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Central extensions and Symplectic Geometry

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Extensões centrais e Geometria Simpléctica

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This work is dedicated to my mother Maria Lucia, to my sister Vióría, and to my girlfriend Isadora Martins.

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“One does not discover new lands without deciding
to lose sight of the shore for a very long time .”
(André Gide)

RESUMO

CARVALHO, P. H. **Extensões centrais e Geometria Simpléctica**. 2020. 83 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

Este trabalho explora um pouco das interações da Teoria de Lie com a geometria simplética. Mais precisamente, estudamos um critério para a integrabilidade de extensões centrais de álgebras de Lie, o qual utiliza conceitos presentes também em geometria simplética e tem aplicação no contexto de ações hamiltonianas. Além disso, discutimos como algumas das estruturas aparecendo nesse primeiro contexto se relacionam com a teoria de groupóides e algebróides de Lie.

Palavras-chave: Teoria de Lie, Extensões Centrais, Geometria Simplética.

ABSTRACT

CARVALHO, P. H. **Central extensions and Symplectic Geometry**. 2020. 83 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

This work explores aspects of Lie theory and its interactions with symplectic geometry. More precisely, we study a criterion to decide about the integrability of a given central extension of Lie algebras, which uses concepts from symplectic geometry in its formulation and finds an application in the context of hamiltonian actions. Futhermore, we discuss how some of the constructions appearing in the first part of this work may relate to the theory of Lie groupoids and Lie algebroids.

Keywords: Lie theory, Central Extensions, Symplectic Geometry.

CONTENTS

CHAPTER

1

INTRODUCTION

In this work, we study some aspects of symplectic geometry and its interactions with Lie theory.

Initially, following (TUYNMAN; W.W.A.J.WIEGERINCK, 1987), we explore under which conditions one can integrate a (one-dimensional) central extension of Lie algebras. More precisely, given a connected Lie group G with Lie algebra $\mathfrak{g}$ and a central extension of $\mathfrak{g}$ by $\mathbb{R}$, we present a criterion to the determine if one can find a central extension of G by a one-dimensional connected abelian Lie group of the form $\mathbb{R}/D$ (D a discrete subgroup of $\mathbb{R}$), which induces the given central extension of $\mathfrak{g}$ by $\mathbb{R}$. It happens that central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are classified by cohomology classes in $H^2(\mathfrak{g},\mathbb{R})$. Then, given a central extension, one can consider the cohomology class $[\omega]$ associated to it. The 2-cocycle ω can be used to define a closed left-invariant 2-form on G, which will be also denoted by ω. These considerations are the starting point to the solution of the already referred integrability problem as presented in (TUYNMAN; W.W.A.J.WIEGERINCK, 1987). Interestingly, the constructions used to solve this problem can be realized in the context of (pre)symplectic geometry. In fact, one of the integrability conditions that a central extension must satisfy is expressed in terms of a moment map for the left action of G on the pair (G,ω); also, the solution passes through the construction of a $\mathbb{R}/D-$principal bundle $p: Y \to G$ with a connection α satisfying $d\alpha = p^*\omega$, which can be constructed in the more general context of a manifold M endowed with a closed 2-form, and it is usually called prequantum bundle. In this context, such a principal bundle may be used to provide a one-dimensional central extension of the Lie algebra of hamiltonian vector fields on M. Moreover, when the 2-form ω is also non-degenerate, i.e, the pair (M,ω) is a symplectic manifold, the same bundle can be used to give a geometric realization of the well-known one-dimensional central extension $C^\infty(M)$ (considered as a Lie algebra by means of the Poisson bracket defined by the symplectic structure) of the Lie algebra of hamiltonian vector fields on M, and this is closely related to the prequantization (in the sense of geometric quantization) of the symplectic manifold (M,ω). Furthermore, the theorem concerning the integrability of central extensions (Theorem 2.4.3) finds an application in the context of hamiltonian actions (Theorem 2.5.8).

Curiously, one of the constructions appearing in the result about integrability of central extension of Lie algebras, also appears when one treats the integrability of a certain Lie algebroid. Indeed, let (M,ω) be a manifold endowed with a closed 2-form. Then, one can use ω to define a Lie algebroid structure on the vector bundle $TM \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the trivial line bundle over M. Let A_ω denote this Lie algebroid. In the case that there exists a principal bundle $p : Y \to M$ with a connection α such that $d\alpha = p^*\omega$, it happens that the Lie algebroid A_ω is integrable. Actually, the converse also holds.

This thesis is organized as follows. Chapter 2 is entirely dedicated to the presentation of what we have described in the second paragraph above. In Chapter 3, we discuss how parts of Chapter 2 relate to the theory of Lie algebroids and Lie groupoids. Finally, in Appendix A, we present the basics of symplectic geometry, which is related to what appears in (TUYNMAN; W.W.A.J.WIEGERINCK, 1987).

CHAPTER

2

ON A INTEGRABILITY PROBLEM OF CENTRAL EXTENSIONS

In this chapter we shall explore, following (TUYNMAN; W.W.A.J.WIEGERINCK, 1987), how one can associate a Lie group central extension to a given Lie algebra central extension. More precisely, given a connected Lie group G with Lie algebra $\mathfrak{g}$ and a central extension of $\mathfrak{g}$ by $\mathbb{R}$, we shall give a criterion to the determine if one can find a central extension of G by the one-dimensional connected abelian Lie group $\mathbb{R}/D$ (D a discrete subgroup of $\mathbb{R}$), which has associated to it the given central extension of $\mathfrak{g}$ by $\mathbb{R}$. As we shall see such central extension determines a cohomology class in $H^2_{\mathrm{al}}(\mathfrak{g},\mathbb{R})$, and the criterion for the integrability of the extension will be given as conditons on the cohomology class that it defines. The solution to this problem passes through the construction of a certain principal bundle with connection over the group G. At first, we carry out the construction of this bundle over a smooth manifold M endowed with a closed 2-form ω. We explore how this bundle provides a central extension of a certain class of vector fields on the base manifold.

2.1 Central Extensions

In this section we shall define what we mean by central extensions of Lie groups and Lie algebras. Also, we will see how these two algebraic constructions can be classified (up to equivalence) by certain cohomology groups and how a central extension of Lie groups determines a central extension of the associated Lie algebras.

We start by defining the notion of central extension for discrete groups.

Definition 2.1.1. Let G be a group, and let A be an abelian group. A group H is called a central extension of G by A if it fits in the short exact sequence of groups

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} G \longrightarrow 1$$

where $\mathrm{im}(i) \subset \mathrm{center}(H)$.

Given two central extensions of a group G by the same abelian group A, there is a notion of equivalence between these two extensions.

Definition 2.1.2. Let H_1 and H_2 be two central extensions of a group G by an abelian group A. We say that H_1 and H_2 are equivalent if there is a morphism of groups $\phi : H_1 \to H_2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, equal] & H_1 \arrow[r, "\pi_1"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, "i_2"] & H_2 \arrow[r, "\pi_2"] & G \arrow[r] & 1
\end{tikzcd}$$

Let us denote by $\mathrm{Ext}(G,A)$ the set of all possible central extensions of G by a fixed abelian group A. It is easy to see that the above notion of equivalence defines an equivalence relation in $\mathrm{Ext}(G,A)$. It happens that the equivalence classes of such a relation are classified by the group cohomology, which we introduce in what follows.

Consider, for each integer $k \geq 1$, $C^k(G,A) := \{\varphi : G^k \to A\}$, which is simply the set of all functions from G^k to A; these are called k-cochains on G with values in the abelian group A. It is clear that under pointwise addition $C^k(G,A)$ becomes an abelian group. Now, we define a map $\delta_k : C^k(G,A) \to C^{k+1}(G,A)$ by putting

$$(\delta_k\varphi)(g_1,\ldots,g_{k+1}) := \varphi(g_2,\ldots,g_{k+1}) + \left(\sum_{j=1}^{k}(-1)^j\varphi(g_1,\ldots,g_{j-1},g_j\cdot g_{j+1},g_{j+2},\ldots,g_{k+1})\right)$$

$$+(-1)^{k+1}\varphi(g_1,\ldots,g_k).$$

It can be shown that $\delta_{k+1}\circ\delta_k = 0$.

Definition 2.1.3. The k-th cohomology group of G with values in A is defined to be

$$H^k_{\mathrm{gr}}(G,A) := \frac{\ker(\delta_k)}{\mathrm{im}(\delta_{k-1})}.$$

Proposition 2.1.4. The equivalence classes of central extensions of a group G by an abelian group A are classified by $H^2_{\mathrm{gr}}(G,A)$.

Proof. Here, what we are willing to do is to stablish a correspondence between $\mathrm{Ext}(G,A)$ and $H^2_{\mathrm{gr}}(G,A)$ in such a way that equivalent extensions are associated to cohomologous cocycles. In other words, each equivalence class of extensions is mapped to a cohomology class. Consider a central extension

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} G \longrightarrow 1$$

Since π is a surjective morphism, there exists a function $s : G \to H$ such that $\pi \circ s = id_H$. For each $g,h \in G$ we have

$$\pi(s(gh)) = gh = \pi(s(g))\pi(s(h)) \Leftrightarrow \pi(s(gh)s(g)^{-1}s(h)^{-1}) = 1.$$

Then, $s(gh)s(g)^{-1}s(h)^{-1} \in \ker(\pi) = \mathrm{im}(i)$. It follows that we can define $\varphi : G \times G \to A$, where $\varphi(g,h)$ is such that

$$s(gh) = s(g)s(h)\varphi(g,h).$$

Notice that φ so defined is a 2-cocycle. To verify this we must show that, for every $g,h,l \in G$, we have

$$0 = (\delta_2\varphi)(g,h,l) = \varphi(h,l) - \varphi(gh,l) + \varphi(g,hl) - \varphi(g,h).$$

By writting

$$s(ghl) = s(gh)s(l)\varphi(gh,l)$$
$$s(ghl) = s(g)s(hl)\varphi(g,hl)$$
$$s(gh) = s(g)s(h)\varphi(g,h)$$
$$s(hl) = s(h)s(l)\varphi(h,l)$$

one can see that $\varphi(h,l)\varphi(g,hl)\varphi(gh,l)^{-1}\varphi(g,h)^{-1} = 1 \in H$, which implies that $(\delta_2\varphi)(g,h,l) = 0 \in A$; then, φ is a 2-cocycle. The cocycle φ has an explicity dependence on the chosen section s. For the association that we are seeking for to be well-defined, we should verify that by choosing another section we end up with a cocycle that is cohomologous to φ. Let $t : G \to H$ a section other than s. For any $g \in G$, we have $\pi(s(g)) = \pi(t(g))$, which implies that $s(g)t(g)^{-1} \in \ker(\pi) = \mathrm{im}(i)$. Then, we can define $\chi : G \to A$ putting $\chi(g) := s(g)t(g)^{-1}$. Let φ' be the 2-cocycle associated to the section t. Carrying out a simple calculation one can conclude that $\varphi(g,h) - \varphi'(g,h) = (\delta_2\chi)(g,h)$, which shows that the cohomology class of φ remains unchanged if we choose another section of $\pi : H \to G$. Thereby, we have shown that there is a well-defined association between Ext(G, A) and $H^2_{\mathrm{gr}}(G,A)$.

Now, we are going to show this is a 1-1 correspondence.

1. Any cohomology class in $H^2_{\mathrm{gr}}(G,A)$ determines an unique equivalence class of central extensions of G by A.

   Indeed, let $\varphi : G \times G \to$ be a 2-cocyle representing a cohomology class in $H^2_{\mathrm{gr}}(G,A)$. Then, we consider $G' := G \times A$ with operation given by $(g,a)(h,b) = (gh, a+b-\varphi(g,h))$. This group will be denoted by $G \times_\varphi A$. From the fact that φ is a cocyle we get the associatity of the operation we have just defined, and also that $\varphi(e,g) = \varphi(e,e) = \varphi(g,e)$, which implies that $(e,\varphi(e,e))$ is the identity of $G \times_\varphi A$. Defining $\pi : G \times_\varphi A \to G$ as $\pi(g,a) = g$ and $i : A \to G \times_\varphi A$ by $i(a) = (e, a+\varphi(e,e))$, we obtain the following short exact sequence

$$0 \longrightarrow A \xrightarrow{i} G \times_{\varphi} A \xrightarrow{\pi} G \longrightarrow 1$$

where $\mathrm{im}(i) \subset \mathrm{center}(G \times_{\varphi} A)$.

If we take another representative of the cohomology class of ϕ, say, φ', then following the steps above we can define a central extension $G \times_{\varphi'} A$. Also, we know that $\varphi' = \varphi + \delta_2 \chi$, for some $\chi \in C^1(G,A)$, and using this we can define $\Phi : G \times_{\varphi} A \to G \times_{\varphi'} A$, which turns out to be an equivalence of extensions.

2. An equivalence class of extensions determines an unique cohomology class in $H^2_{\mathrm{gr}}(G,A)$.

   Notice that, due to the last item, it is enough to prove that each equivalence class of extensions can be represented by an extension of the form $G \times_{\varphi} A$.

   In fact, let (G', i, π) be a central extension representing a certain equivalence class in $\mathrm{Ext}(G,A)$. Remember that by taking a section $s : G \to G'$ of $\pi : G' \to G$ we get a 2-cocycle φ. Using $s : G \to G'$ one can define the function $\Phi : G \times A \to G'$ by putting $\Phi(g,a) = i(a)s(g)$. It turns out that Φ is a bijection. Also, if we consider $\Phi : G \times_{\varphi} A \to G'$ as function from the group $G \times_{\varphi} A$ we see that it is a group morphism. Finally, it can be shown that Φ is actually an equivalence between the two extensions $G \times_{\varphi} A$ and G'.

$\square$

Definition 2.1.5. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{a}$ be an abelian Lie algebra. A Lie algebra $\mathfrak{h}$ is said to be a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ if it fits in the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{a} \xrightarrow{Ti} \mathfrak{h} \xrightarrow{T\pi} \mathfrak{g} \longrightarrow 0$$

where $\mathrm{im}(Ti) \subset \mathrm{center}(\mathfrak{h})$.

Likewise, we have a notion of equivalence between central extensions of Lie algebras.

Definition 2.1.6. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two central extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$. We say that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are equivalent if there is a morphism of Lie algebras $T\phi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "Ti_1"] \arrow[d, equal] & \mathfrak{h}_1 \arrow[r, "T\pi_1"] \arrow[d, "T\phi"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "Ti_2"] & \mathfrak{h}_2 \arrow[r, "T\pi_2"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

We can introduce a cohomology whose cohomology classes are associated to equivalent central extensions of a given Lie algebra. For each integer k ≥ 0, let us put $C^k(\mathfrak{g}, \mathfrak{a}) := \{\omega : \mathfrak{g}^k \to \mathfrak{a} \mid \omega \text{ is multilinear and antisymmetric}\}$ and define $\delta_k : C^k(\mathfrak{g}, \mathfrak{a}) \to C^{k+1}(\mathfrak{g}, \mathfrak{a})$ by

$$(\delta_k\omega)(X_1,\cdots,X_{k+1}) := \sum_{1\leq i\leq j\leq k+1}(-1)^{i+j}\omega([X_i,X_j],\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}).$$

These δ_k's satisfy $\delta_{k+1}\circ\delta_k = 0$. Hence, we can make the following definition.

Definition 2.1.7. The k-th cohomology group of $\mathfrak{g}$ with values in $\mathfrak{a}$ is defined to be

$$H^k_{\text{al}}(\mathfrak{g},\mathfrak{a}) := \frac{\ker(\delta_k)}{\operatorname{im}(\delta_{k-1})}.$$

Analogous to Proposition 2.1.4 we have:

Proposition 2.1.8. The equivalence classes of central extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ are classified by $H^2_{\text{al}}(\mathfrak{g},\mathfrak{a})$.

Proof. Firstly, let us see how a central extension determines a 2-cocycle. Consider the central extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{Ti} \mathfrak{h} \xrightarrow{T\pi} \mathfrak{g} \longrightarrow 0.$$

In particular, it is a short exact sequence of finite dimensional vector spaces, consequently, it splits. Hence, there exists a linear map $\sigma:\mathfrak{g}\to h$ such that $(T\pi)\circ\sigma = id_{\mathfrak{g}}$. We define $\omega:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$ by measuring how much σ deviates from being a Lie morphism, i.e, we put

$$\omega(X,Y) := \sigma([X,Y]) - [\sigma(X),\sigma(Y)].$$

Notice that, actually, $\omega(X,Y)\in\mathfrak{h}$. However, $(T\pi)(\omega(X,Y)) = 0$, since $T\pi$ is a Lie morphism and $(T\pi)\circ\sigma = id_{\mathfrak{g}}$; it follows that $\omega(X,Y)\in\ker(T\pi) = \operatorname{im}(Ti)$, so that we can identify $\omega(X,Y)$ with an unique element of the Lie algebra $\mathfrak{a}$. It is simple to see that ω so defined is multilinear and antisymmetric. Moreover, it follows from Jacobi identity for the Lie algebra $\mathfrak{h}$ and from $\operatorname{im}(\mathrm{T}i)\subset\operatorname{center}(\mathfrak{h})$ that ω is a 2-cocycle. It is easy to see that if we take a section other than σ we end up changing ω by a coboundary. Therefore, we have shown that each central extension determines a cohomology class in $H^2_{\text{al}}(\mathfrak{g},\mathfrak{a})$.

On the other hand, if we are given a cohomology class $[\omega]\in H^2_{\text{al}}(\mathfrak{g},\mathfrak{a})$, we can define a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ as a Lie algebra $\mathfrak{h}$ which has $\mathfrak{g}\oplus\mathfrak{a}$ as underlying vector space and Lie bracket defined by $[(X,v),(Y,w)] = ([X,Y],\omega(X,Y))$. Following a reasoning analogous to the one in Proposition 2.1.4, one can show that different representatives of the cohomology class $[\omega]$ lead to equivalent central extensions. $\square$

Here, we are going to specialize to the case in which $\mathfrak{a}=\mathbb{R}$. Then, looking better to how we have defined δ_k, one can see that it reminds of part of the formula used to define the exterior derivative of a differential k-form on a manifold. It turns out that if we consider a Lie group G with Lie algebra $\mathfrak{g}$, the cohomology groups of Defitinion 2.1.7 are exactly the de Rham cohomology restricted to the left invariant forms on G.

This observation together with the last proposition is the starting point to the solution of our integrability problem (Theorem 2.4.3). In fact, given a central extension of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$, we take G as a Lie group with Lie algebra $\mathfrak{g}$ and consider the $2-$cocycle associated to the given extension as a left invariant 2-form on G. The integrability of the algebra extension will be subjected to conditions that this 2-form must satisfy. Once those conditions are verified, we can construct a certain principal fiber bundle with connection over G, which gives rise to a central extension integrating the central extension of Lie algebra that we started with. In the next section we carry out the construction of this principal bundle by considering a smooth manifold endowed with a closed 2-form.

In the remaning part of this section we define what we mean by a central extension of Lie groups and we point out how a Lie group extension determines a Lie algebra extension.

Definition 2.1.9. Let G be a connected Lie group, and let A be a connected abelian Lie group. A Lie group H is said to be a central extension of G by A if it fits in the following short exact sequence of Lie groups

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} G \longrightarrow 1$$

for which we have $\mathrm{im}(i) \subset \mathrm{center}(H)$ and π admits a smooth local section $s : U \to G$, $\pi \circ s = id_U$, where U is an open neighborhood of the identity e of G.

The existence of a smooth local section defined in a neighborhood of the identity of G implies that H can be seen as a principal fiber bundle over G with structure group A. The action of A on H is defined through the map i. Actually, to ask the existence of the smooth local section $s : U \to G$ is superflous: in fact, it is easy to see that $\pi : H \to G$ appearing in the exact sequence of Definition 2.1.9 is a surjective map of constant rank, hence, by the global rank theorem (Theorem 4.14 in (LEE, 2012)), it is a surjective submersion. Moreover, it is a matter of fact that surjective submersions always admit smooth local sections (Theorem 4.26 in (LEE, 2012)).

The existence of a smooth local section in a neighborhood of the identiy allows us to classify inequivalent Lie group central extensions by a certain cohomology group just as in the case of discrete groups. However, we need to choose carefully what we take as cochains. It is expected that now the k-cochains cannot be simply functions from G^k to A, since we have some differential data involved. At first sight, one may ask these to be smooth. However, as we shall see, the cohomology defined by means of cochains of this kind classifies only those central extensions that are topologically trivial (Proposition 2.5.3). One way out is to take the cochains to be smooth in a neighborhood of the identity of G^k. Then, we take $\varphi : G^k \to A$ to be a locally smooth function (smooth around e^k) and take δ_k defined by the same formula as in the case of discrete groups. In these settings, we define $H^k_{\mathrm{ls}}(G,A)$. Thereby, we can state the following:

Proposition 2.1.10. The inequivalent central extensions of a Lie group G by an abelian Lie group A are classified by $H^2_{\mathrm{ls}}(G,A)$.

For the proof we refer to (TUYNMAN; W.W.A.J.WIEGERINCK, 1987) Proposition 3.11 or (MICHOR, 2008), Ch. 3, Section 15.

The next proposition shows that every Lie group central extension determines a Lie algebra central extension.

Proposition 2.1.11. Every Lie group central extension determines a central extension on the level of the associated Lie algebras.

Proof. In fact, let us consider a Lie group central extension

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} G \longrightarrow 1$$

It is easy to see that $i : A \to H$ and $\pi : H \to G$ are maps of constant rank. Then, since i is injective and π is surjective, the global rank theorem implies that i is a immersion and π is a submersion. Thus, by taking the derivative at the identity of each map in the sequence above, we obtain the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{a} \xrightarrow{T_e i} \mathfrak{h} \xrightarrow{T_e \pi} \mathfrak{g} \longrightarrow 0$$

To show that it is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ we must verify that $\mathrm{im}(T_e i) \subset \mathrm{center}(\mathfrak{h})$. Let $X \in \mathfrak{a}$, and consider $(T_e i)X$ and $Y \in \mathfrak{h}$. Since (H,i,π) is a central extension, the flow of $(T_e i)X$ lies in the center of H, which implies that the flow of Y and the flow of $(T_e i)X$ commute. Therefore, $[(T_e i)X, Y] = 0$. □

2.2 Extensions of (M,ω)

In this section, considering a smooth manifold M endowed with a closed 2-form ω, we want to construct a $\mathbb{R}/D-$principal bundle $p : Y \to M$ with a connection α satisfying $d\alpha = p^*\omega$, to be called an extension of the pair (M,ω). We introduce a notion of equivalence between extensions constructed from distinct initial data, and we classify them by a certain Čech cohomology group.

2.2.1 Group of Periods

Here we introduce one of the concepts that will play a role in the construction of the bundle we are looking for and will also appear as one of the conditions in the integrability criterion that we shall present.

Let M be a smooth manifold. We fix over M a good open cover $\mathscr{U} = \{U_i\}_{i\in I}$, which is an open cover such that every non-empty finite intersection of its elements is contractible. For the existence of such a cover we refer to (RAMANAN, 2005).

To this open cover let us put $\text{Nerve}(\mathscr{U}) = \{(i_0,\cdots,i_k) \in I^{k+1} \mid U_{i_0}\cap\cdots\cap U_{i_k} \neq \emptyset\ \}$. An element $(i_0,\cdots,i_k)$ is called a $k-$chain, and the free $\mathbb{Z}-$module generated by the set of all $k-$chains is denoted by $C_k(\mathscr{U})$. Futhermore, let $\partial : C_k(\mathscr{U}) \to C_{k-1}(\mathscr{U})$ be defined on the basis by

$$\partial(i_0,\cdots,i_k) := \sum_{j=0}^{k}(-1)^j(i_0,\cdots,i_{j-1},i_{j+1},\cdots,i_k).$$

Let ω be a closed 2-form on M. By de Rham theorem, the de Rham cohomology and the singular cohomology are isomorphic (RAMANAN, 2005). Also, singular cohomology of any contractible space is trivial (RAMANAN, 2005). Hence, for the closed 2-form ω one can write

$$\begin{cases} d\theta_i = \omega & \text{on } U_i \\ \theta_i - \theta_j = df_{ij} & \text{on } U_{ij} \\ f_{jk} - f_{ik} + f_{ij} = a_{ijk} & \text{on } U_{ijk} \end{cases}$$

where θ_i are 1-forms, f_{ij} are functions, and a_{ijk} are constants.

Notice that somehow the a_{ijk} are dual to the chains (ijk). Accordingly, $a = (a_{ijk})$ will be referred to as a cochain.

Definition 2.2.1. For the closed 2-form ω on M we define the group of periods of ω as

$$\text{Per}(\omega) := \text{im}\{a : \ker(\partial_2) \to \mathbb{R}\},$$

where a denotes the cochain obtained by solving ω on the open cover $\mathscr{U}$.

As can be seen there are several choices involved in the definition of $\text{Per}(\omega)$. For, we have chosen the good open cover $\mathscr{U}$, the 1-forms θ_i, the functions f_{ij}, and the constants a_{ijk}. It is easy to see that once the good open cover is fixed $\text{Per}(\omega)$ does not depend on the choices of θ_i, f_{ij}, and a_{ijk}. Moreover, $\text{Per}(\omega)$ depends only on the cohomology class of ω. It remains to clarify if $\text{Per}(\omega)$ depends on the fixed cover. This can be done by introducing another definition for $\text{Per}(\omega)$ and proving that it is coincides with the one that we have provided.

Definition 2.2.2. For the closed 2-form ω on M we define the group of periods of ω as

$$\text{Per}(\omega) := \left\{ \int_\sigma \omega \in \mathbb{R} \mid \sigma \text{ 2-cycle in } M \right\}.$$

The equivalence between the two definitions is proved in (WEIL, 1952). Roughly, the idea is to show that once the good open cover is fixed there is a relation between the duality between 2-forms and 2-cycles and the duality between 2-cochains and 2-chains.

We have opted for Definition 2.2.1 because it is suitable for the construction of the $\mathbb{R}/D-$principal bundle we are looking for, and this is due to the following result.

Proposition 2.2.3. Let D be a subgroup of $\mathbb{R}$ containing $\mathrm{Per}(\omega)$. Then, the constants a_{ijk} can be chosen to be in D.

Proof. We start by defining $b:\mathrm{im}(\partial_2)\to\mathbb{R}/D$ as $b:=\pi\circ a\circ\partial_2^{-1}$, where $\pi:\mathbb{R}\to\mathbb{R}/D$ is the natural projection and ∂_2^{-1} means that we choose any element in preimage of a given element in $\mathrm{im}(\partial_2)$. Hence, for b to be well-defined, we need to verify that b does not depend on this choice. For, let (ijk) and (mno) be such that $\partial_2[(ijk)-(mno)]=0$, i.e, $(ijk)-(mno)\in\ker(\partial_2)$. Then, it follows that $a[(ijk)-(mno)]\in\mathrm{Per}(\omega)\subset D$ which implies that b is well-defined.

Now, since $\mathbb{R}/D$ is a divisible $\mathbb{Z}-$module [1], $b:\mathrm{im}(\partial_2)\to\mathbb{R}/D$ can be extended to the whole $C_1(\mathscr{U})$, and this extension will be also denoted by b. Then one can define a map from the basis of $C_1(\mathscr{U})$ to $\mathbb{R}$ simply by taking the preimage by π of any chain (ijk). Since $C_1(\mathscr{U})$ is a free $\mathbb{Z}-$module, we get a morphism $b':C_1(\mathscr{U})\to\mathbb{R}$ such that $\pi\circ b'=b$. Thus, when solving ω on $\mathscr{U}$, we may change f_{ij} by b'_{ij}, i.e, we take $g_{ij}=f_{ij}-b'_{ij}$. As is easily seen, this will change a_{ijk} by $(\delta b')_{ijk}=b'_{jk}-b'_{ik}+b'_{ij}$. It follows from the definition of b' that $\pi[(a-\delta b')_{ijk}]=0$, which means $(a-\delta b')_{ijk}\in D$. □

2.2.2 Construction of the $\mathbb{R}/D-$bundle

Let D be a discrete subgroup of $\mathbb{R}$. Then, $\mathbb{R}/D$ is a Lie group with a unique smooth structure such that the canonical projection $\pi:\mathbb{R}\to\mathbb{R}/D$ is a smooth covering map ((LEE, 2012), Theorem 21.29).

It will be usefull, for what follows, to recall some facts about connections on principal bundles, then we collect these in the following remark.

Remark 2.2.4. Let $p:P\to M$ be a G-principal fiber bundle over a manifold M, and let Ω be a connection 1-form on P. Suppose that $\mathscr{V}=\{V_j\}_{j\in J}$ is an open cover of M trivializing the principal bundle P, i.e, such that for each $j\in J$ there exists a diffeomorphism $\psi_j:p^{-1}(V_j)\to V_j\times G$. It is known that Ω defines 1-forms ω_j on each $V_j\subset M$ satisfying

$$\omega_j=\mathrm{Ad}(\psi_{ij}^{-1})\omega_i+\psi_{ij}^{*}(\omega_{M.C}),\qquad(\dagger)$$

[1] For modules over principal ideal domains divisible modules are the same as injective modules. And injective $\mathbb{Z}-$modules, for example, are defined as follows. Let be M a module over $\mathbb{Z}$. Then, M is said to be injective if given any injective morphism $f:N\to P$ and an arbitrary morphism $g:N\to M$ there exists a morphism $h:P\to M$ such that $g=h\circ f$.

where $\omega_{M.C}$ is the Maurer-Cartan 1-form on G. If instead of the connection 1-form Ω on P we are given 1-forms ω_j on M subjected to the above relation, then we can define a connection 1-form on P as follows. On each $U_j := p^{-1}(V_j)$, we define

$$\Omega|_{U_j}(m) := \mathrm{Ad}(g)p^*(\omega_j) + \psi_j^*(pr_2^*(\omega_{M.C})),$$

where $pr_2 : V_j \times G \to G$ is just the projection on the second factor and $g = pr_2(\psi_j(m))$. One can prove that each $\Omega|_{U_j}$ satisfies the properties of a connection 1-form on U_j. Moreover, it follows from $(\dagger)$ that $\Omega|_{U_i}$ and $\Omega|_{U_j}$ agree on intersections, which implies that Ω is well-defined connection 1-form on P.

Remark 2.2.5. In the next proposition we will need to use explicitly the Maurer-Cartan 1-form in $\mathbb{R}/D$, then we describe it here. The Maurer-Cartan 1-form on the additive Lie group $\mathbb{R}$ is simply dx. Using the unique manifold structure on $\mathbb{R}/D$ that turns $\pi : \mathbb{R} \to \mathbb{R}/D$ into a smooth covering map one can see that the Maurer-Cartan 1-form in $\mathbb{R}/D$ is locally given by dx, where x means the induced coordinate on $\mathbb{R}/D$.

Proposition 2.2.6. Let M be a smooth manifold endowed with a closed 2-form ω, and let $D \subset \mathbb{R}$ be a discrete subgroup such that $\mathrm{Per}(\omega) \subset D$. Then, there exists a smooth $\mathbb{R}/D-$principal bundle $p : Y \to M$ with a connection α satisfyng $d\alpha = p^*(\omega)$.

Proof. Let $\mathscr{U}$ be a good open cover for M. Then, we can solve ω on $\mathscr{U}$ to obtain

$$\begin{cases} d\theta_i = \omega & \text{on } U_i \\ \theta_i - \theta_j = df_{ij} & \text{on } U_{ij} \\ f_{jk} - f_{ik} + f_{ij} = a_{ijk} & \text{on } U_{ijk} \end{cases}$$

Due to Proposition 2.2.3, one can assume that f_{ij} were defined in such a way that $a_{ijk} \in D$. Then, we define $g_{ij} : U_{ij} \to \mathbb{R}/D$ by $g_{ij} := \pi(f_{ij})$. Notice that g_{ij} satisfy the cocycle condition, for

$$g_{ij} - g_{ik} + g_{jk} = \pi(f_{ij} - f_{ik} + f_{jk}) = \pi(a_{ijk}) = 0 \in \mathbb{R}/D.$$

Thereby, the g_{ij} can be taken to be the transition functions of the $\mathbb{R}/D-$principal bundle $p : Y \to M$ defined by

$$Y = \Big(\bigsqcup_{i \in J}(U_i \times \mathbb{R}/D)\Big)/\sim\ ,$$

where for $(m,x) \in U_i \times \mathbb{R}/D$ and $(n,y) \in U_j \times \mathbb{R}/D$ we put $(m,x) \sim (n,y) \Leftrightarrow m = n$, $g_{ij}(m) + x = y$, and $p[(m,x)] := m$.

Now, we would like to define a connection 1-form α on $p : Y \to M$ satisfying $d\alpha = p^*\omega$. We already have the 1-forms $\{\theta_j\}$ on M and, due to Remark 2.2.4, they can be used to define a connection α on Y if they satisfy $(\dagger)$. Firstly, notice that in the case of $\mathbb{R}/D$ the Ad application is trivial, so that $(\dagger)$ reduces to

$$\theta_j = \theta_i + g_{ji}^*(dx).$$

Let us prove that this equally holds. Developing the right-hand side we get

$$\theta_i + g_{ji}^*(dx) = \theta_i + (\pi \circ f_{ji})^*(dx) = \theta_i + f_{ji}^*(\pi^*(dx)) = \theta_i + f_{ji}^*((dx))$$

$$\theta_i + (df_{ji}) = \theta_i + \theta_j - \theta_i = \theta_j,$$

which is exactly $\theta_j = \theta_i + g_{ji}^*(dx)$. Hence, using θ_i one can define α on $U_i \times \mathbb{R}/D$ by

$$\alpha = p^*(\theta_i) + dx.$$

By Remark 2.2.4, α is a global connection on Y. Also, it is easy to see that $d\alpha = p^*\omega$. □

Actually, the condition $\mathrm{Per}(\omega) \subset D$ is also necessary for the existence of (Y, p, α).

Remark 2.2.7. In the next proposition, given a $\mathbb{R}/D-$principal bundle Y, we use that we have trivializations defined over the open sets of the previously fixed good open cover $\mathscr{U}$. This is true because of the following. Suppose $p : E \to X$ is a principal bundle, and let $f, g : Y \to X$ be continuous homotopic functions. Then, f^*E and g^*E are isomorphic (homeomorphic) principal bundles ((STEENROD, 1951), Theorem 11.4). Moreover, if X is a smooth manifold which is contractible as a topological space, then it is smoothly contractible, i.e, there exists a smooth homotopy between the identity map and a constant map ((LEE, 2012), Theorem 6.29). Then, if $p : E \to X$ is a principal bundle over a contractible base one can conclude that E is isomorphic (diffeomorphic) to a principal bundle over a point, which is certainly trivial.

Proposition 2.2.8. Let M be a smooth manifold, and let ω be a closed 2-form. If there exists a $\mathbb{R}/D-$principal bundle $p : Y \to M$ with a connection α satisfying $d\alpha = p^*\omega$, then $\mathrm{Per}(\omega) \subset D$.

Proof. Suppose $p : Y \to M$ is a $\mathbb{R}/D-$principal bundle with a connection α satisfying $d\alpha = p^*\omega$. Recall that we have fixed good cover $\mathscr{U} = \{U_j\}_{j\in J}$ for M. By Remark 2.2.7, we have transition functions $g_{ij} : U_{ij} \to \mathbb{R}/D$. Since U_{ij} is contractible, it is simply connected. Then, since $\pi : \mathbb{R} \to \mathbb{R}/D$ is a smooth covering map, we can lift g_{ij} to $f_{ij} : U_{ij} \to \mathbb{R}$, i.e, we can find f_{ij} satisfying $\pi \circ f_{ij} = g_{ij}$. From the fact that g_{ij} satisfies the cocycle condition, it follows that $f_{jk} - f_{ik} + f_{ij} \in D \subset \mathbb{R}$. Let us define $a_{ijk} := f_{jk} - f_{ik} + f_{ij}$.

As we know the connection α defines 1-forms $\{\theta_j\}$ on the base M, which satisfy $\theta_i = \theta_j + g_{ij}^*(dx)$. From this we deduce

$$\theta_i - \theta_j = g_{ij}^*(dx) = (\pi \circ f_{ij})^*(dx) = df_{ij}.$$

Moreover, locally the connection α can be written as $\alpha = p^*\theta_j + \psi_j^*(dx)$, then the condition $d\alpha = p^*\omega$ implies $d\theta_j = \omega|_{U_j}$. Hence, we have found

$$\begin{cases} d\theta_i = \omega & \text{on } U_i \\ \theta_i - \theta_j = df_{ij} & \text{on } U_{ij} \\ f_{jk} - f_{ik} + f_{ij} = a_{ijk} & \text{on } U_{ijk} \end{cases}$$

with $a_{ijk} \in D$, which implies that $\mathrm{Per}(\omega) \subset D$. $\square$

This proposition tells us that any $\mathbb{R}/D-$principal bundle with connection (Y, p, α) can be constructed following Proposition 2.2.6. From now on the triple (Y, p, α), with $d\alpha = p^*\omega$, will be referred to as an extension of the pair (M, ω); also, the connection α will be said to be compatible with ω.

Remark 2.2.9. It is worth to mention that in the context of symplectic geometry the principal bundle $p : Y \to M$ with a connection α such that $d\alpha = p^*\omega$ is also called prequantum bundle. This due to its relation to the prequantization of the symplectic manifold (M, ω) (see page 54).

In general, two principal bundles with connection (Y, α) and (Y', α') are said to be equivalent if there exists a bundle isomorphism $\varphi : Y \to Y'$ such that $\varphi^*\alpha' = \alpha$. The explicit description of the $\mathbb{R}/D-$principal bundles with connection (Y, p, α) satisfying $d\alpha = p^*\omega$ that we have presented allows us to classify isomorphism classes of these bundles via Čech cohomology group $\check{H}^1(\mathscr{U}, \mathbb{R}/D)$, see Theorem 2.11. However, to classify the central extensions that we are going to construct in Theorem 2.4.3, we shall need a different notion of equivalence between extensions, which we formulate in the next definition.

Definition 2.2.10. Let Θ be a subgroup of the additive group of 1-forms on a manifold M. Also, let ω and ω' be closed 2-forms on M. Two extensions (Y, p, α) and (Y', p', α') of (M, ω) and (M, ω'), respectively, are said to be $\Theta-$equivalent if there exists a bundle isomorphism $\varphi : Y \to Y'$ such that $\varphi^*\alpha' = \alpha + p^*\theta$, where $\theta \in \Theta$.

Notice that the condition $\varphi^*\alpha' = \alpha + p^*\theta$ together with the fact that (Y, p, α) and (Y', p', α') are extensions of (M, ω) and (M, ω'), respectively, implies that $\omega' = \omega + d\theta$. Therefore, we notice that Definition 2.2.10 puts as equivalent extensions constructed from different initial data, i.e, (M, ω) and (M, ω'), where ω' is simply a change in the representative of the cohomology class $[\omega] \in H^2_{dR}(M)$ by a coboundary coming from Θ.

Remark 2.2.11. One can define an explicit isomorphism between $H^1_{dR}(M)$ and $\check{H}^1(\mathscr{U}, \mathbb{R})$. For, given $[\gamma] \in H^1_{dR}(M)$, since each U_j is a contractible set, there exists a function Γ_j on U_j such that $d\Gamma_j = \gamma$. Moreover, on U_{ij} we have $d(\Gamma_i - \Gamma_j) = 0$, which implies that $(\Gamma_i - \Gamma_j)$ is constant on U_{ij}. Then, we can consider the cochain $\Gamma := \{(\Gamma_i - \Gamma_j)\} \in \check{C}^1(\mathscr{U}, \mathbb{R})$. However, it is easy to see that $(\check{\delta}\Gamma)_{ijk}=0$, so that Γ defines a cohomology class in $\check{H}^1(\mathscr{U},\mathbb{R})$. Hence, we have defined a map

$$H^1_{dR}(M)\longrightarrow \check{H}^1(\mathscr{U},\mathbb{R})$$

$$[\gamma]\longmapsto[\{\Gamma_i-\Gamma_j\}]$$

It is clear that if we change the representative for the cohomology class $[\gamma]$ we end up changing $[\Gamma]$ by a coboundary in $\check{C}^1(\mathscr{U},\mathbb{R})$; then, the above map is well-defined. Injectivity follows from the observation we have just made, while surjectivity can be proved by means of a partition of unity subordinate to $\mathscr{U}$.

Notice that the natural projection $\pi:\mathbb{R}\to\mathbb{R}/D$ induces a cochain map $\check{\pi}:\check{C}^\bullet(\mathscr{U},\mathbb{R})\to\check{C}^\bullet(\mathscr{U},\mathbb{R}/D)$, which gives

$$\check{\pi}:\check{H}^1(\mathscr{U},\mathbb{R})\to\check{H}^1(\mathscr{U},\mathbb{R}/D)$$

$$[\{c_{ij}\}]\longmapsto[\{\pi c_{ij}\}].$$

Let Θ_o denote the set of closed 1-forms in Θ. Then, $[\Theta_o]$ will denote the image of Θ_o by the isomorphism of Remark 2.2.11 and $[\Theta_o]_D$ will denote the image of $[\Theta_o]$ by $\check{\pi}$. Given all this, we can enunciate the classification of $\Theta-$equivalent extensions.

Theorem 2.2.12. The $\Theta-$inequivalent extensions of $\{(M,\omega+d\theta)\mid\theta\in\Theta\}$ are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D)\text{mod}[\Theta_o]_D$.

Proof. Let us consider the extension (Y,p,α) constructed from the initial data (M,ω). The idea is to classify all extensions relatively to (Y,p,α). More precisely, if (Y',p',α') is an extension constructed from (M,ω') we proceed as follows.

By Proposition 2.2.8, for the extension (Y,p,α) we can consider θ_i, f_{ij}, and $a_{ijk}\in D$, satisfying

$$\begin{cases} d\theta_i=\omega & \text{on } U_i\\ \theta_i-\theta_j=df_{ij} & \text{on } U_{ij}\\ f_{jk}-f_{ik}+f_{ij}=a_{ijk} & \text{on } U_{ijk}\end{cases}$$

Analogously, for (Y',p',α') we have θ'_i, f'_{ij}, and $a'_{ijk}\in D$. Thus, on U_i we can write

$$d\theta'=\omega'=\omega+d\theta=d\theta_i+d\theta,$$

which gives $d(\theta_i-\theta'_i+\theta)=0$; since U_i is contractible, there exists F_i on U_i satisfying

$$\theta'_i=\theta_i+\theta-dF_i. \tag{2.1}$$

Now, on U_{ij} we have that

$$df_{ij} - df'_{ij} = (\theta_i - \theta_j) - (\theta'_i - \theta'_j).$$

Using (2.1) we conclude that

$$df_{ij} - df'_{ij} = dF_i - dF_j,$$

which implies that $d(f_{ij} - f'_{ij} - F_i + F_j) = 0$ in U_{ij}. From the contractibility of U_{ij}, we get constants c_{ij} such that

$$f_{ij} = f'_{ij} + F_i - F_j + c_{ij}. \tag{2.2}$$

Let $c = \{c_{ij}\} \in \check{C}^2(\mathscr{U}, \mathbb{R})$ be the cochain so defined. We claim that $(\check{\delta}c)_{ijk} \in D$. Indeed,

$$\begin{aligned} f_{jk} &= f'_{jk} + F_j - F_k + c_{jk} \\ f_{ik} &= f'_{ik} + F_i - F_k + c_{ik} \\ f_{ij} &= f'_{ij} + F_i - F_j + c_{ij} \end{aligned}$$

Multiplying the second equation by minus one and summing up the three equations one gets

$$a_{ijk} = a'_{ijk} + c_{jk} - c_{ik} + c_{ij} = a_{ijk} + (\check{\delta}c)_{ijk},$$

which is the same as $(\check{\delta}c)_{ijk} = a_{ijk} - a'_{ijk} \in D$, since $a_{ijk}, a'_{ijk} \in D$. Then, $\breve{\pi}((\check{\delta}c)) = \check{\delta}(\pi c) = 0 \in \check{C}^2(\mathscr{U}, \mathbb{R}/D)$, which means that πc defines a cohomology class in $\check{H}^1(\mathscr{U}, \mathbb{R}/D)$. In constructing this cohomology class we have made choices: firstly, we could have changed θ in $\omega' = \omega + d\theta$ by an element $\theta_o \in \Theta_o$; secondly, we could have taken $F_i + b_i$ instead of F_i. For the latter, it is easy to see that by taking $F_i + b_i$ we will change πc by a coboundary in $\check{C}^1(\mathscr{U}, \mathbb{R}/D)$. For the former, we proceed as follows. Since U_i is contractible, there exists Γ_i in U_i such that $d\Gamma_i = \theta_o$. Taking $\theta + \theta_o$ leads to

$$d(\theta_i - \theta'_i + \theta + \theta_o) = 0 \text{ on } U_i.$$

Then, we have to take $F_i + \Gamma_i$ in order to obtain

$$d(F_i + \Gamma_i) = \theta_i - \theta'_i + \theta + \theta_o \Leftrightarrow \theta'_i = \theta_i + \theta + \theta_o - d(F_i + \Gamma_i).$$

Going back to the definition of c_{ij} (see (2.2)) but now using $F_i + \Gamma_i$ instead of F_i, it is easy to see that we obtain $c_{ij} - \gamma_{ij}$ instead of c_{ij}, where $\gamma_{ij} = \Gamma_i - \Gamma_j$ is constant. Then, we have the cocycle $\breve{\pi}c - \breve{\pi}\gamma \in \check{C}^1(\mathscr{U}, \mathbb{R}/D)$, which means that we have modified the previous $\breve{\pi}c$ by an element $\breve{\pi}\gamma \in [\Theta_o]_D$. Therefore, we have shown that, in order to have a well-defined map, we must associate (Y', p', α') to the cohomology class $(\breve{\pi}c)\mathrm{mod}[\Theta_o]_D \in \check{H}^1(\mathscr{U}, \mathbb{R}/D)\mathrm{mod}[\Theta_o]_D$. Notice that this map will send the extension (Y, p, α) constructed from (M, ω) to the zero class.

To conclude one needs to verify the following three points.

1. To each cohomology class in $\check{H}^1(\mathscr{U},\mathbb{R}/D)\text{mod}[\Theta_o]_D$ we can associate an extension (Y',p',α'): Let $c+\check{\pi}\gamma$ be a cohomology class in $\check{H}^1(\mathscr{U},\mathbb{R}/D)\text{mod}[\Theta_o]_D$. In particular, $c \in \check{H}^1(\mathscr{U},\mathbb{R}/D)$, which means $(\check{\delta}c)_{ijk} = c_{jk} - c_{ik} + c_{ij} = 0 \in \mathbb{R}/D$, and we can consider γ representing a closed 1-form θ_o. Then, we define $f'_{ij} = f_{ij} + c_{ij}$ and $g'_{ij} = \pi(f'_{ij}) = g_{ij}$. Hence, we can define a principal bundle $p' : Y' \to M$ having $g'_{ij} = g_{ij}$ as transition functions and with connection given by $\alpha' = (p')^*(\theta_i + \theta_o) + dx$. It is easy to see that applying the above association to (Y',p',α') leads exactly to the cohomology class $c+\check{\pi}\gamma$.

2. $\Theta-$equivalent extensions go to the same cohomology class.

   Suppose that $\varphi : (Y,p,\alpha) \to (Y',p',\alpha')$ is an $\Theta-$equivalence. On each open set U_i we can write $\varphi(m,x) = (m,x+\varphi_i(m))$, where $\varphi_i : U_i \to \mathbb{R}/D$. Since, $\pi : \mathbb{R} \to \mathbb{R}/D$ is a smooth covering map and U_i is simply connected, we can define $\phi_i : U_i \to \mathbb{R}$ such that $\pi \circ \phi_i = \varphi_i$. If φ is $\Theta-$equivalence, then $\varphi^*\alpha' = \alpha + p^*(\theta'')$, where $\theta'' = \theta + \theta_o$, with $\theta_o \in \Theta_o$. We claim that from this equation it follows that

   $$\theta'_i = \theta_i + \theta + \theta_o - d\phi_i \text{ on } U_i.$$

   Indeed, remember that $\alpha = p^*\theta_i + dx$ and $\alpha' = (p')^*\theta'_i + dx$. Then, we have

   $$\varphi^*\alpha' = \alpha + p^*(\theta'') \Leftrightarrow \varphi((p')^*\theta'_i + dx) = p^*\theta_i + dx + p^*(\theta + \theta_o) \Leftrightarrow$$

   $$\Leftrightarrow p^*(\theta'_i - \theta_i - \theta - \theta_o) = dx - \varphi^*(dx). \tag{2.3}$$

   Now, we calculate explicitly $\varphi^*(dx)$. Firstly, we note that $\varphi^*(dx) = dx + (\varphi_i \circ p)^*dx$. Secondly, we note that by dx on $U_i \times \mathbb{R}/D$ we actually mean $pr_2^*(dx)$, where $pr_2 : U_i \times \mathbb{R}/D \to \mathbb{R}/D$ is projection on the second factor and dx is Maurer-Cartan 1-form in $\mathbb{R}/D$. Given all of this, we can write

   $$\begin{aligned}
   \varphi^*(dx) &= \varphi^*(pr_2^*(dx)) \\
   &= (pr_2^* \circ \varphi)(dx) \\
   &= (pr_2 + (\varphi_i \circ p))^*(dx) \\
   &= pr_2^*(dx) + (\varphi_i \circ p)^*(dx) \\
   &= dx + ((\pi \circ \phi_i) \circ p)^*(dx) \\
   &= dx + p^*(\phi_i^*(\pi^* dx)) \\
   &= dx + p^*(d\phi_i),
   \end{aligned}$$

   which is $\varphi^*(dx) = dx + p^*(d\phi_i)$. Applying this to (2.3), we obtain

   $$p^*(\theta'_i - \theta_i - \theta - \theta_o + d\phi_i) = 0,$$

which implies that

$$\theta_i' = \theta_i + \theta + \theta_o - d\phi_i.$$

From the equations

$$\begin{array}{rcl} \theta_i' & = & \theta_i + \theta + \theta_o - d\phi_i \\ d(F_i + \Gamma_i) & = & \theta_i - \theta_i' + \theta + \theta_o \end{array}$$

one can see that $d(F_i + \Gamma_i - \phi) = 0$, which implies that $(F_i + \Gamma_i - \phi_i)$ is constant on U_i.

Now, we can show that the cohomology class associated to (Y', p', α') is zero. In fact, using the transition functions and the map φ_i one can see that

$$g_{ij}' = g_{ij} + \varphi_j - \varphi_i,$$

which implies that

$$\pi(f_{ij}' - f_{ij} - \phi_j + \phi_i) = 0 \in \mathbb{R}/D.$$

Then, we there exist constants $d_{ij} \in D$ for which we can write

$$f_{ij}' = f_{ij} + \phi_j - \phi_i + d_{ij}.$$

From this equation and $f_{ij} = f_{ij}' - F_i + F_j + c_{ij}$, it follows that

$$c_{ij} - \gamma_{ij} = (\check{\delta}\{F_k + \Gamma_k - \phi_k\})_{ij} + d_{ij}.$$

It means that when we consider the cohomology class defined by $\check{\pi}c$ in $\check{H}^1(\mathscr{U}, \mathbb{R}/D)$ mod$[\Theta_o]_D$ we get zero.

3. Extensions defining the same cohomology class are $\Theta-$equivalent.

Let (Y_1, p_1, α_1) and (Y_2, p_2, α_2) be two extensions constructed from $(M, \omega + d\theta_1)$ and $(M, \omega + d\theta_2)$, respectively. Also, let $\check{\pi}c$ and $\check{\pi}d$ be the cohomology classes associated to (Y_1, p_1, α_1) and (Y_2, p_2, α_2), respectively. Suppose that $\check{\pi}c - \check{\pi}d \mathrm{mod}[\Theta_o] = 0$, i.e, $\check{\pi}c - \check{\pi}d \in [\Theta_o]$, which means that there exists $\gamma \in \check{H}^1(\mathscr{U}, \mathbb{R})$ such that $\check{\pi}\gamma = \check{\pi}c - \check{\pi}d$. By Remark 2.2.11, we know that γ represents a closed 1-form θ, and $\gamma_{ij} = \Gamma_i - \Gamma_j$, where $d\Gamma_i = \theta$. For the extensions (Y_1, p_1, α_1) and (Y_2, p_2, α_2), we have the equations

$$\begin{array}{rcl} f_{ij} & = & f_{ij}' + F_i - F_j + c_{ij} \\ \theta_1' & = & \theta_i + \theta_1 + dF_i \end{array}$$

$$\begin{array}{rcl} f_{ij} & = & h_{ij}' + H_i - H_j + d_{ij} \\ \theta_2' & = & \theta_i + \theta_2 + dF_i \end{array}$$

where f_{ij} is associated to ω, as always, and $d\theta_1' = \omega + d\theta_1$, $d\theta_2' = \omega + d\theta_2$. The bundle Y_1 is constructed using the transition functions $\pi(f_{ij}')$, while Y_2 is contructed using $\pi(h_{ij}')$. Moreover, we know from point (2) above that if there exists a morphism $\varphi : Y_1 \to Y_2$, then the we can write $\pi(h'_{ij}) = \pi(f'_{ij}) + \varphi_j - \varphi_i$. Motivated by this one can calculate $\pi(h'_{ij}) - \pi(f'_{ij})$ and, using the four equations above, one concludes that

$$\pi(h'_{ij}) - \pi(f'_{ij}) = \pi(H_j - F_j - \Gamma_j) - \pi(H_i - F_i - \Gamma_i).$$

It suggests that we define

$$\varphi : Y_1 \to Y_2$$

$$(m,x) \in U_i \times \mathbb{R}/D \longmapsto \left(m, x + \pi\big((H_i - F_i - \Gamma_i)(x)\big)\right).$$

Carrying out a calculation that uses ideas from point (2), one can see that φ so defined is a $\Theta-$equivalence between (Y_1, p_1, α_1) and (Y_2, p_2, α_2).

□

2.3 Lifting infinitesimal symmetries

In this section we shall see how the described extensions of (M,ω) can be used to provide a one-dimensional central extension of a subalgebra of the Lie algebra of infinitesimal symmetries of (M,ω).

Firstly, recall that a vector field X on M is said to be a infinitesimal symmetry of the pair (M,ω) if $\mathscr{L}_X\omega = 0$. The set of all infinitesimal symmetries with the usual bracket of vector fields constitutes a Lie algebra, as can be seen from an application of the formula $\mathscr{L}_{[X,Y]} = \mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X$.

Now, suppose that we have (Y,p,α) an extension of (M,ω). We would like to know if it is possible to lift an infinitesimal symmetry X of (M,ω) to an infinitesimal symmetry X' of (Y,α), i.e, does exist a vector field X' on Y such that $\mathscr{L}_{X'}\alpha = 0$ and $(T_m p)X' = X$? It turns out that we can characterise which vector fields on M satisfy this property, and this is the content of the next proposition.

Proposition 2.3.1. Let X be a vector field on M. Then, a lift X' of X exists if and only if there exists a function $f \in C^\infty(M)$ satisfying $i_X\omega + df = 0$. Moreover, if X' exists, then X' is uniquely determined by f, which will also satisfy $p^*f = \alpha(X')$; also, distinct lifts of X must differ by a constant multiple of ∂_x.

Proof. Suppose that we have X' satisfying $\mathscr{L}_{X'}\alpha = 0$ and $(Tp)X' = X$. Then, we can write

$$0 = \mathscr{L}_{X'}\alpha = 0 = (i_{X'})d\alpha + d(i_{X'}\alpha) = p^*(i_X\omega) + d(\alpha(X')).$$

From this equation, it follows that $d(\alpha(X'))$ is zero when calculated on vertical vector fields, which implies that $\alpha(X')$ is constant along the fibers. Hence, we can define $f : M \to \mathbb{R}$ by

$f(m) = (\alpha(X'))(p^{-1}(m))$, where $p^{-1}(m)$ means that we have chosen any point in the preimage of m. Now, notice that

$$0 = p^*(i_X\omega) + d(\alpha(X)) = p^*(i_X\omega) + d(p^*f) = p^*(i_X\omega + df),$$

which implies that $i_X\omega + df = 0$.

On the other hand, suppose that $i_X\omega + df = 0$, for some function f on M. Then, on $U_i \times \mathbb{R}/D$ we can define

$$X' = X + p^*(f - \theta_i(X))\partial_x,$$

where $d\theta_i = \omega$ and ∂_x is the left invariant vector field on $\mathbb{R}/D$ dual to the Maurer-Cartan 1-form dx. It is clear that $(Tp)(X') = X$. Moreover, using the local expression for α, i.e, $\alpha = p^*(\theta_i) + dx$, one can see that $\alpha(X') = p^*f$. Also, it is easy to see that these two equations determine X' uniquely, which implies that X' is a well-defined global vector field on Y. It remains to show that X' is an infinitesimal symmetry of the pair (Y,α). But this can be seen by calculating

$$\mathscr{L}_{X'}\alpha = (i_{X'})d\alpha + d(\alpha(X')) = p^*(i_X\omega + df) = 0.$$

□

Let G be a connected Lie group and $\phi : G \times M \to M$ a left action of G on M. Recall that G is said to be a symmetry group of the pair (M,ω) if $\phi_g^*\omega = \omega$, for all $g \in G$. We are going to denote by X_M the fundamental vector field associated to the action ϕ, i.e, X_M is the vector whose flow is given by $\phi_{exp(-tX)}$, for $X \in \mathfrak{g} := \mathrm{Lie}(G)$. If G is a symmetry group of (M,ω), it is easy to see that each fundamental vector field X_M is an infinitesimal symmetry of (M,ω).

Remark 2.3.2. The Lie group $\mathbb{R}/D$ is a symmetry group of (Y,α). In fact, recall that the action of $\mathbb{R}/D$ on Y was defined locally as

$$\phi : (U_i \times \mathbb{R}/D) \times \mathbb{R}/D \to U_i \times \mathbb{R}/D$$

$$((m,x),y) \longmapsto (m,x+y).$$

Then, we have to show that $\phi_y^*\alpha = \alpha$, for every $y \in \mathbb{R}/D$. Let $v + w \in T_{(m,x)}(U_i \times \mathbb{R}/D)$, then

$$(\phi_y^*\alpha)_{(m,x)}(v+w) = \alpha_{(m,x+y)}(v + T_{(m,x)}\phi(w)).$$

The local expression for α is $\alpha = p^*(\theta_i) + pr_2^*(dx)$, hence

$$\alpha_{(m,x+y)}(v + T_{(m,x)}\phi(w)) = (p^*_{(m,x+y)}\theta_i)(v) + (p^*_{(m,x+y)}\theta_i)(T\phi(w))$$
$$+((pr_2^*)_{(m,x+y)}dx)(v) + ((pr_2^*)_{(m,x+y)})dx(T\phi(w)).$$

It is easy to see that the first two terms on the righ-hand side coincide if they were calculated at (m,x), and the third term is actually zero. For the fourth, we remember that dx is the Maurer-Cartan form and calculate

$$((pr_2^*)_{(m,x+y)}dx)(T\phi(w)) = (dx)_{x+y}((Tpr_2 \circ T\phi)w) = (dx)_{(x+y)}(TL_y(w))$$

$$= \mathrm{Ad}(-y)((dx)_x(w)) = (dx)_x(w).$$

It follows that

$$(\phi_y^*\alpha)_{(m,x)}(v+w) = \alpha_{(m,x)}(v+w),$$

which shows that $\mathbb{R}/D$ is a symmetry group of (Y,α).

Definition 2.3.3. A hamiltonian vector field on M is an infinitesimal symmetry of (M,ω) for which there exists $f \in C^\infty(M)$ such that $i_X\omega + df = 0$.

Proposition 2.3.4. The hamiltonian vector fields of M constitute a Lie algebra.

Proof. It follows directly from the general formula $i_{[X,Y]} = \mathscr{L}_X i_Y - \mathscr{L}_Y i_X$. □

Let us denote by $\mathscr{H}(M)$ the Lie algebra of hamiltonian vector fields on M and by $\mathrm{sym}(Y,\alpha)$ the Lie algebra of infinitesimal symmetries of the pair (Y,α). Relating these two Lie algebras we have the following result.

Proposition 2.3.5. The Lie algebra $\mathrm{sym}(Y,\alpha)$ is a one-dimensional central extension of the Lie algebra $\mathscr{H}(M)$. More precisely, we have the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R}[\partial_x] \xrightarrow{i} \mathrm{sym}(Y,\alpha) \xrightarrow{Tp} \mathscr{H}(M) \longrightarrow 0$$

Proof. Notice that it follows from Proposition 2.3.1 that Tp is surjective. Now, let us show that Tp applied to $\mathrm{sym}(Y,\alpha)$ has image contained in $\mathscr{H}(M)$. For, let $X' \in \mathrm{sym}(Y,\alpha)$. Then $(\mathscr{L}_{X'})\alpha = 0$, and applying d to this equality, we get

$$0 = d((\mathscr{L}_{X'})\alpha) = d(\,(i_{X'})d\alpha + d((i_{X'})\alpha)\,) = d((i_{X'})d\alpha) = d((i_{X'})(p^*\omega)).$$

Since ω is a closed 2-form, we can write

$$(\mathscr{L}_{X'})(p^*\omega) = d((i_{X'})p^*\omega).$$

Hence, we have $(\mathscr{L}_{X'})(p^*\omega) = 0$. On the other hand,

$$\mathscr{L}_{Tp(X')}(\omega) = (di_{(TpX')} + i_{(TpX')}d)\,\omega = d(i_{(TpX')}\,\omega) = (\mathscr{L}_{X'})(p^*\omega),$$

which implies $\mathscr{L}_{Tp(X')}(\omega) = 0$, i.e, $Tp(X')$ is an infinitesimal symmetry of (M,ω). Moreover, from Proposition 2.3.1, we conclude that $Tp(X') \in \mathscr{H}(M)$.

As we have already observed, $\mathbb{R}/D$ is a symmetry of group of the pair (Y,α), then the fundamental vector fields associated to this action are infinitesimal symmetries. Moreover, since $\mathbb{R}/D$ is one-dimensional, all the fundamental vector fields are multiples of ∂_x. It is clear that all fundamental vector fields and vertical vector fields lie on the kernel of Tp. On the other hand, suppose that $X' \in \mathrm{sym}(Y,\alpha)$ is such that $Tp(X') = 0$. Using the local expression for X' given in Proposition 2.3.1, we can write

$$X' = Tp(X') + (f - \theta_i(Tp(X')))\partial_x = f(\partial_x).$$

However, Proposition 2.3.1 also tell us that the function f is a hamiltonian function of $Tp(X')$. Since $Tp(X') = 0$, it follows that f must be constant.

It remains to show that ∂_x is in the center of $\mathrm{sym}(Y,\alpha)$. Let $X' \in \mathrm{sym}(Y,\alpha)$. Locally it can be expressed as

$$X' = Tp(X) + (f - \theta_i(Tp(X'))).$$

To conclude that $[\partial_x, X'] = 0$, it suffices to notice that the coefficient $(f - \theta_i(Tp(X')))$ is constant along the fibers and $Tp(X)$ is a horizontal vector field in $U_i \times \mathbb{R}/D$. □

We conclude this section by introducing the concept of (co)moment map.[2]

Definition 2.3.6. Let G be a symmetry group of (M,ω). A moment map for the action of G on M is a linear map

$$\mu : \mathfrak{g} \to C^\infty(M)$$
$$X \longmapsto \mu_X$$

where μ_X satisfies $(i_{X_M})\omega + d\mu_X = 0$.

Notice that what moment map does is, essentially, to associate a hamiltonian function to each fundamental vector field.

Remark 2.3.7. When we consider a $G-$action on a symplecitc manifold, the algebra $C^\infty(M)$ can be made into a Poisson algebra, which is in particular a Lie algebra. In this context, the (co)moment map is usually defined to be a Lie morphism between $\mathfrak{g}$ and $C^\infty(M)$ (see Appendix A). However, to prove Theorem 2.4.3 we will not use this property.

As an easy consequence of the Proposition 2.3.1 we have the following.

Proposition 2.3.8. Let G be a symmetry group of (M,ω) and (Y,p,α) an extension of this pair. Then, a moment map μ for the action of G on M exists if and only if each fundamental vector field X_M can be lifted to an infinitesimal symmetry of the pair (Y,α).

[2] Here we chose to maintain the terminology of (TUYNMAN; W.W.A.J.WIEGERINCK, 1987). However, what is called a moment map here is referred to as comoment map in Appendix A.

2.4 The integrability result

In this section, we shall see how the constructions presented in the previous sections may be used to provide an answer to the problem of integrating a central extension of Lie algebras.

Recall from Definition 2.1.9 that a central extension H of a Lie group G by an abelian Lie group A can be seen as a $A-$principal bundle over G. In the following we are going to show that we can define a connection 1-form on H whose curvature is related to the algebra 2-cocyle that H defines.

Proposition 2.4.1. Let H be a central extension of G by A. Then, we can represent explicitly the associated algebra 2-cocycle as a left invariant 2-form on G (with values on $\mathfrak{a}$).

Proof. Let us consider the central extension

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} G \longrightarrow 0$$

with induced Lie algebra central extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{T_e i} \mathfrak{h} \xrightarrow{T_e \pi} \mathfrak{g} \longrightarrow 0.$$

Let $y_1, \dots, y_p \in \mathfrak{a}$ be a basis and define $Y_1 = T_e i(y_1), \dots, Y_p = T_e i(y_p)$. Now, let $\Xi_1, \dots, \Xi_n$ be such that $\Xi_1, \dots, \Xi_n, Y_1, \cdots, Y_p$ form a basis for $\mathfrak{h}$. It follows from the exactness of the algebra sequence that $X_1 = T_e\pi(\Xi_1), \dots, X_n = T_e\pi(\Xi_n)$ form a basis for $\mathfrak{g}$. Now, consider $\beta^1, \dots, \beta^n$ left invariant 1-forms on G defined by the covectors dual to $X_1, \dots, X_n$. The covectors $\pi^*\beta^1, \dots, \pi^*\beta^n$ on $\mathfrak{h}^*$ are linearly independent and dual to $\Xi_1, \dots, \Xi_n$. Now, we complete $\pi^*\beta^1, \dots, \pi^*\beta^n$ to basis of $\mathfrak{h}^*$ by taking $\alpha^1, \dots, \alpha^p$ in such a way that $\pi^*\beta^1, \dots, \pi^*\beta^n, \alpha^1, \dots, \alpha^p$ are dual to $\Xi_1, \dots, \Xi_n, Y_1, \cdots, Y_p$.

Since $\mathrm{im}(i) \subset \mathrm{center}(H)$, it follows that the flow of each Y_i commutes with the flow of each Y_j and Ξ_k (here we are considering each Y_i and Ξ_k as left invariant vector fields on H). Then, we conclude that $[Y_i, Y_j] = 0$ and $[Y_i, \Xi_k] = 0$. Hence, we can write

$$[\Xi_i, \Xi_j] = c_{ij}^k \Xi_k + d_{ij}^r Y_r$$

$$[X_i, X_j] = c_{ij}^k X_k$$

Calculating the exterior derivative of each α^r using the above properties, we conclude that

$$d\alpha^r = \sum_{i \leq j} -d_{ij}^r (\pi^*(\beta_i) \wedge \pi^*(\beta_i)).$$

From this, we see that

$$d\alpha^r = \pi^*\left(\sum_{i\leq j} -d^r_{ij}(\beta_i \wedge \beta_i)\right)$$

Then, we set $\omega^r = \sum_{i\leq j} -d^r_{ij}\beta_i \wedge \beta_i$. Since the d^r_{ij} are constants, it follows that $d\omega^r = 0$.

Now, let us consider the section $\sigma : \mathfrak{g} \to \mathfrak{h}$ of $T_e\pi : \mathfrak{h} \to \mathfrak{g}$ given by $\sigma(X_i) = \Xi_i$. We know that the algebra 2-cocycle defined by this section is $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ given by

$$\omega(X,Y) := \sigma([X,Y]) - [\sigma(X), \sigma(Y)].$$

It is easy to see that, for X_i and X_j as above, we get $\omega(X_i, X_j) = -d_{ij}Y_r$. Hence, $\omega = \omega^r \otimes Y_r$, which is a left invariant 2-form with values in $\mathfrak{a}$ defined on G. □

Proposition 2.4.2. Let $\alpha_1, \ldots, \alpha_r$ be the 1-forms appearing in the proof of Proposition 2.4.1. Then, the 1-form $\alpha := \alpha^r \otimes Y_r$ is a connection 1-form on the $A-$principal bundle H, whose curvature is $\pi^*\omega = \pi^*\omega^r \otimes Y_r$.

Proof. We have to verify that α satisfies the following two conditions:

1. $\alpha(Y_M) = Y$, where $Y \in \mathfrak{a}$ and Y_M is the associated fundamental vector field;
2. $R^*_a\alpha = \mathrm{Ad}(a^{-1})\alpha = \alpha$ (A is an abelian group), where $R_a(h) := h \cdot i(a)$

We use the same notation of Proposition 2.4.1. For $h \in H$, we can calculate

$$Y_M(h) = \left.\frac{d}{dt}\right|_{t=0} (\phi_{exp(-tY)}(h)) = \left.\frac{d}{dt}\right|_{t=0} h(i(exp(-tY)) =$$

$$\left.\frac{d}{dt}\right|_{t=0} L_h(i(exp(-tY))) = (T_eL_h)(T_ei(-Y)) = (T_eL_h)(T_ei(\sum a_j y_j)) =$$

$$\sum \left(a_j(T_eL_h)(T_ei(y_j))\right) = \sum a_jY_j(h).$$

Then, α evaluated in Y_M gives

$$\alpha_h(Y_M(h)) = (\alpha^r_h \otimes Y_r)(\sum a_jY_j(h)) = \sum a_jY_j,$$

which shows that α satisfies (1) above.

For item (2) recall that $\Xi_1, \ldots, \Xi_n, Y_1, \cdots, Y_p$ are left invariant vector fields, which form a basis for each tangent space of H. Hence, given $X \in T_hH$, we can write $X_h = \sum a_jY_j(h) + \sum b_k\Xi_k(h)$. Then, we calculate

$$(R^*_a(\alpha)(X))_h = \alpha_{hi(a)}(T_hR_aX) = \alpha_{hi(a)}\left(T_hR_a\left(\sum a_jY_j(h) + \sum b_k\Xi_k(h)\right)\right) =$$

$$\alpha_{hi(a)}\left(T_hR_a\left(\sum a_jY_j(h)\right)\right) = \alpha_{hi(a)}\left(T_hR_a\left(\sum a_j(Y_j)_M(h)\right)\right) =$$

$$\alpha_{hi(a)}\left(\sum a_j(\mathrm{Ad}(a^{-1})Y_j)_M(hi(a))\right) = \alpha_{hi(a)}\left(\sum a_j(Y_j)_M(hi(a))\right) =$$

$$\sum a_jY_j = (\alpha)_h(X),$$

which shows that α satisfies (2). $\square$

The integrability result proved in (TUYNMAN; W.W.A.J.WIEGERINCK, 1987) is as follows.

Theorem 2.4.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and suppose that ω is a Lie algebra 2-cocycle on $\mathfrak{g}$ with values in $\mathbb{R}$. Then, there is a Lie group central extension H of G by $\mathbb{R}/D$ associated to the Lie algebra extension determined by ω if and only if the following conditions are satisfied:

(i) $\mathrm{Per}(\omega) \subset D$;

(ii) there exists a moment map for the left action of G on (G,ω).

Proof. ($\Rightarrow$) We consider ω as a left invariant closed 2-form on G. Suppose that (H,i,π) is a central extension of G by $\mathbb{R}/D$ associated to the central extension of $\mathfrak{g}$ by $\mathbb{R}$ defined by ω. From Proposition 2.4.2, it follows that there exists a connection 1-form α which is left invariant and satisfies $d\alpha = \pi^*(\omega')$, where ω' is, at least, in the same cohomology class of ω. Then, from Proposition 2.2.8, we conclude that $\mathrm{Per}(\omega) = \mathrm{Per}(\omega') \subset D$. On the other hand, the fundamental vector fields associated to the left action of G on itself are the right invariant vector fields on G, which are infinitesimal symmetries of (G,ω) since ω is left invariant. Then, let X^r be a right invariant vector field on G. One can define a lift of X^r to H as follows. Let $e' \in H$ be the identity of H, and let $X' \in T_{e'}H$ be such that $T_{e'}\pi(X') = X^r_e$. Now, starting from X'_e, define a right invariant vector field $(X')^r$ on H. We claim that $(X')^r$ is a lift of X^r. In fact, we have

$$(T_h\pi)(X')^r = (T_h\pi)(T_{e'}R_h(X')) = T_{e'}(\pi\circ R_h)(X') = T_{e'}(R_{\pi(h)}\circ\pi)(X') =$$

$$T_eR_{\pi(h)}(T_{e'}\pi(X')) = T_eR_{\pi(h)}X^r = (X^r)_{\pi(h)}.$$

Since α is left invariant, $(X')^r$ is a infinitesimal symmetry of the pair (H,α). Hence, we have lifted each fundamental vector field on G to an infinitesimal symmetry of (H,α). Therefore, it follows from Proposition 2.3.8 that there exists a moment map for the left action of G on itself. $\square$

To prove the converse we shall use the condition $\mathrm{Per}(\omega) \subset D$ to construct a extension of (Y,p,α) of the pair (G,ω) and the existence of moment map for the left action is going to be used to turn Y into a Lie group whose Lie algebra is the central extension defined by ω. We are going to divide the proof in several steps. We start with a crucial definition.

Definition 2.4.4. Let $\mathfrak{g}$ be a Lie algebra, let be $X_1, \ldots, X_n$ a basis for $\mathfrak{g}$, and let c_{ij}^k be the associated structural constants. Let M be a manifold and $\tau^1, \ldots, \tau^n$ 1-forms on M. Then, M is called $\mathfrak{g}-$manifold if the follwing condition is fullfilled

(i) at each point $m \in M$ $\tau^1, \ldots, \tau^n$ is a basis for $T_m^* M$;

(ii) $d\tau^k = -\frac{1}{2} c_{ij}^k \tau^i \wedge \tau^j$.

Remark 2.4.5. A diffeomorphism of M which leaves each τ^i invariant is called a Maurer-Cartan automorphism (MC automorphism from now on) of the $\mathfrak{g}-$manifold M. The set of all MC automorphism is clearly a group, which will be denoted by $\mathrm{Aut}_{MC}(M)$. If $\mathrm{Aut}_{MC}(M)$ acts transitively on M, then M is called a complete $\mathfrak{g}-$manifold.

Lemma 2.4.6. A $\mathfrak{g}-$manifold M can be given the structure of Lie group with Lie algebra $\mathfrak{g}$ for which τ_i are left invariant 1-forms if and only if it is a complete $\mathfrak{g}-$manifold.

Proof. Suppose that $\tau^1, \cdots, \tau^n$ are 1-forms that turn M into a $\mathfrak{g}-$complete manifold and let $pr_1, pr_2 : M \times M \to M$ be projections on the first and second factor, respectively. Consider the following n equations

$$pr_1^* \tau^i = pr_2^* \tau^i \quad 1 \leq i \leq n.$$

We claim that these equations define an involutive distribution on $M \times M$. In fact, notice that

$$(pr_1^* \tau^i)_{(m,n)}(v, w) = \tau_m^i(T pr_1(v, w)) = \tau_m^i(v)$$

$$(pr_2^* \tau^i)_{(m,n)}(v, w) = \tau_m^i(T pr_2(v, w)) = \tau_n^i(w),$$

however $pr_1^* \tau^i = pr_2^* \tau^i$, so that $\tau_m^i(v) = \tau_n^i(w)$. Since M is $\mathfrak{g}-$complete manifold the group $\mathrm{Aut}_{MC}(M)$ acts transitively on M. Then, there exists $\phi \in \mathrm{Aut}_{MC}(M)$ such that $\phi(m) = n$. Moreover, there exists $w' \in T_m(M)$ such that $w = T_m \phi(w')$. Then, we have the following

$$\tau_n^i(w) = \tau_{\phi(m)}^i(w) = \tau_{\phi(m)}^i(T_m \phi(w')) = (\phi^* \tau^i)_m(w') = \tau_m^i(w').$$

On the other hand, $\tau_m^i(v) = \tau_n^i(w)$ implies that $\tau_m^i(w') = \tau_m^i(v)$, $\forall i$, i.e, $v - w' \in ker(\tau^i)$, $\forall i$. From condition (i) of Definition 2.4.4, it follows that $v = w'$. It is easy to see this gives $w = T_m \phi(v)$. Hence, we have shown that $(v, w)_{(m,n)}$ satisfies $pr_1^* \tau^i = pr_2^* \tau^i$, $0 \leq i \leq n$, if and only if $(v, w) = (v, T_m \phi(v))$, for some $\phi \in \mathrm{Aut}_{MC}(M)$ such that $\phi(m) = n$. From this we conclude that the equations $pr_1^* \tau^i = pr_2^* \tau^i$ define an involutive distribuiton on $M \times M$. By Frobenius' Theorem, it is an integrable distribuition, and we have only two possible integral manifolds, which are

$$\{(m, m) \in M \times M \mid m \in M\}$$

$$\{(m, \phi(m)) \in M \times M \mid m \in M, \phi \in \mathrm{Aut}_{MC}(M)\}.$$

The uniqueness of integral manifolds assures that if a $MC-$automorphism has a fixed point, then it is the identity automorphism, which means that $\mathrm{Aut}_{MC}(M)$ acts without fixed points. Hence, one can estabilish a bijection between M and $\mathrm{Aut}_{MC}(M)$ simply by fixing a point $m_0 \in M$ and defining

$$\mathrm{Aut}_{MC}(M) \longrightarrow M$$

$$\phi \longmapsto \phi(m_0).$$

A product [3] in M can be defined by $m \cdot n := \phi_m \circ \phi_n(m_0)$, where $\phi_m(m_0) = m$ and $\phi_n(m_0) = n$. □

Remark 2.4.7. Notice that we did not verify the smoothness of the product introduced on M. This verification should rely on a possible smooth structure on $\mathrm{Aut}_{MC}(M)$. Since $\mathrm{Aut}_{MC}(M)$ is the automorphism group of a tensor structure on M (the 1-forms τ^i), we believe that the Theorem IX, Ch. 4, in (PALAIS, 1957) could be used to turn $\mathrm{Aut}_{MC}(M)$ into a Lie group with the property that the evaluation map

$$\mathrm{Aut}_{MC}(M) \times M \to M$$

is smooth. The smoothness of this maps would imply the smoothness of the product. Quoted in terms of the $\mathrm{Aut}_{MC}(M)$, the referred result states that $\mathrm{Aut}_{MC}(M)$ can be given the structure of Lie group if and only if the complete vector fields tangent[4] to $\mathrm{Aut}_{MC}(M)$ form a finite dimensional Lie algebra. We believe that somehow it could be possible to show that, in the specfic case of $\mathrm{Aut}_{MC}(M)$, the complete vector fields tangent to $\mathrm{Aut}_{MC}(M)$ constitue exactly the Lie algebra $\mathfrak{g}$.

In the following we prove that the extension (Y, p, α) constructed from a Lie group G with a Lie algebra 2-cocycle ω seen as left invariant closed to 2-form on G is a $\mathfrak{h}-$manifold, where $\mathfrak{h}$ is a central extension of $\mathrm{Lie}(G) = \mathfrak{g}$ by $\mathbb{R}$ determined by ω.

Lemma 2.4.8. Let G be Lie group with Lie algebra $\mathfrak{g}$, and let ω be a Lie algebra 2-cocyle with values in $\mathbb{R}$. Consider the pair (G, ω) by seeing ω as left invariant closed 2-from on G. Then, the extension (Y, p, α) constructed from this initial data is a $\mathfrak{h}-$manifold, where $\mathfrak{h}$ is the central extension of $\mathfrak{g}$ by $\mathbb{R}$ determined by ω.

Proof. To prove this lemma we need to find $n+1$ one-forms on Y satisfying the conditions of Definition 2.4.4.

Firstly, notice that we already have the 1-form α on Y. Then, let $\beta^1, \ldots, \beta^n$ be left invariant 1-forms on G which form a basis for each cotangent space of G. We claim that $\alpha, p^*\beta^1, \ldots, p^*\beta^n$ are linearly independent at each $y \in Y$. Indeed, suppose that

$$\left(\sum b_i(p^*\beta_i)\right) + a\alpha = 0.$$

Applying it to the vector $(0, v)$, we get $a\alpha(0, v) = 0$; however, we know that locally α is given by $\alpha = p^*\theta_i + dx$, then it follows that $a = 0$. Similarly, we conclude that $b_i = 0$. Thereby, $\alpha, \beta^1, \ldots, \beta^n$ are linearly independent.

3 In (TUYNMAN; W.W.A.J.WIEGERINCK, 1987) the smoothness of this product is said to be merely a technical detail. However, we were not able to elaborate a complete proof of it (see Remark 2.4.7).

4 A vector whose flow always define and element in $\mathrm{Aut}_{MC}(M)$.

Now, we must show that the exterior derivatives of $\alpha, \beta^1, \ldots, \beta^n$ are related to the structual constants of $\mathfrak{h} = g \oplus_\omega \mathbb{R}$. Let $X_1, \ldots, X_n$ be the dual basis of $\beta^1, \ldots, \beta^n$. Then, a basis for $\mathfrak{g} \oplus_\omega \mathbb{R}$ is given by $(X_i, 0), 0 \leq i \leq n$, and $(0,1)$. Recalling that the Lie bracket on $\mathfrak{g} \oplus_\omega \mathbb{R}$ is given by

$$[(X,v),(Y,w)] = ([X,Y], \omega(X,Y)),$$

we can write

$$[(X_i,0),(X_j,0)] = ([X_i,X_j],\, \omega(X_i,X_j)) = \left(\sum c_{ij}^k (X_k,0)\right) + \omega(X_i,X_j)(0,1)$$

$$[(X_i,0),(0,1)] = 0,$$

where c_{ij}^k are the structural constants of the Lie algebra $\mathfrak{g}$. Since $\beta^1, \ldots, \beta^n$ is a basis, we can write $d\beta^k$ as

$$d\beta^k = \sum_{i \leq j} d\beta^k(X_i,X_j)(\beta^i \wedge \beta^j).$$

Then, we get

$$d(p^*\beta^k) = \sum_{i \leq j} d\beta^k(X_i,X_j)(p^*\beta^i \wedge p^*\beta^j).$$

However, we have

$$(d\beta^k)(X_i,X_j) = X_i(\beta^k(X^j)) - X_j(\beta^k(X^i)) - \beta^k([X_i,X_j]) = -c_{ij}^k,$$

Thus,

$$d\beta^k = \sum_{i \leq j} -c_{ij}^k(\beta^i \wedge \beta^j).$$

Finally, for α, we know that $p^*\omega = d\alpha$, but ω can be written as

$$\omega = \sum_{i \leq j} \omega(X_i,X_j)\beta^i \wedge \beta^j,$$

which gives

$$d\alpha = \sum_{i \leq j} \omega(X_i,X_j)(p^*\beta^i) \wedge (p^*\beta^j).$$

Thereby, we have shown that $\alpha, p^*\beta^1, \ldots, p^*\beta^n$ satisfy condition ii. of Definition 2.4.4. □

Recall from Definition 2.4.6 that to prove that Y is a $\mathfrak{h}-$complete manifold we need to verify that the group of diffeomorphism of Y preserving $\alpha, p^*\beta^1, \ldots, p^*\beta^n$ acts transitively on Y. Notice that a diffeomorphism φ_d defined by translation on the fibers by an element of $d \in \mathbb{R}/D$ is a *MC* automorphism. For, it is clear that φ_d preserves each $p^*\beta$; also, we have already shown in the last section that $\mathbb{R}/D$ is a symmetry group of (Y,α), then φ_d preserves α as well. Hence, points on the same fiber can be joined by a *MC* automorphism. Now, to join points lying on distinct fibers, the idea is to lift the left translations L_g defined on G to a *MC* automorphism Λ_g of Y, and this where the existence of a moment map for the left action of G on itself plays a role in the converse of Theorem 2.4.3.

Remark 2.4.9. From the proof of Lemma 2.4.6 one can deduce that any *MC* automorphism of Y will be of the from $\varphi_d \circ \Lambda_g$, where φ_d denotes translation on the fibers by $d \in \mathbb{R}/D$ and Λ_g denotes the lift of L_g.

The existence of a moment map for the left action of G on itself can used to prove the following result.

Lemma 2.4.10. Let G be Lie group with Lie algebra $\mathfrak{g}$, and let ω be a Lie algebra 2-cocyle with values in $\mathbb{R}$. Consider the pair (G,ω) by seeing ω as left invariant closed 2-from on G, and let (Y,p,α) be the extension constructed from this initial data. If there exists a moment map for the left action of G on itself, then the extension (Y,p,α) is a complete $\mathfrak{h}-$manifold, where $\mathfrak{h}$ is the central extension of $\mathfrak{g}$ by $\mathbb{R}$ determined by ω.

Proof. From Lemma 2.4.8 we already know that Y is a $\mathfrak{h}-$manifold. Let $\mu : \mathfrak{g} \to C^\infty(M)$ be a moment map for the left action of G on itself. As we have already noticed, the fundamental vector fields for this action are the right invariant vector fields on G. Following Propositon 2.3.1, considering $X \in \mathfrak{g}$, from μ_X and the right invariant vector field X^r associated to X, we can construct a unique vector field X' on Y, which is an infinitesimal symmetry of (Y,α) (see also Proposition 2.4.12) and satisfies $Tp(X') = X^r$ and $\alpha(X') = p^*\mu_X$. The flow $\mathrm{Fl}_t^{X^r}$ of X^r is given by left multiplication by $exp(-tX)$, then X^r is a complete vector field. By a result analogous to the one concerning the horizontal lift of curves to principal bundles, it follows that the flow of X^r can be lifted to the flow $\mathrm{Fl}_t^{X'}$ of X', which shows that X' is a complete vector field. Alternatively, one can construct the flow of X' explicitly (c.f Remark 2.4.11). Then, the flow of X' satisfies $p \circ Fl_t^{X'} = \mathrm{Fl}_t^{X^r} \circ p$ and $(\mathrm{Fl}_t^{X'})^*\alpha = \alpha$. It is clear that $\mathrm{Fl}_t^{X'}$ also preserves $p^*\beta_j$. Hence, for each $t \in \mathbb{R}$, we have lifted the left multiplication $L_{exp(-tX)}$ to a MC automorphism of Y, which we denote by Λ_{exp-tX}. Since the group G is connected, it can be generated by a symmetric open neighborhood of the identity, which can be taken to be a neighborhood where exp is diffeomorphism. Then, any $g \in G$ can be expressed as $g = exp(t_1X_1)\cdots exp(t_nX_n)$. Thus, the lift of L_g can be defined as the composition of the lifts of each $L_{expt_iX_i}$, i.e, $\Lambda_g := \Lambda_{exp(t_1X_1)} \circ \cdots \circ \Lambda_{exp(t_nX_n)}$. Then, we have shown that we can lift each L_g, $g \in G$, to a *MC* automorphism of Y. And, by Lemma 2.4.6, it follows that Y can be given the structure of Lie group with Lie algebra $\mathfrak{h} = \mathfrak{g} \otimes_\omega \mathbb{R}$. Therefore, we get the following short exact sequence of Lie groups (see Remark 2.4.11)

$$0 \longrightarrow \mathbb{R}/D \xrightarrow{i} Y \xrightarrow{p} G \longrightarrow 0 \qquad (2.4)$$

where i is induced by the action of $\mathbb{R}/D$ on Y. □

Remark 2.4.11. The diffeomorphism $\Lambda_{exp(tX)}$ ocurring in the above proof can be computed explicitly in local coordinates. In fact, let $U \times \mathbb{R}/D$ be a local chart for Y. Then, α can be written as $\alpha = p^*\theta + dx$, for some 1-form θ defined on U and satisfying $d\theta = \omega$. From Proposition

2.3.1, we know that in this chart X' is expressed as $X' = X^r + (\mu_X - \theta(X^r))\partial_x$. Then, the flow $\mathrm{Fl}_t^{X'}$ is given by

$$\mathrm{Fl}_t^{X'}(g,x) = \left(exp(tX)g,\ x + \int_0^t ((\mu_X - \theta(X^r))_{exp(sX)g})ds \right). \tag{2.5}$$

Notice that t must be taken small enough such that $exp(tX)g \in U$. This local expression for the flow of X' proves that X' is a complete vector field. Indeed, one can write it in different charts and verify that they agree on intersections. Moreover, using (2.5), it is easy to see that Y in the short exact sequence (2.4) is actually a central extension of G by $\mathbb{R}/D$.

With Lemma 2.4.10 and Remark 2.4.11 we have completed the proof of the converse of Theorem 2.4.3.

The Proposition 2.3.5 shows that $\mathrm{sym}(Y,\alpha)$ provides a one-dimensional central extension of the Lie algebra of hamiltonian vector fields on (G,ω). It would interesting to understand how this result relates to the Lie group structure that introduced in Y. To this end, for each $X \in \mathfrak{g}$, using the Lie group structure of Y we construct a right invariant vector fields out of $X'_{e'}$, which we denote by $(X')^{right}$, and we state the following.

Proposition 2.4.12. For each $X \in \mathfrak{g}$, the right invariant vector field $(X')^{right}$ is an infinitesimal symmetry of (Y,α). Moreover, the flow $\mathrm{Fl}_t^{(X')^{right}}$ projects onto the flow of X^r.

Proof. For $X \in \mathfrak{g}$, X' is obtained from X^r and μ_X and it is an infinitesimal symmetry of (Y,α). Following the notation introduced in the end of the previous lemma, $\mathrm{Fl}_t^{X'}$ will denoted by $\Lambda_{exp(tX)}$. The multiplication on Y introduced in Lemma 2.4.6 is defined in terms of the *MC* automorphisms, once we have chosen a point in Y. Since we want p to be a Lie morphism, such a point must be chosen over the fiber of $e \in G$. Let e' denote this point. We are going to show that the right invariant vector field generated by $X'_{e'}$ is an infinitesimal symmetry of (Y,α). Let $g' \in Y$ such that $p(g') = g$. Then, the right multiplication by $g' \in Y$ acting on $\Lambda_{exp(tX)}$ is given by $\Lambda_{exp(tX)} \circ \Lambda_g(e')$, where Λ_g denotes the unique MC automorphism that sends e' to g'. Notice that $\Lambda_{exp(tX)} \circ \Lambda_g(e') = L_{\Lambda_{exptX}(e')}(g')$. Let us calculate the differential of the left translation by $\Lambda_{exptX}(e')$ at a point $y \in Y$

$$T_y L_{\Lambda_{exptX}(e')}(v) = \left.\frac{d}{ds}\right|_{s=0} L_{\Lambda_{exptX}(e')}(\gamma(s)) = \left.\frac{d}{ds}\right|_{s=0} \Lambda_{exptX} \circ \Lambda_{p(\gamma(s))}(e') =$$

$$\left.\frac{d}{ds}\right|_{s=0} \Lambda_{exptX}(\gamma(s)) = (T_y\Lambda_{exptX})(v).$$

($\Lambda_{p(\gamma(s))}(e') = \gamma(s)$ because $\Lambda_{p(\gamma(s))}$ denotes the unique *MC* automorphism that sends e' to $\gamma(s)$). Also, we notice that

$$(X')^{right}_{g'} := (T_{e'}R_{g'})X' = \left.\frac{d}{dt}\right|_{t=0} \Lambda_{exp(tX)} \circ \Lambda_g(e') = \left.\frac{d}{dt}\right|_{t=0} L_{\Lambda_{exptX}(e')}(g'),$$

which shows that left multiplication by $\Lambda_{exp(tX)}(e')$ gives the flow of $(X')^{right}$. We claim that $(X')^{right}$ is an infinitesimal symmetry of (Y,α). Indeed, for any vector $v \in T_yY$, we have

$$(T_y \operatorname{Fl}_t^{(X')^{right}})v = \frac{d}{ds}\bigg|_{s=0} \operatorname{Fl}_t^{(X')^{right}}(\gamma(s)) = \frac{d}{ds}\bigg|_{s=0} L_{\Lambda_{exptX(e')}}(\gamma(s)) =$$

$$T_y L_{\Lambda_{exptX(e')}}(v) = (T_y \Lambda_{exptX})(v).$$

Moreover, recall that Λ_{exptX} is the flow of X', which was constructed as a infinitesimal symmetry for α. Thereby, it follows that

$$\left[(\operatorname{Fl}_t^{(X')^{right}})_y^* \alpha\right] v = \alpha_{\operatorname{Fl}_t^{(X')^{right}}(y)} (T_y \operatorname{Fl}_t^{(X')^{right}} v) = \alpha_{\Lambda_{exptX}(y)} (T_y \Lambda_{exptX}) =$$

$$[(\Lambda_{exptX})^* \alpha]_y v = \alpha_y(v),$$

which shows that $(X')^{right}$ is an infinitesimal symmetry of (Y,α). To conclude, notice that

$$p(\Lambda_{exp(tX)} \circ \Lambda_g(e')) = L_{exp(tX)}(p \circ \Lambda_g(e')) = (L_{exp(tX)} \circ L_g)(e) = exp(tX)g$$

which is the flow of X^r. Thefore,

$$Tp\left(\frac{d}{dt}\bigg|_{t=0} \Lambda_{exp(tX)} \circ \Lambda_g(e')\right) = X^r,$$

i.e, the flow of $(X')^{right}$ projects onto the flow of X^r. $\square$

Finally, if we denote by $\mathrm{sym}^{\mathrm{right}}(Y,\alpha)$ the Lie algebra of right invariant vector fields on Y, we conclude that we have the following commutative diagram of Lie algebras

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}[\partial_x] \arrow[r, "Ti"] \arrow[d, equal] & \mathrm{sym}^{\mathrm{right}}(Y,\alpha) \arrow[r, "Tp"] \arrow[d, hook] & \mathfrak{g} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{R}[\partial_x] \arrow[r, "Ti"] & \mathrm{sym}(Y,\alpha) \arrow[r, "Tp"] & \mathscr{H}(G) \arrow[r] & 0
\end{tikzcd}$$

where the first short exact sequence is the Lie algebra sequence associated to the short exact sequence of Lie groups

$$0 \longrightarrow \mathbb{R}/D \xrightarrow{\ i\ } Y \xrightarrow{\ p\ } G \longrightarrow 0$$

which was constructed from the extension (Y,p,α).

2.5 Applications

In this section we provide some applications of what we have done so far. More precisely, assuming the hypothesis of Theorem 2.4.3 we classify inequivalent central extensions of a Lie group G by $\mathbb{R}/D$ and we apply this classification to the cases in which G is taken to be simply connected or semisimple. Moreover, considering the bundle (Y,α) constructed over a symplectic manifold (M,ω), we show how one can lift a hamiltonian action on (M,ω) to an action on (Y,α)

which preserves the connection α; also, we explore how the bundle (Y,α) provides a geometric realization of a well-known central extension appearing in symplectic geometry.

If conditions of Theorem 2.4.3 are fullfilled, we can use Theorem 2.2.12 to classify the inequivalent group extensions constructed following what we have done until now.

Proposition 2.5.1. Let G be a Lie group and $\omega \in H^2_{\mathrm{al}}(\mathfrak{g},\mathbb{R})$. Suppose that

(i) $\mathrm{Per}(\omega) \subset D$;

(ii) there exists a moment map for the left action of G on itself.

Then, the central extensions of G by $\mathbb{R}/D$ are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D)\mathrm{mod}[\Theta_o]_D$, where Θ is the set of all left invariant 1-forms on G.

Proof. We are going to show that (Y_1,p_1,α_1) and (Y_2,p_2,α_2) are $\Theta-$equivalent bundles (in the sense of Definition 2.2.10) if and only if (Y_1,i_1,p_1) and (Y_2,i_2,p_2) are equivalent central extensions of G.

Suppose $\phi : Y_1 \to Y_2$ is a equivalence of extensions. We shall prove that ϕ is a $\Theta-$equivalence, which means that $\alpha_1 - \phi^*\alpha_2$ is the pull back of some left invariant 1-form on G. Notice that it suffices to prove that $\alpha_1 - \phi^*\alpha_2$ vanishes on vectors tangent to the the fibers of Y_1. Consider $Ti_1(\partial_x)$ a tangent vector to the fiber of Y_1. Then, we have

$$\phi^*\alpha_2(Ti_1(\partial_x)) = \alpha_2(T\phi(Ti_1(\partial_x))) = \alpha_2(T(\phi\circ i_1)\partial_x) =$$

$$\alpha_2(T(i_2\circ\phi)\partial_x) = \alpha_2(Ti_2(T\phi(\partial_x)).$$

However, i_1 and i_2 are induced by the action of $\mathbb{R}/D$ on Y_1 and Y_2; also, α_1 and α_2 are dual to the tangent space to the image of $\mathbb{R}/D$ in Y_1 and Y_2. Thus,

$$\alpha_2(Ti_2(T\phi(\partial_x)) = \alpha_2(Ti_2(\partial_x)) = 1$$

$$\alpha_1(Ti_1(\partial_x)) = 1.$$

It follows that $\alpha_1 - \phi^*\alpha_2$ vanishes on each tangent space to the fiber of Y_1. Since α_1 and α_2 are left invariant, it follows that $\alpha_1 - \phi^*\alpha_2$ can be given as the pull-back of some left invariant 1-form on G (it can be written as a linear combination of $p_1^*\beta^1,\ldots,p_1^*\beta^n$).

Let $\phi : Y_1 \to Y_2$ be a $\Theta-$equivalence. Since the action of $\mathbb{R}/D$ on Y_1 gives an $\Theta-$equivalence of Y_1 to itself, we can assume that $\phi(e') = e''$, where e' and e'' are the identities of Y_1 and Y_2, respectively. We shall show that ϕ is a Lie group morphism. Consider the following two diagrams.

$$\begin{tikzcd}
e' \arrow[r, dotted, "\Lambda_g"] \arrow[d, dash] & g' \arrow[r, dotted, "\Lambda_h"] \arrow[d, dash] & (hg)' \arrow[d, dash] \\
e \arrow[r, dotted, "L_g"] & g \arrow[r, dotted, "L_h"] & hg
\end{tikzcd}$$

$$\begin{tikzcd}
e'' \arrow[r, dotted, "\Gamma_g"] \arrow[d, dash] & g'' = \phi(g') \arrow[r, dotted, "\Gamma_h"] \arrow[d, dash] & (hg)'' = \phi((hg)') \arrow[d, dash] \\
e \arrow[r, dotted, "L_g"] & g \arrow[r, dotted, "L_h"] & hg
\end{tikzcd}$$

From these two diagrams we can conclude

$$\phi(h'g') = \phi(\Lambda_h \circ \Lambda_g(e')) = \phi \circ \Lambda_{gh}(e)$$

$$\phi(h')\phi(g') = (\Gamma_h \circ \Gamma_g)(e'') = \Gamma_{hg}(e'').$$

Using the hypothesis that ϕ is a $\Theta-$equivalence (Θ the set of left invariant 1-forms on G), one can see that $\phi^{-1} \circ \Gamma_{hg} \circ \phi$ is MC automorphism. Also, we can see that

$$\phi^{-1} \circ \Gamma_{hg} \circ \phi(e') = \phi^{-1} \circ \Gamma_{hg}(e'') = \phi^{-1}((hg)'') = (hg)',$$

which means that $\phi^{-1} \circ \Gamma_{hg} \circ \phi$ is MC automorphism sending e' to $(hg)'$. Since MC automorphisms do not have fixed points (consequence of the proof of Lemma 2.4.6), it follows that $\phi^{-1} \circ \Gamma_{hg} \circ \phi = \Lambda_{hg}$. Then, we can write

$$\phi(\Lambda_{hg}(e')) = \phi(\Lambda_h \circ \Lambda_g(e')) = \phi(h'g')$$
$$(\Gamma_{hg} \circ \phi)(e'') = \Gamma_{hg}(e'') = (\Gamma_h \circ \Gamma_g)(e'') = h'' \cdot g'' = \phi(h') \cdot \phi(g'),$$

from which we deduce that $\phi(h') \cdot \phi(g') = \phi(h'g')$, i.e, ϕ is a Lie group morphism. Finally, using that i_1 and i_2 are defined by the action of $\mathbb{R}/D$ and that ϕ is equivalence between the bundles Y_1 and Y_2, one can see that ϕ is an equivalence of central extensions (in the sense of Definition 2.1.2). $\square$

To prove the statement of the next proposition we shall need the following result quoted from (ORTEGA; RATIU, 2004).

Theorem 2.5.2. Let G be a simply connected Lie group. Then, both $H^1_{dR}(G)$ and $H^2_{dR}(G)$ vanish.

Proposition 2.5.3. Let G be simply connected Lie group. Then, any cohomology class $[\omega] \in H^2(\mathfrak{g}, \mathbb{R})$ is associated to a class of central extensions of G by $\mathbb{R}/D$. Moreover, there is, up to equivalence, only one such a class and it can be represented by topologically trivial $\mathbb{R}/D-$principal bundle.

Proof. Let $[\omega] \in H^2(\mathfrak{g},\mathbb{R})$ be a cohomology class representing a central extension of $\mathfrak{g}$ by $\mathbb{R}$ and suppose that G is a simply connected Lie group. Firstly, notice that simply connectedness of G guarantees that the cocycle ω satisfies the conditions of Theorem 2.4.3. Indeed, $H^1_{dR}(G) = 0$ implies that $i_{X^r}\omega$ is exact, which means that we have a hamiltonian function associated to X^r, and hence the left action of G on itself admits a comoment map (in the sense of Definition 2.3.6). On the other hand, $H^2_{dR}(G) = 0$ gives $\mathrm{Per}(\omega) = 0 \subset D$. In this manner, we see that any cocycle in $H^2(\mathfrak{g},\mathbb{R})$ defines an algebra central extension that can be integrated. Now, we shall show that the Lie group central extensions obtained in this way are topologically trivial bundles. In fact, since G is simply connected, we have $H^1_{dR}(G) = 0$, which implies that $\check{H}^1(G,\mathbb{R}) = 0$. The natural projection $\pi : \mathbb{R} \to \mathbb{R}/D$ induces a sujerctive map $\check{\pi} : \check{H}^1(G,\mathbb{R}) \to \check{H}^1(G,\mathbb{R}/D)$, and from this we obtain $\check{H}^1(G,\mathbb{R}/D) = 0$. Hence, the classification of the central extensions presented in Theorem 2.5.1 implies that, up to equivalence, there is only one central extension of G by $\mathbb{R}/D$. We claim that it can be represented by a topologically trivial $\mathbb{R}/D-$principal bundle. Indeed, since $H^2_{dR}(G) = 0$, there exists a globally defined 1-form θ such that $d\theta = \omega$. To conclude, we notice that Theorem 2.4.3 assures that the trivial bundle with connection $(G \times \mathbb{R}/D, pr_1^*\theta + dx)$ gives a Lie group central extension associated to that defined by ω.

□

Remark 2.5.4. We notice that the above proposition is a particular case of a theorem presented in (HOCHSCHILD, 1951). There Hochschild shows that any extension of a simply connected group G by any connected Lie group A admits a global smooth section (Theorem 3.1 in (HOCHSCHILD, 1951)).

Definition 2.5.5. A Lie group G is said to be semisimple if its Lie algebra $\mathfrak{g}$ is semisimple.

Proposition 2.5.6. Let G be a semisimple Lie group. Then, the central extensions of G by $\mathbb{R}/D$ are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D)$.

Proof. Suppose that G is a semisimple Lie group with semisimple Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, both $H^1(\mathfrak{g},\mathbb{R})$ and $H^2(\mathfrak{g},\mathbb{R})$ vanish (GUILLEMIN; STERNBERG, 1984). Recall that $H^2(\mathfrak{g},\mathbb{R})$ classify all central extensions of $\mathfrak{g}$ by $\mathbb{R}$. Then $H^2(\mathfrak{g},\mathbb{R}) = 0$ implies that there is, up to equivalence, only one central extension of $\mathfrak{g}$ by $\mathbb{R}$. By Proposition 2.5.1, the Lie group extensions integrating these algebra central extensions are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D)\mathrm{mod}[\Theta_o]_D$, where Θ_o is the vector space of left invariant 1-forms. However, since $H^1(\mathfrak{g},\mathbb{R}) = 0$, we conclude that $\Theta_o = 0$ which implies that $[\Theta_o]_D = 0$. Hence, the Lie group central extensions of G by $\mathbb{R}$ are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D)$. □

Proposition 2.5.7. Let G be a simply connected semisimple Lie group. Then, the only central extension of G by $\mathbb{R}/D$ is the trivial extension (direct product of G by $\mathbb{R}/D$).

Proof. From the last two proposition we conclude that the Lie group central extensions of G by $\mathbb{R}$ are classified by $\check{H}^1(\mathscr{U},\mathbb{R}/D) = 0$. Then, up to equivalence, there is only one such a extension.

Certainly, the trivial extension, i.e, the direct product of G by $\mathbb{R}/D$, can be taken as representative of this class of extensions. $\square$

Let (M,ω) be a symplectic manifold, and let $\psi : G\times M \to M$ be a hamiltonian action. The following result shows that, if $\mathrm{Per}(\omega) \subset D \subset \mathbb{R}$ (D a discrete subgroup), then a central extension of G by $\mathbb{R}/D$ can be taken as symmetry group of the pair (Y,α), where (Y,p,α) is an extension of (M,ω). Moreover, the action of the central extension on Y projects onto the action of G on M, i.e, it provides a lift of the symmetries given by the action of G on M.

Theorem 2.5.8. Let (M,ω) be a symplectic manifold in which a connected Lie group G acts by simplectormorphisms. Suppose that $\mathrm{Per}(\omega)$ is discrete and the action of G is hamiltonian (i.e, it admits a moment map μ). Then, for any discrete subgroup $D\subset\mathbb{R}$ such that $\mathrm{Per}(\omega)\subset D$, we have the following:

(i) There exists an extension (Y,p,α) of (M,ω);

(ii) There exists a central extension G' of G by $\mathbb{R}/D$ acting as symmetry group of (Y,α);

(iii) The action of G' on Y projects onto the action of G on M and the $\ker(G'\to G)$ acts as the structure group $\mathbb{R}/D$.

Proof. Let $\phi : G\times M \to M$ denote the action of G on M, and let $m\in M$. Define an evaluation map $E_m : G\to M$ by $E_m(g) = \phi(g,m) = \phi_g(m)$. Now, let us consider the 2-form $\omega' := E_m^*\omega$. We claim that ω' is a left invariant 2-form on G. For, let $h\in G$, we must verify that $L_h^*\omega' = \omega'$, so we calculate

$$[L_h^*(\omega')(X,Y)]_g = \omega'_{hg}(\,T_gL_h(X),\, T_gL_h(Y)\,) =$$
$$(E_m^*\omega)_{hg}(\,T_gL_h(X),\, T_gL_h(Y)\,) = \omega_{\phi_{hg}(m)}(\,T_{hg}E_m(\,T_gL_h(X)\,),\, T_{hg}E_m(\,T_gL_h(Y)\,)\,) =$$
$$\omega_{\phi_{hg}(m)}(\,T_g(E_m\circ L_h)(X),\, T_g(E_m\circ L_h)(Y)\,).$$

Notice that

$$T_g(E_m\circ L_h)(X) = \left.\frac{d}{dt}\right|_{t=0}(E_m\circ L_h)(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0}(E_m)(h\gamma(t)) =$$
$$\left.\frac{d}{dt}\right|_{t=0}\phi_{h\gamma(t)}(m) = \left.\frac{d}{dt}\right|_{t=0}\phi_h(\phi_{\gamma(t)}(m)) =$$
$$(T_{\phi_g(m)}\phi_h)\left(\left.\frac{d}{dt}\right|_{t=0}\phi_{\gamma(t)}(m)\right) = (T_{\phi_g(m)}\phi_h)(T_gE_m(X)).$$

Since ϕ_g is a symplectormorphism for each $g\in G$, we can write

$$\omega_{\phi_{hg}(m)}(\,T_g(E_m\circ L_h)(X),\, T_g(E_m\circ L_h)(Y)\,) =$$
$$\omega_{\phi_h(\phi_g(m))}(\,(T_{\phi_g(m)}\phi_h)T_gE_m(X),\, (T_{\phi_g(m)}\phi_h)T_gE_m(Y)\,) =$$

$$(\phi_h^*\omega)_{\phi_g(m)}(\,T_gE_m(X),T_gE_m(Y)\,)=\omega_{\phi_g(m)}(\,T_gE_m(X),T_gE_m(Y)\,)=$$
$$(E_m^*\omega)_g(X,Y)=\omega'_g(X,Y),$$

which is

$$[L_h^*(\omega')(X,Y)]_g=\omega'_g(X,Y).$$

Therefore, we can consider G endowed with the left invariant form ω'. Now, let μ denote the moment map for the action of G on M and consider the map $\mu':\mathfrak{g}\to C^\infty(G)$ defined by $\mu'_X:=E_m^*\mu_X$. We claim that μ' is a moment map for the left action of G on itself. To show this we must verify that μ' associates to each fundamental (right invariant) vector field a hamiltonian function. Let X^r be a right invariant vector field. Using the fact that μ is moment map, we can calculate

$$[i_{X^r}\,\omega']_g=\omega_g(X^r,\cdot\,)=(E_m^*\omega)(X^r,\cdot\,)=\omega_{\phi_g(m)}(T_gE_mX^r,T_gE_m(\,\cdot\,))=$$
$$\omega_{\phi_g(m)}(X^r_{gm},\,T_gE_m(\,\cdot\,)\,)=-(d\mu_X)_{\phi_g(m)}(T_gE_m(\,\cdot\,))=-[E_m^*(d\mu_X)]_g(\,\cdot\,)=$$
$$-[d(E_m^*\mu_X)]_g(\,\cdot\,),$$

which is

$$[\,i_{X^r}\,\omega'+d(E_m^*\mu_X)\,]_g=0.$$

Thus, we have shown that $E_m^*\mu_X$ is a hamiltonian function for X^r.

Suppose that $D\subset\mathbb{R}$ is such that $\mathrm{Per}(\omega)\subset D$. It follows from Proposition 2.2.6 that we have (Y,p,α) an extension of (M,ω). Define a $\mathbb{R}/D-$principal bundle G' over G as the pull-back of Y by E_m, i.e, $G':=\{(g,y)\in G\times Y\mid E_m(g)=p(y)\}$. Let pr_1 and pr_2 denote the projections on the first and second factor of G', respectively. Then, $\alpha':=pr_2^*(\alpha)$ is a connection 1-form on G'. We claim that $d\alpha'=pr_1^*\omega'$. Indeed, we have

$$pr_1^*(\omega')=pr_1^*(E_m^*\omega)=(E_m\circ pr_1)^*\omega.$$

On the other hand,

$$d\alpha'=d(pr_2^*\alpha)=pr_2^*(d\alpha)=pr_2^*(p^*\omega)=(p\circ pr_2)^*\omega.$$

It is easy to see that $(p\circ pr_2)(g,y)=(E_m\circ pr_1)(g,y)$, for every $(g,y)\in G'$; then, we conclude that

$$pr_1^*(\omega')=(E_m\circ pr_1)^*\omega=(p\circ pr_2)^*\omega=d\alpha'.$$

Thus, we have shown that (G',pr_1,α') is an extension of the pair (G,ω'). Notice that it follows from Proposition 2.2.8 that $\mathrm{Per}(\omega')\subset D$. Moreover, we have already shown that the left action of G on itself admits a moment map. Therefore, from Theorem 2.4.3, we obtain that the extension (G',pr_1,α') provides a central extension of G by $\mathbb{R}/D$ associated to the 2-cocyle ω'.

It remains to verify that G' can act as a symmetry group of (Y,α). Recall that elements of G' can be obtained by the lift of right invariant vector fields on G and by means of the moment map. More precisely, following Lemma 2.4.10, we see that given a right invariant vector field X^r, we can construct X' which is uniquely determined by X^r and the moment map μ'. The flow of X' was denoted by $\Lambda_{exp(-tX)}$. If $(e,y_0) \in G'$ is a point over the fiber of $e \in G$, then we can consider the element $\Lambda_{exp(-tX)}((e,y_0))$ of G'. Also, recall that using the fundamental vector field X_M and the moment map μ we can obtain a vector field X_M^μ on Y whose flow projects onto the flow of X_M. Then, we define the action of $\Lambda_{exp(-tX)}((e,y_0))$ on Y by means of the flow of X_M^μ, i.e,

$$(\Lambda_{exp(-tX)}(e,y_0),y) \longmapsto \mathrm{Fl}_t^{X_M^\mu}(y)$$

The map $i : \mathbb{R}/D \to G'$ is defined by the action of $\mathbb{R}/D$ on G', which is simply the action $\varphi : \mathbb{R}/D \times Y \to Y$ of $\mathbb{R}/D$ on Y. Hence, any element in $\ker(i) \subset G'$ is of the form $(e,\varphi_d(y_0))$ and we define the action of this element on Y by

$$((e,\varphi_d(y_0)),y) \longmapsto \varphi_d(y).$$

It also follows from Lemma 2.4.10 that any element of G' can be reached from e' by successively following flows of the type $\Lambda_{exp(tX)}$ (and some translation along the fibers). Then, we have defined how G' can act on Y and it is clear that this action is a symmetry group of (Y,α). $\square$

In the following remark we briefly present a setting in which a result analogous to Theorem 2.5.8 appear.

Remark 2.5.9. Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be Hilbert space. In context of quantum mechanics the projective space $\mathbb{P}\mathscr{H}$ represents the space of states of a given system. On $\mathbb{P}\mathscr{H}$ one can consider the function $P : \mathbb{P}\mathscr{H} \times \mathbb{P}\mathscr{H} \to \mathbb{R}_{\geq 0}$ definied by

$$P(\pi x,\pi y) = \frac{\langle x,y\rangle\langle y,x\rangle}{\langle x,x\rangle\langle y,y\rangle},$$

where $\pi : \mathscr{H} \setminus \{0\} \to \mathbb{P}\mathscr{H}$ denotes the projection associated to the construction of $\mathbb{P}\mathscr{H}$. This function P may be seen as the probability transition between two states. A bijection $g : \mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H}$ is said to be a symmetry of the pair $(\mathbb{P}\mathscr{H},P)$ if it preserves P. The Wigner's Theorem (Theorem 2.1 in (TUYNMAN; W.W.A.J.WIEGERINCK, 1987)) assures that any symmetry g of the pair $(\mathbb{P}\mathscr{H},P)$ can be lifted to an unitary or anti-unitary operator $U(g)$ defined on $\mathscr{H}$, which is unique up to a phase factor $e^{i\theta_g}$. In this manner, if a connected Lie group G is the symmetry group of the pair $(\mathbb{P}\mathscr{H},P)$, then one obtains that each $g \in G$ is associated to an unitary operator $U(g)$ on $\mathscr{H}$. The factor $e^{i\theta_g}$, which appears as the freedom to the choice of the lift $U(g)$, is the obstruction for this construction to give a unitary representation of G on $\mathscr{H}$. However, one can define a function $\varphi : G \times G \to U(1)$ by

$$U(gh) = U(g)U(h)\varphi(g,h),$$

since $U(gh)$ and $U(g)U(h)$ are both lifts associated to gh. This function φ defines a cohomology class in the group cohomology of G with coefficients in $U(1)$. Hence, φ defines a central extension of G by $U(1)$, $G' := G \times_\varphi U(1)$, to which we can associate a unitary representaion on $\mathscr{H}$. If G' acts unitarily on $\mathscr{H}$ then, in particular, it preserves the unitary states $S\mathscr{H} = \{x \in \mathscr{H} \mid \langle x,x\rangle = 1\}$. It is simple to see that $S\mathscr{H}$ defines a $U(1)-$bundle $\pi : S\mathscr{H} \to \mathbb{P}\mathscr{H}$ over $\mathbb{P}\mathscr{H}$. In short, we have the following situation: if G acts as the symmetry group of the pair $(\mathbb{P}\mathscr{H},P)$ then one obtains a central extension of G by $U(1)$, $G\times_\varphi U(1)$, which acts as the symmetry group of the pair $(S\mathscr{H},\langle\cdot,\cdot\rangle)$, where $\pi : S\mathscr{H} \to \mathbb{P}\mathscr{H}$ is a $U(1)-$bundle over $\mathbb{P}\mathscr{H}$; besides, it can be shown that the action of $G\times_\varphi U(1)$ on $S\mathscr{H}$ projects onto the action of G on $\mathbb{P}\mathscr{H}$.

Finally, let (M,ω) be symplectic manifold, then we have the following short exact sequence of Lie algebras (see Appendix A)

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\zeta} \mathscr{H}(M) \longrightarrow 0.$$

Suppose that the 2-form ω satisfies the condition $\mathrm{Per}(\omega) \subset D$ for some D a discrete subgroup of $\mathbb{R}$. Then, we have already seen that there exists a $\mathbb{R}/D-$principal bundle $p : Y \to M$ with a connection α satisfying $d\alpha = p^*\omega$ (Proposition 2.2.6). Using this bundle we can represent the Lie algebra $C^\infty(M)$ as infinitesimal symmetries of the pair (Y,α). In fact, to each function f, let X_f be the associated hamiltonian vector field. Then, due to Proposition 2.3.1, one can construct a unique lift of X_f to an infinitesimal symmetry of (Y,α) that satisfies $\alpha(X'_f) = p^*f$ and $\mathrm{T}p(X'_f) = X_f$ Hence, we can define a map

$$C^\infty(M) \longrightarrow \mathrm{sym}(Y,\alpha)$$

$$f \longmapsto X'_f.$$

A simple calculation shows this is an injective Lie algebra morphism. In short, the following is a commutative diagram of Lie algebras

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & C^\infty(M) \arrow[r] \arrow[d, hook] & \mathscr{H}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{R}[\partial_x] \arrow[r] & \mathrm{sym}(Y,\alpha) \arrow[r] & \mathscr{H}(M) \arrow[r] & 0.
\end{tikzcd}$$

2.6 Another proof for the converse of Theorem 2.4.3

The proof for the converse of Theorem 2.4.3 that we have already presented depends heavily on Lemma 2.4.6. However, as we pointed out in Remark 2.4.7, part of the proof of Lemma 2.4.6 remains unclear. Then, in this subsection, we intend to present a different proof for the converse of Theorem 2.4.3, which is based on a result about the integrability of Lie algebra action[5]. We thank Prof. G. Tuynman for indicating this alternative way out.

We start by defining how a Lie algebra $\mathfrak{g}$ can be taken to act on a manifold M.

Definition 2.6.1. Let $\mathfrak{g}$ be a Lie algebra, and let M be a manifold. An action of $\mathfrak{g}$ on M is a Lie algebra morphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of vector fields on M. Moreover, we say that the action is complete if ζ_X is a complete vector field on M for all $X \in \mathfrak{g}$.

We notice that the condition (ii) appearing in Definition 2.4.4 is dual (and equivalent) to the condition used to define a Lie algebra action.

The following result (Theorem III, Ch. 4, in (PALAIS, 1957)) is the base of the proof we shall present here.

Theorem 2.6.2. Let $\mathfrak{g}$ be a Lie algebra, let M be a manifold, and let $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ be a complete action of $\mathfrak{g}$ on M. Then, there exists a unique smooth action $\psi : G \times M \to M$ of the simply connected group G associated to the Lie algebra $\mathfrak{g}$ such that the associated fundamental vector fields are definied by the morphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$.

In the following we sketch the alternative proof that we have mentioned. Suppose $\omega \in H^2(\mathfrak{g}, \mathbb{R})$ is a 2-cocyle satisfyng the conditions of Theorem 2.4.3, and let (Y, p, α) be an extension constructed from the initial data (G, ω). The 1-forms $\alpha, p^*\beta^1, \ldots p^*\beta^n$ from Lemma 2.4.8 provide $n+1$ complete vector fields. Indeed, recall that on Lemma 2.4.10 we constructed a vector field by lifting a right invariant vector field on G. Then, applying this construction to the (right invariant) vector fields which are dual to β^i, one obtains n complete vector $(X^1)', \ldots, (X^n)'$ fields that are dual to the 1-forms $p^*\beta^1, \ldots, p^*\beta^n$. The remaining one is taken to be the fundamental vector field ∂_x associated to the action of $\mathbb{R}/D$ on Y, which will be dual to the 1-form α. Now, recall from Lemma 2.4.8 that the 1-forms $\alpha, p^*\beta^1, \ldots p^*\beta^n$ satisfy condition (ii) of Definition 2.4.4 with the structural constants being those associated to the central extension defined by ω, i.e, $\mathfrak{h} := \mathfrak{g} \oplus_\omega \mathbb{R}$. As we have already noticed, due to those equations one can define a complete Lie algebra action $\zeta : \mathfrak{h} \to \mathfrak{X}(Y)$ of $\mathfrak{h}$ on Y whose fundamental vector fields are $\partial_x, (X^1)', \ldots, (X^n)'$. By Theorem 2.6.2, if H is a simply connected Lie group with Lie algebra $\mathfrak{h}$, then the Lie algebra action ζ can be integrated to an action $\psi : H \times Y \to Y$ of H on Y. From the fact that the fundamental vector fields related to this action are exactly those given by the action $\zeta : \mathfrak{h} \to \mathfrak{X}(Y)$ (and from the connectedness of G), one can conclude that the action $\psi : H \times Y \to Y$ is transitive. Hence, Y can be seen as a $H-$homogenous space. To conclude we need the following two results.

Proposition 2.6.3. Let G be a Lie group, and let M be a $G-$homogeneous space. For any point $m \in M$, let G_m denote its isotropy group (i.e, the group of those elements of G that let m fixed). Then, the group G_m is closed and the map

$$F : G/G_m \longrightarrow M$$

[5] The idea behind this proof was suggested to the author by Prof. G. Tuynmam.

$$g \cdot G_m \longmapsto g \cdot m$$

is an equivariant diffeomorphism.

Proposition 2.6.4. Let G be a Lie group, and let M be a $G-$homogeneous space. For any point $m \in M$, let G_m denote its isotropy group. Then, the Lie algebra of G_m is $\mathfrak{g}_m := \{X \in \mathfrak{g} \mid X_M(m) = 0\}$, where X_M denotes the fundamental vector field generated by X through the action of G on M.

Then, to conclude, let $y \in Y$ such that $p(y) = e$, where e denotes the identity of G. From Proposition 2.6.4, it follows that $\mathfrak{h}_y := \{X \in \mathfrak{h} \mid X_M(y) = 0\}$; however, in the present case, the fundamental vector fields are specified by the morphism $\zeta : \mathfrak{h} \to \mathfrak{X}(M)$, which implies that $\mathfrak{h}_y = 0$. Moreover, from Proposition 2.6.3, one concludes that there is an equivariant diffeomorphism from H/H_y to Y, which already shows that Y can be seen as Lie group (if H_y is a normal subgroup). Finally, notice that the Lie algebra of H/H_y is isomorphic to $\mathfrak{h}/\mathfrak{h}_y$, which is actually $\mathfrak{h}$ since $\mathfrak{h}_y = 0$. Thereby, we have shown that Y can be seen as Lie group with Lie algebra $\mathfrak{h}$.

CHAPTER

3

LIE ALGEBROIDS, LIE GROUPOIDS, AND INTEGRABILITY

In this chapter we shall explore the connections of what we have done so far with the theory of Lie algebroids and Lie groupoids. More precisely, given a manifold M endowed with a closed 2-form ω, we construct a Lie algebroid $A_\omega \to M$, and we prove that the existence of an extension [1] (P, α, π) of (M, ω) is a sufficient condition for the integrability of A_ω. Also, when (M, ω) is a symplectic manifold, we shall see that there exists a Lie subalgebra of the Lie algebra of section $\Gamma(A_\omega)$ which is isomorphic to the underlying Lie algebra of the Poisson algebra $C^\infty(M)$. Furthermore, in the presence of an extension (P, α, π) of (M, ω), we use this to provide another way of constructing the representation of the Poisson algebra $C^\infty(M)$ as (right) invariant vector fields on P preserving the connection α.

This chapter was conceived based on ideas from (ROGERS, 2013), (CRAINIC, 2004).

3.1 Lie algebroids and Lie groupoids

In this section, we introduce the concepts of Lie algebroids and Lie groupoids and provide basic examples. Also, we explain how Lie algebroids are the infinitesimal counterpart of Lie groupoids.

We start by introducing the Lie algebroids. Roughly speaking, a Lie algebroid is a vector bundle with a Lie bracket of on its space of sections. The interesting feature of this is that we may think of Lie algebroids as providing Lie algebras of "geometric type". Formally, we have the following definition.

Definition 3.1.1. Let M be a smooth manifold. A Lie algebroid over M is a triple $(C, \rho, [\cdot,\cdot])$,

[1] Recall that by an extension (P, α, π) of (M, ω) we mean a $\mathbb{R}/D-$principal bundle $\pi : P \to M$ with a connection form α such that $d\alpha = \pi^*\omega$.

where C is a vector bundle over M, $\rho : C \to TM$ is a vector bundle morphism, and $[\cdot,\cdot]_C : \Gamma(C) \times \Gamma(C) \to \Gamma(C)$ is a Lie bracket on the space of sections $\Gamma(C)$ such that, for $\alpha, \beta \in \Gamma(C)$ and $f \in C^\infty(M)$,

1. $[\alpha, f\beta]_C = f[\alpha,\beta]_C + (\mathscr{L}_{\rho(\alpha)}f)\beta$ (Leibniz rule);
2. $[\rho(\alpha), \rho(\beta)] = \rho([\alpha,\beta]_C)$, where $[\cdot,\cdot]$ denotes the usual bracket of vector fields (ρ induces a Lie algebra morphism).

The vector bundle morphism $\rho : C \to TM$ is referred to as the anchor.

The tangent bundle TM of a manifold M provides the simplest example of a Lie algebroid. Indeed, we take the anchor $\rho : TM \to TM$ to be the identity map, and since sections of TM are vector fields, we take the bracket on the space of sections to be the usual bracket of vector fields. It is easy to see that these fulfill the conditions of the above definition. A non-trivial example of Lie algebroid appears in the presence of a Poisson structure on M, i.e, $\pi \in \bigwedge^2(M)$ such that $[\pi,\pi]_{SN} = 0$ or, equivalently, a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$. Indeed, let T^*M be the cotangent bundle of M. Using the Poisson structure π, we define $\rho : T^*M \to TM$ by $\rho(\alpha_x) = \pi(\alpha_x, \cdot)$, for $\alpha_x \in T_x^*M$, while for the bracket $[\cdot,\cdot] : \Gamma(T^*M) \times \Gamma(T^*M) \to \Gamma(T^*M)$, we put

$$[\alpha,\beta]_{T^*M} = \mathscr{L}_{\rho(\alpha)}(\beta) - \mathscr{L}_{\rho(\beta)}(\alpha) - d(\pi(\alpha,\beta)).$$

This bracket is clearly bilinear and antisymmetric, and a direct computation shows that it satisfies the Leibniz rule. However, to check the Jacobi identity for $[\cdot,\cdot]_{T^*M}$ one uses explicitly that $[\pi,\pi]_{SN} = 0$ (see (VAISMAN, 1994)).

Now, we pass to the Lie groupoids which appear as the objects whose infinitesimal counterpart are the Lie algebroids. As we shall see, each Lie groupoid has associated to it a Lie algebroid, which is constructed essentially in the same way as the Lie algebra of a Lie group.

Definition 3.1.2. Let M and $\mathscr{G}$ be smooth manifolds. Then, $\mathscr{G}$ is said to be a Lie groupoid over M if it comes with the following data: two surjective submersions $s,t : \mathscr{G} \to M$ called source map and target map, respectively; a smooth multiplication $m : \mathscr{G}_2 \to M$, where $\mathscr{G}_2 = \{(g,h) \in \mathscr{G} \mid s(g) = t(h)\}$, and a smooth embedding $u : M \to \mathscr{G}$ called unit map; moreover, these maps must satisfy

- $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3))$ (associativity of the multiplication);
- $u(t(g)) \circ g = g = g \circ u(s(g))$;
- for all $g \in \mathscr{G}$ there exists $h \in \mathscr{G}$ such that $s(h) = t(g)$, $t(h) = s(g)$, and $g \circ h$, $h \circ g$ are units,

where, for $g,h \in \mathscr{G}$, $g \circ h := m(g,h)$. The spaces $\mathscr{G}$ and M are callled the space of arrows and the space of objects, respectively. Usually, the Lie groupoid $\mathscr{G}$ over M is denoted by $\mathscr{G} \rightrightarrows M$.

In the following, we list some examples of Lie groupoids.

- A Lie group G is a Lie groupoid over a point, $G \rightrightarrows \{\star\}$.

- Let M be manifold. Then, one can consider $M \times M$ as a Lie groupoid. In fact, the source and target are taken to be projections on the second and first cooordinate, respectively, while the multiplication is given by $(x,y) \circ (y,z) = (x,z)$, and the unit map given by $x \mapsto (x,x)$. The groupoid $M \times M \rightrightarrows M$ is called pair groupoid.

- Let M be a manifold, and G a Lie group. Then, the trivial bundle $M \times G$ can be seen as Lie groupoid. Indeed, source and target are taken to be both projection on the first coordinate, the multiplication given by $(x,g)(x,h) = (x,gh)$, and the unit map given by $x \mapsto (x,e)$, where $e \in G$ is the identity of G.

- Let $\pi : P \to M$ be a $G-$principal bundle over M, and let $\psi : G \times (P \times P) \to P \times P$ be the diagonal action of G on $P \times P$, i.e, $\psi(g,(u,v)) = (gu, gv)$. Then, one can turn $\frac{P \times P}{G}$ into a Lie groupoid over M. In fact, let the source $s : \frac{P \times P}{G} \to M$ be defined by $s([(u,v)]) = \pi(v)$, and let the target $t : \frac{P \times P}{G} \to M$ be defined by $t([(u,v)]) = \pi(u)$; for the multiplication, we put $[(u_1,u_2)] \circ [(u_3,u_4)] = [(u_1, gu_4)]$, where $g \in G$ sends u_2 to u_3 (u_2 and u_3 are both over the same fiber), and the unit map $u : \frac{P \times P}{G} \to M$ we define by $x \mapsto [(u,u)]$, for any $u \in \pi^{-1}(x)$. The groupoid $\frac{P \times P}{G} \rightrightarrows M$ is called the gauge groupoid of P. Equivalently, we may describe the gauge groupoid as the set of triples (m', m, ϕ) such that $m, m' \in M$ and $\phi : \pi^{-1}(m) \to \pi^{-1}(m')$ is an $G-$equivariant map between the fibers over $m, m' \in M$. We set $s(m', m, \phi) := m$, $t(m', m, \phi) := m'$, $u(m) := (m, m, id)$, and we define the multiplication by
$$(m_1', m_1, \phi_1) \circ (m_2', m_2, \phi_2) = (m_1', m_2, \phi_1 \circ \phi_2),$$
whenever $m_1 = m_2'$. Let $\mathscr{G}(P) \rightrightarrows M$ denote the groupoid defined in this way. Given (m', m, ϕ), it is easy to see that by choosing $u \in \pi^{-1}(m)$ and taking the pair $[(\phi(u), u)] \in \frac{P \times P}{G}$, we obtain a groupoid isomorphism
$$\begin{tikzcd} \mathscr{G}(P) \arrow[r, "\Psi"] \arrow[d, shift left] \arrow[d, shift right] & \frac{P \times P}{G} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r] & M \end{tikzcd}$$

The gauge groupoids have an interesting relation with a particular class of Lie groupoids, which are the transitive groupoids. Before introducing them, let us make the following definition.

Definition 3.1.3. Let $\mathscr{G} \rightrightarrows M$ be a groupoid, and let $m \in M$. The orbit of $m \in M$ is $\mathscr{G} \cdot m := t(s^{-1}(m))$, and the isotropy group at $m \in M$ is $\mathscr{G}_m := s^{-1}(m) \cap t^{-1}(m)$

A Lie groupoid $\mathscr{G} \rightrightarrows M$ is said to be transitive if the orbit of any $m \in M$ is the whole M, i.e, $\mathscr{G} \cdot m = M$. Essentially, it says that given any two point $m, m' \in M$, there exists $g \in \mathscr{G}$ such that $s(g) = m$, $t(g) = m'$.

Using the concept of local bisection, it is possible to prove that the isotropy group $\mathscr{G}_m \subset \mathscr{G}$, $m \in M$, is an embedded submanifold. In this way, by restricting the multiplication of $\mathscr{G}$ to $\mathscr{G}_m$, one sees that $\mathscr{G}_m$ is a Lie group (see Proposition 3.7 in (MEINRENKEN, 2007)).

The next result states the aforementioned relation between transitive Lie groupoids and gauge groupoids.

Proposition 3.1.4. Let $\mathscr{G} \rightrightarrows M$ be a transitive Lie groupoid. Then, there exists a $G-$principal bundle $P \to M$ such that $\mathscr{G}$ is isomorphic to $\mathscr{G}(P)$.

Proof. Suppose that $\mathscr{G} \rightrightarrows M$ is a transitive Lie groupoid. Let $m_0 \in M$, and let $G := \mathscr{G}_{m_0}$ be the isotropy group at m_0. Also, let $P := s^{-1}(m_0)$. Since $\mathscr{G} \rightrightarrows M$ is transitive, $\pi := t|_{s^{-1}(m_0)} : P \to M$ is a surjective submersion. One can define an action of G on P by setting $g \cdot p := p \circ g^{-1}$, for $g \in G$, $p \in P$. It is easy to see that the projection π does not feel the action of G. Moreover, this action is free and transitive. Indeed, if $g \cdot p = p$, for $g \in G$ and $p \in P$, we have $p = g \cdot p = p \circ g^{-1}$, and then $g^{-1} = p^{-1} \circ p = u(m_0)$, which shows that $g \in G$ is the identity. Therefore, the action is free. On the other hand, let $p, q \in P$ such that $t(p) = t(q)$. Then, the element $(q^{-1} \circ p) \in G$, and $(q^{-1} \circ p) \cdot p = q$, which shows that the action is transitive. In this manner, we have shown that $\pi : P \to M$ is a $G-$principal bundle over M. In the following, we prove that $\mathscr{G}$ may be identified with $\mathscr{G}(P)$. In fact, let $\phi \in \mathscr{G}$. Notice that left multiplication by ϕ gives a $G-$equivariant map

$$L_\phi : \pi^{-1}(s(\phi)) \to \pi^{-1}(t(\phi)),$$

so that we can define the map

$$F : \mathscr{G} \longrightarrow \mathscr{G}(P)$$

by $F(\phi) := (\pi^{-1}(t(\phi)), \pi^{-1}(s(\phi)), L_\phi)$. Conversely, let $(x, y, \psi) \in \mathscr{G}(P)$. By choosing any $p \in \pi^{-1}(y)$ and using that ψ is a $G-$equivariant morphism, it follows that ψ is the morphism given by the left multiplication by $\psi(p) \circ p^{-1}$, i.e, $\psi = L_{(\psi(p) \circ p^{-1})}$. Moreover, this is independent of the choice of $p \in \pi^{-1}(y)$. This shows that $F : \mathscr{G} \to \mathscr{G}(P)$ has an inverse. It is straightfoward to verify that $F : \mathscr{G} \to \mathscr{G}(P)$ and its inverse are groupoid morphism. Therefore, we have shown that $\mathscr{G}$ is isomorphic to $\mathscr{G}(P)$. $\square$

In the following, we introduce the Lie algebroid of a Lie groupoid. As we have already said, it will be constructed essentially in the same way as the Lie algebra of a Lie group. Indeed, let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid, and let $g \in \mathscr{G}$. Notice that left multiplication by g induces a diffeomorphism

$$L_g : t^{-1}(s(g)) \to t^{-1}(t(g)).$$

Therefore, if we wish to define the notion of left invariant vector field, then these must be necessarily tangent to submanifold of the type $t^{-1}(x)$, $x \in M$. Since $t : \mathscr{G} \to M$ is a surjective submersion, every $x \in M$ is a regular value of t. And it is well-known that the tangent space at a point in a regular level set of a map is exactly the kernel of the differential of this map. Moreover, the fact that t is submersion implies that it has constant rank, then $\ker(Tt)$, given by $(\ker(\mathrm{T}t))_g = \ker(T_g t)$, $g \in \mathscr{G}$, is vector subbundle of $T\mathscr{G}$ (see section 6.6 in (MICHOR PETER; KOLAR, 1993)). Given all this, we make the following definition.

Definition 3.1.5. Let $\mathscr{G} \rightrightarrows M$ be Lie a groupoid. A vector field X on $\mathscr{G}$ is said to left invariant if it takes its values in the subbundle $\ker(Tt) \subset T\mathscr{G}$ and, for any g and $h \in t^{-1}(s(g))$,

$$(T_h L_g)X = X(L_g(h)).$$

The left invariant vector fields constitutes a Lie algebra, which we denote by $\mathfrak{X}_{inv}(\mathscr{G})$.

In analogy with the case of Lie groups, the Lie algebroid of $\mathscr{G} \rightrightarrows M$ will be a vector bundle whose sections are in one-to-one correspondence with left invariant vector fields on $\mathscr{G}$. Indeed, let us denote by $A(\mathscr{G})$ the restriction of $\ker(Tt)$ to M via the map $u : M \to \mathscr{G}$. Hence, the fiber $A(\mathscr{G})_x$, for $x \in M$, is the tangent space at $u(x)$ of $t^{-1}(x)$ (in another words, $A(\mathscr{G})$ is the pullback of $\ker(Tt)$ through the map $u : M \to \mathscr{G}$). Therefore, to take a section of $A(\mathscr{G})$ amounts to choose vector fields tangent to $t^{-1}(x)$ at $u(x)$, for $x \in M$. It easy to see that each section of $A(\mathscr{G})$ generates an unique left invariant vector field, while each left invariant vector field gives a section of $A(\mathscr{G})$ by restriction; also, these procedures are clearly inverse to each other. In this manner, we obtain a one-to-one correspondece between $\Gamma(A(\mathscr{G}))$ and $\mathfrak{X}_{inv}(G)$. Naturally, this correspondence introduces a Lie bracket $[\cdot,\cdot]_{A(\mathscr{G})}$ on the space of sections $\Gamma(A(\mathscr{G}))$. Now, the only thing left to turn $A(\mathscr{G})$ into a Lie algebroid is an anchor $\rho : A(\mathscr{G}) \to TM$, which may be taken to be $\rho := (Ts)|_{A(\mathscr{G})}$, where $s : \mathscr{G} \rightrightarrows M$ is the source map. Therefore, the triple $(A(\mathscr{G}), [\cdot,\cdot], (Ts)|_{A(\mathscr{G})})$ is a Lie algebroid, which is referred to as the Lie algebroid of the Lie groupoid $\mathscr{G} \rightrightarrows M$.

To conclude this section, let us mention that a Lie algebroid $(A, [\![\cdot,\cdot]\!], \rho)$ is said to integrable if it is isomorphic to the Lie algebroid of a Lie groupoid.

3.2 The Lie algebroid A_ω

In this section, considering a manifold M endowed with a closed 2-form ω, we constructed a Lie algebroid A_ω over M. When (M, ω) is a symplectic manifold, we prove that the Poisson algebra $C^\infty(M)$ is isomorphic to a certain Lie subalgebra of the Lie algebra of sections $\Gamma(A_\omega)$.

Let M be a manifold, and let $M \times \mathbb{R}$ be the trivial line bundle. Notice that smooth sections of $M \times \mathbb{R}$ may be identified with smooth funtions on M, i.e, $\Gamma(M \times \mathbb{R}) = C^\infty(M)$. Also, $M \times \mathbb{R}$ can be seen a trivial Lie algebroid if we let both the bracket and the anchor to be identically zero.

Now, let TM be the tangent bundle of M, and let

$$TM \oplus \mathbb{R} := TM \oplus (M \times \mathbb{R})$$

Notice that sections of $TM \oplus \mathbb{R}$ may be identified with $\mathfrak{X}(M) \oplus C^\infty(M)$. Given a 2-form ω on M, we may introduce a bracket on the space of sections $\mathfrak{X}(M) \oplus C^\infty(M)$. Indeed, let $(X,f),(Y,g) \in \mathfrak{X}(M) \oplus C^\infty(M)$, then we define

$$[\![(X,f),(Y,g)]\!] := \big([X,Y], Xg - Yf - \omega(X,Y)\big).$$

This bracket is clearly bilinear and antysimmetric. And using the formula

$$\begin{aligned} d\omega(X,Y,Z) = X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) - \omega([X,Y],Z) \\ + \omega([X,Z],Y) - \omega([Y,Z],X), \end{aligned}$$

it is easy to see that $[\![\cdot,\cdot]\!]$ satisfies Jacobi identity if and only if ω is closed, i.e, $d\omega = 0$. Therefore, when ω is closed, $[\![\cdot,\cdot]\!]$ is a Lie bracket on the space of sections of $TM \oplus \mathbb{R}$. Moreover, taking $\rho : TM \oplus \mathbb{R} \to TM$ as the projection onto TM, one sees that the triple $(TM \oplus \mathbb{R}, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over M. This Lie algebroid will be denoted by A_ω from now on.

Notice that we have the following short exact sequence of Lie algebroids

$$0 \longrightarrow M \times \mathbb{R} \xrightarrow{i} A_\omega \xrightarrow{\rho} TM \longrightarrow 0. \tag{3.1}$$

Let $s : TM \to A_\omega$ be defined by $s(X) = (X,0)$. Clearly, $\rho \circ s = id_{TM}$, so that $s : A_\omega \to TM$ is a left splliting of the sequence (3.1). Notice that

$$s[X,Y]) - [\![s(X), s(Y)]\!] = \omega(X,Y), \tag{3.2}$$

which shows that ω measures how much s deviates from inducing a Lie morphism.

Actually, the short exact (3.1) shows that A_ω is a central extension of the Lie algebroid TM by the trivial Lie algebroid $M \times \mathbb{R}$. Interestingly, as (3.2) indicates, these central extensions are classified, up to isomorphism, by $H^2_{dR}(M)$, which reminds the case of central extensions o Lie algebras.

We may say that a section $(X,f) \in \Gamma(A_\omega)$ preserves the splitting s if, for every $Y \in \mathfrak{X}(M)$,

$$[\![(X,f), s(Y)]\!] = s([X,Y]).$$

Let $\Gamma^s(A_\omega)$ denote the set of all sections of A_ω that preserve the splitting s. The following proposition shows what a section in $\Gamma^s(A_\omega)$ looks like.

Proposition 3.2.1. A section $(X,f) \in \Gamma(A_\omega)$ preserves the splitting $s : TM \to A_\omega$ if and only if X is the hamiltonian vector field of f.

Proof. Let (X,f) be a section in $\Gamma(A_\omega)$, and let $Y \in \mathfrak{X}(M)$. Then, on the one hand,

$$\begin{aligned}\llbracket (X,f),s(Y)\rrbracket &= ([X,Y],X(0)-Y(f)-\omega(X,Y))\\ &= ([X,Y],-Y(f)-\omega(X,Y)).\end{aligned}$$

On the other hand, (X,f) preserves the splitting s if and only if

$$\llbracket (X,f),s(Y)\rrbracket = ([X,Y],0).$$

Thereby, we conclude that (X,f) preserves the splitting s if and only if

$$0 = Y(f)+\omega(X,Y),$$

which is the same as

$$-df(Y) = i_X\omega(Y),$$

for every $Y \in \mathfrak{X}(M)$. Therefore, $df = -i_X\omega$, i.e, X is the hamiltonian vector field of f. □

Using that both $\llbracket\cdot,\cdot\rrbracket$ and $[\cdot,\cdot]$ satisfy the Jacobi identity it is not hard to see that $\Gamma^s(A_\omega)$ constitutes a Lie algebra. Moreover, when (M,ω) is a symplectic manifold, we can state the following.

Proposition 3.2.2. The Lie algebra $(\Gamma^s(A_\omega),\llbracket\cdot,\cdot\rrbracket)$ is isomorphic to the underlying Lie algebra of the Poisson algebra $(C^\infty(M),\{\cdot,\cdot\})$.

Proof. Let $\phi : C^\infty(M) \to \Gamma^s(A\omega)$ be defined by $\phi(f) := (X_f,f)$, where X_f denotes the hamiltonian vector field of f. Due to Proposition 3.2.1, ϕ is surjective. Moreover, since ω is nondegenerate, ϕ is also injective. It remains to check that ϕ preserves the bracket. Let $f,g \in C^\infty(M)$, then

$$\begin{aligned}\llbracket\phi(f),\phi(g)\rrbracket &= \llbracket (X_f,f),(X_g,g)\rrbracket\\ &= ([X_f,X_g],X_f(g)-X_g(f)-\omega(X_f,X_g))\\ &= ([X_f,X_g],\omega(X_f,X_g)).\end{aligned}$$

However, we know that $[X_f,X_g]$ has $\omega(X_f,X_g)$ as hamiltonian function (Appendix A), then it follows that

$$\llbracket\phi(f),\phi(g)\rrbracket = \phi(\{f,g\}),$$

which shows that $\phi : C^\infty(M) \to \Gamma^s(A\omega)$ is a Lie isomorphism. □

Notice how the condition appearing in Proposition 3.2.1 resembles the condition appearing Proposition 2.3.1. Actually, this is not simply a coincidence. In section 3.4, we shall use the isomorphism ϕ for providing another description of the representation of $C^\infty(M)$ as vector fields preserving the connection on a $U(1)-$principal bundle $\pi : P \to M$ with a connection α such that $d\alpha = \pi^*\omega$ (see also page 54).

To conclude this section, it is worth to mention that the construction of A_ω from the closed 2-form ω is just a particular case of a more general construction. In fact, given a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$, one can define the "algebroid deRham complex" associated to A, $(C(A), d_A)$, which has

$$C^p(A) := \{c : \bigotimes_p \Gamma(A) \to C^\infty(M) | \text{ c is antisymmetric and } C^\infty(M)-\text{multilinear}\}$$

and the differential defined by

$$\begin{aligned} d_A(c)(\alpha_0, \cdots, \alpha_p) = & \sum_{i=0}^{p} (-1)^i \mathscr{L}_{\rho(\alpha_i)}(c(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_p)) \\ & + \sum_{i \le j} (-1)^{i+j} c([\alpha_i, \alpha_j], \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_p). \end{aligned}$$

Let $c \in C^2(A)$, and let $A_c = A \oplus \mathbb{R}$. For $(\alpha, f), (\beta, g) \in \Gamma(A_c)$, we can define

$$[\![(\alpha, f), (\beta, g)]\!] := ([\alpha, \beta]_A, \mathscr{L}_{\rho(\alpha)} g - \mathscr{L}_{\rho(\beta)} f - c(\alpha, \beta)),$$

which will be a Lie bracket if and only if $d_A(c) = 0$, i.e, c is a 2-cocycle. When the latter is fulfilled, we obtain the following short exact sequence of Lie algebroids

$$0 \longrightarrow M \times \mathbb{R} \xrightarrow{i} A_c \xrightarrow{\pi} A \longrightarrow 0,$$

where $\pi : A_c \to A$ is just the projection onto A.

Notice how the formula for d_A resembles the formula for the exterior derivative of a p-form. This is not a mere coincidence, because if we take the Lie algebroid A to be TM, then the "algebroid deRham complex" is precisely the deRham complex of M. Then, the "algebroid deRham complex" for a Lie algebroid A may be seen as a generalization of the deRham complex of M.

3.3 The Atiyah algebroid

In this section, we construct the Atiyah algebroid associated to a principal bundle and we prove that it is integrable.

Let $\pi : P \to M$ be $G-$principal bundle over M. Then, the Atiyah algebroid associated to P will be a Lie algebroid over M constructed from the tangent bundle TP. Indeed, since G acts on P, the differential of this action gives an action of G on TP, and the underlying vector bundle of the Atiyah algebroid will the quotient space $\frac{TP}{G}$. Certainly, it is not imediate that $\frac{TP}{G}$ can be seen as vector bundle over M. However, it will be the case due to the following result.

Proposition 3.3.1. ((MACKENZIE, 1987), Appendix A) Let $\pi : P \to M$ be a $G-$principal bundle, and let $p^E : E \to P$ be a vector bundle over P with typical fiber a vector space V. Suppose that $\Psi : E \times G \to E$ is a right action of G on E satisfying the following conditions:

1. For every $g \in G$, $\Psi_g : E \to E$ is a vector bundle isomorphism over $R_g : P \to P$ (the right action of G on P);
2. E is covered by the range of equivariant charts, that is, around each $u_0 \in P$ there are an open set $\mathscr{U} := \pi^{-1}(U)$, $U \subset M$ is open, and
$$\psi : \mathscr{U} \times V \to (p^E)^{-1}(\mathscr{U})$$
an equivariant vector bundle chart for E, i.e, $\psi(ug, v) = \psi(u, v)g$, for $u \in \mathscr{U}, v \in V$, and $g \in G$.

Then, the quotient $\frac{E}{G}$ of E by the action $\Psi : E \times G \to E$ has a unique vector bundle structure over M such that the natural projection $\rho^G : E \to \frac{E}{G}$ is a surjective submersion, and a vector bundle morphism over $\pi : P \to M$.

Let $R_g : P \to P$ denote the right action of $g \in G$ on P. Then, we have the vector bundle isomorphism $TR_g : TP \to TP$ defined by $(TR_g)X_u = (T_uR_g)X_u$. In this way, we obtain an action of G on TP. It is clear that this action fulfills condtion (1) above. Moreover, we claim that TP may be covered by vector bundle charts as condition (2) asks. In fact, let $\theta : \mathbb{R}^n \to U$ be a chart for M, and let $\phi : U \times G \to \mathscr{U} := \pi^{-1}(U)$ be a local trivialization for P. Notice that we have the following identifications $TU \cong U \times \mathbb{R}^n$ and $TG \cong G \times \mathfrak{g}$. Then, we can define
$$\Psi : U \times G \times \mathbb{R}^n \times \mathfrak{g} \longrightarrow T(P)|_{\mathscr{U}}$$
by $\Psi(x, g, t, X) := (T_{(x,g)}\phi)\big((T_{\theta^{-1}(x)}\theta)t, (T_eR_g)X\big)$. Using this and the local trivialization $\phi : U \times G \to \mathscr{U} := \pi^{-1}(U)$, we can define the vector bundle chart $\tilde{\Psi} : \mathscr{U} \times \mathbb{R}^n \times \mathfrak{g} \to T(P)|_{\mathscr{U}}$ by the following commutative diagram
$$\begin{tikzcd}
U \times G \times \mathbb{R}^n \times \mathfrak{g} \arrow[r, "\Psi"] & T(P)|_{\mathscr{U}} \\
\mathscr{U} \times \mathbb{R}^n \times \mathfrak{g} \arrow[u, "(\phi^{-1} \times id)"] \arrow[ur, dashed, "\tilde{\Psi}"'] &
\end{tikzcd}$$

This vector bundle chart will be equivariant in the sense of condition (2) above, because the local trivialization $\phi : U \times G \to \mathscr{U} := \pi^{-1}(U)$ is equivariant, i.e, $\phi(u, gh) = \phi(x, g)h$, for $x \in U$, and $g, h \in G$. Therefore, it follows from Proposition 3.3.1 that $\frac{TP}{G}$ may be seen as a vector bundle over M.

Observe that global sections of TP are simply global vector fields on P. Then, it would interesting to understand sections of $\frac{TP}{G}$ in terms of vector fields on P. For this purpose, let

$\Gamma^G(TP)$ be the set of right-invariant vector fields on P, i.e, $X \in \Gamma^G(TP)$ is such that

$$(T_u R_g)X = X_{ug}.$$

It is easy to see that $\Gamma^G(TP)$ is actually a Lie algebra. Moreover, it can be seen as a $C^\infty(M)-$module. Indeed, for $f \in C^\infty(M)$ and $X \in \Gamma^G(TP)$, we set

$$fX := (\pi^* f)X = (f \circ \pi)X.$$

Notice that $\Gamma(\frac{TP}{G})$, as the set of sections of a vector bundle over M, is also a $C^\infty(M)-$module. Interestingly, it happens that $\Gamma^G(TP)$ and $\Gamma(\frac{TP}{G})$ are isomorphic as $C^\infty(M)-$modules. In fact, if $\rho : TP \to \frac{TP}{G}$ denotes the canonical projection, the maps

$$\begin{aligned} \Gamma^G(TP) &\longrightarrow \Gamma(\frac{TP}{G}) \\ X &\longmapsto \overline{X}, \end{aligned}$$

where $\overline{X}(x) := \rho_u(X(u))$, for any $u \in \pi^{-1}(x)$, and

$$\begin{aligned} \Gamma\left(\frac{TP}{G}\right) &\longrightarrow \Gamma^G(TP) \\ X &\longmapsto \underline{X}, \end{aligned}$$

where $\underline{X}(u) := (\rho_u)^{-1}(X(\pi(u)))$, are morphisms of $C^\infty(M)-$modules and inverse to each other.

Using that $\Gamma(\frac{TP}{G})$ and $\Gamma^G(TP)$ are isomorphic, it is possible to introduce a bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\frac{TP}{G})$. For $\overline{X}, \overline{Y} \in \Gamma(\frac{TP}{G})$, we put

$$[\![\overline{X}, \overline{Y}]\!] := \overline{[X,Y]},$$

where the bracket $[\cdot,\cdot]$ on the right-hand side is the usual bracket of vector fields on P. Clearly, $[\![\cdot,\cdot]\!]$ satisfies Jacobi identity and is antisymmetric, because $[\cdot,\cdot]$ satisfies these properties. Hence, $[\![\cdot,\cdot]\!]$ is a Lie bracket on the space of sections $\Gamma(\frac{TP}{G})$. The only thing left so that we can see $\frac{TP}{G}$ as a Lie algebroid is an anchor. The following calculation indicates what it should be. For $\overline{X}, \overline{Y} \in \Gamma(\frac{TP}{G})$, and $f \in C^\infty(M)$, we have

$$\begin{aligned} [\![\overline{X}, f\overline{Y}]\!] &= \overline{[X, fY]} \\ &= \overline{[X, (f \circ \pi)Y]} \\ &= f\overline{[X,Y]} + \big(X(f \circ \pi)\big)\overline{Y} \end{aligned}$$

however, $X(f \circ \pi) = ((T\pi)X)f$, so that we can write

$$[\![\overline{X}, f\overline{Y}]\!] = f[\![\overline{X}, \overline{Y}]\!] + \big(((T\pi)X)f\big)\overline{Y}.$$

This suggests that anchor should be taken as $T\pi^G : \frac{TP}{G} \to TM$ which is defined as

$$(T\pi^G)\overline{X}_x = (T_u\pi)X,$$

for any $u \in \pi^{-1}(x)$. The map $T^G\pi$ is a vector bundle morphism constructed from the vector bundle morphism $T\pi : TP \to TM$, and the following general result supports its construction.

Proposition 3.3.2. ((MACKENZIE, 1987), Appendix A) Let $\pi : P \to M$ be a $G-$principal bundle, and let $p^E : E \to P$ be a vector bundle over P satisfying the conditions of Proposition 3.3.1. Also, let $p^F : F \to N$ be a vector bundle over another manifold N. Suppose that $\phi : E \to F$ is a vector bundle morphism over a map $f : P \to N$ such that $\phi(\xi g) = \phi(\xi)$, for $\xi \in E$ and $g \in G$. Then, there is a unique vector bundle morphism $\phi^G : \frac{E}{G} \to F$

$$\begin{tikzcd} E \arrow[r, two heads, "\rho^G"] \arrow[d, "p^E"'] & \frac{E}{G} \arrow[r, "\phi^G"] \arrow[d, "p^{\frac{E}{G}}"] & F \arrow[d, "p^F"] \\ P \arrow[r, "\pi"] & M \arrow[r, "f^G"] & N \end{tikzcd}$$

such that $\phi = \phi^G \circ \rho^G$ and $f = f^G \circ \pi$.

In this manner, we have shown that the triple $(\frac{TP}{G}, [\![\cdot,\cdot]\!], T\pi^G)$ is a Lie algebroid over M. This algebroid is the so called *Atiyah algebroid* associated to the principal bundle $\pi : P \to M$.

Interestingly, as the next proposition shows, the Atiyah algebroid is closely related to the gauge groupoid of $\pi : P \to M$.

Proposition 3.3.3. Let $\pi : P \to M$ be a $G-$principal bundle. Then, the Lie algebroid of the gauge groupoid $A(\frac{P\times P}{G})$ is isomorphic to the Atiyah algebroid $(\frac{TP}{G}, [\![\cdot,\cdot]\!], T\pi^G)$.

Proof. Let $\Pi : P \times P \to \frac{P\times P}{G}$ denote the natural projection. This projection is a smooth surjective submersion, then any vector tangent to a point $[(u,v)] \in \frac{P\times P}{G}$ may represented as $(T_{(u,v)}\Pi)(X,Y)$, for $(X,Y) \in T_uP \times T_vP$. Using this, it is easy to see that a vector $(T_{(u,v)}\Pi)(X,Y)$ is in $\ker(Tt)$ if and only if $X \in T_uP$ is vertical, i.e, $(T_u\pi)X = 0$, where $\pi : P \to M$. Hence, for $(T_{(u,u)}\Pi)(X,Y) \in \ker(Tt)$, we can write

$$(T_{(u,u)}\Pi)(X,Y) = (T_{(u,u)}\Pi)(0,Y-X).$$

Motivated by this, we define

$$TP \longrightarrow A\left(\frac{P\times P}{G}\right)$$

$$X_u \longmapsto (T_{(u,u)}\Pi)(0,X).$$

It is easy to see that this map is surjective. Moreover, it is invariant under the right action of G on TP, so that Proposition 3.3.2 guarantees that we can consider the induced map from $\frac{TP}{G}$

$$\frac{TP}{G} \longrightarrow A\left(\frac{P\times P}{G}\right),$$

which will be a Lie algebroid isomorphism. □

Therefore, we have proved that the Atiyah algebroid is integrable.

3.4 Integrability of A_ω

In this section, we shall prove that the existence of an extension [2] (P,α,π) of (M,ω) is a sufficient condition for the integrability of the Lie algebroid A_ω. For simplicity, throughout this section we shall consider $D=\mathbb{Z}$, so that the once called $\mathbb{R}/D-$principal bundles are going to be referred to as $U(1)-$principal bundles.

Let M be a manifold, and let ω be a closed 2-form. Suppose that $\pi : P \to M$ is a $U(1)-$principal bundle with a connection α such that $d\alpha = \pi^*\omega$. Then, as we have already seen, the Atiyah algebroid $\frac{TP}{U(1)}$ associated to the $U(1)-$principal bundle $\pi : P \to M$ will be integrable. Thereby, to prove that A_ω is integrable, we shall show that A_ω is isomorphic to the Atiyah algebroid $\frac{TP}{U(1)}$.

Proposition 3.4.1. Let M be a manifold, and let ω be closed 2-form on M. Suppose that there exists a $U(1)-$principal bundle $\pi : P \to M$ with a connection α such that $d\alpha = \pi^*\omega$. Then, the Lie algebroid A_ω constructed from ω is isomorphic to the Atiyah algebroid $\frac{TP}{U(1)}$.

Proof. Recall that the underlying vector bundle of the Lie algebroid A_ω is $TM\oplus\mathbb{R}$. Then, let $\Psi : TP \to A_\omega$ be defined by $\Psi(X_p) = (T_p\pi(X),\alpha(X_p))$. Clearly, this is a fiberwise isomorphism; moreover, this is invariant under the action of $U(1)$ on TP. From Proposition 3.3.2, we can consider the vector bundle morphism over the identity of M

$$\Psi : \frac{TP}{U(1)} \to A_\omega.$$

To prove that this vector bundle morphism induces a Lie algebra morphism on the level of sections, let us understand how Ψ act on sections of $\frac{TP}{U(1)}$. First of all, notice that a section $\overline{Y} : M \to \frac{TP}{U(1)}$ can be seen as a (right) invariant vector field Y on P. Then, the first coordinate of $\Psi(\overline{Y})$ will be simply $T\pi(Y)$, while the second coordinate will be given by the composition

$$M \xrightarrow{\overline{Y}} \frac{TP}{U(1)} \xrightarrow{\overline{\alpha}} \mathbb{R}$$

which, for $x\in M$, reads $(\overline{\alpha}\circ\overline{Y})(x) = \overline{\alpha}(\overline{Y}(x)) = \alpha(Y_p)$, for any $p\in\pi^{-1}(x)$. Now, we calculate

$$\begin{aligned}
[\![\Psi(\overline{X}),\Psi(\overline{Y})]\!] &= [\![\big((T\pi)X,\overline{\alpha}(\overline{X})\big),\big((T\pi)Y,\overline{\alpha}(\overline{Y})\big)]\!] \\
&= \big([(T\pi)X,(T\pi)Y],\ ((T\pi)X)(\overline{\alpha}(\overline{Y})) - ((T\pi)Y)(\overline{\alpha}(\overline{X})) - \omega(T\pi X, T\pi Y)\big).
\end{aligned}$$

Notice that

[2] Recall that by an extension (P,α,π) of (M,ω) we mean a $\mathbb{R}/D-$principal bundle $\pi : P \to M$ a connection form α such that $d\alpha = \pi^*\omega$.

$$
\begin{aligned}
((T\pi)X)(\overline{\alpha}(\overline{Y})) &= \frac{d}{dt}\bigg|_{t=0} (\overline{\alpha}\circ\overline{Y})(\pi\circ\gamma) \\
&= \frac{d}{dt}\bigg|_{t=0} \overline{\alpha}(\overline{Y}(\pi\circ\gamma(t))) \\
&= \frac{d}{dt}\bigg|_{t=0} \alpha(Y(\gamma(t))) = X(\alpha(Y)).
\end{aligned}
$$

Analogously, for $((T\pi)Y)(\overline{\alpha}(\overline{X}))$, one concludes that $((T\pi)Y)(\overline{\alpha}(\overline{X})) = Y(\alpha(X))$. Thereby, it follows that

$$
[\![\Psi(\overline{X}),\Psi(\overline{Y})]\!] = \big([(T\pi)X,(T\pi)Y],\, X(\alpha(Y)) - Y(\alpha X) - \omega(T\pi X, T\pi Y)\big)
$$

On the other hand,

$$
\Psi\big(\overline{[X,Y]}\big) = (T\pi([X,Y]), \overline{\alpha}\big(\overline{[X,Y]}\big));
$$

however,

$$
d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]),
$$

and since $d\alpha = \pi^*\omega$, we obtain

$$
\alpha([X,Y]) = X\alpha(Y) - Y\alpha(X) - \pi^*\omega(X,Y).
$$

Hence,

$$
\Psi\big(\overline{[X,Y]}\big) = (T\pi([X,Y]), X\alpha(Y) - Y\alpha(X) - \pi^*\omega(X,Y)).
$$

Therefore, from all the above, we obtain

$$
\Psi\big(\overline{[X,Y]}\big) = [\![\Psi(\overline{X}),\Psi(\overline{Y})]\!],
$$

which shows that $\Psi : \frac{TP}{U(1)} \to A_\omega$ is a Lie algebroid (iso)morphism.

To find an inverse for $\Psi : \frac{TP}{U(1)} \to A_\omega$ we may think as follows. Let $(v_x, c) \in T_xM \oplus \mathbb{R}_x$. Define a vector field tangent to the fiber over x by choosing a vector X_u, $u \in \pi^{-1}(x)$, such that $T\pi(X_u) = v_x$ and add to this $c\cdot\partial_u$, where ∂ denotes the fundamental vector vector field. Then, spread $X_u + c\cdot\partial_u$ out by right action of $U(1)$ on P so that we obtain a right invariant vector field $X + c\cdot\partial$ tangent to the fiber $\pi^{-1}(x)$. It indicates that we may send (v_x, c) to $(\overline{X + c\cdot\partial})_x \in \big(\frac{TP}{U(1)}\big)_x$. □

Proposition 3.3.3 together with Proposition 3.4.1 implies that in the presence of an extension (P,α,π) of (M,ω) the Lie algebroid A_ω is integrable. Notice that the condition for the existence of the extension (P, α, π) is $\mathrm{Per}(\omega) \subset \mathbb{Z}$ (Propositions 2.2.6 and 2.2.8) Actually, it can be also proved that if A_ω is integrable then $\mathrm{Per}(\omega) \subset \mathbb{Z}$ (see (CRAINIC, 2004)).

In the context of a symplectic manifold (M, ω), Proposition 3.2.2 shows that the underlying Lie algebra of the Poisson algebra $C^\infty(M)$ is isomorphic to certain subalgebra of sections of A_ω. Then, in the case that there exists a $U(1)-$principal bundle $\pi : P \to M$ with a connection α such that $d\alpha = \pi^*\omega$, using the isomorphism presented in Proposition 3.4.1, we can represent $C^\infty(M)$ as vector fields on P preserving the connection α. This is essentially based on the following result.

Proposition 3.4.2. Let M be a manifold, and let ω be closed 2-form on M. Suppose that there exists a $U(1)-$principal bundle $\pi : P \to M$ with a connection α such that $d\alpha = \pi^*\omega$. Also, let $\overline{X}$ be a section of $\frac{TP}{U(1)}$, and let X denote corresponding (right) invariant vector field on P. Then, X will preserve the connection, i.e, $\mathscr{L}_X\alpha = 0$, if and only if $\Psi(\overline{X}) \in \Gamma^s(A_\omega)$.

Proof. Let $\Psi(X) = (v, f)$, for $v \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$. Then, we calculate

$$\begin{aligned}\mathscr{L}_X\alpha &= di_X\alpha + i_X d\alpha \\ &= d(\pi^* f) + i_X\pi^*\omega \\ &= \pi^*(df + i_v\omega)\end{aligned}$$

From this, we see that $\mathscr{L}_X\alpha = 0$ if and only if $df + i_v\,\omega = 0$. The last equality says precisely that $v \in \mathfrak{X}(M)$ is the hamiltonian vector field of f. Therefore, Proposition 3.2.1 implies that X will preserve the connection if and only if $\Psi(\overline{X}) = (v, f) \in \Gamma^s(A_\omega)$. □

Finally, the representation of $C^\infty(M)$ as (right) invariant vector fields on P may be expressed by the following composition

$$C^\infty(M) \xrightarrow{\phi} \Gamma^s(A_\omega) \xrightarrow{\Psi^{-1}} \Gamma\left(\tfrac{TP}{U(1)}\right).$$

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APPENDIX

A

TOPICS IN SYMPLECTIC GEOMETRY

Here we collect some definitions and results from symplectic geometry. More precisely, considering a symplectic manifold (M,ω), we introduce hamiltonian vector fields on M, the Poisson algebra $C^\infty(M)$, and hamiltonian actions. We also show how $C^\infty(M)$ provides a one-dimensional central extension of the Lie algebra of hamiltonian vector fields. The references for this part are (SILVA, 2001) and (MARSDEN; RATIU, 1994).

Recall that a symplectic manifold is a pair (M,ω), where M is a smooth manifold and ω is a closed nondegenerate 2-form. The 2-form ω induces a skew-symmetric nondegenerate bilinear form on each tangent space T_xM, $x \in M$, so that each T_xM is a symplectic vector space. A standard argument (Theorem 1.1 in (SILVA, 2001)) shows that a symplectic vector space is necessarily even dimensional. Then, T_xM is even dimensional, which shows that the symplectic manifold (M,ω) must be even dimensional.

Example A.0.1. Let N be a smooth manifold. Then, the cotangent bundle $\pi : T^*N \to N$ admits a natural symplectic structure. Indeed, notice that to choose a point $p \in T^*N$ is same as choosing a 1-form on N evaluated in a point $x \in N$, so that we can write $p = (x,\xi)$, where $x \in M$ and $\xi \in T_x^*N$. Then, we can define a 1-form α on T^*N pointwise by

$$\alpha_p := (T_p\pi)^*(\xi) = \xi \circ T_p\pi.$$

The 1-form α is called the tautological 1-form. This terminology becomes clear when we write α in coordinates. In fact, if $(U,x_1,\dots,x_n)$ is a coordinate chart for N, we can write $p = \sum_{i=1}^n \xi_i\, dx_i$, and using the coordinates induced on T^*N, a calculation shows that $\alpha_p = \sum_{i=1}^n \xi_i\, dx_i$. Finally, the canonical symplectic structure ω on T^*N is defined as $\omega = -d\alpha$, and its local expression in coordinates is

$$\omega = \sum_{i=0}^{n} dx_i \wedge d\xi_i$$

Remark A.0.2. The symplectic structure introduced in the last example has some interesting features. For example, let M and N be two smooth manifolds, and let $f : M \to N$ be diffeomorphism. Also, let α_M and α_N be de tautological 1-forms in T^*M and T^*N, respectively. Then, define $f^\sharp : T^*M \to T^*N$ by $f^\sharp(p) = f^\sharp(x,\xi) := (f(x), \xi_1 \circ (T_{f(x)}f^{-1}))$. It is easy to see that $f^\sharp$ is a diffeomorphism. Moreover, it can be proved that $f^\sharp \alpha_N = \alpha_M$, which implies that $f^\sharp$ is actually a symplectomoprhism.

In the following we describe how a symplectic form can be used to define a Poisson structure.

Let (M,ω) be a symplectic manifold. The 2-form ω can be used to assign a vector field to each function in $C^\infty(M)$. In fact, let us define the following map

$$TM \to T^*M$$

$$X \longmapsto \omega(X,\cdot).$$

Since ω is nondegenerate, this is a vector bundle isomorphism. Then, given $f \in C^\infty(M)$, one can consider the 1-form df, which is a section of T^*M; by means of the above map, we see that there exists a unique vector field X_f satisfyng $-i_{X_f}\omega = df$. The vector field X_f is called the hamiltonian vector field of the function f. Let us denote by $\mathscr{H}_M$ the space of all hamiltonian vector fields on M. Since ω is closed, it follows from Cartan's formula, $\mathscr{L}_X = i_X d + d i_X$, that every hamiltonian vector field X_f is an infinitesimal symmetry of the pair (M,ω), i.e, $\mathscr{L}_{X_f}\omega = 0$. Given all this, we can formulate the following proposition.

Proposition A.0.3. The space of all hamiltonian vector field $\mathscr{H}_M$ endowed with usal bracket of vector fields constitues a Lie algebra.

Proof. Let X_f and X_g be the hamiltonian vector fields of $f,g \in C^\infty(M)$, respectively. Notice that it suffices to prove that the bracket $[X_f,X_g]$ is a hamiltonian vector field. Recall the general identity $i_{[X,Y]} = \mathscr{L}_X i_Y - i_Y \mathscr{L}_X$. Then, we can calculate

$$i_{[X_f,X_g]}\omega = (\mathscr{L}_{X_f} i_{X_g})\omega - (i_{X_g}\mathscr{L}_{X_f})\omega = (\mathscr{L}_{X_f} i_{X_g})\omega = (d i_{X_f} + i_{X_f} d)(i_{X_g}\omega) =$$

$$d(i_{X_f} i_{X_g}\omega) + i_{X_f}(d(dg)) = -d(\omega(X_f,X_g)),$$

which shows that $[X_f,X_g]$ is the hamiltonian vector field of the function $\omega(X_f,X_g)$. $\square$

Let us recall the definition of a Poisson algebra.

Definition A.0.4. A Poisson algebra is a commutative associative algebra $\mathscr{P}$ with a Lie bracket $\{\cdot,\cdot\}$ satisfying the following identity (Leibniz rule)

$$\{f,gh\} = \{f,g\}h + g\{f,h\},$$

for all $f,g,h \in \mathscr{P}$.

The symplectic structure on M can be used to turn the algebra $C^\infty(M)$ into a Poisson algebra. Indeed, we define

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$
$$(f,g) \longmapsto \omega(X_f, X_g).$$

We should verify that $\{\cdot,\cdot\}$ defines a Lie bracket satisfying the Leibniz rule. All the properties of Lie bracket, but the Jacobi identity, follow directly from the definiton of $\{\cdot,\cdot\}$. For the Jacobi identity, one can use the following general formulas

$$d\omega(X,Y,Z) = X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)$$
$$(\mathscr{L}_X\omega)(Y,Z) = X(\omega(Y,Z)) + \omega([X,Y],Z) + \omega(Y,[X,Z]),$$

applied to the hamiltonian vector fields X_f, X_g, X_h. Meanwhile, the Leibniz rule follows from a simple calculation once one notices that

$$i_{X_{gh}}\omega = (i_{X_g}\omega)h + g(i_{X_h}\omega).$$

Therefore, $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra, which is, in particular, a Lie algebra.

Notice that we can define a map from $C^\infty(M)$ to $\mathfrak{X}(M)$ simply by putting

$$\zeta : C^\infty(M) \to \mathfrak{X}(M)$$
$$f \longmapsto X_f.$$

It is easy to see that this is a linear map. Also, it follows from Proposition A.0.3 that this is actually a Lie algebra morphism. Moreover, since a constant function has associated to it the vector field identically zero, we can say that $\ker(\zeta) = \mathbb{R} \subset C^\infty(M)$. Futhermore, notice that the $\ker(\zeta)$ lies in the center of the Lie algebra $C^\infty(M)$. Thereby, we have the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\zeta} \mathfrak{X}(M) \longrightarrow 0.$$

Since the image of ζ is equal to $\mathscr{H}(M)$, we can rewrite the above sequence as

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\zeta} \mathscr{H}(M) \longrightarrow 0,$$

which shows that $C^\infty(M)$ is a one-dimensional central extension of the Lie algebra of hamiltonian vector field. We shall refer to this central extension as the Konstant-Soriau central extension of the Lie algebra of hamiltonian vector fields.

As we saw in Proposition 2.1.8, the central extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ are classified by the cohomology group $H^2(\mathfrak{g},\mathfrak{a})$. Let us consider $\mathfrak{g} = \mathscr{H}(M)$ and $\mathfrak{a} = \mathbb{R}$. Hence, there exist a cohomology class in $H^2(\mathscr{H}(M),\mathbb{R})$ which is associated to the Konstant-Soriau central extension. Interestingly, the 2-cocycle representing such a cohomology class can be described by means of the symplectic form ω.

Proposition A.0.5. Let $x \in M$, and assume that M is connected. Then, a 2-cocycle representing the Konstant-Soriau central extension is given by

$$c(X,Y) = \omega_x(X,Y).$$

Proof. Let us consider the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\zeta} \mathscr{H}(M) \longrightarrow 0.$$

We define $s : \mathscr{H}(M) \to C^\infty(M)$ by $s(X) := f$, where f is the unique hamiltonian function of X such that $f(x) = 0$ (notice that here we have used the assumption that M is conneceted). It is easy to see that s is a linear section of ζ. A 2-cocycle c associated to the Konstant-Soriau central extension can be calculated by how much s deviates from being a Lie morphism. More precisely, one can put c as

$$c(X,Y) = \{s(X), s(Y)\} - s([X,Y]).$$

From the definition of $\{\cdot,\cdot\}$, we have $\{s(X), s(Y)\} = \omega(X,Y)$. On the order hand, $s([X,Y])$ is the unique hamiltonian function of the vector field $[X,Y]$ vanishing at x. By Proposition A.0.3, it follows that $s([X,Y]) = \omega(X,Y) - \omega_x(X,Y)$. Finally, we conclude that

$$c(X,Y) = \omega_x(X,Y).$$

□

In the following, we introduce the concepts of moment and comoment map.

Firstly, we need to introduce symplectic actions. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose $\psi : G \times M \to M$ is a left action of G on the symplectic manifold (M, ω). For $X \in \mathfrak{g}$, remind that X_M denotes the fundamental vector field associated to the action ψ : X_M is the vector field whose flow is given by $\psi_{exp(-tX)}$. The action ψ is said to be symplectic if, for each $g \in G$, $\psi_g^* \omega = \omega$, i.e, ψ_g is a symplectormorphism.

Secondly, we notice that throughtout the presentation we shall need some facts about the coadjoint representation of a Lie group and the associate coadjoint representation of its Lie algebra; then, we recall what is needed in the following remark.

Remark A.0.6. Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. As usual, $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$. The coadjoint representation of the Lie group G is defined by

$$\mathrm{Ad}^\sharp : G \longrightarrow GL(\mathfrak{g}^*)$$

$$g \longmapsto \mathrm{Ad}^\sharp_g$$

where $\langle \mathrm{Ad}^\sharp_g(\xi), X\rangle := \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$. The coadjoint representation of the Lie algebra $\mathfrak{g}$ is obtained by taking the derivative of $\mathrm{Ad}^\sharp$ at the identity of G, i.e,

$$\mathrm{ad}^\sharp := T_e(\mathrm{Ad}^\sharp) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*).$$

Using the identity

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{exp(-tX)}(Y) = -[X,Y],$$

one can show that $\mathrm{ad}^{\sharp}X : \mathfrak{g}^* \to \mathfrak{g}^*$ is given by $\langle(\mathrm{ad}^{\sharp}X)(\xi),\cdot\rangle = \langle\xi,-[X,\cdot]\rangle$.

Definition A.0.7. Let $\psi : G\times M \to M$ be a symplectic action on the symplectic manifold (M,ω). A moment map for this action is a map $\mu : M \to \mathfrak{g}^*$ satisfyng the following two properties:

(i) For each $X \in \mathfrak{g}$, the function $\mu^X : M \to \mathbb{R}$ definied by $\mu^X(p) := \langle\mu(p),X\rangle$ satisfies

$$d\mu^X = i_{X_M}\,\omega,$$

so that $-\mu^X$ is a hamiltonian function for fundamental vector field generated by X.

(ii) The μ is equivariant with respect to the action ψ and the coadjoint action of G on $\mathfrak{g}^*$, which means

$$\mu\circ\psi_g = \mathrm{Ad}^{\sharp}_g\circ\mu.$$

A symplectic action that admits a moment map is called a hamiltonian action. We will say that an action is weakly hamiltonian if it admits a map $\mu : M \to \mathfrak{g}^*$ satisfying only condition (i) of Definition A.0.7.

Related to the equivariance mentioned in condition (ii) above, we have the infinitesimal equivariance. Indeed, a map $\mu : M \to \mathfrak{g}^*$ satisfying the condition (i) of Definition A.0.7 is said to be infinitesimal equivariant if

$$T_m\mu(X_M(m)) = \mathrm{ad}^{\sharp}(X)(\mu(m)),$$

for all $X \in \mathfrak{g}$ and $m \in M$. Observe that the infinitesimal equivariance can be obtained from the equivariance. In fact, the equivariance says that

$$\mu\circ\psi_g = \mathrm{Ad}^{\sharp}_g\circ\mu.$$

By taking $g = exp(-tX)$ and differentiating the equality $\mu\circ\psi_{exp(-tX)} = \mathrm{Ad}^{\sharp}_{exp(-tX)}\circ\mu$ at $t = 0$, one sees that equivariance implies infinitesimal equivariance.

Let $\mu : M \to \mathfrak{g}^*$ be a map as in Definition A.0.7. Then, we can define a map $\mu^* : \mathfrak{g} \to C^\infty(M)$ by $(\mu^*(X))(m) = \langle\mu(m),X\rangle$. The next proposition estabislihes a relation bewteen these two maps.

Proposition A.0.8. Let $\psi : G\times M \to M$ be a weakly hamiltonian action, i.e, there exists a map $\mu : M \to \mathfrak{g}^*$ satisfyng condition (i) of Definition A.0.7. Then, the map $\mu : M \to \mathfrak{g}^*$ is infinitesimal equivariant if and only if $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism.

Proof. Suppose that $\mu : M \to \mathfrak{g}^*$ is infinitesimal equivariant. Then, we can calculate

$$\left.\frac{d}{dt}\right|_{t=0} \mu(\psi_{exp(-tX)}(m)) = (T_m \mu)(X_M) = (\mathrm{ad}^\sharp X)(\mu(m)) = \langle \mu(m), -[X,\cdot] \rangle;$$

evaluating this at some vector $Y \in \mathfrak{g}$, we obtain

$$\langle (\mathrm{ad}^\sharp X)(\mu(m)), Y \rangle = \langle \mu(m), -[X,Y] \rangle = -\mu^*([X,Y]).$$

On the other hand, we see that

$$\left\langle \left.\frac{d}{dt}\right|_{t=0} \mu(\psi_{exp(-tX)}(m)), Y \right\rangle = \left.\frac{d}{dt}\right|_{t=0} \langle \mu(\psi_{exp(-tX)}(m)), Y \rangle = \left.\frac{d}{dt}\right|_{t=0} \mu^Y(\psi_{exp(-tX)}(m)) =$$

$$(d\mu^Y)(X_M(m)) = (i_{Y_M}\omega)(X_M(m)) = \omega_m(Y_M, X_M) = -\{\mu^*(X), \mu^*(Y)\}(m).$$

The result follows. □

When $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism we say that it is a comoment map.

The proposition above shows, in particular, that when if $\mu : M \to \mathfrak{g}^*$ is a moment map, then $\mu^* : \mathfrak{g} \to C^\infty(M)$ will be always comoment map. Therefore, a natural question raises: when does $\mu : M \to \mathfrak{g}^*$ infinitesimal equivariant (or, equivalently, $\mu^* : \mathfrak{g} \to C^\infty(M)$ a Lie algebra morphism) implies equivariance of $\mu : M \to \mathfrak{g}^*$? The answer is: whenever G and M are connected.

In fact, let $\psi : G \times M \to M$ be a symplectic action which is weakly hamiltonian. Suppose that the map $\mu : M \to \mathfrak{g}^*$ is infinitesimal equivariant. Let $\Gamma_X : G \times M \to \mathbb{R}$ be defined by

$$\Gamma_X(g,m) := \langle \mu \circ \psi_g(m), X \rangle - \langle \mu(m), \mathrm{Ad}_{g^{-1}} X \rangle.$$

We shall prove that this function vanishes identically. Firstly, let us fix $g \in G$. Then, denote by $\Gamma_{X,g} : M \to \mathbb{R}$ the restriction of Γ_X to $\{g\} \times M$. Notice that

$$\Gamma_{X,g}(g,m) = (\psi_g^* \mu^*(X))(m) - \mu^*(\mathrm{Ad}_{g^{-1}}(X))(m).$$

We want to show that this function is constant; to this end, we shall find what is its hamiltonian vector field. We have

$$X_{\Gamma_{X,g}} = X_{\psi_g^* \mu^*(X)} - X_{\mu^*(\mathrm{Ad}_{g^{-1}})} = X_{\psi_g^* \mu^*(X)} - (\mathrm{Ad}_{g^{-1}}(X))_M,$$

because $\mu^*(\mathrm{Ad}_{g^{-1}})$ is a hamiltonian function for vector field $(\mathrm{Ad}_{g^{-1}}(X))_M$. To find out what is the other term in the last equality we use the following lemma.

Lemma A.0.9. Let $\varphi : M \to M$ be a symplectomorphism, and let $f \in C^\infty(M)$. Then

$$X_{\varphi^* f}(m) = (T_{\varphi(m)} \varphi^{-1}) X_f$$

Proof. We know that $X_{\varphi^* f}$ must satisfy

$$i_{X_{\varphi^* f}}\omega = -d(\varphi^* f) = -\varphi^*(df) = \varphi^*(i_{X_f}\omega).$$

Now, let $m \in M$, and let $Y \in T_m M$. On the one hand,

$$(\varphi^*(i_{X_f}\omega))_m(Y) = \omega_{\varphi(m)}(X_f, (T_m\varphi)Y) = \omega_m((T_{\varphi(m)}\varphi^{-1})X_f, Y)$$

On the other hand,

$$(i_{X_{\varphi^* f}}\omega)_m(Y) = \omega_m(X_{\varphi^* f}, Y).$$

Hence, we have

$$\omega_m(X_{\varphi^* f}, Y) = \omega_m((T_{\varphi(m)}\varphi^{-1})X_f, Y).$$

Since ω is nondegenerate, we conclude that

$$X_{\varphi^* f}(m) = (T_{\varphi(m)}\varphi^{-1})X_f$$

□

From the this lemma, we obatin

$$X_{\psi_g^*\mu^*(X)} = (T_{\psi(m)}\psi_{g^{-1}})(X_{\mu^*(X)}),$$

which is actually $(\mathrm{Ad}_{g^{-1}}X)_M$. Then, we see that

$$X_{\Gamma_{X,g}} = X_{\psi_g^*\mu^*(X)} - (\mathrm{Ad}_{g^{-1}}(X))_M = 0.$$

Therefore, we have shown that the hamiltonian vector field associated to the function $\Gamma_{X,g}$ is zero. Since M is connected, it follows that $\Gamma_{X,g}$ is constant, which means that the function $\Gamma_X : G \times M \to \mathbb{R}$ does not depend on M. In what follows, we prove that $\Gamma_X : G \times M \to \mathbb{R}$ also does not depend on G.

Indeed, let $Y \in \mathfrak{g}$. Notice that

$$(T_e\Gamma_X)(Y) = \frac{d}{dt}\bigg|_{t=0} \Gamma_X(exp(tY)) = \frac{d}{dt}\bigg|_{t=0} \langle \mu(\psi_{exp(-tY)}(m)), X\rangle - \langle \mu(m), \mathrm{Ad}_{exp(-tY)}X\rangle =$$

$$\langle (T_m\mu)(Y_M), X\rangle - \langle \mathrm{ad}^\sharp(Y)\mu(m), X\rangle,$$

which is zero by infinitesimal equivariance, so that we have $(T_e\Gamma_X) = 0$.

To conclude we need one more lemma.

Lemma A.0.10. For any $g, h \in G$ the following equation holds

$$\Gamma_X(gh) = \Gamma_X(g) + \Gamma_{(\mathrm{Ad}_{g^{-1}}X)}(h).$$

Proof. It follows directily from calculating explicitly the left-hand side. □

Applying the above equality for $h = exp(tX)$ leads to

$$\Gamma_X(g(exp(tY))) = \Gamma_X(g) + \Gamma_{(\mathrm{Ad}_{g^{-1}}X)}(exp(tY)).$$

Taking the derivative of this at $t = 0$ gives

$$\left.\frac{d}{dt}\right|_{t=0} \Gamma_X(g(exp(tY))) = \left.\frac{d}{dt}\right|_{t=0} \Gamma_{(\mathrm{Ad}_{g^{-1}}X)}(exp(tY)).$$

However,

$$\left.\frac{d}{dt}\right|_{t=0} \Gamma_X(g(exp(tY))) = (T_g\Gamma_X)((T_eL_g)Y)$$

while

$$\left.\frac{d}{dt}\right|_{t=0} \Gamma_{(\mathrm{Ad}_{g^{-1}}X)}(exp(tY)) = (T_e\Gamma_{\mathrm{Ad}_{g^{-1}}})(Y) = 0,$$

so that we conclude $(T_g\Gamma_X)((T_eL_g)Y) = 0$. Since $(T_eL_g)(Y)$, $Y \in \mathfrak{g}$, generate T_gG, it follows that $T_g\Gamma_X \equiv 0$. Recall that we suppose G to be connected, then it follows that $\Gamma_X : G \to \mathbb{R}$ is constant. Finally, notice that $\Gamma_X(e,m) = 0$, which implies that $\Gamma_X \equiv 0$. However, $\Gamma_X \equiv 0$, for all $X \in \mathfrak{g}$, gives exactly the equivariance of μ.

To summarize, the above discussion shows that hamiltonian actions can be equivalently describe by both a moment map or a comoment map.

The description of hamiltonian actions by comoment maps is suitable to prove Theorem A.0.13, which gives suficient conditions to an action to be hamiltonian. Interestingly, the conditions are given in terms of the Lie algebra cohomology.

Before moving on, let us state (and prove) two lemmas that we shall use in the proof of Theorem A.0.13.

Lemma A.0.11. Let (M,ω) be a symplectic manifold, and let X,Y be two vector fields on M. If $\mathscr{L}_X\omega = \mathscr{L}_Y\omega = 0$, then $[X,Y]$ is a hamiltonian vector field.

Proof. Suppose that $\mathscr{L}_X\omega = \mathscr{L}_Y\omega = 0$. We know that $i_{[X,Y]} = (\mathscr{L}_X i_Y - i_Y\mathscr{L}_X)$, then

$$i_{[X,Y]}\omega = (\mathscr{L}_X i_Y - i_Y\mathscr{L}_X)\omega = (\mathscr{L}_X i_Y)\omega = (di_X + i_Xd)(i_Y\omega) = d(i_Xi_Y\omega) + i_X(d(i_Y\omega));$$

since $\mathscr{L}_Y\omega = 0$, Cartan's formula implies $d(i_Y\omega) = -i_Y(d\omega)$; then, it follows that

$$i_{[X,Y]}\omega = d(i_Xi_Y\omega) + i_X(d(i_Y\omega)) = d(i_Xi_Y\omega) - i_X(i_Y(d\omega)),$$

but ω is a closed 2-form, thus we conclude

$$i_{[X,Y]}\omega = d(i_Xi_Y\omega),$$

which shows that $[X,Y]$ is a hamiltonian vector field with hamiltonian function $\omega(X,Y)$. □

Lemma A.0.12. Let $\mathfrak{g}$ be a Lie algebra. Then, $H^1(\mathfrak{g},\mathbb{R}) = 0$ if and only if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, where $[\mathfrak{g},\mathfrak{g}] = \{\sum_{i,j}[X_i,X_j] \,|\, X_i,X_j \in \mathfrak{g}\}$ is the commmutator ideal of $\mathfrak{g}$.

Proof. Firstly, we notice that from the definition of $H^1(\mathfrak{g},\mathbb{R})$ it is easy to see that $H^1(\mathfrak{g},\mathbb{R})$ is precisely the annihilator of $[\mathfrak{g},\mathfrak{g}]$ (which is denoted by $[\mathfrak{g},\mathfrak{g}]^0$). Thus, $H^1(\mathfrak{g},\mathbb{R})$ is the set of linear functions $c : \mathfrak{g} \to \mathbb{R}$ such that $c([\mathfrak{g},\mathfrak{g}]) = 0$. From this observation, it is clear that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ implies $H^1(\mathfrak{g},\mathbb{R}) = 0$. On the other hand, suppose that $H^1(\mathfrak{g},\mathbb{R}) = 0$. Then, the result follows from

$$\left(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]}\right)^* \cong [\mathfrak{g},\mathfrak{g}]^0$$

□

Theorem A.0.13. Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. If $H^1(\mathfrak{g},\mathbb{R}) = 0$ and $H^2(\mathfrak{g},\mathbb{R}) = 0$, then any symplectic $G-$action is hamiltonian.

Proof. Let $\psi : G \times M \to M$ be a symplectic action of G on the symplectic manifold (M,ω). Let us denote by $T\psi$ the Lie algebra morphism that assigns to each $X \in \mathfrak{g}$ the associated fundamental vector field on M. Since the action is symplectic, for every $X \in \mathfrak{g}$, $T\psi(X) = X_M$ is an infinitesimal symmetry of the pair (M,ω), which means that $\mathscr{L}_{X_M}\omega = 0$ (vector fields satisfying this property are called symplectic vector fields). We claim that, actually, X_M is a hamiltonian vector field. Indeed, from Lemma A.0.12, we know that $H^1(\mathfrak{g},\mathbb{R}) = 0$ is equivalent to $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Therefore, we can write $X = \sum c_{ij}[X_i,X_j]$, for some $X_i,X_j \in \mathfrak{g}$; applying $T\psi$ we obtain

$$X_M = \sum c_{ij}[(X_i)_M,(X_j)_M];$$

however, from Lemma A.0.11, we know that the bracket of two symplectic vector field is a hamiltonian vector field. Then, it follows that X_M is a hamiltonian vector field. Hence, we have shown that $T\psi$ maps $\mathfrak{g}$ into the Lie algebra of hamiltonian vector fields $\mathscr{H}(M)$; so we can write the following diagram

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\zeta"] & \mathscr{H}(M) \\ & \mathfrak{g} \arrow[u, "T\psi"'] \end{tikzcd}$$

Notice that a comoment map associated to the action $\psi : G \times M \to M$ would be $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\zeta"] & \mathscr{H}(M) \\ & \mathfrak{g} \arrow[u, "T\psi"'] \arrow[ul, dashed, "\mu^*"] \end{tikzcd}$$

is a commutative diagram in the category of Lie algebras. To construct this μ^* we proceed as follows. Firstly, we choose $\tau : \mathfrak{g} \to C^\infty(M)$ such that replacing μ^* by τ in the above diagram leads to a commutative diagram of vector spaces. Certainly, such a τ may be not a Lie algebra morphism. However, the hypothesis $H^2(\mathfrak{g},\mathbb{R})=0$ can be used to correct this. Indeed, observe that both $\tau([X,Y])$ and $\{\tau(X),\tau(Y)\}$ will be a hamiltonian function associated to the vector field $[X,Y]_M$. Thus, we can say that $\tau([X,Y])-\{\tau(X),\tau(Y)\}$ is a constant depending only on $X,Y\in\mathfrak{g}$. Hence, we can define $c:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ by

$$c(X,Y)=\tau([X,Y])-\{\tau(X),\tau(Y)\}.$$

Notice that it is antisymmetric. Moreover, since both $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ satisfy Jacobi identity, it follows that c is a 2-cocycle with values in $\mathbb{R}$; which means that it defines a cohomology class in $H^2(\mathfrak{g},\mathbb{R})$. The assumption that $H^2(\mathfrak{g},\mathbb{R})=0$ implies that there exist $b:\mathfrak{g}\to\mathbb{R}$ such that $c(X,Y)=(\delta b)=-b([X,Y])$ (where δ is the differential used to define Lie algebra cohomology). Finally, we define $\mu:\mathfrak{g}\to C^\infty(M)$ by

$$\mu^*(X)=\tau(X)+b(X).$$

It is clear that μ is linear. Futhermore,

$$\mu^*([X,Y])=\tau([X,Y])+b([X,Y])=\{\tau(X),\tau(Y)\}=$$

$$\{\tau(X)+b(X),\tau(Y)+b(Y)\}=\{\mu^*(X),\mu^*(Y)\},$$

which shows that μ^* is a Lie algebra morphism. Therefore, we have proved that the action $\psi:G\times M\to M$ admits a comoment, so it is a hamiltonian action.

□

Example A.0.14. Let $\varphi:G\times N\to N$ be an action of a Lie group G on a manifold N. By Remark A.0.2, it follows that each diffeomorphism $\varphi_g:N\to N$ can be lifted to a diffeomorphism $\varphi_g^\sharp:T^*N\to T^*N$. In this manner, one obtain a lift of the action $\varphi:G\times N\to N$ to an action $\varphi^\sharp:G\times T^*N\to T^*N$. Moreover, if $\pi:T^*N\to N$ is endowed with its canonical symplectic structure ($\omega=d\alpha$), then this lifted action is naturally a symplectic action. Actually, $\varphi^\sharp$ is a hamiltonian action. Indeed, let

$$\mu:T^*N\longrightarrow\mathfrak{g}^*$$

be defined by $\mu((x,\xi))(X):=\langle\xi,X_N(x)\rangle$, for $p=(x,\xi)\in T^*N$ and $X\in\mathfrak{g}$, where X_N is the fundamental vector field associated to $X\in\mathfrak{g}$ and generated by the action $\varphi:G\times N\to N$. Also, let X_{T^*N} denote the fundamental vector field associated to $X\in g$ and generated by the lifted action. For each $g\in G$, the lifted map $\varphi_g^\sharp$ satisfies $(\varphi_g^\sharp)^*\alpha=\alpha$. As a consequece of this, one obtains that $\mathscr{L}_{X_{T^*N}}\alpha=0$. Then, Cartan's formula implies that $d(i_{X_{T^*N}}\alpha)=i_{X_{T^*N}}\omega$, i.e, $i_{X_{T^*N}}\alpha$ is a hamiltonian function for the fundamental vector field X_{T^*N}. Notice that

$$
\begin{aligned}
X_N(x) &= X_N(\pi(x,\xi)) \\
&= \frac{d}{dt}\bigg|_{t=0} \varphi_{exp(-tX)}(\pi(x,\xi)) \\
&= \frac{d}{dt}\bigg|_{t=0} \pi \circ \varphi^{\sharp}_{exp(-tX)}((x,\xi)) \\
&= (T_{x,\xi}\pi)(X_{T^*N}).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mu^X((x,\xi)) &= \mu((x,\xi))(X) \\
&= \xi(X_N(x)) = \xi((T_{x,\xi}\pi)(X_{T^*N})) \\
&= \alpha_{(x,\xi)}(X_{T^*N})
\end{aligned}
$$

which shows that $\mu : T^*N \to \mathfrak{g}^*$ fulfills condition (i) of Definition A.0.7. To prove that $\mu : T^*N \to \mathfrak{g}^*$ is equivariant (condition (ii) of Defitinion A.0.7)), we need to verify that

$$
\mu \circ \varphi^{\sharp}_g = \mathrm{Ad}^{\sharp}_g \circ \mu, \tag{A.1}
$$

for any $g \in G$. Let $(x,\xi) \in T^*N$, and let $X \in \mathfrak{g}$. On the one hand,

$$
\begin{aligned}
(\mathrm{Ad}^{\circ}_g \mu(x,\xi))(X) &= \mathrm{Ad}^{\sharp}_g(\mu(x,\xi)(X)) \\
&= \mu(x,\xi)(\mathrm{Ad}_{g^{-1}}X) \\
&= \langle \xi, (\mathrm{Ad}_{g^{-1}}X)_N \rangle \\
&= \langle \xi, (T_{\varphi_g(x)}\varphi_{g^{-1}})X_N \rangle
\end{aligned}
$$

On the other hand, by Remark A.0.2, we have $\varphi^{\sharp}_g(x,\xi) = (\varphi_g(x), \xi \circ ((T_{\varphi_g(x)}\varphi_{g^{-1}})))$. Therefore,

$$
\begin{aligned}
(\mu \circ \varphi^{\sharp}_g(x,\xi))(X) &= (\mu((\varphi_g(x), \xi \circ ((T_{\varphi_g(x)}\varphi_{g^{-1}})))))(X) \\
&= \langle \xi \circ (T_{\varphi_g(x)}\varphi_{g^{-1}}), (X_N)(\varphi_g(x)) \rangle \\
&= \langle \xi, (T_{\varphi_g(x)}\varphi_{g^{-1}})X_N \rangle.
\end{aligned}
$$

The last two calculations shows that equality (A.1) holds. In this manner, we have proved that $\mu : T^*N \to \mathfrak{g}^*$ is a moment map for the lifted action $\varphi^{\sharp} : G \times T^*N \to T^*N$.

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