UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Some applications of Topological Games

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Dissertação de Mestrado do Programa de Pós-Graduação em Matemática (PPG-Mat)

ICMC USP
SÃO CARLOS

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: ______________________

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Some applications of Topological Games

Dissertation submitted to the Instituto de Ciências Matemáticas e de Computação – ICMC-USP – in accordance with the requirements of the Mathematics Graduate Program, for the degree of Master in Science. *EXAMINATION BOARD PRESENTATION COPY*

Concentration Area: Mathematics

Advisor: Prof. Dr. Leandro Fiorini Aurichi

USP – São Carlos
May 2023

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

R741a Rojas Gutierrez, Walter Angelo
Algumas aplicações dos Jogos Topologicos / Walter Angelo Rojas Gutierrez; orientador Leandro Fiorini Aurichi. -- São Carlos, 2023.
66 p.

Tese (Doutorado - Programa de Pós-Graduação em Matemática) -- Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2023.

1. TOPOLOGIA. 2. JOGOS TOPOLOGICOS. I. Fiorini Aurichi, Leandro, orient. II. Título.

Bibliotecários responsáveis pela estrutura de catalogação da publicação de acordo com a AACR2:
Gláucia Maria Saia Cristianini - CRB - 8/4938
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Walter Angelo Rojas Gutierrez

Algumas aplicações dos Jogos Topológicos

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Leandro Fiorini Aurichi

USP – São Carlos
Maio de 2023

This work is dedicated to my relatives, especially those who are no longer here.

ACKNOWLEDGEMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

I thank my family for supporting me during these difficult years.

I would also like to thank the new friends I met during my stay in Brazil, for brightening my days and allowing me to enjoy the warmth of a family far from my home.

I would also like to thank my advisor Professor Leandro Fiorini Aurichi, for all the teachings and advice he gave me during the writing of this work.

I would also like to thank my teachers and colleagues at FCM-UNMSM, especially those friendships that endure despite time and distance.

“Pure mathematics is, in its form,
the poetry of logical ideas.”
(Albert Einstein)

RESUMO

ROJAS GUTIERREZ, W. A. **Algumas aplicações dos Jogos Topológicos**. 2023. 66 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

Neste artigo, analisaremos alguns jogos topológicos, desde jogos conhecidos como o jogo de Banach ou o jogo de Choquet, até à definição de um novo, O jogo de Čech.
A maioria dos jogos aqui apresentados foram escolhidos pela sua relação total ou parcial com o jogo de Čech. Uma vez que este último será especialmente importante para facilitar algumas provas clássicas, bem como para apresentar novas propriedades de alguns espaços topológicos.

Palavras-chave: Espaço Čech-completo, Jogo do Čech, Espaços paracompactos.

ABSTRACT

ROJAS GUTIERREZ, W. A. **Some applications of Topological Games**. 2023. 66 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

In this paper, we will look at some topological games, from well-known ones like Banach's game or Choquet's game, to defining a new one, Čech's game.
Most of the games presented here were chosen for their complete or partial relationship with Čech's game. Since the latter will be especially important to facilitate some classical proofs as well as to present new properties of some topological spaces.

Keywords: Čech complete space, Čech's game, Paracompact space.

CONTENTS

CHAPTER

1

INTRODUCTION

Topological games have been studied for almost 100 years, with the Banach-Mazur game being the first to be described. In 1935 Stefan Banach started a notebook, called the Scottish Book (MALDUIN, 1981), where the mathematicians residing in or visiting Lwów proposed various mathematical problems (or conjectures) and also indicated their partial or complete solutions. The original version of the Banach game said the following:

" Given a subset X of the unit interval J, he players alternately choose subintervals $J_0, J_1, J_2 \ldots \subset J$; where $J_0 \supset J_1 \supset \ldots$ Player I wins the play if, and only if, $X \bigcap_{n<\omega} J_n \neq \emptyset$."

Čech (CECH, 1937) showed that the Baire Density Theorem is valid for the absolute G_δ spaces, later called Čech complete spaces. Choquet (CHOQUET, 1951) pointed out several difficulties in generalizing completeness so that the Baire Density Theorem would still be valid. In (CHOQUET, 1958) he introduced the siftable and strongly siftable spaces. The strongly siftable spaces are related to game he introduced in (CHOQUET, 1969) called Choquet Game ($Ch(X)$).

In (PORADA, 1979), Porada introduced the following modification of the game $Ch(X)$. Given a subset Y of a topological space X, the play is again $\langle (x_0, U_0), V_0, (x_1, U_1), V_1 \ldots \rangle$, but U_n and V_n are open in X and $x_n \in Y$. Player II wins the game if and only if $\emptyset \neq \bigcap V_n \subset Y$. Denote this game by $P(X,Y)$.

In (TOPSOE, 1982), Topsoe strengthened the condition $\bigcap_{n<\omega} V_n \neq \emptyset$ the game $Ch(X)$ as follows: if $\mathscr{F}$ is a filter base of subsets of X such that for each $n < \omega$ there is an $F \in \mathscr{F}$ with $F \subset V_n$ then $\mathscr{F}$ clusters. We denote this game by $SV(X)$. In Chapter 2, we will present the most relevant definitions that will later be used to define the topological games.

In Chapter 3, we will look at the games described above as well as the ways in which they relate to each other. We will present them divided into 2 groups; in the first place, we will see the Banach, Choquet and Sequential games.And in the second group, we will see the Sieve and Porada games, together with an original game that we will call the Čech game n honor of the space whose definition inspired its creation.

Finally, in Chapter 4, we will prove some properties of the studied games, such as the characterization of those spaces where Player I or Player II have a winning strategy.

CHAPTER

2

PRELIMINARIES

In this chapter we will discuss the most important definitions that we will need throughout the work.

2.1 Basic notions

We will start with the basic concepts that will be relevant.

Definition 2.1.1. A topological space (X,τ) is *Hausdorff* if for every $x,y \in X$ there are $U,V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 2.1.2. Let (X,τ) be a topological space. Let $A \subset X$. Then:

- A is F_σ if A is a countable union of closed sets.
- A is G_δ if A is a countable intersection of open sets.

Let Y be a Hausdorff compact space. Then it is clear that every F_σ subset is a σ-compact subset, and vice versa.

Definition 2.1.3. A topological space X is called a Tychonoff space if it is a Hausdorff space and its points can be separated from closed sets via (bounded) continuous real-valued functions.

Definition 2.1.4. A pair (Y,c) where Y is a compact Hausdorff space and $c : X \to Y$ is a homeomorphic embedding of X in Y such that $\overline{c(X)} = Y$, is called a *compactification* of X.

Although the definition given in 2.1.3 is the best known, in this work we will mainly consider another equivalent definition described in the following theorem.

Theorem 2.1.5. A topological space X is Tychonoff if, and only if, X has a compactification.

Proof.
The complete proof of this theorem can be found in (ENGELKING, 1989), Theorem 3.5.1 □

In (ENGELKING, 1989) There are interesting properties of compactifications, but we will keep the following two:
• Let (Y, c_1) and (Z, c_2) are two compactifications of X. We say that $Y \leq Z$ if and only if there is a function $f : Z \to Y$ such that $f \circ c_1 = c_2$. Let $\mathscr{C}(X)$ be the set of all compactifications of X. Then $(\mathscr{C}(X), \leq)$ is an ordered set.
• There is a largest element in $(\mathscr{C}(X), \leq)$ which we will call βX, the Stone-Čech compactifcation.

From now on, all spaces are considered to be Tychonoff, unless otherwise specified.

Definition 2.1.6. Let $(X, \leq)$ be a partially ordered set. A **filter** $\mathscr{F}$ in X is a subset of $P(X)$ such that:
1) $\mathscr{F}$ is non-empty.
2) For every $x, y \in \mathscr{F}$, there is a $z \in X$ such that $z \leq x$ and $z \leq y$.
3) If $x \in \mathscr{F}$ and $y \in X$ such that $x \leq y$, then $y \in \mathscr{F}$

A **ultrafilter** $\mathscr{U}$ is a maximal filter, that means there is no filter that contains it properly.

2.2 Complete sequence of open covers

The following definition will be fundamental when stating the Čech space and the topological game of Čech.

Definition 2.2.1. A sequence $(\mathscr{C}_n)_{n\in\omega}$ of open covers is complete if for every ultrafilter u (in $\mathscr{P}(X)$) such that $u \cap \mathscr{C}_n \neq \emptyset$ for all n, it is true that $\bigcap_{F\in u} \overline{F} \neq \emptyset$.

Next we will see a way to distinguish compact sets by open complete sequences.

Proposition 2.2.2. Let $(\mathscr{C}_n)_{n\in\omega}$ be a complete sequence of open covers of X. Let K be a closed subset. If for all $n \in \omega$, $K \subset C_n$ for some $C_n \in \mathscr{C}_n$, then K is compact.

Proof.
Let $\{C_n\}_{n\in\omega}$ be a sequence of open sets such that $K \subset C_n$.
Let $\{K_i\}_{i\in I}$ be a family of closed sets in K with a finite intersection property. Since K is closed in X, then, for all $i \in I$, K_i is closed in X as well.
Let u be an ultrafilter containing $\{K_i\}_{i\in I}$. Since $u \cap C_n$ for all $n \in \omega$, then $\emptyset \neq \bigcap_{V\in u} \overline{V} \subset \bigcap_{i\in I} K_i$.
Therefore, K is a compact subset of X. □

Now we will see some ways to obtain new complete sequences of open covers from other complete sequences.

Proposition 2.2.3. Let $(\mathscr{C}_n)_{n\in\omega}$ be a complete sequence of open covers of X. Then:
• Let $\mathscr{R}_n$ be a refinement of $\mathscr{C}_n$, then $(\mathscr{R}_n)_{n\in\omega}$ is a complete sequence of open covers.
• Let $\mathscr{F}_n$ be an open cover formed by finite unions of elements of $\mathscr{C}_n$, then $(\mathscr{F}_n)_{n\in\omega}$ is a complete sequence of open covers.

Proof.
The first result is obvious since for any ultrafilter u, if $u\cap\mathscr{R}_n\neq\emptyset$ then $u\cap\mathscr{C}_n\neq\emptyset$.
For the second result remember that if $A\cup B\in u$ then $A\in u$ or $B\in u$. Therefore if $u\cap\mathscr{F}_n\neq\emptyset$ then $u\cap\mathscr{C}_n\neq\emptyset$. □

With these two properties, we can state a generalization of Proposition 2.2.2.

Corollary 2.2.4. (FROLíK, 1961) Let $(\mathscr{C}_n)_{n\in\omega}$ be a complete sequence of open covers of X. Let K be a closed subset. If for all $n\in\omega$, K is covered by a finite subfamily of $\mathscr{C}_n$, then K is compact.

Let us also see that the characteristic of being a complete sequence of open covers is inherited by operations between complete sequences of open covers.

Definition 2.2.5. Let A,B be open cover of (X,τ). We define

$$A\wedge B=\{V\cap W;\ V\in A,\ W\in B\}$$

and

$$A^E=\{V\in\tau;\ \exists W\in A,\ V\subset W\}$$

Proposition 2.2.6. Let $(\mathscr{C}_n)_{n\in\omega}$ be a complete sequence of open covers of X. Then $(\mathscr{C}_n^E)_{n\in\omega}$ and $(\bigwedge_{i=0}^n\mathscr{C}_i)_{n\in\omega}$ are complete sequences of open covers.

Proof.
By Proposition 2.2.3. □

2.3 Čech-complete space

In this section we will see the definition and some properties of Čech-complete spaces, which will serve us later for the definition of a topological game.

Definition 2.3.1. A Tychonoff space is **Čech-complete** if admits a complete sequence of open covers.

Next we will see equivalent formulation of the definition of Čech complete spaces. for the first equivalence we will need to introduce the following new concept.

Definition 2.3.2. Let X be a topological space. Let $\mathscr{A}$ be an open cover. We say that F, a family of subsets of X, contains **sets of diameter less that** $\mathscr{A}$, if there are $V \in F$ and $A \in \mathscr{A}$ such that $V \subset A$.

The following theorem can be found in (ENGELKING, 1989),

Theorem 2.3.3. A Tychonoff space is Čech-complete if and only if there is a countable family $\{\mathscr{A}_i\}_{i\in\omega}$ of open covers of the space X which has the property that for any family $\mathscr{F}$ of closed subsets of X that has the finite intersection property and contains sets of diameter less that $\mathscr{A}_i$ for any $i \in \omega$, has non-empty intersection.

Proof.
($\Rightarrow$) Suppose that X admits a complete sequence $\{\mathscr{C}_n\}_{n\in\omega}$. We will prove that $\{\mathscr{C}_n\}_{n\in\omega}$ verifies the property. Let $\mathscr{F}$ be a family of closed sets that has the finite intersection property and contains sets of diameter less that $\mathscr{C}_i$ for any $i \in \omega$. Let u be an ultrafilter containing $\mathscr{F}$.
Knowing that for each cover $\mathscr{C}_i$ there is an $F_i \in \mathscr{F}$ such that $F_i \subset C$ for some element $C \in \mathscr{C}_i$. Then $C \in u$, therefore $u \cap \mathscr{C}_i \neq \emptyset$.

$$\emptyset \neq \bigcap_{F\in u} \overline{F} \subset \bigcap_{F\in\mathscr{F}} \overline{F} = \bigcap_{F\in\mathscr{F}} F$$

($\Leftarrow$) Let $\{\mathscr{A}_i\}_{i=1}^{\infty}$ be a family of open covers with the properties described. Let u be an ultrafilter such that $u \cap \mathscr{A}_i \neq \emptyset$ for any $\mathscr{A}_i$.
Let $\overline{u}$ be an ultrafilter containing $\{\overline{F}; F \in u\}$, over the family of closed subsets of X. Then $\overline{u}$ is a family of closed subsets with the finite intersection property. And since for every $\mathscr{A}_i$ we have that $u \cap \mathscr{A}_i \neq \emptyset$, there is a $A \in \mathscr{A}_i$ with diameter less that $\overline{u}$ ($A \subset \overline{A} \in \overline{u}$).

$$\emptyset \neq \bigcap_{F\in\overline{u}} F \subset \bigcap_{F\in u} \overline{F}$$

□

The following is one of the most important, and useful, equivalences to Čech complete spaces that we will deal with in this work.

Theorem 2.3.4. A space X is Čech-complete if and only if $\beta X/X$ is F_δ in βX.

Proof.

$\Rightarrow$) Let $\{A_i\}_{i\in\omega}$ be a complete sequence of open covers. Let $A_i = \{U_{i,s}\}_{s\in S_i}$. Let $V_{i,s}$ be an open subset of βX such that $U_{i,s} = X \cap V_{i,s}$. Clearly

$$X \subset \bigcap_{i\in\omega} \bigcup_{s\in S_i} V_{i,s}$$

Let $x \in \bigcap_{i\in\omega} \bigcup_{s\in S_i} V_{i,s}$. Let $\mathscr{V}$ be the family of open neighborhoods of x in βX. Then $\mathscr{B} = \{X \cap V; V \in \mathscr{V}\}$ is a filter base in $\mathscr{P}(X)$. Let u be an ultrafilter containing $\mathscr{B}$. Then for each A_i there is a $V_{i,s}$ such that $x \in V_{i,s}$. Thus $U_{i,s} = X \cap V_{i,s} \in \mathscr{B} \subset u$. Therefore $U_{i,s} \in A_i \cap u$.

Since $u \cap A_i \neq \emptyset$ for all $i \in \omega$, then

$$\bigcap_{F\in u} Cl_X(F) \neq \emptyset$$

Let $y \in Cl_X(F)$ for all $F \in u$, if $y \neq x$, there is an open neighborhood V of x in βX, such that $y \notin Cl_{\beta X}(V)$. Then $y \notin Cl_X(V \cap X)$, this is a contradiction because $V \cap X \in u$.
Finally,

$$X = \bigcap_{i\in\omega} \bigcup_{s\in S_i} V_{i,s}$$

Therefore X is G_δ in βX or, equivalently, $\beta X / X$ is F_σ.

$\Leftarrow$) Since X is G_δ, there is $\{G_i\}_{i\in\omega}$, a family of open subsets of βX, such that $X = \bigcap G_i$. Since X is Tychonoff, for each $X \in X$ and each G_i there is an open subset $V_{x,i}$ such that $x \in V_{x,i} \subset Cl_{\beta X}(V_{x,i}) \subset G_i$.
We define $A_i = \{X \cap V_{x,i}; x \in X\}$ an open cover of X. Let $\mathscr{A} = \{A_i\}_{i\in\omega}$. We will prove that $\mathscr{A}$ is a complete sequence of open covers of X.

Let u be an ultrafilter in $\mathscr{P}(X)$ such that $u \cap A_i \neq \emptyset$. Let v be an ultrafilter in $\mathscr{P}(\beta X)$ containing u.
Since βX is Hausdorff compact and for each $i \in \omega$ there is $V_{x_i,i} \in v$, then:

$$\{z\} = \bigcap_{F\in v} Cl_{\beta X}(F) \subset \bigcap Cl_{\beta X}(V_{x_i,i}) \subset \bigcap G_i = X.$$

Therefore,

$$z \in \bigcap_{F\in u} Cl_{\beta X}(Cl_X(F))$$

Let V be an open neighborhood of z in X. Then there is W open neighborhood of z in βX such that $W \cap X = V$.
Thus, for any $F \in u$ we have that $W \cap Cl_X(F) \neq \emptyset$. Then $V \cap Cl_X(F) \neq \emptyset$. Therefore $z \in Cl_X(F)$ for any $F \in u$.

In conclusion, $z \in \bigcap_{F\in u} Cl_X(F)$. $\square$

The Proposition 2.3.5 will be important in simplifying proofs of future propositions.

Proposition 2.3.5. Let (X,τ) be a Tychonoff space. Let $\{V_n\}_{n\in\omega} \subset \tau$ such that $\overline{V_{n+1}} \subset V_n$ for each $n \in \omega$. If any ultrafilter u containing $\{V_n\}_{n\in\omega}$ verifies that $\bigcap_{F\in u} \overline{F} \neq \emptyset$. Then $\bigcap_{n\in\omega} V_n$ is compact and $\{V_n\}_{n\in\omega}$ is an local basis for $\bigcap V_n$.

Proof.
Let $\{W_n\}_{n\in\omega}$ be open sets in βX such that $W_n \cap X = V_n$ for each $n \in \omega$. Then $cl_{\beta X}(W_n) \cap X = cl_X(V_n)$.
Let $y \in \bigcap cl_{\beta X}(W_n)$. Then for each U open in βX such that $y \in U$ we have that $U \cap V_n \neq \emptyset$ for all $n \in \omega$. Let u be an ultrafilter containing all open sets in X of the form $U \cap V_n$ for some n and $y \in U$. Then $\bigcap_{F\in u} cl_X(F) \neq \emptyset$.. Since $\bigcap_{F\in u} cl_X(F) \subset \bigcap_{F\in u} cl_{\beta X}(F) \subset \bigcap_{U\ni y} cl_{\beta X}(U) = \{y\}$ we have that $y \in X$. Therefore $\bigcap cl_{\beta X}(W_n) \subset X$.
Then $\bigcap V_n = \bigcap cl_X(V_n) = \bigcap cl_{\beta X}(W_n) \cap X = cl_{\beta X}(W_n) = K$ is compact.
Let U be an open set containing K. Suppose that $V_n \cap U^c \neq \emptyset$ for all $n \in \omega$. Let u be an ultrafilter containing $\{V_n \cap U^c\}$. Then $\bigcap_{F\in u} cl_X(F) \subset \bigcap cl_x(V_n) \cap U^c = K \cap U^c = \emptyset$, contradiction. Therefore there is an $n \in \omega$ such that $V_n \subset U$.

□

The concept of a perfect function, together with the properties that we will see next, will be crucial in the proof of the Theorem 4.2.9.

Definition 2.3.6. Let $f : X \longrightarrow Y$ be a continuous function, f is perfect if:
a)f is closed.
b) $f(X) = Y$.
c)$f^{-1}(y)$ is a compact set for each $y \in Y$.

Proposition 2.3.7. Let X,Y Tychonoff spaces. Let $f : X \longrightarrow Y$ be a perfect function, then:
a) $\overline{f}(\beta X \setminus X) = \beta Y \setminus Y$, where $\overline{f}$ is the continuous extension of f.
b) If Y is G_δ in βY, then X is G_δ in βX.
c) If Y is paracompact, then X is paracompact.

Proof.
a) Since $\overline{f}(\beta X)$ is a compact and dense in βY, then $\beta Y \setminus Y \subset \overline{f}(\beta X \setminus X)$. Now suppose there is $x \in \beta X \setminus X$ such that $\overline{f}(x) = y \in Y$. Since $f^{-1}(y)$ is a compact set and $x \notin f^{-1}(y)$, there is an open set $V \subset X$ such that $f^{-1}(y) \subset V$ $x \notin cl_{\beta X}(V)$.
Then $x \in cl_{\beta X}(X \setminus V)$. Therefore $y \in cl_{\beta Y}\overline{f}(X \setminus V) = cl_{\beta Y} f(X \setminus V)$. Since $f(X \setminus V) \subset Y$ we have that $y \in cl_Y f(X \setminus V) = f(X \setminus V)$. Contradiction with $f^{-1}(y) \subset V$. Therefore $\overline{f}(\beta X \setminus X) = \beta Y \setminus Y$.

b) Note that $\overline{f}(\beta X \setminus X) = \beta Y \setminus Y$ implies $X = \overline{f}^{-1}(Y)$. Let $\{V_n\}_{n\in\omega}$ open sets in βY such that $\bigcap V_n = Y$. Then $X = \overline{f}^{-1}(Y) = \overline{f}^{-1}(\bigcap V_n) = \bigcap \overline{f}^{-1}(V_n)$. Therefor X is G_δ in βX.

c) Let $\overline{f} : \beta X \longrightarrow \beta Y$. Define $F : X \longrightarrow X \times Y$ as $F(x) = (x, \overline{f}(x)) = (x, f(x))$. Then F is injective, Let W be an open subset of $X \times Y$, such that $F^{-1}(W) \neq \emptyset$. For each $(x, f(x)) \in W$ there is $U \subset X$ and $V \subset Y$ open sets such that $(x, f(x)) \in U \times V \subset W$. Therefore $F^{-1}(U \times V) = U \cap f^{-1}(V)$ is an open subset of X. Then F is continuous.
Let U be an open subset of X. Then $F(U) = f(U) \times Y \cap Im(F)$. Then $F : X \longrightarrow Im(F)$ is open bijective continuous map. Therefore X is homeomorphic to $Im(F)$.
Let $(x, y) \in \beta X \times Y$, such that $(x, y) \notin Im(F)$. Since $\overline{f}(x) \neq y$ there is U, V open subset of βY such that $U \cap V = \emptyset$, $\overline{f}(x) \in U$ and $y \in V$. Therefore, $\overline{f}^{-1}(U) \times (V \cap Y)$ is an open subset of $\beta X \times Y$ such that $\overline{f}^{-1}(U) \times (V \cap Y) \cap Im(F) = \emptyset$. Hence, X is homeomorphic to a closed subset of $\beta X \times Y$. In addition, since Y is paracompact, $\beta X \times Y$ is paracompact and then X is too.

□

2.4 Topological Games

The last pillar that we need to see in order to continue is *Topological Games*. In this text, we will not need an exhaustive definition of topological game. So we will settle for understanding the basic notion of how a topological game works. For this purpose, the notion given in "Topological games: On the 50th anniversary of the Banach-Mazur game" will suffice.
In a topological game the players choose some objects related to the topological structure of a space, such as points, closed subsets, open covers, etc., and moreover, the condition on a play to be winning for a player may also involve topological notions such as closure, a convergence, etc. (TELGARSKY, 1987)

Let us now consider the space $\mathbb{R}$ and two players called Player I and Player II. In each round Player I choose a compact set in $\mathbb{R}$, and Player II responds with an open containing the compact set chosen by Player I. In this game, we will say that Player I wins the game if he can "force" Player II to choose open sets that form a cover of $\mathbb{R}$.

If in the first turn Player I choose the compact $[-1, 1]$ and Player II choose an open set U_1, such that $[-1, 1] \subset U_1$. In the next turn Player I choose the compact $[-2, 2]$ and Player II choose an open set U_2. If we continue in this way, we can see that the open sets chosen by Player II necessarily make up an open cover of $\mathbb{R}$.

Notice that in this example, no matter what Player II decides to choose. In the end Player II will always be "forced" to have a cover. This situation is described as **Player I has a winning strategy in the game** and it is represented by $I \uparrow game(\mathbb{R})$.

The strategy described above is called a **Markov's strategy** because only depends on the last play of the rival player and the current turn.

A more dominant strategy is known as **stationary strategy** and is characterized by the fact that it only depends on the last play of the rival.

Definition 2.4.1. Let $G_1(X)$ and $G_2(X)$ be two topological games we will say that:
•$G_1(X)$ and $G_2(X)$ are *equivalent* if $I \uparrow G_1(X)$(resp. $II \uparrow G_1(X)$) implies $I \uparrow G_2(X)$(resp. $II \uparrow G_2(X)$).
•$G_1(X)$ and $G_2(X)$ are *dual* if $I \uparrow G_1(X)$(resp. $II \uparrow G_1(X)$) implies $II \uparrow G_2(X)$(resp. $I \uparrow G_2(X)$).

CHAPTER 3

GAMES AND RELATIONS BETWEEN GAMES

3.1 Sequential, Banach-Mazur and Choquet Games: propierties and relations

In this section we will present the main topological games with which we will work, and we will see the relationships they have among them.

3.1.1 Sequential Games

We will begin by describing a simple game, although important because it will allow us to characterize completely metrizable spaces.
Let (X,d) be a metric space. We define the Sequential Game $G(X,d)$ as follows:

•$T0$: Player I plays $x_0 \in X$, Player II responds with $\varepsilon_0 > 0$.

•$T1$: Player I plays x_1 such that $d(x_0,x_1) < \varepsilon_0$, Player II responds with ε_1.

•Tn : Player I plays x_n such that $d(x_{n-1},x_n) < \varepsilon_{n-1}$, Player II responds with ε_n.

Player II is declared winner if exists $x \in X$ such that $x_n \to x$. In other case, Player I is declared winner.

3.1.2 Banach-Mazur Game

The Banach-Mazur Game is considered the first topological game. Although initially it was described to be played on the real line, here we will describe a modern version that allows us to play it on any topological space.

Let (X,τ) be a topological space. We define the Banach-Mazur Game $BM(X)$ as follows:

•$T0$: Player I plays $U_0 \in \tau$ and Player II responds with $V_0 \in \tau$, such that $V_0 \subset U_0$.

•Tn : Player I plays $U_n \in \tau$, with $U_n \subset V_{n-1}$ and Player II responds with $V_n \in \tau$, such that $V_n \subset U_n$.

Player II is declared winner if $\bigcap V_n \neq \emptyset$, in other case Player I is declared winner.

Let σ be a Player I's strategy. A game $p = \langle U_0, V_0, U_1, V_1 \dots U_n, V_n \dots \rangle$, where each move of Player I is given by σ, is called a $\sigma-$game. Analogously with δ, a Player II's strategy.

The Banach-Mazur game has various applications that we will see later.

3.1.3 Choquet Game

We will now see the Choquet game, described for the first time in (CHOQUET, 1969). Let (X,τ) be a topological space. We define Choquet Game $Ch(X)$ as follows:

•$T0$: Player I plays $(U_0,x_0) \in \tau \times X$ with $x_0 \in U_0$ and Player II responds with $V_0 \in \tau$, such that $x \in V_0 \subset U_0$.

•Tn : Player I plays $(U_n,x_n) \in \tau \times X$, with $x_n \in U_n \subset V_{n-1}$ and Player II responds with $V_n \in \tau$, such that $x_n \in V_n \subset U_n$.

Player II is declared winner if $\bigcap V_n \neq \emptyset$, in other case Player I is declared winner.

Although the Choquet game is very similar to the Banach-Mazur game, the fact that Player I can limit Player II's next move is enough to differentiate the two games.

Clearly, just from the definitions, we can say that
•$I \uparrow BM(X)$ then $I \uparrow Ch(X)$.
•$II \uparrow Ch(X)$ then $II \uparrow BM(X)$.
To demonstrate the difference between the Banach game and the Choquet game we will see an example of a space where $I \uparrow Ch(X)$ but $I \not\uparrow BM(X)$. To this end we will assume a property (proved in 4.1.1) that tells us that in every Baire space it is not possible to find a Player I's winning strategy for $BM(X)$.

Example 3.1.1. Consider the Baire space

$$X = \mathbb{R}^2 \setminus \{(x,0)\ x \notin \mathbb{Q}\}$$

we will see that $I \uparrow Ch(x)$:
Let $\mathbb{Q} = \{q_n, n \in \omega\}$. We define $L_n = \{(x,y), |y| < \frac{1}{n}\} \setminus \{(q_k,0), k < n\}$.
It is clear that $\bigcap L_n = \emptyset$.

Observation: Let A be an open set, if $\mathbb{Q} \times \{0\} \cap A \neq \emptyset$ then $\mathbb{Q} \times \{0\} \cap A$ is infinite.

To begin to define a strategy σ for Player I we will set $\sigma(\langle\rangle) = (L_0, (q_0, 0))$. Let V_0 be a response of Player II. Since $q_0 \in \mathbb{Q} \times \{0\} \cap V_0$ there is $p \neq q_0$ such that $(p,0) \in V_0$. We set $\sigma(\langle V_0 \rangle) = (L_1 \cap V_0, (p,0))$.
In general, let V_n be the last play of Player II. There is $(p_n, 0) \in V_n$ such that $p_n \notin \{q_k, k < n+1\}$.
Then we set $\sigma(\langle V_0, V_1 \dots V_n \rangle) = (L_{n+1} \cap V_n, (p_n, 0))$.
Then we have that $V_{n+1} \subset L_{n+1}$ and therefore $\bigcap V_n \subset \bigcap L_n = \emptyset$.

3.1.4 Properties and relationships among games

Here we will present some properties of the games seen in this section, and we will establish the relationships that exist among them.
To begin we will see the first part of the characterization of completely metrizable spaces, the second part will be seen later in Proposition 4.2.7.

Theorem 3.1.2. If there is an equivalent complete metric to d then Player II has a winning strategy in $G(X,d)$.

Proof. Let c be a complete metric on space X. Then Player II has a winning strategy in $G(X,c)$, playing $\frac{1}{2^n}$ on turn n.

If c is equivalent to d, for any $x \in X$ and $n \in \omega$ there is an $\varepsilon_{x,n} > 0$, such that $B_d(x, \varepsilon_{x,n}) \subset B_c(x, \frac{1}{2^n})$.

Let γ be a winning strategy for Player II in $G(X,c)$ mentioned above. We define σ to be a strategy in $G(X,d)$ by doing the following:

In each turn n, Player I plays x_n and Player II responds with $\sigma(\langle x_0, x_1, \dots x_n \rangle) = \varepsilon_{x_n,n}$. Since $x_{n+1} \in B_d(x_n, \varepsilon_{x_n,n}) \subset B_c(x_n, \frac{1}{2^n})$, then $\{x_n\}_{n\in\omega\}}$ is Cauchy in (X,c). Therefore $\{x_n\}_{n\in\omega\}}$is convergent in (X,c) and convergent in (X,d) because c and d are equivalent. □

Theorem 3.1.3. In a metric space (X,d), Sequential Game and Choquet Game are equivalent.

Proof.

$I \uparrow Ch(X) \Rightarrow I \uparrow G(X,d)$:

Let σ be a Player I's winning strategy in $Ch(X)$. We will define a strategy δ for Player I in $G(X,d)$ by doing the following:
$\delta(\langle\rangle) = x_0$ where $\sigma(\langle\rangle) = (A_0, x_0)$. Let r_0 be Player II's response in $G(X,d)$. We set a $p_0 < r_0$ such that $B_0 = B(x_0, p_0)$ and $\overline{B_0} \subset A_0$.
If $\sigma(\langle B_0 \rangle) = (A_1, x_1)$ then we define $\delta(\langle r_0 \rangle) = x_1$. Let r_1 be Player II's response in $G(X,d)$. We set a $p_1 < r_1$ such that $B_1 = B(x_1, p_1)$ and $\overline{B_1} \subset A_1$. Let $\delta(\langle B_0, B_1 \rangle) = (A_2, x_2)$. We define $\delta(\langle r_0, r_1 \rangle) = x_2$.

If we continue this process repeatedly we will obtain that $\delta(\langle r_0, \dots r_n \rangle) = x_{n+1}$ where $\sigma(\langle B_0, \dots B_n \rangle) = (A_{n+1}, x_{n+1})$ with $B_n = B(x_n, p_n)$ and $p_n < r_n$ such that $\overline{B_n} \subset A_n$.

Observation: We can choose $\{p_n\}_{n\in\omega}$ such that $p_n \longrightarrow 0$.

It is clear that $\overline{B}(x_{n+1}, p_{n+1}) \subset B(x_n, p_n)$. Suppose that $x_n \to x$. If there is an $m \in \omega$ such that $x \notin B_m$. Then $(\overline{B_{m+1}})^C$ is an open set containing x such that $x \notin (\overline{B_{m+1}})^C$ for each $n > m$. This contradicts the initial assumption $x_n \to x$. Therefore $x \in B_n$ for all $n \in \omega$. Then $x \in \bigcap B_n \neq \emptyset$, contradiction with the winning strategy σ. Therefore $\{x_n\}$ is not convergent, that is, δ is a winning strategy for Player I in $G(X,d)$.

$I \uparrow G(X,d) \Rightarrow I \uparrow Ch(X)$:

Let δ be a winning strategy in $G(X,d)$. We will define a strategy σ in $Ch(X)$:
$\sigma(\langle\rangle) = (B(x_0, 1), x_0)$, where $\delta(\langle\rangle) = x_0$, Let B_0 be a response of Player II in $Ch(X)$. Then there is a r_0 such that $\overline{B}(x_0, r_0) \subset B_0$.

Let $\sigma(\langle B_0\rangle) = (B(x_1, p_1), x_1)$ with $x_1 = \delta(\langle r_0\rangle)$ and $p_1 < r_0$, thus $\sigma(\langle B_0, \ldots B_n\rangle) = (B(x_{n+1}, p_{n+1}), x_{n+1})$ where $x_{n+1} = \delta(\langle r_0 \ldots r_n\rangle)$ and $p_{n+1} < r_n$.
It is clear that $\overline{B}(x_{n+1}, p_{n+1}) \subset B(x_n, p_n)$, and since δ is a winning strategy, we have $\bigcap B(x_n, p_n) = \emptyset$. Therefore σ is winning strategy, because $\{x_n\})n \in \omega$ can not be convergence.

$$II \uparrow Ch(X) \Rightarrow II \uparrow G(X,d):$$

Let σ be a winning strategy in $Ch(X)$. Let δ be a strategy in $G(X,d)$, such that:
Let x_0 be the first play of Player I in $G(X,d)$. We define $A_0 = (B(x_0, 1), x_0)$, if $\sigma(\langle A_0\rangle) = B_0$. Then $\delta(\langle x_0\rangle) = r_0$ such that $\overline{B}(x_0, r_0) \subset B_0$.

Let x_1 be a response of Player I, let $A_1 = (B(x_0, r_0), x_1)$ and $B_1 = \sigma(\langle A_0, A_1\rangle)$. We define $\delta(\langle x_0, x_1\rangle) = r_1$ such that $\overline{B}(x_1, r_1) \subset B_1$.

Similarly in turn $n+1$, if Player I choose x_{n+1} in $G(X,d)$, we define $A_{n+1} = (B(x_n, r_n), x_{n+1})$ and let $\sigma(\langle A_0 \ldots A_{n+1}\rangle) = B_{n+1}$. Therefore $\delta(\langle x_0. \ldots x_{n+1}\rangle) = r_{n+1}$ such that $\overline{B}(x_{n+1}, r_{n+1}) \subset B_{n+1}$.

Observation: We can choose $\{r_n\}_{n\in\omega}$ such that $r_n \longrightarrow 0$.

Since σ is a winning strategy for Player II in $Ch(X)$ we have that $\bigcap B(x_n, r_n) \neq \emptyset$. Then $\bigcap B(x_n, r_n) = \{x\}$, that is, $x_n \longrightarrow x$. Therefore δ is a winning strategy for Player II in $G(X,d)$.

$$II \uparrow G(X,d) \Rightarrow II \uparrow Ch(X):$$

Let δ be a winning strategy in $G(X,d)$. Let σ be a strategy in $Ch(X)$, such that:
Let $A_0 = (A_0, x_0)$, and $r_0 = \delta(\langle x_0\rangle)$. We define $\sigma(\langle A_0\rangle) = B(x_0, p_0) \subset A_0$ such that $\overline{B}(x_0, p_0) \subset A_0$ with $p_0 < r_0$.

Let $A_1 = (A_1, x_1)$ be a response of Player I in $Ch(X)$, and $r_1 = \delta(\langle x_0, x_1\rangle)$. We define $\sigma(\langle A_0, A_1\rangle) = B(x_1, p_1)$ such that $\overline{B}(x_1, p_1) \subset A_1$ with $p_1 < r_1$.

Similarly in turn n, if A_n is a play of Player I, and $r_n = \delta(\langle x_0, \ldots x_n\rangle)$, then we define $\sigma(\langle A_0 \ldots A_n\rangle) = B(x_n, p_n) \subset A_n$ with $p_n < r_n$.

Since δ is a winning strategy for Player II in $G(X,d)$, then $x_n \longrightarrow x$, and since $\{x_k\}_{k\in\omega} \subset \overline{B}(x_n, p_n)$ for all $n \in \omega$, then $x \in \bigcap \overline{B}(x_n, p_n)$ and in addition to that, as

$$B(x_0, p_0) \subset \overline{B}(x_0, p_0) \subset B(x_1, p_1) \subset \overline{B}(x_1, p_1) \subset B(x_2, p_2) \ldots$$

we have $\bigcap \overline{B}(x_n, p_n) = \bigcap B(x_n, p_n) \neq \emptyset$. Therefore σ is a winning strategy for Player II in

$Ch(X)$. □

To complement the example given in 2.4, we will see our first stationary strategy described in this text.

Proposition 3.1.4. Let (X,d) be a complete metric space. Player *II* has a stationary winning strategy in $Ch(X)$.

Proof. Let (V_n, x_n) be the Player *I*'s last move. Let $r_n = max\{r \in \mathbb{R},\ B(x_n, 2r) \subset V_n\}_{n\in\omega}$. We define a δ strategy by doing the following: $\delta(\langle(V_n, x_n)\rangle) = B(x_n, r_n)$.
Let $\langle(V_0,x_0), B_0, (V_1,x_1), B_1 \dots\rangle$ be a δ-game. Since $B_n = B(x_n, r_n) \subset B_{n-1}$, $r_n \to 0$ and X is complete, we have that $\bigcap B_n \neq \emptyset$. Therefore δ is a winning strategy.

□

Next we present an important property of the Choquet game that tells us that when Player *I* wins, he wins in the "best way".

Proposition 3.1.5. Let (X,d) be a metric space. If Player *I* has a winning strategy in $Ch(X)$ then Player *I* has a stationary winning strategy in $Ch(X)$.

Proof. Let σ be a winning strategy for Player *I*. Without loss of generality we can assume that if (U_n, x_n) is the last Player *I*'s play, then $x_n \notin \overline{U_{n+1}}$ where (U_{n+1}, x_{n+1}) is the next Player *I*'s play given by σ.

Let's start by considering the following:

Let $\sigma(\langle\ \rangle) = (U_0, x_0)$. Then there is a least $k_0^0 \in \omega$ such that $B(x_0, 1/k_0^0) \subset U_0$.
Let $(U_{0k_0^0}, x_{0k_0^0}) = \sigma(\langle B(x_0, 1/k_0)\rangle)$. Then there is a least $k_1^0 \in \omega$ such that $x_{0k_0^0} \notin \overline{B(x_0, 1/k_1^0)}$.
Let $(U_{0k_1^0}, x_{0k_1^0}) = \sigma(\langle B(x_0, 1/k_1^0)\rangle)$. Then there is a least $k_2^0 \in \omega$ such that $x_{0k_1^0} \notin \overline{B(x_0, 1/k_2^0)}$.

In this way we can define a sequence of natural numbers $\{k_n^0\}_{n\in\omega}$, and $\{x_{0n_1}\}$ where $n_1 \in \{k_n^0\}_{n\in\omega}$.

For any $k_p^0 \in \{k_n^0\}_{n\in\omega}$ we have $\sigma(\langle B(x_0, 1/k_p^0)\rangle) = (U_{0k_p^0}, x_{0k_p^0})$. Then there are a least $k_0^{0k_p^0} \in \omega$ such that $B(x_{0k_p^0}, 1/k_0^{0k_p^0}) \subset U_{0k_p^0} \setminus \overline{B(x_0, 1/k_{p+1}^0)}$.
Let $\sigma(\langle B(x_0, 1/k_p^0), B(x_{0k_p^0}, 1/k_0^{0k_p^0})\rangle) = (U_{0k_p^0 k_0^{0k_p^0}}, x_{0k_p^0 k_0^{0k_p^0}})$. Then there is a least $k_1^{0k_p^0} \in \omega$ such that $x_{0k_p^0 k_0^{0k_p^0}} \notin \overline{B(x_0 k_p^0, 1/k_1^{0k_p^0})}$.

In this way we can define a sequence of natural numbers $\{k_i^{0k_p^0}\}_{i\in\omega}$, and $\{x_{0k_p^0 n_2}\}$ where $n_2 \in \{k_i^{0k_p^0}\}_{i\in\omega}$.

In an analogous way we can continue constructing sequences $\{x_{0n_1n_2\dots n_p}\}$.

In general, $\sigma(\langle B(x_0,1/n_1)\dots B(x_{n_1\dots n_p},1/n_{p+1})\rangle) = (U_{n_1\dots n_{p+1}}, x_{n_1\dots n_{p+1}})$.

Let $T \subset \omega^{<\omega}$, such that for any $s \in T$, x_s is one of the points defined before. In what follows, unless otherwise stated, when we mention the least, we will refer to the set T.

Note that for any $s \in T$, we have that $x_{s^\frown k_n^s} \to x_s$ when $n \to \infty$.

Now we will define a stationary winning strategy γ by doing the following:

$\gamma(\langle\ \rangle) = (U_0, x_0)$.

Let V be a play of Player II. Let $n_1n_2\dots n_p \in T$ be the least such that $x_{n_1n_2\dots n_{p-1}} \in V$.

Case 0:

If $x_0 \in V$, We define $\gamma(\langle V\rangle) = (U_{0n}, x_{0n})$ where n is the least such that $B(x_n,1/n) \subset V$.

With this definition it is obvious that such V can only be played on the turn 0 of Player II.

Case 1:

If $x_{0a} \in V$ is the least.

Suppose that V was played in the turn 1. Then $\gamma(\langle V\rangle) = (U_{0ab}, x_{0ab})$ where $b \in \omega$ is the least such that $0ab \in T$ and $B(x_{0a},1/b) \subset V$.

Suppose that there is a game where V was played in the turn 2. Let W be the Player II's turn 1 move in that game.

Note that W only contain a finite number of x_{0n}, and $B(x_{0a},1/b)$ just contain x_{0a}. Then, by definition of σ, $x_{0a} \notin U_{0ab}$. Since $V \subset U_{0ab}$, V cannot contain points x_{0n}, contradiction.

With this definition, we can notice that any possible play V such that some x_{0n} is their least, only can be played in the turn 1.

Case 2:

If $x_{0ab} \in V$ is the least.

Suppose that V was played in the turn 2.
Then $\gamma(\langle V\rangle) = (U_{0abc}, x_{0abc})$ where $c \in \omega$ is the least such that $0abc \in T$ and $B(x_{0ab}, 1/c) \subset V$.
Analogously to the above, we have that V cannot be played on any turn other than turn 2.

As stated, the reader could understand that this strategy depends on the turn in which V is played. The trick here lies in the fact that each possible move V can be played only in one turn.

In general, if $x_{0n_1n_2\dots n_p}$ is the least in V, then V only can be played in the turn p.

Now let's check that strategy γ is winning:

T0: Player I plays (U_0, x_0), and Player II responds with V_0. Let $n_1 \in \omega$ the least such that $0n_1 \in T$ and $B(x_0, 1/n_1) \subset V_0$.

T1: Player I plays (U_{0n_1}, x_{0,n_1}) and Player II responds with V_1. Let x_{0a} the least such that $x_{0a} \in V_1$. Clearly $a = n_1$.
and let $n_2 \in \omega$ the least such that $0n_1n_2 \in T$ and $B(x_{0n_1}, 1/n_2) \subset V_1$.

$\vdots$

Tp: Player I plays $(U_{0n_1\dots n_p}, x_{0,n_1\dots n_p})$ and Player II responds with V_p.

Therefore, $\bigcap V_n = \bigcap_{p=1}^{\infty} U_{0n_1\dots n_p} = \emptyset$. In conclusion, γ is a winning strategy.

$\square$

Observation: By definition of x_s, note that:

$\bullet B(x_s, 1/n) \cap B(x_r, 1/m) \neq \emptyset \Longleftrightarrow B(x_s, 1/n) \subset B(x_r, 1/m)$ or $B(x_r, 1/m) \subset B(x_s, 1/n)$.

$\bullet B(x_r, 1/k_n^r) \subset B(x_s, 1/k_m^s) \Longleftrightarrow (s = r$ and $m \leq n)$ or $(s^\frown k_n^s$ is a restriction of $r)$

3.2 Čech Game

Motivated by the definition 2.3.1, we will define this new game. This game and its properties will be important tools in the remainder of this text.

Let (X, τ)be a Tychonoff space. We define the game $\check{C}(X)$ as follows:

•$T0$: Player I plays an open cover $\mathscr{C}_0$ of X and Player II responds with $C_0 \in \mathscr{C}_0$.

•Tn : Player I plays an open cover $\mathscr{C}_n$ of C_{n-1} and Player II responds with $C_n \in \mathscr{C}_n$.

Player I is declared winner if for every ultrafilter u such that $u \cap \mathscr{C}_n \neq \emptyset$ for $n \in \omega$, verifies that $\bigcap_{F \in u} \overline{F} \neq \emptyset$.

Now we will see the most basic relationship between the Čech spaces and the Čech Game.

Theorem 3.2.1. Let X be a Čech-complete space. Then $I \uparrow \check{C}(X)$

Proof.
Let $(\mathscr{C}_n)_{n\in\omega}$ be a complete sequence.
We will define a winning strategy σ for Player I :
$\sigma(\langle\rangle) = \mathscr{A}_0 = \mathscr{C}_0$;
$\sigma(\langle C_0 \rangle) = \mathscr{A}_1 = \{C \cap C_0, C \in \mathscr{C}_1 \ e \ C \cap C_0 \neq \emptyset\}$,
$\sigma(\langle C_0, \ldots C_n \rangle) = \mathscr{A}_{n+1} = \{C \cap C_n, C \in \mathscr{C}_{n+1} \ e \ C \cap C_{n+1} \neq \emptyset\}$.
Let be an ultrafilter u such that $u \cap \mathscr{A}_n \neq \emptyset$. Then there is a $C \in \mathscr{C}_n$ such that $C \cap C_{n-1} \in u$. Therefore $C \in u$, that is, $u \cap \mathscr{C}_n \neq \emptyset$. Then, by definition of complete sequence, we have that $\bigcap_{F \in u} \overline{F} \neq \emptyset$. □

The following theorem gives us an important result that allows us to see the compactness of the sets obtained in a Čech game.

Theorem 3.2.2. Let α be a Player I's winning strategy in $\check{C}(X)$ such that for each $n \in \omega$ and $V \in \beta(\langle V_0, V_1 \ldots V_n \rangle)$ we have that $\overline{V} \subset V_n$.
If $\langle V_0, V_1 \ldots \rangle$ is a complete run of $\check{C}(X)$. Then $\bigcap V_n$ is compact and $\{V_n\}_{n\in\omega}$ is an local basis for $\bigcap V_n$.

Proof.
By 2.3.5 we can say that $\{V_0, V_1 \ldots\}$ fulfills the required. □

The following proposition allow us to consider each Player I's winning strategy as one of the form described in 3.2.2.

Proposition 3.2.3. If $I \uparrow \check{C}(X)$. Then there is a Player I's winning strategy β such that for each $n \in \omega$ and $V \in \beta(\langle V_0, V_1 \ldots V_n \rangle)$, $\overline{V} \subset V_n$.

Proof.
Let α be a Player I's winning strategy. We define $\beta(\langle\rangle) = \alpha(\langle\rangle)$. Let $V_0 \in \beta(\langle\rangle)$. Then we will

define $\beta(\langle V_0\rangle)$ as a refinement of $\alpha(\langle V_0\rangle)$ such that for each $V \in \beta(\langle V_0\rangle)$ there is a $W \in \alpha(\langle V_0\rangle)$, fixed for each V, such that $\overline{V} \subset W$.
Let $V_1 \in \beta(\langle V_0\rangle)$ and $W_1 \in \alpha(\langle V_0\rangle)$ as previously chosen. We define $\beta(\langle V_0, V_1\rangle)$ as a refinement of $\{V_1 \cap S,\ S \in \alpha(\langle V_0, W_1\rangle)\}$ such that for each $V \in \beta(\langle V_0, V_1\rangle)$ there is a $W \in \alpha(\langle V_0, W_1\rangle)$, fixed for each V, such that $\overline{V} \subset V_1 \cap W$.
In general, let $V_n \in \beta(\langle V_0, V_1 \dots V_{n-1}\rangle)$ and $W_n \in \alpha(\langle V_0, W_1 \dots W_{n-1}\rangle)$. We define $\beta(\langle V_0, \dots V_n\rangle)$ to be a refinement of $\{V_n \cap S,\ S \in \alpha(\langle V_0, \dots W_n\rangle)\}$ such that for each $V \in \beta(\langle V_0 \dots V_n\rangle)$ there is a $W \in \alpha(\langle V_0 \dots W_n\rangle)$, fixed for each V, such that $\overline{V} \subset V_n \cap W$.

Let u be an ultrafilter containing $\{V_0, V_1 \dots\}$. Since $V_n \subset W_n$ then $\{V_0, W_1 \dots\} \subset u$. Therefore $\bigcap_{F \in u} \overline{F} \neq \emptyset$.

□

With the above, we can now see the relationship of Čech Game with Sequential Game.

Theorem 3.2.4. In a metric space (X,d) the sequential game and the Čech game are dual.

Proof.
1)$I \uparrow \check{C}(X) \to II \uparrow G(X,d)$.

Let δ be a winning strategy for Player I in $\check{C}(X)$.
Now we will define a winning strategy σ for Player II in $G(X,d)$.

$\delta(\langle\rangle) = \mathscr{C}_0$.
Let x_0 be the first play of Player I. Then there is $A_0 \in \mathscr{C}_0$ such that $x_0 \in A_0$, and there is r_0 such that $B(x_0, r_0) \subset A_0$. We define $\sigma(\langle x_0\rangle) = r_0$.

$\delta(\langle A_0\rangle) = \mathscr{C}_1$.
Let x_1 be a response of Player I. Then there are $A_1 \in \mathscr{C}_1$ such that $x_1 \in A_1$ and r_1 such that $B(x_1, r_1) \subset A_1 \cap B(x_0, r_0)$. We define $\sigma(\langle x_0, x_1\rangle) = r_1$.

$\delta(\langle A_0 \dots A_{n-1}\rangle) = \mathscr{C}_n$.
Let x_n be a response of Player I. Then there are $A_n \in \mathscr{C}_n$ such that $x_n \in A_n$ and r_n such that $B(x_n, r_n) \subset A_n \cap B(x_{n-1}, r_{n-1})$. We define $\sigma(\langle x_0 \dots x_n\rangle) = r_n$.

Let u be an ultrafilter containing $\{B(x_n, r_n)\}_{n\in\omega}$. Then $u \cap \mathscr{C}_n \neq \emptyset$ for all $n \in \omega$ and therefore $\bigcap \overline{B}(x_n, r_n) \neq \emptyset$.

We can assume that $r_n \to 0$. Then there is $y \in X$ such that $\bigcap \overline{B}(x_n, r_n) = \{y\}$. Therefore $x_n \to y$.

2)$II \uparrow G(X,d) \to I \uparrow \check{C}(X)$.

Let σ be a winning strategy for Player II in $G(X,d)$, without loss of generality we can assume that every run verifies:
•$r_n \to 0$, and
•$B(x_{n+1}, r_{n+1}) \subset B_(x_n, r_n)$.

Now we will define a winning strategy δ for Player I in $\check{C}(X)$.

We define $\delta(\langle\rangle) = \mathscr{C}_0 = \{B(x, \sigma(\langle x \rangle))\}_{x \in X}$.

Let $B_0 = B(x_0, \sigma(\langle x_0 \rangle))$ be a response of Player II.
We define $\delta(\langle B_0 \rangle) = \mathscr{C}_1 = \{B(x, \sigma(\langle x_0, x \rangle))\}_{x \in B_0}$.

Let $B_{n-1} = B(x_{n-1}, \sigma(\langle x_0 \ldots x_{n-1} \rangle))$ be a response of Player II in turn $n-1$.
We define $\delta(\langle B_0 \ldots B_{n-1} \rangle) = \mathscr{C}_n = \{B(x, \sigma(\langle x_0 \ldots x_{n-1}, x \rangle))\}_{x \in B_{n-1}}$.

Since $x_n \to y$ then $\bigcap \overline{B}(x_n, r_n) = \{y\}$, where $r_n = \sigma(\langle x_0 \ldots x_n \rangle)$.

Let u be an ultrafilter such that $u \cap \mathscr{C}_n \neq \emptyset$, then $B_n \in u$ for all $n \in \omega$.

Let V be an open set such that $y \in V$. Then there are k and r_n such that $B(x_n, r_n) \subset B(y, k) \subset V$. Therefore $V \in u$, that is $u \searrow y$.

3)$I \uparrow G(X,d) \to II \uparrow \check{C}(X)$.

Let σ be a winning strategy for Player I in $G(X,d)$.
Now we will define a strategy δ for Player II in $\check{C}(X)$:

$\sigma(\langle\rangle) = x_0$.
Let $< \mathscr{C}_0 >$ be the first play of Player I. We define $\delta(\langle \mathscr{C}_0 \rangle) = A_0$ where $x_0 \in A_0$, and we choose $r_0 > 0$ such that $B(x_0, r_0) \subset A_0$.

$\sigma(\langle r_0\rangle) = x_1$.
Let $< \mathscr{C}_1 >$ be a response of Player I. We define $\delta(\langle \mathscr{C}_0, \mathscr{C}_1\rangle) = A_1$ where $x_1 \in A_1$, and we choose r_1 such that $B(x_1, r_1) \subset B(x_0, r_0) \cap A_1$.

$\sigma(\langle r_0 \dots r_{n-1}\rangle) = x_n$.
Let $< \mathscr{C}_n >$ be a response of Player I. We define $\delta(\langle \mathscr{C}_0 \dots \mathscr{C}_n\rangle) = A_n$ where $x_n \in A_n$ and we choose r_n such that $B(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap A_n$.

If δ is not a winning strategy then there is a play $< \mathscr{C}_0, A_0, \mathscr{C}_1, A_1 \dots >$ where Player I is declared winner.
In that run, let u be an ultrafilter such that $B(x_n, r_n) \in u$ for all $n \in \omega$, we have that $\{y\} = \bigcap_{F\in u} \overline{F} \subset \bigcap \overline{B}(x_n, r_n)$.

We can assume that $r_n \to 0$, and since $\bigcap \overline{B}(x_n, r_n) \neq \emptyset$ then $\bigcap \overline{B}(x_n, r_n) = \{y\}$. Therefore $x_n \to y$, contradiction.

4)$II \uparrow \check{C}(X) \to I \uparrow G(X,d)$.

Let δ be a winning strategy for Player II in $\check{C}(X)$.

Lemma 3.2.5. Let V be an open set, and $K = \{\mathscr{C}\}$ be the family of open covers of V that verifies $\bigcup_{A\in\mathscr{C}} A = V$ for any $\mathscr{C} \in K$.
There is a $x \in V$ such that for all open set W with $x \in W \subset V$, there is a $\mathscr{C} \in K$ such that $\delta(\langle \mathscr{C}_0, \mathscr{C}_1 \dots \mathscr{C}\rangle) = W$.

Now we will define a strategy σ for Player I in $G(X,d)$:

From now on we will consider p_n to be small enough for $p_n \leq \min\{r_n, \dfrac{1}{2^n}\}$ and $B(x_n, p_n) \subset B(x_{n+1}, p_{n+1})$.

We define $\sigma(\langle\rangle) = x_0$, where x_0 is given by the above lemma with $V = X$.
Let r_0 be a response of Player II. Then we choose $\mathscr{C}_0$ such that $\delta(\langle \mathscr{C}_0\rangle) = B(x_0, p_0)$.

We define $\sigma(\langle r_0\rangle) = x_1$, where x_1 is given by the above lemma with $V = V_0 = B(x_0, p_0)$.
Let r_1 be a response of Player II. Then we choose $\mathscr{C}_1$ open cover of V_0 such that $\delta(\langle \mathscr{C}_0, \mathscr{C}_1\rangle) = B(x_1, p_1)$.

We define $\sigma(\langle r_{n-1}\rangle) = x_n$, where x_n is given by the above lemma with $V = V_n = B(x_{n-1}, p_{n-1})$.
Let r_n be a response of Player II. Then we choose $\mathscr{C}_n$ open cover of V_n such that $\delta(\langle \mathscr{C}_0 \dots \mathscr{C}_n\rangle) = B(x_n, p_n)$.

Since δ is a winning strategy there is an ultrafilter u, such that $B(x_n, p_n) \in u$ and $\bigcap_{F\in u} \overline{F} = \emptyset$. Suppose that $x_n \to y$. Then there are $r > 0$ and $F \in u$ such that $B(y,r) \cap F = \emptyset$.

Since there is an $n \in \omega$ such that $d(x_n, y) < r/2$ and $p_n < r/2$, then $B(x_n, p_n) \subset B(y,r)$. Therefore $B(x_n, p_n), F \in u$ and $B(x_n, p_n) \cap F = \emptyset$, contradiction. □

Proof of Lemma. 3.2.5 Case $\delta(\langle \mathscr{C}\rangle)$:
Suppose it is not true. Then for every $x \in V$ there is $W_x \subset V$, which is not a possible response for any open cover of V.
Let $\mathscr{C}_0 = \{W_x\}_{x\in V}$ be an open cover of V. Then $\delta(\langle \mathscr{C}_0\rangle) = W_x$ for some $x \in X$, contradiction.
In an analogous way we can prove the other cases with two or more covers.□

If together with the previous theorem we consider the theorem 3.1.3, we can affirm the duality relationship between Choquet Game and Čech game in a metric space. In the following proposition, we will try to generalize this relation to a Tychonoff space.

Proposition 3.2.6. Let (X, τ) be a Tychonoff space, then:

- $I \uparrow Ch(X) \Rightarrow II \uparrow \check{C}(X)$.
- $I \uparrow \check{C}(X) \Rightarrow II \uparrow Ch(X)$.

Proof.
1) $I \uparrow Ch(X) \Rightarrow II \uparrow \check{C}(X)$.

Let σ be a winning strategy for Player I, we will define γ a strategy for Player II in $\check{C}(X)$ by doing the following:
Let $\sigma(\langle\rangle) = (A_0, x_0)$ and let $\mathscr{C}_0$ be the first play of Player I in $\check{C}(X)1$. Then there is a $V_0 \in \mathscr{C}_0$ such that $x_0 \in V_0$. We define $\gamma(\langle \mathscr{C}_0\rangle) = V_0$.

Let W_0 be an open such that $\overline{W_0} \subset A_0 \cap V_0$. Let $\sigma(\langle W_0\rangle) = (A_1, x_1)$ and let $\mathscr{C}_1$ be open cover of V_0. Then there is a $V_1 \in \mathscr{C}_1$ such that $x_1 \in V_1$. We define $\gamma(\langle \mathscr{C}_0, \mathscr{C}_1\rangle) = V_1$.

Let $\sigma(\langle W_0 \dots W_n\rangle) = (A_{n+1}, x_{n+1})$ and let $\mathscr{C}_{n+1}$ be open cover of V_n. Then there is a $V_{n+1} \in \mathscr{C}_{n+1}$ such that $x_{n+1} \in V_{n+1}$. We define $\gamma(\langle \mathscr{C}_0, \mathscr{C}_{n+1}\rangle) = V_{n+1}$.

Let u be an ultrafilter containing $\{W_n\}_{n\in\omega}$. Then since $W_n \subset V_n$ we have $u \cap \mathscr{C}_n \neq \emptyset$, in addition $\bigcap \overline{W_n} \subset \bigcap A_n = \emptyset$ because $I \uparrow Ch(X)$.

Therefore $\bigcap_{F\in u} \overline{F} = \emptyset$.

2) $I \uparrow \check{C}(X) \Rightarrow II \uparrow Ch(X)$.

Let σ be a winning strategy for Player I in $\check{C}(X)$, we will define γ a strategy for Player II in $Ch(X)$ by doing the following:

Let (V_0, x_0) be the first play of Player I then there is $C_0 \in \sigma(\langle\rangle)$ such that $x_0 \in C_0$. Since X is a Tychonoff space, there is a open neighborhood K_0 of x_0 such that $\overline{K}_0 \subset C_0 \cap V_0$. We define $\gamma(\langle(V_0, x_0)\rangle) = K_0$.

Let (V_1, x_1) be the play of Player I in turn 1 then there is $C_1 \in \sigma(\langle C_0\rangle)$ such that $x_1 \in C_1$. Since X is a Tychonoff space, there is an open neighborhood K_1 of x_1 such that $\overline{K}_1 \subset C_1 \cap V_1$. We define $\gamma(\langle(V_0, x_0)\rangle) = K_1$.

In general, let (V_n, x_n) be the play of Player I in turn n. Then there are $C_n \in \sigma(\langle C_0 \dots C_{n-1}\rangle)$ such that $x_n \in C_n$ and an open neighborhood K_n of x_n such that $\overline{K}_n \subset C_n \cap V_n$. We define $\gamma(\langle(V_0, x_0) \dots (V_n, x_n)\rangle) = K_n$.

Let u be an ultrafilter such that $\{K_n\}_{n\in\omega} \subset u$. Then $C_n \in u \cap \mathscr{C}_n$ for any n. Since $I \uparrow \check{C}(X)$ then $\bigcap \overline{K}_n \neq \emptyset$ and since $V_{n+1} \subset \overline{K}_n \subset V_n$ for all $n \in \omega$, then $\bigcap V_n = \bigcap \overline{K}_n \neq \emptyset$. Therefore $II \uparrow Ch(X)$.

□

With the following example, we will confirm that the previous result in Proposition 3.2.6 is the best we can obtain to relate the Choquet and Čech games in a general way.

Example 3.2.7. Let $X = \{(m,n) : m, \in \omega\}$ with all singletons except $\{(0,0)\}$ open. Define a set containing $\{(0,0)\}$ to be open if and only if contains all but a finite number of points in all but finitely many columns. Such space is called Arens-Fort space and was defined for the first time in (ARENS, 1950). The Arens-Fort space is Tychonoff but it is not metrizable.
First we will check that $II \uparrow Ch(X)$. We will define a stationary strategy σ, by doing the following: $\sigma(\langle(U,x)\rangle) = U$ if $x = (0,0)$ and $\sigma(\langle(U,x)\rangle) = \{x\}$ if $x \neq (0,0)$. Note that, if Player I chooses

a open set with a point other than $(0,0)$ then his next move can only be a singleton. Therefore, Player *II* wins the game. In case that Player *I* choose an open set with $(0,0)$ in each turn, clearly, each move of Player *II* contains $(0,0)$ and Player *II* wins the game again.
Now we will see that $II \uparrow \check{C}(X)$. Let $\mathscr{C}_0$ be Player *I*'s first move. Define $\sigma(\langle \mathscr{C}_0 \rangle) = C_0$ where $(0,0) \in C_0$. If Player *I* response with $\mathscr{C}_1$, then $\sigma(\langle \mathscr{C}_0, \mathscr{C}_1 \rangle) = C_1$ where $(0,0) \in C_1$ and so on. We will consider the filter base $B = \{(x,y), y \in \omega\}_{x \in \omega}$. Note that $\bigcap_{F \in B} \overline{F} = \emptyset$, and $B \cup \{\sigma(\langle \mathscr{C}_0 \dots \mathscr{C}_n \rangle)\}_{n \in \omega}$ have a finite intersection property. Therefore there is an ultrafilter u containing it. Since $\bigcap_{F \in u} F \subset \bigcap_{F \in B} \overline{F} = \emptyset$, we can say that Player *II* wins the game.

3.2.1 Sieve Game

Since we can't get a complete relationship between Choquet Game and Čech Game, the obvious question is: "what needs to be modified to achieve this relationship?". In (TOPSOE, 1982) we can find an answer to this question, in the form of the following game.

Let (X, τ) be a topological space. We define the Sieve Game $SV(X)$ as follows:

•$T0$: Player *I* plays $(U_0, x_0) \in \tau \times X$ with $x_0 \in U_0$ and Player *II* responds with $V_0 \in \tau$, such that $x \in V_0 \subset U_0$.

•Tn : Player *I* plays $(U_n, x_n) \in \tau \times X$, with $x_n \in U_n \subset V_{n-1}$ and Player *II* responds with $V_n \in \tau$, such that $x_n \in V_n \subset U_n$.

Player *II* is declared winner if any ultrafilter u containing $\{V_n\}_{n \in \omega}$ verifies that $\bigcap_{F \in u} \overline{F} \neq \emptyset$.

Analogously we can define $kSV(X)$ where Player *I* chooses compact subsets instead of points.

Definition 3.2.8. A space X is called sieve complete if $II \uparrow SV(X)$.

With this new game, we can now get what we were looking for. It is expressed in the following theorem.

Theorem 3.2.9. SV(X) and Č(X) are dual games.

Proof.
1) $I \uparrow SV(X) \rightarrow II \uparrow \check{C}(X)$.

Let δ be a Player I's winning strategy for $SV(X)$. We will define a Player II's wining strategy σ for $\check{C}(X)$ by doing the following:
Let $\delta(\langle\rangle) = (V_0, x_0)$.
Let $\mathscr{C}_0$ be Player I's first play then we set $\sigma(\langle\mathscr{C}_0\rangle) = C_0$ where $x_0 \in C_0$.
Let $\delta(\langle V_0 \cap C_0\rangle) = (V_1, x_1)$.
Let $\mathscr{C}_1$ be the Player I's next play, then we set $\sigma(\langle\mathscr{C}_0, \mathscr{C}_1\rangle) = C_1$ where $x_1 \in C_1$.
Let $\delta(\langle V_0 \cap C_0, V_1 \cap C_1 \ldots V_{n-1} \cap C_{n-1}\rangle) = (V_n, x_n)$.
Let $\mathscr{C}_n$ be the Player I's last play, then we set $\sigma(\langle\mathscr{C}_0 \ldots \mathscr{C}_n\rangle) = C_n$ where $x_n \in C_n$.

Since $I \uparrow SV(X)$ there is an ultrafilter u containing $\{V_n\}_{n\in\omega}$ such that $\bigcap_{F\in u} \overline{F} = \emptyset$. Since $V_{n+1} \in u$, then $V_n \cap C_n \in u$. Therefore $C_n \in u$. Since $u \cap \mathscr{C}_n \neq \emptyset$ and $\bigcap_{F\in u} \overline{F} = \emptyset$ so $II \uparrow \check{C}(X)$.

2) $I \uparrow \check{C}(X) \rightarrow II \uparrow SV(X)$.

Let σ be a Player I's winning strategy for $\check{C}(X)$. We will define a Player II's wining strategy for $SV(X)$.
Let $\mathscr{C}_0 = \sigma(\langle\rangle)$. Let (V_0, x_0) be Player I's first play in $SV(X)$. We fix a $C_0 \in \mathscr{C}_0$ such that $x_0 \in C_0$ and set $\delta(\langle(V_0, x_0)\rangle) = V_0 \cap C_0$.
Let $\mathscr{C}_1 = \sigma(\langle C_1\rangle)$. Let, (V_1, x_1) be Player I's next play. We fix a $C_1 \in \mathscr{C}_1$ such that $x_1 \in C_1$ and set $\delta(\langle(V_0, x_0), (V_1, x_1)\rangle) = V_1 \cap C_1$.
Continuing on that way, let $\mathscr{C}_n = \sigma(\langle C_1 \ldots C_n\rangle)$. Let, (V_n, x_n) be Player I's last play. We fix a $C_n \in \mathscr{C}_n$ such that $x_n \in C_n$ and set $\delta(\langle(V_0, x_0), \ldots (V_n, x_n)\rangle) = V_n \cap C_n$.
Let u be an ultrafilter containing $\{V_n \cap C_n\}_{n\in\omega}$, then $C_n \in u$. Since $\mathscr{C}_n \cap u \neq \emptyset$ we have that $\bigcap_{F\in u} \overline{F} = \emptyset$.

3) $II \uparrow SV(X) \rightarrow I \uparrow \check{C}(X)$.

Let δ be a Player II's winning strategy for $SV(X)$. We will define a Player I's wining strategy σ for $\check{C}(X)$.
We set $\sigma(\langle\rangle) = \{\delta(\langle(X, x_0)\rangle)\}_{x_0\in X}$. Let C_0 the Player II's response.
We set $\sigma(\langle C_0\rangle) = \{\delta(\langle(C_0, x_1)\rangle)\}_{x_1\in C_0}$. Let C_1 the Player II's response.
In general, if C_n was a Player II's last response, we set $\sigma(\langle C_0 \ldots C_n\rangle) = \{\delta(\langle(C_n, x_{n+1})\rangle)\}_{x_{n+1}\in C_n}$.
Let $\langle\mathscr{C}_0, C_0, \mathscr{C}_1, C_1 \ldots\rangle$ be a σ-game.
Let u be an ultrafilter such that $u \cap \mathscr{C}_0 \neq \emptyset$ then there are $\{x_n\}_{n\in\omega}$ such that $\langle(X, x_0), C_0, (C_0, x_1) \ldots\rangle$ is a δ-game. As $C_n \in u$ for each $n \in \omega$ then $\bigcap_{F\in u} \overline{F} \neq \emptyset$.

4) $II \uparrow \check{C}(X) \rightarrow I \uparrow SV(X)$.

Let σ be a Player II's winning strategy for $\check{C}(X)$. We will define a Player I's wining strategy δ for $SV(X)$.

By Lemma 3.2.5 there is a x_0 such that for any open neighborhood V_0 of x_0 there is an open cover $\mathscr{C}_0$ of X such that $\sigma(\langle\mathscr{C}_0\rangle) = V_0$.
We set $\delta(\langle\rangle) = (X, x_0)$. Let V_0 be Player II' response. Let $\mathscr{C}_0$ be a open cover of X such that $\sigma(\langle\mathscr{C}_0\rangle) = V_0$.
Again by Lemma 3.2.5, there is a x_1 such that for any open neighborhood $V_1 \subset V_0$ of x_0 there is a open cover $\mathscr{C}_1$ of V_0 such that $\sigma(\langle\mathscr{C}_0, \mathscr{C}_1\rangle) = V_1$.
We set $\delta(\langle V_0\rangle) = (V_0, x_1)$. Let V_1 be Player II's response. Let $\mathscr{C}_1$ be a open cover of V_0 such that $\sigma(\langle\mathscr{C}_0, \mathscr{C}_1\rangle) = V_1$.
In this way we have a δ-game $\langle(X, x_0), V_0, (V_0, x_1), V_1 \ldots\rangle$ and a $\sigma-$game $\langle\mathscr{C}_0, V_0, \mathscr{C}_1, V_1 \ldots\rangle$.
Since $II \uparrow \check{C}(X)$ there is an ultrafilter u such that $u \cap \mathscr{C}_0 \neq \emptyset$ and $\bigcap_{F \in u} \overline{F} = \emptyset$. As $u \cap \mathscr{C}_0 \neq \emptyset$ so $\{V_n\}_{n\in\omega} \subset u$. Therefore $I \uparrow SV(X)$.

□

Additionally, we will now see that $SV(X)$ and $kSV(x)$, although very similar, are not the same game. Which is demonstrated with the following 2 results.

Theorem 3.2.10. The following conditions are equivalent:
i)$II \uparrow SV(X)$.
ii)$II \uparrow kSV(X)$

Proof.
Since a singleton is a compact set, it is clear that $II \uparrow kSV(X)$ implies $II \uparrow SV(X)$.
Let σ be a Player II's wining strategy in $SV(X)$.
We will define a Player II's strategy γ in $KSV(X)$ by doing the following:
Let (V_0, K_0) the first play of Player I.
Then $\{\sigma(\langle(V_0, x_0)\rangle)\}_{x_0\in K_0}$ is a open cover of K_0. Therefore we can set a finite subcover $\{\sigma(\langle(V_0^i, x_0^i)\rangle)\}_{i=0}^{i=n}$.
Let $\gamma(\langle(V_0, K_0)\rangle$ be the union of the open sets in that subcover.
Let (V_1, K_1) the next play of Player I.
Each $x_1 \in K_1$ is contained in one or more open sets of the previous subcover. Then $\{\sigma(\langle(V_0^i, x_0^i), (V_1, x_1 \cap V_0^i)\rangle)\}$, where $x_1 \in V_0^i \cap K_1$, is a open cover of K_1. Therefore we can set a finite subcover. Now we define $\gamma(\langle(V_0, K_0), (V_1, K_1)\rangle$ to be the union of the open sets in the last subcover.
Analogously, we can continue defining Player II's responses.

Note that because of the way we chose Player II's response we have a infinite tree with finite branching where every finite branch is a partial game of $SV(X)$, every infinite branch is a complete game of $SV(X)$ and the nth Player II's response is the union of open sets in the nth tree level.

Let u be an ultrafilter containing every Player II's response in a game of $KSV(X)$. Then u contain at least one open set in every tree level. Therefore we have a infinite sub-tree included in the ultrafilter. Hence u contain a infinite branch. Since $II \uparrow SV(X)$ we have that $\bigcap_{F \in u} \overline{F} \neq \emptyset$. $\square$

Example 3.2.11. Let X be a Bernstein subset of $[0,1]$. Then $I \uparrow kSV(X)$ but $I \not\uparrow SV(X)$. A full proof of this result can be found in (TELGARSKY, 1984).

3.2.2 Porada Game

The Porada game is another way of generalizing the Choquet game in a way that allows us to complete the relationship between Choquet Game and Čech Game.
In this text we will mainly consider the Čech and Sieve games, however, we will verify that both generalization options are equivalent.
Let (X,τ) be a topological space and $Y \subset X$. We define the Porada Game $P(X,Y)$ as follows:

•$T0$: Player I plays $(U_0,x_0) \in \tau \times Y$ with $x_0 \in U_0$ and Player II responds with $V_0 \in \tau$, such that $x \in V_0 \subset U_0$.

•Tn : Player I plays $(U_n,x_n) \in \tau \times Y$, with $x_n \in U_n \subset V_{n-1}$ and Player II responds with $V_n \in \tau$, such that $x_n \in V_n \subset U_n$.

Player II is declared winner if $\emptyset \neq \bigcap V_n \subset Y$.

Analogously we can define $kP(X,Y)$ where Player I chooses compact sets instead of points.
If X is a Tychonoff space, then we define $P(X) := P(\beta X, X)$.

Theorem 3.2.12. Let X be a Tychonoff space. Then $SV(X)$ and $P(X)$ are equivalent.

Proof. $I \uparrow SV(X) \Rightarrow I \uparrow P(X))$
For every U, open set in X, set U^* an open set in βX such that $U^* \cap X = U$.
Let σ be a Player I's winning strategy for $SV(X)$. We will define a Player I's strategy γ for $P(X)$ by doing the following:
Let $\sigma(\langle\rangle) = (U_0,x_0)$ be Player I's first play. We set $\gamma(\langle\rangle) = (W_0,x_0)$ where $W_0 = U_0^*$.
Let V_0 be Player II's response. If $\sigma(\langle V_0 \cap X\rangle) = (U_1,x_1)$, then we set $W_1 \ni x_1$ such that $\overline{W_1} \subset U_1^* \cap V_0$. We define $\gamma(\langle V_0\rangle) = (W_1,x_1)$.

Analogously, we define $\gamma(\langle V_0, V_1 \dots V_n \rangle) = (W_{n+1}, x_{n+1})$ where $\sigma(\langle V_0 \cap X, V_1 \cap X \dots V_n \cap X \rangle) = (U_{n+1}, x_{n+1})$ and $\overline{W_{n+1}} \subset U^*_{n+1} \cap V_n$.

Note that $\overline{V_{n+1}} \subset V_n$ for all $n \in \omega$. Therefore $K = \bigcap V_n$ is a non empty compact subset of βX.

Now suppose $K \subset X$. Then K is compact in X. Since σ is a Player I's winning strategy there is an ultrafilter $u = \{F\}_{F \in u}$ in X containing $\{V_n \cap X\}_{n \in \omega}$ such that $\bigcap_{F \in u} Cl_X(F) = \emptyset$.

Since $Cl_x(V_n \cap X) \subset \overline{V}_n \cap X$ we have that $\bigcap Cl_x(V_n \cap X) \subset \bigcap \overline{V}_n \cap X = K$. Let $x \in K \setminus \bigcap Cl_x(V_n \cap X)$. There is an $n \in \omega$ such that $x \notin Cl_x(V_n \cap X)$. As X is Tychonoff, there is an open neighborhood W of x such that $W \cap (V_n \cap X) = \emptyset$. Since $x \in \overline{V}_n$, we have that $(W^* \cap V_n) \cap X \neq \emptyset$, contradiction. Therefore $\bigcap Cl_x(V_n \cap X) = K$.

Then $\bigcap_{F \in u} Cl_X(F) = \bigcap_{F \in u} (Cl_X(F) \cap K)$. Suppose there are an $F \in u$ such that $Cl_X(F) \cap K = \emptyset$. Since $F \cap V_n \neq \emptyset$ we have $\{\overline{F} \cap \overline{V_n}\}_{n \in \omega}$ is a family of non empty compact sets with the finite intersection property. Therefore $\bigcap(\overline{V}_n \cap \overline{F}) = \overline{F} \cap K = \overline{F} \cap (X \cap K) = Cl_X(F) \cap K \neq \emptyset$, contradiction. Therefore $Cl_X(F) \cap K \neq \emptyset$ for all $F \in u$. Hence $\{Cl_X(F) \cap K\}_{F \in u}$ is a family of compact sets with finite intersection property. Then $\bigcap_{F \in u} Cl_X(F) \cap K \neq \emptyset$, this contradicts Player I's winning strategy in $SV(X)$. Finally, we can state that $K \not\subset X$, that is, γ is a Player I's winning strategy in $P(X)$.

$I \uparrow P(X) \Rightarrow I \uparrow SV(X))$

Let γ be a Player I's winning strategy for $P(X)$. We will define σ, a Player I's strategy, by doing the following:

Let $\gamma(\langle\rangle) = (U_0, x_0)$. We set $\sigma(\langle\rangle) = (U_0 \cap X, x_0)$.

let V_0 be Player II's response. We set an open set W_0 containing x_0 such that $\overline{W}_0 \subset V^*_0 \cap U_0$. Then $\sigma(\langle V_0 \rangle) = (U_1 \cap X, x_1)$ where $\gamma(\langle W_0 \rangle) = (U_1, x_1)$.

In general, if V_n is Player II's last response then $\sigma(\langle V_0 \dots V_n \rangle) = (U_{n+1} \cap X, x_{n+1})$ where $\gamma(\langle W_0 \dots W_n \rangle) = (U_{n+1}, x_{n+1})$.

Since γ is a Player I's winning strategy for $P(X)$ we have that $\bigcap W_n = \bigcap \overline{W}_n = K$ is a non empty compact subset of βX such that $K \not\subset X$.

Let $x \in K \setminus X$. Let us consider a family of open sets in X formed by the intersection of X and open neighborhoods of x in βX. That family have the finite intersection property. Therefore, there is an ultrafilter v containing such family. Since $x \in K \subset W_n \subset V^*_n$ we have that $V_n = V^*_n \cap X \in v$. Let u be an ultrafilter in βX containing v. It is clear that u contains all the open neighborhoods of x in βX. Since u is an ultrafilter in βX, a compact Hausdorff space, we have that $\bigcap_{F \in u} \overline{F} = \{x\}$. Suppose that $\bigcap_{F \in v} Cl_X(F) \neq \emptyset$. Let $y \in \bigcap_{F \in v} Cl_X(F)$. There are V_x and V_y disjoint open neighborhoods in βX of x and y respectively. Therefore $y \in Cl_X(V_x \cap X)$ and $(V_y \cap X) \cap (V_x \cap X) = \emptyset$, contradiction.

In conclusion, there is an ultrafilter v containing $\{V_n\}_{n\in\omega}$ such that $\bigcap_{F\in v} Cl_X(F) = \emptyset$, that is, σ is a Player I's winning strategy for $SV(X)$.

$II \uparrow SV(X) \Rightarrow II \uparrow P(X))$

Let σ be a Player II's winning strategy for $SV(X)$. We will define γ, a Player II's strategy for $P(X)$, by doing the following:

Let (V_0,x_0) be Player I's first play. Let $\sigma(\langle(V_0 \cap X, x_0)\rangle) = U_0$. We set $\gamma(\langle(V_0,x_0)\rangle) = W_0$ such that $x_0 \in W_0 \subset \overline{W_0} \subset U_0^* \cap V_0$.
Let (V_1,x_1) be Player I's response. Since $V_1 \subset W_0 \subset U_0^*$ we have $V_1 \cap X \subset U_0^* \cap X = U_0$. Let $\sigma(\langle(V_0 \cap X, x_0),(V_1 \cap X, x_1\rangle) = U_1$. We set $\gamma(\langle(V_0,x_0),(V_1,x_1)\rangle) = W_1$ such that $x_1 \in W_1 \subset \overline{W_1} \subset U_1^* \cap V_1$.
In general, let (V_n,x_n) be Playe I's last response. Let $\sigma(\langle(V_0 \cap X, x_0)\dots(V_n \cap X, x_n)\rangle) = U_n$. We set $\gamma((V_0,x_0)\dots(V_n,x_n)\rangle) = W_n$ such that $x_n \in W_n \subset \overline{W_n} \subset U_n^* \cap V_n$.
Let $K = \bigcap W_n = \bigcap \overline{W}_n$. Then K is a non empty compact set. Suppose $K \setminus X \neq \emptyset$. Let $y \in K \setminus X$. Let u be an ultafilter in X containing all open set of the form $V \cap X$ where V is an open neighborhood in βX of y.Since any ultrafilter in βX containing u clusters in $\{y\}$, we have that $\bigcap_{F\in u} Cl_X(F) = \emptyset$. This contradicts the fact that $II \uparrow SV(X)$. Therefore $K \subset X$, that is, γ is a winning strategy.

$II \uparrow P(X) \Rightarrow II \uparrow SV(X))$

Let γ be a Player II's winning strategy for $P(X)$. We will define σ, a Player II's strategy for $SV(X)$, by doing the following:

Let (V_0,x_0) be Player I's first play. Let $\gamma(\langle(V_0^*,x_0)\rangle) = U_0$. We set $\sigma(\langle(V_0,x_0)\rangle) = U_0 \cap X$.
Let (V_1,x_1) be Player I's response. We set W_1, an open neighborhood of x_1 in βX, such that $\overline{W}_1 \subset V_1^* \cap U_0$. Let $\gamma(\langle(V_0^*,x_0),(W_1,x_1)\rangle) = U_1$. We define $\sigma(\langle(V_0,x_0),(V_1,x_1)\rangle) = U_1 \cap X$.
In general, let (V_n,x_n) be Player I's last response. We set W_n, an open neighborhood of x_n in βX, such that $\overline{W}_n \subset V_n^* \cap U_{n-1}$. Let $\gamma(\langle(V_0^*,x_0)\dots(W_n,x_n)\rangle) = U_n$. We define $\sigma(\langle(V_0,x_0)\dots(V_n,x_n)\rangle) = U_n \cap X$.
Let $K = \bigcap U_n = \bigcap \overline{U}_n$. Since $II \uparrow P(X)$ we have that K is a non empty compact subset of X. Let u be an ultrafilter in X containing $\{U_n \cap X\}_n$. As $\bigcap Cl_X(U_n \cap X) \subset \bigcap \overline{U}_n = K$.
Suppose that $K \setminus \bigcap Cl_X(U_n \cap X) \neq \emptyset$. Let $y \in K \setminus \bigcap Cl_X(U_n \cap X)$. Then there is a $Cl_X(U_n \cap X)$ such that $y \notin Cl_X(U_n \cap X)$. Since X is a Tychonoff space, let V_y be an open neighborhood of y such that $V_y \cap Cl_X(U_n \cap X) = \emptyset$. Then $V_y \cap U_n = \emptyset$. Since $y \in K \subset \overline{U}_n$, we have that $V_y^* \cap U_n \neq \emptyset$ contradiction. Therefore $K = \bigcap Cl_X(U_n \cap X)$.
Then $\bigcap_{F\in u} Cl_X(F) = \bigcap_{F\in u}(Cl_X(F) \cap K) \neq \emptyset$. That is, σ is a winning strategy.

□

CHAPTER

4

APPLICATIONS OF TOPOLOGICAL GAMES

In this chapter, we will see some properties of topological games that will allow us to characterize some spaces as Baire spaces. We will also see properties that allow us to identify some spaces where Player I or Player II has a winning strategy.

4.1 On the Banach-Mazur Game and the Choquet Game

We begin with the already mentioned result, a characterization of Baire spaces through the use of the Banach game. This result can be found in (OXTOBY, 1957).

Proposition 4.1.1. Let (X,τ) be a topological space. X is not a Baire space if, and only if, $I \uparrow BM(X)$.

Proof.

$\Rightarrow$) Since X is not a Baire space, there is a family $\{A_n; n \in \omega\}$ of dense open sets, such that $A = \bigcap A_n$ is not dense.

We will define a strategy δ by doing the following:
We define $\delta(\langle\rangle) = (U, x_0)$ where $A \cap U = \emptyset$ and $x_0 \in A_0 \cap U$.
Let V_0 be a response of Player II in turn n. We define $\delta(\langle V_0 \rangle) = (A_1 \cap V_0, x_1)$.
Let V_n be a response of Player II in turn n. We define $\delta(\langle V_0, \ldots V_n \rangle) = (A_{n+1} \cap V_0, x_{n+1})$.

Since $V_n \subset U \cap A_n$, then $\bigcap V_n \subset U \cap \bigcap A_n = U \cap A = \emptyset$, therefore $I \uparrow Ch(X)$.

$\Leftarrow$) Let σ be a Player I's winning strategy. Let $\sigma(\langle\rangle) = V_0$.
There is a family $\mathscr{V}_0$, of subsets of V_0, such that $\{\sigma(\langle V \rangle); V \in \mathscr{V}_0\}$ is a maximal pairwise disjoint family. Note that $\bigcup \mathscr{V}_0$ is an open dense subset of V_0.

Analogously, for each $W \in \mathscr{V}_0$ there is a family $\mathscr{V}^W$, of subsets of W, such that $\{\sigma(\langle W,V\rangle);V \in \mathscr{V}^W\}$ is a maximal pairwise disjoint family. We set $\mathscr{V}_1$ as an union of all families $\mathscr{V}^W$. Note that $\bigcup \mathscr{V}_1$ is an open dense subset of V_0.
This way we can define a collection $\{\mathscr{V}_n\}_{n\in\omega}$ of pairwise disjoint open sets, such that $\bigcup \mathscr{V}_n$ is an open dense subset of V_0.
Suppose that $\bigcap_{n\in\omega} \bigcup \mathscr{V}_n \neq \emptyset$. Let $x \in \bigcup \mathscr{V}_n \neq \emptyset$ for all $n \in \omega$. Since $\mathscr{V}_n$ is pairwise disjoint, there is an only $V_n \in \mathscr{V}_n$ such that $x \in V_n$. In addition, there is only one $W_0 \subset V_0$, such that $\sigma(\langle V_0,W_0\rangle) = V_1$. In general there is an only one $W_n \subset V_n$ such that $\sigma(\langle V_0,W_0 \ldots V_n,W_n\rangle) = V_{n+1}$. Since $I \uparrow BM(X)$, then $\bigcap V_n = \emptyset$, contradiction. Therefore, $\bigcap_{n\in\omega} \bigcup \mathscr{V}_n = \emptyset$. Hence, V_0 is not a Baire space, therefore, neither is X. □

Lemma 4.1.2. Let X be a Tychonoff space. Let Y be a dense subspace of X. Let $\{V_n\}_{n\in\omega}$ be a decreasing sequence of open sets in X and define $W_n = V_n|_Y$. If $\bigcap W_n = K$ is compact in X and $\{W_n\}_{n\in\omega}$ is a local base of neighborhoods of K in Y, then $\bigcap V_n = K$ and $\{V_n\}_{n\in\omega}$ is a local base of neighborhoods of K in X.

Proof.
Suppose there is $y \in \bigcap V_n$ such that $y \notin K$. Then there are disjoint open sets A,B such that $y \in A$ and $K \subset B$. Therefore there is a $W_n \subset B|_Y$ and $y \in V_n \cap A$.
Therefore $V_k \cap A$ is a non empty open set. Since Y is dense in X, then $\emptyset \neq V_n \cap A \cap Y = V_n|_Y \cap A|_Y = W_n \cap A|_Y \subset B|_Y \cap A|_Y = \emptyset$, contradiction.
Let V be a neighborhood of K in X. Since K is a compact set and X is a Tychonoff space, we can consider that V is a closed neighborhood. There is a $W_n \subset V|_Y$. Suppose that $V_n \not\subset V$. Since $V \cap V_n$ is open in X and Y is dense in X, there is a $y \in (V_n \cap V^c)|_Y = W_n \cap V^c|_Y = \emptyset$, contradiction. □

The next property to discuss asks us for something more than a winning strategy but also requires that the strategy to be stationary. This property is mentioned in (REVALSKI, 2004), but here we present a different proof of this result.

Proposition 4.1.3. Let X be a Tychonoff space. Then the following assertions are equivalent:
i) The space X contains a dense Čech-complete subspace.
ii) Player II has a complete (stationary) winning strategy in $BM(X)$.

Proof.
$i \Rightarrow ii$) Let Y be a dense Čech-complete subspace of X.
Since Y is a regular space and by Proposition 2.2.6, there is $\{\mathscr{C}_n\}_{n\in\omega}$, a complete sequence of open covers, with the following properties:
1) $\mathscr{C}_0 = \{Y\}$.
2) For each $V \in \mathscr{C}_n$ there is $W \in \mathscr{C}_{n-1}$ such that $\overline{V} \subset W$.
3) For each $U \in \tau \setminus \{\emptyset\}$ and for every $n \in \omega$ there is $V_n \in \mathscr{C}_n$, such that $\overline{V_n} \subset U$.

For each open set $U \neq \emptyset$ we define

$$\lambda(U) = sup\{n \in \omega;\ U \subset V \text{ for some } V \in \mathscr{C}_n\}_{n\in\omega}$$

Note that if $V \subset W$ then $\lambda(V) \geq \lambda(W)$.

Now we will define a stationary strategy δ for Player II.
Let V be a non empty open set. If $\lambda(V) = \infty$ then we set $\sigma(\langle V \rangle) = W$ such that $\overline{W} \subset V$.
If $\lambda(V) < \infty$. We set $\sigma(\langle V \rangle) = W$ such that $W \in \mathscr{C}_{\lambda(U)+1}$ and $\overline{W} \subset V$.
A σ-game have only two possibilities, or every V_n played by Player I is such that $\lambda(V_n) < \infty$ or there is an $m \in \omega$ such that $\lambda(V_n) = \infty$ for all $n \geq m$.
In both cases, let $\langle V_0, W_0, V_1, W_1 \ldots \rangle$ be a σ-game. Note that $\{W_n\}_{n\in\omega}$ is a family with the finite intersection property and $\overline{W}_n \subset W_{n+1}$. Then $\bigcap_{n\in\omega} W_n$ is a closed set contained in some open set of each cover $\mathscr{C}_n$. By definition of complete sequence of open covers and by Proposition 2.2.2, $\bigcap_{n\in\omega} W_n$ is a non empty compact set. And since $\overline{W}_n \subset W_{n+1}$, $\{W_n\}_{n\in\omega}$ is a local base of neighborhoods of $\bigcap W_n$.
In this way we prove that in every Čech-complete space Y, Player II has a complete stationary winning strategy in $BM(Y)$.

For any V open set in Y we can set an open set V' in X, such that $V' \cap Y = V$. Now we can define a strategy σ' for Player II by doing the following: $\sigma'(\langle W \rangle) = \sigma(\langle W \cap Y \rangle)'$ and by Lemma 4.1.2, σ' is a complete stationary strategy for Player II in $BM(X)$.

$ii \Rightarrow i)$ Let σ be a Player II's winning strategy for $BM(X)$.
Let $F_0 = \{\sigma(V),\ V \in \tau\}$. Let $\mathscr{C}_0 \subset F_0$ be a maximal collection of pairwise disjoint sets.
Then $\bigcup_{V\in\mathscr{C}_0} V$ is dense in X.
For every $V \in \mathscr{C}_0$ we set a maximal collection of pairwise disjoint of the form $\{\sigma(W),\ W \subset V\}$.
Let $\mathscr{C}_1$ be the union of all collections define above. Then $\bigcup_{V\in\mathscr{C}_1} V$ is dense in X.
Analogously, we can define a sequence $\{\mathscr{C}_n\}_{n\in\omega}$ of collection of pairwise disjoint open sets such that $\bigcup_{V\in\mathscr{C}_n} V$ is dense in X. Note that, for any $n \in \omega$, $\bigcup_{V\in\mathscr{C}_{n+1}} V \subset \bigcup_{V\in\mathscr{C}_n} V$.

Let

$$Y = \bigcap_{n\in\omega} \bigcup_{V\in\mathscr{C}_n} V$$

Since $II \uparrow BM(X)$, by Proposition 4.1.1, X is a Baire space. Therefore Y is dense in X.

For all $n \in \omega$, we define $\mathscr{C}_n^* = \{V \cap Y,\ V \in \mathscr{C}_n\}_{n\in\omega}$. Note that $\{\mathscr{C}_n^*\}_{n\in\omega}$ is a countable collection of open covers of Y.

Let u be an ultrafilter in Y such that $u \cap \mathscr{C}_n^* \neq \emptyset$ for all $n \in \omega$.
Let $\sigma(\langle V_0 \rangle) \cap Y$ in $u \cap \mathscr{C}_0^*$. Then there is a $\sigma(\langle V_1 \rangle) \cap Y$ in $u \cap \mathscr{C}_1^*$ such that $V_1 \subset \sigma(\langle V_0 \rangle)$. In this way we have a sequence $\{\sigma(\langle V_n \rangle) \cap Y\}_{n \in \omega}$ such that $V_{n+1} \subset \sigma(\langle V_n \rangle)$.

Since σ is a complete winning strategy and $\langle V_0, \sigma(V_0), V_1, \sigma(V_1), V_2 \ldots \rangle$ is a σ-game, we have that $\bigcap \sigma(\langle V_n \rangle) = K$ is a compact. Note that $\sigma(\langle V_n \rangle) \subset \bigcup_{V \in \mathscr{C}_n} V$. Therefore $K \subset Y$.

Let $\overline{u}$ be an ultrafilter in βY containing u. Then $\overline{u} \searrow x$. Suppose that $x \notin K$. Then there is a $\sigma(\langle V_n \rangle)$ such that $x \notin \overline{\sigma(\langle V_n \rangle)}$, contradiction.

In conclusion, for any ultrafilter u such that $u \cap \mathscr{C}_n^* \neq \emptyset$ for all $n \in \omega$, there is $x \in \bigcap_{V \in u} \overline{V} \neq \emptyset$. Therefore Y is Čech-complete. □

In the above result, if in addition to requiring a stationary winning strategy, we add a condition to the target set, we can get a fully metrizable dense subset, instead of just Čech complete, as indicated by the following result.

Proposition 4.1.4. Let X be a Tychonff space. Then the following assertions are equivalent:
i) The space X contains a dense, completely metrizable subspace.
ii) Player II has a complete stationary winning strategy σ in the game $BM(X)$ such that for every σ-game p, the target set $T(p)$ is a singleton.

Proof.
$i \Rightarrow ii$) Let (Y,d) be a dense and complete subspace of X.
By Corollary 4.2.8 and Proposition 3.1.4 Player II has a stationary winning strategy in $Ch(Y)$. Let δ be such strategy.
Note that the open sets played by Player II following the strategy described in Proposition 3.1.4 form a complete sequence.
Let τ be the topology of X. Let $f : \tau \to Y$ be a function such that $f(U) \in U$ and $U|_Y = V|_Y \longrightarrow f(U) = f(V)$.
For each $V \in \tau|_Y$ we set an $V^* \in \tau$ such that $V = V^*|_Y$.
Let σ be a Player II's strategy for $BM(X)$ defined by $\sigma(\langle V \rangle) = V \cap \delta(\langle (V|_Y, f(V)) \rangle)^*$.
Let $p = \langle U_0, V_0, U_1, V_1 \ldots \rangle$ be a $\sigma-$game.
Since δ is a winning strategy in $Ch(Y)$ and $\delta(\langle (U_n|_Y, f(U_n)) \rangle) \subset \sigma(\langle U_n \rangle) = V_n$, we have that $\bigcap V_n \neq \emptyset$.

According to Lemma 4.1.2, since $\bigcap \delta(\langle (U_n|_Y, f(U_n)) \rangle) = \{x\}$ we have that $T(p) = \bigcap \sigma(\langle U_n \rangle) = \{x\}$. And $\{\sigma(\langle U_n \rangle)\}_{n \in \omega}$ is a local base for $x \in X$.

ii $\Rightarrow$ *i*) Similarly to the previous proposition we have a Čech-complete dense subset Y. Note that every $x \in Y$ is a target set for some σ-game (unique).
Now we define a metric in Y by doing the following:

$$d : Y \times Y \to \mathbb{R}$$

$$d(x,y) = \begin{cases} 0 & x = y \\ \frac{1}{n} & x \neq y \end{cases}$$

Where $n \geq 1$ is the least such that x and y are in different sets of $\mathscr{C}_n$.
Obviously $d(x,y) = d(y,x)$. Let $x,y,z \in Y$ such that $d(x,y) = 1/a$. Note that z can not be in the same set of x and y at same time. Therefore $d(x,z) \geq 1/a$ or $d(y,z) \geq 1/a$. Either way we have that $d(x,y) \leq d(x,z) + d(y,z)$.

Since each $V_n \cap Y$ belongs to a local base of neighborhoods of some $x \in Y$, we have that $\tau|_Y = \tau_d$.

Then (Y, τ_d) is a metric Čech-complete space and by Corollary 4.2.8, Y is completely metrizable.

□

Next we will see a characterization of those spaces where Player I has an advantage in the Choquet Game.

Proposition 4.1.5. Let X be a metric space. $I \uparrow Ch(X)$ if, and only if, X contains a copy Y of is $\mathbb{Q}$. such that Y is a G_δ set.

Proof.
($\Rightarrow$) Let σ be a stationary winning strategy for Player I.
With the same points x_s described in Proposition 3.1.5. We will considered $\{k_n^s\}_{n\in\omega}$ the sequence of natural numbers defined above such that $x_{s^\frown k_n^s} \to x_s$.
Now we define a family of open sets by doing the following:

$$A_n = \bigcup_{s \in T} B(x_s, 1/k_n^s)$$

Let $z \in \bigcap A_n$. Suppose that $z \neq x_s$ for all $s \in T$.
Since $z \neq x_0$, there is a k_n^0 such that $z \notin B(x_0, 1/k_n^0)$. Then there is an n_1 such that $z \in B(x_{0n_1}, k_0^{0n_1})$.
Since $z \neq x_{0n_1}$, there is a $k_n^{0n_1}$ such that $z \notin B(x_{0n_1}, 1/k_n^{0n_1})$. Then there is an n_2 such that $z \in B(x_{0n_1n_2}, k_0^{0n_1n_2})$.
Therefore, there is a sequence $\{n_i\}_{i\in\omega^*}$ such that $0n_1 \ldots n_p \in T$ and $z \in B(x_{0n_1\ldots n_p}, 1/n_{p+1})$ for all $p \in \omega^*$.

Since there is a game such that in the turn 0 Player II plays $B(x_0, 1/n_1)$ and in the turn p, Player II plays $B(x_{0n_1\dots n_p}, 1/n_{p+1})$, we have that

$$z \in \bigcap_{p\in\omega^*} B(x_{0n_1\dots n_p}, 1/n_{p+1}) = \emptyset(contradiction)$$

Therefore,

$$\bigcap A_n = \{x_s,\ s \in T\}$$

Then $\{x_s,\ s \in T\}$ is G_δ, countable and without isolate point. Therefore, by Sierpinski Theorem, is homeomorphic to $\mathbb{Q}$.

($\Leftarrow$) To simplify the writing we will consider $\mathbb{Q} \subset X$. Let $\mathbb{Q} = \bigcap V_n$, where V_n is open in X.

Let $U_n = V_n \setminus \{q_k, k < n\}$. We define σ to be a strategy for Player I by doing the following: $\sigma(\langle\rangle) = (U_0, q_0)$.

If Player II chooses V_0, then $\sigma(\langle V_0\rangle) = (U_1 \cap V_0, p_0)$. In general, $\sigma(\langle V_0 \dots V_n\rangle) = (U_{n+1} \cap V_n, p_n)$. Clearly, $\bigcap V_n \subset \bigcap U_0 = \emptyset$. Therefore, σ is a winning strategy. □

To conclude this section, we will see another application that allows us to characterize some spaces where $II \uparrow Ch(X)$. Before seeing such a result, it is necessary to establish some definitions.

Definition 4.1.6. We will say that a Tychonoff space X has a *Base of countable order* if there is a base $\mathscr{B}$ for X such that any sequence $\{B_n\}_{n\in\omega}$ of different members of $\mathscr{B}$ with $B_{n+1} \subset B_n$ is a local base for each point of $\bigcap B_n$. A base of countable order is *monotonically complete* if for each sequence $\{B_n\}_{n\in\omega} \subset \mathscr{B}$ such that $\overline{B_{n+1}} \subset B_n$ then $\bigcap B_n \neq \emptyset$.

Definition 4.1.7. Let (X, τ) be a Tychonoff space. A collection $U \subset \tau \setminus \{\emptyset\}$ is a *regular filter base* if for any $U, V \in U$ there is $W \in U$ such that $\overline{W} \subset U \cap V$. X is called *subcompact* if it has a base $B \subset \tau \setminus \{\emptyset\}$ such that every regular filter base $U \subset B$ has non empty intersection. Such base B is called *subcompact base*.

We will also consider an equivalent definition of bases of countable order, given in (WORRELL; WICKE, 1965).

Proposition 4.1.8. Let (X, τ) be a Tychonoff space. Then X has a base of countable order if , and only if, there is a sequence $\mathscr{B}_n$ of bases for X such that whenever the sets $B_n \in \mathscr{B}_n$ has $\bigcap B_n \neq \emptyset$, then $\{B_n\}_{n\in\omega}$ is a local base for each point of $\bigcap B_n$.

In (GRUENHAGE, 1984) we find another equivalence that will be useful in the proof of the theorem that we mentioned before.

Proposition 4.1.9. A Tychonoff space (X,τ) has a monotonically complete base of countable order if and only if there is a tree $(T,\sqsubset)$ with levels $T_0.T_1,T_2\ldots$ and a function $G:T\to\tau\setminus\{\emptyset\}$ such that:
a) $G(T_n)$ is a cover of X for each n.
b) If $t\in T_n$, then $G(t)=\bigcup\{G(p);p\in T_{n+1},p\sqsubset t\}$.
c) If t_0,t_1,t_2 is a branch of T then $S=\bigcap G(t_n)$ is non empty and $\{G(t_n)\}$ is a local base for each point of S.

With the above we are ready to prove the following result.

Theorem 4.1.10. Let (X,τ) be a Tychonoff space with a base of countable order, then:
a) X has a base of countable order monotonically complete.
b)X is subcompact.
c)Player II have a (stationary) winning strategy in $Ch(X)$.
are equivalent.

Proof.
a) $\Rightarrow$ *b)*
Let $\mathscr{B}$ be a base of countable order monotonically complete for X. Let $U\subset\mathscr{B}$ be a regular filter base. We will suppose that U does not have a minimal element (with respect to set inclusion). Because if U has a minimal element, then there is nothing to prove. Let $\{B_n\}_{n\in\omega}\subset U$ such that $\overline{B_{n+1}}\subset B_n$. Since $\{B_n\}_{n\in\omega}\subset\mathscr{B}$ and X is a Tychonoff space, then $\bigcap B_n=\{x\}$. If $x\notin B\in U$ there is $B'\in U$ such that $x\notin B'$. Therefore there is B_n such that $B_n\cap B'=\emptyset$, this contradicts the fact that U is a filter base.
b) $\Rightarrow$ *c)*
Let B be a subcompact base. Let (U,x) be a pair with $x\in U\in\tau$. We define a Player II's stationary strategy σ by doing the following: $\sigma(\langle U,x\rangle)\subset U$ such that $x\in\overline{\sigma(\langle U,x\rangle)}\subset U$. Let $(\langle(U_0,x_0),(U_1,x_1),(U_2,x_2),\ldots\rangle)$ be a game played with such strategy. Note that $\{\sigma(\langle U_n,x_m\rangle)\}$ is a regular filter base containing in $\mathscr{B}$. Therefore $\bigcap U_n\neq\emptyset$.
c) $\Rightarrow$ *a)*
Since X has a Base of Countable Order, there is a sequence $\{\mathscr{B}_n\}_{n\in\omega}$ as described in Proposition 4.1.8. Let σ be a winning strategy for Player II in $Ch(X)$. We will define $C_0=\{(V_0,x_0);x_0\in V_0\in B_0\}$. Let

$$C_1=\{(V_0,x_0,V_1,x_1);(V_0,x_0)\in C_0,x_1\in V_1\in B_1,V_1\subset\sigma(\langle(V_0,x_0)\rangle)\}$$

and, in general $C_n=\{(V_0,x_0\ldots V_n,x_n);(V_0,x_0\ldots V_{n-1},x_{n-1})\in C_{n-1},x_n\in V_n\in B_n,V_n\in\sigma(\langle(V_0,x_0)\ldots(V_{n_1},x_n$
Now define $T=\bigcup C_n$. Let $t=(V_0,x_0,\ldots V_n,x_n)$ and $p=(W_0,y_0,\ldots W_m,y_m)\in T$. We define the relationship $\sqsubset$ in T by doing the following: $t\sqsubset p$ if and only if $m\geq n$ and $V_i=W_i$, $x_i=y_i$ for each $i\leq n$.
Note that T is a tree and C_n are its levels. We define $G:T\to\tau\setminus\{\emptyset\}$ such that $G(V_0,x_0,V_1,x_1\ldots V_n,x_n)=$

$\sigma(\langle (V_0, x_0), \ldots (V_n, x_n) \rangle)$.
Now we will see that this tree T with the function G meet the conditions given in Proposition 4.1.9.
Then $G(C_n)$ is a cover of X for each n and $G(t) = \bigcup\{G(p); p \in C_{n+1}, p \sqsubset t\}$. Finally, let $\{t_i\}_{i\in\omega}$ be a branch of T. Then there are $\{(V_i, x_i)\}_{i\in\omega}$ such that $t_i = (V_0, x_0, V_1, x_1 \ldots V_i, x_i)$. Therefore $G(t_i) = \sigma(\langle (V_0, x_0), (V_1, x_1) \ldots (V_i, x_i) \rangle)$. Since σ is a winning strategy we have that $\bigcap G(t_i) \neq \emptyset$. Let $x \in \bigcap G(t_i)$. Since $x \in V_i \in \mathscr{B}_i$ for each i and $\bigcap V_i = \bigcap G(t_i) \neq \emptyset$, then $\{V_i\}_{i\in\omega}$ is a local base of x. In addition, since $V_{i+1} \subset G(t_i) \subset V_i$ we can say that $G(t_i)$ is a local base of x as well.

In the following diagrams, we will see a summary of the most important results obtained related to the Choquet game in metric spaces.

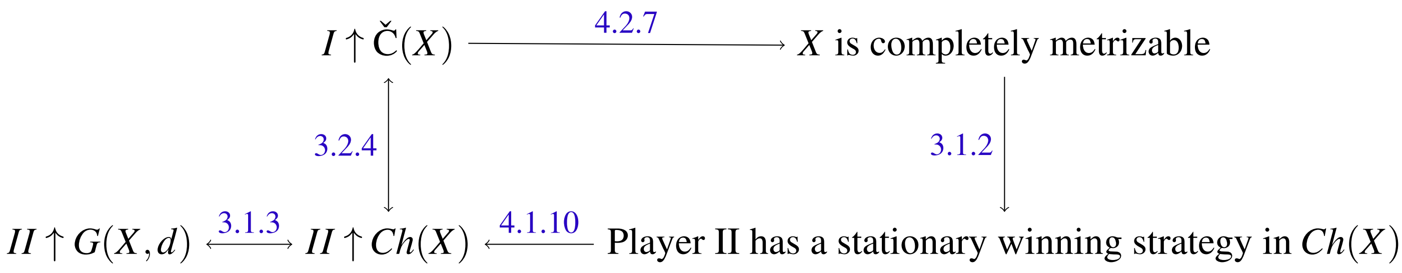

Figure 1

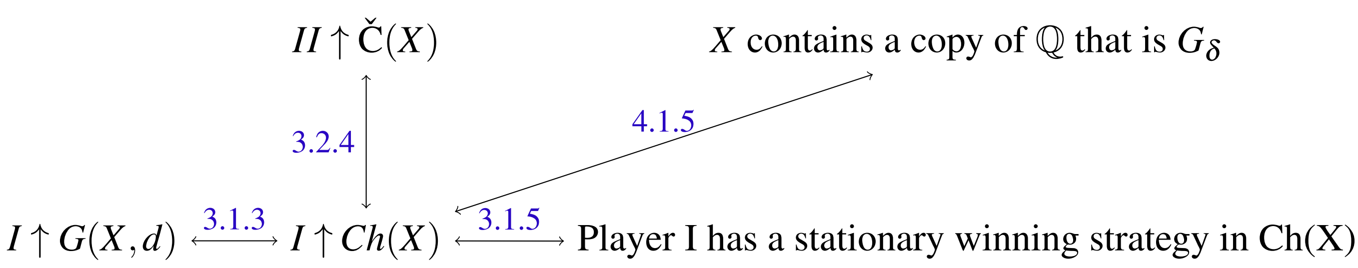

Figure 2

4.2 On the Čech Game

In this section we will see various results related to the Čech game.
Before starting to see these results, it is necessary to know some subcategories of paracompact spaces.

Definition 4.2.1. A space X is hereditary paracompact if every subset of X is paracompact.

Definition 4.2.2. A space X is ultraparacompact if every open cover admits an open refinement made by mutually disjoint open sets.

With the established definitions, we can prove the first proposition that allows us to go from Player I having an advantage in Čech Game to space being Čech complete.

Proposition 4.2.3. Let X be a Tychonoff space. If βX is hereditary paracompact and $I \uparrow \check{C}(X)$ then X is Čech-complete.

Proof.
Let $\mathscr{O} = \{W_k\}_{k\in I}$ be a family of open sets in X. For any $k \in I$, there is a $\tilde{W}_k$ open set in βX such that $\tilde{W}_k|_X = W_k$. We fix $\{\tilde{V}_p\}_{p\in J}$, a locally finite refinement of $\{\tilde{W}_k\}_{k\in I}$, and we define $\mathscr{O}^* = \{\tilde{V}_p|_X\}_{p\in J}$. Therefore $\mathscr{O}^*$ is a locally finite refinement of $\mathscr{O}$.

Let α be a winning strategy for Player I. We will define another winning strategy σ by doing the following:

We define $\sigma(\langle\rangle) = \alpha(\langle\rangle)^*$.

If $V_0 \in \sigma(\langle\rangle)$, there is a $W_0 \in \alpha(\langle\rangle)$ such that $V_0 \subset W_0$. Let $\delta(\langle V_0\rangle) = \{W \cap V_0, W \in \sigma(\langle W_0\rangle)\}$. Then we define $\sigma(\langle V_0\rangle) = \delta(\langle V_0\rangle)^*$.

If $V_{n+1} \in \sigma(\langle V_0 \dots V_n\rangle)$, there is a $W_{n+1} \in \alpha(\langle W_0 \dots W_n\rangle)$ such that $V_{n+1} \subset W_k$. Let $\delta(\langle V_0, \dots V_n\rangle) = \{W \cap V_{n+1}, W \in \sigma(\langle W_0 \dots W_n\rangle)\}$. Then we define $\sigma(\langle V_0 \dots V_{n+1}\rangle) = \delta(\langle V_0 \dots V_{n+1}\rangle)^*$.

Now we are going to show that σ is a winning strategy:
Let u be an ultrafilter such that $u \cap \sigma(\langle V_1 \dots V_n\rangle) \neq \emptyset$ for all $n \in \omega$. Then $V_n \in u$ and $V_n \subset W_n$. Then $u \cap \alpha(\langle W_1 \dots W_n\rangle) \neq \emptyset$ therefore $\bigcap_{F\in u} \overline{F} \neq \emptyset$.

If $\sigma(\langle V_0 \dots V_n\rangle) = \{A_i\}_{i\in I}$, define $\tilde{\sigma}(\langle V_0 \dots V_n\rangle) = \{\tilde{A}_i\}_{i\in I}$ a locally finite family of open sets in βX such that $\tilde{A}_i|_X = A_i$.

Now we define a family of open sets in βX containing X by doing the following:
$\tilde{\mathscr{C}_0} = \tilde{\sigma}(\langle\rangle)$.

$\tilde{\mathscr{C}_1} = \bigcup_{V_0\in\sigma(\langle\rangle)} \tilde{\sigma}(\langle V_0\rangle)$.
$\tilde{\mathscr{C}_2} = \bigcup_{V_0\in\sigma(\langle\rangle)} \bigcup_{V_1\in\sigma(\langle V_0\rangle)} \tilde{\sigma}(\langle V_0, V_1\rangle)$
$\vdots$
$\tilde{\mathscr{C}_n} = \bigcup_{V_0\in\sigma(\langle\rangle)} \bigcup_{V_1\in\sigma(\langle V_0\rangle)} \bigcup_{V_2\in\sigma(\langle V_0,V_1\rangle)} \cdots \bigcup_{V_{n-1}} \tilde{\sigma}(\langle V_0, \dots V_{n-1}\rangle)$.

Now we are going to see that:

$$\bigcap_{n\in\omega} (\bigcup \tilde{\mathscr{C}_n}) = X$$

We already know that $X \subset \bigcap_{n\in\omega}(\bigcup \tilde{\mathscr{C}_n})$.

Let $y \in \bigcap_{n\in\omega}(\bigcup \tilde{\mathscr{C}_n})$.

We are going to consider F an ultrafilter in X containing all restriction neighborhoods of y in βX at X.

Since $\tilde{\mathscr{C}_0}$ is locally finite, there are only a finite $\{\tilde{V}_k\}_{k=1\dots n}$ such that $y \in \tilde{V}_k$.

Now we will prove that there are only finitely many elements of $\tilde{\mathscr{C}_1}$ that contain y.

1° Let $W \in \tilde{\mathscr{C}_1}$ be such that $y \in W$ There is m such that $W \in \tilde{\sigma}(\langle V_m\rangle)$ then $y \in W \subset \tilde{V}_m$, therefore $\tilde{V}_m \in \{\tilde{V}_k\}_{k=1\dots n}$.

2° Let

$$A_k = \{W \in \tilde{\sigma}(\langle V_k\rangle); y \in W\}$$

Since for $k = 1\dots n$ we have that $\tilde{\sigma}(\langle V_k\rangle)$ is locally finite, then A_k is finite.

3°

$$\{W \in \tilde{\mathscr{C}_1}; y \in W\} = \bigcup_{k=1\dots n} \{W \in \tilde{\sigma}(\langle V_k\rangle); y \in W\} = \bigcup_{k=1\dots n} A_k \text{ is finite.}$$

Analogously we have that for every $n \in \omega$, there are only finitely many elements of $\tilde{\mathscr{C}_n}$ that contain y.

Therefore, we have a family of open sets that contain y. That family is an infinite tree with finite branching, and by König's lemma there is an infinite branch $\tilde{V}_0 \supset \tilde{V}_1 \supset \tilde{V}_2 \dots$ where

$\tilde{V}_{n+1} \in \tilde{\sigma}(\langle V_1 \dots V_n \rangle)$.

Now it is clear that $V_n \in F$ for all $n \in \omega$, and since σ is a winning strategy therefore $\bigcap_{f\in F} cl_X f \neq \emptyset$.

If $y \notin X$, for every $x \in X$ there is an open neighborhood of y in βX, V_x, such that $x \notin cl_{\beta X} V_x$, and therefore $x \notin cl_X(V_x|_X) \in F$, this is $\bigcap_{f\in F} cl_X f = \emptyset$, contradiction.

□

Before presenting the following proposition, we will see a property of ultraparacompact spaces that will be useful in the proof.

Lemma 4.2.4. Let X be an ultraparacompact space, if $\{V_i\}_{i\in I}$ is an open cover made by mutually disjoint sets. For any $i \in I$ every open cover of V_i has an open refinement made by mutually disjoint sets.

Proof.
Let $\{A_j\}_{j\in J}$ an open cover of $V_0 = \bigcup_{j\in J} A_j$. Then $\{A_j\}_{j\in J} \cup \{V_i\}_{i\in I\setminus\{0\}}$ is an open cover of X. Then there is a refinement $\{W_p\}$ made by mutually disjoint sets. Let $\mathscr{R} = \{W_p; W_p \cap V_0 \neq \emptyset\}$, $\mathscr{R}$ is a cover of V_0 made by mutually disjoint sets, and therefore is an open refinement of $\{A_j\}_{j\in J}$ made by mutually disjoint sets.

Similarly, we can prove that every open cover cover of $V_1 \in \mathscr{R}$, admits an open refinement made by mutually disjoint sets.

In general every clopen set of an ultraparacompact space is ultraparacompact itself. □

With this we are ready to present and prove the following proposition.

Proposition 4.2.5. If the space X is ultraparacompact and $I \uparrow \check{C}(X)$ then X is Čech-complete.

Proof.
Let σ^* be a winning strategy for Player I. We will define a winning strategy σ where every response is an open cover made by mutually disjoint sets.

Since $\sigma^*(\langle\rangle)$ is an open cover of X, there is a refinement made by disjoint sets. We set that refinement and defined as $\sigma(\langle\rangle)$.

For all $V \in \sigma(\langle\rangle)$, we fix $W_V \in \sigma^*(\langle\rangle)$ such that $V \subset W_V$.
Let $V_0 \in \sigma(\langle\rangle)$. Since $\sigma^*(\langle W_{V_0}\rangle)$ generates an open cover of V_0 and since V_0 is ultraparacompact by Lemma 4.2.4 there is a refinement made by mutually disjoint sets. We fix that refinement and define it as $\sigma(\langle V_0\rangle)$.

In general, for any $V_{n+1} \in \sigma(\langle V_0 \dots V_n\rangle)$ we fix $W_{n+1} \in \sigma^*(\langle w_0 \dots W_{n+1}\rangle)$ such that $V_{n+1} \subset W_{n+1}$. Since $\sigma^*(\langle W_0 \dots W_{n+1}\rangle)$ generates an open cover of V_{n+1} and since V_n is ultraparacompact by Lemma4.2.4 there is a refinement made by mutually disjoint sets. We set that refinement as $\sigma(\langle V_0 \dots V_{n+1}\rangle)$.

Now we will define a family of open covers of X.
$\mathscr{C}_0 = \sigma(\langle\rangle)$.
$\mathscr{C}_1 = \bigcup_{V_0 \in \mathscr{C}_0} \sigma(\langle V_0\rangle)$.
$\mathscr{C}_2 = \bigcup_{V_0 \in \mathscr{C}_0} \bigcup_{V_1 \in \sigma(\langle V_0\rangle)} \sigma(\langle V_0, V_1\rangle)$
$\vdots$.
$\mathscr{C}_n = \bigcup_{V_0 \in \mathscr{C}_0} \bigcup_{V_1 \in \sigma(\langle V_0\rangle)} \bigcup_{V_2 \in \sigma(\langle V_0, V_1\rangle)} \cdots \bigcup_{V_{n-1}} \sigma(\langle V_0, \dots V_{n-1}\rangle)$.

Let u be an ultrafilter such that $u \cap \mathscr{C}_n \neq \emptyset$ for all $n \in \omega$.
Let $\{V_0\} = u \cap \mathscr{C}_0$ (V_0 is unique because the other elements are disjoint of it.)
Let $\{V_1\} = u \cap \mathscr{C}_1$. Then $V_1 \in \sigma(\langle V_0\rangle)$ because the other responses only have open sets disjoint of V_0.
Analogously if $\{V_n\} = u \cap \mathscr{C}_n$ then $V_n \in \sigma(\langle V_1 \dots V_{n-1}\rangle)$.

Therefore there is a play $[\sigma(\langle\rangle), V_0, \sigma(\langle V_0 >, V_1, \sigma(\langle V_0, V_1\rangle) \dots]$, therefore, since $I \uparrow \check{C}(X)$ we have that $\bigcap_{F \in u} \overline{F} \neq \emptyset$. $\square$

With the following result, we will finish those that, by adding conditions, allow us to affirm that a space is Čech complete.

Proposition 4.2.6. Let X be a Tychonoff space where $I \uparrow \check{C}(X)$. If for every compact $K \in \beta X \setminus X$ there is a compact $\tilde{K} \in \beta X \setminus X$, such that $K \subset \tilde{K}$ and $\tilde{K}$ is a G_δ set, then X is a Čech-complete space.

Proof.
Let$\mathscr{C}$ be a collection of compact subsets of $\beta X \setminus X$, and let $\mathscr{K}$ be a collection of G_δ compact subsets of $\beta X \setminus X$.
For each $K \in \mathscr{C}$ let us set a $\tilde{K}$ in $\mathscr{K}$.
For each $\tilde{K}$ let us set $\mathscr{V}_K = \{V_n(K)\}_{n \in \omega}$ such that $\bigcap V_n(K) = \tilde{K}$.

Since $I \uparrow \check{C}(X)$, then $I \uparrow CO(\beta X \setminus X)$.(COSTA, 2019)

If in $CO(\beta X \setminus X)$ we add a restriction for Player II, Player I still wins the game.
Restriction: if Player I chooses K, Player II can response only with elements of $\mathscr{V}_K$.

Playing with that restriction, let a winning strategy δ for Player I in $CO(\beta X \setminus X)$. We define:
$\mathscr{A}_0 = \{\delta(\langle\rangle)\} = \{K\}$,
$\mathscr{A}_1 = \{\delta(\langle V_{m_1}(K)\rangle)\}_{m_1\in\omega} = \{K^{m_1}\}_{m_1\in\omega}$,
$\mathscr{A}_2 = \{\delta(\langle V_{m_1}(K), V_{m_2}(K^{m_1})\rangle\rangle)\}_{m_1,m_2\in\omega} = \{K^{m_1 m_2}\}_{m_1,m_2\in\omega}$, and in general
$\mathscr{A}_n = \{\delta(\langle V_{m_1}(K), V_{m_2}(K^{m_1}), V_{m_3}(K^{m_1 m_2})\dots V_{m_n}(K^{m_1\dots m_{n-1}})\rangle\rangle)\}_{m_i\in\omega}$.

Let $\mathscr{A} = \bigcup \mathscr{A}_n$. Then $\mathscr{A}$ is countable.

Let us suppose there is $y \in \beta X \setminus X$ such that $y \notin \bigcup_{K\in\mathscr{A}} \tilde{K}$. Then for any $K \in A$ there is a $p \in \omega$ such that $y \notin V_p(K)$. Then Player II can respond every choose K_n of Player I with the open set $V_{p_n}(K_n)$ which does not contain y and consequently Player II wins the game, that is a contradiction with $I \uparrow CO(\beta X \setminus X)$.

In conclusion, $\beta X \setminus X = \bigcup_{K\in\mathscr{A}} \tilde{K}$, hence, $\beta X \setminus X$ is $\sigma-$compact and by Proposition 2.1.2 and Theorem 2.3.4, X is Čech-complete. □

With what has been developed so far, we are able to demonstrate the reciprocal of Proposition 3.1.2.

Proposition 4.2.7. Let (X,d) be a metric space. If $I \uparrow \check{C}(X)$ then X is completely metrizable.

Proof.
Let Y be the completion of X. We will prove that X is a G_δ subset of Y. That will ensure what we are looking for.
Since Y is a metric space, Y is a hereditary paracompact.
Analogously at Proposition 4.2.3, we can define a cover of X by opens in Y and prove that intersection of those open is X. We thus prove what we want.

□

Corollary 4.2.8. Let (X,d) be a metric space. Then:

•X is completely metrizable.
•$I \uparrow \check{C}(X)$.

$\bullet II \uparrow G(X,d)$.
$\bullet II \uparrow Ch(X)$.

are equivalent.

Before presenting the last theorem of this text, which allows us to characterize those spaces where Player I always wins the Čech game, we are going to make some observations that will be useful when carrying out the demonstration.

Observations:
• Note that for an arbitrary winning strategy, every move from Player I can be sorted in well order.
• If (in each Player I's response) we remove the elements that are contained in the union of the previous ones, we can ensure that what we are still left with is a winning strategy.

Taking these observations into account, we can establish a winning strategy for Player I with the following characteristics.
Let α be Player I's winning strategy such that:
i) Every Player I respond's is well ordered.
ii) Let $\{W_i\}_{i\in I}$ be a Player I's respond, then for each $i \in I$, $W_i \not\subset \bigcup_{j<i} W_j$.

With all these considerations, we are now ready to prove the following characterization mentioned in (TELGARSKY, 1983), and that we will prove here using the properties of Čech game that we have described throughout this text.

Theorem 4.2.9. The following conditions are equivalent:
$i)X$ is a Sieve complete space.
$ii)X$is an open image of a (paracompact) Čech complete space.

Proof. $i) \Rightarrow ii)$
Since X is sieve complete, then $I \uparrow \check{C}(X)$. Now we will define a family $\{\mathscr{C}_n\}_{n\in\omega}$, of covers of X, as follows:
$\mathscr{C}_0 = \alpha(\langle\rangle)$.
$\mathscr{C}_1 = \bigcup_{V_0\in\alpha(\langle\rangle)} \alpha(\langle V_0\rangle)$.
$\mathscr{C}_2 = \bigcup_{V_0\in\alpha(\langle\rangle)} \bigcup_{V_1\in\alpha(\langle V_0\rangle)} \alpha(\langle V_0, V_1\rangle)$.
$\mathscr{C}_n = \bigcup_{V_0\in\alpha(\langle\rangle)} \cdots \bigcup_{V_{n-1}\in\alpha(\langle V_0 \dots V_{n-2}\rangle)} \alpha(\langle V_0 \dots V_{n-1}\rangle)$.

For each $V_0 \in \mathscr{C}_0$ we can associate with each element W of $\alpha(\langle V_0\rangle)$, an ordered pair (V_0, W).

For each $\alpha(\langle V_0, V_1\rangle)$ we can associate each $W \in \alpha(\langle V_0, V_1\rangle)$ with (V_0, V_1, W). We can continue analogously with each $\mathscr{C}_n$.
Now, we can see $\mathscr{C}_n$ as a subset of τ^{n+1}.
Since $\mathscr{C}_0 = \alpha(\langle\rangle)$ and every $\alpha(\langle V_0\rangle)$ are well ordered, we can induce a well order in $\mathscr{C}_1$ by doing the following: $(V_1, W_1) <_1 (V_2, W_2)$ if $V_1 <_0 V_2$ or $V_1 = V_2$ and $W_1 <_{V_1} W_2$.
Analogously, we can define a well order in every $\mathscr{C}_n$.
Now, for each n, we will remove from $\mathscr{C}_n$ all elements that meet the condition $V \subset \bigcup_{W <_n V} W$.

Let $\mathscr{E}$ denote the set of all sequences $\{(V_0,x),(V_1,x),(V_2,x)\ldots\}$ such that:
(i) $V_n \in \mathscr{C}_n$.
(ii) $(V_0, V_1, V_2 \ldots)$ is a run of $\check{\mathrm{C}}(X)$.
(iii) $x \in \bigcap_{n\in\omega} V_n$.

Let us define $D(n,V,U)$ to be a subset of $\mathscr{E}$ such that:

$$D(n,V,U) = \{\{(V_0,x),(V_1,x),(V_2,x)\ldots\} \in \mathscr{E}, x \in U \subset V = V_n\}$$

Note that, if $n' > n$ and $D(n,V,U) \cap D(n',V',U') \neq \emptyset$, then

$$D(n,V,U) \cap D(n',V',U') = D(n',V',U \cap U')$$

Let us define $\mathscr{D}$ to be the collection of all $D(n,V,U)$. Let Ψ be the collection of all arbitrary unions of elements of $\mathscr{D}$. Then Ψ is a topology in $\mathscr{E}$.

i) $(\mathscr{E}, \Psi)$ is a Hausdorff space.
Let $x' = \{(V_0,x),(V_1,x),(V_2,x)\ldots\}$ and $y' = \{(V_0',y),(V_1',y),(V_2',y)\ldots\}$ be different points of $\mathscr{E}$. If there is an n such that $V_m' = V_m$ for $m < n$ and $V_n' = V_n$. Then $D(n,V_n,V_n)$ contains x' but not y' and $D(n,V_n',V_n')$ contains y' but not x'.
If $V_n' = V_n$ for all n. Then $x \neq y$, therefore exists disjoint open sets V_x, V_y, such that $x \in V_x$ and $y \in V_y$. Then $D(0,V_0,V_0 \cap V_x)$ contains x' but not y' and $D(0,V_0,V_\cap V_y)$ contains y' but not x'.

Let $F : (\mathscr{E},\Psi) \longrightarrow (X,\tau)$ defined by $F(\{(V_0,x),(V_1,x),(V_2,x)\ldots\}) = x$.

ii) F is an open continuous map.
Let $D(n,V,U) \in \mathscr{D}$. Then $y \in F(D(n,V,U))$ if, and only if, there is a $W \in \mathscr{C}_{n+1}$ such that $y \in W \cap U \subset V$. Therefore, $F(D(n,V,U)) = U \cap \bigcup\{W \in \mathscr{C}_{n+1}, W \subset V\} \in \tau$.
Let $V \in \tau$, $F^{-1}(V) = \bigcup\{D(n,V_n,V_n \cap V)\} \in \Psi$.

Let us define an equivalence relation $\simeq$ in $\mathscr{E}$ such that

$$\{(V_0,x),(V_1,x),(V_2,x)\ldots\} \simeq \{(W_0,y),(W_1,y),(W_2,y)\ldots\} \longleftrightarrow V_n = W_n\ \forall n$$

In $\mathscr{E}/\simeq$, let $\overline{V} = [\{(V_0,x),(V_1,x),(V_2,x)\ldots\}]$. We define a metric d by doing the following: $d(\overline{V},\overline{W}) = 1/(n+1)$ where $V_i = W_i$ for $i < n$ and $V_n \neq W_n$. Clearly d is a complete metric.

Finally we will see that $\pi : (\mathscr{E},\psi) \longrightarrow (\mathscr{E}/\simeq, d)$ is a perfect continuous map and by Proposition 2.3.7 we can conclude the proof.

i)π is continuous.
Let $\overline{V} = [\{(V_0,x),(V_1,x),(V_2,x)\ldots\}]$. Let $B(\overline{V},1/(n+1))$ be an open ball in $\mathscr{E}/\simeq$. Then $\{(W_0,y),(W_1,y),(W_2,y),\ldots\} \in \pi^{-1}(B(\overline{V},1/(n+1)))$ if, and only if, $W_i = V_i$ for $i \leq n$. Therefore, $\{(W_0,y),(W_1,y),(W_2,y),\ldots\} \in D(n,V_n,V_n) = \pi^{-1}(B(\overline{V},1/(n+1)))$.

ii) π is closed.
Let C be a closed in $(\mathscr{E},\psi)$. Let $\overline{V} = [\{(V_0,x),(V_1,x),(V_2,x)\ldots\}] \notin \pi(C)$. Then for each $y \in \bigcap V_n$, there is n_y such that $[\{(V_0,x),(V_1,x),(V_2,x)\ldots\} \in D(n_y,V_{n_y},U_y)$ and $D(n_y,V_{n_y},U_y)\cap C = \emptyset$. Since $\{U_y\}_{y\in\bigcap V_n}$ is an open cover of $\bigcap V_n$ and $\bigcap V_n$ is a compact set, there is a finite subcover $\{U_y\}_{y\in F}$. Additionally, since $\{V_n\}_{n\in\omega}$ is a local basis for $\bigcap V_n$, there is $p \geq n_y$ $\forall y \in F$, such that $V_p \subset \bigcup_{y\in F} U_y$.
Now, suppose that $B(\overline{V},1/(p+1))\cap\pi(C) \neq \emptyset$. Then there is a $\{(W_0,a),(W_1,a)\ldots\} \in C$, such that $W_i = V_i$ for $i \leq p$. Since $a \in W_p = V_p$, we have that $a \in U_y$ for some $y \in F$. Then $\{(W_0,a),(W_1,a)\ldots\} \in D(n_y,V_{n_y},U_y)$ for some $y \in F$, contradiction. Therefore, $B(\overline{V},1/(p+1))\cap\pi(C) = \emptyset$. That is, $\pi(C)^c$ is open in $\mathscr{E}/\simeq$.

iii)$\pi^{-1}(\overline{V})$ is a compact set.
Let $\overline{V} = [\{(V_0,x),(V_1,x),\ldots\}]$. Then $\pi^{-1}(\overline{V}) = \{\{(V_0,x),(V_1,x),\ldots\}, x \in \bigcap_{n\in\omega} V_n\}$. Let $\{D(n_i,V_{n_i},U_i)\}_{i\in I}$ be an open cover of $\pi^{-1}(\overline{V})$. Then $\{U_i\}_{i\in I}$ is an open cover of $\bigcap_{n\in\omega} V_n$. Therefore, there is a finite open subcover $\{U_i\}_{i\in F}$.
Let $z \in \bigcap V_n$. Since $z \in U_i$, for some $i \in F$, then $\{(V_0,z),(V_1,z)\ldots\} \in D(n_i,V_{n_i},U_i)$. In conclusion, $\{D(n_i,V_{n_i},U_i)\}_{i\in F}$ is a finite subcover of $\pi^{-1}(\overline{V})$.

ii) $\Rightarrow$ *i)*
Let X be a Tychonoff space. Let Y be a Čech complete space and let $f : Y \longrightarrow X$ be an open continuous function. Since Y is Čech complete, by Theorems 3.2.1 and 3.2.9, we have that $II \uparrow SV(Y)$. Let γ be Player II's winning strategy in $SV(Y)$.
We will define a Player II's strategy σ in $SV(X)$ by doing the following:
Let (U_0,x_0) be Player I's first move in $SV(X)$. We set $y_0 \in f^{-1}(\{x_0\})$. We define $\sigma(\langle(U_0,x_0)\rangle) = f(V_0)$ where $V_0 = \gamma(\langle(f^{-1}(U_0),y_0)\rangle)$.
Let (U_1,x_1) be Player I's response. We set $y_1 \in f^{-1}(\{x_1\})$. We define $\sigma(\langle(U_0,x_0),(U_1,x_1)\rangle) = f(V_1)$ where $V_1 = \gamma(\langle(f^{-1}(U_0),y_0),(f^{-1}(U_1)\cap V_0,y_0)\rangle)$.

Analogously, in the turn n, let (U_n, x_n) be Player I's move. We set $y_n \in f^{-1}(\{x_n\})$ and define $\sigma(\langle (U_0, x_0), (U_1, x_1) \dots (U_n, x_n) \rangle) = f(V_n)$ where $V_n = \gamma(\langle (f^{-1}(U_0), y_0) \dots (f^{-1}(U_n) \cap V_{n-1}, y_n) \rangle)$. Let u be an ultrafilter in X containing $\{f(V_n)\}_{n \in \omega}$. Then $\mathscr{F} = \{f^{-1}(W), W \in u\}$ is a filter base in Y. Let $x \in W \cap f(V_n)$ where $W \in u$. There is $y \in V_n$ such that $f(y) = x \in W$. Therefore $y \in f^{-1}(W)$. Then $\{V_n\}_{n \in \omega} \cup \mathscr{F}$ is a filter base in Y. Since $II \uparrow SV(Y)$ then $\{V_n\}_{n \in \omega} \cup \mathscr{F}$ clusters. Therefore $\mathscr{F}$ clusters as well.

Since $\bigcap Cl_Y(f^{-1}(W)) \neq \emptyset$ we have that $\emptyset \neq \bigcap f^{-1}(Cl_X(W)) = f^{-1}(\bigcap Cl_X(W))$. Therefore, $\bigcap_{W \in u} Cl_X(W) \neq \emptyset$. Then $II \uparrow SV(X)$. □

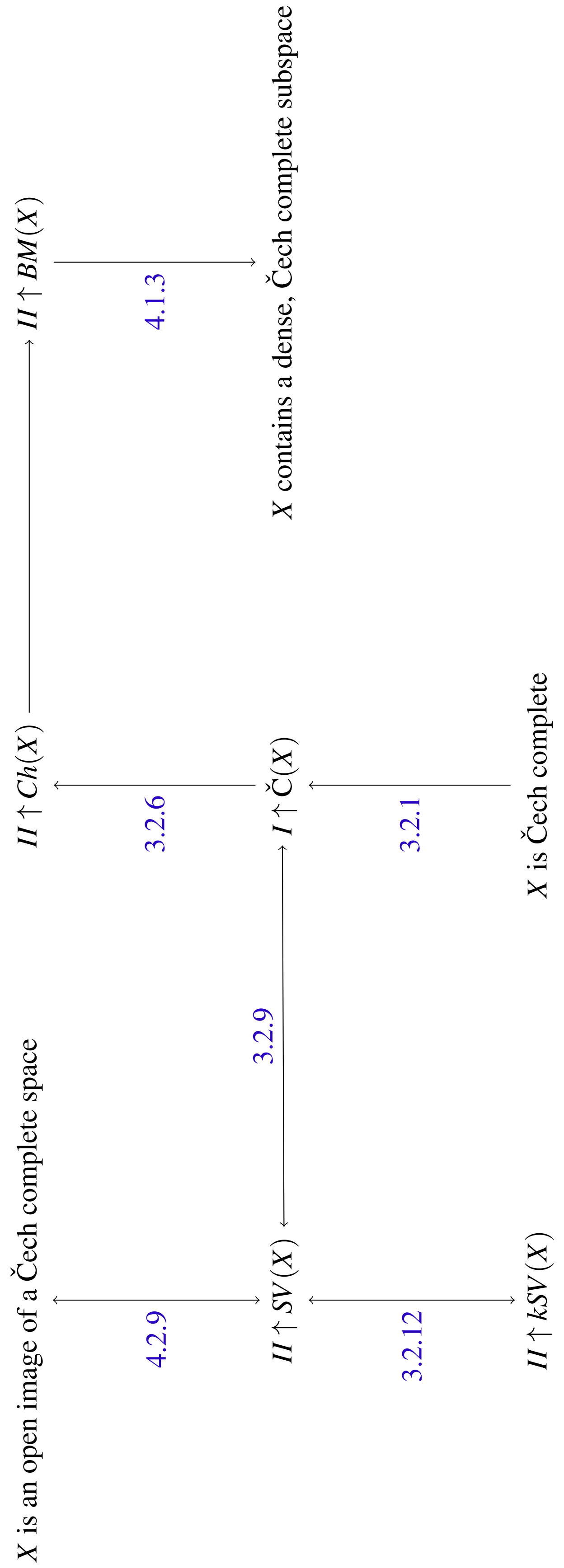
II ↑ BM(X)
4.1.3
X contains a dense, Čech complete subspace
II ↑ Ch(X)
3.2.6
I ↑ Č(X)
3.2.1
X is Čech complete
3.2.9
X is an open image of a Čech complete space
4.2.9
II ↑ SV(X)
3.2.12
II ↑ kSV(X)

Figure 3

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