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Qualitative properties of radial solutions of the Hénon equation

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Propriedades qualitativas de soluções radiais da equação de Hénon

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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To my family, for their ability to believe and invest in me. Mom, your care and dedication are what gave me, in a few moments, the hope to follow. Dad, your presence meant security and the certainty that I am not alone on this walk.

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"Cada um de nós compõe a sua história,
Cada ser em si carrega o dom de ser capaz e ser feliz."
(Almir Sater and Renato Teixeira)

ABSTRACT

LEITE DA SILVA, W. **Qualitative properties of radial solutions of the Hénon equation**. 2020. 78 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

In this work, we study qualitative properties of radial solutions to the Hénon problem

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $B \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 2$, $\alpha \geq 0$ and $p > 1$. We obtained results about the computation of the Morse index and the asymptotic profile, as $\alpha \to \infty$, of both positive and sign changing radial solutions. More precisely, we divided this work into two parts. Firstly, considering the case $N = 2$, we proved that the Morse index of the radial solutions u_α, with the same number of nodal sets, is monotone non-decreasing with respect to α. Moreover, we present a lower bound for the Morse indices $m(u_\alpha)$, which is better than those that already exist in the literature, showing in particular that $m(u_\alpha) \to \infty$ as $\alpha \to \infty$. Secondly, considering $N \geq 3$, we show that the two-dimensional Lane-Emden equation can be seen as a limit problem for the Hénon equation. Finally, we used this fact to obtain some qualitative consequences of these solutions.

Keywords: Semilinear elliptic equations, Hénon equation, nodal radial solutions, Morse index, asymptotic behavior.

RESUMO

LEITE DA SILVA, W. **Propriedades qualitativas de soluções radiais da equação de Hénon**. 2020. 78 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

Neste trabalho, estudamos propriedades qualitativas de soluções radiais para o problema de Hénon

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

onde $B \subset \mathbb{R}^N$ é a bola unitária centrada na origem, $N \geq 2$, $\alpha \geq 0$ e $p > 1$. Obtivemos resultados sobre o cálculo do índice de Morse e o perfil assintótico, quando $\alpha \to \infty$, das soluções radiais, as positivas e também as que trocam de sinal. Mais precisamente, dividimos este trabalho em duas partes. Primeiramente, considerando o caso $N = 2$, provamos que o índice de Morse das soluções radiais u_α, com o mesmo número de conjuntos nodais, é monótono não decrescente com respeito a α. Além disso, apresentamos uma cota inferior para os índices de Morse $m(u_\alpha)$, melhor que aquelas já existentes na literatura, o que mostra em particular que $m(u_\alpha) \to \infty$ quando $\alpha \to \infty$. Segundamente, considerando $N \geq 3$, mostramos que a equação de Lane-Emden bidimensional pode ser vista como um problema limite para a equação de Hénon. Por fim, utilizamos este fato para obter algumas consequências qualitativas destas soluções.

Palavras-chave: Equações elípticas semilineares, equação de Hénon, soluções radiais nodais, índice de Morse, comportamento assintótico..

LIST OF SYMBOLS

$\#S$ — Cardinality of a set S

$\lfloor . \rfloor$ — Floor function: $\lfloor \beta \rfloor = \max\{k \in \mathbb{Z} : k \leq \beta\}$

$\lceil . \rceil$ — Ceiling function: $\lceil \beta \rceil = \min\{k \in \mathbb{Z} : k \geq \beta\}$

$o(g(x))$ as $x \to a$ — Any function $f(x)$ such that $\frac{f(x)}{g(x)} \to 0$ as $x \to a$

$O(g(x))$ as $x \to a$ — Any function $f(x)$ such that $\frac{f(x)}{g(x)}$ is bounded for x in a neighborhood of a and $x \neq a$

$L^q(\Omega)$ — Space of measurable functions $v : \Omega \to \mathbb{R}$ s.t. $\int_\Omega |v|^q dx < \infty$

$L^q(|x|^\alpha, \Omega)$ — Space of measurable functions $v : \Omega \to \mathbb{R}$ s.t. $\int_\Omega |x|^\alpha |v|^q dx < \infty$

$H_0^1(\Omega)$ — $v \in L^2(\Omega)$ s.t. v has first order weak derivatives in $L^2(\Omega)$ and $v = 0$ on $\partial\Omega$

$H_{0,rad}^1(\Omega)$ — $v \in H_0^1(\Omega)$ s.t. v is radially symmetric

$\mathscr{H}_0$ — $v \in H_0^1(\Omega)$ s.t. $\int_\Omega \frac{v^2}{|x|^2} dx < \infty$

$\mathscr{H}_{0,rad}$ — $v \in \mathscr{H}_0$ s.t. v is radially symmetric

$L_M^q(I)$ — Space of measurable functions $v : I \to \mathbb{R}$ s.t. $\int_I |v|^q s^{M-1} ds < \infty$

L_M^q — $L_M^q(0,1)$

H_M^1 — $v \in L_M^2$ s.t. v has first order weak derivative $v' \in L_M^2$

$H_{0,M}^1$ — $v \in H_M^1$ s.t. $v(1) = 0$ and $v \in L_{M-2}^2$

$\mathscr{D}_M^{1,2}$ — $v \in L_M^{\frac{2M}{M-2}}(0,\infty)$ s.t. v has first order weak derivative v' in $L_M^2(0,\infty)$ $(M > 2)$

CONTENTS

CHAPTER 1

INTRODUCTION

In this work, based on the papers (SILVA; SANTOS, 2019) and (SILVA; SANTOS, 2020), we address questions about qualitative properties of solutions of Hénon type equations:

$$\begin{cases} -\Delta u = |x|^{\alpha} f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is either a ball or an annulus centered at the origin, $\alpha \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is $C^{1,\beta}$ on bounded sets of $\mathbb{R}$.

The Hénon equation (HÉNON, 1973) was proposed as a model to study stellar distribution in a cluster of stars with the presence of a black hole located at the center of the cluster. Besides its application to astrophysics, Hénon type equations also model steady-state distributions in other diffusion processes; see the introduction in (SANTOS; PACELLA, 2016) and the references therein for a more precise description on applications.

Apart from its applications, the Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{P_α}$$

where $B \subset \mathbb{R}^N$ is the unit open ball centered at the origin, $N \geq 2$, $\alpha \geq 0$ and $p > 1$, has been investigated by several authors and it is an excellent prototype for the study of some important problems on the qualitative analysis of solutions of elliptic partial differential equations. For example, the results of Gidas, Ni and Nirenberg (GIDAS; NI; NIRENBERG, 1979) show that, if Ω is a ball, every positive solution for (1.1) with $\alpha = 0$ is radially symmetric. Ni (NI, 1982) also showed the existence of a radially symmetric solution for (P_α) when $2 < p+1 < 2^*_\alpha$, where $2^*_\alpha := \infty$ if $N = 2$ and $2^*_\alpha := \frac{2(N+\alpha)}{N-2}$ for $N \geq 3$. When $p \geq 2^*_\alpha - 1$, the Pohozaev identity, as in (FIGUEIREDO; LIONS; NUSSBAUM, 1982, Lemma 1.1), shows that (P_α) has no non-trivial solution. Later on, Nagasaki proved in (NAGASAKI, 1989) that, for each integer $k \geq 1$, (P_α) admits a unique radial solution, up to multiplication by -1, with exactly $k-1$ zeros in (0,1); see also (BARTSCH; WILLEM, 1993), where the authors show this fact using the Nehari's method. Numerical solutions obtained in (CHEN; ZHOU; NI, 2000) show that for each fixed $\alpha > 0$, the ground state solutions (least energy solutions, also characterized as mountain pass solutions) of (P_α) are not radial if $p+1$ is close to $2^* := 2_0^*$. When p is close to 1, results in (SMETS; WILLEM; SU, 2002) show that for each $n \in \mathbb{N}$, there exists $\delta_n > 0$ so that the ground state solution of (P_α) is radial for any $\alpha \leq n$ and $p \in (1, 1+\delta_n]$. Now when the power $p \in (1, 2^*-1)$ is fixed, Smets, Su and Willem (SMETS; WILLEM; SU, 2002) also proved that the least energy solutions of (P_α) are not radial either if $\alpha > 0$ is large enough (see also (SMETS; WILLEM, 2003)). A similar symmetry breaking occurs for least energy nodal solutions, as observed in (BARTSCH; WETH; WILLEM, 2005, Remark 6.4). The main idea used in (SMETS; WILLEM; SU, 2002) is to compare the energy of ground state solutions and that of radially symmetric solutions.

About concentration phenomena, Cao and Peng (CAO; PENG, 2003) proved that for p sufficiently close to 2^*-1, the ground state solutions of (P_α) possess a unique maximum point whose distance from ∂B tends to zero as $p \to 2^*-1$. Later on, these same authors in a work with Yan (CAO; PENG; YAN, 2009) showed that this phenomenon also happens when $p \in (2, 2^*-1)$ is fixed and $\alpha \to \infty$. Results on concentration on spheres also have been established by Moreira dos Santos and Pacella, where they show in (SANTOS; PACELLA, 2016) that if $N = 2m \geq 4$ is even, then the least energy solutions of (P_α) with symmetry in the space $H^1_{0,O(m)\times O(m)}(B)$ concentrate and blow up on the sphere $S^{m-1} \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^m$.

Next, consider a semilinear elliptic equation of the type

$$-\Delta u = g(|x|, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is either a ball or an annulus centered at the origin, $g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is such that $r \mapsto g(r, u)$ is $C^{0,\beta}$ on bounded sets of $[0, +\infty) \times \mathbb{R}$, $u \mapsto g_u(r, u)$ is $C^{0,\gamma}$ on bounded sets of $[0, +\infty) \times \mathbb{R}$, where g_u denotes the derivative of g with respect to the variable u. We recall that the Morse index $m(u)$ of a solution u of (1.2) is the maximal dimension of a subspace of $H_0^1(\Omega)$ in which the quadratic form

$$H_0^1(\Omega) \ni w \mapsto Q_u(w) := \int_\Omega |\nabla w(x)|^2 dx - \int_\Omega g_u(|x|, u(x)) w^2(x) dx$$

is negative definite. Since we are considering the case of bounded domains, $m(u)$ coincides with the number of negative eigenvalues, counted with their multiplicity, of the linearized operator $L_u := -\Delta - g_u(|x|, u)$ in the space $H_0^1(\Omega)$. When the solution u is radial, we will denote by $m_{rad}(u)$ the radial Morse index of u, that is, the maximal dimension of a subspace of $H^1_{0,rad}(\Omega)$ in which the quadratic form Q_u is negative definite or, alternatively, $m_{rad}(u)$ is the number of negative eigenvalues, counted with their multiplicity, of L_u in the space $H^1_{0,rad}(\Omega)$.

The Morse index is, in general, a very important qualitative property of a solution. In particular it helps to classify solutions and study their stability or possible bifurcations, see for instance (AMADORI; GLADIALI, 2014) and more recently (KÜBLER; WETH, 2019). For example, it is proved in (BARTSCH; WETH, 2003, Theorem 1.3), see also (CASTRO; COSSIO; NEUBERGER, 1997), that a ground state solution of (1.1) has Morse index 1, while a least energy nodal solution has Morse index 2, if the nonlinear term f satisfies the following conditions:

(f_1) $f \in C^1(\mathbb{R})$ and $f(0) = 0$.

(f_2) There exist $p \in (1, 2^* - 1)$ and $C > 0$ such that $|f'(t)| \leq C(1 + |t|^{p-1})$ for all $t \in \mathbb{R}$ (subcritical growth condition).

(f_3) $f'(t) > \frac{f(t)}{t}$ for all $t \in \mathbb{R} \backslash \{0\}$ (superlinear growth condition)

(f_4) There exist $R > 0$ and $\theta > 2$ such that $0 < \theta F(t) \leq t f(t)$ for all $|t| \geq R$, where $F(t) = \int_0^t f(s) ds$ (Ambrosetti–Rabinowitz condition).

Note that clearly these assumptions hold for

$$f(t) = \sum_{i=1}^{k} |t|^{p_i - 1} t, \quad 1 \leq p_1 < \cdots < p_k < 2^* - 1.$$

Moreover it is known from (BARTSCH; WETH; WILLEM, 2005) that, under conditions $(f_1)-(f_4)$, both least energy and least energy nodal solutions of (1.1) have a partial symmetry called foliated Schwarz symmetry, and thus these solutions have an $O(N-1)$-symmetry and are monotone with respect to angular coordinate. One good question is to know when a least energy solution (global or nodal) of (1.1) is nonradial. We already commented that for all $\alpha \geq 0$, the least energy solution of (P_α) is radial if p is sufficiently close to 1. Therefore there is only symmetry breaking for ground state solutions of (P_α) when α is large enough with fixed p or p is close to $2^* - 1$ with fixed α. Symmetry breaking of least energy nodal solutions of (1.1) has been studied by several authors. A first result was given by Aftalion and Pacella (AFTALION; PACELLA, 2004), where they obtained some lower bounds on the Morse index which show in particular that every radial nodal solution of an autonomous problem (eq. of type (1.2) with $g(|x|, u) = f(u)$) has Morse index greater than or equal to 3, and this implies that no least energy nodal solution is radial, since they have Morse index 2. For non-autonomous problemas, Moreira dos Santos and Pacella (SANTOS; PACELLA, 2017), for case $N = 2$, and Amadori and Gladiali (AMADORI; GLADIALI, 2018), for case $N \geq 2$, also proved symmetry breaking, via Morse index, for any least energy nodal solution of (1.1).

In this thesis, we present some results on the Morse index and asymptotic behavior of radially symmetric solutions.

Given any continuous function $u : \Omega \to \mathbb{R}$, we will denote by $n(u)$ the number of nodal sets of u, where the nodal sets of u are the connected components of the set $\{x \in \Omega; u(x) \neq 0\}$. In the case of autonomous problems, that is, when the nonlinear term g does not depend on the space variable, Aftalion and Pacella (AFTALION; PACELLA, 2004) obtained some lower bounds on the Morse index of sign changing radial solutions of (1.2), which recently were improved by De Marchis, Ianni and Pacella (MARCHIS; IANNI; PACELLA, 2017b, Theorem 2.1).

Theorem A (**Autonomous problems**). Let u be a radial nodal solution of (1.2) with $g(|x|,u) = f(u)$, $f \in C^1$. Then

$$m_{rad}(u) \geq n(u) - 1 \quad \text{and} \quad m(u) \geq m_{rad}(u) + N(n(u) - 1).$$

Moreover, if f is superlinear, i.e. satisfies the condition (f_3), then

$$m_{rad}(u) \geq n(u) \ \text{ and hence } \ m(u) \geq n(u) + N(n(u) - 1).$$

In chapter 2, we consider a non-autonomous equation of type (1.1) with $N = 2$. We recall the following estimates obtained in (SANTOS; PACELLA, 2017, Theorems 1.1 and 1.2), which were used to prove that least energy nodal solutions of (1.1) are not radially symmetric.

Theorem B. Let u be a radial sign changing solution of (1.1). Then u has Morse index greater than or equal to 3. Moreover, if (f_3) holds, then the Morse index of u is at least $n(u) + 2$. In case α is even, then these lower bounds can be improved, namely they become $\alpha + 3$ and $n(u) + \alpha + 2$, respectively.

Very recently these lower bounds were improved by Amadori and Gladiali (AMADORI; GLADIALI, 2018, Theorem 1.1) by characterizing the Morse index in terms of a singular one dimensional eigenvalue problem. We also mention the paper (LOU; WETH; ZHANG, 2019) where its proved that the Morse index of radial solutions goes to infinity as $\alpha \to \infty$. Given any $\beta \in \mathbb{R}$, we set $\lfloor \beta \rfloor := \max\{n \in \mathbb{Z}; n \leq \beta\}$.

Theorem C. Let $\alpha \geq 0$ and u be a radial nodal solution of (1.1) with $N = 2$. Then

$$m_{rad}(u) \geq n(u) - 1 \quad \text{and} \quad m(u) \geq m_{rad}(u) + 2(n(u) - 1)\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right). \tag{1.3}$$

Moreover, if (f_3) holds, then

$$m_{rad}(u) \geq n(u) \ \text{ and hence } \ m(u) \geq n(u) + 2(n(u) - 1)\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right).$$

We obtain an improvement for these lower bounds.

Theorem 1.1. Let $\alpha \geq 0$ and u be a radial nodal solution of (1.1) with $N = 2$. Then

$$m_{rad}(u) \geq n(u) - 1 \quad \text{and} \quad m(u) \geq m_{rad}(u) + (m(u_0) - m_{rad}(u_0))\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right), \tag{1.4}$$

where u_0 is a radial solution with $n(u_0) = n(u)$ of the autonomous problem

$$-\Delta u_0 = \left(\frac{2}{\alpha + 2}\right)^2 f(u_0) \ \text{ in } \ \Omega_{\frac{2}{\alpha+2}} := \{|x|^{\frac{\alpha}{2}} x; x \in \Omega\}, \quad u_0 = 0 \ \text{ on } \ \partial\Omega_{\frac{2}{\alpha+2}}.$$

Moreover, if (f_3) holds, then

$$m_{rad}(u) \geq n(u) \ \text{ and hence } \ m(u) \geq n(u) + (m(u_0) - m_{rad}(u_0))\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right).$$

Remark 1.2. Observe that the lower bounds (1.4) is greater than or equal to lower bounds in (1.3), that is,

$$m(u_0) - m_{rad}(u_0) \geq 2(n(u) - 1).$$

Indeed, the above inequality is equivalent to

$$m(u_0) \geq m_{rad}(u_0) + 2(n(u) - 1) = m_{rad}(u_0) + 2(n(u_0) - 1),$$

and this is guaranteed by Theorem A. Also observe that this inequality can be strict: consider, for example, the case $f(s) = |s|^{p-1}s$, $p \gg 1$ and $n(u) = 2$, where is shown in (MARCHIS; IANNI; PACELLA, 2017a, Theorem 1.1) that $m(u_0) - m_{rad}(u_0) = 10$.

Next, we consider the particular case of the Hénon equation (P_α), where the weight $\alpha \geq 0$ is a parameter and the power $p > 1$ is fixed. We prove that, if $N = 2$, then fixed the number of nodal sets n, the Morse index of a radial solution of (P_α) with n nodal sets is monotone non-decreasing with respect to α.

Theorem 1.3 (Monotonicity of the Morse indices). Assume $N = 2$ and let u_α and u_β be radial solutions of (P_α) and (P_β), respectively, with the same number of nodal sets. If $0 \leq \alpha \leq \beta$, then $m(u_\alpha) \leq m(u_\beta)$. Moreover, the radial Morse index does not depend on weight, namely we have that $m_{rad}(u_\alpha) = n(u_\alpha)$ for any $\alpha \geq 0$.

The case of $N = 2$ is special. We may use the change of variables (2.9) with $\kappa = \frac{\beta+2}{\alpha+2}$ to establish a correspondence between the radial solutions of (P_α) with the radial solutions of (P_β) with the same number of nodal sets. Although such transformation is not available for dimensions higher than two, we conjecture that Theorem 1.3 should also hold for $N \geq 3$.

The Chapter 3 is focused on asymptotic profile of radial solutions of (P_α) with $N \geq 3$. We recall the following result on asymptotic behavior of radial solutions obtained by Byeon and Wang (BYEON; WANG, 2006, Theorem 3.1-E).

Theorem D. Let $N \geq 1$, $p > 1$ and, for each $\alpha > 0$, be u_α the ground state solution of (P_α) in the space $H^1_{0,rad}(B)$. Then the following transformation

$$W_\alpha(t) = |S^{N-1}|^p \left(\frac{N}{\alpha + N}\right)^{\frac{2}{p-1}} u_\alpha\left(\exp\left(-\frac{N}{\alpha + N}t\right)\right)$$

converges uniformly in $(0,\infty)$ as $\alpha \to \infty$ to a least energy solution (a mountain pass solution) of the problem

$$\begin{cases} W'' + \exp(-Nt)W^p = 0 \quad \text{in } (0,\infty), \\ W > 0 \quad \text{in } (0,\infty), \quad W(0) = 0. \end{cases}$$

Very recently, Kübler and Weth noted that the previous theorem holds also when the solutions u_α are sing changing radial solutions and have the same number of nodal sets. In this case, the limit solution W has the same number of nodal sets (see (KÜBLER; WETH, 2019, Proposition 1.1)). Our main goal on Chapter 3 is to show that a Lane-Emden equation serves as limit problem to (P_α). More precisely, we will prove the following.

Theorem 1.4. Fixed $m \in \mathbb{N}$, let u_α be the radial solution of (P_α) with exactly m nodal sets. Then the following transformed solution

$$v_\alpha(t) := \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha\left(t^{\frac{2}{\alpha+2}}\right), \quad t \in [0,1]$$

converges, as $\alpha \to \infty$, in $C^1([0,1]) \cap C^2([\varepsilon,1])$, for every $\varepsilon \in (0,1)$, to the radial solution of

$$-\Delta w = |w|^{p-1} w \quad \text{in } B^2 \subset \mathbb{R}^2, \quad w = 0 \quad \text{on } \partial B^2,$$

where B^2 is the unit ball in $\mathbb{R}^2$, with precisely m nodal sets.

Apart from being important itself, Theorem 1.4 implies some qualitative properties of the radial solutions of (P_α). For example, we can prove a version of Theorems 1.1 and 1.3 in dimension $N \geq 3$ for α large enough (see Corollaries 3.23 and 3.26).

Finally, we include two appendices which contain some basic results that were used throughout the chapters.

CHAPTER

2

ON THE MORSE INDEX OF RADIAL SOLUTIONS OF THE HÉNON EQUATION IN DIMENSION TWO

In this chapter, we address the question of estimating the Morse index $m(u)$ of a sign changing radial solution u of two-dimensional Hénon type equations

$$\begin{cases} -\Delta u = |x|^{\alpha} f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is either a ball or an annulus centered at the origin, $\alpha \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is $C^{1,\beta}$ on bounded sets of $\mathbb{R}$. In the specific case of the Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-1}u & \text{in } B = B(0,1) \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{P_α}$$

we present even more results. Our main contributions here are in the following theorems.

Theorem 2.1 (Monotonicity of the Morse index). Fixed $n \in \mathbb{N}$ and $p > 1$, let u_α and u_β be radial solutions of (P_α) and (P_β), respectively, with precisely n nodal sets. If $0 \leq \alpha \leq \beta$, then $m(u_\alpha) \leq m(u_\beta)$. Moreover, the radial Morse index does not depend on the weight $|x|^\alpha$, namely $m_{rad}(u_\alpha) = n$ for any $\alpha \geq 0$.

Given $\beta \in \mathbb{R}$, we set the floor function putting $\lfloor \beta \rfloor := \max\{k \in \mathbb{Z} : k \leq \beta\}$.

Theorem 2.2 (Lower bounds for the Morse indices). Let $\alpha \geq 0$ and u be a radial nodal solution of (2.1) with exactly $n \geq 2$ nodal sets. Then

$$m_{rad}(u) \geq n-1 \quad \text{and} \quad m(u) \geq m_{rad}(u) + (m(v_0) - m_{rad}(v_0))\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right),$$

where v_0 is a radial solution with n nodal sets of the autonomous problem

$$-\Delta v_0 = \left(\frac{2}{\alpha+2}\right)^2 f(v_0) \text{ in } \Omega_\kappa, \quad v_0 = 0 \text{ on } \partial\Omega_\kappa, \quad \kappa = \frac{2}{\alpha+2}, \tag{2.2}$$

$\Omega_\kappa := \{|x|^{\frac{\alpha}{2}} x : x \in \Omega\}$. Moreover, if f is a superlinear function, that is, satisfies the condition

$$f'(s) > \frac{f(s)}{s} \quad \forall s \in \mathbb{R}\setminus\{0\}, \tag{2.3}$$

then

$$m_{rad}(u) \geq n \text{ and hence } m(u) \geq n + (m(v_0) - m_{rad}(v_0))\left(\left\lfloor\frac{\alpha}{2}\right\rfloor + 1\right).$$

Corollary 2.3 (Lower bounds for the Morse indices for the Hénon equation). For each $\alpha \geq 0$, let u_α be a radial nodal solution of (P_α) with exactly $n \geq 2$ nodal sets. Then

$$m_{rad}(u_\alpha) = n \text{ and } m(u_\alpha) \geq n + (m(u_0) - m_{rad}(u_0))\left(\left\lfloor\frac{\alpha}{2}\right\rfloor + 1\right). \tag{2.4}$$

2.1 An auxiliary eigenvalue problem

In this section we recall an important decomposition for some singular eigenvalue problems. Let $N \geq 2$ and consider the sphere $S^{N-1} \subset \mathbb{R}^N$. We recall that the spherical harmonics on S^{N-1} are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$. Actually, the operator $-\Delta_{S^{N-1}}$ admits a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \ldots$ and corresponding eigenfunctions (Y_k) which form a complete orthonormal system for $L^2(S^{N-1})$. More precisely, each Y_k satisfies

$$-\Delta_{S^{N-1}} Y_k(\theta) = \lambda_k Y_k(\theta), \quad \text{for } \theta \in S^{N-1},$$

and each eigenvalue λ_k is given by the formula

$$\lambda_k = k(k+N-2), \quad k = 0, 1, \ldots \tag{2.5}$$

whose multiplicity is

$$N_0 = 1 \quad \text{and} \quad N_k = \frac{(2k+N-2)(k+N-3)!}{(N-2)!k!} \text{ for } k \geq 1.$$

Let $\Omega \subset \mathbb{R}^N$ be either a ball or an annulus centered at the origin and consider the problem

$$-\Delta\psi - a(x)\psi = \lambda\frac{\psi}{|x|^2} \text{ in } \Omega\setminus\{0\}, \quad \psi = 0 \text{ on } \partial\Omega, \tag{2.6}$$

where $a(x)$ is a radial function in $L^\infty(\Omega)$. Set

$$\mathscr{H}_0 := \left\{u \in H_0^1(\Omega); \int_\Omega \frac{u^2}{|x|^2} < \infty\right\}.$$

Then, endowed with inner product

$$(u,v)_{\mathscr{H}_0} := \int_\Omega \nabla u \nabla v + \frac{uv}{|x|^2} dx, \quad u, v \in \mathscr{H}_0,$$

$\mathscr{H}_0$ is a Hilbert space. We say that $\psi \in \mathscr{H}_0 \backslash \{0\}$ is an eigenfunction of (2.6), if

$$\int_\Omega \nabla\psi\nabla\varphi - a(x)\psi\varphi dx = \lambda \int_\Omega \frac{\psi\varphi}{|x|^2}dx \quad \forall \varphi \in \mathscr{H}_0.$$

We recall the following result on the decomposition of eigenvalues of (2.6); see for instance (BARTSCH *et al.*, 2012, Lemma 3.1) or Appendix B, Proposition B.10.

Proposition 2.4. Let $\lambda < 0$ be an eigenvalue of (2.6). Then, there exists $k \geq 0$ such that

$$\lambda = \lambda^{rad} + \lambda_k, \tag{2.7}$$

where λ^{rad} is a radial eigenvalue of (2.6) and λ_k as in (2.5). Conversely, if (2.7) holds and ψ^{rad} is an eigenfunction associated to λ^{rad}, then $\psi = \psi^{rad}(r)Y_k(\theta)$ is an eigenfunction of (2.6) associated to λ.

From previous preposition, we have the following corollary.

Corollary 2.5 (A Morse index formula)**.** Let u be a radial solution to

$$-\Delta u = g(|x|,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is either a ball or an annulus. Then

$$m(u) = m_{rad}(u) + 2\sum_{n=1}^{m_{rad}(u)} \#\mathscr{N}_n, \tag{2.8}$$

where, for each $n = 1,\ldots,m_{rad}(u)$, $\mathscr{N}_n := \{k \in \mathbb{N}; \lambda_n^{rad} + k^2 < 0\}$ and λ_n^{rad} is the n^{th} radial eigenvalue of (2.6) with $a(x) = g_u(|x|,u(x))$.

Proof. By Lemma B.8 and Remark B.9, in order to get (2.8), it suffices to count the number of negative nonradial eigenvalues of (2.6). Indeed, by Proposition 2.4, each of these eigenvalues have the decomposition (2.7) for $k \geq 1$. Since radial eigenvalues are simple and, in dimension two, $\lambda_k = k^2$ has multiplicity 2 for $k \geq 1$, we have precisely $2\sum_{n=1}^{m_{rad}(u)} \#\mathscr{N}_n$ negative nonradial eigenvalues. Adding the number of negative radial eigenvalues, which is $m_{rad}(u)$, we get (2.8). □

2.2 Monotonicity of the Morse index

The goal this section is to prove that the Morse index of radial solutions of the two-dimensional Hénon equation

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B = B(0,1) \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{P_α}$$

is nondecreasing with respect to weight α, where $p > 1$ is fixed. For this, given $\kappa > 0$, consider the following transformation

$$T_\kappa : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_\kappa(0) = 0 \text{ and } T_\kappa(y) := |y|^{\kappa-1}y \text{ for } y \neq 0. \tag{2.9}$$

We will perform the change of variable $x \leftrightarrow y$ putting $x = T_\kappa(y)$. By direct computation, observe that T_κ has the following properties:

(i) T_κ is a diffeomorphism from $\mathbb{R}^2\backslash\{0\}$ into $\mathbb{R}^2\backslash\{0\}$ whose inverse is

$$T_\kappa^{-1}(x) = |x|^{\frac{1}{\kappa}-1}x, \text{ i.e. } T_\kappa^{-1} = T_{\frac{1}{\kappa}}. \tag{2.10}$$

(ii) In cartesian coordinates,

$$|\det J_{T_\kappa}(y)| = \kappa|y|^{2\kappa-2}, \ \forall y \neq 0. \tag{2.11}$$

Let $\Omega \subset \mathbb{R}^2$ be either an annulus or a ball centered at the origin and set $\Omega_\kappa := T_\kappa^{-1}(\Omega)$. We will use the following result obtained in (SANTOS; PACELLA, 2017, Lemma 2.4).

Lemma 2.6. The map

$$S_\kappa : H_0^1(\Omega_\kappa) \to H_0^1(\Omega), \ S_\kappa\psi := \psi \circ T_\kappa^{-1},$$

is a continuous linear isomorphism. Moreover, putting $\varphi = \psi \circ T_\kappa^{-1}$,

$$\min\left\{\kappa, \frac{1}{\kappa}\right\} \int_\Omega |\nabla\varphi(x)|^2 dx \le \int_{\Omega_\kappa} |\nabla\psi(y)|^2 dy \le \max\left\{\kappa, \frac{1}{\kappa}\right\} \int_\Omega |\nabla\varphi(x)|^2 dx, \ \forall \psi \in H_0^1(\Omega_\kappa),$$

$$\kappa \int_\Omega |\nabla\varphi(x)|^2 dx = \int_{\Omega_\kappa} |\nabla\psi(y)|^2 dy, \ \forall \psi \in H_{0,rad}^1(\Omega_\kappa).$$

Given a radial function $u : \Omega \to \mathbb{R}$, set $v : \Omega_\kappa \to \mathbb{R}$ by $v(y) = u(T_\kappa y)$. Then v is also radially symmetric and, by (SANTOS; PACELLA, 2017, eq. (2.13)),

$$\Delta v(y) = \kappa^2|x|^{\frac{2\kappa-2}{\kappa}}\Delta u(x). \tag{2.12}$$

Thus, if u is a radial solution of (2.1), then $v : \Omega_\kappa \to \mathbb{R}$ satisfies

$$-\Delta v(y) = \kappa^2|y|^{2\kappa-2+\kappa\alpha} f(v(y)), \quad y \in \Omega_\kappa, \quad v = 0 \quad \text{on } \partial\Omega_\kappa.$$

Now, given any $\beta \ge 0$, we choose κ so that

$$2\kappa - 2 + \kappa\alpha = \beta, \quad \text{i.e.} \quad \kappa := \frac{\beta+2}{\alpha+2}, \tag{2.13}$$

and so

$$-\Delta v(y) = \left(\frac{\beta+2}{\alpha+2}\right)^2 |y|^\beta f(v(y)), \quad y \in \Omega_\kappa, \quad v = 0 \quad \text{on } \partial\Omega_\kappa. \tag{2.14}$$

In the particular case with $f(s) = |s|^{p-1}s$, setting $w(y) = \left(\frac{\beta+2}{\alpha+2}\right)^{\frac{2}{p-1}} v(y)$, we get

$$-\Delta w(y) = |y|^\beta |w(y)|^{p-1} w(y), \quad y \in \Omega_\kappa, \quad v = 0 \quad \text{on } \partial\Omega_\kappa.$$

Therefore, we have proved the following result.

Lemma 2.7. u_α is a radial solution of (P_α) in Ω with n nodal sets if, and only if,

$$u_\beta(y) = \left(\frac{\beta+2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha\left(|y|^{\frac{\beta-\alpha}{\alpha+2}}y\right), \quad y \in \Omega_\kappa, \quad \kappa = \frac{\beta+2}{\alpha+2}, \tag{2.15}$$

is a radial solution of (P_β) in Ω_κ with n nodal sets.

Given $\alpha \geq 0$, $n \in \mathbb{N}$, we know that there exists a unique solution of (P_α) (up to multiplication by -1) with n precisely nodal sets; see for instance (SANTOS; PACELLA, 2017, Theorem 1.3 (i)). Let u_α and u_β be radial solutions of (P_α) and (P_β), respectively, with the same number of nodal sets. Then u_α and u_β are related by (2.15) and $m(u_\alpha)$ is the maximal dimension of a subspace of $H_0^1(B)$ in which the quadratic form

$$H_0^1(B) \ni w \mapsto Q_\alpha(w) := \int_B |\nabla w(x)|^2 dx - p\int_B |x|^\alpha |u_\alpha(x)|^{p-1} w^2(x)dx$$

is negative definite. Similarly we can compute $m(u_\beta)$. We have the following result.

Proposition 2.8. If $0 \leq \alpha \leq \beta$, then

$$Q_\beta(w_\kappa) \leq \kappa Q_\alpha(w), \ \ \forall w \in H_0^1(B), \qquad Q_\beta(w_\kappa) = \kappa Q_\alpha(w), \ \ \forall w \in H_{0,rad}^1(B),$$

where $w_\kappa(y) = w \circ T_\kappa(y)$, T_κ is defined as in (2.9) with $\kappa = \frac{\beta+2}{\alpha+2}$.

Proof. By (2.15), up to multiplication by -1,

$$u_\beta(y) = \kappa^{\frac{2}{p-1}} u_\alpha(T_\kappa y).$$

Hence

$$Q_\beta(w_\kappa) = \int_B |\nabla w_\kappa(y)|^2 dy - \kappa^2 p \int_B |y|^\beta |u_\alpha(T_\kappa y)|^{p-1} w_\kappa^2(y)dy.$$

Since $\kappa \geq 1$, it follows from Lemma 2.6 that

$$\int_B |\nabla w_\kappa(y)|^2 dy \leq \max\left\{\kappa, \frac{1}{\kappa}\right\} \int_B |\nabla w(x)|^2 dx = \kappa \int_B |\nabla w(x)|^2 dx \ \ \forall w \in H_0^1(B)$$

and

$$\int_B |\nabla w_\kappa(y)|^2 dy = \kappa \int_B |\nabla w(x)|^2 dx \quad \forall w \in H_{0,rad}^1(B).$$

Now, putting $x = T_\kappa(y)$, it follows from (2.10) and (2.11) that $dy = \kappa^{-1}|x|^{\frac{2-2\kappa}{\kappa}} dx$. Thus

$$\begin{aligned} \kappa^2 p \int_B |y|^\beta |u_\alpha(T_\kappa y)|^{p-1} w_\kappa^2(y)dy &= \kappa p \int_B |x|^{\frac{\beta-2\kappa+2}{\kappa}} |u_\alpha(x)|^{p-1} w^2(x)dx \\ &= \kappa p \int_B |x|^\alpha |u_\alpha(x)|^{p-1} w^2(x)dx, \end{aligned}$$

since $\frac{\beta-2\kappa+2}{\kappa} = \alpha$. Therefore,

$$Q_\beta(w_\kappa) \leq \kappa \int_B |\nabla w(x)|^2 dx - \kappa p \int_B |x|^\alpha |u_\alpha(x)|^{p-1} w^2(x)dx = \kappa Q_\alpha(w) \ \ \forall w \in H_0^1(B)$$

and

$$Q_\beta(w_\kappa) = \kappa Q_\alpha(w) \ \ \forall w \in H_{0,rad}^1(B).$$

□

As a consequence of the previous proposition, we can prove Theorem 2.1.

Proof of Theorem 2.1. From Proposition 2.8, we have

$$Q_\beta(w_\kappa) \le \kappa Q_\alpha(w), \quad \forall w \in H_0^1(B), \ Q_\beta(w_\kappa) = \kappa Q_\alpha(w), \quad \forall w \in H^1_{0,rad}(B).$$

Therefore, if V is a subspace of $H_0^1(B)$ in which the quadratic form Q_α is negative definite, then Q_β is also negative definite in the subspace $V_\kappa := \{w \circ T_\kappa; w \in V\}$. Moreover, Q_α is negative definite in a subspace V of $H^1_{0,rad}(B)$ if, and only if, Q_β is negative definite in subspace V_κ. Since V and V_κ have the same dimension, we infer that $m(u_\alpha) \le m(u_\beta)$ and $m_{rad}(u_\alpha) = m_{rad}(u_\beta)$. Finally, we know from (HARRABI; REBHI; SELMI, 2011, Proposition 2.9) that $m_{rad}(u_0) = n$ and this shows that $m(u_\alpha) = n$ for all α. □

2.3 Lower bounds for the Morse indices

In this section, we prove some lower bounds for the Morse indices of the radial solutions of semilinear non-autonomous problems. More precisely, we present the

Proof of Theorem 2.2. **Step 1**. We first prove the case where α is an even integer. Write then $\alpha = 2(m-1)$, $m \in \mathbb{N}$, and $\kappa = \frac{1}{m}$. Thus $\kappa = \frac{2}{\alpha+2}$ and, by (2.14) ($\beta = 0$ in this case), the function $v_0 = u \circ T_\kappa$ is a radial nodal solution with n nodal sets of the autonomous problem (2.2). Therefore, by Lemma B.8 and Remark B.9, the singular eigenvalue problem

$$-\Delta\psi - \frac{1}{m^2} f'(v_0)\psi = \lambda \frac{\psi}{|y|^2} \ \text{ in } \ \Omega_\kappa \backslash \{0\}, \quad \psi = 0 \ \text{ on } \ \partial\Omega_\kappa, \tag{2.16}$$

has $m(v_0) - m_{rad}(v_0)$ negative eigenvalues associated to nonradial eigenfunctions, counted with their multiplicity. We count the negative radial eigenvalues of (2.16) as $\lambda_1 = \lambda_1^{rad} < \lambda_2^{rad} < \ldots < \lambda^{rad}_{m_{rad}(v_0)}$ whose corresponding eigenfunctions we denote by ψ_n^{rad}. By Corollary 2.5,

$$m(v_0) - m_{rad}(v_0) = 2 \sum_{n=1}^{m_{rad}(u_0)} \#\mathcal{N}_n, \tag{2.17}$$

where $\mathcal{N}_n := \{k \in \mathbb{N}; \lambda_n^{rad} + k^2 < 0\}$. Moreover, using eq. (2.12) with $\kappa = \frac{2}{\alpha+2}$, we have that the functions $\varphi_n^{rad} = \psi_n^{rad} \circ T_\kappa^{-1}$, $n = 1, \ldots, m_{rad}(v_0)$, are the radial eigenfunctions of

$$-\Delta\varphi - |x|^\alpha f'(u)\varphi = \lambda \frac{\varphi}{|x|^2} \ \text{ in } \ \Omega \backslash \{0\}, \quad \varphi = 0 \ \text{ on } \ \partial\Omega, \tag{2.18}$$

with $\lambda = m^2 \lambda_n^{rad} < 0$. Then $m_{rad}(u) = m_{rad}(v_0)$ and with $\mathcal{N}_n^\alpha := \{k \in \mathbb{N}; m^2\lambda_n^{rad} + k^2 < 0\}$,

$$m(u) - m_{rad}(u) = 2 \sum_{n=1}^{m_{rad}(v_0)} \#\mathcal{N}_n^\alpha. \tag{2.19}$$

We claim that

$$\#\mathcal{N}_n^{\alpha} \geq m(\#\mathcal{N}_n) \quad \forall n = 1, \ldots, m_{rad}(v_0). \tag{2.20}$$

Indeed, if $k \in \mathcal{N}_n$, then $\lambda_n^{rad} + k^2 < 0$, whence $m^2\lambda_n^{rad} + (mk)^2 < 0$. The latter shows that $mk \in \mathcal{N}_n^{\alpha}$ and this proves (2.20). Therefore, from (2.17), (2.19), (2.20), we infer that

$$m(u) - m_{rad}(u) \geq (m(v_0) - m_{rad}(v_0))m = (m(v_0) - m_{rad}(v_0))\left(\frac{\alpha+2}{2}\right).$$

Now, with respect to the radially symmetric eigenfunctions, it is proved (MARCHIS; IANNI; PACELLA, 2017b, Theorem 2.1) that (2.16) has at least $n-1$ negative eigenvalues associated to radial eigenfunctions, and this number becomes n if (2.3) holds. Again using eq. (2.12) with $\kappa = \frac{2}{\alpha+2}$, we have that $\lambda \mapsto m^2\lambda$ is a bijection between radial eigenvalues of (2.16) and (2.18) and hence we obtain the lower bounds for $m_{rad}(u)$.

Step 2. Assume now that α is any positive number. Then, by (2.14), for all $\gamma \geq 0$, the function $v = u \circ T_\kappa$ is a radial nodal solution with n nodal sets of

$$-\Delta v = \kappa^2 |y|^{\gamma} f(v) \;\text{ in }\; \Omega_\kappa, \quad v = 0 \;\text{ on }\; \partial\Omega_\kappa, \quad \text{with }\; \kappa = \frac{\gamma+2}{\alpha+2}.$$

Hence, we can check, as in the proof of Proposition 2.8, that if $\gamma \leq \alpha$, i.e. $\kappa \leq 1$, then it follows from Lemma 2.6 and (2.11) that

$$\int_\Omega |\nabla w(x)|^2 dx - \int_\Omega |x|^{\alpha} f'(u(x)) w^2(x) dx \leq \frac{1}{\kappa}\left[\int_{\Omega_\kappa} |\nabla w_\kappa(y)|^2 dy - \kappa^2 \int_{\Omega_\kappa} |y|^{\gamma} f'(v(y)) w^2(y) dy\right],$$

for all $w \in H_0^1(\Omega)$ and the equality holds for all $w \in H^1_{0,rad}(\Omega)$, where $w_\kappa = w \circ T_\kappa$. Consequently, $m_{rad}(u) = m_{rad}(v)$ and $m(u) \geq m(v)$. In particular, taking $\gamma = 2\lfloor\frac{\alpha}{2}\rfloor$ we may use Step 1 for v to obtain

$$m_{rad}(v) \geq n-1, \;\; m_{rad}(v) \geq n \text{ if (2.3) holds and}$$

$$m(v) \geq m_{rad}(v) + (m(v_0) - m_{rad}(v_0))\left(\frac{\gamma+2}{2}\right) = m_{rad}(u) + (m(v_0) - m_{rad}(v_0))\left(\left\lfloor\frac{\alpha}{2}\right\rfloor + 1\right),$$

where v_0 is a radial solution of (2.2) with $n(v_0) = n$. □

Remark 2.9. (i) Observe that the key argument in the proof of Theorem 2.2 is the monotonicity of the Morse indices $m(u) \geq m(v)$ proved above, thanks to $\gamma \leq \alpha$.

(ii) Our lower bounds for the Morse indices are in function of the lower bounds of radial solutions of the corresponding autonomous problem (2.2). However, lower bounds for the Morse index in autonomous problems have been established in (MARCHIS; IANNI; PACELLA, 2017b), where the authors show that $m(v_0) - m_{rad}(v_0) \geq 2(n-1)$.

(iii) Of course the Morse index of the radial nodal solutions of (2.1) goes to infinite as $\alpha \to \infty$.

Proof of Corollary 2.3. By Theorem 2.2, we have

$$m(u_\alpha) \geq n + (m(v_0) - m_{rad}(v_0))\left(\left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right),$$

where v_0 satisfies the problem

$$-\Delta v_0 = \left(\frac{2}{\alpha+2}\right)^2 |v_0|^{p-1} v_0 \ \text{ in } \ B, \quad v_0 = 0 \ \text{ on } \ \partial B. \tag{2.21}$$

Observe that in this case we have $B = \Omega = \Omega_\kappa$. Putting then $u_0 = \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} v_0$, we can check that u_0 satisfies

$$-\Delta u_0 = |u_0|^{p-1} u_0 \ \text{ in } \ B, \quad u_0 = 0 \ \text{ on } \ \partial B, \tag{2.22}$$

that is, u_0 is a radial solution of (P_0) and $n(u_0) = n$. Observe that, the quadratic form associated to (2.21) is given by

$$H_0^1(B) \ni w \mapsto Q_{v_0}(w) := \int_B |\nabla w(x)|^2 dx - p\left(\frac{2}{\alpha+2}\right)^2 \int_B |v_0|^{p-1} w^2(x) dx,$$

while the quadratic form associated to (2.22) is given by

$$H_0^1(B) \ni w \mapsto Q_{u_0}(w) := \int_B |\nabla w(x)|^2 dx - p \int_B |u_0|^{p-1} w^2(x) dx.$$

Since $u_0 = \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} v_0$, we have $Q_{v_0}(w) = Q_{u_0}(w)$ for all $w \in H_0^1(B)$, which implies that $m(u_0) = m(v_0)$ and we get (2.4). □

2.4 An alternative proof

In this section we present an alternative proof for Theorems 2.1 and 2.2. In fact, we can write them as follows.

Theorem 2.10. Let $n \in \mathbb{N}$ be fixed. For each index $\alpha \geq 0$, we denote by u_α a radial solution of the Hénon type equation (2.1) with n nodal sets. Then, writing $v_\alpha = u_\beta \circ T_\kappa$, $\kappa = \frac{\alpha+2}{\beta+2}$ with $\beta \geq \alpha$, v_α is solution of

$$-\Delta v = \kappa^2 |y|^\alpha f(v) \ \text{ in } \ \Omega_\kappa, \quad v = 0 \ \text{ on } \ \partial\Omega_\kappa,$$

with exactly n nodal sets. Moreover,

$$m(u_\beta) \geq m_{rad}(u_\beta) + (m(v_\alpha) - m_{rad}(v_\alpha))\left\lfloor \frac{\beta+2}{\alpha+2} \right\rfloor.$$

Remark 2.11. (i) Notice that when $\alpha = 0$ in the above inequality, we get Theorem 2.2.

(ii) Furthermore, if $f(s) = |s|^{p-1}s$ and $\Omega = B$ is the unit ball, by Lemma 2.7, the function $w_\alpha = \kappa^{\frac{2}{p-1}} v_\alpha$ is the unique radial solution, up to multiplication by -1, of (P_α) with precisely n nodal sets. Hence $w_\alpha = u_\alpha$. Now, as in the proof of Corollary 2.3, observe that u_α and v_α give rise to the same quadratic form Q_α, which implies that $m(v_\alpha) = m(u_\alpha)$. Consequently,

$$m(u_\beta) \geq m_{rad}(u_\beta) + (m(u_\alpha) - m_{rad}(u_\alpha)) \left\lfloor \frac{\beta+2}{\alpha+2} \right\rfloor.$$

Since $m_{rad}(u_\beta) = m_{rad}(u_\alpha)$ and $\beta \geq \alpha$, we have $\left\lfloor \frac{\beta+2}{\alpha+2} \right\rfloor \geq 1$ and hence $m(u_\beta) \geq m(u_\alpha)$, which shows Theorem 2.1.

Proof. We count the negative radial eigenvalues of the singular problem

$$-\Delta\psi - \kappa^2 |y|^\alpha f'(v_\alpha)\psi = \lambda \frac{\psi}{|y|^2} \text{ in } \Omega_\kappa \backslash \{0\}, \quad \psi = 0 \text{ on } \partial\Omega_\kappa, \;\; \kappa = \frac{\alpha+2}{\beta+2},$$

as $\lambda_1^\alpha < \lambda_2^\alpha < \ldots < \lambda_{m_{rad}(v_\alpha)}^\alpha$ whose corresponding eigenfunctions we denote by ψ_n^α, for $n = 1, 2, \ldots, m_{rad}$. Thus, by Lemma 2.8

$$m(v_\alpha) - m_{rad}(v_\alpha) = 2 \sum_{n=1}^{m_{rad}(v_\alpha)} \#\mathcal{N}_n^\alpha, \tag{2.23}$$

where $\mathcal{N}_n^\alpha := \{k \in \mathbb{N}; \lambda_n^\alpha + k^2 < 0\}$. Moreover, using eq. (2.12) with $\kappa = \frac{\alpha+2}{\beta+2}$, we have that the functions $\varphi_n^\beta = \psi_n^\alpha \circ T_\kappa^{-1}$, $n = 1, \ldots, m_{rad}(v_\alpha)$, are the radial eigenfunctions of

$$-\Delta\varphi - |x|^\beta f'(u_\beta)\varphi = \Lambda \frac{\varphi}{|x|^2} \text{ in } \Omega \backslash \{0\}, \quad \varphi = 0 \text{ on } \partial\Omega,$$

with $\Lambda = \Lambda_n = \kappa^{-2}\lambda_n^\alpha < 0$, and vice-versa. Then $m_{rad}(u_\beta) = m_{rad}(v_\alpha)$ and with $\mathcal{N}_n^\beta := \{k \in \mathbb{N}; \kappa^{-2}\lambda_n^\alpha + k^2 < 0\}$,

$$m(u_\beta) - m_{rad}(u_\beta) = 2 \sum_{n=1}^{m_{rad}(v_\alpha)} \#\mathcal{N}_n^\beta. \tag{2.24}$$

We claim that

$$\#\mathcal{N}_n^\beta \geq \lfloor \kappa^{-1} \rfloor (\#\mathcal{N}_n^\alpha) \quad \forall n = 1, \ldots, m_{rad}(v_\alpha). \tag{2.25}$$

Indeed, if $k \in \mathcal{N}_n^\alpha$, then $\lambda_n^\alpha + k^2 < 0$, meaning that $\kappa^{-2}\lambda_n^\alpha + (\kappa^{-1}k)^2 < 0$, and hence $\kappa^{-2}\lambda_n^\alpha + (\lfloor \kappa^{-1} \rfloor k)^2 < 0$. The latter shows that $\lfloor \kappa^{-1} \rfloor k \in \mathcal{N}_n^\beta$ and this proves (2.25). Therefore, from (2.23), (2.24), (2.25), we infer that

$$m(u_\beta) - m_{rad}(u_\beta) \geq (m(v_\alpha) - m_{rad}(v_\alpha)) \lfloor \kappa^{-1} \rfloor = (m(v_\alpha) - m_{rad}(v_\alpha)) \left\lfloor \frac{\alpha+2}{\beta+2} \right\rfloor,$$

as we wanted to prove. □

2.5 Comments on Chapter 2

Monotonicity of the Morse index in high dimensions

In this chapter, considering the case of two-dimensional problems was crucial, since we may use the change of variables (2.9) to establish a correspondence between radial solutions, with the same number of nodal sets, of (P_α) with different weights $|x|^\alpha$. Actually, in dimensions higher than two, we could not find a transformation that guarantees a relation of laplacians as in (2.12) and, consequently, we could not compare radial solutions of the Hénon equation with different weights. Even so, we conjecture that Theorem 1.3 should also hold for $N \geq 3$.

A path to prove the monotonicity in high dimensions would be the following. Observe that Proposition 2.4 holds in any dimension $N \geq 2$. Therefore, the Morse index of a radial solution u_α of (P_α) just depends on the knowledge of the negative radial singular eigenvalues λ_n^α of the linearized operator L_{u_α}, since the eigenvalues λ_k defined in (2.5) do not depend on the weight $|x|^\alpha$. In fact, using the decomposition (2.7), it is easy to see that if we had the monotonicity of the radial eigenvalues $\lambda_n^\beta \leq \lambda_n^\alpha$ whenever $\alpha < \beta$, then we could get the monotonicity of the Morse index. Note that, in dimension two, the above monotonicity is satisfied because we have the relation $\lambda_n^\beta = \left(\frac{\beta+2}{\alpha+2}\right)\lambda_n^\alpha$ between these eigenvalues (see Section 2.4).

Asymptotic behavior of radial solutions of the Hénon equation

Fixed $m \in \mathbb{N}$, we denote by u_α, for each $\alpha \geq 0$, the unique radial solution with m nodal sets of the two-dimensional Hénon equation (P_α) such that $u_\alpha(0) > 0$. Then u_α has exactly $m-1$ zeros in $(0,1)$. Let $r_{1,\alpha} < \ldots < r_{m,\alpha} = 1$ be the zeros of u_α in $(0,1]$. In each nodal interval $[0, r_{1,\alpha}], \ldots [r_{m-1,\alpha}, r_{m,\alpha}]$, u_α has a global extremum (either a maximum or a minimum depending on the sign of u_α in this interval). For each $i = 1, \ldots, m-1$, we set

$$\mathscr{M}_{1,\alpha} := \max\{u_\alpha(x) : 0 \leq |x| \leq r_{1,\alpha}\}, \quad \mathscr{M}_{i+1,\alpha} := \max\{|u_\alpha(x)| : r_{i,\alpha} \leq |x| \leq r_{i+1,\alpha}\}.$$

We would like to know what happens with sequences $(r_{i,\alpha})$ and $(\mathscr{M}_{i,\alpha})$ as $\alpha \to \infty$. In dimension two, Lemma 2.7 already gives us the answer. Indeed, for all $\alpha > 0$, u_α is given in its radial coordinate as

$$u_\alpha(r) = \left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-1}} u_0\left(r^{\frac{\alpha+2}{2}}\right), \quad r \in (0,1).$$

Hence, we have that

$$r_{i,\alpha} = (r_{i,0})^{\frac{2}{\alpha+2}} \quad \text{and} \quad \mathscr{M}_{i,\alpha} = \left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-1}} \mathscr{M}_{i,0},$$

which shows in particular that

$$r_{i,\alpha} \to 1 \quad \text{and} \quad \mathscr{M}_{i,\alpha} \to \infty, \quad \text{as } \alpha \to \infty$$

for each $i = 1, \cdots, m$. Actually, we have the same asymptotic profile for high dimensions, as we can see in Chapter 3, Lemma 3.14 and Corollary 3.16. This motivates the next chapter.

CHAPTER

3

ASYMPTOTIC PROFILE OF RADIAL SOLUTIONS OF THE HÉNON EQUATION

In this chapter, we consider the N-dimensional Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{P_α}$$

where $B \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 3$, $p > 1$ is a fixed power and $\alpha > 0$ is a parameter. We investigate the asymptotic concentration of the radially symmetric solutions of (P_α) with prescribed number of nodal sets as $\alpha \to \infty$. More precisely, we show that, after a suitable rescaling, the two-dimensional Lane-Emden equation

$$-\Delta w = |w|^{p-1}w \quad \text{in } B^2 \subset \mathbb{R}^2, \quad w = 0 \quad \text{on } \partial B^2, \tag{L}$$

where B^2 is the unit ball, can be seen as a limit problem to (P_α) when α goes to infinity. In fact, our main result here is the following.

Theorem 3.1. For $m \in \mathbb{N}$ and $p > 1$ fixed, let u_α be the radial solution of (P_α) with precisely m nodal sets such that $u_\alpha(0) > 0$. Then

$$v_\alpha(t) := \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha\left(t^{\frac{2}{\alpha+2}}\right), \quad t \in [0,1] \tag{3.1}$$

converges to w in $C^1([0,1]) \cap C^2([\varepsilon,1])$ as $\alpha \to \infty$, for all $\varepsilon \in (0,1)$, where w is the unique radial solution of (L) with exactly m nodal sets such that $w(0) > 0$.

Apart from its applications, Theorem 3.1 implies some qualitative consequences which are presented in Section 3.5.

3.1 Qualitative analysis of solutions to semilinear Sturm-Liouville problems

Motivated by the formula for the Laplacian operator of radial functions, in this section we consider semilinear differential equations of the type

$$\begin{cases} -(t^{M-1}v')' = t^{M-1}g(v) \quad \text{in } (0,1), \\ v(0)>0,\ v'(0)=v(1)=0, \end{cases} \tag{3.2}$$

where $M>1$ is constant (not necessarily an integer), and $g:\mathbb{R}\to\mathbb{R}$ is such that $v\mapsto g'(v)$ is $C^{0,\beta}$ on bounded sets of $\mathbb{R}$. For a solution $v\in C^2([0,1])$ of (3.2) with precisely $m\geq 1$ nodal intervals, we denote by $0<t_1<\cdots<t_m=1$ the zeros of v in $[0,1]$ so that $v(t)=0$ if and only if $t=t_i$ for some i. Assuming $v(0)>0$, we set

$$\mathscr{M}_1 := \max\{v(t): 0\leq t\leq t_1\}, \quad \mathscr{M}_{i+1} := \max\{|v(t)|: t_i\leq t\leq t_{i+1}\}$$

for $i=1,\cdots,m-1$. With respect to the local extrema $\mathscr{M}_i$, one has the following proposition; see also (AMADORI; GLADIALI, 2018, Lemma 5.4).

Proposition 3.2. Suppose in addition that the nonlinear term g satisfies the following sign condition

$$\frac{g(s)}{s} > 0, \quad \forall s\in\mathbb{R}\backslash\{0\}. \tag{3.3}$$

Then v is strictly decreasing in its first nodal interval (starting from 0) so that

$$v(0)=\mathscr{M}_1.$$

Moreover $(-1)^i v$ is positive in (t_i,t_{i+1}) and v has a unique critical point s_i in each nodal interval $[t_i,t_{i+1}]$, $i=1,\cdots,m-1$, with

$$\mathscr{M}_1>\mathscr{M}_3>\cdots \quad \text{and} \quad \mathscr{M}_2>\mathscr{M}_4>\cdots.$$

In particular 0 is the global maximum point and s_1 is the global minimum point of v. Furthermore, if the function $v\mapsto g(v)$ is odd, then

$$\mathscr{M}_1>\mathscr{M}_2>\cdots>\mathscr{M}_m$$

and consequently $v(0)=\|v\|_\infty$.

Proof. Since v is positive in $(0,t_1)$ and g satisfies (3.3), we integrate (3.2) from 0 to s to get

$$v'(s) = -\frac{1}{s^{M-1}}\int_0^s t^{M-1}g(v(t))dt = -\frac{1}{s^{M-1}}\int_0^s t^{M-1}\frac{g(v(t))}{v}v\,dt<0 \quad \forall s\in(0,t_1),$$

which implies that v is strictly decreasing in $[0,t_1)$ and thus $v(0)=\mathscr{M}_1$. Observe that we can write (3.2) as

$$-v''-\frac{M-1}{t}v' = g(v), \quad t\in(0,1). \tag{3.4}$$

Multiplying the above equation by v' and integrating from 0 to s again, we obtain

$$-\int_0^s v''(t)v(t)'dt - (M-1)\int_0^s \frac{(v'(t))^2}{t}dt = \int_0^s g(v(t))v'(t)dt.$$

Note that

$$\int_0^s v''(t)v(t)'dt = \frac{1}{2}\int_0^s [(v'(t))^2]'dt = \frac{1}{2}(v'(s))^2,$$

since $v'(0) = 0$. Moreover, if $G(t) = \int_0^t g(s)ds$, we have that

$$\int_0^s g(v(t))v'(t)dt = \int_0^s [G(v(t))]'dt = G(v(s)) - G(v(0)). \tag{3.5}$$

Hence

$$\frac{1}{2}(v'(s))^2 + (M-1)\int_0^s \frac{(v'(t))^2}{t}dt = G(v(0)) - G(v(s)), \quad \forall s \in (0,1). \tag{3.6}$$

Observe that the left hand side of (3.6) is strictly positive, which shows that $G(v(0)) > G(v(s))$ for all $s \in (0,1)$. It follows then that $v(0) \neq v(s)$ and by continuity of v, we have that either $v(0) > v(s)$ or $v(0) < v(s)$ for all $s \in (0,1)$. Since $v(0) = \mathscr{M}_1$, we conclude that $v(0) > v(s)$ so that 0 is the global maximum point of v.

Now we repeat the same argument in the nodal intervals (t_i, t_{i+1}), $i = 1, \cdots, m-1$, in order to show that $|v|$ is strictly increasing in $[t_i, s_i]$ and it is strictly decreasing in $[s_i, t_{i+1}]$. Here $s_i \in (t_i, t_{i+1})$ is such that $|v(s_i)| = \mathscr{M}_i$. Indeed, integrating (3.2) from s_i to s, we get

$$v'(s) = -\frac{1}{s^{M-1}}\int_{s_i}^s t^{M-1}\frac{g(v(t))}{v}v\,dt \quad \forall s \in (t_i, t_{i+1}),$$

since $v'(s_i) = 0$. Thus for $s \in (t_i, t_{i+1})$, we have

$$(|v(s)|)' = v'(s)\mathrm{sgn}(v) = -\frac{1}{s^{M-1}}\int_{s_i}^s t^{M-1}\frac{g(v(t))}{v}|v|\,dt \begin{cases} >0 & \text{if } t_i < s < s_i, \\ <0 & \text{if } s_i < s < t_{i+1}. \end{cases}$$

This implies that s_i is the unique critical point of v in (t_i, t_{i+1}). Moreover, due to unique continuation principle, notice that the zeros t_i cannot be critical points which implies that $0, s_1, \cdots, s_{m-1}$ are the only critical points of v and $(-1)^i v > 0$ in (t_i, t_{i+1}).

Next, multiplying the equation (3.4) by v' and in integrating from s_i to s, we obtain

$$\frac{1}{2}v'(s)^2 + (M-1)\int_{s_i}^s \frac{(v'(t))^2}{t}dt = G(v(s_i)) - G(v(s)), \quad \forall s \in (0,1),$$

so that $G(v(s_i)) > G(v(s))$ for all $s \in (s_i, 1]$. Then observe that (3.3) implies that

$$G \text{ is increasing on } [0,\infty), \text{ decreasing on } (-\infty,0] \text{ and } G(0) = 0.$$

From these, if $v(s_i) > 0$, then it is the unique global maximum point of v restricted to $[s_i, 1]$. Conversely, if $v(s_i) < 0$, then it is the unique global minimum point of v restricted to $[s_i, 1]$. Consequently,

$$\mathscr{M}_1 > \mathscr{M}_3 > \cdots \quad \text{and} \quad \mathscr{M}_2 > \mathscr{M}_4 > \cdots.$$

Finally, if g is odd, then G is even and hence $G(v(0)) > G(|v(s_i)|) > G(|v(s)|)$ for all $s \in (s_i, 1]$. This implies that s_i is the unique global maximum point of $|v|$ restricted to $[s_i, 1]$ and therefore $\mathscr{M}_1 > \mathscr{M}_2 > \cdots > \mathscr{M}_m$. □

As a consequence of the previous proposition, as observed in (AMADORI; GLADIALI, 2018), we have the following (see Figure 1).

Corollary 3.3. Let u be a radial solution with $m \geq 1$ nodal sets of the equation

$$\begin{cases} -\Delta u = |x|^{\alpha} f(u) \text{ in } B \subset \mathbb{R}^N, \\ u(0) > 0, \quad u = 0 \text{ on } \partial B, \end{cases} \tag{3.7}$$

where $\alpha \geq 0$ and $f \in C^1(\mathbb{R})$ satisfies $f(s)s > 0$ for all $s \neq 0$. Then it holds the following qualitative properties for $u(r) = u(|x|)$:

(i) u has precisely m critical points, namely 0 and $s_i \in (r_i, r_{i+1})$, where s_i is the maximum point of $|u|$ in (r_i, r_{i+1}) for $i = 1, \cdots, m-1$. Here $r_1, \cdots, r_m = 1$ are the zeros of u.

(ii) Setting $\mathscr{M}_1 := \max\{u(r) : 0 \leq r \leq r_1\}$ and $\mathscr{M}_{i+1} := \max\{|u(r)| : r_i \leq r \leq r_{i+1}\}$ for $i = 1, \cdots, m-1$, one has

$$\mathscr{M}_1 > \mathscr{M}_3 > \cdots \quad \text{and} \quad \mathscr{M}_2 > \mathscr{M}_4 > \cdots.$$

(iii) If f is odd, then

$$\mathscr{M}_1 > \mathscr{M}_2 > \cdots > \mathscr{M}_m.$$

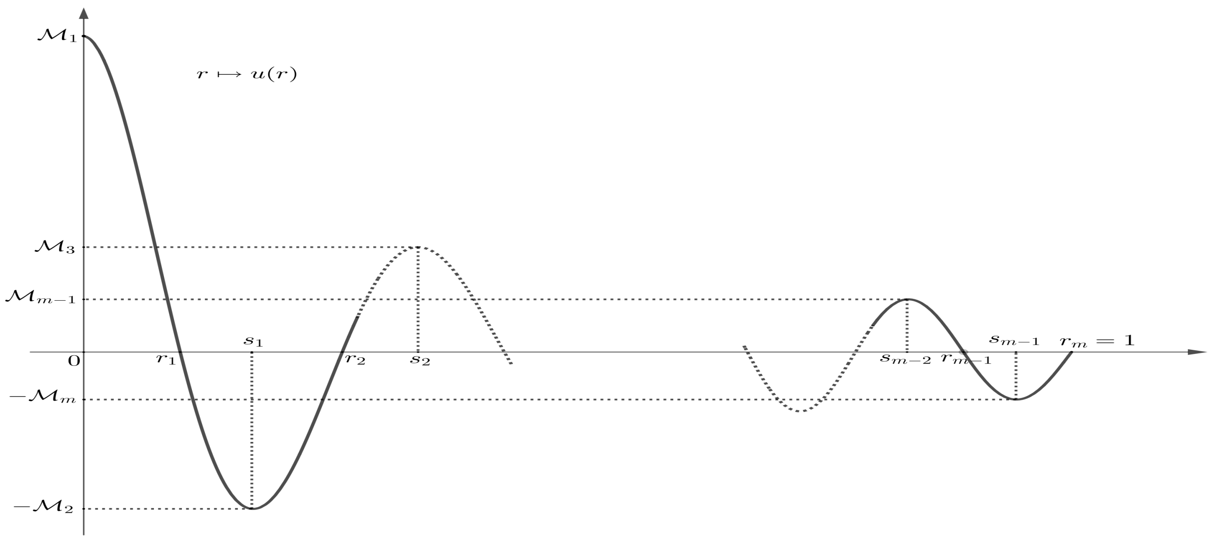

Figure 1 – The radial solution of (3.7) with m nodal sets.

To prove Corollary 3.3, we need to turn equation (3.7) into equation (3.2). Of course every radial solution of (3.7) satisfies in its radial coordinate $r = |x|$ the equation

$$-(r^{N-1}u')' = r^{N-1}g(r,u) \quad \text{in } (0,1), \quad u(0) > 0, \ u'(0) = u(1) = 0 \tag{3.8}$$

with $g(r,u) = r^\alpha f(u)$. However, to prove Proposition 3.2, it was crucial that the nonlinear term g did not depend on the variable r (review how it was checked (3.5) and consequently (3.6)). To overcome this difficulty, we use a change of variable which has been introduced in (COWAN; GHOUSSOUB, 2010).

Proposition 3.4. Let $u \in C^2(\overline{B})$ be a radial solution for (3.7). Then the function v given by transformation

$$v(t) = u(r), \quad t = r^{\frac{\alpha+2}{2}},\ r = |x| \in [0,1] \tag{3.9}$$

belongs to $C^2[0,1]$ and solves (3.2) with $g(s) = \left(\frac{2}{\alpha+2}\right)^2 f(s)$ and $M = M(\alpha,N) = \frac{2(\alpha+N)}{\alpha+2}$.

Proof. It is obvious that $v \in C[0,1] \cap C^2(0,1]$. Performing the change of variable $u(r) = v(t)$, where $t = r^\theta$, $r = |x|$ and $\theta > 0$, we have

$$u_r(r) = \theta r^{\theta-1} v_t(r^\theta) \quad \text{and} \quad u_{rr}(r) = \theta^2 r^{2\theta-2} v_{tt}(r^\theta) + \theta(\theta-1) r^{\theta-2} v_t(r^\theta), \quad r \in (0,1).$$

Since u satisfies $-u_{rr} - \frac{N-1}{r} u_r = r^\alpha f(u)$ in $(0,1)$, we get

$$\begin{aligned} r^\alpha f(u(r)) &= -\theta^2 r^{2\theta-2} v_{tt}(r^\theta) - \theta(\theta-1) r^{\theta-2} v_t(r^\theta) - \frac{N-1}{r}\theta r^{\theta-1} v_t(r^\theta) \\ &= -\theta^2 r^{2\theta-2}\left[v_{tt}(r^\theta) + \frac{(\theta-1)}{\theta} r^{-\theta} v_t(r^\theta) + \frac{N-1}{\theta} r^{-\theta} v_t(r^\theta)\right] \\ &= -\theta^2 r^{2\theta-2}\left[v_{tt}(r^\theta) + \frac{(\theta+N-2)}{\theta} r^{-\theta} v_t(r^\theta)\right], \quad r \in (0,1) \end{aligned} \tag{3.10}$$

that is,

$$-t^{\frac{2\theta-2-\alpha}{\theta}}\left[v_{tt}(t) + \frac{(\theta+N-2)}{\theta t} v_t(t)\right] = \frac{f(v(t))}{\theta^2}, \quad t \in (0,1).$$

Taking then $2\theta - 2 - \alpha = 0$ i.e. $\theta = \frac{\alpha+2}{2}$, we obtain

$$-\left[v_{tt}(t) + \frac{M-1}{t} v_t(t)\right] = \left(\frac{2}{\alpha+2}\right)^2 f(v(t)), \quad t \in (0,1), \tag{3.11}$$

that is

$$-\left(t^{M-1} v'(t)\right)' = t^{M-1}\left(\frac{2}{\alpha+2}\right)^2 f(v(t)), \quad t \in (0,1), \tag{3.12}$$

where $M - 1 = \frac{\theta+N-2}{\theta} = \frac{\alpha+2(N-1)}{\alpha+2}$ i.e. $M = \frac{2(\alpha+N)}{\alpha+2}$.

To conclude the proof, we need to check that v has the second derivative continuous at 0 and satisfies the initial conditions in (3.2). Of course $v(0) > 0$ and $v(1) = 0$. Now using the L'Hôpital rule and integrating (3.8), we get

$$\begin{aligned} v'(0) &= \lim_{t\to 0^+} \frac{v(t)-v(0)}{t} \overset{(3.9)}{=} \lim_{r\to 0^+} \frac{u(r)-u(0)}{r^{\frac{\alpha+2}{2}}} \overset{(L'H)}{=} \frac{2}{\alpha+2} \lim_{r\to 0^+} \frac{u'(r)}{r^{\frac{\alpha}{2}}} \\ &\overset{(3.8)}{=} -\frac{2}{\alpha+2} \lim_{r\to 0^+} \frac{\int_0^r s^{\alpha+N-1} f(u(s))ds}{r^{\frac{\alpha}{2}+N-1}} = 0, \end{aligned}$$

since $|\int_0^r s^{\alpha+N-1} f(u(s))ds| = O(r^{\alpha+N})$ as $r \to 0^+$. Hence integrating (3.12) we have that

$$t^{M-1}v'(t) = -\left(\frac{2}{\alpha+2}\right)^2 \int_0^t s^{M-1} f(v(s))ds \to 0, \quad t \to 0^+.$$

Consequently using the L'Hôpital rule again, we get

$$v''(0) = \lim_{t\to 0^+} \frac{v'(t)-v'(0)}{t} = \lim_{t\to 0^+} \frac{t^{M-1}v'(t)}{t^M} \overset{(L'H)}{=} \lim_{t\to 0^+} \frac{(t^{M-1}v'(t))'}{Mt^{M-1}} \overset{(3.12)}{=} -\left(\frac{2}{\alpha+2}\right)^2 \frac{f(v(0))}{M}.$$

Moreover using (3.11) we have that $\lim_{t\to 0^+} v''(t)$ exists and it is equals to $v''(0)$, which shows that $v \in C^2[0,1]$. □

Remark 3.5. Note that it is an immediate consequence from eq. (3.10) that the transformation (3.9) yields the following relation:

$$\Delta u(x) = u_{rr}(r) + \frac{N-1}{r} u_r(r) = \left(\frac{\alpha+2}{2}\right)^2 r^\alpha \left(v_{tt}(t) + \frac{M-1}{t} v_t(t)\right),$$

where $M = \frac{2(\alpha+N)}{\alpha+2}$.

We can now proceed with the

Proof of Corollary 3.3. Let u be a radial solution of (3.7) with $m \geq 1$ nodal sets. Then Proposition 3.4 implies that the function $v(t) = u(t^{\frac{2}{\alpha+2}})$ solves (3.2) with $g(s) = \left(\frac{2}{\alpha+2}\right)^2 f(s)$ and $M = \frac{2(\alpha+N)}{\alpha+2}$. Of course the properties (i)-(iii) hold for u if and only if they hold for v and the latter is guaranteed by Proposition 3.2. □

3.2 Nodal energy levels and their asymptotic estimates

Consider the N-dimensional Hénon equation

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{P_α}$$

where $B \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 3$, $p > 1$ and $\alpha > 0$ is a parameter. Standard arguments imply that if $p < \frac{N+2}{N-2}$, then the energy functional

$$\varphi_\alpha(u) := \frac{1}{2}\int_B |\nabla u|^2 dx - \frac{1}{p+1}\int_B |x|^\alpha |u|^{p+1} dx$$

is well defined in $H_0^1(B)$ and the weak solutions to (P_α) are precisely the critical points of φ_α. Moreover, by standard elliptic regularity theory, every critical point of φ_α belongs to $C^2(\overline{B})$.

Here, we are interested in the radially symmetric solutions of (P_α). We recall that, for each positive integer m, results in (NAGASAKI, 1989) show that (P_α) admits, up to multiplication by -1, a unique radial solution in $C^2(\overline{B})$ with exactly m nodal sets, that is, $m-1$ zeros in $(0,1)$ with respect to the radial variable $r=|x|$, provided we have the following condition between the power p and the weight α: $1<p<2^*_\alpha-1$, where $2^*_\alpha := \frac{2(N+\alpha)}{N-2}$. Observe that this condition is equivalent to $\alpha > \alpha_p$, where

$$\alpha_p := \max\left\{0, \frac{p(N-2)-(N+2)}{2}\right\}.$$

Since the embedding $H^1_{0,rad}(B) \subset L^q(|x|^\alpha, B)$ is compact for all $1 \le q < 2^*_\alpha$, we have that, for all $p>1$ and $\alpha > \alpha_p$, φ_α is well defined in $H^1_{0,rad}(B)$ and the radial solutions of (P_α) are exactly the critical points of φ_α restricted to $H^1_{0,rad}(B)$. In particular, they lie on the Nehari manifold

$$\mathcal{N}^\alpha := \left\{0 \neq u \in H^1_{0,rad}(B) : \varphi'_\alpha(u)u = 0\right\} = \left\{0 \neq u \in H^1_{0,rad}(B) : \int_B |\nabla u|^2 = \int_B |x|^\alpha |u|^{p+1}\right\}.$$

From now on, for any fixed $m \in \mathbb{N}$ and $p>1$, we will denote by u_α, $\alpha > \alpha_p$, the unique radial solution of (P_α) with precisely m nodal sets such that $u_\alpha(0) > 0$. We also set

$$H_m := \{u \in H^1_{0,rad}(B) : \ u \text{ has precisely } m \text{ nodal sets}\}.$$

For each $u \in H_m$, say that $R_1, \cdots, R_m$ are the nodal regions of u, we will denote by u_i the restriction of u to R_i, that is

$$u_i(x) := \begin{cases} u(x) & \text{if } x \in R_i \\ 0 & \text{if } x \in B \backslash R_i. \end{cases}$$

Since u_α and $-u_\alpha$ are the radial solutions of (P_α) with precisely m nodal sets, we can define simply the level of radial solutions in H_m as

$$C_{\alpha,m} := \varphi_\alpha(u_\alpha).$$

We also set

$$\overline{C}_{\alpha,m} := \inf\{\varphi_\alpha(u) : u \in H_m \text{ and } u_i \in \mathcal{N}^\alpha \ \forall i = 1, \cdots, m\}$$

and

$$\widetilde{C}_{\alpha,m} := \inf_{u \in H_m} \max_{t \in \mathbb{R}^m_+} \sum_{i=1}^m \varphi_\alpha(t_i u_i),$$

where here $\mathbb{R}^m_+ := \{t = (t_1, \cdots, t_m) \in \mathbb{R}^m : t_i > 0 \ \forall i = 1, \cdots, m\}$. We have the following.

Proposition 3.6. $C_{\alpha,m} = \overline{C}_{\alpha,m} = \widetilde{C}_{\alpha,m}$.

To prove the above proposition, we recall how a radial solution of (P_α) with m nodal sets can be produce provided $\alpha > \alpha_p$. The compactness of the previous embedding implies also that the infimum of φ_α on $\mathcal{N}^\alpha$ is achieved at a positive radial solution to (P_α). When $m \ge 2$, one can produce a radial solution to (P_α) with exactly m nodal sets using the so called Nehari method (see (WILLEM, 1996, Chapter 4)). Such method consists of introducing the spaces

$$X_{s,t} := \{u \in H^1_{0,rad}(B) : u(r) = 0 \; \forall r \in [0,s] \cup [t,1]\}, \qquad 0 < s < t \leq 1,$$
$$X_{0,t} := \{u \in H^1_{0,rad}(B) : u(r) = 0 \; \forall r \in [t,1]\}, \qquad 0 < t < 1,$$

and the Nehari sets

$$\mathcal{N}^\alpha_{s,t} := \left\{ 0 \neq u \in X_{s,t} : \int_B |\nabla u|^2 dx = \int_B |x|^\alpha |u|^{p+1} dx \right\}$$

for $0 \leq s < t \leq 1$, to solve the minimization problem

$$\mathscr{C}_{\alpha,m} := \inf \left\{ \sum_{i=1}^m \min_{\mathcal{N}^\alpha(t_{i-1},t_i)} \varphi_\alpha : 0 = t_0 < t_1 < \cdots < t_m = 1 \right\}. \tag{3.13}$$

Actually, it can be checked, using (WILLEM, 1996, Proposition 4.4-(d)), that (3.13) is achieved, say at $0 = r_0 < r_1 < \cdots < r_m = 1$. Moreover, for each $i = 1, \cdots, m$, there exists a nonnegative function $u_i \in X_{r_{i-1},r_i}$ such that

$$\varphi_\alpha(u_i) = \min_{\mathcal{N}^\alpha(r_{i-1},r_i)} \varphi_\alpha.$$

It can then be shown, like in (WILLEM, 1996, Lemma 4.7), that the function $u : B \to \mathbb{R}$ given by $u(x) = (-1)^{i-1} u_i(x)$ if $|x| \in [r_{i-1}, r_i)$ is of class C^2 and therefore it is the radial solution of (P_α) with exactly m nodal sets such that $u(0) > 0$. Consequently, $u = u_\alpha$ and

$$C_{\alpha,m} = \varphi_\alpha(u_\alpha) = \mathscr{C}_{\alpha,m}. \tag{3.14}$$

Now we can prove Proposition 3.6.

Proof of Proposition 3.6. Since $u_\alpha \in H_m$ and the restrictions of u_α to its nodal regions lie on $\mathcal{N}^\alpha$, we have $\overline{C}_{\alpha,m} \leq C_{\alpha,m}$. By contradiction, let us suppose that $\overline{C}_{\alpha,m} < C_{\alpha,m}$. Then, by (3.14), there exists $u \in H_m$ with $u_i \in \mathcal{N}^\alpha$ for $i = 1, \cdots, m$ such that $\varphi_\alpha(u) < \mathscr{C}_{\alpha,m}$. Hence, if $s_1 < \cdots < s_m = 1$ are the zeros of u in $(0,1]$ and $0 = s_0$, then

$$\mathscr{C}_{\alpha,m} > \varphi_\alpha(u) = \sum_{i=1}^m \varphi_\alpha(u_i) \geq \sum_{i=1}^m \min_{\mathcal{N}^\alpha(s_{i-1},s_i)} \varphi_\alpha \geq \mathscr{C}_{\alpha,m},$$

which is a contradiction.

To check the second equality, it suffices to observe that $\max_{t \in \mathbb{R}^m_+} \sum_{i=1}^m \varphi_\alpha(t_i u_i)$ is achieved at a unique point $t \in \mathbb{R}^m_+$ such that $t_i u_i \in \mathcal{N}^\alpha$ for all $i = 1, \cdots, m$ (see (WILLEM, 1996, eq. (4.1))). □

Remark 3.7. Let $u \in H_m$ and $t \in \mathbb{R}^m_+$ such that $t_i u_i \in \mathcal{N}^\alpha$ for all $i = 1, \cdots, m$. Then we have

$$t_i = \left(\frac{\int_B |\nabla u_i|^2}{\int_B |x|^\alpha |u_i|^{p+1}} \right)^{\frac{1}{p-1}} \qquad \forall i = 1, \cdots, m$$

and hence

$$\varphi_\alpha(t_iu_i) = \left(\frac{1}{2}-\frac{1}{p+1}\right)t_i^2\int_B|\nabla u_i|^2 = \left(\frac{1}{2}-\frac{1}{p+1}\right)\frac{\left(\int_B|\nabla u_i|^2\right)^{\frac{p+1}{p-1}}}{\left(\int_B|x|^\alpha|u_i|^{p+1}\right)^{\frac{2}{p-1}}}.$$

Consequently, by Proposition 3.6, we have the following characterization of the level $C_{\alpha,m}$:

$$C_{\alpha,m} = \left(\frac{1}{2}-\frac{1}{p+1}\right)\inf_{u\in H_m}\sum_{i=1}^m\left[\frac{\left(\int_B|\nabla u_i|^2\right)^{\frac{p+1}{p-1}}}{\left(\int_B|x|^\alpha|u_i|^{p+1}\right)^{\frac{2}{p-1}}}\right]. \tag{3.15}$$

Proposition 3.8. There are constants $C_1,C_2>0$ depending just on p, N and m such that

$$C_1\left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}} \le C_{\alpha,m} \le C_2\left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}}, \quad \forall\alpha>\alpha_p.$$

Moreover the function $\alpha\mapsto C_{\alpha,m}$ is strictly increasing in (α_p,∞).

Proof. Given $u\in H_m$, define the rescaled function $v(t)=u(r)$, where $r=t^\beta$ and $\beta=\frac{N}{\alpha+N}\in(0,1]$. Of course v has m nodal sets and $v_i(t)=u_i(r)$ for each $i=1,\cdots,m$. Moreover, we have for each $i=1,\cdots,m$ that

$$\begin{aligned}\int_B|x|^\alpha|u_i(x)|^{p+1}dx &= \omega_{N-1}\int_0^1|u_i(r)|^{p+1}r^{\alpha+N-1}dr = \omega_{N-1}\int_0^1|v_i(t)|^{p+1}t^{\beta(\alpha+N-1)}\beta t^{\beta-1}dt\\ &= \omega_{N-1}\beta\int_0^1|v_i(t)|^{p+1}t^{N-1}dt = \beta\int_B|v_i|^{p+1}dx\end{aligned}$$

and

$$\begin{aligned}\int_B|\nabla u_i(x)|^2dx &= \omega_{N-1}\int_0^1|u_i'(r)|^2r^{N-1}dr = \omega_{N-1}\int_0^1|v_i'(t)|^2\beta^{-2}t^{2-2\beta}t^{\beta(N-1)}\beta t^{\beta-1}dt\\ &= \omega_{N-1}\beta^{-1}\int_0^1|v_i'(t)|^2t^{(1-\beta)(2-N)}t^{N-1}dt = \beta^{-1}\int_B|\nabla v_i(x)|^2|x|^{(2-N)(1-\beta)}dx.\end{aligned}$$

It follows then

$$\frac{\left(\int_B|\nabla u_i|^2\right)^{\frac{p+1}{p-1}}}{\left(\int_B|x|^\alpha|u_i|^{p+1}\right)^{\frac{2}{p-1}}} = \frac{1}{\beta^{\frac{p+3}{p-1}}}\frac{\left(\int_B|\nabla v_i|^2|x|^{(2-N)(1-\beta)}\right)^{\frac{p+1}{p-1}}}{\left(\int_B|v_i|^{p+1}\right)^{\frac{2}{p-1}}}$$

and thus (3.15) yields

$$C_{\alpha,m} = \frac{\left(\frac{1}{2}-\frac{1}{p+1}\right)R_{\beta,m}}{\beta^{\frac{p+3}{p-1}}} = \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}}R_{\beta,m}, \tag{3.16}$$

where

$$R_{\rho,m} := \inf_{v\in H_m}\sum_{i=1}^m\frac{\left(\int_B|\nabla v_i|^2|x|^{(2-N)(1-\rho)}\right)^{\frac{p+1}{p-1}}}{\left(\int_B|v_i|^{p+1}\right)^{\frac{2}{p-1}}}, \quad \rho\in[0,1].$$

Since $\rho \mapsto R_{\rho,m}$ is non-increasing and $0 < R_{1,m} < R_{0,m} < \infty$, we have that

$$C_1 \left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}} \le C_{\alpha,m} \le C_2 \left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}} \quad \forall \alpha > \alpha_p,$$

where $C_1 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \lim_{\rho\to 1^-} R_{\rho,m}$ and $C_2 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \lim_{\rho\to 0^+} R_{\rho,m}$. The strict monotonicity of $C_{\alpha,m}$ follows immediately from (3.16). □

3.3 Estimates for L^∞-norms

In this section, we provide asymptotic estimates for $\|u_\alpha\|_\infty$, where for fixed $m \in \mathbb{N}$, u_α is the radial solution with m nodal sets of (P_α) such that $u_\alpha(0) > 0$. By Corollary 3.3, we have $u_\alpha(0) = \max_{x\in B} u_\alpha(x) = \|u_\alpha\|_\infty$. Moreover, by Proposition 3.8, there exist constants $C_1, C_2 > 0$, that do not depend on α, such that

$$C_1 \left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}} \le \int_B |\nabla u_\alpha|^2 = \int_B |x|^\alpha |u_\alpha|^{p+1} \le C_2 \left(\frac{\alpha+N}{N}\right)^{\frac{p+3}{p-1}}, \quad \forall \alpha > \alpha_p. \tag{3.17}$$

We start with the bound from below.

Lemma 3.9. There exists a constant $C > 0$, that does not depend of α, such that

$$\|u_\alpha\|_\infty \ge C \left(\frac{\alpha+N}{N}\right)^{\frac{2}{p-1}}, \quad \forall \alpha > \alpha_p.$$

Proof. For each $\alpha > \alpha_p$, consider the function

$$\widetilde{u}_\alpha(y) := \beta^{\frac{2}{p-1}} u_\alpha\left(\beta^{\frac{1}{2-N}} y\right), \quad y \in \Omega_\alpha := B\left(0, \beta^{\frac{1}{N-2}}\right),$$

where $\beta = \beta(\alpha) := \frac{N}{\alpha+N}$. Since $\|\widetilde{u}_\alpha\|_\infty = \left(\frac{N}{\alpha+N}\right)^{\frac{2}{p-1}} \|u_\alpha\|_\infty$, it suffices to show that $\|\widetilde{u}_\alpha\|_\infty \ge C$. Performing the change of variables $x = \beta^{\frac{1}{2-N}} y$, we have

$$\begin{aligned}\int_{\Omega_\alpha} |\nabla \widetilde{u}_\alpha(y)|^2 dy &= \int_{\Omega_\alpha} \beta^{\frac{4}{p-1}} |\nabla u_\alpha(\beta^{\frac{1}{2-N}} y)|^2 \beta^{\frac{2}{2-N}} dy \\ &= \int_B \beta^{\frac{4}{p-1}} |\nabla u_\alpha(x)|^2 \beta^{\frac{2}{2-N}} \beta^{\frac{N}{N-2}} dx = \left(\frac{N}{\alpha+N}\right)^{\frac{p+3}{p-1}} \int_B |\nabla u_\alpha(x)|^2 dx.\end{aligned}$$

Hence, by (3.17)

$$C_1 \le \int_{\Omega_\alpha} |\nabla \widetilde{u}_\alpha|^2 dy \le C_2, \quad \forall \alpha > \alpha_p. \tag{3.18}$$

Now, since u_α solves (P_α), we have that $\widetilde{u}_\alpha$ satisfies

$$\begin{aligned}-\Delta \widetilde{u}_\alpha(y) &= \beta^{\frac{2}{p-1}} \beta^{\frac{2}{2-N}} \left[-\Delta u_\alpha\left(\beta^{\frac{1}{2-N}} y\right)\right] \\ &= \beta^{\frac{2}{p-1}} \beta^{\frac{2}{2-N}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha \left|u_\alpha\left(\beta^{\frac{1}{2-N}} y\right)\right|^{p-1} u_\alpha\left(\beta^{\frac{1}{2-N}} y\right) \\ &= \beta^{\frac{2}{p-1}} \beta^{\frac{2}{2-N}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha \beta^{\frac{-2p}{p-1}} |\widetilde{u}_\alpha(y)|^{p-1} \widetilde{u}_\alpha(y), \\ &= \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha |\widetilde{u}_\alpha(y)|^{p-1} \widetilde{u}_\alpha(y), \quad y \in \Omega_\alpha, \quad \widetilde{u}_\alpha = 0 \text{ on } \partial\Omega_\alpha.\end{aligned}$$

Consequently, by Hölder's inequality, we have

$$\begin{aligned}0<C_1 &\overset{(3.18)}{\leq} \int_{\Omega_\alpha} |\nabla \widetilde{u}_\alpha|^2 dy = \int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha |\widetilde{u}_\alpha|^{p+1} dy \\ &\leq \|\widetilde{u}_\alpha\|_\infty^{p-1} \int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha |\widetilde{u}_\alpha|^2 dy \\ &\leq \|\widetilde{u}_\alpha\|_\infty^{p-1} \left(\int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha dy\right)^{\frac{p-1}{p+1}} \left(\int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha |\widetilde{u}_\alpha|^{p+1} dy\right)^{\frac{2}{p+1}} \\ &\overset{(3.18)}{\leq} (C_2)^{\frac{2}{p+1}} \|\widetilde{u}_\alpha\|_\infty^{p-1} \left(\int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha dy\right)^{\frac{p-1}{p+1}}.\end{aligned} \tag{3.19}$$

Note that

$$\begin{aligned}\int_{\Omega_\alpha} \beta^{\frac{-2(N-1)}{N-2}} \left|\beta^{\frac{1}{2-N}} y\right|^\alpha dy &= \omega_{N-1} \beta^{\frac{2(N-1)+\alpha}{2-N}} \int_0^{\beta^{\frac{1}{N-2}}} t^{\alpha+N-1} dt \\ &= \omega_{N-1} \frac{\beta^{\frac{2(N-1)+\alpha}{2-N}} \beta^{\frac{\alpha+N}{N-2}}}{\alpha+N} = \frac{\omega_{N-1}}{N} \beta^{\frac{2(N-1)+\alpha}{2-N}} \beta^{\frac{\alpha+N}{N-2}} \beta = \frac{\omega_{N-1}}{N}.\end{aligned} \tag{3.20}$$

Therefore, from (3.19) and (3.20), there is $C>0$ independent of α such that

$$\|\widetilde{u}_\alpha\|_\infty \geq C, \quad \forall \alpha > \alpha_p,$$

which proves the lemma. □

To prove the reverse estimate, we use the change of variables (3.9) again. So for each $\alpha > \alpha_p$, we set

$$v_\alpha(t) := \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha\left(t^{\frac{2}{\alpha+2}}\right), \quad t \in [0,1]. \tag{3.21}$$

Then by Proposition 3.4, v_α satisfies

$$\begin{cases} -(t^{M_\alpha-1} v_\alpha')' = t^{M_\alpha-1} |v_\alpha|^{p-1} v_\alpha & \text{in } (0,1), \\ v_\alpha(1) = v_\alpha'(0) = 0, \end{cases} \tag{3.22}$$

where $M_\alpha := \frac{2(\alpha+N)}{\alpha+2} \in (2,N)$. Thus integrating from 0 to t, we get

$$v_\alpha'(t) = -\frac{1}{t^{M_\alpha-1}} \int_0^t s^{M_\alpha-1} |v_\alpha(s)|^{p-1} v_\alpha(s) ds \quad \forall t \in (0,1]. \tag{3.23}$$

Moreover performing the change of variables $t \leftrightarrow r$, where $r = t^{\frac{2}{\alpha+2}}$, we have that

$$\begin{aligned}\int_0^1 |v_\alpha'(t)|^2 t^{M_\alpha-1} dt &= \int_0^1 \left[\left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha'\left(t^{\frac{2}{\alpha+2}}\right) \frac{2t^{\frac{-\alpha}{\alpha+2}}}{\alpha+2}\right]^2 t^{M_\alpha-1} dt \\ &= \left(\frac{2}{\alpha+2}\right)^{\frac{4}{p-1}+2} \int_0^1 |u_\alpha'(r)|^2 r^{-\alpha} r^{\frac{(M_\alpha-1)(\alpha+2)}{2}} \left(\frac{\alpha+2}{2}\right) r^{\frac{\alpha}{2}} dr \\ &= \left(\frac{2}{\alpha+2}\right)^{\frac{p+3}{p-1}} \int_0^1 |u_\alpha'(r)|^2 r^{N-1} dr.\end{aligned} \tag{3.24}$$

We recall that the change of variables (3.21) was also used in (SANTOS; PACELLA, 2017) in the case $N=2$ (see also (GLADIALI; GROSSI; NEVES, 2016; SILVA; SANTOS, 2019)). There the authors show that, for all $\alpha>0$, the function v_α is precisely the radial solution with m nodal sets of the Lane-Emden problem

$$-\Delta w=|w|^{p-1}w \text{ in } B^2, \quad w=0 \text{ on } \partial B^2, \quad w(0)>0, \tag{3.25}$$

where B^2 is the unit ball in $\mathbb{R}^2$. In particular, the sequence of functions $(v_\alpha)_{\alpha>0}$ is constant if $N=2$, which is not the case for $N\geq 3$. Our main goal here is to show that, in any dimension $N\geq 3$, v_α is close to the radial solution with m nodal sets w of (3.25), posed in $B^2\subset\mathbb{R}^2$, as $\alpha\to\infty$.

Lemma 3.10. There exist $\alpha_0>\alpha_p$ and $C>0$, that do not depend on α, such that

$$v_\alpha(0)=\|v_\alpha\|_\infty\leq C, \ \forall\alpha\geq\alpha_0$$

that is,

$$\|u_\alpha\|_\infty\leq C\left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-1}} \quad \forall\alpha\geq\alpha_0.$$

Proof. From (3.17) and (3.24), there exists a constant $C_0>0$ independent of α such that

$$\int_0^1|v_\alpha'(t)|^2t^{M_\alpha-1}dt\leq C_0 \quad \forall\alpha>\alpha_p. \tag{3.26}$$

Fix $M>2$ such that $p<\frac{M}{M-2}$ and take $\alpha_0>0$ so that $M_\alpha\leq M$ for $\alpha\geq\alpha_0$. By contradiction, let us suppose that $\|v_\alpha\|_\infty$ is not bounded in $[\alpha_0,\infty)$. Then there exists a sequence $\alpha_n\geq\alpha_0$ such that $\|v_{\alpha_n}\|_\infty\to\infty$ as $n\to\infty$. Set

$$\overline{v}_\alpha(s)=\frac{1}{\|v_\alpha\|_\infty}v_\alpha\left(\|v_\alpha\|_\infty^{1-p}s\right), \quad \text{for } s\in\left[0,\|v_\alpha\|_\infty^{p-1}\right].$$

Setting $t=\|v_\alpha\|_\infty^{1-p}s$, we have

$$\begin{aligned}\int_0^{\|v_{\alpha_n}\|_\infty^{p-1}}|\overline{v}_{\alpha_n}'(s)|^2s^{M-1}ds&=\int_0^{\|v_{\alpha_n}\|_\infty^{p-1}}\left[\frac{1}{\|v_{\alpha_n}\|_\infty}v_{\alpha_n}'\left(\|v_{\alpha_n}\|_\infty^{1-p}s\right)\|v_{\alpha_n}\|_\infty^{1-p}\right]^2s^{M-1}ds\\&=\|v_{\alpha_n}\|_\infty^{-2p}\int_0^1\left|v_{\alpha_n}'(t)\right|^2\|v_{\alpha_n}\|_\infty^{(p-1)(M-1)}t^{M-1}\|v_{\alpha_n}\|_\infty^{p-1}dt\\&=\|v_{\alpha_n}\|_\infty^{(M-2)p-M}\int_0^1|v_{\alpha_n}'(t)|^2t^{M-1}dt\\&\leq\|v_{\alpha_n}\|_\infty^{(M-2)p-M}\int_0^1|v_{\alpha_n}'(t)|^2t^{M_{\alpha_n}-1}dt \quad [M\geq M_\alpha]\\&\leq C_0\|v_{\alpha_n}\|_\infty^{(M-2)p-M}\to 0, \quad [(3.26)\text{ and }(M-2)p-M<0]\end{aligned}$$

and hence

$$\overline{v}_{\alpha_n}\to 0 \text{ in } \mathscr{D}_M^{1,2}:=\left\{v:[0,\infty)\to\mathbb{R}:\int_0^\infty|v'(s)|^2s^{M-1}ds<\infty\right\}.$$

Since the inclusion

$$\mathscr{D}_M^{1,2} \subset L_M^{2_M^*}(0,\infty) := \left\{ v : [0,\infty) \to \mathbb{R} : \int_0^\infty |v(s)|^{\frac{2M}{M-2}} s^{M-1} ds < \infty \right\}$$

is continuous, see Lemma A.3, up to a subsequence, $\overline{v}_{\alpha_n} \to 0$ a.e. in $[0,\infty)$.

On the other hand,

$$\overline{v}_{\alpha_n}(0) = \|\overline{v}_{\alpha_n}\|_\infty = 1,$$

that is, the sequence of functions $(\overline{v}_{\alpha_n})$ is uniformly bounded. Moreover, we have by (3.23) that

$$|\overline{v}_{\alpha_n}'(s)| = \|v_{\alpha_n}\|_\infty^{-p} |v_{\alpha_n}'\left(\|v_{\alpha_n}\|_\infty^{1-p} s\right)| \le \|v_{\alpha_n}\|_\infty^{-p} \|v_{\alpha_n}\|_\infty^{p} = 1.$$

In particular, the sequence of functions $(\overline{v}_{\alpha_n})$ is equicontinuous. Consequently by Arzelà-Ascoli theorem, up to a subsequence, $\overline{v}_{\alpha_n} \to \overline{v}$ uniformly in $[0,1]$ with $\|\overline{v}\|_\infty = 1$, which leads us to a contradiction. □

Remark 3.11. Notice that, unlike Lemma 3.9, the conclusion of previous lemma is not true for all $\alpha > \alpha_p$. Consider for example the case where $p = \frac{N+2}{N-2}$ and u_α is positive, that is $\alpha_p = 0$ and $m = 1$. In this case, we claim that $\|u_\alpha\|_\infty \to \infty$ as $\alpha \to 0$. Indeed, suppose that $\|u_\alpha\|_\infty$ is uniformly bounded as $\alpha \to 0$. Thus, since u_α solves

$$-\Delta u = |x|^\alpha u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

we have that, as $\alpha \to 0$, u_α converges uniformly to a nonnegative solution u of the limit problem

$$-\Delta u = u^{\frac{N+2}{N-2}} \text{ in } B, \quad u = 0 \text{ on } \partial B.$$

This implies that $u = 0$, which is a contradiction because $\|u_\alpha\|_\infty \ge C$ by Lemma 3.9.

3.4 Estimates for the local extrema

Next, if $m \ge 2$, we will denote by $0 < r_{1,\alpha} < \cdots < r_{m,\alpha} = 1$ the zeros of u_α in $(0,1]$ and we set

$$\mathscr{M}_{1,\alpha} := \max\{u_\alpha(r) : 0 \le r \le r_{1,\alpha}\}, \quad \mathscr{M}_{i+1,\alpha} := \max\{|u_\alpha(r)| : r_{i,\alpha} \le r \le r_{i+1,\alpha}\}$$

for $i = 1, \cdots, m-1$. Similarly, we denote by $0 < t_{1,\alpha} < \cdots < t_{m,\alpha} = 1$ the zeros in $(0,1]$ of the function v_α defined in (3.21) so that

$$t_{i,\alpha} = (r_{i,\alpha})^{\frac{\alpha+2}{2}}, \quad \text{for each } i = 1, \cdots, m. \tag{3.27}$$

We also define

$$\mathscr{N}_{1,\alpha} := \max\{v_\alpha(t) : 0 \le t \le t_{1,\alpha}\}, \quad \mathscr{N}_{i+1,\alpha} := \max\{|v_\alpha(t)| : t_{i,\alpha} \le t \le t_{i+1,\alpha}\}$$

so that

$$\mathscr{N}_{i,\alpha} = \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} \mathscr{M}_{i,\alpha}, \quad \text{for each } i = 1, \cdots, m.$$

We have the following.

Lemma 3.12. There is $\delta \in (0,1)$ and $\alpha_0 > \alpha_p$ such that

$$t_{1,\alpha} \geq \delta \quad \forall \alpha \geq \alpha_0.$$

In particular, observing (3.27), we get that $r_{i,\alpha} \to 1$ as $\alpha \to \infty$, for each $i = 1, \cdots, m$.

Proof. Let $\alpha_0 > \alpha_p$ as in Lemma 3.10. Using (3.21) and Lemmas 3.9 and 3.10, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \|v_\alpha\|_\infty \leq C_2, \quad \text{for all } \alpha \geq \alpha_0. \tag{3.28}$$

Moreover, by (3.23) we have

$$\|v_\alpha{}'\|_\infty \leq (C_2)^p, \quad \text{for all } \alpha \geq \alpha_0. \tag{3.29}$$

Suppose, by contradiction, that there exists a sequence $\alpha_n \geq \alpha_0$ such that $t_{1,\alpha_n} \to 0$ as $n \to \infty$. By (3.28) and (3.29), we may use Arzelà-Ascoli theorem to conclude that, up to a subsequence, $v_{\alpha_n} \to v$ uniformly in $[0,1]$ as $n \to \infty$. Note that

$$v(0) = v(0) - v_{\alpha_n}(t_{1,\alpha_n}) = [v(0) - v(t_{1,\alpha_n})] + [v(t_{1,\alpha_n}) - v_{\alpha_n}(t_{1,\alpha_n})] = o(1),$$

that is, $v(0) = 0$. On the other hand

$$0 < C_1 \leq \lim_{n\to\infty} \|v_{\alpha_n}\|_\infty = \lim_{n\to\infty} v_{\alpha_n}(0) = v(0),$$

which is a contradiction. □

Remark 3.13. Observe that the previous lemma guarantees asymptotic concentration, as $\alpha \to \infty$, of the zeros of the radial solutions u_α of (P_α), namely $r_{i,\alpha} \to 1$ for each $i = 1, \cdots, m$. Very recently, in (AMADORI; GLADIALI, 2019), it was studied the asymptotic profile of radial solutions when the weight α is fixed and the power p is close to $2^*_\alpha - 1$. Actually, there the authors show that the zeros of the radial solutions $r_{1,p} < \cdots < r_{m-1,p}$ converge to zero as $p \to 2^*_\alpha - 1$ (see (AMADORI; GLADIALI, 2019, Theorem 1.2)).

Next, we give asymptotic estimates for the local extrema $\mathscr{M}_{i,\alpha}$ of u_α, as $\alpha \to \infty$.

Lemma 3.14. There exist $\alpha_0 > \alpha_p$ and constants $C_1, C_2 > 0$ such that

$$C_1 \leq \mathscr{N}_{i,\alpha} \leq C_2$$

that is,

$$C_1 \left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-1}} \leq \mathscr{M}_{i,\alpha} \leq C_2 \left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-1}}$$

for all $i = 1, \cdots, m$ and $\alpha \geq \alpha_0$.

Proof. By Proposition 3.2, we have $\mathcal{N}_{1,\alpha} > \mathcal{N}_{2,\alpha} > \cdots > \mathcal{N}_{m,\alpha}$ for each $\alpha > \alpha_p$. Therefore, we may take C_2 as the constant C from Lemma 3.10.

Next for simplicity, we denote by $t_\alpha = t_{m-1,\alpha}$ the biggest zero of v_α in (0,1) so that $\mathcal{N}_{m,\alpha} = \max_{t_\alpha<t<1} |v_\alpha(t)|$. Multiplying (3.22) by v_α and integrating by parts from t_α to 1, we get

$$\int_{t_\alpha}^1 |v_\alpha(t)|^{p+1} t^{M_\alpha-1} dt = -\int_{t_\alpha}^1 \left[t^{M_\alpha-1} v_\alpha'(t)\right]' v_\alpha(t) dt$$
$$= -\left[t^{M_\alpha-1} v_\alpha' v_\alpha\right]_{t_\alpha}^1 + \int_{t_\alpha}^1 |v_\alpha'(t)|^2 t^{M_\alpha-1} dt = \int_{t_\alpha}^1 |v_\alpha'(t)|^2 t^{M_\alpha-1} dt,$$

since $v_\alpha(t_\alpha) = v_\alpha(1) = 0$. Now, for every $t \in (t_\alpha, 1)$ we have

$$|v_\alpha(t)| = \left|\int_{t_\alpha}^t v_\alpha'(s) ds\right| \le \int_{t_\alpha}^1 \left(s^{\frac{1-M_\alpha}{2}}\right)\left(|v_\alpha'(s)| s^{\frac{M_\alpha-1}{2}}\right) ds$$
$$\le \left(\int_{t_\alpha}^1 s^{1-M_\alpha} ds\right)^{1/2} \left(\int_{t_\alpha}^1 |v_\alpha'(s)|^2 s^{M_\alpha-1} ds\right)^{1/2}.$$

By Lemma (3.12), there is $\delta > 0$ such that $t_\alpha \ge \delta$ and hence

$$\left(\int_{t_\alpha}^1 s^{1-M_\alpha} ds\right)^{1/2} \le \left(\int_\delta^1 s^{1-N} ds\right)^{1/2} = \left(\frac{\delta^{2-N}-1}{N-2}\right)^{1/2} =: C.$$

Consequently

$$|v_\alpha(t)| \le C\left(\int_{t_\alpha}^1 |v_\alpha'(s)|^2 s^{M_\alpha-1} ds\right)^{1/2} = C\left(\int_{t_\alpha}^1 |v_\alpha(s)|^{p+1} s^{M_\alpha-1} ds\right)^{1/2}, \quad \forall t \in (t_\alpha, 1),$$

which implies, by definition of $\mathcal{N}_{m,\alpha}$, that

$$\mathcal{N}_{m,\alpha} \le C\left(\int_{t_\alpha}^1 |v_\alpha(s)|^{p+1} s^{M_\alpha-1} ds\right)^{1/2} \le C(\mathcal{N}_{m,\alpha})^{(p+1)/2},$$

that is

$$\mathcal{N}_{m,\alpha} \ge \left(\frac{1}{C}\right)^{2/(p-1)} > 0.$$

Taking then $C_1 = C^{2/(1-p)}$ we obtain $\mathcal{N}_{1,\alpha} > \mathcal{N}_{2,\alpha} > \cdots > \mathcal{N}_{m,\alpha} \ge C_1$, which concludes the proof. □

3.5 Asymptotic behavior of radial solutions and consequences

In this section, we prove Theorem 3.1 and present some qualitative applications.

Proof of Theorem 3.1. From Lemma 3.10, we have $\|v_\alpha\|_\infty \le C$ for all $\alpha > \alpha_0$. Using (3.23), this implies that $|v_\alpha'(t)| \le C^p |t|$ for all $t \in (0,1]$, and hence $\|v_\alpha'\|_\infty \le C^p$. Moreover, since v_α satisfies (3.22), we have for all $t \in (0,1]$ and $\alpha > \alpha_0$ that

$$|v_\alpha''(t)| = \left|(M_\alpha - 1)\frac{v_\alpha'(t)}{t} + |v_\alpha(t)|^{p-1} v_\alpha(t)\right| \le (N-1)C^p + C^p = NC^p,$$

and hence $\|v''_\alpha\|_\infty \le NC^p$.

Let (α_n) be any sequence such that $\alpha_n \to \infty$ as $n \to \infty$. Then, by Arzelà-Ascoli theorem, up to a subsequence, $v_{\alpha_n} \to v$ and $v'_{\alpha_n} \to z$ uniformly in $[0,1]$. Then v is differentiable and $v' = z$. We claim that $v \in H^1_{0,rad}(B^2)$ is a weak radial solution to (L). Indeed, let $\psi \in C^1([0,1])$ such that $\psi(1) = 0$. Then multiplying (3.22) by ψ and integrating by parts, we get

$$\int_0^1 v'_{\alpha_n}(t)\psi'(t)t^{M_{\alpha_n}-1}dt = \int_0^1 |v_{\alpha_n}(t)|^{p-1}v_{\alpha_n}(t)\psi(t)t^{M_{\alpha_n}-1}dt, \quad \forall n \in \mathbb{N}.$$

Hence, by uniform convergence, we obtain

$$\int_0^1 v'(t)\psi'(t)t\,dt = \int_0^1 |v(t)|^{p-1}v(t)\psi(t)t\,dt,$$

since $M_{\alpha_n} \to 2$. This implies that v is a weak radial solution of (L) and, by standard elliptic regularity theory, $v \in C^2([0,1])$ and satisfies

$$v'' + \frac{v'}{t} + |v|^{p-1}v = 0 \quad \text{in } (0,1), \quad v'(0) = v(1) = 0.$$

Now given $\varepsilon \in (0,1)$, we have for all $t \in [\varepsilon, 1]$

$$\begin{aligned} |v''_\alpha(t) - v''(t)| &\le \frac{|v'(t) - (M_\alpha - 1)v'_\alpha(t)|}{t} + \big||v(t)|^{p-1}v(t) - |v_\alpha(t)|^{p-1}v_\alpha(t)\big| \\ &\le \frac{|v'(t) - (M_\alpha - 1)v'_\alpha(t)|}{\varepsilon} + \big||v(t)|^{p-1}v(t) - |v_\alpha(t)|^{p-1}v_\alpha(t)\big|, \end{aligned}$$

which shows also that $v_{\alpha_n} \to v$ in $C^1([0,1]) \cap C^2([\varepsilon,1])$.

To conclude the proof, we need to check that $v = w$. Since v and w are radial solutions of (L) and w has precisely m nodal sets, it suffices to show that v has also precisely m nodal sets. Up to a subsequence, we can assume that $t_{i,\alpha_n} \to t_i \in [0,1]$ for each $i = 1, \cdots, m$. Of course $0 < \delta \le t_1 \le t_2 \le \cdots \le t_m = 1$, where δ is given by Lemma 3.12. We claim that $t_i < t_{i+1}$ for all $i = 1, \cdots, m-1$. Indeed, by contradiction, suppose that $t_i = t_{i+1}$ for some i. Let $s_{\alpha_n} \in (t_{i,\alpha_n}, t_{i+1,\alpha_n})$ so that $|v_{\alpha_n}(s_{\alpha_n})| = \mathcal{N}_{i+1,\alpha_n}$. Then by Lemma 3.14

$$\left|\frac{v_{\alpha_n}(s_{\alpha_n}) - v_{\alpha_n}(t_{i,\alpha_n})}{s_{\alpha_n} - t_{i,\alpha_n}}\right| = \frac{|v_{\alpha_n}(s_{\alpha_n})|}{s_{\alpha_n} - t_{i,\alpha_n}} \ge \frac{C_1}{s_{\alpha_n} - t_{i,\alpha_n}} \to \infty,$$

which contradicts $\|v'_\alpha\|_\infty \le C^p$. Finally, since v_{α_n} is positive in $(t_{i,\alpha_n}, t_{i+1,\alpha_n})$ if i is even and it is negative if i is odd, we have that, in (t_i, t_{i+1}), $v \ge 0$ if i is even and $v \le 0$ if i is odd. By Maximum Principle, these inequalities are strict and hence v has m nodal sets. □

Remark 3.15. Notice that, using the same argument, we can prove that, for all $\alpha^* > \alpha_p$, the functions v_α converge in $C^1([0,1]) \cap C^2([\varepsilon,1])$ to the function v_{α^*}, as $\alpha \to \alpha^*$. This can be reformulated as follows. Since v_α is the unique solution with precisely m nodal sets of the equation

$$\begin{cases} -v'' - \frac{M-1}{t}v' = |v|^{p-1}v & \text{in } (0,1), \\ v(0) > 0,\ v'(0) = v(1) = 0, \end{cases} \tag{P_M}$$

with $M = M_\alpha = \frac{2(\alpha+N)}{\alpha+2} \in (2,N)$ (see Proposition 3.4), we can conclude that, for all $M_0 > 2$ such that $p+1 < 2^*_{M_0} := \frac{2M_0}{M_0-2}$, that is, $2 < M_0 < \frac{2(p+1)}{p-1}$, the solution v_M of (P_M) with exactly m nodal sets converges in $C^1([0,1]) \cap C^2([\varepsilon,1])$ to the solution with precisely m nodal sets of (P_{M_0}), as $M \to M_0$.

Next we present some qualitative consequences of Theorem 3.1.

Asymptotic concentration of zeros and blow up

From Section 3.4, Lemma 3.12, we already know that the zeros $r_{i,\alpha} \in (0,1)$ of the radial solution u_α of (P_α) with precisely m nodal sets converge to 1 as $\alpha \to \infty$, $i = 1, \cdots, m-1$ (see Figure 2). Here we give the exact rate at which this convergence occurs, namely, we get the following.

Corollary 3.16. For each $i = 1, \cdots, m-1$, we have

$$\alpha(1 - r_{i,\alpha}) \to -2\log(t_i) \quad \text{as } \alpha \to \infty,$$

where the numbers $t_i \in (0,1)$ are the zeros of the radial solution of (L) with precisely m nodal sets.

Proof. By Theorem 3.1, for each $i = 1, \cdots, m-1$, we have $t_{i,\alpha} \to t_i$, where $t_{i,\alpha}$ are the zeros of v_α defined in (3.1). Since $t_{i,\alpha} = r_{i,\alpha}^{\frac{\alpha+2}{2}}$,

$$\frac{\alpha+2}{2}(1 - r_{i,\alpha}) = \frac{\alpha+2}{2}\left(1 - t_{i,\alpha}^{\frac{2}{2+\alpha}}\right) = \frac{\alpha+2}{2}\left(1 - t_i^{\frac{2}{2+\alpha}} + t_i^{\frac{2}{2+\alpha}} - t_{i,\alpha}^{\frac{2}{2+\alpha}}\right).$$

Note that, by mean value theorem, there is $c_{i,\alpha}$ between t_i and $t_{i,\alpha}$ such that

$$\frac{\alpha+2}{2}\left(t_i^{\frac{2}{\alpha+2}} - t_{i,\alpha}^{\frac{2}{\alpha+2}}\right) = \frac{\alpha+2}{2}\left(\frac{2}{\alpha+2} c_{i,\alpha}^{\frac{-\alpha}{\alpha+2}}(t_i - t_{i,\alpha})\right) = c_{i,\alpha}^{\frac{-\alpha}{\alpha+2}}(t_i - t_{i,\alpha}) \to 0,$$

because $t_{i,\alpha} \to t_i$ and $c_{i,\alpha}^{\frac{-\alpha}{\alpha+2}} \to 1/t_i$. Therefore, as $\alpha \to \infty$

$$\frac{\alpha+2}{2}(1 - r_{i,\alpha}) = \frac{\alpha+2}{2}\left(1 - t_i^{\frac{2}{\alpha+2}}\right) + o(1) \to -\log(t_i),$$

that is,

$$\alpha(1 - r_{i,\alpha}) \to -2\log(t_i) \quad \text{as } \alpha \to \infty.$$

□

We also proved in Lemma 3.14 some asymptotic estimates for the local extrema of u_α, which show in particular that $\|u_\alpha\|_\infty$ blows up in each of its nodal sets. Here we prove that u_α blows up everywhere in B, and we present a sharp rate for the blow up.

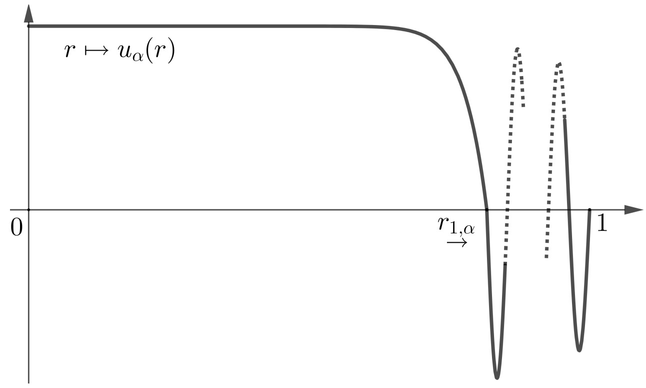

Figure 2 – The zeros of the radial solutions of (P_α) tend to 1 as $\alpha \to \infty$.

Corollary 3.17. Assume $u_\alpha(0) > 0$. Then for each $x \in B$ fixed, one has

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha(x) = w(0) = \|w\|_\infty, \tag{3.30}$$

where w is the unique radial solution of (L) with precisely m nodal sets such that $w(0) > 0$. In particular $u_\alpha(x) \to \infty$ as $\alpha \to \infty$ for all $x \in B$.

Proof. Fixed $x \in B$, set $r_0 = |x| \in [0,1)$. Let v_α be the function defined in (3.1). Since (v_α) converges uniformly to w, we get

$$\left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha(r_0) = v_\alpha(r_0^{\frac{\alpha+2}{2}}) = w(r_0^{\frac{\alpha+2}{2}}) + \left[v_\alpha(r_0^{\frac{\alpha+2}{2}}) - w(r_0^{\frac{\alpha+2}{2}})\right] = w(0) + o(1),$$

as $\alpha \to \infty$. □

Actually, we can prove that the limit (3.30) is uniform for x in each compact subset of B.

Corollary 3.18. Let $0 < R_0 < 1$. Then

$$\left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha(x) \to w(0)$$

uniformly in $B(0,R_0)$ as $\alpha \to \infty$.

Proof. Since u_α satisfies in its radial coordinate $r = |x| \in (0,1)$

$$-(r^{N-1}u_\alpha')' = r^{N-1}|u_\alpha|^{p-1}u_\alpha, \quad u_\alpha(0) > 0, \quad u_\alpha'(0) = u_\alpha(1) = 0,$$

we have that

$$u_\alpha'(r) = -\frac{1}{r^{N-1}}\int_0^r s^{N-1}|u_\alpha(s)|^{p-1}u_\alpha(s)ds, \quad \forall r \in (0,1).$$

By Corollary 3.16, $r_{1,\alpha} \to 1$ as $\alpha \to \infty$ and hence given $0 < R_0 < 1$, there is $\alpha_1 > \alpha_p$ such that $r_{1,\alpha} > R_0$ for all $\alpha > \alpha_1$. Consequently $u_\alpha(r) > 0$ for all $r \in [0,R_0]$ and $\alpha > \alpha_1$ and thus

$u'_\alpha(r) < 0$ for $r \in [0, R_0]$ and $\alpha > \alpha_1$, that is, u_α is strictly decreasing in $[0, R_0]$ for all $\alpha > \alpha_1$. Set $W_\alpha(r) := \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} u_\alpha(r)$. Then given $\varepsilon > 0$, by (3.30), there is $\alpha_2 > \alpha_1$ such that

$$|W_\alpha(0) - W_\alpha(R_0)| < \varepsilon/2, \quad |W_\alpha(R_0) - w(0)| < \varepsilon/2 \quad \forall \alpha > \alpha_2.$$

Hence the monotonicity of W_α in $[0, R_0]$ for $\alpha > \alpha_2$ implies that for any $r \in [0, R_0]$

$$\begin{aligned} |W_\alpha(r) - w(0)| &\le |W_\alpha(r) - W_\alpha(R_0)| + |W_\alpha(R_0) - w(0)| = W_\alpha(r) - W_\alpha(R_0) + |W_\alpha(R_0) - w(0)| \\ &\le W_\alpha(0) - W_\alpha(R_0) + |W_\alpha(R_0) - w(0)| < \varepsilon, \quad \forall \alpha > \alpha_2. \end{aligned}$$

It follows then the uniform convergence, since α_2 does not depend on r. □

Remark 3.19. Corollary 3.18 deserves some comments. Let ω_α be the least energy solution of the Hénon equation (P_α) and $R_0 \in (0, 1)$. It was proved in (BYEON; WANG, 2005, eq. (32)) that

$$\left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}} \omega_\alpha(x) \to 0$$

uniformly in $B(0, R_0)$ as $\alpha \to \infty$. This is in contrast to the radial, either nodal or positive, solutions of (P_α) according to Corollary 3.18.

Asymptotic estimates for the best constants

For any power $p > 1$ and $\alpha > \alpha_p$, we denote by $S^R_{\alpha,p}$ the best constant of the embedding $H^1_{0,rad}(B) \subset L^{p+1}(|x|^\alpha, B)$, that is

$$S^R_{\alpha,p} := \inf_{0 \ne u \in H^1_{0,rad}(B)} \frac{\displaystyle\int_B |\nabla u|^2}{\left(\displaystyle\int_B |x|^\alpha |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

Actually, the condition $\alpha > \alpha_p$ implies that the above embedding is compact and hence standard arguments show that $S^R_{\alpha,p}$ is achieved by a positive radial function u_α, which is also, after scaling, the positive radial solution of (P_α). A known result of Smets, Su and Willem shows asymptotic estimates for $S^R_{\alpha,p}$, namely

$$\lim_{\alpha\to\infty} \left(\frac{N}{\alpha+N}\right)^{\frac{p+3}{p+1}} S^R_{\alpha,p} = C \in (0, \infty),$$

cf. (SMETS; WILLEM; SU, 2002, Theorem 4.1). Here, we give a more precise value for the constant C above.

Corollary 3.20. For any $p > 1$,

$$\lim_{\alpha\to\infty} \left(\frac{2}{\alpha+2}\right)^{\frac{p+3}{p+1}} S^R_{\alpha,p} = \left(\frac{\omega_{N-1}}{2\pi}\right)^{\frac{p-1}{p+1}} S_p,$$

where S_p is the best constant of the embedding $H_0^1(B^2) \subset L^{p+1}(B^2)$, $B^2 \subset \mathbb{R}^2$ is the unit ball, that is

$$S_p := \inf_{0 \neq v \in H_0^1(B^2)} \frac{\int_{B^2} |\nabla v|^2}{\left(\int_{B^2} |v|^{p+1}\right)^{\frac{2}{p+1}}}.$$

Proof. As before, S_p is achieved at a positive solution w of (L), which is also radially symmetric by results in (GIDAS; NI; NIRENBERG, 1979). Moreover, using eq. (3.24) and Theorem 3.1, we have as $\alpha \to \infty$

$$\left(\frac{2}{\alpha+2}\right)^{\frac{p+3}{p-1}} \int_B |\nabla u_\alpha|^2 = \omega_{N-1} \int_0^1 |v_\alpha'(t)|^2 t^{M_\alpha - 1} dt$$
$$\to \omega_{N-1} \int_0^1 |w'(t)|^2 t\, dt = \frac{\omega_{N-1}}{2\pi} \int_{B^2} |\nabla w|^2.$$

Since

$$\int_B |\nabla u_\alpha|^2 = (S_{\alpha,p}^R)^{\frac{p+1}{p-1}} \quad \text{and} \quad \int_{B^2} |\nabla w|^2 = (S_p)^{\frac{p+1}{p-1}},$$

we get

$$\left(\frac{2}{\alpha+2}\right)^{\frac{p+3}{p+1}} S_{\alpha,p}^R \to \left(\frac{\omega_{N-1}}{2\pi}\right)^{\frac{p-1}{p+1}} S_p, \quad \alpha \to \infty,$$

as we wanted. □

Asymptotic distribution for the spectrum of linearized operators

Here we analyze the spectrum of some linear operators, more precisely, we wish to derive asymptotic expansions of the negative eigenvalues of the linearized operators associated to the Hénon equation (P_α) at the radial solution u_α with precisely m nodal sets, as $\alpha \to \infty$. For this, consider the linearized operators $L^\alpha : H^2(B) \cap H_0^1(B) \to L^2(B)$ given by

$$\varphi \mapsto L^\alpha \varphi := -\Delta\varphi - p|x|^\alpha |u_\alpha|^{p-1}\varphi, \quad \alpha > \alpha_p,$$

which are self-adjoint operators in $L^2(B)$ with compact resolvent. In particular, they are Fredholm operators of index zero. In order to obtain a more precise description of the distribution of eigenvalues of L^α as $\alpha \to \infty$, we consider the weighted (singular) eigenvalue problem

$$L^\alpha \varphi = \widehat{\Lambda} \frac{\varphi}{|x|^2}. \tag{3.31}$$

Since we are considering the case $N \geq 3$, we recall that, due to the Hardy inequality, (3.31) is well defined in $H_0^1(B)$. Moreover, $\widehat{\Lambda}$ is an eigenvalue for (3.31) if there exists $0 \neq \varphi \in H_0^1(B)$ such that

$$\int_B \nabla\varphi\nabla\phi - p|x|^\alpha |u_\alpha|^{p-1} \varphi\phi\, dx = \widehat{\Lambda} \int_B \frac{\varphi\phi}{|x|^2} dx, \quad \forall \phi \in H_0^1(B).$$

In addition, each of these negative eigenvalues is given (and vice versa) by the following decomposition

$$\widehat{\Lambda} = \widehat{\Lambda}^{rad} + j(N-2+j), \tag{3.32}$$

where $\widehat{\Lambda}^{rad}$ is a negative radial eigenvalue of (3.31) and j is some nonnegative integer (see Proposition B.10). This implies in particular that the negative eigenvalues for (3.31) can be given in terms of its negative radial eigenvalues. Therefore, we are interested in studying the asymptotic distribution of negative radial eigenvalues for (3.31).

Since u_α has exactly m nodal sets, one has that (3.31) admits precisely m negative radial eigenvalues (see (AMADORI; GLADIALI, 2018, Theorem 1.7)). We denote by $\widehat{\Lambda}_{1,\alpha} < \cdots < \widehat{\Lambda}_{m,\alpha}$ the negative radial eigenvalues for (3.31) and by $\varphi_{i,\alpha}$ their respective eigenfunctions. Now let w be a radial solution for (L) with exactly m nodal sets. Then the singular eigenvalue problem associated to (L) at w is

$$-\Delta\psi - p|w|^{p-1}\psi = \lambda\frac{\psi}{|y|^2}, \quad \psi \in \mathscr{H}_0,$$

see Section 2.1, which has also precisely m negative radial eigenvalues, say $\lambda_1 < \cdots < \lambda_m$. We prove the following.

Corollary 3.21.

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha+2}\right)^2\widehat{\Lambda}_{i,\alpha} = \lambda_i, \quad \forall i = 1,\cdots,m.$$

Proof. As before, we perform the change of variable $t = r^{\frac{\alpha+2}{2}}$, $r = |x|$, and write

$$v_\alpha(t) = \left(\frac{2}{\alpha+2}\right)^{\frac{2}{p-1}}u_\alpha(r), \quad \psi_{i,\alpha}(t) = \varphi_{i,\alpha}(r), \quad t\in(0,1).$$

Since $\varphi_{i,\alpha}$ is radial and solves (3.31) with $\widehat{\Lambda} = \widehat{\Lambda}_{i,\alpha}$, we can check using Remark 3.5 that $\psi_{i,\alpha}$ satisfies

$$-\psi_{i,\alpha}'' - \frac{M_\alpha - 1}{t}\psi_{i,\alpha}' - p|v_\alpha|^{p-1}\psi_{i,\alpha} = \lambda_{i,\alpha}\frac{\psi_{i,\alpha}}{t^2}, \quad t\in(0,1), \quad \psi_{i,\alpha}(1) = 0, \tag{3.33}$$

with $M_\alpha = \frac{2(\alpha+N)}{\alpha+2}$ and $\lambda_{i,\alpha} = \left(\frac{2}{\alpha+2}\right)^2\widehat{\Lambda}_{i,\alpha}$. By Lemma (B.11), we have each $\lambda_{i,\alpha}$ characterized as

$$\lambda_{i,\alpha} = \min_{\substack{Z\subset H^1_{0,M_\alpha}\\ \dim Z = i}}\max_{\substack{z\in Z\\ z\neq 0}}\frac{\int_0^1[|z'(t)|^2 - p|v_\alpha|^{p-1}z^2(t)]t^{M_\alpha-1}dt}{\int_0^1 z^2(t)t^{M_\alpha-3}dt},$$

where for each $M\geq 2$, we set

$$H^1_{0,M} := \left\{ w:[0,1]\to\mathbb{R} \text{ measurable}: \begin{array}{c} w \text{ has first order weak derivative, } w(1)=0 \text{ and} \\ \int_0^1 |w'(t)|^2 t^{M-1}dt + \int_0^1 w^2(t)t^{M-3}dt < \infty \end{array}\right\},$$

which is a Hilbert space with inner product

$$(z,v) \mapsto \int_0^1 z'(t)v'(t)t^{M-1}dt + \int_0^1 z(t)v(t)t^{M-3}dt.$$

Since the space $\mathscr{D} := C_c^\infty(0,1)$ is dense in $H^1_{0,M}$ for all $M \geq 2$ (see Lemma A.6), we have

$$\lambda_{i,\alpha} = \inf_{\substack{Z\subset\mathscr{D}\\ \dim Z=i}} \max_{\substack{z\in Z\\ z\neq 0}} \frac{\int_0^1 [|z'(t)|^2 - p|v_\alpha|^{p-1}z^2(t)]t^{M_\alpha-1}dt}{\int_0^1 z^2(t)t^{M_\alpha-3}dt}, \quad \forall i = 1,\cdots,m.$$

Now by Theorem 3.1, $|v_\alpha|^{p-1} \to |w|^{p-1}$ uniformly in $[0,1]$ as $\alpha \to \infty$. This implies that

$$\lambda_{i,\alpha} \to \inf_{\substack{Z\subset\mathscr{D}\\ \dim Z=i}} \max_{\substack{z\in Z\\ z\neq 0}} \frac{\int_0^1 [|z'(t)|^2 - p|w|^{p-1}z^2(t)]\,t\,dt}{\int_0^1 \frac{z^2(t)}{t}dt}, \quad \forall i = 1,\cdots,m,$$

that is, $\lambda_{i,\alpha} \to \lambda_i$ for all i, as we wanted to prove, since $\lambda_{i,\alpha} = \left(\frac{2}{\alpha+2}\right)^2 \widehat{\Lambda}_{i,\alpha}$. □

Remark 3.22. By the previous corollary, we have

$$\frac{\widehat{\Lambda}_{i,\alpha}}{\alpha^2} \to \frac{\lambda_i}{4}, \quad \alpha\to\infty, \quad \forall i = 1,\cdots,m$$

and therefore the eigenvalues $\widehat{\Lambda}_{i,\alpha}$ satisfy the following asymptotic expansions:

$$\widehat{\Lambda}_{i,\alpha} = \frac{\lambda_i}{4}\alpha^2 + o(\alpha^2), \quad \alpha\to\infty, \quad i = 1,\cdots,m. \tag{3.34}$$

Next the symbols $\lceil . \rceil$ and $\lfloor . \rfloor$ represent respectively the ceiling and floor functions whose definitions are given as $\lceil \beta \rceil := \min\{k\in\mathbb{Z} : k\geq\beta\}$, $\lfloor\beta\rfloor := \max\{k\in\mathbb{Z} : k\leq\beta\}$, for $\beta\in\mathbb{R}$.

Corollary 3.23. Let $m(u_\alpha)$ be the Morse index of the radial solution u_α of (P_α) with fixed m nodal sets. Then

$$m(u_\alpha) \to \infty \quad \text{as } \alpha\to\infty.$$

Indeed, there exists $\alpha^* = \alpha^*(p,N,m) > \alpha_p$, that does not depend on α, such that

$$m(u_\alpha) \geq m + m\sum_{j=1}^{J_\alpha} N_j \quad \forall\alpha\geq\alpha^*, \tag{3.35}$$

where

$$J_\alpha := \left\lceil \frac{\sqrt{(N-2)^2 - (\lambda_m/2)\alpha^2}}{2} \right\rceil \quad \text{and} \quad N_j := \frac{(N+2j-2)(N+j-3)!}{(N-2)!\,j!}.$$

Moreover, assuming $m\geq 2$, for each $\theta > 1$, there exists $\alpha^* = \alpha^*(\theta,p,N,m) > \alpha_p$ independent of α such that

$$m(u_\alpha) \geq m + (m-1)\sum_{j=1}^{K_\alpha(\theta)} N_j \quad \forall\alpha\geq\alpha^*, \tag{3.36}$$

where

$$K_\alpha(\theta) := \left\lceil \frac{\sqrt{(N-2)^2 - (\lambda_{m-1}/\theta)\alpha^2}}{2} \right\rceil.$$

Proof. We first recall that, by Lemma B.8, the Morse index of u_α is precisely the number of negative eigenvalues (with their multiplicity) of (3.31). Moreover, each of these negative eigenvalues is given (and conversely) by the decomposition (3.32). Thus, for each $i = 1, \cdots, m$, we need to know the numbers $j \in \mathbb{N}$ that satisfy

$$\widehat{\Lambda}_{i,\alpha} + j(N-2+j) < 0. \tag{3.37}$$

Indeed, using (3.34), the above inequality is equivalent to

$$j < \frac{\sqrt{(N-2)^2 - (N-2)\lambda_i \alpha^2 + o(\alpha^2)} - (N-2)}{2}.$$

Since $0 < -\lambda_m/2 < -(N-2)\lambda_i$ for $N \geq 3$ and $i = 1, \cdots, m$, there exists $\alpha^* > \alpha_p$ such that

$$J_\alpha \leq \frac{\sqrt{(N-2)^2 - (\lambda_m/2)\alpha^2}}{2} + 1 < \frac{\sqrt{(N-2)^2 - (N-2)\lambda_i \alpha^2 + o(\alpha^2)} - (N-2)}{2},$$

for all $\alpha \geq \alpha^*$ and $i = 1, \cdots, m$. Consequently, whenever $j \leq J_\alpha$ with $\alpha \geq \alpha^*$, we get (3.37) for all $i = 1, \cdots, m$. Thus to show (3.35), it suffices to observe that, by (B.10) and Proposition B.10, each eigenvalue $\widehat{\Lambda}_{i,\alpha} + j(N-2+j)$ has multiplicity N_j. Therefore we conclude that the linearized operator L^α has at least $m\sum_{j=1}^{J_\alpha} N_j$ negative eigenvalues (with their multiplicity) associated to nonradial eigenfunction for all $\alpha \geq \alpha^*$. Adding the number of negative radial eigenvalues, which is m, we obtain (3.35).

Similarly we can check (3.36). Indeed, given $\theta > 1$, we have that $0 < -\lambda_{m-1}/\theta < -(N-2)\lambda_i$ for $N \geq 3$ and $i = 1, \cdots, m-1$. Thus there exists $\alpha^* = \alpha^*(\theta) > \alpha_p$ such that

$$K_\alpha(\theta) < \frac{\sqrt{(N-2)^2 - (N-2)\lambda_i \alpha^2 + o(\alpha^2)} - (N-2)}{2} \quad \forall \alpha \geq \alpha^*, \ \ i = 1, \cdots, m-1.$$

This implies that whenever $j \leq K_\alpha(\theta)$ with $\alpha \geq \alpha^*$, we get (3.37) for all $i = 1, \cdots, m-1$ and thus we conclude (3.36) as before. □

Remark 3.24. Comparing to (AMADORI; GLADIALI, 2018, Theorem 1.1), our contribution here is twofold. Firstly, our result is also valid for positive solutions, while (AMADORI; GLADIALI, 2018) gives no new information for positive solutions. Secondly, for nodal solutions, we improve the lower bounds for the Morse indices for large values of α. Indeed, it is shown in (AMADORI; GLADIALI, 2018, Theorem 1.1) that

$$m(u_\alpha) \geq m + (m-1) \sum_{j=1}^{1+\lfloor \frac{\alpha}{2} \rfloor} N_j \quad \forall \alpha > \alpha_p.$$

Moreover, from (MARCHIS; IANNI; PACELLA, 2017b, Eq. (6.11)) (see also (AMADORI; GLADIALI, 2018, Proposition 3.3)), it is known that $-\lambda_{m-1} > 1$ and hence for fixed $\theta_0 > 1$ close enough to 1, one has $-\lambda_{m-1}/\theta_0 > 1$ and thus $K_\alpha(\theta_0) > 1 + \lfloor \frac{\alpha}{2} \rfloor$ for all α large enough. Theorem 3.23 also presents an improvement on the result (LOU; WETH; ZHANG, 2019, Theorem 1.1 i)), once we have an explicit lower bound. Finally, for nodal solutions and large values of α, it is clear that the lower bound in (3.36) is much better than that in (3.35).

Actually, one can improve the asymptotic expansions (3.34); compare to (KÜBLER; WETH, 2019).

Corollary 3.25. Let $p>1$ and $\alpha>\alpha_p$. Then the negative eigenvalues of (3.31) are C^1-functions $(\alpha_p,\infty)\to(-\infty,0)$, $\alpha\mapsto\widehat{\Lambda}_{i,\alpha}$, $i=1,\cdots,m$, satisfying the following asymptotic expansions

$$\widehat{\Lambda}_i(\alpha):=\widehat{\Lambda}_{i,\alpha}=\frac{\lambda_i}{4}\alpha^2+c_i\alpha+o(\alpha)\quad\text{and}\quad\widehat{\Lambda}_i'(\alpha)=\frac{\lambda_i}{2}\alpha+c_i+o(1)\quad\text{as }\alpha\to\infty,$$

where c_i, $i=1,\cdots,m$, are constants.

Proof. For each $M\geq 2$ such that $p(M-2)<M+2$, let v_M be the solution with precisely m nodal sets of (P_M). Then the eigenvalue problem

$$-\psi''-\frac{M-1}{t}\psi'-p|v_M|^{p-1}\psi=\mu\frac{\psi}{t^2},\quad t\in(0,1),\quad\psi\in H^1_{0,M},\tag{3.38}$$

has exactly m negative eigenvalues, say $\mu_1(M)<\cdots<\mu_m(M)<0$. Using the minimax characterization (B.14) and the differential dependence of eigenvalues with respect to the coefficients, we have by Remark 3.15 that the maps $M\mapsto\mu_i(M)$ are C^1-functions for each $i=1,\cdots,m$. In particular, we may write

$$\mu_i(M)=\mu_i(2)+\mu_i'(2)(M-2)+o(M-2)\quad\text{and}\quad\mu_i'(M)=\mu_i'(2)+o(1)\quad\text{as }M\to2^+.$$

Now, observe that, putting $M=M_\alpha=\frac{2(\alpha+N)}{\alpha+2}$, the problems (3.38) and (3.33) are the same. Thus

$$\mu_i(M_\alpha)=\lambda_{i,\alpha}=\left(\frac{2}{\alpha+2}\right)^2\widehat{\Lambda}_i(\alpha).$$

Note that $M_\alpha\to2^+$ if and only if $\alpha\to\infty$. So performing the transformation $M\leftrightarrow M_\alpha$, a function $o(M-2)$ as $M\to2^+$ corresponds to a function $o(M_\alpha-2)=o(\frac{1}{\alpha})$ as $\alpha\to\infty$. Hence

$$\mu_i(M_\alpha)=\mu_i(2)+\frac{2(N-2)\mu_i'(2)}{\alpha+2}+o\left(\frac{1}{\alpha}\right)\quad\text{and}\quad\mu_i'(M_\alpha)=\mu_i'(2)+o(1)\quad\text{as }\alpha\to\infty.$$

Therefore

$$\begin{aligned}\widehat{\Lambda}_i(\alpha)&=\left(\frac{\alpha+2}{2}\right)^2\mu_i(M_\alpha)=\frac{\mu_i(M_\alpha)}{4}\alpha^2+\mu_i(M_\alpha)\alpha+\mu_i(M_\alpha)\\&=\frac{\mu_i(2)}{4}\alpha^2+\frac{(N-2)\mu_i'(2)}{2}\frac{\alpha^2}{\alpha+2}+\mu_i(2)\alpha+o(\alpha)\\&=\frac{\mu_i(2)}{4}\alpha^2+\left[\frac{(N-2)\mu_i'(2)}{2}+\mu_i(2)\right]\alpha+o(\alpha),\quad\text{as }\alpha\to\infty,\end{aligned}$$

and

$$\begin{aligned}\widehat{\Lambda}_i'(\alpha)&=\frac{\alpha+2}{2}\mu_i(M_\alpha)-\left(\frac{\alpha+2}{2}\right)^2\mu_i'(M_\alpha)\frac{2(N-2)}{(\alpha+2)^2}\\&=\frac{\alpha+2}{2}\mu_i(2)+(N-2)\mu_i'(2)-\frac{(N-2)\mu_i'(2)}{2}+o(1)\\&=\frac{\mu_i(2)}{2}\alpha+\mu_i(2)+\frac{(N-2)\mu_i'(2)}{2}+o(1),\quad\text{as }\alpha\to\infty.\end{aligned}$$

Observe that $\mu_i(2)=\lambda_i$, and taking $c_i=\frac{(N-2)\mu_i'(2)}{2}+\mu_i(2)$ we conclude the proof. □

Monotonicity of the Morse index in dimension $N \geq 3$

In Chapter 2, we showed that, in dimension $N = 2$, the Morse index of the radial solutions of (P_α) with the same number of nodal sets is monotone nondecreasing with respect to α. Here we prove the monotonicity of the Morse index for dimensions $N \geq 3$ and with α large enough.

Corollary 3.26. Let $N \geq 3$ and $p > 1$. Then there exists $\alpha^* > \alpha_p$ such that

$$m(u_\alpha) \leq m(u_\beta), \quad \forall \alpha, \beta \in [\alpha^*, \infty), \ \alpha < \beta.$$

Proof. It is a consequence of equation (3.32) and Corollary 3.25. Indeed, since $\lambda_i < 0$, we have by Corollary 3.25 that

$$\widehat{\Lambda}_i'(\alpha) = \frac{\lambda_i}{2}\alpha + c_i + o(1) \to -\infty, \quad \alpha \to \infty, \quad \forall i = 1, \cdots, m,$$

which shows in particular that there exists $\alpha^* > \alpha_p$ such that $\widehat{\Lambda}_i'(\alpha) < 0$ for all $\alpha \geq \alpha^*$, that is, the function $\alpha \mapsto \widehat{\Lambda}_{i,\alpha}$ is strictly decreasing in $[\alpha^*, \infty)$ for each $i = 1, \cdots, m$. This implies, using (3.32), that the Morse index $m(u_\alpha)$ is nondecreasing with respect to α in $[\alpha^*, \infty)$. □

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APPENDIX

A

ONE-DIMENSIONAL WEIGHTED SOBOLEV SPACES

For any real number $M \geq 1$, we denote by L^q_M the space of mensurable functions $v : (0,1) \to \mathbb{R}$ such that

$$\int_0^1 |v(t)|^q t^{M-1} dt < +\infty.$$

Evidently L^2_M is a Hilbert space with inner product

$$(v,w) \mapsto \int_0^1 v(t)w(t)t^{M-1} dt.$$

Next we denote by H^1_M the space of functions $v \in L^2_M$ such that v has first order weak derivative v' in L^2_M so that

$$\int_0^1 (v^2 + |v'|^2) t^{M-1} dt < +\infty.$$

Then, standard arguments show that H^1_M is a Hilbert space with scalar product

$$(v,w) \mapsto \int_0^1 [v(t)w(t) + v'(t)w'(t)] t^{M-1} dt.$$

Moreover, as in (BREZIS, 2010, Theorem 8.2), for any $v \in H^1_M$, there exists a function $\tilde{v} \in C((0,1])$ such that $v = \tilde{v}$ a.e. in $(0,1)$ and

$$\tilde{v}(t_0) - \tilde{v}(t_1) = \int_{t_0}^{t_1} v'(t) dt \quad \forall t_0, t_1 \in [0,1]. \tag{A.1}$$

Therefore we may always assume w.l.g. that any function $v \in H^1_M$ belongs to $C((0,1])$ and satisfies (A.1). This allows to introduce the space

$$H^1_{0,M} := \{v \in H^1_M : v(1) = 0\}.$$

Observe that if $M = N$ is an integer number, then there exists a natural isometric isomorphism between H^1_N and $H^1_{rad}(B)$, where B denotes the unit ball in $\mathbb{R}^N$. The same holds for spaces $H^1_{0,N}$ and $H^1_{0,rad}(B)$.

Next we present some properties of Sobolev spaces H^1_M and $H^1_{0,M}$ (note the similarity with usual Sobolev spaces).

Lemma A.1 (Poincaré inequality)**.** Let $M > 1$. For any $v \in H^1_{0,M}$

$$\int_0^1 v^2(t)t^{M-1}dt \leq \frac{1}{M-1}\int_0^1 |v'(t)|^2 t^{M-1}dt.$$

Proof. See (AMADORI; GLADIALI, 2018, Lemma 6.1). □

Observe that, due to the Poincaré inequality, the expression $\int_0^1 v'w't^{M-1}dt$ defines an inner product on $H^1_{0,M}$ whose the associated norm $\|v'\|_{L^2_M}$ is equivalent to the usual H^1_M norm.

Lemma A.2 (Hardy inequality)**.** If $M > 2$ then

$$\int_0^1 v^2(t)t^{M-3}dt \leq \left(\frac{2}{M-2}\right)^2 \int_0^1 |v'(t)|^2 t^{M-1}dt$$

for any $v \in H^1_{0,M}$.

Proof. See (AMADORI; GLADIALI, 2018, Lemma 6.5). □

Given $M \geq 1$, we denote $2^*_M := \frac{2M}{M-2}$ if $M > 2$ and $2^*_M := +\infty$ if $1 \leq M \leq 2$. Then, when $M > 2$, the space

$$\mathscr{D}^{1,2}_M := \{v \in L^{2^*_M}_M(0,\infty) : v \text{ has first order weak derivative in } L^2_M(0,\infty)\}$$

is a Hilbert space with scalar product

$$(v,w) \mapsto \int_0^\infty v(t)'w(t)'t^{M-1}dt.$$

Moreover one has the following.

Lemma A.3 (Sobolev embedding)**.** Let $M > 2$. Then the inclusion $\mathscr{D}^{1,2}_M \subset L^{2^*_M}_M(0,\infty)$ is continuous and the best constant

$$S_M := \inf_{0 \neq v \in \mathscr{D}^{1,2}_M} \frac{\int_0^\infty |v'(t)|^2 t^{M-1}dt}{(\int_0^\infty |v(t)|^{2^*_M} t^{M-1}dt)^{2/2^*_M}}$$

is achieved by any Talenti's bubble $U(t) = (a + bt^2)^{-\frac{M-2}{2}}$, where a, b are positive constants.

The just mentioned Sobolev embedding has been established by (TALENTI, 1976). If M is an integer, then the embedding $H^1_M = H^1_{rad}(B)$ into $L^q(B)$ is compact for all $1 \leq q < 2^*_M$. The same arguments can be repeated for any real number M (see for instance (AMADORI; GLADIALI, 2018, Lemma 6.4)).

Lemma A.4 (Rellich-Kondrachov compactness)**.** For any $M \geq 1$, the space H^1_M is compactly embedded in L^q_M for every $q \in [1, 2^*_M)$.

Next, for each $\alpha \geq 0$ and $q \geq 1$, we denote by $L^q(|x|^\alpha, B)$ the space of mensurable functions $u : B \to \mathbb{R}$ such that the norm

$$\|u\|_\alpha := \left(\int_B |x|^\alpha |u|^q dx\right)^{1/q}$$

is finite. Obviously $L^q(B) \subset L^q(|x|^\alpha, B)$. We also denote by $L^q_{rad}(|x|^\alpha, B)$ the space of functions $u \in L^q(|x|^\alpha, B)$ such that u is radially symmetric. Set $M_\alpha := \frac{2(\alpha+N)}{\alpha+2}$. Then it was established in (AMADORI; GLADIALI, 2018, Proposition 5.6) that the map $u \mapsto Au$, where $Au(t) = u(t^{\frac{2}{\alpha+2}})$ yields the following isomorphisms:

$$L^q_{rad}(|x|^\alpha, B) \xrightarrow{A} L^q_{M_\alpha} \quad \text{and} \quad H^1_{rad}(B) \xrightarrow{A} H^1_{M_\alpha}.$$

Hence, if $1 \leq q < 2^*_{M_\alpha}$, we have the following sequence of continuous injections

$$H^1_{rad}(B) \xrightarrow{A} H^1_{M_\alpha} \overset{\text{Sobolev}}{\longrightarrow} L^q_{M_\alpha} \xrightarrow{A^{-1}} L^q(|x|^\alpha, B),$$

where the second arrow is compact by Lemma (A.4). Observe that this is an alternative proof for the following classic result.

Lemma A.5. Let $N \geq 3$ and $1 \leq q < \frac{2(N+\alpha)}{N-2}$. Then the space $H^1_{rad}(B)$ is compactly embedded in $L^q(|x|^\alpha, B)$. If $N = 1, 2$ then $H^1_{rad}(B)$ is compactly embedded in $L^q(|x|^\alpha, B)$ for all $q \in [1, \infty)$.

Now, we show some density properties on weighted Sobolev spaces.

Lemma A.6. Let $M \geq 1$ and $v \in H^1_{0,M}$. Then given $\varepsilon > 0$, there exists a function $\phi \in C^\infty_c(0,1)$ such that

$$\int_0^1 |v'(t) - \phi'(t)|^2 t^{M-1} dt < \varepsilon^2. \tag{A.2}$$

Moreover, if $\int_0^1 v^2(t) t^{M-3} dt < +\infty$, then we can found ϕ so that

$$\int_0^1 |v'(t) - \phi'(t)|^2 t^{M-1} dt + \int_0^1 |v(t) - \phi(t)|^2 t^{M-3} dt < \varepsilon^2. \tag{A.3}$$

Proof. We will check just (A.3), because the proof of (A.2) is similar. To simplify the notation, we write

$$\|w\|_M := \left(\int_0^1 |w'(t)|^2 t^{M-1} dt + \int_0^1 |w(t)|^2 t^{M-3} dt\right)^{1/2}.$$

Note that $\|.\|_M$ is a norm. Choose $0 < \rho < 1/2$ so small that

$$\int_0^{2\rho} |v'(t)|^2 t^{M-1} dt + \int_0^{2\rho} |v(t)|^2 t^{M-3} dt < \frac{\varepsilon^2}{36}. \tag{A.4}$$

Let $\psi \in H^1_M$ be the following function: $\psi(t) = 0$ if $0 \leq t \leq \rho$, $\psi(t) = 1$ if $2\rho \leq t \leq 1$ and ψ restrict to $[\rho, 2\rho]$ is the affine function such that $\psi(\rho) = 0$ and $\psi(2\rho) = 1$. Of course $0 \leq \psi \leq 1$ and $|\psi'|_\infty = 1/\rho$. We first show that

$$\|v(1-\psi)\|_M < \frac{\varepsilon}{2}. \tag{A.5}$$

Indeed, we have that

$$\left|[v(1-\psi)]'\right|^2 \le 2|1-\psi|^2|v'|^2 + 2|v|^2|\psi'|^2 \le 2|v'|^2 + \frac{2}{\rho^2}|v|^2.$$

Hence

$$\begin{aligned}\int_0^1 \left|[v(1-\psi)]'\right|^2 t^{M-1}dt &= \int_0^\rho |v'|^2 t^{M-1}dt + \int_\rho^{2\rho} \left|[v(1-\psi)]'\right|^2 t^{M-1}dt \\ &\le \int_0^\rho |v'|^2 t^{M-1}dt + 2\int_\rho^{2\rho} |v'|^2 t^{M-1}dt + \frac{2}{\rho^2}\int_\rho^{2\rho} |v|^2 t^{M-1}dt \\ &\le 2\int_0^{2\rho} |v'|^2 t^{M-1}dt + 8\int_\rho^{2\rho} |v|^2 t^{M-3}dt \overset{(A.4)}{<} \frac{8\varepsilon^2}{36}.\end{aligned}$$

Furthermore, thanks to (A.4), we also have that

$$\int_0^1 |v(1-\psi)|^2 t^{M-3}dt < \frac{\varepsilon^2}{36},$$

which proves (A.5). Next, we claim that there exists a sequence (ϕ_n) in $C_c^\infty(0,1)$, with support in $(\rho,1)$, such that

$$\|v\psi - \phi_n\|_M \to 0, \quad \text{as } n\to\infty. \tag{A.6}$$

Observe that (A.6) combinend with (A.5) imply that

$$\|v-\phi_n\|_M \le \|v - v\psi\|_M + \|v\psi-\phi_n\|_M < \varepsilon$$

for n large enough, showing (A.3) and concluding the proof.

To prove (A.6), observe that

$$v\psi \in H^1_{0,M}(\rho,1) = \{w\in H^1_{0,M} : w(t)=0 \text{ for } t\le\rho\}$$

and, since $C_c^\infty(\rho,1)$ is dense in $H^1_{0,M}(\rho,1)$, there exists a sequence (ϕ_n) in $C_c^\infty(\rho,1)$ such that $\phi_n \to v\psi$ in $H^1_{M,0}(\rho,1)$. We extend ϕ_n by zero in $[0,\rho]$ so that $\phi_n \in C_c^\infty(0,1)$ and it satisfies

$$\int_0^1 \left|(v\psi-\phi_n)'\right|^2 t^{M-1}dt = \int_\rho^1 \left|(v\psi-\phi_n)'\right|^2 t^{M-1}dt \to 0. \tag{A.7}$$

Then observe that $H^1_{0,M}(\rho,1) \hookrightarrow L^\infty(\rho,1) \hookrightarrow L^2_M(\rho,1)$, we have also that

$$\int_0^1 |v\psi-\phi_n|^2 t^{M-1}dt = \int_\rho^1 |v\psi-\phi_n|^2 t^{M-1}dt \to 0.$$

From this last equality follows also that

$$\int_0^1 |v\psi-\phi_n|^2 t^{M-3}dt = \int_\rho^1 |v\psi-\phi_n|^2 t^{M-3}dt \le \frac{1}{\rho^2}\int_\rho^1 |v\psi-\phi_n|^2 t^{M-1}dt \to 0$$

which together with (A.7) proves (A.6). □

APPENDIX

B

SINGULAR EIGENVALUE PROBLEMS

This appendix is dedicated to the study of singular eigenvalue problems of the type

$$\begin{cases} -\Delta\psi - a(x)\psi = \widehat{\Lambda}\dfrac{\psi}{|x|^2} & \text{in } B\backslash\{0\}, \\ \psi = 0 & \text{on } \partial B, \end{cases} \tag{B.1}$$

where $a(x) \in L^\infty(B)$ and B is the open unit ball in $\mathbb{R}^N$ centered at the origin. Most of all the results in this part are contained in (AMADORI; GLADIALI, 2018).

B.1 Variational characterization

As in the case of regular eigenvalues problems, under suitable hypotheses, the eigenvalues of (B.1) also have a variational characterization. Since, in general, the singular problem (B.1) is not well defined in the space $H_0^1(B)$ because of the singularity at the origin, we need to introduce the Lebesgue space

$$\mathscr{L} := \left\{ u : B \to \mathbb{R} \text{ mensureble s.t. } \int_B \frac{u}{|x|^2} < \infty \right\},$$

which is a Hilbert space with the inner product $\int_B |x|^{-2}uvdx$, for $u,v \in \mathscr{L}$. We also set

$$u \perp v \Longleftrightarrow \int_B |x|^{-2}uvdx = 0, \text{ for } u,v \in \mathscr{L},$$

and the Sobolev spaces

$$\mathscr{H} := H^1(B) \cap \mathscr{L}, \quad \mathscr{H}_0 := H_0^1(B) \cap \mathscr{L} \text{ and } \mathscr{H}_{0,rad} := H^1_{0,rad}(B) \cap \mathscr{L}$$

which are Hilbert spaces with scalar product $\int_B (\nabla u \nabla v + |x|^{-2}uv)dx$.

Remark B.1. Due to the Hardy inequality, in dimension $N \geq 3$, the space $\mathscr{H}_0$ coincides with $H_0^1(B)$. But in dimension two the inclusion $\mathscr{H}_0 \subset H_0^1(B)$ is strict. Consider for example the function $u(x) = 1 - |x|^2$. Indeed in dimension two every continuous function $u \in \mathscr{H}$ satisfies $u(0) = 0$ since $|x|^{-2} \notin L^1(B)$.

Finally we say that $\psi \in \mathscr{H}_0 \backslash \{0\}$ is a eigenfunction of (B.1) if

$$\int_B \nabla\psi\nabla\varphi - a(x)\psi\varphi dx = \widehat{\Lambda}\int_B \frac{\psi\varphi}{|x|^2}dx \quad \forall\, \varphi \in \mathscr{H}_0.$$

Similarly to the regular case, we can try to produce an eigenvalue by minimizing the quotient

$$\widehat{\Lambda}_1 := \inf_{\substack{w\in\mathscr{H}_0\\ w\neq 0}} \frac{\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx}{\int_B |x|^{-2}w^2(x)dx}. \tag{B.2}$$

This method may fail: for example, when $a(x) \equiv 0$ the infimum (B.2) is $\left(\frac{N-2}{2}\right)^2$ and it is not achieved due to the lack of compactness of the embedding of $H_0^1(B)$ into $\mathscr{L}$ (see Proposition B.3). However if the infimum in (B.2) is strictly less than $\left(\frac{N-2}{2}\right)^2$, one has the following.

Proposition B.2. Let $N \geq 2$. If $\widehat{\Lambda}_1 < 0$, then $\widehat{\Lambda}_1$ is achieved at a function $\psi_1 \in \mathscr{H}_0$ which is an eigenfunction of (B.1) corresponding to $\widehat{\Lambda}_1$. Actually $\psi \in C^{1,\alpha}(\overline{B}\backslash\{0\})$ for some $0 < \alpha < 1$ and it is a classical solution whenever $a(x)$ is $C^{0,\beta}$ for some $0 < \beta < 1$.

Proof. Consider a minimizing sequence (w_n) of (B.2), normalized such that

$$\int_B w_n^2 dx = 1, \quad \text{for } n \in \mathbb{N}.$$

So by definition, we can find a sequence (β_n) such that

$$\int_B |\nabla w_n(x)|^2 - a(x)w_n^2(x)dx = \beta_n \int_B |x|^{-2}w_n^2(x)dx \tag{B.3}$$

with $\beta_n < 0$ and $\beta_n \searrow \widehat{\Lambda}_1$ as $n \to \infty$. We claim that

$$\int_B |\nabla w_n|^2 dx < C, \quad \forall n \in \mathbb{N}. \tag{B.4}$$

Since $\beta_n < 0$, by (B.3), it follows that

$$\int_B |\nabla w_n(x)|^2 dx \leq \int_B a(x)w_n^2(x)dx \leq \|a\|_\infty \int_B w_n^2(x)dx = C,$$

and (B.4) follows. Therefore, by (B.4), up to subsequence, $w_n \to w$ weakly in $H_0^1(B)$, strongly in $L^2(B)$, and pointwise almost everywhere in B, in particular

$$\int_B |\nabla w(x)|^2 dx \leq \liminf_{n\to\infty} \int_B |\nabla w_n(x)|^2 dx \quad \text{and} \quad \lim_{n\to\infty}\int_B a(x)w_n^2(x)dx = \int_B a(x)w^2(x)dx.$$

Moreover,

$$\int_B |x|^{-2}w_n^2(x)dx \leq C.$$

Indeed, since $\widehat{\Lambda}_1 < 0$, then from (B.3) and (B.4)

$$\int_B |x|^{-2}w_n^2(x)dx \leq \frac{1}{|\beta_n|}\int_B |\nabla w_n(x)|^2 dx + \frac{\|a\|_\infty}{|\beta_n|}\int_B w_n^2(x)dx \leq C.$$

Next we check that w minimizes the quotient in (B.2). It is enough to show that

$$\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx - \widehat{\Lambda}_1 \int_B |x|^{-2}w^2(x)dx \leq 0. \tag{B.5}$$

Indeed, by Fatou lemma

$$\int_B |x|^{-2}w^2dx \leq \liminf_{n\to\infty} \int_B |x|^{-2}w_n^2dx.$$

Hence,

$$\begin{aligned}\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx - \widehat{\Lambda}_1 \int_B |x|^{-2}w^2(x)dx \\ &\leq \liminf_{n\to\infty} \int_B \left(|\nabla w_n(x)|^2 - \beta_n|x|^{-2}w_n^2(x)\right)dx - \int_B a(x)w^2(x)dx \\ &= \liminf_{n\to\infty} \int_B a(x)w_n^2(x)dx - \int_B a(x)w^2(x)dx = 0,\end{aligned}$$

which shows (B.5).

Now, since w minimizes the quotient (B.2), for any $\phi \in \mathscr{H}_0$ the function

$$F(t) = \frac{\int_B |\nabla w(x) + t\phi(x)|^2 - a(x)(w+t\phi)^2(x)dx}{\int_B |x|^{-2}(w+t\phi)^2(x)dx}$$

has a minimum at $t = 0$ and this implies that $w = \psi_1$ is an eigenfunction to (B.1).

Next, since $a(x) \in L^\infty(B)$ and $\psi_1 \in \mathscr{H}_0$, we have for each $\varepsilon > 0$, $a(x)\psi_1 + \widehat{\Lambda}_1 \frac{\psi_1}{|x|^2} \in L^q(A_{\varepsilon,1})$ for all $q \in (1,\infty)$ when $N = 2$ and for any $q \in (1, \frac{2N}{N-2}]$ if $N \geq 3$. This implies, by elliptic L^q estimates, that $\psi_1 \in W^{2,q}(A_{\varepsilon,1})$ for all $\varepsilon > 0$ and q as before. Using bootstrap arguments, $\psi_1 \in W^{2,q}(A_{\varepsilon,1})$ for all $\varepsilon > 0$ and for any $q \in (1,\infty)$ in any dimension N. The $C^{1,\alpha}(\overline{B}\backslash\{0\})$ regularity follows then by Morrey's Theorem (see (EVANS, 2010, Subsection 5.6.2, Theorem 5)). If in addition $a(x) \in C^{0,\beta}(B)$ for some $0 < \beta < 1$, Schauder estimates yield that $\psi_1 \in C^{2,\beta}(\overline{B}\backslash\{0\})$ so that it is also a classical solution to (B.1), corresponding to $\widehat{\Lambda}_1$. □

The next proposition shows that the condition $\widehat{\Lambda}_1 < 0$ in dimension two is sharp.

Proposition B.3. Let $N = 2$ and assume that in the previous proposition $a \equiv 0$. Then

$$\widehat{\Lambda}_1 = \inf_{\substack{w\in\mathscr{H}_0 \\ w\neq 0}} \frac{\int_B |\nabla w(x)|^2dx}{\int_B |x|^{-2}w^2(x)dx} = 0$$

and it is not achieved.

Proof. Choose the test funcions

$$w_\varepsilon(x) = \begin{cases} 1-|x| & \text{if } \varepsilon \leq |x| \leq 1, \\ \dfrac{2(1-\varepsilon)}{\varepsilon}|x| + \varepsilon - 1 & \text{if } \dfrac{\varepsilon}{2} \leq |x| \leq \varepsilon, \\ 0 & \text{if } |x| \leq \dfrac{\varepsilon}{2}. \end{cases}$$

Hence by integration in polar coordinates

$$\int_B |\nabla w_\varepsilon(x)|^2 dx = 2\pi \int_0^1 (w'_\varepsilon(r))^2 r dr = 2\pi \left(\int_{\varepsilon/2}^{\varepsilon} \frac{4(1-\varepsilon)^2}{\varepsilon^2} r dr + \int_\varepsilon^1 r dr \right) = 4\pi + o(1)$$

and

$$\int_B \frac{w_\varepsilon^2(x)}{|x|^2} dx = 2\pi \int_0^1 \frac{w_\varepsilon^2(r)}{r^2} r dr = 2\pi \left[\int_{\varepsilon/2}^{\varepsilon} \left(\frac{2(1-\varepsilon)}{\varepsilon} r + \varepsilon - 1 \right)^2 r^{-1} dr + \int_\varepsilon^1 (1-r)^2 r^{-1} dr \right]$$
$$= 2\pi \left[(\varepsilon-1)^2 (2+\log(2)) + \frac{4\varepsilon - \varepsilon^2}{2} - \log(\varepsilon) - \frac{3}{2} \right] = 2\pi (o(1)-1) \log \varepsilon$$

which shows

$$\frac{\int_B |\nabla w_\varepsilon(x)|^2 dx}{\int_B |x|^{-2} w_\varepsilon^2(x) dx} \longrightarrow 0, \quad \varepsilon \to 0.$$

Of course $\widehat{\Lambda}_1$ is not achieved. □

When $\widehat{\Lambda}_1 < 0$ and it is achieved at a function $\psi_1 \in \mathscr{H}_0$, we can then define

$$\widehat{\Lambda}_2 := \inf_{\substack{w \in \mathscr{H}_0 \setminus \{0\} \\ w \perp \psi_1}} \frac{\int_B |\nabla w(x)|^2 - a(x) w^2(x) dx}{\int_B |x|^{-2} w^2(x) dx}.$$

Iteratively, if $\widehat{\Lambda}_{i-1} < 0$ and is attained by ψ_{i-1}, we can define

$$\widehat{\Lambda}_i := \inf_{\substack{w \in \mathscr{H}_0 \setminus \{0\} \\ w \perp \{\psi_1, \dots, \psi_{i-1}\}}} \frac{\int_B |\nabla w(x)|^2 - a(x) w^2(x) dx}{\int_B |x|^{-2} w^2(x) dx}. \tag{B.6}$$

As before the numbers $\widehat{\Lambda}_i < 0$ also are eigenvalues of (B.1), indeed one has

Proposition B.4. Whenever the value $\widehat{\Lambda}_i$ defined in (B.6) satisfies $\widehat{\Lambda}_i < 0$, it is achieved at a function $\psi_i \in \mathscr{H}_0$ which is a eigenfunction of (B.1) corresponding to $\widehat{\Lambda}_i$ and satisfies $\psi_i \perp \{\psi_1, \dots, \psi_{i-1}\}$. In addition ψ_i has the same regularity as ψ_1.

Proof. Arguing as in the proof of Proposition B.2, one can see that a minimizing sequence (w_n) of $\widehat{\Lambda}_i$ converges to a function w weakly in $\mathscr{H}_0$, strongly in $L^2(B)$ and pointwise a.e in B. Then we have

$$\int_B |x|^{-2} w \psi_j dx = \lim_{n\to\infty} \int_B |x|^{-2} w_n \psi_j dx = 0$$

for $j = 1, \dots, i-1$, meaning that $w \perp \{\psi_1, \dots, \psi_{i-1}\}$. Then, as before, it follows that $\widehat{\Lambda}_i$ is achieved and $w = \psi_i$ is an eigenfunction of (B.1). □

By Proposition B.4 the numbers $\widehat{\Lambda}_i$ defined in (B.6) are eigenvalues of problem (B.1) when they satisfy $\widehat{\Lambda}_i < 0$. The conserve is also true.

Lemma B.5 (Variational characterization)**.** The negative eigenvalues of problem (B.1) coincide with the numbers $\widehat{\Lambda}_i$ defined in (B.6).

Proof. Counting with their multiplicity, we enumerate the negative eigenvalues of (B.1) as $\mu_1 \leq \ldots \leq \mu_n$. Let $\widehat{\Lambda}_1, \ldots, \widehat{\Lambda}_m$ the negative eigenvalues defined by (B.6). We must check that $n = m$ and $\mu_i = \widehat{\Lambda}_i$ for each $i = 1, \ldots, m$. By definition, $m \leq n$. We recall that the numbers $\widehat{\Lambda}_i$ are eigenvalues to (B.1) whose corresponding eigenfunctions are orthogonal in $\mathscr{L}$. Consequently $\mu_i \leq \widehat{\Lambda}_i$, for $i = 1 \ldots, m$. On the other hand, $\widehat{\Lambda}_1$ is the infimum of the quotient

$$\frac{\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx}{\int_B |x|^{-2}w^2(x)dx}$$

in $\mathscr{H}_0$. When w is the first eigenfunction of (B.1), this quotient becomes μ_1 and hence $\widehat{\Lambda}_1 = \mu_1$ and ψ_1 is the first eigenfunction. Similarly $\widehat{\Lambda}_2$ minimizes the above quotient in $\{w \in \mathscr{H}_0 : w \perp \psi_1\}$ and when w equals the second eigenfunction, this quotient equals μ_2 and hence $\widehat{\Lambda}_2 = \mu_2$ and ψ_2 is the second eigenfunction. Iterating this argument, we have $\widehat{\Lambda}_i = \mu_i$, for $i = 1, \ldots, m$ and $m = n$, since $\widehat{\Lambda}_i < 0$. □

As a consequence of the previous lemma, we have the following.

Lemma B.6 (Minimax characterization)**.** The negative eigenvalues of problem (B.1) are characterized as

$$\widehat{\Lambda}_i := \min_{\substack{W \subset \mathscr{H}_0 \\ \dim W = i}} \max_{\substack{w \in W \\ w \neq 0}} \frac{\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx}{\int_B |x|^{-2}w^2(x)dx}.$$

Remark B.7. As in the regular case, the first eigenvalue $\widehat{\Lambda}_1$ is simple and its corresponding eigenfunction is positive (or negative) in $B \backslash \{0\}$. Moreover when $a(x)$ is radial, $\widehat{\Lambda}_1$ is a radial eigenvalue i.e $\psi_1 = \psi_1^{rad}$ is radially symmetric and the numbers

$$\widehat{\Lambda}_i^{rad} := \inf_{\substack{w \in \mathscr{H}_{0,rad} \backslash \{0\} \\ w \perp \{\psi_1^{rad}, \ldots, \psi_{i-1}^{rad}\}}} \frac{\int_B |\nabla w(x)|^2 - a(x)w^2(x)dx}{\int_B |x|^{-2}w^2(x)dx}$$

are the radial eigenvalues of (B.1), whenever $\widehat{\Lambda}_i^{rad} < 0$.

B.2 Singular and regular negative eigenvalues

The goal of this section is to check that the number of negative eigenvalues of a singular problem coincides with the number of negative eigenvalues of the corresponding regular problem. This is very important and was used throughout Chapters 2 and 3.

Associated to the singular problem (B.1), we consider the regular problem

$$\begin{cases} -\Delta \psi - a(x)\psi = \Lambda \psi & \text{in } B, \\ \psi = 0 & \text{on } \partial B. \end{cases} \tag{B.7}$$

We recall that the problem (B.7) has a sequence of eigenvalues $\Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \ldots$ that converges to infinity. Moreover each eigenvalue Λ_i is given by following minimax characterization

$$\Lambda_i := \min_{\substack{W \subset H_0^1(B) \\ \dim W = i}} \max_{\substack{w \in W \\ w \neq 0}} \frac{\int_B |\nabla w(x)|^2 - a(x) w^2(x) dx}{\int_B w^2(x) dx}. \tag{B.8}$$

Lemma B.8. Let $a(x) \in L^\infty(B)$. Counting with their multiplicities, the number K of negative eigenvalues Λ_i according to (B.8) coincides with the number L of negative eigenvalues $\widehat{\Lambda}_i$ defined in (B.6).

Proof. Case 1: $N \geq 3$. We have, via Hardy inequality, that $\mathscr{H}_0 = H_0^1(B)$. Thus it suffices to observe that by Lemma B.5 and (B.8) K and L coincide with the maximal dimension of a subspace of $H_0^1(B)$ in which the quadratic form

$$Q_a(w) := \int_B |\nabla w(x)|^2 - a(x) w^2(x) dx$$

is negative definite.
Case 2: $N = 2$. Note that the singular eigenfunctions $\widehat{\psi}_1, \ldots, \widehat{\psi}_L$ associated to negative eigenvalues $\widehat{\Lambda}_i$ are test functions in $H_0^1(B)$ in which $Q_a(\widehat{\psi}_i) < 0$. Since $\{\widehat{\psi}_i\}$ is an orthogonal set with respect to both bilinear forms

$$(v, w) \mapsto \int_B |x|^{-2} v w dx \quad \text{and} \quad (V, W) \mapsto \int_B \nabla V \nabla W - a(x) V W dx,$$

Q_a is negative definite in span$\{\widehat{\psi}_i\}$ and hence $L \leq K$.

The reciprocal is not similar because the eigenfunctions $\psi_1, \ldots, \psi_K$ associated to eigenvalues Λ_i are not test functions in $\mathscr{H}_0$. However, suppose by contradiction that $L < K$. This means that the problem (B.7) has at least $L+1$ negative eigenvalues, that is, $\Lambda_{L+1} < 0$. Hence by continuity of eigenvalues with respect to domains, we have that there exists $\varepsilon > 0$ such that the problem

$$-\Delta \psi - a(x) \psi = \Lambda \psi \ \text{ in } \ A_{\varepsilon,1}, \quad \psi = 0 \ \text{ on } \ \partial A_{\varepsilon,1},$$

has at least $L+1$ negative eigenvalues, counting with their multiplicity; cf. (MANES; MICHELETTI, 1973) and (GLADIALI; GROSSI; NEVES, 2016). Consequently, the quadratic form Q_a is negative definite in a subspace of $H_0^1(A_{\varepsilon,1})$ of dimension $L+1$. This contradicts the definition of L, since $H_0^1(A_{\varepsilon,1}) \subset \mathscr{H}_0$. □

Remark B.9. Similarly, when $a(x)$ is a radial function, one has that the number of negative radial eigenvalues of both problems (B.1) and (B.7) coincide.

B.3 Decomposition of singular eigenvalues

In this section, we explain a motivation which leads us to study the singular eigenvalue problem (B.1).

Let us recall that the spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere S^{N-1}. Indeed the operator $-\Delta_{S^{N-1}}$ admits a sequence of eigenvalues $0=\lambda_0<\lambda_1\leq\ldots$ and eigenfunctions $Y_j(\theta)$ which form a complete orthonormal system for $L^2(S^{N-1})$. Namely, each eigenfunction Y_j satisfies

$$-\Delta_{S^{N-1}}Y_j(\theta)=\lambda_jY_j(\theta)\quad\text{for }\theta\in S^{N-1}, \tag{B.9}$$

where the eigenvalue λ_j is given explicitly by expression $\lambda_j=j(N-2+j)$ and has multiplicity

$$N_j=\frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}. \tag{B.10}$$

Moreover these eigenfunctions are bounded in $L^\infty(S^{N-1})$ by standard regularity theory and observe that the first eigenfunction Y_0 is constant and, in dimension two, $Y_j(\theta)=a_j\cos(j\theta)+b_j\sin(j\theta)$ for any $j\geq1$.

Proposition B.10. Let $\widehat{\Lambda}<0$ be an eigenvalue of (B.1). Then there exists $k\geq0$ such that

$$\widehat{\Lambda}=\widehat{\Lambda}^{rad}+\lambda_k, \tag{B.11}$$

where $\widehat{\Lambda}^{rad}$ is a radial eigenvalue of (B.1). Conversely, if (B.11) holds and ψ^{rad} is an eigenfunction associated to $\widehat{\Lambda}^{rad}$, then $\psi=\psi^{rad}(r)Y_k(\theta)$ is an eigenfunction of (B.1) associated to $\widehat{\Lambda}$.

Proof. Let $\psi\in\mathscr{H}_0$ be an eigenfunction of (B.1) associated to the eigenvalue $\widehat{\Lambda}$. Since $\mathscr{H}_0\subset H^1_0(B)$, we can decompose ψ along spherical harmonics $Y_k(\theta)$ as

$$\psi(r,\theta)=\sum_{k=0}^{\infty}\psi_k(r)Y_k(\theta)\quad\text{for }r\in(0,1),\ \theta\in S^{N-1},$$

where

$$\psi_k(r):=\int_{S^{N-1}}\psi(r,\theta)Y_k(\theta)d\sigma(\theta).$$

Since ψ is non-zero, $\psi_k(r)$ is non-zero for some $k\geq0$. We claim that $\psi_k(r)$ is a radial eigenfunction to (B.1) associated to eigenvalue $\widehat{\Lambda}^{rad}=\widehat{\Lambda}-\lambda_k$. Indeed, we first check that $\psi_k\in\mathscr{H}_0$. Of course $\psi_k(1)=0$. Moreover by Jensen inequality we have

$$\begin{aligned}\int_B\frac{\psi_k^2(x)}{|x|^2}dx&=\omega_{N-1}\int_0^1\left(\int_{S^{N-1}}\psi(r,\theta)Y_k(\theta)d\sigma(\theta)\right)^2r^{N-3}dr\\&\leq\omega_{N-1}^2\int_0^1\int_{S^{N-1}}\psi^2(r,\theta)Y_k^2(\theta)d\sigma(\theta)r^{N-3}dr\\&\leq\omega_{N-1}^2\|Y_k\|_\infty^2\int_0^1\int_{S^{N-1}}\psi^2(r,\theta)r^{N-3}d\sigma(\theta)dr=\omega_{N-1}\|Y_k\|_\infty^2\int_B\frac{\psi^2(x)}{|x|^2}dx<\infty\end{aligned}$$

and

$$\begin{aligned}\int_B |\nabla \psi_k(x)|^2 dx &= \omega_{N-1}\int_0^1 \left(\int_{S^{N-1}} \psi'(r,\theta)Y_k(\theta)d\sigma(\theta)\right)^2 r^{N-1}dr\\ &\le \omega_{N-1}^2\int_0^1\int_{S^{N-1}}(\psi'(r,\theta)Y_k(\theta))^2 d\sigma(\theta)r^{N-1}dr\\ &\le \omega_{N-1}^2\|Y_k\|_\infty^2\int_0^1\int_{S^{N-1}}(\psi'(r,\theta))^2 r^{N-1}d\sigma(\theta)dr\\ &\le \omega_{N-1}\|Y_k\|_\infty^2\int_B|\nabla\psi(x)|^2dx<\infty,\end{aligned}$$

where the prime $'$ represents the weak derivate with respect to radial variable r. Next, given $\varphi \in \mathscr{H}_{0,rad}$ we have

$$\omega_{N-1}^{-1}\int_B \nabla\psi_k\nabla\varphi dx = \int_0^1\int_{S^{N-1}}\psi'(r,\theta)(Y_k(\theta)\varphi(r))'r^{N-1}d\sigma(\theta)dr,$$

and using that ψ is an eigenfunction of (B.1), this expression becomes

$$\begin{aligned}&=-\int_0^1\int_{S^{N-1}}\nabla_\theta\psi\nabla_\theta(Y_k(\theta)\varphi(r))r^{N-3}d\sigma(\theta)dr+\int_0^1\int_{S^{N-1}}a(r)\psi Y_k(\theta)\varphi(r)r^{N-1}d\sigma(\theta)dr\\ &\quad+\widehat{\Lambda}\int_0^1\int_{S^{N-1}}\psi Y_k(\theta)\varphi(r)r^{N-3}d\sigma(\theta)dr\\ &=-\int_0^1\varphi(r)r^{N-3}\int_{S^{N-1}}\nabla_\theta\psi\nabla_\theta Y_k(\theta)d\sigma(\theta)dr+\int_0^1 a(r)\varphi(r)r^{N-1}\int_{S^{N-1}}\psi Y_k(\theta)d\sigma(\theta)dr\\ &\quad+\widehat{\Lambda}\int_0^1\varphi(r)r^{N-3}\int_{S^{N-1}}\psi Y_k(\theta)d\sigma(\theta)dr.\end{aligned}$$

Now using that $Y_k(\theta)$ solves (B.9) and that $\{Y_j\}$ is an orthonormal set in $L^2(S^{N-1})$, the last expression reads

$$\begin{aligned}&=-\lambda_k\int_0^1\varphi(r)r^{N-3}\int_{S^{N-1}}\psi Y_k(\theta)d\sigma(\theta)dr+\int_0^1 a(r)\psi_k\varphi r^{N-1}dr+\widehat{\Lambda}\int_0^1\psi_k\varphi r^{N-3}dr\\ &=-\lambda_k\int_0^1\psi_k\varphi r^{N-3}dr+\int_0^1 a(r)\psi_k\varphi r^{N-1}dr+\widehat{\Lambda}\int_0^1\psi_k\varphi r^{N-3}dr\\ &=\omega_{N-1}^{-1}\left[\int_B a(x)\psi_k\varphi dx+(\widehat{\Lambda}-\lambda_k)\int_B\frac{\psi_k\varphi}{|x|^2}dx\right],\end{aligned}$$

and hence ψ_k is a radial eigenfunction of (B.1) corresponding to $\widehat{\Lambda}^{rad}=\widehat{\Lambda}-\lambda_k<0$.

Conversely, suppose that (B.11) holds and ψ^{rad} is a radial eigenfunction to (B.1) associated to $\widehat{\Lambda}^{rad}$. Set $\psi := \psi^{rad}(r)Y_k(\theta)$. Then

$$\int_B\frac{\psi^2}{|x|^2}dx=\int_0^1(\psi^{rad})^2r^{N-3}dr\int_{S^{N-1}}Y_k^2(\theta)d\sigma(\theta)<\infty$$

and

$$\begin{aligned}\int_B|\nabla\psi|^2dx&=\int_0^1\left((\psi^{rad})'\right)^2r^{N-1}dr\int_{S^{N-1}}Y_k^2(\theta)d\sigma(\theta)\\ &\quad+\int_0^1(\psi^{rad})^2r^{N-3}dr\int_{S^{N-1}}|\nabla_\theta Y_k|^2(\theta)d\sigma(\theta)<\infty,\end{aligned}$$

which shows $\psi \in \mathscr{H}_0$. Moreover given $\varphi \in \mathscr{H}_0$

$$\begin{aligned}
\int_B \nabla\psi\nabla\varphi dx &= \int_0^1 \int_{S^{N-1}} \left(\psi'(r,\theta)\varphi'(r,\theta) + \frac{1}{r^2}\nabla_\theta\psi(r,\theta)\nabla_\theta\varphi(r,\theta)\right) r^{N-1} d\sigma(\theta)dr \\
&= \int_{S^{N-1}} Y_k(\theta) \int_0^1 (\psi^{rad})'\varphi'(r,\theta) r^{N-1} dr d\sigma(\theta) \\
&\quad + \int_0^1 \psi^{rad} r^{N-3} \int_{S^{N-1}} \nabla_\theta Y_k(\theta)\nabla_\theta\varphi(r,\theta) d\sigma(\theta)dr \\
&= \int_{S^{N-1}} Y_k(\theta) \int_0^1 \left(a(r) + \frac{\widehat{\Lambda}^{rad}}{r^2}\right) \psi^{rad}\varphi(r,\theta) r^{N-1} dr d\sigma(\theta) \\
&\quad + \lambda_k \int_0^1 \frac{\psi^{rad}}{r^2} \int_{S^{N-1}} Y_k(\theta)\varphi(r,\theta) r^{N-1} d\sigma(\theta)dr \\
&= \int_0^1 \int_{S^{N-1}} \left[\left(a(r) + \frac{\widehat{\Lambda}^{rad}}{r^2}\right)\psi(r,\theta)\varphi(r,\theta) + \lambda_k \frac{\psi(r,\theta)}{r^2}\varphi(r,\theta)\right] r^{N-1} d\sigma(\theta)dr \\
&= \int_B \left[a(x) + \frac{\widehat{\Lambda}^{rad} + \lambda_k}{|x|^2}\right] \psi\varphi dx.
\end{aligned}$$

Therefore ψ is an eigenfunction of (B.1) associated to $\widehat{\Lambda}$. □

B.4 Generalized radial eigenvalues

Assuming that the function $a(x) \in L^\infty(B)$ is radial, let ψ be a radial eigenfunction for (B.1). Then ψ solves in its radial coordinate $r = |x|$ the following eigenvalue problem

$$\begin{cases} -(r^{N-1}\psi')' - r^{N-1}a(r)\psi = \widehat{\Lambda} r^{N-3}\psi, & r \in (0,1), \\ \psi \in \mathscr{H}_{0,rad}. \end{cases} \tag{B.12}$$

The goal here is to generalize the problem (B.12) by replacing the integer N for any real number $M \geq 1$. For this, consider the following eigenvalue problem

$$\begin{cases} -(t^{M-1}\varphi')' - t^{M-1}a(t)\varphi = \lambda t^{M-3}\varphi, & t \in (0,1), \\ \varphi \in H_{0,M}, \end{cases} \tag{B.13}$$

where for each $M \geq 1$, we define the space

$$H^1_{0,M} := \left\{ w : [0,1] \to \mathbb{R} \text{ measurable} : \begin{array}{l} w \text{ has first order weak derivative, } w(1) = 0 \text{ and} \\ \int_0^1 |w'(t)|^2 t^{M-1} dt + \int_0^1 w^2(t) t^{M-3} dt < \infty \end{array} \right\},$$

which is a Hilbert space with scalar product

$$(w,v) \mapsto \int_0^1 w'(t)v'(t)t^{M-1}dt + \int_0^1 w(t)v(t)t^{M-3}dt.$$

We say that λ is an eigenvalue for (B.13) if there exists $0 \neq \varphi \in H^1_{0,M}$ such that

$$\int_0^1 [\varphi'(t)\zeta'(t) - a(r)\varphi(t)\zeta(t)]t^{M-1}dt = \lambda \int_0^1 \varphi(t)\zeta(t)t^{M-3}dt, \quad \forall \zeta \in H^1_{0,M},$$

and, in this case, φ is its corresponding eigenfunction. Similarly to the previous sections, one can prove the following lemma.

Lemma B.11. The negative eigenvalues of (B.13) are characterized as

$$\lambda_i := \min_{\substack{W \subset H^1_{0,M} \\ \dim W = i}} \max_{\substack{w \in W \\ w \neq 0}} \frac{\int_0^1 [|w'(t)|^2 - a(t)w^2(t)]t^{M-1}dt}{\int_0^1 w^2(t)t^{M-3}dt}. \tag{B.14}$$

Moreover the corresponding eigenfunctions are $C^{1,\gamma}$-functions with higher regularity $C^{2,\gamma}(0,1)$, if $a(t) \in C^{0,\gamma}(0,1)$.

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