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Topics in Poisson Geometry

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*Este trabalho é dedicado a minha família,
que sempre me apoiou e foi um refúgio para os momentos mais difíceis.*

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“Sharing tea with a fascinating stranger is one of life’s true delights.”

(Iroh, Episode 2.08 “The Chase”)

“There are good reasons why the theorems should all be easy and the definitions hard.”

(Michael Spivak, “Calculus on Manifolds”)

RESUMO

LUIZ, M. N. **Tópicos em Geometria de Poisson**. 2020. 78 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

Este trabalho é uma revisão bibliográfica nas áreas de geometria de Poisson e teoria de grupoides e algebroides de Lie. Como resultado final desta interação apresentamos a abordagem geométrica para o teorema de linearização de Conn para estruturas de Poisson apresentada em [[Crainic and Fernandes 2011](#)].

Palavras-chave: Geometria de Poisson. Grupoides de Lie. Algebroides de Lie. Linearização.

ABSTRACT

LUIZ, M. N. **Topics in Poisson Geometry**. 2020. 78 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

This monograph is a review of Poisson geometry, Lie groupoids and Lie algebroids theory and they relations. We present a summary of the main known results in this areas and as a return of the striking connection of this topics, we present the geometric approach to Conn's linearization theorem present in [[Crainic and Fernandes 2011](#)].

Keywords: Poisson Geometry. Lie Groupoids. Lie Algebroids. Linearization.

CONTENTS

1	INTRODUCTION	17
2	POISSON GEOMETRY	19
2.1	Poisson manifold	19
2.2	The Schouten-Nijenhuis bracket and Poisson cohomology	25
2.2.1	<i>Lower degree computations</i>	28
2.3	Examples	29
2.3.1	<i>Symplectic manifolds</i>	29
2.3.2	<i>The Lie-Poisson structure on \mathfrak{g}^*</i>	32
2.3.3	<i>The tangent lifts of a Poisson structure</i>	33
2.4	The Poisson category	34
2.5	The splitting theorem for Poisson manifolds	36
2.6	Symplectic foliation	38
2.6.1	<i>Singular foliations and singular distributions</i>	39
2.6.2	<i>The symplectic foliation of a Poisson manifold</i>	41
2.6.3	<i>The symplectic foliation of a linear Poisson structure</i>	43
2.7	Isotropy Lie algebra	44
2.8	Complete Poisson manifold	45
2.9	Symplectic realization	46
3	LIE ALGEBROID AND LIE GROUPOID	49
3.1	Lie algebroid	49
3.2	Relation between Lie algebroid theory and Poisson geometry	52
3.2.1	<i>Isotropy algebras and characteristic foliation of a Lie algebroid</i>	55
3.2.2	<i>Splitting theorem</i>	56
3.2.3	<i>Lie algebroid cohomology</i>	58
3.3	Lie groupoid	61
3.3.1	<i>The Lie algebroid of a Lie groupoid</i>	64
3.4	Integrability problem	65
3.5	Lie groupoid cohomology	67
4	CONN'S LINEARIZATION THEOREM	71
4.1	Compact Lie Algebras	72
4.2	Reduction to a cohomological condition	73

BIBLIOGRAPHY 77

INTRODUCTION

The evolution of a Hamiltonian mechanical system takes place in the so called phase-space which is modeled by a symplectic manifold (M, ω) whose dimension is equal to number of degrees of freedom of the system. The equations describing the time evolution are the so called Hamilton's equations which, in the Darboux's coordinates $(q^1, \dots, q^m, p_1, \dots, p_m)$, assume the following form

$$\begin{aligned}\dot{q}^i &= \frac{\partial h}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial h}{\partial q^i}\end{aligned}$$

for all $i \in 1, \dots, m$. The function $h \in C^\infty(M)$ is called the Hamiltonian of the mechanical system and it plays the role of the total energy when the system is not dissipative. Defining

$$\{f, g\} := \sum_{i=1}^n \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i},$$

for all $f, g \in C^\infty(M)$, and given $h \in C^\infty(M)$ the Hamilton's equations can be written as

$$\begin{aligned}\dot{q}_i &= \{h, q_i\} \\ \dot{p}_i &= \{h, p_i\}\end{aligned}$$

for all $i = 1, \dots, m$.

This bracket satisfies the following properties

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$,
2. Bilinearity: $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$,
3. Jacobi identity : $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,
4. Leibniz rule: $\{f, gh\} = g\{f, h\} + \{f, g\}h$,

for all $f, g, h \in C^\infty(M)$ Abstracting these proprieties one can introduce the concept Poisson manifold, e.g. a pair $(M, \{\cdot, \cdot\})$ where M is a smooth manifold and $\{\cdot, \cdot\}$ is a bracket on $C^\infty(M)$ satisfies the previous proprieties. Poisson structures appear in many different areas of mathematics, for example dynamical systems, differential geometry, abstract algebra and representation theory. In all of these areas the possibility to approximate, in a suitable sense, a Poisson structure with a linear one is of the greatest importance and for this reason the results describing the possible normal forms theorems for these geometric structures, play a fundamental role in Poisson geometry.

The main goal of this dissertation is to present a summary of the main known results in this area, emphasizing the geometrical perspective.

This monograph is organized as follows. In Chapter 2 we introduce the basics of the theory of Poisson manifolds like the Weinstein splitting theorem, describing the local structure of Poisson manifolds and implying the existence of the symplectic foliation. In Chapter 3 we present a summary of the theory of Lie groupoids and Lie algebroids and we introduce the key tools to develop the theory presented in the following chapter. Following the Spivak's philosophy, saying that in a theory we must make the hard work on the definitions and win suitable proves, in Chapter 4 we present a geometric approach to Conn's linearization theorem as explained in [Crainic and Fernandes 2011]. Throughout this work, unless stated otherwise, we assume that all the manifolds are smooth.

POISSON GEOMETRY

Proposed by Poisson as a tool for Lagrangian mechanics, and studied by Jacobi and Lie, the Poisson bracket has connections with different areas such as Lie groups and Lie algebras, classical mechanics, singularity theory, quantum mechanics and others [Weinstein 1998].

The goal of this chapter is to introduce the Poisson Geometry. First, we present the basics facts and definitions. In the Section 2.2 we introduce the Schouten bracket, use it to define the cohomology of Poisson manifolds and compute the lower degree groups. Next, we present some important examples like symplectic manifold and the linear Poisson bracket. In the Section 2.4 we define the notion of Poisson morphism and define the Poisson category, showing that the Hamiltonian vector field, defined from a $f \in C^\infty M$, is invariant under Poisson morphism. In Section 2.5 we present the main result in local theory of Poisson manifolds, the Weinstein's splitting theorem. On the next section, we present the main result about the global structures of a Poisson manifold, the symplectic foliation. At last, in the final three sections we define the isotropy algebra, the concept of complete Poisson map and symplectic realization.

2.1 Poisson manifold

Definition 1. . A smooth **Poisson structure on a finite dimensional manifold** M is an \mathbb{R} -bilinear skew-symmetric operation $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$, $(f, g) \mapsto \{f, g\}$ defined on the algebra of the real-valued smooth functions on M satisfying, the:

1. Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

2. Leibniz identity

$$\{f, gh\} = g\{f, h\} + \{f, g\}h,$$

for all $f, g, h \in C^\infty(M)$.

We call the pair $(M, \{\cdot, \cdot\})$ a **Poisson Manifold**. One can check that any manifold with the bracket $\{f, g\} = 0$ for all $f, g \in C^\infty(M)$ is a Poisson manifold.

An important consequence of the \mathbb{R} -bilinearity and the Leibniz identity is that the Poisson bracket is a local operation, i.e., $\{f, g\}(x)$ depends on the values of the functions f, g in a neighborhood of x . Formally speaking, we have:

Proposition 2. Let $f, f' \in C^\infty(M)$ and $x \in M$, and suppose that there is $U \subset M$, an open neighborhood of x , such that $f|_U = f'|_U$. Then

$$\{f, g\}(x) = \{f', g\}(x), \forall g \in C^\infty(M).$$

Proof. Let $\alpha \in C^\infty(M)$ be a function such that $\alpha(x) = 1$ and $\text{supp}(\alpha) = \{x \in M \mid \alpha(x) \neq 0\} \subset U$, then, $\alpha(f - f') = 0$. Let $g \in C^\infty(M)$, thus,

$$\begin{aligned} 0 &= \{g, \alpha(f - f')\}(x) = \alpha(x)\{g, (f - f')\}(x) + \{g, \alpha\}(x)(f - f')(x) \\ &= \{g, (f - f')\}(x) = \{g, f\}(x) - \{g, f'\}(x). \end{aligned}$$

□

Thanks to proposition (2), we can restrict the Poisson bracket to an open set $U \subset M$. For $f, g \in C^\infty(U)$, take a compact subset K of U and $\alpha \in C^\infty(M)$ such that $\text{supp}(\alpha) \subset K$. Hence, we have two global functions $\alpha f, \alpha g \in C^\infty(M)$, satisfying $\alpha f|_K = f|_K$ and $\alpha g|_K = g|_K$. So, we define

$$\{f, g\}|_U := \{\alpha f, \alpha g\}.$$

By proposition (2) the definition of $\{f, g\}|_U$ does not depend on the function α .

There is an important subalgebra of the Lie algebra of smooth vector fields: the Hamiltonian vector fields.

Definition 3. Given f a smooth function, we define **the Hamiltonian vector field of f** , denoted by X_f , as the unique vector field that satisfies

$$X_f(g) := \{f, g\}, \forall g \in C^\infty(M).$$

It is clear that, by skew-symmetry, we have

$$X_f(g) = \{f, g\} = -X_g(f).$$

The following lemma ensures that the subset of a Hamiltonian vector field is truly a Lie subalgebra.

Proposition 4. The map $f \rightarrow X_f$ is a homomorphism from the Lie algebra $C^\infty(M)$ of smooth functions under the Poisson bracket to the Lie algebra of smooth vector fields under the usual Lie bracket. In other words,

$$[X_f, X_g] = X_{\{f, g\}}.$$

Proof. For any $f, g, h \in C^\infty(M)$ we have

$$\begin{aligned} [X_f, X_g]h &= X_f(X_g(h)) - X_g(X_f(h)) \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\}. \end{aligned}$$

By the Jacobi's identity, we have

$$\begin{aligned} [X_f, X_g]h &= \{\{f, g\}, h\} \\ &= X_{\{f, g\}}(h). \end{aligned}$$

Since h is arbitrary, it means that $[X_f, X_g] = X_{\{f, g\}}$. □

Proposition 5. Let φ_t be the flow of a vector field $X \in \mathfrak{X}(M)$ and let $h \in C^\infty(M)$. Then

1. Let $U \subset M$ be an open subset. For all $f \in C^\infty(U)$,

$$\frac{d}{dt}(f \circ \varphi_t) = \{h, f\} \circ \varphi_t = \{h, f \circ \varphi_t\}$$

if, and only if, φ_t is the flow of X_h .

2. If φ_t is the flow of X_h , then $h \circ \varphi_t = h$.

Proof. Let $x \in M$, then

$$\begin{aligned} \frac{d}{dt}(f \circ \varphi_t(x)) &= df_{(\varphi_t(x))} \left(\frac{d}{dt}(\varphi_t(x)) \right) \\ \{h, f\}(\varphi_t(x)) &= df_{(\varphi_t(x))}(X_h(\varphi_t(x))) \end{aligned}$$

the left-hand side of two equations are equal for any $f \in C^\infty(U)$, if, and only if,

$$\frac{d}{dt}(\varphi_t(x)) = X_h(\varphi_t(x)).$$

On the other hand, if φ_t is the flow of X_h , then

$$X_h(\varphi_t(x)) = d_x \varphi_t(X_h(x)).$$

the chain rule implies that

$$\begin{aligned} \frac{d}{dt}(f(\varphi_t(x))) &= df_{(\varphi_t(x))}(X_h(\varphi_t(x))) \\ &= df_{(\varphi_t(x))}(T_x \varphi_t(X_h(x))) \\ &= d(f \circ \varphi_t)_x(X_h(x)) \\ &= \{f \circ \varphi_t, h\}(x). \end{aligned}$$

The second part is the first one when $h = f$. □

And we have the following corollary.

Corollary 6. Let $g, h \in C^\infty(M)$ such that $\{g, h\} = 0$, then g is constant along the integral curves of X_h .

Is it possible to understand the Poisson bracket, $\{\cdot, \cdot\}$, as a C^∞ tensor field $\Pi \in \mathfrak{X}^2(M)$ $\Pi: T^*M \times T^*M \rightarrow \mathbb{R}$ such that

$$\Pi_x(df_x, dg_x) := \{f, g\}(x).$$

In a local coordinate (x^i) , Π has the following representation

$$\{f, g\} = \Pi(df, dg) = \sum_{i,j} \Pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad \forall f, g \in C^\infty(M),$$

where $\Pi^{ij} = \{x^i, x^j\}$. We call Π the Poisson bivector of $(M, \{\cdot, \cdot\})$.

Conversely, given a bivector field Π , we obtain a skew-symmetric bracket on $C^\infty(M)$ by

$$\{f, g\} := \Pi(df, dg).$$

Since the above bracket is a bi-derivation, to ensure that it is Poisson, it is just necessary to find the conditions which ensure that $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

For example, in terms of the Lie derivative of Π , we can write:

$$\begin{aligned} \mathcal{L}_{X_h} \Pi(df, dg) &= X_h(\{f, g\}) - \{X_h(f), g\} - \{f, X_h(g)\} \\ &= \{\{f, g\}, h\} - \{\{f, h\}, g\} - \{f, \{g, h\}\} \\ &= \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \end{aligned}$$

Thus, the Jacobi identity for $\{\cdot, \cdot\}$ can be replaced by $\mathcal{L}_{X_f} \Pi = 0$, for all f .

Another way is in term of a 3-vector field, called Jac.

Proposition 7. For any smooth 2-vector field Π , we can associate to it a 3-vector field defined by

$$\text{Jac}_\Pi(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$

Proof. Since $\{\cdot, \cdot\}$ is \mathbb{R} -bilinear, so is Jac \mathbb{R} -trilinear. The cyclic sum and skew-symmetry of the bracket gives the skew-symmetry of Jac. We just have to check if it satisfies the Leibniz identity. Indeed,

$$\begin{aligned} \{\{f_1 f_2, g\}, h\} &= f_1 \{\{f_2, g\}, h\} + \{f_2, g\} \{f_1, h\} + f_2 \{\{f_1, g\}, h\} + \{f_1, g\} \{f_2, h\}; \\ \{\{g, h\}, f_1 f_2\} &= f_1 \{\{g, h\}, f_2\} + f_2 \{\{g, h\}, f_1\}; \\ \{\{h, f_1 f_2\}, g\} &= f_1 \{\{h, f_2\}, g\} + \{f_1, g\} \{h, f_2\} + f_2 \{\{h, f_1\}, g\} + \{f_2, g\} \{h, f_1\}. \end{aligned}$$

and we have that,

$$\begin{aligned} \text{Jac}(f_1 f_2, g, h) &= f_1(\{\{f_2, g\}, h\} + \{\{g, h\}f_2\} + \{h, f_2\}, g\}) \\ &\quad + f_2(\{\{f_1, g\}, h\} + \{\{g, h\}f_1\} + \{h, f_1\}, g\}) \\ &= f_1 \text{Jac}(f_2, g, h) + \text{Jac}(f_1, g, h)f_2 \end{aligned}$$

□

In local coordinates (x^1, \dots, x^n) Jac_Π can be write as

$$\text{Jac}_\Pi = \sum_{i,j,k} \left(\oint_{i,j,k} \sum_s \frac{\partial \Pi^{i,j}}{\partial x^s} \Pi^{s,k} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k},$$

where $\oint_{i,j,k} a_{ijk}$ means the cyclic sum $a_{ijk} + a_{jki} + a_{kij}$.

Thus, the following statements are equivalents

- $\{\cdot, \cdot\}$ is a Poisson bracket;
- $[X_f, X_g] = X_{\{f,g\}}$ for all f, g ;
- $\mathcal{L}_{X_f} \Pi = 0$ for all f ;
- $\oint_{i,j,k} \sum_s \frac{\partial \Pi_{i,j}}{\partial x^s} \Pi_{s,k} = 0$ for all i, j, k .

A Poisson structure Π on M , defines a **natural vector bundle morphism**,

$$\#: T^*M \rightarrow TM,$$

which maps each covector $\alpha \in T_x^*M$ to the vector $\#(\alpha) \in T_xM$ such that

$$\langle \alpha \wedge \beta, \Pi \rangle_x = \langle \beta, \#_x(\alpha) \rangle_x, \quad \beta \in T_x^*M,$$

where the $\langle \cdot, \cdot \rangle_x$ is the pering between T_x^*M and T_xM .

Definition 8. Let (M, Π) be a Poisson manifold and $x \in M$. **The characteristic space at x** is $C_x := \text{Im} \#_x$. The dimension of C_x is called the **rank** of Π at x . We call x a **regular point** of (M, Π) if there is a neighborhood U of x such that $\text{Rank}(\Pi_x) = \text{Rank}(\Pi_y)$ for every $y \in U$. Otherwise, we call x a **singular point** of (M, Π) . If $\text{Rank}(\Pi_x)$ does not depend on x , then Π is called a regular Poisson structure. We call $M_{\text{reg}} = \{x \in M \mid x \text{ is a regular point}\}$ the set of regular points and we call $M_{\text{sig}} = \{x \in M \mid x \text{ is a singular point}\}$ the set of singular points.

Proposition 9. Let (M, Π) be a Poisson manifold. For every $x \in M$ the rank of Π at x is an even number.

Proof. Suppose $\dim M = d$ and let (U, φ) be a coordinate neighborhood of $x \in M$. On U , the bivector field can be written as

$$\Pi = \sum_{1 \leq i < j \leq d} \{y^i, y^j\} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}.$$

Evaluating the Poisson structure on $x \in U$, we get a skew-symmetric bilinear map $\Pi_x: T_x^*M \times T_x^*M \rightarrow \mathbb{R}$.

Let $U := \{\mu \in T_x^*M \mid \Pi_x(\mu, \nu) = 0 \ \forall \ \nu \in T_x^*M\}$. Choose a complementary space W in T_x^*M

$$T_x^*M = U \oplus W$$

Take any nonzero $q_1 \in W$. So, there is p_1 such that $\Pi_x(q_1, p_1) \neq 0$. Let

$$W_1 = \text{span}\{p_1, q_1\}$$

$$W'_1 = \{w \in W \mid \pi_x(w, v) = 0 \ \forall \ v \in W_1\}.$$

So, we have that $W_1 \cap W'_1 = \{0\}$ and $W = W_1 \oplus W'_1$. Repeating this process we get

$$V = U \oplus W_1 \oplus W_2 \oplus \dots \oplus W_n$$

Thus the rank of Π_x is $2n$. □

The rank of a Poisson manifold is not necessarily continuous with respect to $x \in M$, however, it varies properly with x . Here, the meaning of "properly" follows the next definition

Definition 10. Let X be a topological space and $f: X \rightarrow \mathbb{N}$. We say that f is lower semi-continuous at $x \in X$ if there is a neighborhood V of x such that $f(y) \geq f(x)$ for all $y \in V$.

This definition is not the usual general definition of lower semi-continuity, but it is an adaptation to our context. Equipped with the above definition, we can prove the following proposition.

Proposition 11. The function $\text{Rank}_\Pi: M \rightarrow \mathbb{N}$ defined by $x \mapsto \text{Rank}(\Pi_x)$ is lower semi-continuous.

Proof. Take some local coordinates (x^1, \dots, x^n) around $x \in M$. If the $\text{Rank}(x) = 2n$ the matrix $\Pi^{ij}(x)$ had Rank $2n$, there would be a submatrix, A , of $\#\Pi^{i,j}$ of size $2n \times 2n$ with non-zero determinant. Since the determinant is a continuous function, there is a neighborhood U of x such that the determinant of $A(y)$ must not vanish for all $y \in U$, and the Rank_Π is at least $2n$ on this neighborhood. □

Using the above proposition, we can prove the following one

Proposition 12. the set M_{reg} , of regular points of a Poisson Manifold, is a dense subset of M and the set M_{sing} , of singular points, is a closed nowhere dense subset of M .

Proof. From the definition, M_{reg} is a union of open sets, therefore open. Since M_{sing} is the complement of M_{reg} , it is closed.

To prove that M_{reg} is a dense subset, suppose that $x_0 \in M$ is a singular point and let V be a neighborhood of x_0 . Since $\text{Rank}(y) \in \{0, 1, \dots, n\}$ there is some $y_0 \in V$ such that $\text{Rank}(y) = n$ and then, by the lower semi-continuity, y_0 is a regular point. Since M_{reg} is an open dense subset, its complement, M_{sing} , is a closed nowhere dense. \square

2.2 The Schouten-Nijenhuis bracket and Poisson cohomology

This section follows closely the formulation presented in [Vaisman 2012] and [Dufour and Zung 2006]. Aiming to define the cohomology of Poisson manifold, we present the Schouten-Nijenhuis bracket. This operation is an extension of the Lie bracket of vector fields to the multivector fields on a smooth manifold.

The following theorem is the key to define the Schouten-Nijenhuis bracket.

Theorem 13. There is a unique bilinear operation $[\cdot, \cdot]: \mathfrak{X}^\bullet(M) \times \mathfrak{X}^\bullet(M) \rightarrow \mathfrak{X}^\bullet(M)$, called **the Schouten bracket**, that satisfies the following properties:

I. For all $f, g \in C^\infty(M)$, $[f, g] = 0$;

II. For all $A, B, C \in \mathfrak{X}^\bullet(M)$ then

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(\deg(A)-1)\deg(B)} B \wedge [A, C] \quad (2.2.1)$$

and $\deg([A, B]) = \deg(A) + \deg(B) - 1$;

III. For functions $f, g \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$, the Schouten bracket coincides with the Lie derivative, i.e.,

a) $[X, f] = X(f)$;

b) $[X, Y]$ is the usual Lie bracket for vector fields.

IV. $[A, B] = -(-1)^{(\deg(A)-1)(\deg(B)-1)} [B, A]$;

V. $(-1)^{\deg(A)\deg(C)} [[A, B], C] + (-1)^{\deg(B)\deg(A)} [[B, C], A] + (-1)^{\deg(C)\deg(B)} [[C, A], B] = 0$.

where $\deg(A)$ is the degree of A .

This operation is defined by

$$[X_1 \wedge \dots \wedge X_p, B] = \sum_{i=1}^p (-1)^{i+1} X_1 \wedge \dots \wedge \hat{X}_i \wedge X_p \wedge [X_i, B], \quad (2.2.2)$$

where $X_1, \dots, X_p \in \mathfrak{X}(M)$, $[X_i, A] := \mathcal{L}_{X_i}(A)$, and the hat, as usual, denotes that the X_i is not present in the wedge product.

Using the above definition and the fact that the Schouten bracket must be a local operator, Vaisman proved the theorem (13) in [Vaisman 2012].

One can get another characterization to the Schouten bracket using the following lemma

Lemma 14. For any $A \in \mathfrak{X}^a(M)$, $B \in \mathfrak{X}^b(M)$, $\omega \in \Omega^{a+b-1}(M)$

$$\begin{aligned} i_{[A,B]}(\omega) &= (-1)^{(a-1)(b-1)} i_A(d(i_B(\omega))) \\ &\quad + (-1)^a i_{A \wedge B}(d(\omega)) + (-1)^b d(i_{A \wedge B}(\omega)). \end{aligned}$$

In local coordinates (y^1, \dots, y^n) , any $A \in \wedge^a TM$, with $a \geq 1$, is given by

$$A = \sum_{i_1 < \dots < i_a} A^{i_1, \dots, i_a} \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_a}}. \quad (2.2.3)$$

When $B = f \in C^\infty(M)$ is a smooth function, the equation (2.2.2) is given in coordinates by

$$[A, f] = \sum_{s=1}^a (-1)^{s+1} \sum_{i_1 < \dots < s < \dots < i_a} A^{i_1, \dots, i_a} \frac{\partial f}{\partial y^s} \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\hat{\partial}}{\partial y^s} \wedge \frac{\partial}{\partial y^{i_a}}. \quad (2.2.4)$$

Aiming to express the Schouten bracket in that local coordinate for two arbitrary multi-vector fields, we should put the expansion (2.2.3) in the equation (2.2.2).

$$\begin{aligned} [A, B] &= \sum_{s=1}^a \sum_{(i_1 < \dots < s < \dots < i_a)} \sum_{(j_1 < \dots < i_b)} A^{(i_1, \dots, i_a)} \frac{\partial B^{(j_1, \dots, j_b)}}{\partial y^s} \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\hat{\partial}}{\partial y^s} \wedge \frac{\partial}{\partial y^{i_a}} \\ &\quad \wedge \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_b}} \\ &\quad - (-1)^{(a-1)(b-1)} \sum_{s=1}^a \sum_{(j_1 < \dots < s < \dots < j_b)} \sum_{(i_1 < \dots < i_a)} B^{(j_1, \dots, j_b)} \frac{\partial A^{(i_1, \dots, i_a)}}{\partial y^s} \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\hat{\partial}}{\partial y^s} \wedge \frac{\partial}{\partial y^{j_b}} \\ &\quad \wedge \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_a}} \end{aligned}$$

Using the expression in coordinates for an arbitrary bivector $\Pi = \sum_{i < j} \pi^{i,j} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$, along with the previous formula, one can show that

$$[\Pi, \Pi] = \text{Jac} = \sum_{i,j,k} \left(\oint_{ijk} \sum_s \frac{\partial \Pi^{i,j}}{\partial y^s} \Pi^{s,k} \right) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \wedge \frac{\partial}{\partial y^k}.$$

In other words:

Proposition 15. A bivector field Π is a Poisson bivector if and only if,

$$[\Pi, \Pi] = 0.$$

Since $[\Pi, \Pi]$ is a trivector field, the previous proposition implies that, if M is a two dimensional manifold, then every bivector field Π on M is Poisson.

Definition 16. We define the Poisson cohomology as the cohomology of the following complex

$$\dots \xrightarrow{d_\Pi} \mathfrak{X}^{p-1}(M) \xrightarrow{d_\Pi} \mathfrak{X}^p(M) \xrightarrow{d_\Pi} \mathfrak{X}^{p+1}(M) \xrightarrow{d_\Pi} \dots$$

Where $d_\Pi(A) = [\Pi, A]$.

It is necessary to show that $d_\Pi \circ d_\Pi = 0$.

Proposition 17. If Π is a Poisson tensor, then for every multivector field A , we have

$$[\Pi, [\Pi, A]] = 0.$$

Proof. Let A be a multivector field of Rank a . By the graded Jacobi identity

$$(-1)^{(a-1)(2-1)}[\Pi, [\Pi, A]] + (-1)^{(2-1)(2-1)}[\Pi, [A, \Pi]] + (-1)^{(a-1)(2-1)}[A, [\Pi, \Pi]] = 0.$$

But $[A, \Pi] = -(-1)^{(a-1)(2-1)}[\Pi, A]$ and, since Π is a Poisson tensor, $[\Pi, \Pi] = 0$, thus

$$(-1)^{(a-1)}[\Pi, [\Pi, A]] + (-1)^{(a+1)} = 2(-1)^{(a-1)}[\Pi, [\Pi, A]] = 0.$$

So, $[\Pi, [\Pi, A]] = 0$, as we wished. □

The Poisson cohomology groups are given by the quotient

$$H_\Pi^p(M) = \frac{\text{Ker}(d_\Pi: \mathfrak{X}^p(M) \rightarrow \mathfrak{X}^{p+1}(M))}{\text{Im}(d_\Pi: \mathfrak{X}^{p-1}(M) \rightarrow \mathfrak{X}^p(M))}.$$

Furthermore, the Schouten bracket can be used to describe the Hamiltonian vector fields.

Proposition 18. If (M, Π) is a Poisson manifold and $f \in C^\infty(M)$, then the corresponding Hamiltonian vector field X_f satisfies the equation

$$X_f = -[\Pi, f].$$

Proof. Setting A as a bivector field on the equation (2.2.4), we have that

$$[\Pi, f] = \sum_{i < j} \pi^{ij} \frac{\partial}{\partial x^i} \frac{\partial f}{\partial x^j} = - \sum_{i < j} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = -X_f.$$

□

2.2.1 Lower degree computations

In general, it is very difficult to compute the H_{Π}^1 , but one can give some interpretation to the firsts Poisson cohomology groups.

Definition 19. Let $f \in C^{\infty}(M)$ be a smooth function on a Poisson manifold. We say that f is a **Casimir function** if $X_f = 0$.

The Proposition (18) shows that the $\text{Ker}(d_{\Pi}: C^{\infty}(M) \rightarrow \mathfrak{X}(M))$ is the vector space of Casimir functions and thus $\text{Im}(d_{\Pi}: C^{\infty}(M) \rightarrow \mathfrak{X}(M))$ is the set of Hamiltonian vector fields. So the zero Poisson cohomology group $H_{\Pi}^0(M)$ is the set of Casimir functions.

Definition 20. Let $X \in \mathfrak{X}(M)$. We call X a Poisson vector field if

$$\mathcal{L}_X(\Pi) = 0.$$

For the first Poisson group, we have that the $\text{Ker}(d_{\Pi}: \mathfrak{X}(M) \rightarrow \mathfrak{X}^2(M))$ is the set of vector field such that $\mathcal{L}_X(\Pi) = -[\Pi, X] = 0$. So H_{Π}^1 is the quotients of the Poisson vector fields by the Hamiltonian vector fields.

Definition 21. Let Π_1, Π_2 be two Poisson bi-vectors to a smooth manifold M . We say that Π_1 and Π_2 are **compatible Poisson structures** if $\Pi_1 + \Pi_2$ is a Poisson structure to M .

One can rewrite this condition in terms of the Schouter bracket using that

$$[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = [\Pi_1, \Pi_1] + [\Pi_2, \Pi_2] + 2[\Pi_1, \Pi_2] = 2[\Pi_1, \Pi_2].$$

So, two Poisson structures are compatible if, and only if,

$$[\Pi_1, \Pi_2] = 0.$$

If Π_1 and Π_2 are compatible Poisson structures, so will do $\varepsilon_1 \Pi_1$ and $\varepsilon_2 \Pi_2$.

Let $\Pi(\varepsilon)$ be a smooth family of Poisson structures. Write

$$\Pi(\varepsilon) = \Pi_0 + \varepsilon \Pi_1 + \varepsilon^2 \Pi_2 \dots$$

as a power series expression. To $\Pi(\varepsilon)$ be a Poisson structure we need that

$$\begin{aligned} [\Pi(\varepsilon), \Pi(\varepsilon)] &= [\Pi_0, \Pi_0] + 2\varepsilon[\Pi_0, \Pi_1] + \varepsilon^2(2[\Pi_0, \Pi_2] + [\Pi_1, \Pi_1]) + \dots \\ &= 0. \end{aligned}$$

The coefficient Π_1 is called an **infinitesimal deformation** of Π_0 when

$$d_{\Pi_0}(\Pi_1) = [\Pi_0, \Pi_1] = 0.$$

Since $d^2 = 0$, every Π defined by

$$\Pi_1 = d_{\Pi_0}(X) = [\Pi_0, V] = -\mathcal{L}_X(\Pi_0)$$

is an infinitesimal deformation. Let φ_t be the flow of $-X$. For each t , the pull-back $\varphi_t^*\Pi_0$ is again a Poisson structure. Furthermore,

$$\left. \frac{d}{dt} \varphi_t^*\Pi_0 \right|_{t=0} = \mathcal{L}_{-X}(\Pi_0)\Pi_1.$$

On this case, we call Π_1 a **trivial infinitesimal deformation** of Π_0 . This deformation is trivial in the sense that $\frac{d}{dt} \varphi_t^*\Pi_0$ are the same Poisson structure up to change of coordinate. So we have that the second Poisson cohomology group is given by

$$H_{\Pi}^2 = \frac{\text{infinitesimal deformation of } \Pi}{\text{trivial infinitesimal deformation of } \Pi}$$

It is possible to extend the definition of $\#$ to forms of any degree by

$$\#(\omega)(\alpha_1, \dots, \alpha_r) = (-1)^r(\#\alpha_1, \dots, \#\alpha_r).$$

We can associate the Poisson cohomology with de Rham cohomology by the next proposition.

Proposition 22. For every smooth differential form η on a Poisson manifold M , we have

$$d_{\pi}(\#\omega) = \#(d\omega),$$

where d_{π} is the differential operation on the Poisson cohomology and d is the differential operation on de Rham cohomology.

The demonstration of the above proposition can be find in [[Dufour and Zung 2006](#)].

2.3 Examples

2.3.1 Symplectic manifolds

Definition 23. Let M be a manifold. A 2-form ω on M is called **symplectic** if it is closed and non-degenerate. We call the pair (M, ω) a **symplectic manifold**.

The notion of symplectormorphism is defined as:

Definition 24. Let (M_1, ω_1) and (M_2, ω_2) be a symplectic manifold and let $\varphi: M_1 \rightarrow M_2$ be a diffeomorphism. We call φ a symplectomorphism if

$$\omega_2(\varphi_*(X), \varphi_*(Y)) = \omega_1(X, Y).$$

For every f smooth function on the symplectic manifold M , we can associate a **Hamiltonian vector field**, X_f , by the only vector field such that, $i_{X_f}\omega = df$, in other words, for every $Y \in \mathfrak{X}(M)$,

$$\omega(X_f, Y) = df(Y) = Y(f).$$

Also, we have another important subset of vector field of a symplectic manifold, which is called locally Hamiltonian.

Definition 25. A vector field X of a symplectic manifold (M, ω) is called **locally Hamiltonian** if $\mathcal{L}_X(\omega) = 0$. Since ω is closed, by the Cartan's magic formula,

$$\mathcal{L}_X(\omega) = i_X d\omega + d(i_X \omega) = d(i_X \omega).$$

Therefore, equivalently, a vector field is locally Hamiltonian if $i_X \omega$ is closed.

By the Poincaré lemma [Abraham, Marsden and Ratiu 1993], if $\omega \in \Omega^k(M)$ is closed, for every $x \in M$ there is an open neighborhood U of x for which $\omega|_U \in \Omega^k(U)$ is exact. As long as $i_X \omega$ is closed, there locally is $h \in C^\infty(M)$ such that $i_X \omega = dh$. Hence, locally $X = X_h$ and therefore it justifies its name.

For every $\omega \in \Omega^k(M)$, and every $X \in \mathfrak{X}(M)$

$$\frac{d}{dt} \varphi_t^X \omega = (\varphi_t^X)^* (\mathcal{L}_X(\omega))$$

where φ_t^X is the flow of the vector field X [Marsden and Ratiu 2002]. If X is locally Hamiltonian, then

$$\frac{d}{dt} \varphi_t^X \omega = 0.$$

So the flow φ_t^X is a local symplectomorphism, if, and only if, X is locally Hamiltonian.

Using the Hamiltonian vector field, we can define a Poisson bracket on the smooth functions of (M, ω) .

Proposition 26. The bilinear, skew-symmetric bracket on $C^\infty(M)$, defined by,

$$\{f, g\} := \omega(X_f, X_g) = X_f(g) = -X_g(f),$$

is a Poisson Bracket.

The bilinearity and the skew-symmetry are inherited from ω , thus is just necessary to check the Leibniz and Jacobi rules. But, before we prove the above proposition, we will discuss some proprieties of this bracket.

Firstly, we have that the above bracket commutes with the flow of locally Hamiltonian vector fields, as can be seen in the next proposition.

Proposition 27. If X is a locally Hamiltonian vector field, then

$$(\varphi_t^X)^* \{f, g\} = \{(\varphi_t^X)^* f, (\varphi_t^X)^* g\}. \quad (2.3.1)$$

Proof. If X is locally Hamiltonian then $(\varphi_t^X)^* \omega = \omega$, so

$$(\varphi_t^X)^* \{f, g\}(x) = (\varphi_t^X)^* (\omega(X_f, X_g)(x)) = \omega(X_f, X_g)(\varphi_t^X(x)) = \{(\varphi_t^X)^* f, (\varphi_t^X)^* g\}.$$

□

Taking the time derivative of the left-hand side of equation (2.3.1) at $t = 0$, one has

$$\left. \frac{d}{dt} \right|_{t=0} ((\varphi_t^X)^* \{f, g\}) = \mathcal{L}_{X_h}(\{f, g\}),$$

while taking the time derivative at $t = 0$ of the the right-hand side, one has

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (\{(\varphi_t^X)^* f, (\varphi_t^X)^* g\}) &= \left\{ \left. \frac{d}{dt} \right|_{t=0} ((\varphi_t^X)^* f, (\varphi_t^X)^* g) \right\} + \left\{ (\varphi_t^X)^* f \Big|_{t=0}, \left. \frac{d}{dt} \right|_{t=0} ((\varphi_t^X)^* g) \right\} \\ &= \{\mathcal{L}_{X_h}(f), g\} + \{f, \mathcal{L}_{X_h}(g)\} \end{aligned}$$

Therefore, we proved the following corollary.

Corollary 28. For every $f, g, h \in C^\infty(M)$ the following identity holds:

$$\mathcal{L}_{X_h}(\{f, g\}) = \{\mathcal{L}_{X_h}(f), g\} + \{f, \mathcal{L}_{X_h}(g)\}.$$

Now, we are ready to prove the Jacobi and Leibniz identities:

Proof. • The Jacobi Identity:

$$\{h, \{f, g\}\} = \mathcal{L}_{X_h}(\{f, g\})$$

thus, by the corollary (28) we have

$$\begin{aligned} \{h\{f, g\}\} &= \mathcal{L}_{X_h}(\{f, g\}) = \{\mathcal{L}_{X_h}(f), g\} + \{f, \mathcal{L}_{X_h}(g)\} \\ &= \{\{h, f\}, g\} + \{f, \{h, g\}\} \\ &= \{g, \{h, f\}\} - \{f, \{g, h\}\}. \end{aligned}$$

• Leibniz Rule

$$\{g, hf\} = X_g(hf) = hX_g(f) + fX_g(h) = h\{g, f\} + f\{g, h\}.$$

□

Note that a necessary condition to the bracket of a symplectic manifold to satisfy the Jacob identity is $\mathcal{L}_{X_h}(\omega) = d(i_{X_h}\omega)$, and this is true for all vector fields only when ω is closed. According to the proposition (9) the bracket defined in the proposition (26) has even Rank, and, therefore, it will be the Rank of the symplectic form ω . Since the ω is non-degenerated, every symplectic manifold must have an even dimension.

In the Section 2.5, we will see that locally, every even dimensional manifold admits a symplectic structure. Although there is no general condition for an arbitrary even-dimensional manifold to admit a symplectic structure, certainly, the possible obstructions have a topological nature. However, in dimension two, there is an answer: a 2-dimensional manifold admits a symplectic structure if, and only if, it is orientable. [Fernandes and Marcut 2015].

The cotangent bundle of every manifold has a special symplectic structure called **canonical form**.

let

$$\pi := T^*M \rightarrow M \quad (2.3.2)$$

$$p = (x, \xi) \mapsto x, \quad (2.3.3)$$

be the canonical projection, thus, pointwise, we define the tautological 1-form by the following

Definition 29. The **tautological 1-form** on T^*M as

$$\Theta_p = \pi_p^* \xi \in T_p^*(T^*M)$$

Let $(U_i, x_1^i, \dots, x_n^i)$ be an atlas to M . One can construct the manifold structure, $(T^*U_i, x_1, \dots, x_n, p_1, \dots, p_n)$, to T^*M using, for a $m \in M$, the basis $\{(dx_1)_m, \dots, (dx_n)_m\}$ of T_m^*M and the relation $\alpha_m = \sum_{i=1}^n \xi_i(dx^i)_m$. Thus, at this coordinates,

$$\Theta = \sum_{i=1}^n p_i dx^i.$$

From this definition, it is possible to induce a symplectic form on the cotangent bundle of M , called **canonical symplectic 2-form** ω , as minus the exterior derivative of Θ , i.e.,

$$\omega = -d\Theta.$$

In the same local coordinates,

$$\omega = \sum_{i=1}^n dx^i \wedge dp^i.$$

2.3.2 The Lie-Poisson structure on \mathfrak{g}^*

A **linear Poisson structure** on a vector space V , is defined by the property that the Poisson bracket of two linear functions is again a linear function. The linear Poisson structures are closely related to Lie algebras structures on the dual space of V .

Let a finite-dimensional Lie algebra, \mathfrak{g} , with bracket $[\cdot, \cdot]$, we denote the dual space of \mathfrak{g} by \mathfrak{g}^* . Every element e of \mathfrak{g} defines a linear functional e^* by

$$e^* : \mathfrak{g}^* \rightarrow \mathbb{R} \quad (2.3.4)$$

$$\xi \mapsto \langle \xi, e \rangle := \xi(e). \quad (2.3.5)$$

If $f \in C^\infty(\mathfrak{g}^*)$, then $d_\xi f : T_\xi \mathfrak{g}^* \rightarrow \mathbb{R}$ and $T_\xi \mathfrak{g}^*$ is naturally isomorphic with \mathfrak{g}^* , one can think of $d_\xi f$ as an element of $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Thus, one can define $\{\cdot, \cdot\} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(\mathfrak{g}^*)$ by

$$\{f, g\}(\xi) := \langle \xi, [d_\xi f, d_\xi g] \rangle.$$

If (e_1, \dots, e_d) is a basis of \mathfrak{g} , and (x_1, \dots, x_d) is the dual basis one can define (x_1, \dots, x_n) global coordinates on \mathfrak{g} by $x_i(v) = \langle e_i^*, v \rangle$ and global coordinates on \mathfrak{g}^* by $y_i(\alpha) = \langle e_i, \alpha \rangle$.

If $f \in C^\infty(\mathfrak{g}^*)$

$$d_\xi f = \frac{d}{dt} \Big|_{t=0} f(\xi + \alpha t) = \sum_{i=1}^n \frac{\partial f}{\partial y_i}(\xi) \alpha_i = \left\langle \sum_{i=1}^n \frac{\partial f}{\partial y_i}(\xi), \alpha \right\rangle$$

and

$$[d_\xi f, d_\xi g] = \sum_{i,j} \frac{\partial f}{\partial y_i}(\xi) \frac{\partial g}{\partial y_j}(\xi) [e_i, e_j] = \sum_{i,j} \frac{\partial f}{\partial y_i}(\xi) \frac{\partial g}{\partial y_j}(\xi) \sum_k c_{i,j}^k e_k \quad (2.3.6)$$

$$= \sum_{i,j} \left(\sum_k c_{i,j}^k y_k \right) \frac{\partial f}{\partial y_i}(\xi) \frac{\partial g}{\partial y_j}(\xi) \quad (2.3.7)$$

By the bilinearity of $\{\cdot, \cdot\}$ and $[\cdot, \cdot]$

$$\{f^*, g^*\} = [g, f]^*, \quad \forall f, g \in \mathfrak{g}, \quad (2.3.8)$$

and also

$$\{\{x^i, x^j\}, x^k\} = [[e_i, e_j], e_k]^*$$

and this proves the Jacobi identity.

On [Laurent-Gengoux, Pichereau and Vanhaecke 2012], one finds the following result

Theorem 30. For every finite-dimensional vector space V , there is a natural one-to-one correspondence between a linear Poisson structures on V^* and Lie algebra structures on V .

2.3.3 The tangent lifts of a Poisson structure

Given a Poisson manifold (M, Π) there is a canonical way to define a Poisson structure, Π_{TM} , on the tangent bundle TM . First of all, we define the tangent lift of a smooth function on M .

Definition 31. Let $f \in C^\infty(M)$, we call the **the tangent lift** of f , the function $\bar{f} \in C^\infty(TM)$ defined as follow: for every $v \in TM$

$$\bar{f}(v) = v(f).$$

using a analogous construction of the cotangent coordinates, one can construct tangent coordinates (x^i, y^i) associated to the bases $\{(\frac{\partial}{\partial x^i})_x, \dots, (\frac{\partial}{\partial x^n})_x\}$ of T^*M . Thus, in this coordinates,

$$\bar{f} = \sum_{i=1}^n \frac{\partial f}{\partial x^i} y^i.$$

Those functions are called **fiber-wise linear function**. Fiber-wise linear in the sense that, for every fiber $\text{proj}^{-1}(x)$ on TM , the function \bar{f} is a linear function.

We define the Poisson structure, Π_{TM} , on TM through the next theorem

Theorem 32. Let (M, Π) be a Poisson manifold. The Poisson bivector field Π_{TM} defined locally by

$$\Pi_{TM} = \sum_{i,j} \Pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} \sum_{i,j,k} \frac{\partial \Pi^{ij}}{\partial x^k} y^k \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j},$$

is the unique Poisson structure such that the associated bracket satisfies

$$\{\bar{f}, \bar{g}\}_{TM} = \{f, g\}, \text{ for all } f, g \in C^\infty(M).$$

The Poisson structure defined on the above theorem is called fiber-wise linear Poisson structure on TM .

2.4 The Poisson category

The Poisson category is the category which objects are Poisson manifolds and which morphisms are defined here bellow.

Definition 33. Let $(M, \{\cdot, \cdot\}_M)$ and $(N, \{\cdot, \cdot\}_N)$ be two Poisson manifolds. A Poisson map, or a Poisson morphism, is a smooth map $\psi: M \rightarrow N$, satisfying:

$$\{f \circ \psi, g \circ \psi\}_M = \{f, g\}_N \circ \psi,$$

for all $f, g \in C^\infty(N)$. Or, equivalently, for the Poisson bivector

$$\psi_*(\Pi_M)_x = (\Pi_N)_x,$$

where Π_M and Π_N are the Poisson bivector associated to $\{\cdot, \cdot\}_M$ and $\{\cdot, \cdot\}_N$, respectively.

The following proposition ensures that we do have a category.

Proposition 34. Let (M, Π_M) , (N, Π_N) and (P, Π_P) be Poisson manifolds, and let $\psi: M \rightarrow N$ and $\varphi: N \rightarrow P$ be differentiable maps.

- I. If ψ and φ are Poisson maps, then $\varphi \circ \psi$ is a Poisson map,
- II. If ψ and $\varphi \circ \psi$ are Poisson maps and, furthermore, if ψ is surjective, then ψ is a Poisson morphism. In particular, if ψ is a Poisson morphism and a diffeomorphism, then ψ^{-1} is a Poisson map. In this case, we say that ψ is a **Poisson diffeomorphism**, or an isomorphism on the category of Poisson manifolds.

Proof. The first claim comes immediately from the definition of the Poisson map. For the second one, let $x \in M$, then

$$\begin{aligned} (\#_P)_{\varphi(\psi(x))} &= T_x(\varphi \circ \psi) \circ (\#_M)_x \circ T_x^*(\varphi \circ \psi) \\ &= T_{\psi(x)}(\varphi) \circ T_x(\psi) \circ (\#_M)_x \circ T_x^*(\psi) \circ T_{\psi(x)}^*(\varphi) \\ &= T_{\psi(x)}(\varphi) \circ (\#_N)_{\psi(x)} \circ T_{\psi(x)}^*(\varphi). \end{aligned}$$

□

The Poisson vector fields are invariant under Poisson morphism.

Proposition 35. Let $\psi: M \rightarrow N$ be a Poisson map between $(M, \{\cdot, \cdot\}_M)$ and $(N, \{\cdot, \cdot\}_N)$. For every $h \in C^\infty N$

$$X_h = T\psi(X_{h \circ \psi}).$$

or, equivalently,

$$(\#_N)_{(\psi(x))} = T_x\psi \circ (\#_M)_x \circ T_x^*\psi, \quad \forall x \in M.$$

Proof. Using the definitions of Hamiltonian vector field and Poisson map, for every $f \in C^\infty N$, we have

$$\begin{aligned} X_{h \circ \psi}(f \circ \psi) &= \{h \circ \psi, f \circ \psi\}_M = \{h, f\}_N \circ \psi \\ &= X_h(f) \circ \psi. \end{aligned}$$

One can understand equivalent equation by the commutativity of the following diagram, and, to prove it is necessary to do nothing but replace the Poisson bracket by the bivector field.

$$\begin{array}{ccc} T_x^*M & \xrightarrow{\#_M(x)} & T_xM \\ T_x^*(\psi) \uparrow & & \downarrow T_x(\psi) \\ T_{\psi(x)}^*N & \xrightarrow{\#_N(\psi(x))} & T_{\psi(x)}N \end{array}$$

□

Furthermore for every $h \in C^\infty M$, $\mathcal{L}_{X_h}(\Pi) = 0$. So, the flow φ_{X_h} is a local Poisson diffeomorphisms and if X_h is a complete vector field, φ_{X_h} is a global Poisson diffeomorphisms.

Example 36. Let $M = \mathbb{R}^{2n}$ with coordinates $(p^1, \dots, p^n, q^1, \dots, q^n)$ and the Poisson bivector field given by

$$\Pi = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i},$$

let $N = \mathbb{R}^n$ with coordinates (q^1, \dots, q^n) and the trivial bracket. The projection $\text{proj}: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$, $(p^1, \dots, p^n, q^1, \dots, q^n) \mapsto (q^1, \dots, q^n)$ is a Poisson map.

Example 37. If there is morphism between two Lie algebras, $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$, the dual transformation, $\varphi^*: \mathfrak{h}^* \rightarrow \mathfrak{g}^*$, is a Poisson morphism in the opposite direction. Indeed, for each $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{h}^*$, we have

$$\begin{aligned} \{f \circ \varphi^*, g \circ \varphi^*\}_{\mathfrak{h}^*}(\xi) &= \langle \xi, [d_\xi(f \circ \varphi^*), d_\xi(g \circ \varphi^*)] \rangle \\ &= \langle \xi, [d_{\varphi^*(\xi)}f \circ d_\xi \varphi^*, d_{\varphi^*(\xi)}g \circ d_\xi \varphi^*] \rangle \\ &= \langle \xi, [d_{\varphi^*(\xi)}f \circ \varphi^*, d_{\varphi^*(\xi)}g \circ \varphi^*] \rangle \\ &= \langle \xi, [\varphi(d_{\varphi^*(\xi)}f), \varphi(d_{\varphi^*(\xi)}g)] \rangle \\ &= \langle \xi, \varphi([d_{\varphi^*(\xi)}f, d_{\varphi^*(\xi)}g]) \rangle \\ &= \langle \xi, [d_{\varphi^*(\xi)}f, d_{\varphi^*(\xi)}g] \circ \varphi^* \rangle \\ &= \langle \varphi^*(\xi), [d_{\varphi^*(\xi)}f, d_{\varphi^*(\xi)}g] \rangle \\ &= \{f, g\}_{\mathfrak{g}^*}(\varphi^*(\xi)). \end{aligned}$$

Since "take the dual" is functorial, so is the above definition. In this way, we have a contravariant functor from the category of Lie algebras to the category of Poisson manifolds.

Definition 38. A **Poisson submanifold** is a submanifold N of a Poisson manifold M such that, N has a Poisson structure that makes the inclusion map a Poisson map.

Lemma 39. $N \subset M$ is a Poisson submanifold if, and only if, each tangent space $T_x N$ contains the image of $\#_x: T_x^* M \rightarrow T_x M$, in other words, if, and only if, all Hamiltonian vector fields are tangent to N .

2.5 The splitting theorem for Poisson manifolds

A classic result of symplectic geometry is the Darboux's theorem [Dufour and Zung 2006]. Roughly speaking, it says that two symplectic manifold of the same dimension locally look like the same. Formally

Theorem 40 (Darboux). Let ω be a non-degenerate two-form on a manifold M . Then $d\omega = 0$ if, and only if, for every $m \in M$, there exists a chart $(U, q^1, \dots, q^n, p^1, \dots, p^n)$ such that $q^i(m) = p^i(m) = 0$ and

$$\omega = \sum_{i=1}^n dq^i \wedge dp^i.$$

The difference between a generic Poisson structure and one that comes from a symplectic form is that the Rank of the generic one does not need to be maximal. In these terms, it is impossible that all the Poisson structures are locally the same. However, we have an analogous result, the Splitting Theorem, which tells us that for a Poisson manifold, at any point $x \in M$, the Poisson structure splits in a non-degenerate part, the symplectic one, and in a new Poisson structure that vanishes at x . The Darboux's theorem is just a corollary of the Splitting theorem. To prove it, we will need the following result [[Laurent-Gengoux, Pichereau and Vanhaecke 2012](#)]:

Theorem 41 (Straightening theorem). Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ and $x \in M$ satisfying $X(x) \neq 0$. Then, there is a coordinate neighborhood (U, y_1, \dots, y_m) such that $X|_U = \frac{\partial}{\partial y_1}$.

Theorem 42 (Splitting theorem). Let (M, Π) be a Poisson manifold, let $x \in M$ and let $2r$ be the Rank of Π at x . There exists a coordinate neighborhood $(U, q^1, \dots, q^r, p^1, \dots, p^n, z^1, \dots, z^s)$, centered at x , such that, on U ,

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} + \sum_{1 \leq k < l \leq s} \Pi^{k,l} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l}.$$

The functions $\Pi^{k,l}$ are smooth functions which depend only on $z = (z_1, \dots, z_s)$, and which vanish at $z = 0$. The local coordinates $(q_1, \dots, q_r, p_1, \dots, p_r, z_1, \dots, z_s)$ are called splitting coordinates centered at x .

In [[Weinstein 1983](#)], Alan Weinstein presents the following demonstration.

Proof. Since this result is local, we can prove it considering $M = \mathbb{R}^n$. We will proceed by induction on r . If $r = 0$, we are done. The induction hypotheses are that the above theorem is true for every Poisson manifold, at every point where the Rank is $2(r-1)$. Take x in a Poisson manifold (M, Π) such that $\text{Rank}_x \Pi = 2r$.

Since $\text{Rank}_x \Pi > 0$, there exists a local function p which Hamiltonian vector field $\#(dp) = X_p$ does not vanish at x . Without losing generality, we may suppose that $p(x) = 0$. Since $X_p(x) \neq 0$, there exists, by the straightening theorem, a system of coordinates (q, y'_2, \dots, y'_d) on a neighborhood U' centered at x , such that $X_p = \frac{\partial}{\partial q}$. For $i = 2, \dots, d$, we have that

$$\{p, q\} = X_p(q) = \frac{\partial q}{\partial q} = 1, \quad [X_p, X_q] = X_{\{p, q\}} = X_1 = 0, \quad X_p(y'_i) = 0.$$

Thus, in terms of these coordinates,

$$X_q = \sum_{i=2}^d \alpha_i \frac{\partial}{\partial y'_i},$$

and the coefficients α_i are independent of q . Indeed, for any smooth function f we have

$$\begin{aligned} X_q(X_p(f)) &= X_p(X_q(f)) \\ \implies \alpha_i \frac{\partial^2 f}{\partial y_i \partial q} &= \frac{\partial \alpha_i}{\partial q} \frac{\partial f}{\partial y_i} + \alpha_i \frac{\partial^2 f}{\partial y_i \partial q} \end{aligned}$$

So X_q is independent of q and does not vanish at x . Applying the straightening theorem once more, on the set (q_0, y'_2, \dots, y'_d) , we may introduce a system of coordinates (q, p', y_3, \dots, y_d) on a neighborhood of x , where (p', y_3, \dots, y_d) depend only on (y'_2, \dots, y'_d) , with

$$\frac{\partial}{\partial p'} = X_q.$$

Substituting p' by p we consider (q, p, y_3, \dots, y_d) , so the Poisson brackets, in these new coordinates, take the following form,

$$\begin{aligned} \{p, q\} &= 1 \\ \{q, y_i\} &= \frac{\partial y_i}{\partial p'} = 0 \\ \{p, y_i\} &= \frac{\partial y_i}{\partial q} = 0 \end{aligned}$$

and we conclude that the Poisson structure Π is given by

$$\Pi = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} + \sum_{3 \leq k < l < d} \{y_k, y_l\} \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_l}.$$

Since $\{\{y_k, y_l\}, p\} = \{\{y_k, y_l\}, q\} = 0$ for all k, l , we have that $\{y_k, y_l\}$ does not depend of p and q .

□

Definition 43. The first term of the above decomposition is called the **symplectic part** while the second term is called the **singular part**.

Thus, we can see the Poisson manifold locally as a product $S \times N$ such that S is symplectic and N is Poisson with $\text{Rank}(x_0) = 0$. When the Poisson structure is also symplectic, that is, it is non-degenerate, we recover the Darboux theorem.

Now, one can reduce the problem of local classification of Poisson structures to classify the Poisson structures Π on \mathbb{R}^n , such that $\Pi(0) = 0$, i.e., we can assume that, locally, every Poisson manifold is \mathbb{R}^n and that $\Pi(0) = 0$.

2.6 Symplectic foliation

The Weinstein's splitting theorem describes the local structure of a Poisson manifold. Globally, roughly speaking, we can think of a Poisson manifold as a disjoint union of symplectic ones, not necessarily of the same dimension.

We begin this section by presenting some definitions and results on the distribution theory – in differential geometry sense. Next, we discuss the integrability of the distribution generated by the Hamiltonian vector fields and how it gives a symplectic structure on each leaf. We conclude presenting the example of the symplectic leaves of a linear Poisson structure. We follow closely [Vaisman 2012, Dufour and Zung 2006].

2.6.1 Singular foliations and singular distributions

A intuitive way to think of a foliation is as a partition of a manifold M in immersed and connected submanifolds, called the leaves of the foliation. We started by defining a regular foliation.

Definition 44. [Moerdijk and Mrcun 2003] Let M be a smooth manifold of dimension n . A **foliation atlas of co-dimension q** on M is an atlas

$$(\varphi_i: U_i \rightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i \in I}$$

of M for which the change of coordinates, φ_{ij} , are locally of the form

$$\varphi_{ij} = (g_{ij}(x, Y), h_{ij}(y))$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$. A **foliation of co-dimension q** on M is a maximal foliation atlas of M .

The above definition includes just the case where the leaves have the same dimension. It is possible to extend this definition to permit leaves of different dimension.

Definition 45. A **smooth singular foliation** on a smooth manifold M , in the sense of Stefan-Sussmann is a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of M into a disjoint union of smooth immersed connected submanifolds \mathcal{F}_α , which satisfies two conditions. For every $x \in M$, let \mathcal{F}_x be the leaf that contains x and suppose that its dimension is d .

- I. There is a smooth local chart of M with coordinates (x_1, \dots, x_m) in a neighborhood U of x , $U = \{-\varepsilon < x_1 < \varepsilon, \dots, -\varepsilon < x_m < \varepsilon\}$, such that the d -dimensional disk $\{y_{d+1} = \dots = y_m = 0\}$ coincides with the path-connected component of the intersection of \mathcal{F}_x with U which contains x ;
- II. Each d -dimensional disk $\{y_{d+1} = c_{d+1}, \dots, y_m = c_m\}$, where c_{d+1}, \dots, c_m are constants, is entirely contained in some leaf \mathcal{F}_α .

It is possible to replace the previous definition using sub-bundles of TM . A **general distribution** on M is a choice for every $x \in M$ of subspace $D_x \subset T_x M$ which dimension may depend on x . A general distribution is called smooth if, for any point $x \in M$, there exists a set

$\{X_1, \dots, X_m\}$ of vector field on M such that $D_x = \text{Span}\{X_1(x), \dots, X_m(x)\}$. Of course, m may depend on the point x . When the dimension of D_x is constant, the distribution is called **regular**, otherwise, it is called **singular**.

Every family C of smooth vector fields on M gives rise to a smooth singular distribution D^C defined as follows: for each $x \in M$, D_x^C is the vector space spanned by the values at x of the vector fields of C .

A distribution D is called **invariant with respect to a family of smooth vector fields C** if, for every $X \in C$, we have

$$(\varphi_X^t)_* D_x = D_{\varphi_X^t(x)},$$

where (φ_X^t) is the local flow of X .

Definition 46. We say that a smooth singular distribution D is **integrable** if every point of M is contained in a maximal immersed submanifold W , such that for every $y \in W$ the tangent space $T_y W = D_y$.

One can ask about the conditions to guarantee that a distribution is integrable. In the regular case, the answer to this question is given by Frobenius theorem [[Warner 1983](#)].

Theorem 47 (Frobenius). The condition to integrability on the regular case is that the distribution is involutive, i. e., if $[X, Y] \in D$ whenever X and Y are smooth vector fields lying in D .

For the singular case, the answer is provided by the Stefan-Sussmann's theorem [[Dufour and Zung 2006](#)].

Theorem 48 (Stefan-Sussmann). Let D be a smooth singular distribution on M . Then the following three conditions are equivalent:

- D is integrable.
- D is generated by a family C of smooth vector fields, and it is invariant with respect to C .
- D is the tangent distribution of a smooth singular foliation \mathcal{F} .

The classical notion of involutivity can be extended to general distributions as follows.

Definition 49. A distribution D is said involutive if there exists a Lie subalgebra $\mathfrak{X}(D)$ of $\mathfrak{X}(M)$ such that the span of the values at x of the fields of $\mathfrak{X}(D)$ equals D_x , for all $x \in M$.

The following theorem [[Vaisman 2012](#)] gives a condition for integrability of general smooth distribution in terms of involutivity.

Theorem 50 (Vilyantsev-Frobenius). A smooth distribution D is integrable if, and only if, it satisfies the following two conditions:

- I. D is involutive;
- II. For all $x \in M$ and for all $v \in D_x$, there is a path $\gamma(t)$, such that $\gamma(0) = x$, $\gamma'(0) = v$ and which $\text{Rank}(D_{\gamma(t)})$ is constant.

2.6.2 The symplectic foliation of a Poisson manifold

Using the characteristic spaces introduced on definition (8), one can define a differentiable singular distribution on every Poisson manifold (M, Π) as follows:

Definition 51. For all $x \in M$,

$$C_x = \text{Span}\{\Pi_x(df) \mid f \in C^\infty(M)\}$$

is called **the characteristic distribution**.

By definition, C is generated by the family of Hamiltonian vector fields and it is invariant with respect to them. Furthermore, by the relation

$$X_{\{f,g\}} = [X_f, X_g],$$

the set of Hamiltonian vector fields is a Lie algebra and C is an involutive distribution.

In the case of regular Poisson structure, the integrability of the characteristic distribution comes from the Frobenius' theorem. Otherwise, since the flow of every Hamiltonian vector field is a Poisson automorphism, one can conclude the integrability of the characteristic distribution using Theorem (48) or (50). We will denote the correspondent singular foliation by \mathcal{F} .

Since for all $x \in M$, the dimension of D_x is even, the leaves of \mathcal{F} are even dimension submanifolds.

For every $x \in M$, one defines a symplectic structure on \mathcal{F}_x as follows. If $f, g \in C^\infty(\mathcal{F}_x)$, we can extend them locally to functions $\tilde{f}, \tilde{g} \in C^\infty(M)$, we define the bracket between f and g as

$$\{\tilde{f}, \tilde{g}\}(x) = \{f, g\} = X_f(g)(x) = -X_g(f)(x).$$

Since the bracket is computed along the flow of X_f and X_g , which belongs to \mathcal{F}_x , the result depends only \tilde{f} and \tilde{g} . Now, we can call this foliation by **symplectic foliation** of the Poisson manifold M .

There is a more explicit construction to the symplectic foliation, which can be seen at [Laurent-Gengoux, Pichereau and Vanhaecke 2012]. In this way, the idea is to construct the manifold using the Hamiltonian vector field. For each $x \in M$, we define the subset of M

$$\mathcal{P}_x(M) := \{x' \in M \mid \exists \text{ piecewise Hamiltonian path in } M \text{ from } x \text{ to } x'\}.$$

Note that the subsets $\mathcal{P}_x(M)$ define a partition of M .

We choose splitting coordinates $\varphi = (q_1, \dots, q_r, p_1, \dots, p_r, z_1, \dots, z_s)$, on a neighborhood U of x , and we define

$$\mathcal{P}_x(U, \varphi) := \{x' \in U \mid z_1(x') = \dots = z_s(x') = 0\}.$$

Thus, $\mathcal{P}_x(U, \varphi)$ is an embedded submanifold of M of dimension $2r$, $(q_1, \dots, q_r, p_1, \dots, p_r)$ forms a system of coordinates on $\mathcal{P}_x(U, \varphi)$ and we have

$$\Pi_{\mathcal{P}} = \sum_{i=1}^r \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

The map $i: (\mathcal{P}_x(U, \varphi), \Pi_{\mathcal{P}}) \rightarrow (M, \Pi)$ is a Poisson map. Indeed, take two functions $f, g \in C^\infty(U)$, for every $x' \in \mathcal{P}_x(U, \varphi)$, we have that

$$\begin{aligned} \{f, g\}(i(x')) &= \{f, g\}x' = \sum_{i=1}^r \left(\frac{\partial f}{\partial q_i}(x') \frac{\partial g}{\partial p_i}(x') - \frac{\partial g}{\partial q_i}(x') \frac{\partial f}{\partial p_i}(x') \right) \\ &= \{f \circ i, g \circ i\}_{\mathcal{P}}(x'). \end{aligned}$$

Proposition 52. Let x be a point in a Poisson manifold (M, Π) and let $\varphi = (q_1, \dots, q_r, p_1, \dots, p_r, z_1, \dots, z_s)$ be the splitting coordinates at x , on a coordinate neighborhood U , with $\varphi(U)$ convex. Then $\mathcal{P}_x(U) = \mathcal{P}_x(U, \varphi)$.

Proof. For $f \in C^\infty(U)$ and $x' \in \mathcal{P}(U, \varphi)$, we have that $\xi_f(z_i)(x') = \sum_{l=1}^s \{z_i, z_l\}(x') \frac{\partial f}{\partial z_l} = 0$ since $\{z_i, z_l\}(x') = 0$. It follows that Hamiltonian vector fields of U are tangent to $\mathcal{P}_x(U, \varphi)$ at points $x' \in \mathcal{P}_x(U, \varphi)$. Thus, a piecewise Hamiltonian path in U , which starts at x , must stay in $\mathcal{P}_x(U, \varphi)$, so $\mathcal{P}_x(U) \subset \mathcal{P}_x(U, \varphi)$. Now, take $x' \in \mathcal{P}$, consider the linear function h on U , given by

$$h := \sum_{i=1}^r (q_i(x')p_i - p_i(x')q_i),$$

the integral curve of ξ_h is given by

$$\begin{aligned} \dot{q}_i &= \frac{\partial h}{\partial p_i} = q_i(x') \\ \dot{p}_i &= \frac{\partial h}{\partial q_i} = -p_i(x') \\ \gamma(t) &= (q_1(x')t, \dots, q_r(x')t, p_1(x')t, \dots, p_r(x')t). \end{aligned}$$

Since U is convex, $\gamma \subset U$. Thus, $x' \in \mathcal{P}_x(U)$. □

We build a topology and a differential structure on each one of the subsets \mathcal{P}_x . A subset $V \subset \mathcal{P}_x$ is, by definition, an open subset if, and only if, for all splitting coordinates (U, φ) at $x \in V$, with $\varphi(U)$ convex, $V \cap U$ is an open subset of $\mathcal{P}_x(U) = \mathcal{P}_x(U, \varphi)$. Similarly, an algebra of functions is defined on $\mathcal{P}_x(U)$ by saying that $f: \mathcal{P}_x(M) \rightarrow \mathbb{R}$ is smooth, when its restriction to every such $\mathcal{P}_x(U) = \mathcal{P}_x(U, \varphi)$ is smooth. Since the topology and the differential structure on

these subsets are intrinsically defined, as embedded submanifolds of M , these definitions yield a well-defined manifold structure on $\mathcal{P}_x(M)$.

On dimension two, the symplectic foliation is quite simple. Basic there are two kinds of leaves: the zero and two-dimensional leaves. The 0-dimensional leaves are the zeros of Π . The 2-dimensional leaves are the connected component of the set $\{x \in M \mid \Pi_x \neq 0\}$.

The corollary (6) implies that every Casimir functions are constant along with the symplectic leaves.

Proposition 53. Let M be a Poisson manifold, $S \subset M$ is a symplectic leaf, and f is a Casimir function, then f is constant on S .

Let M be a manifold and $S(M)$ a general foliation such that

- every leaf S of $S(M)$ is endowed with a symplectic structure W_S ;
- if $f \in C^\infty(M)$, the vector field X_f defined by the Hamiltonian vector field of $f|_{S(x)}$, i. e.,

$$X_f(x) = X_f|_{S(x)},$$

is smooth.

Them we define

$$\{f, g\} = X_f(g),$$

for all $F, G \in C^\infty(M)$. So the Poisson structure can be defined by its symplectic foliation.

2.6.3 The symplectic foliation of a linear Poisson structure

To construct a nontrivial example, we returned to the linear Poisson structures. The goal of this section is to show the following proposition.

Proposition 54. The symplectic leaves of the Lie-Poisson structure of \mathfrak{g}^* are the orbits of the coadjoint representation of any connected Lie group G whose Lie algebra is \mathfrak{g} .

We follow the demonstration presented in [Vaisman 2012], but readers are also referred to [Dufour and Zung 2006, Laurent-Gengoux, Pichereau and Vanhaecke 2012] for further information. In a coordinate system x_i , the equation (2.3.6) implies that one can describe the set of Hamiltonian vector fields of a Poisson linear structure by: let $f \in C^\infty(\mathfrak{g}^*)$, so

$$X_f = \sum_i \frac{\partial f}{\partial x_i} X_{x_i}.$$

The previous equation shows that the characteristic distribution of a Lie-Poisson structure is generated by the set of Hamiltonian vector fields, \mathfrak{X}_{lin} , of the linear functions on \mathfrak{g}^* . Together,

the equations (2.3.6) and (2.3.8) define a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}_{lin}$ which the kernel is the center of \mathfrak{g} .

Supposed that G is a Lie group that \mathfrak{g} is isomorphic to $Lie(G)$. We define a representation of G on G by

$$\begin{aligned} A: G &\rightarrow Aut(G) \\ h &\mapsto A_h(g) = hgh^{-1} \end{aligned}$$

Through the differential of A on the identity element, e , we can define a representation of G on \mathfrak{g}

$$\begin{aligned} Ad: G &\rightarrow Aut(\mathfrak{g}) \\ h &\mapsto A(h)_*(X) \end{aligned}$$

and we define the coadjoint representation by

$$\langle Ad_g^*(\xi), x \rangle = \langle \xi, Ad_{g^{-1}}(x) \rangle$$

The tangent vectors to coadjoint orbits can be describe by the following: let $x \in \mathfrak{g}$, let \mathcal{O} be an orbit of coadjoint action and $\mu \in \mathcal{O}$. It is possible to define a curve on \mathcal{O} as

$$\mu(t) = Ad_{e^{-tx}}^*(\mu).$$

And we have in $y \in \mathfrak{g}$

$$\begin{aligned} \langle \mu'(0), y \rangle &= \left\langle \frac{d}{dt} \Big|_{t=0} \mu(t), y \right\rangle = \frac{d}{dt} \Big|_{t=0} \langle \mu(t), y \rangle = \frac{d}{dt} \Big|_{t=0} \langle \mu, Ad_{e^{tx}}(y) \rangle \\ &= \left\langle \mu, \frac{d}{dt} \Big|_{t=0} Ad_{e^{tx}}(y) \right\rangle = \langle \mu, ad_x(y) \rangle = \langle \mu, [x, y] \rangle = \langle ad_x^* \mu, y \rangle. \end{aligned}$$

So, $\mu(0) = \mu$ and $\mu'(0) = ad_x^*(\mu)$. Thus,

$$T_\mu \mathcal{O} = \{ad_x^*(\mu) | x \in \mathfrak{g}\}.$$

2.7 Isotropy Lie algebra

It is possible to define, algebraically, the cotangent space [Warner 1983] in a point $x \in M$

$$T_x^* M = \frac{m_x}{m_x^2},$$

where m_x is the ideal of germ of functions that vanish in x . If M is a Poisson manifold in a way that the Rank of the Poisson structure is zero at x , m_x^2 is an ideal of the Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$. Indeed, take $fg \in m_x^2$, by Leibniz rule, $\{h, fg\} = f\{h, g\} + \{h, f\}g$, since $\{h, g\}(x) = 0$ for all $f \in C^\infty(M)$, $\{h, fg\} \in m_x^2$. In this way, $T_x^* M$ has a Lie algebra structure, and $T_x M = (T_x^* M)^*$ has a linear Poisson structure.

Explicitly, this structure is given by

$$[d_x f, d_x g] := d_x \{f, g\}.$$

This Lie algebra is called **the isotropy Lie algebra** at x and it is denoted by \mathfrak{g}_x . In a local coordinate (y^i) we have the structure constants

$$[d_x y^i, d_x y^j] = \sum_k \frac{\partial \Pi_{ij}}{\partial y^k}(x) dy^k = \sum_k c_{i,j}^k dy^k. \quad (2.7.1)$$

This definition will play an important role in the linearization problem presented in the chapter 3.

2.8 Complete Poisson manifold

Let us turn back to the example (36). Since \mathbb{R}^{2n} with that Poisson structure given by

$$\Pi = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i}$$

is a symplectic manifold, it is the only symplectic leaf itself. On the other hand, in \mathbb{R}^n with the trivial Poisson structure, every vector field is null, so any point is a symplectic leaf. This simple example shows us that a Poisson morphism does not necessarily map a symplectic leaf into a symplectic leaf. To avoid this kind of situation it is convenient to define the notion of a complete Poisson map.

Definition 55. Let f be a function on a Poisson manifold M . We call f **complete** if the Hamiltonian vector field X_f is a complete vector field, i. e., its flow is globally defined.

If f is a Casimir function of (M, Π) , then X_f is a null vector field, so the flow of X_f is a point and is defined for all times. Thus, f is complete vector field.

Definition 56. A Poisson manifold is called **complete** if every Hamiltonian vector field on M is complete.

On [Marsden and Ratiu 2002] it is presented the following proposition – from the original reference, corollary 4.1.20:

Proposition 57. A C^r vector field with compact support on a manifold M is complete. In particular, a C^r vector field defined on a compact manifold is complete.

In our case, we consider $r = \infty$. The above proposition ensures to us that every support compact function on an arbitrary Poisson manifold is complete. In particular, it ensures that every compact Poisson manifold is complete. Now we can define the notion of complete Poisson maps.

Definition 58. A Poisson map $\phi : M \rightarrow N$ will be called **complete** if the pullback by ϕ of every complete function is complete.

The next proposition tells us that the definition of a complete Poisson map is sufficient to exclude the problem presented at the beginning of the section.

Proposition 59. The image of a complete Poisson map is a union of symplectic leaves.

The following proof is presented in [Silva and Weinstein 1999].

Proof. First, one can see that the condition of "the pullback by ϕ of every complete function is complete" could be replaced by "for each $f \in C^\infty(M)$, if X_f is a complete vector field then X_{ϕ^*f} is also complete".

Now, let $\phi(x) \in N$, it is possible to find a piecewise Hamiltonian composed of complete vector fields X_h 's. By definition, every pullback X_{ϕ^*h} is a complete vector field. Hence, we can integrate them without restriction, and their flows provide a path on M . The image of this path is the original one. So, we conclude that any point of the symplectic leaf of $\phi(x)$ is contained in the image of ϕ . \square

2.9 Symplectic realization

Definition 60. Let (M, Π) be a Poisson manifold. Then, a **full symplectic realization** of M is a symplectic manifold (S, ω) along with a surjective submersion $\mu : S \rightarrow M$ such that it is a Poisson map.

In the paper [Weinstein 1983], Weinstein makes a distinction between symplectic realizations that are surjective submersions and the ones which are not required to be a surjective submersion. He called the first ones by full symplectic realizations and the second ones just by symplectic realizations.

Although the second case is a more general definition, it includes a lot of trivial answers. For example, the inclusion of a symplectic leaf is a symplectic realization. For our purposes, from now on, we will consider just the case of full symplectic realization, and, as made by others, ignore the term "full" [Dufour and Zung 2006, Vaisman 2012].

Definition 61. A symplectic realization of a Poisson manifold M is called **complete** if the surjective submersion $\mu : S \rightarrow M$ is complete.

The existence of complete symplectic realization is introduced in [Crainic and Fernandes 2004] as sufficient and necessary condition to the integrability of the T^*M Lie algebroid.

Example 62. Let \mathbb{R}^2 equipped with the bracket $\{x, y\} = 0$ and let the \mathbb{R}^4 equipped with the Darboux coordinate (q_1, q_2, p_1, p_2) . The map $\mu: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ defined by $\mu(q_1, q_2, p_1, p_2) = (q_1, p_1 q_1 + q_2)$ is a surjective submersion Poisson map, so it is a symplectic realization.

LIE ALGEBROID AND LIE GROUPOID

The concept of a Lie algebroid can be seen as an infinitesimal part of a Lie groupoid, which will be discussed later. This concept unifies several structures, such as foliated manifolds, Poisson manifolds and infinitesimal actions of Lie algebras on a manifold. The aim of this chapter is to present several definitions and results of Lie algebroid and Lie groupoid theory.

First of all, we review the theory for an abstract Lie algebroid and we present some examples. Next, we present the relation between Poisson geometry and Lie algebroids. In section (3.2.2), we prove the splitting theorem for Lie algebroids. At last, we introduce some basic concepts of the Lie groupoid theory. In Section 3.2.1 we introduce the isotropy algebras and the foliation induced by the anchor map. After that we present the analogue of the Weinstein Splitting Theorem for Lie algebroids. We present the concept of Lie algebroid cohomology in the Section 3.2.3. In the subsequent sections we present the Lie groupoid theory and its relation with Lie algebroid theory.

3.1 Lie algebroid

In order to define a Lie algebroid, we start first introduce the concept of an anchored vector bundle.

Definition 63. An **anchored vector bundle** is a vector bundle $A \xrightarrow{proj_A} M$ together with a vector bundle map $\#: A \rightarrow TM$ called **the anchor**. The induced map $\Gamma(A) \rightarrow \mathfrak{X}(M)$ defined pointwise by $\#(X)(x_0) = \#(X(x_0))$, will be denote simply by $\#$.

Roughly speaking, an anchored vector bundle generalizes the idea of the tangent bundle. Precisely, the anchor identify every $\gamma \in \Gamma(A)$ with the derivation $\#(\gamma)$ on $C^\infty(M)$.

Since a Lie algebroid can be thought as a generalization of Lie algebra and of tangent bundle to a given manifold, we need to give a Lie algebra structure on $\Gamma(A)$, the space of global

sections of an anchored vector bundle. Moreover, one needs that the anchor map preserves the Lie structure on $\Gamma(A)$ and on $\Gamma(TM)$. More precisely:

Definition 64. A **Lie algebroid** $(A, [\cdot, \cdot], \#)$ over M is an anchored vector bundle along with a Lie bracket $[\cdot, \cdot]$ in the space of smooth section $\Gamma(A)$. The Lie bracket satisfies the following Leibniz rule

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\#(\alpha)}(f)\beta, \quad \forall \alpha, \beta \in \Gamma(A) \text{ and } f \in C^\infty(M).$$

Note that, by skew-symmetry and Leibniz rule, one has

$$[f\alpha, \beta] = -[\beta, f\alpha] = f[\alpha, \beta] - \mathcal{L}_{\#(\beta)}(f)\alpha.$$

Proposition 65. If $(A, [\cdot, \cdot], \#)$ is a Lie algebroid, then the anchor $\#: \Gamma(A) \mapsto \mathfrak{X}(M)$ is a homeomorphism of Lie algebra, i.e.,

$$\#[\alpha, \beta] = [\#\alpha, \#\beta] \quad \forall \alpha, \beta \in \Gamma(A).$$

Proof. Jacobi identity and the Leibniz rule imply:

$$\begin{aligned} 0 &= [[\alpha, \beta], f\gamma] + [[\beta, f\gamma], \alpha] + [[f\gamma, \alpha], \beta] \\ &= f[[\alpha, \beta], \gamma] + \#[\alpha, \beta](f)\gamma + [f[\beta, \gamma] + \#\beta(f)\gamma, \alpha] + [f[\gamma, \alpha] - \#\alpha(f)[\gamma, \beta]] \\ &= f[[\alpha, \beta], \gamma] + \#[\alpha, \beta](f)\gamma \\ &\quad + f[[\beta, \gamma], \alpha] - \#\alpha(f)[\beta, \gamma] + \#\beta(f)[\gamma, \alpha] - \#\alpha(\#\beta(f)\gamma) \\ &\quad + f[[\gamma, \alpha], \beta] - \#\beta(f)[\gamma, \alpha] + \#\alpha(f)[\gamma, \beta] - \#\beta(\#\alpha(f)\gamma) \\ &= ((\#[\alpha, \beta] - [\#\alpha, \#\beta])(f))\gamma \end{aligned}$$

Since α, β, γ and f are arbitrary, we have $\#[\alpha, \beta] = [\#\alpha, \#\beta]$. □

Example 66. I. Any Lie algebra is a Lie algebroid over a point, where the anchor is the null function.

II. Any vector bundle with the null anchor and trivial bracket is a Lie algebroid.

III. The tangent bundle of any smooth manifold with the standard Lie bracket of vector fields and the identity map as anchor is a Lie algebroid.

A less trivial example is

Example 67. Let ω be a closed 2-form on M . Thus, we give a Lie algebroid structure on M , called A_ω , defined by

- The vector bundle structure

$$A_\omega = TM \otimes (M \times \mathbb{R}).$$

- The anchor map, which is the projection on the first component.
- The Lie bracket $\Gamma(A_\omega) \simeq \mathfrak{X}(M) \times C^\infty(M)$, which is defined by

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X(g) - \mathcal{L}_Y(f) + \omega(X, Y)).$$

Let $X, Y, Z \in \mathfrak{X}(M)$ and $f, g, h \in C^\infty(M)$, so we can prove

- The Jacobi Identity:

$$\begin{aligned} [X + f, [Y + g, Z + h]] &= [X + f, [Y, Z] + \mathcal{L}_Y(h) - \mathcal{L}_Z(g) + w(Y, Z)] \\ &= [X, [Y, Z]] + \mathcal{L}_X(\mathcal{L}_Y(h)) - \mathcal{L}_X(\mathcal{L}_Z(g)) \\ &\quad + \mathcal{L}_X(\omega(Y, Z)) - \mathcal{L}_{[Y, Z]}(f) + \omega(X, [Y, Z]) \end{aligned}$$

$$\begin{aligned} [Y + g, [Z + h, X + f]] &= [Y + g, [Z, X] + \mathcal{L}_Z(f) - \mathcal{L}_X(h) + w(Z, X)] \\ &= [Y, [Z, X]] + \mathcal{L}_Y(\mathcal{L}_Z(f)) - \mathcal{L}_Y(\mathcal{L}_X(h)) \\ &\quad + \mathcal{L}_Y(\omega(Z, X)) - \mathcal{L}_{[Z, X]}(g) + \omega(Y, [Z, X]) \end{aligned}$$

$$\begin{aligned} [Z + h, [X + f, Y + g]] &= [Z + h, [X, Y] + \mathcal{L}_X(g) - \mathcal{L}_Y(f) + w(X, Y)] \\ &= [Z, [X, Y]] + \mathcal{L}_Z(\mathcal{L}_X(g)) - \mathcal{L}_Z(\mathcal{L}_Y(f)) \\ &\quad + \mathcal{L}_Z(\omega(X, Y)) - \mathcal{L}_{[X, Y]}(h) + \omega(Z, [X, Y]) \end{aligned}$$

thus, we have

$$\begin{aligned} &[X + f, [Y + g, Z + h]] + [Y + g, [Z + h, X + f]] + [Z + h, [X + f, Y + g]] = \\ &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &\quad + \mathcal{L}_{[X, Y]}(h) + \mathcal{L}_{[Z, X]}(g) + \mathcal{L}_{[Y, Z]}(f) - \mathcal{L}_{[X, Y]}(h) - \mathcal{L}_{[Z, X]}(g) - \mathcal{L}_{[Y, Z]}(f) \\ &\quad + \mathcal{L}_X(\omega(Y, Z)) - \mathcal{L}_Y(\omega(X, Z)) + \mathcal{L}_Z(\omega(X, Y)) \\ &\quad - \omega([X, Y], Z) - \omega([Y, Z], X) + \omega([X, Z], Y) \\ &= d\omega(x, y, z) \end{aligned}$$

Since ω is closed, $d\omega = 0$ and this bracket satisfies Jacobi.

- Leibniz Rule

$$\begin{aligned} [X + f, hY + hg] &= [X, hY] + \mathcal{L}_X(hg) - \mathcal{L}_{hY}(f) + \omega(X, hY) \\ &= \mathcal{L}_X(g)Y + g\mathcal{L}_X(Y) + \mathcal{L}_X(g)h + g\mathcal{L}_X(h) - g\mathcal{L}_Y(f) + f\omega(X, Y) \\ &= g([X, Y] + \mathcal{L}_X(h) - \mathcal{L}_Y(f) + w(X, Y)) + \mathcal{L}_X(g)(Y + h) \end{aligned}$$

Example 68 (Transformation Algebroid Associated to the Infinitesimal Action). Let M be a manifold equipped with an infinitesimal action of a Lie algebra \mathfrak{g} , i.e, a Lie algebra homomorphism $\phi: \mathfrak{g} \rightarrow \mathfrak{X}(M)$. The vector bundle $M \times \mathfrak{g}$ over M has the following Lie algebroid structure, with anchor and the Lie bracket given by

$$\begin{aligned} \#(x, \xi) &= \phi(\xi)_x \\ [u, v](x) &= [u(x), v(x)] + (\phi(u(x))(v))(x) - (\phi(v(x))(u))(x), \end{aligned}$$

where $u, v \in C^\infty(M, \mathfrak{g}) \cong \Gamma(M, \mathfrak{g} \times M)$ and $x \in M$.

Definition 69. Let $(A_1, \#_1, [\cdot, \cdot]_1)$ and $(A_2, \#_2, [\cdot, \cdot]_2)$ be Lie algebroids over the same manifold M . A **Lie algebroid morphism over the same manifold** is a vector bundle morphism $\Phi: A_1 \rightarrow A_2$, e. g., a functions $\Phi: A_1 \rightarrow A_2$ such that, for every $x \in M$, the map $proj_1^{-1}(x) \rightarrow proj_2^{-1}(x)$ induced by F is a linear map satisfying

$$\text{I } \#_2 \circ \Phi = \#_1$$

$$\text{II } \Phi([\alpha, \beta]_1) = [\Phi(\alpha), \Phi(\beta)]_2, \text{ for all } \alpha, \beta \in A_1.$$

A **Lie algebroid isomorphism** is a morphism that is invertible.

Remark 70. A general definition of Lie algebroid morphism is given in [Crainic and Fernandes 2011], but in this monograph we are just interested in morphism of Lie algebroids over the same manifold.

3.2 Relation between Lie algebroid theory and Poisson geometry

The first way to relate the Poisson geometry with Lie algebroid theory is given by the following important example.

Proposition 71. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Then, there exists a \mathbb{R} -bilinear, skew-symmetric operation $[\cdot, \cdot]: \Gamma(T^*M) \times \Gamma(T^*M) \rightarrow \Gamma(T^*M)$ defined by

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}(\beta) - \mathcal{L}_{\#\beta}(\alpha) - d(\Pi(\alpha, \beta)).$$

So we are ready to give a Lie algebroid structure to every cotangent bundle of a Poisson manifold.

Proposition 72. The cotangent bundle of a Poisson manifold M , with the bracket above and the Poisson bundle morphism $\#$, given by the contraction with Π , is a Lie algebroid over M . It is call this **the cotangent Lie algebroid** of the Poisson manifold (M, Π) .

Proof. To check the Jacobi identity:

$$\begin{aligned}
[\alpha, [\beta, \gamma]] &= [\alpha, \mathcal{L}_{\#\beta}(\gamma) - \mathcal{L}_{\#\gamma}(\beta) - d\Pi(\beta, \gamma)] \\
&= \mathcal{L}_{\#\alpha}(\mathcal{L}_{\#\beta}(\gamma)) - \mathcal{L}_{\#\alpha}(\mathcal{L}_{\#\gamma}(\beta)) - \mathcal{L}_{\#\alpha}(d\Pi(\beta, \gamma)) \\
&\quad - \mathcal{L}_{\#\mathcal{L}_{\#\beta}(\gamma)}(\alpha) + \mathcal{L}_{\#\mathcal{L}_{\#\gamma}(\beta)}(\alpha) + \mathcal{L}_{\#d\Pi(\beta, \gamma)}(\alpha) \\
&\quad d(\Pi(\alpha, \mathcal{L}_{\#\beta}(\gamma)) + d(\Pi(\alpha, \mathcal{L}_{\#\gamma}(\beta)) + d(\Pi(\alpha, d\Pi(\beta, \gamma))) \\
[\beta, [\gamma, \alpha]] &= [\beta, \mathcal{L}_{\#\gamma}(\alpha) - \mathcal{L}_{\#\alpha}(\gamma) - d\Pi(\gamma, \alpha)] \\
&= \mathcal{L}_{\#\beta}(\mathcal{L}_{\#\gamma}(\alpha)) - \mathcal{L}_{\#\beta}(\mathcal{L}_{\#\alpha}(\gamma)) - \mathcal{L}_{\#\beta}(-d\Pi(\gamma, \alpha)) \\
&\quad - \mathcal{L}_{\#\mathcal{L}_{\#\gamma}(\alpha)}(\beta) + \mathcal{L}_{\#\mathcal{L}_{\#\alpha}(\gamma)}(\beta) + \mathcal{L}_{\#d\Pi(\gamma, \alpha)}(\beta) \\
&\quad - d(\Pi(\beta, \mathcal{L}_{\#\gamma}(\alpha)) + d(\Pi(\beta, \mathcal{L}_{\#\alpha}(\gamma)) + d(\Pi(\beta, d\Pi(\gamma, \alpha))) \\
[\gamma, [\alpha, \beta]] &= [\gamma, \mathcal{L}_{\#\alpha}(\beta) - \mathcal{L}_{\#\beta}(\alpha) - d\Pi(\alpha, \beta)] \\
&= \mathcal{L}_{\#\gamma}(\mathcal{L}_{\#\alpha}(\beta)) - \mathcal{L}_{\#\gamma}(\mathcal{L}_{\#\beta}(\alpha)) - \mathcal{L}_{\#\gamma}(d\Pi(\alpha, \beta)) \\
&\quad - \mathcal{L}_{\#\mathcal{L}_{\#\alpha}(\beta)}(\gamma) + \mathcal{L}_{\#\mathcal{L}_{\#\beta}(\alpha)}(\gamma) + \mathcal{L}_{\#d\Pi(\alpha, \beta)}(\gamma) \\
&\quad d(\Pi(\gamma, \mathcal{L}_{\#\alpha}(\beta)) + d(\Pi(\gamma, \mathcal{L}_{\#\beta}(\alpha)) + d(\Pi(\gamma, d\Pi(\alpha, \beta)))
\end{aligned}$$

So, we have that

$$\begin{aligned}
[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] &= \mathcal{L}_{\#[\alpha, \#\beta]}(\gamma) + \mathcal{L}_{\#[\beta, \#\gamma]}(\alpha) + \mathcal{L}_{\#[\gamma, \#\alpha]}(\beta) \\
&\quad - \mathcal{L}_{\#[\beta, \gamma]}(\alpha) - \mathcal{L}_{\#[\alpha, \gamma]}(\beta) - \mathcal{L}_{\#[\alpha, \beta]}(\gamma) \\
&\quad + d(\Pi(\alpha, d\Pi(\beta, \gamma)) + \Pi(\beta, d\Pi(\gamma, \alpha)) + \Pi(\gamma, d\Pi(\alpha, \beta)))
\end{aligned}$$

since, $\#[\alpha, \beta] = [\#\alpha, \#\beta]$ for all $\alpha, \beta \in \Gamma(A)$ and

$$d(\Pi(\alpha, d\Pi(\beta, \gamma)) + \Pi(\beta, d\Pi(\gamma, \alpha)) + \Pi(\gamma, d\Pi(\alpha, \beta))) = 0,$$

we get

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0$$

To check Leibniz's rule:

$$\begin{aligned}
[\alpha, f\beta] &= \mathcal{L}_{\#\alpha}(f\beta) - \mathcal{L}_{\#\beta}(\alpha) - d(\Pi(\alpha, f\beta)) \\
&= fi_{\#\alpha}\beta + dfi_{\#\alpha}\beta + fd(i_{\#\alpha}\beta) - f\mathcal{L}_{\#\beta}(\alpha) - df \wedge i_{\#\beta}\alpha - df\Pi(\alpha, \beta) - f\Pi(\alpha, \beta) \\
&= f(i_{\#\alpha}\beta + d(i_{\#\alpha}\beta) - \mathcal{L}_{\#\beta}(\alpha) - \Pi(\alpha, \beta)) + dfi_{\#\alpha}\beta - dfi_{\#\beta}\alpha - df\Pi(\alpha, \beta) \\
&= f(\mathcal{L}_{\#\alpha}(\beta) - \mathcal{L}_{\#\beta}(\alpha) - \Pi(\alpha, \beta)) + df(i_{\#\alpha}\beta - i_{\#\beta}\alpha - \Pi(\alpha, \beta))
\end{aligned}$$

□

We saw in Section 2.3.2 that the dual space of any Lie algebra has a linear Poisson structure. We have an analogous result for Lie Algebroid, but in this case, the Poisson structure is fiber-wise linear. In this section we follow mostly [MEINRENKEN 2017, Courant 1990, Dufour and Zung 2006].

Let $E \xrightarrow{\pi} M$ be a vector bundle. Let $\mu_t: E \rightarrow E$, $(x, v) \mapsto (x, tv)$ be the fiber-wise scalar multiplication by $t \in \mathbb{R}$. For every $t \neq 0$ this application is a diffeomorphism.

Definition 73. A multivector field $\mathbf{v} \in \mathfrak{X}^k(E)$ is called fiber-wise linear if it is homogeneous of degree $1 - k$, that is,

$$\mu_t^*(\mathbf{v}) = t^{(1-k)}\mathbf{v}, t \neq 0.$$

Let (U, x^1, \dots, x^n) be a chart on M and let (y^1, \dots, y^m) be the fiber coordinates, hence, we have a local coordinate $(\pi^{-1}U, x^1, \dots, x^n, y^1, \dots, y^m)$ such that, a fiber-wise linear multivector field has the form

$$\mathbf{v} = \sum_{(i_1, \dots, i_k)} \sum_j c_j^{(i_1, \dots, i_k)} y^j \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_k}} + \sum_{(i_1, \dots, i_{k-1})} \sum_j a^{(i_1, \dots, i_{k-1}, r)} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_{k-1}}},$$

where $c_j^{(i_1, \dots, i_k)}$ and $a^{(i_1, \dots, i_{k-1}, r)}$ are smooth functions on U . We are interested in fiber-wise linear bi-vector fields.

Let $p: E^* \rightarrow M$ the dual vector bundle of E . Every $\alpha \in \Gamma(E)$ can be seen as a smooth function, ϕ_α on E^* , $\phi_\alpha(x, v) \mapsto \alpha(v)(x)$. Of course, this function is linear on each fiber of E . We have another subset of smooth function on E^* , the functions which are constant on the fibers. It is clear that this functions are in one-to-one correspondence with the smooth function on the base M . In fact, if we have a function $f \in C^\infty(E^*)$ that is constant on the fibers, $f(x) = f(v_x) \in C^\infty(M)$, where $v_x \in E_x^*$. On the other hand, let $f \in C^\infty(M)$, so $f \circ p \in C^\infty(E^*)$ is constant on the fibers.

Now we are ready to prove the following theorem.

Theorem 74. For any Lie algebroid $A \rightarrow M$, the total space of dual bundle $p: A^* \rightarrow M$ has a unique Poisson bracket such that for all $\alpha, \beta \in \Gamma(A)$ and $f, g \in C^\infty(M)$,

$$\begin{aligned} \{\phi_\alpha, \phi_\beta\} &= \phi_{[\alpha, \beta]}, \\ \{\phi_\alpha, f \circ p\} &= \#(\alpha)(f) \circ p, \\ \{f \circ p, g \circ p\} &= 0. \end{aligned}$$

Proof. Let $A|_U = U \times \mathbb{R}^m$ be a local bundle trivialization over open subset $U \subset M$, let (e_1, \dots, e_m) be the corresponding basis of sections of A , and let (x^1, \dots, x^n) be the local coordinates on U . First of all, note that the differentials of the functions $y_i = \phi_{e_i}$ and $x^j y_i = \phi_{x^j y_i}$ span $T^*(A^*)$ everywhere, except where all y^j vanish. Thus,

$$\Pi(d\phi_\alpha, d\phi_\beta) = \phi_{\alpha, \beta}$$

define a bi-vector field on A^* . In local coordinate, it look like

$$\Pi = \frac{1}{2} \sum_{i,j} [e_i, e_j] \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} + \sum_{i,j} (\#(e_i)(x^j)) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x^j}.$$

The Jacobi identity is trivially verified for all functions except for the case when two functions are fiber-wise linear, ϕ_α, ϕ_β , and one is constant on the fibers, $f \circ p$.

$$\{\{\phi_\alpha, \phi_\beta\}, f \circ p\} = \{\phi_{[\alpha, \beta]}, f \circ p\} = \#([\alpha, \beta])(f) = \{\alpha, \{\beta, f\}\} - \{\beta, \{\alpha, f \circ p\}\}.$$

□

Example 75. In Section (2.3.3) we saw the fiber-wise linear Poisson structure defined on the dual of the cotangent bundle with the Lie algebroid structure given in the Section (3.2).

Example 76. In this example we describe the fiber-wise linear Poisson structure on \mathbb{L}_ω . Let x_1, \dots, x_m be local coordinates on M . We have the local frame $(p_1, \dots, p_m, s, x_1, \dots, x_m)$, for $TM \oplus (M \times \mathbb{R})$, where $p_i = \frac{\partial}{\partial x_i}$ and $s = 1$ is the constant function. Thus, we have

$$\begin{aligned} [p_i, p_j] &= \omega(p_i, p_j) \\ [p_i, s] &= \mathcal{L}_{p_i}(s) \\ \#(p_i)(x_j) &= \frac{\partial x_i}{\partial x_j} = \delta_i^j \\ \#(s)(x_i) &= \#(s)(x_i) = 0 \end{aligned}$$

where $\delta_i^j = 1$, if $i = j$, and zero otherwise. Then, in these local coordinates the bivector is given by

$$\Pi = \sum_{i,j} \omega(p_i, p_j) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \sum_i \left(\mathcal{L}_{p_i}(s_x) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial s_x} \right) + \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x_i}.$$

3.2.1 Isotropy algebras and characteristic foliation of a Lie algebroid

Let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid over a manifold M . Given $z \in M$, we will denote by A_z the fiber of A over z , and by $\text{Ker}\#_z$ the kernel of the anchor map

$$\#_z: A_z \rightarrow T_z M. \quad (3.2.1)$$

The kernel $\text{Ker}\#_z$ has a natural Lie algebra structure, defined as the following. For any $\alpha_z, \beta_z \in \text{Ker}\#_z$, let α, β arbitrary sections of A such that $\alpha(z) = \alpha_z$ and $\beta(z) = \beta_z$, so we set

$$[\alpha_z, \beta_z] = [\alpha, \beta](z). \quad (3.2.2)$$

The above bracket is well defined. Indeed, take another section of $A, \tilde{\alpha}$, with $\tilde{\alpha}(z) = \alpha_z$, then, locally, there are functions f_1, \dots, f_n on M which vanish at z and a basis $(\alpha_1, \dots, \alpha_n)$ of A such that $\tilde{\alpha} - \alpha = \sum_{i=1}^n f_i \alpha_i$. By the Leibniz rule, we have $[\tilde{\alpha}, \beta](z) - [\alpha, \beta](z) = \sum_{i=1}^n f_i(z) [\alpha_i, \beta](z) - \sum_{i=1}^n (\# \beta)(f_i)(z) \alpha_i(z) = 0$, because $f_i(z) = 0$ and $\# \beta(z) = 0$.

Definition 77. For any $z \in M$ we call the Lie algebra

$$\mathfrak{g}_z(A) := \text{Ker}\#_z$$

the **isotropy Lie algebra** at z

The distribution $\mathcal{C}: x \in M \mapsto \mathcal{C}_x = \text{Im}\#_x$ is called the characteristic distribution of A , and dimension of $\text{Im}\#_x$ is called the Rank of $\#$ at x .

Let $(\alpha_1, \dots, \alpha_m)$ be a local basis of sections of a Lie algebroid A , then we have that $[\alpha_i, \alpha_j] = \sum_k c_{ij}^k \alpha_k$. Since the anchor is a morphism of Lie algebras we get

$$[\#(\alpha_i), \#(\alpha_j)] = \#([\alpha_i, \alpha_j]) = \sum_k c_{ij}^k \#(\alpha_k).$$

Thus, the characteristic distribution is integrable.

3.2.2 Splitting theorem

Analogue of the Weinstein Splitting Theorem for Poisson manifolds, for all Lie algebroid there are some local coordinates and sections that simplify the Lie algebroid structure.

Theorem 78. Let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid over M . Then A can be decomposed locally into the direct product of a tangent algebroid and a Lie algebroid whose Rank at origin is 0. More precisely, if the Rank of $\#$ at a point z of M is q , then there is a system of local coordinates $(x_1, \dots, x_q, y_1, \dots, y_s)$ on M centered at z , and a basis of local sections $(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_r)$ of A over a neighborhood of z , such that the following conditions are satisfied for all possible indices:

$$[\alpha_i, \alpha_j] = 0, \quad \#\alpha_i = \frac{\partial}{\partial x_i}, \quad (3.2.3)$$

$$[\beta_i, \alpha_j] = 0, \quad \#\beta_i(x_j) = 0, \quad \mathcal{L}_{\frac{\partial}{\partial x_i}} \#\beta_i = 0, \quad (3.2.4)$$

$$[\beta_i, \beta_j] = \sum_k f_{ij}^k(y) \beta_k, \quad (3.2.5)$$

where $f_{ij}^k(y)$ are functions which depend only on the variables $y = (y_1, \dots, y_s)$.

Proof. We will prove by induction on the Rank $\#_x = q$. When $q = 0$, we are done, thus, we will assume that $q > 0$. Suppose that, for any integer d such that $0 \leq d < q$ we have found a local coordinate system $(x_1, \dots, x_d, y_1, \dots, y_{s+q-d})$ of M , and a basis of local sections $(\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_{r+q-d})$ satisfying (3.2.3), (3.2.4) and (3.2.5).

Since $d < q$, there must be some β_i such that $\#_z(\beta) \neq 0$. Without loss of generality, we can assume that $i = 1$. By the straightening theorem, it is possible to assume that $\#\beta_1 = \frac{\partial}{\partial y_1}$.

We want to find a coordinate system with the property that

$$[\beta_1, \beta_i] = 0, \quad \forall i = 2, \dots, (r+q-d), \quad (3.2.6)$$

so we replace β_i by $\sum_{j=1}^{r+q-d} g_i^j \beta_j$, where the functions g_i^j depend only on (y_1, \dots, y_{s+q-d}) . By the Leibniz rule, the condition (3.2.6) is equivalent to

$$[\beta_1, \sum_{j=1}^{(r-q-d)} g_1^j \beta_j] = \sum_{j=1}^{(r-q-d)} [\beta_1, g_1^j \beta_j] = \sum_{j=1}^{(r-q-d)} g_1^j [\beta_1, \beta_j] + \frac{\partial g_1^j}{\partial y_1} \beta_j \quad (3.2.7)$$

$$= \sum_{j=1}^{(r-q-d)} \left(\sum_{k=1}^{(r-q-d)} g_i^j f_{i,j}^k \beta_k + \frac{\partial g_1^j}{\partial y_1} \beta_j \right). \quad (3.2.8)$$

Since the equation (3.2.8) is an ordinary differential equation, we can find a coordinate system satisfying (3.2.6). The equation (3.2.6) implies that, for every $i = 2, \dots, (r+q-d)$

$$\mathcal{L}_{\frac{\partial}{\partial y_1}} (\#\beta_i) = [\#\beta_1, \#\beta_i] = \#[\beta_1, \beta_i] = 0.$$

Note that,

$$[\beta_1, (\#\beta_i(y_1))\beta_1] = \#\beta_i(y_1)[\beta_1, \beta_1] + \mathcal{L}_{\#\beta_1} (\#\beta_i(y_1))\beta_1 = 0.$$

Replacing β_i by $\beta_i - (\#\beta_i(y_1))\beta_1$ for each $i \geq 2$ we have that,

$$\#(\beta_i - (\#\beta_i(y_1))\beta_1)(y_1) = (\#\beta_i - (\#\beta_i(y_1))\#\beta_1)(y_1) = \#\beta_i(y_1) - \#\beta_i(y_1) \frac{\partial y_1}{\partial y_1} = 0 \quad (3.2.9)$$

At last,

$$[\beta_i, \beta_j] = f_{i,j}^1 \beta_1 + \sum_{k=2}^{r+q-d} f_{i,j}^k \beta_k, \quad i, j \geq 2. \quad (3.2.10)$$

Applying the anchor map to the (3.2.10), and valuing it on y_1 , we get

$$0 = [\#\beta_i, \#\beta_j](y_1) = \# \left(f_{i,j}^1 \beta_1 + \sum_{k=2}^{r+q-d} f_{i,j}^k \beta_k \right) = f_{i,j}^1. \quad (3.2.11)$$

Thus, we have

$$[\beta_i, \beta_j] = \sum_{k=2}^{r+q-d} f_{i,j}^k \beta_k. \quad (3.2.12)$$

Now, reindexing the variables, we have the induction procedure. \square

Example 79. When we consider the tangent bundle of a manifold M as a Lie algebroid, the anchor is the identity morphism, so the set of β_i is empty. Let (x_i, \dots, x_n) be local coordinates to M . A clever guess is to set $\alpha_i = \frac{\partial}{\partial x_i}$ and, so, the theorem 78 holds.

Example 80. Let A_ω be the Lie algebroid associated to a closed 2-form as in the Example 67. We can set $\alpha_i = (\frac{\partial}{\partial x_i}, 0)$ and $\beta = (0, 1)$. Let (M, Π) be a Poisson manifold with split coordinates $(p_1, \dots, p_n, q_1, \dots, q_n, z_1, \dots, z_{m-2n})$, so, setting $\alpha_i = dp_i$, $\alpha_{i+n} = dq_i$ and $\beta_i = z_i$, the theorem holds, again.

3.2.3 Lie algebroid cohomology

This section is based on [Crainic 2003]. Aiming to define the Lie algebroid cohomology, we will introduce the concept of A -connection on a vector bundle. Let $E \rightarrow M$ be a C^∞ vector bundle over M and let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid over M . An A -**connection** ∇ on E is a bilinear map

$$\begin{aligned} \nabla: \Gamma(A) \times \Gamma(E) &\rightarrow \Gamma(E) \\ (\alpha, s) &\mapsto \nabla_\alpha s, \end{aligned}$$

such that, for all $f \in C^\infty(M)$, $\alpha \in \Gamma(A)$ and $s \in \Gamma(E)$,

- (i) $\nabla_f \alpha s = f \nabla_\alpha s$;
- (ii) $\nabla_\alpha (fs) = f \nabla_\alpha s + \# \alpha(f)s$.

The **curvature** of the connection is the map

$$\begin{aligned} R_\nabla: \Gamma(A) \times \Gamma(A) &\rightarrow \text{End}(\Gamma(E)), \\ R_\nabla(\alpha, \beta)(X) &= \nabla_\alpha \nabla_\beta X - \nabla_\beta \nabla_\alpha X - \nabla_{[\alpha, \beta]} X. \end{aligned}$$

We say that an A -connection is flat if $R_\nabla = 0$.

Remark 81. When the $A = TM$ we recover the usual notion of connection on vector bundles.

Definition 82. A **representation** of a Lie algebroid A over M consists of a pair (E, ∇) of a vector bundle E over M endowed with a flat connection ∇ .

Given a representation (E, ∇) of a Lie algebroid A we define a **p-cochain on A with values on (E, ∇)** as a multilinear skew-symmetric map

$$\begin{aligned} c: \Gamma(A) \times \dots \times \Gamma(A) &\rightarrow \Gamma(E) \\ (\alpha_1, \dots, \alpha_p) &\mapsto c(\alpha_1, \dots, \alpha_p) \end{aligned}$$

The space $C^p(A, E)$ of p-cochains coincides with the space of sections $\Gamma(\wedge^p A^* \otimes E)$. The family $C^\bullet(A, E)$ turns into a differential complex with the following differential operator $d: C^p(A, E) \rightarrow C^{p+1}(A, E)$

$$\begin{aligned} dc(\alpha_1, \dots, \alpha_{p+1}) &= \sum_{k=1}^{p+1} (-1)^{k+1} \nabla_{\alpha_k} (\omega(\alpha_1, \dots, \hat{\alpha}_k, \dots, \alpha_{p+1})) \\ &\quad + \sum_{k < l} (-1)^{k+l} \omega([\alpha_k, \alpha_l], \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_l, \dots, \alpha_{p+1}). \end{aligned}$$

We call **cohomology of A with coefficients in E** , denoted $H^\bullet(A, E)$, the cohomology of the complex $(C^\bullet(A, E), d)$. When E is the trivial line bundle with $\nabla_\alpha f = \mathcal{L}_{\# \alpha}(f)$, we use the notation $C^\bullet(A)$ and $H^\bullet(A)$. The Lie algebroid cohomology generalizes many cohomology theories [Crainic and Fernandes 2011].

Example 83 (Lie algebra cohomology). When $A = \mathfrak{g}$ is a Lie algebra, the vector bundle $E \rightarrow p$ is nothing but a vector space. A representation is merely a homomorphism $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms of V with the usual commutator. Precisely, ρ is linear and, for all $X, Y \in \mathfrak{g}$

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

An p -chain will be a p -linear alternating mapping $c \in \text{Hom}(\wedge^p \mathfrak{g}, V)$ and the differential operator will be defined by the formula

$$\begin{aligned} d(x)(\alpha_1, \dots, \alpha_{p+1}) &= \sum_{k=1}^{p+1} (-1)^{k+1} \rho(\alpha_k)(c(\alpha_1, \dots, \hat{\alpha}_k, \dots, \alpha_{p+1})) \\ &\quad + \sum_{k < l} (-1)^{k+l} c([\alpha_k, \alpha_l], \alpha_1, \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_l, \dots, \alpha_{p+1}). \end{aligned}$$

This cohomology coincides with the Chevalley-Eilenberg complex [Chevalley and Eilenberg 1948].

Here we are interested in the next two Lie algebra cohomologies.

1. Let $V = \mathbb{R}$ and $\rho(X) = 0$ for all $X \in \mathfrak{g}$. We denote the cohomology simply by $H^\bullet(\mathfrak{g})$ and the differential operator is reduced to

$$d(\omega)(X_1, \dots, X_{q+1}) = \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}).$$

When $q = 1$, we have that $d: C^0(\mathfrak{g}) \rightarrow C^1(\mathfrak{g})$ is null. So, $H^1(\mathfrak{g}) = \text{Ker}(d): C^1(\mathfrak{g}) \rightarrow C^2(\mathfrak{g})$. Since for $\omega \in C^1(\mathfrak{g})$ and $X, Y \in \mathfrak{g}$

$$d\omega(X, Y) = \frac{1}{2} \omega([X, Y])$$

ω is a cocycle if, and only if, it vanishes for all elements on $[\mathfrak{g}, \mathfrak{g}]$. Thus,

$$H^1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

2. We can set $V = \mathfrak{g}$ and the adjoint representation $\rho(X)(Y) = ad(X, Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$. On this case the differential operator of the cohomology $H^\bullet(\mathfrak{g}, \mathfrak{g})$ is given by

$$\begin{aligned} d(\omega)(X_1, \dots, X_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} [X_i, \omega(X_1, \dots, \hat{X}_i, \dots, X_{q+1})] \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}). \end{aligned}$$

Example 84 (Poisson cohomology). The Poisson cohomology, discussed in Section 2.2, can be seen as a special case of the Lie algebroid cohomology. Let $A = T^*M$ be the cotangent algebroid of the Poisson manifold (M, Π) and $E = M \times \mathbb{R}$ be the trivial line bundle. Choosing the following representation

$$\begin{aligned} \#: \Gamma(T^*M) \times C^\infty(M) &\rightarrow C^\infty(M) \\ (\omega, f) &\mapsto \#(\omega)(f) \end{aligned}$$

A p-cochain is nothing but a p-vector field.

Proposition 85. The differential operator d of the cotangent algebroid of the Poisson manifold (M, Π) with the above representation is given by

$$d = d_\Pi = [\Pi, \cdot].$$

Proof. For a vector $X \in \mathfrak{X}(M)$ and $\alpha, \beta \in \Gamma(T^*M)$ we have

$$\begin{aligned} d(X)(\alpha, \beta) &= \mathcal{L}_{\#(\alpha)}(X(\beta)) - \mathcal{L}_{\#(\beta)}(X(\alpha)) - \langle [\alpha, \beta], X \rangle \\ &= \mathcal{L}_{\#(\alpha)}(\beta)(X) + \langle \beta, \mathcal{L}_{\#(\alpha)}(X) \rangle - \mathcal{L}_{\#(\beta)}(\alpha)(X) - \langle \alpha, \mathcal{L}_{\#(\beta)}(X) \rangle \\ &\quad - \mathcal{L}_{\#(\alpha)}(\beta)(X) + \mathcal{L}_{\#(\beta)}(\alpha)(X) + X(\Pi(\alpha, \beta)) \\ &= \langle \beta, \mathcal{L}_{\#(\alpha)}(X) \rangle - \langle \alpha, \mathcal{L}_{\#(\beta)}(X) \rangle + X(\Pi(\alpha, \beta)) \end{aligned}$$

On the other hand,

$$\begin{aligned} d_{\Pi(X)}(\alpha, \beta) &= [\Pi, X](\alpha, \beta) = \mathcal{L}_X(\Pi) = -\mathcal{L}_X(\Pi(\alpha, \beta)) + \Pi(\mathcal{L}_X(\alpha \wedge \beta)) \\ &= -X(\Pi(\alpha, \beta)) + \Pi(\mathcal{L}_X(\alpha), \beta) + \Pi(\alpha, \mathcal{L}_X(\beta)) \\ &= -X(\Pi(\alpha, \beta)) - \mathcal{L}_X(\alpha)(\#(\beta)) + \mathcal{L}_X(\beta)(\#(\alpha)) \\ &= -X(\Pi(\alpha, \beta)) - X(\Pi(\beta, \alpha)) + \langle \alpha, \mathcal{L}_X(\#(\beta)) \rangle + X(\Pi(\alpha, \beta)) - \langle \beta, \mathcal{L}_X(\#(\alpha)) \rangle \\ &= \langle \beta, \mathcal{L}_{\#(\alpha)}(X) \rangle - \langle \alpha, \mathcal{L}_{\#(\beta)}(X) \rangle + X(\Pi(\alpha, \beta)). \end{aligned}$$

Therefore, the result holds for the case $X \in \mathfrak{X}(M)$. Now suppose the result has been proved for $A \in \mathfrak{X}^a(M)$, and let $X \in \mathfrak{X}(M)$. From theorem 13

$$[\Pi, A \wedge X] = [\Pi, A] \wedge X + (-1)^a A \wedge [\Pi, X].$$

On Section 7.5 from [Mackenzie 2005], Mackenzie proves that

$$d(A^X) = d(A) \wedge X + (-1)^a A \wedge d(X)$$

Since the result holds for $X \in \mathfrak{X}(M)$ and $A \in \mathfrak{X}^a(M)$, thus it also holds for $\xi \in \mathfrak{X}^{a+1}(M)$. \square

3.3 Lie groupoid

Definition 86. A groupoid $\mathcal{G} \rightrightarrows M$ over a set M , called objects set, is a set \mathcal{G} , called set of arrows, together with:

1. Two surjective maps $s, t : \mathcal{G} \rightarrow M$, associating to each arrow g its source object $s(g)$ and its target object $t(g)$. We call them, respectively, the source and the target map.

$$t(g) \longleftarrow s(g)$$

2. The partial composition map, $m : \mathcal{G}_2 \rightarrow \mathcal{G}$, $(g, h) \mapsto gh$

$$\begin{array}{ccccc} & & gh & & \\ & \swarrow & \text{---} & \searrow & \\ z & \xleftarrow{g} & y & \xleftarrow{h} & x \end{array}$$

where $\mathcal{G}_2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$, satisfying

- a) Compatibility with s e t :

$$s(gh) = s(h), \quad t(gh) = t(g) \quad \forall (g, h) \in \mathcal{G}_2.$$

- b) Associativity:

$$(hg)f = h(gf) \quad \forall f, g, h \in \mathcal{G} \text{ with } s(g) = t(h) \text{ e } s(h) = t(k).$$

$$\begin{array}{ccccccc} & & hg & & f & & \\ & \swarrow & \text{---} & \searrow & \swarrow & \searrow & \\ w & \xleftarrow{h} & z & \xleftarrow{g} & y & \xleftarrow{f} & x \\ & \swarrow & \text{---} & \searrow & \swarrow & \searrow & \\ & & h & & gf & & \end{array}$$

3. The inclusion map

$$e : M \rightarrow \mathcal{G}, \quad x \mapsto 1_x,$$

such that, for any $g : x \rightarrow y$,

$$g1_x = g = 1_y g.$$

4. The inverse map,

$$i : \mathcal{G} \rightarrow \mathcal{G}, \quad g \mapsto g^{-1},$$

such that $s(i(g)) = t(g)$, $t(i(g)) = s(g)$, and

$$g^{-1}g = 1_{s(g)}, \quad gg^{-1} = 1_{t(g)}.$$

$$\begin{array}{ccc} & g & \\ y & \xrightarrow{\quad} & x \\ & g^{-1} & \end{array}$$

Example 87. Let B be a set and G be a group. We define a groupoid structure on $B \times G \times B$ as follows.

- s, t are defined, respectively, as the projections onto the first and third factor;
- The identity is the map $x \mapsto (x, 1, x)$, where 1 is the group identity;
- The multiplication operation $(z, h, y)(y', g, x) = (z, hg, x)$, is defined only if $y' = y$;
- The inverse map of (y, g, x) is (x, g^{-1}, y) .

For example, every group is a groupoid over $B = \{p\}$.

Example 88. Let B be a topological space. We define the space of arrows between every $x, y \in B$ as the homotopy-classes $\langle c \rangle$ of continuous maps $c : I \rightarrow B$ which endpoints are x and y . The groupoid structure over B is given by

- $s(\langle c \rangle) = c(0)$;
- $t(\langle c \rangle) = c(1)$;
- $e(x) = \langle c \rangle = x$;
- composition

$$\langle c_1 \rangle \langle c_2 \rangle = \begin{cases} c_1(2t), & 0 \leq t \leq \frac{1}{2} \\ c_2(2t), & \frac{1}{2} \leq t \leq 1 \end{cases} ;$$

- $\langle c \rangle^{-1} = \langle c(1-t) \rangle$.

Now, we present the concept of morphism between two groupoids, $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$.

Definition 89. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ be groupoids. A morphism between \mathcal{G} and \mathcal{G}' is a pair of maps $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$, $\varphi : M \rightarrow M'$, such that,

I $\varphi(s(g)) = s'(\Phi(g))$ and $\varphi(t(g)) = t'(\Phi(g))$, that is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{G}' \\ s \downarrow & & \downarrow s' \\ M & \xrightarrow{\varphi} & M' \end{array} \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{G}' \\ t \downarrow & & \downarrow t' \\ M & \xrightarrow{\varphi} & M' \end{array}$$

II $\Phi(g_1 g_2) = \Phi(g_1) \Phi(g_2)$

for all $g, g_1, g_2 \in \mathcal{G}$ with $s(g_1) = t(g_2)$. We also say that Φ is a **morphism over φ** .

Remark 90. The groupoid definitions is the same of a category which all the arrows are mesomorphs. The definition of groupoid morphs is a functor.

Definition 91. A **Lie groupoid** is a groupoid $\mathcal{G} \rightrightarrows M$ which \mathcal{G} and M are smooth manifolds, the structural functions are smooth and s, t are submersion.

Remark 92. It is usual to permit that the space of arrows be a non-Hausdorff manifold [Crainic and Fernandes 2011]. When it is Hausdorff it is called **Hausdorff groupoid** but during this monograph we assume that all groupoids are Hausdorff groupoids.

A **morphism between Lie groupoids** is a morphism of groupoids (Φ, φ) where both Φ and φ are smooth. We say that a morphism between two Lie groupoids is a **isomorphism** if both (Φ, φ) are diffeomorphisms.

Example 93. Let $E \xrightarrow{\pi} M$ be a vector bundle. Let $\Phi(E)$ be the set of all isomorphisms $\eta: E_x \rightarrow E_y$, for all $x, y \in M$. Then $\Phi(E)$ is a Lie groupoid over M .

- The source and target are given by $E_x \xrightarrow{\psi} E_y, s(\psi) = x$ and $t(\psi) = y$.
- The indent map of x is $1_x = Id: E_x \rightarrow E_x$.
- The multiplication is defined by the composition of linear maps.
- The inverse of $\psi: E_x \rightarrow E_y$ is the inverse map $\psi^{-1}: E_y \rightarrow E_x$

Example 94. Let $G \times M \rightarrow M$ be a smooth action of a Lie group G in a manifold M . We define $G \times M \rightrightarrows M$ by:

Given any arrow $(g, x) \in G \times M$

$$s(g, x) = x, t(g, x) = gx.$$

Let $(g_2, y), (g_1, x) \in G \times M$, such that $y = g_1x$, so

$$(g_2, y)(g_1, x) = (g_2g_1, x).$$

$$(g, x)^{-1} = (g^{-1}, gx),$$

$$1_x = (x, e), \text{ where } e \text{ is the identity of } G.$$

We denote $G \times M \rightrightarrows M$ by $G \ltimes M$.

Example 95. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We define a Lie groupoid over \mathcal{G} as

- The arrows manifold is $\mathcal{G} \times_s \mathcal{G} = \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid s(h) = s(g)\}$.
- The composition is defined by

$$(h, g) \circ (g, f) = (h, f).$$

We define a groupoid morphism

$$\begin{array}{ccc} \mathcal{G} \times_s \mathcal{G} & \xrightarrow{\delta} & \mathcal{G} \\ \text{Proj}_1 \downarrow & & \downarrow s \\ \mathcal{G} & \xrightarrow{t} & M \end{array} \quad \begin{array}{ccc} \mathcal{G} \times_s \mathcal{G} & \xrightarrow{\delta} & \mathcal{G} \\ \text{Proj}_2 \downarrow & & \downarrow t \\ \mathcal{G} & \xrightarrow{t} & M \end{array}$$

where $\delta(h, g) = hg^{-1}$.

Definition 96. For a groupoid $\mathcal{G} \rightrightarrows M$ and $x \in M$ we define

- I **The s -fiber of x** by $\mathcal{G}_x := s^{-1}(x)$.
- II **The t -fiber of x** by $\mathcal{G}^x := t^{-1}(x)$.
- III **The isotropy group at x** by $\mathcal{G}_x^x := s^{-1}(x) \cap t^{-1}(x)$.

3.3.1 The Lie algebroid of a Lie groupoid

If G is a Lie Group the vector space of left (right) invariant vector fields is a Lie subalgebra of $\mathfrak{X}(G)$. This subalgebra is finite dimensional and carries a lot of information about the Lie group. Indeed, it is isomorphic to the tangent space of the group at identity. This construction defines a functor from the category of Lie groups to the category of Lie algebras. In this section, we will show how a Lie groupoid gives rise to a Lie algebroid. To this purpose, it is necessary to generalize the idea of right-invariant vector fields.

Definition 97. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Let $T^s\mathcal{G}$ be the vector bundle over \mathcal{G} defined by

$$T^s\mathcal{G} := \text{Ker}(ds) \subset T\mathcal{G}.$$

We define $\text{Lie}(\mathcal{G})$ whose fiber in $x \in M$ are the same of the tangent space in 1_x of $T^s(\mathcal{G})$, more precisely, the total space of the Lie algebroid would be

$$\text{Lie}(\mathcal{G}) := T^s\mathcal{G}|_M.$$

When we try to extend the idea of right action of arrow of $\mathcal{G} \rightrightarrows M$ to M , this action is not everywhere defined.

Definition 98. Let $y \xleftarrow{g} x$ be a arrow and define the **right action of g** by

$$\begin{aligned} R_g : s^{-1}(y) &\rightarrow s^{-1}(x) \\ h &\mapsto hg, \end{aligned}$$

and the differential of R_g on $h \in \mathcal{G}$ gives us a linear isomorphism, defined by

$$dR_g : T_h\mathcal{G} \rightarrow T_{hg}\mathcal{G}.$$

Now, we are ready to define a good notion of right invariant vector field on \mathcal{G} .

Definition 99. We define the space of **right invariant vector fields of \mathcal{G}** as

$$\mathfrak{X}_{inv}^s(\mathcal{G}) = \{X \in \Gamma(T^s\mathcal{G}) : X_{hg} = dR_g(X_h), \forall (g, h) \in \mathcal{G}_{(2)}\}.$$

The next proposition ensures that $Lie(\mathcal{G})$ is a Lie algebroid over M .

Proposition 100. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid and $\mathfrak{X}_{inv}^s(\mathcal{G})$ the space of right invariant vector bundles of \mathcal{G} , then

1. The vector bundle morphism

$$\begin{aligned} \Gamma(Lie(\mathcal{G})) &\rightarrow \mathfrak{X}_{inv}^s(\mathcal{G}) \\ \alpha &\mapsto \tilde{\alpha}_g = dR_g(\alpha_t(g)) \end{aligned}$$

is an isomorphism.

2. $\mathfrak{X}_{inv}^s(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$ is a Lie subalgebra.
3. The derivative of the target map induces a Lie algebra homomorphism $dt : \mathfrak{X}_{inv}^s(\mathcal{G}) \rightarrow \mathfrak{X}(M)$.
4. For all $\alpha, \beta \in \Gamma(A)$ and for any $f \in C^\infty$.

$$[\alpha, f\beta] = f[\alpha, \beta] + dt(\alpha)(f)\beta$$

Thanks to the above proposition, the vector bundle $Lie(\mathcal{G})$ is a Lie algebroid over M whose the anchor map is the differential of the target map, $\# = dt$, and the bracket is given by the restriction of the vector fields bracket in $T\mathcal{G}$ to $\mathfrak{X}_{inv}^s(\mathcal{G})$.

Example 101. When the groupoid is a Lie group, $Lie(\mathcal{G})$ is the Lie algebra associated to \mathcal{G} .

Definition 102. When a Lie algebroid $(A, [\cdot, \cdot], \#) \cong Lie(\mathcal{G})$, for some Lie groupoid, we say that \mathcal{G} integrates A .

Proposition 103. Let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid and $x_0 \in A$ a fixed point. If U is a neighborhood of x_0 which integrates to a Lie groupoid \mathcal{G} , then the fiber of the source map $s : \mathcal{G} \rightrightarrows U$ above x_0 is a Lie group integrating \mathfrak{g}_{x_0} .

3.4 Integrability problem

We have seen in the previous section how to construct a Lie algebroid, starting from a Lie groupoid. A natural question one is led to ask is: under what conditions is it possible to do

the opposite process, i. e., when is it possible to get a Lie groupoid that integrates a given Lie algebroid?

In the case of Lie groups and Lie algebras, the answer is given by the Lie's third theorem. A classic and famous proof is obtained as a consequence of the Ado's theorem [Tao 2011]. A different approach is presented in [Duistermaat and Kolk 1999], which is based on a path space approach to construct the Lie group integrating the original Lie algebra.

The integrability problem of Lie algebroids was solved by Marius Crainic and Rui Loja Fernandes both in the general case and for the Poisson one, and this was presented in [Crainic and Fernandes 2003] and [Crainic and Fernandes 2004], respectively. Adapting the idea used by Duistermaat and Kolk for the Lie's third theorem, Crainic and Fernandes showed that it is possible to define a groupoid, associated with a general Lie algebroid, called Weinstein groupoid. This section presents a brief discussion of the Lie third theorem for Lie algebroids. We followed the construction of [Crainic and Fernandes 2011], to which we refer the reader for further details.

To define this groupoid it is necessary to define the concept of A -path and homotopy of A -path.

Definition 104. Let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid. We call a C^∞ curve $a: I \rightarrow A$ an A -path if

$$\#a(t) = \frac{d}{dt}(\pi(a(t)))$$

where $\pi(a(t))$ is the base path of $a(t)$. We denote the space of A -paths, endowed with C^1 topology, by $P(A)$.

The notion of A -path is summarized by the diagram bellow

$$\begin{array}{ccccc} TI & \xrightarrow{Ta} & TA & \xrightarrow{T\pi} & TM \\ \downarrow Prj & & & \nearrow \# & \\ I & \xrightarrow{a} & A & & \end{array}$$

Let ∇ be an usual TM -connection on A . We can associate a contravariant A -connection, denoted by $\bar{\nabla}$, as

$$\bar{\nabla}_\alpha \beta = \nabla_{\# \beta} \alpha + [\alpha, \beta].$$

Proposition 105. Let $(A, [\cdot, \cdot], \#)$ be a Lie algebroid and $a_\varepsilon: I \times I \rightarrow A$ a variation of A -paths. If ∇ is a TM -connection on A with torsion T_∇ , then the solution $b = b(\varepsilon, t)$ of the differential equation

$$\partial_t b - \partial_\varepsilon a = T_\nabla(a, b), \quad b(\varepsilon, 0) = 0,$$

does not depend on ∇ . Moreover, $\#b = \frac{d}{d\varepsilon}(\pi(a))$.

So, we define the homotopic of A -paths as:

Definition 106. We say that two A -paths a_0 and a_1 are homotopic if there exist a variation a_ε such that the function b , defined on the proposition above, satisfies $b(\varepsilon, 1) = 0$.

The space of A -paths is a foliated manifold where two paths a_0 and a_1 belong to the same leaf if they are homotopic. We define the Weinstein groupoid, $\mathcal{G}(A)$, as a set by the leaf space of the foliation above. The composition map is given by the concatenation of paths and the structural maps are given by $s(a) = \pi(a(0))$ and $t(a) = \pi(a(1))$.

Given a Lie algebroid, it is always possible to define the corresponding Weinstein groupoid. However, it is not always true that the Weinstein groupoid admits a structure of Lie groupoid, i. e., sometimes it is impossible to provide $\mathcal{G}(A)$ with a smooth structure that makes it into a Lie groupoid. Finding a necessary and sufficient condition for the Weinstein groupoid to admit a structure of a Lie groupoid is the main result of [Crainic and Fernandes 2003].

3.5 Lie groupoid cohomology

This section is based on [Crainic 2003]. We denote by $\mathcal{G}(p)$ the space of p -composable strings $x \xleftarrow{g_1} \xleftarrow{g_2} \dots \xleftarrow{g_p} y$.

Definition 107. A left action of $\mathcal{G} \rightrightarrows M$ on a manifold X consists of a smooth map $\mu : X \rightarrow M$, called moment map of the action, together with a smooth map

$$\begin{aligned} \mathcal{G} \times_M X &\rightarrow X \\ (g, x) &\mapsto gx, \end{aligned}$$

where $\mathcal{G} \times_M X = \{(g, x) \in \mathcal{G} \times X \mid s(g) = \mu(x)\}$ and such that, for all $(g, x) \in \mathcal{G} \times_M X$,

$$\mu(gx) = t(g)$$

and which satisfies the usual identities of an action, i. e.,

- i $g(hx) = (gh)x$, for all $g, h \in \mathcal{G}$ and $X \in X$ for which it is defined.
- ii $1_{\mu(x)}x = x$, for all $x \in X$

we call X left \mathcal{G} -space.

Definition 108. A left representation of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a vector bundle $\text{proj} : E \rightarrow M$, endowed with an action of \mathcal{G} whose moment map is precisely proj and which is fiber-wise linear.

Definition 109. An E -valued differentiable p -cochain on \mathcal{G} is a smooth map c which associates to a string an element $c(g_1, \dots, g_p) \in E_x$. We denote the space of E -valued p -cochain by $C_d^p(\mathcal{G}, E)$.

We can define a differential operator $d: C_d^p(\mathcal{G}, E) \rightarrow C_d^{p+1}(\mathcal{G}, E)$

$$d(c)(g_1, \dots, g_p, g_{p+1}) = g_1 c(g_2, \dots, g_p) + \sum_{i=1}^p (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1}) + (-1)^{p+1} c(g_1, \dots, g_p).$$

Definition 110. We define **the differentiable cohomology of \mathcal{G} with coefficients in E** , $H_d^*(\mathcal{G}, E)$, as the differential cohomology of the complex $(C_d^*(\mathcal{G}, E), d)$. When E is the trivial line bundle, we use the notation $C_d^*(\mathcal{G})$ and $H^*(\mathcal{G})$.

Proposition 111. For any proper Lie groupoid \mathcal{G} , and E , for all $k \geq 1$,

$$H_d^K(\mathcal{G}, E) = 0.$$

Let P be a right \mathcal{G} -space. We call it **proper** if the map

$$\begin{aligned} \mathcal{G} \times_M P &\rightarrow P \times P \\ (g, p) &\mapsto (gp, p) \end{aligned}$$

is proper, i. e., if the inverse images of compact subsets are compact.

Theorem 112. Let $\mathcal{G} \rightrightarrows M$ be a s -connected Lie groupoid, and let E be a representation of \mathcal{G} . Then there exists a map

$$\Phi: H_d^p(\mathcal{G}, E) \rightarrow H_{\mathcal{G}}^p(\text{Lie}(\mathcal{G}), E),$$

which is compatible with the product structures. Moreover, if the s -fibers are homologically n -connected, then Φ is an isomorphism in degrees $p \leq n$, and it is injective in degree $p = n + 1$.

Example 113 (Lie Group Cohomology). Let G be a Lie group and V a vector space. A **representation** of G on V is a smooth group homomorphism $\rho: G \rightarrow GL(V)$, where $GL(V)$ is the general linear group of all invertible linear transformations of V .

Similar to the Lie algebra cohomology, given a Lie group G and a representation $\rho: G \rightarrow GL(V)$, we define the cochains as

$$C^q(G, V) = \{c: G^q \rightarrow V \mid c \text{ is smooth}\}.$$

We define $d: C^q(G, V) \rightarrow C^{q+1}(G, V)$ by

$$\begin{aligned} (d\omega)(g_0, \dots, g_q) &= \rho(g_0)\omega(g_1, \dots, g_q) + \sum_{i=0}^{q-1} (-1)^{i+1} \omega(g_0, \dots, g_i g_{i+1}, \dots, g_q) \\ &\quad + (-1)^{q+1} \omega(g_0, \dots, g_{q-1}). \end{aligned}$$

Then, the Lie group cohomology is defined as follows.

Definition 114. Given a Lie group G with a representation $\rho: G \rightarrow GL(V)$. Using the previous definition of $C^q(G, V)$ and $d: C^q(G, V) \rightarrow C^{q+1}(G, V)$, we define the **Lie group cohomology with coefficients in V** by

$$H^q(G, V) := \frac{\text{Ker}(d: C^q(G, V) \rightarrow C^{q+1}(G, V))}{\text{Im}(d: C^{q-1}(G, V) \rightarrow C^q(G, V))}.$$

When $V = \mathbb{R}$ and ρ is trivial, we denote $C^q(G, \mathbb{R})$ and $H^q(G, \mathbb{R})$ only by $C^q(G)$ and $H^q(G)$.

In [Est 1953, Est 1953], Van Est presented the following result that permits us to relate Lie group cohomology with Lie algebra cohomology.

Theorem 115 (Van Est). Let G be a n -connected Lie Group and $\rho: G \rightarrow GL(\mathfrak{v})$ a representation. If \mathfrak{g} is the Lie algebra of G , then $H^i(G, V)$ is isomorphic with $H^i(\mathfrak{g}, V)$ for all $i = 0, 1, \dots, m$.

CONN'S LINEARIZATION THEOREM

Let (M, Π) be a Poisson manifold and let $x_0 \in M$ such that $\Pi(x_0) = 0$. Let $\Pi = \Pi^1 + \Pi^2 + \dots$ be the Taylor expansion of Π with respect to a local coordinate centered on x_0 . From the condition that the Schouten bracket of Π with itself vanishes, one deduces

$$\begin{aligned} 0 = [\Pi, \Pi] &= \left[\sum_k \Pi^k, \sum_k \Pi^k \right] \\ &= [\Pi^1, \Pi^1] + 2 [\Pi^1, \Pi^2] + (2 [\Pi^1, \Pi^3] + [\Pi^2, \Pi^2]) + \dots \end{aligned}$$

Since every coefficient on the above expansion is a polynomial function of the local coordinates, considering the terms of same degree, we get

$$\begin{aligned} [\Pi^1, \Pi^1] &= 0, \\ [\Pi^1, \Pi^2] &= 0, \\ 2 [\Pi^1, \Pi^3] + [\Pi^2, \Pi^2] &= 0, \\ \text{and so on } &\dots \end{aligned}$$

So, by $[\Pi^1, \Pi^1] = 0$, the linear term of the expansion is a Poisson structure.

Our interest is to study the case in which Π^1 is non-null. According to equation (2.7.1) we can notice that $\Pi^{(1)}$ is the linear Poisson structure associated to the isotropy Lie algebra \mathfrak{g}_{x_0} .

It is always possible to truncate the series expansion and take an approximation of a bivector field by a linear one. But now, we may ask if there is any condition for a Poisson structure to be locally isomorphic to the linear approximation. The following definition summarizes the core of the linearization problem.

Definition 116. We say that Π is locally linearizable if there is a locally defined diffeomorphism Φ such that $\Phi_*\Pi = \Pi^{(1)}$.

Theorem 118 gives some conditions to the isotropy Lie algebra to carry all the information, up to local isomorphisms, about the Poisson structure near x_0 such that $\Pi(x_0) = 0$. The following definition formalizes this idea.

Definition 117. A finite-dimensional Lie algebra \mathfrak{g} is called non-degenerate if every Poisson structure Π , which vanishes at a point x_0 and whose $\Pi^{(1)}$ is the linear Poisson structure corresponding to \mathfrak{g} , is linearizable.

In [Weinstein 1983], Alan Weinstein proposed the following problem: how to classify the Lie algebras that are non-degenerate? In that paper, Weinstein conjectured that every compact semi-simple Lie algebra is non-degenerate. This claim was proved by Conn using non-trivial analytic arguments [Conn 1985].

Recently, a more geometric approach to Conn's theorem was presented by Marius Crainic and Rui Loja Fernandes on [Crainic and Fernandes 2011].

The goal of this chapter is to present a discussion, based on [Crainic and Fernandes 2011] as the main reference, about the proof of the following theorem.

Theorem 118 (Conn). Let (M, Π) be a Poisson manifold and $x_0 \in M$ such that $\Pi(x_0) = 0$. If the isotropy Lie algebra \mathfrak{g}_x is semi-simple of compact type, then Π is linearizable around x_0 .

4.1 Compact Lie Algebras

The studies of this section is presented in [Duistermaat and Kolk 1999].

Definition 119. For every Lie algebra \mathfrak{g} , we can associate a bilinear form, called **Killing form**, defined by

$$k(x, y) = \text{tr}(adX \circ adY), X, Y \in \mathfrak{g}.$$

Corollary 120 (3.6.3 [Duistermaat and Kolk 1999]). Let \mathfrak{g} be a finite-dimensional Lie algebra. Then the following conditions are equivalent:

- \mathfrak{g} is compact and semi-simple.
- \mathfrak{g} is compact and its center is zero.
- \mathfrak{g} has a negative definite Killing form.
- Every connected Lie group with Lie algebra isomorphic to \mathfrak{g} is compact. In any of these cases, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

In [Dufour and Zung 2006] the next theorem is presented.

Theorem 121 (Whitehead). If \mathfrak{g} is semi-simple and V is a finite-dimensional \mathfrak{g} -module, then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$.

4.2 Reduction to a cohomological condition

By a Poisson version of Morse's path method it is possible to find a sufficient condition to a Poisson structure to be linearizable around x_0 . This condition involves only the local cohomology of the Poisson manifold and the cohomology of the isotropy Lie algebra at x_0 .

Proposition 122. Let $(M, \{\cdot, \cdot\})$ be a Poisson Manifold such that Π vanishes at x_0 . Assume that the Lie algebra cohomology groups $H^1(\mathfrak{g}_{x_0}) = H^1(\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0}) = 0$. If $H_{\Pi}^1(M, x_0) = 0$, then $\{\cdot, \cdot\}$ is linearizable at x_0 .

Proof. Since this is a local fact, without loss of generality, one can assume that $M = \mathbb{R}^n$ and $x_0 = 0$. We define a path of Poisson structures by

$$\Pi_t(x) = \frac{1}{t}\Pi(tx), t \in [0, 1].$$

We will show that there exists a vector field X in an open neighborhood of 0 whose linearization vanishes at the origin and

$$\mathcal{L}_X(\Pi) = -\dot{\Pi}_1. \quad (4.2.1)$$

Indeed, taking the time derivative of $[\Pi_t, \Pi_t] = 0$, at $t = 1$, one has

$$\left. \frac{d}{dt} [\Pi_t, \Pi_t] \right|_{t=1} = [\dot{\Pi}_1, \Pi_1] + [\Pi_1, \dot{\Pi}_1] = [\Pi, \dot{\Pi}_1] - (-1^{(2-1)(2-1)}) [\Pi, \dot{\Pi}_1] = 0$$

so $d_{\Pi}(\dot{\Pi}_1) = [\Pi, \dot{\Pi}_1] = 0$, and since $H_{\Pi}^2(M, 0) = 0$, there is an open neighborhood U of the origin and a vector field on U such that

$$\dot{\Pi}_1 = d_{\Pi}Y = -\mathcal{L}_Y(\Pi).$$

Since $\dot{\Pi}_1 = 0$ at x_0 , if we evaluate both sides of the above expression on a pair of 1-forms α, β

$$Y_{x_0}([\alpha, \beta]) = \mathcal{L}_Y(\Pi)(\alpha, \beta)|_{x=0} = \dot{\Pi}_1(\alpha, \beta)|_{x_0} = 0.$$

By hypothesis, $H^1(\mathfrak{g}_{x_0}) = 0$, so $[\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0}] = \mathfrak{g}_{x_0}$ and then, we have that $Y_{x_0} = 0$.

Let Y_0 be the linearization of Y at the origin. Since $\dot{\Pi}_1 = 0$ has zero linearization at the origin, if we compare the linear terms of the both sides at x_0

$$d_{\Pi_{x_0}}Y_0 = 0.$$

Hence, Y_0 is a 1-cocycle for the adjoint representation of \mathfrak{g}_{x_0} . Since $H^1(\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0}) = 0$, there is $v \in \mathfrak{g}_{x_0}$ such that $Y_0 = d_{\Pi_{x_0}}v$. Since $(d_{\Pi})^2 = 0$, if we set $X = Y - d_{\Pi}v$ we have that

$$\mathcal{L}_X(\Pi) = 0$$

and the linearization of X at the origin is zero.

Now we can construct a time dependent vector field defined by $X_t(x) := \frac{1}{t^2}X(tx)$. So, by 4.2.1,

$$\mathcal{L}_{X_t}(\Pi_t) = -\dot{\Pi}_t.$$

Let Φ_t be the flow in the X_t such that it is defined in some neighborhood V of the origin for $t \in [0, 1]$. So

$$\frac{d}{dt}(\Phi_t)_*\Pi_t = (\Phi_t)_* \left(\mathcal{L}_{X_t}(\Pi_t) + \frac{d\Pi_t}{dt} \right) = 0.$$

So, $(\Phi_t)_*\Pi_t$ is constant in relation to t and Φ_1 is the desired diffeomorphism. \square

Thus, we have a cohomological condition that implies in the linearization. By the Whitehead theorem (121), the condition of semi-simplicity implies that $H^1(\mathfrak{g}_{x_0}) = H^1(\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0}) = 0$. With the purpose of trying to change this condition to one that refers to the isotropy Lie algebra in x_0 only, we will go back to the integrability problem for associated Lie algebroid of M .

Theorem 123. Let (M, Π) be a Poisson manifold, $x_0 \in M$ such that \mathfrak{g}_{x_0} is semi-simple of compact type. If some neighborhood of x_0 is integrable by a Lie groupoid with 1-connected s-fibers, then

$$H_{\Pi}^2(M, x_0) = 0.$$

Now we reduce the proof of Coinn's linearization theorem to local integrability around x_0 . On [Crainic and Fernandes 2004], Crainic and Fernandes, show the relation of the integrability of Poisson manifolds and symplectic realization.

Theorem 124. Let (M, Π) be a Poisson manifold and $x_0 \in M$ a fixed point. If a neighborhood U of x_0 admits a symplectic realization $\nu: S \rightarrow U$ with the property that $\nu^{-1}(x_0)$ is simply connected and compact, then there exists a neighborhood of x_0 which is integrable by a Hausdorff Lie groupoid with simple connected s-fibers.

The following theorem shows that, if the hypotheses of Conn's linearization theorem are true, the hypotheses of the previous theorem are also true. Thus, the statements of Conn's linearization theorem holds.

Theorem 125. Let (M, Π) be a Poisson manifold, $x_0 \in M$ a fixed point, such that, the isotropy Lie algebra \mathfrak{g} at x_0 is semi-simple of compact type with associated simply connected Lie group G . Then there exists a symplectic realization $\nu: S \rightarrow U$ of some open neighborhood U of x_0 such that $\nu^{-1}(x_0) = G$.

Example 126 (A non-linearizable Poisson structure). Consider the linear Poisson structure on $\mathfrak{sl}^*(2) \simeq \mathbb{R}^3$ given by $\{x, y\}_0 = -z$, $\{y, z\}_0 = x$, $\{z, x\}_0 = y$. The function $f(x, y, z) = x^2 + y^2 - z^2$ is in the center of $\{\cdot, \cdot\}$. So, f is a Casimir function whose symplectic leaves are hyperboloids with vertex in the origin of \mathbb{R}^3 . Let $g(f)$ be such that $g(f) = 0$ if $f \leq 0$ and $g(f) > 0$ if

$f > 0$. Consider the perturbation to $\{\cdot, \cdot\}_0$ given by $\{x, y\} = -z$, $\{y, z\} = x - \frac{y}{x^2+y^2}g(f(x, y, z))$, $\{z, x\} = y + \frac{x}{x^2+y^2}g(f(x, y, z))$.

The Hamiltonian vector field

$$X_z = - \left(y + \frac{x}{x^2+y^2}g(f(x, y, z)) \right) \frac{\partial}{\partial x} + \left(x - \frac{y}{x^2+y^2}g(f(x, y, z)) \right) \frac{\partial}{\partial y}.$$

Since $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is a periodic vector field, the integral curves of X_z in the region out of the cone $\{x^2 + y^2 - z^2 = 0\}$ will spiral in the direction of the cone. Thus, while the symplectic leaves of $\{\cdot, \cdot\}$ are closed, the symplectic leaves of $\{\cdot, \cdot\}$ in the region contain the cone $\{x^2 + y^2 - z^2 = 0\}$ in their closure.

On [Weinstein 1987], Weinstein presented the following theorem.

Theorem 127. Let \mathfrak{g} be a semi-simple Lie algebra of non-compact type with rank at least two. Then \mathfrak{g} is degenerate.

The reader can refer to [Dufour and Zung 2006, Weinstein 1987] for a more detailed discussion.

Lemma 128. Let X, Y be two Poisson vector fields of a Poisson manifold (M, Π) . $\Pi + X \wedge Y$ is a Poisson structure if, and only if, X, Y and $[X, Y]$ are linearly dependent or $[X, Y] = 0$.

Proof. Calculating the Schouten bracket of $\Pi + X \wedge Y$ with itself, we have

$$\begin{aligned} [\Pi + X \wedge Y, \Pi + X \wedge Y] &= [\Pi, \Pi] + 2[\Pi, X \wedge Y] + [X \wedge Y, X \wedge Y] \\ &= 2([\Pi, X] \wedge Y - X \wedge [\Pi, Y]) + [X \wedge Y, X \wedge Y] \\ &= [X \wedge Y, X] \wedge Y - X \wedge [X \wedge Y, Y] \\ &= X \wedge [Y, X] \wedge Y - X \wedge [X, Y] \wedge Y \\ &= 2X \wedge [Y, X] \wedge Y. \end{aligned}$$

The previous expression is equal to zero just when $[X, Y] = 0$ or X, Y , and $[X, Y]$ are linearly dependent. \square

The following theorem says that for any non-compact real semi-simple Lie algebra of real rank at least 2, it is possible to find two vector fields satisfying the conditions of the lemma above.

Theorem 129. Let \mathfrak{g} be a real semi-simple Lie algebra of real rank $r \geq 1$. Then for the linear Poisson structure Π_{lin} on the dual space \mathfrak{g}^* , there are r smooth Poisson vector fields X_1, \dots, X_r which are flat at the origin and which commute pair-wise, and such that, in an open subset of \mathfrak{g}^* whose closure contains the origin. These vector fields span a r -dimensional distribution whose intersection with characteristic distribution is trivial.

So, taking two vector fields, X and Y , given by the theorem above, the Poisson structure $\Pi = \Pi_{lin} + X \wedge Y$ has linear part equal to Π_{lin} . Since X and Y are linearly independent and are transversal to the symplectic leaves of Π_{lin} , the rank of Π is equal to the rank of $\Pi_{lin} + 2$. Thus, Π_{lin} can not be isomorphic to Π .

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