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Chern classes via differential forms

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Classes de Chern via formas diferenciais

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This work is dedicated to my lovely dog, Buddy,
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"Above all, don't lie to yourself. The man who lies to himself and listens to his own lie comes to a point that he cannot distinguish the truth within him, or around him, and so loses all respect for himself and for others. And having no respect he ceases to love."
(Fyodor Dostoevsky, The Brothers Karamazov)

RESUMO

TEZÔTO, I. T. O. **Classes de Chern via formas diferenciais**. 2022. 134 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

O objetivo dessa dissertação é apresentar algumas das bases matemáticas necessárias para a construção das classes de Chern em fibrados vetoriais complexos $\pi : E \to M$, com M uma variedade diferenciável, a partir da topologia diferencial. No trabalho abordamos alguns tópicos preliminares de álgebra multilinear, topologia geral, álgebra comutativa e teoria de categorias com o fim de apresentar as bases necessárias para desenvolver os conceitos presentes aqui. Em seguida, fazemos uma discussão sobre a teoria de variedades diferenciáveis necessária, como definições básicas, espaço tangente, diferenciabilidade, orientação e fronteira. A partir da noção de variedades, introduzimos as **formas diferenciais** e suas principais propriedades, que nos permite trabalhar com integração em variedades diferenciáveis de maneira simplificada devido às propriedades algébricas que o espaço graduado $\Omega^*(M)$ possui. Usando a teoria de formas diferenciais construímos uma teoria de cohomologia, chamada **Cohomologia de DeRham**, que é feita a partir dos espaços vetoriais das formas diferenciais. Os grupos de cohomologia são essenciais no presente trabalho, pois a partir deles temos as bases para apresentar diversos dos resultados importantes na tese como a **Dualidade de Poincaré**, a **Fórmula de Künneth** e o **Teorema de Leray-Hirsch**. Além disso, as classes de cohomologia são usadas para definir a classe de Euler nos fibrados vetoriais reais de rank 2 e, por consequência, a definição da primeira classe de Chern nos fibrados vetoriais complexos de rank 1. Depois, apresentamos de forma simplicada a construção geral das classes de Chern e algumas de suas propriedades. Por fim, é importante ressaltar a importância do conceito topológico de fibrados vetoriais no trabalho, tanto reais como complexos, tendo em vista sua relevância para definir as classes desejadas.

Palavras-chave: Variedades Diferenciáveis, Formas Diferenciais, Cohomologia, Fibrados Vetoriais, Classes Características.

ABSTRACT

TEZÔTO, I. T. O. **Chern classes via differential forms**. 2022. 134 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

The objective of this dissertation is to present, through differential topology, some of the mathematical foundations to construct the Chern classes on complex vector bundles $\pi : E \to M$, where M is a differentiable manifold. In this work we cover some preliminary topics of multilinear algebra, general topology, homological algebra and category theory in order to present the necessary background to develop the concepts here present. Next, we discuss the theory of differentiable manifolds needed, such as basic definitions, tangent space, differentiability, orientation and boundary. From the notion of manifolds, we introduce **differential forms** and their main properties, which allows us to work with integration on differentiable manifolds in a simplified way due to the algebraic properties that the graded space $\Omega^*(M)$ possesses. Using the theory of differential forms we construct a cohomology theory, called **de Rham's Cohomology**, which is defined from the vector spaces of differential forms. The cohomology groups are essential in this work, because from them we have the basis to present several of the important results in the thesis such as the **Poincaré duality**, the **Künneth formula** and the **Leray-Hirsch theorem**. Also, they are important for the definition of Euler classes on real vector bundles of rank 2 and, consequently, the definition of the first Chern class on complex line bundles. We then give an overview of the general construction of Chern classes and give some of its properties. Finally, it is important to emphasize the importance of the topological concept of vector bundles in the work, both real and complex, in view of its relevance to define the desired classes.

Keywords: Smooth Manifolds, Differential Forms, Cohomology, Vector Bundles, Characteristic Classes.

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CHAPTER

1

INTRODUCTION

The main objective of this work is to establish the mathematical foundations in order to study and define Chern classes using differential topology. These classes have not only mathematical meaning but historical meaning for mathematics. Shiing-Shen Chern was able to unify different characteristic classes of complex vector bundles, on what was later called Chern classes, among other notable achievements, such as Chern-Gauss-Bonnet theorem and Chern-Simons theory.

Using these characteristic classes we are able, in informal mathematical statement, to tell how much a complex vector bundle is trivial, i.e., how distant it is from a trivial bundle (or product bundle) $B \times F$.

Throughout this work we study many different subjects with more or less deepening, depending on its necessity to reach our final objective, which is to define Chern classes using the construction via differential forms and see examples for this theory. Our work is divided into 9 chapters, where

- the first chapter is dedicated to this introduction;
- the second chapter is used to present some of the necessary knowledge needed as a basis for the content in this work, such as multilinear algebra, general topology, commutative algebra and the very basics of category theory;
- the third chapter we work through the minimum necessary topics on differential

manifolds, such as the introduction to manifolds, its topology and basic concepts, differentiation and tangent spaces, orientation, manifolds with boundary (in order to study integration later on) and partition of unity;

- the fourth chapter is used to define and understand the importance of differential forms on multivariable calculus, passing through topics such as differentiation, integration over manifolds and the Stokes' theorem;
- the fifth chapter is dedicated to the study of de Rham cohomology, which is a particular type of cohomology, made from the vector spaces of the differential forms on a certain manifold M. We discuss the concept of de Rham cohomology and compactly supported cohomology and, then, we work with plenty of examples on the Euclidean space. After that, we discuss and prove both Poincaré lemmas. In the end of this chapter we introduce and use the Mayer-Vietoris sequence, both for de Rham cohomology and compactly supported cohomology;
- the sixth chapter is the discussion of the main theorems of this work, namely, the Poincaré duality, the Künneth formula and the Leray-Hirsch theorem;
- the seventh chapter is an introduction to the concepts of Vector bundles. We work with a detailed example and some propositions and lemmas. Also, here we work through the proof of an important theorem about vector bundles, the Homotopy Property of Vector Bundles;
- the eighth chapter is an overview on the final part of the theory used to define Chern classes, such as the compact vertical cohomology, the Thom class and the Euler class. At the end of the chapter we present the definition of Chern classes and give some of its properties;
- the ninth chapter is the conclusion of this work.

CHAPTER

2

PRELIMINARIES

2.1. Multilinear algebra

In this section we recall definitions and facts about multilinear algebra, such as the definition of a multilinear function and of alternate transformations, we give an explicit basis for the space of alternate transformations and, at the end of the section, we recall the definition of the pullback of linear functions. Those concepts are the base to understand differential forms, since a differential k-form is function ω given by $\omega : x \in M \mapsto \omega(x) \in \Lambda^k(T_xM)$, where M is an n-manifold. The content presented in this section is based on (LIMA, 2014) and (MELO, 2019).

Given two vector spaces E and F over a field K, where $\dim(E) < \infty$, we denote the space of k-linear functions between E and F as $L^k(E,F)$. This is a vector space with natural operations $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda \cdot f(x)$, for $x \in E$ and $\lambda \in K$. Unless stated otherwise, we always consider $K = \mathbb{R}$. We remember that a k-linear function T between E and F is a function

$$T : \underbrace{E \times \cdots \times E}_{k-times} \to F,$$

where T is linear in each of its coordinates independently from one another.

For $F = \mathbb{R}$ we denote $L^k(E,\mathbb{R})$ as $L^k(E)$. We assume the fact that if E is finite dimensional then $L^k(E,F)$ is finite dimensional, regardless of the dimension of F. If no mention is made, we consider our vector space E as $\mathbb{R}^m$.

Definition 2.1.1. We say that a k-linear transformation $T \in L^k(E,F)$ is alternate if it satisfies at least one of the following conditions:

- $T(v_1,\dots,v_k) = 0$, whenever $v_i = v_j$ for $i \neq j$;
- T is antisymmetric, *i.e.*, $T(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = -T(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$ for every $v_1,\dots,v_k \in E$ and $i < j$.

In fact, it is possible to show that the conditions on the previous definition are equivalent. We denote the subvector space composed by alternate transformations of $L^k(\mathbb{R}^m)$ as $\Lambda^k(\mathbb{R}^m)$. We consider $\Lambda^0(\mathbb{R}^m) = \mathbb{R}$ as a convention.

Example 2.1.2. Some examples of alternate linear transformations are:

- Every linear transformation between two vector spaces is alternate, since any of the two conditions on definition 2.1.1 cannot be unfulfilled by a linear transformation. Therefore, $\Lambda^1(\mathbb{R}^m,F) = L(\mathbb{R}^m,F)$, which give us $\Lambda^1(\mathbb{R}^m) = (\mathbb{R}^m)^*$.
- The determinant of square matrices is a m-linear alternate transformation from $M_m(\mathbb{R})$ to $\mathbb{R}$. Here we can identify $M_m(\mathbb{R})$ as $\mathbb{R}^{m^2}$.
- Given $f_1,\dots,f_r \in (\mathbb{R}^m)^*$ we can define $f_1 \wedge \cdots \wedge f_r : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}$ as a k-linear transformation by making

$$(f_1 \wedge \cdots \wedge f_r)(v_1,\dots,v_r) = \det(f_i(v_j)).$$

We define a linear function $\mathrm{Alt} : L^k(\mathbb{R}^m) \to \Lambda^k(\mathbb{R}^m)$, called *alternator*, given by

$$Alt(T)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} sign(\sigma)T(v_{\sigma(1)},\dots,v_{\sigma(k)}),$$

where S_k is the permutation group of the set $\{1,2,\dots,k\}$ and the function $sign : S_k \to \{-1,1\}$ is equal to 1 if the permutation can be decomposed in an even number of transpositions and it is equal to -1 if it can be decomposed in an odd number of transpositions.

We define the exterior product $\wedge : \Lambda^k(\mathbb{R}^m) \times \Lambda^l(\mathbb{R}^m) \to \Lambda^{k+l}(\mathbb{R}^m)$ by making

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta).$$

which gives us

$$\omega \wedge \eta(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sign(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

Remark 2.1.3. The previous definition allows us to work with alternate transformations that are expressed in a more abstract manner than the ones defined in the third item of the example 2.1.2. However, theorem 2.1.5 show us that every alternate transformation in $\Lambda^k(\mathbb{R}^m)$ can be written as a finite linear combination of $f_{i1} \wedge \cdots \wedge f_{ik}$, where $\{f_1, \ldots, f_m\}$ is a basis for $(\mathbb{R}^m)^*$. After proving this theorem, we can work with alternate transformations by not having to resort to this last definition.

Proposition 2.1.4. Given ω, η, θ multilinear alternate transformations from $\mathbb{R}^m$ to $\mathbb{R}$ then

- $(\omega + \theta) \wedge \eta = \omega \wedge \eta + \theta \wedge \eta$, where ω and θ are both k-linear;
- $\eta \wedge (\omega + \theta) = \eta \wedge \omega + \eta \wedge \theta$, where ω and θ are both k-linear;
- $a(\omega \wedge \eta) = (a\omega) \wedge \eta = \omega \wedge (a\eta)$, for all $a \in \mathbb{R}$;
- $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, for $\omega \in \Lambda^k(\mathbb{R}^m)$ and $\eta \in \Lambda^l(\mathbb{R}^m)$;
- $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$.

We denote the canonical basis of $\mathbb{R}^m$ as $\{e_1, \ldots, e_m\}$, where $e_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{mj})$ is a vector in $\mathbb{R}^m$ such that $\delta_{ij} = 0$, for $i \neq j$, and $\delta_{jj} = 1$. The dual basis of the dual space $(\mathbb{R}^m)^*$, which is the algebraic dual space of $\mathbb{R}^m$, is the subset of continuous linear functions denoted by $\{dx^1, \ldots, dx^m\} \subset (\mathbb{R}^m)^*$, where $dx^i(v) = v_i$ for all $v \in \mathbb{R}^m$. Therefore, we get $dx^i(e_j) = \delta_{ij}$.

The set $\{dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} | I = \{0 < i_1 < \cdots < i_k \leq m\}\}$ is a basis for the vector space $\Lambda^k(\mathbb{R}^m)$, when all possible combinations in the index subset $I \subset \{1, \ldots, m\}$ are done. For elements dx^I we are using the same definition given in the third item of example 2.1.2. Therefore, making all possible combinations of k elements on a set of m elements we get $\dim(\Lambda^k(\mathbb{R}^m)) = \dfrac{m!}{k!(m-k)!}$.

Theorem 2.1.5. For all possible $I = \{i_1 < \cdots < i_k\} \subset \{1,\ldots,m\}$, the set $\{dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}$ is a basis for $\Lambda^k(\mathbb{R}^m)$. Moreover, $\dim(\Lambda^k(\mathbb{R}^m)) = \dfrac{m!}{k!(m-k)!}$.

Proof. Let $\omega \in \Lambda^k(\mathbb{R}^m)$ and define $\varphi = \sum \alpha_I dx^I = \sum \omega(e_{i_1},\ldots,e_{i_k})dx^I$, for all subsets of integers $I = \{i_1 < \cdots < i_k\} \subset \{1,2,\ldots,m\}$. Consider the subset of integers $J = \{j_1,\ldots,j_k\}$, then

$$dx^I(e_{j_1},\ldots,e_{j_k}) = \begin{cases} 0, & \text{if } I \neq J \\ 1, & \text{if } I = J \end{cases} \tag{2.1}$$

Applying equality (2.1), we obtain

$$\varphi(e_{j_1},\ldots,e_{j_k}) = \sum_I \alpha_I dx^I(e_{j_1},\ldots,e_{j_k}) = \alpha_J = \omega(e_{j_1},\ldots,e_{j_k}).$$

A theorem from multilinear algebra states that if k-linear transformations coincide over the elements from a basis of our vector space $\mathbb{R}^m$, then they are the same. Therefore, $\varphi = \omega$. Moreover, if $\varphi = \sum_I \alpha_I dx^I = 0$, then for $J = \{j_1,\ldots,j_k\}$, we get

$$0 = \varphi(e_{j_1},\ldots,e_{j_k}) = \alpha_I dx^I(e_{j_1},\ldots,e_{j_k}) = \alpha_J \cdot 1 \Rightarrow \alpha_J = 0.$$

Varying for every possible set J, we show that the elements dx^I are linear independent. This finishes our proof. $\square$

Remark 2.1.6. The theorem used on the preceding proof is stated as: Let $T, S : E \times \cdots \times E \to F$ be two alternate k-linear transformations and $\{e_1,\ldots,e_m\}$ a basis for E. If $T(e_{i_1},\ldots,e_{i_k}) = S(e_{i_1},\ldots,e_{i_k})$ for all possible index sets $I = \{i_1 < \cdots < i_k\} \subset \{1,\ldots m\}$, then $T = S$.

Using the previous depicted basis of $\Lambda^k(\mathbb{R}^m)$ and the property that the wedge product between two alternate transformation is an alternate transformation we have

Corollary 2.1.7. If $k > \dim(\mathbb{R}^m)$, then $\Lambda^k(\mathbb{R}^m) = \{0\}$.

Remark 2.1.8. The theorem of multilinear algebra used on 2.1.5 to imply that $\varphi = \omega$ is a generalization from a theorem of linear algebra, which states that two linear transformations are equal if they coincide over a basis of their domain, that must be a finite basis.

Any given linear transformation $A : \mathbb{R}^m \to \mathbb{R}^p$ gives us a linear application between the spaces of alternate transformations. For each k we get

$$A^* : \Lambda^k(\mathbb{R}^p) \to \Lambda^k(\mathbb{R}^m)$$

defined by

$$(A^*\omega)(v_1, \ldots, v_k) = \omega(Av_1, \ldots, Av_k).$$

The alternate transformation $A^*\omega$ is called pullback of ω by A. If $k = 1$ we are looking at the transpose application of A.

2.2. General Topology

In this section we recall basic definitions and facts about topology that are recurrent throughout this work, given its importance to understand the differential structures of manifolds and the structures of vector bundles. The content presented in this section was based on (MUNKRES, 2000).

We begin from the definition of a topological space, which is an ordered pair (X, τ), with X being any set satisfying the ZFC and τ being a collection of subsets of X, which satisfies the following axioms

- $\emptyset, X \in \tau$;
- Any finite intersections of elements of τ must be in τ;
- Any arbitrary unions of τ must be in τ.

The collection τ is called a topology in X an its elements are defined as being the *open sets* of X. Any given $F = A^c = X \setminus A$ of any open set $A \in \tau$ is called a *closed set* of X. More precisely, a subset $F \subset X$ is said to be *closed* in the topological space (X, τ) if, and only if, it is the complement of an open subset in (X, τ).

Definition 2.2.1. A subset $\mathscr{B} \subset \tau$ is said to be a basis of the topological space (X,τ) if for any open subset $A \in \tau$ and for any point $x \in A$ there is a element $B \in \mathscr{B}$ such that $x \in B \subset A$. A topological space is said to be second countable if it exists a countable basis made of open sets for him.

Definition 2.2.2. A topological space (X,τ) is said to be a Hausdorff space, or T_2 space, if for any given distinct points $x,y \in X$ there is $A,B \in \tau$, $A \cap B = \emptyset$, such that $x \in A$ and $y \in B$.

Theorem 2.2.3 (Lindelöf's Lemma)**.** Let (X,τ) be a second countable topological space. If $\mathscr{C}$ is an open cover for X, then there is $\mathscr{C}' \subset \mathscr{C}$ a countable subcover for X.

Definition 2.2.4. A topological space (X,τ) is said to be paracompact if every open cover has an open refinement that is locally finite, *i.e.*, for every open cover $\{U_\alpha \in \tau : \alpha \in A\}$ there is $\{V_\beta \in \tau : \beta \in B\}$, where for every $\beta \in B$ there is an $\alpha \in A$ such that $V_\beta \subset U_\alpha$. Moreover, for every $x \in X$, there is an open neighbourhood V_x of x, such that $V_x \cap V_\beta \neq \emptyset$ for only finitely many $\beta \in B$.

Definition 2.2.5. Let (X,τ) and (Y,ρ) be two topological spaces and $f : X \to Y$ be a function between them. We say that f is a continuous function if, for every open set $A \in \rho$, then $f^{-1}(A)$ is an open set of X, *i.e.*, $f^{-1}(A) \in \tau$. Here $f^{-1}(A)$ is the preimage of the set A.

Definition 2.2.6. Let (X,τ) and (Y,ρ) be two topological spaces and $f,g : X \to Y$ two continuous functions. We say that $F : X \times [0,1] \to Y$ is an homotopy between f and g if F is a continuous function and $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$.

Definition 2.2.7. Let (X,τ) be a topological space, V a vector space and $f : X \to V$ a function between them. We define the support of f, denoted by $\operatorname{supp}(f)$, as the set

$$\operatorname{supp}(f) := \overline{\{x \in X | f(x) \neq 0\}}$$

2.3. Homological Algebra

Since we work with cohomology, we recall some basic concepts about homological algebra, especially those related with exact sequences. The content presented in this section was based on (TENGAN; BORGES, 2015).

Denote C^q as vector spaces indexed in the integers. We call the direct sum $C = \oplus_{q\in\mathbb{Z}} C^q$ a differential complex if there are homomorphisms

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

satisfying $d^2 = 0$. The operator d is called the differential operator of the complex C. We define the cohomology of C as the direct sum $H(C) = \oplus_{q\in\mathbb{Z}} H^q(C)$ where, for each $q \in \mathbb{Z}$, we have the quotient group $H^q(C) = (\text{Ker } d)/(\text{Im} d)$, for Ker $d : C^q \to C^{q+1}$ and $\text{Im} d : C^{q-1} \to C^q$.

Definition 2.3.1. We call a map $f : A \to B$ a *chain map* between differential complexes if it commutes between both differential operators of A and B, d_A and d_B, *i.e.*, $f d_A = d_B f$.

The most important concept of homological algebra for us in this work is the one of exact sequence. We say that the following sequence of vector spaces

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \dots$$

is an *exact sequence* if Ker $(f_i) = \text{Im}(f_{i-1})$ for all i. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

If we take a short exact sequence of chain maps f and g between differential complexes given by

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0,$$

then we can induce a long exact sequence on the cohomology groups

$$\cdots \longrightarrow H^q(A^\bullet) \xrightarrow{f^*} H^q(B^\bullet) \xrightarrow{g^*} H^q(C^\bullet) \xrightarrow{d^*} H^{q+1}(A^\bullet) \longrightarrow \dots,$$

where d^* is a connecting homomorphism. An important fact about exact sequences, which is extremely useful to us in order to compute cohomology groups, is the next theorem.

Theorem 2.3.2. Let $0 \overset{f_0}{\to} V^1 \overset{f_1}{\to} V^2 \overset{f_2}{\to} V^3 \dots \overset{f_{k-1}}{\to} V^k \overset{f_k}{\to} 0$ be an exact sequence, where each V^q is a finite dimensional vector space, then

$$\sum_{q=1}^{k} (-1)^q dim(V^q) = 0.$$

Proof. For every vector space V^q we can write that $\dim V^q = \dim \operatorname{Ker} f_i + \dim \operatorname{Im} f_i$. Then,

$$\sum_{q=1}^{k}(-1)^q dim(V^q) = \sum_{q=1}^{k}(-1)^q \dim \operatorname{Ker} f_i + \sum_{q=1}^{k}(-1)^q \dim \operatorname{Im} f_i.$$

Since $0 \xrightarrow{f_0} V^1 \xrightarrow{f_1} V^2 \xrightarrow{f_2} V^3 \dots \xrightarrow{f_{k-1}} V^k \xrightarrow{f_k} 0$ is an exact sequence, then $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$. Therefore,

$$\begin{aligned}
\sum_{q=1}^{k}(-1)^q \dim(V^q) &= \sum_{q=1}^{k}(-1)^q \dim \operatorname{Ker} f_i + \sum_{q=1}^{k}(-1)^q \dim \operatorname{Im} f_i \\
&= \sum_{q=1}^{k}(-1)^q \dim \operatorname{Ker} f_i + \sum_{q=1}^{k}(-1)^q \dim \operatorname{Ker} f_{i+1} \\
&= 0
\end{aligned}$$

$\square$

Also, the lemma known as Five Lemma is of importance to us.

Lemma 2.3.3 (Five Lemma)**.** Let the following be a commutative diagram of Abelian groups and groups homomorphisms

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "g_1"'] & B \arrow[r, "f_2"] \arrow[d, "g_2"'] & C \arrow[r, "f_3"] \arrow[d, "g_3"'] & D \arrow[r, "f_4"] \arrow[d, "g_4"'] & E \arrow[d, "g_5"'] \\
A' \arrow[r, "f_1'"'] & B' \arrow[r, "f_2'"'] & C' \arrow[r, "f_3'"'] & D' \arrow[r, "f_4'"'] & E'
\end{tikzcd}$$

in which the rows are exact and the maps g_1, g_2, g_4, g_5 are isomorphisms. Then g_3 is an isomorphism.

A proof for this lemma can be found in (MASSEY, 1991).

2.4. Category theory

In order to finish this chapter we discuss some category theory. Note that we are only working with basic definitions and names, because is what we need for our work.

Also, we draw attention for the concept of functor, both covariant and contravariant, which is the most important concept we use in the cohomology chapter. The content presented in this section is taken from (AWODEY, 2010).

We define a category as being a class consisting of the following:

- *Objects:* $A, B, C, \ldots$
- *Arrows:* $f, g, h, \ldots$
- For each arrow f there are given objects

$$dom(f), \quad cod(f)$$

  which are the *domain* and *codomain* of f. Those are written as

$$f : A \to B$$

  to indicate that $A = dom(f)$ and $B = cod(f)$.
- Given two arrows $f : A \to B$ and $g : B \to C$, where $cod(f) = dom(g)$, then there is a given arrow

$$g \circ f : A \to C$$

  called the *composite* of f and g.
- For each object A, there is a given arrow called the *identity arrow* of A

$$1_A : A \to A.$$

The arrows on a category should satisfy the two following axioms:

- Associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

  for any given arrows $f : A \to B, g : B \to C, h : C \to D$.
- Unit:

$$f \circ 1_A = f = 1_B \circ f$$

  for any given arrow $f : A \to B$.

Given two objects inside a category we denote the set of arrows between A and B as $Hom(A,B)$.

Definition 2.4.1. Given two categories $\mathbf{C},\mathbf{D}$ we define a functor between them as

$$F : \mathbf{C} \to \mathbf{D}$$

a mapping of objects to objects and arrows to arrows, which satisfies the following:

- $F(f : A \to B) = F(f) : F(A) \to F(B)$;
- $F(1_A) = 1_{F(A)}$;
- $F(g \circ f) = F(g) \circ F(f)$, for arrows $f : A \to B$ and $g : B \to C$.

The functor above is called *covariant functor*. In the first item, if we have reversed arrows, *i.e.*, $F(f : A \to B) = F(f) : F(B) \to F(A)$, instead of $F(f : A \to B) = F(f) : F(A) \to F(B)$, then F would be called *contravariant functor*. Also, the third item should be replaced by $F(g \circ g) = F(f) \circ F(g)$.

CHAPTER

3

SMOOTH MANIFOLDS

Smooth manifolds as a concept is a very useful way to work with a big variety of geometrical objects. Usually, we think of a smooth manifold M, often called just manifold, as a geometric object that locally looks like the Euclidean space and have a differentiable structure induced from $\mathbb{R}^n$. Throughout this chapter we work and discuss several definitions and results for the development of our theory. We recall what a smooth manifold is, discuss some of its structure, such as its differentiable structure and the tangent space, define orientation and partition of unity for manifolds. We also recall manifolds with boundary so we are able to talk about the Stokes' theorem. The content developed in this chapter was based on (LEE, 2013), (MANFIO, 2021), (LIMA, 2014) and (MELO, 2019).

3.1. Smooth Manifolds

Definition 3.1.1. Let M be a set. We define a *local chart* for M as a bijection $\varphi : U \to \varphi(U)$ where $U \subset M$ and $\varphi(U) \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. We denote our local chart as (U,φ) or $(U, x^1, \ldots, x^n)$.

Definition 3.1.2. Let M be a set. If (U,φ) and (V,ψ) are any two local charts for M, then they are said to be compatible if $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open sets of $\mathbb{R}^n$ and the transition function $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Definition 3.1.3. We define a n-dimensional *atlas* for M as a collection of local charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$, where every two of them are compatible, for every $\alpha \in I$ we have that $\varphi_\alpha(U_\alpha)$ is an open set of $\mathbb{R}^n$ and $M = \cup_{\alpha\in I} U_\alpha$, *i.e.*, the collection $(U_\alpha)_{\alpha\in I}$ is a cover for M.

Given an atlas $\mathscr{A}$ for a set M and a local chart (U, φ) of M, then we say that φ is compatible with $\mathscr{A}$ if it is compatible with every other local chart $\psi \in \mathscr{A}$.

Lemma 3.1.4. Let $\mathscr{A}$ be an atlas for M. If (U, φ) and (V, ψ) are local charts in M, both of them being compatible with $\mathscr{A}$, then φ and ψ are compatible.

Proof. Let us suppose that $U \cap V \neq \emptyset$, since otherwise it would be trivial. We can write $U = \cup_{\alpha\in I}(U \cap U_\alpha)$ where I is the index set for $\mathscr{A}$. Therefore,

$$\varphi(U \cap V) = \cup_{\alpha\in I}\varphi(U \cap V \cap U_\alpha).$$

$$\psi(U \cap V) = \cup_{\alpha\in I}\psi(U \cap V \cap U_\alpha).$$

We shall prove that $\varphi(U \cap V \cap U_\alpha)$, $\psi(U \cap V \cap U_\alpha)$ are open sets of $\mathbb{R}^n$ and $\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap U_\alpha)}$ is differentiable for every $\alpha \in I$. Indeed, given the fact that (U, φ) and (V, ψ) are compatible with every $(U_\alpha, \varphi_\alpha)$, it follows that $\varphi_\alpha(U_\alpha \cap U)$ and $\varphi_\alpha(U_\alpha \cap V)$ are open sets of $\mathbb{R}^n$. Moreover, $\varphi \circ \varphi_\alpha^{-1}$ and $\psi \circ \varphi_\alpha^{-1}$ are diffeomorphisms. Therefore, given that φ_α is a bijection, we get

$$\begin{aligned}\varphi(U \cap V \cap U_\alpha) &= (\varphi \circ \varphi_\alpha^{-1})(\varphi_\alpha(U \cap V \cap U_\alpha)) \\ &= (\varphi \circ \varphi_\alpha^{-1})(\varphi_\alpha(U_\alpha \cap U) \cap \varphi_\alpha(U_\alpha \cap V))\end{aligned}$$

is an open set of $\mathbb{R}^n$ and

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap U_\alpha)} = (\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1})|_{\varphi(U\cap V\cap U_\alpha)}$$

is a diffeomorphism, which proves that $\psi(U \cap V \cap U_\alpha)$ is an open set of $\mathbb{R}^n$. $\square$

Lemma 3.1.5. Given a set M and an atlas $\mathscr{A}$ for M, then it exists a single maximal atlas for M which contains $\mathscr{A}$.

Proof. Denote $\mathscr{A}_{max}$ as the set of all local charts in M that are compatible with $\mathscr{A}$. It is straightforward that $\mathscr{A} \subset \mathscr{A}_{max}$. The last lemma grants us that $\mathscr{A}_{max}$ is an atlas for M.

In order to prove that $\mathscr{A}_{\max}$ is the maximal atlas, we must take another atlas $\mathscr{B}$ for M which contains $\mathscr{A}$. Given that $\mathscr{B}$ is an atlas, then every one of its elements are compatible with elements of $\mathscr{A}$, since $\mathscr{A}$ is a subset of $\mathscr{B}$. Therefore $\mathscr{B} \subset \mathscr{A}_{\max}$. The uniqueness of such atlas is given by a similar argument. We suppose now $\mathscr{B}$ is another maximal atlas for M which contains $\mathscr{A}$. From this we can imply that every element of $\mathscr{B}$ is compatible with every element of $\mathscr{A}$. Therefore, $\mathscr{B} \subset \mathscr{A}_{\max}$, and given that they are both maximals, then $\mathscr{B} = \mathscr{A}_{\max}$. □

Theorem 3.1.6. Let M be a set and $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ an atlas for M. Then there is a unique topology in M which turns every U_α in an open set in M and every φ_α in an homeomorphism. This topology is given by

$$\tau_{\mathscr{A}} = \{V \subset M : \varphi_\alpha(U_\alpha \cap V) \text{ is an open set of } \mathbb{R}^n, \forall \alpha \in I\}.$$

Proof. By definition of $\tau_{\mathscr{A}}$ we get $\varphi_\alpha(U_\alpha \cap \emptyset) = \emptyset$, $\varphi_\alpha(U_\alpha \cap M) = \varphi_\alpha(U_\alpha)$, which proves that $\emptyset, M \in \tau_{\mathscr{A}}$. Moreover, we recall that φ_α is a bijection for every $\alpha \in I$, then $\varphi_\alpha(U_\alpha \cap V_1 \cap V_2) = \varphi_\alpha((U_\alpha \cap V_1) \cap (U_\alpha \cap V_2)) = \varphi_\alpha(U_\alpha \cap V_1) \cap \varphi_\alpha(U_\alpha \cap V_2)$, *i.e.*, finite intersections of elements of $\tau_{\mathscr{A}}$ are its own elements, and, similarly, $\varphi_\alpha(U_\alpha \cap (\cup_\beta V_\beta)) = \cup_\beta \varphi_\alpha(U_\alpha \cap V_\beta)$, *i.e.*, arbitrary unions remain in $\tau_{\mathscr{A}}$. This proves that $\tau_{\mathscr{A}}$ is a topology on M.

Notice that $\varphi_\alpha(U_\alpha)$ is an open set of $\mathbb{R}^n$ by definition. Let (U_β, φ_β) be another local chart of M, then, by definition of compatible charts, $\varphi_\beta(U_\beta \cap U_\alpha)$ is an open set of $\mathbb{R}^n$. Therefore, $U_\alpha \in \tau_{\mathscr{A}}$. Also, for $V \in \tau_{\mathscr{A}}$, $\varphi_\alpha(V) = \varphi_\alpha(V \cap U_\alpha)$ which is an open set of $\mathbb{R}^n$. In order to finish proving that φ_α is an homeomorphism, let A be an open set of $\mathbb{R}^n$ and consider $V = \varphi_\alpha^{-1}(A)$. Let $\beta \in I$ and notice that $\varphi_\beta(V \cap U_\beta) = \varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha(V \cap U_\beta) = \varphi_\beta \circ \varphi_\alpha^{-1}(A \cap \varphi_\alpha(U_\beta))$, where $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism and $(A \cap \varphi_\alpha(U_\beta)) = (A \cap \varphi_\alpha(U_\alpha \cap U_\beta))$ is an open set of $\mathbb{R}^n$. Therefore, φ_α is an homeomorphism for every $\alpha \in I$.

Finally, let τ be another topology on M such that every U_α is an open set and every φ_α is an homeomorphism. Let $V \in \tau$. By definition of τ, $V \cap U_\alpha \in \tau$, then $\varphi_\alpha(U_\alpha \cap V)$ is an open set on $\mathbb{R}^n$ for every $\alpha \in I$. Therefore, $\tau \subset \tau_{\mathscr{A}}$. On the other hand, let $V \in \tau_{\mathscr{A}}$, then $\varphi_\alpha(V \cap U_\alpha)$ is an open set of $\mathbb{R}^n$ for every $\alpha \in I$. Since φ_α is an homeomorphism on τ, then $\varphi_\alpha^{-1}(\varphi_\alpha(V \cap U_\alpha)) = V \cap U_\alpha \in \tau$ for every $\alpha \in I$. We can write $V = \cup_\alpha (V \cap U_\alpha)$, then $V \in \tau$ and, consequently, $\tau_{\mathscr{A}} \subset \tau$. $\square$

Definition 3.1.7. We define a *differentiable manifold* with dimension n as a pair $(M, \mathscr{A})$, where M is a set and $\mathscr{A}$ is a maximal atlas of dimension n, where the induced topology $\tau_{\mathscr{A}}$ is Hausdorff and second countable.

Example 3.1.8. The Euclidean space $\mathbb{R}^n$ is a manifold and one possible atlas is composed by a single chart $(\mathbb{R}^n, Id)$. More generally, every finite dimensional real vector space is a manifold and has a possible atlas composed by a single chart given by (V, T), where $T : V \to \mathbb{R}^n$ is the linear transformation that takes a basis of V to a basis of $\mathbb{R}^n$.

Example 3.1.9. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is a differentiable manifold. An atlas for it is given by $\mathscr{A} = \{(S^n \setminus \{N\}, \pi_N), (S^n \setminus \{S\}, \pi_S)\}$, where π_N and π_S are the stereographic projections. We have

$$\pi_N : S^n \setminus \{N\} \to \mathbb{R}^n$$
$$x \mapsto \pi_N(x) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n),$$

$$\pi_N^{-1} : \mathbb{R}^n \to S^n \setminus \{N\}$$
$$y \mapsto \pi_N^{-1}(y) = \Big(\frac{2y_1}{||y||^2+1}, \ldots, \frac{2y_n}{||y||^2+1}, \frac{||y||^2-1}{||y||^2+1}\Big),$$

$$\pi_S : S^n \setminus \{S\} \to \mathbb{R}^n$$
$$x \mapsto \pi_S(x) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n),$$

$$\pi_S^{-1} : \mathbb{R}^n \to S^n \setminus \{S\}$$
$$y \mapsto \pi_S^{-1}(y) = \Big(\frac{2y_1}{||y||^2+1}, \ldots, \frac{2y_n}{||y||^2+1}, \frac{||y||^2+1}{||y||^2+1}\Big),$$

To prove that S^n is a n-dimensional manifold it suffices to prove now that the stereographic projections are compatible charts. First of all, notice that $\pi_N(S^n \setminus \{N, S\}) =$

$\mathbb{R}^n \setminus \{0\}$ and $\pi_S(S^n \setminus \{N,S\}) = \mathbb{R}^n \setminus \{0\}$, which is an open set of $\mathbb{R}^n$. Moreover, $\pi_S \circ \pi_N^{-1}$: $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, $\pi_N \circ \pi_S^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ and, for $x \neq 0$, we get

$$(\pi_S \circ \pi_N^{-1})(x) = \frac{1}{||x||^2}(x_1, \ldots, x_n),$$

$$(\pi_N \circ \pi_S^{-1})(x) = \frac{1}{||x||^2}(x_1, \ldots, x_n),$$

i.e., $\pi_N \circ \pi_S^{-1}$ is a diffeomorphism. This proves that S^n is n-dimensional manifold.

Proposition 3.1.10. Every manifold M is paracompact.

Proof. We know that M is a Hausdorff, second countable and regular space space. By the Urysohn's metrization theorem (which can be found at (MUNKRES, 2000)), M is metrizable. Therefore, M is paracompact. □

Definition 3.1.11. Let N be a n-manifold. A subset $M \subset N$ is a submanifold of dimension m of N if for every $p \in M$ there is a local chart (U, φ) of N, where $p \in U$, such that

$$\varphi(U \cap M) = \varphi(U) \cap \mathbb{R}^m.$$

Notice that M can be seen as a m-manifold by taking a local chart (U, φ) of N, with $p \in U$, and defining $\overline{\varphi} : M \cap U \to \varphi(U) \cap \mathbb{R}^m$, where $\overline{\varphi} = \varphi|_{M \cap U}$.

3.2. Smooth maps and the tangent space

In this section we recall the notion of a differentiable function between manifolds. This idea allows us to work with tangent spaces and allows us to prove that every local chart is, in fact, a diffeomorphism. As everything related to manifolds, differentiability between manifolds is a concept that strongly relies on the notion of differentiability on the Euclidean space.

Definition 3.2.1. Let M be a m-dimensional manifold, N be a n-dimensional manifold and $f : M \to N$ be a function between manifolds. We say that f is *differentiable* at a point $p \in M$ if there are local charts, (U, φ) for M and (V, ψ) for N, where $p \in U$ and $f(U) \subset V$, such that the composite $\psi \circ f \circ \varphi^{-1}$ is differentiable at the point $\varphi(p) \in \varphi(U)$.

We say that the composite $\psi \circ f \circ \varphi^{-1}$ is the representation of f according to the local charts (V, ψ) and (U, φ). Moreover, f is said to be differentiable if it is differentiable at every point of M. Lastly, f is said to a be a diffeomorphism if it is a differentiable bijection with inverse $f^{-1} : N \to M$ also differentiable.

Proposition 3.2.2. Let M be a m-dimensional manifold, N a n-dimensional manifold and $f : M \to N$ a differentiable function at a point $p \in M$. Then the differentiability of f at p does not depend on the choice of the local charts as given in definition 3.2.1.

Proof. Let (U, φ), (U', φ') be local charts for M and (V, ψ), (V', ψ') be local charts for N, where $p \in U \cap U'$ and $f(U) \subset V$, $f(U') \subset V'$. Suppose $\psi \circ f \circ \varphi^{-1}$ and $\psi' \circ f \circ \varphi'^{-1}$ are the representations of f according to the local charts (V, ψ), (U, φ) and (V', ψ'), (U', φ') respectively. Consider, without loss of generality, that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$, then

$$\begin{aligned}\psi' \circ f \circ \varphi'^{-1}(\varphi'(p)) &= (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1})(\varphi'(p)) \\ &= (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1})(\varphi(p))\end{aligned}$$

is differentiable at the point $\varphi'(p)$, since $\varphi \circ \varphi'^{-1}$, $\psi' \circ \psi^{-1}$ are diffeomorphisms and $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$. $\square$

Proposition 3.2.3. Let $(M, \mathscr{A})$ be a differentiable manifold, a subset $U \subset M$, an open set $V \subset \mathbb{R}^n$ and a bijection $\varphi : U \to V$. Then $(U, \varphi) \in \mathscr{A}$ if, and only if, $U \in \tau_{\mathscr{A}}$ and φ is a diffeomorphism.

Proof. Suppose that φ is a diffeomorphism and U is an open set of M. We already know that $\varphi(U)$ is an open set of $\mathbb{R}^n$, then we just need to prove that the local chart (U, φ) is compatible with every local chart of $\mathscr{A}$. Let $(U', \varphi') \in \mathscr{A}$, then U' is an open set of M and φ' is an homeomorphism. We have that $\varphi(U \cap U')$ and $\varphi'(U \cap U')$ are open sets of $\mathbb{R}^n$, since $U \cap U'$ is an open set of M. Now, since φ is a diffeomorphism, the representation $\varphi' \circ \varphi^{-1}$ of φ^{-1}, according to the local charts $Id : \varphi(U \cap U') \subset \mathbb{R}^n \to \varphi(U \cap U') \subset \mathbb{R}^n$ and $\varphi'|_{U \cap U'} : U \cap U' \to \varphi'(U \cap U')$, is differentiable. In the same way $\varphi \circ \varphi'^{-1}$ is differentiable, proving that $(U, \varphi) \in \mathscr{A}$.

On the other hand, suppose $(U,\varphi) \in \mathscr{A}$. By definition of $\tau_{\mathscr{A}}$, U is an open set of M. Let $Id : V \to V$, then

$$Id \circ \varphi \circ \varphi^{-1} = Id : V \to V,$$

$$\varphi \circ \varphi^{-1} \circ Id = Id : V \to V$$

are the representations of φ and φ^{-1} according to local charts (V, Id) and (U,φ), respectively. Since $Id : V \to V$ differentiable, then φ is differentiable and has a differentiable inverse, *i.e.*, φ is a diffeomorphism. □

Now, we define the tangent space of a manifold M at a point $p \in M$. This space must have a structure of real vector space and it is useful for us to work with the differential of a differentiable function between manifolds.

We define the set $C_p := \{\lambda : (-\varepsilon,\varepsilon) \to M \mid \lambda(0) = p, \lambda \text{ is differentiable}\}$ and the following equivalence relation onto it: let $\lambda,\mu \in C_p$, we say that they are equivalent, which is denoted by $\lambda \sim \mu$, if there is a local chart (U,φ) on M, $p \in U$, such that $(\varphi\circ\lambda)'(0) = (\varphi\circ\mu)'(0)$.

Proposition 3.2.4. The relation $\sim$ given in last paragraph does not depend on the local chart.

Proof. Indeed, let (V,ψ) be another local chart on M such that $p \in V$, then

$$\begin{aligned}(\psi\circ\lambda)'(0) &= (\psi\circ\varphi^{-1}\circ\varphi\circ\lambda)'(0)\\ &= d(\psi\circ\varphi^{-1})(\varphi(p))\cdot(\varphi\circ\lambda)'(0)\\ &= d(\psi\circ\varphi^{-1})(\varphi(p))\cdot(\varphi\circ\mu)'(0)\\ &= (\psi\circ\mu)'(0).\end{aligned}$$

Since maps of the form $\varphi\circ\lambda$ are maps between Euclidean spaces, therefore, we can apply the chain rule. □

Proposition 3.2.5. The relation $\sim$ is an equivalence relation on C_p.

Proof. The reflexivity and symmetry property are immediate. Now, on the transition property, let $\lambda,\mu,\gamma \in C_p$, such that $\lambda \sim \mu$ and $\mu \sim \gamma$. Therefore, there are local charts

(U,φ), (V,ψ) on M, where $p \in U \cap V$, such that

$$(\varphi \circ \lambda)'(0) = (\varphi \circ \mu)'(0),$$

$$(\psi \circ \mu)'(0) = (\psi \circ \gamma)'(0).$$

We get

$$\begin{aligned}(\varphi \circ \lambda)'(0) &= (\varphi \circ \mu)'(0) \\ &= d(\varphi \circ \psi^{-1})(\psi(\mu(0))) \cdot (\psi \circ \mu)'(0) \\ &= d(\varphi \circ \psi^{-1})(\psi(\gamma(0))) \cdot (\psi \circ \gamma)'(0) \\ &= (\varphi \circ \psi^{-1} \circ \psi \circ \gamma)'(0) \\ &= (\varphi \circ \gamma)'(0),\end{aligned}$$

by working on the local chart $(U \cap V, \varphi)$. □

Definition 3.2.6. We define T_pM, the tangent space of M at the point p, by making $T_pM := C_p/\sim$.

In order to prove that T_pM has a real vector space structure we define an application induced from a local chart. Let (U,φ) be a local chart of M, where $p \in M$. We define

$$\begin{aligned}\overline{\varphi} : T_pM &\to \mathbb{R}^n \\ [\lambda] &\mapsto (\varphi \circ \lambda)'(0)\end{aligned}$$

which is well-defined thanks to proposition 3.2.4. Moreover, this map is easily seen to be injective. In order to prove that $\overline{\varphi}$ is surjective we take, for every $v \in \mathbb{R}^n$, the composition $\lambda = \varphi^{-1} \circ \alpha$, where $\alpha : (-\varepsilon,\varepsilon) \to \varphi(U)$ is given by $\alpha(t) = \varphi(p) + t \cdot v$. Given that $\overline{\varphi}$ is a bijection we can now give a space vector structure on T_pM by inducing operations from the Euclidean space, where $\overline{\varphi}$ becomes a linear isomorphism, namely

$$[\lambda] + [\mu] = \overline{\varphi}^{-1}(\overline{\varphi}([\lambda]) + \overline{\varphi}([\mu])),$$

$$a \cdot [\lambda] = \overline{\varphi}^{-1}(a \cdot \overline{\varphi}([\lambda])),$$

for every $[\lambda], [\mu] \in T_pM$ and $a \in \mathbb{R}$.

Remark 3.2.7. Similarly to proposition 3.2.4 we can verify that the vector space structure on T_pM does not depend on the local chart we are defining it.

Definition 3.2.8. Let M be a m-manifold, N be a n-manifold and $f: M \to N$ be a differentiable application at a point $p \in M$. We define the differential of f at p as an application $df(p): T_pM \to T_{f(p)}N$, by making $df(p)\cdot[\lambda] = [f \circ \lambda]$, for every $[\lambda] \in T_pM$.

Proposition 3.2.9. The differential $df(p)$ is well-defined and it is a linear map between tangent spaces.

Proof. Let (U,φ) be a local chart on M, with $p \in U$, and (V,ψ) be a local chart on N, where $f(U) \subset V$. Let $[\lambda] \in T_pM$, then

$$\overline{\psi}([f\circ\lambda]) = (\psi\circ f\circ\lambda)'(0) = (\psi\circ f\circ\varphi^{-1}\varphi\circ\lambda)'(0) = d(\psi\circ f\circ\varphi^{-1})(\varphi(p))\cdot\overline{\varphi}([\lambda]).$$

By applying $\overline{\psi}^{-1}$ on both sides we get

$$df(p)\cdot[\lambda] = \overline{\psi}^{-1}\circ(d(\psi\circ f\circ\varphi^{-1})(\varphi(p))\cdot\overline{\varphi}([\lambda])).$$

This shows that $df(p)$ is well defined since its definition depends only on the equivalence class $[\lambda]$ and that it is linear, since it can be written as a composite of linear operators. $\square$

Remark 3.2.10. Let (U,φ) be a local chart on a m-dimensional manifold M and $p \in U$. We say that the following set

$$\left\{\frac{\partial}{\partial x_1}(p),\dots,\frac{\partial}{\partial x_m}(p)\right\}$$

is a basis for the tangent space T_pM where $\frac{\partial}{\partial x_i}(p) = \overline{\varphi}^{-1}(e_i)$.

Let $f: M \to N$ be a differentiable application at p and (V,ψ) be a local chart for N, such that $f(U) \subset V$, where

$$\left\{\frac{\partial}{\partial y_1}(f(p)),\dots,\frac{\partial}{\partial y_n}(f(p))\right\}$$

is the basis for $T_{f(p)}N$. If $A = (a_{ij})$ is the matrix representation for the differential $df(p)$, then

$$df(p)\cdot\frac{\partial}{\partial x_i}(p) = \sum_{j=1}^{n} a_{ij}\frac{\partial}{\partial y_j}(f(p)).$$

We know that $\dfrac{\partial}{\partial x_i}(p) = \overline{\varphi}^{-1}(e_i)$ and $\dfrac{\partial}{\partial y_j}(f(p)) = \overline{\psi}^{-1}(e_j)$. Therefore, by the linearity of $\overline{\psi}$ we have

$$\overline{\psi}(df(p)\cdot\overline{\varphi}^{-1}(e_i)) = \sum_{j=1}^{n} a_{ij}e_j,$$

which implies that

$$d(\psi\circ f\circ\varphi^{-1})(\varphi(p))\cdot e_i = \sum_{j=1}^{n} a_{ij}e_j.$$

Therefore, A is the same as the jacobian matrix at the point $\varphi(p)$ of $\psi\circ f\circ\varphi^{-1}$, the representation of f according to φ and ψ.

Proposition 3.2.11. [Chain Rule] Let M, N and P be manifolds and $f : M \to N$ and $g : N \to P$ differentiable at p and $f(p)$ respectively. Then $g\circ f$ is differentiable at p and $d(g\circ f)(p) = dg(f(p))\circ df(p)$.

Proof. Let (U,φ), (V,ψ) and (W,η) be local charts on M, N and P, respectively, such that $p\in U$ and $f(U)\subset V$ and $g(V)\subset W$. Then

$$\begin{aligned}\eta\circ g\circ f\circ\varphi^{-1} &= \eta\circ g\circ(\psi^{-1}\circ\psi)\circ f\circ\varphi^{-1}\\ &= (\eta\circ g\circ\psi^{-1})\circ(\psi\circ f\circ\varphi^{-1})\end{aligned}$$

where $\psi\circ f\circ\varphi^{-1}$ is differentiable at $\varphi(p)$ and $\eta\circ g\circ\psi^{-1}$ is differentiable at $\psi(f(p))$. Therefore, $g\circ f$ is differentiable at p.

Now, for the formula, let $[\lambda]\in T_pM$, then

$$d(g\circ f)(p)\cdot[\lambda] = [(g\circ f)\circ\lambda] = [g\circ(f\circ\lambda)] = dg(f(p))\cdot[f\circ\lambda] = dg(f(p))\circ df(p)\cdot[\lambda].$$

□

3.3. Orientation

In this section we recall the notion of orientation on a manifold. Besides its importance to integration of forms, orientation plays an important role on cohomology and characteristic classes, since we need it to prove Poincaré's Duality and to define the Euler class on vector bundles of rank 2.

Let (U,φ) and (V,ψ) be two local charts of a manifold M. We say that they are orientation preserving when

- $U\cap V=\emptyset$ or
- $U\cap V\neq\emptyset$ and the matrix representation of $d(\varphi\circ\psi^{-1})(x)$ has a positive determinant for every $x\in\psi(U\cap V)$.

Note that $\varphi\circ\psi^{-1}$ is a diffeomorphism, therefore $d(\varphi\circ\psi^{-1})(x)$ has a non-zero determinant at every point $x\in\psi(U\cap V)$. Moreover, since the determinant function is continuous, then the determinant of the differential $d(\varphi\circ\psi^{-1})$ has a constant sign in every connected component of $\psi(U\cap V)$.

We call an atlas $\mathscr{A}$ of M to be oriented whenever two random local charts belonging to it are orientation preserving. Moreover, $\mathscr{A}$ is said to be a maximal oriented atlas if it is not a proper subset of any other oriented atlas of M. As in lemma 3.1.5 we can prove that every oriented atlas is contained in a maximal oriented atlas.

Definition 3.3.1. Let M be a manifold. We say that M is an orientable manifold if there exists an oriented atlas for it. We say that an oriented manifold is a pair $(M,\mathscr{A})$, where M is a manifold and $\mathscr{A}$ is a maximal oriented atlas. $\mathscr{A}$ is called an orientation for M and every local chart $(U,\varphi)\in\mathscr{A}$ is called positive.

Example 3.3.2. The manifold $(\mathbb{R}^n,\{(\mathbb{R}^n,Id)\})$ is an orientable manifold. In fact, every finite dimensional vector space V is an orientable manifold. We can consider the atlas $\mathscr{A}=\{(V,T)\}$ where T is a linear isomorphism to $\mathbb{R}^n$, $T(v_i)=e_i$ for a basis $\{v_1,v_2,\dots,v_n\}$ of V, which is orientation preserving. In case where T is not orientation preserving, we can make the composite $T'=T\circ\xi$, where $\xi(v_1)=-v_1$ and $\xi(v_i)=v_i$. Therefore, $\det(T')=\det(T\circ\xi)=\det(T)\cdot\det(\xi)>0$.

Example 3.3.3. Computations made in example 3.1.9 show us the n-sphere S^n is an orientable manifold with atlas $\mathscr{A}=\{(S^n\setminus\{N\},\pi_N),(S^n\setminus\{S\},\pi_S)\}$.

Definition 3.3.4. Let M be a manifold. We define the tangent bundle of M as $TM=\sqcup_{p\in M}T_xM=\cup_{p\in M}(\{p\}\times T_pM)=\{(p,v)\mid p\in M,v\in T_pM\}$.

Proposition 3.3.5. The tangent bundle of a n-manifold M is a $2n$-manifold. Moreover, TM is an orientable manifold.

Sketch of proof. We set the function $\pi : TM \to M$ defined as $\pi(p,v) = p$. Let (U,φ) be a local chart on M. We define a map $\overline{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by setting

$$\overline{\varphi}(p,v) = (\varphi(p), d\varphi(p) \cdot v),$$

for every $p \in U$ and $v \in T_pM$.

The set $\mathscr{A} = \{(\pi^{-1}(U), \overline{\varphi}) : (U,\varphi) \text{ are local charts of } M\}$ is an atlas of TM, *i.e.*, its elements are bijections, where the induced topology of $\mathscr{A}$, $\tau_{\mathscr{A}}$, is Hausdorff and second countable.

Moreover, notice that the transition functions $\overline{\psi} \circ \overline{\varphi}^{-1}$ have a Jacobian matrix

$$\begin{pmatrix} D(\psi \circ \varphi^{-1})(x) & 0 \\ 0 & D(\psi \circ \varphi^{-1})(x) \end{pmatrix}$$

with positive determinant for every $x \in \varphi(U \cap V)$.

The complete proof can be found on (MANFIO, 2021). $\square$

3.4. Manifolds with boundary

A semi-space on the Euclidean space is a subset $H \subset \mathbb{R}^n$ defined by means of a linear functional, *i.e.*, $H = \{x \in \mathbb{R}^n : \alpha(x) \geq 0\}$, where $\alpha \in (\mathbb{R}^n)^* \setminus \{0\}$. For instance, $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$ are semi-spaces of $\mathbb{R}^n$. The boundary of H, denoted by ∂H, is defined as $\partial H = \{x \in \mathbb{R}^n : \alpha(x) = 0\} = \text{Ker}\ (\alpha)$.

Let $\text{int}(H)$ be the interior of H according to the usual topology of $\mathbb{R}^n$, then $H = \text{int}(U) \cup \partial H$, where the union is disjoint. Furthermore, an open set $A \subset H$ is of two types

- $A \subset \text{int}(H)$, which is an open subset of $\mathbb{R}^n$, or
- $A \cap \partial H \neq \emptyset$, which is not an open subset of $\mathbb{R}^n$.

We define $\partial A = A \cap \partial H$.

Definition 3.4.1. Let H be a semi-space of $\mathbb{R}^n$ and $A \subset H$ an open subset of H. Let $f : A \to \mathbb{R}^m$ be a map from A. We say that f is differentiable on A if f is the restriction on A of a differentiable application $F : U \to \mathbb{R}^m$ on an open subset $U \subset \mathbb{R}^n$ such that $A \subset U$. Also, we define the differential of f as $Df(p) = DF(p)$ for every $p \in A$.

Remark 3.4.2. An important remark is that the definition of differentiability of $f : A \to \mathbb{R}^m$ on the previous definition does not depend on the choice of F, *i.e.*, for another extension F' of f then $DF'(p) = DF(p)$, for $p \in A$. Moreover, all definitions and propositions in past sections for manifolds with no boundary remain valid.

Proposition 3.4.3. Let $A \subset H$ and $B \subset K$ be open subsets of semi-spaces on $\mathbb{R}^n$. Let $f : A \to B$ be a diffeomorphism. Then $f(\partial A) = \partial B$ and the restriction $f|_{\partial A}$ is a diffeomorphism between the boundaries ∂A and ∂B.

Proof. Let $x \in \text{int}(A) \subset \text{int}(H)$. The subset $\text{int}(A)$ is an open subset of $\mathbb{R}^n$, therefore, there is an open subset $U \subset \mathbb{R}^n$, such that $x \in U \subset A$, such that $f|_U : U \to f(U) \subset B$ is a diffeomorphism between open sets of $\mathbb{R}^n$ by the inverse function theorem. Then $f(x) \in f(U)$, which is an open set, implies that $f(x) \in \text{int}(B)$ and, consequently, $f(\text{int}(A)) \subset \text{int}(B)$. Therefore, $f^{-1}(\partial B) \subset \partial A$. By the same reasoning applied to $f^{-1} : B \to A$ we can prove the inverse inclusion and arrive at the conclusion that $f(\partial A) = \partial B$. Moreover, by the same definition of differentiability applied on f, we can see that $f|_{\partial A}$ is differentiable. For this we use the same extension used for f. $\square$

Let M be a set, we say that (U, φ), $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^n$, is a local chart if it is a bijection and $\varphi(U)$ is an open set of a semi-space H of $\mathbb{R}^n$. Moreover, if (U, φ) and (V, ψ) are two local charts for M, then we say they are compatible if

- $U \cap V = \emptyset$ or
- $U \cap V \neq \emptyset$ and $\varphi(U \cap V)$ is an open subset of H and $\psi(U \cap V)$ is an open subset of K, where both H and K are semi-spaces on $\mathbb{R}^n$. Also, the application $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ must be a diffeomorphism everywhere on $\varphi(U \cap V)$.

Definition 3.4.4. A manifold with boundary M of dimension n is a pair $(M, \mathscr{A})$, where $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ is a maximal atlas for M and $\varphi_\alpha(U_\alpha)$ is an open set of a semi-space of $\mathbb{R}^n$. Moreover, the induced topology $\tau_{\mathscr{A}}$ on M, as in theorem 3.1.6, must be Hausdorff and second countable.

Definition 3.4.5. Let M be a manifold with boundary. We define the set of points p that are on the boundary of M, denoted by ∂M, as the points $p \in M$ such that there is a local chart (U, φ), $p \in U$, satisfying $\varphi(p) \in \partial H$ for the semi-space $H \supset \varphi(U)$.

Note that by proposition 3.4.3 we have that if (V, ψ) is another local chart, where $p \in V$, then $\psi \circ \varphi^{-1}$ is a diffeomorphism between the boundaries of open sets of semi-spaces $\partial \varphi(U \cap V) \subset \partial H$ and $\partial \psi(U \cap V) \subset \partial K$, *i.e.*, $\psi(p) \in \partial K$.

Definition 3.4.6. Let M and N be manifolds with boundary and $f : M \to N$ an application between them. As in definition 3.2.1 we say that f is differentiable at a point $p \in M$, if there are local charts (U, φ) of M and (V, ψ) of N, $f(U) \subset V$, such that the composite

$$\psi \circ f \circ \varphi^{-1}$$

is differentiable everywhere on $\varphi(U)$ as in definition 3.4.1.

Theorem 3.4.7. Let M be a manifold with boundary of dimension n and atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$. Then there is a maximal atlas $\mathscr{B}$ induced by $\mathscr{A}$ that turns ∂M into a differentiable manifold of dimension $n-1$.

Proof. Indeed, define $B = \{(V_\alpha, \psi_\alpha) : \alpha \in I\}$ where $V_\alpha = U_\alpha \cap \partial M$ and $\psi_\alpha = \varphi_\alpha|_{V_\alpha}$. Each pair (V_α, ψ_α) is a local chart for ∂M, since ψ_α is a bijection and

$$\psi_\alpha(V_\alpha) = \varphi_\alpha(U_\alpha \cap \partial M) = \varphi_\alpha(U_\alpha) \cap \partial H, \quad \text{for some semi-space } H,$$

which is an open set of $\mathbb{R}^{n-1}$. Since $H = \mathrm{Ker}\,(\alpha)$ for some $\alpha \in (\mathbb{R}^n)^* \setminus \{0\}$, then $H \simeq \mathbb{R}^{n-1}$. Moreover, the compatibility of any two local charts $(V_\alpha, \psi_\alpha), (V_\beta, \psi_\beta)$ of B comes from the fact that $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta)$ stands for $\varphi_\beta|_{U_\beta \cap \partial M} \circ \varphi_\alpha^{-1}|_{U_\alpha \cap \partial M} : \varphi_\alpha(U_\alpha \cap U_\beta) \cap \partial H \to \varphi_\beta(U_\alpha \cap U_\beta) \cap \partial K$, which is a diffeomorphism by proposition 3.4.3.

Finally, B is an atlas for ∂M, since

$$\cup_{\alpha \in I} V_\alpha = \cup_{\alpha \in I} U_\alpha \cap \partial M = (\cup_{\alpha \in I} U_\alpha) \cap \partial M = M \cap \partial M = \partial M.$$

We then take the maximal atlas $\mathscr{B}$ on ∂M that contains B. The induced topology $\tau_{\mathscr{B}}$ is Hausdorff and second countable since $\tau_{\mathscr{B}}$ is the same as the subspace topology induced by $\tau_{\mathscr{A}}$, which is Hausdorff and second countable. $\square$

Definition 3.4.8. Let M be a manifold with boundary of dimension n. Let $p \in M$ and (U, φ) be a local chart of M, such that $p \in U$ and $\varphi(U)$ is an open subset of the semi-space H. We define

- $T_pM = d\varphi(p)^{-1}(\mathbb{R}^n) = d\varphi(p)^{-1}(T_{\varphi(p)}H)$, for $\varphi(p) \in \text{int}(H)$, and
- $T_p\partial M = d\varphi(p)^{-1}(\partial H)$, for $\varphi(p) \in \partial H$.

3.5. Partition of Unity

Partitions of unity play a fundamental role in our theory, since we are able to define and work with integration on manifolds properly. Furthermore, we use partitions of unity throughout this work in order to define functions in cohomology. In some sense, the concept of partition of unity is the tool enabling us to expand notions from local open sets on a given manifold to the whole manifold. In this section we recall the basic definitions and most important results for us in this work.

Definition 3.5.1. Let M be a manifold. We say that a family of subsets of M, denoted by $\mathscr{C} = (C_\lambda)_{\lambda \in L}$, is locally finite if, for every point $p \in M$, there is at least one open neighbourhood V of p such that $V \cap C_\lambda \neq \emptyset$ for only a finite number of λ. If $\mathscr{C}$ can cover all M, we call it a locally finite cover of M.

Examples 3.5.2. Some examples of locally finite families.

- Every finite family $\mathscr{C}$ is a locally finite family;
- Every open cover $\mathscr{C} = (C_\lambda)_{\lambda \in L}$ of M where every C_λ intersects only a finite number of C_α is a locally finite family.
- The family $\mathscr{C} = (C_i)_{i \in \mathbb{N}}$ of $\mathbb{R}^m$, defined by

$$C_i = \{x \in \mathbb{R}^m; i-1 < |x| < i+1\}$$

  is a locally finite family;

- The family $\mathscr{C} = (C_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^m$ given by $C_1 = \mathbb{R}^m$ and $C_{i+1} = \mathbb{R}^m \setminus B(0,i)$, for every $i \in \mathbb{N}$, is a locally finite family.

The following proposition is of extreme importance for the definition of partition of unity.

Proposition 3.5.3. Let M be a manifold. If $\mathscr{C} = (C_\lambda)_{\lambda\in L}$ is a locally finite open cover of M, then $\mathscr{C}$ can be considered to be countable.

Proof. We need only to use theorem 2.2.3 on the open cover $\mathscr{C}$ and then use the fact that the topology on M is Hausdorff. □

Definition 3.5.4. Let M be a manifold. We define the partition of unity as a family of differentiable functions $(\psi_\lambda)_{\lambda\in L}$, $\psi_\lambda : M \to \mathbb{R}$, satisfying

- $\psi_\lambda(x) \geq 0$, for every $\lambda \in L$ and $x \in M$;
- The family $\mathscr{C} = (\operatorname{supp}(\psi_\lambda))_{\lambda\in L}$ is a locally finite family;
- For every $x \in M$, we have $\sum_{\lambda\in L} \psi_\lambda(x) = 1$.

Note that the second condition allows the third condition to be well-defined, since $\sum_{\lambda\in L} \psi_\lambda$ turns into a finite sum when applying it at a specific point $x \in M$. Also, from the first and third condition we obtain that $0 \leq \psi_\lambda(x) \leq 1$, for every $\lambda \in L$ and $x \in M$. We usually denote the partition of unity as $\sum_{\lambda\in L} \psi_\lambda = 1$.

Example 3.5.5. Let $\mathbb{R}$ be a manifold with usual topology being induced by $\mathscr{A} = \{(\mathbb{R}, Id)\}$. Let $\varepsilon \in (0,\infty)$ and define the open cover $\mathscr{U} = \{(n-1-\varepsilon, n+1+\varepsilon)\}_{n\in\mathbb{Z}}$. Notice that this open cover is a locally finite open cover for $\mathbb{R}$, since the set $(n-1-\varepsilon, n+1+\varepsilon)$ intersects only a finite number of other open sets, namely $\{(m-1-\varepsilon, m+1+\varepsilon)\}$, where $m-1-\varepsilon \leq n+1+\varepsilon$ or $m+1+\varepsilon \geq n-1-\varepsilon$.

We define a partition of unity $\rho_n : \mathbb{R} \to [0,1]$ subordinate to $\mathscr{U}$, *i.e.*, $\operatorname{supp}(\rho_n) \subset (n-1-\varepsilon, n+1+\varepsilon)$, by making

$$\rho_n(x) = \begin{cases} x-(n-1), & n-1 \leq x \leq n \\ 1-(x-n), & n \leq x \leq n+1 \\ 0, & \text{otherwise} \end{cases}$$

Notice $(\mathrm{supp}(\rho_n))$ is a locally finite family and that $0 \leq \rho_n(x) \leq 1$ for every $x \in \mathbb{R}$. Also, let $x \in \mathbb{R}$, such that $n-1 \leq x \leq n$ for some $n \in \mathbb{Z}$, then

$$\sum_{m \in \mathbb{Z}} \rho_m(x) = \rho_n(x) + \rho_{n-1}(x) = x - (n-1) + 1 - (x - (n-1)) = 1.$$

Remark 3.5.6. This example was taken from the website <https://ncatlab.org/nlab/show/partition+of+unity> at the time this work was being written.

We state the most important result for us in this section without proof, which can be found on (LEE, 2013).

Theorem 3.5.7. Let M be a differentiable manifold. If $\mathscr{C} = (C_\lambda)_{\lambda \in L}$ is an open cover of M, then there exists a partition of unity $\sum_{\lambda \in L} \psi_\lambda = 1$ that is strictly subordinated to the open cover $\mathscr{C}$, *i.e.*, $\mathrm{supp}(\psi_\lambda) \subset C_\lambda$ for every $\lambda \in L$.

CHAPTER

4

DIFFERENTIAL FORMS

Differential forms are objects defined on manifolds that allow us to work with multivariable calculus in an easier manner since their vector space have good algebraic properties, such as an exterior product. Usually, these properties help us define and work with integration on manifolds. Besides understanding what is a differentiable form and how to integrate it, we also recall a very important operator for our theory, which is the exterior derivative, a higher degree analogue of the usual differentiation. Finally, we end this chapter by proving Stokes theorem. A result whose proof shows us the benefits by using this new language. We note that the content presented in this chapter was based on (LIMA, 2014), (MELO, 2019), (TAO, 2018) and (BOTT; TU, 1982).

4.1. Introduction to differential forms on manifolds

Definition 4.1.1. Given an open subset $U \subset \mathbb{R}^m$ we define a differential k-form ω as a C^∞-function $\omega : U \to \Lambda^k(\mathbb{R}^m)$. The set of all differential k-forms is denoted by $\Omega^k(U)$. Also, we write $\omega(x) = \sum_I \alpha_I(x)dx^I$, where each $\alpha_I : U \to \mathbb{R}$ is C^∞ and $\alpha_I(x) = \omega(x)\cdot(e_{i_1},\dots,e_{i_k})$ for $I = \{i_1 < \cdots < i_k\} \subset \{1,\dots,m\}$.

Example 4.1.2. Let $U \subset \mathbb{R}^3$ be an open subset, then

- real C^∞-functions $f : U \to \mathbb{R}$ are the differential 0-forms;

- the differential 1-forms are applications $\omega = adx + bdy + cdz$ where $a,b,c : U \to \mathbb{R}$ are C^∞-functions, *i.e.*, differential 0-forms;
- the differential 2-forms are applications $\omega = adx \wedge dy + bdy \wedge dz + cdx \wedge dz$, where $a,b,c : U \to \mathbb{R}$ are C^∞-functions;
- the differential 3-forms are applications $\omega = adx \wedge dy \wedge dz$, where $a : U \to \mathbb{R}$ is a C^∞-function.

Given the nature of a differential form we are able to define the exterior product between two differential forms

$$\wedge : \Omega^k(\mathbb{R}^m) \times \Omega^l(\mathbb{R}^m) \to \Omega^{k+l}(\mathbb{R}^m),$$

by making a pointwise definition $(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x)$. We can see that $\omega \wedge \eta$ is a bilinear form because it is bilinear at every point $x \in U$. The properties from proposition 2.1.4 remain valid for differential forms.

Remark 4.1.3. The set $\Omega^k(U)$ is a real vector space and a $C^\infty(U)$-module over $\mathbb{R}$.

Let $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ be a C^∞-function between open sets of the Euclidean space. We can define the pullback f^* between the vector spaces of differential k-forms of V and U, denoted by $f^* : \Omega^k(V) \to \Omega^k(U)$, by making

$$(f^*\omega)(x)(v_1, \dots, v_k) = \omega(f(x))(Df(x) \cdot v_1, \dots, Df(x) \cdot v_k)$$

for every $\omega \in \Omega^k(V)$ and $x \in U$. Notice that for any given vectors $v_1, \dots v_k \in \mathbb{R}^m$ we have that $Df(x) \cdot v_1, \dots, Df(x) \cdot v_k \in \mathbb{R}^n$. Therefore, $f^*\omega$ is well-defined.

Proposition 4.1.4. Let $\omega, \overline{\omega} \in \Omega^k(V)$ and $\eta \in \Omega^l(V)$ and assume the last definitions regarding the function f. Then the following properties for pullbacks are valid

- $f^*(\omega + \overline{\omega}) = f^*\omega + f^*\overline{\omega}$;
- $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$;
- $f^*(\phi \cdot \omega) = (\phi \circ f) \cdot f^*\omega$, for every $\phi \in C^\infty(V, \mathbb{R})$;

- $(g \circ f)^* = f^* \circ g^*$, for a C^∞-function $g : V \subset \mathbb{R}^n \to W \subset \mathbb{R}^p$.

Proof. In order to prove the first and second items we need only to apply the definitions of sum and wedge product, respectively, combined with the definition of pullback at each point $x \in U$.

For the third item, let $x \in U$ and $v_1, \ldots, v_k$ vectors on $\mathbb{R}^m$. Then

$$\begin{aligned} f^*(\phi \cdot \omega)(x) \cdot (v_1, \ldots, v_k) &= (\phi \cdot \omega)(f(x)) \cdot (Df(x) \cdot v_1, \ldots, Df(x) \cdot v_k) \\ &= (\phi(f(x))) \cdot (\omega(f(x))) \cdot (Df(x) \cdot v_1, \ldots, Df(x) \cdot v_k) \\ &= (\phi \circ f)(x) \cdot (\omega(f(x))) \cdot (Df(x) \cdot v_1, \ldots, Df(x) \cdot v_k). \end{aligned}$$

For the fourth and last item, notice that $(g \circ f)^* \omega$ and $f^*(g^* \omega)$ are differential k-forms on $\Omega^k(U)$, since $g^* \omega \in \Omega^k(V)$. Let $\omega \in \Omega^k(W)$, $x \in U$ and $v_1, \ldots v_k, \in \mathbb{R}^m$, then

$$\begin{aligned} [(g \circ f)^*(\omega)(x)](v_1, \ldots, v_k) &= \omega(g(f(x))) \cdot (D(g \circ f)(x) \cdot v_1, \ldots, D(g \circ f)(x) \cdot v_k) \\ &= \omega(g(f(x))) \cdot (Dg(f(x))Df(x) \cdot v_1, \ldots, Dg(f(x))Df(x) \cdot v_k). \end{aligned}$$

Define $y = f(x)$ and $u_i = Df(x) \cdot v_i$ for $i \in \{1, \ldots, k\}$, then

$$(g^* \omega)(y)(u_1, \ldots, u_k) = \omega(g(y)) \cdot (Dg(y) \cdot u_1, \ldots, Dg(y) \cdot u_k), \quad \text{and, consequently,}$$

$$\begin{aligned} f^*(g^* \omega)(x)(v_1, \ldots, v_k) &= (g^* \omega)(f(x)) \cdot (Df(x) \cdot v_1, \ldots, Df(x) \cdot v_k) \\ &= (g^* \omega)(y)(u_1, \ldots, u_k). \end{aligned}$$

The chain rule finishes the proof. □

We now define a differential k-form on a manifold M. It is important to notice that every definition and proposition we have done until now continue to be valid in this case.

Definition 4.1.5. We define a differential k-form on a manifold M as a function associating each point $x \in M$ to a k-linear alternate transformation on T_xM, *i.e.*,

$$\omega : x \in M \mapsto \omega(x) \in \Lambda^k(T_xM).$$

If $\{du^1(x),\dots,du^m(x)\}$ is the dual basis of $\left\{\dfrac{\partial}{\partial u_1}(x),\dots,\dfrac{\partial}{\partial u_m}(x)\right\}$, induced by a certain local chart (U,φ) around x on T_xM, then we write $\omega(x)=\sum_I \alpha_I(\varphi(x))du^I(x)$. Each $\alpha_I:\varphi(U)\subset\mathbb{R}^m\to\mathbb{R}$ is a C^∞-function. Moreover, the space of differential forms on M is denoted by $\Omega^k(M)$.

Since φ is a bijection we simply write $\omega(x)=\sum_I \alpha_I(x)du^I$, given that the context makes it clear that for every $x\in U$ the set $\{du^1(x),\dots,du^m(x)\}=\{du^1,\dots,du^m\}$ is a basis for $(T_xM)^*$.

Furthermore, let (V,ψ) be another local chart such that $x\in V$, then $\psi\circ\varphi^{-1}$ is a diffeomorphism from $\varphi(U\cap V)$ to $\psi(U\cap V)$. If $\left\{\dfrac{\partial}{\partial v_1}(x),\dots,\dfrac{\partial}{\partial v_m}(x)\right\}$ is the basis for T_xM, induced by (V,ψ), with dual basis $\{dv^1,\dots,dv^m\}$, we can write $\omega(x)=\sum_J \beta_J(x)dv^J$. For every set J we can relate

$$\beta_J(x)=\sum_I \det\left(\frac{\partial u_I}{\partial v_J}\right)\alpha_I(x),$$

where $\left(\dfrac{\partial u_I}{\partial v_J}\right)=\left(\dfrac{\partial u_I}{\partial v_J}(\varphi(x))\right)$ is the $k\times k$ submatrix of the matrix representation of $d(\psi\circ\varphi^{-1})(\varphi(x))$, indexed by $\{i_1,\dots,i_k\}$ on the rows and by $\{j_1,\dots,j_k\}$ on the columns. Note that $\alpha_I(x)$ stands for $\alpha_I(\varphi(x))$ and $\beta_J(x)$ stands for $\beta_J(\psi(x))$. Therefore, the differential k-form ω on definition 4.1.5 does not depend on the local chart (U,φ) taken.

Example 4.1.6. Let M be a m-manifold and $\omega\in\Omega^k(M)$. Let (U,φ) be a local chart such that $\mathrm{supp}(\omega)\subset U$. Let $\left\{\dfrac{\partial}{\partial u_1}(x),\dots,\dfrac{\partial}{\partial u_m}(x)\right\}$ be the basis for T_xM induced by (U,φ), where $x\in U$, with dual basis $\{du^1,\dots,du^m\}$. We can write $\omega(x)=\sum_I \alpha_I(x)du^I$. We prove that $((\varphi^{-1})^*\omega)(\varphi(x))=\sum_I \alpha_I(x)dx^I$, where $\alpha_I(x)$ still stands for $\alpha_I(\varphi(x))$. Indeed,

$$\begin{aligned}((\varphi^{-1})^*\omega)(\varphi(x))\cdot(e_{i_1},\dots,e_{i_k}) &= \omega(\varphi^{-1}(\varphi(x)))\cdot(D\varphi^{-1}(\varphi(x))\cdot e_{i_1},\dots,D\varphi^{-1}(\varphi(x))\cdot e_{i_k})\\ &= \omega(x)\cdot\left(\frac{\partial}{\partial u_{i_1}}(x),\dots,\frac{\partial}{\partial u_{i_k}}(x)\right)\\ &= \alpha_I(\varphi(x))=\alpha_I(x).\end{aligned}$$

We denote the set of differential k-forms on M with compact support as $\Omega^k_c(M)$.

Proposition 4.1.7. Let M be a m-manifold, N be a n-manifold and $f : M \to N$ be a differentiable proper function, *i.e.*, a differentiable function satisfying the property that inverse images of compact sets are compact sets, then

$$f^*(\Omega_c^k(N)) \subset \Omega_c^k(M).$$

Proof. Let $\omega \in \Omega_c^k(N)$. Then $f^{-1}(\text{supp}(\omega))$ is compact and we only need to prove that $\text{supp}(f^*\omega) \subset f^{-1}(\text{supp}(\omega))$, since the support is a closed set. Let $x \in M \setminus f^{-1}(\text{supp}(\omega))$ and $v_1, \ldots, v_k \in T_xM$, then

$$0 = \omega(f(x)) \cdot (Df(x) \cdot v_1, \ldots, Df(x) \cdot v_k) = (f^*\omega)(x) \cdot (v_1, \ldots, v_k).$$

This means $x \in M \setminus \text{supp}(f^*\omega)$. Therefore, $\text{supp}(f^*\omega) \subset f^{-1}(\text{supp}(\omega))$ and the inclusion stated in the proposition statement is proved. $\square$

Now, after introducing the basic notions of differential forms on manifolds we can present a proposition that relates differential forms and the orientability of a manifold.

Proposition 4.1.8. An n-manifold M is orientable if and only if it has a global nowhere vanishing n-form.

Proof. Let M be an orientable manifold with oriented atlas $\mathscr{A} = \{U_\alpha, \varphi_\alpha : \alpha \in I\}$. For all transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ we write

$$(\varphi_\beta \circ \varphi_\alpha^{-1})^*(dx^1 \wedge \cdots \wedge dx^n) = \lambda \cdot dx^1 \wedge \cdots \wedge dx^n,$$

where λ is some positive function on $\varphi_\alpha(U_\alpha \cap U_\beta)$, since $D(\varphi_\beta \circ \varphi_\alpha^{-1})$ is a linear isomorphism at every point of $\varphi_\alpha(U_\alpha \cap U_\beta)$ and its matrix representation has a positive determinant everywhere. Also, note that $\dim(\Lambda^n(T_xM)) = 1$. Using properties of pullback functions we then have

$$\varphi_\beta^* dx^1 \wedge \cdots \wedge dx^n = (\varphi_\alpha^*\lambda)(\varphi_\alpha^* dx^1 \wedge \cdots \wedge dx^n).$$

Write $\varphi_\alpha^* dx^1 \wedge \cdots \wedge dx^n = \omega_\alpha \in \Omega^n(M)$, for each α, and $f = \varphi_\alpha^*\lambda = \lambda \circ \varphi_\alpha$. We obtain $\omega_\beta = f\omega_\alpha$, where f is a positive function in the intersection $U_\alpha \cap U_\beta$.

Now, define $\omega = \sum_\alpha \rho_\alpha \omega_\alpha$, where $\{\rho_\alpha\}$ is a partition of unity subordinated to the open cover $\{U_\alpha\}$. Then ω is a non-vanishing n-form since for each point $p \in M$, every ω_α defined on this point are multiples of one another by means of the positive function f and there is at least one $\rho_\alpha \geq 0$ that does not vanish at this point.

Conversely, let ω be a global nowhere vanishing n-form on M and let $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ be a local chart. We may assume that $\varphi_\alpha(U_\alpha)$ is homeomorphic to $\mathbb{R}^n$. By applying the pullback of φ_α we obtain $\varphi_\alpha^* dx^1 \wedge \cdots \wedge dx^n = f_\alpha \omega$, since $\dim(\Lambda^n(T_xM)) = 1$ at every $x \in M$, where f_α is a non-vanishing real valued function on U_α. Given that U_α is a connected open set, then f_α must be either positive over U_α or negative over U_α. There is no loss in generality to assume that f_α is positive. Otherwise, we would only need to compose $T \circ \varphi_\alpha$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation reversing diffeomorphism, such as $T(x_1, \ldots, x_n) = (-x_1, \ldots, x_n)$, which would give us $-f_\alpha$. Since we may assume f_α to be positive for all α, then any transition function $\varphi_\beta \circ \varphi_\alpha^{-1}$ takes $dx^1 \wedge \cdots \wedge dx^n$ to a positive multiple of itself by means of the pullback $(\varphi_\beta \circ \varphi_\alpha^{-1})^*$, which would mean that $D(\varphi_\beta \circ \varphi_\alpha^{-1})$ has a positive determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$. Therefore, $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ is an oriented atlas. □

Given two global non-vanishing n-forms ω and ω' on an n-manifold M, which is orientable, then they differ by a nowhere vanishing function f: $\omega = f \cdot \omega'$. This is true because $\dim \Lambda^n(T_xM) = 1$ at every point $x \in M$. Suppose M is a connected manifold, then f must be everywhere positive or everywhere negative. If f is positive then we say that ω and ω' are equivalent. This fact allows us to define an equivalence relation that gives rise to two equivalence classes. Since last proposition is a way to relate orientability on M with global non-vanishing differential n-forms, then, when M is connected, it only has two possible orientations.

4.2. Integration on manifolds

Integration of forms is the concept that shows all the potential of this language, since we can unify integrals of various types of functions in one definition. We use integration extensively on cohomology and to prove important theorems on chapter 6. Also, we use integration in order to work with compact vertical cohomology, so we can discuss the Thom isomorphism and, then, define the Euler class and first Chern class.

In order to define integrals on manifolds we initially suppose that $\omega \in \Omega_c^m(V)$ for an open subset $V \subset \mathbb{R}^m$ and that ω can be written as

$$\omega = a(x)dx^1 \wedge \cdots \wedge dx^m.$$

We then define the integral of ω in V as $\int_V \omega = \int_V a(x)dx$.

Let $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^m$ be a diffeomorphism and $a : V \to \mathbb{R}$ be an integrable function. Then the Variable Change theorem states that

$$\int_V a(x)dx = \int_U (a \circ f) \cdot |Df(x)|dx.$$

Looking at the pull-back of ω by f we got $f^*\omega(x) = (a \circ f)(x) \cdot \det(Df(x))dx^1 \wedge \cdots \wedge dx^m \in \Omega_c^m(U)$. Then from the definition of integral we obtain that $\int_U f^*\omega = \pm \int_V \omega$, where the sign depends on whether f preserves or inverses orientation on the connected components of U.

Now, saying that ω has a compact support means that the function a has a compact support. Therefore, the concepts in the beginning of this section is well-defined and the integration map is a linear functional on $(\Omega_c^m(V))^*$. We are now able to define integration of a differential form over an oriented manifold.

Definition 4.2.1. Let M be an oriented m-manifold. We define the integration as a linear application over the elements du^I

$$\int_M : \Omega_c^m(M) \to \mathbb{R}$$

for every given $\omega \in \Omega_c^m(M)$ satisfying $\mathrm{supp}(\omega) \subset U$, where $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^m$ is a local chart of M. We make

$$\int_M \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^*\omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

By the Variable Change theorem the preceding definition is well defined, since transition functions $\psi \circ \varphi^{-1}$ preserve orientation. Here, being well-defined, means that the sign remains positive if we took another local chart (V, ψ), $p \in V$, and apply the last definition. Note that, by doing this we have made a choice on the orientation of M.

We now define integration of forms with compact support not contained in any domain of any local chart. In order to do this we choose $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ an oriented atlas for M and $\{\lambda_i\}$ a partition of unity that is strictly subordinated to our atlas. Notice we take our index i in $\mathbb{N}$ due to proposition 3.5.3 and theorem 3.5.7. We make

$$\int_M \omega = \sum_i \int_M \lambda_i \omega.$$

Since $\operatorname{supp}(\omega)$ is a compact subset of M then the sum above is a finite sum and $\operatorname{supp}(\omega_i) \subset U_i$ for every $i \in I$. Moreover, the integral definition is not dependent from the partition of unity $\{\lambda_i\}$ we have chosen nor from the initial oriented atlas $\mathscr{A}$ picked, which proves that the integral is a well-defined linear functional.

Proposition 4.2.2. The integral $\int_M \omega$ does not have a definition dependent on the oriented atlas nor on the partition of unity.

Proof. Suppose $\{V_j\}$ is another oriented atlas for M and $\{\rho_j\}$ a partition of unity subordinate to $\{V_j\}$. Given that $\sum_j \rho_j = 1$ and that the sum $\sum_i \int_{U_i} \lambda_i \omega$ is not dependant on the index j, then

$$\sum_i \int_{U_i} \lambda_i \omega = \sum_{i,j} \int_{U_i} \lambda_i \rho_j \omega.$$

For each i and j we have $\operatorname{supp}(\lambda_i \rho_j \omega) \subset U_i \cap V_j$. Therefore,

$$\int_{U_i} \lambda_i \rho_j \omega = \int_{V_j} \lambda_i \rho_j \omega.$$

Finally, we have

$$\sum_i \int_{U_i} \lambda_i \omega = \sum_{i,j} \int_{V_j} \lambda_i \rho_j \omega = \sum_j \int_{V_j} \rho_j \omega.$$

□

Proposition 4.2.3. Let M be an oriented and compact m-manifold. If $\omega \in \Omega^m(M)$, then ω is integrable.

As a linear function on $(\Omega_c^m(M))^*$ we expect our integral to satisfy linear properties. Also, we expect it to satisfy usual properties of integrals from integration theory.

Proposition 4.2.4. Let $\omega, \eta \in \Omega_c^m(M)$ where M is an oriented m-dimensional manifold. Then the following are true

- $\int_M (\omega + \eta) = \int_M \omega + \int_M \eta$;

- $\int_M c \cdot \omega = c \cdot \int_M \omega$, for every number $c \in \mathbb{R}$;
- If $\omega \geq 0$ and $\omega(x) > 0$ for at least one $x \in M$, then $\int_M \omega > 0$;
- If $\omega \in \Omega_c^m(N)$, where N is an oriented m-manifold, and $f : M \to N$ is a diffeomorphism preserving orientation, then $\int_M f^*\omega = \int_N \omega$.

Proof. For the first item, suppose that $\text{supp}(\omega), \text{supp}(\eta) \subset U$, where $\varphi : U \to \varphi(U)$ is a local chart of M. Then,

$$\int_M (\omega + \eta) = \int_{\varphi(U)} (\varphi^{-1})^*(\omega + \eta) = \int_{\varphi(U)} (\varphi^{-1})^*\omega + \int_{\varphi(U)} (\varphi^{-1})^*\eta = \int_M \omega + \int_M \eta.$$

Suppose now that $\text{supp}(\omega) \subset U$ and $\text{supp}(\eta) \subset V$, for local charts (U, φ) and (V, ψ), where it is not important whether $U \cap V = \emptyset$ or $U \cap V \neq \emptyset$. Let $\{\rho_U, \rho_V\}$ be a partition of unity for $\{U, V\}$. Then,

$$\begin{aligned}
\int_M (\omega + \eta) &= \int_{\varphi(U)} (\varphi^{-1})^*(\rho_U[\omega + \eta]) + \int_{\psi(V)} (\psi^{-1})^*(\rho_V[\omega + \eta]) \\
&= \int_{\varphi(U)} (\varphi^{-1})^*(\rho_U \omega) + \int_{\psi(V)} (\psi^{-1})^*(\rho_V \omega) + \\
&\quad + \int_{\varphi(U)} (\varphi^{-1})^*(\rho_U \eta) + \int_{\psi(V)} (\psi^{-1})^*(\rho_V \eta) \\
&= \int_M (\rho_U + \rho_V) \cdot \omega + \int_M (\rho_U + \rho_V) \cdot \eta \\
&= \int_M \omega + \int_M \eta.
\end{aligned}$$

The general case comes now from similar arguments.

The second item in this proposition follows the same line of reasoning of the first item combined with the third item of proposition 4.1.4, where c is seen as a constant C^∞-function.

In order to prove the third item let (U_i) be an atlas for M and $\{\lambda_i\}$ be a partition of unity for (U_i). Then $\omega = \sum_i \lambda_i \omega$ and $\int_M \omega = \sum_i \int_M \lambda_i \omega$. Also, $\omega = \sum_I \alpha_I du^I \geq 0$ means that $a_I \geq 0$ for every I and, consequently, $\int_M \lambda_i \omega = \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \lambda_i \omega \geq 0$. Let i_0 be the index such that $\omega(x) = (\lambda_{i_0}\omega)(x) > 0$, then $\int_M \lambda_{i_0} \omega = \int_{\varphi_{i_0}(U_{i_0})} (\varphi_{i_0}^{-1})^* \lambda_{i_0} \omega > 0$. Therefore,

$$\int_M \omega = \sum_i \int_M \lambda_i \omega \geq \int_M \lambda_{i_0} \omega > 0.$$

The fourth and last item in this proposition is, in reality, a general statement of the ideas in the beginning of this section. We would only need to adapt the context of the arguments for general manifolds and the fact that a differential form is usually written as a sum $\omega = \sum_I \alpha_I du^I$. □

4.3. Exterior derivative

The exterior derivative is an important linear operator for the theory of differential forms and for de Rham cohomology. From an analytic point of view it grants us the generalization of the notion of derivatives of 0-forms, which is the usual derivative. This application basically takes as input differential k-forms and gives back as output differential $(k+1)$-forms by making the gradient of the α_J functions. We begin in the Euclidean space by looking at $\Omega^0(U)$, which is basically the space $C^\infty(U)$, for U an open subset of $\mathbb{R}^m$.

Initially, we define the exterior derivate $d : \Omega^0(U) \to \Omega^1(U)$ simply as being the gradient, that is

$$f \mapsto df = \sum_j \frac{\partial f}{\partial x_j} dx^j$$

For $\omega \in \Omega^k(U)$, written as $\omega = \sum_J \alpha_J dx^J$, we take the gradient of the functions α_J and define

$$d\omega = \sum_J da_J \wedge dx_J = \sum_j \sum_J \frac{\partial a_J}{\partial x_j} dx^j \wedge dx^J \in \Omega^{k+1}(U).$$

Example 4.3.1. Let $\omega \in \Omega^1(\mathbb{R}^2)$ given by $\omega = adx + bdy$, then

$$\begin{aligned} d\omega = da \wedge dx + db \wedge dy &= \Big(\frac{\partial a}{\partial x} dx + \frac{\partial a}{\partial y} dy\Big) \wedge dx + \Big(\frac{\partial b}{\partial x} dx + \frac{\partial b}{\partial y} dy\Big) \wedge dy \\ &= \frac{\partial a}{\partial y} dy \wedge dx + \frac{\partial b}{\partial x} dx \wedge dy = \Big(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\Big) dx \wedge dy. \end{aligned}$$

Now, let $\omega \in \Omega^1(\mathbb{R}^3)$ given by $\omega = adx + bdy + cdz$, then

$$d\omega = \Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big) dx \wedge dy + \Big(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\Big) dy \wedge dz + \Big(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\Big) dx \wedge dz.$$

Finally, for $\omega \in \Omega^2(\mathbb{R}^3)$, defined as $\omega = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$, we have

$$d\omega = \Big(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\Big) dx \wedge dy \wedge dz.$$

We have some important facts about exterior derivatives that provide us the algebraic basis for our cohomology theory. Note that the second item is know as the Leibniz rule.

Proposition 4.3.2. Let $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, for an open subset $U \subset \mathbb{R}^m$. Also, let $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ be a C^∞-function between open sets of the Euclidean space. Then

- $d(\omega + \eta) = d\omega + d\eta$;
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\text{degree}(\omega)}\omega \wedge d\eta$;
- $d(f^*\omega) = f^*(d\omega)$, where $\omega \in \Omega^k(V)$;

Proof. In order to prove the first item we need only to assume $\omega = adx^I$ and $\eta = bdx^J$. The general case $\omega = \sum_I \alpha_I dx^I$ and $\eta = \sum_J \beta_J dx^J$ comes from the fact that d is linear over the elements dx^I.

For the second item we may assume the same initial conditions for ω and η, then the general case is the outcome from the fact that d is linear over the elements dx^I. Therefore

$$\begin{aligned} d(\omega \wedge \eta) &= d(adx^I \wedge bdx^J) = d(abdx^I \wedge dx^J) = (bda + adb) \wedge dx^I \wedge dx^J \\ &= bda \wedge dx^I \wedge dx^J + adb \wedge dx^I \wedge dx^J \\ &= (da \wedge dx^I) \wedge bdx^J + (-1)^{\text{degree}(\omega)} adx^I \wedge (db \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^{\text{degree}(\omega)} \omega \wedge d\eta. \end{aligned}$$

For the third item, let $\omega = g : V \to \mathbb{R}$ be a 0-form. Therefore, if $x \in U$ e $w \in \mathbb{R}^m$, then

$$f^*(dg)(x) \cdot w = dg(f(x)) \cdot f'(x) \cdot w = d(g \circ f)(x) \cdot w = d(f^*g)(x) \cdot w.$$

Now,let $\omega = adx^I = adx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be a differential k-form on V, where $a : V \to \mathbb{R}$ is a 0-form. Consider $adg_1 \wedge \cdots \wedge dg_k$, where a is at least C^1 and the g_i's are at least C^2, all defined in V. Using the second property and the hypothesis of induction on k, we obtain

$$d(adg_1 \wedge \cdots \wedge dg_k) = da \wedge dg_1 \wedge \cdots \wedge dg_k.$$

Also,

$$f^*\omega = f^*a \cdot f^*dx^{i_1} \wedge \cdots \wedge f^*dx^{i_k} = f^*a \cdot d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_k} \circ f).$$

Therefore,

$$\begin{aligned} f^*(d\omega) &= f^*(da \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= f^*da \wedge f^*dx^{i_1} \wedge \cdots \wedge f^*dx^{i_k} \\ &= d(f^*a) \wedge d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_k} \circ f) \\ &= d(f^*a \cdot d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_k} \circ f)) \\ &= d(f^*\omega). \end{aligned}$$

□

Theorem 4.3.3. For every $\omega \in \Omega^k(U)$, we have $d(d\omega) = 0$.

Proof. Consider that $\omega = adx^I$, which gives us $d\omega = \sum_{j=1}^m \dfrac{\partial a}{\partial x_j} dx^j \wedge dx^I$. Therefore,

$$d(d\omega) = \Big[\sum_{k,j=1}^m \frac{\partial^2 a}{\partial x_k \partial x_j} dx^k \wedge dx^j\Big] \wedge dx^I = \Big[\sum_{j<k} \Big(\frac{\partial^2 a}{\partial x_j \partial x_k} - \frac{\partial^2 a}{\partial x_k \partial x_j}\Big) dx^j \wedge dx^k\Big] \wedge dx^I = 0.$$

The third piece of the expression above grants us the equality is equal 0 due the fact that a is a function that is at least C^2, then we can apply Schwartz theorem. The general case comes from this computation and the linearity of the d over the elements dx^I. □

Let $x_1, \ldots, x_n$ be the standard coordinate system in $\mathbb{R}^n$ and $y_1, \ldots, y_n$ be another coordinate system for $\mathbb{R}^n$, then there is a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ changing variables, *i.e.*, $y_i = x_i \circ f = f^*(x_i)$. Let g be a smooth function on $\mathbb{R}^n$, *i.e.*, $g \in \Omega^0(\mathbb{R}^n)$, then, by the chain rule, we get

$$dg = \sum_i \frac{\partial g}{\partial y_i} dy_i = \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j} dx_j = \sum_j \frac{\partial g}{\partial x_j} dx_j,$$

which proves that exterior derivative is independent of the coordinate system on $\mathbb{R}^n$.

We now define the exterior derivative on a manifold M. It is important to notice that all properties for the exterior derivative on the Euclidean case remain true for manifolds.

Definition 4.3.4. Let M be a n-manifold. We define the linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ as

$$x \in M \mapsto (d\omega)(x) = (d_\varphi \omega)(x)$$

where $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a local chart around x and $d_\varphi \omega$ stands for $d((\varphi^{-1})^*\omega) = \sum_I d\alpha_I \wedge dx^I$.

Let (V, ψ) be another local chart, where $U \cap V \neq \emptyset$, then $d_\psi \omega$ and $d_\varphi \omega$ coincide by means of a diffeomorphism on $U \cap V$. Indeed, $\xi = \psi \circ \varphi^{-1}$ is a diffeomorphism, then $(\varphi^{-1})^* = (\psi^{-1} \circ \xi)^* = \xi^* \circ (\psi^{-1})^*$, where ξ^* is a linear isomorphism between $\Omega^k(\varphi(U \cap V))$ and $\Omega^k(\psi(U \cap V))$. Therefore, the last definition is well-defined.

4.4. Stokes Theorem

Theorem 4.4.1 (Stokes theorem)**.** Let M be an oriented m-manifold with boundary. Then for every $\omega \in \Omega_c^{m-1}(M)$ we get

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Suppose, without loss of generality, that $\omega \in \Omega_c^{m-1}(M)$ has support $\operatorname{supp}(\omega) \subset U$, where $\varphi : U \to \varphi(U) \subset H$ for a semi-space H of $\mathbb{R}^m$, and $\omega = a(x) du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^m$. Otherwise, ω could be rewritten as a finite sum $\sum_{i=1}^k \lambda_i \omega$ for a partition of unity $\{\lambda_i\}$ subordinate to the open cover $\{U_i\}_{i \in \mathbb{N}}$ of M and $\omega = \sum_I \alpha_I du^I$. In any case, this assumptions would fall in the previous case.

By applying the exterior derivative d on ω we obtain

$$d\omega = \frac{\partial a}{\partial u_j} du^j \wedge du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^m.$$

We assume that are our semi-space H is as follows $H = \{(x_1, \ldots, x_m) \in \mathbb{R}^m | x_m \geq 0\}$, since given any semi-space K, then there is a diffeomorphism between K and H. Moreover, proposition 3.4.3 shows that this diffeomorphism takes one boundary to another boundary. Therefore, by example 4.1.6, we get

$$\int_M d\omega = \int_{\mathbb{R}^m} (\varphi^{-1})^* d\omega = \int \left(\int_H \frac{\partial a}{\partial x_j} dx^j \right) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m.$$

Now, the function $a : U \to \mathbb{R}$ has a compact support. By integrating $\int \frac{\partial a}{\partial x_j} dx^j$ for $j \neq m$ we have that the integral has value zero, because x_j varies freely on the real line $\mathbb{R}$ and supp $\left(\frac{\partial a}{\partial x_j}\right)|_{x_j} \subset [a,b]$. If $j = m$, then

$$\int_{x,x_m=0}^{\infty} \frac{\partial a}{\partial x_m} dx^m = -a(x_1, \ldots, x_{m-1}, 0).$$

Conversely, computing $\int_{\partial M} \omega$, for $j \neq m$, we have that

$$\int_{\partial M} \omega = \int_{\partial H} (\varphi|_{\partial U}^{-1})^* \omega = \int_{\partial H} a(x) dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m = 0,$$

since x_j can vary freely on the real line $\mathbb{R}$ and $a : U \to \mathbb{R}$ has a compact support. Also, if $j = m$, then

$$\int_{\partial M} \omega = \int \left(\int_{\partial H} a(x) dx^m \right) dx^1 \wedge \cdots \wedge dx^{m-1},$$

where $\int_{\partial H} a(x) dx^m = -a(x_1, \ldots, x_{m-1}, 0)$, by making the induced orientation on ∂H. $\square$

CHAPTER 5

DE RHAM COHOMOLOGY

In this chapter we recall a specific cohomology theory, called de Rham cohomology, where our vector spaces are $\Omega^k(M)$ and our differential operator is the exterior derivative d. Throughout the first section of this chapter we explore the properties of de Rham cohomology and compute some examples. After that, we recall another type of cohomology, called de Rham cohomology for compact supports or, simply, compactly supported cohomology, which is basically de Rham cohomology, but restricted to differential forms, which supports are compact sets. Then we prove the Poincaré lemmas, which are the explicit computation of the cohomology groups of the Euclidean space $\mathbb{R}^n$. We end this chapter by working with Mayer-Vietoris sequence, both usual and compactly supported, which is a technique used to compute cohomology groups of manifolds that are not diffeomorphic to $\mathbb{R}^n$ themselves. In this work cohomology is a essential piece to understand characteristic classes, since a characteristic class is a cohomology class for us. We note that the content developed in this chaper was based on (MELO, 2019) and (BOTT; TU, 1982).

5.1. De Rham Cohomology

Suppose that M is m-manifold. In the past chapter we have proven that the exterior derivative d is a linear operator between vector spaces. Therefore, we can talk about its kernel

$$Z^k(M) = \{\omega \in \Omega^k(M); d\omega = 0\}$$

which is the space of the differential forms called *closed forms*, *i.e.*, the differential k-forms with exterior derivative equal to 0. We can also speak of the image of d, denoted here by $B^k(M)$. Differential forms on $B^k(M)$ are called *exact forms*. We have

$$B^k(M) = \{\omega \in \Omega^k(M) : \text{where } d\eta = \omega \text{ for some } \eta \in \Omega^{k-1}(M)\}.$$

When $k = 0$, we define $B^k(M) = \{0\}$. More specifically, we have subgroups of $\Omega^k(M)$ where the group operation is the sum of differential forms

$$Z^k(M) = \operatorname{Ker} d : \Omega^k(M) \to \Omega^{k+1}(M),$$

$$B^k(M) = \operatorname{Im} d : \Omega^{k-1}(M) \to \Omega^k(M)$$

By establishing the sequence

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0$$

we have a complex of cochains, since $\dim(M) = m$ and $d^2 = 0$. The property $d^2 = 0$ also implies that $B^k(M) \subset Z^k(M)$, which then make it possible to define the quotient group $H^k_{dR}(M) := Z^k(M)/B^k(M)$. Those are called the **de Rham cohomology groups**. Given $\omega \in Z^k(M)$ we usually denote its class in $H^k_{dR}(M)$ by $[\omega]$. Usually, we can write $H^k(M)$ instead of $H^k_{dR}(M)$, since there is no space for misunderstanding.

Examples 5.1.1. We give some examples of de Rham cohomology groups of the Euclidean space $\mathbb{R}^n$.

- Suppose $\mathbb{R}^0 = \{0\}$, where 0 is the zero vector. We made this assumption in order to make $\mathbb{R}^0$ a vector space. Therefore,

$$H^q(\mathbb{R}^0) = \begin{cases} \mathbb{R}, & q = 0; \\ 0, & q > 0. \end{cases}$$

For $q = 0$ the set of all closed forms is $\mathbb{R}$. Indeed, let $\omega \in \Omega^0(\mathbb{R}^0)$ be a random differential 0-form. For the alternate functions of degree 0 we assume as a convention that $\Lambda^0(E) = \mathbb{R}$ for any vector space E. From this we have $\Lambda^0(\mathbb{R}^0) = \mathbb{R}$ and then $\omega : \{0\} \to \mathbb{R}$. This means that ω is constant and, therefore, is a closed form, since the exterior derivative of constant functions is zero. We can make the association $\omega \in \Omega^0(\mathbb{R}^0) \mapsto \omega(0) \in \mathbb{R}$ to identify $\Omega^0(\mathbb{R}^0)$ with $\mathbb{R}$. The only option for a exact form is the element 0. From this,

$$H^0(\mathbb{R}^0) = \frac{\text{closed 0-forms}}{\text{exact 0-forms}} = \frac{\Omega^0(\mathbb{R}^0)}{\{0\}} = \frac{\mathbb{R}}{\{0\}} = \mathbb{R}.$$

Suppose $q > 0$. Then, by corollary 2.1.7, $\Lambda^q(\mathbb{R}^0) = \{0\}$, which implies $H^q(\mathbb{R}^0) = 0$.

- Suppose $n = 1$. The space $\Omega^0(\mathbb{R}^1)$ is the space of smooth functions, since $\omega \in \Omega^0(\mathbb{R}^1)$ is a function $\omega : \mathbb{R}^1 \to \Lambda^0(\mathbb{R}^1) = \mathbb{R}^1$. Therefore, since $\mathbb{R}$ is a connected topological space, then Ker $(d) \subset \Omega^0(\mathbb{R}^1)$ is composed only by constant functions, given they are the only real functions to real values where the exterior derivative returns zero. We get

$$\{\text{closed 0-forms}\} = \text{Ker }(d) = \text{constant functions} = \mathbb{R}.$$

Since $\{0\} = \{\text{exact 0-forms}\}$, then

$$H^0_{dR}(\mathbb{R}^1) = \{\text{closed 0-forms}\}/\{\text{exact 0-forms}\} \simeq \mathbb{R}.$$

Again, here we identify the differential 0-forms $\omega \in$ Ker (d) to the constant value $x_\omega \in \mathbb{R}$ they assume.

In order to compute $H^1_{dR}(\mathbb{R}^1)$ we first notice that $\Omega^1(\mathbb{R}^1) =$ Ker (d), since $d : \Omega^1(\mathbb{R}^1) \to \Omega^2(\mathbb{R}^1) = \{0\}$. Moreover, we prove that every 1-form is exact. Indeed, let $\omega = g(x)dx$ be a differential 1-form. Define $f = \int_0^x g(u)du$. By the fundamental theorem of calculus we know that f is a differentiable real valued function. Therefore, $f \in \Omega^0(\mathbb{R}^1)$. Also from the fundamental theorem of calculus we can take the derivative of f, which is of the form $df = g(x)dx$.

This proves that $\{\text{closed 1-forms}\} = \{\text{exact 1-forms}\}$, which implies $H^1_{dR}(\mathbb{R}^1) = 0$. This means that the only equivalence class in $H^1_{dR}(\mathbb{R}^1)$ is the equivalence class of the element 0, which can be identified with the element 0 of differential 1-forms: $0dx$.

- A more general result, with proof found in proposition 5.3.2, called the Poincaré lemma, is given by

$$H^q(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & q = 0; \\ 0, & q > 0. \end{cases}$$

Example 5.1.2. Let U be an open set on $\mathbb{R}^1$ given by a disjoint union of m open intervals I_j, then

$$H^0(U) = \mathbb{R}^m,$$
$$H^1(U) = 0.$$

Indeed, we can establish an isomorphism

$$\varphi : \Omega^k(U) \to \bigoplus_{j=1}^{m} \Omega^k(I_j)$$

by putting

$$\varphi(\omega) = (\omega_{|I_1}, \ldots, \omega_{|I_m}). \tag{5.1}$$

Each $\omega|_{I_j}$ is a differential form because is a restriction of a differential form on an open subset. The isomorphism φ induces an isomorphism over the cohomology groups $H^k_{dR}(U) \simeq \bigoplus_{j=1}^m H^k_{dR}(I_j)$ by working with the equivalence classes of closed forms in equation (5.1). Indeed, if $\omega \in \Omega^k(U)$ is a closed form, then $d\omega = 0$ and for every $I_j \subset U$ we get that $d(\omega|_{I_j}) = 0$. Also, if $\omega = d\eta$, then $\omega|_{I_j} = (d\eta)|_{I_j} = d(\eta|_{I_j})$.

Let $I \subset \mathbb{R}$ be an open interval. We study the cohomology groups $H^q(I)$. First of all, we set that $\{\text{exact 0-forms}\} = \{0\}$. Also, let $\omega \in \Omega^0(I)$, then $\omega(x) \in \Lambda^0(T_xI) = \mathbb{R}$, for every $x \in I$. This means that we can look at $\omega \in \Omega^0(I)$ as a function $\omega : I \to \mathbb{R}$. Suppose that ω is closed, *i.e.*, $d\omega = 0$. Then ω is a constant function defined on the interval I, since I is a connected set. From this we get

$$\{\text{closed 0-forms}\} = \{\text{constant functions on } I \text{ to } \mathbb{R}\} = \mathbb{R}.$$

Therefore, $H^0(I) = \{\text{closed 0-forms}\}/\{\text{exact 0-forms}\} = \mathbb{R}/\{0\} = \mathbb{R}$.

For $\omega \in \Omega^1(I)$, then $d\omega = 0$, since $d : \Omega^1(I) \to \Omega^2(I) = \{0\}$ and I is a manifold of dimension 1. Indeed, for $\eta \in \Omega^2(I)$, then $\eta : x \mapsto \eta(x) \in \Lambda^2(T_xI) = \Lambda^2(\mathbb{R}) = \{0\}$ by corollary 2.1.7. Moreover, every $\omega \in \Omega^1(I)$ is exact by the same reasoning found on the second case of last example. This gives us that $H^1(I) = \{0\}$, which finishes this example.

Remark 5.1.3. A generalization for this last example is: Let M be a manifold composed by m connected components, then we have an isomorphism $H^0_{dR}(M) \simeq \mathbb{R}^m$.

Let $f : U \subset M \to V \subset N$ be a C^∞-function between open sets of manifolds, then

$$d(f^*\omega) = f^*d(\omega). \tag{5.2}$$

From (5.2) we have that if $f : M \to N$ then $f^*(Z^k(N)) \subset Z^k(M)$. Also, the inclusion $f^*(B^k(N)) \subset B^k(M)$ is valid. Indeed, given $\omega \in B^k(N)$ then $\omega = d\eta$, $\eta \in \Omega^{k-1}(N)$. Since $f^*(\omega) = f^*(d\eta) = d(f^*\eta)$ and $f^*\eta \in \Omega^{k-1}(M)$, the property is proved.

Because of those inclusions we can define a function between the cohomology classes

$$f^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

which works simply as the pullback, but now with equivalence classes of closed forms, considering the calculations we made above. As pullbacks, we have the following properties

- $(f \circ g)^* = g^* \circ f^*$;
- $(id_M)^* = id_{H^k_{dR}(M)}$.

Definition 5.1.4. Let M be a m-manifold. We define the algebra $\Omega^*(M) = \oplus_{k=0}^m \Omega^k(M)$, with sum operation being the natural sum of differential forms and the product operation being the wedge product.

Remark 5.1.5. The algebra $\Omega^*(M)$ is naturally graded. By definition $\Omega^*(M)$ is a direct sum. To conclude that this is a graded algebra we remember that the wedge product, which is our ring multiplicative function, is given by $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$, for any given manifold. Therefore, $\Omega^*(M)$ is a graded algebra.

Definition 5.1.6. We call the combination of the complex $\Omega^*(M)$ with the operator d as the *de Rham complex on M*.

According to (BOTT; TU, 1982): "The de Rham complex may be viewed as a God-given set of differential equations, whose solutions are the closed forms." An example to this statement is the 1-form $fdx + gdy \in \Omega^1(\mathbb{R}^2)$, which solution resides in solving the differential equation $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$.

Definition 5.1.7. Let M be a manifold and $U \subset M$ an open subset of M. We define the graded algebra $\Omega^*(U)$ as follows

$$\Omega^*(U) = \{\omega|_U : \omega \in \Omega^*(M)\}$$

and we denote its de Rham cohomology by $H^*_{dR}(U)$.

We can also talk about the cohomology ring $H^*(M)$ which is given by $H^*(M) = \oplus_{k=0}^m H^k(M)$ and product operation $[\omega] \wedge [\eta] = [\omega \wedge \eta]$.

Proposition 5.1.8. The wedge product $\wedge$ on the cohomology level is well-defined.

Proof. In order to prove that $\wedge$ is well-defined on the level of cohomology we need to check two properties. First of all, let ω and η be two closed differential forms. Then $\alpha \wedge \beta$ is closed. Indeed,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\text{degree}(\omega)} \omega \wedge d\eta = 0.$$

Also, suppose that ω or η is an exact differential form, besides being closed forms, then $\omega \wedge \eta$ must be an exact differential form. Suppose that $\omega = d\gamma$, then

$$\omega \wedge \eta = d\gamma \wedge \eta = d(\gamma \wedge \eta) - (-1)^{\text{degree}(\gamma)} \gamma \wedge d\eta = d(\gamma \wedge \eta).$$

This proves that the wedge product on the cohomology level $[\omega] \wedge [\eta] = [\omega \wedge \eta]$ is well-defined. $\square$

5.2. Compactly supported cohomology

Another important type of cohomology is the cohomology group of differential forms with compact support. We start by defining the graded algebra $\Omega^*_c(M)$ and then we give some examples.

Definition 5.2.1. Let M be a manifold. We define the cohomology of compact support on M by making

$$\Omega_c^*(M) = \{\omega \in \Omega^*(M) : \text{such that } \omega \text{ has compact support}\}.$$

The cohomology ring for differential forms with compact support is denoted by $H_c^*(M)$. Note that each cohomology group $H_c^k(M)$ is still given by the same definition of quotient groups, but now we are restricting ourselves by working only with differential forms with compact support. Moreover, $\mathrm{supp}(d\omega) \subset \mathrm{supp}(\omega)$ for any $\omega \in \Omega_c^*(M)$, showing that the exterior derivative is well-defined for compact support forms.

Examples 5.2.2. In this example we compute some compactly supported cohomology for the Euclidean space $\mathbb{R}^n$.

- Suppose $n = 0$ and $\mathbb{R}^0 = \{a\}$. Then

$$H_c^q(\{a\}) = \begin{cases} \mathbb{R}, & \text{for } q = 0 \\ 0, & \text{elsewhere} \end{cases}$$

  Remind that $T_a\{a\}$ can be identified as the trivial vector space $\{0\}$. Therefore, for $q > 0$ we have $\Omega^q(\{a\}) = \{0\}$, since for $\omega \in \Omega^q(\{a\})$ we have $\omega : \{a\} \to \Lambda^q(T_a\{a\}) = \{0\}$. Restricting to $\Omega_c^q(\{a\})$ gives us $H_c^q(\{a\}) = \{0\}$. For $q = 0$ we know that $\Lambda^0(T_a\{a\}) = \mathbb{R}$, therefore $\Omega^0(\{a\}) = \mathbb{R}$. Since, any differential 0-form on $\Omega^0(\{a\})$ has a finite domain, then it has a compact support. Therefore $H_c^0(\{a\}) = \mathbb{R}$.

- Suppose $n = 1$. Then

$$H_c^q(\mathbb{R}^1) = \begin{cases} 0, & \text{for } q = 0 \\ \mathbb{R}, & \text{for } q = 1 \end{cases}$$

  Suppose $q = 0$. From example 5.1.1 the closed 0-forms are constant functions on $\mathbb{R}^1$. Since there are no constant functions with compact support, besides the 0 function, then $H_c^0(\mathbb{R}^1) = 0$.

  Suppose $q = 1$ and define the integration map

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \longrightarrow \mathbb{R}^1$$

which is a surjective and well-defined map. It is well-defined because is defined on forms with compact support and it is surjective because we can apply it on the 1-forms $\omega_a = \chi_{[0,a]}dx$, where $\chi_{[0,a]}$ is the characteristic function of the interval $[0,a]$. This map vanishes on the exact 1-forms df where f is a function (0-form) with compact support. Suppose $\mathrm{supp}(f)$ is a subset on the interior of some interval $[a,b]$, then

$$\int_{\mathbb{R}^1} \frac{df}{dx}dx = \int_a^b \frac{df}{dx}dx = f(b) - f(a) = 0.$$

This proves that $df \in \mathrm{Ker}\ (\int_{\mathbb{R}^1})$. On the other hand, take $g(x)dx \in \mathrm{Ker}\ (\int_{\mathbb{R}^1}) \subset \Omega^1_c(\mathbb{R}^1)$. The function

$$f(x) = \int_{-\infty}^{x} g(u)du$$

has a compact support, since $g(x)dx \in \mathrm{Ker}\ (\int_{\mathbb{R}^1})$. From $df = g(x)dx$ we get that $\mathrm{Ker}\ \int_{\mathbb{R}^1} = \{\text{exact 1-forms with compact support}\}$. Then

$$H^1_c(\mathbb{R}^1) = \frac{\text{closed 1-forms with compact support}}{\text{exact 1-forms with compact support}} = \frac{\Omega^1_c(\mathbb{R}^1)}{\mathrm{Ker}\ \int_{\mathbb{R}^1}} = \mathbb{R}^1.$$

The last equality is given by the first isomorphism theorem of group theory.

- More generally, we have the Poincaré lemma for cohomology with compact support, which states that

$$H^q_c(\mathbb{R}^n) = \begin{cases} 0, & \text{for } q < n \\ \mathbb{R}, & \text{for } q = n \end{cases}$$

Notice in the last item that we obtain a major difference between the compactly supported cohomology and the usual de Rham cohomology, where $H^0(\mathbb{R}^n)$ is always $\mathbb{R}$ for any n and $H^q(\mathbb{R}^n) = 0$ is always 0 for any n and for $q > 0$.

Let $f : M \to N$ be a proper C^∞-function. We are able to also induce a function $f^* : H^k_c(N) \to H^k_c(M)$, by restricting our application to the equivalence classes of closed forms with compact support. Notice that if $\omega \in \Omega^k_c(N)$ is a closed form, then $d(\omega) = 0$. Therefore, $d(f^*\omega) = f^*(d\omega) = 0$. The same property shows us that exact forms are taken to exact forms. Moreover, since f is a proper function, then the compact support is preserved.

5.3. Poincaré Lemmas

Consider the functions $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the projection on the first factor, and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$, the zero section $s(x) = (x,0)$. The pullback functions are of the form $\pi^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n \times \mathbb{R})$ and $s^* : \Omega^*(\mathbb{R}^n \times \mathbb{R}) \to \Omega^*(\mathbb{R}^n)$. Both of these maps are assumed to be C^∞. We use these pullbacks to prove that $H^*(\mathbb{R}^{n+1}) \simeq H^*(\mathbb{R}^n)$.

Proposition 5.3.1. The maps $H^*(\mathbb{R}^n \times \mathbb{R}^1) \xrightarrow{s^*} H^*(\mathbb{R}^n)$ and $H^*(\mathbb{R}^n) \xrightarrow{\pi^*} H^*(\mathbb{R}^n \times \mathbb{R}^1)$ are isomorphisms.

Proof. We know that $\pi \circ s = 1$. By applying the pullback we get $(\pi \circ s)^* = s^* \circ \pi^* = 1$. On the other hand, $s \circ \pi(x,t) = (x,0) \neq (x,t) = Id(x,t)$, which means that $(s \circ \pi)^* = \pi^* \circ s^* \neq 1$ on the level of forms. For a given function $f(x,t) \in \Omega^0(\mathbb{R}^n \times \mathbb{R}^1)$ we have that $(\pi^* \circ s^*)(f)(x,t) = (s \circ \pi)^*(f)(x,t) = f(s \circ \pi(x,t)) = f(x,0)$, make f constant on the second factor.

We want to show that $\pi^* \circ s^*$ is the identity on the cohomology level. In order to do this we must show that there is a map K defined on $\Omega^*(\mathbb{R}^n \times \mathbb{R}^1)$, called homotopy operator, that satisfies $1 - \pi^* \circ s^* = \pm(dK \pm Kd)$. Notice the expression $dK \pm Kd$ is able to take closed forms to exact forms, since for a closed form ω we get $(dK \pm Kd)(\omega) = dK(\omega)$. This allows us to induce the zero element in cohomology and show that $1 = \pi^* \circ s^*$ on the cohomology level. Moreover, in order to obtain $dK \pm Kd$ well-defined, K must decrease the degree of forms by 1.

Given $\eta \in \Omega^*(\mathbb{R}^n \times \mathbb{R}^1)$ we can write it as a linear combination of the subset $\{(\pi^*\phi)f(x,t), (\pi^*\phi)f(x,t)dt\}$, for a given $\phi \in \Omega^*(\mathbb{R}^n)$ and $f \in \Omega^0(\mathbb{R}^n \times \mathbb{R})$. We are able to define the map K as a linear map, by deciding how it operates on which type of q-form. Therefore, we define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by establishing

$$
\begin{aligned}
&(\pi^*\phi)f(x,t) \mapsto 0 \\
&(\pi^*\phi)f(x,t)dt \mapsto (\pi^*\phi)\int_0^t f(x,u)du
\end{aligned}
$$

For the first case, suppose $\omega = (\pi^*\phi) \cdot f(x,t) \in \Omega^q(\mathbb{R}^n \times \mathbb{R})$. We want to prove that $1-\pi^*\circ s^* = \pm(dK \pm Kd)$. Applying the left side of this equation on ω we get

$$\begin{aligned}(1-\pi^*\circ s^*)\omega &= (\pi^*\phi)\cdot f(x,t) - (\pi^*\circ s^*)((\pi^*\phi)\cdot f(x,t))\\ &= (\pi^*\phi)\cdot f(x,t) - (\pi^*\circ(s^*\circ\pi^*)\phi)\cdot(\pi^*\circ s^*)f(x,t)\\ &= (\pi^*\phi)\cdot f(x,t) - (\pi^*\phi)\cdot f(x,0)\end{aligned}$$

since $\pi^*\circ s^*$ takes functions $f(x,t)$ to $f(x,0)$. On the other hand, now applying the right side of the equation on ω, which is of the first type, we get

$$\begin{aligned}(dK-Kd)\omega = -Kd\omega &= -K\Big((d\pi^*\phi)\cdot f(x,t) + (-1)^q(\pi^*\phi)\cdot\Big(\Big(\sum\frac{\partial f}{\partial x_i}dx_i\Big) + \frac{\partial f}{\partial t}dt\Big)\Big)\\ &= (-1)^{q-1}(\pi^*\phi)\cdot\int_0^t\frac{\partial f}{\partial u}du\\ &= (-1)^{q-1}(\pi^*\phi)(f(x,t)-f(x,0)).\end{aligned}$$

Therefore, for differential forms of the first type we have that $1-\pi^*\circ s^* = \pm(dK \pm Kd)$, showing that K is the operator we desire on this first case.

Now suppose, ω is of the second type, *i.e.*, $\omega = (\pi^*\phi)f(x,t)dt \in \Omega^*(\mathbb{R}^n\times\mathbb{R})$. First of all, $(1-\pi^*\circ s^*)\omega = \omega$, *i.e.*, $(\pi^*\circ s^*)\omega = 0$. Indeed, $f(x,t)dt \in \Omega^1(\mathbb{R}^n\times\mathbb{R})$, then

$$\begin{aligned}(\pi^*\circ s^*)\omega &= (\pi^*\circ s^*)((\pi^*\phi)f(x,t)dt)\\ &= (\pi^*\phi)\cdot(f(x,0))\cdot(\pi^*\circ s^*)dt\\ &= (\pi^*\phi)\cdot(f(x,0))\cdot(\pi^*\circ d(s^*t))\\ &= (\pi^*\phi)\cdot(f(x,0))\cdot(\pi^*\circ d(t\circ s))\\ &= (\pi^*\phi)\cdot(f(x,0))\cdot(\pi^*\circ 0)\\ &= 0\end{aligned}$$

where, from the third for the fourth line we are facing t as the projection on the second factor of $\mathbb{R}^n\times\mathbb{R}$. On the other hand,

$$d\omega = (\pi^*d\phi)f(x,t)dt + (-1)^{q-1}(\pi^*\phi)\cdot\Big(\sum_i\frac{\partial f}{\partial x_i}dx_i\wedge dt\Big),$$

then computing $Kd\omega$ and $dK\omega$ we get

$$\begin{aligned}
dK\omega &= d\Big((\pi^*\phi)\int_0^t f(x,u)du\Big) \\
&= (\pi^* d\phi)\int_o^t f(x,u)du + (-1)^{q-1}(\pi^*\phi)\cdot\Big[\int_0^t \frac{\partial f}{\partial x}dx + f dt\Big],
\end{aligned}$$

$$\begin{aligned}
Kd\omega &= K\Big((\pi^* d\phi) f(x,t)dt + (-1)^{q-1}(\pi^*\phi)\cdot\Big(\sum_i \frac{\partial f}{\partial x_i}dx_i \wedge dt\Big)\Big) \\
&= (\pi^* d\phi)\int_0^t f(x,u)du + (-1)^{q-1}(\pi^*\phi)\cdot\int_0^t \frac{\partial f}{\partial x}dx,
\end{aligned}$$

where $\dfrac{\partial f}{\partial x}dx = \sum_i \dfrac{\partial f}{\partial x_i}dx_i$. Then $(dK - Kd)\omega = (-1)^{q-1}\omega = (-1)^{q-1}(1 - \pi^* \circ s^*)\omega$, which now proves the proposition by looking this expression on the cohomology level. □

Corollary 5.3.2. [Poincaré Lemma] The cohomology group of the Euclidean space $\mathbb{R}^n$ is the same as the cohomology group of the point, *i.e.*,

$$H^q(\mathbb{R}^n) = H^q(\{0\}) = \begin{cases} \mathbb{R}, & q = 0 \\ 0, & \text{elsewhere} \end{cases}$$

As a matter of fact, this last proposition can be generalized for a generic manifold M. By applying the same methods used in proposition 5.3.1 we can prove the following

Proposition 5.3.3. Let M be a manifold. Then $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$ is an isomorphism via π^* and s^*.

Corollary 5.3.4 (Homotopy Axiom for de Rham Cohomology)**.** Any two homotopic maps induce the same map in cohomology.

Proof. Let $F : M \times \mathbb{R}^1 \to N$ be a homotopy between two maps $f, g : M \to N$. Take $s_0, s_1 : M \to M \times \mathbb{R}^1$ the 0-section and the 1-section respectively, *i.e.*, $s_0(x) = (x, 0)$ and $s_1(x) = (x, 1)$. We can easily see that

$$g = F \circ s_0$$

$$f = F \circ s_1$$

Applying the pullback on both sides of the equations we have

$$g^* = (F \circ s_0)^* = s_0^* \circ F^*$$

$$f^* = (F \circ s_1)^* = s_1^* \circ F^*$$

As we saw in the main proposition, s_0^* is the inverse for π^* in the cohomology. Changing s_0^* for s_1^* does not cause any changes in the proof. Therefore, both functions work as inverse for π^* on cohomology, which means they are equal. Then $f^* = g^*$. $\square$

Definition 5.3.5. Let M and N two manifolds. They are said to have the same **homotopy type** if there are two C^∞ maps, $f : M \to N$ and $g : N \to M$ such that $g \circ f$ and $f \circ g$ are C^∞ homotopic to the identity on M and N respectively. In particular, when a manifold has the same homotopy type of a point then it is said to be **contractible**.

Definition 5.3.6. Let M be a manifold. Let $A \subset M$ and $i : A \to M$ the inclusion. Let $r : M \to A$ be a retraction of M onto A, *i.e.*, r is at least a continuous map and its restriction to A is the identity map on A: $r|_A = Id_A$. From this we have that $r \circ i : A \to A$ is the identity. Moreover, if $i \circ r : M \to M$ is homotopic to the identity on M, then r is said to be a deformation retraction of M onto A.

Corollary 5.3.7. Two manifolds with the same homotopy type have the same de Rham cohomology.

Proof. Let $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to id_M and $f \circ g$ is homotopic to id_N. Therefore,

$$id_M^* = (g \circ f)^* = f^* \circ g^*,$$

$$id_N^* = (f \circ g)^* = g^* \circ f^*,$$

implying that f^* and g^* are isomorphisms on the level of cohomology. $\square$

Corollary 5.3.8. If A is a deformation retract of M, then A and M have the same de Rham cohomology.

Theorem 5.3.9 (Poincaré Lemma for Compactly Supported Cohomology)**.** The compactly supported cohomology group of the Euclidean space is given by

$$H_c^q(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & \text{for } q = n \\ 0, & \text{for } q < n \end{cases}$$

Proof. We start the proof by looking at the case where $q \in \{1,\dots,n-1\}$. Let $\omega \in \Omega_c^q(\mathbb{R}^n)$ and $\varepsilon > 0$ such that $\mathrm{supp}(\omega) \subset \{x \in \mathbb{R}^n; ||x|| < \varepsilon/2\} = V$. Let $U = \overline{V}^c$. Consider the function $\pi : U \to S = \{x \in \mathbb{R}^n; ||x|| = 2\varepsilon\}$ given by $\pi(x) = 2\varepsilon \dfrac{x}{||x||}$. If $i : S \to U$ is the inclusion, then $\pi \circ i$ is homotopic equivalent to the identity $Id|_U$ on U by means of the homotopy $F(x,t) = t\pi(x) + (1-t)x$. Also, by the same reason, $i \circ \pi$ is homotopic equivalent to the identity $Id|_S$. Therefore, by corollary 5.3.7, $H^q(U) = H^q(S)$, which clearly have the same cohomology of S^{n-1}, *i.e.*, $H^q(U) = H^q(S^{n-1})$. Poincaré Lemma tell us that there is $(q-1)$-form η_1 such that $\omega = d\eta_1$. Since $\omega = 0$ on U, then $d\eta_1|_U = 0$. By the fact that $H^{q-1}(U) = H^{q-1}(S^{n-1}) = 0$, there is $\eta_2 \in \Omega^{q-2}(U)$ such that $d\eta_2 = \eta_1$ on U. There is $f : \mathbb{R}^n \to [0,1]$, a C^∞-function, that values 1 in $\{x \in \mathbb{R}^n; ||x|| \geq 2\}$ and 0 in $\{x \in \mathbb{R}^n; ||x|| \leq 3/2\}$. Define $\eta_3 = d(f \cdot \eta_2)$ on U and $\eta_3 = 0$ on U^c. Notice this proves that η_3 is a closed C^∞-form on $\mathbb{R}^n$. We get that $\eta = \eta_1 - \eta_3$ is a form with compact support ($\mathrm{supp}(\eta) \subset \overline{B_r}$, where $r = \max\{\varepsilon/2, 2\}$) and $\omega = d\eta$. Therefore, every form $\Omega_c^q(\mathbb{R}^n)$ is exact, which proves that $H_c^k(\mathbb{R}^n) = 0$. From examples 5.2.2 we can replicate the argument to prove that $H_c^0(\mathbb{R}^n) = 0$ for any natural n.

Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \omega = 0$. Let the number ε and the set U be as before. Once again, $\omega = d\eta_1$ on $\mathbb{R}^n$ and, therefore, $d\eta_1|_U = 0$. By Stokes theorem,

$$0 = \int_{\mathbb{R}^n} \omega = \int_{B_{2\varepsilon}} \omega = \int_{B_{2\varepsilon}} d\eta_1 = \int_S \eta_1.$$

Then $\eta_1|_S$ is an exact form, since $\int_S \eta_1 = 0$. We saw that U and S have the same cohomology groups, then η_1 is an exact form on U as well, which means that there is η_2 such that $d\eta_2 = \eta_1$ on U. As before, let $\eta_3 = d(f \cdot \eta_2)$ on U and $\eta_3 = 0$ on U^c. Again, $\eta = \eta_1 - \eta_3$ is a form with compact support such that $\omega = d\eta$, which means that $[\omega] = 0 \in H_c^n(\mathbb{R}^n)$. Since integration $\int_{\mathbb{R}^n} : \Omega_c^n(\mathbb{R}^n) \to \mathbb{R}$ is a surjective linear application, then

$$H_c^n(\mathbb{R}^n) = \frac{\text{closed } n\text{-forms with compact support}}{\text{exact } n\text{-forms with compact support}} = \frac{\Omega_c^n(\mathbb{R}^n)}{\mathrm{Ker}\ \int_{\mathbb{R}^n}} = \mathbb{R}.$$

□

Remark 5.3.10. The fact that $H^q(S^{n-1}) = 0$ has a proof in the next section. Also, since $H^{n-1}(S) = H^{n-1}(S^{n-1})$, given by the fact that S and S^{n-1} are diffeomorphic, and $H(S^{n-1}) = \mathbb{R}$, which also has proof in the next section, proves that Ker $\int_{S^{n-1}}$ is composed by the exact forms on S^{n-1}.

5.4. Mayer-Vietoris Sequence

Using category theory language we say that Ω^* is a contravariant functor from the category of Euclidean spaces, which consists of $\{\mathbb{R}^n\}_{n\in\mathbb{Z}}$ as objects and smooth maps $f:\mathbb{R}^m\to\mathbb{R}^n$ as arrows, and the category of commutative differential graded algebras, with them being the objects and their homomorphisms being the arrows.

The Mayer-Vietoris sequence is an algebraic technique used to compute the cohomology groups of manifolds that can be written as finite unions of its open sets. Initially, we suppose M as a union of two open sets $M = U\cup V$. When an open cover for M is given by n sets we can use a simple induction argument that is presented in the next section in order to prove important theorems in this work. Let $U\sqcup V$ be the disjoint union of U and V and ∂_0, ∂_1 the inclusions of $U\cap V$ into U and V, respectively, then we can establish the inclusions

$$M \leftarrow U\bigsqcup V \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} U\cap V,$$

and by applying our contravariant functor Ω^* gives us a sequence of restrictions of differential forms

$$\Omega^*(M)\to\Omega^*(U)\oplus\Omega^*(V) \overset{-\partial_0^*}{\underset{\partial_1^*}{\rightrightarrows}} \Omega^*(U\cap V).$$

The restriction of forms to submanifolds are meant to be its image under the pullback map induced by the inclusion.

Here we need more explanation of what is happening under those inclusions. First of all, let us write $U=U_1$ and $V=U_2$ to make our notations easier. We recall that the disjoint unions of sets is given by $\{A_i : i\in I\}$ a family of sets indexed by I and is defined as

$$\bigsqcup_{i\in I} A_i = \{(x,i) : x\in A_i\}.$$

Therefore, for only two sets we obtain $U_1\sqcup U_2 = \cup_{i\in\{1,2\}}\{(u,i) : u\in U_i\}$, which now gives meaning to both of our inclusions ∂_1,∂_2, *i.e.*,

$$U_1\cap U_2 \xrightarrow{\partial_i} X_i = \{(x,i) : x\in U_1\cap U_2\subset U_i\}\subset U_1\bigsqcup U_2.$$

On the other hand, when applying the functor Ω^* on the inclusion $M \leftarrow U_1 \bigsqcup U_2$ we obtain

$$\Omega^*(M) \to \Omega^*(U_1 \bigsqcup U_2).$$

However, $\Omega^*(U_1 \bigsqcup U_2)$ is isomorphic to $\Omega^*(U_1) \bigoplus \Omega^*(U_2)$ by establishing the map $(\omega, \tau) \mapsto \eta$, where $\eta(x,1) = \omega(x)$ and $\eta(x,2) = \tau(x)$.

Then, the Mayer-Vietoris sequence is given by

$$0 \longrightarrow \Omega^*(M) \xrightarrow{(*)} \Omega^*(U) \bigoplus \Omega^*(V) \xrightarrow{(**)} \Omega^*(U \cap V) \longrightarrow 0 \tag{5.3}$$

by considering the restrictions on each U and V on $(*)$ and by putting $(\omega, \tau) \mapsto \tau - \omega$ on (**), where each τ and ω is considered to be restricted to $U \cap V$.

Proposition 5.4.1. The Mayer-Vietoris sequence is exact.

Proof. The function on $(*)$ in equation (5.3) is clearly injective, since a differential form defined on M is 0 when each of its restriction to U and V is 0. Moreover, the difference δ on $(**)$ is surjective. Indeed, let $\{\rho_U, \rho_V\}$ be a partition of unity subordinated to the cover $\{U,V\}$ and let $\omega \in \Omega^*(U \cap V)$. By putting $(-\rho_V \omega, \rho_U \omega)$ we have an element of $\Omega^*(U) \oplus \Omega^*(V)$, since $-\rho_V \omega$ becomes well-defined on $U \setminus V$ and $\rho_U \omega$ becomes well-defined on $V \setminus U$. Then

$$\delta(-\rho_V \omega, \rho_U \omega) = \rho_U \omega - (-\rho_V \omega) = (\rho_U + \rho_V)\omega = \omega.$$

Finally, the image of the function on $(*)$ is the same as the kernel of the difference δ. Let $\omega \in \Omega^*(M)$, then by making the restrictions on U and V we have $(\omega|_U, \omega|_V) \in \Omega^*(U) \oplus \Omega^*(V)$. Then, $\delta(\omega|_U, \omega|_V) = \omega|_V - \omega|_U = 0$, since we are working with differential forms restricted to $U \cap V$ and $\omega|_U$, $\omega|_V$ are the same on $U \cap V$. $\square$

We know that a short exact sequence gives rise to a long exact sequence on the cohomology level. Therefore, the Mayer-Vietoris short exact sequence gives rise to the Mayer-Vietoris long exact sequence on the cohomology level which is given by

$$\begin{aligned} \ldots H^q(M) \longrightarrow H^q(U) \oplus H^q(V) \longrightarrow H^q(U \cap V) \xrightarrow{d^*} \\ \xrightarrow{d^*} H^{q+1}(M) \longrightarrow H^{q+1}(U) \oplus H^{q+1}(V) \longrightarrow H^{q+1}(U \cap V) \ldots \end{aligned}$$

The operator d^* is called the coboundary operator and it is given by the following

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)], & \text{on } U \\ [d(\rho_U \omega)], & \text{on } V \end{cases}$$

for $\omega \in \Omega^q(U \cap V)$ being a closed form and $\{\rho_U, \rho_V\}$ a partition of unity for $\{U, V\}$. The operator d^* is well-defined. Indeed, since the following Mayer-Vietoris short sequence is exact

$$0 \to \Omega^q(M) \to \Omega^q(U) \oplus \Omega^q(V) \to \Omega^q(U \cap V) \to 0,$$

then, given a closed form $\omega \in \Omega^q(U \cap V)$, there is $\eta \in \Omega^q(U) \oplus \Omega^q(V)$ that maps to ω through δ, which is $\eta = (-\rho_V \omega, \rho_U \omega)$. This definition allow us to define $\rho_V \omega$ in all U since for $x \in U \cap V^c$, $\rho_V(x)\omega(x) = 0 \cdot \omega(x) = 0$. The same is valid for $\rho_U \omega$. The following diagram is commutative

$$\begin{tikzcd} \Omega^q(U) \oplus \Omega^q(V) \arrow[r, "\delta"] \arrow[d, "d"'] & \Omega^q(U \cap V) \arrow[d, "d"'] \\ \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) \arrow[r, "\delta"] & \Omega^{q+1}(U \cap V) \end{tikzcd}$$

The application d on the left side maps $\eta \mapsto (-d(\rho_V \omega), d(\rho_U \omega))$ which is mapped through δ to $d(\rho_U \omega) - (-d(\rho_V \omega)) = d((\rho_U + \rho_V)\omega) = d(\omega) = 0$. Therefore, the applications $d(\rho_U \omega)$ and $-d(\rho_V \omega)$ coincide over $U \cap V$. This means that in the exact sequence

$$0 \to \Omega^{q+1}(M) \xrightarrow{f} \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) \xrightarrow{\delta} \Omega^{q+1}(U \cap V) \to 0$$

we have $d(\eta) \in ker(\delta) = Im(f)$. Therefore, exists only one $\psi \in \Omega^{q+1}(M)$ such that $f(\psi) = d(\eta)$.

Example 5.4.2 (Cohomology group of S^1)**.** Consider an open covering $\{U, V\}$ of S^1 as in figure 1, which is the geometric representation of the open cover given in example 3.1.9. Also, notice that $U \cap V$, on figure 2, is made up of two connected components. Recall that their image by any of the local charts π_N, π_S is equal to $\mathbb{R} \setminus \{0\}$.

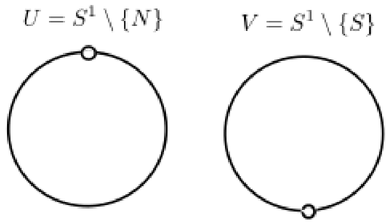

Figure 1 – Geometric representation of an open cover for S^1.

Source: Elaborated by the author.

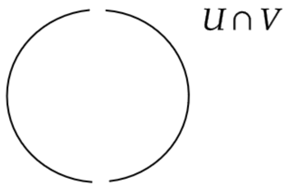

Figure 2 – Geometric representation of the intersection of U and V.

Source: Elaborated by the author.

The short Mayer-Vietoris sequence induces the long Mayer-Vietoris sequence

$$0 \to H^0(S^1) \xrightarrow{r} H^0(U) \oplus H^0(V) \xrightarrow{\delta} H^0(U \cap V) \xrightarrow{d^*}$$
$$\xrightarrow{d^*} H^1(S^1) \xrightarrow{w} H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to 0$$

By the fact that the open sets U and V are connected sets diffeomorphic to $\mathbb{R}^1$, then $H^0(S^1) = H^0(U) = H^0(V) = \mathbb{R}$ and $H^q(U) = H^q(V) = H^q(\mathbb{R}^1) = 0$ for $q > 0$. Also, by the figure we see that $U \cap V$ has two connected pieces, each one of them being connected sets diffeomorphic to $\mathbb{R}^1$, then $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$ and $H^1(U \cap V) = 0$. Therefore, Mayer-Vietoris long exact sequence looks like

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow$$
$$\longrightarrow H^1\left(S^1\right) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

By applying theorem 2.3.2 we are able to compute $H^1(S^1) = \mathbb{R}$.

Now, let us look more closely what is the meaning of the identifications we have made above on the language of forms. For both $H^0(U)$ and $H^0(V)$ we are considering the equivalence classes of constant functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ and establishing the relation

$$([f],[g]) \mapsto (x,y)$$

in order to say that $H^0(U) \oplus H^0(V) = \mathbb{R}^2$, where $f(u) = x$, for all $u \in U$, and $g(v) = y$, for all $v \in V$.

Also, let $[f] \in H^0(S^1)$, then $r([f]) = ([f|_U],[f|_V])$. Notice that both $f|_U$ and $f|_V$ are constant functions. Since $U \cap V \neq \emptyset$, then $(f|_U)|_V = (f|_V)|_U = f|_{U\cap V}$, which means that $f|_U$ and $f|_V$ have the same constant value as image on $\mathbb{R}$. Therefore, the function f is a constant function, which allow us to identify $H^0(S^1) = \mathbb{R}$. Note that the same argument can be repeated to $H^0(S^n)$ for every n.

Now let us understand what is happening to identifications made for δ on the language of forms. By definition of the Mayer-Vietoris sequence we get that $\delta(\omega,\tau) = \tau|_{U\cap V} - \omega|_{U\cap V}$. We can write $U \cap V = A \bigsqcup B$, where each A and B are connected subsets of S^1 and $A \cap B = \emptyset$. This allows us to identify $H^0(U\cap V) = H^0(A \bigsqcup B) = H^0(A) \oplus H^0(B)$. This shows us that

$$\delta(\omega,\tau) = \tau|_{U\cap V} - \omega|_{U\cap V} = \tau|_{A\bigsqcup B} - \omega|_{A\bigsqcup B} \mapsto (\tau|_A - \omega|_A, \tau|_B - \omega|_B).$$

From a geometrical point of view, when we have $f \in H^0(U\cap V)$, then f can be defined to reach two different values on $\mathbb{R}$, where f is constant on each of the connected components A and B of $U \cap V$. This also gives us that δ has a one-dimensional image on $H^0(U\cap V)$, because for $(\omega,\tau) \in H^0(U) \oplus H^0(V)$, then both ω and τ have constant values, which means that

$$(\tau|_A - \omega|_A, \tau|_B - \omega|_B)$$

have the same constant values on each coordinate.

At last, we study the identifications for the homomorphism d^*. First of all, notice that

$$\operatorname{Im} d^* = \operatorname{Ker} w,$$

$$\operatorname{Im} \delta = \operatorname{Ker} d^*,$$

$$H^0(U \cap V) \simeq \text{Ker } d^* \oplus \text{Im}\, d^*,$$

and, since $H^0(U \cap V)$ is 2-dimensional, then $\text{Im}\, d^*$ is 1-dimensional. Consequently, Ker w is 1-dimensional and $H^1(S^1)$ is 1-dimensional, since $\text{Im}\, w$ is 0-dimensional. In order to find a generator for $H^1(S^1)$ we need to find a differential form $[\alpha] \in H^0(U \cap V)$ such that $[\alpha] \notin \text{Im}\, \delta$ for a closed form, because, otherwise, we would get that $[\alpha] \in \text{Ker } d^*$ and, consequently, $d^*[\alpha] = 0$, which would not provide us a generator for the desired cohomology group.

Therefore, α must admit different constant values for each connected component A and B of $U \cap V$. We can choose $[\alpha] = ([\omega], [\tau]) \in H^0(A) \oplus H^0(B)$, where ω is the constant function equal to 1 and τ is the constant function equal to 0. With this definition α is a closed form on $U \cap V$. Then

$$d^*[\alpha] = \begin{cases} [-d(\rho_V \alpha)], & \text{on } U \\ [d(\rho_U \alpha)], & \text{on } V \end{cases}$$

which gives us

$$d^*[\alpha] = \begin{cases} ([-d(\rho_V \omega)], [-d(\rho_V \tau)]), & \text{on } U \\ ([d(\rho_U \omega)], [d(\rho_U \tau)]), & \text{on } V \end{cases}$$

where $[-d(\rho_V \tau)] = 0$ on $B \subset U$ and $[d(\rho_U \tau)] = 0$ on $B \subset V$. On the other hand $[-d(\rho_V \omega)] = [-d\rho_V]$ on $A \subset U$ and $[d(\rho_U \omega)] = [d\rho_U]$ on $A \subset V$. Since $[d(\rho_U \alpha)]$ and $[-d(\rho_V \alpha)]$ coincide on the intersection $U \cap V$ for each of the subsets A and B, then $d^*[\alpha]$ is a global closed form on S^1, which generates $H^1(S^1)$.

Example 5.4.3 (Cohomology groups of S^2)**.** Take U an open set of S^2 which is the north hemisphere passing a bit after the equator and V as another open set doing the same thing but starting by the south hemisphere. Therefore, $U \cap V$ is homeomorphic to $S^1 \times \mathbb{R}$. The short Mayer-Vietoris sequence induces the long Mayer- Vietoris sequence

$$\begin{aligned} 0 \to H^0(S^2) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \\ \to H^1(S^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \\ \to H^2(S^2) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to 0 \end{aligned}$$

By the fact that the open sets U and V are connected sets diffeomorphic to $\mathbb{R}^2$, then $H^0(S^2) = H^0(U) = H^0(V) = H^0(U \cap V) = \mathbb{R}$ and $H^q(U) = H^q(V) = H^q(\mathbb{R}^2) = 0$

for $q > 0$. Also, we know that $U \cap V$ is homeomorphic to $S^1 \times \mathbb{R}$ and the latter is homotopy equivalent to S^1, then $H^q(U \cap V) = H^q(S^1)$. Therefore, Mayer-Vietoris long exact sequence looks like

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \\ \longrightarrow H^1\left(S^2\right) \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \\ \longrightarrow H^2\left(S^2\right) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \end{array}$$

Applying theorem 2.3.2 to parts of our Mayer-Vietoris long exact sequence gives us that $H^1(S^2) = 0$ and $H^2(S^2) = \mathbb{R}$.

Example 5.4.4. In order to generalize the computation of the cohomology groups of the n-sphere S^n we suppose the covering of S^n with open sets U and V as before, then $U \cap V$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ and, consequently, of the same homotopy type as before. Therefore, by following the same steps in the previous two examples, we get

$$H^q(S^n) = \begin{cases} \mathbb{R}, & q = 0, n \\ 0, & \text{otherwise} \end{cases}$$

5.5. Mayer-Vietoris sequence for compact support

First of all, Ω_c^* is not a functor from the category of manifolds and smooth maps, because it does not take necessarily forms with compact support to forms with compact support. Indeed, consider the pullback of forms under the projection $\pi : M \times \mathbb{R} \to M$. Let $\omega \in \Omega_c^*(M)$ and let $x = (p, a) \in M \times \mathbb{R}$. We get

$$(\pi^*\omega)(x)(v_1, \ldots, v_m) = \omega(p)(D\pi(x) \cdot v_1, \ldots, D\pi(x) \cdot v_m),$$

which shows that $a \in \mathbb{R}$ varies all over the real line. Therefore, $\pi^*\omega$ does not have compact support.

In order to work with Ω_c^* as a functor, we need to consider only a subset of the set of smooth maps. If we consider

- proper maps, then we have that Ω_c^* is a contravariant functor;

- inclusions of open sets, then we have that Ω_c^* is a covariant functor.

We work with the second case in this work. Let $i : U \to M$ be an inclusion of an open set U of M. Then by applying the functor Ω_c^* we obtain $i^* : \Omega_c^*(U) \to \Omega_c^*(M)$ as a map which extends any form on U to a form on M by making this form evaluates as 0 on $M \setminus U$.

The covariant functor Ω_c^* gives rise to a Mayer-Vietoris sequence as well. Suppose $M = U \cup V$ being covered by two open sets. Since the functor Ω_c^* is a covariant functor for inclusions of open sets then the inclusions

$$M \leftarrow U \bigsqcup V \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} U \cap V$$

define a sequence between the graded algebras

$$\Omega_c^*(M) \xleftarrow{\text{sum}} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{f} \Omega_c^*(U \cap V)$$

where f is the signed inclusion given by $\omega \mapsto (-j^*\omega, i^*\omega)$, with j^* expanding the definition of ω for the rest of U by 0 and i^* doing the same expansion on V. Also, when applying the function *sum*, we are extending the definition of forms to 0 on the rest of M.

Proposition 5.5.1. The Mayer-Vietoris short sequence for compactly supported cohomology

$$0 \leftarrow \Omega_c^*(M) \xleftarrow{\text{sum}} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{f} \Omega_c^*(U \cap V) \leftarrow 0$$

is exact.

Proof. The function f is injective since $(-j^*\omega, i^*\omega) = (0,0)$ implies that $i^*\omega = 0$ and $j^*\omega = 0$, then, consequently, $\omega = 0$. Moreover, the sum from $\Omega_c^*(U) \oplus \Omega_c^*(V) \to \Omega_c^*(M)$ is surjective, by taking $\omega \in \Omega_c^*(M)$ and defining $(\rho_U \omega, \rho_V \omega) \in \Omega_c^*(U) \oplus \Omega_c^*(V)$. Finally, a differential form on the image of f is of the form $(-j^*\omega, i^*\omega)$ for a function $\omega \in \Omega_c^*(U \cap V)$, which is equal to zero when applying the sum function, since $-j^*\omega$ and $i^*\omega$ coincide on the intersection $U \cap V$, except for their sign, and they have value 0 at every other point of M. Therefore the image of f is the same as the kernel of the sum. $\square$

As before the Mayer-Vietoris short exact sequence gives rise to a Mayer-Vietoris long exact sequence

$$\cdots \leftarrow H_c^{q+1}(M) \leftarrow H_c^{q+1}(U) \oplus H_c^{q+1}(V) \leftarrow H_c^{q+1}(U \cap V) \overset{d_*}{\leftarrow}$$
$$\overset{d_*}{\leftarrow} H_c^q(M) \leftarrow H_c^q(U) \oplus H_c^q(V) \leftarrow H_c^q(U \cap V) \leftarrow \dots$$

through means of a connecting homomorphism d_*. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinated to $\{U,V\}$. Then for $[\omega] \in H_c^k(M)$ we define $d_* : H_c^k(M) \to H_c^{k+1}(U \cap V)$ as

$$d_*([\omega]) = [d(\rho_U \omega|_{U\cap V})].$$

Notice that d_* is well-defined this way, because $\rho_U \omega \in \Omega_c^k(U)$ and $\rho_V \omega \in \Omega_c^k(V)$. Therefore, for $[\omega] \in H_c^K(M)$, *i.e.*, ω is a closed differential form, then

$$0 = d(\omega) = d(\rho_U \omega + \rho_V \omega) \Rightarrow d(\rho_U \omega) = d(-\rho_V \omega),$$

which means that $\text{supp}(d(\rho_U \omega)) \subset U \cap V$. Moreover, notice that $d(\rho_U \omega|_{U\cap V}) = (d\rho_U)\omega|_{U\cap V}$ is a closed form, since

$$d^2(\rho_U \omega|_{U\cap V}) = \underbrace{d^2(\rho_U)}_{=0} \omega|_{U\cap V} - d(\rho_U) \underbrace{d\omega|_{U\cap V}}_{=0} = 0.$$

Example 5.5.2 (Compactly supported cohomology for the circle S^1)**.** A simple way to look at these cohomology groups is to recall that S^1 is already a compact manifold, therefore compactly supported cohomology should be the same as the usual de Rham cohomology groups for S^1. On the other hand, using the Mayer-Vietoris sequence argument to compute these groups, consider the open covering for S^1 as in example 5.4.2, with geometric representations as in figures 1 and 2.

The Mayer-Vietoris long exact sequence induced by the Mayer-Vietoris short exact sequence for compactly supported cohomology is

$$\leftarrow H_c^1(S^1) \leftarrow H_c^1(U) \oplus H_c^1(V) \leftarrow H_c^1(U \cap V) \leftarrow$$
$$\leftarrow H_c^0(S^1) \leftarrow H_c^0(U) \oplus H_c^0(V) \leftarrow H_c^0(U \cap V) \leftarrow$$

By using the Poincaré lemma for compactly supported cohomology we know that $H_c^0(U) = H_c^0(V) = 0$ and $H_c^1(U) = H_c^1(V) = \mathbb{R}$ and, since $U \cap V$ is composed by two connected components each one diffeomorphic to $\mathbb{R}$, then $H_c^0(U \cap V) = 0$ and $H_c^1(U \cap V) = \mathbb{R} \oplus \mathbb{R}$. Since the sequence above is a long exact sequence, then $H_c^0(S^1) = \operatorname{Ker} f$, for $f : H_c^1(U \cap V) \to H_c^1(U) \oplus H_c^1(V)$, which is 1-dimensional. Therefore, by applying theorem 2.3.2, we are able to compute $H_c^1(S^1) = \mathbb{R}$.

CHAPTER

6

POINCARÉ DUALITY, KÜNNETH FORMULA AND LERAY-HIRSCH THEOREM

In this chapter we discuss some very important results for our theory, such as the Poincaré Duality, the Künneth formula and the Leray-Hirsch theorem. We begin by presenting the Mayer-Vietoris argument as a form to compute de Rham cohomology groups for generic manifolds that have a finite good cover. In other words, the Mayer-Vietoris is a induction process into the cardinality of those covers. For this we need to use the five lemma, recall some concepts of Riemannian geometry for manifolds, as well as other set theory and homological algebra concepts. We begin by proving a proposition that illustrates the Mayer-Vietoris argument, by showing the finitude of dimensions of the cohomology groups of a manifold under certain conditions. After that, we understand what is a pairing and establish important diagrams in order to understand and to prove Poincaré's Duality. We note that we prove Poincaré's Duality not only for oriented manifolds with finite good cover, but for any oriented manifolds. We end the chapter by showing the Künneth formula, using very similar algebraic arguments, and making an initial sketch for the proof of Leray-Hirsch theorem. The content in this chapter is mainly based on (BOTT; TU, 1982), with additional resources taken on (MELO, 2019) and (HAFKENSCHEID, 2020). Also, there are references to (SPIVAK, 1979) and (GARZA, 2019).

6.1. The Mayer-Vietoris argument

Definition 6.1.1. Let M be a n-manifold. An open cover $\{V_\alpha\}_{\alpha\in I}$ of M, is called a good cover for M, if any finite intersections $V_{\alpha_1} \cap \cdots \cap V_{\alpha_p} \neq \emptyset$ are diffeomorphic to the Euclidean space $\mathbb{R}^n$. Such a manifold M is said of finite type.

Definition 6.1.2. Let M be a n-manifold. We define a Riemannian structure on M as a smoothly varying metric $\langle -,-\rangle$ on the tangent space T_pM of M at each point $p \in M$. More precisely, a Riemannian metric on M is a function g that takes $p \in M \mapsto g_p$, where each g_p is a positive-definite inner product on T_pM, *i.e.*, $g_p : T_pM \times T_pM \to \mathbb{R}$. Another way to think about this definition is given X,Y smooth vector fields on M, then $\langle X,Y\rangle$ is a smooth function on M, *i.e.*, we are applying $\langle X,Y\rangle$ on each point $p \in M$ smoothly.

Proposition 6.1.3. Any n-manifold M has a Riemannian structure.

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be a locally finite open cover of M, where each U_α is a coordinate open set of M, *i.e.*, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is a diffeomorphism for each $\alpha \in I$. There is $\{\rho_\alpha\}_{\alpha\in I}$ a partition of unity subordinate to the atlas $\{U_\alpha\}_{\alpha\in I}$. On each U_α we can define a metric $g_\alpha = \varphi_\alpha^* g^{Euc}$ where g^{Euc} is the Euclidean metric on $\mathbb{R}^n$ and $\varphi_\alpha^* g^{Euc}$ is the pullback of g^{Euc} along the local chart φ_α. We define

$$g := \sum_{\alpha\in I} \rho_\alpha \cdot g_\alpha,$$

where the hypothesis of locally finiteness from the partition of unity allows g to be well-defined. □

Theorem 6.1.4. Any n-manifold M has a good cover. Moreover, if M is compact, then it has a finite good cover.

Proof. By the previous proposition M has a Riemannian structure. By (SPIVAK, 1979), every manifold M with a Riemannian structure can be covered with geodesically convexed neighborhoods, which are neighborhoods where every two points are connected by means of a arc path. Any finite intersection of geodesically convex neighborhoods is geodesically convex. Every geodesically convex neighborhood in M is diffeomorphic to $\mathbb{R}^n$. Therefore, the open cover consisting of geodesically convex neighborhoods is a good cover. $\square$

Definition 6.1.5. Let M be a manifold and $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$, $\mathscr{V} = \{V_\beta\}_{\beta\in J}$ be two covers of M. We say that $\mathscr{V}$ is a refinement of $\mathscr{U}$, denoted by $\mathscr{U} < \mathscr{V}$, if every $V_\beta \subset U_\alpha$ for some α. More precisely, a refinement is a map $\phi : J \to I$ that takes $V_\beta \subset U_{\phi(\beta)}$.

Definition 6.1.6. A directed set is a set I combined with a relation $<$ satisfying

- *Reflexivity:* $a < a$ for every $a \in I$;
- *Transitivity:* if $a < b$ and $b < c$, then $a < c$;
- *Upper bound:* For any $a, b \in I$, there is an element $c \in I$, satisfying $a < c$ and $b < c$.

Proposition 6.1.7. The set of open covers on a manifold is a directed set.

Proof. The reflexivity and transitivity properties are immediate. For the upper bound property we take the family $\{U_\alpha \cap V_\beta\}_{\alpha\in I,\beta\in J}$. Such a family is an open cover, since every U_α and every V_β are open sets of M, and for any given point $p \in M$ there is $\alpha' \in I$ and $\beta' \in J$ such that $p \in U_{\alpha'}$ and $p \in V_{\beta'}$. Therefore, $p \in U_{\alpha'} \cap V_{\beta'}$. $\square$

Proposition 6.1.8. Let M be a manifold. If M has a finite good cover, then its cohomology group is finite dimensional. Therefore, every compact manifold M has a finite dimensional cohomology group.

Proof. Let U and V be two open sets of M. The Mayer-Vietoris long exact sequence for $U \cup V$ is

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \to \ldots.$$

From linear algebra and the fact that this is a long exact sequence we get

$$H^q(U \cup V) \simeq \mathrm{Ker}\, r \oplus \mathrm{Im}\, r \simeq \mathrm{Im}\, d^* \oplus \mathrm{Im}\, r,$$

where the first isomorphism is a direct implication of the linearity of d^* and by the exactness of the Mayer-Vietoris long sequence we get $\mathrm{Ker}\, r \simeq \mathrm{Im}\, d^*$.

Looking at the long exact sequence and the relations we establish between kernels and images we can state the following: If the cohomology groups $H^q(U)$, $H^q(V)$ and $H^{q-1}(U\cap V)$ are finite dimensional, then $H^q(U\cup V)$ is finite dimensional.

Now the proof proceeds by induction on the cardinality of the good cover. Suppose that our good cover is only given by M, *i.e.*, the case $n=1$. Therefore, M is a manifold diffeomorphic to $\mathbb{R}^n$ and, by corollary 5.3.7 $H^*(M)=H^*(\mathbb{R}^n)$, which is finite dimensional by the Poincaré Lemma. At the beginning of the proof we have made the arguments for the case $n=2$, which aids us to prove the general case. Suppose the hypothesis is valid for a manifold with good cover given by p elements, *i.e.*, a manifold M with a good cover $\{U_0,U_1,\ldots,U_{p-1}\}$ composed by p elements has a finite dimensional cohomology. Let N be a manifold with good cover $\{U_0,U_1,\ldots,U_p\}$ composed by $p+1$ elements. The manifold $(U_0\cup\cdots\cup U_{p-1})\cap U_p$ has a good cover $\{U_{0p},U_{1p},\ldots,U_{(p-1)p}\}=\{U_0\cap U_p,U_1\cap U_p,\ldots,U_{p-1}\cap U_p\}$ with p elements. By the hypothesis of induction the q-th cohomology group of $U_0\cup\cdots\cup U_{p-1}$, U_p and $(U_0\cup\cdots\cup U_{p-1})\cap U_p$ are finite dimensional. Therefore, by the Mayer-Vietoris long exact sequence

$$\cdots\to H^{q-1}((U_0\cup\cdots\cup U_{p-1})\cap U_p)\xrightarrow{d^*}H^q(N)\xrightarrow{r}H^q(U_0\cup\cdots\cup U_{p-1})\oplus H^q(U_p)\to\ldots,$$

we prove that $H^q(U_0\cup\cdots\cup U_p)=H^q(N)$ is finite dimensional. □

Proposition 6.1.9. Let M be a manifold with a finite good cover. Then $H_c^*(M)$ is finite dimensional.

Proof. For the compactly supported case, recall that the Mayer-Vietoris long exact sequence is as follows

$$\ldots H_c^q(U)\oplus H_c^q(V)\xrightarrow{g}H_c^q(U\cup V)\xrightarrow{d_*}H_c^{q+1}(U\cap V)\to\ldots$$

Then from linear algebra and from the fact that the sequence is exact we get that

$$H_c^q(U\cup V)\simeq \operatorname{Ker} d_*\oplus \operatorname{Im} d_*\simeq \operatorname{Im} g\oplus \operatorname{Im} d_*,$$

where g is given by the sum of the extended differential forms on each $H^q(U)$ and $H^q(V)$.

From this point the proof proceeds as in the usual cohomology case. □

Remark 6.1.10. The induction argument made in proposition 6.1.8 is called *Mayer-Vietoris argument*. The same argument is used in the proof of the Poincaré Duality and the Künneth formula.

6.2. Poincaré Duality on an Orientable Manifold

Let V and W be finite-dimensional vector spaces and the function $\langle -,- \rangle : V \otimes W \to \mathbb{R}$ be a pairing (real linear map) from their tensor product to the real line. We say $\langle -,- \rangle$ to be non-degenerate if

- $\langle v,w \rangle = 0$ for all $w \in W$ implies that $v = 0$, and
- $\langle v,w \rangle = 0$ for all $v \in V$ implies $w = 0$.

Equivalently, we are able to define $\langle -,- \rangle$ as non-degenerate if

- the map $v \mapsto \langle v,- \rangle$ defines an injection $V \hookrightarrow W^*$ and
- the map $w \mapsto \langle -,w \rangle$ defines an injection $W \hookrightarrow V^*$.

More precisely, we are able to state the following lemma.

Lemma 6.2.1. Let V and W be finite dimensional vector spaces. The pairing $\langle -,- \rangle : V \otimes W \to \mathbb{R}$ is non-degenerate if, and only if, the map $v \mapsto \langle v,- \rangle$ is an isomorphism $V \xrightarrow{\sim} W^*$.

Proof. Suppose the pairing $\langle -,- \rangle$ is non-degenerate. This implies that we have injections $V \hookrightarrow W^*$ and $W \hookrightarrow V^*$. Taking into account that those vector spaces are finite dimensional, then $\dim V = \dim V^*$ and $\dim W = \dim W^*$. Therefore, $\dim V = \dim W^*$, which means that $v \mapsto \langle v,- \rangle$ is an isomorphism in $V \xrightarrow{\sim} W^*$.

On the other hand, suppose that the map $v \mapsto \langle v,- \rangle$ is an isomorphism from V to W^*. Therefore, the kernel of this map is made only of $v = 0$, *i.e.*, $\langle v,w \rangle = 0$ for all $w \in W$ implies that $v = 0$. The other condition for non-degeneracy is derived from the fact that $W \to V^*$ is also an isomorphism. $\square$

Define a map $\mathscr{P}\mathscr{D} : \Omega^q(M) \to (\Omega_c^{n-q}(M))^*$ given by

$$\mathscr{P}\mathscr{D}(\omega)(\eta) = \int_M \omega \wedge \eta$$

for every form $\omega \in \Omega^q(M)$ and for every form $\eta \in \Omega_c^{n-q}(M)$. For every ω we have that $\mathscr{P}\mathscr{D}(\omega)$ is a linear map due to bilinearity of the wedge product. We have the following property

$$\begin{aligned}
\mathscr{P}\mathscr{D}(d\omega)(\eta) &= \int_M d\omega \wedge \eta \\
&= \int_M d(\omega \wedge \eta) - \omega \wedge (-1)^q d\eta \\
&= \underbrace{\int_M d(\omega \wedge \eta)}_{=0} - \int_M \omega \wedge (-1)^q d\eta \\
&= \mathscr{P}\mathscr{D}(\omega)((-1)^{q+1} d\eta) \\
&= (-1)^{q+1} \mathscr{P}\mathscr{D}(\omega)(\eta)
\end{aligned}$$

where $d(\omega \wedge \eta) = 0$ is a $(n+1)$-form on a n-manifold. Also, this shows us that $\mathscr{P}\mathscr{D}$ is sign-commutativity, *i.e.*, $\mathscr{P}\mathscr{D}$ commutes except for the sign. Moreover, $\mathscr{P}\mathscr{D}$ induces a pairing on the cohomology level, which is the following integration map

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R},$$

by putting $[\omega] \otimes [\eta] \mapsto \int_M \omega \wedge \eta$, which is well-defined. We omit, for the rest of this work, the notation of the equivalence classes that concerns the cohomology spaces.

By proving that this integration map is non-degenerate, whenever M is orientable and has a finite good cover, then, equivalently by lemma 6.2.1, we have that $H^q(M) \simeq (H_c^{n-q}(M))^*$, which is the finite version of the Poincaré Duality. In order to prove it we need the following lemma.

Lemma 6.2.2. The Mayer-Vietoris sequences given by the usual cohomology groups and the compactly support cohomology can be paired to form a sign-commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cup V) \arrow[r, "\text{restriction}"] & H^q(U) \oplus H^q(V) \arrow[r, "\text{difference}"] & H^q(U \cap V) \arrow[r, "d^*"] & H^{q+1}(U \cup V) \arrow[r] & \cdots \\
& \otimes & \otimes & \otimes & \otimes & \\
\cdots & H_c^{n-q}(U \cup V) \arrow[l] \arrow[d, "\int_{U \cup V}"] & H_c^{n-q}(U) \oplus H_c^{n-q}(V) \arrow[l, "\text{sum}"'] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U \cap V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q-1}(U \cup V) \arrow[l, "d_*"'] \arrow[d, "\int_{U \cup V}"] & \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

For $\omega \in H^q(U \cap V)$ and $\eta \in H_c^{n-q-1}(U \cup V)$ the sign commutativity is

$$\int_{U \cap V} \omega \wedge d_* \eta = \pm \int_{U \cup V} (d^* \omega) \wedge \eta$$

which is induced from the map $\mathscr{PD}$. Recall that $d_* : H_c^{n-q-1}(U\cup V) \to H_c^{n-q}(U\cap V)$ and $d^* : H^q(U\cap V) \to H^{q+1}(U\cup V)$ are the connecting homomorphisms for each type of cohomology. By lemma 6.2.1 is equivalent to say that the following diagram is commutative. Notice that we compensate the sign-commutativity by the multiplication $(-1)^{q+1}d^*$, which gives us an adapted signed commutativity diagram. This will allows us to further apply this proposition using the Five lemma in the next theorem. Also, the column on the right is the dual sequence derived from the Mayer-Vietoris sequence for compactly supported cohomology as in proposition 5.5.1.

$$\begin{tikzcd}
H^q(U\cup V) \arrow[r, "\mathscr{PD}|_{U\cup V}"] \arrow[d, "{([\omega_U],\ [\omega_V])}"'] & H_c^{n-q}(U\cup V)^* \arrow[d, "sum^*"'] \\
H^q(U)\oplus H^q(V) \arrow[r, "\mathscr{PD}|_U \oplus \mathscr{PD}|_V"] \arrow[d, "\eta-\omega"'] & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^* \arrow[d, "f^*"'] \\
H^q(U\cap V) \arrow[r, "\mathscr{PD}|_{U\cap V}"] \arrow[d, "(-1)^{q+1}d^*"'] & H_c^{n-q}(U\cap V)^* \arrow[d, "(d_*)^*"'] \\
H^{q+1}(U\cup V) \arrow[r, "\mathscr{PD}|_{U\cup V}"] & H_c^{n-q-1}(U\cup V)^*
\end{tikzcd}$$

Proof. In order to prove the commutativity of the third square recall that $d^* : H^q(U\cap V) \to H^{q+1}(U\cup V)$, where for all $[\omega]\in H^q(U\cap V)$, then $d^*[\omega]\in H^{q+1}(U\cup V)$ is defined by

$$d^*[\omega] = \begin{cases} [-d(\rho_V\omega)], & \text{on } U \\ [d(\rho_U\omega)], & \text{on } V \end{cases}$$

where we have verified in both cases they coincide over the intersection $U\cap V$. Secondly, $d_* : H_c^{n-q-1}(U\cup V) \to H_c^{n-q}(U\cap V)$, where for all $[\eta]\in H_c^{n-q-1}(U\cup V)$, we have $d_*[\eta] = [d(\rho_U\eta|_{U\cap V})]$. Moreover, given $d_*[\eta]\in H_c^{n-q}(U\cap V)$ then, for a pair of partition of unity $\{\rho_U,\rho_V\}$, we have

$$\begin{aligned}(-(\text{extension by 0 of } d_*\eta \text{ to } U), (\text{extension by 0 of } d_*\eta \text{ to } V) &= (d(\rho_U\eta), d(\rho_V\eta)) \\ &= ((d\rho_U)\eta, (d\rho_V)\eta)\end{aligned}$$

Notice the last equality comes from the fact that the functions ρ_U, ρ_V are differential 0-forms, therefore, for ρ_V, we get

$$d(\rho_V \eta) = (d\rho_V) \wedge \eta + (-1)^{\text{degree}(\rho_V)} \rho_V \wedge d(\eta) \overset{d(\eta)=0}{=} (d\rho_V)\eta$$

due to the fact that η is a closed form. We have not used any property related to the fact that η has a compact support, so this equality is equally valid for differential forms without support compact, such as ω. We get

$$\int_{U\cap V} \omega \wedge d_* \eta = \int_{U\cap V} \omega \wedge (d\rho_V)\eta = (-1)^{\text{degree}(\omega)} \int_{U\cap V} (d\rho_V)\omega \wedge \eta, \tag{6.1}$$

where the second equality comes from the fact we are commuting the forms ω and $(d\rho_V)$. On the other hand, remember that $\text{supp}(d^*\omega) \subset U \cap V$, because, by definition, we are extending the definition of ω to $U \cup V \setminus U \cap V$ by putting $d^*\omega$ as zero in this previous set. Since in $U \cap V$ there is no differentiation between $d(\rho_U \omega)$ and $d(\rho_V \omega)$, then we can choose $\rho_V \omega$ as our representative in the following

$$\int_{U\cup V} d^*\omega \wedge \eta = -\int_{U\cap V} (d\rho_V)\omega \wedge \eta. \tag{6.2}$$

Therefore, using (6.1) and (6.2), we get

$$\begin{aligned}
\int_{U\cap V} \omega \wedge d_*\eta &= (-1)^{\text{degree}(\omega)} \int_{U\cap V} (d\rho_V)\omega \wedge \eta \\
&= (-1)^{\text{degree}(\omega)+1} \left(-\int_{U\cap V} (d\rho_V)\omega \wedge \eta \right) \\
&= (-1)^{\text{degree}(\omega)+1} \int_{U\cup V} d^*\omega \wedge \eta \\
&= (-1)^{q+1} \int_{U\cup V} d^*\omega \wedge \eta,
\end{aligned}$$

proving that the diagram is commutative in the third square. The first two squares are easily seen to be commutative. □

Theorem 6.2.3 (Poincaré Duality - Finite Case)**.** *Let M be an oriented n-manifold with a finite good cover. Then*

$$H^q(M) \simeq (H_c^{n-q}(M))^*$$

Proof. The proof follows by induction on the number of open sets of a good cover of M. For $n = 1$, our good cover is given by $\{M\}$ which means that M is diffeomorphic to $\mathbb{R}^n$.

According to the Poincaré lemmas, we get

$$0 = H^q(M) \simeq H_c^{n-q}(M) = 0, \quad \text{for } q = 1, 2, \ldots, n,$$

Since $(H_c^{n-q}(M))^* \simeq H_c^{n-q}(M)$, because according to proposition 6.1.9 $H_c^{n-q}(M)$ is finite dimensional, then $H^q(M) \simeq (H_c^{n-q}(M))^*$, for every $q \in \{1, \ldots, n\}$.

For $q = 0$, let $f \in \Omega^0(M)$, such that $\text{supp}(f)$ is compact and $\int_M f = 1$. Then

$$\mathscr{P}\mathscr{D}(1)([f(x)dx^1 \wedge \cdots \wedge dx^n]) = \int_M f = 1,$$

and the constant function 1 and $[f dx^1 \wedge \cdots \wedge dx^n]$ are generators for $H^0(M)$ and $H_c^n(M)$, respectively, then $\mathscr{P}\mathscr{D}$ is an isomorphism between $H^0(M)$ and $(H_c^n(M))^*$. This can be proven the same way as in the first argument.

For the case $n = 2$ let $\{U, V\}$ be a good open cover for M. Both U and V are diffeomorphic to $\mathbb{R}^n$, then by the case $n = 1$ Poincaré Duality holds for them. Since $\{U, V\}$ is a good cover, then $U \cap V$ is diffeomorphic to $\mathbb{R}^n$, which means that Poincaré Duality holds for $U \cap V$ as well. Applying the Five lemma (lemma 2.3.3) on the following diagram, which is commutative and has exact rows by the preceding lemma, proves that the Poincaré Duality holds for the union $U \cup V = M$.

$$\begin{tikzcd}
H^{q-1}(U) \oplus H^{q-1}(V) \arrow[r] \arrow[d] & H^{q-1}(U \cap V) \arrow[r] \arrow[d] & H^q(M) \arrow[r] \arrow[d] & H^q(U) \oplus H^q(V) \arrow[r] \arrow[d] & H^q(U \cap V) \arrow[d] \\
(H_c^{n-q+1}(U))^* \oplus (H_c^{n-q+1}(V))^* \arrow[r] & (H_c^{n-q+1}(U \cap V))^* \arrow[r] & (H_c^{n-q}(M))^* \arrow[r] & (H_c^{n-q}(U))^* \oplus (H_c^{n-q}(V))^* \arrow[r] & (H_c^{n-q}(U \cap V))^*
\end{tikzcd}$$

For the general case where M has a good cover composed by $p+1$ elements we use the Mayer-Vietoris argument as used in proposition 6.1.8. □

We can extend the Poincaré Duality to a manifold M which does not need to have necessarily a finite good cover. In order to prove this, we prove a lemma.

Lemma 6.2.4. Let M be a manifold. There are two open sets U and V such that $M = U \cup V$ where each one of those open sets can be written as a disjoint union of open sets.

Proof. We start the proof by writing M as a union of compact sets $M = \cup_{i \in \mathbb{N}} K_i$, where each K_i is a compact set and $K_i \subset \text{int} K_{i+1}$. Put $\{W_i = A_{1i} \cup \cdots \cup A_{ni} | i \in \mathbb{N}\}$ as the set of finite open covers for each of the compact sets $K_i \setminus \operatorname{int} K_{i-1}$, where every open set $A_{1i}, \ldots, A_{ni} \subset W_i$ is such that $\overline{A}_{ji} \subset \operatorname{int}(K_{i+1}) \setminus K_{i-2}$. One geometric example for such a cover is seen at figure 3.

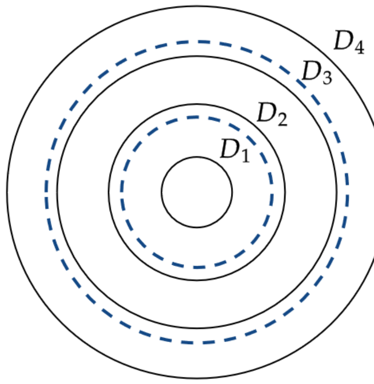

Figure 3 – For $M = \mathbb{R}^2$ and each $K_i = D_i$ as the disc of radius i centered at the origin, we get that the region between the two dotted blue circles is an open set $W_3 = A_{13}$ covering $D_3 \setminus \operatorname{int}(D_2)$ such that $\overline{A}_{13} \subset \operatorname{int}(D_4) \setminus D_1$.

Source: Elaborated by the author.

The family $\{W_i\}_{i\in\mathbb{N}}$ satisfies that $W_i \cap W_{i+j} = \emptyset$ for every $j \geq 2$. Finally, we make $U = \cup_i W_{2i}$ and $V = \cup_i V_{2i+1}$. □

Lemma 6.2.5. Let M be a n-manifold and $\mathscr{B}$ a basis of open sets of M satisfying

- for $U, V \in \mathscr{B}$, then $U \cap V \in \mathscr{B}$, and
- $\mathscr{P}\mathscr{D}|_U : H^q(U) \to (H_c^{n-q}(U))^*$ is an isomorphism for every $U \in \mathscr{B}$,

then $\mathscr{P}\mathscr{D} : H^q(M) \to (H_c^{n-q}(M))^*$ is an isomorphism.

Proof. Let $\mathscr{F}$ be the family of finite unions of elements of $\mathscr{B}$. Notice that if $\overline{U} = U_1 \cup \cdots \cup U_p \in \mathscr{F}$, then by the Poincaré Duality (finite case) we have that $\mathscr{P}\mathscr{D}|_{\overline{U}}$ is an isomorphism. Notice that $\mathscr{B}$ is not demanded to be a good cover for M, but is satisfies the same hypothesis of a good cover that are needed in the last lemmas and Poincaré Duality.

From lemma 6.2.4 we get that $M = U \cup V$ where each U and V can be written as disjoint unions of elements of $\mathscr{F}$. We write $U = \cup_i W_{2i}$ from the previous lemma, where each W_{2i} is, by last paragraph, such that $\mathscr{P}\mathscr{D}|_{W_{2i}}$ is an isomorphism. Then we have the identifications

$$H^q(U) = \prod_i H^q(W_{2i}), \quad \text{and}$$
$$H_c^{n-q}(U) = \bigoplus_i H_c^{n-q}(W_{2i}),$$

which maintains our differential forms transitioning from one identification to another well-defined. Also, we get that $(H_c^{n-q}(U))^* = \prod_i H_c^{n-q}(W_{2i})^*$ and $\mathscr{P}\mathscr{D}|_U(([\omega_i])_i) = (\mathscr{P}\mathscr{D}|_{W_{2i}}([\omega_i]))_i$. This gives us that $\mathscr{P}\mathscr{D}|_U : H^q(U) \to H_c^{n-q}(U)$ is an isomorphism.

Since the same arguments can be repeated with V, then $\mathscr{P}\mathscr{D}|_V$ is also an isomorphism. By the fact that $M = U \cup V$, then $\mathscr{P}\mathscr{D}$ is an isomorphism by the finite case of Poincaré Duality. □

Theorem 6.2.6 (Poincaré Duality)**.** Let M be an oriented n-manifold. Then

$$H^q(M) \simeq (H_c^{n-q}(M))^*$$

Proof. Using the last lemma, then we only need to recall that every manifold M has a good cover by theorem 6.1.4 and that good covers satisfies the hypothesis of our previous lemma. □

6.3. Künneth Formula

A fiber bundle is a whole structure (E, B, π, F) satisfying some conditions

- The sets E, B and F are topological spaces, called total space, base space and fiber, respectively;
- The map $\pi : E \to B$ is a continuous surjection, called projection map or bundle projection;
- Every $x \in B$ has an open neighborhood U such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ where the composition $U \times F \xrightarrow{\varphi^{-1}} \pi^{-1}(U) \xrightarrow{\pi} U$ is locally a projection in the first factor $(\pi \circ \varphi^{-1})(x, y) = x$ for all $(x, y) \in U \times F$.

Some important remarks are that $\pi^{-1}(U)$ holds the subspace topology, the set of all homeomorphisms $\{(U_i, \varphi_i)\}$ are called a local trivialization of the bundle and the fact that the preimage $\pi^{-1}(\{p\})$ is homeomorphic to F for every $p \in B$, because $(\pi \circ \varphi^{-1})^{-1}(\{p\})$ is homeomorphic to F for every $p \in U$. Moreover, the set $\pi^{-1}(\{p\})$ is called a fiber over the point p. Finally, we say that E is a fiber bundle over M with fiber F.

Due to the nature of our theory we work with smooth fiber bundles, which means our topological spaces E, B and F are smooth manifolds and the local trivialization are smooth homeomorphisms. In our particular case, they are C^∞ manifolds and C^∞ diffeomorphisms. Therefore, we usually denote B as M.

Now, we introduce the structure group in our fiber bundles by using a set of local trivializations. Let G be a topological group that acts effectively on F on the left. We say that our fiber bundle has a structure group G if it has a set of local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ where, for any two homeomorphisms φ_α and φ_β, the composite

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

satisfies

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x) \cdot v),$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is a continuous function called **transition function**. Note that for $v \in F$ everything is coherent with the definitions above because G is a group acting on F on the left.

Note that the action of G over F is effective when the only element that acts trivially on F is the identity 1_G, *i.e.*, if $g \in G$ and $g \cdot v = v$ for all $v \in F$, then $g = 1_G$. In order to explain the meaning of this property we shall restrict the functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ to $\{x\} \times F$. We get

$$\varphi_\alpha \circ \varphi_\beta^{-1}|_{\{x\} \times F}(x, v) = (x, g_{\alpha\beta}(x) \cdot v),$$

for all $v \in F$, which can be seen as a diffeomorphism on F belonging to $\operatorname{Diff} F$. Indeed, we can establish a relation $G \to \operatorname{Diff} F$ by means of the map $g_{\alpha\beta}(x) \mapsto \varphi_\alpha \circ \varphi_\beta^{-1}|_{\{x\} \times F}$. Since the group action is effective then the kernel of this last map is only the element 1_G, which means that every diffeomorphism of F of the form $\varphi_\alpha \circ \varphi_\beta^{-1}|_{\{x\} \times F}$ is associated with only one element of G without incoherence.

Transition functions must satisfy three properties

- $g_{\alpha\alpha}(x) = 1_G$, for every $x \in U_\alpha$;
- $g_{\alpha\beta}(x) = g^{-1}_{\beta\alpha}(x)$;
- $g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)g_{\beta\gamma}(x)$. This last property is called the cocycle condition which applies on the overlap of three open neighborhoods $U_\alpha \cap U_\beta \cap U_\gamma$.

Example 6.3.1. Some examples of fiber bundles are:

- The trivial bundle $E = B \times F$ with the projection bundle being only the projection on the first factor;
- If M is a n-manifold, then the tangent bundle TM is a fiber bundle with fiber being the Euclidean space $\mathbb{R}^n$.

Theorem 6.3.2 (Künneth Formula)**.** Let M and F be manifolds, M with a finite good cover, then

$$H^*(M \times F) = H^*(M) \otimes H^*(F),$$

which in more detail is written as

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F)$$

for all non-negative integer n.

Proof. We face the product $M \times F$ as the trivial bundle over M, also called the product bundle. We can establish projections on the first and second factors, given by π and ρ, respectively. We have the diagram

$$\begin{array}{ccc} M \times F & \xrightarrow{\rho} & F \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

By means of the pullbacks $\pi^* : \Omega^*(M) \to \Omega^*(M \times F)$ and $\rho^* : \Omega^*(F) \to \Omega^*(M \times F)$ we can define a map $\Psi : \Omega^k(M) \otimes \Omega^l(F) \to \Omega^{k+l}(M \times F)$ given by

$$\omega \otimes \eta \mapsto \pi^*\omega \wedge \rho^*\eta.$$

The map Ψ is well-defined since pullbacks take n-forms to n-forms, then $\pi^*\omega$ is a k-form on $\Omega^k(M\times F)$ and $\rho^*\eta$ is a l-form on $\Omega^l(M\times F)$. Therefore, $\pi^*\omega\wedge\rho^*\eta\in\Omega^{k+l}(M\times F)$. From Ψ we can induce a map in cohomology by considering only closed forms $\omega\in\Omega^k(M)$ and $\eta\in\Omega^l(F)$ as representatives of equivalence classes. We denote this map as $\psi: H^k(M)\otimes H^l(F)\to H^{k+l}(M\times F)$, which is given by $[\omega]\otimes[\eta]\mapsto[\pi^*\omega\wedge\rho^*\eta]$.

The map ψ is also well-defined. Indeed, if ω and η are closed forms, then, by the fact that exterior derivative commutes with pullbacks, we get

$$\begin{aligned}
d(\pi^*\omega\wedge\rho^*\eta) &= d(\pi^*\omega)\wedge\rho^*\eta+(-1)^{\text{degree}(\omega)}\pi^*\omega\wedge d(\rho^*\eta)\\
&= \pi^*(d\omega)\wedge\rho^*\eta+(-1)^{\text{degree}(\omega)}\pi^*\omega\wedge\rho^*(d\eta)\\
&= 0.
\end{aligned}$$

Also, suppose that ω or η are exact as well. Let $\omega=d\gamma$ without loss of generality. Then

$$\begin{aligned}
\pi^*\omega\wedge\rho^*\eta &= \pi^*(d\gamma)\wedge\rho^*\eta\\
&= d(\pi^*\gamma)\wedge\rho^*\eta\\
&= d(\pi^*\gamma\wedge\rho^*\eta)+(-1)^{\text{degree}(\gamma)+1}\pi^*\gamma\wedge d(\rho^*\eta)\\
&= d(\pi^*\gamma\wedge\rho^*\eta)+(-1)^{\text{degree}(\gamma)+1}\pi^*\gamma\wedge\rho^*(d\eta)\\
&= d(\pi^*\gamma\wedge\rho^*\eta).
\end{aligned}$$

Notice those are similar steps as the ones given in proposition 5.1.8.

The proof proceeds by induction on the cardinality of the good cover of M. For each induction step we aim to prove that $\psi:\oplus_{p=0}^n H^p(M)\otimes H^{n-p}(F)\to H^n(M\times F)$ is an isomorphism.

The first part of the inductive argument is to consider that M has a good cover composed by only one element. Thus, M is diffeomorphic to $\mathbb{R}^m$ and by the Poincaré Lemma we know its cohomology groups. The general statement of the Poincaré Lemma says that for a manifold N it is possible to relate $H^*(N\times\mathbb{R})\simeq H^*(N)$. We can extend this by induction to $H^*(N\times\mathbb{R}^n)\simeq H^*(N)$ on n. Therefore, since $H^*(M)\otimes H^*(F)\simeq\mathbb{R}\otimes H^*(F)\simeq H^*(F)$ and $H^*(M\times F)=H^*(\mathbb{R}^m\times F)\simeq H^*(F)$ we obtain the isomorphism desired.

For the case $n = 2$, consider U and V open sets of M and p a fixed integer. From the Mayer-Vietoris short exact sequence we can induce the Mayer-Vietoris long exact sequence. Tensoring the Mayer-Vietoris long exact sequence with $H^{n-p}(F)$ we obtain the following sequence

$$\begin{aligned}\cdots \to H^p(U \cup V) \otimes H^{n-p}(F) \to \\ \to (H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \to \\ \to H^p(U \cap V) \otimes H^{n-p}(F) \to \dots\end{aligned}$$

This sequence is an exact sequence and this can be shown by simply taking the functions connecting the original long exact sequence and tensoring it with the identity. The exactness is preserved in the new following exact sequence by making the sum of all functions from one space to another

$$\begin{aligned}\cdots \to \oplus_{p=0}^{n}(H^p(U \cup V) \otimes H^{n-p}(F)) \to \\ \to \oplus_{p=0}^{n}((H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \to \\ \to \oplus_{p=0}^{n}(H^p(U \cap V) \otimes H^{n-p}(F)) \to \dots\end{aligned}$$

By means of the function ψ we can establish a relation between two long exact sequences as in the following diagram

$$\begin{tikzcd}
\oplus_{p=0}^{n} H^p(U \cup V) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"'] & \oplus_{p=0}^{n}((H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \arrow[r] \arrow[d, "\psi"'] & \oplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \arrow[d, "\psi"'] \\
H^n((U \cup V) \times F) \arrow[r] & H^n(U \times F) \oplus H^n(V \times F) \arrow[r] & H^n((U \cap V) \times F)
\end{tikzcd}$$

Now we have in the previous diagram that both rows are exact sequences, then we only need to prove that every square is commutative in order to apply the five lemma on it. Most of the squares can be shown that are commutative directly by the fact that we are working with linear or multi-linear functions. The problem may arise when we work with the square in our diagram that is "from p to $p+1$", *i.e.*,

$$\begin{tikzcd}
\bigoplus_{p=0}^{n} (H^p(U \cap V) \otimes H^{n-p}(F)) \arrow[r, "d^*"] \arrow[d, "\psi"'] & \bigoplus_{p=0}^{n} \left(H^{p+1}(U \cup V) \otimes H^{n-p}(F)\right) \arrow[d, "\psi"'] \\
H^n(U \cap V) \times F) \arrow[r, "d^*"] & H^{n+1}(U \cup V) \times F)
\end{tikzcd}$$

Let $\omega \otimes \eta \in H^p(U \cap V) \otimes H^{n-p}(F)$. Notice that in this case d^* stands for $d^* \otimes Id$. By applying ψd^* and $d^* \psi$ we get

$$\psi d^*(\omega \otimes \eta) = \pi^*(d^*\omega) \wedge \rho^*\eta, \quad \text{and,}$$

$$d^*\psi(\omega \otimes \eta) = d^*(\pi^*\omega \wedge \rho^*\eta).$$

If $\{\rho_U, \rho_V\}$ is a partition of unity subordinated to $\{U,V\}$, then $\{\pi^*\rho_U, \pi^*\rho_V\}$ is a partition of unity on $(U \cup V) \times F = M \times F$ subordinated to $\{U \times F, V \times F\}$, given that $\pi : M \times F \to M$. Indeed, ρ_U is a 0-form, then $\pi^*\rho_U \in \Omega^0(U \times F)$, $(\pi^*\rho_U)(x) = \rho_U(\pi(x)) = \rho_U(p) \geq 0$, for $\pi(x) = p \in M$, and $\operatorname{supp}(\pi^*\rho_U) = \operatorname{supp}(\rho_U)$. Since the same can be said for $\pi^*\rho_V$, then we have that $\pi^*\rho_U(x) + \pi^*\rho_V(x) = \rho_U(p) + \rho_V(p) = 1$.

Notice that $\psi : \oplus_{p=0}^n (H^p(U \cap V) \otimes H^{n-p}(F)) \to H^n((U \cap V) \times F)$, then, when restricted to V, we get

$$\begin{aligned}
d^*(\pi^*\omega \wedge \rho^*\eta) &= d((\pi^*\rho_U)(\pi^*\omega \wedge \rho^*\eta)) \\
&= d((\pi^*\rho_U \pi^*\omega) \wedge \rho^*\eta) \\
&= (d\pi^*(\rho_U\omega)) \wedge \rho^*\eta + (-1)^{\text{degree}(\omega)}\pi^*(\rho_U\omega) \wedge d\eta \\
&= \pi^*(d^*\omega) \wedge \rho^*\eta.
\end{aligned}$$

Applying the definition of d^* when working on U we can get get the same equality as above. In the equality above we use the facts that η is a closed form and d commutes with pullbacks. Also, we use that pullbacks are linear over wedge product, *i.e.*, $(f^*\omega) \wedge (f^*\eta) = f^*(\omega \wedge \eta)$. This completes the proof that the diagram is commutative.

By applying the case $n = 1$ on U, V, and $U \cap V$ and by applying the five lemma on the diagram (the rows are exact), we prove that

$$\psi : \oplus_{p=0}^n (H^p(U \cup V) \otimes H^{n-p}(F)) \to H^n((U \cup V) \times F)$$

is an isomorphism. Notice that ψ was firstly defined not on a direct sum but between spaces $H^p(U \cup V) \otimes H^{n-p}(F) \to H^n((U \cup V) \times F)$. However, it remains to be well-defined on the direct sum. The rest of the proof proceeds by Mayer-Vietoris argument as stated in the beginning. □

Example 6.3.3. By using the Künneth formula we can compute the cohomology of the 2-torus more easily. Let $T^2 = S^1 \times S^1$ be the 2-torus. The last theorem gives us the equality

$$H^*(T^2) = H^*(S^1 \times S^1) = H^*(S^1) \otimes H^*(S^1),$$

since the open cover given in example 3.1.9 is a finite good cover for S^1. On example 5.4.2 we saw that $H^0(S^1) = H^1(S^1) = \mathbb{R}$. Therefore, by the Künneth formula we have

- $H^0(T^2) = H^0(S^1) \otimes H^0(S^1) = \mathbb{R} \otimes \mathbb{R} \simeq \mathbb{R}$;
- $H^1(T^2) = (H^0(S^1) \otimes H^1(S^1)) \oplus (H^1(S^1) \otimes H^0(S^1)) = (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) \simeq \mathbb{R}^2$;
- $H^2(T^2) = H^1(S^1) \otimes H^1(S^1) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}$;

We obtain that $H^n(T^2) = \mathbb{R}^{\binom{2}{n}}$. More generally, we can compute $H^n(T^k) = \mathbb{R}^{\binom{k}{n}}$, where $T^k = S^1 \dots S^1$ is the k-torus.

Similarly we can prove the next theorem.

Theorem 6.3.4 (Künneth Formula for Compatly Supported Cohomology)**.** Let M and F be manifolds with a finite good cover, then

$$H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N).$$

The next theorem, Leray-Hirsch Theorem, can be seen as generalization of the Künneth Formula, since for this statement we work with a general fiber bundle. Let $\pi : E \to M$ be a fiber bundle over M with fiber F. If we suppose that there are cohomology classes $e_1, \dots, e_r$ on E that can be restricted to a basis of the cohomology of each fiber, then we can define

$$\psi : H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \to H^*(E)$$

similarly as on the proof of the Künneth Formula. Before stating the theorem and giving a sketch of its proof, let us understand in more depth the paragraph above.

When demanded that the set $\{e_1, \dots, e_r\}$ is composed of global cohomology classes on E that when restricted to each fiber is a basis for the cohomology of the fiber, means that if $E_p = \pi^{-1}(\{p\})$ is the fiber over the point p, then there is an inclusion

$i_p : E_p \to E$ and the set $\{i_p^* e_1, \dots, i_p^* e_r\}$ generates the cohomology ring $H^*(E_p)$, *i.e.*, it generates every cohomology group $H^q(E_p)$ of every degree q. Notice that every E_p is diffeomorphic to the fiber F, which allows us to induce an inclusion $i : F \to E$ and then making the set $\{i^* e_1, \dots, i^* e_r\}$ the one that generates $H^*(F)$ and it is denoted, by means of an abuse of notation, by the set $\mathbb{R}\{e_1, \dots, e_r\}$.

Therefore, we define the function $\psi : H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \to H^*(E)$ as

$$\omega \otimes \eta \mapsto \pi^* \omega \wedge \Big(\sum_{k=0}^{r} \alpha_k e_k \Big),$$

where η is written as a linear combination of the set $\{i^* e_1, \dots, i^* e_r\}$.

Theorem 6.3.5 (Leray-Hirsch)**.** Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover. If there are global cohomology classes $e_1, \dots, e_r$ on E which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \dots, e_r\}$, *i.e.*

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \simeq H^*(M) \otimes H^*(F).$$

Proof. The proof is on the cardinality of the good cover of M. When proved the case $n = 2$ we can generalize by using the Mayer-Vietoris argument. For $n = 1$ the theorem reduces to the Künneth Formula. For $n = 2$ we have $\{U, V\}$ a good cover for M. E is a fiber bundle over M, therefore exists diffeomorphisms $\varphi_U : \pi^1(U) \to U \times F$ and $\varphi_V : \pi^{-1}(V) \to V \times F$, where $\pi \circ \varphi_U^{-1}$ and $\pi \circ \varphi_V^{-1}$ are projections on the first factor of $U \times F$ and $V \times F$, respectively. From the fact that $E|_{\pi^{-1}(U)}$ and $E|_{\pi^{-1}(V)}$ have the same homotopy type of $U \times F$ and $V \times F$, respectively, and the Künneth Formula, then

$$H^*(E|_{\pi^{-1}(U)}) \simeq H^*(U \times F) \simeq H^*(U) \otimes H^*(F),$$
$$H^*(E|_{\pi^{-1}(V)}) \simeq H^*(V \times F) \simeq H^*(V) \otimes H^*(F).$$

We then establish the following diagram

$$\begin{tikzcd}
\oplus_{p=0}^n H^p(U \cup V) \otimes H^{n-p}(F) \arrow[r] \arrow[d, "\psi"'] & \oplus_{p=0}^n ((H^p(U) \otimes H^{n-p}(F)) \oplus (H^p(V) \otimes H^{n-p}(F)) \arrow[r] \arrow[d, "\psi"'] & \oplus_{p=0}^n H^p(U \cap V) \otimes H^{n-p}(F) \arrow[d, "\psi"'] \\
H^n(E|_{\pi^{-1}(U \cup V)}) \arrow[r] & H^n(E|_{\pi^{-1}(U)}) \oplus H^n(E|_{\pi^{-1}(V)}) \arrow[r] & H^n(E|_{\pi^{-1}(U \cap V)})
\end{tikzcd}$$

so that the rest of the proof follows the same reasoning as in the proof of Künneth Formula. $\square$

Remark 6.3.6. Check (GARZA, 2019) for the general statement of the Leray-Hirsch theorem.

CHAPTER

7

VECTOR BUNDLES

In this chapter we recall what is a vector bundle, by understanding its structure through the definition of fiber bundles given in Künneth formula's section. We present examples and a great number of results regarding vector bundles. We draw attention to two particular results, the "Homotopy Property of Vector Bundles" and its corollary, which states that any vector bundle over a contractible manifold is trivial. This corollary has a central role for proving Thom isomorphism. We develop the content in this chapter based on (BOTT; TU, 1982), (HUSEMOLLER, 1993) and (MELO, 2019).

We say that a fiber bundle $\pi : E \to M$ is a vector bundle of rank n when its fiber F is a finite n-dimensional vector space V^n, where $V^n = \mathbb{R}^n$ or $V^n = \mathbb{C}^n$. More precisely, $\pi^{-1}(\{p\})$ has a vector space structure for every $p \in M$ and there is a open cover $\{U_\alpha\}$ of M such that

$$E|_{\pi^{-1}(U_\alpha)} \simeq U_\alpha \times V^n$$

and diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V^n$, which for every $p \in M$, we get $\pi^{-1}(\{p\})$ diffeomorphic to V^n and $v \mapsto \varphi_\alpha^{-1}(p,v)$ is a linear isomorphism for every $p \in U_\alpha$. We call $\pi : E \to M$ a real vector bundle when $V = \mathbb{R}$ and a complex vector bundle when $V = \mathbb{C}$.

The structure group of our vector bundle is $G = GL(n,V)$ because the composites

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\alpha) \times V^n \to (U_\alpha \cap U_\alpha) \times V^n$$

induce vector space automorphisms of V^n and the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,V)$$

can be identified as the matrices making the change of variables between systems of coordinates given by φ_α and φ_β, *i.e.*,

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x,v)|_{\{x\}\times V^n} = (x, g_{\alpha\beta}(x)\cdot v).$$

Example 7.0.1. Consider V^n a n-dimensional vector space and its projectivization given by $P(V^n) = \{$1-dimensional subspaces of $V^n\}$, which is a manifold. We can derive some types of vector bundles using the projectivization, such as

- The product bundle $\hat{V} = P(V^n) \times V^n$, which is a trivial bundle;
- The universal subbundle $S = \{(l,v) \in \hat{V} | v \in l\}$.
- The universal quotient bundle Q given by the exact sequence, called tautological exact sequence over $P(V)$, $0 \to S \to \hat{V} \to Q \to 0$.

The fiber of S above each point l in $P(V)$ consists of all the points in l, for l being a line on V^n. The fibers in Q are the quotient between the related fibers on $\hat{V}$ and S.

Definition 7.0.2. Let $\pi : E \to M$ be a vector bundle over M with fiber V^n and U an open set of M. We define $s : U \to E$ to be a section of the vector bundle E over U if $\pi \circ s = Id|_U$.

Proposition 7.0.3. Every vector bundle has a well-defined global zero section.

Proof. For every $p \in M$ there is a fiber over it $E_p = \pi^{-1}(\{p\})$. We define a global zero section $s : M \to E$ as a map $p \mapsto 0_{E_p} \in E_p \subset E$. Since trivializations induces isomorphisms $v \mapsto \varphi_\alpha^{-1}(p,v)$ for every fixed $p \in U_\alpha$, then $\pi \circ s(p) = \pi(0_{E_p}) = p$. □

Transition functions on fiber bundles satisfy the cocycle property on triple intersections. This remains true on vector bundles.

Lemma 7.0.4. Let $\pi : E \to M$ be a vector bundle over M. Let $\{g_{\alpha\beta}\}$ be the transition functions given by the trivializations φ_α and φ_β. If the cocycle $\{g'_{\alpha\beta}\}$ comes from the trivializations φ'_α and φ'_β, then there are maps $\lambda_\alpha : U_\alpha \to GL(n,V)$ such that

$$g_{\alpha\beta} = \lambda_\alpha \cdot g'_{\alpha\beta} \cdot \lambda_\beta^{-1}$$

on $U_\alpha \cap U_\beta$, where λ_α^{-1} means $\lambda_\alpha^{-1}(p)$ for $p \in U_\alpha$, *i.e.*, an inverse matrix at each point $p \in U_\alpha$.

Proof. For every two trivializations of E, φ_α and φ'_α, defined on $\pi^{-1}(U_\alpha)$, then $\varphi'_\alpha \circ \varphi_\alpha^{-1} : U_\alpha \times V^n \to U_\alpha \times V^n$ is a diffeomorphism and, for each $p \in U_\alpha$ there is a non-singular transformation of V^n, which can be seen as a change of variables between $\varphi_\alpha(\pi^{-1}(p))|_{\{p\}\times V^n}$ and $\varphi'_\alpha(\pi^{-1}(p))|_{\{p\}\times V^n}$. We denote the set of these non-singular transformations as a function $\lambda_\alpha : U_\alpha \to GL(n,V)$. Therefore,

$$\varphi_\alpha(\pi^{-1}(p)) = \lambda_\alpha(p) \cdot \varphi'_\alpha(\pi^{-1}(p)), \quad \text{for every } p \in U_\alpha$$

which implies

$$\begin{aligned} g_{\alpha\beta}(x) \cdot v &= \varphi_\alpha \cdot \varphi_\beta^{-1}|_{\{x\}\times V^n}(x,v) \\ &= (\lambda_\alpha(x) \cdot [\varphi'_\alpha \cdot {\varphi'}_\beta^{-1}|_{\{x\}\times V^n}] \cdot \lambda_\beta^{-1}(x)) \cdot v \\ &= (\lambda_\alpha(x) g'_{\alpha\beta}(x) \lambda_\beta^{-1}(x)) \cdot v, \end{aligned}$$

for every $x \in U_\alpha \cap U_\beta$. □

We say that two cocycles with the relation expressed in the last lemma are equivalent. This concept leads to an important property of a vector bundle E, which is the possibility to reduce its structure group $GL(n,V)$ to one of its subgroups, *i.e.*, given the cocycle $\{g_{\alpha\beta}\}$, we can find an equivalent cocycle $\{g'_{\alpha\beta}\}$ with values only in this subgroup. Let H be such a subgroup, then we say that the structure group of E can be reduced to H.

Definition 7.0.5. Let $\pi : E \to M$ and $\pi' : E' \to N$ be two vector bundles. We say that there is a bundle homomorphism between E and E' when there are continuous functions $f : E \to E'$ and $g : M \to N$ and f is fiber-preserving, which is linear on the corresponding fibers, *i.e.*, $f|_{\pi^{-1}(p)} : \pi^{-1}(p) \subset E \to (\pi')^{-1}(g(p)) \subset E'$ is a linear map. We say that E and E' are isomorphic when f admits an inverse which is bundle homomorphism from E' to E.

7.1. Real Vector Bundles

Example 7.1.1. The tangent bundle of S^2, $TS^2 = \bigcup_{p\in S^2}(\{p\}\times T_pS^2)$, is a real vector bundle. Geometrically, TS^2 can be seen as the 2-sphere and all its tangent spaces.

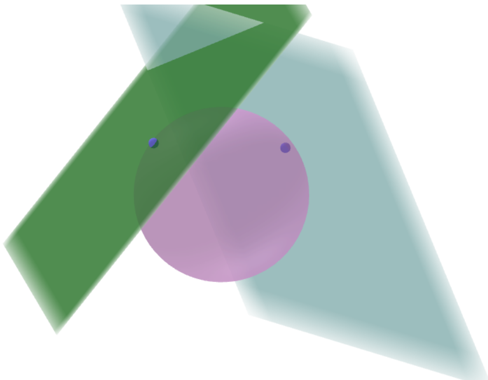

Figure 4 – Geometric representation of TS^2.

Source: Elaborated by the author.

Let $U_N = S^2 \setminus \{N\}$ and $U_S = S^2 \setminus \{S\}$. The stereographic projections φ_N, φ_S compose an atlas for S^2, $\mathscr{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ and the bundle projection $\pi : TS^2 \to S^2$ is given by $\pi(p,v) = p$, for every $p \in S^2$ and every $v \in T_pS^2$. Therefore, $\pi^{-1}(\{p\}) = T_pS^2$, which has a real vector space structure.

Denote the local chart $\varphi_N : U_N \to \mathbb{R}^2$ as (U_N, x^1, x^2). We can induce a function $T\varphi_N : U_N \times \mathbb{R}^2 \to \pi^{-1}(U_N)$ given by

$$(p, v_1, v_2) \mapsto \Big(p, \underbrace{\Big(v_1 \frac{\partial}{\partial x_1}(p), v_2 \frac{\partial}{\partial x_2}(p)\Big)}_{\in TS^2_p}\Big)$$

which is a linear isomorphism by fixating the point $p \in U_N$.

Moreover, for all $p \in U_N$, $(\pi \circ T\varphi_N)(p, (v_1, v_2)) = p$ for every $(v_1, v_2) \in \mathbb{R}^2$. The computations just made can be repeated with (U_S, φ_S). Then, to finish the proof that this tangent bundle is a real vector bundle we need just to compute its transition functions and analyse is structure group. We derive the transition functions from the local trivializations induced from the local charts in $\mathscr{A}$.

Let the 2-sphere be with polar coordinates and tangent spaces with a given basis as in figure 5, where the green tangent space is over the point B in S^2 and the blue tangent space is over the point A in S^2.

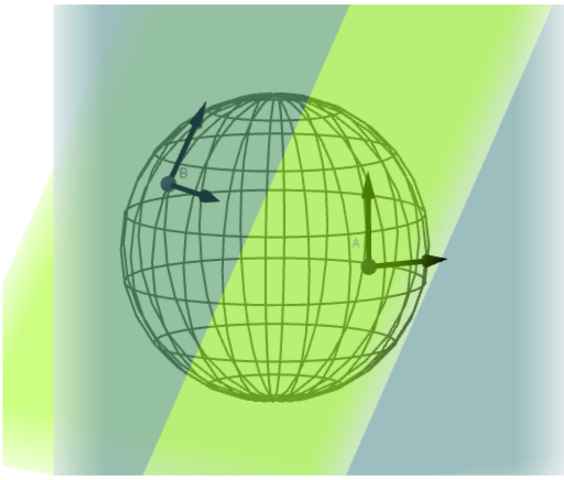

Figure 5 – Vector fields on the 2-sphere.

Source: Elaborated by the author.

Looking at a net section of the sphere planified and with cartesian coordinates we obtain a collection of pairs of the vectors that generates each tangent space, which may have different rotation angles, as in figure 6. The transition functions are the functions that adjust the rotation between the coordinate systems given in each tangent space, presented in a net section of the 2-sphere.

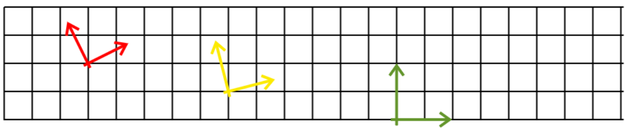

Figure 6 – Net section of the 2-sphere with cartesian coordinates.

Source: Elaborated by the author.

In the 2-sphere, with local charts $(U_N,\varphi_N) = (U_N, x^1, x^2)$ and $(U_S,\varphi_S) = (U_S, y^1, y^2)$, given $(p, w_1, w_2) \in (U_N \cap U_S) \times \mathbb{R}^2$ we have

$$T\varphi_S^{-1} \circ T\varphi_N(p, w_1, w_2) = T\varphi_S^{-1}\left(p, w_1\frac{\partial}{\partial x_1}(p), w_2\frac{\partial}{\partial x_2}(p)\right)$$

$$= T\varphi_S^{-1}\Big(p, \Big[w_1\frac{\partial y^1}{\partial x_1}(p) + w_2\frac{\partial y^1}{\partial x_2}(p)\Big]\frac{\partial}{\partial y_1}(p), \Big[w_1\frac{\partial y^2}{\partial x_1}(p) + w_2\frac{\partial y^2}{\partial x_2}(p)\Big]\frac{\partial}{\partial y_2}(p)\Big)$$
$$= \Big(p, \Big[w_1\frac{\partial y^1}{\partial x_1}(p) + w_2\frac{\partial y^1}{\partial x_2}(p)\Big], \Big[w_1\frac{\partial y^2}{\partial x_1}(p) + w_2\frac{\partial y^2}{\partial x_2}(p)\Big]\Big)$$

This means that for a point $p \in U_N \cap U_S$ we get that the transition function $g_{U_N U_S} : U_N \cap U_S \to GL(2,\mathbb{R})$ is given by

$$g_{U_N U_S}(p) = \begin{pmatrix} \frac{\partial y^1}{\partial x_1}(p) & \frac{\partial y^1}{\partial x_2}(p) \\ \frac{\partial y^2}{\partial x_1}(p) & \frac{\partial y^2}{\partial x_2}(p) \end{pmatrix}$$

where the matrix above is the rotation necessary to adjust between the coordinate systems induced on T_pS^2 by each of the local charts (U_N, φ_N) and (U_S, φ_s).

Remark 7.1.2. The images on this last example have as basis drawings firstly developed by Matthias Zach, postdoctoral researcher at Leibniz Universitat Hannover and participant on the defense committee, on a reunion with Timo Essig, postdoctoral researcher at Christian-Albrechts-Universitat zu Kiel, and myself. This is the reason why the sources were named after me, since there is no document or work by Matthias Zach available for me to reference.

Example 7.1.3 (Tangent bundle)**.** Inspired by the last example we can prove that the tangent bundle TM of any smooth manifold M is a real vector bundle. We can induce a function $U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$ from a local chart $(U_\alpha, \varphi_\alpha)$ of M the same way we induced the function $T\varphi_N : U_N \times \mathbb{R}^2 \to \pi^{-1}(U_N)$ from the local chart (U_N, φ_N) of TS^2.

Definition 7.1.4. Let E be a real vector bundle. We say that E is orientable as a vector bundle if its structure group can be reduced to the subgroup of linear transformations with positive determinant $GL^+(n,\mathbb{R})$.

Definition 7.1.5. Let E be a real vector bundle and $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialization for E. We define $\{(U_\alpha, \varphi_\alpha)\}$ to be oriented if for every $\alpha, \beta \in I$, the transition function $g_{\alpha\beta}$ has positive determinant at every point $x \in U_\alpha \cap U_\beta$.

Proposition 7.1.6. Let E be a real vector bundle. Then E is orientable as a vector bundle if, and only if, has an oriented trivialization $\{(U_\alpha, \varphi_\alpha)\}$.

Proof. It comes straightforward out of the two previous definitions. □

Definition 7.1.7. Let E be a real vector bundle and $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ two oriented trivializations for E. We say that both these trivializations are equivalent if for every point $x \in U_\alpha \cap V_\beta$, the linear transformation $(\varphi_\alpha \circ \psi_\beta^{-1})(x) = \varphi_\alpha \circ \psi_\beta^{-1}|_{\{x\}\times\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ has a positive determinant.

Proposition 7.1.8. The definition above gives us an equivalence relation on the set of oriented trivializations.

Proof. In order to prove equivalence relations we have to show the reflexive, the symmetric and transition properties.

The reflexive property comes straight from the fact that we are relating a oriented trivialization $\{(U_\alpha, \varphi_\alpha)\}$ with itself. Therefore, from the definition of the equivalence relation we obtain the transition functions $g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}(x)$, which have positive determinant since $\{(U_\alpha, \varphi_\alpha)\}$ is an oriented trivialization.

The symmetric property comes from the property that the matrix representation of $(\varphi_\alpha \circ \psi_\beta^{-1})(x)$ has an inverse with positive determinant.

At last, for the transitive property, take three trivializations $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$, $\mathscr{V} = \{(V_\beta, \psi_\beta)\}$ and $\mathscr{W} = \{(W_\gamma, \eta_\gamma)\}$, where $\mathscr{U}$ and $\mathscr{V}$ are related and $\mathscr{V}$ and $\mathscr{W}$ are related. By definition of the equivalence relation we get

$$(\varphi_\alpha \circ \psi_\beta^{-1})(x) : \mathbb{R}^n \to \mathbb{R}^n, \quad \text{has a positive determinant for every } x \in U_\alpha \cap V_\beta$$

$$(\psi_\beta \circ \eta_\gamma^{-1})(x) : \mathbb{R}^n \to \mathbb{R}^n \quad \text{has a positive determinant for every } x \in V_\beta \cap W_\gamma.$$

Therefore, every composite $(\varphi_\alpha \circ \eta_\gamma^{-1})(x) : \mathbb{R}^n \to \mathbb{R}^n$ has a positive determinant for every $x \in U_\alpha \cap W_\gamma$, since $(\varphi_\alpha \circ \eta_\gamma^{-1})(x) = (\varphi_\alpha \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ \eta_\gamma^{-1})(x)$ for a trivialization (V_β, ψ_β) such that $x \in V_\beta$. □

At the end of section 4.1 we saw that connected orientable manifolds have only two orientations. Similarly, we can prove, using oriented trivializations, the following

Proposition 7.1.9. Let E be a real vector bundle over a connected manifold M. Then the equivalence relation over the set of oriented trivializations is given by two elements called *orientations* for E.

Theorem 7.1.10. Let $\pi : E \to M$ be real a vector bundle over M. Suppose E is orientable as a vector bundle and M is an orientable manifold. Then E is orientable as a manifold.

Proof. Let $\{(U_\alpha, \psi_\alpha)\}$ be an oriented atlas for M with transition functions $h_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$. Remember that $h_{\alpha\beta}$ must be orientation preserving, which means its Jacobian matrix $J(h_{\alpha\beta})$ must have a positive determinant in every point. For a set of oriented local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ for E, the transition functions $g_{\alpha\beta}$ must have a positive determinant at every point $x \in U_\alpha \cap U_\beta$.

We can establish an atlas for E by making the compositions $(\psi_\alpha \times Id_{\mathbb{R}^n}) \circ \varphi_\alpha$, which is denoted by $\mathscr{A} = \{(\pi^{-1}(U_\alpha), (\psi_\alpha \times Id_{\mathbb{R}^n}) \circ \varphi_\alpha)\}$. Notice that we can cover E because given any $x \in E$ we get $\pi(x) \in M$, then there is α such that $\pi(x) \in U_\alpha$, which implies $x \in E|_{\pi^{-1}(U_\alpha)}$. Notice that each local chart from $\mathscr{A}$ gives us the diffeomorphic sets

$$E|_{\pi^{-1}(U_\alpha)} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m \times \mathbb{R}^n.$$

The transition functions for $\mathscr{A}$ are given by the compositions

$$[(\psi_\alpha \times Id_{\mathbb{R}^n}) \circ \varphi_\alpha] \circ [\varphi_\beta^{-1} \circ (\psi_\beta^{-1} \times Id_{\mathbb{R}^n})] : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

where

$$\begin{aligned}
[(\psi_\alpha \times Id_{\mathbb{R}^n}) \circ \varphi_\alpha] \circ [\varphi_\beta^{-1} \circ (\psi_\beta^{-1} \times Id_{\mathbb{R}^n})](x,y) &= (\psi_\alpha \times Id_{\mathbb{R}^n}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})(\psi_\beta^{-1}(x), y) \\
&= (\psi_\alpha \times Id_{\mathbb{R}^n})(\psi_\beta^{-1}(x), g_{\alpha\beta}(\psi_\beta^{-1}(x)) \cdot y) \\
&= (h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\beta^{-1}(x)) \cdot y).
\end{aligned}$$

Therefore, the Jacobian matrix for this transition function is

$$\begin{pmatrix} J(h_{\alpha\beta}) & 0 \\ 0 & g_{\alpha\beta}(\psi_\beta^{-1}(x)) \end{pmatrix}$$

which has a positive determinant. Therefore, $\mathscr{A}$ is an oriented atlas for E. □

Given the general linear group of square matrices with order n and entries determined by real values, $GL(n, \mathbb{R})$, we can determine two subgroups, defined by

$O(n) = \{A \in GL(n, \mathbb{R}) | A^T \cdot A = A \cdot A^T = I\}$, with A^T being the transpose matrix of A.

$$SP(n) = \{A \in GL(n,\mathbb{R}) | v^T \cdot A \cdot v > 0, \forall v \in \mathbb{R}^n \setminus \{0\} \text{ and } A = A^T\}$$

where $SP(n)$ are the positive definite symmetric matrices. Using polar decomposition for matrices we obtain the direct product $GL(n,\mathbb{R}) = O(n) \times SP(n)$, where direct product in this case means exactly multiplication of matrices. Since $g_{\alpha\beta}(x) \in GL(n,\mathbb{R})$, then by last proposition $g_{\alpha\beta}(x) = U_{x,\alpha\beta} \cdot P_{x,\alpha\beta}$, which proves the following

Proposition 7.1.11. Let E be a real vector bundle with rank n and structure group $G = GL(n,\mathbb{R})$. Then G can be reduced to $O(n)$.

7.2. Complex Vector Bundles

We recall that $\pi : E \to M$ is a complex vector bundle of rank n over M, when it is a fiber bundle with fiber $V^n = \mathbb{C}^n$ and a structure group $GL(n,\mathbb{C})$. If $n = 1$ we call it a complex line bundle.

Remark 7.2.1. Since the structure group of a real vector bundle of rank n can be reduced to the orthogonal group $O(n)$, then, by the Hermitian analogue, the complex vector bundle of rank n can be reduced to the unitary group $U(n) = \{A \in GL(n,\mathbb{C}) : A^*A = AA^* = AA^{-1} = I\}$, where A^* is the conjugate transpose of A, *i.e.*, $\langle Ax, y\rangle = \langle x, A^*y\rangle$.

Proposition 7.2.2. Given a certain complex vector bundle $\pi : E \to M$ of rank n, then it is also a real vector bundle $\pi : E_{\mathbb{R}} \to M$ of rank $2n$.

Proof. We can simply forget the complex structure of our fiber $\mathbb{C}^n$ by using the forgetful functor. □

Proposition 7.2.3. Every complex vector bundle E is oriented as a real vector bundle.

Proof. According to remark 7.2.1 the structure group of E can be reduced to $U(n)$. If A is a matrix belonging to the structure group of E, then we can write as a real matrix of order $2n$ composed by the blocks

$$\begin{bmatrix} A_r & -A_i \\ A_i & A_r \end{bmatrix}$$

where A_r is a real matrix of order n composed by the real part of the components of the matrix A and A_i is a real matrix of order n composed by the complex part of the components of the matrix A. We denote this matrix as $A_{\mathbb{R}}$. Since we can rewrite A^* following the same reasoning and the multiplication of matrices is preserved in this new form, then $A_{\mathbb{R}}A_{\mathbb{R}}^* = A_{\mathbb{R}}^*A_{\mathbb{R}} = A_{\mathbb{R}}A_{\mathbb{R}}^{-1} = I$. This proves that $A_{\mathbb{R}}$ has a positive determinant for every $A \in U(n)$, which shows that E is oriented as a real vector bundle by definition 7.1.4. □

7.3. Operations on vector bundles

Given real vector bundles E and E' over a manifold M, of ranks n and m respectively, we can define operations between them as if they were vector spaces.

Definition 7.3.1. Let $\pi : E \to M$ and $\pi' : E' \to M$ be real vector bundles over M, of ranks n and m respectively, with fibers at each point x denoted by E_x and E'_x respectively. For an open cover $\{U_\alpha\}$ of M, let $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U_\alpha, \varphi'_\alpha)\}$ be local trivializations for E and E', respectively. We define

- The direct sum $E \oplus E'$ as a real vector bundle over M, with fiber $E_x \oplus E'_x$ at each point x. Its local trivializations are given by $\varphi_\alpha \oplus \varphi'_\alpha : E|_{\pi^{-1}(U_\alpha)} \oplus E'|_{\pi'^{-1}(U_\alpha)} \to U_\alpha \times (\mathbb{R}^n \oplus \mathbb{R}^m)$. The transition functions for this vector bundle are given by

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{pmatrix}$$

- The tensor product $E \otimes E'$ as a real vector bundle over M, with fiber $E_x \otimes E'_x$ at each point x. Its local trivializations are given by $\varphi_\alpha \otimes \varphi'_\alpha : E|_{\pi^{-1}(U_\alpha)} \otimes E'|_{\pi'^{-1}(U_\alpha)} \to U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m)$. The transition functions for this vector bundle are given by $g_{\alpha\beta} \otimes g'_{\alpha\beta}$;
- The dual bundle E^* as a real vector bundle over M, with fiber E_x^* at each point x. Recall that given a linear transformation between real vector spaces $T : V \to W$ we can define a linear map $T^* : W^* \to V^*$ given by the transpose of the matrix of T. Then, for the trivializations of E^* we put $(\varphi_\alpha^*)^{-1} : E^*|_{\pi(U_\alpha)} \to U_\alpha \times (\mathbb{R}^n)^*$,

for each trivialization $\varphi_\alpha : E|_{\pi(U_\alpha)} \to U_\alpha \times \mathbb{R}^n$. For each point $x \in U_\alpha$, we are, in fact, establishing a relation between $(\mathbb{R}^n)^*$ and the fiber E_x^* by means of $\varphi_\alpha^*(x)$, which is the dual transformation of the isomorphism $v \mapsto \varphi_\alpha^{-1}(x,v)$. The transition functions of E^* are given by

$$(\varphi_\alpha^*)^{-1} \circ \varphi_\beta^* = ((\varphi_\alpha \circ \varphi_\beta^{-1})^*)^{-1} = (g_{\alpha\beta}^*)^{-1}.$$

- The Hom-bundle $Hom(E,E')$ as a real vector bundle over M, with fiber $Hom(E_x,E_x')$, the space of linear maps from E_x to E_x', at each point $x \in M$. In order to make easier for us to work with this space we identify it with $E^* \otimes E'$.

- The pullback bundle. Let M and N be manifolds and $\pi : E \to M$ a vector bundle over M. Let $f : N \to M$ be a map between manifolds. We define the vector bundle $f^{-1}E$ on N, called the pullback of E by f, as a subset of $N \times E$ that is given by the set

$$f^{-1}E = \{(n,e) \in N \times E | f(n) = \pi(e)\} \subset N \times E.$$

The projection bundle $\pi' : f^{-1}E \to N$ is given by the projection on the first coordinate $\pi'(n,e) = n$. A fiber $f^{-1}E_y$ over a point $y \in N$ is isomorphic to the fiber E_x, where $f(y) = x$.The trivialization set for $f^{-1}E$ is given by the pullback functions $f^*\varphi_\alpha$ for, the trivialization set $\{(U_\alpha, \varphi_\alpha)\}$ of E, and trivializing sets $\{f^{-1}U_\alpha\}$, which gives an open cover for $f^{-1}E$ over N. Also, we can identify $(f^{-1}E)|_{f^{-1}U_\alpha} \simeq f^{-1}(E|_{\pi^{-1}(U_\alpha)})$. This identification shows us that product bundle pulls back to product bundle, making $f^{-1}E$ locally trivial. Following the same reasoning, we have that transition functions are given by the pullback $f^*g_{\alpha\beta}$ of the transition functions $g_{\alpha\beta}$ of E by f. Also, it can be easily seen by definition that given a composition of maps between manifolds $P \xrightarrow{g} N \xrightarrow{f} M$, then $(f \circ g)^{-1}E = g^{-1}(f^{-1}E)$. The properties for this pullback bundle derives from the pullback property $(f \circ g)^* = g^* \circ f^*$. Finally, if $h : f^{-1}E \to E$ is projection on the second factor, then the pullback bundle is the maximal subset of $N \times E$ that makes the diagram commutative

$$\begin{tikzcd} f^{-1}E \arrow[r, "h"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

- The projectivization of a vector bundle $\rho : E \to M$. Suppose E has transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,V)$, where E_p is the fiber over each point $p \in M$. The projectivization of E is a vector bundle $\pi : P(E) \to M$ over M, where each fiber at a point $p \in M$ is the projective space $P(E_p)$. Moreover, the transition functions $\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to PGL(n,V)$ are induced from the transition functions of E, $g_{\alpha\beta}$, and $PGL(n,V) = GL(n,V)/\{\text{scalar matrices}\}$.

Remark 7.3.2. A point in $P(E)$ is the coordinate made of point and a line, (p,l_p), where l_p is in the fiber E_p. For instance, for $E = TS^2$ we have (p,l_p) where $l_p \subset T_pS^2$. Other types of vector bundles can be derived from the projectivization.

We define the pullback bundle $\pi^{-1}E$ as a vector bundle over $P(E)$, whose fiber at every point l_p of $P(E)$ is E_p. A set definition for this is

$$\pi^{-1}E = \{((x,l),(y,\vec{v})) \in P(E) \times E | \pi(x,l) = x = y = \rho(y,\vec{v})\}.$$

Also, we can define a universal subbundle over $P(E)$ as well, which is given by

$$S = \{(l_p,v) \in \pi^{-1}E | v \in l_p\},$$

where each fiber over l_p is all of the points in l_p, where l_p is viewed as 1-dimensional subvector space of E_p.

Remark 7.3.3. On what follows until the end of this chapter is taken from (HUSEMOLLER, 1993). From the statement of the results to the complete ideas of their proofs are fully taken from this source. Adaptations were made for this work mainly related to the language used throughout this dissertation, which is slightly different from Husemoller's book.

We want to prove the next theorem in order to prove the next corollary, which are important facts about vector bundles, especially the corollary.

Theorem 7.3.4 (Homotopy Property of Vector Bundles)**.** Let M and N be manifolds and E be a vector bundle on M. If f_0 and f_1 are homotopic maps from N to M, then $f_0^{-1}E$ is isomorphic to $f_1^{-1}E$ as vector bundles over N.

Corollary 7.3.5. Any vector bundle over a contractible manifold is trivial.

For this we work through some lemmas and propositions, where our vector bundles have the base space as a topological space B and fiber is a vector space V.

Lemma 7.3.6. Let $\pi : E \to B$ be a vector bundle over B of rank n. Suppose that $B = B_1 \cup B_2$, where $B_1 = A \times [a,c]$ and $B_2 = A \times [c,b]$, for $a < c < b$. If our vector bundle E restricted to B_1 and B_2, *i.e.*, $E|_{\pi^{-1}(B_1)}$ and $E|_{\pi^{-1}(B_2)}$, are trivial, then E is trivial.

Proof. Let $\varphi_i^{-1} : B_i \times V \to E|_{\pi^{-1}(B_i)}$ be the inverse of trivializations φ_i for $i = 1,2$. Denote $v_i = \varphi_i|_{((B_1 \cap B_2)\times V)}$ for $i = 1,2$. The composition $h = v_2^{-1} \circ v_1$ is an isomorphism of trivial bundles over $A \times \{c\}$. Therefore, $h(x,y) = (x, g(x)\cdot y)$, for $(x,y) \in (B_1 \cap B_2) \times V$, and g is a map $g : A \to GL(n,V)$. By defining $w : B_2 \times V \to B_2 \times V$ by putting $w(x,t,y) = (x,t,g(x)\cdot y)$, where $x \in A$, $t \in [c,b]$ and $y \in V$, then we are able to extend the definition of h for every point $x \in A$ to B_2 (remember that $B_2 = A \times [c,b]$).

Using this extension we see that the bundle isomorphisms $\varphi_1^{-1} : B_1 \times V \to E|_{\pi^{-1}(B_1)}$ and $\varphi_2^{-1} \circ w : B_2 \times V \to E|_{\pi^{-1}(B_2)}$ coincide on the closed set $(B_1 \cap B_2) \times V$. Therefore, by topological properties we can extend to a global trivialization $\varphi : B \times V \to E$, where the restrictions $\varphi|_{B_1 \times V}$ and $\varphi|_{B_2 \times V}$ are equal to φ_1^{-1} and $\varphi_2 \cdot w$ respectively. $\square$

Lemma 7.3.7. Let $\pi : E \to B \times [0,1]$ be a vector bundle over $B \times I$ with fiber V. Then there exists an open covering $\{U_i\}_{i\in[0,1]}$ of B, such that $E|_{\pi^{-1}(U_i \times [0,1])}$ is trivial.

Proof. Since E is a vector bundle over $B \times [0,1]$, for every $b \in B$ and $t \in [0,1]$ there are open neighborhoods $U(t)$ of b and $V(t)$ of t such that we find a trivialization $\varphi_t : E|_{\pi^{-1}(U(t)\times V(t))} \to (U(t) \times V(t)) \times V$. Fix $b \in B$. Notice that $[0,1]$ is a compact space, therefore exists a finite sequence of numbers $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ where for each t_i we have local trivializations $\varphi_{t_i} : E|_{\pi^{-1}(U(t_i)\times[t_{i-1},t_i])} \to (U(t_i) \times [t_{i-1},t_i]) \times V$, for open neighborhoods $U(t_i)$ of $b \in B$. Applying the preceding lemma inductively on the open set $U = \cap_{1\le i\le n} U(t_i)$ we prove that $E|_{\pi^{-1}(U_b\times[0,1])}$ is locally trivial. Repeating this step for every $b \in B$ we can find an open covering $\{U_i\}_{i\in[0,1]}$ of B, such that $E|_{\pi^{-1}(U_i\times[0,1])}$ is locally trivial. $\square$

Definition 7.3.8. Let (B,τ) be a topological space and $\{B_i\}_{i\in L}$ be a locally finite open cover for B. We say that a family of continuous functions $(\eta_i)_{i\in L}$, $\eta_i : B \to [0,1]$, is an envelope of unity subordinated to B_i if $\mathrm{supp}(\eta_i) \subset B_i$ and for each $b \in B$ we get $\max_{i\in L} \eta_i(b) = 1$.

Proposition 7.3.9. Let $\pi : E \to B\times[0,1]$ be vector bundle over $B\times[0,1]$ of rank n and B a paracompact topological space. Define $r : B\times[0,1] \to B\times[0,1]$ by $r(b,t) = (b,1)$, *i.e.*, constant on the second factor. Then, there exists a map $u : E \to E$ such that $(u,r) : E \to E$ is a morphism of vector bundles and u is an isomorphism on each fibre.

Proof. By lemma 7.3.7 we can find an open cover $\{U_i\}_{i\in[0,1]}$ of B such that we have the trivializations $\varphi_i : E|_{\pi^{-1}(U_i\times[0,1])} \to (U_i\times[0,1])\times V$. Since B is assumed to be paracompact, then $\{U_i\}_{i\in[0,1]}$ is assumed to be locally finite. According to (HUSEMOLLER, 1993) there is an envelope of unity $\{\rho_i\}_{i\in[0,1]}$ subordinate to the open covering $\{U_i\}_{i\in[0,1]}$. Notice that each φ_i can be seen as a isomorphism between vector bundles over $(U_i\times[0,1])$. Denote $h_i = \varphi_i^{-1}$.

Define $(u_i,r_i) : E \to E$ as a morphism between vector bundles by putting

$$r_i(b,t) = (b, \max(\rho_i(b),t)), \text{for } (b,t) \in U_i\times[0,1], \quad \text{and}$$

$$u_i = \begin{cases} Id, \text{on } E \setminus \{\pi^{-1}(U_i\times[0,1])\} \\ u_i(\varphi_i^{-1}(b,t,x)) = h_i(b,\max(\rho_i(b),t),x), \text{on } (b,t,x) \in (U_i\times[0,1])\times V. \end{cases}$$

We now make use of set theory techniques. We can apply a well order on the set $[0,1]$, *i.e.*, we can put a total order on $[0,1]$ such that every non-empty subset S has a least element in this ordering. Notice that this is not the usual order in $\mathbb{R}$ and that this statement is equivalent to the axiom of choice.

Let $b \in B$. Since B is paracompact there is an open neighborhood $U(b)$ of b such that $U_i \cap U(b) \neq \emptyset$ only for $i \in I(b)$, where $I(b) \subset [0,1]$ is finite. Denote $I(b) = \{i_1,\dots,i_{n(b)}\}$, where $i_1 < i_2 < \cdots < i_{n(b)}$ is given by the well order. For each $b \in B$, define the composition $r = r_{i_{n(b)}} \dots r_{i_1}$ on $U(b)\times[0,1]$ and define the composition $u = u_{i_{n(b)}} \dots u_{i_1}$ on $\pi^{-1}(U(b)\times[0,1])$. The map r is well defined, since each r_i has ρ_i in its definition, with support as a subset of U_i, and $U(b) \subset \cup_{i_j} U_{i_j}$. The same reasoning proves that u is well-defined.

Let $i \notin I(b)$, then $U_i \cap U(b) = \emptyset$ and $\mathrm{supp}(\rho_i) \cap U(b) = \emptyset$, which means that $\rho_i(c) = 0$ for $c \in U(b)$. Therefore, for $i \notin I(b)$, we have

$$r_i(b,t) = (b, \max(\rho_i(b),t)) = (b, \max(0,t)) = (b,t), \quad \text{for } (b,t) \in U(b) \times [0,1], \text{ and}$$

$$\begin{aligned} u_i(h_i(b,t,x)) &= h_i(b, \max(\rho_i(b),t),x) \\ &= h_i(b, \max(0,t),x) \\ &= h_i(b,t,x), \quad \text{for points } h_i(b,t,x) \in \pi^{-1}(U(b) \times [0,1]). \end{aligned}$$

This means we can see u and r as a composition of infinite maps, where all except a finite number of maps are the identity near a point. This makes our functions r and u defined on $B \times [0,1]$ and on all vector bundle E, respectively. Moreover, since each u_i is an isomorphism on each fibre, then u is an isomorphism on each fibre. □

Corollary 7.3.10. Using the same notation and definitions of the last theorem, then there is an isomorphism $(u,r) : E|_{\pi^{-1}(B\times\{0\})} \to E|_{\pi^{-1}(B\times\{1\})}$ of vector bundles.

Proof. First of all, making the identification $B \times \{0\} = B \times \{1\} = B$ does not lose any key information on the topology of the preceding spaces. By identifying the function $r(b,0) = (b,1) = b$, we obtain from last theorem a morphism $(u,r) : E|_{\pi^{-1}(B\times\{0\})} \to E|_{\pi^{-1}(B\times\{1\})}$ of vector bundles, where u is an isomorphism on each fiber. Moreover, (u,r) is an isomorphism because of the identification $r(b,0) = (b,1) = b$ we made. □

We now proceed to the proofs of theorem 7.3.4 and corollary 7.3.5.

Proof of theorem 7.3.4. Note that every isomorphism stated here is a isomorphism between vector bundles and remember that all manifolds are paracompacts. Let $F : N \times [0,1] \to M$ be an homotopy between functions f_0 and f_1. Then $f_0^{-1}E$ and $F^{-1}(E)|_{N\times\{0\}}$ are isomorphic and $f_1^{-1}E$ and $F^{-1}(E)|_{N\times\{1\}}$ are isomorphic. By last corollary, $F^{-1}(E)|_{N\times\{0\}}$ and $F^{-1}(E)|_{N\times\{1\}}$ are isomorphic. Therefore, $f_0^{-1}E$ and $f_1^{-1}E$ are isomorphic. □

Proof of Corollary 7.3.5. Let $\pi : E \to M$ be a vector bundle over M, a contractible manifold. There are maps f and g such that $f : M \to \{x\}$ and $g : \{x\} \to M$ such that the composition $g \circ f$ is homotopic to the identity Id_M on M. Therefore, by theorem 7.3.4, E and $(g \circ f)^{-1}E$ are isomorphic. Note that $(g \circ f)^{-1}E$ is the same as $f^{-1}(g^{-1}E)$. Since $g : \{x\} \to M$, then $g^{-1}E$ is vector bundle over the point $\{x\}$, which makes this vector bundle trivial and isomorphic to $\{x\} \times V$. Since, pullback preserves trivial vector bundles, then $f^{-1}(g^{-1}E)$ is also a trivial vector bundle. Therefore, E is a trivial vector bundle. $\square$

CHAPTER

8

CHERN CLASSES

In this chapter we give an overview on the content left necessary to define Chern classes and present some of its properties. This overview guide is based entirely on (BOTT; TU, 1982).

First of all, we need to recall the compact vertical cohomology on a vector bundle, which is denoted by $H^*_{cv}(E)$.

Definition 8.0.1. Let $\pi : E \to M$ be a rank n vector bundle over a m-manifold M. We start by defining the graded algebra $\Omega^*_{cv}(E)$, which is composed by forms $\omega \in \Omega^n(E)$, such that for every compact set $K \subset M$, then $\pi^{-1}(K) \cap \operatorname{supp}(\omega)$ is compact. Notice that $\operatorname{supp}(\omega)$ might not be a compact set on E, but $\operatorname{supp}(\omega|_{\pi^{-1}(x)}) \subset \pi^{-1}(x) \cap \operatorname{supp}(\omega)$ is a compact set, *i.e.*, its support restricted into each fiber is compact. The cohomology ring of $\Omega^*_{cv}(E)$ is the set $H^*_{cv}(E)$.

Let $\omega \in \Omega^*_{cv}(E)$ and $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented trivialization for E, which is now supposed to be oriented. For an open set U_α, there are coordinate functions $x_1, \ldots, x_m$. For $E_{\pi^{-1}(U_\alpha)}$, there are fiber coordinates $t_1, \ldots, t_n$. Restricting ω to $\pi^{-1}(U_\alpha)$, denoted by ω_α, then it can be locally written as one of the two following types

$$(\pi^*\phi) f(x_1, \ldots, x_m, t_1 \ldots, t_n) dt_{i_1} \ldots dt_{i_r}, \quad \text{for } r < n, \text{ or}$$

$$(\pi^*\phi) f(x_1, \ldots, x_m, t_1, \ldots, t_n) dt_1 \ldots dt_n,$$

where $f(x,-)$ is a function of compact support for every fixed $x=(x_1,\dots,x_m)$ and $\phi\in\Omega^*(M)$.

We define a map $\pi_*:\Omega^*_{cv}(E)\to\Omega^{*-n}(M)$, called integration along the fiber, by making $\pi_*\omega_\alpha=0$, when ω_α is of the first type, and

$$\pi_*\omega_\alpha=\phi\int_{\mathbb{R}^n} f(x,t)dt_1\dots dt_n,$$

when ω_α is of the second type.

This map is well-defined. Indeed, for forms of the first type is immediate to see that the map is well-defined. For forms of second type, since $\operatorname{supp}(f(x,-))$ is compact for each x, then $\int_{\mathbb{R}^n} f(x,t)dt_1\dots dt_n<\infty$. Moreover, the form

$$(\pi^*\phi)f(x_1,\dots,x_m,t_1,\dots,t_n)dt_1\wedge\dots\wedge dt_n$$

turns into a form with degree subtracted by n, then $\phi\int_{\mathbb{R}^n} f(x,t)dt_1\dots dt_n\in\Omega^{*-n}(M)$, since now it depends only from the point x.

For another open set U_β, then the coordinates related to each ω_α and ω_β can be related by an element of $GL^+(n,\mathbb{R})$, then $\pi_*\omega_\alpha=\pi_*\omega_\beta$. Also, the operator π_* commutes with the exterior derivative.

An important fact about the compact vertical cohomology is the Thom isomorphism. This theorem is what makes possible for us to define the Euler class for vector bundles of rank 2 and, then, the first Chern class. This isomorphism can be seen as a Poincaré Duality, but now we are working with the compact vertical cohomology.

Theorem 8.0.2 (Thom Isomorphism)**.** Let $\pi:E\to M$ be a vector bundle of rank n over M. If E is oriented, then

$$H^*_{cv}(E)\simeq H^{*-n}(M).$$

In order to prove the Thom isomorphism for a manifold M of finite type we establish the following Mayer-Vietoris sequence from the graded algebras

$$0\to\Omega^*_{cv}(E|_{U\cup V})\to\Omega^*_{cv}(E|_U)\oplus\Omega^*_{cv}(E|_V)\to\Omega^*_{cv}(E|_{U\cap V})\to 0.$$

From this sequence we establish the following diagram, using the induced map from the integration along the fiber π_* on the cohomology level

$$\begin{tikzcd}
H^*(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"'] & H^*(E|_U)\oplus H^*(E|_V) \arrow[r] \arrow[d, "\pi_*"'] & H^*(E|_{U\cap V}) \arrow[r] \arrow[d, "\pi_*"'] & H^{*+1}(E|_{U\cup V}) \arrow[d, "\pi_*"'] \\
H^{*-n}(U\cup V) \arrow[r] & H^{*-n}(U)\oplus H^{*-n}(V) \arrow[r] & H^{*-n}(U\cap V) \arrow[r] & H^{*+1-n}(U\cup V)
\end{tikzcd}$$

The rest of the proof follows the same reasoning applied for the finite case of Poincaré's Duality. The general case uses arguments from Riemannian metrics and presheaves theory, where the latter was not introduced in this work.

Definition 8.0.3. Let $\mathscr{T} : H^{*-n}(M) \to H^*_{cv}(E)$ be the Thom Isomorphism in the last theorem. Let $[1] \in H^0(M)$ be the cohomology class given by the constant function 1 on M, then we define $\mathscr{T}([1]) \in H^n_{cv}(E)$ to be the Thom class of the oriented vector bundle E.

Now, we take the next step, which is to define the Euler class on an oriented vector bundle of rank 2, in order to define the first chern class. Therefore, let $\pi : E \to M$ be a vector bundle of rank 2 over M and $\{U_\alpha\}$ a coordinate open cover of M trivializing E. A differential form of maximum degree on an oriented manifold is called positive if it is in the chosen orientation class of the manifold. If σ is the positive differential form that generates the cohomology group $H^{n-1}(S^{n-1})$, then the form $\psi = \mu^*\sigma$ is the angular form on $\mathbb{R}^n \setminus \{0\}$, where $\mu : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the deformation retraction given by $\dfrac{x}{||x||}$.

Let there be a Riemannian structure on E. Consider E^0 the complement of the zero section of E. There is a global angular form ψ on E^0, where ψ restricted to each fiber is the angular form on $\mathbb{R}^n \setminus \{0\}$. This form gives us the Thom class by making $d(\rho \cdot \psi)$, where ρ is the radius function on E. In our particular case ψ is found from the construction of the Euler class.

We start by considering an orthonormal frame over each U_α. From this we are able to define polar coordinates r_α and θ_α on $E^0|_{\pi^{-1}(U_\alpha)}$. Notice that for coordinates $x_1,\dots,x_n$ we have that $\pi^*x_1,\dots,\pi^*x_n, r_\alpha, \theta_\alpha$ are coordinates on $E^0|_{\pi^{-1}(U_\alpha)}$. Between two different open sets , U_α and U_β, the radius r_α and r_β coincide on $U_\alpha \cap U_\beta$, however θ_α and θ_β differ by a rotation. The 0-form that is defined as the angle rotation in the counterclockwise direction from α-coordinate system to a β-coordinate system is denoted by $\varphi_{\alpha\beta}$. We make

$$\theta_\beta = \theta_\alpha + \pi^*\varphi_{\alpha\beta}, \quad \text{for } \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}.$$

For each $\varphi_{\alpha\beta}$ there are 1-forms ξ_α on U_α such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

We define the Euler class as a 2-form on M, denoted by e, which is a global form on M by glueing all $d\xi_\alpha$, since for any α, β, $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$. Note that e is a closed form and can be often denoted by $e(E)$. This class is functorial, which means that for $f : N \to M$, a C^∞ map, then $e(f^{-1}E) = f^*e(E)$. The global angular form on E^0 satisfies $d\psi = -\pi^*e$, where ψ is given by the glueing of $\dfrac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha$.

Remark 8.0.4. In order to see this construction in more detail and with more geometric intuition check (BOTT; TU, 1982), pages $70 - 74$.

Definition 8.0.5. Let M be a manifold. We define the first Chern class, denoted as c_1, of a complex line bundle L over M as the Euler class of the underlying real bundle $L_\mathbb{R}$, *i.e.*, $c_1(L) = e(L_\mathbb{R}) \in H^2(M)$.

Notice that by proposition 7.2.3 every complex vector bundle is oriented as a real vector bundle, then last definition is well-defined and we can apply the Euler class in $L_\mathbb{R}$.

Let L and L' two complex line bundles. Then $c_1(L \otimes L') = c_1(L) + c_1(L')$. We will assume this last equality, but it can be proved by using a relation between the transition functions of these complex line bundles, since we can write the Euler class of a real vector bundle of rank 2 using its transition functions. This expression can be found in (BOTT; TU, 1982) in the end of page 73. Also, if L^* is the dual space of L, then $L^* \otimes L = Hom(L, L)$, which has a nowhere vanishing given by the identity map. Therefore, $L^* \otimes L$ is a trivial bundle and its first Chern class is zero, then $c_1(L^*) = -c_1(L)$.

Let $\rho : E \to M$ be a complex vector bundle and let $\pi : P(E) \to M$ be its projectivization. From now on, we work with some vector bundles on $P(E)$. First of all, the pullback bundle $\pi^{-1}E$, with fiber E_p over each point l_p. Also, we work with the universal subbundle $S = \{(l_p, v) \in \pi^{-1}E | v \in l_p\}$, where the fiber over each point l_p consists of all the points of l_p. If S^* is the dual bundle of S, then set $x = c_1(S^*)$.

The element x is a cohomology class in $H^2(P(E))$. The restriction of S to a fiber $P(E_p)$ is the universal subbundle $\overline{S}$ of $P(E_p)$. Then $c_1(\overline{S})$ is the restriction of $-x = c_1(S)$ to $P(E_p)$ by the functorial property of the Euler class $e(f^{-1}S_{\mathbb{R}}) = f^*e(S_{\mathbb{R}})$, where f is the inclusion $i : P(E_p) \hookrightarrow P(E)$. Therefore, the cohomology classes $1, x, \dots, x^{n-1}$ are global classes on $P(E)$, whose restriction to each fiber $P(E_p)$ freely generate the cohomology of the fiber. Using the Leray-Hirsch theorem we are able to see that the cohomology $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1, x, \dots, x^{n-1}\}$. This means that the class x^n can be written as unique linear combination of this basis with coefficients in $H^*(M)$. We call these coefficients the Chern classes of the complex vector bundle E, satisfying the equation

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0, \quad \text{for } c_i(E) \in H^{2i}(M), \tag{8.1}$$

where $c_i(E)$, the i-th Chern class of E, stands for $\pi^*c_i(E)$, the pullback of the coefficients from $H^{2i}(M)$. Moreover, we define the total Chern class, given by

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M).$$

Now we got two definitions of the first Chern class. Denote $c_1(-)$ as the definition coming from the Leray-Hirsch theorem and $e(-)$ the definition from the Euler class. Let L be a complex line bundle. We have that the projectivization and the pullback of L are M and L, respectively. Also, the universal sub-bundle S on $P(L)$ is L. Therefore, $x = e(S^*_{\mathbb{R}}) = -e(S_{\mathbb{R}}) = -e(L_{\mathbb{R}}) \Rightarrow x + e(L_{\mathbb{R}}) = 0$. Then from the coefficients uniqueness of equation (8.1) we get that $c_1(L) = e(L_{\mathbb{R}})$. Also, for a trivial bundle $E = M \times V$, then $P(E) = M \times P(V)$, which implies. $x^n = 0$. This means that $c_i(E) = 0$ for all i, since x^n is an unique combination of the basis $\{1, x, \dots, x^{n-1}\}$ where coefficients are $c_i(E)$.

Proposition 8.0.6. The Chern classes have the following important properties

- Let E be a complex vector bundle of rank n, then $c_i(E) = 0$ for $i > n$;
- Let E be a complex vector bundle of rank n, then $c_0(E) = 1$;
- On trivial bundles all Chern classes $c_i(E)$, $i > 0$, are zero;

- (Naturality) Let $f : Y \to X$ be a map and E a complex vector bundle over X, then $c(f^{-1}E) = f^*c(E)$;
- Let S^* be the dual bundle of the universal sub-bundle $S \subset P(V^n) \times V^n$, then $c_1(S^*)$ generates $H^*(P(V^n))$;
- (Whitney Product Formula) If E and E' are complex vector bundles over M, then $c(E \oplus E') = c(E)c(E')$;
- If E is a complex vector bundle of rank n and E has a non-vanishing section, then $c_n(E) = 0$.
- If E is a complex vector bundle of rank n, then $c_n(E) = e(E_{\mathbb{R}})$.

We give an outline on the ideas for most properties. We start by stating that the first two properties are considered to be definitions.

The naturality comes from the properties of pullback bundle $f^{-1}P(E) = P(f^{-1}E)$ and $f^{-1}S_E^* = S_{f^{-1}E}^*$, where S_X is the universal sub-bundle over $P(X)$.

The Whitney product formula derives from the Splitting principle, which states that for a complex vector bundle $\pi : E \to M$ of rank n, there is a manifold $F(E)$ and a map $\sigma : F(E) \to M$, such that the pullback bundle $\sigma^{-1}E = L_1 \oplus \cdots \oplus L_n$ is a complex vector bundle over $F(E)$ and we have the embedding $\sigma^* : H^*(M) \hookrightarrow H^*(F(E))$.

In order to prove that $c_n(E) = 0$ whenever $\rho : E \to X$ is a complex vector bundle of rank n with a non-vanishing section, we must take the mentioned section s from the hypothesis and induce a section $\bar{s}$ on $P(E)$ by doing $\bar{s} : X \to P(E)$ where $\bar{s}(p) = l_{s(p)} \subset E_p$. We then take the line bundle $\bar{s}^{-1}S_E$ over X. Its fibers at points p are $\bar{s}^{-1}E_p = l_{s(p)} \subset E_p$. The section $\bar{s}$ is a non-vanishing section on the line bundle $\bar{s}^{-1}S_E$. Then $\bar{s}^{-1}S_E$ is isomorphic to the trivial line bundle $X \times \mathbb{C}$. From the naturality we get that $\bar{s}^*c_1(S_E) = 0$ and $\bar{s}^*c_1(S_E^*) = \bar{s}^*x = 0$. Applying the pullback $\bar{s}^*$ on the Chern polynomial we get

$$\bar{s}^*(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)) = 0 \Longrightarrow \bar{s}^*x^n + \bar{s}^*c_1(E)\bar{s}^*x^{n-1} + \cdots + \bar{s}^*c_n(E) = 0.$$

This gives us $\bar{s}^*c_n(E) = 0$, which really means $(\bar{s}^* \circ \pi^*)c_n(E) = c_n(E) = 0$.

The last property follows by Splitting principle on the sequence of equalities

$$\begin{aligned}
\sigma^* c_n(E) &= c_n(\sigma^{-1}E) \\
&= c_1(L_1)\cdots c_1(L_n) \\
&= e((L_1)_{\mathbb{R}})\cdots e((L_n)_{\mathbb{R}}) \\
&= e((L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}}) \\
&= e((\sigma^{-1}E)_{\mathbb{R}}) \\
&= \sigma^* e(E_{\mathbb{R}})
\end{aligned}$$

The conclusion follows from the injectivity of σ^*.

Definition 8.0.7. A set M is said to be a complex manifold if it satisfies all definitions and has all the same properties of a smooth manifold with the exception that the local charts for M are holomorphic functions $\varphi_\alpha : U_\alpha \to \varphi(U) \subset \mathbb{C}^n$ for an open cover $\{U_\alpha\}_{\alpha\in A}$ for M.

Example 8.0.8. The complex projective space $P(\mathbb{C}^n)$ is assumed to be a complex manifold for this example. Let $E = P(\mathbb{C}^{n+1}) \times \mathbb{C}^{n+1}$ be a trivial complex vector bundle of rank $n+1$ over $P(\mathbb{C}^{n+1})$. We have the tautological sequence

$$0 \to S \to E \to Q \to 0. \tag{8.2}$$

We may represent the tangent bundle $TP(\mathbb{C}^{n+1})$ over $P(\mathbb{C}^{n+1})$ as

$$TP(\mathbb{C}^{n+1}) \simeq Hom(S,Q) = Q \otimes S^*.$$

We can tensor (8.2) with S^*, which provide us

$$0 \to S^* \otimes S \to S^* \otimes E \to S^* \otimes Q \to 0.$$

The bundle $S^* \otimes S$ remains to be a line bundle. Then, by making $x = c_1(S^*)$, we get

$$\begin{aligned}
c(P(\mathbb{C}^{n+1})) &= c(TP(\mathbb{C}^{n+1})) = c(S^* \otimes Q) \\
&= c(S^* \otimes E) = c(S^* \oplus \cdots \oplus S^*) \\
&= c(S^*)^{n+1} = (c_0(S^*) + c_1(S^*))^{n+1} \\
&= (1+x)^{n+1}
\end{aligned}$$

Example 8.0.9. For a vector bundle $E = L_1 \oplus \cdots \oplus L_n$, where each L_i is a line bundle, we define

$$\Lambda^p E = \bigoplus_{1 \le i_1 < \cdots < i_p \le n} (L_{i_1} \otimes \cdots \otimes L_{i_p}).$$

Using the Whitney product formula we get

$$\begin{aligned} c(\Lambda^p E) &= c\Big(\bigoplus_{1 \le i_1 < \cdots < i_p \le n} (L_{i_1} \otimes \cdots \otimes L_{i_p}) \Big) \\ &= \prod_{1 \le i_1 < \cdots < i_p \le n} c(L_{i_1} \otimes \cdots \otimes L_{i_p}) \\ &= \prod_{1 \le i_1 < \cdots < i_p \le n} (c_0(L_{i_1} \otimes \cdots \otimes L_{i_p}) + c_1(L_{i_1} \otimes \cdots \otimes L_{i_p})) \\ &= \prod_{1 \le i_1 < \cdots < i_p \le n} (1 + x_{i_1} + \cdots + x_{i_p}) \end{aligned}$$

where $x_i = c_1(L_i)$ and for line bundles we may assume $c_1(L \otimes L') = c_1(L) + c_1(L')$.

Example 8.0.10. Let L be a complex line bundle. The tensor bundle $L \otimes L^*$ is a line bundle with a non-vanishing section given by the identity map, then

$$c_1(L \otimes L^*) = 0 \Rightarrow c_1(L^*) = -c_1(L) \tag{8.3}$$

If $E = L_1 \oplus \cdots \oplus L_n$ and each $c(L_i) = (1 + c_1(L_i))$, then by the Whitney product formula we get

$$c(E) = \prod_{1 \le i \le n} (1 + c_1(L_i)).$$

Also, the dual bundle is given by $E^* = L_1^* \oplus \cdots \oplus L_n^*$. By (8.3) we get

$$c(E^*) = \prod_{1 \le i \le n} (1 - c_1(L_i)).$$

Then $c_q(E^*) = (-1)^q c_q(E)$.

CHAPTER

9

CONCLUSION

Throughout this work, we approach a great amount of mathematical concepts from many different fields of mathematics, passing through analysis, topology, algebra and geometry, in order to achieve the main goal of these studies, *i.e.*, understand the mathematics needed to have a basic comprehension of the Chern classes construction using differential topology. The topics studied are smooth manifolds, differential forms, which includes integration and Stokes theorem, de Rham cohomology and Vector bundles. Moreover, we have presented some very important results in this text, such as the Poincaré Duality, the Künneth formula and the Homotopy Property of Vector Bundles. Finally, in chapter 8, we reach our objective, which is to present a simplified introduction for the construction of Chern classes, starting by the geometric construction of the first Chern class and, then, generalizing it with Leray-Hirsch theorem.

Furthermore, we notice that in order to comprehend in full extent the concept of Chern classes using differential topology one would still need to look in more detail to Cech cohomology, generalizations of Mayer-Vietoris and Künneth formula theorem, more profound category theory, such as sheaves and presheaves, and Thom Isomorphism. Also, Chern classes applications are many and important to mathematics and physics, since it provides us key topological information about complex vector bundles in the following sense: let be two complex vector bundles with different Chern classes, then they are different vector bundles. Ultimately, this work may be seen as an introduction for the field of Chern classes using differential topology, which purpose is to give the most complete as possible and coherent basis for the basics of this subject.

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