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$;
4. $C_{\mathfrak{B}}(X)$ is Fréchet.

Proof. The implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (4)$ are immediate. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{O}^X_{\mathfrak{B}}$. Note that Observation 2.41 is also valid for $\Gamma_{\mathfrak{B}}$. So, for any $n \in \omega$, there is a $\mathscr{V}_n \subseteq \mathscr{U}_n$ countable such that $\mathscr{V}_n \in \Gamma_{\mathfrak{B}}$. Choosing, for all $n \in \omega$, any $V_n \in \mathscr{V}_n$, it follows that $\{V_n : n \in \omega\} \in \Gamma_{\mathfrak{B}}$. Then (1) and (2) are equivalent. As $C_{\mathfrak{B}}(X)$ is homogeneous, it suffices to show the rest of the implications considering $g = o$.

$(2) \Rightarrow (3)$. Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements in Ω_o. Define, for all $n \in \omega$ the set:

$$\mathscr{U}_n(A) = \{g^{-1}(\langle -\frac{1}{2^n}, \frac{1}{2^n} \rangle) : g \in A\}.$$

We claim that $\mathscr{U}_n(A) \in \mathscr{O}^X_{\mathfrak{B}}$, for all $n \in \omega$. Indeed, let $B \in \mathfrak{B}$. As $o \in \overline{A}$, it follows that there is $h \in A \cap [o, B, \frac{1}{2^n}]$. So, $B \subseteq h^{-1}(\langle -\frac{1}{2^n}, \frac{1}{2^n} \rangle) \in \mathscr{U}_n(A)$.

So, since $\langle \mathscr{U}_n(A_n) : n \in \omega \rangle$ is a sequence of elements in $\mathscr{O}^X_{\mathfrak{B}}$, by (2), it follows that, for all $n \in \omega$, there is $U_n = g_n^{-1}(\langle -\frac{1}{2^n}, \frac{1}{2^n} \rangle) \in \mathscr{U}_n(A_n)$, with $g_n \in A_n$, such that $\{U_n : n \in \omega\} \in \Gamma_{\mathfrak{B}}$.

We claim that $\{g_n : n \in \omega\} \in \Gamma_o$. Indeed let $[o, B, \varepsilon]$ be a neighborhood of o, with $B \in \mathfrak{B}$ and $\varepsilon > 0$. Then, there is $n_0 \in \omega$ such that $B \subset U_n = g_n^{-1}(\langle -\frac{1}{2^n}, \frac{1}{2^n} \rangle)$, for all $n \geq n_0$. Choose

$m \geq n_0$ such that $\frac{1}{2^m} < \varepsilon$. Then $B \subset U_n \subseteq g_n^{-1}(\langle -\varepsilon, \varepsilon \rangle)$, for all $n \geq m$. Therefore, $g_n \in [o, B, \varepsilon]$, for all $n \geq m$.

(4) $\Rightarrow$ (2). Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of elements of $\mathscr{O}^X_{\mathscr{B}}$. Define, for all $n \in \omega$, the set:

$$A(\mathscr{U}_n) = \{g \in C_{\mathfrak{B}}(X) : \exists U_g \in \mathscr{U} \ (g(X \backslash U_g) = \{1\})\}.$$

We claim that $A(\mathscr{U}_n) \in \Omega_o$, for all $n \in \omega$. Indeed, consider a basic neighborhood $[o, B, \varepsilon]$ of o, with $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $\overline{B} \in \mathfrak{B}$, there is $U \in \mathscr{U}$ such that $\overline{B} \subset U$. As $X \backslash U$ is a closed subset that is disjoint of the compact set $\overline{B}$ (this set is compact by the hypotheses that $\mathfrak{B}$ has a compact base), by Theorem 2.28, there is $h \in C_{\mathfrak{B}}(X)$ such that $h(x) = 0$, for all $x \in B$, e $h(X \backslash U) = \{1\}$. So $h \in [o, B, \varepsilon] \cap A(\mathscr{U}_n)$. That is, $o \in \overline{A(\mathscr{U}_n)}$.

Now, since $\langle A(\mathscr{U}_n) : n \in \omega \rangle$ is a sequence of elements in Ω_o, by (4), it follows that for all $n \in \omega$, there is $f_n \in A(\mathscr{U}_n)$, such that $\{f_n : n \in \omega\} \in \Gamma_o$.

We claim that $\{U_n : n \in \omega\} \in \Gamma_{\mathscr{B}}$, where, for any $n \in \omega$, $U_n \in \mathscr{U}_n$ is such that $f_n(X \backslash U_n) \equiv 1$. Indeed, let $B \in \mathscr{B}$. Consider an open neighborhood of o, $[o, B, \frac{1}{2}]$. Then, there is $n_0 \in \omega$ such that $B \subset f_n^{-1}(\langle -\frac{1}{2}, \frac{1}{2} \rangle)$, for all $n \geq n_0$. So $B \cap (X \backslash U_n) = \emptyset$, for all $n \geq n_0$. Therefore, $B \subset U_n$, for all $n \geq n_0$. □

With a few modifications to the previous theorem, we can obtain the following results:

Theorem 2.44. Let (X, τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base and $f : \omega \to \omega \backslash \{0\}$ is a function. The following statements are equivalent:

1. $\begin{pmatrix} \mathscr{O}^X_{\mathfrak{B}} \\ \Gamma^X_{\mathfrak{B}} \end{pmatrix}$ holds;
2. $S_f(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds;
3. $S_f(\Omega_g, \Gamma_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$;
4. $C_{\mathfrak{B}}(X)$ is Fréchet.

Theorem 2.45. Let (X, τ) be a Tychonoff space and $\mathscr{B}$ is a bornology with compact base. The following statements are equivalent:

1. $\begin{pmatrix} \mathscr{O}^X_{\mathfrak{B}} \\ \Gamma^X_{\mathfrak{B}} \end{pmatrix}$ holds ;
2. $S_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds ;
3. $S_{fin}(\Omega_g, \Gamma_g)$ holds, for all $g \in C_{\mathfrak{B}}(X)$;
4. $C_{\mathfrak{B}}(X)$ is Fréchet.

On the other hand, when dealing with spaces of functions, we have the following equivalence between selection principles and games:

Theorem 2.46. (SCHEEPERS, 1997)(SCHEEPERS, 1999) Let (X,τ) be a separable metric space and $g \in C_p(X)$. The following statements are equivalent:

1. $S_1(\Omega_g, \Omega_g)$ holds;
2. $I \not\uparrow G_1(\Omega_g, \Omega_g)$;
3. $I \not\uparrow G_1(\mathscr{D}_{C_p(X)}, \mathscr{D}_{C_p(X)})$;
4. $S_1(\mathscr{D}_{C_p(X)}, \mathscr{D}_{C_p(X)})$ holds.

Theorem 2.47. (SCHEEPERS, 1997)(SCHEEPERS, 1999) Let (X,τ) be a separable metric space and $g \in C_p(X)$. The following statements are equivalent:

1. $S_{fin}(\Omega_g, \Omega_g)$;
2. $I \not\uparrow G_{fin}(\Omega_g, \Omega_g)$;
3. $I \not\uparrow G_{fin}(\mathscr{D}_{C_p(X)}, \mathscr{D}_{C_p(X)})$;
4. $S_{fin}(\mathscr{D}_{C_p(X)}, \mathscr{D}_{C_p(X)})$.

2.4 Duality

We will start by defining the equivalence and duality of selective topological games:

Definition 2.48. Let G_1 and G_2 be two selective topological games. The games are called equivalent if:

- $I \uparrow G_1$ if, and only if, $I \uparrow G_2$;
- $II \uparrow G_1$ if, and only if, $II \uparrow G_2$.

Definition 2.49. Let G_1 and G_2 be two selective topological games. The games are said to be dual if:

- $I \uparrow G_1$ if, and only if, $II \uparrow G_2$;
- $I \uparrow G_2$ if, and only if, $II \uparrow G_1$.

We will define

$$G_1(\mathscr{A}, \neg\mathscr{B}) = G_1\left(\mathscr{A}, \wp\left(\bigcup\mathscr{A}\right) \setminus \mathscr{B}\right),$$

that is, in this game Player *II* wins if $\{B_n : n \in \omega\} \notin \mathscr{B}$.

For a set X, we will denote by $\mathscr{C}(X) := \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ the set of all the choice functions of X.

Definition 2.50. Let X and Y be two sets. X is called coinitial in Y with respect to $\subseteq$, and we denote by $X \preceq Y$, if $X \subseteq Y$ and for all $y \in Y$, there is a $x \in X$ such that $x \subseteq y$.

Definition 2.51. A set $\mathscr{R}$ is called a reflection of a class $\mathscr{A}$ if $\{range(f) : f \in \mathscr{C}(\mathscr{R})\} \preceq \mathscr{A}$.

The following result describes, in general, the dual game for certain classes of sets.

Theorem 2.52. (CLONTZ, 2020) Let $\mathscr{R}$ be a reflection of a class $\mathscr{A}$. Then $G_1(\mathscr{A}, \mathscr{B})$ and $G_1(\mathscr{R}, \neg\mathscr{B})$ are dual games.

2.4.1 Examples

Below, we will present some examples of duality. Let (X, τ) be a topological space. For $x \in X$, denote by $\tau_x = \{U \in \tau : U$ is neighborhood of $x\}$ and $P_X = \{\tau_x : x \in X\}$.

Proposition 2.53. (CLONTZ, 2020) P_X is a reflection of $\mathscr{O}_X$.

Proof. Let $\mathscr{U} \in \mathscr{O}_X$ and $\tau_x \in P_X$. So, there is $U_x \in \mathscr{U}$ such that $x \in U_x$. Define $f_{\mathscr{U}}(\tau_x) = U_x$. So $f_{\mathscr{U}} \in \mathscr{C}(P_X)$. It is easy to see that $range(f_{\mathscr{U}}) \in \mathscr{O}_X$ and $range(f_{\mathscr{U}}) \subseteq \mathscr{U}$. □

Definition 2.54. Let (X, τ) be a topological space. The point-open game $G^{\mathscr{O}_X}(X)$ is played as follows: In each inning $n \in \omega$, Player *I* chooses $x_n \in X$. Then, Player *II* chooses $U_n \in \tau$ such that $x_n \in U_n$. Player *I* is the winner if $\{U_n : n \in \omega\} \in \mathscr{O}_X$. Otherwise, Player *II* wins.

Corollary 2.55. (GALVIN, 1978) Let (X, τ) be a topological space. The games $G_1(\mathscr{O}_X, \mathscr{O}_X)$ and $G^{\mathscr{O}_X}(X)$ are dual.

Proof. Note that the point-open game is equivalent to $G_1(P_X, \neg\mathscr{O}_X)$. The result is derived from Proposition 2.53 and Theorem 2.52. □

We can consider the following variation of the point-open game.

Definition 2.56. Let (X, τ) be a topological space. The finite open game is $G_F^{\mathscr{O}_X}(X)$ and is played as: In each inning $n \in \omega$, Player *I* chooses $F_n \in [X]^{<\aleph_0}$. Then, Player *II* chooses $U_n \in \tau$ such that $F_n \subset U_n$. Player *I* is the winner if $\{U_n : n \in \omega\} \in \mathscr{O}_X$. Otherwise, Player *II* wins.

Interestingly, this change in the number of elements that Player I chooses does not make a difference in the existence of winning strategies.

Proposition 2.57. (TELGÁRSKY, 1975) Let (X,τ) be a topological space. The games $G^{\mathscr{O}_X}(X)$ and $G_F^{\mathscr{O}_X}(X)$ are equivalent.

For $F \in [X]^{<\aleph_0}$, we denote by $\tau_F = \{U \in \tau : F \subseteq U\}$ and $\mathscr{F}_X = \{\tau_F : F \in [X]^{<\aleph_0}\}$.

Proposition 2.58. (CLONTZ, 2020) $\mathscr{F}_X$ is a reflection of Ω_X.

Proof. Let $\mathscr{U} \in \Omega_X$ and $\tau_F \in \mathscr{F}_X$, with $F \in [X]^{<\aleph_0}$. Then, there is $U_F \in \mathscr{U}$ such that $F \subseteq U_F$. Define $f_{\mathscr{U}}(\tau_F) = U_F$. So $f_{\mathscr{U}} \in \mathscr{C}(\mathscr{F}_X)$. It is immediate that $range(f_{\mathscr{U}}) \in \Omega_X$ and $range(f_{\mathscr{U}}) \subseteq \tau_F$. □

Definition 2.59. Let (X,τ) be a topological space. The Ω-finite-open game $\Omega G_F(X)$ is played as follows: In the inning $n \in \omega$, Player I chooses $F_n \in [X]^{<\aleph_0}$. Next, Player II responds with $U_n \in \tau$ such that $F_n \subset U_n$. Player I is a winner if $\{U_n : n \in \omega\} \in \Omega_X$. Otherwise, the winner is Player II.

Corollary 2.60. (CLONTZ, 2020) Let (X,τ) be a topological space. The games $G_1(\Omega_X,\Omega_X)$ and $\Omega G_F(X)$ are dual.

Proof. Note that the game Ω-finite-open is equivalent to the game $G_1(\mathscr{F}_X,\neg\Omega_X)$. The conclusion is derived from Proposition 2.58 and Theorem 2.52. □

Proposition 2.61. (CLONTZ, 2020) For any $x \in X$, τ_x is a reflection of Ω_x.

Proof. Let $Y \in \Omega_x$ and $U \in \tau_x$. So, there is $y_U \in U \cap Y$. Define $f_Y(U) := y_U$. Then $f_Y \in \mathscr{C}(\tau_x)$. It is immediate that $range(f_Y) \in \Omega_x$ and $range(f_Y) \subseteq Y$. □

Definition 2.62. Let (X,τ) be a topological space and $x \in X$. The neighborhood-point game $G(X,x)$ is played as follows: In the inning $n \in \omega$, Player I chooses $V_n \in \tau_x$. Next, Player II chooses $x_n \in V_n$. Player I is the winner if $\{x_n : n \in \omega\} \in \Omega_x$. Otherwise, Player II is the winner.

Corollary 2.63. (GALVIN, 1978) Let (X,τ) be a topological space and $x \in X$. The games $G_1(\Omega_x,\Omega_x)$ and $G(X,x)$ are dual.

Proof. Note that the neighborhood-point game is equivalent to $G_1(\tau_x,\neg\Omega_x)$. The result is derived from Proposition2.61 and Theorem 2.52. □

The following game is a small variation of the neighborhood-point game.

Definition 2.64. Let (X,τ) be a topological space and $x \in X$. The convergent neighborhood-point $G^{\rightarrow}(X,x)$ is played as follows: In the inning $n \in \omega$, Player I chooses $U_n \in \tau_x$. Next, Player II chooses $x_n \in U_n$. Player I is the winner if $x_n \to x$, when $n \to \infty$. Otherwise, Player II is the winner.

This change in the winning criteria does not make a difference for Player I.

Theorem 2.65 ((GRUENHAGE, 1976))**.** Let $x \in X$. Then $I \uparrow G(X,x) \Leftrightarrow I \uparrow G^{\rightarrow}(X,x)$.

Proposition 2.66 ((CLONTZ, 2020))**.** τ is a reflection of $\mathscr{D}_X$.

Proof. Let $D \in \mathscr{D}_X$ and $U \in \tau$. So, there is $x_U \in U \cap D$. Define $f_D(U) = x_U$. Then $f_D \in \mathscr{C}(\mathscr{R})$. It is clear that $range(f_D) \in \mathscr{D}_X$ and $range(f_D) \subseteq U$. □

Definition 2.67. Let (X,τ) be a topological space. The point picking game $G^{\mathscr{D}_X}(X)$ is played as follows: In the inning $n \in \omega$, Player I chooses $U_n \in \tau$. Next, Player II chooses $x_n \in U_n$. Player I is the winner if $\{x_n : n \in \omega\} \in \mathscr{D}_X$. Otherwise, Player II is the winner.

Corollary 2.68 ((SCHEEPERS, 1999))**.** Let (X,τ) be a topological space. The games $G_1(\mathscr{D}_X,\mathscr{D}_X)$ and $G^{\mathscr{D}_X}(X)$ are dual.

Proof. Note that the point picking game is equivalent to the game $G_1(\tau,\neg\mathscr{D}_X)$. The result follows from Proposition 2.66 and Theorem 2.52. □

CHAPTER

3

EQUIVALENCES IN VARIATIONS OF SELECTION PRINCIPLES

The work of (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995) shows an equivalence of the selection principle $S_f(\Omega_x, \Omega_x)$ in the following result:

Corollary 3.1. (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995) If X is a T_1 space, $x \in X$ and $f : \omega \to \omega \setminus \{0\}$ is unbounded, then $S_f(\Omega_x, \Omega_x)$ is equivalent to $S_S(\Omega_x, \Omega_x)$, where $S : \omega \to \omega \setminus \{0\}$ is the function given by $S(n) = n + 1$, for all $n \in \omega$.

Lemma 3.2. (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995) If X is a T_1 space, $x \in X$ and $f : \omega \to \omega \setminus \{0\}$ is bounded, then $S_f(\Omega_x, \Omega_x)$ is equivalent to $S_1(\Omega_x, \Omega_x)$.

The above function S is called the successor function. We will use this notation throughout this chapter. In (AURICHI; BELLA; DIAS, 2018) it is proved a similar equivalence to the tightness selective topological game in the following result:

Proposition 3.3. (AURICHI; BELLA; DIAS, 2018) If X is a T_1 space $x \in X$ then:

1. If $f : \omega \to \omega \setminus \{0\}$ is bounded, then the games $G_f(\Omega_x, \Omega_x)$ and $G_k(\Omega_x, \Omega_x)$ are equivalent, where $k = \limsup_{n \in \omega} f(n) \in \omega \setminus \{0\}$;
2. If $f : \omega \to \omega \setminus \{0\}$ is unbounded, then the games $G_f(\Omega_x, \Omega_x)$ and $G_S(\Omega_x, \Omega_x)$ are equivalent.

The properties of Ω_x used in the above proofs can be summarized as follows.

(P1) If $B \in \mathscr{B}$ e $F \in [B]^{<\aleph_0}$ then $B \setminus F \in \mathscr{B}$;

(P2) If $B \in \mathscr{B}$ and $A \supseteq B$, then $A \in \mathscr{B}$, where $\mathscr{B}$ is a class of families of subsets of X.

In this chapter, we use the properties $(P1)$ and $(P2)$ to obtain generalizations of the results presented previously and some other similar results.

3.1 Equivalences of selection principles with arbitrary classes of families

We begin by generalizing one of the equivalences mentioned above.

Proposition 3.4. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X and $f : \omega \to \omega\setminus\{0\}$ be a limited function. Suppose that $\mathscr{B}$ satisfies properties $(P1)$ and $(P2)$. The following statements are equivalent:

1. $S_f(\mathscr{A},\mathscr{B})$ holds;
2. $S_k(\mathscr{A},\mathscr{B})$ holds, where $k = \limsup_{n\in\omega} f(n) \in \omega\setminus\{0\}$.

Proof. $(1) \Rightarrow (2)$. Let $\langle A_n : n \in \omega\rangle$ be a sequence of elements in $\mathscr{A}$. Let $n_0 \in \omega$ be such that $f(n) \leq k$, for all $n \geq n_0$. Consider the finite set $H = \{n \in \omega : f(n) > k\}$. By (1), for all $n \in \omega$, there is $F_n \in [A_n]^{\leq f(n)}$ such that $\bigcup_{n\in\omega} F_n \in \mathscr{B}$.

So, by $(P1)$, $\bigcup_{n\in\omega\setminus H} F_n \in \mathscr{B}$. Define $G_n = F_n$, if $n \in \omega\setminus H$, and take $G_n \in [A_n]^{\leq k}$ arbitrarily, for $n \in H$. Then $\bigcup_{n\in\omega\setminus H} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (2) is true.

$(2) \Rightarrow (1)$. Let $\langle A_n : n \in \omega\rangle$ be a sequence of elements in $\mathscr{A}$. Note that the set $N := \{n \in \omega : f(n) = k\}$ is infinite. Consider the sequence $\langle A_m : m \in N\rangle$. By (2), for all $m \in N$, there is an $F_m \in [A_m]^{\leq k}$ such that $\bigcup_{m\in N} F_m \in \mathscr{B}$.

Define $G_n = F_n$, if $n \in N$ and take $G_n \in [A_n]^{\leq f(n)}$ arbitrarily, for $n \in \omega\setminus N$. Then $\bigcup_{n\in N} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (1) is true. □

Now, the case where f is unbounded:

Proposition 3.5. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X and an unlimited function $f : \omega \to \omega\setminus\{0\}$. Suppose that $\mathscr{B}$ satisfies property $(P2)$. The following statements are equivalent:

1. $S_f(\mathscr{A},\mathscr{B})$ holds;
2. $S_S(\mathscr{A},\mathscr{B})$ holds.

Proof. (1) $\Rightarrow$ (2). Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements of $\mathscr{A}$. There is an increasing sequence $\langle k_m : m \in \omega \rangle$ such that for all $m \in \omega$, $f(m) \leq k_m + 1$. Consider the sequence $\langle A_{k_m} : m \in \omega \rangle$. By (1), for all $m \in \omega$, there is an $F_{k_m} \in [A_{k_m}]^{\leq f(m)}$ such that $\bigcup_{m\in\omega} F_{k_m} \in \mathscr{B}$.

Define $G_n = F_n$, if $n \in M := \{k_m : m \in \omega\}$ and take $G_n \in [A_n]^{\leq n+1}$ arbitrarily, for $n \in \omega \backslash M$. Then $\bigcup_{n\in\omega\backslash M} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (2) is true.

(2) $\Rightarrow$ (1). Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{A}$. As f is unlimited, we can obtain an increasing sequence $\langle k_m : m \in \omega \rangle$ such that, for all $m \in \omega$, $m+1 \leq f(k_m)$. Consider the sequence $\langle A_{k_m} : m \in \omega \rangle$. By (2), for all $m \in \omega$, there is $F_{k_m} \in [A_{k_m}]^{\leq m+1}$ such that $\bigcup_{m\in\omega} F_{k_m} \in \mathscr{B}$.

Define $G_n = F_n$, if $n \in M := \{k_m : m \in \omega\}$ and take $G_n \in [A_n]^{\leq f(n)}$ arbitrarily, for $n \in \omega \backslash M$. Then $\bigcup_{n\in\omega\backslash M} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (1) is true. □

Let $f : \omega \to (\omega\backslash\{0,1\}) \cup \{\aleph_0\}$ be a function. It is natural to consider the following selection principle $S_f^<(\mathscr{A},\mathscr{B})$, defined as follows: for all sequences $\langle A_n : n \in \omega \rangle$ in $\mathscr{A}$, there is an $F_n \in [A_n]^{<f(n)}$, for all $n \in \omega$, such that $\bigcup_{n\in\omega} F_n \in \mathscr{B}$. In the presence of properties $(P1)$ and $(P2)$, this selection principle collapses to the classical selection principles.

Proposition 3.6. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X and let $f : \omega \to (\omega\backslash\{0\}) \cup \{\aleph_0\}$ be a function such that the set $W = \{n \in \omega : f(n) = \aleph_0\}$ is finite. Suppose that $\mathscr{B}$ satisfies properties $(P1)$ and $(P2)$. The following statements are equivalent:

1. $S_f^<(\mathscr{A},\mathscr{B})$ holds;
2. $S_h(\mathscr{A},\mathscr{B})$ holds, where a function $h : \omega \to \omega\backslash\{0\}$ satisfies $h(m) = f(m) - 1$, for all $m \in \omega\backslash W$

Proof. (1) $\Rightarrow$ (2). Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{A}$. By (1), for all $n \in \omega$, there is $F_n \in [A_n]^{<f(n)}$ such that $\bigcup_{n\in\omega} F_n \in \mathscr{B}$. So, by $(P1)$, $\bigcup_{n\in\omega\backslash W} F_n \in \mathscr{B}$.

Now, define $G_n = F_n$, if $n \in \omega\backslash W$ and take $G_n \in [A_n]^{\leq h(n)}$ arbitrarily for $n \in W$. Then $\bigcup_{n\in\omega\backslash H} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (2) is true.

(2) $\Rightarrow$ (1). Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{A}$. Consider the sequence $\langle A_m : m \in \omega\backslash W \rangle$. By (2), for all $m \in \omega\backslash W$, there is $F_m \in [A_m]^{\leq h(m)}$ such that $\bigcup_{m\in\omega\backslash W} F_m \in \mathscr{B}$.

Define $G_n = F_n$, if $n \in \omega\backslash W$ and take $G_n \in [A_n]^{<\aleph_0}$ arbitrarily, for $n \in W$. Then $\bigcup_{n\in\omega\backslash W} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (1) is true. □

We also have the case where W is infinite:

Proposition 3.7. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X, and let $f : \omega \to (\omega\setminus\{0\}) \cup \{\aleph_0\}$ be a function such that the set $W = \{n \in \omega : f(n) = \aleph_0\}$ is infinite. Suppose that $\mathscr{B}$ satisfies property $(P2)$. The following statements are equivalent:

1. $S_f^{<}(\mathscr{A},\mathscr{B})$ holds;
2. $S_{fin}(\mathscr{A},\mathscr{B})$ holds.

Proof. $(1) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (1)$. Let $\langle A_n : n \in \omega\rangle$ be a sequence of elements in $\mathscr{A}$. Consider the sequence $\langle A_m : m \in W\rangle$. By (2), for all $m \in W$, there is an $F_m \in [A_m]^{<\aleph_0}$ such that $\bigcup_{m\in W} F_m \in \mathscr{B}$.

Define $G_n = F_n$, if $n \in W$ and take $G_n \in [A_n]^{<f(n)}$ arbitrarily, for $n \in \omega\setminus W$. Then $\bigcup_{n\in W} F_n \subseteq \bigcup_{n\in\omega} G_n$. So, by $(P2)$, $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, (1) is true. □

From the results obtained previously, it follows that, in the study of the selection principles for classes $\mathscr{A}$ and $\mathscr{B}$, with $\mathscr{B}$ satisfying $(P1)$ and $(P2)$, we can restrict ourselves, for now on, to the study of the selection principles S_1, S_k, with $k \in \omega\setminus\{0\}$, S_S and S_{fin}.

On the other hand, in (AURICHI; DUZI, 2021) the following selection principle has been defined:

Definition 3.8. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of a set X. $S_{bnd}(\mathscr{A},\mathscr{B})$ is the following selection principle: for all sequences $\langle A_n : n \in \omega\rangle$ of elements in $\mathscr{A}$, there is a sequence $\langle B_n : n \in \omega\rangle$ and $k \in \omega\setminus\{0\}$ such that for all $n \in \omega$, $B_n \in [A_n]^{<\aleph_0}$, $|B_n| \leq k$ and $\bigcup_{n\in\omega} B_n \in \mathscr{B}$.

For classes $\mathscr{B}$ with properties $(P1)$ and $(P2)$, we can also obtain the following implication:

Proposition 3.9. Let $\mathscr{A}$ e $\mathscr{B}$ be classes of families of subsets of a set X. Suppose that $\mathscr{B}$ has properties $(P1)$ and $(P2)$. If $S_{bnd}(\mathscr{A},\mathscr{B})$ holds, then $S_S(\mathscr{A},\mathscr{B})$ holds.

Proof. Let $\langle A_n : n \in \omega\rangle$ be a sequence of elements in $\mathscr{A}$. By hypothesis, there is $k \in \omega\setminus\{0\}$ and, for all $n \in \omega$, there is $B_n \in [A_n]^{<\aleph_0}$ such that $|B_n| \leq k$ and $\bigcup_{n\in\omega} B_n \in \mathscr{B}$. By $(P1)$, it follows that $\bigcup_{n\geq k-1} B_n \in \mathscr{B}$. Define $G_n = B_n$, for $n \geq k-1$ and take $G_n \in [A_n]^{\leq n+1}$ arbitrarily for $0 \leq n < k-1$. So $\bigcup_{n\geq k-1} B_n \subseteq \bigcup_{n\in\omega} G_n$. By $(P2)$, it follows that $\bigcup_{n\in\omega} G_n \in \mathscr{B}$. Therefore, $S_S(\mathscr{A},\mathscr{B})$ is true. □

In summary, we have:

Proposition 3.10. Let (X,τ) be a topological space, $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of X, with $\mathscr{B}$ satisfying the properties $(P1)$ and $(P2)$. Then, if $f:\omega\to\omega\setminus\{0\}$ is a bounded function, $k=\limsup_{n\in\omega} f(n)$ and $h:\omega\to\omega\setminus\{0\}$ is an unbounded function, the following implications are true:

$$S_1(\mathscr{A},\mathscr{B})\Rightarrow S_f(\mathscr{A},\mathscr{B})\Leftrightarrow S_k(\mathscr{A},\mathscr{B})\Rightarrow S_{bnd}(\mathscr{A},\mathscr{B})\Rightarrow S_S(\mathscr{A},\mathscr{B})\Leftrightarrow S_h(\mathscr{A},\mathscr{B})\Rightarrow S_{fin}(\mathscr{A},\mathscr{B}).$$

We can see that the properties $(P1)$ and $(P2)$ are sufficient conditions for us to obtain equivalences as the ones we got before. Later, we will see that these conditions are not a necessary condition for obtaining equivalences.

Problem 3.11. What conditions, including conditions $(P1)$ and $(P2)$, are necessary and sufficient in $\mathscr{A}$ and $\mathscr{B}$ to obtain the equivalences mentioned here?

3.2 Some specific cases

3.2.1 Tightness

Note that in the case of Ω_x, the principle S_f collapses to S_1, but this is possible using a particular topological property:

Proposition 3.12. (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995) Let (X,τ) be a T_1 space, $x\in X$, and let $f:\omega\to\omega\setminus\{0\}$ be a limited function. The following statements are equivalent:

1. $S_f(\Omega_x,\Omega_x)$ holds;
2. $S_1(\Omega_x,\Omega_x)$ holds.

Proof. Use Proposition 3.10 and $\overline{\bigcup_{i=1}^{n} F_n}=\bigcup_{i=1}^{n}\overline{F_n}$. □

In (AURICHI; DUZI, 2021) it was shown that, practically with the same proof as above, $S_{bnd}(\Omega_x,\Omega_x)$ is equivalent to $S_1(\Omega_x,\Omega_x)$.

On the other hand, Examples 3.7 and 3.8 in (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995), show that, if (X,τ) be a T_1 space, $x\in X$ and $\mathscr{A}=\mathscr{B}=\Omega_x$, there are three different types of selection principles: $S_1(\Omega_x,\Omega_x)$, $S_S(\Omega_x,\Omega_x)$ and $S_{fin}(\Omega_x,\Omega_x)$.

3.2.2 Open covers

Note that the family $\mathscr{O}_X$, for a topological space (X,τ), does not satisfy the property $(P1)$. But we see that most variations collapse to $S_1(\mathscr{O}_X,\mathscr{O}_X)$. We include the following proof for the interest of the reader:

Proposition 3.13. (GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995) Let (X,τ) be a topological space and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{O}_X,\mathscr{O}_X)$ holds;

2. $S_1(\mathscr{O}_X,\mathscr{O}_X)$ holds.

Proof. $(2)\Rightarrow(1)$. It is clear from the Observation 2.13.

$(1)\Rightarrow(2)$. Let $\langle\mathscr{U}_n : n\in\omega\rangle$ be a sequence of elements in $\mathscr{O}_X$. Define

$$\mathscr{V}_0=\left\{\bigcap_{i=0}^{f(0)-1}U_i:\ U_i\in\mathscr{U}_i,\ 0\leq i\leq f(0)-1\right\}$$

and, for $n\geq 1$,

$$\mathscr{V}_n=\left\{\bigcap_{i=\sum_{j=0}^{n-1}f(j)}^{\sum_{j=0}^{n}f(j)-1}U_i:\ U_i\in\mathscr{U}_i,\ \sum_{j=0}^{n-1}f(j)\leq i\leq\sum_{j=0}^{n}f(j)-1\right\}.$$

So $\langle\mathscr{V}_n : n\in\omega\rangle$ is a sequence of elements in $\mathscr{O}_X$. By (1), for all $n\in\omega$, there is an $\mathscr{F}_n\in[\mathscr{V}_n]^{\leq f(n)}$ such that $\bigcup_{n\in\omega}\mathscr{F}_n\in\mathscr{O}_X$. We can suppose that

$$\mathscr{F}_0=\{H_i:\ 0\leq i\leq f(0)-1\}$$

and, for $n\geq 1$,

$$\mathscr{F}_n=\left\{H_i:\ \sum_{j=0}^{n-1}f(j)\leq i\leq\sum_{j=0}^{n}f(j)-1\right\}.$$

From the definition of the families $\mathscr{V}_n$ it follows that for all $n\in\omega$, there is $U_n\in\mathscr{U}_n$ such that $H_n\subseteq U_n$. Then $\bigcup_{n\in\omega}\left(\bigcup\mathscr{F}_n\right)\subseteq\bigcup_{n\in\omega}U_n$. Therefore, $\{U_n:\ n\in\omega\}\in\mathscr{O}_X$. □

Observation 3.14. Note that for a topological space (X,τ), Proposition 3.6 is still valid. Indeed, just ignore all the $n\in W$ innings, do as in the previous proof, and finally add open sets in the $n\in W$ innings.

Observation 3.15. Any topological space (X,τ) that is σ-compact is such that $S_{fin}(\mathscr{O}_X,\mathscr{O}_X)$ holds. Indeed, let $\langle\mathscr{U}_n : n\in\omega\rangle$ be a sequence of elements in $\mathscr{O}_X$ and $X=\bigcup_{n\in\omega}C_n$, with all C_n's compact. So, for all $n\in\omega$, there is $\mathscr{F}_n\in[\mathscr{U}_n]^{<\aleph_0}$ such that $C_n\subseteq\bigcup\mathscr{F}_n$. Then $\bigcup_{n\in\omega}\mathscr{F}_n\in\mathscr{O}_X$.

Note that $II\uparrow G^{\mathscr{O}_\mathbb{R}}(\mathbb{R})$ (just note that, in all inning $n\in\omega$, Player II chooses an open interval with length $\frac{1}{2^n}$, the union of all intervals cannot cover $\mathbb{R}$.). So, by Corollary 2.55, player $I\uparrow G_1(\mathscr{O}_\mathbb{R},\mathscr{O}_\mathbb{R})$. Then, according to the previous observation and Theorem 2.23, $S_{fin}(\mathscr{O}_\mathbb{R},\mathscr{O}_\mathbb{R})$ holds, but $S_1(\mathscr{O}_\mathbb{R},\mathscr{O}_\mathbb{R})$ fails.

Furthermore, in (AURICHI; DUZI, 2021) it is observed that in any compact space (X,τ), $S_{bnd}(\mathcal{O}_X,\mathcal{O}_X)$ is true, but in the space 2^ω (that is, the countable product of discrete space $2 := \{0,1\}$) $S_1(\mathcal{O}_{2^\omega},\mathcal{O}_{2^\omega})$ fails, and in $\mathbb{R}$, $S_{fin}(\mathcal{O}_\mathbb{R},\mathcal{O}_\mathbb{R})$ is true but $S_{bnd}(\mathcal{O}_\mathbb{R},\mathcal{O}_\mathbb{R})$ fails.

So, for a topological space (X,τ) and $\mathscr{A}=\mathscr{B}=\mathcal{O}_X$, there are three different selection principles: $S_1(\mathcal{O}_X,\mathcal{O}_X)$, $S_{bnd}(\mathcal{O}_X,\mathcal{O}_X)$ and $S_{fin}(\mathcal{O}_X,\mathcal{O}_X)$.

3.2.3 Ω-covers

It is clear that the family Ω_X satisfies the property $(P2)$. Using Lemma 1 in (SCHEEPERS, 1994) we have that Ω_X satisfies property $(P1)$. Then, Propositions 3.6, 3.7, and 3.10 are valid. The following results provide us with a translation of the selection principle in $\mathcal{O}_X$ to Ω_X.

We begin with the following lemmas:

Lemma 3.16. (SCHEEPERS, 1996) Let (X,τ) be a topological space. If $S_1(\Omega_X,\Omega_X)$ holds, then $S_1(\Omega_{X^n},\Omega_{X^n})$ holds, for all $n\in\omega\backslash\{0\}$.

Lemma 3.17. (SCHEEPERS, 1996) Let (X,τ) be a topological space. If $S_{fin}(\Omega_X,\Omega_X)$ holds, then $S_{fin}(\Omega_{X^n},\Omega_{X^n})$ is true for all $n\in\omega\backslash\{0\}$.

With a few modifications to the proof above, the following result can be obtained:

Lemma 3.18. Let (X,τ) be a topological space. If $S_{bnd}(\Omega_X,\Omega_X)$ holds, then $S_{bnd}(\Omega_{X^n},\Omega_{X^n})$ holds for all $n\in\omega\backslash\{0\}$.

Proof. Fix $n\in\omega\backslash\{0\}$ and let $\langle\mathscr{U}_m : m\in\omega\rangle$ be a sequence of elements in Ω_{X^n}. Define, for all $m\in\omega$, $\mathscr{V}_m=\{V\in\tau\ :\ V^n\subseteq U \text{ for some } U\in\mathscr{U}_m\}$. We claim that $\mathscr{V}_m\in\Omega_X$, for all $m\in\omega$.

Indeed, let $F\in[X]^{<\aleph_0}$. As $F^n\in[X^n]^{<\aleph_0}$, it follows that there is an open $U\in\mathscr{U}_m$ such that $F^n\subset U$. Let $z=(x_1,\dots,x_n)\in F^n$. Then, for all $i\in\{1,\dots,n\}$, there are $U_i(z)\in\tau$ such that $(x_1,\dots,x_n)\in\prod_{i=1}^n U_i(z)\subseteq U$. So, for all $x\in F$, consider $U_x=\bigcap\{U_i(z) : z$ having x as an element in the coordinate $i\in\{1,\dots,n\}\}\in\tau$. Consider $V_F=\bigcup_{x\in F}U_x\in\tau$. Note that $F\subset V_F$. So $F^n\subset V_F^n\subseteq U$ and $V_F\in\mathscr{V}_m$.

It follows that for all $m\in\omega$, there are $V_m\in\mathscr{V}_m$ and $k\in\omega\backslash\{0\}$, such that $|V_m|\leq k$ and $\bigcup_{m\in\omega}V_m\in\Omega_X$. Choose, for all $m\in\omega$, a set $\mathscr{W}_m\in[\mathscr{U}_m]^{\leq k}$ such that there is, for each $Z\in V_m$, an element $W\in\mathscr{W}_m$ such that $Z^n\subseteq W$. Therefore, we conclude that $\bigcup_{m\in\omega}\mathscr{W}_m\in\Omega_{X^n}$. That is, $S_{bnd}(\Omega_{X^n},\Omega_{X^n})$ holds. □

With practically the same proof as above, we obtain:

Lemma 3.19. Let (X,τ) be a topological space and let $f:\omega\to\omega\backslash\{0\}$ be a function. If $S_f(\Omega_X,\Omega_X)$ holds, then $S_f(\Omega_{X^n},\Omega_{X^n})$ holds for all $n\in\omega\backslash\{0\}$.

From the previous Lemmas 3.16 and 3.17, we have the following results:

Theorem 3.20. (SAKAI, 1988) Let (X,τ) be a topological space. The following statements are equivalent:

1. $S_1(\Omega_X,\Omega_X)$ holds;
2. $S_1(\mathscr{O}_{X^n},\mathscr{O}_{X^n})$ holds for all $n\in\omega\setminus\{0\}$.

Theorem 3.21. (JUST *et al.*, 1996) Let (X,τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\Omega_X,\Omega_X)$ holds;
2. $S_{fin}(\mathscr{O}_{X^n},\mathscr{O}_{X^n})$ holds for all $n\in\omega$.

Using Lemma 3.19, and with few modifications in the proof of the previous theorems, we can obtain the following result:

Theorem 3.22. Let (X,τ) be a topological space and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. If $S_f(\Omega_X,\Omega_X)$ holds, then $S_f(\mathscr{O}_{X^n},\mathscr{O}_{X^n})$ holds, for all $n\in\omega$
2. If $S_g(\mathscr{O}_{X^n},\mathscr{O}_{X^n})$ holds for all $n\in\omega$ and all infinite subsequences g of f, then $S_f(\Omega_X,\Omega_X)$ holds.

Proof. (1). By the Lemma 3.19, is enough to show the case $n=1$.

Indeed, let $\langle\mathscr{U}_m : m\in\omega\rangle$ be a sequence of elements in $\mathscr{O}_X$. Re-indexing the sequence $\langle\mathscr{U}_m : m\in\omega\rangle=\langle\mathscr{U}_{m,k} : m\in\omega, k\in\omega\setminus\{0\}\rangle$ and considering $g:\omega\times\omega\setminus\{0\}\to\omega\setminus\{0\}$ such that $g(m,k)$ is $f(n)$, where $n\in\omega$ corresponds to the number, in an order fixed by $\omega\times\omega\setminus\{0\}$, of the pair (m,k). Define, for all $m\in\omega$ and $k\in\omega\setminus\{0\}$,

$$\mathscr{V}_{m,2k-1}=\left\{\bigcup_{i=1}^{2k-1}U_i : U_i\in\mathscr{U}_{m,(k-1)(2k-1)+i}\right\}$$

and

$$\mathscr{V}_{m,2k}=\left\{\bigcup_{i=1}^{2k}U_i : U_i\in\mathscr{U}_{m,(k(2k)-(k-1))+(i-1)}\right\}.$$

Define, for all $m\in\omega$, $\mathscr{V}_m=\bigcup_{k\in\omega\setminus\{0\}}\mathscr{V}_{m,k}$. Let $F\in[X]^{<\aleph_0}$ adm suppose that $|F|=r$. As every element of $\mathscr{V}_{m,r}$ is the union of r elements selected from r open covers of X, it follows that there is $U \in \mathscr{V}_{m,r} \subseteq \mathscr{V}_m$ such that $F \subset U$. That is, for all $m \in \omega$, $\mathscr{V}_m \in \Omega_X$. By hypothesis, for all $m \in \omega$, there is a $\mathscr{W}_m \in [\mathscr{V}_m]^{\leq f(m)}$ such that $\bigcup_{m\in\omega} \mathscr{W}_m \in \Omega_X$. We suppose that

$$\mathscr{W}_0 = \{V_i \;:\; 0 \leq i \leq f(0)-1\}$$

and, for $m \geq 1$,

$$\mathscr{W}_m = \left\{V_i : \sum_{j=0}^{m-1} f(j) \leq i \leq \sum_{j=0}^{m} f(j) - 1\right\}.$$

Then there is a k_0^i, with $V_i \in \mathscr{V}_{0,k_0^i}$, for $0 \leq i \leq f(0)-1$, and a k_m^i with $V_i \in \mathscr{V}_{m,k_m^i}$, for $m \geq 1$ and $\sum_{j=0}^{m-1} f(j) \leq i \leq \sum_{j=0}^{m} f(j) - 1$.

By the definition of the families $\mathscr{V}_{m,k}$ we find that there is $\mathscr{F}_{0,k_0^i} \in [\mathscr{U}_{0,l_i(k_o^i)}]^{\leq g(0,l_i(k_0^i))}$, for $0 \leq i \leq f(0)-1$, and there is $\mathscr{F}_{m,k_m^i} \in [\mathscr{U}_{m,l_i(k_m^i)}]^{\leq g(m,l_i(k_m^i))}$ for $m \geq 1$ and $\sum_{j=0}^{m-1} f(j) \leq i \leq \sum_{j=0}^{m} f(j) - 1$, such that $\bigcup \mathscr{W}_0 = \bigcup_{i=1}^{f(0)-1} \mathscr{F}_{0,k_0^i}$ and $\bigcup \mathscr{W}_m = \bigcup_{i=\sum_{j=0}^{m-1} f(j)}^{\sum_{j=0}^{m} f(j)-1} \mathscr{F}_{0,k_m^i}$, for $m \geq 1$. Therefore,

$$\bigcup_{i=1}^{f(0)-1} \mathscr{F}_{0,k_0^i} \cup \bigcup_{m\in\omega} \bigcup_{i=\sum_{j=0}^{m-1} f(j)}^{\sum_{j=0}^{m} f(j)-1} \mathscr{F}_{0,k_m^i} \in \mathscr{O}_X.$$

Choosing arbitrary $f(m)$ elements, with m that were not considered in the construction above and joining them to the previous family, we conclude that (1) is true.

(2). Let $\langle \mathscr{U}_m : m \in \omega\rangle$ be a sequence of elements in Ω_X. Re-index the sequence $\langle \mathscr{U}_m : m \in \omega\rangle = \langle \mathscr{U}_{m,n+1} : m,n \in \omega\rangle$ and consider $g : \omega \times \omega\backslash\{0\} \to \omega\backslash\{0\}$ so that $g(m,n+1)$ is $f(k)$, where $k \in \omega$ corresponds to the number, in an order fixed by $\omega \times \omega\backslash\{0\}$, of the pair $(m,n+1)$.

Define, for all $m,n \in \omega$, $\mathscr{V}_{m,n+1} = \{U^{n+1} : U \in \mathscr{U}_{m,n+1}\}$. We claim that, for all $m,n \in \omega$, $\mathscr{V}_{m,n+1} \in \mathscr{O}_{X^{n+1}}$. Indeed, let $(x_1,...,x_{n+1}) \in X^{n+1}$. As $F = \{x_1,...,x_{n+1}\} \in [X]^{<\aleph_0}$, it follows that there is $U \in \mathscr{U}_{m,n+1}$ such that $F \subset U$. So, $(x_1,...,x_{n+1}) \in U^{n+1}$.

Since, for all $n \in \omega$, $\langle \mathscr{V}_{m,n+1} : m \in \omega\rangle$ is a sequence of elements in $\mathscr{O}_{X^{n+1}}$, by hypothesis it follows that, for all $m \in \omega$, there is an $\mathscr{F}_{m,n+1} \in [\mathscr{V}_{m,n+1}]^{\leq g(m,n+1)}$ such that $\bigcup_{m\in\omega} \mathscr{F}_{m,n+1} \in \mathscr{O}_{X^{n+1}}$. Let $F \in [X]^{<\aleph_0}$. Suppose that $F = \{x_1,...,x_k\}$, for some $k \in \omega\backslash\{0\}$.

Since $(x_1,..,x_k) \in X^k$, there is $m \in \omega$ such that $(x_1,..,x_k) \in \bigcup \mathscr{F}_{m,k}$. Then $(x_1,..,x_k) \in U^k$, for some $U \in \mathscr{U}_{m,k}$. Then $F \subset U$. So

$$\bigcup_{m\in\omega} \bigcup_{n\in\omega} \{U : U \in \mathscr{H}_{m,n+1}\} \in \Omega_X,$$

where $\mathscr{H}_{m,n+1} = \{U : U^{n+1} \in \mathscr{F}_{m,n+1}\}$. Therefore, (2) is true. □

With the previous theorem, Proposition 3.13 and Theorem 3.20, we can obtain the following result:

Proposition 3.23. Let (X,τ) be a topological space and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. $S_f(\Omega_X,\Omega_X)$ holds;
2. $S_1(\Omega_X,\Omega_X)$ holds.

On the other hand, note that by Theorem 3.21, $S_{fin}(\Omega_{\mathbb{R}},\Omega_{\mathbb{R}})$ holds (because, for all $n\in\omega\setminus\{0\}$, $\mathbb{R}^n$ is $\sigma-$compact, and so $S_{fin}(\mathscr{O}_{\mathbb{R}^n},\mathscr{O}_{\mathbb{R}^n})$ holds). But, by Theorem 3.20, $S_1(\Omega_{\mathbb{R}},\Omega_{\mathbb{R}})$ fails (because $S_1(\mathscr{O}_{\mathbb{R}},\mathscr{O}_{\mathbb{R}})$ fails).

Then, by Proposition 3.9 for a topological space (X,τ) and $\mathscr{A}=\mathscr{B}=\Omega_X$, there are two selection principles: $S_1(\Omega_X,\Omega_X)$ and $S_{fin}(\Omega_X,\Omega_X)$.

3.2.4 Bornologies families

The Proposition 3.23 is also true in a more general way:

Proposition 3.24. (MEZABARBA; AURICHI, 2019) Let (X,τ) be a topological space, $\mathfrak{B}$ be a family of subsets of X, and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$ holds;
2. $S_1(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$ holds.

From this result and Theorem 2.39, we can obtain the following results for continuous function spaces:

Proposition 3.25. Let (X,τ) be a Tychonoff space, $\mathscr{B}$ be a bornology with a compact base, and let $f:\omega\to\omega$ be a function. The following statements are equivalent:

1. $S_f(\Omega_g,\Omega_g)$ holds, in $C_{\mathfrak{B}}(X)$, for all $g\in C_{\mathfrak{B}}(X)$;
2. $S_1(\Omega_g,\Omega_g)$ holds in $C_{\mathfrak{B}}(X)$, for all $g\in C_{\mathfrak{B}}(X)$.

Proposition 3.26. Let (X,τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base, and let $f:\omega\to\omega$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{D}_{C_{\mathfrak{B}}(X)},\Omega_g)$ is true in $C_{\mathfrak{B}}(X)$, for all $g\in C_{\mathfrak{B}}(X)$;
2. $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)},\Omega_g)$ is true in $C_{\mathfrak{B}}(X)$, for all $g\in C_{\mathfrak{B}}(X)$.

3.2.5 Dense subsets

To consider the case $\mathscr{A} = \mathscr{B} = \mathscr{D}_X$, with (X, τ) being a topological space, we can assume the following restrictions:

(1) If X has a dense non-countable subset D, with the property that no countable subset of it is dense, then $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$ fails (consider the constant sequence D). Therefore, we can always assume that all dense subsets of X have a countable dense subset.

(2) Let I be the set of all isolated points of X. Note that $I \subseteq D$, for all $D \in \mathscr{D}_X$. When considering (1), it follows that I is countable. If I is dense, then $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$ is always true because for any sequence in $\mathscr{D}_X$ just take, in any inning $n \in \omega$, an element of I different from the previous one. Then we can assume that I is not dense. Therefore, $X \backslash \bar{I}$ is an open, non-empty subset of X.

(3) Finally, note that $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$ holds if and only if $S_{fin}(\mathscr{D}_{X\backslash \bar{I}}, \mathscr{D}_{X\backslash \bar{I}})$ holds (Indeed, this result is true for any open non-empty subset of X). Therefore, we can assume that X has no isolated points.

We will start with the following equivalence:

Theorem 3.27. (BARMAN; DOW, 2011) Let (X, τ) be a separable space. The following statements are equivalent:

1. $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$ holds;
2. $S_{fin}(\mathscr{D}_X, \Omega_x)$ holds, for all $x \in X$.

Again, without much complication and without changing the proof of the previous theorem, we obtain the following result.

Theorem 3.28. Let (X, τ) be a separable space and let $f : \omega \to \omega \backslash \{0\}$ be an increasing function. The following statements are equivalent:

1. $S_f(\mathscr{D}_X, \mathscr{D}_X)$ holds;
2. $S_f(\mathscr{D}_X, \Omega_x)$ holds for all $x \in X$.

We can prove an analogous result of Theorem 3.12, for the family of dense subsets:

Theorem 3.29. Let (X, τ) be a separable space and $k \in \omega \backslash \{0, 1\}$ The following statements are equivalent:

1. $S_k(\mathscr{D}_X, \mathscr{D}_X)$ holds;

2. $S_1(\mathscr{D}_X, \mathscr{D}_X)$ holds.

Observation 3.30. Note that if X is not separable, none of the selection principle is satisfied.

Proof. It is sufficient to prove $(1) \Rightarrow (2)$.

Let $\langle D_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{D}_X$. Consider the subsequence $\langle D_{2m} : m \in \omega \rangle$. By (1), there is a sequence $\langle F_{2m} : m \in \omega \rangle$ such that, for all $m \in \omega$, $F_{2m} \in [D_{2m}]^{\leq k}$ and $\bigcup_{m \in \omega} F_{2m} \in \mathscr{D}_X$.

Denote for $\{a_m : m \in \omega\}$, with $a_m \in F_{2m}$, the set of elements arbitrarily chosen to satisfy (2), and let $\{b_m : m \in \omega\}$ be an enumeration of the remaining elements of the sets F_{2m}, for all $m \in \omega$.

Consider $\{p_m : m \in \omega\}$ an increasing enumeration of prime numbers minus 2. Consider the sequence $\langle D_{p_0^i} : i \in \omega \rangle$. Using (1) and Theorem 3.28, it follows that there is a sequence $\langle F_{p_0^i} : i \in \omega \rangle$ such that, for all $i \in \omega$, $F_{p_0^i} \in [D_{p_0^i}]^{\leq k}$ and $\bigcup_{i \in \omega} F_{p_0^i} \in \Omega_{b_0}$.

We can suppose that, for all $i \in \omega$, $F_{p_0^i} = \{c_{0,i}^j : 1 \leq j \leq k\}$. Consider, for any $1 \leq j \leq k$, the sets $C_0^j = \{c_{0,i}^j : i \in \omega\}$. So,

$$\bigcup_{i \in \omega} F_{p_0^i} = \bigcup_{j=1}^{k} C_0^j$$

As $b_0 \in \overline{\bigcup_{i \in \omega} F_{p_0^i}} = \overline{\bigcup_{j=1}^{k} C_0^j} = \bigcup_{j=1}^{k} \overline{C_0^j}$, we see that there is a $j_0 \in \{1, \ldots, k\}$ such that $b_0 \in \overline{C_0^{j_0}}$.

Analogously, for $m \geq 1$, consider the sequence $\langle D_{p_m^i} : i \geq 1 \rangle$. Using (1) and Theorem 3.28 again, it follows that there is a sequence $\langle F_{p_m^i} : m \geq 1 \rangle$ such that for all $i \geq 1$, $F_{p_m^i} \in [D_{p_m^i}]^{\leq k}$ and $\bigcup_{i \geq 1} F_{p_m^i} \in \Omega_{b_m}$. We can assume that for all $i \geq 1$, $F_{p_m^i} = \{c_{m,i}^j : 1 \leq j \leq k\}$.

Consider the sets, for all $1 \leq j \leq k$, $C_m^j = \{c_{m,i}^j : i \geq 1\}$. So,

$$\bigcup_{i \geq 1} F_{p_m^i} = \bigcup_{j=1}^{k} C_m^j$$

As $b_m \in \overline{\bigcup_{i \geq 1} F_{p_m^i}} = \overline{\bigcup_{j=1}^{k} C_m^j} = \bigcup_{j=1}^{k} \overline{C_m^j}$ it follows that there is a $j_m \in \{1, \ldots, k\}$ such that $b_m \in \overline{C_m^{j_m}}$.

We claim that $D = \{a_m : m \in \omega\} \cup \left(\bigcup_{m \in \omega} C_m^{j_m} \right) \in \mathscr{D}_X$. Indeed, let U be a nonempty element in τ. Since $\bigcup_{m \in \omega} F_{2m} \in \mathscr{D}_X$, it follows that there is $m \in \omega$ such that $a_m \in U$, or there is $m' \in \omega$ such that $b_{m'} \in U$.

If the first case happens, we are done. If it is the second case, it follows that $U \cap C^{j_{m'}}_{m'} \neq \emptyset$ because $b_{m'} \in \overline{C^{j_{m'}}_{m'}}$. In any case, we conclude that $U \cap D \neq \emptyset$. So $D \in \mathscr{D}_X$. We conclude that (2) is satisfied. $\square$

Since we assume that the topological spaces (X,τ) considered have no isolated points, it follows that $\mathscr{D}_X$ satisfies the properties $(P1)$ and $(P2)$, if (X,τ) is at least T_1. (For T_0 spaces this is not true: for example, if $\mathbb{R}$ is equipped with the topology $\{(-n,n) : n \in \mathbb{N}\} \cup \{\mathbb{R}\}$. This space has no isolated points, but $D = \{0\}$ is dense).

Therefore Proposition 3.10 is valid in this case. Additionally, with a few modifications in Theorem 3.29, we can obtain that $S_1(\mathscr{D}_X, \mathscr{D}_X)$ is equivalent to $S_f(\mathscr{A}, \mathscr{B})$, with $f : \omega \to \omega \backslash \{0\}$ be a limited function.

By Proposition 3.26 and Theorem 3.28, we can obtain the following result:

Proposition 3.31. Let (X,τ) be a separable metric space, $\mathfrak{B}$ be a bornology with a compact basis, and let $f : \omega \to \omega \backslash \{0\}$ be an increasing function. The following statements are equivalent:

1. $S_f(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ hols in $C_{\mathfrak{B}}(X)$;
2. $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ is true in $C_{\mathfrak{B}}(X)$.

Example 3.32. (BONANZINGA *et al.*, 2014) We have already seen that $S_{fin}(\Omega_{\mathbb{R}}, \Omega_{\mathbb{R}})$ holds, but $S_1(\Omega_{\mathbb{R}}, \Omega_{\mathbb{R}})$ fails.

Since $\mathbb{R}$ is a separable metric space, from Theorems 2.35, 2.36, 2.46 and 2.47 it follows that $S_{fin}(\mathscr{D}_{C_p(\mathbb{R})}, \mathscr{D}_{C_p(\mathbb{R})})$ holds; but $S_1(\mathscr{D}_{C_p(\mathbb{R})}, \mathscr{D}_{C_p(\mathbb{R})})$ fails.

Another example is given in (BELLA; BONANZINGA; MATVEEV, 2009), and a direct proof that the selection principle $S_1(\mathscr{D}_X, \mathscr{D}_X)$ fails is given in (CAMARGO; UZCÁTEGUI, 2018). The specific space is $X = CL(2^\omega)$, the set of all clopen subsets of 2^ω, when this is considered as a subset of 2^{2^ω}.

Note that by Proposition 3.31, $S_S(\mathscr{D}_{C_p(\mathbb{R})}, \mathscr{D}_{C_p(\mathbb{R})})$ also fails.

Example 3.33. Consider the set $X = \omega \times \omega$. We provide X with a topology τ whose basic open sets are of the form:

$$V_H = X \backslash \bigcup_{h \in H} \{(n, h(n)) : n \in \omega\}, \text{ where } H \in [{}^{\omega}\omega]^{<\aleph_0}.$$

Note that (X,τ) is a T_1 space, but it is not a Hausdorff space.

We have that $S_1(\mathscr{D}_X, \mathscr{D}_X)$ fails. Indeed, note that, for all $n \in \omega$, $D_n = \{(n,m) : m \in \omega\} \in \mathscr{D}_X$. Then consider the sequence $\langle D_n : n \in \omega \rangle$ of elements in $\mathscr{D}_X$. So, for all $n \in \omega$, for any choice $d_n = (n, k_n) \in D_n$, we have $\{d_n \in \omega\} \notin \mathscr{D}_X$ (taking $H = \{g\}$, where g is a function given by $g(n) = k_n$, it follows that $V_H \cap \{d_n : n \in \omega\} = \emptyset$).

On the other hand, $S_S(\mathscr{D}_X, \mathscr{D}_X)$ holds. This follows directly from the following observation:

Observation 3.34. For $D \subseteq X$, we have $D \in \mathscr{D}_X$ if and only if, for all $n \in \omega \backslash \{0\}$, there is $k_n \in \omega$ such that $|D \cap C_{k_n}| > n$, where $C_n = \{(n,m) : m \in \omega\}$. Indeed, for sufficiency, suppose otherwise. Then, there is an $n \in \omega \backslash \{0\}$ such that for all $k \in \omega$, $|D \cap C_k| \leq n$. Note that we can assume that $D \cap C_k = \{c_k^i : 1 \leq i \leq n\}$. Define, for all $1 \leq i \leq n$, the function $f_i : \omega \to \omega$ given by $f_i(k) = c_k^i$. Consider the finite set $H = \{f_i : \omega \to \omega : 1 \leq i \leq n\}$. Then $D \cap V_H = \emptyset$, that is, $D \notin \mathscr{D}_X$.

Reciprocally, let V_H be a basic open with $H \in [{}^{\omega}\omega]^{<\aleph_0}$. Suppose that $|H| = m \in \omega \backslash \{0\}$. Then, there is a $k_m \in \omega$ such that $|D \cap C_{k_m}| > m$. So $D \cap V_H \neq \emptyset$. Since this is true for an arbitrary open basic, we conclude $D \in \mathscr{D}_X$.

Then, let $\langle D_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{D}_X$. Based on the previous observation, for all $n \in \omega$, we can chooses $F_n \in [D_n \cap C_{k_{n+1}}]^{n+1}$. We claim that $\bigcup_{n\in\omega} F_n \in \mathscr{D}_X$. Indeed, let V_H be an element of τ, with $H \in [{}^{\omega}\omega]^{<\aleph_0}$. Suppose that $|H| = m \in \omega \backslash \{0\}$. It is clear that $|F_{2m} \cap \{(k_{2m}, f_j(k_{2m})) : 1 \leq j \leq m\}| \leq m$. Since $|F_{2m}| = 2m$, it follows that $F_{2m} \cap V_H \neq \emptyset$. We conclude that $S_S(\mathscr{D}_X, \mathscr{D}_X)$ holds. Note that, by a similarly argument in the case of $S_1(\mathscr{D}_X, \mathscr{D}_X)$, we obtain that $S_{bnd}(\mathscr{D}_X, \mathscr{D}_X)$ fails.

Therefore, for a T_1 space (X, τ) and $\mathscr{A} = \mathscr{B} = \mathscr{D}_X$, there are three different selection principles: $S_1(\mathscr{D}_X, \mathscr{D}_X)$, $S_S(\mathscr{D}_X, \mathscr{D}_X)$, and $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$. Is still a open question if $S_{bnd}(\mathscr{D}_X, \mathscr{D}_X)$ is different or not of $S_1(\mathscr{D}_X, \mathscr{D}_X)$.

Class	(P1)	(P2)	Propositions 3.6, 3.7	Proposition 3.10	Additional Implications
Ω_x (if X is T_1)	✓ (×: Consider $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$)	✓	✓	✓	$S_f \implies S_1$, with f limited ((GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995)).
$\mathscr{O}_X$	× (Consider a open cover $\{\{x\} : x \in X\}$ in a not unitary discrete space X)	✓	✓ (proof of Proposition 3.6 is the same of Proposition 3.13)	× (because is not valid the Proposition 3.9)	$S_f \implies S_1$, f function ((GARCÍA-FERREIRA; TAMARIZ-MASCARÚA, 1995)).
Ω_X	✓ ((SCHEEPERS, 1994))	✓	✓	✓	$S_f \implies S_1$, f function (Proposition 3.23)
$\mathscr{K}_X$	✓	✓	✓	✓	$S_f \implies S_1$, f function.(Theorem 4.9)
$\mathscr{O}^X_{\mathscr{B}}$	✓	✓	✓	✓	$S_f \implies S_1$, f function. ((MEZABARBA; AURICHI, 2019))
Λ_X	✓	✓	✓	✓	$S_f \implies S_1$ (because that principles are equivalent to selection principle in the case of open covers)
Γ_X	✓	× (In $\mathbb{R}$ be can obtain a element in Γ_X such that with additional elements it not in Γ_X)	✓ (It is implication of the additional implication)	✓ (It is implication of the additional implication)	$S_{fin} \implies S_1$ ((SCHEEPERS, 1996))
$\mathscr{D}_X$ (if X is T_1)	✓	✓	✓	✓	$S_k \implies S_1$ (Theorem 3.29)

CHAPTER

4

EQUIVALENCES IN GAME VARIATIONS FOR DENSE CLASSES IN SPACES OF THE FORM $C_k(X)$

In (AURICHI; BELLA; DIAS, 2018) it is investigated the difference between certain selective topological games that involve tightness. In that work, the following problem was proposed:

Problem 4.1 (((AURICHI; BELLA; DIAS, 2018), Problem 4.4.))**.** What can be said about the relation between the various games $G_k(\mathscr{A},\mathscr{B})$, $G_f(\mathscr{A},\mathscr{B})$ and $G_{fin}(\mathscr{A},\mathscr{B})$-and their associated selective properties- for other pairs $(\mathscr{A},\mathscr{B})$?

In Chapter 3, we have already seen some relationship between the variation of some selection principles towards Problem 4.1. In this chapter, we focus on the relations of some selective topological games with other classes $\mathscr{A}$, $\mathscr{B}$, and we obtain an equivalence result in the case $\mathscr{A} = \mathscr{B} = \mathscr{D}_X$.

We begin by presenting the version of Theorem 3.27 for games.

Theorem 4.2 ((CLONTZ, 2019))**.** Let (X,τ) be a separable space. The following statements are equivalent:

1. $II \uparrow G_{fin}(\mathscr{D}_X,\mathscr{D}_X)$;
2. $II \uparrow G_{fin}(\mathscr{D}_X,\Omega_x)$, for all $x \in X$.

With a few modifications to the proof of the previous theorem, we can also obtain a weak version of Theorem 3.28 for games:

Theorem 4.3. Let (X,τ) be a separable space and let $f : \omega \to \omega\backslash\{0\}$ be an increasing function. The following statements are equivalent:

1. $II \uparrow G_f(\mathscr{D}_X, \mathscr{D}_X)$;

2. $II \uparrow G_f(\mathscr{D}_X, \Omega_x)$, for all $x \in X$.

Proof. It is sufficient to prove $(2) \Rightarrow (1)$.

To show that, consider a dense subset $\{d_i : i \in \omega\}$ of X. For all $i \in \omega$, let $\sigma_i : {}^{<\omega}\mathscr{D}_X \to [\bigcup \mathscr{D}_X]^{<\aleph_0}$ be a winning strategy for Player *II* in $G_f(\mathscr{D}_X, \Omega_{d_i})$.

Let $\{A_i : i \in \omega\}$ be a partition of ω in infinite subsets. Given a finite sequence $t = \langle D_0, .., D_n\rangle \in {}^{<\omega}\mathscr{D}_X$, we define

$$\rho(t) = \sigma_i(t'),$$

where $i \in \omega$ is such that $n \in A_i$ and t' is a subsequence of t obtained by removing all elements with index not belonging to A_i. As the latest elements of t and t' are the same and f is increasing, $\rho : {}^{<\omega}\mathscr{D}_X \to [\bigcup \mathscr{D}_X]^{<\aleph_0}$ define a strategy for Player *II* in $G_f(\mathscr{D}_X, \mathscr{D}_X)$.

We claim that ρ is a winning strategy. Indeed, consider the following play in $G_f(\mathscr{D}_X, \mathscr{D}_X)$:

$$\langle D_0, \rho(D_0), D_1, \rho(D_0, D_1), ..., D_n, \rho(D_0, ..., D_n), ...\rangle.$$

For all $i \in \omega$, consider an increasing enumeration $A_i = \{m^i_k : k \in \omega\}$. So,

$$\langle D_{m^i_0}, \sigma_i(D_{m^i_0}), ..., D_{m^i_k}, \sigma_i(D_{m^i_0}, ..., D_{m^i_k}), ...\rangle,$$

is a play in $G_f(\mathscr{D}_X, \Omega_{d_i})$. As σ_i is a winning strategy, it follows that

$$d_i \in \overline{\bigcup_{k\in\omega} \sigma_i(D_{m^i_0}, ..., D_{m^i_k})} \subseteq \overline{\bigcup_{n\in\omega} \rho(D_0, ..., D_n)}$$

Then,

$$X = \overline{D} \subseteq \overline{\bigcup_{n\in\omega} \rho(D_0, ..., D_n)}.$$

Therefore, $II \uparrow G_f(\mathscr{D}_X, \mathscr{D}_X)$. ☐

4.1 Hurewicz's and Pawlikowski's Theorems versions for $\mathscr{O}^X_{\mathfrak{B}}$

In (SZEWCZAK; TSABAN, 2020) conceptual proofs of the Hurewicz and Pawlikowski theorems are obtained. These results can be generalized to a more general form.

First, we will need the following definition:

Definition 4.4. A countable $\mathfrak{B}$-cover $\mathscr{U}$ is called a $\mathfrak{B}$-tail cover if the family of intersections of cofinite subsets of $\mathscr{U}$ is a $\mathfrak{B}$-cover of X.

Equivalently, $\mathscr{U} = \{U_i : i \in \omega\} \in \mathcal{O}_{\mathfrak{B}}^{X}$ is a $\mathfrak{B}$-tail cover if and only if the family

$$\left\{ \bigcap_{i=k}^{\infty} U_i : k \in \omega \right\} \in \mathcal{O}_{\mathfrak{B}}^{X}.$$

Theorem 4.5. Let (X, τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. The following statements are equivalent:

1. $S_{fin}(\mathcal{O}_{\mathfrak{B}}^{X}, \mathcal{O}_{\mathfrak{B}}^{X})$ holds;
2. $I \not\uparrow G_{fin}(\mathcal{O}_{\mathfrak{B}}^{X}, \mathcal{O}_{\mathfrak{B}}^{X})$.

Proof. It is sufficient to prove the implication $(1) \Rightarrow (2)$.

Let $\sigma : {}^{<\omega}[\bigcup \mathcal{O}_{\mathfrak{B}}^{X}]^{<\aleph_0} \to \mathcal{O}_{\mathfrak{B}}^{X}$ be a strategy for Player I in $G_{fin}(\mathcal{O}_{\mathfrak{B}}^{X}, \mathcal{O}_{\mathfrak{B}}^{X})$. We can assume that, for any $k \in \omega$ and using the strategy σ, Player II fails to cover X with the choices made until round k (otherwise, Player II wins, so σ is not a winning strategy).

Furthermore, we can assume that the strategy σ chooses countable and increasing families in $\mathcal{O}_{\mathfrak{B}}^{X}$ (the first statement is based on the fact that $S_{fin}(\mathcal{O}_{\mathscr{B}}^{X}, \mathcal{O}_{\mathscr{B}}^{X})$ is valid, the second statement is satisfies by the property $(P2)$ on the page (37)).

Also, we can suppose that Player II chooses a single element in each inning (because if Player II chooses finitely many elements, we can instead consider that he chooses the union of those elements).

Note that for any reply $\{U_i : i \in \omega\} \in \mathcal{O}_{\mathfrak{B}}^{X}$ made with the strategy σ to the choice U of Player II, we can suppose $U = U_0$. Indeed, note that $\{U, U \cup U_0, U \cup U_1, ...\} \in \mathcal{O}_{\mathfrak{B}}^{X}$. Then, if U is the element chosen by Player II of the family $\{U, U \cup U_0, U \cup U_1, ...\}$, we suppose that U_0 is the element chosen by Player II of the initial family $\{U_i : i \in \omega\}$. On the other hand if for some $n \in \omega$, $U \cup U_n$ is the element chosen by Player II, then we suppose that U_n is the element chosen by Player II in the initial family. As the union, in both cases of the definition of the strategy for Player II, does not vary, we obtain the result.

Denote by

$$\sigma(\emptyset) = \mathscr{U}_{\emptyset} = \{U_{\langle n \rangle} : n \in \omega\} \in \mathcal{O}_{\mathfrak{B}}^{X}.$$

If $U_{\langle n_1 \rangle}$ is the element chosen by Player II, denote by

$$\sigma(\langle U_{\langle n_1 \rangle} \rangle) = \mathscr{U}_{\langle n_1 \rangle} = U_{\langle n_1, n \rangle} : n \in \omega\} \in \mathcal{O}_{\mathfrak{B}}^{X}.$$

In general, if U_{ρ} is a choice made by Player II, with $\rho \in {}^{k}\omega$ and $k \geq 1$, denote by

$$\sigma(\langle U_{\rho \upharpoonright 1}, U_{\rho \upharpoonright 2}, ..., U_{\rho \upharpoonright k} \rangle) = \mathscr{U}_{\rho} = \{U_{\rho \frown n} : n \in \omega\} \in \mathcal{O}_{\mathfrak{B}}^{X}.$$

We define $\mathscr{V}_0 = \mathscr{U}_{\emptyset}$ and, for $n \geq 1$, $\mathscr{V}_n = \bigcup_{\rho \in {}^{n}\omega} \mathscr{U}_{\rho}$. We claim that $\mathscr{V}_n$ is a $\mathfrak{B}$-tail cover for any $n \in \omega$. Indeed, note that

$$\left\{\bigcap_{i=n}^{\infty} U_i : n \in \omega\right\} = \mathscr{V}_0 \in \mathscr{O}_{\mathfrak{B}}^X.$$

So $\mathscr{V}_0$ is a $\mathfrak{B}$-tail cover.

Suppose that for any $n \geq 0$, $\mathscr{V}_n$ is a $\mathfrak{B}$-tail cover. Now, we claim that $\mathscr{V}_{n+1}$ is a $\mathfrak{B}$-tail cover. Indeed, consider the enumerations $\mathscr{V}_n = \{V_m : m \in \omega\}$ and

$$\mathscr{V}_{n+1} = \bigcup_{\rho \in {}^{n+1}\omega} \mathscr{U}_\rho = \bigcup_{m=0}^{\infty} \{V_{\langle m,k\rangle} : k \in \omega\},$$

with $V_m = V_{\langle m,0\rangle} \subseteq V_{\langle m,1\rangle} \subseteq \ldots$

Let $\mathscr{V}$ be a co-final subset of $\mathscr{V}_{n+1}$. For all $m \in \omega$, let k_m be the minimum in ω such that $V_{\langle m,k_m\rangle} \in \mathscr{V}$. So, for each $m \in \omega$, $\bigcap(\mathscr{V} \cap \{V_{\langle m,k\rangle} : k \in \omega\}) = V_{\langle m,k_m\rangle}$. Furthermore, because $\mathscr{V}$ is co-finite, we have $k_m = 0$, for all but finitely many natural numbers m. Consider $I = \{m \in \omega : k_m = 0\}$, which will be co-final. So

$$\bigcap \mathscr{V} = \bigcap(\mathscr{V} \cap \{V_{\langle m,k\rangle} : k \in \omega\}) = \bigcup_{m\in\omega} V_{\langle m,k_m\rangle} = \bigcap_{m\in I} V_m \cap \bigcap_{m\in\omega\setminus I} V_{\langle m,k_m\rangle}.$$

Since $\mathscr{V}_n$ is a $\mathfrak{B}$-tail cover, it follows that $\bigcap_{k\in I} V_k$ is open, and then $\bigcap \mathscr{V}$ is open.

Now, let $B \in \mathfrak{B}$. Since $\mathscr{V}_n$ is a $\mathfrak{B}$-tail cover, the set $J = \{m \in \omega : B \subset V_m\}$ is co-finite. For all $m \in \omega\setminus J$, let $k_m \in \omega$ be the minimum number such that $B \subset V_{\langle m,k_m\rangle}$. Then

$$B \subset \bigcap_{m\in J} V_m \cap \bigcap_{m\in\omega\setminus J} V_{\langle m,k_m\rangle}.$$

This implies that $\mathscr{V}_{n+1}$ is a $\mathfrak{B}$-tail cover, because we see that any co-finite intersection has the form of the last statement.

For any $n \in \omega$, we denote by $\mathscr{V}_n'$ the family of intersections of co-finite segments of $\mathscr{V}_n$. Applying $S_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$ to the sequence $\langle \mathscr{V}_n' : n \in \omega\rangle$, there is, for any $n \in \omega$, a $\mathscr{W}_n \in [\mathscr{V}_n']^{<\aleph_0}$ and $k_n \in \omega\setminus\{0\}$ such that $|\mathscr{W}_n| = k_n$ and $\bigcup_{n\in\omega} \mathscr{W}_n \in \mathscr{O}_{\mathfrak{B}}^X$. Then, there are, for $n \in \omega$ and $1 \leq j \leq k_n$, co-final families $\mathscr{W}_j'$ of $\mathscr{V}_n$ such that $\bigcap \mathscr{W}_j'$ is an element in $\mathscr{W}_n$.

Finally, for any $n \in \omega$ and $1 \leq j \leq k_n$, let $W_j \in \mathscr{W}_j' \cap \mathscr{U}_\rho$ be an arbitrary element, with $\rho \in {}^n\omega$. In each round $n \in \omega$, Player *II* chooses $\mathscr{U}_n' = \{W_j : 1 \leq j \leq k_n\}$. If $B \in \mathfrak{B}$, there is a $V \in \bigcup_{n\in\omega} \mathscr{W}_n$ such that $B \subset V = \bigcap \mathscr{W}_j'$, for any $1 \leq j \leq k_m$ and $m \in \omega$. So, we see that Player *II* wins the game $G_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$. Therefore, $I \not\uparrow G_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$. □

We can use the previous result to obtain the following:

Proposition 4.6. Let (X,τ) be a topological space such that $S_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$ holds, where $\mathfrak{B}$ is a family of subsets of X. Then, for any strategy σ for Player *I* in $G_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$, there is a play

$$\langle \mathscr{U}_0, \mathscr{F}_0, \mathscr{U}_1, \mathscr{F}_1, \ldots\rangle$$

following the strategy σ such that for all $B \in \mathfrak{B}$, $B \subset U_m$, with $U_m \in \mathscr{F}_m$, for infinitely many $m \in \omega$.

Proof. Fix a strategy $\sigma : {}^{<\omega}\left([\bigcup \mathscr{O}_{\mathfrak{B}}]^{<\aleph_0}\right) \to \mathscr{O}_{\mathfrak{B}}$ for Player I in $G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$. For all $n \in \omega$, denote X_n as the topological space $X \times \{n\}$ equipped with the topology $\tau_n = \{U \times \{n\} : U \in \tau\}$. Then, for any $n \in \omega$, $S_{fin}(\mathscr{O}^{X_n}_{\mathfrak{B}_n}, \mathscr{O}^{X_n}_{\mathfrak{B}_n})$ is true, where $\mathfrak{B}_n = \{B \times \{n\} : B \in \mathfrak{B}\}$.

Consider $Y = \bigcup_{n\in\omega} X_n = X \times \omega$, equipped with the topology generated by $\bigcup_{n\in\omega} \tau_n$. Then, $S_{fin}(\mathscr{O}^Y_{\mathfrak{C}}, \mathscr{O}^Y_{\mathfrak{C}})$ holds, where $\mathfrak{C} = \bigcup_{n\in\omega} \mathfrak{B}_n$ (We can divide a sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ in $\mathscr{O}^Y_{\mathfrak{C}}$ into infinitely many disjoint subsequences. Therefore, use each subsequence to obtain a $\mathfrak{B}_n$-cover, for all $n \in \omega$).

Let us define a strategy φ for Player I in the game $G_{fin}(\mathscr{O}^Y_{\mathfrak{C}}, \mathscr{O}^Y_{\mathfrak{C}})$. Let $\sigma(\emptyset) \in \mathscr{O}^X_{\mathfrak{B}}$. Define:

$$\varphi(\emptyset) = \{U \times \{n\} : U \in \sigma(\emptyset), n \in \omega\} \in \mathscr{O}^Y_{\mathfrak{B}}.$$

Suppose that Player II chooses $\widetilde{\mathscr{F}_0} \in [\varphi(\emptyset)]^{<\aleph_0}$. Then, there is a $k_0 \in \omega$ such that

$$\widetilde{\mathscr{F}_0} = \left\{U^0_j \times \{n^0_j\} : 0 \le j \le k_0\right\}.$$

Define $\mathscr{F}_0 = \left\{U^0_j : 0 \le j \le k_0\right\} \in [\sigma(\emptyset)]^{<\aleph_0}$. Consider $\sigma(\langle \mathscr{F}_0 \rangle) \in \mathscr{O}^X_{\mathfrak{B}}$. So, define:

$$\varphi(\langle \widetilde{\mathscr{F}_0} \rangle) = \{U \times \{n\} : U \in \sigma(\langle \mathscr{F}_0 \rangle), n_i \in \omega\} \in \mathscr{O}^Y_{\mathfrak{B}}.$$

Suppose that Player II chooses $\widetilde{\mathscr{F}_1} = \left\{U^1_j \times \{n^1_j\} : 0 \le j \le k_1\right\}$, for some $k_1 \in \omega$. Define $\mathscr{F}_1 = \left\{U^1_j : 0 \le j \le k_1\right\}$, and so on in all the next innings. By the previous theorem, it follows that there is a play

$$\langle \varphi(\emptyset), \widetilde{\mathscr{F}_0}, \varphi(\langle \widetilde{\mathscr{F}_0} \rangle), \widetilde{\mathscr{F}_1}, \varphi(\langle \widetilde{\mathscr{F}_0}, \widetilde{\mathscr{F}_1} \rangle), \ldots \rangle$$

in $G_{fin}(\mathscr{O}^Y_{\mathfrak{C}}, \mathscr{O}^Y_{\mathfrak{C}})$, such that $\bigcup_{n\in\omega} \widetilde{\mathscr{F}_n} \in \mathscr{O}^Y_{\mathfrak{C}}$.

Consider the respective play

$$\langle \sigma(\emptyset), \mathscr{F}_0, \sigma(\langle \mathscr{F}_0 \rangle), \mathscr{F}_1, \sigma(\langle \mathscr{F}_0, \mathscr{F}_1 \rangle), \ldots \rangle$$

in $G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$. Let $B \in \mathfrak{B}$. There is $m_0 \in \omega$, such that $B \times \{0\} \subset A_{m_0}$, with $A_{m_0} \in \widetilde{\mathscr{F}_{m_0}}$. Then $A_{m_0} = U^{m_0}_{j_0} \times \{0\}$, with $0 \le j_0 \le k_0$. Put $U_0 = U^{m_0}_{j_0} \in \mathscr{F}_{m_0}$. Then $B \subset U_0$.

Now, consider:

$$F_0 = \left\{k \in \omega : \textit{ there is } U \textit{ such that } U \times \{k\} \in \bigcup_{i=0}^{m_0} \widetilde{\mathscr{F}_i}\right\}.$$

Choose the first natural number n_1 such that $n_1 > \max F$. Consider $B \times \{n_1\}$. Then, there is $m_1 \in \omega$ such that $B \subset U_{m_1}$, with $U_{m_1} = U^{m_1}_{j_1} \in \mathscr{F}_{m_1}$ and $0 \le j_1 \le k_{m_1}$. Note that $m_0 < m_1$.

Consider

$$F_1 = \left\{ k \in \omega : \textit{ there is } U \textit{ such that } U \times \{k\} \in \bigcup_{i=0}^{m_1} \widetilde{\mathscr{F}_i} \right\}$$

and choose the first natural number n_2 such that $n_2 > \max F_2$. Then, there is $m_2 \in \omega$ such that $B \subset U_{m_2}$, with $U_{m_2} \in \mathscr{F}_{m_2}$. Continuing in this same way, we obtain the result. □

Additionally, we state the following result.

Lemma 4.7. Let (X, τ) be a topological space such that $S_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$ holds. Let $\{\mathscr{F}_n\}_{n\in\omega}$ be a finite non-empty open families satisfying that, for all $B \in \mathfrak{B}$, we have that $B \subset U_m$, with $U_m \in \mathscr{F}_m$, for infinitely many $m \in \omega$. Then, for all $n \in \omega$ there is $U_n \in \mathscr{F}_n$ such that $\{U_n : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}$.

Proof. For any $n \in \omega$, define $\mathscr{V}_n$ as the family of all intersections of n elements taken from distinct elements of the families $\mathscr{F}_n$. By hypothesis, for all $n \in \omega$, $\mathscr{V}_n \in \mathscr{O}^X_{\mathfrak{B}}$. Then, by $S_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$, it follows that for all $n \in \omega$, there is a $V_n \in \mathscr{V}_n$ such that $\{V_n : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}$.

Since each V_n is the intersection of n elements taken from distinct elements of the families $\mathscr{F}_n$, we obtain a unique element of infinitely many $\mathscr{F}_n$. Taking arbitrary elements from $\mathscr{F}_n$ that were not considered in the previous choice, we conclude the result. □

For all $k \in \omega$ and families $\mathscr{U}_1, ..., \mathscr{U}_k$, we define

$$\mathscr{U}_1 \wedge ... \wedge \mathscr{U}_k := \{U_1 \cap ... \cap U_k : U_1 \in \mathscr{U}_1, ..., U_k \in \mathscr{U}_k\}.$$

Theorem 4.8. Let (X, τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. The following statements are equivalent:

1. $S_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$ holds;
2. $I \not\uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$.

Proof. It is sufficient to prove the implication $(1) \Rightarrow (2)$. Let $\sigma : {}^{<\omega}\left(\bigcup \mathscr{O}^X_{\mathfrak{B}}\right) \to \mathscr{O}^X_{\mathfrak{B}}$ be a strategy for Player I in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$. By hypothesis, we can assume that the choices by σ are countable. Define

$$\sigma(\emptyset) = \mathscr{U}_\emptyset = \{U_{\langle n \rangle} : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}.$$

If $U_{\langle n_1 \rangle}$ is the choice of Player II, define

$$\sigma(\langle U_{\langle n_1 \rangle} \rangle) = \mathscr{U}_{\langle n_1 \rangle} = \{U_{\langle n_1, n \rangle} : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}.$$

In general, if U_ρ is the choice of Player II, with $\rho \in {}^k\omega$ and $k \geq 1$, let

$$\sigma(\langle U_{\rho \restriction 1}, U_{\rho \restriction 2}, ..., U_{\rho \restriction dom(\rho)} \rangle) = \mathscr{U}_\rho = \{U_{\rho^\frown n} : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}.$$

Now, we define a strategy for Player I in $G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$ as follows. We begin by defining $\varphi(\emptyset) = \sigma(\emptyset) \in \mathscr{O}_{\mathfrak{B}}$. Suppose that Player II chooses $\mathscr{F}_0 \in [\varphi(\emptyset)]^{<\aleph_0}$. Let m_0 be the minimum element in ω such that $\mathscr{F}_0 \subseteq \{U_{\langle 0\rangle}, ..., U_{\langle m_0\rangle}\}$. Then, define

$$\varphi(\langle \mathscr{F}_0 \rangle) = \mathscr{U}_{\langle 0\rangle} \wedge ... \wedge \mathscr{U}_{\langle m_0\rangle} \in \mathscr{O}^X_{\mathfrak{B}}.$$

Suppose that Player II chooses $\mathscr{F}_1 \in [\varphi(\langle \mathscr{F}_0\rangle)]^{<\aleph_0}$. Let m_1 be the minimum element in ω such that for all $k \leq m_0$, $\mathscr{F}_1$ is a refinement of $\{U_{\langle k,0\rangle}, ..., U_{\langle k,m_1\rangle}\}$. For $\rho, \lambda \in {}^n\omega$ we write $\rho \preceq \lambda$ if $\rho(i) \leq \lambda(i)$, for all $0 \leq i \leq n$. Define

$$\varphi(\langle \mathscr{F}_0, \mathscr{F}_1\rangle) = \bigwedge_{\rho \preceq \langle m_0, m_1\rangle} \mathscr{U}_\rho \in \mathscr{O}^X_{\mathfrak{B}}.$$

Suppose that Player II chooses $\mathscr{F}_2 \in [\varphi(\langle \mathscr{F}_0, \mathscr{F}_1\rangle)]^{<\aleph_0}$. Let m_2 be the minimum element in ω such that, for any $\rho \preceq \langle m_0, m_1\rangle$, $\mathscr{F}_2$ is a refinement of $\{U_{\rho^\frown 0}, ..., U_{\rho^\frown m_2}\}$. Define

$$\varphi(\langle \mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_2\rangle) = \bigwedge_{\rho \preceq \langle m_0, m_1, m_2\rangle} \mathscr{U}_\rho \in \mathscr{O}^X_{\mathfrak{B}},$$

and so on in all the next innings.

By Proposition 4.6, it follows that there is a play

$$\langle \varphi(\emptyset), \mathscr{F}_0, \varphi(\langle \mathscr{F}_0\rangle), \mathscr{F}_1, \varphi(\langle \mathscr{F}_0, \mathscr{F}_1\rangle), ...\rangle$$

such that, for all $B \in \mathfrak{B}$, there are infinitely many $m \in \omega$ with $B \subset U_m$, where $U_m \in F_m$. Applying Lemma 4.7 to the family $\{\mathscr{F}_n : n \in \omega\}$, it follows that, for all $n \in \omega$, there is $U_n \in \mathscr{F}_n$ such that $\{U_n : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}$. Choose $k_0 \leq m_0$ so that U_{k_0} contains U_0. So, consider $k_1 \leq m_1$ satisfying that $U_{\langle k_0,k_1\rangle}$ contains U_1, and so on. Then, the play

$$\langle \sigma(\emptyset), U_{k_0}, \sigma(\langle U_{k_0}\rangle) = \mathscr{U}_{k_0}, U_{\langle k_0,k_1\rangle}, \sigma(\langle U_{k_0}, U_{\langle k_0,k_1\rangle}\rangle) = \mathscr{U}_{\langle k_0,k_1\rangle}, ...\rangle$$

is a winning strategy for Player II. Therefore, $I \not\uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$. □

4.2 Equivalences of Games in $\mathscr{K}_X$

With the same ideas used in the Proposition 3.13, we can obtain the following version of the Proposition 3.24, but now in the game version.

Theorem 4.9. Let (X, τ) be a topological space, $\mathfrak{B}$ be a family of subsets of X and let $f : \omega \to \omega \backslash \{0\}$ be a function. The following statements are equivalent:

1. $I \uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$;

2. $I \uparrow G_2(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$.

Proof. It is clear that $I \uparrow G_2(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$ implies $I \uparrow G_1(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. Reciprocally, suppose that $I \uparrow G_1(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. By Theorem 4.8, there is a sequence $\langle \mathscr{V}_n : n \in \omega \rangle$ of elements in $\mathcal{O}^X_{\mathfrak{B}}$ such that $\{V_n : n \in \omega\} \notin \mathcal{O}^X_{\mathfrak{B}}$, where $V_n \in \mathscr{V}_n$, for all $n \in \omega$.

In the inning 0 in $G_2(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$, suppose that Player I chooses $\mathscr{U}_0 = \{V_0 \cap V_1 : V_0 \in \mathscr{V}_0,\ V_1 \in \mathscr{V}_1\}$. Note that $\mathscr{U}_0 \in \mathcal{O}^X_{\mathfrak{B}}$, because $\mathscr{V}_0$ and $\mathscr{V}_1$ belong to $\mathcal{O}^X_{\mathfrak{B}}$. Let $\{U^0_0, U^1_0\}$ be the set chosen by Player II. Then $U^0_0 \subset V_0$ and $U^1_0 \subset V_1$, for some $V_0 \in \mathscr{V}_0$ and $V_1 \in \mathscr{V}_1$.

In the inning $k \in \omega$, Player I chooses $\mathscr{U}_k = \{V_{2k} \cap V_{2k+1} : V_{2k} \in \mathscr{V}_{2k},\ V_{2k+1} \in \mathscr{V}_{2k+1}\} \in \mathcal{O}^X_{\mathfrak{B}}$. Let $\{U^{2k}_k, U^{2k+1}_k\}$ be the choice of Player II. Then $U^{2k}_k \subset V_{2k}$ e $U^{2k+1}_k \subset V_{2k+1}$, for some $V_{2k} \in \mathscr{V}_{2k}$ and $V_{2k+1} \in \mathscr{V}_{2k+1}$.

So $\bigcup_{k\in\omega}\{U^{2k}_k, U^{2k+1}_k\} \notin \mathcal{O}^X_{\mathfrak{B}}$. Indeed, otherwise, for all $B \in \mathfrak{B}$, there is a $k \in \omega$ such that $B \subset U^{2k}_k$ or $B \subset U^{2k+1}_k$. Then $B \subset V_{2k}$ or $B \subset V_{2k+1}$. But this would imply that $\{V_m : m \in \omega\} \in \mathcal{O}^X_{\mathfrak{B}}$, which contradicts our initial hypothesis. Then $I \uparrow G_2(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. □

Observation 4.10. The previous result is also valid for $G_f(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$, where $f : \omega \to \omega \setminus \{0\}$ is a function, instead of $G_2(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$.

In particular, $I \uparrow G_1(\mathscr{K}_X, \mathscr{K}_X)$ is equivalent to $I \uparrow G_2(\mathscr{K}_X, \mathscr{K}_X)$.

Let $\mathfrak{B}$ be a bornology. For $B \in \mathfrak{B}$, we define $\tau_B = \{U \in \tau : B \subseteq U\}$ and $\mathbf{B}_{\mathfrak{B}} = \{\tau_B : B \in \mathfrak{B}\}$.

Definition 4.11. The game $\mathfrak{B}$-open is played as follows: in each inning $n \in \omega$, Player I chooses $B \in \mathfrak{B}$ and Player II responds with $U_n \in \tau_B$. Player I wins if $\{B_n : n \in \omega\} \in \mathcal{O}_{\mathfrak{B}}$. Otherwise, Player II is the winner.

Proposition 4.12. $\mathbf{B}_{\mathfrak{B}}$ is a reflection of $\mathcal{O}^X_{\mathfrak{B}}$

Proof. Let $\mathscr{U} \in \mathcal{O}^X_{\mathfrak{B}}$ and $\tau_B \in \mathbf{B}$, with $B \in \mathfrak{B}$. So, there is $U_B \in \mathscr{U}$ such that $B \subseteq U$. Define $f(\tau_B) = U_B \in \tau_B$. It is clear that $range(f) \in \mathcal{O}^X_{\mathfrak{B}}$ and $range(f) \subseteq \mathscr{U}$. □

Theorem 4.13. The games $\mathfrak{B}$-open and $G_1(\mathcal{O}_{\mathfrak{B}}, \mathcal{O}_{\mathfrak{B}})$ are dual.

Proof. Note that the game $\mathfrak{B}$-open is equivalent to $G_1(\mathbf{B}_{\mathfrak{B}}, \neg\mathcal{O}^X_{\mathfrak{B}})$. The result follows from the previous proposition and Theorem 2.52. □

In particular, if $\mathfrak{B} = \{A \subset X : \overline{A} \text{ is compact }\}$, we call the game $\mathfrak{B}-$open of $\mathscr{K}_X-$open.

Before proving the next result, we recall the following characterization of regular spaces:

Proposition 4.14. A topological space (X, τ) is regular if, and only if, for all $x \in X$ and any neighborhood V of x there is a neighborhood U of x such that $\overline{U} \subset V$.

We have

Lemma 4.15. Let (X,τ) be a regular space. The following statements are equivalent:

1. X is compact;

2. For all $\mathcal{U} \in \mathcal{K}_X$, there is $\mathcal{U}' \subset \mathcal{U}$ finite such that $X \subset \overline{\bigcup \mathcal{U}'}$.

Proof. (1) implies (2) follows from the fact that $\mathcal{U} \in \mathcal{K}_X$ is an open cover of X.

Reciprocally, let $\mathcal{U} \in \mathcal{O}_X$ and $K \subset X$ compact. Using Proposition 4.14, for all $x \in K$, with $x \in U_x \in \mathcal{U}$, there is $V_x \in \tau$, such that $x \in V_x \subset \overline{V}_x \subset U_x$.

Consider $\mathcal{V}_K = \{V_x : x \in K\}$. We see that $\mathcal{V}_K$ is an open cover of K. By the compactness of K, there is

$$\mathcal{V}'_K = \{V_{x_i} : 1 \leq i \leq r, \{x_i : 1 \leq i \leq r\} \in [K]^{<\aleph_0}\} \subset \mathcal{V}_K,$$

for some $r \in \omega$, such that $K \subset \bigcup \mathcal{V}'_K$.

Consider $\mathcal{W} = \{\bigcup \mathcal{V}'_K : K \subset X \text{ is compact}\}$. Note that $\mathcal{W} \in \mathcal{K}_X$. By (2), it follows that there is

$$\mathcal{W}' = \{\bigcup \mathcal{V}'_{K_j} : K_j \subset X \; compact, \; 1 \leq j \leq m\} \subset \mathcal{W},$$

for some $m \in \omega$, such that

$$X \subset \overline{\bigcup_{j=1}^{m} \bigcup \mathcal{V}'_{K_j}} = \bigcup_{j=1}^{m} \overline{\bigcup_{i=1}^{r_j} V_{x_i^j}} = \bigcup_{j=1}^{m} \bigcup_{i=1}^{r_j} \overline{V_{x_i^j}} \subset \bigcup_{j=1}^{m} \bigcup_{i=1}^{r_j} U_{x_i^j},$$

where

$$\mathcal{V}'_{K_j} = \{V_{x_i^j} : 1 \leq i \leq r_j, \{x_i^j : 1 \leq i \leq r_j\} \in [K]^{<\aleph_0}\},$$

with $r_j \in \omega$ e $1 \leq j \leq m$. So X is compact. □

Observation 4.16. Note that the previous result is true if we change, in the statement (2), $\mathcal{K}_X$ by $\mathcal{O}^X_{\mathfrak{B}}$, where $\mathfrak{B}$ is a bornology with a compact base.

Lemma 4.17. Let (X,τ) be a regular space. Let σ be a strategy of Player *II* in $G_2(\mathcal{K}_X, \mathcal{K}_X)$. For all $s \in {}^{<\omega}\mathcal{K}_X$, define:

$$C_s = \bigcap_{\mathcal{U} \in \mathcal{K}_X} \overline{\bigcup \sigma(s^\frown \langle \mathcal{U} \rangle)}$$

Then C_s is a compact subset of X.

Proof. Let $\mathcal{U} \in \mathcal{K}_{C_s}$. According to the Lemma 4.15, it suffices to prove that there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $C_s \subset \overline{\bigcup \mathcal{U}'}$. First, note that C_s is a closed subset of X. Let $K \subset X$ be a compact. Then $K \cap C_s$ is compact in C_s. So, there is $U_K \in \mathcal{U}$ such that $K \cap C_s \subset U_K$.

On the other hand, for all $x \in K \cap (X \backslash C_s)$, from the fact that X is regular and from Proposition 4.14, it follows that there is $A_x \in \tau$ such that $x \in A_x$ and the closure of A_x is disjoint from C_s. Therefore, $\{A_x : x \in K \cap (X \backslash C_s)\}$ is an open cover of $K \cap (X \backslash C_s)$. Then,

$$K = (K \cap C_s) \cup [K \cap (X \backslash C_s)] \subset U_K \cup \left(\bigcup_{x \in K \cap (X \backslash C_s)} A_x \right).$$

By the compactness of K, there is $r_K \in \omega \backslash \{0\}$ such that

$$K \subset U_K \cup \left(\bigcup_{i=1}^{r_K} A_{x_i} \right),$$

with $\{x_i : 1 \leq i \leq r_K\} \in [K \cap (X \backslash C_s)]^{<\aleph_0}$. Then,

$$\mathscr{V} = \left\{ U_K \cup \left(\bigcup_{i=1}^{r_K} A_{x_i} \right) : K \subset X \text{ } compact, \{x_i : 1 \leq i \leq r_K\} \in [K \cap (X \backslash C_s)]^{<\aleph_0}, r_K \in \omega \backslash \{0\} \right\}$$

is an element of $\mathscr{K}_X$.

So, $C_s \subset \overline{\bigcup \sigma(s^\frown \mathscr{V})}$. As $\bigcup_{i=1}^{r_{K_1}} \overline{A_{x_i^1}}$ and $\bigcup_{i=1}^{r_{K_2}} \overline{A_{x_i^2}}$ are disjoint of C_s, these elements can be excluded from the set $\overline{\bigcup \sigma(s^\frown \mathscr{V})}$. So, $C_s \subset \overline{U_{K_1} \cup U_{K_2}}$. Therefore, C_s is a compact subset of X. □

Lemma 4.18. Suppose that a topological space (X, τ) satisfies the requirement that for all $\mathscr{U} \in \mathscr{K}_X$ there is a countable $\mathscr{U}' \in \mathscr{K}_X$ such that $\mathscr{U}' \subseteq \mathscr{U}$. If $A \subseteq X$ is closed, then A satisfies that for all $\mathscr{V} \in \mathscr{K}_A$ there is a $\mathscr{V}' \in \mathscr{K}_A$ countable such that $\mathscr{V}' \subseteq \mathscr{V}$.

Proof. Let $\mathscr{V} \in \mathscr{K}_A$. So, every $K \subseteq A$ that is compact is contained in some $U_K \in \mathscr{V}$.

On the other hand, let $C \subseteq X$ be a compact. As $C \cap A$ is a compact subset in A, then C is contained in some set of the form $U_{C \cap A} \cup (X \backslash A)$ (which is open in X, because A is closed), where $U_{C \cap A} \in \mathscr{V}$. So

$$\mathscr{U} = \{U_{C \cap A} \cup (X \backslash A) : C \subseteq X \text{ } is \text{ } compact, \text{ } U_{C \cap A} \in \mathscr{V}\} \in \mathscr{K}_X.$$

Then, there is $\mathscr{U}' \in \mathscr{K}_X$ countable such that it is contained in $\mathscr{U}$, we can say

$$\mathscr{U}' = \{U_{C_n \cap A} \cup (X \backslash A) : U_{C_n \cap A} \in \mathscr{V}, \text{ } n \in \omega\}.$$

We claim that $\mathscr{V}' = \{U_{C_n \cap A} : n \in \omega\} \in \mathscr{K}_A$. Indeed, let $K \subseteq A$ be a compact. So, there is $n \in \omega$ such that $K \subset U_{C_n \cap A} \cup (X \backslash A)$. Then $K \subset U_{C_n \cap A}$. This concludes the proof. □

Observation 4.19. The previous lemma is also valid in the following form: If, in a topological space (X, τ), for all $\mathscr{U} \in \mathscr{O}^X_{\mathfrak{B}}$ there is a countable $\mathscr{U}' \in \mathscr{O}^X_{\mathfrak{B}}$ (respectively, $\mathscr{O}_X$) with $\mathscr{U}' \subseteq \mathscr{U}$, then any closed subset A of X has the following property: for any $\mathscr{V} \in \mathscr{O}^A_{\mathfrak{C}}$, there is a countable $\mathscr{V}' \in \mathscr{O}^A_{\mathfrak{C}}$ (respectively, $\mathscr{O}_A$) with $\mathscr{V}' \subseteq \mathscr{V}$ (here $\mathfrak{B}$ is a bornology with compact base and $\mathfrak{C} = \{B \cap A : B \in \mathfrak{B}\}$).

The following proof is inspired in Theorem 2.2 of (CRONE *et al.*, 2019)].

Theorem 4.20. Let (X,τ) be a regular space. Then $G_1(\mathscr{K}_X,\mathscr{K}_X)$ and $G_2(\mathscr{K}_X,\mathscr{K}_X)$ are equivalent.

Proof. By Theorem 4.9, it is sufficient to prove $II\uparrow G_2(\mathscr{K}_X,\mathscr{K}_X)\Rightarrow II\uparrow G_1(\mathscr{K}_X,\mathscr{K}_X)$.

Let σ be a winning strategy for Player II in the game $G_2(\mathscr{K}_X,\mathscr{K}_X)$ We define a winning strategy ρ for Player I in the game $\mathscr{K}_X-$open as follows. Consider $C_0 := C_\emptyset$, where $C_\emptyset$ is, as in Lemma 4.17, the first move of the strategy ρ in the game $\mathscr{K}_X-$open.

Suppose that Player II responds with $V_0\in\tau$ such that $C_0\subseteq V_0$. Then $X\backslash V_0\subseteq X\backslash C_0$. Let $K\subset X\backslash V_0$ be a compact set. For all $x\in K$, there is $\mathscr{U}_x\in\mathscr{K}_X$ such that $x\in X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_x\rangle)}$. So

$$K\subseteq\bigcup_{x\in K}X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_x\rangle)}.$$

By compactness, there is $F_K\in[\omega]^{<\aleph_0}$ such that

$$K\subseteq\bigcup_{i\in F_K}X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_i^K\rangle)}.$$

Then,

$$\left\{\bigcup_{i\in F_K}X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_i^K\rangle)} : K\subseteq X\backslash V_0\ compact,\ F_K\in[\omega]^{<\aleph_0}\right\}\in\mathscr{K}_{X\backslash V_0}.$$

By the previous Lemma, we can fix an countable subset, namely

$$\left\{\bigcup_{i\in F_{(m)}}X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_i^{(m)}\rangle)} : m\in\omega,\ F_{(m)}\in[\omega]^{<\aleph_0}\right\}\in\mathscr{K}_{X\backslash V_0}.$$

Fix any bijection $\varphi : {}^{<\omega}\omega\longrightarrow\omega$ such that if $s\subset t$ then $\varphi(s)\leq\varphi(t)$. Suppose that up to the inning $n\in\omega$ in the game $\mathscr{K}_X$-open, the sequence C_0, V_0,...,C_{n-1}, V_{n-1} has been played, where V_j is an open set that contains C_j, for all $0\leq j\leq n-1$, and $\mathscr{U}_i^{\varphi^{-1}(j)^\frown m}$ were also defined, for all $m\in\omega$ and $i\in\bigcup_{m\in\omega}F_{\varphi^{-1}(j)^\frown m}$. If $s=\varphi^{-1}(j)$, we assume that:

1. $C_j^{i_{dom(s)},\ldots,i_2,i_1} = \bigcap_{\mathscr{U}\in\mathscr{K}_X}\overline{\bigcup\sigma(\langle\mathscr{U}_{i_1}^{s\restriction 1},\mathscr{U}_{i_2,i_1}^{s\restriction 2},\ldots,\mathscr{U}_{i_{dom(s)},\ldots,i_2,i_1}^{s\restriction dom(s)},\mathscr{U}\rangle)}$, for $0\leq j\leq n-1$. Note that this set is a compact subset of X by Lemma 4.17. So,

$$C_j=\bigcup_{i_k\in F_{s\restriction k},1\leq k\leq dom(s)}C_j^{i_{dom(s)},\ldots,i_2,i_1}$$

   is a compact subset of X.

2. By Lemma 4.18, there is

$$\left\{\bigcup_{i\in F_{s^\frown m}}\ \bigcap_{i_k\in F_{s\restriction k},1\leq k\leq dom(s)}X\backslash\overline{\bigcup\sigma(\langle\mathscr{U}_{i_1}^{s\restriction 1},\mathscr{U}_{i_2,i_1}^{s\restriction 2},\ldots,\mathscr{U}_{i_{dom(s)},\ldots,i_2,i_1}^{s\restriction dom(s)},\mathscr{U}_{i,i_{dom(s)},\ldots,i_2,i_1}^{s^\frown m}\rangle)}\right\}_{m\in\omega}\in\mathscr{K}_{X\backslash V_j}.$$

Now, we define the choice of Player I using ρ in this inning. Let $t = \varphi^{-1}(n)$, and define:

$$C_n^{i_{dom(t)},\dots,i_2,i_1} = \bigcap_{\mathscr{U}\in\mathscr{K}_X} \overline{\bigcup \sigma(\langle \mathscr{U}_{i_1}^{t\restriction 1}, \mathscr{U}_{i_2,i_1}^{t\restriction 2}, \dots, \mathscr{U}_{i_{dom(t)},\dots,i_2,i_1}^{t\restriction dom(t)}, \mathscr{U}\rangle)}.$$

Note that this set is compact by Lemma 4.17. So,

$$C_n = \bigcup_{i_k\in F_{t\restriction k}, 1\leq k\leq dom(s)} C_n^{i_{dom(t)},\dots,i_2,i_1}$$

is a compact subset.

If V_n is a choice of Player II, by Lemma 4.18, it follows that there is:

$$\left\{ \bigcup_{i\in F_{t^\frown m}} \bigcap_{i_k\in F_{t\restriction k}, 1\leq k\leq dom(t)} X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{i_1}^{t\restriction 1}, \mathscr{U}_{i_2,i_1}^{t\restriction 2}, \dots, \mathscr{U}_{i_{dom(t)},\dots,i_2,i_1}^{t\restriction dom(t)}, \mathscr{U}_{i,i_{dom(t)},\dots,i_2,i_1}^{t^\frown m}\rangle)} \right\}_{m\in\omega} \in \mathscr{K}_{X\setminus V_n}.$$

This completes the definition of the strategy $\rho : {}^{<\omega}(\mathbf{B_K}) \to \mathbf{K}$ for Player I in the game $\mathscr{K}_X$-open. We now prove that ρ is a winning strategy. Indeed, suppose that $C_0, V_0, C_1, V_1,\dots$ is a play in the compact open game, where Player I uses strategy ρ.

Suppose that $\{V_n : n\in\omega\} \notin \mathscr{K}_X$. Then, there is a $K\subset X$ compact such that $K \not\subset V_n$, for all $n\in\omega$. In particular, there are $x_0\in K$ and $x_0\notin V_0$. So, there is $m_0\in\omega$ such that

$$x_0 \in \bigcup_{i\in F_{(m_0)}} X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_i^{(m_0)}\rangle)}.$$

Then, there is a $i_0\in F_{(m_0)}$ such that

$$x_0 \in X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{i_0}^{(m_0)}\rangle)}.$$

In addition, there is $x_1\in K$ such that $x_1\notin V_1$. So, there is $m_1\in\omega$ such that

$$x_1 \in \bigcup_{i\in F_{(m_0,m_1)}} \bigcap_{j_0\in F_{(m_0)}} X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{j_0}^{(m_0)}, \mathscr{U}_{i,j_0}^{(m_0,m_1)}\rangle)}.$$

So, there is $i_1\in F_{(m_0,m_1)}$ such that

$$x_1 \in \bigcap_{j_0\in F_{(m_0)}} X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{j_0}^{(m_0)}, \mathscr{U}_{i_1,j_0}^{(m_0,m_1)}\rangle)}.$$

In particular:

$$x_1 \in X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{i_0}^{(m_0)}, \mathscr{U}_{i_1,i_0}^{(m_0,m_1)}\rangle)}.$$

In general, suppose that we have defined $m_0, m_1, \dots, m_{n-1}\in\omega$ and $i_0, i_1, \dots, i_{n-1}$, with $i_l \in F_{(m_0,m_1,\dots,m_l)}$, such that

$$x_l \in X\setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{i_0}^{(m_0)}, \mathscr{U}_{i_1,i_0}^{(m_0,m_1)}, \dots, \mathscr{U}_{i_l,\dots,i_1,i_0}^{(m_0,m_1..,m_l)}\rangle)},$$

for all $0 \leq l \leq n-1$. Since there is $x_n \in K$ such that $x_n \notin V_n$, it follows that there is $m_n \in \omega$ such that

$$x_n \in \bigcup_{i \in F_{(m_0,m_1,\ldots,m_n)}} \bigcap_{j_k \in F_{(m_0,m_1,\ldots,m_k)}, 0 \leq k \leq n-1} X \setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{j_0}^{(m_0)}, \ldots, \mathscr{U}_{j_{n-1},\ldots,j_1,j_0}^{(m_0,m_1,\ldots,m_{n-1})}, \mathscr{U}_{i,j_{n-1},\ldots,j_1,j_0}^{(m_0,m_1,\ldots,m_n)} \rangle)}.$$

Then there is $i_n \in F_{(m_0,\ldots,m_{n-1},m_n)}$ such that

$$x_n \in \bigcap_{j_k \in F_{(m_0,m_1,\ldots,m_k)}, 0 \leq k \leq n-1} X \setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{j_0}^{(m_0)}, \ldots, \mathscr{U}_{j_{n-1},\ldots,j_1,j_0}^{(m_0,m_1,\ldots,m_{n-1})}, \mathscr{U}_{i_n,j_{n-1},\ldots,j_1,j_0}^{(m_0,m_1,\ldots,m_n)} \rangle)}.$$

In particular:

$$x_n \in X \setminus \overline{\bigcup \sigma(\langle \mathscr{U}_{i_0}^{(m_0)}, \ldots, \mathscr{U}_{i_{n-1},\ldots,i_1,i_0}^{(m_0,m_1,\ldots,m_{n-1})}, \mathscr{U}_{i_n,i_{n-1},\ldots,i_1,i_0}^{(m_0,m_1,\ldots,m_n)} \rangle)}.$$

So, we obtain $\{\mathscr{U}_{i_n,\ldots,i_1,i_0}^{(m_0,m_1\ldots,m_n)}\}_{n\in\omega}$, a sequence of $\mathscr{K}_X$-covers such that there is a $K \subset X$ compact, with the property that

$$K \not\subset \bigcup \sigma(\langle \mathscr{U}_{i_0}^{(m_0)}, \ldots, \mathscr{U}_{i_n,\ldots,i_0}^{(m_0,\ldots,m_n)} \rangle),$$

for all $n \in \omega$. That is, this sequence defines a strategy of Player I to defeat σ in the game $G_2(\mathscr{K}_X, \mathscr{K}_X)$. But this contradicts the fact that σ is a winning strategy for Player II in the game $G_2(\mathscr{K}_X, \mathscr{K}_X)$.

Therefore, ρ is a winning strategy for Player I in the compact-open game. By duality, there is a winning strategy for Player II in $G_1(\mathscr{K}_X, \mathscr{K}_X)$. This concludes the proof. □

By Observation 4.16 we can obtain the following results

Lemma 4.21. Let (X, τ) be a regular space and $\mathfrak{B}$ be a bornology with a compact base. Let σ be a strategy for Player II in $G_{fin}(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$. For $s \in {}^{<\omega}\mathscr{O}_{\mathfrak{B}}^X$, define:

$$C_s = \bigcap_{\mathscr{U} \in \mathscr{O}_{\mathfrak{B}}^X} \overline{\bigcup \sigma(s^\frown \langle \mathscr{U} \rangle)}.$$

Then C_s is a compact subset of X.

Lemma 4.22. Let (X, τ) be a regular space, $\mathfrak{B}$ be a bornology with a compact base, and let $f : \omega \to \omega \setminus \{0\}$ be a function. Let σ be a strategy for Player II in $G_f(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$. For all $s \in {}^{<\omega}\mathscr{O}_{\mathfrak{B}}^X$, define:

$$C_s = \bigcap_{\mathscr{U} \in \mathscr{O}_{\mathfrak{B}}^X} \overline{\bigcup \sigma(s^\frown \langle \mathscr{U} \rangle)}.$$

Then C_s is a compact subset of X.

From these results, and with a few modifications to the proof of Theorem 4.20, we obtain the following results:

Corollary 4.23. Let (X,τ) be a regular space. Then the games $G_{fin}(\mathscr{K}_X,\mathscr{K}_X)$ and $G_1(\mathscr{K}_X,\mathscr{K}_X)$ are equivalent for Player *II*.

Corollary 4.24. Let (X,τ) be a regular space and let $f:\omega\to\omega\setminus\{0\}$ be a function. Then, the games $G_f(\mathscr{K}_X,\mathscr{K}_X)$ and $G_1(\mathscr{K}_X,\mathscr{K}_X)$ are equivalent.

In addition, we obtain the following result.

Theorem 4.25. Let (X,τ) be a separable metrizable space and $\mathfrak{B}$ be a bornology with compact base. If $II\uparrow G_{fin}(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$ then X is σ-compact.

Proof. Let $\mathscr{C}$ be a countable basis of X and σ be a winning strategy of Player *II* in $G_{fin}(\mathscr{O}^X_{\mathfrak{B}},\mathscr{O}^X_{\mathfrak{B}})$. We denote by $O_{\mathscr{C}}$ the family of all families in $\mathscr{O}^X_{\mathfrak{B}}$ whose elements belong to $\mathscr{C}$. Note that $\{\sigma(\langle\mathscr{U}\rangle):\mathscr{U}\in O_{\mathscr{C}}\}$ is countable. In the same way as in the proof of Lemma 4.22, we can prove that

$$C_\emptyset=\bigcap_{n\in\omega}\overline{\bigcup\sigma(\langle\mathscr{U}_{\langle n\rangle}\rangle)}$$

is a compact subset of X.

For all $m\in\omega$ fixed, we see that $\{\sigma(\langle\mathscr{U}_{\langle m\rangle},\mathscr{U}\rangle):\mathscr{U}\in O_{\mathscr{C}}\}$ is countable. Then

$$C_{\langle m\rangle}=\bigcap_{n\in\omega}\overline{\bigcup\sigma(\langle\mathscr{U}_{\langle m\rangle},\mathscr{U}_{\langle m,n\rangle}\rangle)}$$

is a compact subset of X.

In general, given $s=\langle s_0,\ldots,s_k\rangle\in^{<\omega}\omega$, with $k\in\omega\setminus\{0\}$, we have that the following set

$$\{\sigma(\langle\mathscr{U}_{\langle s_0\rangle},\mathscr{U}_{\langle s_0,s_1\rangle},\ldots,\mathscr{U}_s,\mathscr{U}\rangle):\mathscr{U}\in O_{\mathscr{C}}\}$$

is countable. Then

$$C_s=\bigcap_{n\in\omega}\overline{\bigcup\sigma(\langle\mathscr{U}_{\langle s_0\rangle},\mathscr{U}_{\langle s_0,s_1\rangle},\ldots,\mathscr{U}_s,\mathscr{U}_{s\frown n}\rangle))}$$

is a compact subset of X.

We claim that $X=\bigcup_{s\in^{<\omega}\omega}C_s$. Indeed, suppose that there is $x\in X\setminus(\bigcup_{s\in^{<\omega}\omega}C_s)$. In particular, $x\notin C_\emptyset$. So, there is $n_0\in\omega$ such that

$$x\notin\overline{\bigcup\sigma(\langle\mathscr{U}_{\langle n_0\rangle}\rangle)}.$$

Also, $x\notin C_{\langle n_0\rangle}$. Then, there is $n_1\in\omega$ such that

$$x \notin \overline{\bigcup \sigma(\langle \mathscr{U}_{\langle n_0 \rangle}, \mathscr{U}_{\langle n_0, n_1 \rangle} \rangle)}.$$

Suppose that, for all $k \in \omega \backslash \{0\}$, we have defined $n_0, \ldots, n_k \in \omega$. As $x \notin C_{\langle n_0, \ldots, n_k \rangle}$, it follows that there is $n_{k+1} \in \omega$ such that

$$x \notin \overline{\bigcup \sigma(\langle \mathscr{U}_{\langle n_0 \rangle}, \ldots, \mathscr{U}_{\langle n_0, \ldots, n_k \rangle}, \mathscr{U}_{\langle n_0, \ldots, n_k, n_{k+1} \rangle} \rangle)}.$$

Then

$$\mathscr{U}_{\langle n_0 \rangle}, \mathscr{U}_{\langle n_1 \rangle}, \ldots, \mathscr{U}_{\langle n_0, \ldots, n_k \rangle}, \ldots$$

is a play by Player I in $G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$ that defeats σ, a contradiction. Therefore, X is σ-compact. □

4.3 A weak result for Player II.

As a break in the study of equivalences, we present a little result of equivalence about open covers, but restricted to Player II. In first place, the following selection principles and games were defined in (AURICHI; DUZI, 2021).

Definition 4.26. Let (X, τ) be a topological space, $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of X. $S_f(\mathscr{A}, \mathscr{B})\text{mod}fin$ is the following selection principle: for any sequence $\langle A_n : n \in \omega \rangle$ of elements of $\mathscr{A}$, there is a sequence $\langle B_n : n \in \omega \rangle$ such that, for all $n \in \omega$, $B_n \in [A_n]^{<\aleph_0}$, $\{n \in \omega : |B_n| > f(n)\}$ is finite and $\bigcup_{n \in \omega} B_n \in \mathscr{B}$.

When $f \equiv k$, with $k \in \omega \backslash \{0\}$, we simply write $S_k(\mathscr{A}, \mathscr{B})\text{mod}fin$. $S_f(\mathscr{A}, \mathscr{B})\text{mod}1$ is defined similarly to $S_f(\mathscr{A}, \mathscr{B})\text{mod}fin$, with the difference that $|B_n| \leq f(n)$, for all $n \geq 1$.

Definition 4.27. Let (X, τ) be a topological space, $\mathscr{A}$ and $\mathscr{B}$ be classes of families of subsets of X. The game $G_f(\mathscr{A}, \mathscr{B})\text{mod}fin$ is defined as follows: In any inning $n \in \omega$, Player I chooses $A_n \in \mathscr{A}$. Player II responds with $B_n \in [A_n]^{<\aleph_0}$. Player II wins if $\{n \in \omega : |B_n| > f(n)\}$ is finite and $\bigcup_{n \in \omega} B_n \in \mathscr{B}$. Otherwise, Player I wins.

When $f \equiv k$, with $k \in \omega \backslash \{0\}$, we simply write $G_k(\mathscr{A}, \mathscr{B})\text{mod}fin$. The game $G_f(\mathscr{A}, \mathscr{B})\text{mod}1$ is defined similar to $G_f(\mathscr{A}, \mathscr{B})\text{mod}fin$, with the difference that, Player II wins if $|B_n| \leq f(n)$, for all $n \geq 1$.

In that same work, the following result is proved:

Theorem 4.28 (((AURICHI; DUZI, 2021))). Let (X, τ) be a regular space. The following statements are equivalent:

1. $II \uparrow G_1(\mathscr{O}_X, \mathscr{O}_X)\text{mod}1$;
2. there is a compact set $K \subset X$ such that, for every open set V with $K \subset V$, we have $II \uparrow G_1(\mathscr{O}_{X\setminus V}, \mathscr{O}_{X\setminus V})$.

We can obtain a different version of that result:

Theorem 4.29. Let (X,τ) be a regular space and $\mathfrak{B}$ be a bornology with a compact base. The following statements are equivalent:

1. $II \uparrow G_1(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_X)\text{mod}1$;
2. there is a compact set $K \subset X$ such that, for every open set V with $K \subset V$, we have $II \uparrow G_1(\mathscr{O}_{\mathfrak{C}}^{X\setminus V}, \mathscr{O}_{X\setminus V})$, where $\mathfrak{C} = \{B \cap (X\setminus V) : B \in \mathfrak{B}\}$.

Proof. $(2) \Rightarrow (1)$. Let K be the compact set such that K satisfies (2). We define a strategy σ for Player *II* in the game $G_1(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_X)\text{mod}1$ as follows:

- Suppose that, in the first inning, Player *I* chose $\mathscr{U}_0 \in \mathscr{O}_{\mathfrak{B}}^X$. Define $\sigma(\langle \mathscr{U}_0 \rangle)$ as a finite sub-cover of $\mathscr{U}_0$ for K. Let $V = \bigcup \sigma(\langle \mathscr{U}_0 \rangle)$ and let σ^V be a winning strategy for Player *II* in the game $G_1(\mathscr{O}_{\mathfrak{C}}^{X\setminus V}, \mathscr{O}_{X\setminus V})$.
- In the innings $n \geq 1$, suppose that Player *I* chooses $\mathscr{U}_n \in \mathscr{O}_{\mathfrak{B}}^X$. Consider

$$\mathscr{U}_n' := \{U \cap (X\setminus V) : U \in \mathscr{U}_n\}.$$

Note that $\mathscr{U}_n' \in \mathscr{O}_{\mathfrak{C}}^{X\setminus V}$. Then define:

$$\sigma(\langle \mathscr{U}_0, \mathscr{U}_1 ..., \mathscr{U}_n \rangle) = \sigma^V(\langle \mathscr{U}_1, ..., \mathscr{U}_n \rangle)$$

It is clear that σ is a winning strategy for Player *II* in $G_1(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_X)\text{mod}1$.

$(1) \Rightarrow (2)$. Let σ be a winning strategy for Player *II* in the game $G_1(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_X)\text{mod}1$. By Lemma 4.22 (which is valid if we consider the game $G_1(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_X)\text{mod}1$), we have that

$$K := \bigcap_{\mathscr{U} \in \mathscr{O}_{\mathfrak{B}}^X} \overline{\bigcup \sigma(\langle \mathscr{U} \rangle)}$$

is a compact set.

Now, let V be an open set containing K. Since $\mathfrak{B}$ has a compact base then $\mathfrak{C}$ has too. Let $\mathfrak{C}'$ a compact base of $\mathfrak{C}$, $D \in \mathfrak{C}'$ and $x \in D$. Then, there is $\mathscr{U}_x \in \mathscr{O}_{\mathfrak{B}}^X$ such that $x \in X\setminus\overline{\bigcup \sigma(\langle \mathscr{U}_x \rangle)}$.

So, $D \subset X \setminus \bigcap_{x \in D} \overline{\bigcup \sigma(\langle \mathscr{U}_x \rangle)}$. As D is compact, there is $F_D \in [\omega]^{<\aleph_0}$ such that

$$D \subset X \setminus \bigcap_{i \in F_D} \overline{\bigcup \sigma(\langle \mathscr{U}_i \rangle)}.$$

Then

$$\left\{X\setminus\bigcap_{i\in F_D}\overline{\bigcup\sigma(\langle\mathscr{U}_i\rangle)} : D\in\mathfrak{C}', F_D\in[\omega]^{<\aleph_0}\right\}\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}.$$

As $X\setminus V$ is closed, by Observation 4.19, we have that

$$\left\{X\setminus\bigcap_{i\in F_{D_n}}\overline{\bigcup\sigma(\langle\mathscr{U}_i\rangle)} : n\in\omega, D_n\in\mathfrak{C}', F_{D_n}\in[\omega]^{<\aleph_0}\right\}\in\mathscr{O}_{X\setminus V}.$$

On the other hand, if $\mathscr{V}\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$ then the family $\mathscr{V}'=\{U\cup V : U\in\mathscr{V}\}\in\mathscr{O}_{\mathfrak{B}}^{X}$. Let $\{p_n : n\in\omega\setminus\{0\}\}$ be an enumeration of prime numbers, and let $\{m_k : k\in\omega\setminus\{0\}\}$ be an enumeration of $\bigcup_{n\in\omega}F_{D_n}$. Now, we define a strategy σ^V for Player *II* in the game $G_1(\mathscr{O}_{\mathfrak{C}}^{X\setminus V},\mathscr{O}_{X\setminus V})$ as follows:

- In innings $n\leq 1$, suppose that Player *I* chooses $\mathscr{V}_n\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$. Player *II* chooses any element $U_n\in\mathscr{V}_n$.
- In the next inning, suppose that Player *I* chose $\mathscr{V}_2\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$. Then define
$$\sigma^V(\langle\mathscr{V}_0,\mathscr{V}_1,\mathscr{V}_2\rangle)=\sigma(\langle\mathscr{U}_{m_1},\mathscr{V}_2'\rangle).$$
- In the next inning, if Player *I* chose $\mathscr{V}_3\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$ then define
$$\sigma^V(\langle\mathscr{V}_0,...,\mathscr{V}_3\rangle)=\sigma(\langle\mathscr{U}_{m_2},\mathscr{V}_3'\rangle).$$
- In the next inning, if Player *I* chose $\mathscr{V}_4\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$ then define
$$\sigma^V(\langle\mathscr{V}_0,...,\mathscr{V}_4\rangle)=\sigma(\langle\mathscr{U}_{m_1},\mathscr{V}_2',\mathscr{V}_4'\rangle).$$
- In general, in the innings $n=p_k^l$, with $l\geq 1$ and $k\geq 1$, if Player *I* chose $\mathscr{V}_n\in\mathscr{O}_{\mathfrak{C}}^{X\setminus V}$ then define
$$\sigma^V(\langle\mathscr{V}_0,...,\mathscr{V}_n\rangle)=\sigma(\langle\mathscr{U}_{m_k},\mathscr{V}'_{p_k^1},...,\mathscr{V}'_{p_k^l}\rangle).$$
In the innings remaining (that is, if $n\notin\{p_n : n\in\omega\setminus\{0\}\}$), if Player *I* chose $\mathscr{V}_n$ then Player *II* chooses any element $U_n\in\mathscr{V}_n$.

Finally, we prove that σ^V is a winning strategy. Indeed, let $y\in X\setminus V$ and let $\langle\mathscr{V}_n : n\in\omega\rangle$ be a play by Player *I* in the game $G_1(\mathscr{O}_{\mathfrak{C}}^{X\setminus V},\mathscr{O}_{X\setminus V})$. Then, there are $k\in\omega$ and $j\in F_{D_k}$ such that $y\notin\overline{\bigcup\sigma(\langle\mathscr{U}_j\rangle)}$. But, since σ is a winning strategy in $G_1(\mathscr{O}_{\mathfrak{B}}^{X},\mathscr{O}_X)\text{mod}1$, y must be in some response of σ when consider the following play for to Player *I*

$$\langle\mathscr{U}_j\rangle^\frown\langle\mathscr{V}'_{p_j^l} : l\in\omega\setminus\{0\}\rangle.$$

So, y is contained in some response made by σ^V. This concludes the proof. □

4.4 Equivalent games in $C_k(X)$

The game version of Theorem 2.39 is given in the following result:

Theorem 4.30. (MEZABARBA; AURICHI, 2019) Let (X,τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base and let $f:\omega\to\omega\backslash\{0\}$ be a function. Then, the game $G_f(\mathscr{O}_{\mathfrak{B}},\mathscr{O}_{\mathfrak{B}})$, and the games $G_f(\Omega_g,\Omega_g)$ and $G_f(\mathscr{D}_{C_{\mathfrak{B}}(X)},\Omega_g)$ in $C_{\mathfrak{B}}(X)$ are equivalent for all $g\in C_{\mathfrak{B}}(X)$.

In particular, it follows that the game $G_f(\mathscr{K}_X,\mathscr{K}_X)$ is equivalent to $G_f(\Omega_o,\Omega_o)$ in $C_k(X)$, for any function $f:\omega\to\omega\backslash\{0\}$.

Therefore, we can obtain the following result:

Theorem 4.31. Let (X,τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base, $f:\omega\to\omega\backslash\{0\}$ be a function and $g\in C_{\mathfrak{B}}(X)$. The following statements are equivalent:

1. $S_f(\Omega_g,\Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;
2. $I\not\uparrow G_f(\Omega_g,\Omega_g)$ in $C_{\mathfrak{B}}(X)$.

Proof. The result follows from Theorems 2.39, 4.5 and 4.30. □

In addition, from Corollary 4.24 and Theorem 4.30, the following result follows:

Corollary 4.32. Let (X,τ) be a Tychonoff space and let $f:\omega\to\omega\backslash\{0\}$ be a function. Then, the games $G_1(\Omega_g,\Omega_g)$ and $G_f(\Omega_g,\Omega_g)$ are equivalent in $C_k(X)$, for all $g\in C_k(X)$.

From this last result, it also follows:

Corollary 4.33. Let (X,τ) be a Tychonoff space and let $f:\omega\to\omega\backslash\{0\}$ be a function. Then, the games $G_1(\mathscr{D}_{C_k(X)},\Omega_g)$ and $G_f(\mathscr{D}_{C_k(X)},\Omega_g)$ are equivalent in $C_k(X)$, for all $g\in C_k(X)$.

On the other hand, we can obtain the versions of Theorems 2.47 and 2.46 for $C_{\mathfrak{B}}(X)$, with $\mathfrak{B}$ a bornology with a compact base.

First, we recall the following definitions:

Definition 4.34. Let (X,τ), (Y,σ) be topological spaces. We say that Y is a continuous one-to-one image of X, if there is a function $f:X\to Y$ bijective and continuous.

Definition 4.35. Let (X,τ) be a topological space. The i-weight of X is the smallest cardinality $w(Y)$, where Y is a continuous one-to-one image of X. Denote the i-weight of X as $iw(X)$.

Definition 4.36. Let (X,τ) be a topological space. A subset $D\subset C(X)$ separates points if any $x,y\in X$ with $x\neq y$, there is $f\in D$ such that $f(x)\neq f(y)$.

We need to Remember the following result

Theorem 4.37. (NOBLE, 1974) Let (X,τ) be a Tychonoff space. Then $d(C_p(X)) = d(C_k(X)) = i\omega(X)$.

With a few modifications to the proof, we can obtain the following result.

Theorem 4.38. Let (X,τ) be a Tychonoff space and let $\mathfrak{B}$ be a bornology with a compact base. Then $d(C_{\mathfrak{B}}(X)) = i\omega(X)$.

For the proof we need the following

Theorem 4.39. Let (X,τ) be a Tychonoff space and let $\mathfrak{B}$ be a bornology with a compact base. Let $D \subseteq C_{\mathfrak{B}}(X)$ be a family that separates points and contains the constant function 1. Then the subalgebra generated by D is dense in $C_{\mathfrak{B}}(X)$.

This is obtained from:

Theorem 4.40. (Stone-Weierstrass) Let (X,τ) be a compact Hausdorff space. If $D \subset C(X)$ separates points and contains the constant function 1, then the algebra generated by D is dense in $C(X)$ (if $C(X)$ with the uniform topology).

Observation 4.41. Note that by this theorem, if $i\omega(X) = \aleph_0$, then $C_{\mathfrak{B}}(X)$ is a separable space. Also, note that if (X,τ) is a separable metrizable space, then $i\omega(X) = \aleph_0$.

Lemma 4.42. Let (X,τ) be a Tychonoff space such that $S_1(\Omega_o,\Omega_o)$ holds in $C_{\mathfrak{B}}(X)$. Then, for all sequences $\langle A_n : n \in \omega\rangle$ of elements in Ω_o, there is a pairwise disjoint sequence $\langle B_n : n \in \omega\rangle$ of elements in Ω_o such that $B_n \subseteq A_n$.

Proof. Let $\langle A_n : n \in \omega\rangle$ be a sequence of elements in Ω_o. By hypothesis, we can assume that each A_n is countable. Note that $f \in \Omega_o$ if, and only if, $|f| \in \Omega_o$. Then we can assume that the elements of any A_n are positive. Suppose that, for all $n \in \omega$, $A_n = \{f^n_m : m \in \omega\}$.

We define a strategy σ for I in $G_1(\Omega_o,\Omega_o)$. In the first inning, define $\sigma(\emptyset) = A_0 \in \Omega_0$. Suppose that Player II chooses the element $f^0_{m^0_0}$.

Define

$$\sigma(\langle f^0_{m^0_0}\rangle) = \{f^0_{k_0} + f^1_{k_1} : k_o, k_1 \in \omega, |\{f^0_{m^0_0}, f^0_{k_0}, f^1_{k_1}\}| = 3\}.$$

To see that it belongs to Ω_o, let $[o,B,\varepsilon]$ be a basic neighborhood, with $B \in \mathfrak{B}$ e $\varepsilon > 0$. So, there are $f^0_{k_0} \in (A_0 \backslash \{f^0_{m_0}\}) \cap [o,B,\frac{\varepsilon}{2}]$ and $f^1_{k_1} \in (A_1 \backslash \{f^0_{m_0}, f^0_{k_0}\}) \cap [o,B,\frac{\varepsilon}{2}]$. Then $f^0_{k_0} + f^1_{k_1} \in [o,B,\varepsilon]$. Suppose that Player II chose the element $f^0_{m^1_0} + f^1_{m^1_1}$.

Now, define

$$\sigma(\langle f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}\rangle) = \{f^0_{k_0} + f^1_{k_1} + f^2_{k_2} : k_o, k_1, k_2 \in \omega, |\{f^0_{m^0_0}, f^0_{m^1_0}, f^1_{m^1_1}, f^0_{k_0}, f^1_{k_1}, f^2_{k_2}\}| = 6\}.$$

Let $[o,B,\varepsilon]$ be a basic neighborhood , with $B \in \mathfrak{B}$ and $\varepsilon > 0$. Then, there are

$$f^0_{k_0} \in (A_0 \setminus \{f^0_{m_0}.f^0_{m^1_0}, f^1_{m^1_1}\}) \cap [o,B,\tfrac{\varepsilon}{3}],\ f^1_{k_1} \in (A_1 \setminus \{f^0_{m_0}, f^0_{m^1_0}, f^0_{k_0}, f^1_{m^1_1}\}) \cap [o,B,\tfrac{\varepsilon}{3}] \text{ and}$$
$$f^2_{k_2} \in (A_2 \setminus \{f^0_{m_0}, f^0_{m^1_0}, f^0_{k_0}, f^1_{m^1_1}, f^1_{k_1}\}) \cap [o,B,\tfrac{\varepsilon}{3}].$$

So, $f^0_{k_0} + f^1_{k_1} + f^2_{k_2} \in [o,B,\varepsilon]$. Then, $\sigma(\langle f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}\rangle) \in \Omega_o$. This way we define all innings $n \in \omega$.

By Theorem 4.31, we have that σ is not a winning strategy. So, there is a set C in Ω_o, with elements of the form

$$f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}, f^0_{m^2_0} + f^1_{m^2_1} + f^2_{m^2_2}, \ldots$$

Then we can consider, for all $n \in \omega$, the sets $B_n = \{f^n_{m^i_n} : i \geq n\}$. As $C \in \Omega_o$, it follows that, for all $n \in \omega$, $B_n \in \Omega_o$, and by the construction done, all sets B_n are pairwise disjoint. □

Theorem 4.43. Let (X,τ) be a Tychonoff space with $i\omega(X) = \aleph_0$. Let $g \in C_{\mathfrak{B}}(X)$. The following statements are equivalent:

1. $S_1(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;
2. $I \not\uparrow G_1(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;
3. $I \not\uparrow G_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$;
4. $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ holds.

Proof. $(1) \Leftrightarrow (2)$. Follows from Theorem 4.31. As $C_{\mathfrak{B}}(X)$ is homogeneous, it follows that it is sufficient to prove $(2) \Rightarrow (3)$ and $(4) \Rightarrow (1)$, for the case $g = o$.

$(2) \Rightarrow (3)$. Let σ be a strategy for Player I in game $G_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$. By Observation 4.41, we can assume that σ chooses countable subsets, and we fix $\{g_n : n \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$. We define a strategy ρ for Player I in the game $G_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$.

Suppose that $\sigma(\emptyset) = \{f_n : n \in \omega\} \in \mathscr{D}_{C_{\mathscr{B}}(X)}$. Define $\rho(\emptyset) = \{|f_n - g_0| : n \in \omega\}$. We claim that $\rho(\emptyset) \in \Omega_o$. Indeed, let $[o,B,\varepsilon]$ be a basic neighborhood with $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $[g_0,B,\varepsilon]$ is an open subset of $C_{\mathfrak{B}}(X)$ and $\sigma(\emptyset) \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$, it follows that there is a $k \in \omega$ such that $f_k \in [g_0,B,\varepsilon]$. So, $|f_k - g_0| \in [o,B,\varepsilon]$.

Suppose that Player II chooses, in the game $G_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$, the element $|f_{n^0_0} - g_0|$, and that $\sigma(\langle f_{n^0_0}\rangle) = \{f_{n^0_0,n} : n \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$. We define

$$\rho(\langle |f_{n^0_0} - g_0|\rangle) = \{|f_{n^0_0,i} - g_0| + |f_{n^0_0,j} - g_1| : i,j \in \omega\}.$$

Similarly to the previous case (in this case, consider the basic open $[g_i,B,\frac{\varepsilon}{2}]$, with $B \in \mathfrak{B}$ e $i = 0,1$), it follows that $\rho(\langle |f_{n^0_0} - g_0|\rangle) \in \Omega_o$.

Suppose that Player II chooses, in the game $G_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$, the element $|f_{n^0_0,n^1_0} - g_0| + |f_{n^0_0,n^1_1} - g_1|$. Also, suppose that

$$\sigma(\langle f_{n_0^0}, f_{n_0^0,n_0^1}\rangle) = \{f_{n_0^0,n_0^1,n} : n \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)} \text{ and}$$
$$\sigma(\langle f_{n_0^0}, f_{n_0^0,n_1^1}\rangle) = \{f_{n_0^0,n_1^1,n} : n \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}.$$

So, we can define

$$\rho(\langle |f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|\rangle) = \{|f_{n_0^0,n_0^1,i_1} - g_0| + |f_{n_0^0,n_0^1,i_2} - g_1| +$$
$$+ |f_{n_0^0,n_0^1,i_3} - g_2| + |f_{n_0^0,n_1^1,j_1} - g_0| + |f_{n_0^0,n_1^1,j_2} - g_1| + |f_{n_0^0,n_1^1,j_3} - g_2| : i_1, i_2, i_3, j_1, j_2, j_3 \in \omega\}.$$

Similarly to the previous case (in this case, consider the open $[g_i, B, \frac{\varepsilon}{6}]$, with $B \in \mathfrak{B}$ and $i = 0,1,2$), it follows that $\rho(\langle |f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|\rangle) \in \Omega_o$.

Following the construction above in all innings $n \in \omega$, it follows that $\rho : {}^{<\omega}(\bigcup \Omega_o) \to \Omega_o$ is a strategy of Player I in the game $G_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$. By (2), we can choose a sequence of Player II choices that form a set $C \in \Omega_o$, with elements of the form:

$$|f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|, |f_{n_0^0,n_0^1,n_0^2} - g_0| + |f_{n_0^0,n_0^1,n_1^2} - g_0| +$$
$$|f_{n_0^0,n_0^1,n_2^2} - g_2| + |f_{n_0^0,n_1^1,n_3^2} - g_0| + |f_{n_0^0,n_1^1,n_4^2} - g_1| + |f_{n_0^0,n_1^1,n_5^2} - g_2|, \ldots$$

Then, by Lemma 4.42, we can obtain a partition of C in countable many pairwise disjoint of sets $B_n \in \Omega_o$. For all $n \in \omega$, we define J_n as the set of all $m \in \omega$ such that Player II chose an element of B_n in the inning m. Note that these sets are pairwise disjoint and we can assume, for all $n \in \omega$, that $\min(J_n) \geq n$.

So, we define $m_0 = n_0^0$. Now, since the only possibilities are $1 \in I_0$ or $1 \in J_1$, define $m_1 = n_j^1$, where $j \in \{0,1\}$ is the term $|f_{m_0,n_j^1} - g_j|$ of the choice of Player II in the inning 1. In general, for all $k \geq 2$, since the unique possibilities are $k \in J_i$, with $i \leq k$, we define $m_k = n_j^k$, where $j \leq k$ is the term $|f_{m_0,\ldots,m_{k-1},n_j^k} - g_j|$ of the choice of Player II in the inning k. So, for all $k \in \omega$, $\{|f_{m_0,\ldots,m_j} - g_k| : j \in I_k\} \in \Omega_0$. Indeed, let $[o, B, \varepsilon]$ be a neighborhood, with $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $B_k \in \Omega_o$, it follows that there is a $r \in \omega$ such that $|f_{m_1,..,m_r} - g_k| \in [o, B, \varepsilon]$.

Finally, we claim that $\{f_{m_0,\ldots,m_j} : j \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$. Indeed, let $[h, B, \varepsilon]$ be a basic neighborhood, with $h \in C_{\mathfrak{B}}(X)$, $B \in \mathfrak{B}$ e $\varepsilon > 0$. As $\{g_k : k \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$, it follows that there is $l \in \omega$ such that $g_l \in [h, B, \frac{\varepsilon}{2}]$. So, there is $r \in \omega$, such that $|f_{m_0,\ldots,m_r} - g_l| \in [o, B, \frac{\varepsilon}{2}]$. Therefore, $f_{m_0,\ldots,m_r} \in [h, B, \varepsilon]$. So, we obtain a sequence of choices for Player II in the game $G_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$ that defeats a strategy σ.

(4) $\Rightarrow$ (1). It follows from the implication (3) $\Rightarrow$ (1) in Theorem 2.39, changing Ω_o by $\mathscr{D}_{C_{\mathfrak{B}}}(X)$. □

With a few modifications to the previous theorem, we can obtain the following results:

Theorem 4.44. Let (X, τ) be a Tychonoff space such that $i\omega(X) = \aleph_0$. Let $f : \omega \to \omega \backslash \{0\}$ be a function and $g \in C_{\mathfrak{B}}(X)$. The following statements are equivalent:

1. $S_f(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;

2. $I \not\uparrow G_f(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;

3. $I \not\uparrow G_f(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$;

4. $S_f(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ holds.

Theorem 4.45. Let (X,τ) be a Tychonoff space such that $i\omega(X) = \aleph_0$. Let $g \in C_{\mathfrak{B}}(X)$. The following statements are equivalent:

1. $S_{fin}(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;

2. $I \not\uparrow G_{fin}(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;

3. $I \not\uparrow G_{fin}(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$;

4. $S_{fin}(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ holds in $C_{\mathfrak{B}}(X)$.

Finally, from Corollary 4.32, Corollary 4.33, Theorem 4.3 and Theorem 4.44, the following result follows:

Corollary 4.46. Let (X,τ) be a Tychonoff space such that $i\omega(X) = \aleph_0$. Let $f : \omega \to \omega \setminus \{0\}$ be an increasing function. Then the games $G_f(\mathscr{D}_{C_k(X)}, \mathscr{D}_{C_k(X)})$ and $G_1(\mathscr{D}_{C_k(X)}, \mathscr{D}_{C_k(X)})$ are equivalent.

CHAPTER

5

ADDITIONAL TRANSLATIONS IN GAMES AND SELECTION PRINCIPLES

5.1 Translations on the pair $(\mathscr{D}_X, \mathscr{S}_X)$

In (OSIPOV, 2018a) it is shown a list of similarities between the different properties involving dense families.

Theorem 5.1. ((OSIPOV, 2018a)) Let X be a Tychonoff space and $i\omega(X) = \aleph_0$. The following statements are equivalent:

1. $S_1(D_{C_p(X)}, \mathscr{S}_{C_p(X)})$ holds;
2. $C_p(X)$ is strongly sequentially dense;
3. $S_1(\Omega_X, \Gamma_X)$;
4. $S_1(\Omega_0, \Gamma_0)$ holds in $C_p(X)$;
5. $S_1(\mathscr{D}_{C_p(X)}, \Gamma_0)$ holds;
6. $S_{fin}(\mathscr{D}_{C_p(X)}, \mathscr{S}_{C_p(X)})$ holds;
7. $S_{fin}(\Omega_X, \Gamma_X)$ holds;
8. $S_{fin}(\Omega_0, \Gamma_0)$ holds in $C_p(X)$;
9. $S_{fin}(\mathscr{D}_{C_p(X)}, \Gamma_0)$ holds.

In this section, we make a generalization of the equivalences of this theorem.

Theorem 5.2. Let (X, τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with compact base. The following statements are equivalent:

1. $C_{\mathfrak{B}}(X)$ is strongly sequentially dense
2. $S_1(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds.

Proof. (1) $\Rightarrow$ (2). By Theorem 2.43, is sufficient to prove that $\begin{pmatrix} \mathcal{O}^X_{\mathfrak{B}} \\ \Gamma^X_{\mathfrak{B}} \end{pmatrix}$ holds. Let $\mathscr{U} \in \mathcal{O}^X_{\mathfrak{B}}$ and $D \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$. We define $D(\mathscr{U}) = \{f \in C(X) : f(X \backslash U) \equiv 1$, for some $U \in \mathcal{O}^X_{\mathfrak{B}}\}$. We claim that $D(\mathscr{U})$ is dense. Indeed, let $g \in C_{\mathscr{D}}(X)$ and $W = \langle g, B, \varepsilon \rangle$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $\overline{B} \in \mathfrak{B}$ is compact and X is Tychonoff, it follows from Theorem 2.28 that there is $h \in W \cap D(\mathscr{U})$.

By (1), $D(\mathscr{U})$ is sequentially dense. Thus, there is a sequence $\langle f_n : n \in \omega \rangle$ of elements in $D(\mathscr{U})$ that converges to o. Then, for all $n \in \omega$, $f_n(X \backslash U_n) \equiv 1$, for some $U_n \in \mathscr{U}$.

We claim that $\{U_n : n \in \omega\} \in \Gamma^X_{\mathfrak{B}}$. Indeed, let $B \in \mathfrak{B}$, and consider $W = \langle o, B, 1 \rangle$. Then there is $n_0 \in \omega$ such that $f_n \in W$, for all $n \geq n_0$. Thus $|f_n(x)| < 1$, for all $x \in B$ and $n \geq n_0$. So, we have $B \cap (X \backslash U_n) = \emptyset$. Then $B \subset U_n$, for all $n \geq n_0$.

(2) $\Rightarrow$ (1): By Theorem 2.43, $C_{\mathfrak{B}}(X)$ is Fréchet. Let $D \subset X$ be dense. Let $g \in C_{\mathfrak{B}}(X)$. As $D \in \Omega_g$, there is a sequence $\langle f_n : n \in \omega \rangle$ in D that converges to g. Thus $X = [D]_{seq}$. Therefore, $C_p(X)$ is strongly sequentially dense. □

Observation 5.3. It is clear that $S_{fin}(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ implies that $\begin{pmatrix} \mathcal{O}^X_{\mathfrak{B}} \\ \Gamma^X_{\mathfrak{B}} \end{pmatrix}$ holds. Therefore, by Theorem 2.43, we have that $S_{fin}(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}}) = S_1(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$.

By the previous observation, the following result follows:

Theorem 5.4. Let (X, τ) be a Tychonoff space with $i\omega(X) = \aleph_0$, $\mathfrak{B}$ be a bornology with compact base and $g \in C_{\mathfrak{B}}(X)$. The following statements are equivalent:

1. $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{S}_{C_{\mathfrak{B}}(X)})$ holds;
2. $C_{\mathfrak{B}}(X)$ is strongly sequentially dense;
3. $S_1(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds;
4. $S_1(\Omega_g, \Gamma_g)$ holds in $C_{\mathfrak{B}}(X)$;
5. $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \Gamma_g)$ holds;
6. $S_{fin}(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{S}_{C_{\mathfrak{B}}(X)})$ holds;
7. $S_{fin}(\mathcal{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds;

8. $S_{fin}(\Omega_g, \Gamma_g)$ holds in $C_{\mathfrak{B}}(X)$;

9. $S_{fin}(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \Gamma_g)$ holds.

Proof. $(1) \Rightarrow (2)$. Let D be a dense subset of $C_{\mathfrak{B}}(X)$. Let $\langle D_n : n \in \omega \rangle$ be the sequence such that $D_n = D$, for all $n \in \omega$. It follows from (1) that there is a sequence $\langle f_n : n \in \omega \rangle$ of elements in D such that $\{f_n : n \in \omega\} \in \mathscr{S}_{C_{\mathfrak{B}}(X)}$. This implies that D is sequentially dense in X.

$(2) \Rightarrow (1)$. Let $\langle D_n : n \in \omega \rangle$ be a sequence of elements in $\mathscr{D}_{C_{\mathfrak{B}}(X)}$. By Theorem 5.2, $S_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds. It is clear that $\Gamma^X_{\mathfrak{B}} \subset \mathscr{O}^X_{\mathfrak{B}}$. Therefore, $S_1(\mathscr{O}^X_{\mathfrak{B}}, \mathscr{O}^X_{\mathfrak{B}})$ holds. By Theorem 4.43, it follows that $S_1(\mathscr{D}_{C_{\mathfrak{B}}(X)}, \mathscr{D}_{C_{\mathfrak{B}}(X)})$ holds. Then, there is a sequence $\langle f_n : n \in \omega \rangle$ of elements such that, for all $n \in \omega$, $f_n \in D_n$ and $\{f_n : n \in \omega\} \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$. Finally, by (2), $\{f_n : n \in \omega\}$ is sequentially dense, that is, $\{f_n : n \in \omega\} \in \mathscr{S}_{C_{\mathfrak{B}}(X)}$.

$(2) \Leftrightarrow (3)$. It is Theorem 5.2.

$(3) \Leftrightarrow (4)$. It is part of Theorem 2.43.

$(4) \Rightarrow (5)$ and $(8) \Rightarrow (9)$. It follows directly from the fact that $\mathscr{D}_{C_{\mathfrak{B}}(X)} \subseteq \Omega_g$.

$(5) \Rightarrow (2)$. Let $D \in \mathscr{D}_{C_{\mathfrak{B}}(X)}$ and $g \in C_{\mathfrak{B}}(X)$. Using (5) for the constant sequence whose only element is D, there is a sequence $\langle f_n : n \in \omega \rangle$ of elements in $\mathscr{D}_{C_{\mathfrak{B}}(X)}$ that converges to g. Therefore, D is sequentially dense.

$(7) \Leftrightarrow (8)$. This is part of Theorem 2.45.

The implications $(1) \Rightarrow (6)$ and $(3) \Rightarrow (7)$ are immediate.

$(9) \Rightarrow (2)$. Similar to $(5) \Rightarrow (2)$.

$(6) \Rightarrow (2)$. Similar to $(1) \Rightarrow (2)$. $\square$

5.2 Other equivalences of Games and selection principle

We denote, for all $n \in \omega$, $I_n = \langle -\frac{1}{n+1}, \frac{1}{n+1} \rangle$. We can obtain the game version of Theorem 2.43:

Theorem 5.5. Let (X,τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with a compact base. Then $II \uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ if, and only if, $II \uparrow G_1(\Omega_g, \Gamma_g)$ in $C_{\mathfrak{B}}(X)$, for all $g \in C_{\mathfrak{B}}(X)$.

Proof. As $C_{\mathfrak{B}}(X)$ is a homogeneous space, it suffices to prove the case $g = o$.

Let σ be a winning strategy for Player II in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$. For any $A \in \Omega_o$, define $\mathscr{U}_n(A) = \{f^{-1}(I_n) : f \in A\}$. We have $\mathscr{U}_n(A) \in \mathscr{O}^X_{\mathfrak{B}}$. We define the strategy ρ, for Player II in the game $G_1(\Omega_o, \Gamma_o)$, in the following way: $\rho(\langle A_0, .., A_n\rangle) = f_n \in A_n$, where $f_n^{-1}(I_n) = \sigma(\langle \mathscr{U}_0(A_0), ..., \mathscr{U}_n(A_n)\rangle)$.

We claim that ρ is a winning strategy. Indeed, consider

$$\langle A_0, \rho(\langle A_0\rangle), ..., A_n, \rho(\langle A_0, ..., A_n\rangle), ...\rangle$$

a play in $G_1(\Omega_o, \Gamma_o)$. Then

$$\langle \mathscr{U}_0(A_0), \sigma(\langle \mathscr{U}_0(A_0)\rangle), ..., \mathscr{U}_n(A_n)), \sigma(\langle \mathscr{U}_0(A_0), ..., \mathscr{U}_n(A_n)\rangle), ...\rangle$$

is a play in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$. As σ is a winning strategy, it follows that $\{C_n : n \in \omega\} \in \Gamma^X_{\mathfrak{B}}$, where $C_n = \sigma(\langle \mathscr{U}_0(A_0), ..., \mathscr{U}_n(A_n)\rangle) = f_n^{-1}(I_n)$, for some $f_n \in A_n$ and for all $n \in \omega$.

We claim that $\{f_n : n \in \omega\} \in \Gamma_o$. Indeed, we prove that $o = \lim\{f_n : n \in \omega\}$. This set is infinite, because $\{C_n : n \in \omega\}$ is infinite. Also, $o \notin \{f_n : n \in \omega\}$, because $o \notin A_n$, for all $n \in \omega$. Let $\langle o, B, \varepsilon\rangle$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Choose an $n_1 \in \omega$ such that $\frac{1}{n_1+1} < \varepsilon$.

So, there is $n_2 \in \omega$ such that $B \subset C_k$, for all $k \geq n_2$. Taking $n_0 = \max\{n_1, n_2\}$, we have $B \subset C_k = f_k^{-1}(I_n) \subset f_k^{-1}(\langle -\varepsilon, \varepsilon\rangle)$, for all $k \geq n_0$. This is, $f_k \in \langle o, B, \varepsilon\rangle$. for all $k \geq n_0$. Therefore, $\{f_n : n \in \omega\} \setminus \langle o, B, \varepsilon\rangle$ is finite.

Reciprocally, let σ be a winning strategy for player II in game $G_1(\Omega_o, \Gamma_o)$. Let $\mathscr{U} \in \mathscr{O}^X_{\mathfrak{B}}$, define $A(\mathscr{U}) = \{f \in C(X) : f(X \setminus U) \equiv 1$, for some $U \in \mathscr{O}^X_{\mathfrak{B}}\}$.

Using that X is Tychonoff and that $\overline{B} \in \mathfrak{B}$ is compact, by Theorem 2.28, it follows that $A(\mathscr{U}) \in \Omega_o$. We define a strategy ρ for Player II in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ as follows: $\rho(\langle \mathscr{U}_0, ..., \mathscr{U}_n\rangle) = U_n \in \mathscr{U}_n$, where $f_n(X \setminus U_n) \equiv 1$ and $f_n = \sigma(\langle A_0(\mathscr{U}_0), ..., A_n(\mathscr{U}_n)\rangle)$.

Let

$$\langle \mathscr{U}_0, \rho(\langle \mathscr{U}_0\rangle), ..., \mathscr{U}_n, \rho(\langle \mathscr{U}_0, ..., \mathscr{U}_n\rangle), ...\rangle$$

be a play in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$. Then

$$\langle A_0(\mathscr{U}_0), \sigma(\langle A_0(\mathscr{U}_0)\rangle), ..., A_n(\mathscr{U}_n), \sigma(\langle A_0(\mathscr{U}_0), ..., A_n(\mathscr{U}_n)\rangle), ...\rangle$$

is a play in the game $G_1(\Omega_o, \Gamma_o)$. As σ is a winning strategy, we have $\{f_n : n \in \omega\} \in \Gamma_o$, where $f_n = \sigma(\langle A_0(\mathscr{U}_0), ..., A_n(\mathscr{U}_n)\rangle) \in A_n(\mathscr{U}_n)$, for all $n \in \omega$.

We claim that $\{\tau(\langle \mathscr{U}_0, ..., \mathscr{U}_n\rangle) = U_n : n \in \omega\} \in \Gamma^X_{\mathfrak{B}}$. Indeed, the set is infinite, because $\{f_n : n \in \omega\}$ is infinite. Let $B \in \mathfrak{B}$. As $\{f_n : n \in \omega\} \setminus \langle o, B, 1\rangle$ is finite, it follows that there is $n_0 \in \omega$ such that $f_k \in \langle o, B, 1\rangle$, for all $k \geq n_0$. Then $|f_k(x)| < 1$, for all $k \geq n_0$ and $x \in B$. Thus, $B \cap (X \setminus U_k) = \emptyset$, for all $k \geq n_0$. Therefore, $B \subset U_k$, for all $k \geq n_0$. Then $\{U \in \{U_n : n \in \omega\} : B \not\subset U\}$ is finite. This concludes the proof. □

With few modifications, we can obtain the following results:

Theorem 5.6. Let (X, τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base, and let $f : \omega \to \omega \setminus \{0\}$ be a function. Then $II \uparrow G_f(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ if, and only if, $II \uparrow G_f(\Omega_g, \Gamma_g)$ in $C_{\mathfrak{B}}(X)$, for all $g \in C_{\mathfrak{B}}(X)$.

Theorem 5.7. Let (X, τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with a compact base. Then $II \uparrow G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ if, and only if, $II \uparrow G_{fin}(\Omega_g, \Gamma_g)$ in $C_{\mathfrak{B}}(X)$, for all $g \in C_{\mathfrak{B}}(X)$.

Let us remember the following result of the equivalence between selection principle and game:

Theorem 5.8. (SCHEEPERS, 1994) Let (X, τ) be a ω-Lindelöf space. The following statements are equivalent:

1. $S_1(\Omega_X, \Gamma_X)$ holds;
2. $I \not\uparrow G_{fin}(\Omega_X, \Gamma_X)$.

Using that $S_1(\Omega, \Gamma) = S_{fin}(\Omega, \Gamma)$, we obtain the following results,

Corollary 5.9. Let (X, τ) be a ω-Lindelöf space. The following statements are equivalent:

1. $S_{fin}(\Omega_X, \Gamma_X)$ holds;
2. $I \not\uparrow G_{fin}(\Omega_X, \Gamma_X)$.

Corollary 5.10. Let (X, τ) be a ω-Lindelöf space. The following statements are equivalent:

1. $I \not\uparrow G_1(\Omega_X, \Gamma_X)$;
2. $I \not\uparrow G_{fin}(\Omega_X, \Gamma_X)$.

The previous results can be extended for the context of more general covering families

Theorem 5.11. Let (X, τ) be a $\mathfrak{B}$-Lindelöf space, with $\mathfrak{B}$ a family of subsets of X. The following statements are equivalent:

1. $S_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds;

2. $I \not\uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$.

Proof. The implication $(2) \Rightarrow (1)$ is clear. Therefore, it is sufficient to prove $(1) \Rightarrow (2)$. Let σ be a strategy for Player I in $G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$. We define specific elements of $\mathscr{O}^X_{\mathfrak{B}}$ as follows:

- Let $\mathscr{U}_\emptyset = \{U_{\langle n \rangle} : n \in \omega\}$ be an enumeration of $\sigma(\emptyset) \in \mathscr{O}^X_{\mathfrak{B}}$.
- For all $m \geq 1$, we assume that U_α, is defined for all $\alpha \in {}^{<\omega}\omega$ such that $|\alpha| = k$. We can write $\alpha = \langle m_1, .., m_k \rangle$. We define

$$\mathscr{U}_\alpha = \sigma(\langle U_{\langle m_1 \rangle}, U_{\langle m_1, m_2 \rangle} \dots, U_\alpha \rangle) \setminus \{U_{\langle m_1 \rangle}, U_{\langle m_1, m_2 \rangle}, \dots, U_\alpha\} = \{U_{\alpha \frown n} : n \in \omega\} \in \mathscr{O}^X_{\mathfrak{B}}$$

.

- Consider $\langle \mathscr{U}_\alpha : \alpha \in {}^{<\omega}\omega \rangle$. By (2), it follows that for all α, there is n_α such that $\mathscr{W} = \{U_{\alpha \frown n_\alpha} : \tau \in {}^{<\omega}\omega\} \in \Gamma^X_{\mathfrak{B}}$. Define a sequence of positive integers $n_1, n_2, \dots$ such that $n_1 = n_\emptyset$ and $n_{k+1} = n_{\langle n_1, \dots, n_k \rangle}$, for all $k \geq 1$. Then, the sequence:

$$U_{\langle n_1 \rangle}, U_{\langle n_1, n_2 \rangle}, \dots$$

is a strategy of Player *II* that defeats σ, because $\mathscr{V} = \{U_{\langle n_1, \dots, n_k \rangle} : k \geq 1\} \in \Gamma^X_{\mathfrak{B}}$, since it is an infinite subset of $\mathscr{W}$. Thus, σ is not a winning strategy.

□

With few modifications to the previous theorem, we can obtain the following result:

Theorem 5.12. Let (X, τ) be a $\mathfrak{B}$-Lindelöf space, where $\mathfrak{B}$ is a family of subsets of X. The following statements are equivalent:

1. $S_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$ holds;
2. $I \not\uparrow G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$.

By the Observation 5.3, we can obatin the following result.

Corollary 5.13. Let (X, τ) be a $\mathfrak{B}$-Lindelöf space, where $\mathfrak{B}$ is a family of subsets of X. The following statements are equivalent:

1. $I \not\uparrow G_1(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$;
2. $I \not\uparrow G_{fin}(\mathscr{O}^X_{\mathfrak{B}}, \Gamma^X_{\mathfrak{B}})$.

5.3 Games about n-denses sets

Definition 5.14. A subset $A \subseteq C_p(X)$ is called $n-$dense in $C_p(X)$, if for all $n-$finite set $\{x_1,...,x_n\} \subset X$ such that $x_i \neq x_j$, for $i \neq j$, and opens sets $W_1,...,W_n$ in $\mathbb{R}$ there is a $g \in A$ such that $g(x_i) \in W_i$, for all $i = 1,...,n$.

Note that if A is n-dense, for all $n \in \omega$, then A is dense.

Denote by $\mathscr{A}_n$ the family of n dense subsets of $C_p(X)$ and $\mathscr{A}$ instead of $\mathscr{A}_1$.

Definition 5.15. Let $f \in C(X)$. A subset $B \subseteq C_p(X)$ is called n-dense at a point g, if for all n-finite sets $\{x_1,...,x_n\} \subset X$ and $\varepsilon > 0$ there is $h \in B$ such that $h(x_i) \in (g(x_i) - \varepsilon, g(x_i) + \varepsilon)$, for $i \in \{1,...,n\}$.

Note that if, for all $n \in \omega$, B is n-dense in a point g, then $f \in \overline{B}$.

Denote by $\mathscr{B}_{n,g}$ the family of all n-dense sets at a point g, and write $\mathscr{B}_g$ instead of $\mathscr{B}_{1,g}$.

Let $\mathscr{U}$ be an open cover of X and $n \in \omega$. We say that $\mathscr{U}$ is a n-cover of X if, for any $F \subset X$ with $|F| \leq n$, there is $U \in \mathscr{U}$ such that $F \subset U$.

Denote by $\mathscr{O}_n$ the family of all n-cover of X.

In (OSIPOV, 2018b), the following results were obtained from the equivalences of the selection principles with respect to the families defined above,

Theorem 5.16. Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $S_1(\mathscr{A},\mathscr{A})$ holds;

2. $S_1(\mathscr{O}_X,\mathscr{O}_X)$ holds;

3. $S_1(\mathscr{B}_g,\mathscr{B}_g)$ holds for all $g \in C_p(X)$;

4. $S_1(\mathscr{A},\mathscr{B}_g)$ holds, for all $g \in C_p(X)$.

Theorem 5.17. Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $S_{fin}(\mathscr{A},\mathscr{A})$ holds;

2. $S_{fin}(\mathscr{O}_X,\mathscr{O}_X)$ holds;

3. $S_{fin}(\mathscr{B}_g,\mathscr{B}_g)$ holds for all $g \in C_p(X)$;

4. $S_{fin}(\mathscr{A},\mathscr{B}_g)$ holds, for all $g \in C_p(X)$.

We can obtain the version of the previous theorems in its game version:

Theorem 5.18. Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $II \uparrow G_1(\mathscr{A}, \mathscr{A})$;

2. $II \uparrow G_1(\mathscr{O}_X, \mathscr{O}_X)$;

3. $II \uparrow G_1(\mathscr{B}_g, \mathscr{B}_g)$, for all $g \in C_p(X)$;

4. $II \uparrow G_1(\mathscr{A}, \mathscr{B}_g)$, for all $g \in C_p(X)$.

Proof. (1) $\Rightarrow$ (2). Let σ be a winning strategy for Player *II* in $G_1(\mathscr{A}, \mathscr{A})$. We define a strategy for Player *II* in $G_1(\mathscr{O}_X, \mathscr{O}_X)$.

First, note that for any $\mathscr{U} \in \mathscr{O}_X$ we have $A(\mathscr{U}) = \{f \in C_p(X) : f(X \backslash U) \equiv 1$ and $f \restriction K \equiv q$, for some $U \in \mathscr{U}, K \subset U$ finite, and $q \in \mathbb{Q}\} \in \mathscr{A}$. Indeed, let $x \in X$ and $W \subset \mathbb{R}$ be an open. Let $q \in \mathbb{Q} \cap W$. As $\mathscr{U} \in \mathscr{O}_X$, there is $U \in \mathscr{U}$ such that $x \in U$. Because $X \backslash U$ is closed and X is Tychonoff, it follows that there is $h \in C(X)$ such that $h(X \backslash U) \equiv 1$ and $h(x) = q \in W$. That is, $h \in A(\mathscr{U})$.

In the inning $0 \in \omega$ suppose that Player *I* in $G_1(\mathscr{O}_X, \mathscr{O}_X)$ chooses $\mathscr{U}_0 \in \mathscr{O}_X$. Then, suppose that Player *I*, in $G_1(\mathscr{A}, \mathscr{A})$, chooses $A(\mathscr{U}_0) \in \mathscr{A}$. Consider $\sigma(\langle A(\mathscr{U}_0)\rangle) := f_0 \in A(\mathscr{U}_0)$. So, we define $\rho(\langle \mathscr{U}_0 \rangle) = U_0$, where U_0 is open such that $f_0(X \backslash U_0) \equiv 1$.

In the inning $n \geq 1$ suppose that Player *I* in $G_1(\mathscr{O}_X, \mathscr{O}_X)$, choose $\mathscr{U}_n \in \mathscr{O}_X$. Then, suppose that Player *I* in $G_1(\mathscr{A}, \mathscr{A})$, choose $A(\mathscr{U}_n) \in \mathscr{A}$. Consider $\sigma(\langle A(\mathscr{U}_0), ..., A(\mathscr{U}_n)\rangle) := f_n \in A(\mathscr{U}_n)$. So, we define $\rho(\langle \mathscr{U}_0, ..., \mathscr{U}_n \rangle) = U_n$, where U_n is open such that $f_n(X \backslash U_n) \equiv 1$.

As σ is a winning strategy, we have $\{f_n : n \in \omega\} \in \mathscr{A}$. We claim that $\{U_n : n \in \omega\} \in \mathscr{O}_X$. Indeed, let $x \in X$. Consider $W = (-\frac{1}{2}, \frac{1}{2})$. Then, there is $m \in \omega$ such that $f_m(x) \in W$. So, $x \notin X \backslash U_m$, that is, $x \in U_m$. Thus, ρ is a winning strategy for Player *II* in $G_1(\mathscr{O}_X, \mathscr{O}_X)$.

(2) $\Rightarrow$ (3). Due to the homogeneity of $C_p(X)$, we can assume that $g \equiv 0$. Let σ be a winning strategy for Player *II* in $G_1(\mathscr{O}_X, \mathscr{O}_X)$. We define a strategy ρ for Player *II* in $G_1(\mathscr{B}_0, \mathscr{B}_0)$.

First, note that if $B \in \mathscr{B}_0$ we have that, for any $n \in \omega$, $\mathscr{U}_n(B) = \{g^{-1}((-\frac{1}{n}, \frac{1}{n})) : g \in B\} \in \mathscr{O}_X$. Indeed, let $x \in X$. Taking into account $\varepsilon = \frac{1}{n}$, we see that there is a $g \in B$ such that $g(x) \in (-\frac{1}{n}, \frac{1}{n})$. This is, $x \in g^{-1}((-\frac{1}{n}, \frac{1}{n})) \in \mathscr{U}_n(B)$.

Let $\{M_k : k \in \omega\}$ be a partition of ω in infinite sets and $\{p_k : k \in \omega\}$ an increasing sequence of prime numbers. Suppose that, for all $k \in \omega$, $M_k = \{m_{i,k} : i \in \omega\}$.

For any sequence $\alpha = \langle B_0, ..., B_k \rangle \in {}^{<\omega}\mathscr{B}_0$, with $k \in \omega$, consider the sequence α' whose elements are $\mathscr{U}_{p_{k_m}}(B_i)$, where B_i are elements of α with sub-indices in M_{m_k} and $m_k \in \omega$ is such that $k \in M_{m_k}$. We define:

$$\rho(\alpha) = g_k,$$

where g_k is a function such that $\sigma(\alpha') = g_k^{-1}((-\frac{1}{p_{m_k}}, \frac{1}{p_{m_k}})) \in \mathscr{U}_{p_{m_k}}(B_k)$.

Then $\rho : {}^{<\omega}\mathscr{B}_0 \to \bigcup \mathscr{B}_0$ is a strategy for Player *II* in $G_1(\mathscr{B}_0, \mathscr{B}_0)$.

We claim that ρ is a winning strategy. Indeed, consider the following play

$$\langle B_0, \rho(\langle B_0\rangle), B_1, \rho(\langle B_0, B_1\rangle), ..., B_n, \rho(\langle B_0, ..., B_n\rangle), ...\rangle,$$

in $G_1(\mathscr{B}_0, \mathscr{B}_0)$.

Consider a re-indexing of the sequence $\langle B_n : n \in \omega\rangle = \langle B_{i,k} : i,k \in \omega\rangle$, where $B_n = B_{i,k}$ if $n = m_{i,k}$, with $i,k \in \omega$. For all $k \in \omega$, consider the subsequences $\alpha_k = \langle B_{i,k} : i \in \omega\rangle$. As σ is a winning strategy, it follows that for all $k \in \omega$, $\mathscr{V}_k = \{\sigma(\alpha_k \upharpoonright i) = g_{i,k}^{-1}((\frac{1}{p_k}, \frac{1}{p_k})) : i \in \omega\} \in \mathscr{O}_X$. We claim that $\{g_n : n \in \omega\} \in \mathscr{B}_0$.

Indeed, let $x \in X$ and $W = (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$. Choose $k \in \omega$ such that $\frac{1}{p_k} < \varepsilon$. So, as $\mathscr{V}_k \in \mathscr{O}_X$, there is $i \in \omega$ such that $x \in g_{i,k}^{-1}((-\frac{1}{p_k}, \frac{1}{p_k}))$. That is $g_{i,k}(x) \in (-\frac{1}{p_k}, \frac{1}{p_k}) \subset (-\varepsilon, \varepsilon)$. Therefore, ρ is a winning strategy for Player *II* in the game $G_1(\mathscr{B}_0, \mathscr{B}_0)$ on $C_p(X)$.

(3) $\Rightarrow$ (4). It is evident from that $\mathscr{A} \subseteq \mathscr{B}_g$, for all $g \in C_p(X)$.

(4) $\Rightarrow$ (1). Let σ_g be a winning strategy for Player *II* in $G_1(\mathscr{A}, \mathscr{B}_g)$. Consider $\{M_k : k \in \omega\}$ be a partition of natural numbers in infinite sets and $\{q_k : k \in \omega\}$ is an enumeration of rational numbers $\mathbb{Q}$. Suppose that for all $k \in \omega$, $M_k = \{m_{i,k} : i \in \omega\}$.

For any sequence $\alpha = \langle A_0, ..., A_k\rangle \in {}^{<\omega}\mathscr{A}$, with $k \in \omega$, consider the sequence α' whose elements are elements of α with sub-indices in M_{m_k} e $m_k \in \omega$ such that $k \in M_{m_k}$. With the final elements of α and α' being the same, we can define:

$$\rho(\alpha) = \sigma_{f_k}(\alpha') \in A_k,$$

where f_k is a constant function equal to q_k.

Then $\rho : {}^{<\omega}\mathscr{A} \to \bigcup \mathscr{A}$ is a strategy for Player *II* in $G_1(\mathscr{B}_0, \mathscr{B}_0)$.

We claim that ρ is a winning strategy. Indeed, consider the following play

$$\langle A_0, \rho(\langle A_0\rangle), A_1, \rho(\langle A_0, A_1\rangle), ..., A_n, \rho(\langle A_0, ..., A_n\rangle), ...\rangle,$$

in $G_1(\mathscr{A}, \mathscr{A})$.

Consider a re-indexing of sequence $\langle A_n : n \in \omega\rangle = \langle A_{i,k} : i,k \in \omega\rangle$, where $A_n = A_{i,k}$ if $n = m_{i,k}$, with $i,k \in \omega$. For all $k \in \omega$, consider the subsequences $\alpha_k = \langle A_{i,k} : i \in \omega\rangle$.

Since, for all $k \in \omega$, σ_{f_k} is a winning strategy, it follows that $\{\sigma_{f_k}(\alpha_k \upharpoonright i) = f_{i,k} : i \in \omega\} \in \mathscr{B}_{f_k}$. We claim that $\{f_{i,k} : i \in \omega, k \in \omega\} \in \mathscr{A}$. Indeed, let $x \in X$ and $W \subset \mathbb{R}$ be an open. Choose $q_k \in \mathbb{Q} \cap W$. Since W is open, there is $\varepsilon > 0$ such that $(q_k - \varepsilon, q_k + \varepsilon) \subset W$. So, there is $i \in \omega$ such that $f_{i,k}(x) \in (q_k - \varepsilon, q_k + \varepsilon) \subset W$.

Therefore, ρ is a winning strategy for Player *II* in $G_1(\mathscr{A}, \mathscr{A})$. $\square$

With a few modifications, we can obtain the following additional results:

Theorem 5.19. Let (X,τ) be a Tychonoff space and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. $II\uparrow G_f(\mathscr{A},\mathscr{A})$;
2. $II\uparrow G_f(\mathscr{O}_X,\mathscr{O}_X)$;
3. $II\uparrow G_f(\mathscr{B}_g,\mathscr{B}_g)$, for all $g\in C_p(X)$;
4. $II\uparrow G_f(\mathscr{A},\mathscr{B}_f)$, for all $g\in C_p(X)$.

Theorem 5.20. Let (X,τ) be a Tychonoff space. The following statements are equivalent:

1. $II\uparrow G_{fin}(\mathscr{A},\mathscr{A})$;
2. $II\uparrow G_{fin}(\mathscr{O}_X,\mathscr{O}_X)$;
3. $II\uparrow G_{fin}(\mathscr{B}_g,\mathscr{B}_g)$, for all $g\in C_p(X)$;
4. $II\uparrow G_{fin}(\mathscr{A},\mathscr{B}_f)$, for all $g\in C_p(X)$.

CHAPTER

6

VARIATIONS IN SELECTIVELY CCC PROPERTY AND GAMES

6.1 Selectively ccc

In (AURICHI, 2013) it is introduced a selectively-ccc property.

This property is equivalent to the selection principle $S_1(\mathscr{M}_c, \mathscr{D}_0)$. In this section, we study different connections between the selection principle and our game version, in order to obtain, in the following section, the same equivalences in the star version. We start with the following:

Observation 6.1. For a topological space (X, τ), note that $\mathscr{M}_c \subset \mathscr{D}_0$. Indeed, let $\mathscr{A} \in \mathscr{M}_c$ and suppose that $\bigcup \mathscr{A}$ is not dense. Then there is $B \in \tau$ such that $B \cap (\bigcup \mathscr{A}) = \emptyset$. In particular, $B \notin \mathscr{A}$. So, $\mathscr{A} \subsetneq \mathscr{A} \cup \{B\}$ and $\mathscr{A} \cup \{B\}$ is a cellular family. This contradicts the maximality of $\mathscr{A}$. Therefore $\mathscr{A} \in \mathscr{D}_0$.

Therefore, in addition to the fact that every element in $\mathscr{D}_0$ admits a refinement that is a maximal cellular family, the following result follows.

Lemma 6.2. (AURICHI, 2013) Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_1(\mathscr{M}_c, \mathscr{D}_0)$ holds;
2. $S_1(\mathscr{D}_0, \mathscr{D}_0)$ holds.

More generally, we have the following.

Theorem 6.3. Let (X, τ) be a topological space and let $f : \omega \to \omega \setminus \{0\}$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{M}_c, \mathscr{D}_0)$ holds;

2. $S_f(\mathscr{D}_0, \mathscr{D}_0)$ holds.

Theorem 6.4. Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\mathscr{M}_c, \mathscr{D}_0)$ holds;

2. $S_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ holds.

Taking into account the game $G_1(\mathscr{M}_c, \mathscr{D}_0)$, by the same observations made, we can obtain the following results:

Theorem 6.5. Let(X, τ) be a topological space and let $f : \omega \to \omega \setminus \{0\}$ be a function. The following statements are equivalent:

1. $II \uparrow G_f(\mathscr{M}_c, \mathscr{D}_0)$ $(I \not\uparrow G_f(\mathscr{M}_c, \mathscr{D}_0))$;

2. $II \uparrow G_f(\mathscr{D}_0, \mathscr{D}_0)$ $(I \not\uparrow G_f(\mathscr{M}_c, \mathscr{D}_0))$.

Theorem 6.6. Let (X, τ) be a topological space and let $f : \omega \to \omega \setminus \{0\}$ be a function. The following statements are equivalent:

1. $II \uparrow G_{fin}(\mathscr{M}_c, \mathscr{D}_0)$ $(I \not\uparrow G_{fin}(\mathscr{M}_c, \mathscr{D}_0))$;

2. $II \uparrow G_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ $(I \not\uparrow G_{fin}(\mathscr{M}_c, \mathscr{D}_0))$.

In addition, we have the following characterizations of the game version of the selection principle. First, we show the fin case.

Theorem 6.7. (SCHEEPERS, 2000) Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ holds;

2. $I \not\uparrow G_{fin}(\mathscr{D}_0, \mathscr{D}_0)$.

Proof. Is sufficient to prove that $(1) \Rightarrow (2)$. Let σ be a strategy for Player I in $G_{fin}(\mathscr{D}_0, \mathscr{D}_0)$. By hypothesis, we can assume that σ chooses only countable many elements of $\mathscr{D}_0$.

Also, we can assume that σ chooses families containing the family of the previous inning. Indeed, if σ is a normal strategy in $G_{fin}(\mathscr{D}_0, \mathscr{D}_0)$, we define a new increasing strategy φ as follows. Put $\varphi(\emptyset) = \sigma(\emptyset) \in \mathscr{D}_0$. Suppose that Player II chooses $\mathscr{V}_0 \in [\varphi(\emptyset)]^{<\aleph_0}$. Suppose that Player II chooses $\mathscr{V}_0' = \mathscr{V}_0$ in the game where Player I uses strategy σ.

Define $\varphi(\langle \mathscr{V}_0 \rangle) = \sigma(\emptyset) \cup \sigma(\langle \mathscr{V}_0' \rangle) \in \mathscr{D}_0$. Player *II* chooses $\mathscr{V}_1 \in [\varphi(\mathscr{V}_0)]^{<\aleph_0}$. Suppose that Player *II* chooses $\mathscr{V}_1' = \mathscr{V}_1 \cap \sigma(\langle \mathscr{V}_0' \rangle)$ in the game where Player *I* uses the strategy σ. Define

$$\varphi(\langle \mathscr{V}_0, \mathscr{V}_1 \rangle) = \sigma(\emptyset) \cup \sigma(\langle \mathscr{V}_0' \rangle) \cup \sigma(\langle \mathscr{V}_0', \mathscr{V}_1' \rangle).$$

Player *II* chooses $\mathscr{V}_2 \in [\varphi(\langle \mathscr{V}_0, \mathscr{V}_1 \rangle)]^{<\aleph_0}$. Suppose that Player *II* chooses $\mathscr{V}_2' = \mathscr{V}_2 \cap \sigma(\langle \mathscr{V}_0', \mathscr{V}_1' \rangle)$ in the game where Player *I* uses the strategy σ. Define

$$\varphi(\langle \mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2 \rangle) = \sigma(\emptyset) \cup \sigma(\langle \mathscr{V}_0' \rangle) \cup \sigma(\langle \mathscr{V}_0', \mathscr{V}_1' \rangle) \cup \sigma(\langle \mathscr{V}_0', \mathscr{V}_1', \mathscr{V}_2' \rangle),$$

and so on, for all inning $n \in \omega$. Then, if Player *I* has no winning strategy using increasing sequences, we would have $\bigcup_{n\in\omega} \mathscr{V}_n \notin \mathscr{D}_0$.

Note that $\bigcup_{n\in\omega} \mathscr{V}_n' \subset \bigcup_{n\in\omega} \mathscr{V}_n$. Then $\bigcup_{n\in\omega} \mathscr{V}_n' \notin \mathscr{D}_0$, that is, *I* has no winning strategy using normal strategies.

Finally, we can assume that σ has the following property: for all sequences $\langle \mathscr{V}_0, ..., \mathscr{V}_n \rangle$, where $\mathscr{V}_j$ is a finite family, we have $\bigcup_{j=0}^{n} \bigcup \mathscr{V}_j \subset A$, for all $A \in \sigma(\langle \mathscr{V}_0, ..., \mathscr{V}_n \rangle)$. Indeed, let σ be a normal strategy for Player *I*. Define $\varphi(\emptyset) = \sigma(\emptyset) \in \mathscr{D}_0$. Suppose that Player *II* chooses $\mathscr{V}_0 \in [\varphi(\emptyset)]^{<\aleph_0}$. Now, suppose that Player *II* chooses $\mathscr{V}_0' = \mathscr{V}_0$ in the game where Player *I* uses a strategy σ. Define $\varphi(\langle \mathscr{V}_0 \rangle) = \{A \cup (\bigcup \mathscr{V}_0') : A \in \sigma(\langle \mathscr{V}_0' \rangle)\} \in \mathscr{D}_0$.

Suppose that Player *II* chooses $\mathscr{V}_1 \in [\varphi(\mathscr{V}_0)]^{<\aleph_0}$. So, suppose that Player *II* chooses $\mathscr{V}_1' = \{A : A \cup (\bigcup \mathscr{V}_0') \in \mathscr{V}_1\}$ in the game where Player *I* uses the strategy σ. Define

$$\varphi(\langle \mathscr{V}_0, \mathscr{V}_1 \rangle) = \{A \cup \left(\bigcup_{j=0}^{1} \mathscr{V}_j' \right) : A \in \sigma(\langle \mathscr{V}_0', \mathscr{V}_1' \rangle)\},$$

and so on, for all innings $n \in \omega$. Then, if Player *I* does not have a winning strategy with the above property, we would have $\bigcup_{n\in\omega} \mathscr{V}_n \notin \mathscr{D}_0$. But $\bigcup_{n\in\omega} \mathscr{V}_n = \bigcup_{n\in\omega} \mathscr{V}_n'$. Therefore, Player *I* will not have a winning strategy using normal strategies.

Thus, we can consider σ, the strategy of Player *I*, who chose increasing sequences of countable elements in $\mathscr{D}_0$ such that the union of the elements played by Player *II*, until the inning $n \in \omega$, is contained in each of the elements of the move made by Player *I* in the inning $n+1$. We denote by $\sigma(\emptyset) = \{U_n : n \in \omega\}$. For all $n_1 \in \omega$, $\sigma(\langle U_{n_1} \rangle) = \{U_{\langle n_1 \rangle ^\frown n} : n \in \omega\}$. For any $n_2 \in \omega$, $\sigma(\langle U_{n_1}, U_{(n_1)^\frown n_2} \rangle) = \{U_{\langle n_1, n_2 \rangle ^\frown n} : n \in \omega\}$, and so on. Thus, for all $\gamma \in^{<\omega} \omega$, we have:

1. If $m < n$, $U_{\gamma^\frown m} \subseteq U_{\gamma^\frown n}$;

2. For all $n \in \omega$, $U_\gamma \subseteq U_{\gamma^\frown n}$;

3. $\{U_{\gamma^\frown n} : n \in \omega\} \in \mathscr{D}_0$.

For any $n, k \in \omega$, we define:

$$U_k^n = \begin{cases} U_k, & \text{if } n = 0; \\ \left(\bigcap\{U_{\gamma^\frown n} : \gamma \in {}^{n-1}\omega\}\right) \cap U_k^{n-1}, & \text{otherwise.} \end{cases}$$

By the properties above, it follows that, for each $n \in \omega$, $\mathscr{U}_n = \{U_k^n : k \in \omega\}$ is an increasing family of open sets. Also, for each $n \in \omega$, $\mathscr{U}_n \in \mathscr{D}_0$. Indeed, let U be a non-empty element in τ. By property 3 above, we have that there are finite $\gamma \in {}^n\omega$ such that $U \cap U_\gamma = \emptyset$. So, there is a $k \in \omega$ such that $U_k^n \cap U \neq \emptyset$.

Applying $S_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ to the sequence $\langle \mathscr{U}_n : n \in \omega \rangle$, there are $\mathscr{V}_n \in [\mathscr{U}_n]^{<\aleph_0}$ such that $\bigcup_{n\in\omega} \mathscr{V}_n \in \mathscr{D}_0$. As $\mathscr{U}_n$ is an increasing family, it follows that there is $k_n \in \omega$ such that $\bigcup \mathscr{V}_n = U_{k_n}^n$ and $\{U_{k_n}^n : n \in \omega\} \in \mathscr{D}_0$.

By the construction of the open sets U_k^n, we have $U_{k_n}^n \subseteq U_{\langle k_1,\dots,k_n\rangle}$, for all $n \in \omega$. Thus, $\{U_{\langle k_1,\dots,k_n\rangle} : n \in \omega\} \in \mathscr{D}_0$. So if, in each inning $n \in \omega$ of the game where Player *I* uses the strategy σ, Player *II* chooses the open set $U_{\langle k_1,\dots,k_n\rangle}$ then Player *II* is the winner. Therefore, $I \not\uparrow G_1(\mathscr{D}_0, \mathscr{D}_0)$. □

Using the previous result, we can now show the result for selectively-ccc spaces:

Theorem 6.8. (SCHEEPERS, 2000) Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_1(\mathscr{D}_0, \mathscr{D}_0)$ holds;
2. $I \not\uparrow G_1(\mathscr{D}_0, \mathscr{D}_0)$.

Proof. Again, it is sufficient to prove that $(1) \Rightarrow (2)$. Let σ be a strategy for Player *I* in the game $G_1(\mathscr{D}_0, \mathscr{D}_0)$. By hypothesis, we can consider that σ chooses countable families in $\mathscr{D}_0$. Let $\sigma(\emptyset) = \{U_n : n \in \omega\}$. For $n_1 \in \omega$, $\sigma(\langle U_{n_1}\rangle) = \{U_{\langle n_1\rangle^\frown n} : n \in \omega\}$, and so on. Thus, for each $\gamma \in {}^{<\omega}\omega$, we have $\{U_{\gamma^\frown n} : n \in \omega\} \in \mathscr{D}_0$.

Fix $m \in \omega$, and let $j \in \omega$ and $\rho : \{1, .., j^m\} \to \omega$. Define:

$$U_\rho(m, j) = \bigcap_{\gamma \in {}^m\{1,\dots,j\}} \left(\bigcup\{U_{\gamma^\frown \rho\restriction i} : i \leq j^m\}\right).$$

Note that the above sets are open. Furthermore, for each $m, j \in \omega$, $\mathscr{U}(m, j) := \{U_\rho(m, j) : \rho : \{1, \dots, j^m\} \to \omega\} \in \mathscr{D}_0$. Indeed, let U be a non-empty element in τ. Let $\{\gamma_i : 1 \leq i \leq j^m\}$ be a numeration of ${}^m\{1, \dots, j\}$. As $\{U_{\gamma_1{}^\frown n} : n \in \omega\} \in \mathscr{D}_0$, there is a k_1 such that $U \cap U_{\gamma_1{}^\frown k_1} \neq \emptyset$. As $\{U_{\langle\gamma_2,k_1\rangle^\frown n} : n \in \omega\} \in \mathscr{D}_0$, there is a k_2 such that $U \cap U_{\langle\gamma_2,k_1\rangle^\frown k_2} \neq \emptyset$. So, for each $1 \leq i \leq j^m$, there is a k_i such that $U \cap U_{\langle\gamma_i,k_1,\dots,k_{i-1}\rangle^\frown k_i} \neq \emptyset$. Define $\rho : \{1, \dots, j^m\} \to \omega$ such that $\rho(i) = k_i$. Then $U \cap U_\rho(m, j) \neq \emptyset$.

Using the same construction as in Lemma 2 in (PAWLIKOWSKI, 1994), there are increasing sequences $\langle j_n : n \in \omega \rangle$ and $\langle m_n : n \in \omega \rangle$ such that, for all $U \in \tau$ not empty, there are infinitely many $n \in \omega$ and functions $\rho : \{1, ..., m_{n+1} - m_n\} \to j_{n+1}$ in such a way that $U \cap U_\rho(m_n, j_n) \neq \emptyset$. Fixed sequences $\langle j_n : n \in \omega \rangle$ e $\langle m_n : n \in \omega \rangle$, for all $k_1 < k_2 < ... < k_n$ in ω and $\rho_1, ..., \rho_n$, where each $\rho_i : \{1, ..., m_{k_i+1} - m_{k_i}\} \to j_{k_i+1}$, we define the open sets:

$$W(k_1, ..., k_n; \rho_1, ..., \rho_n) = \bigcap_{i \leq n} U_{\rho_i}(m_{k_i}, j_{k_i}).$$

Consider the family $\mathscr{W}_n = \{W(k_1, ..., k_n; \rho_1, ..., \rho_n) : k_1 < k_2 < ... < k_n$ in ω and $\rho_1, ..., \rho_n$, where $\rho_i : \{1, ..., m_{k_i+1} - m_{k_i}\} \to j_{k_i+1}\}$. By the property of the sequences j_n's and m_n's, we see that $\langle \mathscr{W}_n : n \in \omega \rangle$ is a sequence in $\mathscr{D}_0$. Applying $S_1(\mathscr{D}_0, \mathscr{D}_0)$, it follows that there is, for all $n \in \omega$, $S_n = W(k_1^n, ..., k_n^n; \rho_1^n, ..., \rho_n^n)$ such that $\{S_n : n \in \omega\} \in \mathscr{D}_0$.

For all $n \in \omega$, choose $l_n \in \{k_1^n, ..., k_n^n\} \setminus \{l_i : i < n\}$ and $\rho_n = \rho_{i_n}^n$, where i_n is such that $l_n = k_{i_n}^n$. From the definition of the sets S_n and the choice of indexes, it follows that, for all $n \in \omega$, $S_n \subseteq U_{\rho_n}(m_{l_n}, j_{l_n})$. Consider the function $f : \omega \to \omega$ given by $f(i) \leq j_{l_1}$, for all $i \leq j_{l_1}^{m_{l_1}}$; and $f(m_{l_n} + i) = \rho_n(i)$, for $i \leq m_{l_n+1} - m_{l_n}$, with $n \geq 1$. Then, the game:

$$\sigma(\emptyset), U_{f(1)}, \sigma(U_{f(1)}), U_{(f(1), f(2)}, \sigma(U_{f(1)}, U_{(f(1), f(2))}), ...$$

is defeated by Player *II*, because

$$\bigcup_{n \in \omega} S_n \subseteq \bigcup_{n \in \omega} U_{\rho_n}(m_{l_n}, j_{l_n}) \subseteq \bigcup_{n \in \omega} U_{(f(1), ..., f(n))}.$$

Therefore, $I \not\uparrow G_1(\mathscr{D}_0, \mathscr{D}_0)$. □

With few modifications, we obtain:

Theorem 6.9. Let (X, τ) be a topological space and let $f : \omega \to \omega \setminus \{0\}$ be a function. The following statements are equivalent:

1. $S_f(\mathscr{D}_0, \mathscr{D}_0)$ holds;
2. $I \not\uparrow G_f(\mathscr{D}_0, \mathscr{D}_0)$.

Theorem 6.10. Let (X, τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ holds;
2. $I \not\uparrow G_{fin}(\mathscr{D}_0, \mathscr{D}_0)$.

6.2 Star selectively-ccc

Let (X,τ) be a topological space. For $A \subseteq X$ and $\mathscr{P} \subseteq \wp(X)$, we define the following set:

$$St(A,\mathscr{P}) = \bigcup\{P \in \mathscr{P} : P \cap A \neq \emptyset\}.$$

In (BAL; KOCINAC, 2020) a selection star-ccc property is introduced as follows:

Definition 6.11. We say that a topological space (X,τ) is star selectively-ccc if for all $U \in \mathscr{O}_X$, and any sequence $\langle \mathscr{A}_n : n \in \omega \rangle$, with $\mathscr{A}_n \in \mathscr{M}_c$, there is a sequence $\langle A_n : n \in \omega \rangle$ such that $A_n \in \mathscr{A}_n$, for all $n \in \omega$, and $St(B,\mathscr{U}) = X$, where $B = \bigcup_{n\in\omega} A_n$. This property is denoted by $SS^*_{\mathscr{O}_x,1}(\mathscr{M}_c,\mathscr{O}_X)$.

Theorem 6.12. (BAL; KOCINAC, 2020) Any selectively-ccc space is star selectively-ccc. There is a star selectively-ccc space that is not selectively-ccc space.

Problem 6.13. (BAL; KOCINAC, 2020) There exists a game-theoretic characterization of selectively star-ccc?

In direction of this problem, we introduced the following game:

Definition 6.14. For each $\mathscr{U} \in \mathscr{O}_X$, define the following game $G_1^{\mathscr{U}}(\mathscr{M}_c,\mathscr{O}_X)$: in each inning $n \in \omega$, Player I chooses $\mathscr{A}_n \in \mathscr{M}_c$. Then Player II chooses $A_n \in \mathscr{A}_n$. Player II wins the game if $St(B,\mathscr{U}) = X$, where $B = \bigcup_{n\in\omega} A_n$. Otherwise, Player I wins.

From the same observations made for selectively-ccc spaces, we have:

Lemma 6.15. Let (X,τ) be a topological space. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,1}(\mathscr{M}_c,\mathscr{O}_X)$ holds;
2. $SS^*_{\mathscr{O}_X,1}(\mathscr{D}_0,\mathscr{D}_0)$ holds.

Proof. We just need to include the following observation: if $B \subseteq A$, then $St(B,\mathscr{U}) \subseteq St(A,\mathscr{U})$. □

In general, we obtain the following:

Theorem 6.16. Let (X,τ) be a topological space and let $f : \omega \to \omega\backslash\{0\}$ be a function. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,f}(\mathscr{M}_c,\mathscr{O}_X)$ holds;
2. $SS^*_{\mathscr{O}_X,f}(\mathscr{D}_0,\mathscr{D}_0)$ holds.

Theorem 6.17. Let (X,τ) be a topological space. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,fin}(\mathscr{M}_c,\mathscr{O})$ holds;

2. $SS^*_{\mathscr{O}_X,fin}(\mathscr{D}_0,\mathscr{D}_0)$ holds.

Additionally, we have the following:

Theorem 6.18. Let (X,τ) be a topological space and let $f:\omega\to\omega\setminus\{0\}$ be a function. For all $\mathscr{U}\in\mathscr{O}_X$, the following statements are equivalent:

1. $II\uparrow G_f^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0)$ $(I\not\uparrow G_f^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0))$;

2. $II\uparrow G_f^{\mathscr{U}}(\mathscr{D}_0,\mathscr{D}_0)$ $(I\not\uparrow G_f^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0))$.

Theorem 6.19. Let (X,τ) be a topological space. For all $\mathscr{U}\in\mathscr{O}_X$, the following statements are equivalent:

1. $II\uparrow G_{fin}^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0)$ $(I\not\uparrow G_{fin}^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0))$;

2. $II\uparrow G_{fin}^{\mathscr{U}}(\mathscr{D}_0,\mathscr{D}_0)$ $(I\not\uparrow G_{fin}^{\mathscr{U}}(\mathscr{M}_c,\mathscr{D}_0))$.

Finally, with practically the same proof of Theorem 6.7 and Theorem 6.8, only by making a change in the winning criterion and using the observation in the proof of Lemma 6.15, we obtain the following results:

Theorem 6.20. Let (X,τ) be a topological space. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,fin}(\mathscr{D}_0,\mathscr{O}_X)$ holds;

2. $I\not\uparrow G_{fin}^{\mathscr{U}}(\mathscr{D}_0,\mathscr{D}_0)$, for all $\mathscr{U}\in\mathscr{O}_X$.

Theorem 6.21. Let (X,τ) be a topological space. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,1}(\mathscr{D}_0,\mathscr{O}_X)$ holds;

2. $I\not\uparrow G_1^{\mathscr{U}}/(\mathscr{D}_0,\mathscr{D}_0)$, for all $\mathscr{U}\in\mathscr{O}_X$.

Theorem 6.22. Let (X,τ) be a topological space and let $f:\omega\to\omega\setminus\{0\}$ be a function. The following statements are equivalent:

1. $SS^*_{\mathscr{O}_X,f}(\mathscr{D}_0,\mathscr{O}_X)$ holds;

2. $I\not\uparrow G_f^{\mathscr{U}}(\mathscr{D}_0,\mathscr{D}_0)$, for all $\mathscr{U}\in\mathscr{O}_X$.

6.3 Equivalences in selectively-ccc spaces

In the direction of the Problem 4.1 we study the case of the class $\mathscr{M}_c$. We introduce the following.

Definition 6.23. We say that a topological space (X,τ) is open-separable if there is a $\mathscr{B} \in \mathscr{D}_0$ countable.

Definition 6.24. Let $A \subseteq X$, we define, $\Omega_A = \{\mathscr{A} \subseteq \tau : \forall\, U \text{ open in } A, \exists\, B \in \mathscr{A}, U \cap B \neq \emptyset\}$.

Lemma 6.25. Let (X,τ) be a topological space. The following statements are equivalent:

1. $S_1(\mathscr{D}_0,\mathscr{D}_0)$ holds;
2. X is open-separable and $S_1(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \tau$;
3. X have a countable family $\mathscr{A} \in \mathscr{D}_0$ such that $S_1(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \mathscr{A}$.

Proof. $1) \Rightarrow 2) \Rightarrow 3)$ are clear, because $\mathscr{D}_0 \subseteq \Omega_A$, for all $A \in \tau$. It is sufficient to prove the implication $3) \Rightarrow 1)$. Let $\mathscr{A} = \{A_n : n \in \omega\} \in \mathscr{D}_0$ such that $S_1(\mathscr{D}_0,\Omega_{A_n})$ holds for all $n \in \omega$. Let $\langle \mathscr{A}_n : n \in \omega\rangle$ be a sequence with elements in $\mathscr{D}_0$ and $\{F_n : n \in \omega\}$ be a partition of ω in infinitely disjoint sets.

Applying $S_1(\mathscr{D}_0,\Omega_{A_n})$, for all $n \in \omega$, in the sequence $\langle \mathscr{A}_m : m \in F_n\rangle$, we find that there are, for all $m \in F_n$, $B^n_m \in \mathscr{A}_m$ such that $\{B^n_m : m \in F_n\} \in \Omega_{A_n}$. We claim that $\bigcup_{n\in\omega}\{B^n_m : m \in F_n\} \in \mathscr{D}_0$. Indeed, let $U \in \tau$. As $\mathscr{A} \in \mathscr{D}_0$, there is a $k \in \omega$ such that $U \cap A_k \neq \emptyset$. As $\{B^k_m : m \in F_k\} \in \Omega_{A_k}$, there is $l \in F_k$ such that $B^k_l \cap (U \cap A_k) \neq \emptyset$. Thus $B^k_l \cap U \neq \emptyset$, which concludes the proof. □

With few modifications, we obtain:

Lemma 6.26. Let (X,τ) be a topological space. The following statements are equivalent:

1. $S_f(\mathscr{D}_0,\mathscr{D}_0)$ holds;
2. X is open-separable and $S_f(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \tau$;
3. X have a countable family $\mathscr{A} \in \mathscr{D}_0$ such that $S_f(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \mathscr{A}$.

Lemma 6.27. Let (X,τ) be a topological space. The following statements are equivalent:

1. $S_{fin}(\mathscr{D}_0,\mathscr{D}_0)$;
2. X is open-separable and $S_{fin}(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \tau$;
3. X have a countable family $\mathscr{A} \in \mathscr{D}_0$ such that $S_{fin}(\mathscr{D}_0,\Omega_A)$ holds, for all $A \in \mathscr{A}$.

Observation 6.28. Let $\mathscr{C}, \mathscr{B} \subset \tau$ and $A \subset X$. If $\mathscr{C} \cup \mathscr{B} \in \Omega_A$ then $\mathscr{C} \in \Omega_A$ or $\mathscr{B} \in \Omega_A$. Indeed, suppose that $\mathscr{C} \notin \Omega_A$ and $\mathscr{B} \notin \Omega_A$. So, there are open sets U, V in A such that $C \cap U = \emptyset$, for all $C \in \mathscr{C}$ and $B \cap V = \emptyset$, for all $B \in \mathscr{B}$. Then $U \cap V$ is an open subset in A, and $D \cap (U \cap V) = \emptyset$ for all $D \in \mathscr{C} \cup \mathscr{B}$, that is, $\mathscr{C} \cup \mathscr{B} \notin \Omega_A$.

From this observation, we obtain the following result:

Theorem 6.29. Let (X, τ) be an open-separable space. If $S_2(\mathscr{D}_0, \mathscr{D}_0)$ holds, then $S_1(\mathscr{D}_0, \mathscr{D}_0)$ holds.

Proof. Let $(\mathscr{A}_n : n \in \omega)$ be a sequence in $\mathscr{D}_0$. Applying $S_2(\mathscr{D}_0, \mathscr{D}_0)$ to the sequence $(\mathscr{A}_{2n} : n \in \omega)$, we have that there are $\{A_n, B_n\} \subset \mathscr{A}_{2n}$, for all $n \in \omega$, such that $\bigcup_{n\in\omega} \{A_n, B_n\} \in \mathscr{D}_0$. Choose $A_n \in \mathscr{A}_{2n}$, for all $n \in \omega$. Let $\{p_i : i \in \omega\}$ be an increasing enumeration of the odd number primes. Then, for all $i \in \omega$, consider $(\mathscr{A}_{p_i^n} : n \in \omega)$. By $S_2(\mathscr{D}_0, \Omega_{B_i})$, it follows that there are $\{C_n^i, D_n^i\} \subset \mathscr{A}_{p_i^n}$ such that $\bigcup_{n\in\omega} \{C_n^i, D_n^i\} \in \Omega_{B_i}$. By the above observation, we have for all $i \in \omega$, $\{C_n^i : n \in \omega\} \in \Omega_{B_i}$ or $\{D_n^i : n \in \omega\} \in \Omega_{B_i}$. For each $n, i \in \omega$, define:

$$F_n^i = \begin{cases} C_n^i, & \{C_n^i : n \in \omega\} \in \Omega_{B_i} \\ D_n^i, & \{D_n^i : n \in \omega\} \in \Omega_{B_i} \end{cases}$$

Note that $F_n^i \in \mathscr{A}_{p_i^n}$, for all $n, i \in \omega$. Then $\{A_n : n \in \omega\} \cup \bigcup_{i\in\omega} \{F_n^i : n \in \omega\} \in \mathscr{D}_0$. Indeed, let $U \in \tau$. As $\bigcup_{n\in\omega} \{A_n, B_n\} \in \mathscr{D}_0$, there is $m \in \omega$ such that $U \cap A_m \neq \emptyset$ or $U \cap B_m \neq \emptyset$. If the first case is true, we are done. In the other case, $U \cap B_m$ is an open subset of B_m. As $\{F_n^m : n \in \omega\} \in \Omega_{B_m}$, there is $l \in \omega$ such that $F_l^m \cap (U \cap B_m) \neq \emptyset$. Then $U \cap F_l^m \neq \emptyset$. This concludes the proof. □

Using the same argument, we obtain the following result:

Theorem 6.30. Let (X, τ) be an open-separable space and $k \geq 1$. All selection principles $S_k(\mathscr{D}_0, \mathscr{D}_0)$ are equivalent.

Observation 6.31. In Theorem 2.17. in (BONANZINGA *et al.*, 2014), we see that the Pixley-Roy space $PR(\mathbb{R})$ is such that $S_{fin}(\mathscr{D}_0, \mathscr{D}_0)$ holds, but $S_1(\mathscr{D}_0, \mathscr{D}_0)$ fails.

CHAPTER

7

FINAL CONSIDERATIONS

In Chapter 3 we obtain, with certain conditions in the classes $\mathscr{A}$ and $\mathscr{B}$, equivalences in some variation of selection principles. We must mention that the implications and equivalences of Proposition 3.10 are valid in the game version (see (AURICHI; BELLA; DIAS, 2018)).

Our main objective in Chapter 4 was to obtain possible equivalences between selective topological games considering the class of dense subsets of a topological space X.

We see that, in general, selective topological games in the class of dense subsets in topological spaces are different. Indeed, consider the space $X = {}^{<\omega}\omega$ with the topology generated by the basis

$$\mathfrak{B} = \{X \setminus \bigcup_{f\in F} \{f \upharpoonright n : n \in \omega\} : F \subset {}^{\omega}\omega \; is \; finite\}.$$

Firstly, let $k_0, k_1, ..., k_{m-1} \in \omega$, with $m \in \omega$ (here $k_{-1} = \emptyset$), we have that the set $D = \{(k_0, k_1 ..., k_{m-1}, k) : k \in \omega\}$ is dense in X, because for any $F \subset {}^{\omega}\omega$ finite the set $\{f \upharpoonright m : f \in F\}$ is finite, and then there is $(k_0, .., k_{m-1}, k_m) \in D$ such that $(k_1, .., k_{n-1}, k_m) \neq f \upharpoonright m$, for all $f \in F$. So $D \cup \left(X \setminus \bigcup_{f\in F}\{f \upharpoonright n : n \in \omega\}\right) \neq \emptyset$.

Now, in the game $G_1(\mathscr{D}_X, \mathscr{D}_X)$, in the inning 0, Player I chooses $D_0 = \{(k) : k \in \omega\}$. If Player II chooses $x_0 = (k_0) \in D_0$, then Player I chooses $D_1 = \{(k_0, k) : k \in \omega\}$. If Player II chooses $x_1 = (k_0, k_1) \in D_1$, then Player I chooses $D_2 = \{(k_0, k_1, k) : k \in \omega\}$, and so on. Taking $f = (k_0, ..., k_n, ...)$ we have $\{x_n : n \in \omega\} \cap X \setminus \{f \upharpoonright n : n \in \omega\} = \emptyset$, that is, $\{x_n : n \in \omega\} \notin \mathscr{D}_X$. So, $I \uparrow G_1(\mathscr{D}_X, \mathscr{D}_X)$, and then $II \not\uparrow G_1(\mathscr{D}_X, \mathscr{D}_X)$.

On the other hand, suppose that Player I chooses $A_0 \in \mathscr{D}_X$ in the inning 0 in the game $G_2(\mathscr{D}_X, \mathscr{D}_X)$. As A_0 is dense, we can choose $a_1^0, a_2^0 \in A_1$ such that they do not belong to a same branch (branch is a set of the form $\{f \upharpoonright n : n \in \omega\}$, with $f \in {}^{\omega}\omega$). Then, Player II chooses $\{a_1^0, a_2^0\}$. It is clear that $\{a_1^0\}$ or $\{a_2^0\}$ is a set such that no branch contains two elements of it. Let $\{t_0\}$ be the set $\{a_1^0\}$.

In the next inning, suppose that Player I chooses $A_1 \in \mathscr{D}_X$. If there is an element a_1^1 in A_1

such that it is not in any branch that does not intersect $\{t_0\}$, then Player *II* chooses $\{a_1^1, a_2^1\}$, with a_2^1 an arbitrary element in A_1. Note that the set $\{t_0, a_1^1\}$ is a set such that no branch contains two elements of it. If all the elements in A_1 are in a branch that intersects $\{t_0\}$, since A_1 is dense, we can choose incompatible elements a_1^1 and a_2^1 (that is, $a_1^1 \not\subset a_2^1$ and $a_2^1 \not\subset a_1^1$) in A_2 such that $t_0 \subset a_1^1$ and $t_0 \subset a_2^1$. So, Player *II* chooses $\{a_1^1, a_2^1\}$. Note that the set is such that no branch contains two elements of it. So, in any of the cases, we have a set with 2 elements, namely $\{t_0, t_1\}$, such that no branch contains two elements of it.

In the next inning, suppose that Player *I* chooses $A_2 \in \mathscr{D}_X$. If there is an element a_1^2 in A_2 such that it is not in any branch that does not intersect $\{t_0, t_1\}$, then Player *II* chooses $\{a_1^2, a_2^2\}$, with a_2^2 an arbitrary element in A_1. Note that the set $\{t_0, t_1, a_1^2\}$ is a set such that no branch contains two elements of it. If all elements in A_1 are in a branch that intersects $\{t_0, t_1\}$, since A_1 is dense, there is t_i such that we can choose a_1^2 and a_2^2 incompatible elements in A_2 with $t_i \subset a_1^2$ and $t_i \subset a_2^2$. So, Player *II* chooses $\{a_1^2, a_2^2\}$. Note that for $j \neq i$, the set $\{t_j, a_1^2, a_2^2\}$ is such that no branch contains two elements of it. So, in any of the cases, we have a set with 3 elements, namely $\{t_0, t_1, t_2\}$, such that no branch contains two elements of it.

In general, in the inning $n \geq 1$, suppose that Player *I* chooses $A_n \in \mathscr{D}_X$. If there is an element a_1^n in A_n such that it is not in any branch that does not intersect $\{t_0, t_1, .., t_{n-1}\}$, then Player *II* chooses $\{a_1^n, a_2^n\}$, with a_2^n an arbitrary element in A_n. Note that the set $\{t_0, t_1, ..., t_{n-1}, a_1^n\}$ is a set such that no branch contains two elements of it. If all the elements in A_n are in a branch that intersects $\{t_0, t_1, ..., t_{n-1}\}$, since A_n is dense, there is t_i such that we can choose a_1^n and a_2^n incompatible elements in A_n with $t_i \subset a_1^n$ and $t_i \subset a_2^n$. So, Player *II* chooses $\{a_1^n, a_2^n\}$. Note that the set $\{t_j : j \neq i\} \cup \{a_1^n, a_2^n\}$ is such that no branch contains two elements of it. So, in any of the cases, we have a set with $n+1$ elements, namely $\{t_0, t_1, ..., t_n\}$, such that no branch contains two elements of it.

In summary, we obtain a strategy σ for Player *II* such that, for each $n \in \omega$, the set of answers played includes a set $\{t_0, ..., t_n\}$ with the property that no branch contains two elements of it.

We have that σ is a winning strategy. Indeed, let D be the set of all the answers of Player *II* in a play using σ. If $D \notin \mathscr{D}_X$, then there is a basic open $U = X \setminus \bigcup_{F \in {}^{\omega}\omega} \{f \restriction n : n \in \omega\}$, with F finite, such that $D \cap U = \emptyset$. So, $D \subset \bigcup_{F \in {}^{\omega}\omega} \{f \restriction n : n \in \omega\}$. Suppose that $|F| = m$. Then, since the set $\{t_0, ..., t_m\} \subset D$ is such that no branch contains two elements of it, there is a t_i such that $t_i \notin \bigcup_{F \in {}^{\omega}\omega} \{f \restriction n : n \in \omega\}$. So, $D \not\subset \bigcup_{F \in {}^{\omega}\omega} \{f \restriction n : n \in \omega\}$, a contradiction. Then $D \in \mathscr{D}_X$. Therefore, $II \uparrow G_2(\mathscr{D}_X, \mathscr{D}_X)$.

We can see that X is a T_1 space that is not a Hausdorff space. The following question is still open:

Problem 7.1. Restricted to Hausdorff spaces, the selective topological games $G_1(\mathscr{D}_X, \mathscr{D}_X)$ and $G_2(\mathscr{D}_X, \mathscr{D}_X)$ are equivalent?

Furthermore, if X is a P-space and 1st-countable, then Problem 7.1 has a positive answer. Indeed, if X is a Hausdorff space, then X is a discrete space and therefore all games $G_k(\mathscr{D}_X, \mathscr{D}_X)$, $G_f(\mathscr{D}_X, \mathscr{D}_X)$ and $G_{fin}(\mathscr{D}_X, \mathscr{D}_X)$ are equivalents, where $k \in \omega$ and $f : \omega \to \omega \setminus \{0\}$ is a function.

On the other hand, if X is not a Hausdorff space, then we have $II \uparrow G_2(\mathscr{D}_X, \mathscr{D}_X) \Rightarrow II \uparrow G_1(\mathscr{D}_X, \mathscr{D}_X)$. Indeed, let σ be a winning strategy for Player II in $G_2(\mathscr{D}_X, \mathscr{D}_X)$. Suppose that Player I chooses $D_0 \in \mathscr{D}_X$ in the first inning of the game $G_1(\mathscr{D}_X, \mathscr{D}_X)$. Suppose that $\sigma(\langle D_0 \rangle) = \{x_0, y_0\}$. So, define $\varphi(\langle D_0 \rangle) = x_0$. Next, Player I chooses $D_1 \in \mathscr{D}_X$.

Recall that

$$II \uparrow G_2(\mathscr{D}_X, \mathscr{D}_X) \Rightarrow S_2(\mathscr{D}_X, \mathscr{D}_X) \Rightarrow S_1(\mathscr{D}_X, \mathscr{D}_X) \Rightarrow S_1(\mathscr{D}_X, \Omega_x),$$

for all $x \in X$.

Then, using that $S_1(\mathscr{D}_X, \Omega_{y_0})$ holds, there is $\{z_n^1 : n \in \omega\} \subset D_1$ such that $\{z_n^1 : n \in \omega\} \in \Omega_{y_0}$. Let $\{U_n^0 : n \in \omega\}$ be a local base for y_0. Then $\bigcap_{n\in\omega} U_n^0$ is an open (because X is P-space) and contains y_0. Therefore, there is m_1 such that $z_{m_1}^1 \in \bigcap_{n\in\omega} U_n^0$. Define $\varphi(\langle D_0, D_1 \rangle) = z_{m_1}^1$. Thus, in each inning $2n$, $n \in \omega$, we define $\varphi(\langle D_0, D_1, ..., D_{2n} \rangle) = x_n$, where $\sigma(\langle D_0, D_2, ..., D_{2n} \rangle) = \{x_n, y_n\}$. On the other hand, in each inning $2n+1$, with $n \in \omega$, we define $\varphi(\langle D_0, D_1, ..., D_{2n+1} \rangle) = z_{m_{2n+1}}^{2n+1}$, where

$$z_{m_{2n+1}}^{2n+1} \in \bigcap_{m\in\omega} U_m^n$$

and $\{U_m^n : m \in \omega\}$ is a local base of y_n (here we use $S_1(\mathscr{D}_X, \Omega_{y_n})$).

Then, we claim that $\{x_n : n \in \omega\} \cup \{z_{m_{2n+1}}^{2n+1} : n \in \omega\} \in \mathscr{D}_X$. Indeed, let $U \in \tau$. As σ is a winning strategy, we see that there is $k \in \omega$ such that $x_k \in U$ or $y_k \in U$. If the first case is true, we are done. Suppose that $y_k \in U$. Then, there is $l \in \omega$ such that $y_k \in U_l^k \in U$. So $z_{m_{2k+1}}^{2k+1} \in \bigcap_{m\in\omega} U_m^k \subset U$. Therefore, φ is a winning strategy for Player II in the game $G_1(\mathscr{D}_X, \mathscr{D}_X)$.

Due to the impossibility, for now, of answering positively Question 7.1 we focus on function spaces $C(X)$. However, we had to restrict ourselves due to the following problems, which are still open:

Problem 7.2. If (X, τ) is a regular space and $f : \omega \to \omega \setminus \{0\}$ is a function. The games $G_f(\Omega_X, \Omega_X)$ and $G_1(\Omega_X, \Omega_X)$ are equivalent?

Problem 7.3. Let (X, τ) be a regular space. If $s \in {}^{<\omega}\Omega_X$ and σ is a strategy in $G_f(\Omega_X, \Omega_X)$, then

$$C_s = \bigcap_{\mathscr{U} \in \Omega_X} \overline{\bigcup \sigma(s^\frown \mathscr{U})}$$

is finite?

And, more generally:

Problem 7.4. Let (X,τ) be a regular or Tychonoff space and $\mathscr{B}$ a bornology with a compact basis. If $s \in {}^{<\omega}\mathscr{O}_{\mathfrak{B}}^X$ and σ is a strategy in $G_f(\mathscr{O}_{\mathfrak{B}}^X, \mathscr{O}_{\mathfrak{B}}^X)$, then

$$C_s = \bigcap_{\mathscr{U} \in \mathscr{O}_{\mathfrak{B}}^X} \overline{\bigcup \sigma(s^\frown \mathscr{U})}$$

is an element of $\mathscr{B}$?

If the statements in either of the first two problems are true, we can obtain a version of Corollary 4.46 for the function space $C_p(X)$. If the statement of Problem 7.4 is true, we can probably obtain a version of Corollary 4.46 by $C_{\mathscr{B}}(X)$.

In particular, when X is a P-space, Problems 7.2 and 7.3 have positive answers, and therefore we have an equivalence of the topological games $G_1(\mathscr{D}_X, \mathscr{D}_X)$ and $G_f(\mathscr{D}_X, \mathscr{D}_X)$, with $f : \omega \to \omega \setminus \{0\}$ an increasing function.

In Chapter 5 we obtain generalizations of equivalences and translations, about selection principles and selective topological games, in other classes of dense subsets.

In Chapter 6 we obtain equivalences in variations of selection principles associated with the selectively-ccc property. An interesting question is about the variations in the selective topological games associated with that property.

BIBLIOGRAPHY

ARKHANGEL'SKII, A. V. Hurewicz spaces, analytic sets and fan tightness of function spaces. **Soviet Math. Dokl.**, v. 33, p. 396–399, 1986. Citation on page 28.

AURICHI, L.; DUZI, M. Topological games of bounded selections. **Topology and its Applications**, v. 291, n. 107449, 2021. Citations on pages 40, 41, 43, and 67.

AURICHI, L. F. Selectively c.c.c. spaces. **Topology and its Applications**, v. 160, n. 1, p. 2243–2250, 2013. Citation on page 85.

AURICHI, L. F.; BELLA, A.; DIAS, R. R. Tightness games with bounded finite selections. **Israel Journal Math.**, v. 224, n. 1, p. 133–158, 2018. Citations on pages 37, 53, and 95.

AURICHI, L. F.; DIAS, R. R. A minicourse on topological games. **Topology and its Applications**, v. 258, p. 305–335, 2019. Citation on page 23.

BAL, P.; KOCINAC, L. D. R. On selectively star-ccc spaces. **Topology and its applications**, n. Article in press: 107184, 2020. Citation on page 90.

BARMAN, D.; DOW, A. Selective separability and S^+. **Topology proceedings**, v. 37, p. 181–204, 2011. Citation on page 47.

BELLA, B.; BONANZINGA, M.; MATVEEV, M. Variations of selective separability. **Topology and its applications**, v. 156, p. 1241–1252, 2009. Citation on page 49.

BONANZINGA, M.; CAMMAROTO, F.; PANSERA, B. A.; TSABAN, B. Diagonalizations of dense families. **Topology and its applications**, v. 165, p. 12–25, 2014. Citations on pages 49 and 93.

CAMARGO, J.; UZCÁTEGUI, C. Selective separability on spaces with an analytic topology. **Topology and its applications**, v. 248, p. 176–191, 2018. Citation on page 49.

CLONTZ, S. Relating games of menger, countable fan tightness, and selective separability. **Topology and its applications**, v. 258, p. 215–228, 2019. Citations on pages 28 and 53.

____. Dual selection games. **Topology and its applications**, v. 272, n. 107056, 2020. Citations on pages 33, 34, and 35.

CRONE, L.; FISHMAN, L.; HIERS, N.; JACKSON, S. Equivalence of the rothberger, k-rothberger, and restricted menger games. **Topology and its applications**, v. 258, n. 172-176, 2019. Citation on page 63.

ENGELKING, R. **General Topology**. Heldermann, Berlin: Sigma series in pure mathematics, 1989. 540 p. Citation on page 21.

GALVIN, F. Indeterminacy of point-open games. **Bull. Acad. Pol. Sci.**, v. 26, p. 445–449, 1978. Citations on pages 33 and 34.

GARCÍA-FERREIRA, S.; TAMARIZ-MASCARÚA, A. Some generalizations of rapid ultrafilters in topology and id-fan tightness. **Tsukuba Journal Math.**, v. 19, n. 1, p. 173–185, 1995. Citations on pages 37, 41, 42, and 51.

GERLITS, J.; NAGY, Z. The combinatorics of open covers *I*. **Topology and its applications**, v. 14, p. 151–161, 1982. Citation on page 30.

GRUENHAGE, G. Infinite games and generalizations of first-countable spaces. **Gen. Topol. Appl.**, v. 6, n. 3, p. 339–352, 1976. Citation on page 35.

HOGBE-NLEND, H. **Bornologies and Functional Analysis**. North-Holland, Amsterdam: Mathematics studies, 1977. 156 p. Citation on page 27.

HUREWICZ, W. Über eine verallgemenerung des borelschen theorems. **Mat. Z.**, v. 24, p. 401–421, 1925. Citation on page 26.

JUST, W.; MILLER, A. W.; SCHEEPERS, M.; SZEPTYCKI, P. J. The combinatorics of open covers *II*. **Topology and its applications**, v. 73, p. 241–266, 1996. Citation on page 44.

KOCINAC, L. D. R. Closure properties of function spaces. **Applied General Topology**, v. 4, n. 2, p. 255–261, 2003. Citation on page 29.

LIN, S.; LIU, C.; TENG, H. On fan tightness and strong fréchet property for $C_k(X)$ (in chinese). **Advances in Mathematics (China)**, v. 23, n. 3, p. 234–237, 1994. Citation on page 29.

MCCOY, R. A.; NTANTU, I. **Topological properties of spaces of continuous functions**. Springer-Verlag. Berlin: Lecture Notes in Mathematics, 1988. 127 p. Citations on pages 27 and 30.

MEZABARBA, R. M.; AURICHI, L. F. Bornologies and filters in selection principles on function spaces. **Topology and its applications**, v. 258, p. 187–201, 2019. Citations on pages 28, 29, 46, 51, and 70.

NOBLE, N. The density character of function spaces. **Proceedings of American Mathematical Society**, v. 42, n. 1, p. 228–233, 1974. Citation on page 71.

NOKHRIN, S. E.; OSIPOV, A. V. On the coincidence of set-open and uniform topologies. **Proceedings of the Steklov Institute of Matematics**, v. 267, n. 1, p. 184–191, 2009. Citation on page 28.

OSIPOV, A. V. Classification of selectors for sequences of dense sets of $C_p(X)$. **Topology and its Applications**, v. 242, p. 20–30, 2018. Citation on page 75.

____. The functional characterizations of the rothberger and menger properties. **Topology and its Applications**, v. 243, p. 146–152, 2018. Citation on page 81.

____. Selection principles in function spaces with the compact-open topology. **Filomat**, v. 32, n. 15, p. 5403–5413, 2018. Citation on page 29.

PAWLIKOWSKI, J. Undetermined sets of point-open games. **Fundamenta Mathematicae**, v. 144, p. 279–285, 1994. Citations on pages 26 and 89.

SAKAI, M. Property C'' and function spaces. **Proceedings of the American Mathematical Society**, v. 104, n. 3, p. 917–919, 1988. Citations on pages 28 and 44.

SCHEEPERS, M. Combinatorics of open covers I: Ramsey theory. **Topology and its applications**, v. 69, p. 31–62, 1994. Citations on pages 43, 51, and 79.

____. The combinatorics of open covers. II. **Topology and its applications**, v. 73, p. 241–266, 1996. Citations on pages 43 and 51.

____. Combinatorics of open covers. III: games, $C_p(X)$. **Fundamenta Mathematicae**, v. 152, p. 231–254, 1997. Citations on pages 26 and 32.

____. Combinatorics of open covers. VI: Selectors for sequences of dense sets. **Quaestiones Mathematicae**, v. 22, n. 1, p. 109–130, 1999. Citations on pages 32 and 35.

____. Combinatorics of open covers (V): Pixley–roy spaces of sets of reals, and ω-covers. **Topology and its applications**, v. 102, p. 13–31, 2000. Citations on pages 86 and 88.

SZEWCZAK, P.; TSABAN, B. Conceptual proofs of the menger and rothberger games. **Topology and its Applications**, v. 272, n. 107048, 2020. Citations on pages 26 and 54.

TELGÁRSKY, R. Spaces defined by topological games. **Fundam. Math.**, v. 88, p. 193–223, 1975. Citation on page 34.

INDEX

ICMC USP
SÃO CARLOS