UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Integrable Systems and Partition Functions of Random Matrix Models

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Dissertação de Mestrado do Programa de Pós-Graduação em Matemática (PPG-Mat)

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Carla Mariana da Silva Pinheiro

Integrable Systems and Partition Functions of Random Matrix Models

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Sistemas Integráveis e Funções Partição de Modelos de Matrizes Aleatórias

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestra em Ciências – Matemática. *VERSÃO REVISADA*

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"Fazer pesquisa é como observar pássaros.
Estudar artigos é adentrar uma floresta em busca
de algum espécime novo, colorido e empolgante.
São horas desbravando as páginas de outros trabalhos
na esperança de encontrar uma frase, uma imagem,
alguma coisa que você ainda não sabe o que é.
E, de repente, você liga os pontos e ganha uma
resposta. Às vezes, é apenas um susto, apenas um
pardal que de um certo ângulo parecia um pássaro
novo. Mas às vezes, bem às vezes, é tudo
o que você precisa para o próximo passo."
(Carla Silva)

RESUMO

PINHEIRO, C. M. S. **Sistemas Integráveis e Funções Partição de Modelos de Matrizes Aleatórias**. 2022. 123 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

A Teoria de Matrizes Aleatórias é um tópico bem atual devido à sua ampla gama de aplicações em diferentes áreas, como mecânica quântica, aprendizado de máquinas, sistemas dinâmicos, entre outros. O presente trabalho começa com algumas das aplicações mais conhecidas. Em seguida, dá-se especial atenção à enumeração de mapas através da esperança do traço de matrizes aleatórias em um Ensemble Gaussiano Unitário. Posteriormente, desenvolve-se uma expansão assintótica da função partição, o que permite contar mapas através da conexão entre a esperança do traço e a função partição. Tal expansão é explorada em detalhes e os cálculos envolvendo um importante problema de Riemann-Hilbert são explicitamente elaborados. Por fim, conexões entre matrizes aleatórias e sistemas integráveis são abordadas de dois modos diferentes. Quando a dimensão das matrizes é fixa, a função partição de um modelo de matrizes aleatórias é uma função tau da hierarquia KP, enquanto que no limite em que a dimensão vai para o infinito recupera-se soluções de equações de Painlevé.

Palavras-chave: Sistemas Integráveis, Matrizes Aleatórias, Função Partição, Problemas de Riemann-Hilbert, Equações de Painlevé.

ABSTRACT

PINHEIRO, C. M. S. **Integrable Systems and Partition Functions of Random Matrix Models**. 2022. 123 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Random Matrix Theory is a hot topic nowadays given its wide range of applications in different areas, such as quantum mechanics, machine learning, dynamical systems, among others. The present work begins reviewing some of the most famous applications. Then, particular attention is given to the enumeration of maps through the expectation of the trace of random matrices in a Gaussian Unitary Ensemble. Latter, an asymptotic expansion of the partition function is developed, which allows one to count maps by the connection between the expectation of the trace and the partition function. Such expansion is explored in full details and the calculations involving an important Riemann-Hilbert problem are explicitly worked out. At last, connections between random matrices and integrable systems are explored in two different ways. When the dimension is fixed, the partition function of a random matrix model is a tau-function of the KP hierarchy, while in the limit where the dimension goes to infinity one recovers Painlevé solutions.

Keywords: Integrable Systems, Random Matrices, Partition Function, Riemann-Hilbert Problems, Painlevé Equations.

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1

INTRODUCTION

Random Matrix Theory made a modest debut in mathematical statistics in the 1930s, with the work of Hsu and Wishart. However, it only attracted considerable attention after the discovery of a connection between nuclear physics and eigenvalues statistics by Wigner in the 1950s.

Since then, a large number of applications in physics and mathematics were discovered, what motivated the study of different aspects of such theory. Random matrices appeared in the characterization of chaotic systems, in the studies on conductivity of disordered metals, quantum gravity and string theory, on the theory of sound waves in quartz crystals and in the distribution of values of Riemann zeta function, to mention only a few (MEHTA, 2004).

The theory of the Painlevé equations has also born in the early 1900s, but only attracted a substantial attention after the discovery of applications in physics in the second half of the 20th century. In an attempt to classify all of the second order ordinary differential equations with some special properties, Paul Painlevé elaborated a list of equations that included the six so-called canonical Painlevé equations (Equations (5.2)-(5.7)). Such objects came together with random matrices when, in the 1980s, studies on the behaviour of eigenvalues led to unexpected connections to some integrable hierarchies of Painlevé II and V (see (TRACY; WIDOM, 2002) and (JIMBO *et al.*, 1980)). By integrable hierarchies we mean a generalization of integrable systems, that is, systems with enough conservated quantities. In the last decade solutions to the Painlevé XXXIV were also found in the context of matrix models (ITS; KUIJLAARS; ÖSTENSSON, 2008).

In the 1960s the Feymann path integral developed as a tool for quantum field theory gave rise to questions about the enumeration of the so-called Feymann diagrams. A diagram is a particular case of a map, and the latter is defined as a graph embedded to a compact surface with some special features. Later, the Feymann path integral was

identified as a matrix integral, and the problem of map enumeration became a problem of random matrix models (SCHWINGER, 1958).

The two latter topics we mentioned, namely integrable systems and map enumeration, do not seem to be connected at all at the first sight. However, a great deal of this text is devoted to explaining how they both relate to random matrices, the central object of this dissertation.

The Chapter 2 is probably the most random of them all. It begins with the definition of probability, passes through random variables and end up defining a *random matrix*. After defining one of the most important objects in this text, the linearity is thrown out the window and we present briefly a set of applications not necessarily ordered by importance or date of discover. Despite that may be a little bit random, it has a noble purpose. For those who find the concept of random matrices kind of abstract, it brings some light on its connection with concrete problems in an attempt to motivate them. In the last section we provide some tools for the reader interested in getting to know random matrices better.

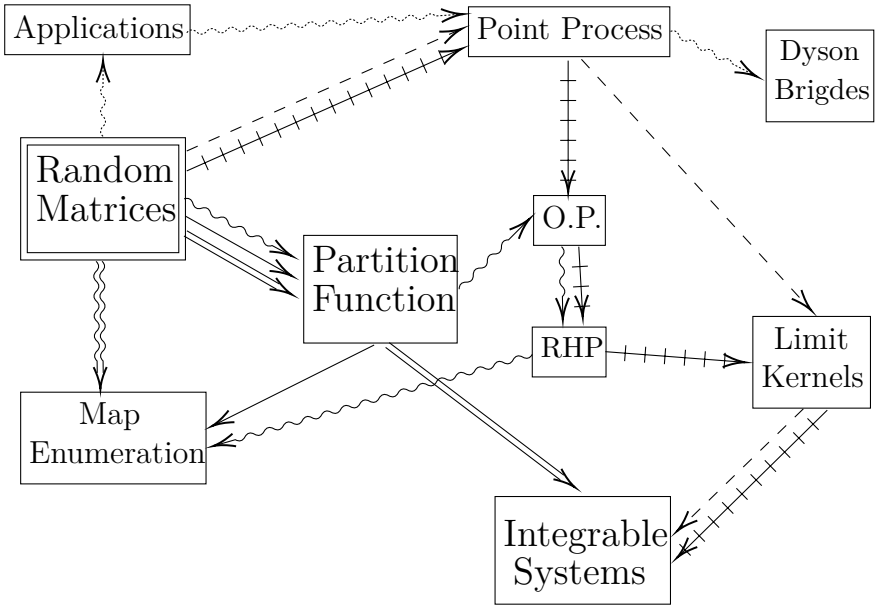
Chapter 3 is all about random matrices. We start with a section defining a Gaussian measure in a space of hermitian matrices. Once the measure is built, we take the Gaussian expectation of the trace of a matrix. In the following section we explore graphs and maps and establish a relation between map enumeration and the expectation of the trace in a random matrices ensemble.

The Chapter 4 is probably the most scary to the unwary. It is where one can find the most technical calculations on this dissertation, and (or but) the deepest results. But make no mistake: it is also the most linear one. It starts with the definition of a partition function and a result about its asymptotic behaviour. In the following section we explain why the partition function is important: because we can recover the expectation of the trace of a random matrix from it. But, as one may recall from the previous chapter, this quantity relates directly to the enumeration of maps. So, once we know the asymptotic expansion for the partition function we can, with the help of the trace of a matrix, count maps. The last sections are devoted to the proof of the asymptotic expansion claimed in the first section.

The Chapter 5 starts with a brief introduction to integrable systems. Then, the connection between random matrices and integrable hierarchies is explored in two different ways. At first, we see how the partition function of a random matrix model relates to the KP hierarchy. Then, in the following section we detail the behaviour of eigenvalues for matrices of large dimension, which leads to some results involving Painlevé II and V. Then, we have some final comments in Chapter 6 and some Python codes in the Appendix.

In case one gets lost between the lines of a proof or maybe gets distracted by a

Figure 1 – Diagram of the connections between the main topics of the text.



Source: Elaborated by the author.

fascinating detail, the guide below might be helpful to get you back on track. Enjoy the journey!

1.1 User guide

The section 2.1 is recommended to everyone. It is where, starting from the definition of random variables, we establish the concept of *random matrices*. Moreover, section 2.8 brings an overview of the rest of the dissertation. After that, there are some choices of paths depending on the topics in which the reader is most interested in.

- *Applications* trail:

  ⇝ If your main interest is in discovering some applications of Random Matrix Theory, the first sections of Chapter 2 are a good option. The section 2.2 shows an application of random matrices in neutron ressonance theory and the section 2.3 illustrate the Wigner semicircle law. The section 2.4 brings the Marchenko-Pastur distribution and the section 2.5 presents an application of such distribution in Big Data. Moreover, the section 2.6 introduces the concept of Point Process, culminating in the construction of Dyson brigdes in the subsection 2.6.2, which appears in quantum chaos in the subsection 2.6.3.

- *Connection between Random Matrices and Integrable Systems...*

  - *... through the partition function:*

    ⟹ Start with some definitions in integrable systems theory in section 5.1.

Then, explore the connection between partition functions and KP hierarchy in section 5.2.

- *... through the limit kernel:*
  - - - - ➢ If you do not want to give a chance to the beauty of Riemann-Hilbert Problems, start with point process in the section 2.6 and gap probability in the subsection 2.7.1. Read briefly the definition of some important auxiliary function in subsection 4.3.1. Accept Theorem 10 and Equations (4.67)-(4.68), concerning to the formula for asymptotics of orthogonal polynomials. Then, after the introduction to integrable systems in the section 5.1, see the development of limit kernels in the section 5.3.

  ++++➢ For a full experience, start with point process in the section 2.6 and the subsection 2.7.1. Then, go to the asymptotics for orthogonal polynomials in the section 4.3. Enjoy the details in the Riemann-Hilbert construction of such asymptotics, and once you reach the final expression for the polynomials in the subsection 4.3.6, go to the introduction to integrable systems in the section 5.1 and, finally, to the section 5.3 of limit kernels.

- *Connection between Random Matrices and Map Enumeration...*
  - *... purely:*
    ≈≈≈≈➢ The Chapter 3 is enough to provide an overview of the relation between the expectation of the trace in random matrix models and the enumeration of g-maps.
  - *... through the Partition Function:*
    ——➢ After the contact with map enumeration in Chapter 3, explore the introduction of Chapter 4 and the section 4.1.
  - *... through the Partition Function and RHP:*
    ~~~➢ For the ones brave enough to face (almost) all the technical details, enjoy Chapter 3 and Chapter 4.
    ~~~

CHAPTER

2

RANDOM MATRICES

The approach throughout the present chapter is rather superficial, as the focus is on motivating our later work. Do not expect rigorous calculations nor proofs, but a simple overview of the main applications developed in almost a century of Random Matrix Theory.

2.1 Introduction

Probability is a common and natural concept in everyday life. You wake up, turn on the television and hear about "the 50 percent chance of rain", "the probabilities in election" and "the chance to win the lottery". Before commuting to work you try to do some predictions about other probabilistic events. What is the chance that you are going to be late because of traffic?

Nature is plenty of non-deterministic events, and the formal study of them gave rise to the *Probability Theory*. The essential building blocks of probability theory are the so called *random variables*, i.e., functions that assign to the result of a probabilistic event a quantity which can be treated mathematically. But before going into more details about random variables we need to define a probability space.

Given a probabilistic experiment, like roll dices or tosses of a coin, one can think of the probability as the ratio between the frequency at which each possible result occurs and the number of experiments. The set of all possible results is called *sample space* ($\mathbf{\Omega}$). The subsets of $\mathbf{\Omega}$ for which one can assign a probability are called *events*.

By means of Measure Theory, a probability is a finite and normalized measure in the sigma-algebra of the events. Analogously to a measure space, we define a **probability space** by a triple $(\mathbf{\Omega}, \mathcal{A}, P)$, where $\mathbf{\Omega}$ is the sample space, $\mathcal{A}$ is the *sigma-algebra* of the measurable events and P is the probability measure.

The sigma-algebra $\mathcal{A}$ has the three following properties:

i. $\emptyset \in \mathcal{A}$, that is, the impossible event $A = \emptyset$, is measurable;

ii. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;

iii. If $\{A_i\}_{i=1}^{\infty} \in \mathcal{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$.

It holds that $\mathcal{A} \subset \wp(\Omega)$ but the inverse inclusion usually does not hold. Now the probability P is, mathematically, a function $P : \mathcal{A} \mapsto [0,1]$ such that

i. $P(A) \geq 0$ for all $A \in \mathcal{A}$;

ii. $P(\Omega) = 1$;

iii. For a sequence of mutually exclusive events $\{A_i\}_{i=1}^{\infty}$ it follows that

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

For example, if a fair coin is tossed once the probability space is characterized by $\Omega = \{C, K\}$, $\mathcal{A} = \{\emptyset, \{C\}, \{K\}, \Omega\}$ and $P[\{C\}] = \frac{1}{2}$, where C corresponds to the event of getting a head and K to the event of getting a tail.

A random variable is a numeric description of a probabilistic event. Some examples are the number of heads in n tosses of a coin or the sum of the faces obtained when two dice are rolled. Formally, a random variable in a probability space is a real function

$$\begin{array}{rccc} X: & \Omega & \to & \mathbb{R} \\ & w & \mapsto & X(w), \end{array} \tag{2.1}$$

such that for all $x \in \mathbb{R}$

$$(X \leq x) := \{w \in \Omega : X(w) \leq x\} \in \mathcal{A}.$$

A quantity of interest when dealing with random variables is its *distribution* $F : \mathbb{R} \to [0,1]$ defined by

$$F(x) := P(\{w \in \Omega : X(w) \leq x\}). \tag{2.2}$$

The two most common types of random variables are the *discrete* ones, whose range is a countable set, and the *continuous* ones, whose distribution function may be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u, \tag{2.3}$$

for some probability density function f. Some famous discrete distributions are the Binomial, the Bernoulli and the Poisson distributions. The continuous ones include the Normal

distribution $N(m,\sigma)$ with mean m and standard deviation σ, whose probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right). \tag{2.4}$$

Instead of a single random variable, one could choose to work with a list of random variables, which is known as a *random vector*. Moreover, one could deal with a "list of lists" of random variables, which is called *random matrix*. So, for now on we refer to a *random matrix* as a matrix whose entries are random variables. A natural question that arises is about the behaviour of the eigenvalues of a random matrix. Recall that the eigenvalues of a complex matrix M are the elements $\lambda \in \mathbb{C}$ for which

$$\det(M - \lambda I) = 0.$$

Despite its simple formulation, the pertinence of such question is easily justified by its relation to a wide range of applications, as illustrated by the next sections.

2.2 Neutron resonance

At a microscopic scale the interactions between atoms, electrons, photons and neutrons lead to important phenomena. Take, for example, the interaction between neutrons and atoms.

Since neutrons have no charge and, consequently, no Coulomb interaction with the atom's electrosphere, they can reach the nucleus more easily than electrons and photons. However, the occurrence of an effective interaction is a probabilistic event which depends on two non-deterministic quantities, namely the energy levels of the atom and the energy of the neutron.

We divide these interactions into *fast reactions*, in which a neutron is absorbed and another is rejected with almost the same energy, and the *slow reactions*, in which the interaction takes longer and the energy is distributed among more constituents of the nucleus, giving rise to several effects. In the 1950s some physicists studied the relation between the statistical behaviour of slow neutron resonances in nuclear interactions and random matrices (MEHTA, 2004).

The dependence of the interaction on the energy of the incoming neutron gives rise to a probability density called *cross section*. The cross section σ_n is usually decomposed into functions known as partial cross sections as follows

$$\sigma_n = \sigma_a + \sigma_s + \sigma_f + \sigma_p + \sigma_\gamma,$$

where σ_a is the contribution due to neutron absorption by the nucleus, σ_s is the contribution of scattering events, σ_f is the fission cross section, σ_p is the contribution of proton

emission and σ_γ is the neutron radiative cross section. Each of such functions refers to one of the most important interactions between a nucleus and a neutron (MUGHABGHAB; GARBER, 1973).

The experimental data related to the dependence of the cross section on the energy of the neutron reveals the presence of peaks of resonance due to the excited states. In short, the kinetic energy of the neutron and the binding energy of the nucleus combine and originate an excitation energy called energy of resonance, which is observed by the presence of peaks. Due to the random character of these peaks, exact predictions cannot be obtained.

As the energy of the neutron increases, the level distance decreases, resonance in cross sections appears and all we can observe is a continuum. Although the states become dense we still can evaluate some average properties like its distribution.

Recall that in Quantum Mechanics the energy levels of a system are the eigenvalues of the Hamiltonian (an Hermitian operator) in an infinite dimensional Hilbert space. Usually we approximate the problem by a large but finite dimensional space, choose a basis and solve the equation

$$H\Psi_i = E_i\Psi_i,$$

where E_i are the energy levels and Ψ_i are the eigenfunctions (wave functions). Since the entries of H are random variables, due to the random nature of the interactions and the symmetry properties of the problem, Wigner proposed that the statistical distribution of the energy levels is identical to the distribution of the eigenvalues of a random matrix (MEHTA, 2004). In fact, the behaviour observed experimentally suggested a distribution of levels given by

$$\rho(s)\,\mathrm{d}s = c_\beta s^\beta \,\mathrm{e}^{-\alpha s^2}\,\mathrm{d}s,$$

where c_β is the normalization constant and the so-called *Dyson index* $\beta \in \{1,2,4\}$ is determined by nucleus symmetries. Such distribution is the same that appears for the eigenvalues of random matrix ensembles. Later, Monte Carlo analysis carried out by Porter and Rosenzweig (1960) corroborated Wigner hypothesis.

2.3 Semicircle law

In this section we explore one universality distribution law for eigenvalues.

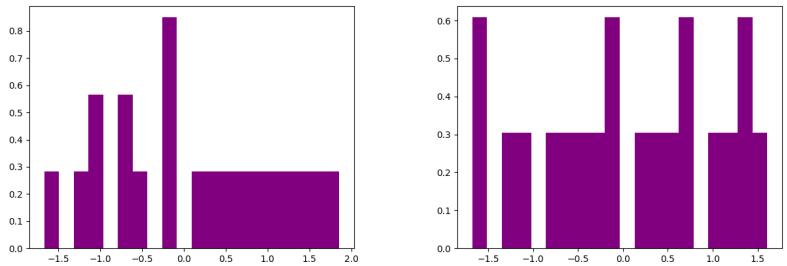
We start with a random selfadjoint matrix $M = (m_{ij})_{i,j=1}^n$. One could take, for example, $m_{ij} \in \{1,-1\}$ being independent and identically distributed entries. The main question is how the eigenvalues behave as $n \to \infty$. With a normalization by $\frac{1}{\sqrt{n}}$, the elements of our space of matrices can be computed explicitly for n small. Some of the matrices

obtained for $n = 1, 2$ are listed below.

Dimension	*Matrices*	*Eigenvalues*	*Probability*
$n = 1$	(1)	$+1$	$1/2$
	(-1)	-1	$1/2$
$n = 2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$0, \sqrt{2}$	$1/8$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$0, \sqrt{2}$	$1/8$
	$\dots$	$\dots$	$\dots$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$1, -1$	$1/8$

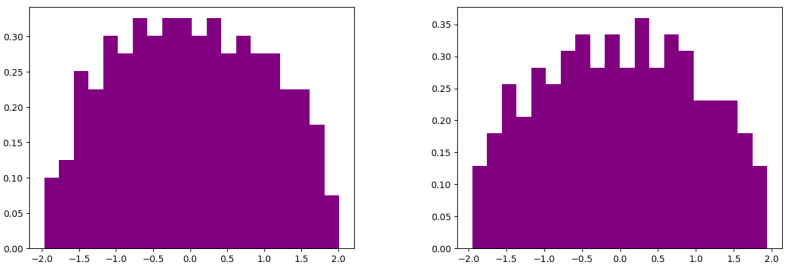
For each fixed n one can construct $2^{\frac{n(n+1)}{2}}$ matrices satisfying the conditions above. As the dimension of the matrices increases, the behaviour of the eigenvalues becomes more regular, see Figures 2-4.

Figure 2 – Histogram for eigenvalues of two 20×20 random matrices with entries $m_{ij} \in \{-1, 1\}$.



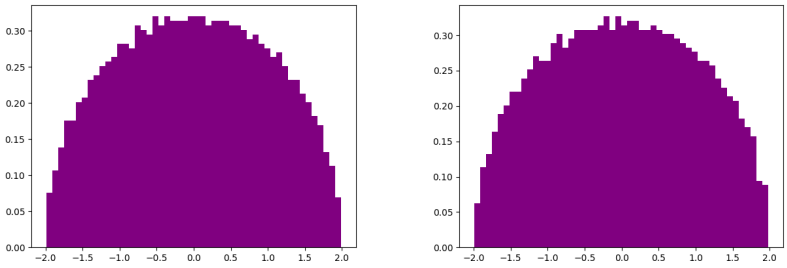
Source: Elaborated by the author.

Figure 3 – Histogram for eigenvalues of two 200×200 random matrices with entries $m_{ij} \in \{-1, 1\}$.



Source: Elaborated by the author.

Figure 4 – Histogram for eigenvalues of two 2000×2000 random matrices with entries $m_{ij} \in \{-1, 1\}$.



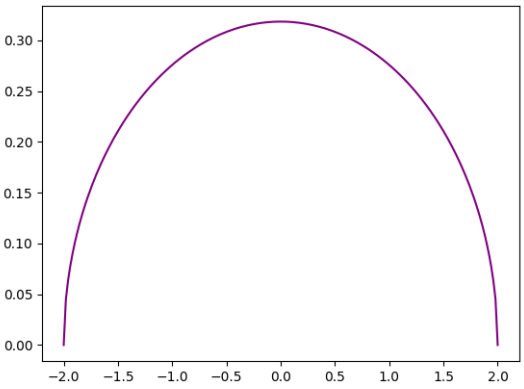
Source: Elaborated by the author.

The limit behaviour that one can observe in Figures 2, 3 and 4 turns out to be the Wigner's semicircle law (GINIBRE, 1965). That is, in the appropriate sense

$$\frac{\{\text{number of eigenvalues in } [a,b]\}}{n} \xrightarrow{n\to\infty} \frac{1}{2\pi}\int_a^b \chi_{[-2,2]}(x)\sqrt{4-x^2}\,\mathrm{d}x. \tag{2.5}$$

One of the most interesting properties of the *Wigner semicircle distribution* $\mu_W(x)\,\mathrm{d}x = \frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-x^2}\,\mathrm{d}x$ is its universality. The convergence above holds in much more general conditions. One could take $\{m_{ij}\}_{i,j=1}^n$ with independent and identically distributed entries with any "reasonable" distribution and the distribution still converges to the semicircle distribution (TAO; VU, 2010).

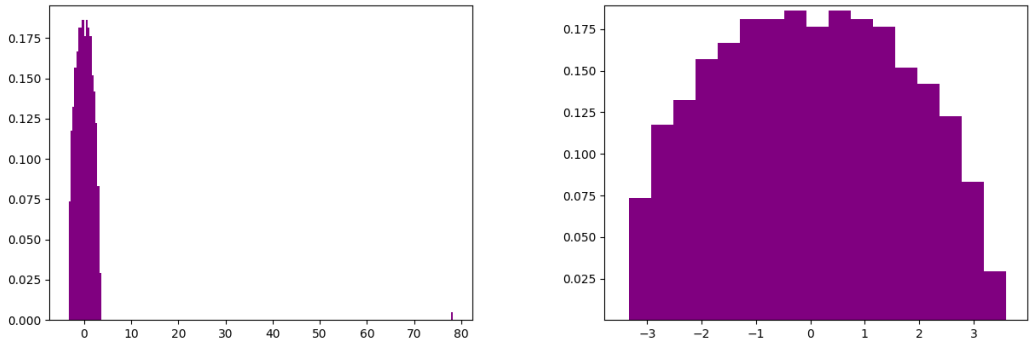
Figure 5 – The Semicircle Distribution.



Source: Elaborated by the author.

Take, for example, $m_{ij} \in \{1,2,3,4,5,6\}$ independent identically distributed entries. The eigenvalues of a computational simulation of a 500×500 matrix in such conditions returns the distribution shown in Figure 6a. Notice that we have some eigenvalues near 80. Those are known as atypical eigenvalues.

Figure 6 – Histogram of eigenvalues for a 500×500 random matrix with entries $m_{ij} \in \{1,2,3,4,5,6\}$.



(a) With atypical eigenvalues. (b) Without atypical eigenvalues.

Source: Elaborated by the author.

Throwing away the atypical eigenvalues one obtains, once again, the semicircle distribution except for a normalization factor, as shown in Figure 6b. An interesting fact about the semicircle distribution is that its moments are well known by means of the *Catalan numbers*.

Definition 1. The *Catalan numbers* are a sequence $\{C_k\}_{k\in\mathbb{N}}$ defined by

$$C_k = \frac{1}{k+1}\binom{2k}{k}. \tag{2.6}$$

Some algebraic manipulation leads to the following result:

Theorem 1. The moments of the semicircle distribution are given by

$$\frac{1}{2\pi}\int_{-2}^{2} x^n \sqrt{4-x^2}\,\mathrm{d}x = \begin{cases} 0, & n \text{ odd}, \\ C_k, & n = 2k. \end{cases}$$

2.4 Marchenko Pastur Distribution

The first appearance of random matrices in statistics is due to Wishart (1928). Let M be a rectangular matrix of dimension $n \times p$, that is, $M = (m_{ij})_{i,j}$ for $1 \le i \le n$ and $1 \le j \le p$. We also require that the rows $M_i := (m_{ij})_{1\le j\le p}$ are independent identically distributed random vectors with real entries and multivariate normal distribution,

$$N(0,V)(M_i) = c\,\mathrm{e}^{-\frac{1}{2}\operatorname{tr} M_i V^{-1} M_i^T} \prod_{j=1}^{p} \mathrm{d}m_{ij},$$

where V is the covariance matrix and $c > 0$ is the normalization constant. This model known as the Wishart random matrix model. The first studies on Wishart matrices concerned the behaviour of the $p \times p$ correlation matrix $X = (x_{ij})_{1\le i,j\le p}$ given by

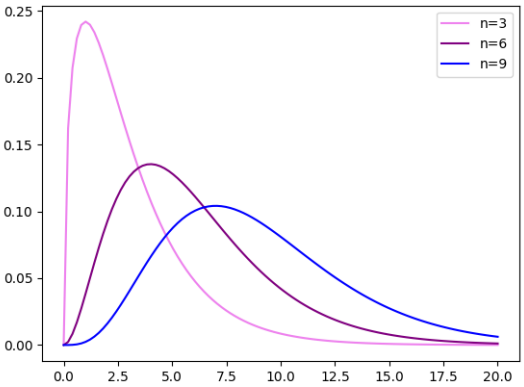
$$X := M^T M.$$

Notice that such matrix is symmetric, i.e., $x_{ij} = x_{ji}$. It was found that the joint probability density of the $p(p+1)/2$ elements x_{ij} for $i \leq j$ is given by the Wishart distribution

$$\rho(X)\,\mathrm{d}X = \frac{(\det X)^{\frac{n-p-1}{2}}}{2^{\frac{np}{2}}(\det V)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\,\mathrm{e}^{-\frac{1}{2}\operatorname{tr}V^{-1}X}\,\mathrm{d}X, \tag{2.7}$$

where $\mathrm{d}X = \prod_{i=1}^{n}\mathrm{d}x_{ii}\prod_{i<j}\mathrm{d}x_{ij}$. The dimension n of the matrix M is called degree of freedom. Taking $p = 1$ we obtain the distribution in Figure 7.

Figure 7 – Wishart distribution.



Source: Elaborated by the author.

Later, Marchenko and Pastur (1967) studied what happens to the $n \times n$ matrix

$$Y := \frac{1}{p}MM^T,$$

as n grows, if one takes the entries m_{ij} as independent identically distributed random variables with normal distribution $N(0,\sigma^2)$. They concluded that if the ratio $\frac{n}{p} = u < \infty$ is fixed then the eigenvalue distribution converges to a function proportional to the Marchenko-Pastur distribution.

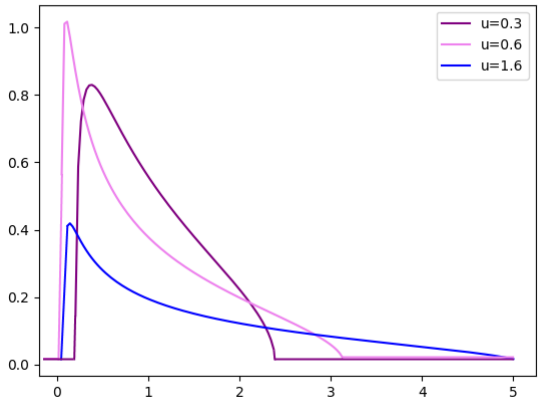
In fact, set

$$\mathrm{d}\mu_{MP}(y) := \frac{1}{2\pi\sigma^2}\frac{\sqrt{(u_+ - y)(y - u_-)}}{uy}\,\mathrm{d}y, \tag{2.8}$$

the Marchenko Pastur distribution, where $u_\pm = \sigma^2(1 \pm \sqrt{u})^2$. The limit distribution of eigenvalues of the matrix Y is given by

$$\mathrm{d}\rho(y) = \begin{cases} \left(1 - \frac{1}{u}\right)\delta(y)\,\mathrm{d}y + \mathrm{d}\mu_{MP}(y), & u > 1, \\ \mathrm{d}\mu_{MP}(y), & 0 \leq u \leq 1. \end{cases}$$

Figure 8 – Marchenko Pastur distribution.



Source: Elaborated by the author.

2.5 Big data

An important subfield of artificial intelligence is the one of automatic data processing, known as *machine learning*. Given a learning task and a set of data, the machine learning can extract the important information without new instructions.

The construction of such algorithms deals with massive amounts of data. For example, in order to train the machine into recognizing a car, lots of samples of car images are used to "teach" the pattern. It is not an easy task, since the machine is not able to "see" forms but just measure the distance of the values in each matrix entry corresponding to a pixel.

Therefore, thanks to *big data* a machine can classify images, identify objects and recognize faces and speeches. However, there are two problems with the big data approach. At first, the access to data in certain areas is not enough to the training process. Moreover, large samples of data come with a big computational cost.

In order to find more efficient approaches it is essential to understand the process better. Since learning algorithms are non-linear, when dealing with large dimension data unique challenges show up. For example, as the dimension of data grows the phenomenon of *concentration of distances* appears, i.e., the distance between the input vectors becomes indistinguishable, and since learning algorithms rely on geometric proximity of the data the results become unpredictable (MAI, 2019).

The learning outcomes are random vectors which depend on the input data, and this is the moment when the importance of random matrix theory shows up. Given the distribution of the input vectors, the data sample can be interpreted as a random matrix whose rows are given by such vectors.

Therefore, the classical results for random matrices apply. For example, suppose that both the number of vectors n and their length p grows at the same rate, i.e., $\frac{n}{p} = u < \infty$. If the input vectors possess multivariate normal distribution, then the density distribution of eigenvalues of the covariance matrix converges to the Marchenko-Pastur distribution (see section 2.4).

2.6 Point Process

A point process can be seen as a set of random points in $\mathbb{R}$.

The term *point process* can refer either to the set of random points or to the probability measure associated to such set. We call each enumerable subset of $\mathbb{R}$ with no accumulation points that constitutes the point process a *configuration*. If

$$P(\#X = n) = 1,$$

we call the process a *n-point process*. In particular, a stochastic process defined in $\mathbb{R}^n$ induces a n-point process in $\mathbb{R}$.

In this new formulation, the expectation of the random variable $N(A)$, defined as the number of points of the process belonging to a certain set $A \subset \mathbb{R}$, is given by

$$\mathbb{E}(N(A)) = \int_A \rho_1(x)\,\mathrm{d}x, \tag{2.9}$$

where ρ_1 is the *one-point correlation function*. Analogously, one can define a *n-th correlation function* for a n-point process.

When the n-point process is induced by a joint probability density P in $\mathbb{R}^n$, the k-th correlation function is given by

$$\rho_k(x_1,\cdots,x_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} P(x_1,\cdots,x_n)\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n. \tag{2.10}$$

2.6.1 Determinantal Point Process

Definition 2. A *kernel* is a Borel-measurable function $K : X \times X \to \mathbb{C}$ such that

$$||K|| := \sup_{(x_1,x_2)\in X\times X} |K(x_1,x_2)| < \infty.$$

A special case of point process occurs when the correlation function can be expressed as the determinant of a kernel,

$$\rho_n(x_1,\cdots,x_n) = \det[K(x_i,x_j)]_{i,j=1}^n, \tag{2.11}$$

where K is the *correlation kernel* and the matrix is explicitly given by

$$[K(x_i,x_j)]_{i,j=1}^n = \begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_n) \\ K(x_2,x_1) & K(x_2,x_2) & \cdots & K(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,x_1) & K(x_n,x_2) & \cdots & K(x_n,x_n) \end{pmatrix}.$$

Definition 3. A sequence of polynomials $\{p_j(x)\}_{j=1}^n$, $p_j : \mathbb{R} \to \mathbb{R}$, $\deg p_j = j$ and positive leading coefficients is said to be a set of *orthogonal polynomials* with respect to the weight $w(x)$ if, for every $1 \leq i,j \leq n$,

$$\int_{\mathbb{R}} p_i(x)p_j(x)w(x)\,\mathrm{d}x = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The orthogonal polynomials kernel, also known as *Christoffel-Darboux* kernel, is given by

$$K_n(x_i,x_j) := \sqrt{w(x_i)w(x_j)} \sum_{k=0}^{n-1} p_k(x_i)p_k(x_j).$$

Now, let $\mathcal{H}_n$ denote the space of hermitian matrices of size $n \times n$ and define a measure

$$\mathrm{d}\mu = f(M)\,\mathrm{d}M,$$

where $\mathrm{d}M = \prod_{i=0}^n \mathrm{dRe}\,M_{ii} \prod_{i<j} \mathrm{dRe}\,M_{ij}\,\mathrm{dIm}\,M_{ij}$ is a Lebesgue measure and $f(M)$ is a density function invariant by unitary transformations, that is, $\mathrm{d}\mu$ defines an *unitary ensemble* (see section 2.8). The eigenvalues under such ensemble constitute a determinantal point process with a Christoffel-Darboux kernel. Moreover, the distribution on the space of eigenvalues induced by the distribution over the space of matrices can be related to the latter by the *Weyl integration formula.*

Theorem 2. Let $f : \mathcal{H}_n \to \mathbb{C}$ be a *class function*, that is, a function invariant by unitary transformation. Then,

$$\int_{\mathcal{H}_n} f(M)\,\mathrm{d}M = c_n \int_{\mathbb{R}^n} f(\lambda_1,\cdots,\lambda_n) \prod_{i<j} (\lambda_j - \lambda_i)^2\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n, \tag{2.12}$$

where

$$c_n = \frac{\pi^{n(n-1)/2}}{\prod_{j=1}^n j!} \tag{2.13}$$

For a proof of such result, see Anderson, Guionnet and Zeitouni (2010).

2.6.2 Dyson bridges

The problem of the non-intersecting Brownian bridges, also known as *Dyson bridges* due to the works of Dyson (1962), deals with the construction of n Brownian bridges $X_i(t)$, starting at $X_1(0) = \cdots = X_n(0) = a$ and such that for some $T > 0$,

$X_1(T) = \cdots = X_n(T) = b$. Moreover, we require that their paths do not intersect for any $t \in (0,T)$. Such construction can be obtained from random matrices.

We start with a result about the joint probability distribution of n bridges.

Theorem 3 ((KARLIN; MCGREGOR, 1959)). Take n independent copies $X_1(t), X_2(t), \cdots, X_n(t)$ of an unidimensional and continuous Markov process such that

$$X_j(0) = a_j,$$

where $a_1 < a_2 < \cdots < a_n$. Let $p_t(x,y)$ be the density of the probability and $E_1, \cdots, E_n$ Borel sets with $\sup E_j < \inf E_{j+1}$ for $j = 1, \cdots, n-1$. Then,

$$\int_{E_n} \cdots \int_{E_1} \det[p_t(a_i, x_j)]_{i,j=1} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

is the probability that such paths do not intersect on the interval $[0,t]$ and $X_j(t) \in E_j$.

We start adding the condition that for a $T > 0$,

$$X_j(T) = b_j,$$

where $b_1 < \cdots < b_n$. The non-intersection condition is now extended to the set $[0,T]$. In each time step t, the joint probability distribution of the n paths is given by

$$\frac{1}{Z_n} \det[p_t(a_i, x_j)]_{i,j=1}^n \det[p_{T-t}(x_i, b_j)]_{i,j=1}^n,$$

where Z_n is the normalization constant. Since we want Brownian bridges, the transition density is Gaussian, and therefore is given by

$$p_t(x_1, x_2) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x_2 - x_1)^2}{2t}}.$$

We take the limit $a_i \to a$, $b_i \to b$ (for a more general case where not all b_i converge to the same point we refer to Adler *et al.* (2011)). But then we have a $0/0$ limit and the L'Hopital rule can be applied. Therefore, the first determinant becomes

$$\det[p_t(a, x_j)]_{i,j=1}^n = c \det[x_j^{i-1}]_{i,j=1}^n \prod_{j=1}^n \mathrm{e}^{-\frac{\left(x_j^2 - 2a x_j\right)}{2t}},$$

for some $c > 0$. Analogously,

$$\det[p_{T-t}(x_i, b)]_{i,j=1}^n = c' \det[x_j^{i-1}]_{i,j=1}^n \prod_{j=1}^n \mathrm{e}^{-\frac{\left(x_j^2 - 2b x_j\right)}{2(T-t)}},$$

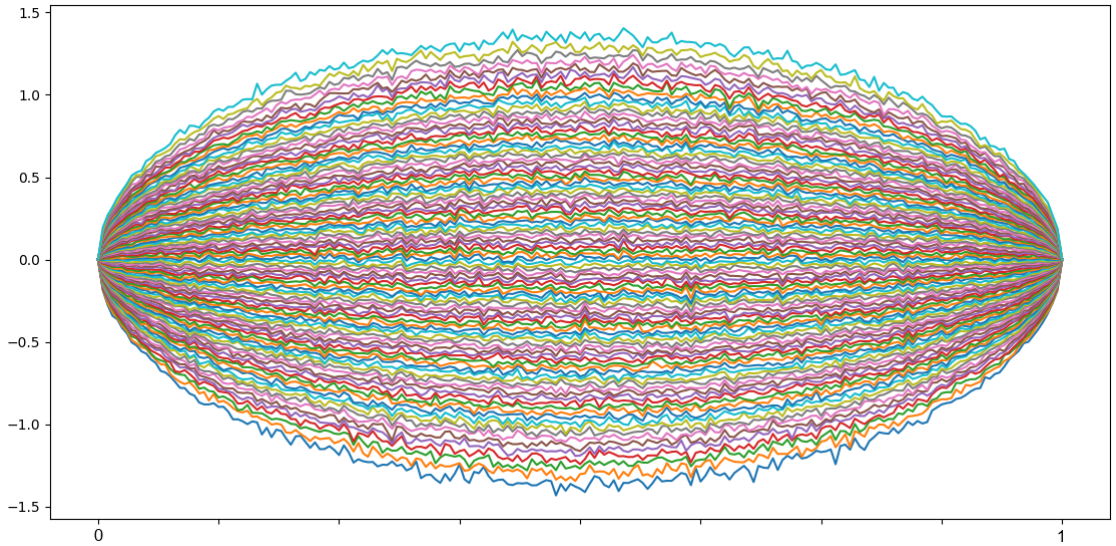
for a constant $c' > 0$. Thus, in the particular case where $a = b = 0$,

$$\det[p_t(a, x_j)]_{i,j=1}^n \det[p_{T-t}(x_i, b)]_{i,j=1}^n = C \prod_{i<j} (x_j - x_i)^2 \prod_{j=1}^n \mathrm{e}^{-\frac{T x_j^2}{2t(T-t)}},$$

and by the Weyl integration formula, Equation 2.12, it is equivalent to the distribution of eigenvalues induced by a probability measure over the space of hermitian matrices. Our problem is then reduced to the evaluation of the eigenvalues of such matrices. The algorithm is the following

1. Set $0 < t_1 < \cdots < t_n < 1$ a partition of the interval $[0,1]$.
2. Construct a matrix of dimension k.
3. Set the diagonal entries of the matrix to be random variables with Gaussian distribution, mean $\mu = 0$ and variance $\sigma^2 = 2t_i(1-t_i)$.
4. Set each non-diagonal entry to be $X + iY$ where both X, Y are random variables with Gaussian distribution, mean $\mu = 0$ and variance $\sigma^2 = t_i(1-t_i)$.
5. Evaluate the eigenvalues and save in a ordered list.
6. For each t_i, repeat steps 2 to 5.
7. Obtain n lists of k elements each and the linear interpolation between the i-th elements of the lists for each $1 \leq i \leq n$ converges to the Brownian bridges desired.

Figure 9 – A hundred non-intersecting Brownian bridges.



Source: Elaborated by the author.

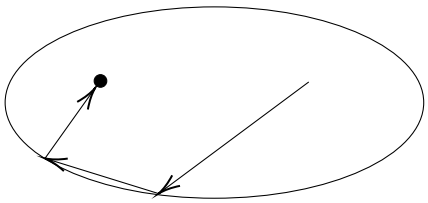
2.6.3 Quantum chaos

The Brownian bridges appear naturally in some physical problems. The experiments of Berry and Tabor (1977) were the first to corroborate with the idea that random

matrices can describe the statistics of the energy levels in quantum chaos. The characterization of quantum spectra by means of random matrices was conjectured by Bohigas, Giannoni and Schmit (1984).

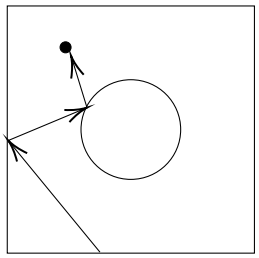
Start with a billiard game where a particle is constrained into a two-dimensional shape. Suppose that the particle is free and, once it reaches the boundary, it is reflected (see Figure 10). We also require our particle to be in the quantum regime.

Figure 10 – A particle in an ellipse.



Source: Elaborated by the author.

Figure 11 – Sinai's billiard.



Source: Elaborated by the author.

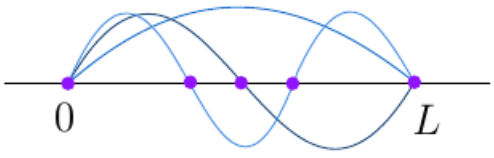
Since the particle is in the quantum regime, its behaviour is described by the eigenfunctions of the Schrödinger operator in dimension 2 (for a nice introduction into the quantum mechanics universe, we recommend Griffiths (1994)).

To clarify some concepts, consider the simplest case of the Schrödinger operator. Take a free particle in a box of length L and solve the Schrödinger equation

$$\ddot{\psi}(x) = -\frac{2mE}{\hbar^2}\psi(x),$$

with Dirichlet condition $\psi(0) = \psi(L) = 0$. The eigenfunctions are given by trigonometric functions. Notice that such solutions possess zeros (node points) as in Figure 12.

Figure 12 – The node points of the first eigenfunctions.



Source: Elaborated by the author.

When dealing with a two-dimensional problem we have a generalization of nodal points called *nodal lines*. The nodal lines are the curves where some characteristic such as velocity does not oscillate. The Uhlenbeck's Theorem says that the nodal lines of a generic

wave function do not intersect. Moreover, their relevant statistical characteristics can be obtained from random matrix ensembles. In the particular case of a Sinai billiard (Figure 11), the nodal lines are described by non-intersecting Brownian paths (MONASTRA; SMILANSKY; GNUTZMANN, 2003).

2.7 Some general results

This Section contains a list of general results that will be useful throughout the present work.

2.7.1 Gap probability

In the context of point process the correlation function gives us the expectation of the number of points belonging to a certain set. But one can be interested in the probability that no point belongs to such set. This concept is known as *gap probability*.

Before going into the details of the formula of the gap probability, we prove a general result for expectations in point processes.

Theorem 4. Given a determinantal point process with correlation kernel K and a measurable function f defined on a Borel set $A \subset \mathbb{R}$, one has

$$\mathbb{E}\left[\prod_{x\in X}(1+f(x))\right] = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{A^n}\det[K(x_i,x_j)f(x_j)]_{i,j=1}^{n}\,\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{2.14}$$

The proof is quite simple. Denote by $\{x_i\}_{i\in I}$ the set of points of the configuration X. Recall that a configuration is enumerable, therefore $I \subset \mathbb{N}$. We start expanding the product $\prod_{i\in I}(1+f(x_i))$. A direct calculation implies that

$$\begin{aligned}\prod_{i\in I}(1+f(x_i)) &= 1+\sum_i f(x_i)+\frac{1}{2!}\sum_{i\neq j}f(x_i)f(x_j)+\frac{1}{3!}\sum_{i\neq j\neq k}f(x_i)f(x_j)f(x_k)+\cdots\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum\prod_{j=1}^{n}f(x_j),\end{aligned}$$

where the inner sum is taken over all the sets of n distinct elements. Therefore,

$$\begin{aligned}\mathbb{E}\left[\prod_{i\in I}(1+f(x_i))\right] &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\mathbb{E}\left[\sum\prod_{j=1}^{n}f(x_j)\right]\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{A^n}\left[\prod_{j=1}^{n}f(x_j)\right]\rho_n(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{A^n}\left[\prod_{j=1}^{n}f(x_j)\right]\det[K_n(x_i,x_j)]_{i,j=1}^{n}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{A^n}\det[K_n(x_i,x_j)f(x_j)]_{i,j=1}^{n}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,\end{aligned}$$

where the first equality follows from the linearity of the expectation, the second one follows from Equation (2.10) and the third one comes from Equation (2.11). ■

Now, the gap probability is equivalent to the special case where $f(x) = -\chi_A$. Therefore,

$$P(\text{there is no point of X in A}) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} \det[K(x_i, x_j)]_{i,j=1}^n \, dx_1 \cdots dx_n. \quad (2.15)$$

The right-hand side side of Equation (2.15) is known in functional analysis as a *Fredholm determinant* and is denoted by

$$\det[1 - K]|_{A^n} := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{A^n} \det[K(x_i, x_j)]_{i,j=1}^n \, dx_1 \cdots dx_n, \quad (2.16)$$

is the generalization of the determinant of a finite dimentional linear operator (see (SIMON, 2005)).

2.7.2 Gaussian Measure

The general Gaussian measure on the real line $\mathbb{R}$ with mean μ and variance σ^2 has a probability density function given by

$$\rho(x)\, dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx. \quad (2.17)$$

In order to construct a Gaussian measure in higher dimensions, say $\mathbb{R}^n$, we need another characterization.

Definition 4. A measure μ on $\mathbb{R}^n$ is a Gaussian measure if its characteristic function, i.e., its Fourier transform, is given by

$$\phi(t) = \int_{\mathbb{R}^n} e^{i(t,x)} \, d\mu(x) = e^{i(m,t) - \frac{1}{2}(Ct,t)},$$

where $C = (c_{ij})_{i,j=1}^n$ is a non-negative and non-degenerate matrix of quadratic form called *covariance matrix* and m is the *mean vector*.

Taking the mean vector $m = 0$ one obtains that

$$\phi(t) = e^{-\frac{1}{2}(Ct,t)},$$

and, consequently, the Gaussian measure is given by

$$d\mu(x) = \frac{(\det B)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(x,Bx)} \, dx, \quad (2.18)$$

where $B = C^{-1}$ and $x := (x_1, x_2, \cdots, x_n)$. It means that in order to construct a Gaussian measure over a space isomorphic to $\mathbb{R}^n$, we just need to obtain the matrix B from a non-degenerate quadratic form. Such construction will be useful in section 3.1, where we construct a Gaussian measure over a space of matrices. Also notice that the expectation with respect to this Gaussian measure is given by $\langle x_i \rangle = 0$ and $\langle x_i x_j \rangle = c_{ij}$.

2.7.3 Wick formula

The goal in this subsection is to present a formula that allows us to apply the results $\langle x_i \rangle = 0$ and $\langle x_i x_j \rangle = c_{ij}$ from previous section to evaluate the expectation with respect to the Gaussian measure of much more complicated products.

At first, notice that since $\langle x \rangle = 0$ and any odd degree monomial can be reduced by integration by parts to a monomial of degree 1, $\langle x^n \rangle = 0$ for all odd n. Wick formula will allow us to deal with the monomials of even degree. We define the *Wick couplings* as the possible pairings $(p_1, q_1), (p_2, q_2), \cdots, (p_k, q_k)$ of the indices $1, 2, \cdots, 2k$ such that $p_1 < p_2 < \cdots < p_k$ and $p_1 < q_1, \cdots, p_k < q_k$.

Theorem 5 (Wick Theorem)**.** Let $\{f_i\}_{i=1}^{2k}$ be a set of linear functions of $x_1, \cdots, x_n$, then

$$\langle f_1 \cdots f_n \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_k} f_{q_k} \rangle$$

where the sum is over all the $(2k-1)!! = (2k-1) \times \cdots 3 \times 1$ *Wick couplings*.

Example: To illustrate the use of such result, take the Gaussian standard measure on the real line and apply Wick formula to $\langle x^4 \rangle$. In the notation of Theorem 5,

$$f_1 = f_2 = f_3 = f_4 = x.$$

Therefore,

$$\begin{aligned} \langle f_1 f_2 f_3 f_4 \rangle &= \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle \\ &= 3 \langle x^2 \rangle \langle x^2 \rangle = 3, \end{aligned}$$

and we conclude that $\langle x^4 \rangle = 3$.

2.7.4 Heine formula

A classical result in Determinantal Point Process is the so-called Andréev identity.

Theorem 6 (Andréev identity)**.** Given a measurable space (X, μ) and $f_i, g_i \in L^2(X, \mu)$, $1 \leq i, j \leq n$, we have

$$\int_X \cdots \int_X \det_{1 \leq i,j \leq n} (f_i(x_j)) \det_{1 \leq i,j \leq n} (g_i(x_j)) \, \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_n) = n! \det_{1 \leq i,j \leq n} \left(\int_X f_i(x) g_j(x) \, \mathrm{d}\mu(x) \right).$$

A special case of Theorem 6 is when one has a positive Borel measure $\mathrm{d}\mu$ on the complex plane such that its absolute moments converge, i.e.,

$$\int_X |z|^i \, \mathrm{d}\mu(z) < \infty \quad \text{for every } i = 0, 1, 2, \cdots.$$

Thus, setting $f_i(z) = z^i$ and $g_i(z) = \bar{z}^i$, Andréev identity says that

$$\det_{0\le i,j\le n-1}\left(\int_X z^i \bar{z}^j \,\mathrm{d}\mu(z)\right) = \frac{1}{n!}\int_X \cdots \int_X \Delta(z)\Delta(\bar{z})\,\mathrm{d}\mu(z_1)\cdots \mathrm{d}\mu(z_n), \tag{2.19}$$

where $\Delta(z) = \prod_{i<j}(z_j - z_i)$. Equation (2.19) is known as **Heine formula**, and will be useful in Chapter 5.

Proof of Theorem 6: The idea is to prove that Fubini Theorem and apply it to obtain a proof by induction.

Laplace expansion allow us to write

$$\det_{1\le i,j\le n}(f_i(x_j)) = \sum_{i=1}^n (-1)^{i+1} f_i(x_1) \det_{\ell\neq i, m\neq 1}(f_\ell(x_m))$$
$$\det_{1\le i,j\le n}(g_i(x_j)) = \sum_{i=1}^n (-1)^{i+1} g_i(x_1) \det_{\ell\neq i, m\neq 1}(g_\ell(x_m)).$$

Therefore,

$$\begin{aligned}
I &= \int_X \cdots \int_X |\det_{1\le i,j\le n}(f_i(x_j)) \det_{1\le i,j\le n}(g_i(x_j))|\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)\\
&= \int_X \cdots \int_X \left|\left(\sum_{i=1}^n (-1)^{i+1} f_i(x_1) \det_{\ell\neq i,m\neq 1}(f_\ell(x_m))\right)\left(\sum_{i=1}^n (-1)^{i+1} g_i(x_1) \det_{\ell\neq i,m\neq 1}(g_\ell(x_m))\right)\right| \times\\
&\quad \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)\\
&= \int_X \cdots \int_X \left|\sum_{i,j=1}^n (-1)^{i+j+2} f_i(x_1)g_j(x_1) \det_{\ell\neq i,m\neq 1}(f_\ell(x_m)) \det_{\ell\neq j,m\neq 1}(g_\ell(x_m))\right| \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)\\
&\le \int_X \cdots \sum_{i,j=1}^n \left|\det_{\ell\neq i,m\neq 1}(f_\ell(x_m)) \det_{\ell\neq j,m\neq 1}(g_\ell(x_m))\right| \left(\int_X |f_i(x_1)g_j(x_1)|\,\mathrm{d}\mu(x_1)\right)\cdots\mathrm{d}\mu(x_n).
\end{aligned}$$

But since $f_i, g_j \in L^2(X,\mu)$, by Hölder inequality the integral in parentheses converges,

$$I_{ij} := \int_X |f_i(x_1)g_j(x_1)|\,\mathrm{d}\mu(x_1) \le \left(\int_X |f_i(x_1)|^2\,\mathrm{d}\mu(x_1)\right)^{1/2}\left(\int_X |g_j(x_1)|^2\,\mathrm{d}\mu(x_1)\right)^{1/2} < \infty.$$

Therefore, we are left with

$$\int_X \cdots \sum_{i,j=1}^n I_{ij} \int_X \left|\det_{i_2\neq i, j_2\neq 1}(f_{i_2}(x_{j_2})) \det_{i_2\neq j, j_2\neq 1}(g_{i_2}(x_{j_2}))\right| \mathrm{d}\mu(x_2)\cdots\mathrm{d}\mu(x_n),$$

where the indices ℓ, m were replaced by i_2, j_2 respectively. Such expression is analogous to our initial equation, and therefore the Laplace expansion can be applied once again in order to isolate the x_2-dependent terms. Thus,

$$I \le \int_X \cdots \sum_{i,j=1}^n I_{ij} \int_X \left|\det_{i_2\neq i, j_2\neq 1}(f_{i_2}(x_{j_2})) \det_{i_2\neq j, j_2\neq 1}(g_{i_2}(x_{j_2}))\right| \mathrm{d}\mu(x_2)\cdots\mathrm{d}\mu(x_n),$$

and since

$$\det_{1\le i_2,j_2\le n, i_2\neq i, j_2\neq 1}(f_{i_2}(x_{j_2})) = \sum_{i_2=1, i_2\neq i}^{n}(-1)^{i_2+2} f_{i_2}(x_2) \det_{\ell\neq\{i_2,i\}, m\neq\{2,1\}}(f_\ell(x_m))$$
$$\det_{1\le i_2,j_2\le n, i_2\neq j, j_2\neq 1}(g_{i_2}(x_{j_2})) = \sum_{i_2=1, i_2\neq i}^{n}(-1)^{i_2+2} g_{i_2}(x_2) \det_{\ell\neq\{i_2,i\}, m\neq\{2,1\}}(g_\ell(x_m)),$$

it follows that,

$$\begin{aligned}
I &\le \int_X \cdots \sum_{i,j=1}^{n} I_{ij} \int_X \left| \left(\sum_{i_2=1, i_2\neq i}^{n}(-1)^{i_2+2} f_{i_2}(x_2) \det_{\ell\neq\{i_2,i\}, m\neq\{2,1\}}(f_\ell(x_m)) \right) \times \right. \\
&\qquad \left. \left(\sum_{i_2=1, i_2\neq i}^{n}(-1)^{i_2+2} g_{i_2}(x_2) \det_{\ell\neq\{i_2,i\}, m\neq\{2,1\}}(g_\ell(x_m)) \right) \right| \mathrm{d}\mu(x_2)\cdots\mathrm{d}\mu(x_n) \\
&= \int_X \cdots \int_X \sum_{i,j=1}^{n} I_{ij} \int_X \left| \sum_{i_2\neq i, j_2\neq j}^{n}(-1)^{i_2+j_2+2} f_{i_2}(x_2) g_{j_2}(x_2) \det_{\ell\neq i_2, m\neq 1}(f_\ell(x_m)) \times \right. \\
&\qquad \left. \det_{\ell\neq j_2, m\neq 1}(g_\ell(x_m)) \right| \mathrm{d}\mu(x_2)\cdots\mathrm{d}\mu(x_n) \\
&\le \int_X \cdots \int_X \sum_{i,j=1}^{n} I_{ij} \sum_{i_2\neq i, j_2\neq j}^{n} \left| \det_{\ell\neq i_2, m\neq 1}(f_\ell(x_m)) \det_{\ell\neq j_2, m\neq 1}(g_\ell(x_m)) \right| \times \\
&\qquad \left(\int_X \left| f_{i_2}(x_2) g_{j_2}(x_2) \right| \mathrm{d}\mu(x_2) \right) \mathrm{d}\mu(x_3)\cdots\mathrm{d}\mu(x_n).
\end{aligned}$$

Since $f_{i_2}, g_{j_2} \in L^2(X,\mu)$, Hölder inequality guarantees the convergence of the integral in parentheses. Setting

$$I_{i_2 j_2} := \int_X \left| f_{i_2}(x_2) g_{j_2}(x_2) \right| \mathrm{d}\mu(x_2),$$

one obtains

$$\begin{aligned}
I \le \int_X \cdots \int_X \sum_{i,j=1}^{n} I_{ij} \sum_{i_2\neq i, j_2\neq j}^{n} I_{i_2 j_2} \int_X \left| \det_{i_3\neq\{i,i_2\}, j_2\neq\{1,2\}}(f_{i_2}(x_{j_2})) \times \right. \\
\left. \det_{i_2\neq\{j,j_2\}, j_2\neq\{1,2\}}(g_{i_2}(x_{j_2})) \right| \mathrm{d}\mu(x_3)\cdots\mathrm{d}\mu(x_n).
\end{aligned}$$

Recursively, we obtain that

$$I \le \sum_{i,j=1}^{n} \left(I_{ij} \sum_{i_2\neq i, j_2\neq j} \left(I_{i_2 j_2} \sum_{i_3\neq\{i,i_2\}, j_3\neq\{j,j_2\}} I_{i_3 j_3} \cdots \sum_{i_n\neq\{i,i_2,\cdots,i_{n-1}\}, j_n=n} I_{i_n j_n} \right) \cdots \right) < \infty,$$

because it is a finite sum of products of finite terms. Moreover, the product of measurable functions with same domain is mensurable. Therefore $\det_{1\le i,j\le n}(f_i(x_j)) \det_{1\le i,j\le n}(g_i(x_j)) \in L^1(X^n, \mu(x_1)\times\cdots\mu(x_n))$ and the Fubini Theorem can be applied.

Now, we proceed by induction. If $n = 1$,

$$\int_X \det(f_1(x_1))\det(g_1(x_1))\,\mathrm{d}\mu(x_1) = \int_X f_1(x_1)g_1(x_1)\,\mathrm{d}\mu(x_1)$$
$$= 1!\det\left(\int_X f_1(x_1)g_1(x_1)\,\mathrm{d}\mu(x_1)\right).$$

Suppose it holds for n. By the previous calculation, one can apply Fubini. Therefore,

$$\int_X \cdots \int_X \det_{1\le i,j\le n+1}(f_i(x_j))\det_{1\le i,j\le n+1}(g_i(x_j))\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_{n+1})$$
$$= \int_X \cdots \int_X \left(\sum_{i=1}^{n+1}(-1)^{i+n+1} f_i(x_{n+1}) \det_{\ell\neq i, m\le n}(f_\ell(x_m))\right)\left(\sum_{i=1}^{n+1}(-1)^{i+n+1} g_i(x_{n+1})\times\right.$$
$$\left.\det_{\ell\neq i, m\le n}(g_\ell(x_m))\right)\mathrm{d}\mu(x_{n+1})\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$
$$= \sum_{i,j=1}^{n+1}(-1)^{i+j+2n+2}\int_X\cdots\left(\int_X f_i(x_{n+1})g_j(x_{n+1})\,\mathrm{d}\mu(x_{n+1})\right)\det_{\ell\neq i, m\le n}(f_\ell(x_m))\times$$
$$\det_{\ell\neq j, m\le n}(g_\ell(x_m))\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$
$$= \sum_{i,j=1}^{n+1}(-1)^{i+j+2n+2}\left(\int_X f_i(x_{n+1})g_j(x_{n+1})\,\mathrm{d}\mu(x_{n+1})\right) n!\det_{\ell\neq i, m\neq j}\left(\int_X f_\ell(x)g_m(x)\,\mathrm{d}\mu(x)\right)$$
$$= n!\sum_{i,j=1}^{n+1}(-1)^{i+j+2n+2}\left(\int_X f_i(x_{n+1})g_j(x_{n+1})\,\mathrm{d}\mu(x_{n+1})\right)\det_{\ell\neq i, m\neq j}\left(\int_X f_\ell(x)g_m(x)\,\mathrm{d}\mu(x)\right),$$

and since for each fixed j the sum over i is the Laplace expansion of the desired determinant, one has

$$\int_X \cdots \int_X \det_{1\le i,j\le n+1}(f_i(x_j))\det_{1\le i,j\le n+1}(g_i(x_j))\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_{n+1}) =$$
$$(n+1)!\det_{1\le i,j\le n+1}\left(\int_X f_i(x)g_j(x)\,\mathrm{d}\mu(x)\right),$$

and the identity is proved. ■

2.8 What comes next

We have shortly discussed several applications of random matrix theory and presented some important mathematical tools. But before we move forward, it is time to present an overview of the next steps and introduce some important objects that we will be handling for the rest of the dissertation.

We start with some definitions on unitary random matrix models. Given $V : \mathbb{C} \to \mathbb{C}$ a polynomial function,

$$V(x) = \sum_{j=1}^{n} a_j x^j, \tag{2.20}$$

one can define the operator $V(H)$, where H is a square matrix, by

$$V(H) := \sum_{j=1}^{n} a_j H^j. \tag{2.21}$$

Since the product of matrices is well-defined, also is $V(H)$. A general unitary ensemble consists of the space $\mathcal{H}_n$ of $n \times n$ hermitian matrices with a probability density

$$\rho(H)\,\mathrm{d}H := \frac{1}{Z_\rho}\,\mathrm{e}^{-\operatorname{tr} V(H)}\,\mathrm{d}H, \tag{2.22}$$

where the function V is called potential, and is chosen such that the integral

$$\int_{\mathcal{H}_n} \mathrm{e}^{-\operatorname{tr} V(H)}\,\mathrm{d}H \tag{2.23}$$

converges, and $\mathrm{d}H = \prod_{i=0}^{n} \mathrm{d}h_{ii} \prod_{i<j} \mathrm{dRe}\, h_{ij}\, \mathrm{dIm}\, h_{ij}$. The explicit construction of such measure will be carried out in section 3.1.

In particular, if one takes $V : \mathbb{C} \to \mathbb{C}$ as $x \mapsto \frac{1}{2}x^2$, the *Gaussian Unitary Ensemble* (GUE) is obtained, with probability density given by

$$\rho(H) = \frac{2^{(n^2-n)/2}}{(2\pi)^{n/2}}\,\mathrm{e}^{-\frac{1}{2}\operatorname{tr}(H^2)}. \tag{2.24}$$

Throughout Chapter 3, the measure given by Equation (2.24) will play an important role. In section 3.2 we define g-maps as graphs embedded in compact bidimensional surfaces of genus g and relate the enumeration of g-maps to the expectation with respect to the Gaussian measure (2.24) of the trace in a random matrix ensemble.

This matrix model is called unitary because the measure defined above is invariant under unitary transformations. From linear algebra, every hermitian matrix is diagonalizable under conjugacy by some unitary matrix, that is, given any matrix H there exists a unitary matrix U and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ such that

$$H = UDU^*,$$

where the choice of U is not unique. Such aspect of the model also indicates that many useful statistics on the matrices should actually be dependent only on the underlying eigenvalues.

Notice that the diagonalization of H allows us to express the potential $V(H)$ as $V(\lambda_1, \cdots, \lambda_n)$. Therefore, the Weyl formula (2.12) tells us that for any unitarily invariant function $f : \mathcal{H}_n \to \mathbb{C}$,

$$\frac{1}{Z_\rho}\int_{\mathcal{H}_n} f(H)\,\mathrm{e}^{-\operatorname{tr} V(H)}\,\mathrm{d}H = \frac{1}{Z_n}\int_{\mathbb{R}^n} f(\lambda_1, \cdots, \lambda_n) \prod_{i<j}(\lambda_j - \lambda_i)^2 \prod_{j=1}^{n} \mathrm{e}^{-V(\lambda_j)}\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n.$$

This means that the probability measure over the space of matrices induces a joint probability density on the space of eigenvalues given by

$$\frac{1}{Z_n}\prod_{i<j}(\lambda_j-\lambda_i)^2\prod_{j=1}^{n}\mathrm{e}^{-V(\lambda_j)}. \tag{2.25}$$

The new constant Z_n, known as *partition function*, is the great protagonist of Chapter 4 and relates to the previous Z_ρ via

$$Z_\rho = Z_n\frac{\pi^{n(n-1)/2}}{\prod_{j=1}^{n} j!}.$$

It also plays a role in section 5.2, where we relate it to a celebrated integrable hierarchy, namely the KP hierarchy.

In Chapter 4 we will see that the partition function Z_n, for large n, admits an asymptotic expansion in inverse powers of n, with coefficients that are connected with the map enumeration problem from Chapter 3. In order to obtain such expansion, we will explore the relation between the partition function and the correlation function for a kernel of orthogonal polynomials. Then, such expansion is achieved after some work on the asymptotics for orthogonal polynomials with respect to the weight of the measure generated by Z_n. We now elaborate a little more on this approach.

As we just explained, the probability density $\rho(H)$ on the matrix space induces the probability density (2.25) on the space of eigenvalues. Furthermore, one can show that the eigenvalues form a determinantal point process with a kernel that can be chosen to be the Christoffel-Darboux kernel. Recall from subsection 2.6.1 that such kernel is build through a set of polynomials $\{p_j(x)\}_{j=1}^{n}$ with leading coefficients $\gamma_n > 0$, which are orthogonal with respect to the weight

$$w(x) := \mathrm{e}^{-V(x)}.$$

The expectation of a function of the eigenvalues is then given in terms of the *one point function* (see Equation (2.10))

$$\rho_1(\lambda) = K_n(\lambda,\lambda) \tag{2.26}$$

$$= \frac{\mathrm{e}^{-V(\lambda)}}{n}\sum_{k=0}^{n-1} p_j^2(\lambda). \tag{2.27}$$

But, taking the logarithm derivative of the partition function, one obtains the expectation of the trace in a random matrix model, and since the expectation is evaluated through the one point function, one obtains a relation between $\rho_1(\lambda)$ and Z_n (see section 4.2).

Equation (2.27) has two important consequences. First of all, the asymptotic expansion claimed for the partition function can be achieved through the asymptotic expansion for a set of orthogonal polynomials. This asymptotic analysis takes most of this

dissertation, and it is also the part that we devote the most mathematical energy to fill out all the details in section 4.3. We take the route of the characterization of orthogonal polynomials by means of a Riemann-Hilbert problem found by Fokas, Its and Kitaev in 1993 and improved by the asymptotic techniques developed by Deift and Zhou. In this context, our major mathematical effort will be in explaining the Riemann-Hilbert method in the context of orthogonal polynomials. Once the asymptotics for p_n is obtained, the asymptotics for the one point function is accomplished in details in section 4.4 and we conclude the asymptotic expansion for the partition function.

Second, in virtue of (2.27) statistics of eigenvalues of large random matrices can be studied through asymptotic formulas of K_n for large n. This means that all we need to understand eigenvalues is reduced to orthogonal polynomials of large degree. This conclusion leads us to the core of section 5.3, where we will see that given the right normalization, the distribution of the eigenvalues converges weakly in probability to the equilibrium measure defined in subsection 4.3.1. Moreover, in subsection 5.3.2 it is shown that after rescaling the kernel, the behaviour of the eigenvalues near a edge point of the support of the equilibrium measure is given by the Airy kernel

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}.$$

On the other hand, an analogous analysis in subsection 5.3.1 shows that the behaviour of the eigenvalues in the bulk of the support of the equilibrium measure after rescaling is given by the Sine kernel

$$\frac{\sin[\pi(x-y)]}{\pi(x-y)}.$$

The gap probabilities resulting from the Airy kernel and the Sine kernel will lead to connections with the Painlevé II and the Painlevé V, respectively. At last, we discuss briefly a limiting kernel related to the PXXXIV, which brings us to another connection with integrable systems and also motivates further work.

Finally, we bring an appendix with some Python codes developed throughout the present work, in order to allow the reader to play a little with some famous distributions and build the Dyson bridges presented in subsection 2.6.2.

CHAPTER

3

MAP ENUMERATION

In the present chapter we explore the bridge between map enumeration and random matrices in an attempt to justify our interest in evaluating the expectation of the trace in a random matrix ensemble. Instead of rigorous proofs, we focus on examples and explore the geometric intuition in order to clarify the main ideas behind such connections.

3.1 Space of Hermitian Matrices

Let $\mathcal{H}_n$ the space of the $n \times n$ random selfadjoint complex matrices, that is,

$$H = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{pmatrix} = \begin{pmatrix} \bar{h}_{11} & \bar{h}_{21} & \cdots & \bar{h}_{n1} \\ \bar{h}_{12} & \bar{h}_{22} & \cdots & \bar{h}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{h}_{1n} & \bar{h}_{2n} & \cdots & \bar{h}_{nn} \end{pmatrix} = H^{\dagger}.$$

Therefore, the entries of the diagonal are real random variables and $\bar{h}_{ij} = h_{ji}$ for every $i \neq j$. As a consequence, H can be identified with the n^2 vector

$$(h_{11}, h_{22}, \cdots, h_{nn}, \operatorname{Re} h_{12}, \operatorname{Im} h_{12}, \operatorname{Re} h_{13}, \operatorname{Im} h_{13} \cdots),$$

and $\mathcal{H}_n$ is isomorphic to $\mathbb{R}^{n^2}$. Such isomorphism suggests a natural Lebesgue measure in $\mathcal{H}_n$ given by

$$\mathrm{d}H = \prod_{i=0}^{n} \mathrm{d}x_{ii} \prod_{i<j} \mathrm{d}x_{ij}\, \mathrm{d}y_{ij}, \tag{3.1}$$

where $x_{ij} = \operatorname{Re} h_{ij}$ and $y_{ij} = \operatorname{Im} h_{ij}$. We want to introduce a Gaussian density as in subsection 2.7.2. By Equation (2.18) all we need to do is to specify the correlation matrix C. Let us introduce a non-degenerate quadratic form. Take $\operatorname{tr}(H^2)$. An explicitly calculation

shows that

$$\operatorname{tr}(H^2) = \operatorname{tr}\begin{pmatrix} \sum_{j=1}^{n} h_{1j}h_{j1} & * & * & \cdots \\ * & \sum_{j=1}^{n} h_{2j}h_{j2} & * & \cdots \\ * & * & \sum_{j=1}^{n} h_{3j}h_{j3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}h_{ji}.$$

Since h_{ij} may be expressed as $h_{ij} = x_{ij} + iy_{ij}$, we have

$$\begin{aligned} \operatorname{tr}(H^2) &= \sum_{i,j=1}^{n} h_{ij}h_{ji} \\ &= x_{11}^2 + x_{22}^2 + \cdots + x_{nn}^2 + 2x_{12}^2 + 2y_{12}^2 + x_{13}^2 + 2y_{13}^2 + 2x_{23}^2 + 2y_{23}^2 + \cdots \\ &= (x_{11}\ \ x_{22}\ \ \cdots\ \ x_{nn}\ \ x_{12}\ \ y_{12}\cdots)\, B \begin{pmatrix} x_{11} \\ x_{22} \\ \vdots \\ x_{nn} \\ x_{12} \\ \vdots \end{pmatrix}. \end{aligned}$$

Therefore B is a diagonal matrix with n terms equal to 1 and the rest of the diagonal entries equal to 2. Consequently the covariance matrix $C = B^{-1}$ is also diagonal with the first n terms equal to 1 and the rest of them equal to $\frac{1}{2}$. The first consequence is that

$$\langle h_{ij}h_{ji}\rangle = \langle x_{ij}^2\rangle + \langle y_{ij}^2\rangle = 1, \qquad \langle h_{ij}h_{kl}\rangle = 0 \text{ for all } (k,l) \neq (i,j). \tag{3.2}$$

Taking the determinant of B we obtain the following formula for the Gaussian measure:

$$\mathrm{d}\mu = \frac{2^{(n^2-n)/2}}{(2\pi)^{n/2}}\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(H^2)}\, \mathrm{d}H. \tag{3.3}$$

3.1.1 Expectation of the trace

As an application of the Wick formula introduced in subsection 2.7.3 we evaluate the expectation of the trace in the Gaussian Unitary Ensemble with respect to the measure in Equation (3.3). Since the trace is a polynomial of the terms h_{ij}, we can apply Wick formula. Take H^r for $r \in \mathbb{N}$. If $r = 2k+1$ then every Wick coupling has the form

$$\langle h_{i_1 i_2}h_{i_2 i_3}\rangle \cdots \langle h_{i_{2k-1} i_{2k}}h_{i_{2k} i_{2k+1}}\rangle \langle h_{i_{2k+1} i_1}\rangle,$$

and since $\langle h_{i_{2k+1} i_1}\rangle = 0$, it follows that

$$\langle \operatorname{tr}(H^{2k+1})\rangle = 0.$$

Now take H^{2k}, $k \in \mathbb{N}$. Then

$$\langle \mathrm{tr}(H^{2k}) \rangle = \sum \langle h_{i_1 i_2} h_{i_2 i_3} \rangle \cdots \langle h_{i_{2k-1} i_{2k}} h_{i_{2k} i_1} \rangle, \tag{3.4}$$

where the sum is taken over all the n^{2k} combinations of indices.

Example: Take $k = 2$. Therefore $r = 2k = 4$. Then we have n^4 products of the form

$$h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_1}. \tag{3.5}$$

The application of the Wick formula returns $(r-1)!! = 3!! = 3$ possible Wick couplings, given by

$$\begin{aligned} &\langle h_{i_1 i_2} h_{i_4 i_1} \rangle \langle h_{i_2 i_3} h_{i_3 i_4} \rangle \\ &\langle h_{i_1 i_2} h_{i_3 i_4} \rangle \langle h_{i_2 i_3} h_{i_4 i_1} \rangle \\ &\langle h_{i_1 i_2} h_{i_2 i_3} \rangle \langle h_{i_3 i_4} h_{i_4 i_1} \rangle. \end{aligned}$$

Taking the first Wick coupling,

$$\langle h_{i_1 i_2} h_{i_4 i_1} \rangle \langle h_{i_2 i_3} h_{i_3 i_4} \rangle, \tag{3.6}$$

the Equation (3.2) implies that

$$\begin{aligned} \langle h_{i_1 i_2} h_{i_4 i_1} \rangle = 1 \;\; &\Leftrightarrow \;\; i_1 = i_1,\; i_2 = i_4 \\ \langle h_{i_2 i_3} h_{i_3 i_4} \rangle = 1 \;\; &\Leftrightarrow \;\; i_3 = i_3,\; i_2 = i_4. \end{aligned}$$

Thus, the coupling contribution is n^3 because we have three free indices $\{i_1, i_2, i_3\}$ each one varying from 1 to n. Taking another Wick coupling

$$\langle h_{i_1 i_2} h_{i_3 i_4} \rangle \langle h_{i_2 i_3} h_{i_4 i_1} \rangle, \tag{3.7}$$

we obtain

$$\begin{aligned} \langle h_{i_1 i_2} h_{i_3 i_4} \rangle = 1 \;\; &\Leftrightarrow \;\; i_1 = i_4,\; i_2 = i_3 \\ \langle h_{i_2 i_3} h_{i_4 i_1} \rangle = 1 \;\; &\Leftrightarrow \;\; i_3 = i_4,\; i_2 = i_1. \end{aligned}$$

Therefore, $i_1 = i_2 = i_3 = i_4$ and the contribution of this pair is n, since it has only one free index. Finally, taking

$$\langle h_{i_1 i_2} h_{i_2 i_3} \rangle \langle h_{i_3 i_4} h_{i_4 i_1} \rangle, \tag{3.8}$$

we have,

$$\begin{aligned} \langle h_{i_1 i_2} h_{i_2 i_3} \rangle = 1 \;\; &\Leftrightarrow \;\; i_1 = i_2,\; i_3 = i_1, \\ \langle h_{i_3 i_4} h_{i_4 i_1} \rangle = 1 \;\; &\Leftrightarrow \;\; i_4 = i_4,\; i_3 = i_1, \end{aligned}$$

and the contribution of this pair is n^3, since the only restriction over its indices is $i_1 = i_3$. Thus,

$$\begin{aligned}\langle \mathrm{tr}(H^4)\rangle &= \langle h_{i_1 i_2} h_{i_4 i_1}\rangle \langle h_{i_2 i_3} h_{i_3 i_4}\rangle + \langle h_{i_1 i_2} h_{i_3 i_4}\rangle \langle h_{i_2 i_3} h_{i_4 i_1}\rangle + \langle h_{i_1 i_2} h_{i_2 i_3}\rangle \langle h_{i_3 i_4} h_{i_4 i_1}\rangle \\ &= 2n^3 + n.\end{aligned}$$

One question that naturally arises is how much we can predict about the coefficients of the powers of n after evaluate $\langle \mathrm{tr}(H^{2k})\rangle$. In order to answer this question we first normalize our Hermitian matrix mapping $h_{ij} \mapsto \frac{1}{\sqrt{n}} h_{ij}$ as in section 2.3. Then, the following result holds

Theorem 7 (Wigner semicircle law for GUE (GINIBRE, 1965))**.** For all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{n} \left\langle \mathrm{tr}\left(\frac{1}{\sqrt{n}} H\right)^{2k} \right\rangle = \frac{1}{2\pi} \int_{-2}^{2} x^{2k} \sqrt{4 - x^2}\, \mathrm{d}x = C_k,$$

where C_k are the Catalan numbers defined in section 2.3.

From the linearity of the expectation it follows that

$$\left\langle \mathrm{tr}\left(\frac{1}{\sqrt{n}} H\right)^{2k} \right\rangle = \frac{1}{n^k} \langle \mathrm{tr} H^{2k}\rangle,$$

and by Theorem 7,

$$\langle \mathrm{tr} H^{2k}\rangle = C_k n^{k+1} + O\left(n^k\right).$$

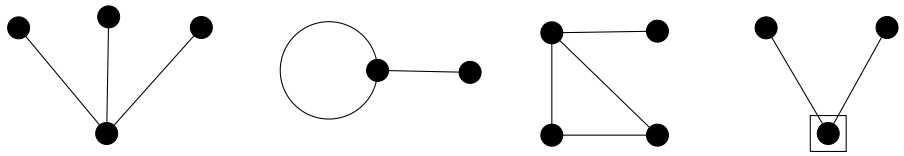
Thus, the major contribution in the expansion of $\langle \mathrm{tr} H^{2k}\rangle$ in powers of n has a Catalan number as its coefficient.

3.2 Map enumeration

A *graph* is a set of objects that might be related somehow. Graphically they are often represented by a set of points, called *vertices*, connected by lines, called *edges*. The edges represent the relation between the objects.

A graph is said to be *labelled* if their vertices are distinguishable. We say that a graph is *rooted* if one of its nodes, called root, is labelled in a special way. When the graph is *directed*, that is, its edges points towards some direction, one can think of the root as a starting point. We usually denote the root by a point inside a square.

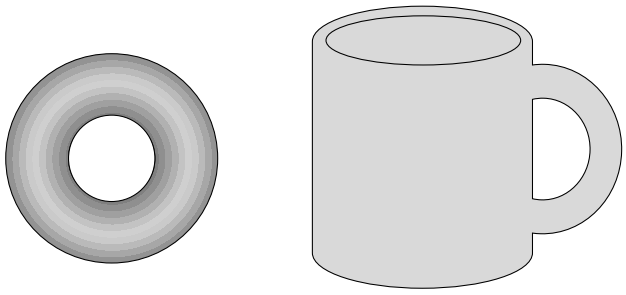
Figure 13 – Examples of graphs, including a rooted one.



Source: Elaborated by the author.

Graphs are allowed to have loops and multiple edges, but usually we require them to be connected. We also deal with a specific type of surface, namely a compact oriented bidimensional manifold with no boundary and genus $g \geq 0$. Up to homeomorphism such surfaces are uniquely determined by their genus. In this context, a well-known topological anecdote says that a donut and a cup are the same, since they have the same genus, i.e., the same number of "holes" (see Figure 14).

Figure 14 – A donut and a cup.



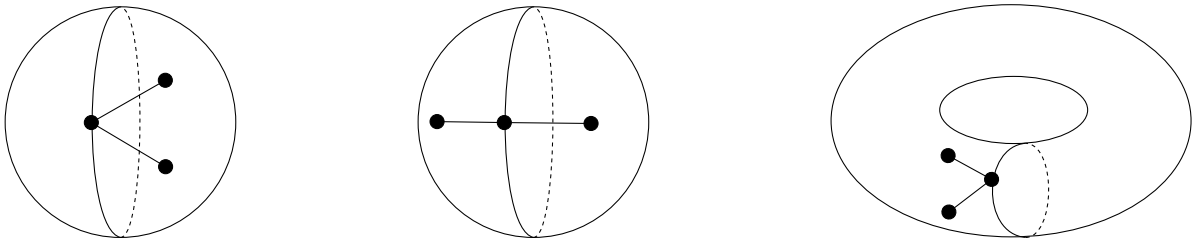
Source: Elaborated by the author.

Now we are able to define a *g-map*.

Definition 5. A *g-map* is a graph embedded to a compact bidimensional surface of genus g such that the edges do not intersect. Moreover, if one cuts along the edges the sets obtained are homeomorphic to open disks. This sets are called *faces*.

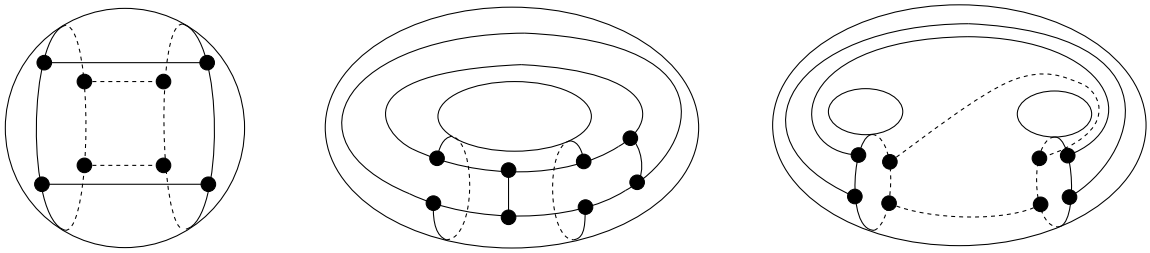
In Figure **??** and Figure **??** the sets of graph and surface define a 0-map, but the Figure **??** is not a 1-map. It occurs because if one cuts along the edges of the last graph a cylinder like surface is obtained. But such a surface can not be homeomorphic to a disk.

Figure 15 – Two examples and a counterexample of g-maps.



Source: Elaborated by the author.

Figure 16 – A cube embedded to a sphere, a torus and a bi-torus.



Source: Elaborated by the author.

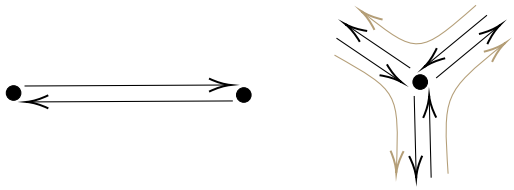
Since the same pair of graph and surface can generate different g-maps (see **??**) and a graph can be embedded to different surfaces (see Figure 16), one may ask what additional information we need to determine uniquely a g-map. The following proposition gives the answer.

Proposition 1 ((EDMONDS, 1960))**.** Any given cyclic order of edges of a graph around each vertex (chosen arbitrarily and independently at each vertex) determines uniquely the embedding of the graph into a surface.

Despite we do not present a formal proof, we give a geometric construction that shows how to recover the map given a cyclic order of the edges around each vertex. Such a construction also allow us to find all the possible g-maps given a graph.

At first, we represent an edge as a *two-way* street. The resulting object is called *ribbon graph*. We choose the cyclic order to be counter-clockwise, that is, arriving to a vertex by a street, the next edge to be travelled is the one on your right.

Figure 17 – A *two-way* street and a cyclic order around a vertex.



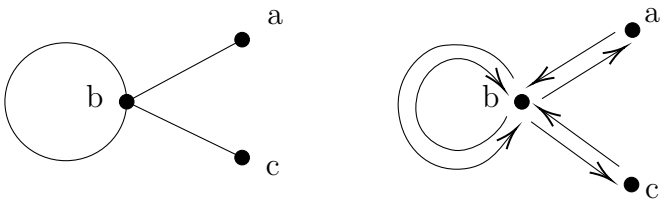
Source: Elaborated by the author.

The possible routes form a set of disjoint cycles, which corresponds to the faces in the definition of a g-map. The boundary of these faces is oriented clockwise. Gluing their sides such that the arrows glued point towards opposite directions, one recovers the g-map.

Notice that if the degree of a vertex v is denoted by d_v, the number of cyclic orders allowed at each vertex is $(d_v - 1)!$ and therefore the total number of possible g-maps is $\prod(d_v - 1)!$, where the product is taken over all the vertices.

Example: Let us start with the graph in Figure 18.

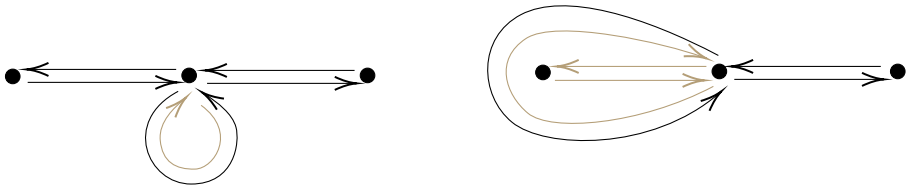
Figure 18 – A graph and its ribbon graph.



Source: Elaborated by the author.

Around the vertex a there is only one possible cyclic order. Around the vertex c the same holds. Thus the different g-maps comes from the possible different cyclic orders of the vertices around b. By the previous discussion, since b has degree 3, the number of maps is $(3-1)! = 2$. In order to construct them, notice that the only choice concerns to the step after the loop: coming from, say a, will it return for the vertex it came from or will it go to c once the loop is done? The two possible choices of cyclic orders are represented in Figure 19.

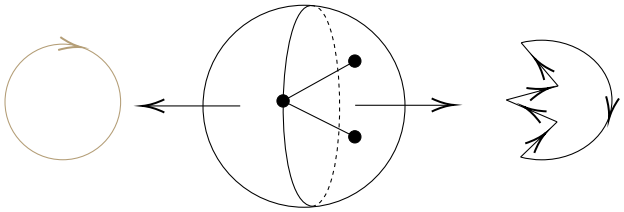
Figure 19 – Choices of cyclic order around vertex b.



Source: Elaborated by the author.

Each of these cyclic order defines uniquely a 0-map. Take, for example, the left one on Figure 19. The black arrows define a pentagon like shape and the gray arrow define a disk like shape. The only way we can glue these two surfaces around the graph is as shown in Figure 20.

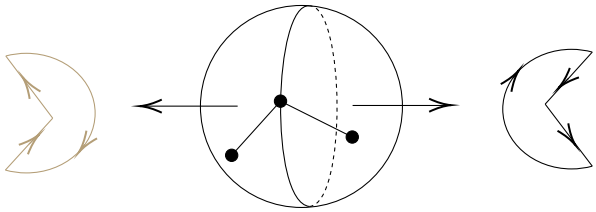
Figure 20 – The resulting g-map.



Source: Elaborated by the author.

Analogously, the cyclic order on the right side of Figure 19 generate the 0-map in Figure 21.

Figure 21 – The other g-map.



Source: Elaborated by the author.

3.2.1 One face map

One way to construct all the possible labelled g-maps from a connected graph with k edges is by a construction called *one face map*. In order to construct an one face map we start with a polygon of $2k$ labelled vertices. After splitting the indices into pairs we glue the corresponding sides two by two and obtain a g-map.

Given the first index we have $2k-1$ options for its pair. After this step there are $2k-2$ indices left. Given another index, we have $2k-3$ options for its pair, and so on, giving us a total of $(2k-1)!!$ possible maps, where the double factorial means $(2k-1)!! = (2k-1)\times(2k-3)\times\cdots\times 3\times 1$. The objects obtained constitute the set of the connected labelled g-maps with k edges.

The genus of an one face map is the greater number $g \in \mathbb{N}$ such that the graph can be embedded to a surface of genus g. Euler equation tell us that

$$\chi = V - E + F = 2 - 2g, \tag{3.9}$$

where χ is the Euler characteristic, V is the number of vertices, E is the number of edges and F is the number of faces. Since we have $F = 1$ and $E = k$, the number of vertices is given by

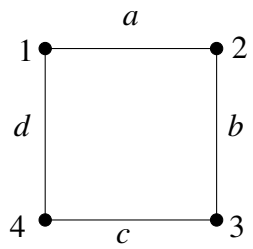
$$V = 1 - 2g + k. \tag{3.10}$$

We know that $V \geq 1$, thus $g \leq k/2$. So we can express the sum of all maps indexed by genus as

$$\sum_{g=0}^{[k/2]} \varepsilon_g = (2k-1)!!.$$

Example: Take $k = 2$. Our polygon is the square in Figure 22

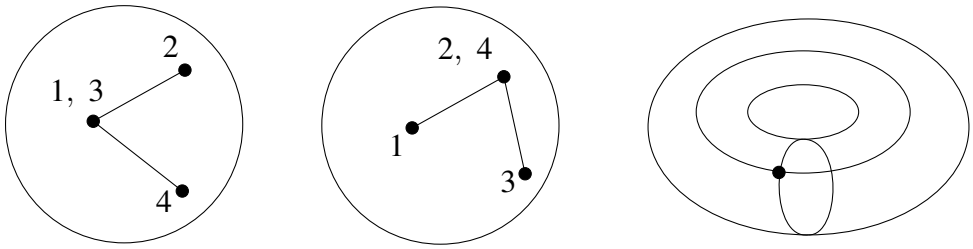
Figure 22 – A square with labelled vertices.



Source: Elaborated by the author.

We have a total of $(4-1)!! = 3 \times 1 = 3$ different maps, obtained by the pairings $(ab)(cd)$, $(ad)(bc)$ and $(ac)(bd)$. The last one has genus 1, while the others have genus 0.

Figure 23 – Labelled g-maps for a square.



Source: Elaborated by the author.

In Figure 23 one has, from the left to the right, the resulting labelled g-maps for the pairs $(ad)(bc)$, $(ac)(bd)$ and $(ac)(bd)$ respectively. Observe that the number of maps with genus 0 equals $C_2 = 2$, a Catalan number. It is not just a coincidence but a well-know result that

$$\varepsilon_0(k) = C_k = \frac{1}{k+1}\binom{2k}{k}.$$

3.2.2 Back to Wick couplings

Back to the example in subsection 3.1.1 one can see that the one face map approach provides a geometric interpretation for the Wick couplings. The set of Wick couplings equals the number of labelled maps with k edges. Recall that the possible Wick couplings in that example were given by

$$\langle h_{i_1 i_2} h_{i_4 i_1} \rangle \langle h_{i_2 i_3} h_{i_3 i_4} \rangle$$
$$\langle h_{i_1 i_2} h_{i_3 i_4} \rangle \langle h_{i_2 i_3} h_{i_4 i_1} \rangle$$
$$\langle h_{i_1 i_2} h_{i_2 i_3} \rangle \langle h_{i_3 i_4} h_{i_4 i_1} \rangle.$$

The set of indices $\{i_1, i_2, i_3, i_4\}$ can be geometrically represented by the vertices of a square, and the Wick couplings above correspond to the three maps in Figure 23.

Let us build one more example. Take $k = 4$. Then the trace is

$$\mathrm{tr}(H^{2k}) = \sum h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_5} h_{i_5 i_6} h_{i_6 i_7} h_{i_7 i_8} h_{i_8 i_1},$$

where the sum is taken over the n^8 possible products. Applying Wick formula we obtain

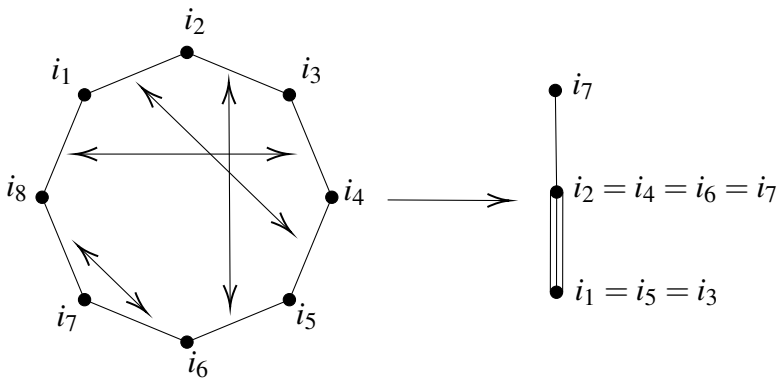
$$\langle \mathrm{tr}(H^{2k}) \rangle = \sum \langle h_{i_1 i_2} h_{i_2 i_3} \rangle \langle h_{i_3 i_4} h_{i_4 i_5} \rangle \langle h_{i_5 i_6} h_{i_6 i_7} \rangle \langle h_{i_7 i_8} h_{i_8 i_1} \rangle,$$

where the sum is now over the $7!! = 105$ possible Wick couplings. Take, for example, the coupling given by

$$\langle h_{i_1 i_2} h_{i_4 i_5} \rangle \langle h_{i_2 i_3} h_{i_5 i_6} \rangle \langle h_{i_3 i_4} h_{i_8 i_1} \rangle \langle h_{i_6 i_7} h_{i_7 i_8} \rangle.$$

It corresponds to the pairing $(i_1 i_2, i_4 i_5)(i_2 i_3, i_5 i_6)(i_3 i_4, i_8 i_1)(i_6 i_7, i_7 i_8)$ of the sides of a octagon, where $i_j i_k$ stands by the edge connecting the vertices i_j and i_k. The geometric representation of such a pairing is given in Figure 24.

Figure 24 – Graph representation for a Wick coupling.



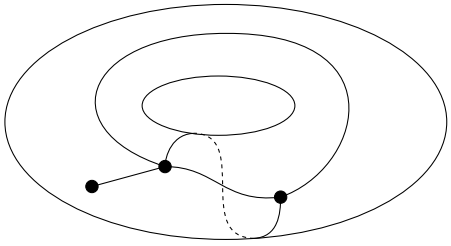
Source: Elaborated by the author.

Therefore, the indices restrictions are given by

$$\begin{aligned} & i_7, \\ & i_2 = i_4 = i_6 = i_8, \\ & i_1 = i_5 = i_3, \end{aligned}$$

and the respective one face map is given by the Figure 25.

Figure 25 – One face map for the Wick coupling.



Source: Elaborated by the author.

The relation between Wick couplings and one face maps is formalized by the following result:

Theorem 8 ((ZVONKIN, 1997)). Given $k \in \mathbb{N}$ and

$$\langle \mathrm{tr}(H^{2k}) \rangle = \int_{\mathcal{H}_n} \mathrm{tr}(H^{2k})\, \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is the Gaussian measure on the space of Hermitian matrices. Then

$$\langle \mathrm{tr}(H^{2k}) \rangle = \sum_{g=0}^{[k/2]} \varepsilon_g(k) n^{k+1-2g} \tag{3.11}$$

where $\varepsilon_g(k)$ is the number of labelled g-maps with k edges.

The results of Theorem 8 can be extended to arbitrary products of matrices, leading to

$$\langle \prod_{j=1}^{\eta} (\mathrm{tr} H^j)^{v_j} \rangle = \sum_g \varepsilon'_g n^{2-2g+\sum_{j=1}^{\eta}(j/2-1)v_j}, \tag{3.12}$$

where ε'_g equals the number of g-maps with v_j j-valent vertices, where $j = 1, \cdots, \eta$. In fact, for any $j \in \mathbb{N}$ we have

$$\mathrm{tr}(H^j) = \sum_{i^1,\cdots,i^j=1}^{n} m_{i^1 i^2} m_{i^2 i^3} \cdots m_{i^j i^1}$$

$$(\mathrm{tr}(H^j))^{v_j} = \sum m_{i_1^1 i_1^2} m_{i_1^2 i_1^3} \cdots m_{i_1^j i_1^1} m_{i_2^1 i_2^2} m_{i_2^2 i_2^3} \cdots m_{i_{v_j}^j i_{v_j}^1},$$

where the sum is over all the n^{jv_j} combinations. Thus,

$$\prod_{j=1}^{\eta} (\mathrm{tr}(H^j))^{v_j} = \prod_{j=1}^{\eta} \sum m_{i_1^1 i_1^2} m_{i_1^2 i_1^3} \cdots m_{i_1^j i_1^1} m_{i_2^1 i_2^2} \cdots m_{i_{v_j}^j i_{v_j}^1},$$

and the resulting sum is over the $n^{\sum_{j=1}^{\eta} jv_j}$ combinations. We can apply the Wick formula to the expectation of this new quantity. The contribution of each free index is n. The number of free indices such that the coupling result is 1, given a genus g, equals the number of faces F in the following geometric problem

$$\langle \prod_{j=1}^{\eta} (\mathrm{tr}(H^j))^{v_j} \rangle = \sum_g \#\{\text{g-map } v_j \text{ j-valent vertices}\} n^F.$$

By Euler characteristic formula we have that the number of faces is given by

$$F = \chi - V + E = 2 - 2g - V + E.$$

A simple calculation leads to

$$\left.\begin{aligned} V &= \sum_{j=1}^{\eta} v_j \\ E &= \sum_{j=1}^{\eta} \frac{jv_j}{2} \end{aligned}\right\} \Rightarrow n^F = n^{2-2g+\sum_{j=1}^{\eta}(\frac{j}{2}-1)v_j},$$

and, therefore, (3.12) holds.

CHAPTER

4

THE PARTITION FUNCTION

In this chapter we start with a family of integrals Z_n known as *partition functions* in statistical mechanics. Our final goal is to relate the number of connected maps by genus and the coefficients appearing in the asymptotic expansion of the logarithm of Z_n.

Except for some classical results in equilibrium measures and conformal maps in subsection 4.3.1, all of the remaining results are presented in full details.

Take the measure

$$\rho(H)\,\mathrm{d}H := \frac{1}{Z_\rho}\,\mathrm{e}^{-n\,\mathrm{tr}\,V(H)}\,\mathrm{d}H, \tag{4.1}$$

in the matrix space $\mathcal{H}_n$. This is the same measure of section 2.8, except for a normalization factor n. By the Weyl formula (2.12), $Z_\rho = c_n Z_n(\mathbf{t})$ where $Z_n(\mathbf{t})$ is the partition function for this random matrix model. Take $\mathbf{t} = (t_1, \cdots, t_m)$ and $\{t_1, \cdots, t_m\}$ real parameters and define the potential $V : \mathbb{R} \to \mathbb{R}$ as

$$V(\lambda; \mathbf{t}) = \frac{1}{2}\lambda^2 + \sum_{k=1}^{m} t_k \lambda^k. \tag{4.2}$$

Thus, $V(H) = V(H; \mathbf{t})$ is defined by

$$V(H; \mathbf{t}) = \frac{1}{2}M^2 + \sum_{k=1}^{m} t_k M^k. \tag{4.3}$$

The partition function is given by

$$Z_n(\mathbf{t}) := \int_{\mathbb{R}^n} \exp\left\{-n^2\left[\frac{1}{n}\sum_{j=1}^{n} V(\lambda_j; \mathbf{t}) - \frac{1}{n^2}\sum_{j\neq l}\log|\lambda_j - \lambda_l|\right]\right\}\mathrm{d}\lambda, \tag{4.4}$$

where the parameters $\{t_1, \cdots, t_m\}$ are such that the integral converges. Due to the logarithm term in Equation (4.4), $Z_n(\mathbf{t})$ can be written as

$$Z_n(\mathbf{t}) = \int_{\mathbb{R}^n} \prod_{j<l}(\lambda_j - \lambda_l)^2 \exp\left\{-n\sum_{j=1}^{n} V(\lambda_j; \mathbf{t})\right\}\mathrm{d}\lambda. \tag{4.5}$$

The Weyl Theorem allows us to think about the partition function as the joint probability density induced on the eigenvalues by an unitary ensemble.

As a consequence of this interpretation, asymptotic results for (4.4) provide information concerning statistics of eigenvalues of random matrices. Moreover, $Z_n(\mathbf{t})$ can be related to the expectation of the trace of a matrix. In order to illustrate how the relation between so different objects appears naturally when we differentiate $Z_n(\mathbf{t})$, take, for instance, the case where $t_i = 0 \ \forall\, i \neq 4$ and $t_4 = t$.

It allows us to relate $Z_n(\mathbf{t})$ to the counting of a specific type of map called *diagram*. By definition, a diagram is a labelled 4-valent g-map. Set

$$\hat{Z}_n(\mathbf{t}) := \frac{Z_n(\mathbf{t})}{Z_n(0)}. \tag{4.6}$$

By Weyl integration formula, Equation (2.12), one can express $\hat{Z}_n(\mathbf{t})$ in terms of a unitary ensemble for some matrix H with eigenvalues $\{\lambda_j\}_{j=1}^n$.

$$\begin{aligned}\hat{Z}_n(\mathbf{t}) &= \frac{1}{Z_n(0)} \int \cdots \int \prod_{j<l} (\lambda_j - \lambda_l)^2 \exp\left\{ -n \sum_{j=1}^{n} (\lambda_j^2 + \lambda_j^4 t) \right\} \mathrm{d}\lambda \\ &= \frac{c_0}{Z_n(0)} \int \cdots \int \exp\left\{ -\frac{n}{2} \operatorname{tr}(H^2) - nt \operatorname{tr}(H^4) \right\} \mathrm{d}H,\end{aligned}$$

where $c_0 > 0$ is a real constant. Since by section 3.2 we know how to relate the number of g-maps and the expectation of the trace of a matrix for a Gaussian Unitary Ensemble, we change variables $\lambda \mapsto \mu/\sqrt{n}$ in order to recover a Gaussian measure,

$$\begin{aligned}\hat{Z}_n(\mathbf{t}) &= \frac{1}{Z_0'} \int \cdots \int \exp\left\{ -\frac{t}{n} \operatorname{tr}(M^4) \right\} \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(M^2)} \mathrm{d}M \\ &= \left\langle \exp\left\{ -\frac{t}{n} \operatorname{tr}(M^4) \right\} \right\rangle,\end{aligned} \tag{4.7}$$

where $\{\mu_i\}_{i=1}^n$ are the eigenvalues of the new matrix M. Unfortunately the integral in (4.7) is not one of explicit solution. Even if $n = 1$ we have convergence issues in any neighbourhood of $t = 0$, because the expression diverges for negative values of t.

In order to simplify the calculations we begin by taking the Taylor series for the exponential, which is given by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{4.8}$$

If we could exchange the sum and the integral, then

$$\hat{Z}_n(\mathbf{t}) = \left\langle \sum_{k=0}^{\infty} \frac{(-\frac{t}{n}\operatorname{tr}(H^4))^k}{k!} \right\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-t}{n} \right)^k \langle (\operatorname{tr}(H^4))^k \rangle. \tag{4.9}$$

Since we know how the expected value of the trace relates to map enumeration by Equation (3.11), if Equation (4.9) were to hold, one could relate the coefficients in the asymptotic expansion of $\hat{Z}_n$ to the number of g-maps with k 4-valent vertices.

Fortunately, there exists a result that asserts the existence of a series expansion for the logarithm of the partition function for a certain set of parameters. Define

$$\mathbb{T}(T,\gamma)=\left\{t\in\mathbb{R}^{2m}:|t|\leq T, t_{2m}>\gamma\sum_{j=1}^{2m-1}|t_j|\right\}, \tag{4.10}$$

where $T>0$, $\gamma>0$ are given constants. The following holds.

Theorem 9 ((ERCOLANI; MCLAUGHLIN, 2003)). There exist $T>0$, $\gamma>0$ such that for every $\mathbf{t}\in\mathbb{T}(T,\gamma)$ and n large enough the asymptotic expansion

$$\log\left(\frac{Z_n(\mathbf{t})}{Z_n(0)}\right)\sim n^2e_0(\mathbf{t})+e_1(\mathbf{t})+\sum_{g=2}^{\infty}\frac{1}{n^{2g-2}}e_g(\mathbf{t}) \tag{4.11}$$

is valid, where $e_j(\mathbf{t})$ is an analytic function of $\mathbf{t}$ for each j. Furthermore, the expansion can be differentiated term-by-term.

Such expansion holds in the sense that if terms up to order n^{-2m} are kept, the error is bounded by kn^{-2m-2}, where k is a constant which does not depends on $\mathbf{t}$. Despite an explicitly motivation was presented just for the quartic case, we are interested in the proof of the general case. Such result is developed in the following sections.

4.1 Counting maps by partition function

Before going into the details of the proof, we explore the counting of g-maps in terms of the coefficients in the expansion of the logarithm. In order to do so, assume Theorem 9 holds, that is,

$$\log\left(\frac{Z_n(\mathbf{t})}{Z_n(0)}\right)\sim n^2e_0(\mathbf{t})+e_1(\mathbf{t})+\sum_{g=2}^{\infty}\frac{1}{n^{2g-2}}e_g(\mathbf{t}).$$

Taking the exponential in each side, one obtains

$$\frac{Z_n(\mathbf{t})}{Z_n(0)}\sim\exp\left\{n^2e_0(\mathbf{t})+e_1(\mathbf{t})+\sum_{g=2}^{\infty}\frac{1}{n^{2g-2}}e_g(\mathbf{t})\right\}. \tag{4.12}$$

Differentiating the left side of (4.12) we find

$$\left.\frac{\partial^m}{\partial t_1^{m_1}\cdots\partial t_\nu^{m_\nu}}\frac{Z_n(\mathbf{t})}{Z_n(0)}\right|_{t=0}=(-1)^m n^m\mathbb{E}\left[\prod_{j=1}^{\nu}(\operatorname{tr}H^j)^{m_j}\right], \tag{4.13}$$

where $\mathbb{E}$ denotes the expectation with respect to the probability measure given by Equation (4.6). The same change of variables of the previous section can be carried out in order to recover the Gaussian measure. Take the map $\lambda_j\mapsto\mu_j/\sqrt{n}$, where $\{\lambda_j\}_{j=1}^n$ are

the eigenvalues of a matrix H and $\{\mu_j\}_{j=1}^n$ are eigenvalues of a new matrix M. Equation (4.13) becomes

$$\frac{\partial^m}{\partial t_1^{m_1}\cdots\partial t_v^{m_v}}\frac{Z_n(\mathbf{t})}{Z_n(0)}\bigg|_{t=0} = (-1)^m n^m n^{-\Sigma_{j=1}^v j/2}\left\langle\prod_{j=1}^v(\mathrm{tr}M^j)^{m_j}\right\rangle \tag{4.14}$$

$$= (-1)^m\sum_g \#\{g-\text{map with } m_j \text{ j-valent vertices}\}n^{2-2g} \tag{4.15}$$

$$= \frac{\partial^m}{\partial t_1^{m_1}\cdots\partial t_v^{m_v}}\exp\left\{n^2e_0(\mathbf{t})+e_1(\mathbf{t})+\sum_{g=2}^\infty\frac{1}{n^{2g-2}}e_g(\mathbf{t})\right\}\bigg|_{t=0}, \tag{4.16}$$

where $\langle . \rangle$ is the expectation with respect to the Gaussian measure. The second equality comes from Equation (3.12). Now, starting from (4.16), one can recover the enumeration of maps by genus from the coefficients $\{e_j(\mathbf{t})\}_{j\in\mathbb{N}}$.

For example, take $m_k = m$. Thus

$$\frac{\partial^m}{\partial t_k^m}\exp\left\{\sum_{g=0}^\infty\frac{1}{n^{2g-2}}e_g(\mathbf{t})\right\}\bigg|_{t=0} = (-1)^m\sum_{g=0}\#\{g-\text{map with } m \text{ k-valent vertices}\}n^{2-2g},$$

that is,

$$\frac{\partial^m}{\partial t_k^m}\left\{\sum_{g=0}^\infty\frac{1}{n^{2g-2}}e_g(0)\right\} = (-1)^m\sum_{g=0}\#\{g-\text{map with } m \text{ k-valent vertices}\}n^{2-2g}.$$

If $m=1$, we have

$$(-1)\frac{\partial}{\partial t_k}e_0(0) = \#\{0-\text{map with 1 k-valent vertex}\},$$
$$(-1)\frac{\partial}{\partial t_k}e_1(0) = \#\{1-\text{map with 1 k-valent vertex}\},$$

and so on. Therefore, the coefficients $e_g(\mathbf{t})$ encode information about the number of labelled connected g-maps.

4.2 The partition function and orthogonal polynomials

In order to develop the expansion in Theorem 9, we start with the asymptotics for orthogonal polynomials with respect to the weight of the measure generated by Z_n. Then, we achieve the asymptotic expansion for the partition function by its relation with the *one point function* depending on a parameter n. Recall from section 2.8 that

$$\rho_n(\lambda) = \frac{\mathrm{e}^{-nV(\lambda)}}{n}\sum_{k=0}^{n-1}p_j^2(\lambda), \tag{4.17}$$

where V is the same as in Equation (4.2).

Now, taking the logarithm derivative of the partition function, one obtains

$$\begin{aligned}\frac{\partial}{\partial t_j}\log(Z_n(\mathbf{t})) =& \frac{1}{Z_n(\mathbf{t})}\int_{\mathbb{R}^n}\left(-n\sum_{j=1}^{n}\lambda_j^l\right)\times \\ & \exp\left\{-n^2\left[\frac{1}{n}\sum_{j=1}^{n}V(\lambda_j;\mathbf{t})-\frac{1}{n^2}\sum_{j\neq l}\log|\lambda_j-\lambda_l|\right]\right\}\mathrm{d}\lambda \qquad (4.18)\\ =& -n\mathbb{E}(\mathrm{tr}H^l) \qquad (4.19)\\ =& -n^2\int_{-\infty}^{\infty}\lambda^j\rho_n(\lambda)\,\mathrm{d}\lambda, \qquad (4.20)\end{aligned}$$

where $\mathbb{E}$ is the expectation with respect to the measure defined by (4.4) and the last equality follows from the relation between this expectation and the one-point function (see Equation (2.9)).

By the Equation (4.17), we have

$$\frac{\partial}{\partial t_j}\log(Z_n(\mathbf{t})) = -n\int_{-\infty}^{\infty}\lambda^j\mathrm{e}^{-nV(\lambda)}\sum_{j=0}^{n-1}p_j^2(\lambda)\,\mathrm{d}\lambda. \qquad (4.21)$$

Thus, the asymptotic expansion in Theorem 9 is obtained by the asymptotic behaviour of an integral of orthogonal polynomials, which we now study.

4.3 Asymptotics for orthogonal polynomials

The Riemann-Hilbert problem (RHP) approach developed by Fokas, Its and Kitaev and improved by Deift and Zhou (1992) with the development of the *non-linear steepest descent method* led to an asymptotic expansion for orthogonal polynomials (DEIFT *et al.*, 1999a).

In this section we replicate the asymptotic expansion of Deift *et al.* (1999a) in detail.

We look for a 2×2 matrix $Y(z)$ such that

i. $Y(z)$ is analytic at every $z\in\mathbb{C}/\mathbb{R}$.

ii. Y has boundary values as z approaches $s\in\mathbb{R}$

$$Y_\pm(s) := \lim_{\substack{z\to s\\ z \text{ on } \pm\text{side of } \mathbb{R}}} Y_\pm(z)$$

related by

$$Y_+(s) = Y_-(s)\begin{pmatrix}1 & w(s)\\ 0 & 1\end{pmatrix}, \qquad s\in\mathbb{R}$$

where $w(s)=\mathrm{e}^{-nV(s)}$.

iii. As $z \to \infty$,

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n} \end{pmatrix} = I + O\left(\frac{1}{z}\right).$$

The RHP for $Y(z)$ possesses a unique solution, which is intimately related to the orthogonal polynomials, as stated in the following result.

Theorem 10 ((FOKAS; ITS; KITAEV, 1992))**.** The matrix

$$Y(z) = \begin{pmatrix} \pi_n(z) & \displaystyle\int_{\mathbb{R}} \frac{\pi_n(s)w(s)\,\mathrm{d}s}{2\pi i(s-z)} \\ -2\pi i\gamma_{n-1}^2\pi_{n-1}(z) & \displaystyle\int_{\mathbb{R}} \frac{-\gamma_{n-1}^2\pi_{n-1}(s)w(s)\,\mathrm{d}s}{s-z} \end{pmatrix}, \tag{4.22}$$

where $\pi_n(z)$ is the n-th monic orthogonal polynomial with respect to the weight $w(z)$, is the unique solution to the Riemann-Hilbert problem stated previously for $Y(z)$.

We present a proof of such result in order to illustrate some common techniques in Riemann-Hilbert problems. First, we prove the uniqueness. Since $Y(z)$ is analytic for every $z \in \mathbb{C}\backslash\mathbb{R}$, so is $\det Y(z)$. Moreover, the jump condition for the $\det$ as $z \to s \in \mathbb{R}$ is such that

$$(\det Y(s))_+ = \det(Y_+(s)) = \det(Y_-(s))\det\begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix} = \det(Y_-(s)) = (\det Y(s))_-.$$

Thus, Morera's Theorem implies that $\det$ is also analytic for every $s \in \mathbb{R}$ and, consequently, $\det Y(z)$ is an entire function. Moreover, analyticity implies continuity and, therefore, the determinant is bounded on any compact set. Since $\det(Y(z)) \to I$ as $z \to \infty$, by continuity we obtain that it is bounded on the whole complex plane. By Liouville Theorem, any entire bounded function is constant. Consequently, $\det(Y(z)) = 1$ for all $z \in \mathbb{C}$.

Notice that $Y^{-1}(z)$ may be expressed as

$$Y^{-1}(z) = \frac{1}{\det Y(z)}\begin{pmatrix} Y_{22} & Y_{12} \\ -Y_{21} & Y_{11} \end{pmatrix}.$$

Since $\det Y(z) = 1$ and the entries of $Y(z)$ are analytic for every $z \in \mathbb{C}\backslash\mathbb{R}$, it follows that $Y^{-1}(z)$ is also analytic for every $z \in \mathbb{C}\backslash\mathbb{R}$.

Now, suppose that we have another solution to the RHP, say $\tilde{Y}$, and define $W := \tilde{Y}Y^{-1}$. Then, $W(z)$ is analytic for every $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, $W(z)$ satisfies the following jump condition as $z \to s \in \mathbb{R}$:

$$W_+(s) = (\tilde{Y}(s)Y^{-1}(s))_+ = \tilde{Y}_-(s)\begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -w(s) \\ 0 & 1 \end{pmatrix}Y_-^{-1}(s) = W_-(s).$$

Therefore, Morera's Theorem allows us to conclude that $W(z)$ is entire. But, since $W(z)$ is continuous, it follows that $W(z)$ is bounded in any compact. Moreover, since $W(z) \to I$ as $z \to \infty$, it follows that $W(z)$ is bounded in the whole complex plane and Liouville Theorem implies that $W = I$. Therefore, the solution, if it exists, is unique.

Now, we look for a solution entrywise. The jump condition implies that

$$Y_{11,+}(s) = Y_{11,-}(s) \tag{4.23}$$
$$Y_{21,+}(s) = Y_{21,-}(s) \tag{4.24}$$
$$Y_{12,+}(s) = Y_{12,-}(s) + w(s)Y_{11}(s), \tag{4.25}$$
$$Y_{22,+}(s) = Y_{22,-}(s) + w(s)Y_{21}(s), \tag{4.26}$$

and the asymptotic condition gives us that

$$Y(z) = \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}. \tag{4.27}$$

By Equations (4.23) and (4.24) plus Morera's Theorem, we know that $Y_{11}(z)$ and $Y_{21}(z)$ are entire functions. Moreover, Equation (4.27) implies that $Y_{11}(z)$ is a monic polynomial of degree n and $Y_{21}(z)$ is a polynomial of degree at most $n-1$.

On the other hand, Equations (4.25) and (4.26) give rise to linear RHP's. Their solution is expressed by the Plemelj formula for every $z \in \mathbb{C}\backslash\mathbb{R}$,

$$Y_{12}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(s)Y_{11}(s)}{s-z}\,\mathrm{d}s, \tag{4.28}$$
$$Y_{22}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(s)Y_{21}(s)}{s-z}\,\mathrm{d}s. \tag{4.29}$$

The geometric series expansion for $\frac{1}{s-z}$ allows us to expand Equations (4.28) and (4.29) as $z \to \infty$ and obtain

$$Y_{12}(z) = -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\frac{1}{z^{j+1}}\int_{-\infty}^{\infty} s^j w(s)Y_{11}(s)\,\mathrm{d}s + O(z^{-n-1}), \tag{4.30}$$
$$Y_{22}(z) = -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\frac{1}{z^{j+1}}\int_{-\infty}^{\infty} s^j w(s)Y_{21}(s)\,\mathrm{d}s + O(z^{-n-1}). \tag{4.31}$$

Therefore, comparing the asymptotic condition (4.27) with (4.30), one obtain that

$$\int_{-\infty}^{\infty} s^j w(s)Y_{11}(s)\,\mathrm{d}s = 0, \quad \text{for } j = 0, \cdots, n-1,$$

and since $Y_{11}(s)$ is a monic polynomial of degree n we conclude that $Y_{11}(s)$ is precisely the n-th monic orthogonal polynomial with respect to the weigh $w(s)$. Analogously, one obtains that $Y_{21}(s) = c\pi_{n-1}(z)$, where $\pi_{n-1}(z)$ is the $(n-1)$-th monic orthogonal polynomial with

respect to the weigh $w(s)$ and $c = -2\pi i \gamma_{n-1}^2$ is a constant factor. Therefore, the solution of the RHP for $Y(z)$ is given by Equation (4.22). ■

Once Theorem 10 is proved, one can obtain an explicitly n-dependent asymptotics for the orthogonal polynomials through achieving an expansion for the solution $Y(z)$ of the above RHP. The algorithm to be followed starts with a sequence of transformations in order to simplify the initial RHP, as described below.

- $Y(z) \mapsto M(z)$: the first transformation involves the so called *equilibrium measure* and is intended to normalize the asymptotic condition of the initial RHP, that is, to turn the asymptotic condition for the initial RHP into something of the form
$$I + O(z^{-1}), \quad \text{as } z \to \infty;$$
- $M(z) \mapsto S(z)$: we open lenses around the support of the measure to turn the oscillatory terms into exponential decaying ones;
- Then, a parametrix solution $S^{(par)}$ is constructed;
- $S(z)$ is related to the error matrix $R(z)$ with the help of $S^{(par)}$ and the so-called *small norm theory* leads to the desired asymptotic behaviour.

The goal of the following sections is to provide this detailed analysis. We begin with the definition of some important functions which will be crucial for the next steps.

4.3.1 Important objects

In this subsection we discuss several objects that we need. The focus is on introducing them and we do not provide proofs.

4.3.1.1 Equilibrium measure

Before our first transformation we must introduce an important object in approximation theory, the so called *equilibrium measure* μ_V (see Saff and Totik (1997)) defined as the unique minimizer of the operator

$$E = \int_{\mathbb{R}^2} \log|x-y|^{-1} \, d\mu(x) \, d\mu(y) + \int_{\mathbb{R}} V(x) \, d\mu(x), \tag{4.32}$$

over is the space of all probability measures on $\mathbb{R}$. Under some assumptions on the potential $V(x)$, the equilibrium measure uniquely exists and is supported on one single interval.

The set of available values for $\mathbf{t}$ in Theorem 9 appears in order to assure that the measure is well-behaved in the sense that its support is a compact interval and its density is a non-vanishing analytic function, which behaves at worst as a square root at the end-points of the support.

Theorem 11 ((DEIFT *et al.*, 1999a)). There exists $t_0 > 0$, $\gamma_0 > 0$ real numbers such that for every $\mathbf{t} \in \mathbb{T}(t_0, \gamma_0)$ it follows that $\mathrm{d}\mu_V = \hat{\psi}(x)\,\mathrm{d}x$, where

$$\hat{\psi}(x) = \frac{\chi_{[a,b]}(x)}{2\pi}\sqrt{(b-x)(x-a)}h(x), \tag{4.33}$$

with $h(x)$ a polynomial of degree $2m-2$ given by

$$h(x) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{V'(s)}{\sqrt{(s-a)(b-s)}}\frac{\mathrm{d}s}{s-x} \tag{4.34}$$

where Γ is any contour that encloses a, b and x. Moreover, the endpoints of the support are determined by the equations

$$\int_a^b \frac{V'(s)}{\sqrt{(s-a)(b-s)}}\,\mathrm{d}s = 0, \quad \text{and} \quad \int_a^b \frac{sV'(s)}{\sqrt{(s-a)(b-s)}}\,\mathrm{d}s = 2\pi. \tag{4.35}$$

From the characterization in Theorem 11 we have, for example, that $\mathbf{t} = 0$ leads to the Semicircle distribution (see section 2.3). In fact, if one takes $\mathbf{t} = 0$ in Equation (4.2) and perform the change of variables $s \mapsto y := \frac{2s-a-b}{b-a}$, the conditions in (4.35) become

$$\int_{-1}^{1}\frac{(b-a)y+(b+a)}{2\sqrt{1-y^2}}\,\mathrm{d}y = 0,$$
$$\int_{-1}^{1}\frac{[(b-a)y+(b+a)]^2}{4\sqrt{1-y^2}}\,\mathrm{d}y = 2\pi.$$

By trigonometric substitution we obtain $a+b=0$ and $b-a=4$. Moreover, similar calculations shows that $h(x)=1$ and the density obtained is the Semicircle distribution.

Another characterization to the equilibrium measure is given by the Euler-Lagrange equations: there exists a constant $\ell \in \mathbb{R}$ such that (see Saff and Totik (1997))

$$2\int \log|x-y|\,\mathrm{d}\mu_V(y) - V(x) - \ell = 0 \qquad x \in \mathrm{supp}(\mu_V), \tag{4.36}$$
$$2\int \log|x-y|\,\mathrm{d}\mu_V(y) - V(x) - \ell \leq 0 \qquad x \in \mathbb{R}. \tag{4.37}$$

4.3.1.2 The g-function

In the solution of the RHP for $Y(z)$, it will be required a *g-function*. The set of conditions imposed over $g(z)$ is called *phase conditions*.

i. $g(z)$ is analytic at every $z \in \mathbb{C}\backslash\mathbb{R}$.

ii. As $z \to \infty$, $g(z) = \log z + O(1)$.

iii. There exists a closed interval $I \subset \mathbb{R}$ such that $g_+(s) - g_-(s)$ is purely imaginary for $s \in I$ and

$$g_+(s) + g_-(s) - V_n(s) - \ell = 0 \qquad s \in I \tag{4.38}$$

$$i\frac{\mathrm{d}}{\mathrm{d}s}(g_+(s) - g_-(s)) > 0, \qquad s \in I \tag{4.39}$$

$$g_+(s) + g_-(s) - V_n(s) - \ell < 0 \qquad s \in \mathbb{R}\backslash I \tag{4.40}$$

$$\mathrm{e}^{g_+(s)-g_-(s)} = 1 \qquad s \in \mathbb{R}\backslash I. \tag{4.41}$$

Set

$$g(z) = \int \log(z-x)\,\mathrm{d}\mu(x), \quad z \in \mathbb{C}\backslash(-\infty, b]. \tag{4.42}$$

If μ is compactly supported in a single interval, then the phase conditions hold if and only if the Euler-Lagrange Equations (4.36) and (4.37) are satisfied. That is, $g(z)$ given by Equation (4.42) solves phase conditions if and only if $\mu = \mu_V$, where μ_V is the equilibrium measure defined in the previous section.

In fact, take μ_V the equilibrium measure. Then, for every $s \in (a, b)$,

$$\begin{aligned} g_+(s) &= \lim_{\varepsilon\to 0}\int_a^b \log(s + i\varepsilon - x)\,\mathrm{d}\mu_V(x) \\ &= \lim_{\varepsilon\to 0}\left(\int_a^s \log(\sqrt{(s-x)^2+\varepsilon^2}) + i\arg(|s-x| + i\varepsilon)\,\mathrm{d}\mu_V(x) + \right. \\ &\quad \left. \int_s^b \log(\sqrt{(s-x)^2+\varepsilon^2}) + i\arg(-|s-x| + i\varepsilon)\,\mathrm{d}\mu_V(x)\right). \end{aligned}$$

It is straightforward that $\lim_{\varepsilon\to 0}\log(\sqrt{(s-x)^2+\varepsilon^2}) = \log(|s-x|)$. Moreover, $\arg(|s-x| + i\varepsilon) = \theta$ such that

$$\begin{aligned} \cos\theta &= \frac{|s-x|}{\sqrt{(s-x)^2+\varepsilon^2}} \\ \sin\theta &= \frac{\varepsilon}{\sqrt{(s-x)^2+\varepsilon^2}}. \end{aligned}$$

Taking the limits $\sin\theta \downarrow 0$ and $\cos\theta \uparrow 1$ one concludes that $\theta = 0$. Analogously, $\arg(-|s-x| + i\varepsilon) = \theta$ such that

$$\begin{aligned} \cos\theta &= \frac{-|s-x|}{\sqrt{(s-x)^2+\varepsilon^2}} \\ \sin\theta &= \frac{\varepsilon}{\sqrt{(s-x)^2+\varepsilon^2}}, \end{aligned}$$

and taking the limits $\sin\theta \downarrow 0$ and $\cos\theta \downarrow -1$ one obtains $\theta = \pi$. Then,

$$g_+(s) = \int_a^b \log(|s-x|)\,\mathrm{d}\mu_V(x) + \int_s^b i\pi\,\mathrm{d}\mu_V(x). \tag{4.43}$$

Proceeding analogously for $g_-(s)$ one obtains

$$g_-(s) = \int_a^b \log(|s-x|)\,\mathrm{d}\mu_V(x) - \int_s^b i\pi\,\mathrm{d}\mu_V(x). \tag{4.44}$$

Such characterization of g_- and g_+ implies (4.39). Moreover, Equations (4.43) and (4.44) together with (4.36) and (4.37) imply (4.38) and (4.40). Furthermore, if $s < a$ then $\operatorname{Im} g_+ = \pi$, $\operatorname{Im} g_- = -\pi$ and (4.41) holds. If $s > b$ it is analogous.

Now suppose that the phase conditions hold. The characterization of g_+ and g_- provided by Equations (4.43) and (4.44) together with (4.38) and (4.40) implies the Euler Lagrange Equations (4.36) and (4.37). Therefore, $\mathrm{d}\mu = \mathrm{d}\mu_V$.

4.3.1.3 Conformal maps

The construction of a local parametrix around the endpoints of the support of the equilibrium measure involves a *conformal map*, i.e., an invertible analytic transform that takes the original problem into a new one centered in the origin. We start defining a map in a neighbourhood U_δ of b given by

$$\begin{aligned} f(z) &= \left(\frac{3n}{4}\right)^{2/3} \left(i\int_b^z (b-y)^{1/2}(y-a)^{1/2}h(y)\,\mathrm{d}y\right)^{2/3} \\ &= \left(\frac{3n}{4}\right)^{2/3} (-2g(z)+V(z)+\ell)^{2/3}. \end{aligned}$$

If one defines $\psi(z)$ as the analytic continuation of $\hat{\psi}(x)$ to $\mathbb{C}\backslash((-\infty,a]\cup[b,\infty))$ then $f(z)$ can be expressed as

$$f(z) = \left(\frac{3n}{4}\right)^{2/3} \left(2\pi i\int_b^z \psi(y)\,\mathrm{d}y\right)^{2/3}.$$

The measure $\psi(z)$ vanishes as square-root, therefore its integral behaves like a $\frac{3}{2}$ power and it follows that $f(z)$ is analytic in z in a neighbourhood of b. Furthermore, there exists a real constant $c > 0$ such that

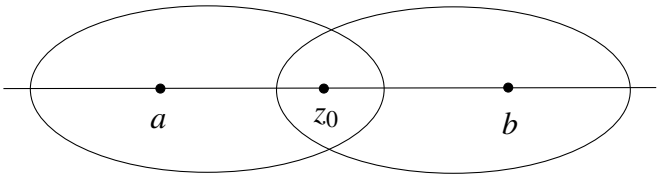
$$f(z) = cn^{2/3}[(z-b) + O((z-b)^2)].$$

Analogously, one can define a conformal map $\tilde{f}$ in a neighbourhood $\tilde{U}_\delta$ of a by

$$\begin{aligned} \tilde{f}(z) &= \left(\frac{3n}{4}\right)^{2/3} \left(i\int_b^z (b-y)^{1/2}(y-a)^{1/2}h(y)\,\mathrm{d}y - i\int_b^a (b-y)^{1/2}(y-a)^{1/2}h(y)\,\mathrm{d}y\right)^{2/3} \\ &= \left(\frac{3n}{4}\right)^{2/3} (2g(a)-V(a)-\ell-(2g(z)-V(z)-\ell))^{2/3}. \end{aligned}$$

Near the endpoint a one has $\tilde{f}(z) = -cn^{2/3}[(z-a) + O((z-a)^2)]$ for some real constant $c > 0$. The neighbourhoods where f and $\tilde{f}$ hold can be extended to overlapping regions as in Figure 26.

Figure 26 – New domains of definition for f and $\tilde{f}$.



Source: Elaborated by the author.

Lemma 1 (Lemma 4.3, (ERCOLANI; MCLAUGHLIN, 2003)). There exists a unique z_0 in (a,b) such that $f(z_0) = \tilde{f}(z_0)$. Moreover, there exist T_b and γ_b such that for $t \in \mathbb{T}(T_b, \gamma_b)$ there exists a neighbourhood U of $[z_0, b]$ independent of n in which f is a conformal map. An analogous result holds to $\tilde{f}(z)$.

The analytical issue in here is whether $f(z)$ remains a conformal map from a fixed neighbourhood of $[z_0, b]$ into its range. For analyticity notice that for $\mathbf{t} = 0$

$$f(z)|_{t=0} = \left(\frac{3n}{4}\right)^{2/3} \left(\int_2^z \sqrt{s^2-4}\,\mathrm{d}s\right)^{2/3},$$

which is analytic for an open neighbourhood of $(-2,2]$. Since f depends analytically on the parameter $\mathbf{t}$, the existence of an open and connected set of values of $\mathbf{t}$ for which it remains analytic follows.

4.3.2 First transformation: $Y(z) \to M(z)$

The standard procedure to normalize the asymptotic condition is to choose $p(z)$ a function and α a constant such that the transformation

$$Y(z) \mapsto \mathrm{e}^{\alpha\sigma_3} Y(z) \mathrm{e}^{p(z)\sigma_3} =: M(z), \tag{4.45}$$

leads to the desired asymptotic behaviour,

$$M(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

Thus, α and $p(z)$ are chosen in such a way that

$$\mathrm{e}^{\alpha\sigma_3}\left[I + O\left(\frac{1}{z}\right)\right]\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}\mathrm{e}^{p(z)\sigma_3} = I + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

This condition is equivalent to

$$p(z) = n\log z + o(\log z), \quad z \to \infty. \tag{4.46}$$

We want to keep the domain of analyticity, i.e.,

$$p(z) \text{ is analytic at every } z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.47}$$

Moreover, we take this opportunity to turn the jump into an exponentially decaying one. To do so, we impose that

$$\mathrm{e}^{-p_-(s)\sigma_3}\begin{pmatrix}1 & \mathrm{e}^{-nV(s)}\\ 0 & 1\end{pmatrix}\mathrm{e}^{p_+(s)\sigma_3} \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad s\to\infty.$$

One way to achieve this is to require the existence of a compact set $I \subset \mathbb{R}$ such that for every $s \notin I$,

$$p_+(s) + p_-(s) + nV(s) > 0. \tag{4.48}$$

The jump matrix for $s \in I$ is given by

$$\begin{pmatrix}\mathrm{e}^{p_+(s)-p_-(s)} & \mathrm{e}^{-(p_+(s)+p_-(s)+nV(s))}\\ 0 & \mathrm{e}^{-(p_+(s)-p_-(s))}\end{pmatrix}.$$

A simplification of the problem for $s \in I$ is obtained by the technique of *opening of lenses*, what can be done in the next step if an extra condition is imposed over $p(z)$: the jump matrix needs to be oscillatory. Therefore, for every $s \in I$ we impose

$$\mathrm{Re}(p_+(s) - p_-(s)) = 0, \tag{4.49}$$

and, to deal with the 1,2-entry, we require that

$$p_+(s) + p_-(s) + nV(s) = c, \tag{4.50}$$

for some constant c. The conditions (4.46), (4.47), (4.48), (4.49) and (4.50) lead to the choice

$$p(z) = -n\left(g(z) - \frac{\ell}{2}\right),$$
$$\alpha = -n\frac{\ell}{2},$$

where $g(z)$ is the g-function defined in subsection 4.3.1.2. We introduce the new information into Equation (4.45) and obtain the first transformation

$$M(z) = \mathrm{e}^{-n\frac{\ell}{2}\sigma_3} Y(z)\, \mathrm{e}^{-n(g(z)-\frac{\ell}{2})\sigma_3}, \tag{4.51}$$

where σ_3 is the Pauli matrix given by

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The matrix $M(z)$ is the unique solution for the new RHP

i. $M(z)$ is analytic at every $z \in \mathbb{C}\backslash\mathbb{R}$.

ii. M has boundary values as z approaches $\mathbb{R}$ related by

$$M_+(s) = M_-(s) \begin{pmatrix} e^{-n(g_+(s)-g_-(s))} & e^{n(g_+(s)+g_-(s)-V(s)-\ell)} \\ 0 & e^{n(g_+(s)-g_-(s))} \end{pmatrix}, \qquad s \in \mathbb{R}.$$

iii. As $z \to \infty$,

$$M(z) = I + O\left(\frac{1}{z}\right).$$

4.3.3 Second transformation: $M(z) \to S(z)$

Some of the properties required in the construction of the g-function, allow us to turn the oscillatory terms in the jump matrix into exponentially decaying terms for $s \in I := [a,b]$ by splitting the jump matrix as follows

$$\begin{pmatrix} e^{-n(g_+(s)-g_-(s))} & 1 \\ 0 & e^{n(g_+(s)-g_-(s))} \end{pmatrix} = \tag{4.52}$$

$$\begin{pmatrix} 1 & 0 \\ e^{n(g_+(s)-g_-(s))} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n(g_+(s)-g_-(0))} & 1 \end{pmatrix} =: v_- v_0 v_+. \tag{4.53}$$

Since $g_+ + g_- - V - \ell = 0$ in the support of the equilibrium measure, Equation (4.33), it follows that $g_+ - g_- = -2g_- + V + \ell = 2g_+ - V - \ell$. Thus,

$$h(z) := g_+(z) - g_-(z)$$

possesses analytic continuation above and below $[a,b]$. The second phase condition in subsection 4.3.1.2 plus the Cauchy-Riemann equations implies that

$$\begin{aligned} h(x,y) &= u(x,y) + iv(x,y), \\ v(x,y) &= g_+(x,y) - g_-(x,y), \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} < 0, \end{aligned}$$

and therefore $\mathrm{Re}\, h(z) > 0$ above $[a,b]$ and $\mathrm{Re}\, h(z) < 0$ below $[a,b]$. Thus $\exp(g_+ - g_-)$ and $\exp(-[g_+ - g_-])$ decay exponentially in the lower and in the upper half-plane respectively and we can open lenses around the interval $[a,b]$. Moreover, $h(x)$ can be extended to $\mathbb{C}\backslash((-\infty,a] \cup [b,\infty))$.

Take a lens-shaped region around $[a,b]$ such that its contour contains the endpoints of the interval. Define the new matrix $S(z)$ as follows.

$$S(z) = M(z) \times \begin{cases} I & \text{outside the lens-shaped region,} \\ v_+^{-1} & \text{in the upper lens,} \\ v_- & \text{in the lower lens.} \end{cases} \tag{4.54}$$

The new RHP is given as follows

i. $S(z)$ is analytic at every $z \in \mathbb{C}\backslash\Sigma_s$, where Σ_s is the contour in Figure 27.

ii. S has boundary values as z approaches $s \in \Sigma_s$ related by

$$S_+(s) = S_-(s) \times \begin{cases} v_+ & s \in \Sigma_1 \\ v_0 & s \in \Sigma_2 \\ v_- & s \in \Sigma_3 \\ v_s & s \in \Sigma_4 \cup \Sigma_5 \end{cases}$$

where v_+, v_-, v_0 are the same as in Equation (4.53) and

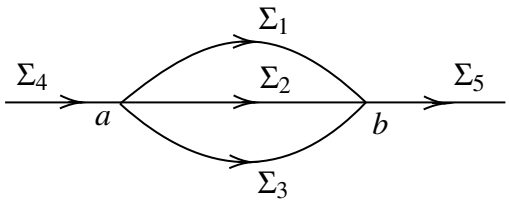
$$v_s = \begin{pmatrix} 1 & e^{n(g_+(s)+g_-(s)-V(s)-\ell)} \\ 0 & 1 \end{pmatrix}.$$

iii. As $z \to \infty$,

$$S(z) = I + O\left(\frac{1}{z}\right).$$

iv. $S(z) = O(1)$ as $z \to a, b$.

Figure 27 – Contour of the RHP for S.



Source: Elaborated by the author.

On the one hand, we have a more complicated contour, but on the other hand, all jump matrices go to the identity, except for the one in the support of the equilibrium measure. Therefore, when n is large enough, the problem becomes close to a limiting RHP whose solution is called *Global Parametrix*.

4.3.4 Global Parametrix

The only jump that does not goes to the identity when n grows, is the one over the contour $\Sigma_2 = [a, b]$. Therefore, the Global Parametrix must solve the following RHP

i. $S^{(1)}(z)$ is analytic at every $z \in \mathbb{C}\backslash[a, b]$.

ii. $S^{(1)}$ has boundary values as z approaches (a,b) related by

$$S^{(1)}_+(s) = S^{(1)}_-(s)v_0 = S^{(1)}_-(s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad s \in (a,b).$$

iii. As $z \to \infty$,

$$S^{(1)}(z) = I + O\left(\frac{1}{z}\right).$$

iv. $S^{(1)}$ possesses at worst square-integrable singularities as $z \to a, b$.

Since the jump matrix is now constant, the standard procedure is to diagonalize such matrix, solve the new problem and perform an unitary transform back to the original problem. Thus, take U_0 such that

$$U_0 v_0 U_0^{-1} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The transformation

$$S^{(1)}(z) \mapsto U_0 S^{(1)}(z) U_0^{-1} =: N(z),$$

gives rise to the RHP below.

i. $N(z)$ is analytic at every $z \in \mathbb{C}\backslash[a,b]$.

ii. N has boundary values as z approaches (a,b) related by

$$N_+(s) = N_-(s)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

iii. As $z \to \infty$,

$$N(z) = I + O\left(\frac{1}{z}\right).$$

iv. $N(z)$ possesses at worse square-integrable singularities as $z \to a, b$.

Looking at the jump entrywise, one obtains for every $s \in (a,b)$

$$\begin{aligned} N_{11,+}(s) &= (-i)N_{11,-}(s) \\ N_{21,+}(s) &= (-i)N_{21,-}(s) \\ N_{12,+}(s) &= iN_{12,-}(s) \\ N_{22,+}(s) &= iN_{22,-}(s). \end{aligned}$$

These are scalar RHP's that we solve one by one. Taking the logarithm of the $1,1$-entry,

$$\log(N_{11,+}(s)) = \log(N_{11,-}(s)) + \log(-i),$$

which is the jump for a scalar RHP whose solution is well-known by means of the *Cauchy transform* as

$$\begin{aligned}\log(N_{11}(z)) &= \frac{1}{2\pi i}\int_a^b \frac{\log(-i)}{s-z}\,\mathrm{d}s \\ &= -\frac{1}{4}\log\left(\frac{z-b}{z-a}\right) \\ \therefore N_{11}(z) &= \left(\frac{z-b}{z-a}\right)^{-1/4}.\end{aligned}$$

Proceeding analogously to the other entries, one obtains that

$$N(z) = \begin{pmatrix} \alpha(z)^{-1} & 0 \\ 0 & \alpha(z) \end{pmatrix},$$

where

$$\alpha(z) = \frac{(z-b)^{1/4}}{(z-a)^{1/4}}. \tag{4.55}$$

Consequently,

$$S^{(1)}(z) = \frac{1}{2}\begin{pmatrix} \alpha(z)+\alpha(z)^{-1} & i(\alpha(z)^{-1}-\alpha(z)) \\ i(\alpha(z)-\alpha(z)^{-1}) & \alpha(z)+\alpha(z)^{-1} \end{pmatrix}. \tag{4.56}$$

Notice that one could have chosen the transformation $S^{(1)}(z) \mapsto U_0^{-1}S^{(1)}(z)U_0$, what lead us to the conclusion that $S^{(1)}(z) = U_0^{-1}N(z)U_0 = U_0N^{-1}(z)U_0^{-1}$. Unfortunately, this solution does not behave well in the neighbourhood of the endpoints of the support of the equilibrium measure. Therefore, we must construct an explicit solution for small neighbourhoods of a,b and estimate the difference between the approximated solution and the original one.

4.3.5 Local Parametrix

Take $U_\delta, \tilde{U}_\delta$ circles of ratio δ around b and a respectively. The local solutions $P(z)$ and $\tilde{P}(z)$ need to match $S^{(1)}(z)$ in the boundary of U_δ and $\tilde{U}_\delta$, to satisfy the same jump conditions and to be analytic in $U_\delta \backslash \Sigma_S$ and $\tilde{U}_\delta \backslash \Sigma_S$.

The conditions over the solution $P(z)$ around b leads to the following RHP:

i. $P(z)$ is analytic at every $z \in U_\delta \backslash (\Sigma_S \cap U_\delta)$;

ii. P has boundary values as z approaches $s \in \Sigma_S$ related by

$$P_+(s) = P_-(s) \times \begin{cases} v_+ & s \in \Sigma_1 \cap U_\delta \\ v_0 & s \in \Sigma_2 \cap U_\delta \\ v_- & s \in \Sigma_3 \cap U_\delta \\ v_s & s \in \Sigma_5 \cap U_\delta \end{cases}$$

where

$$v_s = \begin{pmatrix} 1 & \mathrm{e}^{n(g_+(s)+g_-(s)-V(s)-\ell)} \\ 0 & 1 \end{pmatrix},$$

and Σ_i's are the same of Figure 27.

iii. If $|z-b| = \delta$,

$$P(z) = [I + o(1)]S^{(1)}(z).$$

iv. $P(z)$ remains bounded as $z \to b$.

At first, we turn the jump matrices into constant ones with the transformation

$$P(z) \mapsto P(z)\,\mathrm{e}^{-\frac{2}{3}f(z)^{3/2}\sigma_3} =: W(z),$$

where $f(z)$ is the conformal map defined in subsection 4.3.1.3. The new RHP is given by

i. $W(z)$ is analytic at every $z \in U_\delta \backslash (\Sigma_S \cap U_\delta)$;

ii. W has boundary values as z approaches $s \in \Sigma_S \cap U_\delta$ related by

$$W_+(s) = W_-(s) \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & s \in \Sigma_1 \cap U_\delta \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & s \in \Sigma_2 \cap U_\delta \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & s \in \Sigma_3 \cap U_\delta \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & s \in \Sigma_5 \cap U_\delta \end{cases}.$$

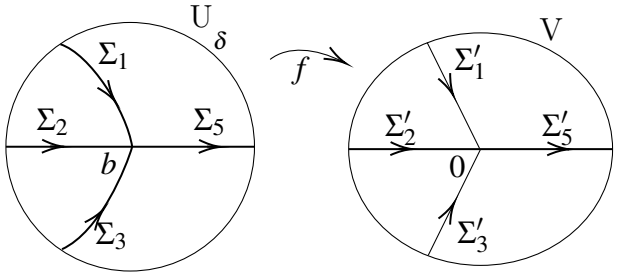
iii. If $|z-b| = \delta$,

$$W(z) = [I + o(1)]S^{(1)}(z)\,\mathrm{e}^{-\frac{2}{3}f(z)^{3/2}\sigma_3}.$$

iv. $W(z)$ remains bounded as $z \to b$.

Choose the ratio $\delta > 0$ such that $f(z)$ is an analytic invertible function from U_δ onto a neighbourhood V of zero. Notice that f maps the contour $\Sigma_S \cap U_\delta$ into a new one, say Σ'. The asymptotic condition for the RHP to $W(z)$ suggests a solution by means of the Airy function. Choose the new contour to be the union of the four rays in Figure 28 for some $\sigma \in (\pi/3, \pi)$.

Figure 28 – Transformation of the contour under the action of f.



Source: Elaborated by the author.

The contours Σ_1 and Σ_3 are still to be defined, and we choose them as the pre-images of Σ_1' and Σ_3' under $f(z)$, respectively. Notice that $\Sigma' = \Sigma_5' \cup \cup_{i=1}^{3} \Sigma_i'$ delimits four regions $I' = \{\zeta \in V : \arg(\zeta) \in (0,\sigma)\}$, $II' = \{\zeta \in V : \arg(\zeta) \in (\sigma,\pi)\}$, $III' = \{\zeta \in V : \arg(\zeta) \in (\pi, 2\pi - \sigma)\}$ and $VI' = \{\zeta \in V : \arg(\zeta) \in (2\pi - \sigma, 2\pi)\}$. Naturally, the corresponding regions I, II, III, IV in U_δ are the pre-images under $f(z)$ of I', II', III', IV' respectively. The RHP in U_δ induces a new RHP in V in terms of the variable $\zeta := f(z)$ and it can be extended to the whole plane and is given as follows.

i. $L(\zeta)$ is analytic at every $z \in V \backslash \Sigma_S'$;

ii. L has boundary values as ζ approaches $s \in \Sigma_S'$ related by

$$L_+(\zeta) = L_-(\zeta) \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \zeta \in \Sigma_1' \cup \Sigma_3' \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \zeta \in \Sigma_2' \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \zeta \in \Sigma_5' \end{cases} .$$

iii. As $\zeta \to \infty$,

$$L(\zeta) = (I + o(1))\, \zeta^{-\sigma_3/4} U_0 \, \mathrm{e}^{-\frac{2}{3}\zeta^{3/2}\sigma_3} .$$

iv. $L(\zeta)$ remains bounded as $\zeta \to 0$.

Thus, the solution to $W(z)$ has the form $E(z)L(f(z))$, where E is some pre-factor to be determined later. The asymptotic condition on the RHP for L comes from the behaviour of $f(z)$ as $z \to b$. In fact, previously we had that

$$W(z) = [I + o(1)] S^{(1)}(z) \, \mathrm{e}^{-\frac{2}{3} f(z)^{3/2}\sigma_3} .$$

But we know the explicit formula of $S^{(1)}(z)$,

$$W(z) = [I + o(1)] U_0^{-1} \begin{pmatrix} \left(\frac{z-a}{z-b}\right)^{-1/4} & 0 \\ 0 & \left(\frac{z-a}{z-b}\right)^{1/4} \end{pmatrix} U_0 \, \mathrm{e}^{-\frac{2}{3} f(z)^{3/2}\sigma_3} .$$

Since $f(z) = cn(z-b) + O(z-b)^2$ as $z \to b$ for some constant $c > 0$, it follows that changing variables $z \mapsto \zeta = f(z) \approx cn(z-b)$,

$$\begin{aligned} L(\zeta) :&= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \\ &= (I + o(1))\, \zeta^{-\sigma_3/4} U_0\, \mathrm{e}^{-\frac{2}{3}\zeta^{3/2}\sigma_3} \\ &= \begin{pmatrix} \zeta^{-1/4} \dfrac{\mathrm{e}^{-\frac{2}{3}\zeta^{3/2}}}{\sqrt{2}} & \zeta^{-1/4} \dfrac{i\mathrm{e}^{\frac{2}{3}\zeta^{3/2}}}{\sqrt{2}} \\ \zeta^{1/4} \dfrac{i\mathrm{e}^{-\frac{2}{3}\zeta^{3/2}}}{\sqrt{2}} & \zeta^{1/4} \dfrac{\mathrm{e}^{\frac{2}{3}\zeta^{3/2}}}{\sqrt{2}} \end{pmatrix}, \end{aligned}$$

as $\zeta \to \infty$. By the asymptotics for the Airy function,

$$\begin{aligned} \mathrm{Ai}(z) &\approx \frac{z^{-1/4}\, \mathrm{e}^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}} \\ \mathrm{Ai}'(z) &\approx -\frac{z^{1/4}\, \mathrm{e}^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}}, \qquad \text{as } z \to \infty, \end{aligned}$$

it follows that for $\zeta \in I'$,

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{2\pi}\,\mathrm{Ai}(\zeta) & -w^2\sqrt{2\pi}\,\mathrm{Ai}(w^2\zeta) \\ -i\sqrt{2\pi}\,\mathrm{Ai}'(\zeta) & iw\sqrt{2\pi}\,\mathrm{Ai}'(w^2\zeta) \end{pmatrix}.$$

Multiplying by the jumps and using the connection formula

$$\mathrm{Ai}(z) + w\,\mathrm{Ai}(wz) + w^2\,\mathrm{Ai}(w^2 z) = 0,$$

we find that the solution to this new problem is given by

$$L(\zeta) = \sqrt{2\pi} \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & -w^2\,\mathrm{Ai}(w^2\zeta) \\ -i\,\mathrm{Ai}'(\zeta) & iw\,\mathrm{Ai}'(w^2\zeta) \end{pmatrix} & \zeta \in I', \\ \begin{pmatrix} -w\,\mathrm{Ai}(w\zeta) & -w^2\,\mathrm{Ai}(w^2\zeta) \\ iw^2\,\mathrm{Ai}'(w\zeta) & iw\,\mathrm{Ai}'(w^2\zeta) \end{pmatrix} & \zeta \in II', \\ \begin{pmatrix} -w^2\,\mathrm{Ai}(w^2\zeta) & w\,\mathrm{Ai}(w\zeta) \\ iw\,\mathrm{Ai}'(w^2\zeta) & -iw^2\,\mathrm{Ai}'(w\zeta) \end{pmatrix} & \zeta \in III', \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & w\,\mathrm{Ai}(w\zeta) \\ -i\,\mathrm{Ai}'(\zeta) & -iw^2\,\mathrm{Ai}'(w\zeta) \end{pmatrix} & \zeta \in IV', \end{cases} \tag{4.57}$$

where $w = \mathrm{e}^{\frac{2}{3}\pi i}$. The local RHP possesses one degree of freedom left, and we must explore it in order to match the boundary condition. The pre-factor $E(z)$ must be chosen such that for every $z \in U_\delta$, we have $P(z)(S^{(1)}(z))^{-1} = (1 + o(1))$ as $n \to \infty$. That is, as $n \to \infty$,

$$\begin{aligned} E(z)L(f(z))\,\mathrm{e}^{\frac{2}{3}f(z)^{3/2}\sigma_3} &= (1 + o(1))S^{(1)}(z) \\ \implies E(z)\,(I + o(1))\,(f(z))^{-\sigma_3/4} U_0 &= (1 + o(1))S^{(1)}(z) \\ \implies E(z) &= S^{(1)}(z) U_0^{-1} (f(z))^{\sigma_3/4}. \end{aligned}$$

But since $S^{(1)}(z) = U_0^{-1} N(z) U_0$, one has

$$E(z) = U_0^{-1} (\alpha(z)^{-1} f(z))^{\sigma_3/4}, \tag{4.58}$$

and the solution for $z \in U_\delta$ is given by

$$P(z) = U_0^{-1} (\alpha(z)^{-1} f(z))^{\sigma_3/4} L(f(z)) \, \mathrm{e}^{\frac{2}{3} f(z)^{3/2} \sigma_3}. \tag{4.59}$$

A similar construction allows us to define $\tilde{P}(z)$ for $z \in \tilde{U}_\delta$ satisfying the following RHP:

i. $\tilde{P}(z)$ is analytic at every $z \in \tilde{U}_\delta \backslash (\Sigma_S \cap \tilde{U}_\delta)$.

ii. $\tilde{P}$ has boundary values as z approaches $s \in \Sigma_S \cap \tilde{U}_\delta$ related by

$$\tilde{P}_+(s) = \tilde{P}_-(s) \times \begin{cases} v_+ & s \in \Sigma_1 \cap \tilde{U}_\delta \\ v_0 & s \in \Sigma_2 \cap \tilde{U}_\delta \\ v_- & s \in \Sigma_3 \cap \tilde{U}_\delta \\ v_s & s \in \Sigma_4 \cap \tilde{U}_\delta \end{cases},$$

where

$$v_s = \begin{pmatrix} 1 & \mathrm{e}^{n(g_+(s)+g_-(s)-V(s)-\ell)} \\ 0 & 1 \end{pmatrix}.$$

iii. If $|z-a| = \delta$,

$$\tilde{P}(z) = [I + o(1)] S^{(1)}(z).$$

iv. $\tilde{P}(z)$ remains bounded as $z \to a$.

The solution is given by

$$\tilde{P}(z) = \tilde{E}(z) \tilde{L}(\tilde{f}(z)) \, \mathrm{e}^{\frac{2}{3} \tilde{f}^{3/2} \sigma_3}, \tag{4.60}$$

where $\tilde{L}$ is constructed by Airy functions, $\tilde{f}$ is the conformal map from subsection 4.3.1.3 and $\tilde{E}$ is the new pre-factor.

4.3.6 One last transformation: $S(z) \to R(z)$

The parametrix solution $S^{(par)}$ is given by

$$S^{(par)}(z) = \begin{cases} P(z) & z \in U_\delta \backslash \Sigma_s \\ \tilde{P}(z) & z \in \tilde{U}_\delta \backslash \Sigma_s \\ S^{(1)}(z) & z \in \mathbb{C} \backslash (U_\delta \cup \tilde{U}_\delta \cup \Sigma_s) \end{cases}.$$

In order to quantify how close the parametrix is of the solution to the original RHP, define the error matrix by

$$R(z) = S(z)(S^{(par)})^{-1}(z) = \begin{cases} S(z)(S^{(1)})^{-1}(z) & z \in \mathbb{C}\backslash(\Sigma_S \cup \overline{U_\delta} \cup \overline{\tilde{U}_\delta}), \\ S(z)P^{-1}(z) & z \in U_\delta \backslash \Sigma_S, \\ S(z)\tilde{P}^{-1}(z) & z \in \tilde{U}_\delta \backslash \Sigma_S. \end{cases} \tag{4.61}$$

Then, $R(z)$ satisfies the following RHP

i. $R(z)$ is analytic at every $z \in \mathbb{C}\backslash\Sigma_R$, where $\Sigma_R = \cup_{i=1}^{6}\Sigma_{R,i}$ is the contour in Figure 29. Notice that $\Sigma_{R,5} = \partial\tilde{U}_\delta$ and $\Sigma_{R,6} = \partial U_\delta$.

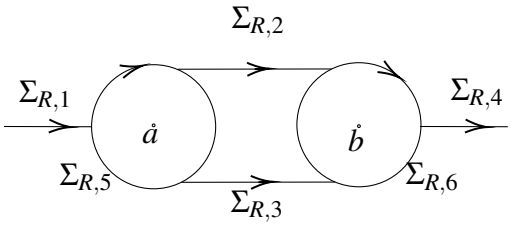
ii. R has boundary values as z approaches Σ_R related by

$$R_+(s) = R_-(s) \times \begin{cases} S^{(1)}(s)\begin{pmatrix} 1 & e^{n(g_+(s)+g_-(s)-V(s)-\ell)} \\ 0 & 1 \end{pmatrix}(S^{(1)})^{-1}(s), & s \in \Sigma_{R,1} \cup \Sigma_{R,4} \\ S^{(1)}(s)\begin{pmatrix} 1 & 0 \\ e^{-n(g_+(s)-g_-(s))} & 1 \end{pmatrix}(S^{(1)})^{-1}(s), & s \in \Sigma_{R,2} \\ S^{(1)}(s)\begin{pmatrix} 1 & 0 \\ e^{n(g_+(s)-g_-(s))} & 1 \end{pmatrix}(S^{(1)})^{-1}(s), & s \in \Sigma_{R,3} \\ P(s)(S^{(1)})^{-1}(s), & s \in \Sigma_{R,6} \\ \tilde{P}(s)(S^{(1)})^{-1}(s), & s \in \Sigma_{R,5}. \end{cases}$$

iii. As $z \to \infty$,

$$R(z) = \left[I + O\left(\frac{1}{z}\right)\right].$$

Figure 29 – Contour in the RHP for the error matrix.



Source: Elaborated by the author.

In order to estimate the norm of the jump matrices we define

$$\Delta_{R,i} = v_{R,i} - I, \quad i = 1, 2, \cdots, 6. \tag{4.62}$$

Thus,

$$\begin{aligned}
\Delta_{R,1}(z) &= S^{(1)}(z)\begin{pmatrix} 1 & \mathrm{e}^{n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & 1 \end{pmatrix}(S^{(1)})^{-1}(z) = \Delta_{R,4}(z), \\
\Delta_{R,2}(z) &= S^{(1)}(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix}(S^{(1)})^{-1}(z), \\
\Delta_{R,3}(z) &= S^{(1)}(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{n(g_+(z)-g_-(z))} & 1 \end{pmatrix}(S^{(1)})^{-1}(z).
\end{aligned} \tag{4.63}$$

Since both $S^{(1)}, (S^{(1)})^{-1}$ are uniformly bounded outside a fixed neighbourhood of the endpoints, the following estimate holds

$$||\Delta_{R,i}||_{L^\infty(\Sigma_{R,i})\cap L^2(\Sigma_{R,i})} = O(\mathrm{e}^{-cn}), \tag{4.64}$$

for $c>0$ a real constant and $i=1,2,3,4$. Here, given a matrix $M=(m_{ij})_{i,j=1}^2$, the L^2-norm is given by $||M||_{L^2} := \sum ||m_{ij}||_{L^2}$ and the L^∞-norm is given by $||M||_{L^\infty} := \sum ||m_{ij}||_{L^\infty}$.

Moreover, from the complete asymptotic expansion for Airy function, the jump matrices $v_{R,5}$ and $v_{R,6}$ possess asymptotic expansions in powers of n given by

$$\begin{aligned}
v_{R,5} &= I + \sum_{k=1}^{\infty} \frac{v_k^a}{n^k}, \\
v_{R,6} &= I + \sum_{k=1}^{\infty} \frac{v_k^b}{n^k},
\end{aligned}$$

where v_k^b and v_k^a can be determined explicitly in terms of the coefficients of asymptotic expansion of the Airy function. Such expansion leads to the following estimate

$$||\Delta_{R,i}||_{L^\infty(\Sigma_{R,i})\cap L^2(\Sigma_{R,i})} = O\left(\frac{1}{n}\right), \tag{4.65}$$

for $i=5,6$. It follows from (4.65) and (4.64) that $R_+ = R_-(I+O(\mathrm{e}^{-cn}))$ in both L^2 and L^∞ norms, and by the *small norm theorem*

$$R(z) = I + O\left(\frac{1}{n(|z|+1)}\right), \quad \text{as } n\to\infty.$$

Furthermore, $R(z)$ is uniformly bounded and has a complete asymptotic expansion in powers of n^{-1}

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k}. \tag{4.66}$$

Inserting (4.66) into the RHP for the error with jump on $\partial U_\delta \cup \partial \tilde{U}_\delta$ denoted by $v_d = v_d^a \cup v_d^b$ we obtain a different expression for $R(z)$ in terms of a recursive set of linear

RHPs. In fact,

$$R_+(s) = R_-(s)v_d,$$

which implies,

$$\left[I + \sum_{j=1}^{\infty} n^{-j} R_j(s)\right]_+ = \left[I + \sum_{j=1}^{\infty} n^{-j} R_j(s)\right]_- \left[I + \sum_{j=1}^{\infty} n^{-j} v_j(s)\right].$$

Collecting terms of order n^{-j} we finally have

$$(R_j)_+ = (R_j)_- + v_j + \sum_{i=1}^{j-1} (R_i)_- v_{j-i},$$

and therefore the following result holds.

Theorem 12. The matrices in the right-hand side of Equation (4.66) can be obtained as follows. If $k = 1$,

i. $R_1(z)$ is analytic at every $z \in \mathbb{C}\backslash(\Sigma_{R,5} \cup \Sigma_{R,6})$, where $\Sigma_{R,5} = \partial\tilde{U}_\delta$ and $\Sigma_{R,6} = \partial U_\delta$.

ii. R_1 has boundary values as z approaches $s \in \Sigma_{R,5} \cup \Sigma_{R,6}$ related by

$$(R_1(s))_+ - (R_1(s))_- = v_1.$$

iii. As $z \to \infty$,

$$R_1(z) = O\left(\frac{1}{z}\right).$$

If $k > 1$

i. $R_k(z)$ is analytic at every $z \in \mathbb{C}\backslash(\Sigma_{R,5} \cup \Sigma_{R,6})$, where $\Sigma_{R,5} = \partial\tilde{U}_\delta$ and $\Sigma_{R,6} = \partial U_\delta$.

ii. R_k has boundary values as z approaches $s \in \Sigma_{R,5} \cup \Sigma_{R,6}$ related by

$$(R_k(s))_+ - (R_k(s))_- = v_k + \sum_{j=1}^{k-1} (R_j(s))_- v_{j-i}.$$

iii. As $z \to \infty$,

$$R_k(z) = O\left(\frac{1}{z}\right).$$

Each one of the problems stated in Theorem (12) possesses unique solution by means of Cauchy transform. Combining the solutions for all the RHPs defined so far we obtain the following formula for the initial problem on the upper half plane

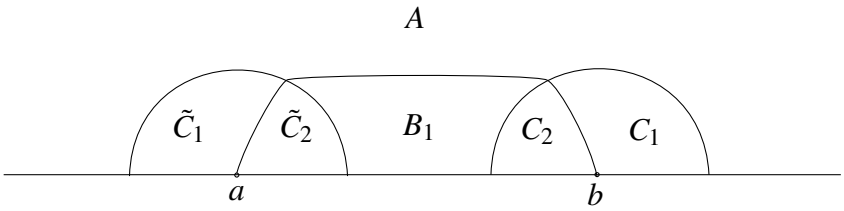
$$Y(z) = \mathrm{e}^{n\frac{\ell}{2}\sigma_3} R(z) T(z) \mathrm{e}^{n(g(z)-\frac{\ell}{2})\sigma_3}, \tag{4.67}$$

where

$$T(z) = \begin{cases} S^{(1)}(z) & \text{for } z \in A \\ P(z) & \text{for } z \in C_1 \\ \tilde{P}(z) & \text{for } z \in \tilde{C}_1 \\ P(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix} & \text{for } z \in C_2 \\ \tilde{P}(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix} & \text{for } z \in \tilde{C}_2 \\ S^{(1)}(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix} & \text{for } z \in B_1, \end{cases} \tag{4.68}$$

where $R(z)$ is given by Equation (4.66), $P(z)$ by Equation (4.59), $\tilde{P}(z)$ by Equation (4.60) and $S^{(1)}(z)$ by Equation (4.56). The regions in Equation (4.68) are the ones in Figure 30.

Figure 30 – Regions of definition of $Y(z)$.



Source: Elaborated by the author.

Each term in Equation (4.67) depends analytically on $\mathbf{t}$ for $\mathbf{t} \in \mathbb{T}(T, \gamma)$. Since $p(z) = \overline{p(\bar{z})}$, it suffices to stablish the asymptotic expansion of the orthogonal polynomials in the upper plane. Moreover, by analytic continuation the asymptotic expression developed in the upper half plane extends naturally to the real line.

4.4 Asymptotic behaviour of the one-point function

The expression to be evaluated is

$$\int_{-\infty}^{\infty} \lambda^{\ell} \rho_n(\lambda)\, \mathrm{d}\lambda, \tag{4.69}$$

where ρ_n stands for the one-point correlation function depending on a parameter n. In order to obtain a complete asymptotic expansion for (4.69) we will work with a more general expression. Let $q(\lambda)$ be any C^{∞} function that grows faster than any polynomial. Then we will prove that

$$\int_{-\infty}^{\infty} q(\lambda)\rho_n(\lambda)\, \mathrm{d}\lambda \tag{4.70}$$

possesses asymptotic expansion in even powers of n. Recall that

$$\rho_n(z) = \frac{\mathrm{e}^{-nV(z)}}{-2\pi i n}[Y_{11}'(z)Y_{21}(z) - Y_{11}(z)Y_{21}'(z)]. \tag{4.71}$$

Notice that for $x < a-\delta$ and for $x > b+\delta$ the expansion of the orthogonal polynomials $p(x)$ decays exponentially with x and is exponentially small in n. So it suffices to evaluate

$$\int_{a-\delta}^{b+\delta} q(\lambda)\rho_n(\lambda)\,\mathrm{d}\lambda, \tag{4.72}$$

where δ is a real constant. Since we have different expressions for the asymptotics of the orthogonal polynomials corresponding to the intervals $(a-\delta,a)$, $(a,a+\delta)$, $(a+\delta,b-\delta)$, $(b-\delta,b)$ and $(b-\delta,b+\delta)$, one needs to evaluate the integral over each interval separately.

A simplification of the problem is obtained by extending the local solutions in the neighbourhoods U_δ and $\tilde{U}_\delta$ of the endpoints to solutions covering the whole interval, as observed by Ercolani and McLaughlin (2003). To do so, the conformal maps are extended as detailed in Lemma 1. Keep the notation U_δ and $\tilde{U}_\delta$ to the new neighbourhoods, which are deformed ellipsis as in Figure 26. We introduce $T_a, T_b, \gamma_a, \gamma_b$ such that for $T < \min\{T_a, T_b\}$ and $\gamma > \max\{\gamma_a, \gamma_b\}$ the neighbourhoods of a, b in which the local solution holds have an overlap.

In order to split the integral of ρ_n into an expression that holds for all $z \in (z_0 - \varepsilon, b+\delta)$ and another that holds for $z \in (a-\varepsilon, z_0+\delta)$, set $\{\chi_a, \chi_b\}$ a partition of unity such that

i. χ_a, χ_b are smooth functions with range $[0,1]$;

ii. $\overline{\operatorname{supp}\chi_b} \subset (z_0-\varepsilon, \infty)$ and $\overline{\operatorname{supp}\chi_a} \subset (-\infty, z_0+\varepsilon)$;

iii. for every $z \in (z_0+\varepsilon, \infty)$, $\chi_b = 1$ and for every $z \in (-\infty, z_0-\varepsilon)$, $\chi_a = 1$;

iv. and $\chi_a(z) + \chi_b(z) = 1$ for every $z \in \mathbb{R}$.

Let ρ_n^b and ρ_n^a be the corresponding expressions for $\rho_n(z)$ in U_δ and $\tilde{U}_\delta$ respectively. Therefore, ρ_n is expressed as

$$\rho_n = \chi_a \rho_n^a + \chi_b \rho_n^b. \tag{4.73}$$

The original integral (4.72) assumes the form

$$\int_{a-\delta}^{b+\delta} q(\lambda)\rho_n(\lambda)\,\mathrm{d}\lambda = \int_{a-\delta}^{z_0+\varepsilon} q(\lambda)\chi_a(\lambda)\rho_n^a(\lambda)\,\mathrm{d}\lambda + \int_{z_0-\varepsilon}^{b+\delta} q(\lambda)\chi_b(\lambda)\rho_n^b(\lambda)\,\mathrm{d}\lambda. \tag{4.74}$$

Due to the similarity of the two integrals on the right side, it suffices to develop an asymptotic expansion for the last one. For the first one, it follows analogously.

From Equations (4.71) and (4.67), in the neighbourhood of b it holds that

$$\begin{aligned} n\rho_n^{(b)}(z) = & \left(\frac{f'(z)}{4f(z)} - \frac{\alpha'(z)}{\alpha(z)}\right)[2\operatorname{Ai}(f(z))\operatorname{Ai}'(f(z))] \\ & + f'(z)[(\operatorname{Ai}'(f(z)))^2 - f(z)(\operatorname{Ai}(f(z)))^2] \\ & + \frac{ni}{2}\left\{\sum_{j \text{ even}\geq 2} n^{-j}\tilde{a}_j(z)\Psi_{11}^2(f(z))\frac{\sqrt{f(z)}}{\alpha(z)^2}\right. \\ & + \sum_{j \text{ even}\geq 2} n^{-j}\tilde{b}_j(z)\Psi_{21}^2(f(z))\frac{\alpha^2}{\sqrt{f(z)}} \\ & \left. + \sum_{j \text{ odd}\geq 3} n^{-j}\tilde{c}_j(z)\Psi_{11}(f(z))\Psi_{21}(f(z))\right\}, \end{aligned} \tag{4.75}$$

where $\Psi(z)$ is the auxiliary matrix given by

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} \operatorname{Ai}(\zeta) & \operatorname{Ai}(w^2\zeta) \\ \operatorname{Ai}'(\zeta) & w^2\operatorname{Ai}'(w^2\zeta) \end{pmatrix} & \zeta \in \mathbb{C}_+ \\ \begin{pmatrix} \operatorname{Ai}(\zeta) & -w^2\operatorname{Ai}(w\zeta) \\ \operatorname{Ai}'(\zeta) & -\operatorname{Ai}'(w\zeta) \end{pmatrix} & \zeta \in \mathbb{C}_- \end{cases}. \tag{4.76}$$

Moreover, $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j$ are analytic on z and $\mathbf{t}$ in a fixed neighbourhood of $[z_0, b]$ and $\mathbf{t} \in \mathbb{T}(\gamma_b, T_b)$. An analogous relation holds for a neighbourhood of a.

From the previous analysis it suffices to show that

$$\int_{z_0-\varepsilon}^{b+\delta} q(\lambda)\chi_b(\lambda)\rho_n^b(\lambda)\,\mathrm{d}\lambda \sim \sum_{k=0}^{\infty} q_k n^{-2k}. \tag{4.77}$$

Set

$$F_0(\zeta) := [\operatorname{Ai}']^2(\zeta) - \zeta\operatorname{Ai}^2(\zeta). \tag{4.78}$$

Each of the terms in (4.75) gives rise to one of the four following types of integral

$$\frac{1}{n}\int_{z_0-\varepsilon}^{b+\delta} g(z)F_0(f(z))f'(z)\,\mathrm{d}z, \tag{4.79}$$

$$\int_{z_0-\varepsilon}^{b+\delta} g(z)\operatorname{Ai}(f(z))\operatorname{Ai}'(f(z))\,\mathrm{d}z, \tag{4.80}$$

$$n^{1/3}\int_{z_0-\varepsilon}^{b+\delta} \tilde{g}(z)\operatorname{Ai}^2(f(z))\,\mathrm{d}z, \tag{4.81}$$

$$n^{-1/3}\int_{z_0-\varepsilon}^{b+\delta} \tilde{g}(z)(\operatorname{Ai}'(f(z)))^2\,\mathrm{d}z, \tag{4.82}$$

where $g(z)$ is an infinitely differentiable function compactly supported within $(z_0-\varepsilon, b+\delta)$. Showing that (4.72) possesses an asymptotic expansion in even powers of n is now

equivalent to prove that (4.80) possesses an expansion in odd powers of n and that (4.79), (4.81), (4.82) posses expansions in even powers of n.

It is a piece of luck that the asymptotic expansion of one of the Equations (4.79)-(4.82) implies the asymptotic expansion for the others. In fact, suppose that (4.80) possesses an expansion in odd powers of n. Integration by parts plus the assumption that $g(z)$ is compactly supported in $(z_0-\varepsilon, b+\delta)$ implies that

$$\begin{aligned} 2\int_{z_0-\varepsilon}^{b+\delta} g(z)\,\mathrm{Ai}(f(z))\,\mathrm{Ai}'(f(z))\,\mathrm{d}z &= \frac{g(z)\,\mathrm{Ai}(f(z))}{f'(z)}\Big|_{z_0-\varepsilon}^{b+\delta} - \int_{z_0-\varepsilon}^{b+\delta}\left(\frac{g(z)}{f'(z)}\right)'\mathrm{Ai}^2(f(z))\,\mathrm{d}z, \\ &= -\frac{1}{n^{2/3}}\int_{z_0-\varepsilon}^{b+\delta}\left(\frac{g(z)}{\phi_b'(z)}\right)'\mathrm{Ai}^2(f(z))\,\mathrm{d}z \\ &= -\frac{1}{n}\int_{z_0-\varepsilon}^{b+\delta} n^{1/3}\left(\frac{g(z)}{\phi_b'(z)}\right)'\mathrm{Ai}^2(f(z))\,\mathrm{d}z, \end{aligned}$$

where $\phi_b(z)$ is the n-independent function determined by $f(z)=n^{2/3}\phi_b(z)$. Then (4.81) possesses expansion in even powers. The same equation also provides a proof that the expansion in even powers for (4.81) implies an expansion in odd powers of n for (4.80), since

$$\left[\frac{g(z)}{\phi_b'(z)}\right]',$$

is a new C^∞ function of z compactly supported in $(z_0-\varepsilon, b+\delta)$.

Suppose the required expansion holds for (4.81). Integration by parts shows that

$$n^{1/3}\int_{z_0-\varepsilon}^{b+\delta} g(z)\,\mathrm{Ai}^2(f(z))\,\mathrm{d}z = \frac{1}{n}\int_{z_0-\varepsilon}^{b+\delta}\left(\frac{g(z)}{\phi_b'(z)}\right)'\frac{1}{\phi_b'(z)}F_0(f(z))f'(z)\,\mathrm{d}z. \tag{4.83}$$

Thus, (4.79) has asymptotic expansion in even powers of n. Analogously to the previous case, the equality also proves that the expansion for (4.79) implies the expansion for (4.81).

Finally, if (4.79) and (4.80) have expansions in even powers, an integration by parts leads to

$$\begin{aligned} n^{-1/3}\int_{z_0-\varepsilon}^{b+\delta} g(z)[\mathrm{Ai}'(f(z))]^2(f(z))\,\mathrm{d}z &= -\frac{2}{3n}\int_{z_0-\varepsilon}^{b+\delta}\left[\frac{g(z)}{\phi_b'(z)}\right]'\mathrm{Ai}'(f(z))\,\mathrm{Ai}'(f(z))\,\mathrm{d}z \\ &= -\frac{1}{3n}\int_{z_0-\varepsilon}^{b+\delta}\left[\frac{g(z)}{\phi_b'(z)}\right]'\frac{\phi_b(z)}{\phi_b'(z)}F_0(f(z))f'(z)\,\mathrm{d}z, \end{aligned}$$

and thus (4.82) has an expansion in even powers of n.

Consequently, it suffices to prove that (4.79) possesses an asymptotic expansion in even powers of n. The proof relies on the existence of $\{F_i(\zeta)\}_{i=0}^\infty$ and $\{G_i(\zeta)\}_{i=0}^\infty$ sequences of functions such that for every $i\geq 1$

$$F_i'(\zeta) = F_{i-1}(\zeta), \tag{4.84}$$

$G_i(0) = F_i(0)$, F_i decays exponentially for positive values and G_i possesses an asymptotic expansion for negative values with no constant term.

Since g is C^∞, it can be differentiated as many times as we wish. So the trick is to integrate by parts repeatedly always integrating the term related to F_i or G_i and taking the derivative of the term related to g. The fact that $f(b) = 0$ and that g is compactly supported on $(z_0 - \varepsilon, b + \delta)$ assure us that the boundary terms in the integration by parts always vanish and what is left is a new integral to be evaluated.

The asymptotic expansion of the Airy function implies that

$$\begin{aligned} F_0(\zeta) = &\sqrt{-\zeta}(a_0 + a_1(-\zeta)^{-3} + a_2(-\zeta)^{-6} + \cdots) \\ &+ \sqrt{-\zeta}(b_0(-\zeta)^{-3} + b_1(-\zeta)^{-6} + \cdots)\sin\left(\frac{4}{3}(-\zeta)^{3/2}\right) \\ &+ (c_0(-\zeta)^{-1} + c_1(-\zeta)^{-4} + \cdots)\cos\left(\frac{4}{3}(-\zeta)^{3/2}\right), \end{aligned}$$

as $\zeta \to -\infty$ and that $F_0(\zeta)$ decays exponentially as $\zeta \to \infty$. Thus, $F_0(\zeta) - (-\zeta)^{1/2}a_0 \to 0$ when $\zeta \to -\infty$, what suggests us an auxiliary function $G_1(\zeta)$ given by

$$G_1'(\zeta) = F_0(\zeta) - (-\zeta)^{1/2}a_0. \tag{4.85}$$

The asymptotic expansion of F_0 implies that G_1 possesses an asymptotic expansion given by

$$\begin{aligned} G_1(\zeta) = \; &(a_0^{(1)}(-\zeta)^{-3/2} + a_1^{(1)}(-\zeta)^{-9/2} + \cdots) \\ &+ (b_0^{(1)}(-\zeta)^{-3/2} + b_1^{(1)}(-\zeta)^{-9/2} + \cdots)\sin\left(\frac{4}{3}(-\zeta)^{3/2}\right) \\ &+ (c_0^{(1)}(-\zeta)^{-3} + c_1^{(1)}(-\zeta)^{-6} + \cdots)\cos\left(\frac{4}{3}(-\zeta)^{3/2}\right). \end{aligned} \tag{4.86}$$

Analogously we have well-behaved asymptotic expansions for

$$\begin{aligned} G_2'(\zeta) &:= G_1(\zeta), \\ G_3'(\zeta) &:= G_2(\zeta). \end{aligned}$$

Straightforward calculations show that F_i, $i = 1,2,3$ defined by

$$F_1(\zeta) := G_1(\zeta) - \frac{2a_0}{3}(-\zeta)^{3/2} \tag{4.87}$$

$$F_2(\zeta) := G_2(\zeta) + \frac{2^2 a_0}{5\times 3}(-\zeta)^{3/2} \tag{4.88}$$

$$F_3(\zeta) := G_3(\zeta) - \frac{2^3 a_0}{7\times 5\times 3}(-\zeta)^{3/2}, \tag{4.89}$$

satisfy (4.84). Notice that by construction $f(b) = 0$, $f(z) < 0$ if $z < b$, $z \in U_\delta$ and $f(z) > 0$ if $z > b$, $z \in U_\delta$. Moreover G_i has a well-defined asymptotic expansion for negative values

that tends to zero when $\zeta \to -\infty$ and F_i has an exponential decay for positive values that also goes to zero when $\zeta \to \infty$. Therefore we must split our integral once again in such a way that we can express the integrand by means of G_i in the interval $(z_0-\varepsilon, b)$ and F_i in the interval $(b, b+\delta)$. Thus, given the definitions and asymptotic expansions of F_1, F_2, F_3, G_1, G_2 and G_3 we are able to perform the first integration by parts over (4.79)

$$\begin{aligned}\frac{1}{n}\int_{z_0-\varepsilon}^{b+\delta} g(z)F_0(f(z))f'(z)\,\mathrm{d}z =& \frac{1}{n}\int_{z_0-\varepsilon}^{b+\delta} g(z)(F_0(f(z))-a_0(-f(z))^{1/2}+a_0(-f(z))^{1/2})f'(z)\,\mathrm{d}z\\ =& \frac{1}{n}\int_{z_0-\varepsilon}^{b} g(z)a_0(-f(z))^{1/2}f'(z)\,\mathrm{d}z+\frac{1}{n}\int_{z_0-\varepsilon}^{b} g(z)G_1'(f(z))\times\\ & f'(z)\,\mathrm{d}z+\frac{1}{n}\int_{b}^{b+\delta} g(z)F_0(f(z))f'(z)\,\mathrm{d}z\\ =& \hat{e}_0+A_1,\end{aligned}$$

where

$$\hat{e}_0 = \frac{1}{n}\int_{z_0-\varepsilon}^{b} g(z)a_0(-f(z))^{1/2}f'(z)\,\mathrm{d}z, \tag{4.90}$$

$$A_1 = \frac{1}{n}\int_{z_0-\varepsilon}^{b} g(z)G_1'(f(z))f'(z)\,\mathrm{d}z+\frac{1}{n}\int_{b}^{b+\delta} g(z)F_0(f(z))f'(z)\,\mathrm{d}z. \tag{4.91}$$

A simple calculation shows that $\hat{e}_0$ is independent of n since $f(z) = n^{2/3}\phi_b(z)$:

$$\begin{aligned}\hat{e}_0 &= -\frac{1}{n}\int_{z_0-\varepsilon}^{b} g(z)a_0\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{2}{3}(-f(z))^{3/2}\right)\mathrm{d}z,\\ &= -\frac{a_0}{2}\int_{z_0-\varepsilon}^{b} g(z)(b-z)^{1/2}(z-a)^{1/2}h(z)\,\mathrm{d}z.\end{aligned}$$

Now we deal with A_1. Two more integrations by parts lead us to

$$A_1 = -\frac{1}{n}\int_{z_0-\varepsilon}^{b}\left[\frac{1}{f'(z)}\left(\frac{g'(z)}{f'(z)}\right)'\right]' G_3(f(z))\,\mathrm{d}z-\frac{1}{n}\int_{b}^{b+\delta}\left[\frac{1}{f'(z)}\left(\frac{g'(z)}{f'(z)}\right)'\right]' F_3(f(z))\,\mathrm{d}z. \tag{4.92}$$

The asymptotic behaviour of G_3 suggests that we may sum and subtract $a_0^{(3)}(-f(z))^{1/2}$ from the first integrand. Therefore, we obtain

$$A_1 = \hat{e}_1 n^{-2}+A_2 \tag{4.93}$$

$$\hat{e}_1 = -n\int_{z_0-\varepsilon}^{b}\left[\frac{1}{f'(z)}\left(\frac{g'(z)}{f'(z)}\right)'\right]' a_0^{(3)}(-f(z))^{1/2}\,\mathrm{d}z \tag{4.94}$$

$$\begin{aligned}A_2 =& -\frac{1}{n}\int_{z_0-\varepsilon}^{b}\left[\frac{1}{f'(z)}\left(\frac{g'(z)}{f'(z)}\right)'\right]' (G_3(f(z))-a_0^{(3)}(-f(z))^{1/2})\,\mathrm{d}z\\ & -\frac{1}{n}\int_{b}^{b+\delta}\left[\frac{1}{f'(z)}\left(\frac{g'(z)}{f'(z)}\right)'\right]' F_3(f(z))f'(z)\,\mathrm{d}z.\end{aligned} \tag{4.95}$$

Notice that $\hat{e}_1$ is n-independent. In fact,

$$\begin{aligned}\hat{e}_1 &= -n\int_{z_0-\varepsilon}^{b}\left[\frac{1}{n^{2/3}\phi_b'(z)}\left(\frac{g'(z)}{n^{2/3}\phi_b'(z)}\right)'\right]' a_0^{(3)}(-n^{2/3}\phi_b(z))^{1/2}\,\mathrm{d}z\\ &= -\int_{z_0-\varepsilon}^{b}\left[\frac{1}{\phi_b'(z)}\left(\frac{g'(z)}{\phi_b'(z)}\right)'\right]' a_0^{(3)}(-\phi_b(z))^{1/2}\,\mathrm{d}z.\end{aligned}$$

The factor n^{-2} comes from $f(z)$ and A_2 is the sum of two integrals that can be expressed by means of F_3 and G_3.

The inductive step proves that the existence and the asymptotic behaviour of F_i, G_i, $i \in \{1, 3m\}$ implies the existence of $G_{3m+1}, G_{3m+2}, G_{3m+3}$ such that

$$G_{3m+1}(\zeta) = \int_{-\infty}^{\zeta}(G_{3m}(s) - a_0^{(3m)}(-s)^{1/2})\,\mathrm{d}s \tag{4.96}$$

$$G_{3m+2}(\zeta) = \int_{-\infty}^{\zeta} G_{3m+1}(s)\,\mathrm{d}s \tag{4.97}$$

$$G_{3m+3}(\zeta) = \int_{-\infty}^{\zeta} G_{3m+2}(s) - a_0^{(3m+2)}(-s)^{-1/2}\,\mathrm{d}s - 2a_0^{(3m+2)}(-\zeta)^{1/2}, \tag{4.98}$$

and G_i possesses an asymptotic expansion that goes to zero when $\zeta \to -\infty$. Furthermore, set $F_{3m+1}, F_{3m+2}, F_{3m+3}$ as

$$F_{3m+1} = \int_{\infty}^{\zeta} F_{3m}(s)\,\mathrm{d}s, \tag{4.99}$$

$$F_{3m+2} = \int_{\infty}^{\zeta} F_{3m+1}(s)\,\mathrm{d}s \tag{4.100}$$

$$F_{3m+3} = \int_{\infty}^{\zeta} F_{3m+2}(s)\,\mathrm{d}s. \tag{4.101}$$

By definition, $F_{3m+1}, F_{3m+2}, F_{3m+3}$ satisfy (4.84) and decay exponentially for $\zeta \to \infty$. Because of the recursive definitions of G_i, F_i it follows that

$$\begin{aligned}F_{3m+1} =& G_{3m+1} - \frac{2a_0^{(3m)}}{3}(-\zeta)^{\frac{3}{2}} + \frac{2^4 a_0^{(3m-3)}}{9\times 7\times 5\times 3}(-\zeta)^{\frac{9}{2}}\\ &+\cdots+\frac{(-1)^{3m+1}2^{3m+1}a_0}{\prod_{j=1}^{3m+1}(2j+1)}(-\zeta)^{\frac{6m+3}{2}},\end{aligned} \tag{4.102}$$

$$\begin{aligned}F_{3m+2} =& G_{3m+2} + \frac{2^2 a_0^{(3m)}}{5\times 3}(-\zeta)^{5/2} + \frac{2^5 a_0^{(3m-3)}}{11\times 9\times 7\times 5\times 3}(-\zeta)^{11/2}\\ &+\cdots+\frac{(-1)^{3m+2}2^{3m+2}a_0}{\prod_{j=1}^{3m+2}(2j+1)}(-\zeta)^{(6m+5)/2},\end{aligned} \tag{4.103}$$

$$\begin{aligned}F_{3m+3} =& G_{3m+3} - \frac{2^3 a_0^{(3m)}}{7\times 5\times 3}(-\zeta)^{\frac{7}{2}} + \frac{2^6 a_0^{(3m-3)}}{13\times 11\times 9\times 7\times 5\times 3}(-\zeta)^{\frac{13}{2}}\\ &+\cdots+\frac{(-1)^{3m+3}2^{3m+3}a_0}{\prod_{j=1}^{3m+3}(2j+1)}(-\zeta)^{\frac{6m+7}{2}}.\end{aligned} \tag{4.104}$$

Now, by inductive hypothesis suppose that after $3m$ integrations by parts we come up with the expression

$$A = \hat{e}_1 n^{-2} + \hat{e}_2 n^{-4} + \cdots + \hat{e}_m n^{-2m} + A_{m+1}, \tag{4.105}$$

where $\hat{e}_i$, $i = 1,2,\cdots,m$, does not depend on n and A_{m+1} is the sum of integrals depending on G_{3m} and F_{3m}. More precisely,

$$\begin{aligned} A_{m+1} = &\frac{(-1)^m}{n}\int_{z_0-\varepsilon}^{b+\delta} \frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}\cdots\frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}g'\right)\cdots\right)\right)\times \\ &(G_{3m}(f(z)) - a_0^{(3m)}(-f(z))^{1/2})\,\mathrm{d}z + \\ &\frac{(-1)^m}{n}\int_{z_0-\varepsilon}^{b+\delta} \frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}\cdots\frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}g'\right)\cdots\right)\right)F_{3m}(f(z))\,\mathrm{d}z. \end{aligned} \tag{4.106}$$

In each nested set of derivatives the operator

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((f')^{-1}.\right), \tag{4.107}$$

appears $(3m-1)$ times. Integrating by parts three more times and applying the information provided by Equations (4.98) and (4.101), it is straightforward to show that

$$A_{m+1} = \hat{e}_{m+1} n^{-2(m+1)} + A_{m+2}, \tag{4.108}$$

where A_{m+2} is the sum of integrals depending on G_{3m+3}, F_{3m+3}.

So far, we have that for a smooth function $g(z)$ compactly supported on $(z_0-\varepsilon, b+\delta)$ the integral (4.79) has asymptotic expansion in even powers of n. Consequently, the expansion desired for Equations (4.80)-(4.82) also holds. Then, an asymptotic expansion in even powers of n is valid for the second integral in (4.74). For the integral in $(a-\delta, z_0+\varepsilon)$ analogous calculations hold.

The only thing left to do is to show that the asymptotic expansion depends analytically on the parameter $\mathbf{t}$. Remember that the coefficients in the expansion were of the form

$$\hat{e}_i = \int_{z_0-\varepsilon}^{b} \chi_b^{(j)}(z) F(z,\mathbf{t})(-\phi_b(z))^{1/2}\,\mathrm{d}z, \tag{4.109}$$

where $F(z,\mathbf{t})$ gathers the terms which do not dependent on $\chi_b(z)$, $\chi_b^{(j)}$ denotes the j-th derivative and ϕ_b is the n-independent part of the conformal map. Notice that $\phi_b(z)$ is analytic with branch cut emanating from b. By the definition of χ_b we can decompose the integral into a sum

$$\hat{e}_i = \int_{z_0-\varepsilon}^{z_0+\varepsilon} \chi_b^{(j)}(z) F(z,\mathbf{t})(-\phi_b(z))^{1/2}\,\mathrm{d}z + \int_{z_0+\varepsilon}^{b} F(z,\mathbf{t})(-\phi_b(z))^{1/2}\,\mathrm{d}z. \tag{4.110}$$

The first integral is analytic on $\mathbf{t}$ because the integrand is. For the second one, take the branch cut starting on b, passing by $z_0+\varepsilon$. Thus, such integral is half the contour integral encircling $(z_0+\varepsilon, b)$, since the integrand is analytic on $\mathbf{t}$ and posses a square-root branch point.

In particular,

$$\int_{-\infty}^{\infty} \lambda^{\ell} \rho_n(\lambda)\, \mathrm{d}\lambda \tag{4.111}$$

possesses an asymptotic expansion in even powers of n such that the coefficients are analytic on $\mathbf{t}$.

4.5 Final comments

The technical details explored in previous sections can now be summarized into simple steps in order to explain the proof of Theorem 9. We used the fact that our partition function is a matrix model based one to connect it to a random matrix ensemble and express it in terms of the so-called one point correlation function, as in Equation (4.20). That is,

$$\frac{\partial}{\partial t_j} \log(Z_n(\mathbf{t})) = -n^2 \int_{-\infty}^{\infty} \lambda^j \rho_n(\lambda)\, \mathrm{d}\lambda .$$

The section 4.3 provided an expression for the one point function where the n-dependence becomes explicit and section 4.4 built an expansion in even powers of n for the integral of the correlation function.

All together, such results leads us to the conclusion that

$$\frac{\partial}{\partial t_j} \frac{1}{n^2} \log(Z_n(\mathbf{t})) = \hat{e}_0 + \frac{1}{n^2}\hat{e}_1 + \frac{1}{n^4}\hat{e}_2 + \cdots .$$

Now, since it is uniformly valid for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$, one can integrate and obtain the asymptotic expansion for the logarithm of the partition function. Therefore, Theorem 9 is proved and all the calculations in section 4.1 are legitimate.

CHAPTER

5

INTEGRABLE SYSTEMS IN MATRIX MODELS

In this chapter, we explore some connections between random matrix models and integrable systems. At first, we present a short discussion on the definition of integrable systems. Then, we construct the bridge between the two theories by different approaches. When n is fixed, we explore the partition function in order to reach the KP-hierarchy, and for $n \to \infty$ we recover some solutions to a Painlevé equation through the point process described by the eigenvalues of random matrices. The first connection will be dealt with briefly, while the second one will be discussed in detail with the help of the asymptotics obtained in section 4.3.

5.1 Integrable systems

A **system** is a common concept in physics, and refers to the mathematical model that gives the approximate behaviour of a physical object.

Take, for instance, the one dimensional harmonic oscillator. If one needs to describe the position $x(t)$ of a mass m in the endpoint of a spring with constant $k > 0$ starting in the point x_0 with velocity 0, Newton's equations say that $x(t)$ must solve

$$\begin{aligned} m\frac{\mathrm{d}^2 x(t)}{\mathrm{d}^2 t} &= -kx(t) \\ x(0) &= x_0 \\ x'(0) &= 0. \end{aligned}$$

That is, our problem reduces to an ordinary differential equation (ODE), which is easily solved by means of trigonometric functions.

This famous problem possesses a remarkable characteristic: it is **integrable**. But what does integrable mean in the context of dynamical systems? In order to bring some light into this concept, we start with a classical system and adopt the Hamiltonian formalism. The state of the system at a given time is then specified by a point (q_i, p_i) in the *phase space*, where q_i stands for i-th coordinate of the position and p_i stands for i-th coordinate of the momentum ($p_i = mx_i''$). The information of the system is then encoded by a function $H(q_i, p_i)$ called Hamiltonian, and the motion equations are first order ODEs

$$\begin{aligned}\frac{\mathrm{d}q_i}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i}\\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\partial H}{\partial q_i}.\end{aligned}$$

Given a function $f(q,p)$ in the phase space,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}\\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\\ &=: \{H, f\},\end{aligned}$$

where $\{.,.\}$ denote the Poisson bracket,

$$\{f,g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}. \tag{5.1}$$

Thus, a function is conserved under time evolution if and only if $\{H, f\} = 0$. A system on a phase space of dimension $2n$ is said to be *(Liouville) integrable* if there exist n independent functions f_i on phase space in involution, that is,

$$\{f_i, f_j\} = 0.$$

When such condition is satisfied, Liouville Theorem (see Babelon, Bernard and Talon (2003)) implies that the system has a solution expressed in terms of a finite numbers of integrals (a solution by the quadrature method).

Back to the harmonic oscillator, its Hamiltonian is given by

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2m},$$

where $q := x$ and $p := mv$. Since the phase space is bidimensional and the energy $E := H$ is conserved, the system is integrable.

In 1965, Zabusky and Kruskal (1965) notice that the *solitons*, modelled by the solutions to the KdV equation, suffer no distortion under collision, what suggests a strong

stability of the system. It marks the beginning of the modern theory of integrable systems. A huge breakthrough came with the development of an important tool for the non-linear ODEs when Gardner *et al.* (1967) started their work on the *scattering - inverse scattering method*, also known as *Riemann-Hilbert method*. Some years latter, Deift and Zhou (1992) improved the technique with the development of the *non-linear steepest descent method*. The Riemann-Hilbert method provided the construction of an infinity hierarchy of recursive relations. Since at a finite dimension it implies that the system is Liouville integrable, such hierarchies became known as integrable hierarchies.

It was the beginning of the modern theory of integrable systems, which include equations such as the Painlevé transcendents, mKdV, Sine-Gordon, dNLS, KP and so on. An important tool in this context are the so-called τ-functions, which provide solutions to all the equations of a hierarchy and are usually expressed as a determinant of a matrix whose entries are elementary functions.

In the present work some important hierarchies are the Painlevé ones. The history of Painlevé equations begin in the early years 1900's, when the french mathematician Paul Painlevé classified all the second order non-linear ODEs of the form

$$u_{xx}(x) = F(x, u, u_x),$$

where $F(x)$ is a rational function and the ODE possesses the *Painlevé Property*, that is, the only movable singularities are poles. The sixty equations obtained by Painlevé and Gambier were summarized by Ince (1944). It is noteworthy that the fifth Painlevé equation was independently discovered by Fuchs. Each of such equations either possesses a closed form to its solution or is solvable by quadrature or can be reduced to one of the following six canonical Painlevé equations:

i. Painlevé I:
$$u'' = 6u^2 + x, \tag{5.2}$$

ii. Painlevé II:
$$u'' = 2u^3 + xu + \alpha, \tag{5.3}$$

iii. Painlevé III:
$$u'' = \frac{1}{xu}[x(u')^2 - uu' + \beta u + \alpha u^3 + \gamma x u^4 + \delta x], \tag{5.4}$$

iv. Painlevé IV:
$$u'' = \frac{1}{u}\left[\frac{1}{2}(u')^2 + 2(x^2 - \alpha)u^2 + 4xu^3 + \frac{3}{2}u^4 + \beta\right], \tag{5.5}$$

v. Painlevé V:
$$u'' = \left(\frac{1}{u-1} + \frac{1}{xu}\right)(u')^2 - \frac{u' - \gamma u}{x} + \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \delta\frac{u(u+1)}{u-1}, \tag{5.6}$$

vi. Painlevé VI:

$$\begin{aligned} u'' = \; & \frac{1}{2}\left(\frac{1}{u-x}+\frac{1}{u-1}+\frac{1}{u}\right)(u')^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right)u'+ \\ & \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\delta\frac{x(x-1)}{(u-x)^2}+\gamma\frac{x-1}{(u-1)^2}+\beta\frac{x}{u^2}+\alpha\right), \end{aligned} \tag{5.7}$$

where α, β, γ and δ are complex parameters.

5.2 Partition Function and KP Hierarchy

The Kadomtsev-Petviashvili (KP) hierarchy is defined by an infinite set of evolution equations and their compatibility conditions. One approach to construct such hierarchy is through any pseudo-differential operator

$$\mathcal{L} = \partial + \sum_{k=1}^{\infty} \alpha_k \partial^{-k}, \tag{5.8}$$

where $\alpha_k = \alpha_k(x, \mathbf{t})$ and $\mathbf{t} = (t_1, t_2, \cdots)$ are the KP *flow-parameters* or the KP *times*.

Definition 6. Consider the infinite set of equations in the Lax form of the KP equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = [\mathcal{D}_k, \mathcal{L}], \;\; k = 1, 2, 3, \cdots, \tag{5.9}$$

$$\mathcal{D}_k := (\mathcal{L}^k)_+ = \partial^k + \sum_{j=2}^{k} w_{k,j}\partial^{k-j}, \tag{5.10}$$

where $(.)_+$ means that we only take the terms of non-negative order in ∂ and $w_{k,j}$ is a polynomial in α_j and its derivatives. Then the **KP hierarchy** arises from the set of infinite compatibility conditions given by

$$\frac{\partial \mathcal{D}_k}{\partial t_\ell} - \frac{\partial \mathcal{D}_\ell}{\partial t_k} + [\mathcal{D}_k, \mathcal{D}_\ell] = 0. \tag{5.11}$$

For instance, taking $k = 2$ we have

$$\begin{aligned} \mathcal{D}_2 =& (\mathcal{L}^2)_+ \\ =& \left[\left(\partial + \sum_{k=1}^{\infty}\alpha_k\partial^{-k}\right)\left(\partial + \sum_{k=1}^{\infty}\alpha_k\partial^{-k}\right)\right]_+ \\ =& \left[\partial^2 + 2\alpha_1 + 2\sum_{k=2}^{\infty}\alpha_k\partial^{-k+1} + \sum_{k=1}^{\infty}\partial\alpha_k\partial^{-k} + \left(\sum_{k=1}^{\infty}\alpha_k\partial^{-k}\right)^2\right]_+ \\ =& \partial^2 + 2\alpha_1. \end{aligned}$$

Proceeding analogously we conclude that the first $\mathcal{D}_i$'s are given by

$$\begin{aligned}\mathcal{D}_1 &= \partial,\\ \mathcal{D}_2 &= \partial^2 + 2\alpha_1,\\ \mathcal{D}_3 &= \partial^3 + 3\alpha_1\partial + 3\alpha_1' + 3\alpha_2,\end{aligned}$$

and so on. Setting $\alpha_1 = u$, $\alpha_2 = v$, $t_2 = y$ and $t_3 = t$ the pair $\mathcal{D}_2, \mathcal{D}_3$ originates two equations:

$$\begin{aligned}3u_{xy} + 6v_y - 2u_t &= u_{xxx} + 6v_{xx} - 3uu_x\\ u_y &= u_{xx} + 4v_x.\end{aligned}$$

Eliminating v, one obtains the KP equation

$$3u_{yy} = (4u_t - u_{xxx} - 6uu_x)_x, \tag{5.12}$$

that names the entire hierarchy.

The operator $\mathcal{L}$ can be conjugated by another pseudodifferential operator $\mathcal{W}$ known as *dressing operator*, such that

$$\mathcal{L} = \mathcal{W}\partial\mathcal{W}^{-1}. \tag{5.13}$$

An important result states that the Wronskian determinant constructed by the dressing operator is a τ-function of solutions to the KP-hierarchy. In fact, take $\{x_i(\mathbf{t})\}_{i=1}^n$ linearly independent functions satisfying

$$\frac{\partial x_i}{\partial t_j} = \frac{\partial^j x_i}{\partial t_1^j}, \tag{5.14}$$

where $i = 1, \cdots, n$ and $j = 1, 2, \cdots$. We assume the following result.

Theorem 13 (Theorem 3.8.2, (HARNAD; BALOGH, 2021))**.** The Wronskian determinant

$$\tau(t) := \det\begin{bmatrix} x_1 & x_2 & \cdots & x_n\\ \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} & \cdots & \dfrac{\partial x_n}{\partial t_1}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial x_1}{\partial t_{n-1}} & \dfrac{\partial x_2}{\partial t_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial t_{n-1}}\end{bmatrix}, \tag{5.15}$$

is a τ-function for the KP-hierarchy.

Take $\mathcal{H}_n$ the space of Hermitian matrices $n \times n$, just as in section 2.8. The partition function of the matrix model determined by a measure $\mathrm{d}\nu = \rho(H)\,\mathrm{d}H$, where $\mathrm{d}H = \prod_{i=1}^n \mathrm{d}h_{ii} \prod_{i<j} \mathrm{dRe}\, h_{ij}\, \mathrm{dIm}\, h_{ij}$, is given by

$$Z_\rho = \int_{\mathcal{H}_n} \rho(H)\,\mathrm{d}H, \tag{5.16}$$

that is, the partition function is the quantity that normalizes the measure. If ρ is invariant by unitary transformation, the Weyl formula (2.12) still holds and Z_ρ can be expressed in terms of the eigenvalues of the matrix H. Taking

$$\rho(H) = \mathrm{e}^{-\operatorname{tr}(V(H))},$$

we have

$$\begin{aligned} Z_\rho &= \int_{\mathcal{H}_n} \mathrm{e}^{-\operatorname{tr}(V(H))}\,\mathrm{d}H \\ &= c_n \int_{\mathbb{R}^n} \Delta(\lambda)^2\,\mathrm{e}^{-\sum_i V(\lambda_i)}\,\mathrm{d}\lambda \\ &=: c_n Z_n(V). \end{aligned}$$

The term $Z_n(V)$ may be recognized as the partition function from Chapter 4. Take the potential $V(\lambda,\mathbf{t}) := V(\lambda;t_1,t_2,\cdots) = \frac{\lambda^2}{2} - \sum_{k=1}^{\infty} t_k\lambda^k$. Therefore,

$$Z_n(V,\mathbf{t}) := Z_n(V(\lambda,\mathbf{t})) = \int_{\mathbb{R}^n} \Delta(\lambda)^2 \prod_{i=1}^{n} \mathrm{e}^{-V(\lambda_i,\mathbf{t})}\,\mathrm{d}\lambda.$$

The Heine formula (see subsection 2.7.4) implies that

$$Z_n(V,\mathbf{t}) = n!\det(M_n(\mathbf{t})),$$

where $M_n(\mathbf{t})$ is the $n\times n$ matrix of moments, that is,

$$M_n(\mathbf{t}) := \begin{pmatrix} m_0(\mathbf{t}) & m_1(\mathbf{t}) & \cdots & m_{n-1}(\mathbf{t}) \\ m_1(\mathbf{t}) & m_2(\mathbf{t}) & \cdots & m_n(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1}(\mathbf{t}) & m_n(\mathbf{t}) & \cdots & m_{2n-2}(\mathbf{t}) \end{pmatrix}, \qquad m_k(\mathbf{t}) := \int_{\mathbb{R}} x^k\,\mathrm{e}^{-V(x,\mathbf{t})}\,\mathrm{d}x.$$

But notice that $\{m_k(\mathbf{t})\}_{i=0}^{2n-2}$ satisfy the relation (5.14), that is,

$$\begin{aligned} \frac{\partial m_k(\mathbf{t})}{\partial t_j} &= \frac{\partial}{\partial t_j}\int_{\mathbb{R}} x^k\,\mathrm{e}^{-\frac{x^2}{2}+\sum_{k=1}^{\infty} t_k x^k}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} x^{k+j}\,\mathrm{e}^{-\frac{x^2}{2}+\sum_{k=1}^{\infty} t_k x^k}\,\mathrm{d}x \\ &= \frac{\partial^j m_k(\mathbf{t})}{\partial t_1^j}. \end{aligned}$$

Thus, the partition function may be expressed as

$$Z_n(V,\mathbf{t}) = n!\det\left(\frac{\partial^j m_k(\mathbf{t})}{\partial t_1^j}\right)_{0\le k,j\le n-1},$$

and by Theorem 13,

$$\frac{Z_n(V,\mathbf{t})}{Z_n(V,0)} = \det\left(\frac{\partial^j m_k(\mathbf{t})}{\partial t_1^j}\right)_{0\le k,j\le n-1},$$

is a τ-function of the KP hierarchy.

5.3 Limit kernels

Another situation where the relation between integrable hierarchies and random matrix models naturally shows up is when dealing with the behaviour of the eigenvalues as the dimension of the matrix grows.

The Gaussian measure on the space of matrices induces a probability measure over the space of eigenvalues whose law depends on the ensemble. For the Gaussian Unitary Ensemble (GUE), Weyl formula (2.12) tells us that the induced density has the form

$$c_n \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^n \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^2.$$

In another ensembles, the general law has the form

$$c_n \mathrm{e}^{-\frac{\alpha}{2}\sum_{i=1}^n \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta,$$

where α is some constant and β is the so-called *Dyson index*. When dealing with the space of real symmetric matrices, i.e., the Gaussian Orthogonal Ensemble (GOE), we have $\beta = 1$. In the space of complex hermitian matrices, i. e., in the Gaussian Unitary Ensemble, we have $\beta = 2$ and, at last, for the quartenionic hermitian matrices where the Gaussian measure defines the Gaussian Symplectic Ensemble (GSE), we have $\beta = 4$.

Our main interest is in the Unitary ensemble. From section 2.6, we know that given a Gaussian ensemble, the eigenvalues behave like a determinantal point process with a Christoffel-Darboux kernel. Therefore, if anyone wonders about the distribution of eigenvalues as the dimension goes to infinity, the answer is in the limit behaviour of such kernel.

Recall the definition of the Christoffel-Darboux kernel

$$K_n(x,y) = \sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}} \sum_{k=0}^{n-1} p_k(x)p_k(y).$$

The Christoffel-Darboux formula for orthogonal polynomials (SZEGö, 1939) states that

$$\sum_{k=0}^{n-1} p_k(x)p_k(y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}. \tag{5.17}$$

Moreover, we know that

$$Y_{11}(z) = \frac{p_n(z)}{\gamma_n}$$
$$Y_{12}(z) = -2\pi i \gamma_{n-1} p_{n-1}(z),$$

where $Y(z)$ is the solution to the Riemann-Hilbert problem in section 4.3. Therefore,

$$\begin{aligned} K_n(x,y) &= \sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}}\frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)p_{n-1}(y)-p_n(y)p_{n-1}(x)}{x-y} \\ &= \sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}}\frac{Y_{11}(x)Y_{12}(y)-Y_{11}(y)Y_{12}(x)}{(x-y)(-2\pi i)} \\ &= \frac{\sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}}}{2\pi i(x-y)}[Y_{11}(y)Y_{12}(x)-Y_{11}(x)Y_{12}(y)], \end{aligned}$$

and the kernel can be written as

$$K_n(x,y) = \frac{\sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}}}{2\pi i(x-y)}\begin{pmatrix}0 & 1\end{pmatrix}Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\end{pmatrix}. \tag{5.18}$$

The *global/macroscopic* limit for the kernel is given by the equilibrium measure for the potential V. In fact, from (5.18) we have

$$\begin{aligned} K_n(x,y) = \frac{\sqrt{\mathrm{e}^{-nV(x)}}\sqrt{\mathrm{e}^{-nV(y)}}}{2\pi i(x-y)}&\begin{pmatrix}0 & 1\end{pmatrix}\mathrm{e}^{-n(g_+(y)-\frac{\ell}{2})\sigma_3}T_+^{-1}(y)R^{-1}(y) \\ &\times R(x)T_+(x)\,\mathrm{e}^{n(g_+(x)-\frac{\ell}{2})\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}. \end{aligned} \tag{5.19}$$

Define

$$\xi(z) := -2\pi i\int_b^z \hat{\psi}(y)\,\mathrm{d}y, \tag{5.20}$$

where $\hat{\psi}(x)$ is the density of the equilibrium measure $\mu_V(x)$ from subsection 4.3.1.1. Therefore, by the definition of the g-function (see subsection 4.3.1.2) one has

$$\xi(z) = g_+(z) - g_-(z), \qquad z\in(a,b), \tag{5.21}$$

and the kernel can be expressed as

$$K_n(x,y) = \frac{1}{2\pi i(x-y)}\begin{pmatrix}0 & \mathrm{e}^{n\frac{\xi(y)}{2}}\end{pmatrix}T_+^{-1}(y)R^{-1}(y)R(x)T_+(x)\begin{pmatrix}\mathrm{e}^{n\frac{\xi(x)}{2}}\\0\end{pmatrix}.$$

Letting $y\to x$ and applying L'Hopital rule, leads to

$$\lim_{n\to\infty}\frac{1}{n}K_n(x,x) = \hat{\psi}(x).$$

In short, the following result holds.

Theorem 14 (Theorem 6.96, (DEIFT, 1999))**.** The unique Borel measure

$$\mathrm{d}\mu_V(x) := \hat{\psi}(x)\,\mathrm{d}x$$

minimizing the operator

$$E = \int_{\mathbb{R}^2} \log|x-y|^{-1}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_{\mathbb{R}} V(x)\,\mathrm{d}\mu(x), \tag{5.22}$$

over the space of all probability measures on $\mathbb{R}$, is such that

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \hat{\psi}(x). \tag{5.23}$$

The Gaussian case of Theorem 14 was proved by Ginibre (1965) and is the Semi-circle law (see section 2.3).

The so-called *local* limit is a little more involved. Since the expression for the orthogonal polynomials changes in different regions of the complex plane, the behaviour of the kernel depends on the location of x, y as well.

5.3.1 Bulk limit

To determine the distribution of eigenvalues near $x^* \in \mathbb{R}$, we need to center and rescale the correlation kernel. Suppose there exists a fixed $\delta > 0$ such that $x, y, x^* \in (a-\delta, b-\delta)$. Moreover, choose the ratio of the local parametrix domain to be $\frac{\delta}{2}$, i.e., $P(z)$ is the parametrix for $z \in U_{\delta/2}$ given by Equation (4.59) and $\tilde{P}(z)$ is the parametrix for $z \in \tilde{U}_{\delta/2}$ given by Equation (4.60). The behaviour of the orthogonal polynomials in $(a-\delta, b-\delta)$ is given by the limit of the solution in region B_1 (see Figure 30) when $z \downarrow \mathbb{R}$. From Equations (4.67) and (4.68) we have

$$\begin{aligned} Y_+(z) &= \mathrm{e}^{n\frac{\ell}{2}\sigma_3} R(z) T(z) \mathrm{e}^{n(g_+(z)-\frac{\ell}{2})\sigma_3} \\ &= \mathrm{e}^{n\frac{\ell}{2}\sigma_3} R(z) S^{(1)}(z) v_+ \mathrm{e}^{n(g_+(z)-\frac{\ell}{2})\sigma_3}. \end{aligned}$$

Therefore,

$$\begin{aligned} Y^{-1}(y)Y(x) = \mathrm{e}^{-n(g_+(y)-\frac{\ell}{2})\sigma_3} v_+^{-1} (S^{(1)}(y))^{-1} R^{-1}(y) \times \\ R(x) S^{(1)}(x) v_+ \mathrm{e}^{n(g_+(x)-\frac{\ell}{2})\sigma_3}, \end{aligned}$$

which implies that

$$\begin{aligned} \mathrm{e}^{-n\frac{V(y)}{2}} \mathrm{e}^{-n\frac{V(x)}{2}} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(y)Y(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{e}^{-n\frac{V(y)}{2}} \mathrm{e}^{-n(g_+(y)-\frac{\ell}{2})\sigma_3} v_+^{-1} (S^{(1)}(y))^{-1} \\ \times R^{-1}(y) R(x) S^{(1)}(x) v_+ \mathrm{e}^{n(g_+(x)-\frac{\ell}{2})\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-n\frac{V(x)}{2}}. \end{aligned}$$

In terms of the function $\xi(z)$ defined by Equation (5.20),

$$\begin{aligned}\begin{pmatrix}0 & 1\end{pmatrix} \mathrm{e}^{-n\frac{V(y)}{2}} \mathrm{e}^{-n(g_+(y)-\frac{\ell}{2})\sigma_3} v_+^{-1} &= \begin{pmatrix}0 & 1\end{pmatrix} \mathrm{e}^{-n\frac{V(y)}{2}} \mathrm{e}^{-n(g_+(y)-\frac{\ell}{2})\sigma_3} \begin{pmatrix}1 & 0\\ -\mathrm{e}^{-n\xi(y)} & 1\end{pmatrix} \\ &= \begin{pmatrix}-\mathrm{e}^{n(g_+(y)-\frac{\ell}{2}-\frac{V(y)}{2}-\xi(y))} & \mathrm{e}^{n(g_+(y)-\frac{\ell}{2}-\frac{V(y)}{2})}\end{pmatrix} \\ &= \begin{pmatrix}-\mathrm{e}^{-n\frac{\xi(y)}{2}} & \mathrm{e}^{n\frac{\xi(y)}{2}}\end{pmatrix}.\end{aligned}$$

Analogously,

$$\begin{aligned}v_+ \mathrm{e}^{n(g_+(x)-\frac{\ell}{2})\sigma_3} \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-n\frac{V(x)}{2}} &= \begin{pmatrix}1 & 0\\ \mathrm{e}^{-n\xi(x)} & 1\end{pmatrix} \mathrm{e}^{n(g_+(x)-\frac{\ell}{2})\sigma_3} \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-n\frac{V(x)}{2}} \\ &= \begin{pmatrix}\mathrm{e}^{n\frac{\xi(x)}{2}}\\ \mathrm{e}^{-n\frac{\xi(x)}{2}}\end{pmatrix}.\end{aligned}$$

At last, we need to evaluate

$$(S^{(1)}(y))^{-1} R^{-1}(y) R(x) S^{(1)}(x). \tag{5.24}$$

Because of Equation (4.66), that is,

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k}{n^k},$$

we have that

$$R^{-1}(y)R(x) = \begin{pmatrix}1+\frac{1}{n}(R_{1,11}(x)+R_{1,22}(y)) & \frac{1}{n}(R_{1,12}(x)-R_{1,12}(y))\\ \frac{1}{n}(R_{1,21}(x)-R_{1,21}(y)) & 1+\frac{1}{n}(R_{1,11}(y)+R_{1,22}(x))\end{pmatrix} + O\left(\frac{1}{n^2}\right)$$

Due to the symmetry of the jump matrices in the problem summarized by Theorem 12, it follows that $R_{1,11}(z) = -R_{1,22}(z)$ and $R_{1,12}(z) = R_{1,21}(z)$. Therefore, it suffices to estimate two terms. Define $B = \partial U_{\delta/2} \cup \partial \tilde{U}_{\delta/2}$. Then,

$$\begin{aligned}R_{1,21}(x) - R_{1,12}(y) &= \frac{1}{2\pi i} \int_B \left[\frac{V_{12}(s)}{s-x} - \frac{V_{12}(s)}{s-y}\right] \mathrm{d}s \\ \implies |R_{1,21}(x) - R_{1,12}(y)| &\le \frac{1}{2\pi} \int_B \left|\frac{V_{12}(s)(x-y)}{(s-x)(s-y)}\right| |\mathrm{d}s| \\ &= \frac{|x-y|}{2\pi} \int_B \left|\frac{V_{12}(s)}{(s-x)(s-y)}\right| |\mathrm{d}s| \\ &\le \frac{|x-y|}{2\pi} \frac{4|B|}{\delta^2} \max_{s\in B} |V_{12}(s)| \\ &= |x-y| C_1,\end{aligned}$$

where C_1 is independent of x, y. Analogously,

$$\begin{aligned}|R_{1,11}(x) + R_{1,22}(y)| &= \left|\frac{1}{2\pi i} \int_B \left[\frac{V_{11}(s)}{s-x} - \frac{V_{11}(s)}{s-y}\right] \mathrm{d}s\right| \\ &\le |x-y| C_2.\end{aligned}$$

Thus, $R^{-1}(y)R(x) = I + O\left(\frac{x-y}{n}\right)$. By the expression for $S^{(1)}(z)$ in Equation (4.56), it is straightforward that Equation (5.24) equals to

$$\frac{1}{2}\begin{pmatrix} \alpha(y)\alpha^{-1}(x)+\alpha(x)\alpha^{-1}(y) & -i(\alpha(x)\alpha^{-1}(y)-\alpha(y)\alpha^{-1}(x)) \\ i(\alpha(x)\alpha^{-1}(y)-\alpha(y)\alpha^{-1}(x)) & \alpha(y)\alpha^{-1}(x)+\alpha(x)\alpha^{-1}(y) \end{pmatrix} + O\left(\frac{1}{n}\right).$$

where $\alpha(z)$ is the same as in Equation (4.55). By the Taylor expansion for the binomial function, we obtain

$$\begin{aligned} \alpha(y)\alpha^{-1}(x) &= \left(\frac{y-b}{y-a}\right)^{1/4}\left(\frac{x-a}{x-b}\right)^{1/4} \\ &= \left(1+(y-x)\frac{(b-a)}{(y-a)(x-b)}\right)^{1/4} \\ &= 1+O(x-y), \\ \alpha(x)\alpha^{-1}(y) &= \left(1+(x-y)\frac{(b-a)}{(x-a)(y-b)}\right)^{1/4} \\ &= 1+O(x-y), \end{aligned}$$

and, consequently,

$$(S^{(1)}(y))^{-1}R^{-1}(y)R(x)S^{(1)}(x) = I + O(x-y).$$

Therefore, by Equation (5.18) the Kernel can be expressed as

$$\begin{aligned} K_n(x,y) &= \frac{(e^{\frac{n}{2}(\xi(y)-\xi(x))} - e^{-\frac{n}{2}(\xi(y)-\xi(x))} + O(x-y))}{2\pi i(x-y)} \\ &= \frac{\sin(\frac{n}{2i}(\xi(y)-\xi(x)))}{\pi(x-y)} + O(x-y). \end{aligned}$$

Now we perform a rescaling and a centralization to obtain the limit kernel. Start taking $x \mapsto x^* + \frac{x}{n\hat{\psi}(x^*)}$ and $y \mapsto x^* + \frac{y}{n\hat{\psi}(x^*)}$, where $\hat{\psi}(z)$ is the density of the equilibrium measure. The kernel is then given by

$$\frac{K_n\left(x^*+\frac{x}{n\hat{\psi}(x^*)}, x^*+\frac{y}{n\hat{\psi}(x^*)}\right)}{n\hat{\psi}(x^*)} = \frac{\sin\left\{\frac{n}{2i}\left[\xi\left(x^*+\frac{y}{n\hat{\psi}(x^*)}\right)-\xi\left(x^*+\frac{x}{n\rho_V(x^*)}\right)\right]\right\}}{\pi(x-y)} + O\left(\frac{x-y}{n}\right)$$

Taking the integral formula for $\xi(z)$ (Equation (5.20)), some algebraic manipulations lead to

$$\begin{aligned} \frac{n}{2i}\left[\xi\left(x^*+\frac{y}{n\hat{\psi}(x^*)}\right)-\xi\left(x^*+\frac{x}{n\hat{\psi}(x^*)}\right)\right] &= \frac{n}{2i}\left[-2\pi i\int_b^{x^*+\frac{y}{n\hat{\psi}(x^*)}}\hat{\psi}(s)\,ds + \right. \\ &\quad \left. 2\pi i\int_b^{x^*+\frac{x}{n\hat{\psi}(x^*)}}\hat{\psi}(s)\,ds\right] \\ &= -\pi\left[n\int_{x^*+\frac{x}{n\hat{\psi}(x^*)}}^{x^*+\frac{y}{n\hat{\psi}(x^*)}}\hat{\psi}(s)\,ds\right] \\ &= -\pi\left[\frac{1}{\hat{\psi}(x^*)}\int_x^y\hat{\psi}\left(x^*+\frac{t}{n\hat{\psi}(x^*)}\right)dt\right], \end{aligned}$$

and since by the Dominated Convergence Theorem

$$\lim_{n\to\infty}\frac{1}{\hat{\psi}(x^*)}\int_x^y \hat{\psi}\left(x^*+\frac{t}{n\hat{\psi}(x^*)}\right)\mathrm{d}t = \frac{1}{\hat{\psi}(x^*)}\int_x^y \hat{\psi}(x^*)\,\mathrm{d}t$$
$$= y-x,$$

it follows that the rescaled kernel converges, as $n\to\infty$, to

$$\frac{\sin[\pi(x-y)]}{\pi(x-y)}. \tag{5.25}$$

In short, we just verified the following result.

Theorem 15 ((PASTUR; SHCHERBINA, 1997))**.** Let V be a real analytic potential for which the equilibrium measure $\mathrm{d}\mu_V$ satisfies the conditions from Theorem 11 and $x^*\in\mathbb{R}$ such that $\hat{\psi}(x^*)>0$, where $\hat{\psi}$ is the density of $\mathrm{d}\mu_V$. Then, uniformly for x and y in compact subsets of $\mathbb{R}$, it holds that

$$\lim_{n\to\infty}\frac{K_n\left(x^*+\frac{x}{n\hat{\psi}(x^*)},x^*+\frac{y}{n\hat{\psi}(x^*)}\right)}{n\hat{\psi}(x^*)} = \frac{\sin[\pi(x-y)]}{\pi(x-y)}. \tag{5.26}$$

It is noteworthy to say that the Gaussian case $V(x)=\frac{x^2}{2}$ of Theorem 15 was discovered earlier by Forrester (1993). The general case enunciated above was first studied by Pastur and Shcherbina (1997) and rediscovered with the RHP approach by Deift *et al.* (1999b).

5.3.2 Limit kernel at endpoints

Take the formula for $Y(z)$ in the neighbourhood U_δ of b. Proceeding analogously as in the previous section,

$$\mathrm{e}^{-n\frac{V(y)}{2}}\mathrm{e}^{-n\frac{V(x)}{2}}\begin{pmatrix}0 & 1\end{pmatrix}Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\mathrm{e}^{-n\frac{\xi(y)}{2}} & \mathrm{e}^{n\frac{\xi(y)}{2}}\end{pmatrix}(P(y))^{-1}\times$$
$$R^{-1}(y)R(x)P(x)\begin{pmatrix}\mathrm{e}^{n\frac{\xi(x)}{2}}\\ \mathrm{e}^{-n\frac{\xi(x)}{2}}\end{pmatrix}.$$

We already know that $R^{-1}(y)R(x)=I+O\left(\frac{x-y}{n}\right)$. Writing the conformal map $f(z)$ in terms of the auxiliary function $\xi(z)$, the local parametrix takes the form

$$P(z)=E(z)L(f(z))\,\mathrm{e}^{-n\frac{\xi(z)}{2}\sigma_3}.$$

Therefore,

$$\mathrm{e}^{-n\frac{V(y)}{2}}\mathrm{e}^{-n\frac{V(x)}{2}}\begin{pmatrix}0 & 1\end{pmatrix}Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1 & 1\end{pmatrix}L^{-1}(f(y))E^{-1}(y)\times$$
$$R^{-1}(y)R(x)E(x)L(f(x))\begin{pmatrix}1\\1\end{pmatrix}.$$

It follows from the explicit formula for $E(z)$ (Equation (4.58)) that

$$E^{-1}(y)R^{-1}(y)R(x)E(x) = [f^{-1/4}(y)\alpha(y)\alpha^{-1}(x)f^{1/4}(x)]^{\sigma_3} + O\left(\frac{x-y}{n}\right).$$

Moreover, since the equilibrium measure vanishes as a square-root at b, there exists a constant $c > 0$ such that as $z \to b$,

$$\hat{\psi}(z) = \frac{c}{\pi}(b-z)^{1/2}(1+o(1)).$$

Therefore, from the integral representation for $f(z)$,

$$f(z) = \left(\frac{3n}{2}\right)^{2/3}\left(-2\pi i\int_b^z \hat{\psi}(s)\,\mathrm{d}s\right)^{2/3},$$

it follows that

$$\begin{aligned}
f(y) &= \left(\frac{3n}{2}\right)^{2/3}\left(-2\pi i\int_b^y \frac{c}{\pi}(b-s)^{1/2}(1+o(1))\,\mathrm{d}s\right)^{2/3}\\
&= \left(\frac{3n}{2}\right)^{2/3}\left(\frac{4ic}{3}\right)^{2/3}[(b-y)^{3/2}+o(b-y)^{3/2}]^{2/3}\\
&= \left(\frac{3n}{2}\right)^{2/3}\left(\frac{4ic}{3}\right)^{2/3}(b-y)[1+o(1)]^{2/3}\\
f^{-1/4}(y)\alpha(y)\alpha^{-1}(x)f^{1/4}(x) &= (b-y)^{-1/4}\left(\frac{y-b}{y-a}\right)^{1/4}\left(\frac{x-a}{x-b}\right)^{1/4}(b-x)^{1/4}\left[\frac{1+o(1)}{1+o(1)}\right]^{1/6}\\
&= \left(\frac{x-a}{y-a}\right)^{1/4}(1+o(1))\\
&= \left[1+\frac{x-y}{y-a}\right]^{1/4}(1+o(1)) = 1+O(x-y).
\end{aligned}$$

Thus, Equation (5.18) becomes

$$K_n(x,y) = \frac{1}{2\pi i(x-y)}\begin{pmatrix}-1 & 1\end{pmatrix}L_+^{-1}(f(y))\,(I+O(x-y))\,L_+(f(x))\begin{pmatrix}1\\1\end{pmatrix},$$

where the n-dependence of K_n comes from the n-dependence of the conformal map f.

We rescale and centralize the kernel taking $x \mapsto b + \frac{x}{(cn)^{2/3}} =: x_n$ and $y \mapsto b + \frac{y}{(cn)^{2/3}} =: y_n$. Therefore, as $n \to \infty$, a straightforward calculation implies that $f(x_n) \to x$ and $f(y_n) \to y$. Thus,

$$\frac{1}{(cn)^{3/2}}K_n(x_n,y_n) = \frac{1}{2\pi i(x-y)}\begin{pmatrix}-1 & 1\end{pmatrix}L_+^{-1}(f(y_n))(I+O(n^{-2/3}))L_+(f(x_n))\begin{pmatrix}1\\1\end{pmatrix}.$$

Now, by the definition of $L(f(z))$ in the sector II (see Equation (4.57) and Figure 28), we have

$$L_+(z) = \begin{pmatrix} -w\,\mathrm{Ai}(wz)\sqrt{2\pi} & -w^2\,\mathrm{Ai}(w^2z)\sqrt{2\pi} \\ iw^2\,\mathrm{Ai}'(wz)\sqrt{2\pi} & iw\,\mathrm{Ai}'(w^2z)\sqrt{2\pi} \end{pmatrix}$$
$$L_+^{-1}(z) = \frac{1}{\det L_+(z)} \begin{pmatrix} iw\,\mathrm{Ai}'(w^2z)\sqrt{2\pi} & w^2\,\mathrm{Ai}(w^2z)\sqrt{2\pi} \\ -iw^2\,\mathrm{Ai}'(wz)\sqrt{2\pi} & -w\,\mathrm{Ai}(wz)\sqrt{2\pi} \end{pmatrix},$$

where $\det L_+(z) = 2\pi i(w\,\mathrm{Ai}(w^2z)\,\mathrm{Ai}'(wz) - w^2\,\mathrm{Ai}'(w^2z)\,\mathrm{Ai}(wz) = 1$, by the Wronskian relations for the solutions to the Airy equation. Moreover, since $\mathrm{Ai}(z)+w\,\mathrm{Ai}(wz)+w^2\,\mathrm{Ai}(w^2z) = 0$, the rescaled kernel becomes

$$\frac{1}{(cn)^{3/2}}K_n(x_n,y_n) = \frac{1}{(x-y)}(\mathrm{Ai}(f(x_n))\,\mathrm{Ai}'(f(y_n)) - \mathrm{Ai}(f(y_n))\,\mathrm{Ai}'(f(x_n))) + O\left(\frac{x-y}{n^{2/3}}\right).$$

Therefore, the limit kernel is given by

$$\lim_{n\to\infty}\frac{1}{(cn)^{2/3}}K_n(x_n,y_n) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}, \tag{5.27}$$

and we just verified the following Theorem:

Theorem 16 ((DEIFT; GIOEV, 2007)). Let V be a real analytic potential for which the equilibrium measure $\mathrm{d}\mu_V$ satisfies the conditions from Theorem 11 and let b be the right edge point of the support of $\mathrm{d}\mu_V$, such that for some constant $c > 0$

$$\hat{\psi}(x) = \frac{c}{\pi}(b-x)^{1/2}(1+o(1)) \qquad \text{as } x \uparrow b.$$

Then, uniformly for x and y in compact subsets of $\mathbb{R}$, it holds that

$$\lim_{n\to\infty}\frac{1}{(cn)^{2/3}}K_n(x_n,y_n) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}. \tag{5.28}$$

The Gaussian case of Theorem 16 is due to Forrester (1993). For the general case, the relevant asymptotic estimates of orthogonal polynomials were fully developed by Deift *et al.* (1999b), but the formal statement as above only came in Deift and Gioev (2007).

5.3.3 Largest eigenvalue

Let $\lambda^{\max}$ be the largest eigenvalue. Given a constant t, the probability that $\lambda^{\max} \leq t$ is a Gap Probability (see subsection 2.7.1), since

$$P(\lambda^{\max} \leq t) = P(\text{there is no eigenvalue in } (t,\infty)).$$

We know from section 5.3 that given the right normalization, the distribution of the eigenvalues converges weakly in probability to the equilibrium measure. For instance,

we know that the distribution of the normalized eigenvalues of a matrix whose entries are independent identically distributed random variables converges to the Wigner semicircle law. That is, as $n \to \infty$ the distribution of eigenvalues of $H/\sqrt{n}$ goes to $\frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2,2]}$. Consequently, $\frac{\lambda^{\max}}{\sqrt{n}} \to 2$ almost surely.

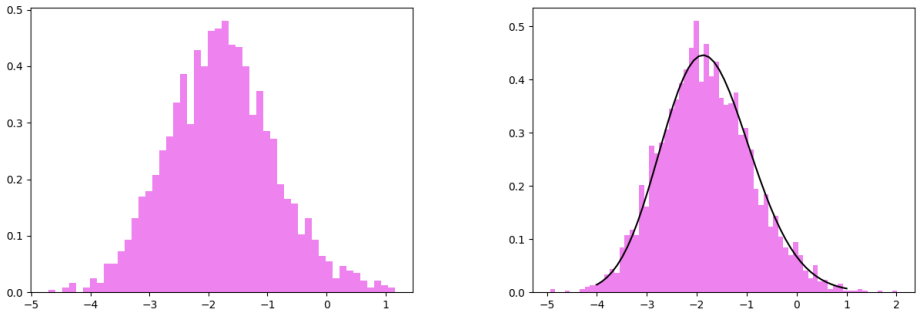
In the general case, the normalized largest eigenvalue goes to the endpoint b. Following the approach for λ in a neighbourhood of b, we rescale and centralize the normalized eigenvalue as follows,

$$\lambda \mapsto (\lambda - b)(cn)^{2/3}.$$

Now look at the distribution function

$$\begin{aligned} F_n(t) &:= P\left[(\lambda^{\max} - b)(cn)^{2/3} \leq t\right] \\ &= P\left[\lambda^{\max} \leq \frac{t}{(cn)^{2/3}} + b\right]. \end{aligned}$$

Figure 31 – Tracy-Widom function.[1]



Source: Elaborated by the author.

From Theorem 4, the distribution function is a Fredholm determinant,

$$F_n(t) = \det[I - K^{(n)}_{(t,\infty)}],$$

and as $n \to \infty$, in virtue of Theorem 16 it is natural to expect that $F_n(x) \to F(x)$ where

$$\begin{aligned} F(t) &= \det[I - \mathbb{A}_{(t,\infty)}], \\ &= 1 - \int_t^\infty \mathbb{A}(x_1, x_2)\,\mathrm{d}x + \frac{1}{2!}\int_t^\infty \int_t^\infty \det[\mathbb{A}(x_i, x_j)]_{i,j=1}^2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \cdots, \end{aligned}$$

[1] In Figure 31, at the left one has the distribution of 2000 matrices of dimension $n = 100$ and at the right, the distribution of 3000 matrices of dimension $n = 100$ plus the plot of the Tracy-Widom function. The Tracy-Widom trace comes from the MIT licensed package obtained from https://github.com/yymao/TracyWidom/. Such package generates the plot for the Tracy-Widom function using the interpolation tables available on Borot and Nadal (2012).

where $\mathbb{A}_{(t,\infty)}$ is the Airy kernel. The function $F(t)$ is called *Tracy-Widom distribution*. Despite it can not be expressed in a closed form it can be related to an integrable system.

Theorem 17 ((TRACY; WIDOM, 2002)). The function $F(t)$ is given by

$$F(t) = \exp\left(-\int_t^\infty (s-t)u^2(s)\,\mathrm{d}s\right), \tag{5.29}$$

where $u(s)$ is a solution to the Painlevé II equation

$$u''(x) = 2u(x)^3 + xu(x).$$

In particular, $u(s)$ in equation (5.29) is the so-called Hasting-Mcleod solution and

$$u(s) \to \mathrm{Ai}(s),$$

as $s \to \infty$.

Now consider the bulk limit. Let x^* in the bulk and $s > 0$. Take a enumeration of the normalized eigenvalues $\{\lambda_k\}_{k=1}^n$. Once again consider the centralized and rescaled eigenvalues

$$\lambda_k \mapsto n\hat{\psi}(x^*)(\lambda_k - x^*).$$

Thus,

$$\lambda_k \notin \left(x^* - \frac{s}{2n\hat{\psi}(x^*)}, x^* + \frac{s}{2n\hat{\psi}(x^*)}\right), \text{ where } 1 \le k \le n,$$

is equivalent to

$$n\hat{\psi}(x^*)(\lambda_k - x^*) \notin \left(-\frac{s}{2}, \frac{s}{2}\right), \text{ where } 1 \le k \le n.$$

Define the function

$$H_n(s) := P\left[n\hat{\psi}(x^*)(\lambda_k - x^*) \notin \left(-\frac{s}{2}, \frac{s}{2}\right), \text{ where } 1 \le k \le n\right].$$

Since it is a gap probability and the limit kernel in the support of the measure is the sine kernel, in virtue of Theorem 15 it is natural to expect that $H_n(s) \to H(s)$ where

$$H(s) = \det[I - \mathbb{S}_{(-\frac{s}{2},\frac{s}{2})}], \tag{5.30}$$

$$= 1 - \int_{-\frac{s}{2}}^{\frac{s}{2}} \mathbb{S}(x_1,x_2)\,\mathrm{d}x + \frac{1}{2!}\int_{-\frac{s}{2}}^{\frac{s}{2}}\int_{-\frac{s}{2}}^{\frac{s}{2}} \det[\mathbb{S}(x_i,x_j)]_{i,j=1}^2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \cdots, \tag{5.31}$$

where $\mathbb{S}_{(-\frac{s}{2},\frac{s}{2})}$ is the Sine kernel. Despite $H(s)$ has no closed form, it can be expressed as a solution to the Painlevé V.

Theorem 18 ((JIMBO *et al.*, 1980)). For every $s > 0$,

$$H(s) = \exp\left(\int_0^s \frac{\sigma(t)}{t}\,\mathrm{d}t\right), \tag{5.32}$$

where $\sigma(t)$ satisfies the σ-form of Painlevé V equation

$$(t\sigma'')^2 + 4(t\sigma' - \sigma)(t\sigma' - \sigma + (\sigma')^2) = 0,$$

with the following expansion around $t = 0$

$$\sigma(t) = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3} + O(t^4).$$

5.3.4 Conclusion

Different measures over a space of matrices may lead to connections to different integrable systems/hierarchies. In the present work we took specifically the Gaussian measure, and ended up finding the sine kernel, the Airy kernel and the Tracy-Widom distribution.

It is a noteworthy fact that the limit kernel at the endpoints of the measure does not depend on its exact form, rather on some general properties about its behaviour. In our specific case, the important point was that the equilibrium measure vanishes as a square-root at the endpoints, what leads us to a *Airy kernel* (DEIFT; GIOEV, 2007).

If the equilibrium measure vanishes as $z^{-1/2}$ we end up with a *Bessel kernel* (TRACY; WIDOM, 1994). When the measure vanishes as $z^{3/2}$ we have the *pure gravity case* and the limit kernel is related to the Painlevé I equation, and so on (BERGERE; EYNARD, 2009).

For instance, Its, Kuijlaars and Östensson (2008) found out that the measure

$$p(M)\,\mathrm{d}M = \frac{1}{Z_{n,N}} |\det M|^{2\alpha}\, \mathrm{e}^{-N \operatorname{tr} V(M)}\,\mathrm{d}M.$$

where $N > 0, \alpha > -1/2$, defined over the space of hermitian matrices of dimension n, i. e., $\mathcal{H}_n$, gives rise to an Unitary ensemble. Moreover, when $n, N \to \infty$, $n/N \to 1$, the global eigenvalue regime is determined by the equilibrium measure and the one-parameter family of limiting kernels at the endpoint of the support, say

$$K_\alpha^{edge}(x, y; s),$$

is expressed as

$$K_\alpha^{edge}(x, y; s) = \frac{u_2(x; s)u_1(y; s) - u_1(x; s)u_2(y; s)}{2\pi i(x - y)},$$

where u_1, u_2 are related to special solutions to the Painlevé *XXXIV* (ITS; KUIJLAARS; ÖSTENSSON, 2008).

CHAPTER

6

FINAL COMMENTS

The first Chapter brought plenty of applications, some of them quite old, such as the Wigner Semicircle law, some of them very modern, such as quantum chaos and big data. All of them justify the interest in the behaviour of eigenvalues in random matrix ensembles. Moreover, we discussed how a measure over a space of matrices induces a distribution over the space of eigenvalues, and how the point process theory can be applied to stablish formally such connection.

Latter, we chose one among the possible applications, namely the enumeration of g-maps, to motivate the calculations in the following sections. After presenting some results about the relation between the trace of matrices in random ensembles and the enumeration of maps, we explored the details of a well-known Riemann-Hilbert problem, namely the one related to orthogonal polynomials. The asymptotic information obtained through such calculations were applied in order to achieve an asymptotic expansion to the partition function and, consequently, to obtain a way to count maps by the coefficients in the partition function expansion.

Moreover, we also discussed briefly about how the partition function of random matrix models relates to the KP hierarchy, illustrating a connection between random matrices and integrable systems when the dimension is fixed. Then, we explored the relation that arises between random matrix ensembles and integrable systems when the dimension of the matrices goes to infinity.

Through the calculations in section 5.3, we concluded that in order to understand the behaviour of the eigenvalues one must understand the limit of the kernel that describes the point process associated to the eigenvalues. In such context we saw some limit behaviours in details, what led us to solutions to Painléve II and V, while in the last Section we discussed briefly about an Ensemble that leads to a kernel whose behaviour is expressed in terms of the Painlevé XXXIV. Therefore, we have a reason to be interested

in the behaviour of the solutions to such equation. Thus, in further works, we aim at a better asymptotic characterization to the solutions of Painlevé XXXIV.

It is important to notice that, despite no original calculations were presented, a lot of techniques which will be essential in the future were fully explored throughout the dissertation. All the work in understanding and replicating the calculations originally developed by Deift, Zhou, McLaughlin and others in the context of Riemann-Hilbert problems will be applied in the development of an asymptotic analysis to the Painlevé XXXIV in the next steps of our academic journey.

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APPENDIX

A

SOME PYTHON CODES

For the ones who have enjoyed the journey, we have some good news. You may have finished the reading, but the fun is not over yet!

Some figures along the text were generated through Python codes. In this Appendix we invite the reader to play a little bit with matrices eigenvalues through the codes behind such figures. The comments in the codes are intended to make it possible to have fun with random matrices even for the ones who are not very fond of programming.

We start with Section 2.3 where we saw how the eigenvalues of a normalized matrix with entries $\{-1, 1\}$ are distributed in a semicircle shape when the dimension of the matrix grows. The Python code to generate images similar to the Figures 3 and 4 is as follows.

Source code 1 – Empirical Semicircle distribution.

```
% First of all, we need some packages, namely 'random' to build
    random variables, 'numpy' to construct matrices and 'matplotlib
    .pyplot' to generate images.
import random
import numpy
import matplotlib.pyplot as plt

%Then, set a variable 'k' to be the dimension of the matrix.
k = 1000

%Now we create a list where the eigenvalues will be stored.
List = []

%It is time to construct the matrix.
w, h = k, k
```

```
14: Mat = [[0 for x in range(w)] for y in range(h)]
15:
16: for x in range(w):
17:   for y in range(h):
18:     if x == y:
19:         Mat[x][y] =  (1/(k**(1/2)))*random.choice([-1,1])
20:     if x<y:
21:         Mat[x][y] =(1/(k**(1/2)))*random.choice([-1,1])
22:         Mat[y][x] = Mat[x][y]
23:
24: % We evaluate its eigenvalues and save them in the list initiated
      before.
25: val, vet = numpy.linalg.eigh(Mat)
26: List.append(val)
27:
28: % At last, choose the number of bins and plot the histogram.
29: plt.hist(List, density=True, bins=40, color='purple', rwidth=1)
30: plt.show()
```

One can experiment other choices of random entries. For example, take complex random variables with normal distribution.

Source code 2 – Empirical Semicircle distribution for hermitian matrices.

```
1: % Just change the random variables in the construction of the
     matrix by complex normal ones
2: if x == y:
3:         Mat[x][y] =  random.normalvariate(0,1)/((k)**(1/2))
4:     if x<y:
5:         Mat[x][y] = complex(random.normalvariate(0,((1/2)**(1/2)))
     , random.normalvariate(0,((1/2)**(1/2))))/((k)**(1/2))
6:         Mat[y][x] = Mat[x][y].conjugate()
```

The package *random* offers even more options to explore. One can choose variables with uniform (*random.uniform(a,b)*), exponential (*random.expovariate(lambda)*) or gamma distribution (*random.gammavariate(a,b)*) and see what happens.

Although less exciting than the previous code, the semicircle distribution plot allows comparisons between the empirical result with the limit distribution.

Source code 3 – Semicircle law.

```
% In order to invoke the constant pi, one needs the package 'math'.
import math
import numpy as np
import matplotlib.pyplot as plt

%We determine the interval in the real axis for which we want to evaluate the semicircle distribution and plot the image.
x= np.linspace(-2,2,200)
y=((4-x**2)**(1/2))/(2*math.pi)

plt.plot(x, y, color='purple')
plt.show()
```

Latter, in Section 2.6.2, we constructed the so-called "Dyson bridges", i.e, non-intersecting Brownian bridges. We also suggested an algorithm that now we implement in Python to generate images such as Figure 9.

Source code 4 – A simple code for non-intersecting Brownian bridges.

```
% Once again, we need some packages to build random variables, construct matrices and generate images, respectively.
import random
import numpy
import matplotlib.pyplot as plt

%Set k to be the number of matrices and n the number of steps desired.
k, n = 1, 1

% We initiate a matrix setting every entry as zero. It is where we store the eigenvalues of each matrix.
List = [[0 for i in range(n)] for j in range(k)]

% Now, it is time to construct each matrix and evaluate its eigenvalues, as in the algorithm of Chapter 1.
for i in range(n):

    w, h = k, k
    Matrix = [[0 for x in range(w)] for y in range(h)]

    for x in range(w):
```

```
19:       for y in range(h):
20:         if x == y:
21:           Matrix[x][y] = (1/(k**(1/2)))*random.normalvariate
      (0,(2*(i/n)*(1-i/n))**(1/2))
22:         if x<y:
23:           Matrix[x][y] = (1/(k**(1/2)))*complex(random.
      normalvariate(0,(((i/n)*(1-i/n))**(1/2))), random.normalvariate
      (0,(((i/n)*(1-i/n))**(1/2))))
24:           Matrix[y][x] = Matrix[x][y].conjugate()
25:     val, vet = numpy.linalg.eigh(Matrix)
26:
27: %Once we have the eigenvalues, we ordenate them.
28:     Alist = sorted(val)
29:
30: %Now, save the eigenvalues of the n-th matrix in the n-th row of
      the first matrix we built.
31:     for j in range(k):
32:       List[j][i] = Alist[j]
33:
34: % To construct the bridges, we set the endpoints as zero.
35: NewList = [[0 for i in range(n+1)] for j in range(k)]
36: for j in range(k):
37:   for i in range(n+1):
38:     if i in range(n):
39:       NewList[j][i] = List[j][i]
40:     else:
41:       NewList[j][i] = 0
42:
43: %Well-done. Just plot your brownian bridges and have fun
      increasing the number of steps and/or bridges and observing
      what happens.
44: for a in range(k):
45:   plt.plot(NewList[a])
46: plt.show()
```

Some interesting behaviour appears in the Dyson bridges plot as the number of paths increase. Let a small number of steps, say $n = 10$, and let k be 50, then 500 and, at last, 1000. How do the paths behave? What happens to the largest eigenvalue? And the smallest?

Increasing the number of steps also leads to interesting results, but be careful. The number of steps is the dimension of a matrix for which you must evaluate the eigenvalues.

And since it is a non-linear process, the computational cost increases drastically as n grows.

And what about to watch the "particles" which describe non-intersecting Brownian bridge trajectories running in real time? In order to do so, all we need is a Python package able to animate figures. Bellow we exemplify how to animate each bridge.

Source code 5 – Animated non-intersecting Brownian bridges.

```
% We need one auxiliary package to animate the plot.
from matplotlib.animation import FuncAnimation

%Here the construction of the matrices is exactly the same as in
    the previous code. However, after the construction of the
    object 'NewList', instead of 'plt.plot(NewList[a])' we animate
    each path as follows.
fig, ax = plt.subplots()

%First we initiate the bridge and the animation.
xdata, bridge1 = [], []
path1 = plt.plot([], [], animated=True)

%Then, we initiate the animation.
def init():
    ax.set_xlim(0,n)
    ax.set_ylim(-1.7,1.7)
    path1.set_data(xdata, bridge1)

    return path1,

%Now, we update the frame for each time step.
def update(frame):
    xdata.append(frame)
    bridge1.append(NewList[0][frame])
    path1.set_data(xdata, bridge1)

    return path1,

animation = FuncAnimation(fig, update, frames=f, init_func=init,
    blit=True, interval=100, repeat=False)

animation.save('1bridge.gif')
plt.show()
```

At last, we present the distribution of the largest eigenvalue in the unitary ensemble. In order to compare the empirical distribution and the Tracy-Widom distribution, we suggest installing the Tracy-Widom package (<https://github.com/yymao/TracyWidom/>).

Source code 6 – Empirical Tracy-Widom distribution.

```
% Notice that now we call the function TracyWidom.
import numpy
import random
import matplotlib.pyplot as plt
from TracyWidom import TracyWidom

%Let n be the number of matrices and k their dimension. Also,
    create a list to store the eigenvalues.
k, n = 500, 2000
List = []

%Now we create each matrix, evaluate its eigenvalues, ordenate
    them and pick only the largest one, saving it in the list
    created before.
for i in range(n):

    w, h = k, k
    Mat = [[0 for x in range(w)] for y in range(h)]

    for x in range(w):
            for y in range(h):
                    if x == y:
                        Mat[x][y] = random.normalvariate(0,1)/((k)
    **(1/2))
                    if x<y:
                        Mat[x][y] = complex(random.normalvariate
    (0,((1/2)**(1/2))), random.normalvariate(0,((1/2)**(1/2))))/((k
    )**(1/2))
                        Mat[y][x] = Mat[x][y].conjugate()
    val, vet = numpy.linalg.eigh(Mat)
    A_list = sorted(val)
    List.append((k**(2/3))*(val[k-1]-2))

% Once the list of largest eigenvalues is done, just plot the
```

```
      histogram.
29: plt.hist(List, density=True, bins=40, color='violet', rwidth=1)
30:
31: %To compare with the Tracy-Widom function, call its distribution
      and choose beta=2.
32: s = numpy.linspace(-5,3)
33: tw1 = TracyWidom(beta=2)  # allowed beta values are 1, 2, and 4
34: pdf = tw1.pdf(s)
35:
36: plt.plot(s,pdf, color ='black')
37: plt.show()
```

The code results in images such as Figure 31.


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