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Obstruction theory, characteristic classes and applications

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Teoria de obstrução, classes características e aplicações

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP – como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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Este trabalho é dedicado à minha avó, Rosa,
que me ajudou a plantar muitas sementes,
mas que infelizmente não teve tempo
de me ver colher os frutos que delas nasceram.

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"Um coração é um fardo pesado."
(Sophie, no filme "O Castelo Animado", dirigido por Hayao Miyazaki)

RESUMO

MARTINS, E. B. C. **Teoria de obstrução, classes características e aplicações**. 2022. 264 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Este trabalho tem como objetivo estudar e demonstrar alguns dos principais resultados da Teoria de Obstrução, assim como apresentar algumas possíveis aplicações. A demonstração de tais resultados depende do desenvolvimento de diversos pré-requisitos ao longo do caminho, como as noções de homotopia livre e pontuada, H-grupos e H-cogrupos, grupos de homotopia e fibrados localmente triviais. Esse desenvolvimento culmina com a demonstração de que o problema de estender mapas e seções ao longo dos esqueletos de um CW-complexo é controlado por um invariante cohomológico. Esse resultado é então usado para construir as classes características associadas a um fibrado vetorial, e também para definir a obstrução local de Euler em um ponto de um espaço singular.

Palavras-chave: Teoria de Obstrução, classes características, homotopia, fibrados, obstrução local de Euler.

ABSTRACT

MARTINS, E. B. C. **Obstruction theory, characteristic classes and applications**. 2022. 264 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

The goal of this work is to study and prove some of the main results of Obstruction Theory, as well as to present some possible applications. The proof of these results depends on the development of several prerequisites along the way, like the notions of free and pointed homotopy, H-groups and H-cogroups, homotopy groups and locally trivial bundles. This development culminates in the proof that the problem of extending maps and sections over the skeletons of a CW-complex is controlled by a cohomological invariant. This result is then used to construct the characteristic classes associated with a vector bundle, and also to define the local Euler obstruction of a point in a singular space.

Keywords: Obstruction Theory, characteristic classes, homotopy, bundles, local Euler obstruction.

LIST OF FIGURES

LIST OF SYMBOLS

$\mathsf{C}(a,b)$ — Set of morphisms from an object a to an object b in the category C

$\mathsf{C}(a,g) : \mathsf{C}(a,x) \to \mathsf{C}(a,c)$ — Pushforward function along morphism $g : b \to c$

$\mathsf{C}(f,c) : \mathsf{C}(b,c) \to \mathsf{C}(a,c)$ — Pullback function along morphism $f : a \to b$

$a \times b$ — Categorical product of the objects a and b

$(f,g) : a \to b \times c$ — Morphism induced by $f : a \to b$ and $g : a \to c$

$f \times g : a \times b \to c \times d$ — Categorical product of morphisms $f : a \to c$ and $g : b \to d$

$a \sqcup b$ — Categorical coproduct of the objects a and b

$\langle f,g \rangle : a \sqcup b \to c$ — Morphism induced by $f : a \to c$ and $g : b \to c$

$f \sqcup g : a \sqcup b \to c \sqcup d$ — Categorical coproduct of morphisms $f : a \to c$ and $g : b \to d$

$\theta : F \Rightarrow G$ — Natural transformation between functors F, $G : \mathsf{C} \to \mathsf{D}$

Top — Category of spaces and maps

HoTop — Homotopy category

Top_* — Category of pointed spaces and pointed maps

HoTop_* — Pointed homotopy category

Grp — Category of groups and group homomorphisms

Ab — Category of abelian groups and group homomorphisms

$\mathrm{Map}(X,Y)$ — Space of maps from X to Y with the compact-open topology

$\mathrm{Map}_*(X,Y)$ — Space of pointed maps from (X,x_0) to (Y,y_0) with the compact-open topology

$[X,Y]$ — Set of homotopy classes of maps $X \to Y$

$f \simeq g$ — The maps f and g are homotopic

$[X,Y]_*$ — Set of pointed homotopy classes of pointed maps $(X,x_0) \to (Y,y_0)$

$f \simeq_* g$ — The pointed maps f and g are pointed homotopic

CX — Cone (resp. reduced cone) over a space X (resp. a pointed space (X,x_0))

ΣX — Reduced suspension of a pointed space (X,x_0)

ΩX — Loop space of pointed space (X, x_0)

$X \rtimes I$ — Reduced cylinder over a pointed space (X, x_0)

$X \vee Y$ — Wedge sum of pointed spaces (X, x_0) and (Y, y_0)

$X \wedge Y$ — Smash product of pointed spaces (X, x_0) and (Y, y_0)

$\lambda f : X \to \mathrm{Map}(Y, Z)$ — Exponential adjoint of a map $f : X \times Y \to Z$

$\lambda^* f : X \to \mathrm{Map}_*(Y, Z)$ — Pointed exponential adjoint of a pointed map $f : (X \wedge Y, *) \to (Z, z_0)$

$X \rightarrowtail Y$ — Some kind of inclusion or injection

$X \twoheadrightarrow Y$ — Some kind of quotient or surjection

$\pi_0(X)$ — Set of path-components of a space X

$\pi_0(X, x_0)$ — Pointed set of path-components of space X with distinguished element $[x_0]$

$\pi_n(X, x_0)$ — n-th homotopy group of pointed space (X, x_0)

$t_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ — Transport map induced by path $\gamma : I \to X$ from x_0 to x_1

$\mathrm{ct}_{X,y} : X \to Y$ — Map of type $X \to Y$ which is constant and equal to $y \in Y$ (sometimes the domain X is omitted)

CONTENTS

INTRODUCTION

The goal of this dissertation is to study and prove the main results of Obstruction Theory. Broadly speaking, Obstruction Theory is a subarea of Algebraic Topology concerned with studying the following type of problem: suppose we are given space E, B and X, a subspace $A \subseteq X$, and maps $f : A \to E$, $g : X \to B$ and $p : E \to B$ which fit together in the commutative square below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

Under what conditions can we find a diagonal map $h : X \to E$ such that the resulting diagram is still commutative?

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

If such a map h exists, then the commutativity conditions imply the two following equalities:

1. $h|_A = f$;
2. $p \circ h = g$.

The first of these says that h is an *extension* of f, while the second says that h is a *lift* of g through p. Due to this, problems of this type are called *extension-lifting problems.*

There is no hope of solving a completely general extension-lifting problem, so in this work we will restrict ourselves to two classes of problems:

1. We consider the extension-lifting problem posed by the square below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & X \times Y \arrow[d, "p_1"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

The map $f : A \to X \times Y$ can simply be identified with a map $A \to Y$, and if $h : X \to X \times Y$ solves the problem, then it can be identified with a map $X \to Y$

extending the given map f. In other words, this particular instance of the extension-lifting problem is concerned with extending a map from a subspace to the whole space.

2. We consider the extension-lifting problem posed by the square below,

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, tail] & E \arrow[d, "p"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

where $p : E \to B$ is a *locally trivial bundle*. The commutativity condition means that $s : A \to E$ is a *section* of p over A, and if there is a diagonal map $S : X \to E$ solving the problem, then it will extend s to the whole space X, and *still* be a section of p. In other words, this particular extension-lifting problem is concerned with extending partial sections of bundles to the whole space.

In both of these classes of problems, we will also assume that the pair (X, A) is a relative CW-complex. It is not surprising that this class of spaces is useful for Algebraic Topology, since in some sense they carry two potential algebraic structures:

- the cells that are attached during the construction of the complex carry *homotopical structure* when mapped to other spaces;
- the way higher-dimensional cells intersect lower-dimensional cells carries (co)homological structure.

In the end, we will see that the possibility of solving these two classes of extension-lifting problem is controlled by an algebraic invariant which involving both cohomological and homotopical structures. This is one of the main results of Chapter 7.

Of course, in order to get there, we must develop a bunch of auxiliary theory. This dissertation in particular is mainly concerned with developing the homotopical structure necessary to understand the main results of Obstruction Theory. We start from the most basic definitions of Homotopy Theory, and slowly develop more concepts until we are able to define the homotopy groups, and also obtain some computational tools for studying them.

In the end, after studying the obstruction-theoretic results, we briefly present two possible applications of these ideas: one of them is a construction of the infamous characteristic classes of vector bundles, and the other is the definition of the local Euler obstruction at a point of a singular space, which is in some sense connected to the theory of characteristic classes too, although in the substantially different context of singular spaces.

About the text

The development of ideas in the text is more or less linear, with each chapter building up on the previous ones. It has a somewhat categorical flavor to it, with a frequent use of categorical ideas like adjoint functors, natural transformations, pullbacks and pushouts, group and cogroup objects, and so on. This categorical approach is especially evident when we have to construct a map of some kind. In these cases, the actual definition of the map usually comes only after some “categorical preparation” which consists of understanding how the objects in question are built: are they products? Coproducts? Pushouts? Pullbacks? What are the morphisms coming into or out of these objects? I do not think this is the most succinct approach, but it does help to have a sort of “organizational principle” for constructing maps, and in a few occasions it also helps us circumvent some silly topological problems.

This text once contained a sizeable appendix on the basics of Category Theory, which got eventually removed due to size limitations. Nonetheless, it still contains an appendix on group and cogroup objects, two concepts which are not always included in introductory discussions. As for the rest of the categorical concepts used, some of them are briefly described and discussed in the text when they are first needed, and on these occasions we also give pointers to easily accessible references on the subject. There is also a list of symbols which also contains some of the categorical symbols that appear more frequently in the text.

Now we give a brief overview of each of the chapters. See the individual introductions to each of them for more detailed overviews. The first chapter is of a preliminary nature, being devoted to the study of some properties of the category spaces, as well as to the introduction of an important class of spaces for us: the CW-complexes. The second chapter is where we really start our study of Homotopy Theory, we introduce the basic notions, discuss the “algebraic properties” of homotopy, and prove the basic, but important, result connecting null homotopic maps on the sphere to extensions to the disk.

The third chapter is concerned with pointed spaces. In it, we study some properties of the category of pointed spaces and pointed maps, and introduce important constructions like the wedge sum, the smash product, the reduced suspension and the loop space. Chapter 4 then uses some of these concepts to adapt the results of unpointed homotopy theory to the pointed case.

In chapter 5 we finally define the homotopy groups of a pointed space using the machinery of group and cogroup objects contained in the appendix. The application of these concepts in Homotopy Theory is in the form of H-groups and H-cogroups, and their construction occupies a good deal of the chapter.

Chapter 6 introduces locally trivial bundles. These maps will be important for us for two reasons: studying their sections is one of our goals in Obstruction Theory, and they interact nicely with homotopy, which allows us to obtain computational results for homotopy groups. This chapter contains both results and also several examples that will reappear later on.

In chapter 7 we finally get to Obstruction Theory. We focus mainly on the extension problem for maps, only mentioning the analogous results for the extension problem for sections. Nonetheless, in both cases we stress the subtleties that naturally arise, and the possible ways in which we can deal with them. Finally, the last chapter quickly describes some possible applications of the results of Obstruction Theory. It is more concerned with discussing the relevant ideas, containing almost no proofs.

CHAPTER

1

TOPOLOGICAL PRELIMINARIES

This preliminary chapter describes some topological constructions and results that will be used throughout most of the text. The first section describes the construction of a space of maps $\mathrm{Map}(X, Y)$ between any two topological spaces X and Y by equipping the set of maps $\mathsf{Top}(X, Y)$ with the compact-open topology. The goal is to study the categorical properties of this construction, in particular its functoriality and its relation with the cartesian product via the exponential adjunction. We then exploit this relation to prove some internalization results and also some results concerning products of quotient maps which will be used often. The second section then describes the inductive construction of CW-complexes via the notion of cell attachment. The approach used here is somewhat categorical, cell attachments are defined via pushout diagrams, and a CW-complex is defined as a colimit of a sequence of subspaces where each one is obtained from the previous one by a cell attachment.

1.1 Spaces of maps

In this section we define a topology on the set $\mathsf{Top}(X, Y)$ of maps between two spaces, and prove that this topology satisfies many useful properties. In particular, we prove that there is an adjunction between mapping spaces and products. This adjunction will be used all throughout the text, especially for the study of Homotopy Theory.

1.1.1 Definition. Given space X and Y, the **compact-open topology** on the set of maps $\mathsf{Top}(X, Y)$ is the topology having as sub-basis the subsets of the form

$$S(C, V) := \{f \in \mathsf{Top}(X, Y) \mid f(C) \subseteq V\},$$

where $C \subseteq X$ is a compact subspace and $V \subseteq Y$ is an open subset. Whenever we consider this topology on the set $\mathsf{Top}(X, Y)$ we, denote the resulting space by $\mathrm{Map}(X, Y)$.

Intuitively, two points of $\mathrm{Map}(X,Y)$, that is, two maps f, $g : X \to Y$ are closer the more their values are closer along compact subspaces of the domain. This topology shows up under different names in other places. For example, if Y is a metric space, then the compact-open topology on $\mathrm{Map}(X,Y)$ is usually called the *topology of uniform convergence on compact subspaces*, because in this case a sequence $(f_n : X \to Y)_{n\in\mathbb{N}}$ converges to a map f if and only if, for every compact subspace $C \subseteq X$, the sequence of restrictions $(f_n|_C)_{n\in\mathbb{N}}$ converges uniformly to $f|_C$.

The compact-open topology is certainly not the only topology we can define on the set $\mathsf{Top}(X,Y)$. For example, we could also define a topology by considering the sub-basic sets defined as

$$S(\{x\},V) \coloneqq \{f : X \to Y \mid f(x) \in V\},$$

where x is any point of X, and V is any open subset of Y. It turns out that, with this topology, convergence of a sequence of maps $(f_n)_{n\in\mathbb{N}}$ is the same as pointwise convergence. At least in the context of metric spaces, this already tells us that this topology is not as useful as the compact-open topology, since the notion of uniform convergence is far more useful than pointwise convergence.

But how do we judge the usefulness of the compact-open topology on our purely topological context? Taking into account the categorical approach we are taking in this text, the usefulness of a certain construction is related to its categorical properties, like the existence of maps (morphisms), functors and adjunctions relating this construction to others. Thus, we devote the rest of the section to the study of the main categorical properties satisfies by the compact-open topology.

The first collection of maps we can define are pushforwards and pullbacks. Recall that, since Top is a locally small category, given spaces X, Y and Z, and maps $f : X \to Y$ and $g : Y \to Z$, we have the corresponding pushforward along g function

$$\mathsf{Top}(X,g) : \mathsf{Top}(X,Y) \to \mathsf{Top}(X,Z),$$

and also the corresponding pullback along f function

$$\mathsf{Top}(g,Z) : \mathsf{Top}(Y,Z) \to \mathsf{Top}(X,Z).$$

The next results show that using the compact-open topology we can "topologize" these constructions.

1.1.2 Proposition. Let X, Y and Z be spaces.

1. If $g : Y \to Z$ is a map, then the pushforward along g

$$\mathsf{Top}(X,g) : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z)$$

defines a continuous function.

2. If $f : X \to Y$ is a map, then the pullback along f

$$\mathsf{Top}(f, Z) : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

defines a continuous function.

Proof. 1. Given a sub-basic open set $S(C, V) \subseteq \mathrm{Map}(X, Z)$, a straightforward argument shows that $\mathrm{Map}(X, g)^{-1}(S(C, V)) = S(C, g^{-1}(V))$, therefore $\mathrm{Map}(X, f)^{-1}(S(C, V))$ is an open subset of $\mathrm{Map}(X, Y)$.

2. Given a sub-basic open set $S(C, V) \subseteq \mathrm{Map}(X, Z)$, another straightforward argument shows that $\mathrm{Map}(f, Z)^{-1}(S(C, V)) = S(f(C), V)$, therefore $\mathrm{Map}(f, Z)^{-1}(S(C, V))$ is an open subset of $\mathrm{Map}(Y, Z)$. ■

From now on, whenever we want to regard the pushforward $\mathsf{Top}(X, g)$ as a *map*, and not just a mere function, we will use the notation $\mathrm{Map}(X, g)$. Similarly, the pullback map will be denoted by $\mathrm{Map}(g, Z)$.

Using the pushforward map we can define a functor $\mathrm{Map}(X, -) : \mathsf{Top} \to \mathsf{Top}$ which sends a space Y to the space of maps $\mathrm{Map}(X, Y)$, and which sends a map $g : Y \to Z$ to the induced pushforward map $\mathrm{Map}(X, g) : \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$. Similarly, we also have the pullback functor $\mathrm{Map}(-, Z) : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Top}$ which sends a space X to the space of maps $\mathrm{Map}(X, Z)$, and which sends a map $f : X \to Y$ to the induced pullback map $\mathrm{Map}(f, Z) : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$. It is interesting to remark that these functors "upgrade" the usual representable functors $\mathsf{Top}(X, -) : \mathsf{Top} \to \mathsf{Top}$ and $\mathsf{Top}(-, Z) : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Top}$ to the category of spaces. More precisely, if $U : \mathsf{Top} \to \mathsf{Set}$ denotes the forgetful functor sending a space to its underlying set, then we have the commutative diagrams of functors below.

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\mathrm{Map}(X,-)"] \arrow[dr, "\mathsf{Top}(X,-)"'] & \mathsf{Top} \arrow[d, "U"] \\ & \mathsf{Set} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Top}^{\mathsf{op}} \arrow[r, "\mathrm{Map}(-,Z)"] \arrow[dr, "\mathsf{Top}(-,Z)"'] & \mathsf{Top} \arrow[d, "U"] \\ & \mathsf{Set} \end{tikzcd}$$

Our next goal is to show that the functors $- \times Y$ and $\mathrm{Map}(Y, -)$ are *adjoint* to one another under some topological conditions on Y.

Recall that a pair of opposite functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$, we say that F is **left adjoint** to G, or that G is **right adjoint** to F, if there exists

- a natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ called the **unit** of the adjunction,
- a natural transformation $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ called the **counit** of the adjunction,

which satisfy the following commutativity conditions, called the *triangle identities*:

$$
\begin{tikzcd}
F \arrow[rr, Rightarrow, "\mathrm{id}_F"] \arrow[dr, Rightarrow, "F\eta"'] & & F & G \arrow[rr, Rightarrow, "\mathrm{id}_G"] \arrow[dr, Rightarrow, "\eta G"'] & & G \\
& F \circ G \circ F \arrow[ur, Rightarrow, "\varepsilon F"'] & & & G \circ F \circ G \arrow[ur, Rightarrow, "G\varepsilon"'] &
\end{tikzcd}
\tag{1.1}
$$

In the two diagrams above, the natural transformations $F\eta$, εF, ηG and $G\varepsilon$ are called **whiskerings**. The component of the whiskering $F\eta$ at an object $c \in \mathsf{C}$, for example, is defined as

$$(F\eta)_c \coloneqq F(\eta_c) : F(c) \to F(G(F(c))),$$

while the component at c of εF is defined as

$$(\varepsilon F)_c \coloneqq \varepsilon_{F(c)} : F(G(F(c))) \to F(c).$$

The components of ηG and $G\varepsilon$ at an object $d \in \mathsf{D}$ are defined analogously. The commutativity of the two triangles above is then equivalent to saying that, for any objects $c \in \mathsf{C}$ and $d \in \mathsf{D}$, we have the equations:

$$
\begin{cases}
\varepsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}_{F(c)}; \\
G(\varepsilon_d) \circ \eta_{G(d)} = \mathrm{id}_{G(d)}.
\end{cases}
\tag{1.2}
$$

We start by defining the unit transformations of the adjunction we would like to obtain.

1.1.3 Lemma. Let Y be any topological spaces.

1. For any other space X, the function

   $$\iota_X : X \to \mathrm{Map}(Y, X \times Y)$$

   given by the formula

   $$\iota_X(x) \coloneqq (\mathrm{ct}_{Y,x}, \mathrm{id}_Y) \quad \forall\, x \in X$$

   is continuous.

2. The collection of maps

   $$\{\iota_X : X \to \mathrm{Map}(Y, X \times Y)\}_{X \in \mathsf{Top}}$$

   defines a natural transformation of functors

   $$\iota : \mathrm{id}_{\mathsf{Top}} \Rightarrow \mathrm{Map}(Y, - \times Y).$$

Proof. 1. Let $V \subseteq X \times Y$ be an open subset, $C \subseteq Y$ a compact subspace, and consider the sub-basic open subset $S(C,V) \subseteq \mathrm{Map}(Y, X \times Y)$. If $x \in \iota_X^{-1}(S(C,V))$, then $\iota_X(x) = (\mathrm{ct}_{Y,x}, \mathrm{id}_Y) \in S(C,V)$, which means that $(\mathrm{ct}_{Y,x}, \mathrm{id}_Y)(C) \subseteq V$. This means that the relation $(x,y) \in V$ holds for every $y \in C$, or in other words, we have the inclusion $\{x\} \times C \subseteq V$. Since C is compact, it follows from the Tube Lemma that there exists a neighborhood U of x such that $U \times C \subseteq V$, so, for any $x' \in U$, the relation $\{x'\} \times C = [\iota_X(x')](C) \subseteq V$ holds, therefore $U \subseteq \iota_X^{-1}(S(C,V))$; proving the continuity of ι_X.

2. Let A and B be spaces, and consider a map $f : A \to B$. We need to show the commutativity of the square below,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota_A"'] & B \arrow[d, "\iota_B"] \\
\mathrm{Map}(Y, A \times Y) \arrow[r, "\mathrm{Map}(Y, f \times \mathrm{id}_Y)"'] & \mathrm{Map}(Y, B \times Y)
\end{tikzcd}$$

or in other words, that for any $a \in A$ we have the equality

$$(f \times \mathrm{id}_Y) \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y) = (\mathrm{ct}_{Y,f(a)}, \mathrm{id}_Y). \tag{1.3}$$

This can be shown pretty easily directly. For any $y \in Y$, unpacking the definitions we have

$$\begin{aligned}
((f \times \mathrm{id}_Y) \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y))(y) &= (f \times \mathrm{id}_Y)(a,y) \\
&= (f(a), y) \\
&= (\mathrm{ct}_{Y,f(a)}, \mathrm{id}_Y)(y).
\end{aligned}$$

We give a second proof showing how this equality also follows from the universal properties defining the maps involved. Consider the canonical projections below

$$\begin{aligned}
\pi_1 &: A \times Y \to A, \\
\pi_2 &: A \times Y \to Y, \\
\pi_1 &: B \times Y \to B, \\
\pi_2' &: B \times Y \to Y.
\end{aligned}$$

Since $(f \times \mathrm{id}_Y) \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y)$ is a map into the product $B \times Y$, it is completely determined by the projections π_1' and π_2', that is, we can prove the equality (1.3) by showing that both sides coincide when composed with π_1' and π_2'.

For the first projection, if we recall that $f \times \mathrm{id}_Y$ satisfies the equations

$$\pi_1' \circ (f \times \mathrm{id}_Y) = f \circ \pi_1 \quad \text{and} \quad \pi_2' \circ (f \times id_Y) = \mathrm{id}_Y \circ \pi_2 = \pi_2,$$

then we have the chain of equalities

$$\begin{aligned}\pi_1' \circ (f \times \mathrm{id}_Y) \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y) &= f \circ \pi_1 \circ (\mathrm{ct}_{Y,a}) \\ &= f \circ \mathrm{ct}_{Y,a} \\ &= \mathrm{ct}_{Y,f(a)}.\end{aligned}$$

Similarly, for the second projection we have the equalities

$$\begin{aligned}\pi_2' \circ (f \times \mathrm{id}_Y) \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y) &= \pi_2 \circ (\mathrm{ct}_{Y,a}, \mathrm{id}_Y) \\ &= \mathrm{id}_Y.\end{aligned}$$

This ends the proof because by definition the induced map $(\mathrm{ct}_{Y,f(a)}, \mathrm{id}_Y)$ also satisfies the equalities

$$\pi_1' \circ (\mathrm{ct}_{Y,f(a)}, \mathrm{id}_Y) = \mathrm{ct}_{Y,f(a)} \quad \text{and} \quad \pi_2 \circ (\mathrm{ct}_{Y,f(a)}, \mathrm{id}_Y) = \mathrm{id}_Y. \qquad \blacksquare$$

Now we turn to the collection of maps that defines the counit of our soon-to-be adjunction. It is at this point that we need to make additional assumptions on the spaces involved.

1.1.4 Lemma. Let Y be a locally compact Hausdorff space.

1. For any other space Z, the evaluation function

   $$\mathrm{ev}_{Y,Z} : \mathrm{Map}(Y, Z) \times Y \to Z$$

   defined by the formula

   $$\mathrm{ev}_{Y,Z}(f, z) := f(z) \quad \forall\, (f, z) \in \mathrm{Map}(Y, Z) \times Z$$

   is continuous.

2. The collection of maps

   $$\{\mathrm{ev}_{Y,Z} : \mathrm{Map}(Y, Z) \times Y \to Z\}_{Z \in \mathsf{Top}}$$

   defines a natural transformation of functors

   $$\mathrm{ev}_Y : \mathrm{Map}(Y, -) \times Y \Rightarrow \mathrm{id}_{\mathsf{Top}}.$$

Proof. 1. Let $V \subseteq Z$ be an open subset, and suppose $(f, y) \in \mathrm{ev}_{Y,Z}^{-1}(V)$, which means that $f(y) \in V$, or in other words, that $y \in f^{-1}(V)$. Since Y is locally compact Hausdorff by hypothesis, we can find an open neighborhood U of y such that $\overline{U}$ is a compact subspace satisfying the condition $\overline{U} \subseteq f^{-1}(V)$. The product $S(\overline{U}, V) \times U$ then defines a neighborhood of (f, y) in $\mathrm{Map}(Y, Z) \times Y$, and, if $(g, y') \in S(\overline{U}, V) \times U$, then $\mathrm{ev}_{Y,Z}(g, y') =$

$g(y') \in V$; which shows that $S(\overline{U}, V) \times U \subseteq \mathrm{ev}_{Y,Z}^{-1}(V)$, therefore proving the continuity of $\mathrm{ev}_{Y,Z}$.

2. Let A and B be spaces, and consider a map $g : A \to B$. We need to show the commutativity of the square below.

$$\begin{tikzcd} \mathrm{Map}(Y,A) \times Y \arrow[r, "\mathrm{Map}(Y,g)\times \mathrm{id}_Y"] \arrow[d, "\mathrm{ev}_{Y,A}"'] & \mathrm{Map}(Y,B) \times Y \arrow[d, "\mathrm{ev}_{Y,B}"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

This follows by direct computation. For any pair $(f, y) \in \mathrm{Map}(Y, A) \times Y$ we have the following chain of equalities:

$$\begin{aligned} (\mathrm{ev}_{Y,B} \circ (\mathrm{Map}(Y,g) \times \mathrm{id}_Y))(f,y) &= \mathrm{ev}_{Y,B}(g \circ f, y) \\ &= (g \circ f)(y) \\ &= g(f(y)) \\ &= g(\mathrm{ev}_{Y,A}(f,y)) \\ &= (g \circ \mathrm{ev}_{Y,A})(f,y). \end{aligned}$$

■

We have all the ingredients necessary for defining an adjunction between products and spaces of maps.

1.1.5 Theorem (Exponential adjunction)**.** If Y is a locally compact Hausdorff space, then the product functor $- \times Y$ is left-adjoint to the pushforward functor $\mathrm{Map}(Y, -)$.

Proof. Our candidates for unit and counit transformations of the adjunction are the transformations

$$\iota : \mathrm{id}_{\mathsf{Top}} \Rightarrow \mathrm{Map}(Y, - \times Y) \quad \text{and} \quad \mathrm{ev}_Y : \mathrm{Map}(Y, -) \times Y \Rightarrow \mathrm{id}_{\mathsf{Top}}$$

of Lemma 1.1.3 and Lemma 1.1.4, respectively.

We just need to show that these transformations satisfy the triangle identities. The first of these identities says that the triangle below commutes for every space $X \in \mathsf{Top}$.

$$\begin{tikzcd} X \times Y \arrow[rr, "\mathrm{id}_{X\times Y}"] \arrow[dr, "\iota_X \times \mathrm{id}_Y"'] & & X \times Y \\ & \mathrm{Map}(Y, X \times Y) \times Y \arrow[ur, "\mathrm{ev}_{Y,X\times Y}"'] & \end{tikzcd}$$

This is just a matter of computation. Given any $(x, y) \in X \times Y$, we have

$$\begin{aligned} (\mathrm{ev}_{Y,X\times Y} \circ (\iota_X \times \mathrm{id}_Y))(x,y) &= \mathrm{ev}_{Y,X\times Y}(\iota_X(x), y) \\ &= \mathrm{ev}_{Y,X\times Y}((\mathrm{ct}_{Y,x}, \mathrm{id}_Y), y) \\ &= (\mathrm{ct}_{Y,X}, \mathrm{id}_Y)(y) \\ &= (x, y). \end{aligned}$$

The second triangle identity, on the other hand, says that the triangle below commutes for every space $Z \in \mathsf{Top}$.

$$\begin{tikzcd}
\mathrm{Map}(Y,Z) \arrow[rr, "\mathrm{id}_{\mathrm{Map}(Y,Z)}"] \arrow[dr, "\iota_{\mathrm{Map}(Y,Z)}"'] & & \mathrm{Map}(Y,Z) \\
& \mathrm{Map}(Y, \mathrm{Map}(Y,Z) \times Y) \arrow[ur, "\mathrm{Map}(Y, \mathrm{ev}_{Y,Z})"'] &
\end{tikzcd}$$

Given any $f \in \mathrm{Map}(Y,Z)$ and any $y \in Y$, we have

$$\begin{aligned}
[(\mathrm{Map}(Y, \mathrm{ev}_{Y,Z}) \circ \iota_{\mathrm{Map}(Y,Z)})(f)](y) &= [\mathrm{Map}(Y, \mathrm{ev}_{Y,Z})((\mathrm{ct}_{Y,f}, \mathrm{id}_Y))](y) \\
&= (\mathrm{ev}_{Y,Z} \circ (\mathrm{ct}_{Y,f}, \mathrm{id}_Y))(y) \\
&= \mathrm{ev}_{Y,Z}(f, y) \\
&= f(y),
\end{aligned}$$

and since this holds for any $y \in Y$, we have the equality of maps

$$(\mathrm{Map}(Y, \mathrm{ev}_{Y,Z}) \circ \iota_{\mathrm{Map}(Y,Z)})(f) = f,$$

but this holds for every $f \in \mathrm{Map}(Y,Z)$, therefore we have the desired equality

$$\mathrm{Map}(Y, \mathrm{ev}_{Y,Z}) \circ \iota_{\mathrm{Map}(Y,Z)} = \mathrm{id}_{\mathrm{Map}(Y,Z)}. \qquad \blacksquare$$

One important thing about categorical adjunctions in general is that they can also be described by a certain natural bijection between sets of morphisms. More precisely, we have the following result, whose proof can be found in either (RIEHL, 2017, Proposition 4.2.6) or (LEINSTER, 2014, Theorem 2.2.5).

1.1.6 Theorem. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be a pair of opposing functors between locally small categories. The following are equivalent:

1. F is left adjoint to G;
2. for each $c \in \mathsf{C}$ and each $d \in \mathsf{D}$ there exists a bijection

$$\lambda_{c,d} : \mathsf{D}(F(c), d) \xrightarrow{\cong} \mathsf{C}(c, G(d))$$

   which depends naturally on both c and d.

Concretely, the bijection $\lambda_{c,d}$ assigns to each morphism $\alpha : F(c) \to d$ the composite morphism

$$\lambda_{c,d}(\alpha) := G(\alpha) \circ \eta_c : c \to G(d)$$

as shown below.

$$F(c) \xrightarrow{\alpha} d \quad \rightsquigarrow \quad \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[rr, bend right, dashed, "\lambda_{c,d}(\alpha)"'] & G(F(c)) \arrow[r, "G(\alpha)"] & G(d) \end{tikzcd}$$

The inverse bijection $\lambda^{-1}_{c,d}$ has a dual description: it assigns to a morphism $\beta : c \to G(d)$ the composite morphism

$$\lambda^{-1}_{c,d}(\beta) \coloneqq \varepsilon_d \circ F(\beta) : F(c) \to d$$

as shown below.

$$\begin{tikzcd} c \arrow[r, "\beta"] & G(d) & \rightsquigarrow & F(c) \arrow[r, "F(\beta)"] \arrow[rr, bend right, dashed, "\lambda^{-1}_{c,d}(\beta)"'] & F(G(d)) \arrow[r, "\varepsilon_d"] & d \end{tikzcd}$$

We can of course apply this alternative description to the exponential adjunction between $- \times Y$ and $\mathrm{Map}(Y, -)$ described above. If Y is locally compact Hausdorff, then for any two spaces X and Z we have the bijection

$$\lambda_{X,Z} : \mathsf{Top}(X \times Y, Z) \xrightarrow{\cong} \mathsf{Top}(X, \mathrm{Map}(Y, Z))$$

which associates to every map $f : X \times Y \to Z$ the corresponding map $\lambda_{X,Z} f : X \to \mathrm{Map}(Y, Z)$ defined as the composition

$$\lambda_{X,Z} f \coloneqq \mathrm{Map}(Y, f) \circ \iota_X$$

as shown in the diagram below.

$$\begin{tikzcd} X \times Y \arrow[r, "f"] & Z & \rightsquigarrow & X \arrow[r, "\iota_X"] \arrow[rr, bend right, dashed, "\lambda_{X,Z} f"'] & \mathrm{Map}(Y, X \times Y) \arrow[r, "\mathrm{Map}(Y,f)"] & \mathrm{Map}(Y, Z) \end{tikzcd}$$

For any $x \in X$, the value $\lambda_{X,Z} f(x)$ is itself a map of type $Y \to Z$. If we then evaluate this map at a point $y \in Y$, we see that

$$\begin{aligned} [\lambda_{X,Z} f(x)](y) &= [(\mathrm{Map}(Y, f) \circ \iota_X)(x)](y) \\ &= [\mathrm{Map}(Y, f)((\mathrm{ct}_{Y,x}, \mathrm{id}_Y))](y) \\ &= (f \circ (\mathrm{ct}_{Y,x}, \mathrm{id}_Y))(y) \\ &= f(x, y). \end{aligned}$$

Throughout the text, we call the map $\lambda_{X,Z} f$ the **exponential adjoint of** f, or also the **exponential transpose of** f. Sometimes we also say that $\lambda_{X,Z} f$ is obtained from f by **currying** the second variable.

We can also describe the inverse function

$$\lambda^{-1}_{X,Z} : \mathsf{Top}(X, \mathrm{Map}(Y, Z)) \to \mathsf{Top}(X \times Y, Z)$$

by using the counit and the adjunction together with the product functor:

$$\lambda^{-1}_{X,Z} g \coloneqq \mathrm{ev}_{Y,Z} \circ (g \times \mathrm{id}_Z)$$

for every $g : X \to \mathrm{Map}(Y, Z)$, as shown below.

$$X \xrightarrow{g} \mathrm{Map}(Y,Z) \quad \rightsquigarrow \quad \begin{tikzcd} X \times Y \arrow[r, "g\times \mathrm{id}_Y"] \arrow[rr, dashed, bend right, "\lambda^{-1}_{X,Z} g"'] & \mathrm{Map}(Y,Z) \times Y \arrow[r, "\mathrm{ev}_{Y,Z}"] & Z \end{tikzcd}$$

A direct computation using the definitions shows that the formula

$$\lambda^{-1}_{X,Z} g(x, y) = [g(x)](y)$$

holds for every $g : X \to \mathrm{Map}(Y, Z)$ and every $(x, y) \in X \times Y$. We sometimes say that $\lambda^{-1}_{X,Z} g$ is obtained by **uncurrying** the map g.

1.1.7 Remark. In Category Theory, we say an object c of a category C is *exponentiable* if it satisfies the two following conditions:

1. the product $a \times c$ exists for every other object $a \in \mathsf{C}$, so that we can define a product functor $- \times c : \mathsf{C} \to \mathsf{C}$;

2. the product functor $- \times a$ has a right-adjoint functor $R : \mathsf{C} \to \mathsf{C}$.

Under these conditions, the value of R on an object $b \in \mathsf{C}$ is usually denoted as b^a, and a similar notation is used for the value on morphisms.

This terminology allows us to restate Theorem 1.1.5 in the following way: *every locally compact Hausdorff space is exponentiable in the category* Top.

1.1.1 Convenient consequences

In this subsection we use our previous results on the exponential result to deduce some useful consequences. Most of the results have to do with topologizing some categorical notions. We have already seen an example of this when we saw how the compact-open topology allowed us to upgrade the representable functors $\mathsf{Top}(X, -)$ and $\mathsf{Top}(-, Y)$ to the corresponding functors of spaces $\mathrm{Map}(X, -)$ and $\mathrm{Map}(Y, -)$.

Most of the results of this section, however, involve upgrading bijections coming from categorical constructions to homeomorphisms. The tool used for this is the Yoneda Embedding, since it allows us to deduce "internal" isomorphisms from natural isomorphisms between functors.

Our first result of this type concerns understanding spaces of the form $\mathrm{Map}(W, X \times Y)$. Recall that general categorical products are defined as limits of particularly simple diagrams. More precisely, two objects a and b of a (locally small) category have a product if the functor $\mathsf{C}(-, a) \times \mathsf{C}(-, b) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ is representable, and the product $a \times b$ is precisely a choice of representing object for this functor, so that for any object $x \in \mathsf{C}$ we have a natural bijection of sets of morphisms:

$$\mathsf{C}(x, a \times b) \cong \mathsf{C}(x, a) \times \mathsf{C}(x, b).$$

In the category Top, this means that for any three spaces X, A and B we have a bijection

$$\mathsf{Top}(X, A \times B) \cong \mathsf{Top}(X, A) \times \mathsf{Top}(X, B)$$

which is natural in X. The compact-open topology gives us an internal version of the sets of maps $\mathsf{Top}(X, Y)$: the *spaces* of maps $\mathrm{Map}(X, Y)$. It is then natural to wonder if the bijection above can be upgraded to a homeomorphism

$$\mathrm{Map}(X, A \times B) \cong \mathrm{Map}(X, A) \times \mathrm{Map}(X, B).$$

A general categorical tool to deduce "internal" isomorphisms from natural isomorphisms between functors is the infamous *Yoneda Embedding* which we now briefly recall. Given a locally small category C, each object $a \in \mathsf{C}$ gives rise to a functor $\mathsf{C}(a, -) : \mathsf{C} \to \mathsf{Set}$ by sending an object $x \in \mathsf{C}$ to the set of morphisms $\mathsf{C}(a, x)$, and by sending a morphism $\phi : x \to y$ to the function $\mathsf{C}(a, \phi) : \mathsf{C}(a, x) \to \mathsf{C}(a, y)$ - called the **pushforward along** ϕ - defined as

$$\mathsf{C}(a, \phi)(f) := \phi \circ f \quad \forall\, f \in \mathsf{C}(a, x). \tag{1.4}$$

It is also possible to define a dual construction, i.e, a functor $\mathsf{C}(-, a) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ sending an object $x \in \mathsf{C}$ to the set of morphisms $\mathsf{C}(x, a)$, and sending a morphism $\psi : x \to y$ to the function $\mathsf{C}(\psi, a) : \mathsf{C}(y, a) \to \mathsf{C}(x, y)$ - called the **pullback along** ψ - in the opposite direction defined as

$$\mathsf{C}(\psi, a)(g) := g \circ \psi \quad \forall\, g \in \mathsf{C}(y, a). \tag{1.5}$$

These two constructions depend functorially on a. If b is another object, and $\theta : a \to b$ is a morphism in C, then on the hand we have a natural transformation

$$\mathsf{C}(\theta, -) : \mathsf{C}(b, -) \Rightarrow \mathsf{C}(a, -),$$

called the **pullback transformation**, in the opposite direction whose component at an object $x \in \mathsf{C}$ is the pullback function

$$\mathsf{C}(\theta, x) : \mathsf{C}(x, b) \to \mathsf{C}(x, a);$$

while on the other we have a natural transformation

$$\mathsf{C}(-, \theta) : \mathsf{C}(-, a) \Rightarrow \mathsf{C}(-, b)$$

called the **pushforward transformation** whose component at x is the pushforward function

$$\mathsf{C}(x,\theta) : \mathsf{C}(x,a) \to \mathsf{C}(x,b).$$

These two dual constructions give rise to two related functors:

1. The functor $\mathcal{Y} : \mathsf{C} \to [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ sending an object $a \in \mathsf{C}$ to the functor $\mathsf{C}(-,a) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, and sending a morphism $\theta : a \to b$ to the transformation $\mathsf{C}(-,\theta) : \mathsf{C}(-,a) \Rightarrow \mathsf{C}(-,b)$.
2. The functor $\overline{\mathcal{Y}} : \mathsf{C}^{\mathsf{op}} \to [\mathsf{C}, \mathsf{Set}]$ sending an object $a \in \mathsf{C}$ to the functor $\mathsf{C}(a,-) : \mathsf{C} \to \mathsf{Set}$, and sending a morphism $\theta : a \to b$ to the transformation $\mathsf{C}(\theta,-) : \mathsf{C}(b,-) \Rightarrow \mathsf{C}(a,-)$.

The Yoneda Embedding says that these two functors are embeddings of categories in a precise sense. See (RIEHL, 2017, Corollary 2.2.8) or (LEINSTER, 2014, Corollary 4.3.7) for proofs.

1.1.8 Theorem (Yoneda Embedding)**.** Let C be a locally small category. The functor $\mathcal{Y} : \mathsf{C} \to [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ is full and faithful, that is, it satisfies the two following conditions:

1. For any two objects $a, b \in \mathsf{C}$, if the morphisms $\theta, \theta' : a \to b$ are such that the transformations $\mathsf{C}(-,\theta), \mathsf{C}(-,\theta') : \mathsf{C}(-,a) \Rightarrow \mathsf{C}(-,b)$ are equal, then θ and θ' are themselves equal. In other words, $\mathcal{Y}$ induces an injection between sets of morphisms

$$\mathsf{C}(a,b) \rightarrowtail [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](\mathsf{C}(-,a), \mathsf{C}(-,b)).$$

2. For any two objects $a, b \in \mathsf{C}$, if $\lambda : \mathsf{C}(-,a) \Rightarrow \mathsf{C}(-,b)$ is a natural transformation, then there exists a morphism $\theta : a \to b$ such that $\lambda = \mathsf{C}(-,\theta)$. In other words, $\mathcal{Y}$ induces a surjection between sets of morphisms

$$\mathsf{C}(a,b) \twoheadrightarrow [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](\mathsf{C}(-,a), \mathsf{C}(-,b)).$$

Similarly, the functor $\overline{\mathcal{Y}} : \mathsf{C}^{\mathsf{op}} \to [\mathsf{C}, \mathsf{Set}]$ is also full and faithful.

The main result underpinning the proof of the Yoneda Embedding is the also famous *Yoneda Lemma*, which completely characterizes natural transformations of type $\mathsf{C}(-,a) \Rightarrow F$. A proof of this result can be found in (RIEHL, 2017, Theorem 2.2.4) or also in (LEINSTER, 2014, Theorem 4.2.1).

1.1.9 Theorem (Yoneda Lemma)**.** Let C be a locally small category. For any object $c \in \mathsf{C}$ and any functor $F : \mathsf{C} \to \mathsf{Set}$, there is a bijection

$$\Psi_{c,F} : [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](\mathsf{C}(-,c), F) \xrightarrow{\cong} F(c)$$

defined as

$$\Psi_{c,F}(\theta) := \theta_c(\mathrm{id}_c) \tag{1.6}$$

for every natural transformation $\theta : \mathsf{C}(-,c) \Rightarrow F$.

With these tools at our disposal, the comparison between $\mathrm{Map}(X, A \times B)$ and $\mathrm{Map}(X, A) \times \mathrm{Map}(X, B)$ is just a matter of using the exponential adjunction and the Yoneda Embedding.

1.1.10 Proposition. Let A and B be spaces. If X is a locally compact Hausdorff space, then there is a homeomorphism

$$\mathrm{Map}(X, A \times B) \cong \mathrm{Map}(X, A) \times \mathrm{Map}(X, B).$$

Proof. It suffices to show that there is a natural isomorphism of functors

$$\mathsf{Top}(-, \mathrm{Map}(X, A \times B)) \cong \mathsf{Top}(-, \mathrm{Map}(X, A) \times \mathrm{Map}(X, B)),$$

and then apply the Yoneda Embedding.

We begin by noting that, since X is locally compact Hausdorff, the exponential adjunction implies there is a natural isomorphism of functors

$$\mathsf{Top}(-, \mathrm{Map}(X, A \times B)) \cong \mathsf{Top}(- \times X, A \times B).$$

As we remarked above, the definition of the product is such that there is a natural isomorphism

$$\mathsf{Top}(-, A \times B) \cong \mathsf{Top}(-, A) \times \mathsf{Top}(-, B),$$

and then precomposing both sides with the product functor $- \times X : \mathsf{Top} \to \mathsf{Top}$ gives us the natural isomorphism

$$\mathsf{Top}(- \times X, A \times B) \cong \mathsf{Top}(- \times X, A) \times \mathsf{Top}(- \times X, B).$$

Applying the exponential adjunction twice gives the natural isomorphism

$$\mathsf{Top}(- \times X, A) \times \mathsf{Top}(- \times X, B) \cong \mathsf{Top}(-, \mathrm{Map}(X, A)) \times \mathsf{Top}(-, \mathrm{Map}(X, B)),$$

and then using the universal property of the product again we get

$$\mathsf{Top}(-, \mathrm{Map}(X, A)) \times \mathsf{Top}(-, \mathrm{Map}(X, B)) \cong \mathsf{Top}(- \mathrm{Map}(X, A) \times \mathrm{Map}(X, B)).$$

Following this chain of natural isomorphisms gives us the desired natural isomorphism of functors

$$\mathsf{Top}(-, \mathrm{Map}(X, A \times B)) \cong \mathsf{Top}(-, \mathrm{Map}(X, A) \times \mathrm{Map}(X, B)). \qquad \blacksquare$$

The proof above shows the existence of a homeomorphism, but it does not exhibit one explicitly. If

$$\theta : \mathsf{Top}(-, \mathrm{Map}(X, A \times B)) \cong \mathsf{Top}(-, \mathrm{Map}(X, A) \times \mathrm{Map}(X, B))$$

denotes the natural isomorphism we described, the proof of the Yoneda Embedding shows that we can recover one direction of the homeomorphism

$$\mathrm{Map}(X, A \times B) \xrightarrow{\cong} \mathrm{Map}(X, A) \times \mathrm{Map}(X, B)$$

by consider the map given by

$$\theta_{\mathrm{Map}(X, A\times B)}(\mathrm{id}_{\mathrm{Map}(X, A\times B)}) : \mathrm{Map}(X, A \times B) \to \mathrm{Map}(X, A) \times \mathrm{Map}(X, B).$$

Unpacking this expression shows that this homeomorphism is given by the map

$$(\mathrm{Map}(X, \pi_1), \mathrm{Map}(X, \pi_2)) : \mathrm{Map}(X, A \times B) \to \mathrm{Map}(X, A) \times \mathrm{Map}(X, B),$$

where π_1 and π_2 are the canonical projections out of $A \times B$. In other words, this homeomorphism sends a map $f : X \to A \times B$ to the pair of maps $(\pi_1 \circ f, \pi_2 \circ f)$.

We can also recover the inverse homeomorphism. In terms of the Yoneda Embedding, it can be obtained by using the inverse natural isomorphism to the identity map of $\mathrm{Map}(X, A) \times \mathrm{Map}(X, B)$, which results in a map

$$\theta^{-1}_{\mathrm{Map}(X,A)\times \mathrm{Map}(X,B)}(\mathrm{id}_{\mathrm{Map}(X,A)\times \mathrm{Map}(X,B)}) : \mathrm{Map}(X, A) \times \mathrm{Map}(X, B) \to \mathrm{Map}(X, A \times B)$$

Explicitly, this homeomorphism sends a pair of maps $(f : X \to A, g : X \to B)$ to the map $(f, g) : X \to A \times B$ induced by the universal property of the product.

There is also an internalization result for mapping spaces of the form $\mathrm{Map}(A \sqcup B, Y)$. Categorically, the $a \sqcup b$ of two objects of a category C, if it exists, is a representing object for the functor $\mathsf{C}(a, -) \times \mathsf{C}(b, -) : \mathsf{C} \to \mathsf{Set}$. This means that, for any other object $y \in \mathsf{C}$, there is a natural bijection

$$\mathsf{C}(a \sqcup b, y) \cong \mathsf{C}(a, y) \times \mathsf{C}(b, y).$$

The next result shows how this can be interpreted inside the category of spaces.

1.1.11 Proposition. If A and B are locally compact Hausdorff spaces, then for any other space Y there is a homeomorphism

$$\mathrm{Map}(A \sqcup B, Y) \cong \mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y).$$

Proof. The strategy of proof is to use the Yoneda Embedding again. Since A and B are locally compact Hausdorff, the same is true of the disjoint union $A \sqcup B$, so we can use the exponential adjunction to obtain the natural isomorphism

$$\mathsf{Top}(-, \mathrm{Map}(A \sqcup B, Y)) \cong \mathsf{Top}(- \times (A \sqcup B), Y).$$

In Top, products distribute over coproducts naturally, that is, there is a natural isomorphism of functors

$$- \times (A \sqcup B) \cong (- \times A) \sqcup (- \times B),$$

and this implies the isomorphism

$$\mathsf{Top}(- \times (A \sqcup B), Z) \cong \mathsf{Top}((- \times A) \sqcup (- \times B), Y).$$

Here we have to be careful. The universal property of the coproduct gives us a natural isomorphism

$$\mathsf{Top}(A \sqcup B, -) \cong \mathsf{Top}(A, -) \times \mathsf{Top}(B, -)$$

for any two spaces A and B. However, the situation at hand is different, because the functor

$$\mathsf{Top}((- \times A) \sqcup (- \times B), Y)$$

has a fixed target space, but a varying coproduct as source. We know that, for each choice of space $W \in \mathsf{Top}$, we have a *bijection*

$$\mathsf{Top}((W \times A) \sqcup (W \times B), Y) \cong \mathsf{Top}(W \times A, Y) \times \mathsf{Top}(W \times B, Y),$$

and it is not unreasonable to expect this bijection to depend naturally on W, so that we end up with a natural isomorphism

$$\mathsf{Top}((- \times A) \sqcup (- \times B), Y) \cong \mathsf{Top}(- \times A, Y) \times \mathsf{Top}(- \times B, Y).$$

This is indeed true, and it follows from a general categorical fact. Given a small category J and a locally small category C, if *every* functor $F : \mathsf{J} \to \mathsf{C}$ has a colimit, then the diagonal functor $\Delta : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$ has a left adjoint $\operatorname{colim} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}$ which assigns to every functor $F : \mathsf{J} \to \mathsf{C}$ its colimit $\operatorname{colim} F \in \mathsf{C}$. The adjointness relations means that, for every object $c \in \mathsf{C}$, and every functor $F : \mathsf{J} \to \mathsf{C}$, we have a bijection

$$\mathsf{C}(\operatorname{colim} F, c) \cong \mathsf{C}^{\mathsf{J}}(F, \Delta(c))$$

which depends naturally on *both* the object c and the functor F. A proof of this fact be found in either (RIEHL, 2017, section 4.5) or (LEINSTER, 2014, section 6.1).

Coproducts of spaces are colimits of functors of type $\mathsf{2} \to \mathsf{Top}$, and since Top admits all coproducts, for any three spaces W, X and Y we have a bijection

$$\mathsf{Top}(W \sqcup X, Y) \cong \mathsf{Top}(W, Y) \times \mathsf{Top}(X, Y)$$

depending naturally on all the spaces. In particular, if we fix Y, and let W and X vary, we get the natural isomorphism of two-variable functors

$$\mathsf{Top}(- \sqcup -, Y) \cong \mathsf{Top}(-, Y) \times \mathsf{Top}(-, Y),$$

and if we precompose this with the functor

$$(- \times A, - \times B) : \mathsf{Top} \to \mathsf{Top} \times \mathsf{Top}$$

we obtain the desired natural isomorphism

$$\mathsf{Top}((- \times A) \sqcup (- \times B), Y) \cong \mathsf{Top}(- \times A, Y) \times \mathsf{Top}(- \times B, Y).$$

Continuing, if we apply the exponential adjunction twice, we obtain

$$\mathsf{Top}(- \times A, Y) \times \mathsf{Top}(- \times B, Y) \cong \mathsf{Top}(-, \mathrm{Map}(A, Y)) \times \mathsf{Top}(-, \mathrm{Map}(B, Y)),$$

and then using the universal property of the product once again we get

$$\mathsf{Top}(-, \mathrm{Map}(A, Y)) \times \mathsf{Top}(-, \mathrm{Map}(B, Y)) \cong \mathsf{Top}(-, \mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y)).$$

If we go over the chain of natural isomorphisms, we see that at the end we obtained the isomorphism

$$\mathsf{Top}(- \mathrm{Map}(A \sqcup B, Y)) \cong \mathsf{Top}(-, \mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y)),$$

which implies the desired homeomorphism. ■

Similarly to what happened in the previous result, the proof given above does not explicitly provide us with a homeomorphism

$$\mathrm{Map}(A \sqcup B, Y) \cong \mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y),$$

but it can be recovered by "running" the proof of the Yoneda Embedding. On the one hand, we have the map

$$(\mathrm{Map}(j_1, Y), \mathrm{Map}(j_2, Y)) : \mathrm{Map}(A \sqcup B, Y) \to \mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y)$$

induced by the universal map of the product, where j_1 and j_2 are the canonical injections into the disjoint union. Explicitly, this sends a map $f : A \sqcup B \to Y$ to the pair $(f \circ j_1, f \circ j_2)$. The map in the opposite direction

$$\mathrm{Map}(A, Y) \times \mathrm{Map}(B, Y) \to \mathrm{Map}(A \sqcup B, Y)$$

sends a pair of maps $(f : A \to Y, g : B \to Y)$ to the map $\langle f, g \rangle : A \sqcup B \to Y$ induced by the universal property of the coproduct.

Another result which can be internalized under some topological conditions is the exponential adjunction itself. We saw in Theorem 1.1.5 that, if Y is locally compact Hausdorff, then for any two spaces X and Z there is a natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, \mathrm{Map}(Y, Z)).$$

A natural question is: can we replace the sets of maps by spaces of maps an upgrade the above bijection to a homeomorphism? The answer is yes, but we must be careful with the required conditions.

1.1.12 Proposition. Let X and Y be locally compact Hausdorff spaces. If Z is any other space, there is a natural homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Proof. We use the Yoneda Embedding once again. Since X is locally compact Hausdorff, by the exponential adjunction we have the natural isomorphism

$$\mathsf{Top}(-, \mathrm{Map}(X, \mathrm{Map}(Y, Z))) \cong \mathsf{Top}(- \times X, \mathrm{Map}(Y, Z)).$$

Similarly, we also have the natural isomorphism

$$\mathsf{Top}(-, \mathrm{Map}(Y, Z)) \cong \mathsf{Top}(- \times Y, Z),$$

and if we precompose this with the product functor $- \times X : \mathsf{Top} \to \mathsf{Top}$, we obtain the isomorphism

$$\mathsf{Top}(- \times X, \mathrm{Map}(Y, Z)) \cong \mathsf{Top}((- \times X) \times Y, Z).$$

At this point we have to deal with a small subtlety. The product operation on spaces is associative up to homeomorphism, that is, given spaces W, X and Z, there is an *associator homeomorphism*

$$(W \times X) \times Y \cong W \times (X \times Y)$$

which is natural in all three variables. In particular, if we fix X and Y, but let W vary, we obtain a natural isomorphism of functors

$$(- \times X) \times Y \cong -(X \times Y),$$

which implies the natural isomorphism

$$\mathsf{Top}((- \times X) \times Y, Z) \cong \mathsf{Top}(- \times (X \times Y), Z).$$

The property of being locally compact Hausdorff is preserved by products, therefore $X \times Y$ is a locally compact Hausdorff space, and using the exponential adjunction one more time gives us the isomorphism

$$\mathsf{Top}(- \times (X \times Y), Z) \cong \mathsf{Top}(-, \mathrm{Map}(X \times Y, Z)).$$

Overall, this reasoning shows the existence of a natural isomorphism of functors

$$\mathsf{Top}(-, \mathrm{Map}(X, \mathrm{Map}(Y, Z))) \cong \mathsf{Top}(-, \mathrm{Map}(X \times Y, Z)),$$

and the desired homeomorphism then follows from the Yoneda Embedding. ∎

We end this section with results of a more point-set nature instead of internalization results.

1.1.13 Proposition. Let X and Y be spaces, and consider a quotient map $p : X \to Y$. If Z is a locally compact Hausdorff space, then the product $p \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is still a quotient map.

Proof. Let A be another space, and suppose $f : Y \times Z \to A$ is a function such that the composition $f \circ (p \times \mathrm{id}_Z) : X \times Z \to A$ is continuous. We claim that the exponential adjoint function

$$\lambda_{Y,A} f : Y \to \mathrm{Map}(Z, A)$$

is continuous. Indeed, from the naturality of the exponential adjunction applied to the map $p : X \to Y$, we know that the square below commutes.

$$\begin{tikzcd}
\mathsf{Top}(Y \times Z, A) \arrow[r, "\mathsf{Top}(p \times \mathrm{id}_Z, A)"] \arrow[d, "\lambda_{Y,A}"'] & \mathsf{Top}(X \times Z, A) \arrow[d, "\lambda_{X,A}"] \\
\mathsf{Top}(Y, \mathrm{Map}(Z, A)) \arrow[r, "\mathsf{Top}(p, \mathrm{Map}(Z,A))"'] & \mathsf{Top}(X, \mathrm{Map}(Z, A))
\end{tikzcd}$$

In particular, chasing $f \in \mathsf{Top}(Y \times Z, A)$ around the squares yields the equation

$$\lambda_{X,A}(f \circ (p \times \mathrm{id}_Z)) = \lambda_{Y,A} f \circ p.$$

Since $f \circ (p \times \mathrm{id}_Z)$ is continuous by hypothesis, so is its adjoint $\lambda_{X,A}(f \circ (p \times \mathrm{id}_Z))$, and the equality above then implies the continuity of the composition $\lambda_{Y,A} f \circ p$; but this implies the continuity of $\lambda_{Y,A} f$ because p is a quotient map by hypothesis.

Now, since Z is locally compact Hausdorff, we have a well-defined inverse exponential transformation

$$\lambda_{Y,A}^{-1} : \mathsf{Top}(Y, \mathrm{Map}(Z, A)) \to \mathsf{Top}(Y \times Z, A),$$

and by applying it to the map $\lambda_{Y,A} f \in \mathsf{Top}(Y, \mathrm{Map}(Z, A))$ we deduce the continuity of f, as required. ■

1.1.14 Corollary. Let $f : W \to X$ and $g : Y \to Z$ be quotient maps. If X and Y are locally compact Hausdorff, then the product $f \times g : W \times Y \to X \times Z$ is also a quotient map.

Proof. Applying Proposition 1.1.13 twice we deduce that the maps $f \times \mathrm{id}_Y : W \times Y \to X \times Y$ and $\mathrm{id}_X \times g : X \times Y \to X \times Z$ are both quotient maps. The result then follows from the equality

$$f \times g = (\mathrm{id}_X \times g) \circ (f \times \mathrm{id}_Y)$$

and the fact that composition preserves quotient maps. ■

1.2 CW-complexes

In this section we define the notion of a CW-complex and mention some of its properties that will be important later on.

The basic idea is that a CW-complex is a space obtained by repeatedly gluing disks of increasing dimensions along their boundary spheres. In order to formalize this, we first define what we mean by gluing disks.

1.2.1 Definition. We say that a pair (X, A), where $A \subseteq X$ is a subspace, is an n-**cellular pair** if there exists a collection of maps $\{\Phi_e : D^n \to X\}_{e\in\mathcal{E}}$ satisfying the following conditions:

1. $\Phi_e(S^{n-1}) \subseteq A$ for every $e \in \mathcal{E}$;

2. the diagram below is a pushout square.

$$\begin{tikzcd}
\coprod_{e\in\mathcal{E}} S^{n-1} \arrow[r, "\langle \Phi_e|_{S^{n-1}} \rangle_{e\in\mathcal{E}}"] \arrow[d, tail] & A \arrow[d, tail] \\
\coprod_{e\in\mathcal{E}} D^n \arrow[r, "\langle \Phi_e \rangle_{e\in\mathcal{E}}"'] & X
\end{tikzcd}$$

In the literature, it is common to say that X **is obtained from** A **by attaching** n**-cells**. In order to simplify the notation a bit, we denote the restricted maps $\Phi_e|_{S^{d(e)-1}}$ by φ_e, and we follow this convention throughout: upper-case for maps on the disk and the corresponding lower-case for its restriction to the boundary sphere. The maps $\Phi_e : D^n \to X$ are called the **characteristic maps**, while the restrictions $\varphi_e : S^{n-1} \to A$ are called **attaching maps**.

We could consider pairs (X, A) where X is obtained from A by simultaneously gluing disks of varying dimensions to the subspace, and many of the required results would still hold. Since in the examples we will encounter the disks glued have all the same dimension, we decided to work with the slightly less general notion of Definition 1.2.1.

The cellular pairs we have introduced are an auxiliary notion used to give an inductive definition of CW-complexes, and thus many properties of CW-complexes actually follow from analogous properties of cellular pairs. We will not need many deep topological properties of cellular pairs and topological spaces, but some basic ones will be useful at several points, so we leave them registered here for later referencing.

1.2.2 Proposition. Let (X, A) be an n-cellular pair, and let $\{\Phi_e : D^n \to X\}_{e\in\mathcal{E}}$ be its family of characteristic maps. The following properties hold:

1. A is a closed subset of X;

2. the map $\langle \Phi_e \rangle_{e \in \mathcal{E}}$ induces by restriction a homeomorphism $X \setminus A \cong \bigsqcup_{e \in \mathcal{E}} (D^n \setminus S^{n-1})$;
3. if A is Hausdorff, then so is X.

It follows from the second item that, for each $e \in \mathcal{E}$, the restriction $\Phi_e|_{D^n \setminus S^{n-1}} : D^n \setminus S^{n-1} \to X$ is an open embedding, and its image $\Phi_e(D^n \setminus S^{n-1}) \subseteq X$ is called an **open n-cell** of (X, A). If e_1, $e_2 \in \mathcal{E}$ are different indices, then the corresponding open n-cells $\Phi_{e_1}(D^n \setminus S^{n-1})$ and $\Phi_{e_2}(D^n \setminus S^{n-1})$ are disjoint, therefore the open n-cells of X constitute a partition of $X \setminus A$ into open subsets.

We now give some examples of cellular pairs that will be useful later.

1.2.3 Example. For every integer $n \geq 0$, the pair (D^n, S^{n-1}) is n-cellular with a single characteristic map given by the identity $\mathrm{id}_{D^n} : D^n \to D^n$, since the diagram below is trivially a pushout.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\mathrm{id}_{S^{n-1}}"] \arrow[d, tail] & S^{n-1} \arrow[d, tail] \\
D^n \arrow[r, "\mathrm{id}_{D^n}"'] & D^n
\end{tikzcd}$$

1.2.4 Example. For any integer $n \geq 1$, regard S^{n-1} as a subspace of S^n via the embedding at the equator $i_{n-1} : S^{n-1} \to S^n$ defined as

$$i_{n-1}(x_1, \ldots, x_n) \coloneqq (x_1, \ldots, x_n, 0)$$

for every $(x_1, \ldots, x_n) \in S^{n-1}$. We claim that the pair (S^n, S^{n-1}) is n-cellular. Consider the maps Φ_n^+, $\Phi_n^- : D^n \to S^n$ defined as follows: given $x = (x_1, \ldots, x_n) \in D^n$

$$\begin{aligned}
\Phi_n^+(x) &\coloneqq \left(x_1, \ldots, x_n, \sqrt{1 - \|x\|^2} \right), \\
\Phi_n^-(x) &\coloneqq \left(x_1, \ldots, x_n, -\sqrt{1 - \|x\|^2} \right).
\end{aligned}$$

Since the restrictions φ_n^+, $\varphi_n^- : S^{n-1} \to S^n$ take values in the subspace S^{n-1} (seen as the image $i_{n-1}(S^{n-1})$), we have the commutative diagram below.

$$\begin{tikzcd}
S^{n-1} \sqcup S^{n-1} \arrow[r, "\langle \varphi_n^+, \varphi_n^- \rangle"] \arrow[d, tail] & S^{n-1} \arrow[d, tail] \\
D^n \sqcup D^n \arrow[r, "\langle \Phi_n^+, \Phi_n^- \rangle"'] & S^n
\end{tikzcd} \tag{1.7}$$

The attaching maps Φ_n^+ and Φ_n^- are embeddings whose images are the north and south hemispheres of S^n, respectively, which intersect along the embedded S^{n-1}. If we are given maps $f : S^{n-1} \to X$ and $g : D^n \sqcup D^n$ satisfying the equality $g|_{S^{n-1} \sqcup S^{n-1}} = f \circ \langle \varphi_n^+, \varphi_n^- \rangle$, we can define a map $h : S^n \to X$ using the formula

$$h(x_1, \ldots, x_{n+1}) \coloneqq \begin{cases} g(i_1(x_1, \ldots, x_n)), & \text{if } x_{n+1} \geq 0, \\ g(i_2(x_1, \ldots, x_n)), & \text{if } x_{n+1} \leq 0, \end{cases}$$

where i_1, $i_2 : D^n \to D^n \sqcup D^n$ are the canonical injections. This is well-defined, because if $x_{n+1} = 0$, then $(x_1, \dots, x_n) \in S^{n-1}$, and using the commutativity condition of g, as well as the equality $\varphi^+ = \varphi^-$, we see that

$$g(i_1(x_1, \dots, x_n)) = f(\varphi^+(x_1, \dots, x_n)) = f(\varphi^-(x_1, \dots, x_n)) = g(i_2(x_1, \dots, x_n)).$$

It follows from the Pasting Lemma that h is continuous, and by definition it satisfies the equalities $h \circ \langle \Phi_n^+, \Phi_n^- \rangle = g$ and $h|_{S^{n-1}} = f$.

Now suppose $h' : S^n \to X$ is another map satisfying the equations $h \circ \langle \Phi_n^+, \Phi_n^- \rangle = g$ and $h'|_{S^{n-1}} = f$. Given $(x_1, \dots, x_{n+1}) \in S^n$, if $x_{n+1} \geq 0$, then

$$(x_1, \dots, x_{n+1}) = \Phi_n^+(x_1, \dots, x_n) = \langle \Phi_n^+, \Phi_n^- \rangle (i_1(x_1, \dots, x_n)),$$

and from this it follows that

$$\begin{aligned} h'(x_1, \dots, x_{n+1}) &= h'(\langle \Phi_n^+, \Phi_n^- \rangle (i_1(x_1, \dots, x_n))) \\ &= g(i_1(x_1, \dots, x_n)) \\ &= h(x_1, \dots, x_{n+1}). \end{aligned}$$

A similar reasoning shows that $h'(x_1, \dots, x_{n+1}) = h(x_1, \dots, x_{n+1})$ also holds if $x_{n+1} \leq 0$, therefore h and h' coincide; showing that (1.7) is a pushout diagram.

1.2.5 Example. Consider the quotient space D^n/S^{n-1} obtained by collapsing the boundary sphere of the disk to a single point, and let $p : D^n \to D^n/S^{n-1}$ be the canonical projection. We will show that there exists a homeomorphism $D^n/S^{n-1} \cong S^n$.

The map $q : S^{n-1} \times I \to D^n$ defined as

$$q(x, t) \coloneqq (1 - t) \cdot x + t \cdot *_{S^{n-1}},$$

where $*_{S^{n-1}} \coloneqq (1, 0, \dots, 0)$, is a quotient map (see Lemma 4.3.4). Let $T : S^{n-1} \times I \to S^n$ be defined as

$$T(x, t) \coloneqq \begin{cases} \Phi_n^+((2t) \cdot x + (1 - 2t) \cdot *_{S^{n-1}}), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \Phi_n^-((2 - 2t) \cdot x + (2t - 1) \cdot *_{S^{n-1}}), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that T is well-defined since, for $t = \frac{1}{2}$ and for any $x \in S^{n-1}$, we have on the one hand

$$\Phi_n^+ \left(\left(2 \cdot \tfrac{1}{2}\right) \cdot x + \left(1 - 2 \cdot \tfrac{1}{2}\right) \cdot *_{S^{n-1}} \right) = \Phi_n^+(x),$$

while on the other

$$\Phi_n^- \left(\left(2 - 2 \cdot \tfrac{1}{2}\right) \cdot x + \left(2 \cdot \tfrac{1}{2} - 1\right) \cdot *_{S^{n-1}} \right) = \Phi_n^-(x),$$

and Φ_n^+ and Φ_n^- agree on the points of $S^{n-1} \subseteq D^n$. There is a strong geometric meaning behind the definition of T. The first expression stretches the lower half $S^{n-1} \times \left[0, \frac{1}{2}\right]$ of the cylinder $S^{n-1} \times I$ upwards to cover the whole cylinder, maps it to the disk via q, and then uses this disk to cover the north hemisphere of S^n via Φ_n^+. Similarly, the second expression stretches the upper half $S^{n-1} \times \left[\frac{1}{2}, 1\right]$ of the cylinder downwards, maps it to the disk via q, and then uses this disk to cover the south hemisphere of S^n using the map Φ_n^-. In this process, the central slice $S^{n-1} \times \left\{\frac{1}{2}\right\}$ is mapped precisely to the equator $S^{n-1} \subseteq S^n$.

The only non-trivial fiber of the quotient map q is $q^{-1}(*_{S^{n-1}}) = (S^{n-1} \times \{1\}) \cup (\{*_{S^{n-1}}\} \times I)$, and a direct computation shows that T maps all points of this fiber to $*_{S^n}$, therefore T induces a map $\overline{T} : D^n \to S^n$ such that $\overline{T}(*_{S^{n-1}}) = *_{S^n}$. Now, since the boundary sphere S^{n-1} is the image $q(S^{n-1} \times \{0\})$, and $T(S^{n-1} \times \{0\}) \subseteq \{*_{S^n}\}$, we have

$$\overline{T}(S^{n-1}) = \overline{T}(q(S^{n-1} \times \{0\})) = T(S^{n-1} \times \{0\}) \subseteq \{*_{S^n}\}.$$

This means that $\overline{T}$ is constant and equal to $*_{S^n}$ on S^{n-1}, thus it can be further factored through p to define a map $\Theta : D^n/S^{n-1} \to S^n$ mapping $[*_{S^{n-1}}]$ to $*_{S^n}$.

$$\begin{tikzcd}
S^{n-1} \times I \arrow[d, "q"'] \arrow[ddr, bend left, "T"] & \\
D^n \arrow[d, "p"'] \arrow[dr, "\overline{T}"] & \\
D^n/S^{n-1} \arrow[r, dashed, "\Theta"'] & S^n
\end{tikzcd}$$

Explicitly, Θ can be described as follows: given $z \in D^n$, if $(x, t) \in S^{n-1} \times I$ is such that $z = (1 - t) \cdot x + t \cdot *_{S^{n-1}}$, with $x = (x_1, \ldots, x_n)$, then

$$\Theta([z]) \coloneqq \begin{cases} (2tx_1 + 1 - 2t, \ldots, 2tx_n, \sqrt{1 - \|(2t) \cdot x + (1 - 2t) \cdot *_{S^{n-1}}\|^2}), \\ ((2 - 2t)x_1 + 2t - 1, \ldots, (2 - 2t)x_n, \sqrt{1 - \|(2 - 2t) \cdot x + (2t - 1) \cdot *_{S^{n-1}}\|^2}), \end{cases} \tag{1.8}$$

where the first expression used for $0 \le t \le \frac{1}{2}$, and the second one for $\frac{1}{2} \le t \le 1$.

Now we construct an inverse for the map Θ. We know from Example 1.2.4 that the pair (S^n, S^{n-1}) is n-cellular with characteristic maps Φ_n^+, $\Phi_n^- : D^n \to S^n$. This means that the square below is a pushout.

$$\begin{tikzcd}
S^{n-1} \arrow[r, tail] \arrow[d, tail] & D^n \arrow[d, "\Phi_n^+"] \\
D^n \arrow[r, "\Phi_n^-"'] & S^n
\end{tikzcd}$$

We will use the universal property of this pushout to induce a map $S^n \to D^n/S^{n-1}$ using certain maps $D^n \to D^n/S^{n-1}$. Consider first $\Psi^+ : S^{n-1} \times I \to D^n/S^{n-1}$ defined as

$$\Psi^+(x, t) \coloneqq p\left((1 - t) \cdot \left(\tfrac{1}{2} \cdot x + \tfrac{1}{2} \cdot *_{S^{n-1}}\right) + t \cdot *_{S^{n-1}}\right) \quad \forall\, (x, t) \in S^{n-1} \times I.$$

A direct computation shows that

$$\Psi^+(q^{-1}(*_{S^{n-1}})) = \Psi^+((S^{n-1} \times \{1\}) \cup (\{*_{S^{n-1}}\} \times I)) \subseteq \{[*_{S^{n-1}}]\},$$

so Ψ^+ descends to a map $\psi^+ : D^n \to D^n/S^{n-1}$.

$$\begin{tikzcd} S^{n-1} \times I \arrow[d, "q"'] \arrow[dr, "\Psi^+"] & \\ D^n \arrow[r, dashed, "\psi^+"'] & D^n/S^{n-1} \end{tikzcd}$$

Similarly, let $\Psi^- : S^{n-1} \times I \to D^n/S^{n-1}$ be defined as

$$\Psi^-(x,t) \coloneqq p\left((1-t)\cdot\left(\tfrac{1}{2}\cdot x + \tfrac{1}{2}\cdot *_{S^{n-1}}\right) + t\cdot x\right) \quad \forall\,(x,t) \in S^{n-1} \times I.$$

This map also sends the whole fiber $q^{-1}(*_{S^{n-1}})$ to $[*_{S^{n-1}}]$, therefore it can be factored through q to define a map $\psi^- : D^n \to D^n/S^{n-1}$.

$$\begin{tikzcd} S^{n-1} \times I \arrow[d, "q"'] \arrow[dr, "\Psi^-"] & \\ D^n \arrow[r, dashed, "\psi^-"'] & D^n/S^{n-1} \end{tikzcd}$$

Notice that ψ^+ and ψ^- coincide on S^{n-1}, since, for any $x \in S^{n-1}$, we have on the one hand

$$\psi^+(x) = \psi^+(q(x,0)) = \Psi^+(x,0) = p\left(\tfrac{1}{2}\cdot x + \tfrac{1}{2}\cdot *_{S^{n-1}}\right),$$

and on the other

$$\psi^-(x) = \psi^-(q(x,0)) = \Psi^-(x,0) = p\left(\tfrac{1}{2}\cdot x + \tfrac{1}{2}\cdot *_{S^{n-1}}\right).$$

The universal property of the pushout then gives us a map $\Psi : S^n \to D^n/S^{n-1}$ that fits in the commutative diagram below.

$$\begin{tikzcd} S^{n-1} \arrow[r, tail] \arrow[d, tail] & D^n \arrow[d, "\Phi_n^+"] \arrow[ddr, bend left, "\psi^+"] & \\ D^n \arrow[r, "\Phi_n^-"'] \arrow[drr, bend right, "\psi^-"'] & S^n \arrow[dr, dashed, "\Psi"] & \\ & & D^n/S^{n-1} \end{tikzcd}$$

It is then a matter of direct computation to show that Θ and Ψ are inverse to one another. Moreover, Θ maps the point $[*_{S^{n-1}}] \in D^n/S^{n-1}$ to the corresponding point $*_{S^n} = (1,0,\ldots,0) \in S^n$. In the language of pointed spaces of Chapter 3, Θ defines a *pointed homeomorphism* $(D^n/S^{n-1}, [*_{S^{n-1}}]) \cong (S^n, *_{S^n})$.

Equipped with such homeomorphism, we can modify the pushout square

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\mathrm{ct}_{S^{n-1},\mathrm{pt}}"] \arrow[d, tail] & \{\mathrm{pt}\} \arrow[d, "\mathrm{ct}_{\{\mathrm{pt}\},[*_{S^{n-1}}]}"] \\
D^n \arrow[r, "p"'] & D^n/S^{n-1}
\end{tikzcd}$$

characterizing the quotient D^n/S^{n-1} to obtain the pushout square below,

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\mathrm{ct}_{S^{n-1},*_{S^n}}"] \arrow[d, tail] & \{*_{S^n}\} \arrow[d, tail] \\
D^n \arrow[r, "\Theta\circ p"'] & S^n
\end{tikzcd}$$

which shows that the pair $(S^n, \{*_{S^n}\})$ is n-cellular.

1.2.6 Definition. A pair (X, A), where $A \subseteq X$ is a subspace, is called a **relative CW-complex** if there exists a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

satisfying the following conditions:

1. $X = \bigcup_{n\geq -1} X_n$;
2. X is the colimit of the subspaces X_n for $n \geq -1$;
3. for every integer $n \geq 0$, (X_n, X_{n-1}) is an n-cellular pair.

If $A = \varnothing$, then we simply say that X is a **CW-complex**.

The subspace X_n appearing in the filtration above is called the n**-skeleton of** X. Each n-skeleton is a closed subset of the next skeleton X_{n+1} by virtue of Proposition 1.2.2. This means that A is closed in X_0, but X_0 is closed in X_1, so A is closed in X_1 too. Continuing inductively we see that A is closed in all the skeletons, therefore A is closed in X itself because the latter is the colimit of the skeletons. A similar argument inductive argument shows that any skeleton X_k is closed in X_n for $n \geq k$, and the intersections $X_k \cap X_j$ are closed in X_j for $j \leq k$ because X_k is closed in X_{k+1} and X_j is a subspace of X_{j+1}; therefore X_k is also closed in X. This shows that all the skeletons of a (relative) CW-complex are automatically closed.

The second property of Proposition 1.2.2 implies that each of the differences $X_n \setminus X_{n-1}$ is a disjoint union of open n-cells. More precisely, if $\{\Phi_e : D^n \to X_{n-1}\}_{e\in\mathcal{E}_n}$ are the attachment maps used to build the n-skeleton, then we have a homeomorphism $X_n \setminus X_{n-1} \cong \bigsqcup_{e\in\mathcal{E}_n} D^n \setminus S^{n-1}$ obtained by restricting $\langle\Phi_e\rangle_{e\in\mathcal{E}_n}$. It follows that the restriction

$\Phi_e|_{D^n\setminus S^{n-1}} : D^n \setminus S^{n-1} \to X_n \setminus X_{n-1}$ is an embedding and its image is open in X for every $n \geq 0$. This allows us to partition $X \setminus A$ into open cells of varying dimensions.

One interesting consequence of this decomposition is that a CW-complex is automatically Hausdorff. Indeed, given two distinct points $x_1, x_2 \in X$, we can find an open n_1-cell X_{n_1} such that $x_1 \in X_{n_1}$, and an open n_2-cell X_{n_2} such that $x_2 \in X_{n_2}$. If these two cells are different, then they separated the two points since they are disjoint and open in X. If $X_{n_1} = X_{n_2}$, since these cells are homeomorphic to the interior of a disk, which is a Hausdorff space, we can separate the two points inside the cell, and this also separates them in X because again the cell is itself open in X.

1.2.1 Some results on products

In this subsection we briefly mention some results concerning products of absolute and relative CW-complexes. We first show that the operation of attaching cells to a subspace interacts nicely with products. The next example is the guiding principle this result.

1.2.7 Example. We have seen in Example 1.2.3 that the disk D^n can be obtained from its boundary sphere S^{n-1} by a single n-cell attachment. Given two disks D^m and D^n, can the cell attachments of each one be combined to describe the product $D^m \times D^n$ as being obtained from a subspace by cell attachments? We will see that this is true, but the subspace in question is not the most obvious choice of the product $S^{m-1} \times S^{n-1}$ of the two boundary spheres.

Consider the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ equipped with their usual euclidean norms. We then consider the corresponding maximum norm on the product $\mathbb{R}^m \times \mathbb{R}^n$, which induces a norm on $\mathbb{R}^{m+n}$ via the homeomorphism $\alpha : \mathbb{R}^{m+n} \to \mathbb{R}^m \times \mathbb{R}^n$ defined as

$$\alpha(x_1, \ldots, x_m, y_1, \ldots, y_n) := ((x_1, \ldots, x_m), (y_1, \ldots, y_n)).$$

Explicitly, this norm, which we denote by $\|-\|_{m,n}$ is given by

$$\|(x_1, \ldots, x_m, y_1, \ldots, y_n)\|_{m,n} := \max\{\|(x_1, \ldots, x_m)\|, \|(y_1, \ldots, y_n)\|\}.$$

Let D denote the unit disk with respect to the norm $\|-\|_{m,n}$, and let $S := \partial D$ denote its boundary, which is the unit sphere with respect to the norm $\|-\|_{m,n}$. Recall that all norms on $\mathbb{R}^{m+n}$ are equivalent, so there is a homeomorphism of pairs $(\mathbb{R}^{m+n}, S^{m+n-1}) \cong (D, S)$. If we use the simplified notation $(x, y) := (x_1, \ldots, x_m, y_1, \ldots, y_n)$, then this homeomorphism is given by the map $\beta : D^{m+n} \to D$ defined as

$$\beta(x, y) := \begin{cases} \mathbf{0}, & \text{if } (x, y) = \mathbf{0}, \\ \dfrac{\|(x, y)\|}{\|(x, y)\|_{m,n}} \cdot (x, y), & \text{otherwise.} \end{cases}$$

Geometrically, this first normalizes the point (x, y) with respect to the norm $\|-\|_{m,n}$ sending it to the sphere S, and then moves it inside the disk D by scaling it by $\|(x, y)\|$. The inverse has a very similar description:

$$\beta^{-1}(x,y) := \begin{cases} \mathbf{0}, & \text{if } (x,y) = \mathbf{0}, \\ \dfrac{\|(x,y)\|_{m,n}}{\|(x,y)\|} \cdot (x,y), & \text{otherwise.} \end{cases}$$

Now, the homeomorphism $\alpha : \mathbb{R}^{m+n} \to \mathbb{R}^m \times \mathbb{R}^n$ restricts to a homeomorphism between D and $D^m \times D^n$. Notice that $\|(x, y)\|_{m,n} = 1$ if and only if $\|x\| = 1$ or $\|y\| = 1$, therefore we have a homeomorphism of pairs

$$\alpha|_D : (D, S) \xrightarrow{\cong} (D^m \times D^n, (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n)).$$

By composition, we then have a homeomorphism of pairs

$$\Delta_{m,n} : (D^{m+n}, S^{m+n-1}) \to (D^m \times D^n, (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n))$$

as shown below.

$$\begin{tikzcd} (D^{m+n}, S^{m+n-1}) \arrow[r, "\beta"] \arrow[rr, bend right, dashed, "\Delta_{m,n}"'] & (D, S) \arrow[r, "\alpha|_D"] & (D^m \times D^n, (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n)). \end{tikzcd}$$

Since $\Delta_{m,n}$ is a homeomorphism, the square below is a pushout, showing that the pair $(D^m \times D^n, (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n))$ is $(m + n)$-cellular.

$$\begin{tikzcd} S^{m+n-1} \arrow[r, "\delta_{m,n}"] \arrow[d, tail] & (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n) \arrow[d, tail] \\ D^{m+n} \arrow[r, "\Delta_{m,n}"'] & D^m \times D^n \end{tikzcd}$$

This example allows us to prove a more general result on products of cellular pairs under some finiteness conditions.[1]

1.2.8 Proposition. If (X, A) a finite m-cellular pair, and (Y, B) is a finite n-cellular pair, with both A and B Hausdorff, then $(X \times Y, (X \times B) \cup (A \times Y))$ is a finite $(m+n)$-cellular pair.

Proof. Let $\{\Phi_i : D^m \to X\}_{i \in I}$ and $\{\Psi_j : D^n \to X\}_{j \in J}$ be the two finite families of characteristic maps for the two cellular pairs. For each pair of indices $(i, j) \in I \times J$, let $\Theta_{(i,j)} : D^{m+n} \to X \times Y$ be the composite map

$$\Theta_{(i,j)} := (\Phi_i \times \Psi_j) \circ \Delta_{m,n},$$

[1] On a personal remark, most versions of this statement that I could find in the literature were formulated for relative CW-complexes, which is a bit stronger than what we will need, so I decided to include a more or less complete proof of this simpler result.

where $\Delta_{m,n} : D^{m+n} \to D^m \times D^n$ is the homeomorphism discussed in Example 1.2.7. We would like to show that the diagram

$$\begin{tikzcd}
\coprod_{(i,j)\in I\times J} S^{m+n-1} \arrow[r, "\langle\theta_{(i,j)}\rangle"] \arrow[d, tail] & (X\times B)\cup(A\times Y) \arrow[d, tail] \\
\coprod_{(i,j)\in I\times J} D^{m+n} \arrow[r, "\langle\Theta_{(i,j)}\rangle"'] & X\times Y
\end{tikzcd}$$

is a pushout square. Since we already have the pushout square

$$\begin{tikzcd}
\coprod_{(i,j)\in I\times J} S^{m+n-1} \arrow[r, "\coprod_{(i,j)}\delta_{m,n}"] \arrow[d, tail] & \coprod_{(i,j)\in I\times J} (D^m\times S^{n-1})\cup(S^{m-1}\times D^n) \arrow[d, tail] \\
\coprod_{(i,j)\in I\times I} D^{m+n} \arrow[r, "\coprod_{(i,j)}\Delta_{m,n}"'] & \coprod_{(i,j)\in I\times J} D^m\times D^n
\end{tikzcd}$$

of Example 1.2.7, it suffices to show that the square below is also a pushout,

$$\begin{tikzcd}
\coprod_{(i,j)\in I\times J} (D^m\times S^{n-1})\cup(S^{m-1}\times D^n) \arrow[r] \arrow[d, tail] & (X\times B)\cup(A\times Y) \arrow[d, tail] \\
\coprod_{(i,j)\in I\times J} D^m\times D^n \arrow[r, "\langle\Phi_i\times\Psi_j\rangle"'] & X\times Y
\end{tikzcd}$$

and then use the Pasting Law for pushouts.[2]

We now sketch the proof that the diagram above is really a pushout. According to Proposition 1.2.2, the spaces X and Y can be covered by the finite families of closed subsets

$$\{A\}\cup\{\Phi_i(D^m)\}_{i\in I} \quad \text{and} \quad \{B\}\cup\{\Psi_j(D^n)\}_{j\in J},$$

respectively. The product $X\times Y$ can then be covered by the finite family of closed subsets

$$\{(X\times B)\cup(A\times Y)\}\cup\{\Phi_i(D^m)\times\Psi_j(D^n)\}_{(i,j)\in I\times J}.$$

Let Z be another space, and suppose we are given a map $f : (X\times B)\cup(A\times Y) \to Z$ and a family of maps $\{g_{(i,j)} : D^m\times D^n \to Z\}_{(i,j)\in I\times J}$ satisfying the following commutativity condition: the equation

$$f\circ(\Phi_i\times\Psi_j)|_{(D^m\times S^{n-1})\cup(S^{m-1}\times D^n)} = g_{(i,j)}|_{(D^m\times S^{n-1})\cup(S^{m-1}\times D^n)} \tag{1.9}$$

holds for every pair of indices $(i,j)\in I\times J$. Let $h : X\times Y \to Z$ be the map defined as follows: given $(x,y)\in X\times Y$, if either $x\in A$ or $y\in B$ we set $h(x,y) := f(x,y)$; otherwise

[2] See (NLAB, 2021, Proposition 3.3) for a proof of the dual result on pasting pullback diagrams.

there is a unique $(i,j) \in I \times J$ and a unique $(p,q) \in (D^m \setminus S^{m-1}) \times (D^n \setminus S^{n-1})$ such that $(x,y) = (\Phi_i(p), \Psi_j(y))$, and we then set $h(x,y) \coloneqq g_{(i,j)}(p,q)$.

So far we have only defined h as a function, we still need to verify its continuity, and this is where the covering by closed sets comes in. The restriction of h to $(X \times B) \cup (A \times Y)$ coincides with f, so its is continuous. Now, given a pair $(i,j) \in I \times J$, we claim that the diagram below is commutative.

$$\begin{tikzcd} D^m \times D^n \arrow[r, "g_{(i,j)}"] \arrow[d, "\Phi_i \times \Psi_j"'] & Z \\ \Phi_i(D^m) \times \Psi_j(D^n) \arrow[ur, "h|_{\Phi_i(D^m) \times \Psi_j(D^n)}"'] & \end{tikzcd} \tag{1.10}$$

We split this in two cases:

1. If $(p,q) \in (D^m \setminus S^{m-1}) \times (D^n \setminus S^{n-1})$, then the equality $h(\Phi_i(p), \Psi_j(q)) = g_{(i,j)}(p,q)$ follows from the very definition of h.

2. If either $p \in S^{m-1}$, or $q \in S^{n-1}$, then $(\Phi_i(p), \Psi_j(q)) \in (X \times B) \cup (A \times Y)$, then combining the definition of h with the commutativity condition of (1.9) we see that

$$h(\Phi_i(p), \Psi_j(q)) = f(\Phi_i(p), \Psi_j(q)) = g_{(i,j)}(p,q).$$

Now, since A and B are Hausdorff by hypothesis, so are X and Y according to Proposition 1.2.2, and therefore so is the product $X \times Y$ and its subspace $\Phi_i(D^m) \times \Psi_j(D^n)$. The map $\Phi_i \times \Psi_j : D^m \times D^n \to \Phi_i(D^m) \times \Psi_j(D^n)$ is then a quotient map, and the commutativity condition of (1.10) then implies the continuity of $h|_{\Phi_i(D^m) \times \Psi_j(D^n)}$.

All of this argument so far has shown that h is continuous when restricted to each of the elements of the closed cover $\{(X \times B) \cup (A \times Y)\} \cup \{\Phi_i(D^m) \times \Psi_j(D^n)\}_{(i,j) \in I \times J}$. Since this cover has only finitely many elements, it follows from the Pasting Lemma that h is continuous, and by construction it satisfies the equalities $h|_{(X \times B) \cup (A \times Y)} = f$, and $h \circ \langle \Phi_i \times \Psi_j \rangle = \langle g_{(i,j)} \rangle$. Lastly, arguing by cases we see that, if $h' : X \times Y \to Z$ satisfies these same two equalities, then we must have $h = h'$. ■

1.2.9 Remark. It is important to notice that the finiteness conditions of Proposition 1.2.8 is only used at the very end to deduce the continuity of $h : X \times Y \to Z$ from the continuity of its restrictions to a certain cover by closed subsets. The finiteness condition is necessary to use the Pasting Lemma, but the result still holds under weaker assumptions, and the proof is essentially the same.

The idea behind this more general version is the following result from General Topology: *if $\{C_i\}_{i \in I}$ is a locally finite covering of a space X by closed subsets, then a subset $A \subseteq X$ is closed if and only if the intersection $A \cap C_i$ is closed in C_i for every $i \in I$.*

It follows from the above result that a function $h : X \to Y$ is continuous if and only if the restriction $h|_{C_i} : C_i \to Y$ is continuous for every $i \in I$. The proof we gave for Proposition 1.2.8 still works if we replace the finiteness conditions on (X, A) and (Y, B) by the condition that at least one of them is *locally finite*, which means that each of its vertices is contained only in a finite number of cells.

CHAPTER

2

BASIC NOTIONS OF HOMOTOPY THEORY

In this chapter we begin our study of Homotopy Theory with some basic definitions and results. In the first section, we define the classical notion of homotopy and prove its main properties that are later used to define the homotopy category. This initial section also contains an explanation of how the exponential law allows us to give two alternative formulations of the notion of homotopy, one of which is always equivalent to the classical one, and the other only under some topological conditions.

The second section introduces the infamous homotopy category. This is the category where many of the important functors of Algebraic Topology are "naturally" defined. We also show how some common functors of spaces like products and coproducts interact with homotopies.

In the last section, we introduce the notions of contractible spaces and null homotopic maps, and then prove how these two are related. We then investigate a particular instance of this relation and prove an important result which can be seen as one of the cornerstones of Obstruction Theory.

2.1 Different notions of homotopy

This section defines the classical notion of homotopy and also gives alternative equivalent definitions based on the exponential adjunction. The material of this section is mostly based on the exposition contained in (ARKOWITZ, 2011).

2.1.1 Definition. Let X and Y be spaces. Two maps $f, g : X \to Y$ are said to be **homotopic** if there is a map $H : X \times I \to Y$ satisfying the following properties:

1. $H(x, 0) = f(x)$ for every $x \in X$;
2. $H(x, 1) = g(x)$ for every $x \in X$.

The map H is said to be a **homotopy from f to g**. We use the notation $f \simeq g$ to simply say that two maps are homotopic, and, if we want to consider an explicit homotopy, we either use the notation $H : f \simeq g$ or $H : f \Rightarrow g$.

2.1.2 Remark. The reader may wonder why we used the notation $f \Rightarrow g$ for homotopies, since we already use this for natural transformations. This is because there are similarities between homotopies and natural transformations. We can transform a category using functors just as we can transform a space using maps; but we can also transform a functor between two categories using natural transformations just as we can transform a map between two spaces using homotopies. In other words, in both the category Cat of all (small) categories and the category Top of all topological spaces, there is a notion of deformation between two morphisms with the same domain and codomain. This hints at the fact that Cat and Top are not just categories, but in fact *2-categories*, a notion which is (sadly) beyond the scope of this text.

The definition of homotopy can also be rewritten diagrammatically. For any space X, let $i_{X,0} : X \to X \times I$ be the map defined as $i_{X,0}(x) := (x, 0)$ for every $x \in X$. We also have the analogously defined map $i_{X,1} : X \to X \times I$ which maps X to the top face of the cylinder $X \times I$. Two maps $f, g : X \to Y$ are then homotopic if and only if there is a map $H : X \times I \to Y$ which makes the diagram below commute.

$$\begin{tikzcd} X \arrow[d, "i_{X,0}"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, dashed, "H"] & Y \\ X \arrow[u, "i_{X,1}"] \arrow[ur, "g"'] & \end{tikzcd} \tag{2.1}$$

Geometrically, a homotopy from f to g deforms the image of f into the image of g inside the space Y. This is a deformation that happens along a family of paths parameterized by the points of X. The next result makes this idea precise and shows that it in fact works both ways.

2.1.3 Proposition. Given any spaces X and Y, two maps $f, g : X \to Y$ are homotopic if and only if there is a map $D : X \to \mathrm{Map}(I, Y)$ satisfying the following properties:

1. $\mathrm{ev}_{0,Y} \circ D = f$, that is, $[D(x)](0) = f(x)$ for every $x \in X$.

2. $\mathrm{ev}_{1,Y} \circ D = g$, that is, $[D(x)](1) = g(x)$ for every $x \in X$.

In other words, $D(x)$ is a path in from $f(x)$ to $g(x)$ in Y, and these paths depend continuously on x.

Proof. Suppose $H : X \times I \to Y$ defines a homotopy from f to g. Let $D \coloneqq \lambda H : X \to \mathrm{Map}(I, Y)$ be the exponential adjoint of H. For every $x \in X$

$$\begin{aligned} \mathrm{ev}_{0,Y}(D(x)) &= [D(x)](0) \\ &= [\lambda H(x)](0) \\ &= H(x, 0) \\ &= f(x), \end{aligned}$$

thus $\mathrm{ev}_{0,Y} \circ D = f$, and by a completely analogous reasoning

$$\mathrm{ev}_{1,Y} \circ D = g;$$

therefore D satisfies the required conditions.

Conversely, suppose there exists a map $D : X \to \mathrm{Map}(I, Y)$ satisfying the properties stated above. Since the unit interval I is locally compact Hausdorff, by the exponential adjunction there exists a map $H : X \times I \to Y$ such that $\lambda H = D$. This map H then satisfies, for every $x \in X$,

$$\begin{aligned} H(x, 0) &= [\lambda H(x)](0) \\ &= [D(x)](0) \\ &= (\mathrm{ev}_{0,Y} \circ D)(x) \\ &= f(x), \end{aligned}$$

i.e., the equality $H \circ i_{X,0} = f$ holds, and by an analogous argument so does the equality $H \circ i_{X,1} = g$; proving that H defines a homotopy $f \simeq g$. ∎

The conditions satisfied by the map D in Proposition 2.1.3 are equivalent to the commutativity of the diagram below,

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, dashed, "D"] \arrow[dr, "g"'] & \mathrm{Map}(I, Y) \arrow[u, "\mathrm{ev}_{0,Y}"'] \arrow[d, "\mathrm{ev}_{1,Y}"] \\ & Y \end{tikzcd} \tag{2.2}$$

which is in some sense dual to diagram (2.1).

Although Proposition 2.1.3 is a rather simple results, sometimes thinking of a homotopy as a parameterized family of paths instead of as a deformation defined on a cylinder either helps us give a simpler proof of a result, or it gives an alternative interpretation of a classical result. We illustrate this second advantage by proving that homotopy gives rise to an equivalence relation.

2.1.4 Proposition. Let X and Y be arbitrary topological spaces. The homotopy relation $\simeq$ between maps is an equivalence relation on the set $\mathsf{Top}(X, Y)$.

Proof. We first show that $\simeq$ is reflexive. Let $f : X \to Y$ be any map. The first projection $\pi_1 : X \times I \to X$ induces by currying a map

$$C \coloneqq \lambda\pi_1 : X \to \mathrm{Map}(I, X).$$

Geometrically, for any $x \in X$, $C(x)$ is the constant path at x. Using f we can also define the pushforward map

$$\mathrm{Map}(I, f) : \mathrm{Map}(I, X) \to \mathrm{Map}(I, Y),$$

and then define $D : X \to \mathrm{Map}(I, Y)$ as the composition shown below.

$$\begin{tikzcd} X \arrow[r, "C"] \arrow[rr, bend right, dashed, "D"'] & \mathrm{Map}(I, X) \arrow[r, "\mathrm{Map}(I,f)"] & \mathrm{Map}(I, Y) \end{tikzcd}$$

Unwrapping the definitions we see that $[D(x)](0) = [D(x)](1) = f(x)$ for every $x \in X$, thus $D(x)$ defines a path from $f(x)$ to itself, which implies that $f \simeq f$ by Proposition 2.1.3.

Now we prove that $\simeq$ is symmetric. Given maps f, $g : X \to Y$, suppose $f \simeq g$, and then let $D : X \to \mathrm{Map}(I, Y)$ be a family of paths from $f(x)$ to $g(x)$ for every $x \in X$. In order to obtain a reverse homotopy $g \simeq f$, we want to reverse the paths $D(x)$ so that they now start at $g(x)$ and $f(x)$. We can reverse all the paths simultaneously and continuously by considering the reversal map $r : I \to I$ defined as $r(t) \coloneqq 1 - t$ for every $t \in I$, forming the pullback map

$$\mathrm{Map}(r, Y) : \mathrm{Map}(I, Y) \to \mathrm{Map}(I, Y),$$

and then defining a map $\overline{D} : X \to \mathrm{Map}(I, Y)$ by the composition

$$\overline{D} \coloneqq \mathrm{Map}(r, Y) \circ D.$$

For any $x \in X$ we have

$$[\overline{D}(x)](0) = [\mathrm{Map}(r, Y)(D(x))](0) = [D(x) \circ r](0) = [D(x)](1) = g(x),$$

and similarly, we also have

$$[\overline{D}(x)](1) = [\mathrm{Map}(r, Y)(D(x))](1) = [D(x) \circ r](1) = [D(x)](0) = f(x);$$

therefore $\overline{D}$ induces the desired homotopy $f \simeq g$.

Lastly, we have to show the transitivity of $\simeq$. Here, we just give the classical proof using maps defined on cylinders. Consider three maps f, g, $h : X \to Y$ together with homotopies $H_1 : f \Rightarrow g$ and $H_2 : g \Rightarrow h$. Define $H : X \times I \to Y$ by the formula

$$H(x, t) \coloneqq \begin{cases} H_1(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

which is well-defined by the Pasting Lemma because at $t = \frac{1}{2}$ we have $H_1(x, 1) = g(x) = H_2(x, 0)$. We can verify directly that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for every $x \in X$, therefore we have the relation $f \simeq h$. ■

Of course, the proofs above are not better than the classical ones, and one might even argue that they are more complicated. Nonetheless, they serve at least two purposes:

1. They illustrate the algebraic manipulation we can perform with tools like push-forwards, pullbacks and adjunctions, which help us construct maps of a specified type.

2. As we had already remarked before, these proofs can be seen as alternative to the classical ones, and in fact, we can recover them by simply uncurrying. In the proof of symmetry, for example, if we let H, $\overline{H} : X \times I \to Y$ be the unique maps such that $\lambda H = D$ and $\lambda H = \overline{D}$, then we have the equalities

$$\overline{H}(x, t) = [\lambda\overline{H}(x)](t) = [\overline{D}(x)](t) = [D(x)](1 - t) = H(x, 1 - t),$$

and this is how we classically obtain a homotopy $\overline{H} : g \Rightarrow f$ from a homotopy $H : f \Rightarrow g$.

We finish this section introducing yet another way of thinking about homotopies. Given a map $H : X \times I \to Y$, instead of currying with respect to the second variable to obtain a map of type $X \to \mathrm{Map}(I, Y)$, we can curry with respect to the first variable to obtain a map of type $I \to \mathrm{Map}(X, Y)$, which we denote by $\lambda' H$ to differentiate from the usual adjoint λH that we have used so far. This other adjoint $\lambda' H$ defines a path in the space of maps $\mathrm{Map}(X, Y)$. If H is in fact a homotopy from f to g, then we have

$$[\lambda' H(0)](x) = H(x, 0) = f(x),$$

so the path $\lambda' H$ starts at the map f, and an analogous calculation shows that $\lambda' H$ ends at g.

This means that a homotopy between two maps connects them by a path inside the space $\mathrm{Map}(X, Y)$. Sadly, this intuition does not work in the reverse direction in general, because without additional hypothesis on X, the construction of this other exponential adjoint λ' might not give rise to a bijection

$$\mathsf{Top}(X \times I, Y) \cong \mathsf{Top}(I, \mathrm{Map}(X, Y)),$$

so a path $I \to \mathrm{Map}(X, Y)$ from f to g might not correspond to a homotopy $X \times I \to Y$ from f to g.

There is a way to circumvent this difficulty. We admit to ourselves that the category Top is too pathological to develop Homotopy Theory comfortably, and we work with a more well-behaved subcategory. One typical choice of such subcategory is the category of *compactly generated spaces*. This class of spaces admits all the usual constructions of Topology, that is, it admits all limits and colimits. Moreover, all objects of this category are exponentiable, so any two compactly generated spaces X and Y have a compactly generated space of maps between them, and there is an exponential adjunction relating products and spaces of maps that *always* works. This allows us to identify homotopies with paths in the space of maps, something which is very useful for developing homotopy theory. Notice, for example, that with this identification the fact that the homotopy relation is an equivalence relation follows from the fact that the "connected by a path" relation is an equivalence relation.[1]

2.2 The homotopy category

In the end of the previous section we showed that the homotopy relation is an equivalence relation on the set of all maps between two spaces. We can therefore consider the quotient by this relation.

2.2.1 Definition. Let X and Y be spaces. The quotient *set* $\mathsf{Top}(X,Y)/\simeq$ is denoted by $[X,Y]$, and its elements are called **homotopy classes of maps**. If $f : X \to Y$ is a map, its corresponding equivalence class in $[X,Y]$ is denoted by $[f]$.

2.2.2 Remark. It is important to stress that we are considering a quotient *set*. We could in principle consider the quotient *space* $\mathrm{Map}(X,Y)/\simeq$ equipped with the quotient topology. This is an interesting idea and could lead to a topological version of the homotopy groups for example, but sadly this quotient space is not well-behaved in general with respect to the usual operations that we perform on homotopy classes.

We now investigate how the notion of homotopy interacts with composition of maps. First we investigate the compatibility with composition on one side at a time.

2.2.3 Proposition. Let W, X, Y and Z be spaces, and consider maps $\alpha : W \to X$, $f, g : X \to Y$ and $\beta : Y \to Z$ as shown in the (not necessarily commutative) diagram below.

$$W \xrightarrow{\alpha} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\beta} Z$$

If $f \simeq g$, then $f \circ \alpha \simeq g \circ \alpha$ and $\beta \circ f \simeq \beta \circ g$.

[1] This text once contained a whole chapter dedicated to compactly generated spaces, as well as explanations of how to develop classical Homotopy Theory inside this subcategory. However, since we will not need very deep notions of Homotopy Theory, I eventually decided that compactly generated spaces would not be very useful for the purposes of this text and removed this chapter from the text.

Proof. Let $D : X \to \mathrm{Map}(I, Y)$ be the family of paths induced by a homotopy $H : f \Rightarrow g$. The composite map

$$D \circ \alpha : W \to \mathrm{Map}(I, Y)$$

satisfies

$$\mathrm{ev}_{0,Y} \circ D \circ \alpha = f \circ \alpha$$

and similarly,

$$\mathrm{ev}_{1,Y} \circ D \circ \alpha = g \circ \alpha,$$

therefore it induces a homotopy $f \circ \alpha \simeq g \circ \alpha$.

For the second part, let $H' : X \times I \to Z$ be defined as the composition

$$H' \circ \beta \circ H.$$

On the one hand H' satisfies the equality

$$H' \circ i_{X,0} = \beta \circ H \circ i_{X,0} = \beta \circ f,$$

and on the other it satisfies

$$H' \circ i_{X,1} = \beta \circ H \circ i_{X,1} = \beta \circ g;$$

therefore H' defines a homotopy $\beta \circ f \simeq \beta \circ g$. ■

Using Proposition 2.2.3 together with the transitivity of homotopy we prove that homotopy is preserved by composition on both sides.

2.2.4 Corollary. Let X, Y and Z be spaces, and consider two pairs of maps f_1, $g_1 : X \to Y$ and f_2, $g_2 : Y \to Z$ as in the (not necessarily commutative) diagram below.

$$X \underset{g_1}{\overset{f_1}{\rightrightarrows}} Y \underset{g_2}{\overset{f_2}{\rightrightarrows}} Z$$

If f_1 and g_1 are homotopic, and f_2 and g_2 are homotopic, then the composite maps $f_2 \circ f_1$ and $g_2 \circ g_1$ are also homotopic.

Proof. We know from Proposition 2.2.3 that homotopies are preserved by pullbacks and pushforward. If we pushforward the relation $f_1 \simeq g_1$ along f we deduce that $f_2 \circ f_1 \simeq f_2 \circ g_1$, while if we pullback the relation $f_2 \simeq g_2$ along g_1 we deduce that $f_2 \simeq g_1 \simeq g_2 \circ g_1$. Transitivity of homotopy then implies the desired relation $f_2 \circ f_1 \simeq g_2 \circ g_1$. ■

This result on the compatibility of homotopies with composition allows us to define a composition operation on the level of homotopy classes. Given spaces X, Y and Z, define a composition $[Y, Z] \times [X, Y] \to [X, Z]$ by the rule

$$[g] \circ [f] \coloneqq [g \circ f]$$

for any $[f] \in [X, Y]$ and $[g] \in [Y, Z]$. This composition is independent of the actual maps used to represent the homotopy classes by virtue of Corollary 2.2.4. Since this composition of classes is defined in terms of the usual composition of maps, it satisfies the usual conditions of associativity and existence of identities, where the identity of $[X, X]$ is given by $[\mathrm{id}_X]$.

2.2.5 Definition. We define a category HoTop whose objects are topological spaces, and whose set of morphisms between two objects X and Y is by definition the set of homotopy classes of maps $\mathsf{HoTop}(X, Y) := [X, Y]$, with composition defined as above. This is called the **classical homotopy category**.

Now that we have a category, we can specialize many concepts of Category Theory to it, and of particular important is the specialization of the concept of isomorphism.

2.2.6 Definition. A map $f : X \to Y$ between topological spaces is said to be **homotopy equivalence** if $[f]$ is an isomorphism in HoTop. In this case we either say that X and Y are **homotopy equivalent** or that they are **of the same homotopy type** and denote this by $X \simeq Y$.

Unpacking the definition, if $f : X \to Y$ is a homotopy equivalence, then there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We say that the map g is a **homotopy inverse** of f.

We now investigate how some common functors of topological spaces interact with the notion of homotopy.

2.2.7 Proposition. Given spaces X, Y_1 and Y_2, and given maps $f_1, g_1 : X \to Y_1$ and $f_2, g_2 : X \to Y_2$, if $f_1 \simeq g_1$ and $f_2 \simeq g_2$, then the induced maps $(f_1, f_2), (g_1, g_2) : X \to Y_1 \times Y_2$ are homotopic.

Proof. We prove this working with homotopies as maps out of a cylinder. Let $H_1 : X \times I \to Y_1$ be a homotopy from f_1 to g_1, and let $H_2 : X \times I \to Y_2$ be a homotopy from f_2 to g_2. We have the induced map

$$H := (H_1, H_2) : X \times I \to Y_1 \times Y_2$$

which is described explicitly as

$$H(x, t) := (H_1(x, t), H_2(x, t)) \quad \forall\, (x, t) \in X \times I.$$

We wish to show that (H_1, H_2) defines a homotopy $(f_1, f_2) \simeq (g_1, g_2)$. The first step is proving the equality

$$(H_1, H_2) \circ i_{X,0} = (f_1, f_2).$$

If $\pi_1' : Y_1 \times Y_2 \to Y_1$ and $\pi_2' : Y_1 \times Y_2 \to Y_2$ denote the canonical projections, we have the equalities

$$\begin{aligned} \pi_1' \circ (H_1, H_2) \circ i_{X,0} &= H_1 \circ i_{X,0} \\ &= f_1, \end{aligned}$$

and a similar computation shows that we also have the equality

$$\pi_2' \circ (H_1, H_2) \circ i_{X,1} = f_2.$$

The universal property of the product then implies the equality

$$(H_1, H_2) \circ i_{X,0} = (f_1, f_2),$$

and an analogous reasoning shows that

$$(H_1, H_2) \circ i_{X,1} = (g_1, g_2);$$

therefore (H_1, H_2) defines the desired homotopy. ■

2.2.8 Corollary. The product functor $\times : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ respects homotopies. More precisely, for any spaces X_1, X_2, Y_1 and Y_2, and for any maps f_1, $g_1 : X_1 \to Y_1$ and f_2, $g_2 : X_2 \to Y_2$, if $f_1 \simeq g_1$ and $f_2 \simeq g_2$, then we also have a homotopy $f_1 \times f_2 \simeq g_1 \times g_2$

Proof. If π_1 and π_2 are the canonical projections out of the product, recall that the product $f_1 \times f_2$ is by definition the map induced by $f_1 \circ \pi_1 : X_1 \times X_2 \to Y_1$ and $f_2 \circ \pi_2 : X_2 \to Y_2$, that is, we have an equality

$$f_1 \times f_2 = (f_1 \circ \pi_1, f_2 \circ \pi_2),$$

and a similar one holds for the product $g_1 \times g_2$.

The hypothesis $f_1 \simeq g_1$ implies $f_1 \circ \pi_1 \simeq g_1 \circ \pi_1$, and similarly, $f_2 \circ \pi_2 \simeq g_2 \circ \pi_2$. Using the equalities of the previous paragraph together with the Proposition 2.2.7 we get

$$f_1 \times f_2 = (f_1 \circ \pi_1, f_2 \circ \pi_2) \simeq (g_1 \circ \pi_1, g_2 \circ \pi_2) = g_1 \times g_2.$$ ■

We have dual results for coproducts too.

2.2.9 Proposition. Let X_1, X_2 and Y be spaces, and consider maps f_1, $g_1 : X_1 \to Y$ and f_2, $g_2 : X_2 \to Y$. If $f_1 \simeq g_1$ and $f_2 \simeq g_2$, then the induced maps $\langle f_1, f_2\rangle$, $\langle g_1, g_2\rangle : X_1 \sqcup X_2 \to Y$ are homotopic.

Proof. In this proof we regard homotopies as families of paths. Let $D_1 : X_1 \to \mathrm{Map}(I, Y)$ and $D_2 : X_2 \to \mathrm{Map}(I, Y)$ be the families of paths induced by the homotopies $f_1 \simeq g_1$ and $f_2 \simeq g_2$, respectively. The universal property of the coproduct then gives us the map

$$\langle D_1, D_2\rangle : X_1 \sqcup X_2 \to \mathrm{Map}(I, Y).$$

We need to check that the diagram below commutes.

$$\begin{tikzcd}
 & & Y \\
X_1 \sqcup X_2 \arrow[urr, "f_1 \sqcup f_2"] \arrow[rr, "\langle D_1, D_2 \rangle"] \arrow[drr, "g_1 \sqcup g_2"'] & & \mathrm{Map}(I, Y) \arrow[u, "\mathrm{ev}_{0,Y}"'] \arrow[d, "\mathrm{ev}_{1,Y}"] \\
 & & Y
\end{tikzcd}$$

If j_1 and j_2 are the canonical injections into $X_1 \sqcup X_2$, by the defining properties of $\langle D_1, D_2 \rangle$ we have

$$\begin{aligned} \mathrm{ev}_{0,Y} \circ \langle D_1, D_2 \rangle \circ j_1 &= \mathrm{ev}_{0,Y} \circ D_1 \\ &= f_1, \end{aligned}$$

and similarly,

$$\mathrm{ev}_{0,Y} \circ \langle D_1, D_2 \rangle \circ j_2 = f_2.$$

These two equalities together imply

$$\mathrm{ev}_{0,Y} \circ \langle D_1, D_2 \rangle = \langle f_1, f_2 \rangle,$$

and by a completely analogous reasoning we also have the equality

$$\mathrm{ev}_{1,Y} \circ \langle D_1, D_2 \rangle = \langle g_1, g_2 \rangle;$$

showing that $\langle D_1, D_2 \rangle$ induces a homotopy $\langle f_1, f_2 \rangle \simeq \langle g_1, g_2 \rangle$. ■

2.2.10 Corollary. The coproduct functor $\sqcup : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ respects homotopies. More precisely, for any spaces X_1, X_2, Y_1 and Y_2, and for any maps f_1, $g_1 : X_1 \to Y_1$ and f_2, $g_2 : X_2 \to Y_2$, if $f_1 \simeq g_1$ and $f_2 \simeq g_2$, then we also have a homotopy $f_1 \sqcup f_2 \simeq g_1 \sqcup g_2$.

Proof. Let j_1' and j_2' be the canonical injections into $Y_1 \sqcup Y_2$. The coproduct $f_1 \sqcup f_2$ is the universal map induced by the maps $j_1' \circ f_1 : X_1 \to Y_1 \sqcup Y_2$ and $j_2' \circ f_2 : X_2 \to Y_1 \sqcup Y_2$, that is, we have an equality

$$f_1 \sqcup f_2 = \langle j_1' \circ f_1, j_2' \circ f_2 \rangle,$$

and there is also a similar description of the coproduct $g_1 \sqcup g_2$.

The hypothesis $f_1 \simeq g_1$ implies $j_1' \circ f_1 \simeq j_1' \circ g_1$, and similarly, $j_2' \circ f_2 \simeq j_2' \circ g_2$. Using these relations, together with the equalities of the previous paragraph, and applying the result of Proposition 2.2.9, we see that

$$f_1 \sqcup f_2 = \langle j_1' \circ f_1, j_2' \circ f_2 \rangle \simeq \langle j_1' \circ g_1, j_2' \circ g_2 \rangle = g_1 \sqcup g_2.$$ ■

Another important class of functor to consider is the one formed by the functors which identify homotopic maps, or in other words, functors that transform homotopy relations into equality relations. We will refer to these functors as **homotopy functors**.

The most important homotopy functors that we will deal with in this text are the homotopy group functors, but there is a simpler example that we can talk about now.

2.2.11 Example (The path-components functor)**.** Given any space X, we denote by $\pi_0(X)$ its set of path-components. Given another space Y together with a map f, if x_1, $x_2 \in X$ are two points which belong to the same path-component, then there is a path $\gamma : I \to X$ from x_1 to x_2. Using f we can define a path $f \circ \gamma : I \to Y$ joining $f(x_1)$ to $f(x_2)$, so these points belong to the same path-component of Y. This shows that we have a well-defined function $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ given by $\pi_0(f)([x]) \coloneqq [f(x)]$.

A straightforward calculation shows that this construction preserves identities and composition, so it gives rise to a functor $\pi_0 : \mathsf{Top} \to \mathsf{Set}$ called the **path-components functor**.

We now show that π_0 is a homotopy functor, that is, if f, $g : X \to Y$ are homotopic maps, we must show that the induced functions coincide $\pi_0(f)$, $\pi_0(g) : \pi_0(X) \to \pi_0(Y)$ are equal. Let $D : X \to \mathrm{Map}(I, Y)$ be the family of paths induced by a homotopy $f \simeq g$. For any point $x \in X$, since $D(x) : I \to Y$ is a path from $f(x)$ to $g(x)$, these points belong to the same path-component of Y, therefore we can write the equalities

$$\pi_0(f)([x]) = [f(x)] = [g(x)] = \pi_0(g)([x]).$$

Since this holds for any point $x \in X$, we deduce that $\pi_0(f) = \pi_0(g)$.

The fact that π_0 identifies homotopical maps means that it can be thought of as a functor of homotopy types instead of a functor of spaces. More precisely, π_0 induces a functor $\overline{\pi_0} : \mathsf{HoTop} \to \mathsf{Set}$ which sends a space X to $\overline{\pi_0}(X) \coloneqq \pi_0(X)$, and which sends a homotopy class $[f] : X \to Y$ to the induced function $\overline{\pi_0}([f]) \coloneqq \pi_0(f)$. In particular $\overline{\pi_0}$ transforms homotopy equivalences (isomorphisms in HoTop) into bijections (isomorphisms in Set).

2.3 Contractions, null homotopies and extensions

In this section we are interested in maps which are homotopic to constant ones. We first present a particular case of this notion and then generalize it and show how the generalization relates to the particular one. Using this relation we prove an important result that can be regarded as a first step towards Obstruction Theory.

We begin by introducing a convenient notation.

2.3.1 Definition. Given two spaces X and Y and a point $y \in Y$, we denote by $\mathrm{ct}_{X,y} : X \to Y$ the constant map defined as $\mathrm{ct}_{X,y}(x) \coloneqq y$ for every $x \in X$.

We use the rather verbose notation $\mathrm{ct}_{X,y}$ to avoid confusion when working with multiple constant maps defined on different spaces. If the domain of the constant map has a complicated notation (like a product of mapping spaces), then we usually drop it from the notation.

2.3.2 Definition. A space X is said to be **contractible** if there exists a point $x_0 \in X$ such that $\mathrm{id}_X \simeq \mathrm{ct}_{X,x_0}$. A particular choice of homotopy $H : \mathrm{id}_X \Rightarrow \mathrm{ct}_{X,x_0}$ is called a **contraction** of X.

Unpacking the definition, if X is contractible, then there exists a point $x_0 \in X$ and a map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 0) = x_0$ for every $x \in X$. Geometrically, this means that, as time passes by, the map H deforms the whole space X to the single point x_0, that is, the space is *contracted* to this point.

The next result shows that, from the point of view of homotopy theory, a contractible space is indistinguishable from a point.

2.3.3 Proposition. A space is contractible if and only if it is homotopy equivalent to a point.

Proof. Suppose X is contractible, and let $H : \mathrm{id}_X \Rightarrow \mathrm{ct}_{X,x_0}$ be a contraction to a point $x_0 \in X$. Let $\mathrm{ct}_{X,\mathrm{pt}} : X \to \{\mathrm{pt}\}$ be the unique map to the singleton space, and consider also the map $\mathrm{ct}_{\{\mathrm{pt}\},x_0} : \{\mathrm{pt}\} \to X$ mapping pt to x_0. These two maps are homotopy inverses. On the one hand, the composition $\mathrm{ct}_{X,\mathrm{pt}} \circ \mathrm{ct}_{\{\mathrm{pt}\},x_0}$ is already equal to the identity of $\{\mathrm{pt}\}$, and on the other hand, the composition $\mathrm{ct}_{\{\mathrm{pt}\},x_0} \circ \mathrm{ct}_{X,\mathrm{pt}}$ is equal to the constant map ct_{X,x_0}, which is homotopic to id_X by hypothesis.

Conversely, suppose X is homotopy equivalent to $\{\mathrm{pt}\}$, and let $f : X \to \{\mathrm{pt}\}$ and $g : \{\mathrm{pt}\} \to X$ be homotopy inverses. We claim that X can be contracted to the point $x_0 \coloneqq g(\mathrm{pt})$. Indeed, since f and g are homotopy inverses, the composition $g \circ f$ is homotopic to the identity id_X, but $g \circ f$ is equal to ct_{X,x_0}, thus we have the desired homotopy relation $\mathrm{id}_X \simeq \mathrm{ct}_{X,x_0}$. ■

2.3.4 Corollary. Every contractible space is path-connected.

Proof. We give two different proofs of this simple result using different tools we have developed so far. For the first proof, let $H : \mathrm{id}_X \Rightarrow \mathrm{ct}_{X,x_0}$ be a contraction of X, and let $D : X \to \mathrm{Map}(I, X)$ be the adjoint family of paths induced by this contraction. For every $x \in X$, $D(x)$ is a path from $\mathrm{id}_X(x) = x$ to $\mathrm{ct}_{X,x_0}(x) = x_0$, so every point can be joined to x_0 by a path, which means that X is path-connected.

Alternatively, Proposition 2.3.3 implies that we have a homotopy equivalence $X \simeq \{\mathrm{pt}\}$. Since the path-components functor $\pi_0 : \mathsf{Top} \to \mathsf{Set}$ of Example 2.2.11 is a homotopy functor, we have a bijection $\pi_0(X) \cong \pi_0(\{\mathrm{pt}\})$. Since $\pi_0(\{\mathrm{pt}\})$ has only one element, it follows from the previous bijection that $\pi_0(X)$ also has only one element, or in other words, X has only one path-component, which means precisely that X is path-connected. ■

The concept of contraction can be generalized if we replace id_X by an arbitrary map.

2.3.5 Definition. A map $f : X \to Y$ between spaces is **null homotopic** if there exists a point $y \in Y$ such that $f \simeq \mathrm{ct}_{X,y}$. A particular choice of homotopy $H : f \Rightarrow \mathrm{ct}_{X,y} y$ is called a **null homotopy**.

Unpacking the definition, if $f : X \to Y$ is null homotopic, then there exists a point $y \in Y$ and a map $H : X \times I \to Y$ such that

1. $H(x, 0) = f(x)$ for every $x \in X$;
2. $H(x, 1) = y$ for every $x \in X$.

Notice then that a space X is contractible precisely if $\mathrm{id}_X : X \to X$ is null homotopic.

Thinking of $X \times I$ geometrically as a cylinder, the null homotopy H glues the lower face $X \times \{0\}$ to the image of f and collapses the upper face $X \times \{1\}$ to a single point which is then glued to the point y. This collapsing process suggests the following construction.

2.3.6 Definition. Given a space X, its **cone**, denoted by CX, is the quotient space

$$CX := \frac{X \times I}{X \times \{1\}}.$$

The point $* \in CX$ obtained by collapsing the subspace $X \times \{1\}$ is called the **vertex** of the cone.

There is an important connection between the cone construction and null homotopic maps, but in order to understand this connection we first need a result on the homotopical properties of cones.

2.3.7 Proposition. The cone CX of any space X is contractible.

Proof. Since CX is obtained from $X \times I$ by collapsing the subspace $X \times \{1\}$, the diagram below is a pushout of spaces,

$$\begin{tikzcd}
X \times \{1\} \arrow[r, "\mathrm{ct}_{\mathrm{pt}}"] \arrow[d, tail] & \{\mathrm{pt}\} \arrow[d, "\mathrm{ct}_*"] \\
X \times I \arrow[r, "q"'] & CX
\end{tikzcd}$$

where q is the canonical projection, $X \times \{1\} \rightarrowtail X \times I$ denotes an inclusion, and the two other maps are the appropriate constant ones.

The exponential adjunction tells us that the functor $- \times I$ is a left adjoint, therefore it preserves colimits, and since a pushout is a particular instance of a colimit, the square below is still a pushout of spaces.

$$\begin{tikzcd}
(X \times \{1\}) \times I \arrow[r, "\mathrm{ct}_{\mathrm{pt}} \times \mathrm{id}_I"] \arrow[d, tail] & \{\mathrm{pt}\} \times I \arrow[d, "\mathrm{ct}_* \times \mathrm{id}_I"] \\
(X \times I) \times I \arrow[r, "q \times \mathrm{id}_I"'] & CX \times I
\end{tikzcd}$$

This is useful because we can exploit the universal property of the pushout to obtain a map out of type $CX \times I \to CX$ out of simpler maps $(X \times I) \times I \to CX$ and $\{\mathrm{pt}\} \times I \to CX$.

Consider then the map $T : (X \times I) \times I \to CX$ defined as

$$T((x, s), t) \coloneqq q(x, (1 - t)s + t)$$

for every $((x, s), t) \in (X \times I) \times I$. Given a point $((x, 1), t) \in (X \times \{1\}) \times I$, we have

$$T((x, 1), t) = q(x, 1 - t + t) = q(x, 1) = *.$$

It follows that the "outer shell" of the diagram below commutes, and the universal property of the pushout then gives us the map $H : CX \times I \to CX$ depicted.

$$\begin{tikzcd}
(X \times \{1\}) \times I \arrow[r, "\mathrm{ct}_{\mathrm{pt}} \times \mathrm{id}_I"] \arrow[d, tail] & \{\mathrm{pt}\} \times I \arrow[d, "\mathrm{ct}_* \times \mathrm{id}_I"] \arrow[rdd, bend left, "\mathrm{ct}_*"] & \\
(X \times I) \times I \arrow[r, "q \times \mathrm{id}_I"'] \arrow[rrd, bend right, "T"'] & CX \times I \arrow[rd, dashed, "H"] & \\
& & CX
\end{tikzcd}$$

Given any point $q(x, s) \in CX$, we have

$$\begin{aligned}
H(q(x, s), 0) &= (H \circ (q \times \mathrm{id}_I))((x, s), 0) \\
&= T((x, s), 0) \\
&= q(x, (1 - 0)s + 0) \\
&= q(x, s),
\end{aligned}$$

therefore $H \circ i_{CX,0} = \mathrm{id}_{CX}$. Now, for the final stage of H we have

$$\begin{aligned} H(q(x,s),1) &= (H \circ (q \times \mathrm{id}_I))((x,s),1) \\ &= T((x,s),1) \\ &= q(x,(1-1)s+1) \\ &= q(x,1) \\ &= *, \end{aligned}$$

therefore $H \circ i_{CX,1} = \mathrm{ct}_*$. The map H then defines a homotopy from id_{CX} to the constant map $\mathrm{ct}_* : CX \to CX$, proving the contractibility of CX. ■

Using this we can prove an important theorem relating null homotopic maps and cones.

2.3.8 Theorem. A map $f : X \to Y$ between spaces is null homotopic if, and only if, there is a map $F : CX \to Y$ such that $f = F \circ i$.

$$\begin{tikzcd} CX \arrow[dr, "F"] & \\ X \arrow[u, "i"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Suppose first that a map $F : CX \to Y$ satisfying the stated conditions exist. Using Proposition 2.3.7 we know that $\mathrm{id}_{CX} \simeq \mathrm{ct}_{CX,v}$, where v is the vertex of CX, and since homotopies are compatible with function composition, we have

$$\begin{aligned} f &= F \circ i \\ &= (F \circ \mathrm{id}_{CX}) \circ i \\ &\simeq (F \circ \mathrm{ct}_{CX,v}) \circ i \\ &= \mathrm{ct}_{CX,F(v)} \circ i \\ &= \mathrm{ct}_{X,F(v)}; \end{aligned}$$

therefore f is null homotopic.

Conversely, suppose f is null homotopic and consider a null homotopy $H : f \Rightarrow \mathrm{ct}_{X,y}$ for some point $y \in Y$. The map $H : X \times I \to Y$ sends any point of the form $(x,1) \in X \times I$ to y, therefore it can be factored through the cone CX giving rise to a map $F : CX \to Y$. This is desired map because

$$\begin{aligned} F \circ i &= F \circ p \circ i_{X,0} \\ &= H \circ i_{X,0} \\ &= f. \end{aligned}$$

■

We are especially interested in applying the result of Theorem 2.3.8 in the case where $X = S^n$, but for this we need to better understand the cone CS^n. The next lemma will help us with this.

2.3.9 Lemma. The map $p : S^n \times I \to D^{n+1}$ given by $p(x,t) := (1-t)x$ for every $(x,t) \in S^n \times I$ is a quotient map.

Proof. The product $S^n \times I$ is compact, and D^{n+1} is a Hausdorff space, therefore p is a closed map. Moreover, p is also surjective, since $\mathbf{0} = p(x,1)$ for any $x \in S^n$, and, if $y \in D^{n+1} \setminus \{\mathbf{0}\}$, then

$$y = \|y\|\frac{y}{\|y\|} = (1-(1-\|y\|))\frac{y}{\|y\|} = p\left(\frac{y}{\|y\|}, 1-\|y\|\right).$$

The result then follows from the fact that every closed and surjective map is a quotient map. ■

Using this we compare the cone over a sphere to another familiar space.

2.3.10 Lemma. For every $n \geq 0$ there is a homeomorphism $CS^n \cong D^{n+1}$ which maps the vertex of CS^n to the center of D^{n+1}.

Proof. Consider the map $\alpha : S^n \times I \to D^{n+1}$ defined as

$$\alpha(x,t) := (1-t)\cdot x.$$

This maps the subset $S^n \times \{1\}$ to $\mathbf{0}$ - the center of the disk D^{n+1} - therefore it can be factored through the cone to define a map $\overline{\alpha} : CS^n \to D^{n+1}$. Explicitly, $\overline{\alpha}([x,t]) = (1-t)x$ for every $[x,t] \in CS^n$.

We now construct an inverse to $\overline{\alpha}$. Consider the function $\beta : D^{n+1} \to CS^n$ defined as follows:

$$\beta(x) := \begin{cases} v, & x = \mathbf{0}; \\ \left[\frac{x}{\|x\|}, 1-\|x\|\right], & x \neq \mathbf{0}. \end{cases}$$

A direct calculation shows that β is exactly the inverse of $\overline{\alpha}$, but we still need to show that β is continuous. Consider then the composite map $\beta \circ p : S^{n+1} \times I$, where $p : S^n \times I \to D^{n+1}$ is the quotient map of Lemma 2.3.9. Another direct calculation shows that

$$(\beta \circ p)(x,t) = \begin{cases} v, & (x,t) \in S^n \times \{1\}; \\ [x,t], & (x,t) \in S^n \times I \setminus S^n \times \{1\}. \end{cases}$$

But this means that the composition $\beta \circ p$ coincides with the canonical projection $q : S^n \times I \to CS^n$, so $\beta \circ p$ is continuous. The continuity of β then follows from the continuity of $\beta \circ p$ and from the fact that p is a quotient map. ■

2.3.11 Corollary. Let X be an arbitrary space. A map $f : S^n \to X$ is null homotopic if, and only if, it admits an extension $F : D^{n+1} \to X$.

Proof. We know from Theorem 2.3.8 that f is null homotopic if, and only if, there exists a map $G : CS^n \to X$ factoring f through the inclusion $i : S^n \to CS^n$. We claim that this map G exists if, and only if, there exists an extension $F : D^{n+1} \to X$ of f. Indeed, if $\overline{\alpha} : CS^n \to D^{n+1}$ is the homeomorphism constructed in Lemma 2.3.10, then given an extension $F : D^{n+1} \to X$, the composition $F \circ \overline{\alpha} : CS^n \to X$ is the desired factorization of f trough $i : S^n \to CS^n$; and if we are given $G : CS^n \to X$ factoring f, then $F \circ \overline{\alpha}^{-1} : D^{n+1} \to X$ is the desired extension of f. ■

This result can be regarded as the origin of Obstruction Theory. It basically says that the topological question of the existence of an extension of a map of type $S^n \to X$ to a map of type $D^{n+1} \to X$ depends on the homotopical properties of the map f. This may not seem like a very useful result because so far we have not yet developed techniques allowing us to study the homotopical properties of maps, but over the course of the next chapters we will see that these homotopical properties have an algebraic side that allows us to better understand them.

CHAPTER 3

POINTED SPACES

This chapter introduces the notion of a pointed space, which is the natural context for the definition of the homotopy groups later on. The goal of the chapter is to understand better the category Top_* of pointed space and pointed maps, and also to adapt some results and construction from unpointed spaces to the pointed context.

The first section contains the basic definitions and examples of pointed spaces and pointed maps. In the second section, we introduce the *space* of pointed maps $\mathrm{Map}_*(X, Y)$ between two pointed spaces, and we prove a pointed version of the exponential adjunction. In the course of this section we are naturally led to the smash product operation between two pointed spaces. The third section contains further results of a categorical nature concerning the smash product. We prove, in particular, that it satisfies some "algebraic" properties similar to the ones satisfied by the usual cartesian product. In the final section, we explicitly calculate some smash products, and in the process introduce the reduced suspension and loop space constructions, which are related by an adjunction relation known as Eckmann-Hilton Duality.

3.1 The category of pointed spaces

This first section is devoted to basic definitions and examples.

3.1.1 Definition. A **pointed space** is a pair $(X, *)$ where X is a topological space and $*$ is a point of X called the **basepoint**.

Of course any space can be seen as a pointed space if we arbitrarily choose one of its points as basepoint, but without extra information there is no way to naturally make such a choice. However, for some families of spaces, there are useful choices of basepoints that are frequently made.

3.1.2 Example. The n-dimensional sphere S^n is often seen as a pointed space by choosing the point $(1, 0, \dots, 0)$ as basepoint.

3.1.3 Example. The unit interval I is often regarded as a pointed space by choosing either 0 or 1 as basepoint. When we need to consider I as a pointed space, we always make our choice of basepoint explicit.

3.1.4 Example. If G is a topological group, there's the pointed space (G, e), where e is the identity element.

3.1.5 Example. Let (X, A) be a pair of spaces. A reasonable choice of basepoint for the quotient X/A is the point to which the subspace A is identified, that is, we consider the pointed space $(X/A, [a])$ where $a \in A$ is arbitrary.

We now define a notion of transformation between pointed spaces.

3.1.6 Definition. Let (X, x_0) and (Y, y_0) be pointed spaces. A map $f : X \to Y$ is **pointed** if it satisfies $f(x_0) = y_0$. We also say that f **preserves basepoints** and write $f : (X, x_0) \to (Y, y_0)$.

3.1.7 Example. If $(X, *)$ is any pointed space, then the identity map $\mathrm{id}_X : X \to X$ is pointed.

3.1.8 Example. If $i : S^n \to S^{n+1}$ is the embedding of S^n as the equator of S^{n+1}, then i is pointed map, where the spheres are pointed as in Example 3.1.2.

3.1.9 Example. Regard the circle S^1 as the space of unitary complex numbers. The operation of multiplication of complex numbers defines a pointed map $m : (S^1 \times S^1, (1, 1)) \to (S^1, 1)$. More generally, if (G, e) is a topological group, then its multiplication defines a pointed map $m : (G \times G, (e, e)) \to (G, e)$.

We saw in Example 3.1.7 that, if $(X, *)$ is a pointed space, then the identity id_X defines a pointed map $(X, *) \to (X, *)$. Moreover, a straightforward argument shows that the composition of two pointed maps is again pointed, and this composition is of course associative. It follows that we have a category whose objects are pointed spaces and whose morphisms are pointed maps. This category will be denoted by Top_*. Following our notational convention for collections of morphisms in a category, given two pointed spaces (X, x_0) and (Y, y_0), $\mathsf{Top}_*((X, x_0), (Y, y_0))$ denotes the *set* of pointed maps from (X, x_0) to (Y, y_0). We sometimes omit the basepoints of X and Y and write simply $\mathsf{Top}_*(X, Y)$, but we always include the asterisk lest we forget about the poor basepoints.

3.1.10 Example. In a topological vector space X, since translation by a vector defines a map $X \to X$, given two vectors $x, x' \in X$, translation by $x' - x$ defines an isomorphism $(X, x) \simeq (X, x')$ in Top_*.

3.1.11 Example. If X is a disconnected space, and $x, x' \in X$ are not in the same connected component, then $\mathsf{Top}_*((X,x),(X,x')) = \varnothing$. In particular, the pointed spaces (X,x) and (X,x') are not isomorphic.

We now discuss some properties of the category Top_*.

3.1.12 Proposition. Let $\{\mathrm{pt}\}$ denote any singleton space. The pointed space $(\{\mathrm{pt}\}, \mathrm{pt})$ is both an initial and terminal object of Top_*.

Proof. Let (X, x_0) be a pointed space. Any map defined on $\{\mathrm{pt}\} \to X$ is continuous, and if we ask for it to be pointed, then there is a unique one given by mapping $*$ to x_0; showing that $(\{\mathrm{pt}\}, \mathrm{pt})$ is an initial object in Top_*.

There is a single map from X to $\{\mathrm{pt}\}$ given by the constant map $\mathrm{ct}_{X,\mathrm{pt}}$, and since this map is pointed, it follows that there is a unique pointed map $(X, x_0) \to (\{pt\}, \mathrm{pt})$, so this pointed space is a terminal object in Top_*. ■

This is an interesting distinction of Top and Top_*. While the former has distinct initial and terminal objects, in the latter these two notions coincide.

Let us now analyze products in Top_*. Given two pointed spaces (X, x_0) and (Y, y_0), the usual product $X \times Y$ has a reasonable choice of basepoint: the ordered pair (x_0, y_0) combining the two basepoints. This is a good choice, because it turns the canonical projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ into pointed maps. Moreover, given another pointed space (Z, z_0) and pointed maps $f : (Z, z_0) \to (X, x_0)$ and $g : (Z, z_0) \to (Y, y_0)$, the usual induced map $(f, g) : Z \to X \times Y$ is pointed, since

$$(f, g)(z_0) = (f(z_0), g(z_0)) = (x_0, y_0).$$

This proves the next result stating that the pointed space $(X \times Y, (x_0, y_0))$ behaves as it should.

3.1.13 Proposition. The category Top_* has all binary products. More precisely, given two pointed spaces (X, x_0) and (Y, y_0), the pointed space $(X \times Y, (x_0, y_0))$ together with the usual canonical projections defines a product of (X, x_0) and (Y, y_0) in Top_*.

Lastly, the situation with coproducts in Top_* is more interesting and plays an important role in the next sections. In Top, coproducts are given by disjoint unions. If we try to do this in Top_* we immediately run into a problem: given pointed spaces (X, x_0) and (Y, y_0), what is the right choice of basepoint for the disjoint union $X \sqcup Y$? The two choices that first come to mind are to choose either x_0 or y_0 as basepoint. However, if we choose x_0, then the canonical injection $i_2 : Y \to X \sqcup Y$ does not preserve basepoints, while if we choose y_0, then an analogous problem occurs with the injection $i_1 : X \to X \sqcup Y$.

The solution of this problem is not to choose one or the other, but to modify the space $X \sqcup Y$ in a way that allows us to make *both* choices at once.

3.1.14 Definition. Given two pointed spaces $(X, *_X)$ and $(Y, *_Y)$, their **wedge sum** is the space defined as

$$X \vee Y \coloneqq (X \sqcup Y)/\{i_1(*_X), i_2(*_Y)\},$$

where i_1 and i_2 are the canonical injections into the disjoint union. It becomes a pointed space by defining its basepoint as

$$*_{X \vee Y} \coloneqq [i_1(*_X)] = [i_2(*_Y)].$$

If $q : X \sqcup Y \to X \vee Y$ denotes the canonical projection, we define the maps $j_1 : X \to X \vee Y$ and $j_2 : Y \to X \vee Y$ by the formulas $j_1 \coloneqq q \circ i_1$ and $j_2 \coloneqq q \circ i_2$.

$$\begin{tikzcd}
X \arrow[dr, "i_1"'] \arrow[drr, bend left, "j_1"] & & \\
& X \sqcup Y \arrow[r, "q"] & X \vee Y \\
Y \arrow[ur, "i_2"] \arrow[urr, bend right, "j_2"'] & &
\end{tikzcd}$$

Geometrically, the wedge sum glues the pointed spaces $(X, *_X)$ and $(Y, *_Y)$ by their basepoints. This can also be stated categorically by saying that the wedge sum fits into the pushout square below.

$$\begin{tikzcd}
\{\mathrm{pt}\} \arrow[r, "\mathrm{ct}_{\{\mathrm{pt}\},*_X}"] \arrow[d, "\mathrm{ct}_{\{\mathrm{pt}\},*_Y}"'] & X \arrow[d, "j_1"] \\
Y \arrow[r, "j_2"'] & X \vee Y
\end{tikzcd}$$

Indeed, if Z is another space, and $f : X \to Z$ and $g : Y \to Z$ are maps satisfying $f \circ \mathrm{ct}_{\{\mathrm{pt}\},*_X} = g \circ \mathrm{ct}_{\{\mathrm{pt}\},*_Y}$, which means that $f(*_X) = g(*_Y)$, then the induced map $\langle f, g\rangle : X \sqcup Y \to Z$ satisfies $\langle f, g\rangle(i_1(*_X)) = \langle f, g, \rangle(i_2(*_Y))$, so it factors through the quotient to define a map $h : X \vee Y \to Z$ which satisfies

$$h \circ j_1 = h \circ q \circ i_1 = \langle f, g, \rangle \circ i_1 = f,$$

and also

$$h \circ j_2 = h \circ q \circ i_2 = \langle f, g\rangle \circ i_2 = g.$$

This property, however, only concerns the space $X \vee Y$, it does not involve the choice of basepoint that we made. The next result characterizes the pointed space $(X \vee Y, *_{X \vee Y})$ by a universal property in Top_*.

3.1.15 Proposition. Let $(X, *_X)$ and $(Y, *_Y)$ be pointed spaces. The triple

$$((X \vee Y, *_{X \vee Y}), j_1, j_2)$$

defines a coproduct for $(X, *_X)$ and $(Y, *_Y)$ in Top_*.

Proof. Let $(Z, *_Z)$ be a pointed space, and let $f : (X, *_X) \to (Z, *_Z)$ and $g : (Y, *_Y) \to (Z, *_Z)$ be pointed maps. The induced map $\langle f, g \rangle : X \sqcup Y \to Z$ satisfies

$$\langle f, g \rangle (j_1(*_X)) = f(*_X) = *_Z,$$

and also

$$\langle f, g \rangle (j_2(*_Y)) = g(*_Y) = *_Z,$$

so it factors through the quotient and defines a map $\overline{\langle f, g \rangle} : X \vee Y \to Z$. This is a pointed map because

$$\overline{\langle f, g \rangle}(*_{X \vee Y}) = \overline{\langle f, g, \rangle}(q(j_1(*_X))) = \langle f, g \rangle (j_1(x)) = *_Z.$$

It also factors f and g through j_1 and j_2 since

$$\overline{\langle f, g, \rangle} \circ j_1 = \overline{\langle f, g, \rangle} \circ q \circ i_1 = \langle f, g \rangle \circ i_1 = f,$$

and, similarly, $\overline{\langle f, g \rangle} \circ j_2 = g$.

Now, if $h : (X \vee Y, *_{X \vee Y}) \to (Z, *_Z)$ is another pointed map satisfying $h \circ j_1 = f$ and $h \circ j_2 = g$, then the composition $h \circ q$ satisfies $(h \circ q) \circ i_1 = f$ and $(h \circ q) \circ i_2 = g$, so $h \circ q = \langle f, g \rangle$ by the universal property of the coproduct in Top. This last equality implies $h = \overline{\langle f, g \rangle}$ because by the universal property of the quotient space, $\overline{\langle f, g \rangle}$ is the *only* map that factors $\langle f, g \rangle$ through the quotient map q. We conclude that the map $\overline{\langle f, g \rangle}$ we defined is the only one factoring f and g through j_1 and j_2. ∎

3.2 The return of the exponential adjunction

In Section 1.1 we learned that products and mapping spaces are related by an adjunction. Formally, we proved two different versions of this result: the first one is a natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, \mathrm{Map}(Y, Z)),$$

when Y is locally compact Hausdorff, and the second one is a homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

when both X and Y are locally compact Hausdorff.

In the present section we would like to adapt these results to the category Top_* of pointed spaces. The first question we must answer in order to obtain this generalization is:

what is the substitute for the space $\mathrm{Map}(X, Y)$ in the pointed case? Since for any pointed spaces (X, x_0) and (Y, y_0) there is an inclusion $\mathsf{Top}_*(X, Y) \subseteq \mathsf{Top}(X, Y)$, it makes sense to turn the set of pointed maps into a space by considering it as a subspace of $\mathrm{Map}(X, Y)$.

3.2.1 Definition. Given pointed spaces (X, x_0) and (Y, y_0), we define a topology on the set of pointed maps $\mathsf{Top}_*(X, Y)$ by considering the subspace topology inherited from $\mathrm{Map}(X, Y)$, and we denote the resulting space of pointed maps by $\mathrm{Map}_*(X, Y)$. This becomes a pointed space itself by choosing the constant map $\mathrm{ct}_{X,y_0} : (X, x_0) \to (Y, y_0)$ as a basepoint.

The main functorial properties of $\mathrm{Map}_*(X, Y)$ are summarized in the next result. They essentially follow from the analogous properties satisfied by $\mathrm{Map}(X, Y)$.

3.2.2 Proposition. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces, and let $f : X \to Y$ and $g : Y \to Z$ be pointed maps.

1. The pushforward function along g

$$\mathsf{Top}_*(X, g) : \mathsf{Top}_*(X, Y) \to \mathsf{Top}_*(X, Z)$$

induces a pointed map

$$\mathrm{Map}_*(X, g) : \mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(X, Z).$$

2. The pullback function along f

$$\mathsf{Top}_*(f, Z) : \mathsf{Top}_*(Y, Z) \to \mathsf{Top}_*(X, Z)$$

induces a pointed map

$$\mathrm{Map}_*(f, Z) : \mathrm{Map}_*(Y, Z) \to \mathrm{Map}_*(X, Z).$$

Proof. 1. We know the usual pushforward $\mathrm{Map}(X, g) : \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ is continuous, so by restriction we also have a map $\mathrm{Map}(X, g)|_{\mathrm{Map}_*(X,Y)} : \mathrm{Map}_*(X, Y) \to \mathrm{Map}(X, Z)$. Since this map takes values in the subspace $\mathrm{Map}_*(X, Z) \subseteq \mathrm{Map}(X, Z)$, we may then regard this pushforward as a map of type $\mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(X, Z)$. Lastly, since for any $x \in X$ we have the equality

$$[\mathrm{Map}_*(X, g)(\mathrm{ct}_{X,y_0})](x) = (g \circ \mathrm{ct}_{X,y_0})(x) = g(y_0) = z_0,$$

it follows that $\mathrm{Map}_*(X, g)(\mathrm{ct}_{X,y_0}) = \mathrm{ct}_{X,z_0}$; therefore $\mathrm{Map}_*(X, g)$ defines a pointed map.

2. The proof of this item is analogous to the proof of the previous one. ■

Now we turn our attention to adapting the exponential adjunction to the pointed case. In an ideal world, the adjunction would work straight away, that is, a pointed map $f : X \times Y \to Z$ would give rise to a pointed adjoint map $\lambda f : X \to \mathrm{Map}_*(Y, Z)$, and this rule would define a bijection under suitable topological conditions. The next example, however, shatters our expectations about this ideal world.

3.2.3 Example. Regard S^1 as the space of unitary complex numbers and consider the pointed map $m : (S^1 \times S^1, (1,1)) \to (S^1, 1)$ given by multiplication. Let $\lambda m : S^1 \to \mathrm{Map}(S^1, S^1)$ be the exponential adjoint. Can λm be seen as a pointed map $(S^1, 1) \to (\mathrm{Map}_*(S^1, S^1), \mathrm{ct}_{S^1,1})$? The short answer is no, and this is due to two reasons:

1. If $z \in S^1$ is such that $\lambda m(z) \in \mathrm{Map}_*(S^1, S^1)$, then we must have

$$z = [\lambda m(z)](1) = 1.$$

This shows that the image of λm is not even contained in the subspace of pointed maps.

2. For any $z \in S^1$ we have

$$[\lambda m(1)](z) = 1z = z,$$

therefore $\lambda m(1) = \mathrm{id}_{S^1}$ which is different from the constant map $\mathrm{ct}_{S^1,1}$; showing that λm does not preserve basepoints.

In summary, the example above shows that the exponential adjunction can fail miserably in the pointed case. Luckily, the two ways in which the example fails are the only ones possible, and by examining how to avoid these problems we are naturally led to a new construction which will let us take back the power of the exponential adjunction.

3.2.4 Lemma. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces. Given a map $f : X \times Y \to Z$, its exponential adjoint $\lambda f : X \to \mathrm{Map}(Y, Z)$ defines a pointed map of type $(X, x_0) \to (\mathrm{Map}(Y, Z), \mathrm{ct}_{Y,z_0})$ if and only if it satisfies the condition

$$f(X \times \{y_0\} \cup \{x_0\} \times Y) \subseteq \{z_0\}.$$

Proof. The adjoint λf defines a pointed map of the required type if and only if it satisfies the two following conditions:

1. $\lambda f(x)$ is a pointed map of type $(Y, y_0) \to (Z, z_0)$ for every $x \in X$, which means that the equality

$$f(x, y_0) = [\lambda f(x)](y_0) = z_0$$

must hold for every $x \in X$;

2. λf preserves basepoints, that is, it must satisfy the condition $\lambda f(x_0) = \mathrm{ct}_{Y,z_0}$, which means that the equality

$$f(x_0, y) = [\lambda f(x_0)](y) = \mathrm{ct}_{Y,z_0} = z_0$$

must hold for every $y \in Y$.

These two conditions mean precisely that the inclusion $f(X \times \{y_0\} \cup \{x_0\} \times Y) \subseteq \{z_0\}$ must hold. ■

This suggests that the natural domain for a pointed version of the exponential adjunction is not the set of maps defined on the product $X \times Y$, but on a certain quotient of it.

3.2.5 Definition. Given pointed spaces (X, x_0) and (Y, y_0), their **smash product** is the space defined by

$$X \wedge Y := \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

It becomes a pointed space by choosing $[x_0, y_0] := q(x_0, y_0)$ as its basepoint, where $q : X \times Y \to X \wedge Y$ denotes the canonical projection.

The construction of the smash product can also be performed on pairs of pointed maps. Given pointed maps $f : (X_1, *_{X_1}) \to (X_2, *_{X_2})$ and $g : (Y_1, *_{Y_1} \to (Y_2, *_{Y_2})$, if $q_1 : X_1 \times Y_1 \to X_1 \wedge Y_1$ and $q_2 : X_2 \times Y_2 \to X_2 \wedge Y_2$ denote the canonical projections, then the composition

$$q_2 \circ (f \times g) : X_1 \times Y_1 \to X_2 \wedge Y_2$$

satisfies the condition

$$(q_2 \circ (f \times g))(X_1 \times \{y_1\} \cup \{x_1\} \times Y_1) \subseteq \{*_{X_2 \wedge Y_2}\},$$

therefore it can be factored through the projection q_1 to define a map

$$f \wedge g : X_1 \wedge Y_1 \to X_2 \wedge Y_2$$

which we call the **smash product of f and g**.

Notice that by construction the smash product $f \wedge g$ is the *only* map from $X_1 \wedge Y_1$ to $X_2 \wedge Y_2$ making the square below commute.

$$\begin{tikzcd}
X_1 \times Y_1 \arrow[r, "f \times g"] \arrow[d, "q_1"'] & X_2 \times Y_2 \arrow[d, "q_2"] \\
X_1 \wedge Y_1 \arrow[r, dashed, "f \wedge g"'] & X_2 \wedge Y_2
\end{tikzcd}$$

From this commutativity we can obtain an explicit formula for $f \wedge g$: given a point of $X_1 \wedge Y_1$ of the form $[x_1, y_1] = q_1(x_1, y_1)$ for some point $(x_1, y_1) \in X_1 \times Y_1$, we have

$$(f \wedge g)([x_1, y_1]) = (f \wedge g)(q(x_1, y_1)) = q_2((f \times g)(x_1, y_1)) = q_2(f(x_1), g(y_1)) = [f(x_1), g(y_1)].$$

This also implies that $f \wedge g$ is automatically a pointed map since

$$(f \wedge g)(*_{X_1 \wedge Y_1}) = (f \wedge g)([*_{X_1}, *_{Y_1}]) = [f(*_{X_1}), g(*_{Y_1})] = [*_{X_2}, *_{Y_2}] = *_{X_2 \wedge Y_2}.$$

As with most of the constructions we have considered so far, the smash product is functorial.

3.2.6 Proposition. The construction of the smash product of pointed spaces and of pointed maps defines a functor $\wedge : \mathsf{Top}_* \times \mathsf{Top}_* \to \mathsf{Top}_*$.

Proof. We first need to show that the smash product preserves identity maps, i.e., that for any two pointed spaces (X, x_0) and (Y, y_0) the equality

$$\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}$$

holds. Recall that $\mathrm{id}_X \wedge \mathrm{id}_Y$ is the only map of its type that fits in the commutative square below.

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id}_X \times \mathrm{id}_Y"] \arrow[d, "q"'] & X \times Y \arrow[d, "q"] \\ X \wedge Y \arrow[r, "\mathrm{id}_X \wedge \mathrm{id}_Y"'] & X \wedge Y \end{tikzcd}$$

Since the product $\times$ is itself functorial, the map $\mathrm{id}_X \times \mathrm{id}_Y$ on the first line is equal to $\mathrm{id}_{X \times Y}$, and then it is clear that the square below also commutes,

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id}_{X \times Y}"] \arrow[d, "q"'] & X \times Y \arrow[d, "q"] \\ X \wedge Y \arrow[r, "\mathrm{id}_{X \wedge Y}"'] & X \wedge Y \end{tikzcd}$$

from which the desired equality follows.

Now we need to check the compatibility of the smash product with compositions. Suppose we are given the following pointed spaces and maps

$$\begin{aligned} f_1 &: (X_1, *_{X_1}) \to (X_2, *_{X_2}) \\ f_2 &: (X_2, *_{X_2}) \to (X_3, *_{X_3}) \\ g_1 &: (Y_1, *_{Y_1}) \to (Y_2, *_{Y_2}) \\ g_2 &: (Y_2, *_{Y_2}) \to (Y_3, *_{Y_3}). \end{aligned}$$

We need to prove the equality

$$(f_2 \circ f_1) \wedge (g_2 \circ g_1) = (f_2 \wedge g_2) \circ (f_1 \wedge g_1),$$

which is equivalent to proving the commutativity of the square below.

$$\begin{tikzcd} X_1 \times Y_1 \arrow[r, "(f_2\circ f_1)\times(g_2\circ g_1)"] \arrow[d, "q_1"'] & X_3 \times Y_3 \arrow[d, "q_3"] \\ X_1 \wedge Y_1 \arrow[r, "(f_2\wedge g_2)\circ(f_1\wedge g_1)"'] & X_3 \wedge Y_3 \end{tikzcd}$$

Since the product $\times$ is itself a functor, we can rewrite the product on the first line as a composition

$$(f_2 \circ f_1) \times (g_2 \circ g_1) = (f_2 \times g_2) \circ (f_1 \times g_1),$$

so that our goal now is to prove the equality

$$q_3 \circ (f_2 \times g_2) \circ (f_1 \times g_1) = (f_2 \wedge g_2) \circ (f_1 \wedge g_1) \circ q_1.$$

This follows by simply combining the two commutative diagrams below coming from the definitions of $f_1 \wedge g_1$ and $f_2 \wedge g_2$.

$$\begin{tikzcd} X_1 \times Y_1 \arrow[r, "f_1\times g_1"] \arrow[d, "q_1"'] & X_2 \times Y_2 \arrow[r, "f_2\times g_2"] \arrow[d, "q_2"] & X_3 \times Y_3 \arrow[d, "q_3"] \\ X_1 \wedge Y_1 \arrow[r, "f_1\wedge g_1"'] & X_2 \wedge Y_2 \arrow[r, "f_2\wedge g_2"'] & X_3 \wedge Y_3 \end{tikzcd}$$ ■

As with any functor of two variables, we can fix any one of them to obtain a functor of a single variable. In particular, for any pointed space (Y, y_0) we have the functor $-\wedge Y : \mathsf{Top}_* \to \mathsf{Top}_*$ which sends a pointed space $(X, *_X)$ to the smash product $(X \wedge Y, *_{X\wedge Y})$, and which sends a pointed map $f : (X_1, *_{X_1}) \to (X_2, *_{X_2})$ to the smash product map $f \wedge \mathrm{id}_Y : (X_1 \wedge Y, *_{X_1\wedge Y}) \to (X_2 \wedge Y, *_{X_2\wedge Y})$.

The result of Lemma 3.2.4 suggests that a pointed version of the exponential adjunction should relate the pointed mapping space functor $\mathrm{Map}_*(Y, -)$ with the smash product functor $- \wedge Y$ instead of the usual product functor $- \times Y$. In order to do this, we introduce a pointed version of the exponential adjoint of a map.

3.2.7 Definition. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces, and let $q : X \times Y \to X \wedge Y$ be the canonical projection. Given a pointed map $f : (X \wedge Y, *_{X\wedge Y}) \to (Z, z_0)$, the pointed map

$$\lambda^*_{X,Z} f : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$$

defined by the formula

$$\lambda^*_{X,Z} f \coloneqq \lambda_{X,Z}(f \circ q) \tag{3.1}$$

is called the **pointed exponential adjoint of** f.

Since q maps the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ to the basepoint $*$ of $X \wedge Y$, and f by hypothesis maps this basepoint to z_0, the composition $f \circ q$ maps $X \times \{y_0\} \cup \{x_0\} \times Y$

to z_0, therefore $\lambda^*_{X,Z} f$ really defines a pointed map of type $(X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$ according to Lemma 3.2.4.

Using the explicit formula for the unpointed exponential adjunction, we see that, for any $x \in X$, the pointed map $\lambda^*_{X,Z} f(x) : (Y, y_0) \to (Z, z_0)$ is given by the formula

$$[\lambda^*_{X,Z} f(x)](y) = f([x, y])^{1} \tag{3.2}$$

for every $y \in Y$.

3.2.8 Theorem (Pointed exponential adjunction)**.** If (Y, y_0) is a locally compact Hausdorff space, then the smash product functor $- \wedge Y$ is left adjoint to the pointed mapping space functor $\mathrm{Map}_*(Y, -)$.

Proof. We will show that the collection of functions

$$\{\lambda^*_{X,Z} : \mathsf{Top}_*(X \wedge Y, Z) \to \mathsf{Top}_*(X, \mathrm{Map}_*(Y, Z))\}_{(X,x_0),(Z,z_0) \in \mathsf{Top}_*}$$

defines a natural isomorphism of functors (of two variables)

$$\mathsf{Top}_*(- \wedge Y, -) \cong \mathsf{Top}_*(-, \mathrm{Map}_*(Y, -)).$$

Suppose f, $g : (X \wedge Y, *_{X \wedge Y}) \to (Z, z_0)$ are two maps such that $\lambda^*_{X,Z} f = \lambda^*_{X,Z} g$. This means that the equality $\lambda_{X,Z}(f \circ q) = \lambda_{X,Z}(g \circ q)$ holds, but since $\lambda_{X,Z}$ defines an injection

$$\lambda_{X,Z} : \mathsf{Top}(X \times Y, Z) \to \mathsf{Top}(X, \mathrm{Map}(Y, Z)),$$

we must also have the equality $f \circ q = g \circ q$; therefore $f = g$ because q is a surjective map.

Now consider a pointed map $g : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$. Since $\mathrm{Map}_*(Y, Z)$ is by definition a subspace of $\mathrm{Map}(Y, Z)$, we may also regard g as a map of type $X \to \mathrm{Map}(Y, Z)$, and then, using the usual exponential adjunction, we can find a map $F : X \times Y \to Z$ such that $\lambda_{X,Z} F = g$. This means that, for any $(x, y) \in X \times Y$, we have the equality

$$F(x, y) = [\lambda F(x)](y) = [g(x)](y).$$

Using this formula we see that

$$F(x, y_0) = [g(x)](y_0) = z_0,$$

because $g(x) \in \mathrm{Map}_*(Y, Z)$, and also that

$$F(x_0, y) = [g(x_0)](y) = \mathrm{ct}_{Y,z_0}(y) = z_0,$$

[1] Notice that the brackets on both sides of the equation have very different meanings. The brackets on the left are merely for avoiding confusion, we could just as well write the left-hand side as $\lambda^*_{X,Z} f(x)(y)$. The brackets on the right-hand side are a way to denote an element of the form $q(x, y)$ in the smash product.

because g is itself pointed. It follows that we can factor F through the quotient map q to obtain a pointed map

$$f : (X \wedge Y, *_{X\wedge Y}) \to (Z, z_0)$$

which satisfies

$$\lambda^*_{X,Z} f = \lambda_{X,Z}(f \circ q) = \lambda_{X,Z} F = g;$$

proving the surjectivity of $\lambda^*_{X,Z}$.

We have shown that $\lambda^*_{X,Z}$ is a bijection, so the only thing left is to show that it depends naturally on both (X, x_0) and (Y, y_0). Given two pointed maps $f : (X', x_0') \to (X, x_0)$ and $g : (Z, z_0) \to (Z', z_0')$, we must show the commutativity of the square below.

$$\begin{tikzcd}
\mathsf{Top}_*(X \wedge Y, Z) \arrow[r, "\mathsf{Top}_*(f\wedge \mathrm{id}_Y, g)"] \arrow[d, "\lambda^*_{X,Z}"'] & \mathsf{Top}_*(X' \wedge Y, Z') \arrow[d, "\lambda^*_{X',Z'}"] \\
\mathsf{Top}_*(X, \mathrm{Map}_*(Y, Z)) \arrow[r, "\mathsf{Top}_*(f, \mathrm{Map}_*(Y,g))"'] & \mathsf{Top}_*(X', \mathrm{Map}_*(Y, Z'))
\end{tikzcd}$$

Given $\alpha \in \mathsf{Top}_*(X \wedge Y, Z)$, on the one hand we have

$$\begin{aligned}
(\lambda^*_{X',Z'} \circ \mathsf{Top}_*(f \wedge \mathrm{id}_Y, g))(\alpha) &= \lambda^*_{X',Z'}(g \circ \alpha \circ (f \wedge \mathrm{id}_Y)) \\
&= \lambda_{X',Z'}(g \circ \alpha \circ (f \wedge \mathrm{id}_Y) \circ q') \\
&= \lambda_{X',Z'}(g \circ \alpha \circ q \circ (f \times \mathrm{id}_Y)) \\
&\qquad\qquad \text{(by the definition of } f \wedge \mathrm{id}_Y) \\
&= (\lambda_{X',Z'} \circ \mathsf{Top}(f \times \mathrm{id}_Y, g))(\alpha \circ q)
\end{aligned}$$

Now, on the other hand, we have

$$\begin{aligned}
(\mathsf{Top}_*(f, \mathrm{Map}_*(Y, g)) \circ \lambda^*_{X,Z})(\alpha) &= \mathsf{Top}_*(f, \mathrm{Map}_*(Y, g))(\lambda_{X,Z}(\alpha \circ q)) \\
&= \mathrm{Map}_*(Y, g) \circ \lambda_{X,Z}(\alpha \circ q) \circ f \\
&= \mathrm{Map}(Y, g) \circ \lambda_{X,Z}(\alpha \circ q) \circ f \\
&= (\mathsf{Top}(f, \mathrm{Map}(Y, g)) \circ \lambda_{X,Z})(\alpha \circ q).
\end{aligned}$$

Notice that, when passing from the second to the third line, we used the fact that $\mathrm{Map}_*(Y, g)$ is obtained by simply restricting $\mathrm{Map}(Y, g)$.

After these computations, we see that the naturality we are trying to prove is equivalent to the equality

$$(\lambda_{X',Z'} \circ \mathsf{Top}(f \times \mathrm{id}_Y, g))(\alpha \circ q) = (\mathsf{Top}(f, \mathrm{Map}(Y, g)) \circ \lambda_{X,Z})(\alpha \circ q),$$

but this holds due to the naturality of the usual (unpointed) exponential adjunction. ■

When we proved the unpointed exponential adjunction (Theorem 1.1.5), we used the unit and counit transformations, but now for the pointed version we directly defined a natural bijection between the two relevant sets of pointed maps. Notice also that directly using the fact that the unpointed exponential transformations $\lambda_{X,Z}$ are invertible allowed us to show that the pointed exponential transformations $\lambda^*_{X,Z}$ are also invertible without actually exhibiting its inverse.

Nevertheless, it is useful to have the unit and counit transformations associated with the pointed exponential adjunction, as well as the inverse of the exponential transformations, and we now focus on constructing these objects.

Given a pointed space (X, x_0), the component of the unit transformation at (X, x_0) - which we denote by ι^*_X - is a pointed map of type $(X, x_0) \to (\mathrm{Map}_*(Y, X \wedge Y), \mathrm{ct}_{Y,*_{X\wedge Y}})$ obtained by applying the pointed exponential transformation $\lambda^*_{X,X\wedge Y}$ to the identity map of $X \wedge Y$.

$$X \wedge Y \xrightarrow{\mathrm{id}_{X\wedge Y}} X \wedge Y \overset{\lambda^*_{X,Z}}{\rightsquigarrow} X \xrightarrow{\iota^*_X} \mathrm{Map}_*(Y, X \wedge Y)$$

Using equation (3.2), we see that, for any $x \in X$, $\iota^*_X(x) : (Y, y_0) \to (X \wedge Y, *_{X\wedge Y})$ is the pointed map given by

$$[\iota^*_X(x)](y) \coloneqq [x, y] \quad \forall\, y \in Y. \tag{3.3}$$

Now we describe the inverse to $\lambda^*_{X,Z}$. If $g : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$ is a pointed map, since $\mathrm{Map}_*(Y, Z)$ is a subspace of $\mathrm{Map}(Y, Z)$, we may regard g also as a map of type $X \to \mathrm{Map}(Y, Z)$ and apply the inverse of the unpointed exponential adjunction to obtain a map

$$\lambda^{-1}_{X,Z} g : X \times Y \to Z.$$

This map satisfies the following properties:

1. $\lambda^{-1}_{X,Z} g(x, y_0) = [g(x)](y_0) = z_0$ for every $x \in X$ because $g(x)$ is a pointed map;

2. $\lambda^{-1}_{X,Z} g(x_0, y) = [g(x_0)](y) = \mathrm{ct}_{Y,z_0}(y) = z_0$ for every $y \in Y$ because g is itself pointed.

We can then factor $\lambda^{-1}_{X,Z} g$ through the quotient $q : X \times Y \to X \wedge Y$ to obtain a pointed map

$$\overline{\lambda^{-1}_{X,Z} g} : (X \wedge Y, *_{X\wedge Y}) \to (Z, z_0).$$

This map is in fact the inverse $(\lambda^*_{X,Z})^{-1} g$ we were looking for, since

$$\begin{aligned}
\lambda^*_{X,Z}(\overline{\lambda^{-1}_{X,Z} g}) &= \lambda_{X,Z}(\overline{\lambda^{-1}_{X,Z} g} \circ q) && \text{(by Definition 3.2.7)}\\
&= \lambda_{X,Z}(\lambda^{-1}_{X,Z} g) && \text{(by the construction of } \overline{\lambda^{-1}_{X,Z} g})\\
&= g;
\end{aligned}$$

which implies the desired equality

$$(\lambda^*_{X,Z})^{-1} g = \overline{\lambda^{-1}_{X,Z} g}.$$

In summary, the inverse exponential transformation

$$(\lambda^*_{X,Z})^{-1} : \mathsf{Top}_*(X, \mathrm{Map}_*(Y,Z)) \to \mathsf{Top}_*(X \wedge Y, Z)$$

sends a pointed map $g : X \to \mathrm{Map}_*(Y,Z)$ to the pointed map $(\lambda^*_{X,Z})^{-1}g : X \wedge Y \to Z$ defined by the formula

$$(\lambda^*_{X,Z})^{-1}g([x,y]) \coloneqq [g(x)](y) \quad \forall\, [x,y] \in X \wedge Y. \tag{3.4}$$

With the inverse pointed exponential adjunction at our disposal, we can also describe the associated counit transformation. Recall that its component at the pointed space (Z, z_0) is the pointed map

$$\mathrm{ev}^*_{Y,Z} : (\mathrm{Map}_*(Y,Z) \wedge Y, *_{\mathrm{Map}_*(Y,Z)\wedge Y}) \to (Z, z_0)$$

obtained by applying the inverse pointed exponential transformation $(\lambda^*_{X,Z})^{-1}$ to the identity map of $\mathrm{Map}_*(Y,Z)$.

$$\mathrm{Map}_*(Y,Z) \xrightarrow{\mathrm{id}_{\mathrm{Map}_*(Y,Z)}} \mathrm{Map}_*(Y,Z) \quad \overset{(\lambda^*_{\mathrm{Map}_*(Y,Z),Z})^{-1}}{\rightsquigarrow} \quad \mathrm{Map}_*(Y,Z) \wedge Y \overset{\mathrm{ev}^*_{Y,Z}}{\dashrightarrow} Z$$

Using equation (3.4) we see that the counit $\mathrm{ev}^*_{Y,Z}$ is given explicitly by the formula

$$\mathrm{ev}^*_{Y,Z}([f,y]) \coloneqq f(y) \quad \forall [f,y] \in \mathrm{Map}_*(Y,Z) \wedge Y, \tag{3.5}$$

which justifies our choice of notation, since $\mathrm{ev}^*_{Y,Z}$ is essentially a pointed version of the usual evaluation $\mathrm{ev}_{Y,Z}$. In fact, using the construction of the inverse transformation $(\lambda^*_{\mathrm{Map}_*(Y,Z),Z})^{-1}$ given above, one can show that $\mathrm{ev}^*_{Y,Z}$ is obtained precisely by factoring the evaluation map $\mathrm{ev}_{Y,Z}$ (suitably restricted to $\mathrm{Map}_*(Y,Z) \times Y \subseteq \mathrm{Map}(Y,Z) \times Y$) through the quotient map defining the smash product $\mathrm{Map}_*(Y,Z) \wedge Y$.

As with any unit and counit transformations associated with an adjunction, they define natural transformations between certain functors. The result below simply records this fact for the particular case of the unit and counit associated with the pointed exponential adjunction we have described above.

3.2.9 Corollary. Let (Y, y_0) be a locally compact Hausdorff pointed space.

1. The collection of pointed maps

$$\{\iota^*_X : (X, x_0) \to (\mathrm{Map}_*(Y, X \wedge Y), \mathrm{ct}_{Y,*_{X\wedge Y}})\}_{(X,x_0)\in\mathsf{Top}_*}$$

defines a natural transformation of functors $\iota^* : \mathrm{id}_{\mathsf{Top}_*} \Rightarrow \mathrm{Map}_*(Y, - \wedge Y)$.

2. The collection of pointed maps

$$\{\mathrm{ev}^*_{Y,Z} : (\mathrm{Map}_*(Y,Z) \wedge Y, *_{\mathrm{Map}_*(Y,Z)\wedge Y}) \to (Z, z_0)\}_{(Z,z_0)\in\mathsf{Top}*}$$

defines a natural transformation of functors $\mathrm{ev}^*_Y : \mathrm{Map}_*(Y,-) \wedge Y \Rightarrow \mathrm{id}_{\mathsf{Top}_*}$.

3.3 More on the smash product

In the previous section we introduced the smash product of two pointed spaces in order to obtain a pointed version of the exponential adjunction, but if we want to use this adjunction, it would be good to know some properties satisfied by the smash product construction, so the goal of this section is to prove some "algebraic" properties satisfied by the smash product functor.

3.3.1 Proposition. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces. The smash product satisfies the following properties:

1. The 0-dimensional sphere $(S^0, +1)$ is a unit for the smash product, that is, there are pointed homeomorphisms
$$S^0 \wedge X \cong X \cong X \wedge S^0$$
which are natural in X.
2. The smash product is commutative, that is, there is a pointed homeomorphism
$$X \wedge Y \cong Y \wedge X$$
which is natural in both X and Y.
3. The smash product is associative under some topological conditions, that is, if X and Z are locally compact and Hausdorff, then there is pointed homeomorphism
$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$$
which is natural in X, Y and Z.

Proof. 1. The product $S^0 \times X$ is just the disjoint union $\{-1\} \times X \cup \{+1\} \times X$ of two copies of X, so the function $L_X : S^0 \times X \to X$ defined as

$$L(t, x) := \begin{cases} x, & \text{if } t = -1; \\ x_0, & \text{if } t = +1, \end{cases}$$

is continuous by the universal property of the coproduct. The map L_X sends the subset

$$S^0 \times \{x_0\} \cup \{+1\} \times X = \{(-1, x_0)\} \cup \{+1\} \times X$$

to the basepoint x_0 of X, so it induces a pointed map $\ell_X : (S^0 \wedge X, *_{S^0 \wedge X}) \to (X, x_0)$. This map is a homeomorphism, since its inverse $\ell_X^{-1} : (X, x_0) \to (S^0 \wedge X, *_{S^0 \wedge X})$ can be defined explicitly as

$$\ell_X^{-1}(x) := [-1, x] \quad \forall x \in X.$$

We still need to check the naturality statement. This means that, if$f : (X, x_0) \to (Y, y_0)$ is a pointed map, then the square below commutes.

$$\begin{array}{ccc} S^0 \wedge X & \xrightarrow{\mathrm{id}_{S^0} \wedge f} & S^0 \wedge Y \\ {\scriptstyle \ell_X}\big\downarrow & & \big\downarrow{\scriptstyle \ell_Y} \\ X & \xrightarrow[f]{\quad\quad} & Y \end{array}$$

This is just a matter of straightforward calculations, since for every $x \in X$ we have

$$f(\ell_X([-1, x])) = f(x) = \ell_Y([-1, f(x)]) = \ell_Y((\mathrm{id}_{S^0} \wedge f)([-1, x])),$$

and also

$$f(\ell_X([1, x])) = f(x_0) = y_0 = \ell_Y([1, f(x)]) = \ell_Y((\mathrm{id}_{S^0} \wedge f)([1, x])).$$

The proof that S^0 is also a right unit is very similar. We define $R_X : X \times S^0 \to X$ by the formula

$$R_X(x, t) \coloneqq \begin{cases} x, & \text{if } t = -1; \\ x_0, & \text{if } t = +1. \end{cases}$$

Factoring this through the quotient defining the smash product we obtain a pointed map $r_X : (X \wedge S^0, *_{X \wedge S^0}) \to (X, x_0)$. Like in the first case, this map is a homeomorphism that depends naturally on (X, x_0).

2. Let $\tau_{X,Y} : X \times Y \to Y \times X$ be the permutation map that swaps the two coordinates. The composition $q' \circ \tau_{X,Y} : X \times Y \to Y \wedge X$, where $q' : Y \times X \to Y \wedge X$ is the canonical projection, maps $X \times \{y_0\} \cup \{x_0\} \times Y$ to the basepoint $*_{Y \wedge X}$, so it induces a pointed map $s_{X,Y} : (X \wedge Y, *_{X \wedge Y}) \to (Y \wedge X, *_{Y \wedge X})$ which can be described explicitly as

$$s_{X,Y}([x, y]) \coloneqq [y, x] \quad \forall\, [x, y] \in X \wedge Y.$$

If we do the same construction starting with the other permutation map $\tau_{Y,X} : Y \times X \to X \times Y$, then we obtain the analogous pointed map $s_{Y,X} : (Y \wedge X, *_{Y \wedge X}) \to (X \wedge Y, *_{X \wedge Y})$ given explicitly by

$$s_{Y,X}([y, x]) \coloneqq [x, y] \quad \forall\, [y, x] \in Y \wedge X.$$

A simple direct calculation then shows that $s_{X,Y}$ and $s_{Y,X}$ are inverse maps, therefore they define pointed homeomorphisms.

Now let (X', x'_0) and (Y, y'_0) be two other pointed spaces, and consider pointed maps $f : X \to X'$ and $g : Y \to Y'$. We need to show that the square below is commutative.

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{f \wedge g} & X' \wedge Y' \\ {\scriptstyle r_{X,Y}}\big\downarrow & & \big\downarrow{\scriptstyle r_{X',Y'}} \\ Y \wedge X & \xrightarrow[g \wedge f]{} & Y' \wedge X' \end{array}$$

For any $[x,y] \in X \wedge Y$ we have

$$\begin{aligned} r_{X',Y'}(f \wedge g([x,y])) &= r_{X',Y'}([f(x),g(y)]) \\ &= [g(y),f(x)] \\ &= g \wedge f([y,x]) \\ &= g \wedge f(r_{X,Y}([x,y])). \end{aligned}$$

3. This one is a bit more complicated. We first show the following auxiliary result: if Z is locally compact Hausdorff, then the function $\alpha_{X,Y,Z} : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$ defined as

$$\alpha_{X,Y,Z}([[x,y],z]) \coloneqq [x,[y,z]]$$

for every $[[x,y],z] \in (X \wedge Y) \wedge Z$ is a pointed map.

Consider first the *associator homeomorphism*

$$A : (X \times Y) \times Z \to X \times (Y \times Z)$$

defined as

$$A((x,y),z) \coloneqq (x,(y,z)).$$

We may form the adjoint map

$$\lambda A : X \times Y \to \mathrm{Map}(Z, X \times (Y \times Z)).$$

If $p : X \times (Y \times Z) \to X \wedge (Y \wedge Z)$ denotes the projection given by $(x,(y,z)) \mapsto [x,[y,z]]$, we have the induced pushforward

$$\mathrm{Map}(Z,p) : \mathrm{Map}(Z, X \times (Y \times Z)) \to \mathrm{Map}(Z, X \wedge (Y \wedge Z)).$$

We claim that the composition

$$\mathrm{Map}(Z,p) \circ \lambda A : X \times Y \to \mathrm{Map}(Z, X \wedge (Y \wedge Z))$$

maps the subspace $X \times \{*_Y\} \cup \{*_X\} \times Y$ to the basepoint of $\mathrm{Map}(Z, X \wedge (Y \wedge Z))$, which is by definition the constant map $\mathrm{ct}_{Z,*_{X\wedge(Y\wedge Z)}}$. Given a point $(x,y) \in X \times Y$, we have

$$(\mathrm{Map}(Z,p) \circ \lambda A)(x,y) = p \circ \lambda A(x,y),$$

therefore for any $z \in Z$ we have

$$\begin{aligned} [(\mathrm{Map}(Z,p) \circ \lambda A)(x,y)](z) &= p([\lambda A(x,y)](z)) \\ &= p(A((x,y),z)) \\ &= p(x,(y,z)) \\ &= [x,[y,z]]. \end{aligned}$$

In particular, if $x = x_0$, then

$$[(\mathrm{Map}(Z, p) \circ \lambda A)(x_0, y)](z) = [x_0, [y, z]] = *_{X\wedge(Y\wedge Z)},$$

and if $y = y_0$, then

$$[(\mathrm{Map}(Z, p) \circ \lambda A)(x, y_0)](z) = [x, [y_0, z]] = [x, *_{Y\wedge Z}] = *_{X\wedge(Y\wedge Z)}.$$

It follows by passing to the quotient that the composite $\mathrm{Map}(Z, p) \circ \lambda A$ induces a pointed map

$$\beta_{X,Y,Z} : X \wedge Y \to \mathrm{Map}_*(Z, X \wedge (Y \wedge Z)),$$

and by Theorem 3.2.8 there exists a pointed map

$$\alpha_{X,Y,Z} : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$$

such that $\beta_{X,Y,Z} = \lambda_*\alpha_{X,Y,Z}$. Using this relation we can deduce that $\alpha_{X,Y,Z}$ is precisely given by the formula

$$\alpha_{X,Y,Z}([x, y], z]) = [x, [y, z]]$$

that we stated in the beginning.

We can construct the inverse map of $\alpha_{X,Y,Z}$ by an analogous procedure with the difference that we start with the inverse associator homeomorphism

$$A^{-1} : X \times (Y \times Z) \to (X \times Y \times Z),$$

and use the fact that the functor $X \wedge -$ is also left adjoint to $\mathrm{Map}_*(X, -)$ due to X being locally compact and Hausdorff. This is true because we have already proved that the smash product is symmetric, therefore $X \wedge -$ is naturally isomorphic to $- \wedge X$.

There is, however, another way to obtain the inverse of $\alpha_{X,Y,Z}$ using the constructions we have described so far. The definition of the pointed map $\alpha_{X,Y,Z}$ works for any triple (X, Y, Z) of pointed spaces as long as Z - a.k.a. the third component of the triple - is locally compact and Hausdorff. Since X is also locally compact and Hausdorff by hypothesis, applying the construction to the triple (Z, Y, X) gives us a pointed map

$$\alpha_{Z,Y,X} : (Z \wedge Y) \wedge X \to Z \wedge (Y \wedge X)$$

which is given explicitly by $\alpha_{Z,Y,X}([[z, y], x]) = [z, [y, x]]$ for every $[[z, y], x] \in (Z \wedge Y) \wedge X$.

Consider then the pointed map of type $X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$ defined by the composition depicted below.

$$\begin{tikzcd}
X \wedge (Y \wedge Z) \arrow[r, "s_{X,Y\wedge Z}"] \arrow[d, dashed] & (Y \wedge Z) \wedge X \arrow[r, "s_{Y,Z}\wedge \mathrm{id}_X"] & (Z \wedge Y) \wedge X \arrow[d, "\alpha_{Z,Y,X}"] \\
(X \wedge Y) \wedge Z & Z \wedge (X \wedge Y) \arrow[l, "s_{Z,X\wedge Y}"] & Z \wedge (Y \wedge X) \arrow[l, "\mathrm{id}_Z \wedge s_{Y,X}"]
\end{tikzcd}$$

If we start with an element $[x, [y, z]] \in X \wedge (Y \wedge Z)$ and chase it around this diagram, we see that it gets mapped to $[[x, y], z]$.

$$\begin{tikzcd}
{[x,[y,z]]} \arrow[r, mapsto] \arrow[d, dashed, mapsto] & {[[y,z],x]} \arrow[r, mapsto] & {[[z,y],x]} \arrow[d, mapsto] \\
{[[x,y],z]} & {[z,[x,y]]} \arrow[l, mapsto] & {[z,[y,x]]} \arrow[l, mapsto]
\end{tikzcd}$$

In other words, this big composition defines a map which is precisely the inverse map of $\alpha_{X,Y,Z}$, proving therefore that we have a pointed homeomorphism

$$\alpha_{X,Y,Z} : (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z).$$

The only thing left is showing that this homeomorphism depends naturally on its three variables. Given three other pointed spaces (X', x_0'), (Y', y_0') and (Z', z_0') as well as pointed maps $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$; we must show the commutativity of the square depicted below.

$$\begin{tikzcd}
(X \wedge Y) \wedge Z \arrow[r, "(f\wedge g)\wedge h"] \arrow[d, "\alpha_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha_{X',Y',Z'}"] \\
X \wedge (Y \wedge Z) \arrow[r, "f\wedge(g\wedge h)"'] & X' \wedge (Y' \wedge Z')
\end{tikzcd}$$

Again, a direct calculation using the definitions of the maps in question shows that both of the compositions defined by the square above map an arbitrary element $[[x, y], z] \in (X \wedge Y) \wedge Z$ to $[f(x), [g(y), h(z)]] \in X' \wedge (Y' \wedge Z')$. ■

3.4 Reduced suspensions and loop spaces

This section introduces two very important constructions of Homotopy Theory: the reduced suspension and the loop space. There is an important duality between these two constructions which is made possible by the pointed exponential adjunction and by a comparison of the reduced suspension with a certain smash product. While we navigate towards this duality, we use the algebraic properties of the smash product proved in the previous section to compute some important explicit examples of smash products.

3.4.1 Definition. Let (X, x_0) be a pointed space. The quotient space

$$\Sigma X := \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I}$$

is called the **reduced suspension** of X. We regard it as a pointed space by considering the image of the subset $X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I$ under the canonical projection to the quotient as basepoint.

Geometrically, the reduced suspension ΣX is constructed by first forming a "double cone" over X by means of collapsing the top and bottom parts of the cylinder $X \times I$, and then collapsing the line segment on the surface of this cylinder that connects the two vertices and passes through the basepoint of X.

We can not only suspend spaces, but also maps.

3.4.2 Definition. Let (X, x_0) and (Y, y_0) be pointed maps. Denote by $q_X : X \times I \to \Sigma X$ and $q_Y : Y \times I \to \Sigma Y$ the canonical projections. Given a pointed map $f : X \to Y$, the **reduced suspension of** f is the unique pointed map $\Sigma f : \Sigma X \to \Sigma Y$ that fits into the commutative square below.

$$\begin{tikzcd} X \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "q_X"'] & Y \times I \arrow[d, "q_Y"] \\ \Sigma X \arrow[r, dashed, "\Sigma f"'] & \Sigma Y \end{tikzcd}$$

This definition makes sense because the composite $q_Y \circ (f \times \mathrm{id}_I) : X \times I \to \Sigma Y$ maps the subset $X \times \{0\} \cup X \times \{1\} \cup \{*_X\} \times I$ to the basepoint $*_{\Sigma Y}$, and Σf is the unique pointed map that factors this composition through the quotient q_X. The commutativity of the square means that Σf has an explicit formula given by

$$\Sigma f([x, t]) = [f(x), t]$$

for all points $[x, t] \in \Sigma X$.

As with all constructions that we have introduced so far, the reduced suspension gives rise to a functor $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$ unsurprisingly called the **reduced suspension functor**. This functoriality property can be proved in a manner similar to the proof of the functoriality of the smash product (Proposition 3.2.6) by pasting together suitable commutative diagrams, and using the already known functoriality of the product functor $- \times I$.

Our next goal is to show how reduced suspension can be regarded as a convenient way to represent some smash products. The result is essentially a consequence of the next simple lemma.

3.4.3 Lemma. There exists a pointed homeomorphism $(I/\partial I, *) \cong (S^1, (1, 0))$.

Proof. Consider the exponential map $\exp : I \to S^1$ defined by the formula

$$\exp(t) := (\cos 2\pi t, \sin 2\pi t) \quad \forall\, t \in I.$$

This is a surjective map, and it only looses injectivity at the endpoints 0 and 1 of I, which get both mapped to the basepoint of S^1. If we then factor exp through the quotient $I \to I/\partial I$ we obtain the desired pointed homeomorphism. ■

Using this homeomorphism and the universal property of quotient spaces we can interpret this result as saying that, in order to define a map of type $S^1 \to Y$, it suffices to define a map of type $I \to Y$ which sends the endpoints of the interval to the same point in Y.

Using this homeomorphism we can express the reduced suspension construction in terms of the smash product.

3.4.4 Proposition. The reduced suspension functor Σ is naturally isomorphic to the smash product functor $- \wedge S^1$.

Proof. Consider an arbitrary pointed space (X, x_0). If $q : X \times S^1 \to X \wedge S^1$ denotes the canonical projection, then the composite map

$$q \circ (\mathrm{id}_X \times \exp) : X \times I \to S \wedge S^1$$

maps all points of the subspace $X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I \subseteq X \times I$ to the basepoint of $X \wedge S^1$, therefore it can be factored through the quotient $p : X \times I \to \Sigma X$ to yield a pointed map

$$\theta_X : (\Sigma X, *_{\Sigma X}) \to (X \wedge S^1, *_{X \wedge S^1})$$

like shown in the commutative square below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times \exp"] \arrow[d, "p"'] & X \times S^1 \arrow[d, "q"] \\ \Sigma X \arrow[r, dashed, "\theta_X"'] & X \wedge S^1 \end{tikzcd} \tag{3.6}$$

Explicitly, θ_X satisfies the equation

$$\theta_X([x, t]) = [x, \exp(t)] = [x, (\cos(2\pi t), \sin(2\pi t))] \tag{3.7}$$

for every $x \in X$ and every $t \in I$.

We now construct an inverse to θ_X. The map $\exp : I \to S^1$ is closed, since its domain is compact and its codomain is Hausdorff. Moreover, exp has compact fibers because $\exp^{-1}(z)$ either consists of a single point when $z \neq *_{S^1}$, or it consists of the points 0 and 1 when $z = *_{S^1}$. It follows from these two properties that exp is a *proper map* (see (BROWN, 2006, result 3.6.3)), and thus the product $\mathrm{id}_X \times \exp$ defines a closed map, but since this product is also surjective, it is in fact a quotient map.

Using the description of the fibers of exp given above, we see that the canonical projection $p : X \times I \to \Sigma X$ is constant on the fibers of the quotient product map $\mathrm{id}_X \times \exp$, therefore it can be factored through it to define a map $\overline{p} : X \times S^1 \to \Sigma X$ that fits in the commutative diagram below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times \exp"] \arrow[d, "p"'] & X \times S^1 \arrow[dl, dashed, "\overline{p}"] \\ \Sigma X & \end{tikzcd}$$

Explicitly, $\overline{p}$ is described as follows: given a point $(x, z) \in X \times S^1$, if we write $z = \exp(t)$ for some $t \in I$, then

$$\overline{p}(x, z) = [x, t].$$

We see from this formula that $\overline{p}$ satisfies the property

$$\overline{p}(X \times \{(1, 0)\} \cup \{x_0\} \times S^1) \subseteq \{*_{\Sigma X}\},$$

therefore it can be factored through the quotient $q : X \times S^1 \to X \wedge S^1$ to define a pointed map

$$\psi_X : (X \wedge S^1, *_{X \wedge S^1}) \to (\Sigma X, *_{\Sigma X}),$$

and we end up with the commutative diagram below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times \exp"] \arrow[d, "p"'] & X \times S^1 \arrow[dl, "\overline{p}"'] \arrow[d, "q"] \\ \Sigma X & X \wedge S^1 \arrow[l, dashed, "\psi_X"] \end{tikzcd} \tag{3.8}$$

Explicitly, the map ψ_X defined like this satisfies the equation

$$\psi_X([x, \exp(t)]) = [x, t] \tag{3.9}$$

for every $x \in X$ and every $t \in I$.

Let us show that θ_X and ψ_X are inverse maps. On the one hand we have

$$\begin{aligned} \psi_X \circ \theta_X \circ p &= \psi_X \circ q \circ (\mathrm{id}_X \times \exp) && \text{(by (3.6))} \\ &= p, && \text{(by (3.8))} \end{aligned}$$

and since p is surjective, it can be cancelled on both sides to yield the equality

$$\psi_X \circ \theta_X = \mathrm{id}_{\Sigma X};$$

and on the other hand

$$\begin{aligned} \theta_X \circ \psi_X \circ q \circ (\mathrm{id}_X \times \exp) &= \theta_X \circ \overline{p} \circ (\mathrm{id}_X \times \exp) \\ &= \theta_X \circ p \\ &= q \circ (\mathrm{id}_X \times \exp), \end{aligned}$$

and by cancelling $q \circ (\mathrm{id}_X \times \exp)$ on both sides we arrive at the equality

$$\theta_X \circ \psi_X = \mathrm{id}_{X \wedge S^1}.$$

This proves that θ_X defines a pointed homeomorphism $\Sigma X \cong X \wedge S^1$ for every pointed space (X, x_0). The only thing left is showing the naturality of this homeomorphism. Given a pointed map $f : (X, x_0) \to (Y, y_0)$, we need to prove the equality

$$\theta_Y \circ \Sigma f = (f \wedge \mathrm{id}_{S^1}) \circ \theta_X.$$

Let $p_X : X \times I \to \Sigma X$ and $q_X : X \times S^1 \to X \wedge S^1$ denote the canonical projections, with analogous notations p_Y and q_Y for the space (Y, y_0). Recall that the maps Σf and $f \wedge \mathrm{id}_{S^1}$ are constructed in such a way as to fit in the commutative diagrams below.

$$\begin{tikzcd} X \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "p_X"'] & Y \times I \arrow[d, "p_Y"] \\ \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \end{tikzcd} \qquad \begin{tikzcd} X \times S^1 \arrow[r, "f \times \mathrm{id}_{S^1}"] \arrow[d, "q_X"'] & Y \times S^1 \arrow[d, "q_Y"] \\ X \wedge S^1 \arrow[r, "f \wedge \mathrm{id}_{S^1}"'] & Y \wedge S^1 \end{tikzcd}$$

Using these diagrams together with the defining properties of the homeomorphisms θ_X we see that

$$\begin{aligned} \theta_Y \circ \Sigma f \circ p_X &= \theta_Y \circ p_Y \circ (f \times \mathrm{id}_I) \\ &= q_Y \circ (\mathrm{id}_X \times \exp) \circ (f \times \mathrm{id}_I) && \text{(by (3.6))} \\ &= q_Y \circ (f \times \exp) && \text{(by functoriality)} \\ &= q_Y \circ (f \times \mathrm{id}_{S^1}) \circ (\mathrm{id}_X \times \exp) \\ &= (f \wedge \mathrm{id}_{S^1}) \circ q_X \circ (\mathrm{id}_X \times \exp) \\ &= (f \wedge \mathrm{id}_{S^1}) \circ \theta_X \circ p_X, && \text{(by (3.6))} \end{aligned}$$

and then cancelling p_X on both sides yields the desired equality. ■

The interesting part about this comparison of the reduced suspension with the smash product is that it allows a further comparison with a certain mapping space.

3.4.5 Definition. Let (X, x_0) be a pointed space. The mapping space

$$\Omega X := \mathrm{Map}_*(S^1, X) \tag{3.10}$$

is called the **loop space of** X. It is a pointed space whose basepoint is given by the constant loop $\mathrm{ct}_{S^1, x_0} : S^1 \to X$ at the basepoint of X.

The adjunction between smash products and mapping spaces gives us an adjunction between reduced suspension and loop spaces.

3.4.6 Corollary (Eckmann-Hilton Duality)**.** The reduced suspension functor Σ is left adjoint to the loop space functor Ω, or in other words, for any pointed spaces (X, x_0) and (Y, y_0) there is a natural bijection

$$\mathsf{Top}_*(\Sigma X, Y) \cong \mathsf{Top}_*(X, \Omega Y).$$

Proof. In Proposition 3.4.4 we saw that the reduced suspension functor is naturally isomorphic to the smash product functor $- \wedge S^1$ by means of the family of pointed homeomorphisms

$$\{\psi_X : (X \wedge S^1, *_{X\wedge S^1}) \to (\Sigma X, *_{\Sigma X})\}_{(X,x_0)\in\mathsf{Top}_*}$$

It follows that the Set-valued functors

$$\mathsf{Top}_*(- \wedge S^1, -),\ \mathsf{Top}(\Sigma -, -) : \mathsf{Top}_*^{\mathsf{op}} \times \mathsf{Top}_* \to \mathsf{Set}$$

are naturally isomorphic too, with component bijections given by the pullbacks

$$\mathsf{Top}_*(\psi_X, Y) : \mathsf{Top}_*(\Sigma X, Y) \xrightarrow{\cong} \mathsf{Top}_*(X \wedge S^1, Y).$$

Combining this with the natural isomorphism

$$\mathsf{Top}_*(- \wedge S^1, -) \cong \mathsf{Top}_*(-, \mathrm{Map}_*(S^1, -)) = \mathsf{Top}_*(-, \Omega -)$$

coming from the pointed exponential adjunction yields the desired natural isomorphism

$$\mathsf{Top}_*(\Sigma -, -) \cong \mathsf{Top}_*(-, \Omega -). \qquad \blacksquare$$

We can make the component bijections of the adjunction above more explicit. Given a pointed map $f : (\Sigma X, *) \to (Y, y_0)$, its adjunct map $\widetilde{f} : (X, x_0) \to (\Omega Y, \mathrm{ct}_{S^1, y_0})$ associates to each point $x \in X$ the loop $\widetilde{f}(x) : S^1 \to Y$ based at y_0 which satisfies the equation

$$[\widetilde{f}(x)](\exp(t)) = f([x, t]) \tag{3.11}$$

for every $t \in I$.

We finish the chapter with a description of the smash product of spheres. We already know how to describe this if one of the factors is the 0-sphere S^0, so we start by understanding the smash product of a sphere with the circle, or in other words, the reduced suspension of spheres.

3.4.7 Proposition. There is a pointed homeomorphism $\Sigma S^n \cong S^{n+1}$ for every $n \in \mathbb{N}$.

Proof. The definition of the homeomorphism has to do with the geometric meaning of the construction of the reduced suspension. When applying the canonical projection $p : X \times I \to \Sigma X$, for each $x \in X$, the end points $(x, 0)$ and $(x, 1)$ of the line segment $\{x\} \times I$

on the surface of the cylinder get identified, so this line segment gets transformed into a circle. In other words, the reduced suspension ΣX is "foliated" by circles, but there is a catch: since *all* points in the line $\{x_0\} \times I$ get identified, not just the endpoints, this line gets transformed into a point - the basepoint of the reduced suspension - which we may regard as a degenerate circle. See Figure 1 for a visualization of this foliation by circles in the case $n = 1$.

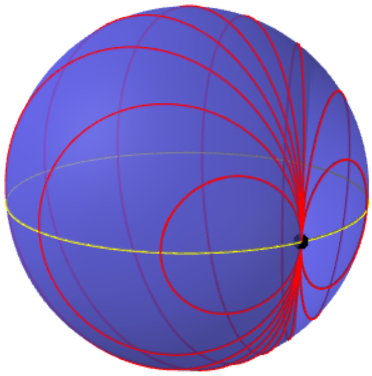

Figure 1 – The sphere is the reduced suspension of the circle.

The idea is then to realize S^{n+1} as also being "foliated" by a family of circles parameterized by S^n, where the circle associated with $*_{S^n}$ is just the basepoint $*_{S^{n+1}}$. This is done as follows: each point $x \in S^n$ gives rise to a point $(x, 0)$ on the equator of S^{n+1}, and the plane spanned by $*_{S^{n+1}} - (x, 0)$ and $(0, \dots, 0, 1)$ intersects S^{n+1} along a circle. This is the family of circles we will use to construct our homeomorphism.

Consider first the map $c : S^n \to \mathbb{R}^{n+2}$ defined as follows: given $x = (x_1, \dots, x_{n+1}) \in S^n$,

$$c(x) = \tfrac{1}{2} \cdot (*_{S^{n+1}} + (x, 0)) = \tfrac{1}{2} \cdot (1 + x_1, x_2, \dots, x_{n+1}, 0).$$

The circle we want to define has the form

$$c(x) + (\|*_{S^{n+1}} - c(x)\| \cos(2\pi t)) \cdot u(x) + (\|*_{S^{n+1}} - c(x)\| \sin(2\pi t)) \cdot (0, \dots, 0, 1)$$

In this formula, $u(x)$ is a *unit* vector which together with $(0, \dots, 0, 1)$ spans the plane whose intersection with S^{n+1} is the circle we are trying to parameterize. It is important to remark that $u(x)$ *must* have unit norm, otherwise the expression above describes an ellipse, and not a circle. The number $\|*_{S^{n+1}} - c(x)\|$ appearing in the formula is exactly the radius of the circle we are parameterizing.

We can define $u : S^n \to \mathbb{R}^{n+2}$ to be

$$u(x) = \frac{1}{\|*_{S^{n+1}} - c(x)\|} \cdot (*_{S^{n+1}} - c(x)).$$

Having defined $u(x)$, the expression for the circle can be rewritten as

$$c(x) + \cos(2\pi t) \cdot (*_{S^{n+1}} - c(x)) + (\|*_{S^{n+1}} - c(x)\| \sin(2\pi t)) \cdot (0, \dots, 0, 1).$$

Grouping together the first two terms, we are led to define $F : S^n \times I \to S^{n+1}$ by the formula

$$\begin{aligned} F(x,t) \coloneqq \left(\frac{1+\cos(2\pi t)}{2}\right) \cdot *_{S^{n+1}} + \left(\frac{1-\cos(2\pi t)}{2}\right) \cdot (x,0) \\ + (\|*_{S^{n+1}} - c(x)\| \sin(2\pi t)) \cdot (0, \dots, 0, 1). \end{aligned}$$

If $x = (x_1, \dots, x_{n+1})$, taking into account that

$$\|*_{S^{n+1}} - c(x)\| = \frac{1}{2}\|*_{S^{n+1}} - (x,0)\| = \frac{1}{2}\sqrt{(1-x_1)^2 + x^2 + \cdots + x_{n+1}^2} = \frac{1}{2}\sqrt{2-2x_1},$$

we have a more explicit (and more terrifying) formula

$$\begin{aligned} F(x,t) = \frac{1}{2} \cdot \big(1+\cos(2\pi t) + (1-\cos(2\pi t))x_1, (1-\cos(2\pi t))x_2, \cdots, (1-\cos(2\pi t))x_{n+1}, \\ \sin(2\pi t)\sqrt{2-2x_1}\big). \quad (3.12) \end{aligned}$$

These formulas show that $F(x,t) = *_{S^n}$ if either $x = *_{S^n}$ or $t \in \{0,1\}$. This means that F induces a pointed map $W : (\Sigma S^n, *_{\Sigma S^n}) \to (S^{n+1}, *_{S^{n+1}})$, which is the desired homeomorphism.

$$\begin{tikzcd} S^n \times I \arrow[d, "p"'] \arrow[dr, "F"] & \\ \Sigma S^n \arrow[r, dashed, "W"'] & S^{n+1} \end{tikzcd}$$

■

Before continuing, let us analyze the homeomorphism $W : \Sigma S^0 \cong S^1$ constructed above. Since $S^0 = \{-1, +1\}$, the product $S^0 \times I$ consists of two disjoint copies $\{-1\} \times I$ and $\{+1\} \times I$ of the interval I. The projection $p : S^0 \times I \to \Sigma S^0$ identifies one of the copies $\{+1\} \times I$, as well as the two endpoints $(-1,0)$ and $(-1,1)$ of the other copy. The map $F : S^0 \times I \to S^1$ is particularly simple in this case:

$$F(s,t) = \begin{cases} *_{S^1}, & \text{if } s = +1, \\ (\cos(2\pi t), \sin(2\pi t)), & \text{if } s = -1. \end{cases}$$

In other words, the composition $W \circ p : S^0 \times I \to S^1$ is constant on the copy $\{+1\} \times I$ of the interval, and it is equal to the exponential $\exp : I \to S^1$ on the copy $\{-1\} \times I$.

We are now able to describe the smash product of any two spheres.

3.4.8 Corollary. There is an isomorphism of pointed spaces $S^m \wedge S^n \cong S^{m+n}$ for any $m, n \in \mathbb{N}$.

Proof. We fix m and perform an induction on n. For the base case $n = 0$, the isomorphism $S^m \wedge S^0 \cong S^m$ follows from the fact that S^0 is a unit for the smash product according to Proposition 3.3.1. If the isomorphism holds for n, using the associativity of the smash product and Proposition 3.4.7, we have

$$S^m \wedge S^{n+1} \cong S^m \wedge (S^n \wedge S^1) \cong (S^m \wedge S^n) \wedge S^1 \cong S^{m+n} \wedge S^1 \cong S^{m+n+1}. \qquad \blacksquare$$

CHAPTER

4

POINTED HOMOTOPY

This chapter is devoted to developing the basic notions of *pointed homotopy*, which is the version of homotopy well-adapted to the category Top_* of pointed spaces. It follows the same thread of ideas of Chapter 2. The first section introduces the notion of pointed homotopy in different equivalent forms making use of the pointed exponential adjunction. The next section then introduces the pointed version of the homotopy category, and studies some of its properties, like how some functors of pointed spaces interact with pointed homotopies. The last section adapts the results relating null homotopic maps and contractible spaces to the context of pointed spaces. Overall, the structure of this chapter is completely analogous to that of the second one.

4.1 Different notions of pointed homotopy

We first define a pointed homotopy as a map defined on a cylinder. It is an ordinary homotopy satisfying an extra condition related to the basepoints.

4.1.1 Definition. Consider pointed spaces (X, x_0) and (Y, y_0). We say that two pointed maps $f, g : (X, x_0) \to (Y, y_0)$ are **pointed homotopic** if there is a map $H : X \times I \to Y$ satisfying the following conditions:

(i) $H(x, 0) = f(x) \quad \forall\, x \in X$;

(ii) $H(x, 1) = g(x) \quad \forall\, x \in X$;

(iii) $H(x_0, t) = y_0 \quad \forall\, t \in I$.

The map H is called a **pointed homotopy** from f to g.

Throughout the text we use the notation $f \simeq_* g$ to denote that f and g are *pointed* homotopic. Occasionally, we also use the notation $H : f \Rightarrow_* g$ in case we want to make the pointed homotopy explicit.

In Chapter 2 we saw that a homotopy $H : X \times I \to Y$ can also be thought of as a family of paths, that is, as a map of type $X \to \mathrm{Map}(I, Y)$, connecting the images of two maps. We would like to do the same for pointed homotopies, since in particular this would allow us to recycle many of the proofs we gave for results about unpointed homotopies. This relation between homotopies and families of paths was obtained by means of the exponential adjunction

$$\mathsf{Top}(X \times I, Y) \cong \mathsf{Top}(X, \mathrm{Map}(I, Y)).$$

In order to be able to describe a pointed homotopy as a certain family of paths, we introduce a modified version of the usual cylinder $X \times I$. The idea is that, since a pointed homotopy $H : X \times I \to Y$ is constant along the subspace $\{x_0\} \times I$, it makes sense to collapse this subspace to a point, and regard the condition on H as a pointed condition on a map defined on the quotient.

4.1.2 Definition. Given a pointed space (X, x_0), the quotient space

$$X \rtimes I := \frac{X \times I}{\{x_0\} \times I}$$

is called the **reduced cylinder** over (X, x_0). It becomes a pointed space if we choose the image of $x_0 \times I$ under the canonical projection $q : X \times I \to X \rtimes I$ as basepoint.

The usual cylinder $X \times I$ comes equipped with an inclusion $i_{X,0} : X \to X \times I$ into the “lower face” of the cylinder. If we compose it with the canonical projection $q : X \times I \to X \rtimes I$, we obtain an analogous inclusion $j_{X,0} : X \to X \rtimes$ into the reduced cylinder. Notice that

$$j_{X,0}(x_0) = q(i_{X,0}(x_0)) = q(x_0, 0) = *_{X \rtimes I},$$

so $j_{X,0}$ actually defines a pointed map $(X, x_0) \to (X \rtimes I, *_{X \rtimes I})$. See Figure 2 for a representation of the reduced cylinder over the circle S^1 along with the inclusions $j_{S^1,0}$ and $j_{S^1,1}$.

In the definition of an unpointed homotopy, the inclusions $i_{X,0}$, $i_{X,1} : X \to X \times I$ allow us to specify the starting and ending points of a homotopy. More precisely, we saw that, if f, $g : X \to Y$ are maps, then the data of a homotopy $H : f \Rightarrow g$ can be encoded

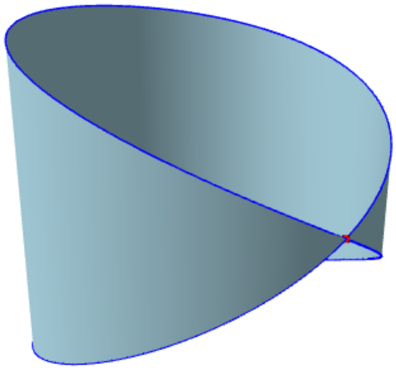

Figure 2 – The reduced cylinder over the circle S^1. The bottom and top blue circles are the images of the inclusions $j_{S^1,0}$ and $j_{S^1,1}$ respectively.

in the diagram below.

$$\begin{tikzcd}
X \arrow[d, "i_{X,0}"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, "H"] & Y \\
X \arrow[u, "i_{X,1}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

If X, Y, f and g are pointed, then the commutativity of this diagram does not guarantee that H is a pointed homotopy, since it does not enforce any conditions on the values of $H(x_0, t)$ for $t \in I$. The next result shows that by working with the reduced cylinder we can give a similar diagrammatic description of a pointed homotopy with the $j_{X,0}$ and $j_{X,1}$ playing the roles of $i_{X,0}$ and $i_{X,1}$.

4.1.3 Proposition. Two pointed maps f, $g : (X, x_0) \to (Y, y_0)$ are pointed homotopic if and only if there exists a pointed map $K : (X \rtimes I, *) \to (Y, y_0)$ that fits in the commutative diagram below.

$$\begin{tikzcd}
X \arrow[d, "j_{X,0}"'] \arrow[dr, "f"] & \\
X \rtimes I \arrow[r, dashed, "K"] & Y \\
X \arrow[u, "j_{X,1}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. Suppose first that there exists a pointed map $K : (X \rtimes I, *) \to (Y, y_0)$ as depicted above. Let $H : X \times I \to Y$ be defined via the composition $H := K \circ q$, where $q : X \times I \to X \rtimes I$ denotes the canonical projection.

We claim that H defines a pointed homotopy $f \simeq_* g$. Indeed, on the one hand we have

$$H \circ i_{X,0} = K \circ q \circ i_{X,0} = K \circ j_{X,0} = f,$$

while on the other

$$H \circ i_{X,1} = K \circ q \circ i_{X,1} = K \circ j_{X,1} = g,$$

so H defines at least an unpointed homotopy from f to g. In order to see that H is actually pointed, we just note that, for any $t \in I$, there is an equality

$$H(x_0, t) = K(q(x_0, t)) = K(*) = y_0.$$

Conversely, suppose $H : X \times I \to Y$ is a pointed homotopy starting at f and ending at g. Since by hypothesis H maps the whole line segment $\{x_0\} \times I$ to the basepoint y_0 of Y, it can be factored through the canonical projection q to define a pointed map $K : (X \rtimes I, *) \to (Y, y_0)$ as shown below.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "q"'] & Y \\ X \rtimes I \arrow[ur, dashed, "K"'] & \end{tikzcd}$$

This is the desired map, since on the one hand

$$K \circ j_{X,0} = K \circ q \circ i_{X,0} = H \circ i_{X,0} = f,$$

while on the other

$$K \circ j_{X,1} = K \circ q \circ i_{X,1} = H \circ i_{X,1} = g. \qquad \blacksquare$$

Having described pointed homotopies in diagrammatic terms, we would now like to obtain a description in terms of families of paths similar to Proposition 2.1.3 for unpointed homotopies. In order to do this, we first prove that the reduced cylinder fits into a sort of exponential adjunction relation.

The construction of the reduced cylinder can be easily made functorial. If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, then the product $f \times \mathrm{id}_I : X \times I \to Y \times I$ maps the line segment $\{x_0\} \times I$ to the corresponding segment $\{y_0\} \times I$, so the composition $q_Y \circ (f \times \mathrm{id}_I) : X \times I \to Y \rtimes I$ is constant and equal to $*_{Y \rtimes I}$ on $\{x_0\} \times I$; therefore by passing to the quotient we obtain a pointed map $f \rtimes \mathrm{id}_I : (X \rtimes I, *_{X \rtimes I}) \to (Y \rtimes I, *_{Y \rtimes I})$.

$$\begin{tikzcd} X \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "q_X"'] & Y \times I \arrow[d, "q_Y"] \\ X \rtimes I \arrow[r, dashed, "f \rtimes \mathrm{id}_I"'] & Y \rtimes I \end{tikzcd}$$

Using the functorial properties of $- \times I$ it is straightforward to show that the assignment $(X, x_0) \mapsto (X \rtimes I, *_{X \rtimes I})$ and $(f : (X, x_0) \to (Y, y_0)) \mapsto (f \rtimes \mathrm{id}_I : (X \rtimes I, *_{X \rtimes I}) \to (Y \rtimes I, *_{Y \rtimes I}))$ defines a functor $- \rtimes I : \mathsf{Top}_* \to \mathsf{Top}_*$.

We will show that $- \rtimes I$ is a left adjoint functor just like $- \times I$. The right adjoint in this case is the space of paths functor $\mathrm{Map}(I, -)$. Notice that, even without choosing a particular basepoint for I, for any pointed space (Y, y_0), the space of paths $\mathrm{Map}(I, Y)$ has a natural basepoint: the constant path $\mathrm{ct}_{I,y_0} : I \to Y$. If $g : (Y, y_0) \to (Z, z_0)$ is pointed, then the pushforward $\mathrm{Map}(I, g) : \mathrm{Map}(I, Y) \to \mathrm{Map}(I, Z)$ sends the constant path ct_{I,y_0} to the corresponding constant path $g \circ \mathrm{ct}_{I,y_0} = \mathrm{ct}_{I,z_0}$, so we can regard it as a pointed map

$$\mathrm{Map}(I, g) : (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0}) \to (\mathrm{Map}(I, Z), \mathrm{ct}_{I,z_0}).$$

We can then consider the space of paths functor $\mathrm{Map}(I, -) : \mathsf{Top}_* \to \mathsf{Top}_*$.

The next result concerns the adjointness relation which is behind the two possible descriptions of a pointed homotopy.

4.1.4 Proposition. The reduced cylinder functor $- \rtimes I : \mathsf{Top}_* \to \mathsf{Top}_*$ is left adjoint to the space of maps functor $\mathrm{Map}(I, -) : \mathsf{Top}_* \to \mathsf{Top}_*$.

Sketch of proof. Given pointed spaces (X, x_0) and (Y, y_0), let

$$\lambda^* : \mathsf{Top}_*(X \rtimes I, Y) \to \mathsf{Top}_*(X, \mathrm{Map}(I, Y))$$

be the function defined in terms of the usual exponential adjunction as

$$\lambda^* f := \lambda(f \circ q_X),$$

where $q_X : X \times I \to X \rtimes I$ denotes the canonical projection. Notice that λ^* really takes values in $\mathsf{Top}_*(X, \mathrm{Map}(I, Y))$ since for any $f \in \mathsf{Top}_*(X \rtimes I, Y)$ we have the equality

$$[\lambda^* f(x_0)](t) = [\lambda(f \circ q)(x_0)](t) = (f \circ q)(x_0, t) = f(*_{X \rtimes I}) = y_0,$$

proving that $\lambda^* f(x_0) = \mathrm{ct}_{I,y_0}$.

The injectivity of λ^* follows from the injectivity of the usual exponential adjunction, because if $\lambda^* f = \lambda^* g$, then $\lambda(f \circ q_X) = \lambda(g \circ q_X)$, which implies $f \circ q_X = g \circ q_X$, and therefore $f = g$ due to the surjectivity of q_X. The surjectivity of λ^* similarly follows from the analogous property of the normal exponential adjunction. Given a pointed map $g : (X, x_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$, if we forget for a moment that it is pointed, and regard it simply as a map $g : X \to \mathrm{Map}(I, Y)$, then by the usual exponential adjunction there is a map $f : X \times I \to Y$ such that $\lambda f = g$. Notice that, for any $t \in I$, since g is pointed we have

$$f(x_0, t) = [\lambda f(x_0)](t) = [g(x_0)](t) = \mathrm{ct}_{I,y_0}(t) = y_0.$$

This means that f is constant and equal y_0 on the line segment $\{x_0\} \times I$, thus it can be factored through q_X to define a pointed map $\overline{f} : (X \rtimes I, *_{X \rtimes I}) \to (Y, y_0)$. Applying λ^* to $\overline{f}$ we find that

$$\lambda^* \overline{f} = \lambda(\overline{f} \circ q_X) = \lambda f = g,$$

proving that λ^* is a surjective function.

We have proved so far that λ^* defines a bijection of sets $\mathsf{Top}_*(X \rtimes I, Y) \cong \mathsf{Top}_*(X, \mathrm{Map}(I, Y))$. The naturality of this bijection follows from the naturality of the usual exponential adjunction λ. ■

Equipped with this adjunction we can finally describe a pointed homotopy in terms of a suitable family of paths.

4.1.5 Proposition. Two pointed maps f, $g : (X, x_0) \to (Y, y_0)$ are pointed homotopic if and only if there exists a pointed map $D : (X, x_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$ that fits in the commutative diagram below.

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, dashed, "D"] \arrow[dr, "g"'] & \mathrm{Map}(I, Y) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & Y \end{tikzcd}$$

Proof. Suppose first that f and g are pointed homotopic. According to Proposition 4.1.3, this means that there exists a pointed map $K : (X \rtimes I, *) \to (Y, y_0)$ such that $K \circ j_{X,0} = f$ and $K \circ j_{X,1} = g$.

Applying the adjunction of Proposition 4.1.4 we obtain the pointed map $D := \lambda^* K : (X, x_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$. For any $x \in X$ we have

$$\begin{aligned} \mathrm{ev}_0(D(x)) &= [D(x)](0) \\ &= [\lambda^* K(x)](0) \\ &= [\lambda(K \circ q_X)(x)](0) \\ &= K(q_X(x, 0)) \\ &= K(q_X(i_{X,0}(x))) \\ &= K(j_{X,0}(x)) \\ &= f(x), \end{aligned}$$

and a completely analogous reasoning shows that we also have $\mathrm{ev}_1(D(x)) = g(x)$ for every $x \in X$; proving that D satisfies the required commutativity conditions.

Conversely, given a pointed map $D : (X, x_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$ as in the diagram, by applying Proposition 4.1.4 we obtain a pointed map $K : (X \rtimes Y, *) \to (Y, y_0)$

such that $D = \lambda^* K$. For any $x \in X$ we have the equalities

$$\begin{aligned} K(j_{X,0}(x)) &= (K \circ q_X)(x, 0) \\ &= [\lambda(K \circ q_X)(x)](0) \\ &= [\lambda^* K(x)](0) \\ &= [D(x)](0) \\ &= \mathrm{ev}_0(D(x)) \\ &= f(x); \end{aligned}$$

therefore $K \circ j_{X,0} = f$. Analogously, we also have $K \circ j_{X,1} = g$, thus K defines a pointed homotopy $f \simeq_* g$ according to Proposition 4.1.3. ■

4.2 The pointed homotopy category

In the previous section we introduced the concept of a pointed homotopy between two pointed maps, and we then studied how it can be described in three equivalent ways:

1. as an ordinary homotopy satisfying an extra condition;
2. as a pointed map out of the reduced cylinder satisfying some conditions;
3. as a pointed family of paths satisfying some conditions.

In the present section, we introduce the analogue of the homotopy category HoTop in the pointed case, and we exploit the three characterizations above to deduce some useful properties of this category. This section is basically an adaptation of Section 2.2 to the context of pointed spaces.

Given two pointed spaces (X, x_0) and (Y, y_0), the notion of pointed homotopy introduces a relation $\simeq_*$ in the set $\mathsf{Top}{*}(Y, Y)$ of pointed maps. It turns out that this relation is in fact an equivalence relation. In order to prove this, we can think of homotopy as maps of type $X \times I \to Y$, and then use the exact same formulas obtained in Proposition 2.1.4 taking care to check that they actually define pointed homotopies.

We can then form the quotient set

$$[X, Y]_* := \frac{\mathsf{Top}_*(X, Y)}{\simeq_*}$$

whose elements are called **pointed homotopy classes of maps**. If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, then its equivalence class in $[X, Y]_*$ will be denoted by $[f]_*$.

The next step is to analyze how pointed homotopies interact with compositions of pointed maps.

4.2.1 Proposition. Let (W, w_0), (X, x_0), (Y, Y_0) and (Z, z_0) be pointed spaces, and consider pointed maps $\alpha : (W, w_0) \to (X, x_0)$, f, $g : (X, x_0) \to (Y, y_0)$ and $\beta : (Y, y_0) \to (Z, z_0)$ as shown in the diagram below.

$$W \xrightarrow{\alpha} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\beta} Z$$

If $f \simeq_* g$, then $f \circ \alpha \simeq_* g \circ \alpha$ and $\beta \circ f \simeq_* \beta \circ g$.

Sketch of proof. We give a sketch of the proof to illustrate how the diagrammatic formulations of homotopy allow us to essentially reuse the proofs of the unpointed case.

Let $D : (X, x_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$ be the pointed family of paths associated to the pointed homotopy $f \simeq_* g$. The composite pointed map

$$D \circ \alpha : (W, w_0) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$$

satisfies $\mathrm{ev}_{0,Y} \circ D \circ \alpha = f \circ \alpha$ and $\mathrm{ev}_{1,Y} \circ D \circ \alpha = g \circ \alpha$, therefore it induces a pointed homotopy $f \circ \alpha \simeq_* g \circ \alpha$.

Now for the other homotopy, we regard the homotopy $f \simeq_* g$ as being given by a pointed map $K : (X \rtimes I, *) \to (Y, y_0)$ satisfying the equalities $K \circ j_{X,0} = f$ and $K \circ j_{X,1} = g$. It is then a matter of simple computation to show that the composite pointed map

$$\beta \circ K : (X \rtimes I, *) \to (Z, z_0)$$

satisfies the equalities $\beta \circ K \circ j_{X,0} = \beta \circ f$ and $\beta \circ K \circ j_{X,1} = \beta \circ g$, inducing thus a pointed homotopy $\beta \circ f \simeq_* \beta \circ g$. ■

As in the unpointed case, this result implies that pointed homotopies are preserved by compositions.

4.2.2 Corollary. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces, and consider two pairs of pointed maps f_1, $g_1 : (X, x_0) \to (Y, y_0)$ and f_2, $g_2 : (Y, y_0) \to (Z, z_0)$ as shown in the diagram below.

$$X \underset{g_1}{\overset{f_1}{\rightrightarrows}} Y \underset{g_2}{\overset{f_2}{\rightrightarrows}} Z$$

If $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then $f_2 \circ f_1 \simeq_* g_2 \circ g_1$.

Using this result we can define a composition of pointed homotopy classes. Given pointed spaces (X, x_0), (Y, y_0) and (Z, z_0), we define a composition function

$$\circ : [Y, Z]_* \times [X, Y]_* \to [X, Z]_*$$

by the formula

$$[g]_* \circ [f]_* := [g \circ f]_*$$

for any pointed homotopy classes $[f]_* \in [X, Y]_*$ and $[g]_* \in [Y, Z]_*$. This function is well-defined precisely due to the compatibility of pointed homotopies with the usual composition of pointed maps.

This composition operation is associative by virtue of the associativity of the usual composition of pointed maps. Moreover, for any pointed space $(X, x_0)_*$, the pointed homotopy class $[\mathrm{id}_X]_*$ defines a unit for the composition operation of $[X, X]_*$. We can then consider the category whose objects are pointed spaces, and whose morphisms are pointed homotopy classes with composition described as above. The category defined like this is called the **pointed homotopy category** and is denoted as HoTop_*.

Like in the unpointed case, we can specialize categorical notions to the category HoTop_*. For example, a pointed map $f : (X, x_0) \to (Y, y_0)$ is said to be a **pointed homotopy equivalence** if its pointed homotopy class $[f]_*$ defines an isomorphism in HoTop_*, that is, if there exists a pointed map $g : (Y, y_0) \to (X, x_0)$ such that $g \circ f \simeq_* \mathrm{id}_X$ and $f \circ g \simeq_* \mathrm{id}_Y$. If this is the case, we also say that (X, x_0) and (Y, y_0) are of the same **pointed homotopy type**.

We now study some categorical properties of HoTop_*. We are especially interested in how some constructions made in Top_* behave when interpreted in HoTop_*.

First, recall that, given two pointed spaces (X, x_0) and (Y, y_0), the usual cartesian product $X \times Y$ can be regarded as a pointed space by choosing (x_0, y_0) as basepoint. This is in fact a very nice choice, because the canonical projection $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ become pointed maps, and, given another pointed space (W, w_0) together with pointed maps $f : (W, w_0) \to (X, x_0)$ and $g : (W, w_0) \to (Y, y_0)$, the usual induced map $(f, g) : W \to X \times Y$ also becomes a pointed map. In summary, this choice of basepoint on $X \times Y$ ensures that the triple $((X \times Y, (x_0, y_0)), \pi_1, \pi_2)$ defines a product of (X, x_0) and (Y, y_0) in the category Top_*.

We saw in Proposition 2.2.7 that the formation of the induced map into a product is compatible with homotopies, that is, two pairs of homotopic maps give rise to two homotopic maps into the product. The next result shows that this is also true for pointed homotopies.

4.2.3 Proposition. Given pointed spaces (X, x_0), (Y_1, y_1) and (Y_2, y_2), and given pointed maps f_1, $g_1 : (X, x_0) \to (Y_1, y_1)$ and f_2, $g_2 : (X, x_0) \to (Y_2, y_2)$, if $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then the induced pointed maps (f_1, f_2), $(g_1, g_2) : (X, x_0) \to (Y_1 \times Y_2, (y_1, y_2))$ are pointed homotopic.

Proof. The proof is analogous to that of Proposition 2.2.7. Suppose that the pointed homotopies $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$ are given by pointed maps $H_1 : (X \rtimes I, *) \to (Y_1, y_1)$ and $H_2 : (X \rtimes I, *) \to (Y_2, y_2)$ respectively.

The two homotopies together define a pointed map

$$(H_1, H_2) : (X \rtimes I, *) \to (Y_1 \times Y_2, (y_2, y_2)).$$

If π'_1 and π'_2 are the canonical projections out of $Y_1 \times Y_2$, then computations analogous to those of Proposition 2.2.7 show that

$$\pi'_1 \circ (H_1, H_2) \circ j_{X,0} = f_1 \quad \text{and} \pi_2 \circ (H_1, H_2) \circ j_{X,0} = f_2,$$

therefore the universal property of the product implies the equality $(H_1, H_2) \circ j_{X,0} = (f_1, f_2)$. A similar reasoning shows that we also have $(H_1, H_2) \circ j_{X,1} = (g_1, g_2)$, thus (H_1, H_2) defines a pointed homotopy from (f_1, f_2) to (g_1, g_2). ■

From this we obtain the pointed analogue of Corollary 2.2.8.

4.2.4 Corollary. The product functor $\times : \mathsf{Top}_* \times \mathsf{Top}_* \to \mathsf{Top}_*$ respects pointed homotopies. More precisely, for any pointed spaces (X_1, x_1), (X_2, x_2), (Y_1, y_1) and (Y_2, y_2), and for any pointed maps f_1, $g_1 : (X_1, x_1) \to (Y_1, y_1)$ and f_2, $g_2 : (X_2, x_2) \to (Y_1, y_2)$, if $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then we also have pointed homotopy $f_1 \times f_2 \simeq_* g_1 \times g_2$.

Sketch of proof. The proof is the same as that of Corollary 2.2.8. If π_1 and π_2 are the canonical projections out of $X_1 \times X_2$, then the products $f_1 \times f_2$ and $g_1 \times g_2$ are the induced maps

$$f_1 \times f_2 = (f_1 \circ \pi_1, f_2 \circ \pi_2) \quad \text{and} \quad g_1 \times g_2 = (g_1 \circ \pi_1, g_2 \circ \pi_2)$$

into the product $Y_1 \times Y_2$. We just need to use the compatibility of composition with pointed homotopies and the result of Proposition 4.2.3. ■

We saw in Proposition 2.2.9 and Corollary 2.2.10 that the coproduct in Top_*, that is, the disjoint union, also interacts nicely with homotopy. These results also have analogues in the pointed case, but we must remember that the coproduct in Top_* is given by the *wedge sum* of Definition 3.1.14.

4.2.5 Proposition. Let (X_1, x_1), (X_2, x_2) and (Y, y_0) be pointed space, and consider pointed maps f_1, $g_1 : (X_1, x_1) \to (Y, y_0)$ and f_2, $g_2 : (X_2, x_2) \to (Y, y_0)$. If $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then the induced pointed maps $\langle f_1, f_2 \rangle$, $\langle g_1, g_2 \rangle : (X_1 \vee X_2, *) \to (Y, y_0)$ are pointed homotopic.

Proof. The proof is analogous to that of Proposition 2.2.9. Suppose that the pointed homotopies $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$ are given by pointed families of paths $D_1 : (X_1, x_1) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$ and $D_2 : (X_2, x_2) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0})$ respectively.

These two maps together induce a pointed map

$$\langle D_1, D_2\rangle : (X_1 \vee X_2, *) \to (\mathrm{Map}(I, Y), \mathrm{ct}_{I,y_0}),$$

since the wedge sum is a coproduct in Top_*. If j_1 and j_2 are the canonical injections into $X_1 \vee X_2$, then following the computations of Proposition 2.2.9 we see that

$$\mathrm{ev}_{0,Y} \circ \langle D_1, D_2\rangle \circ j_1 = f_1 \quad \text{and} \quad \mathrm{ev}_{0,Y} \circ \langle D_1, D_2\rangle \circ j_2 = f_2,$$

therefore the universal property of the coproduct implies the equality $\mathrm{ev}_{0,Y} \circ \langle D_1, D_2\rangle = \langle f_1, f_2\rangle$. Similarly, we also have the equality $\mathrm{ev}_{1,Y} \circ \langle D_1, D_2\rangle = \langle g_1, g_2\rangle$, thus the map $\langle D_1, D_2\rangle$ induces a pointed homotopy $\langle f_1, f_2\rangle \simeq_* \langle g_1, g_2\rangle$. ■

We then obtain the analogue of Corollary 2.2.10 for the wedge sum of maps.

4.2.6 Corollary. The wedge sum functor $\vee : \mathsf{Top}_* \times \mathsf{Top}_* \to \mathsf{Top}_*$ respects pointed homotopies. More precisely, given pointed spaces (X_1, x_1), (X_2, x_2), (Y_1, y_1) and (Y_2, y_2), and given pointed maps f_1, $g_1 : (X_1, x_1) \to (Y_1, y_1)$ and f_2, $g_2 : (X_2, x_2) \to (Y_2, y_2)$, if $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then we also have a pointed homotopy $f_1 \vee f_2 \simeq_* g_1 \vee g_2$.

Proof. Let j_1' and j_2' denote the canonical injections into $Y_1 \vee Y_2$. The wedge sums $f_1 \vee f_2$ and $g_1 \vee g_2$ are then the induced maps

$$f_1 \vee f_2 = \langle j_1' \circ f_1, j_2' \circ f_2\rangle \quad \text{and} \quad g_1 \vee g_2 = \langle j_1' \circ g_1, j_2' \circ g_2\rangle,$$

therefore the result follows by combining the compatibility of composition with pointed maps and the result of Proposition 4.2.5. ■

These results on the compatibility of products and wedge sums with homotopies can be reinterpreted as saying that these functors descend to functors defined on the homotopy category.

4.2.7 Corollary. Let (X, x_0) and (Y, y_0) be pointed spaces.

1. The triple $((X \times Y, (x_0, y_0)), [\pi_1]_*, [\pi_2]_*)$ defines a product for (X, x_0) and (Y, y_0) in HoTop_*, where π_1 and π_2 are the canonical projections out of $X \times Y$.
2. The triple $((X \vee Y, *), [j_1]_*, [j_2]_*)$ defines a coproduct for (X, x_0) and (Y, y_0) in HoTop_*, where j_1 and j_2 are the canonical injections into $X \vee Y$.

Proof. 1. Given another pointed space (W, w_0) equipped with two pointed homotopy classes $[f]_* : (W, w_0) \to (X, x_0)$ and $[g]_* : (W, w_0) \to (Y, y_0)$, we need to show that these two morphisms can be factored through the product $X \times Y$, that is, we need to show that there exists a unique pointed homotopy class $[h]_* : (W, w_0) \to (X \times Y, (x_0, y_0))$

satisfying the conditions $[\pi_1]_* \circ [h]_* = [f]_*$ and $[\pi_2]_* \circ [h]_* = [g]_*$. Thinking in terms of the representing pointed maps instead of the pointed homotopy classes, we need to show that there exists a pointed map $h : (W, w_0) \to (X \times Y, (x_0, y_0))$ satisfying the homotopical conditions $\pi_1 \circ h \simeq_* f$ and $\pi_2 \circ h \simeq_* g$, and moreover, we also need to show that any other pointed map satisfying these two conditions is in fact pointed homotopic to h.

We claim that $h \coloneqq (f, g)$ is the desired map. Since the equalities $\pi_1 \circ (f, g) = f$ and $\pi_2 \circ (f, g) = g$ hold strictly, they also hold up to pointed homotopy, so (f, g) satisfies the two required homotopical conditions. Now, if $h' : (W, w_0) \to (X \times Y, (x_0, y_0))$ also satisfies $\pi_1 \circ h' \simeq_* f$ and $\pi_2 \circ h' \simeq_* g$, then by Proposition 4.2.3 we know that the induced map $(\pi_1 \circ h', \pi_2 \circ h')$ is pointed homotopic to (f, g), but $(\pi_1 \circ h', \pi_2 \circ h')$ is precisely the map h'.

2. The proof is similar to that of the previous item. Working directly with pointed maps, we need to show that, given a pointed space (Z, z_0) and pointed maps $f : (X, x_0) \to (Z, z_0)$ and $g : (Y, y_0) \to (Z, z_0)$, there exists a pointed map $h : (X \vee Y, *) \to (Z, z_0)$ such that $h \circ j_1 \simeq_* f$, $h \circ j_2 \simeq_* g$, and any other pointed map satisfying these two conditions must be pointed homotopic to h.

We claim that $h \coloneqq \langle f, g \rangle : (X \vee Y, *) \to (Z, z_0)$ is the desired map. It satisfies the equations $\langle f, g \rangle \circ j_1 = f$ and $\langle f, g \rangle \circ j_2 = g$, thus it satisfies the two required homotopical conditions. If $h' : (X \vee Y, *) \to (Z, z_0)$ also satisfies $h' \circ j_1 \simeq_* f$ and $h' \circ j_2 \simeq_* g$, then according to Proposition 4.2.5 the induced map $\langle h' \circ j_1, h' \circ j_2 \rangle$ is pointed homotopic to $\langle f, g \rangle$, but $\langle h' \circ j_1, h' \circ j_2 \rangle$ is equal to the map h' itself. ■

In the case of pointed spaces, there is yet another construction whose interaction with pointed homotopies is worth studying: the smash product. Since the smash product of two maps is obtained by factoring a certain map through a quotient map, it is useful to have a slightly more general result about factoring two homotopic maps through a quotient.

4.2.8 Proposition. Let X and Y be spaces, and consider subspaces $A \subseteq X$ and $B \subseteq Y$. Denote by $q_A : X \to X/A$ and $q_B : Y \to Y/B$ the quotient maps obtained by collapsing the subspaces to a single point. Given two maps $f, g : X \to Y$ such that $f(A) \subseteq B$ and $g(A) \subseteq B$, let $\overline{f}, \overline{g} : (X/A, *_{X/A}) \to (Y/B, *_{Y/B})$ be the pointed maps obtained by factoring $q_B \circ f$ and $q_B \circ g$ through q_A as shown in the diagrams below.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q_A"'] & Y \arrow[d, "q_B"] \\ X/A \arrow[r, "\overline{f}"'] & Y/B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q_A"'] & Y \arrow[d, "q_B"] \\ X/A \arrow[r, "\overline{g}"'] & Y/B \end{tikzcd}$$

Suppose there is a homotopy $H : f \Rightarrow g$ with the added property that, for any

$x \in A$ and $t \in I$, the relation $H(x,t) \in B$ holds. Then the two induced maps $\overline{f}$ and $\overline{g}$ are pointed homotopic.

Proof. By definition of the quotient space X/A, the diagram below is a pushout square in Top,

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\mathrm{ct}_{A,\mathrm{pt}}"'] & X \arrow[d, "q_A"] \\ \{\mathrm{pt}\} \arrow[r, "\mathrm{ct}_{\{\mathrm{pt}\},*_{X/A}}"'] & X/A \end{tikzcd}$$

with the first horizontal map being the inclusion map. Since the functor $- \times I$ is a left adjoint, because I is locally compact Hausdorff, the diagram below is also a pushout square.

$$\begin{tikzcd} A \times I \arrow[r, tail] \arrow[d, "\mathrm{ct}_{A,\mathrm{pt}} \times \mathrm{id}_I"'] & X \times I \arrow[d, "q_A \times \mathrm{id}_I"] \\ \{\mathrm{pt}\} \times I \arrow[r, "\mathrm{ct}_{\{\mathrm{pt}\},*_{X/A}} \times \mathrm{id}_I"'] & X/A \times I \end{tikzcd}$$

If we take into account that there is a homeomorphism $\{\mathrm{pt}\} \times I$, then we can modify the diagram above to obtain the pushout square below.

$$\begin{tikzcd} A \times I \arrow[r, tail] \arrow[d, "\pi_2"'] & X \times I \arrow[d, "q_A \times \mathrm{id}_I"] \\ I \arrow[r, "(\mathrm{ct}_{I,*_{X/A}}, \mathrm{id}_I)"'] & X/A \times I \end{tikzcd}$$

The goal of this setup is to give us a universal property that we can use to map out of the cylinder $X/A \times I$. Consider the maps $q_B \circ H : X \times I \to Y/B$ and $\mathrm{ct}_{I,*_{Y/B}} : I \to Y/B$. For any $(a,t) \in A \times I$, we have

$$q_B(H(a,t)) = *_{Y/B} = \mathrm{ct}_{I,*_{Y/B}}(\pi_2(a,t)),$$

since H maps $A \times I \to B$, and q_B maps B to $\{*_{Y/B}\}$. It follows from the universal property of the pushout that there is an induced map $\overline{H} : X/A \times I \to Y/B$ making the diagram below commute.

$$\begin{tikzcd} A \times I \arrow[r, tail] \arrow[d, "\pi_2"'] & X \times I \arrow[d, "q_A \times \mathrm{id}_I"] \arrow[ddr, bend left, "q_B \circ H"] & \\ I \arrow[r, "(\mathrm{ct}_{I,*_{X/A}}, \mathrm{id}_I)"'] \arrow[drr, bend right, "\mathrm{ct}_{I,*_{Y/B}}"'] & X/A \times I \arrow[dr, dashed, "\overline{H}"] & \\ & & Y/B \end{tikzcd}$$

We claim that $\overline{H}$ is the desired pointed homotopy. First notice that, for any $t \in I$, by a simple computation we have the equality

$$i_{X/A,t} \circ q_A = (q_A \times \mathrm{id}_I) \circ i_{X,t}.$$

Using this we obtain the following chain of equalities:

$$\begin{aligned}\overline{H} \circ i_{X/A,0} \circ q_A &= \overline{H} \circ (q_A \times \mathrm{id}_I) \circ i_{X,0} \\ &= q_B \circ H \circ i_{X,0} \\ &= q_B \circ f \\ &= \overline{f} \circ q_A,\end{aligned}$$

and by cancelling the projection q_A we deduce that

$$\overline{H} \circ i_{X/A,0} = \overline{f}.$$

By an analogous reasoning we can also show the equality

$$\overline{H} \circ i_{X/A,1} = \overline{g},$$

therefore H defines at least a homotopy $\overline{f} \simeq \overline{g}$. This homotopy is in fact pointed because, for any $t \in I$, by the commutativity of the diagram above we see that

$$\begin{aligned}\overline{H}(*_{X/A}, t) &= (\overline{H} \circ (\mathrm{ct}_{I,*_{X/A}}, \mathrm{id}_I))(t) \\ &= \mathrm{ct}_{I,*_{Y/B}}(t) \\ &= *_{Y/B}.\end{aligned}$$

■

4.2.9 Remark. In Proposition 4.2.8 we considered pairs (X, A), and maps $f : (X, A) \to (Y, B)$ satisfying the condition $f(A) \subseteq B$, which are called *maps of pairs*. There is a category Top_2 whose objects are pairs, and whose morphisms are maps of pairs. Every pair (X, A) gives rise to the pointed quotient $(X/A, *_{X/A})$, and if $f : (X, A) \to (Y, B)$ is a map of pairs, we have the induced pointed map $f : (X/A, *_{X/A}) \to (Y/B, *_{Y/B})$, and this gives us a functor $Q : \mathsf{Top}_2 \to \mathsf{Top}_*$.

The category Top_2 has its own notion of homotopy: two maps of pairs $f, g : (X, A) \to (Y, B)$ are said to be homotopic if there is an ordinary homotopy $H : X \times I \to Y$ from f to g satisfying the additional condition $H(A \times I) \subseteq B$. This notion of homotopy satisfies properties analogous to those satisfied by ordinary or pointed homotopies, and we can then define the homotopy category of pairs HoTop_2.

Using this language of pairs, Proposition 4.2.8 can be rephrased as saying that the functor Q turns homotopies of pairs into pointed homotopies, and therefore it induces a quotient functor $\mathsf{Ho}(Q) : \mathsf{HoTop}_2 \to \mathsf{HoTop}_*$.

4.2.10 Corollary. Let (X_1, x_1), (X_2, y_2), (Y_1, y_1) and (Y_2, y_2) be pointed spaces, and consider pointed maps $f_1, g_1 : (X_1, x_1) \to (Y_1, y_1)$ and $f_2, g_2 : (X_2, x_2) \to (Y_2, y_2)$. If $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$, then we also have a pointed homotopy $f_1 \wedge f_2 \simeq_* g_1 \wedge g_2$.

Proof. Following the terminology of Proposition 4.2.8, the product maps $f_1 \times f_2$ and $g_1 \times g_2$ both map the subspace $X_1 \times \{x_2\} \cup \{x_1\} \times X_2$ to the subspace $Y_1 \times \{y_2\} \cup \{y_1\} \times Y_2$. If $q : X_1 \times X_2 \to X_1 \wedge X_2$ and $q' : Y_1 \times Y_2 \to Y_1 \wedge Y_2$ denote the canonical projections, then the smash products $f_1 \wedge f_2$ and $g_1 \wedge g_2$ are defined by factoring the composites $q' \circ (f_1 \times f_2)$ and $q' \circ (g_1 \times g_2)$, respectively, through q.

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "f_1 \times f_2"] \arrow[d, "q"'] & Y_1 \times Y_2 \arrow[d, "q'"] \\ X_1 \wedge X_2 \arrow[r, dashed, "f_1 \wedge f_2"'] & Y_1 \wedge Y_2 \end{tikzcd} \qquad \begin{tikzcd} X_1 \times X_2 \arrow[r, "g_1 \times g_2"] \arrow[d, "q"'] & Y_1 \times Y_2 \arrow[d, "q'"] \\ X_1 \wedge X_2 \arrow[r, dashed, "g_1 \wedge g_2"'] & Y_1 \wedge Y_2 \end{tikzcd}$$

We already know from Proposition 4.2.3 that the hypothesis $f_1 \simeq_* g_1$ and $f_2 \simeq_* g_2$ imply that $f_1 \times f_2 \simeq_* g_1 \times h_2$. More precisely, if $H_1 : f_1 \Rightarrow_* g_1$ and $H_2 : f_2 \Rightarrow_* g_2$ are pointed homotopies, then $H : (X_1 \times X_2) \times I \to Y_1 \times Y_2$ given by the formula

$$H((x, x'), t) \coloneqq (H_1(x, t), H_2(x', t))$$

defines a pointed homotopy from $f_1 \times f_2$ to $g_1 \times g_2$. Notice that, for any $t \in I$, if $x \in X_1$, then

$$H((x, x_2), t) = (H_1(x, t), H_2(x_2, t)) = (H_1(x, t), y_2) \in Y_1 \times \{y_2\},$$

while if $x' \in X_2$, then

$$H((x_1, x'), t) = (H_1(x_1, t), H_2(x', t)) = (y_1, H_2(x', t)) \in \{y_1\} \times Y_2.$$

This show that, at every instant $t \in I$, the homotopy H maps the subspace $X_1 \times \{x_2\} \cup \{x_1\} \times X_2$ into the subspace $Y_1 \times \{y_2\} \cup \{y_1\} \times Y_2$, therefore Proposition 4.2.8 implies the existence of a pointed homotopy $f_1 \wedge f_2 \simeq_* f_2 \wedge g_2$. ∎

We end this section by studying the compatibility with pointed homotopies of two other closely related functors: the pushforward and pullback functors. These results also hold in the unpointed case, but we decided to only prove them in the pointed cases since this is the only context in which we will need them.

4.2.11 Proposition. Consider pointed spaces (X, x_0), (Y, y_0) and (Z, z_0).

1. If the pointed maps α, $\beta : (X, x_0) \to (Y, y_0)$ are pointed homotopic, and Y is locally compact Hausdorff, then the pullback maps

   $$\mathrm{Map}_*(\alpha, Z),\ \mathrm{Map}_*(\beta, Z) : (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0}) \to (\mathrm{Map}_*(X, Z), \mathrm{ct}_{X,z_0})$$

   are pointed homotopic.

2. If the pointed maps α, $\beta : (Y, y_0) \to (Z, z_0)$ are pointed homotopic, and X is locally compact Hausdorff, then the pushforward maps

$$\mathrm{Map}_*(X, \alpha),\ \mathrm{Map}_*(X, \beta) : (\mathrm{Map}_*(X, Y), \mathrm{ct}_{X,y_0}) \to (\mathrm{Map}_*(X, Z), \mathrm{ct}_{X,z_0})$$

are pointed homotopic.

Proof. 1. Let $H : X \times I \to Y$ be a *pointed* homotopy from α to β. Consider the map $\Phi : (\mathrm{Map}_*(Y, Z) \times I) \times X \to Z$ defined by the formula

$$\Phi((f, t), x) \coloneqq f(H(x, t)).$$

Notice that part of the definition of Φ involves the evaluation map $\mathrm{Map}_*(Y, Z) \times Y \to Z$, so we need the local compactness and Hausdorff properties on Y to ensure its continuity. Direct computations show that Φ satisfies the following properties:

(i) $\Phi((f, 0), x) = [\mathrm{Map}_*(\alpha, Z)(f)](x)$ for every $f \in \mathrm{Map}_*(Y, Z)$ and every $x \in X$;

(ii) $\Phi((f, 1), x) = [\mathrm{Map}_*(\beta, Z)(f)](x)$ for every $f \in \mathrm{Map}_*(Y, Z)$ and every $x \in X$;

(iii) $\Phi((f, t), x_0) = z_0$ for every $f \in \mathrm{Map}_*(Y, Z)$ and every $t \in I$;

(iv) $\Phi((\mathrm{ct}_{Y,z_0}, t), x) = \mathrm{ct}_{X,z_0}(x)$ for every $t \in X$ and every $x \in X$.

We then let

$$\overleftarrow{H} \coloneqq \lambda\Phi : \mathrm{Map}_*(Y, Z) \times I \to \mathrm{Map}_*(X, Z)$$

be the usual exponential adjoint of Φ. Conditions (iii) above guarantees that $\overleftarrow{H}$ really takes values in $\mathrm{Map}_*(X, Z)$ and not just in $\mathrm{Map}(X, Z)$. We then notice that conditions (i) and (ii) above imply that $\overleftarrow{H}$ defines a homotopy from $\mathrm{Map}_*(\alpha, Z)$ and $\mathrm{Map}_*(\beta, Z)$, while condition (iv) implies that this homotopy is in fact pointed.

2. The proof is similar to that of the previous item. Let $H : Y \times I \to Z$ be a *pointed* homotopy from α to β. Consider the map

$$\Psi : (\mathrm{Map}_*(X, Y) \times I) \times X \to Z$$

defined by the formula

$$\Psi((f, t), x) \coloneqq H(f(x), t)$$

for every $(f, t) \in \mathrm{Map}_*(X, Y) \times I$ and every $x \in X$. Notice that, since part of the definition of Ψ involves evaluating the map $f \in \mathrm{Map}_*(X, Y)$ on the point $x \in X$, we need the local compactness and Hausdorff conditions on X to ensure its continuity. Direct computations show that Ψ satisfies the following properties:

(i) $\Psi((f,0),x) = [\mathrm{Map}_*(X,\alpha)(f)](x)$ for every $f \in \mathrm{Map}_*(X,Y)$ and every $x \in X$;

(ii) $\Psi((f,1),x) = [\mathrm{Map}_*(X,\beta)(f)](x)$ for every $f \in \mathrm{Map}_*(X,Y)$ and every $x \in X$;

(iii) $\Psi((f,t),x_0) = z_0$ for every $f \in \mathrm{Map}_*(X,Y)$ and every $t \in I$;

(iv) $\Psi((\mathrm{ct}_{X,y_0},t),x) = \mathrm{ct}_{X,z_0}(x)$ for every $t \in I$ and every $x \in X$.

We then let

$$\overrightarrow{H} \coloneqq \lambda\Psi : \mathrm{Map}_*(X,Y) \times I \to \mathrm{Map}_*(X,Z)$$

be the usual exponential adjunct, which in this case really takes values in $\mathrm{Map}_*(X,Z)$ by virtue of property (iii) above. We then notice that properties (i) and (ii) imply that $\overrightarrow{H}$ defines a homotopy from $\mathrm{Map}_*(X,\alpha)$ to $\mathrm{Map}_*(X,\beta)$, while property (iv) implies that this homotopy is in fact pointed. ■

4.3 Pointed contractions and pointed null homotopies

This last section is an adaptation of Section 2.3 to the context of pointed spaces. We define the notions of pointed contractibility and pointed null homotopy, and we then show that these two concepts are related to one another by proving that a pointed map is pointed null homotopic if and only if it can be extended to a pointed contractible space. This proof requires adapting the construction of the cone over a space to the category of pointed spaces. After proving the mentioned equivalence, we apply it to the study of extending pointed maps from the n-sphere to the $(n+1)$-disk.

A pointed space (X,x_0) is said to be **pointed contractible** if its identity map id_X is pointed homotopic to the constant map ct_{X,x_0} from X to its basepoint, and a particular choice of pointed homotopy $H : \mathrm{id}_X \Rightarrow_* \mathrm{ct}_{X,x_0}$ is then called a **pointed contraction**. At first, this might seem equivalent to contractibility, because if X is contractible, then for *any* point $x \in X$ we can find a homotopy $\mathrm{id}_X \simeq \mathrm{ct}_{X,x}$, so in particular we can find a homotopy $\mathrm{id}_X \simeq \mathrm{ct}_{X,x_0}$. Notice, however, that there is no guarantee that this homotopy is pointed, and this is the crucial difference. While a contractible space can be shrunk to any particular basepoint, in a pointed contractible space the chosen basepoint cannot move during the shrinking process. Similar to the unpointed case, the notion of pointed contractibility can also be stated in terms of pointed homotopy types: a pointed space (X,x_0) is pointed contractible if and only if it is pointed homotopy equivalent to the singleton space $(\{\mathrm{pt}\},\mathrm{pt})$.

The notion of null homotopic map also has a straightforward generalization to the pointed case. A pointed map $f : (X,x_0) \to (Y,y_0)$ is **pointed null homotopic** if it is pointed homotopic to the constant map ct_{X,y_0}. In this case, a choice of pointed homotopy $H : f \Rightarrow_* \mathrm{ct}_{X,y_0}$ is called a **pointed null homotopy** for f.

In the unpointed case, we saw that a map $f : X \to Y$ was null homotopic if and only it could be factored through a contractible space, namely the cone CX over X. We would like to obtain an analogous result relating pointed null homotopic maps and pointed contractible spaces, and in order to do this we need to adapt the construction of the cone to the pointed category.

Throughout the rest of the chapter, we will denote by I_1 the pointed space $(I, 1)$.

4.3.1 Definition. Given a pointed space (X, x_0), the pointed space defined as the smash product

$$CX := X \wedge I_1$$

is called the **reduced cone over** X.

Explicitly, the reduced cone CX is the quotient space

$$CX = \frac{X \times I}{X \times \{1\} \cup \{x_0\} \times I}.$$

The construction of the reduced cone over X can be seen as a two-step process: first we construct the usual (unreduced) cone over X by collapsing the subset $X \times \{1\} \subseteq X \times I$, and then, if we identify X with the basis of the cone, we further collapse the line segment joining x_0 to the vertex of the cone. See Figure 3 for a visualization of this construction when $X = S^1$.

(a) Cylinder over S^1 (b) Unreduced cone over S^1 (c) Reduced cone over S^1

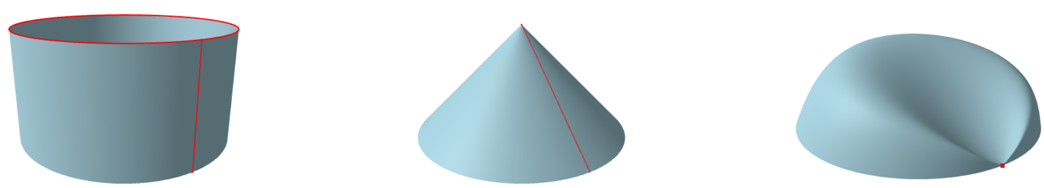

Figure 3 – Construction of the reduced cone over the circle.

Composing the usual inclusion $i_{X,0} : X \to X \times I$ with the canonical projection $p : X \times I \to CX$ gives us a map

$$i_{CX} := p \circ i_{X,0} : X \to CX,$$

and since the image of $i_{X,0}$ is contained in the subspace which is collapsed to the basepoint of $*$ of CX by p, i_{CX} actually defines a pointed map $(X, x_0) \to (CX, *)$.

We want to show that, just like the usual cone allows us to relate null homotopic maps to contractible spaces, the reduced cone allows us to relate pointed null homotopic maps to pointed contractible spaces. In the unpointed case, in order to establish this connection we first had to prove the contractibility of the cone itself in Proposition 2.3.7, so we first adapt this result to the pointed case.

4.3.2 Proposition. The reduced cone over any pointed space (X, x_0) is pointed contractible.

Proof. We need to show the existence of a pointed homotopy $\mathrm{id}_{CX} \simeq_* \mathrm{ct}_{CX,*}$. The trick to make this easier is to notice that both of these maps can be obtained by factoring through the quotient $p : X \times I_1 \to CX$. More precisely, id_{CX} and $\mathrm{ct}_{CX,*_{CX}}$ can be obtained by factoring the composites $p \circ \mathrm{id}_{X\times I}$ and $p \circ (\mathrm{id}_X \times \mathrm{ct}_{I,1})$, respectively, through the quotient map p, so that we have the two commutative squares below.

$$\begin{tikzcd} X \times I_1 \arrow[r, "\mathrm{id}_{X\times I_1}"] \arrow[d, "p"'] & X \times I_1 \arrow[d, "p"] \\ CX \arrow[r, "\mathrm{id}_{CX}"'] & CX \end{tikzcd} \qquad \begin{tikzcd} X \times I_1 \arrow[r, "\mathrm{id}_X \times \mathrm{ct}_{I,1}"] \arrow[d, "p"'] & X \times I_1 \arrow[d, "p"] \\ CX \arrow[r, "\mathrm{ct}_{CX,*}"'] & CX \end{tikzcd}$$

We can then try to use Proposition 4.2.8 to obtain the desired pointed homotopy.

Consider the map $H : (X \times I_1) \times I \to X \times I_1$ defined by the formula

$$H((x, s), t) := (x, (1 - t)s + t)$$

for every $(x, s) \in X \times I_1$ and every $t \in I$. Straightforward computations show that H satisfies the following properties:

1. $H((x, s), 0) = \mathrm{id}_{X\times I}(x, s)$ for every $(x, s) \in X \times I_1$;
2. $H((x, s), 1) = (\mathrm{id}_X \times \mathrm{ct}_{I,1})(x, s)$ for every $(x, s) \in X \times I_1$,

which means that H defines a homotopy $\mathrm{id}_{X\times I} \simeq \mathrm{id}_X \times \mathrm{ct}_{I,1}$. Moreover, for any $t \in I$, if $x \in X$, then

$$H((x, 1), t) = (x, 1) \in X \times \{1\} \cup \{x_0\} \times I,$$

and if $s \in I$, then

$$H((x_0, s), t) = (x_0, (1 - t)s + t) \in X \times \{1\} \cup \{x_0\} \times I.$$

This means that not only does H define a homotopy from $\mathrm{id}_{X\times I_1}$ to $\mathrm{id}_X \times \mathrm{ct}_{I,1}$, but also that, for every "instant" $t \in I$, this homotopy maps the subspace $X \times \{1\} \cup \{x_0\} \times I$ to itself. It then follows from Proposition 4.2.8 that there is an induced pointed homotopy $\mathrm{id}_{CX} \simeq_* \mathrm{ct}_{CX,*}$. ■

Now we are able to adapt Theorem 2.3.8 to the context of pointed spaces.

4.3.3 Theorem. A pointed map $f : (X, x_0) \to (Y, y_0)$ is pointed null homotopic if and only there exists a pointed map $F : (CX, *) \to (Y, y_0)$ satisfying the equation $F \circ i_{CX} = f$.

$$\begin{tikzcd} CX \arrow[r, dashed, "F"] & Y \\ X \arrow[u, "i_{CX}"] \arrow[ur, "f"'] & \end{tikzcd}$$

In other words, f is pointed null homotopic if and only if it can be extended to the reduced cone.

Proof. Suppose first that the extension $F : (CX, *) \to (Y, y_0)$ exists. Using that CX is contractible, and that pointed homotopies are compatible with composition we see that

$$f = F \circ i_{CX} = F \circ \mathrm{id}_{CX} \circ i_{CX} \simeq_* F \circ \mathrm{ct}_{CX,*} \circ i_{CX} = \mathrm{ct}_{X,y_0},$$

which means that f is pointed null homotopic.

Conversely, suppose f is pointed null homotopic, and let $H : f \Rightarrow_* \mathrm{ct}_{X,y_0}$ be a pointed null homotopy regarded as a usual homotopy $H : X \times I \to Y$ mapping the whole line segment $\{x_0\} \times I$ to y_0. This homotopy then maps the subspace $X \times \{1\} \cup \{x_0\} \times I$ to the basepoint y_0 of Y, therefore it can be factored through p to define a pointed map $F : (CX, *) \to (Y, y_0)$.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "p"'] & Y \\ CX \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

This map F is the extension we are looking for since

$$F \circ i_{CX} = F \circ p \circ i_{X,0} = H \circ i_{X,0} = f. \qquad \blacksquare$$

Like in the unpointed case, the reduced cone CS^n over the n-sphere can be compared with the $(n+1)$-disk. In order to do this, we introduce an auxiliary map from the cylinder to the disk which “interacts” nicely with the basepoint of D^{n+1}.

4.3.4 Lemma. For every $n \geq 0$, the map $\pi : S^n \times I \to D^{n+1}$ given by the formula

$$\pi(x, t) \coloneqq (1 - t) \cdot x + t \cdot *_{S^n},$$

where $*_{S^n} = (1, 0, \ldots, 0)$ is the basepoint of the n-sphere, is a quotient map.

Proof. The map π can be interpreted geometrically as follows: for each $x \in S^n$, if we let t vary in I, then π transforms the segment from $(x, 0)$ to $(x, 1)$ on the surface of the cylinder into the segment from x to $*_{S^n}$ on the disk like shown in Figure 4.

We want to show that π is a surjective map. Notice first that the basepoint $*_{S^n}$ surely belongs to the image of π because $*_{S^n} = \pi(x, 1)$ for any $x \in S^n$. Now, if $p \in D^{n+1}$

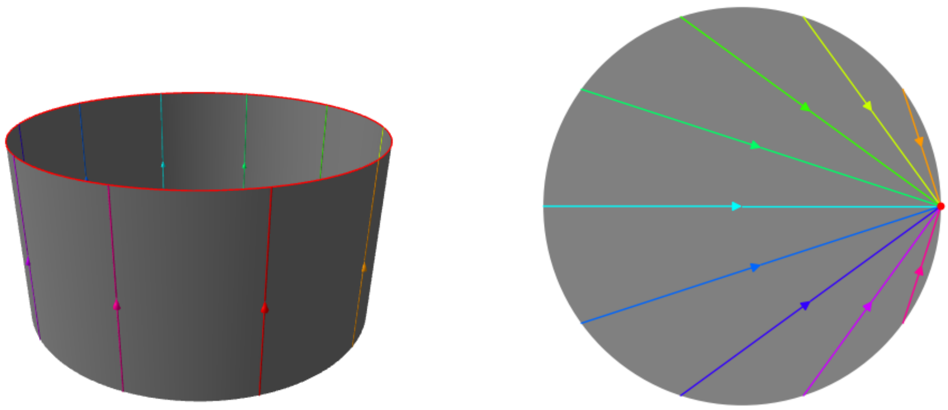

Figure 4 – Each colored line segment on the cylinder is mapped to the segment of the same color on the disk.

is different from $*_{S^n}$, consider the ray emanating from $*_{S^n}$ and passing through p, which is given by the curve $\gamma : [0, +\infty) \to \mathbb{R}^{n+1}$ defined as

$$\gamma(t) := (1 - t) \cdot *_{S^n} + t \cdot p.$$

Using this curve define $d : [0, +\infty) \to [0, +\infty)$ as

$$d(t) := \|\gamma(t)\|^2 = \langle \gamma(t), \gamma(t) \rangle,$$

which can be written explicitly as

$$d(t) = \|p - *_{S^n}\|^2 t^2 + 2(\langle p, *_{S^n} \rangle - 1)t + 1.$$

At $t = 1$ we have

$$d(t) = \|\gamma(1)\|^2 = \|p\|^2 \leq 1$$

because $\|p\| \leq 1$. Moreover, since d is a quadratic polynomial with positive leading coefficient, $\lim_{t \to +\infty} d(t) = +\infty$, and so by the Intermediate Value Theorem we can find a number $\widehat{t} > 0$ such that $d(\widehat{t}) = 1$. This means that the point $x_p := \gamma(\widehat{t})$ belongs to the sphere S^n.

We claim that the number $\widehat{t}$ so defined satisfies the inequality $\widehat{t} \geq 1$. In order to show this, we consider the number $t_{\min}$ of $[0, +\infty)$ which minimizes $d(t)$. This point is given explicitly by the formula

$$t_{\min} = \frac{1 - \langle p, *_{S^n} \rangle}{\|p - *_{S^n}\|^2}.$$

We claim that the inequality $t_{\min} < \widehat{t}$ holds. One way to see this is to note that, since the graph of $d(t)$ is symmetric with respect to the vertical line $t = t_{\min}$, and $d(0) = d(T) = 1$, we actually have an equality

$$t_{\min} = \frac{T}{2},$$

and this implies the desired inequality.

We will use this to prove that the inequality $\widehat{t} \geq 1$ holds. We split the proof into two cases.

1. If $t_{\min} \geq 1$, then by transitivity we immediately conclude that $\widehat{t} \geq 1$ also holds.

2. If $t_{\min} < 1$, and we also suppose that $\widehat{t} < 1$, then $\widehat{t}$ must lie on the open interval $(t_{\min}, 1)$, but since d is *strictly* increasing on this interval, we would have $d(\widehat{t}) < d(1)$, which is another contradiction since $d(\widehat{t}) = 1$ and $d(1) = \|p\|^2 \leq 1$.

Having obtained $\widehat{t}$ as above, we define

$$t_p \coloneqq 1 - \frac{1}{\widehat{t}},$$

and notice that $t_p \in I$ due to the fact that $\widehat{t} \geq 1$. These choices of $x_p \in S^n$ and $t_p \in I$ satisfy

$$\begin{aligned}
\pi(x_p, t_p) &= (1 - t_p) \cdot x_p + t_p \cdot *_{S^n} \\
&= \frac{1}{\widehat{t}} \cdot x_p + \left(1 - \frac{1}{\widehat{t}}\right) \cdot *_{S^n} \\
&= \frac{1}{\widehat{t}} \cdot ((1 - \widehat{t}) \cdot *_{S^n} + \widehat{t} \cdot p) + \left(1 - \frac{1}{\widehat{t}}\right) \cdot *_{S^n} \\
&= \frac{1}{\widehat{t}} \cdot *_{S^n} - *_{S^n} + p + *_{S^n} - \frac{1}{\widehat{t}} \cdot *_{S^n} \\
&= p;
\end{aligned}$$

showing at last that p belongs to the image of π.

So far we have shown that π is a surjective map, but since $S^n \times I$ is a compact space, and D^{n+1} is a Hausdorff space, π is also a closed map. The result then follows from the fact that every surjective and closed map is a quotient map. ■

Another way to visualize the map π defined above is to study its restriction to the "horizontal slices" $S^n \times \{t\}$ that make up the cylinder $S^n \times I$. Given any $x \in S^n$, we have

$$\begin{aligned}
\|\pi(x, t) - t \cdot *_{S^n}\| &= \|(1 - t) \cdot x + t \cdot *_{S^n} - t \cdot *_{S^n}\| \\
&= \|(1 - t) \cdot x\| \\
&= |1 - t| \|x\| \\
&= 1 - t.
\end{aligned}$$

This means that the sphere $S^n \times \{t\}$ is mapped to a sphere of radius $1 - t$ centered at the point $t \cdot *_{S^n}$ like shown in Figure 5

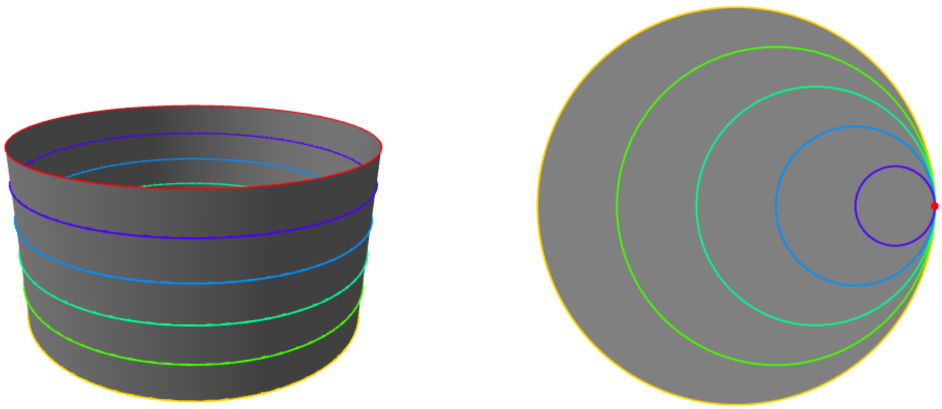

Figure 5 – Each colored circle on the cylinder is mapped to the circle of the same color on the disk.

4.3.5 Proposition. For every $n \geq 0$, there is a pointed homeomorphism

$$(CS^n, *) \cong (D^{n+1}, *_{S^n}).$$

Proof. Let $p : S^n \times I_1 \to CS^n = S^n \wedge I_1$ be the canonical projection, and consider the map $\pi : S^n \times I \to D^{n+1}$ of Lemma 4.3.4. A direct calculation shows that

$$\pi(S^n \times \{1\} \cup \{*^n_S\} \times I) \subseteq \{*_{S^n}\},$$

so by factoring π through p we obtain a pointed map $\overline{\pi} : (CS^n, *) \to (D^{n+1}, *_{S^n})$.

$$\begin{tikzcd}
& S^n \times I \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\
CS^n \arrow[rr, dashed, "\overline{\pi}"'] & & D^{n+1}
\end{tikzcd}$$

Given a point $x \in D^{n+1}$, if $x \neq *_{S^n}$, then the fiber $\pi^{-1}(x)$ consists of a single point, and if $x = *_{S^n}$, then $\pi^{-1}(*_{S^n}) = S^n \times \{1\} \cup \{*_{S^n}\} \times I$. It follows that p is constant on the fibers of π, therefore it can be factored through this quotient to define a map $\overline{p} : D^{n+1} \to CS^n$.

$$\begin{tikzcd}
& S^n \times I \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\
CS^n & & D^{n+1} \arrow[ll, dashed, "\overline{p}"]
\end{tikzcd}$$

Notice that $\overline{p}(*_{S^n}) = \overline{p}(\pi(*_{S^n}, 1)) = p(*_{S^n}, 1) = *$, thus $\overline{p}$ actually defines a pointed map

$$\overline{p} : (D^{n+1}, *_{S^n}) \to (CS^n, *).$$

We now show that $\overline{\pi}$ and $\overline{p}$ are inverse maps. On the one hand,

$$\overline{p} \circ \overline{\pi} \circ p = \overline{p} \circ \pi = p,$$

and by cancelling p we obtain $\overline{p} \circ \overline{\pi} = \mathrm{id}_{CS^n}$; while on the other

$$\overline{\pi} \circ \overline{p} \circ \pi = \overline{\pi} \circ p = \pi,$$

and then by cancelling π we obtain the other equality $\overline{\pi} \circ \overline{p} = \mathrm{id}_{D^{n+1}}$. ■

This allows us to obtain a pointed version of Corollary 2.3.11.

4.3.6 Corollary. For every $n \geq 0$, a pointed map $f : (S^n, *_{S^n}) \to (X, x_0)$ is pointed null homotopic if and only if it can be extended to the disk, that is, if and only if there exists a map $F : (D^{n+1}, *_{S^n}) \to (X, x_0)$ satisfying $F|_{S^n} = f$.

Proof. If a map F like above exists, then the fact that $(D^{n+1}, *_{S^n})$ is contractible together with the relation $F|_{S^n} = f$ imply that f is pointed null homotopic.

Conversely, if f is pointed null homotopic, then by Theorem 4.3.3 there exists a pointed map $G : (CS^n, *) \to (X, x_0)$ such that $G \circ i_{CX} = f$. Let $\overline{p} : (D^{n+1}, *_{S^n}) \to (CS^n, *_{CS^n})$ be the pointed homeomorphism of Proposition 4.3.5, and define

$$F := G \circ \overline{p} : (D^{n+1}, *_{S^n}) \to (X, x_0).$$

As we remarked above, the inclusion $i : S^n \hookrightarrow D^{n+1}$ coincides with the composition $\pi \circ i_{S^n,0}$ of the inclusion $i_{S^n,0} : S^n \to S^n \times I$ with the quotient map $\pi : S^n \times I \to D^{n+1}$. Using this we see that

$$\begin{aligned} F|_{S^n} &= F \circ i \\ &= G \circ \overline{p} \circ \pi \circ i_{S^n,0} \\ &= G \circ p \circ i_{S^n,0} \\ &= G \circ i_{CX} \\ &= f, \end{aligned}$$

therefore F is an extension of f. ■

CHAPTER

5

HOMOTOPY GROUPS

In this chapter we finally introduce the homotopy groups of a pointed space. The approach is somewhat categorical, with homotopy groups being obtained via cogroup objects in the pointed homotopy category HoTop_*. These cogroup objects are called H-cogroups, and the first section is devoted to the study of their properties, and also the construction of important families of examples. The second section then makes use of the categorical machinery of cogroup objects to define the homotopy groups of a pointed space, and also deduce some of its basic properties like functoriality with respect to pointed maps.

The third section dualizes the concepts of the first one by introducing H-groups. The main goal of this section is to construct a family of examples of H-groups. In the fourth section, the concepts of H-cogroup and H-group come together and allows us to deduce some interesting basic properties of the higher homotopy groups. The Eckmann-Hilton Duality plays a particularly important role here. Lastly, the fifth section analyzes the dependence of the homotopy groups on the chosen basepoints by introducing the transport maps along paths. We also define the concept of n-simple space which will be crucial in our study of Obstruction Theory.

5.1 H-cogroups

In this first section we introduce the notion of H-cogroups. We first study their general properties, and then later we give concrete example which will be useful later for defining the homotopy groups and for proving some of their basic properties.

We begin by introducing H-cogroups.

5.1.1 Definition. An **H-cogroup** consists of a pointed space (X, x_0) together with the following data:

- A pointed map $\mu : (X, x_0) \to (X \vee X, *_{X\vee X})$ called the **H-comultiplication**;
- A pointed map $\nu : (X, x_0) \to (X, x_0)$ called the **H-co-inversion**.

These maps are required to satisfy the following "commutativity up to homotopy" conditions:

1. $\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0}\rangle \circ \mu \simeq_* \mathrm{id}_X \simeq \langle \mathrm{ct}_{X,x_0}, \mathrm{id}_X\rangle \circ \mu$;
2. $\langle \mathrm{id}_X, \nu\rangle \circ \mu \simeq_* \mathrm{ct}_{X,x_0} \simeq_* \langle \nu, \mathrm{id}_X\rangle \circ \mu$;
3. $(\mathrm{id}_X \vee \mu) \circ \mu \simeq_* A \circ (\mu \vee \mathrm{id}_X) \circ \mu$, where α denotes the associator homeomorphism
$$A : (X \vee X) \vee X \xrightarrow{\cong} X \vee (X \vee X)$$

The next theorem gives us a whole family of H-cogroups. Its proof is rather long, but we give plenty of details. It is essentially a more refined version of the proof that the concatenation product on loops satisfies the group axioms.

5.1.2 Theorem. If (X, x_0) is any pointed space, then its reduced suspension $(\Sigma X, *_{\Sigma X})$ admits an H-cogroup structure.

Proof. Throughout the proof, we use the notation

$$j_1,\ j_2 : \Sigma X \to \Sigma X \vee \Sigma X$$

to denote the canonical injections.

Let $d : X \times I \to \Sigma X \vee \Sigma X$ be defined as

$$d(x, s) := \begin{cases} j_1([x, 2s]), & \text{if } 0 \le t \le \frac{1}{2} \\ j_2([x, 2s-1]), & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

where $[x, t]$ denotes the image of $(x, s) \in X \times I$ under the canonical projection $q : X \times I \to \Sigma X$. This is well-defined because setting $s = \frac{1}{2}$ in the first line of the definition give us $j_1([x, 1]) = j_1(*_{\Sigma X})$ while the second line gives us $j_2([x, 0]) = j_2(*_{\Sigma X})$, and these two define the same point of $\Sigma X \vee \Sigma X$.

Given any $x \in X$, for $s = 0$ we have

$$d(x, 0) = j_1([x, 2 \cdot 0]) = j_1([x, 0]) = j_1(*_{\Sigma X}) = *_{\Sigma X\vee\Sigma X},$$

while for $s = 1$ we have

$$d(x, 1) = j_2([x, 2 \cdot 1 - 1]) = j_2([x, 1]) = j_2(*_{\Sigma X}) = *_{\Sigma X\vee\Sigma X}.$$

Moreover, for any $s \in I$, if $s \leq \frac{1}{2}$ we have

$$d(x_0, s) = j_1([x_0, 2s]) = j_1(*_{\Sigma X}) = *_{\Sigma X \vee \Sigma X},$$

while if $s \geq \frac{1}{2}$ we have

$$d(x_0, s) = j_2([x_0, 2s-1]) = j_2(*_{\Sigma X}) = *_{\Sigma X \vee \Sigma X}.$$

This means that d satisfies the relation

$$d(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I) \subseteq \{*_{\Sigma X \vee \Sigma X}\},$$

therefore we can factor it through the canonical projection $q : X \times I \to \Sigma X$ to obtain the pointed map

$$\mu : (\Sigma X, *_{\Sigma X}) \to (\Sigma X \vee \Sigma X, *_{\Sigma X \vee \Sigma X}).$$

Explicitly, μ is described by the formula

$$\mu([x, s]) = \begin{cases} j_1([x, 2s]), & \text{if } 0 \leq s \leq \frac{1}{2} \\ j_2([x, 2s-1]), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Let us begin by proving that the basepoint $*_{\Sigma X}$ acts as a counit up to homotopy for μ, which means that we must construct pointed homotopies

$$\langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle \circ \mu \simeq_* \mathrm{id}_{\Sigma X} \simeq_* \langle \mathrm{ct}_{\Sigma X, *_{\Sigma X}}, \mathrm{id}_{\Sigma X} \rangle \circ \mu.$$

Using the definition of μ, as well as the relation between the induced map $\langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle$ and the canonical injections j_1 and j_2, we obtain the following formula:

$$(\langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle \circ \mu)([x, s]) = \begin{cases} [x, 2s], & \text{if } 0 \leq s \leq \frac{1}{2} \\ *_{\Sigma X}, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.1}$$

The important thing to notice in this equation is that, for a fixed $x \in X$, when we vary $s \in I$, the only thing varying in the image of $[x, s]$ is the "second component": in the first line it goes from 0 to 1, and then in the second line it remains constant at 1, since $*_{\Sigma X} = [x, 1]$. More precisely, if we define a map $\alpha : I \to I$ by the formula

$$\alpha(s) := \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

then it fits into the commutative diagram below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times \alpha"] \arrow[d, "q"'] & X \times I \arrow[d, "q"] \\ \Sigma X \arrow[r, "\langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle \circ \mu"'] & \Sigma X \end{tikzcd}$$

It seems reasonable to expect that the homotopical properties of α affect the homotopical properties of the bottom map, and the next lemma makes this relation precise.

5.1.3 Lemma (Reparameterization I)**.** Consider a pointed space (X, x_0) and a pointed map $f : (\Sigma X, *_{\Sigma X}) \to (\Sigma X, *_{\Sigma X})$. Suppose there exists a map $\alpha : I \to I$ that fits into a commutative diagram like the one below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times \alpha"] \arrow[d, "q"'] & X \times I \arrow[d, "q"] \\ \Sigma X \arrow[r, "f"'] & \Sigma X. \end{tikzcd}$$

1. If $\alpha(0) = 0$ and $\alpha(1) = 1$, then $f \simeq_* \mathrm{id}_{\Sigma X}$.

2. If $\alpha(0) = \alpha(1) = 0$, or $\alpha(0) = \alpha(1) = 1$, then $f \simeq_* \mathrm{ct}_{\Sigma X, *_{\Sigma X}}$.

Proof of the Lemma. 1. The conditions on α ensure that the product $\mathrm{id}_X \times \alpha$ can be seen as a map of pairs

$$\mathrm{id}_X \times \alpha : (X \times I, A) \to (X \times I, A),$$

where, in order to simplify the notation, we have denoted by A the subspace

$$X \times \{0, 1\} \cup \{x_0\} \times I \subseteq X \times I$$

which gets collapsed to a point in the construction of the reduced suspension. Following the notation of Proposition 4.2.8, the commutativity hypothesis implies that f is the map $\overline{\mathrm{id}_X \times \alpha}$ obtained by factoring the composition $q \circ (\mathrm{id}_X \times \alpha)$ through the quotient q.

Consider the map $H : (X \times I) \times I \to X \times I$ defined by the formula

$$H((x, s), t) := (x, (1 - t)\alpha(s) + ts)$$

for all $((x, s), t) \in (X \times I) \times I$. By direct computation we see that the map H satisfies the following properties:

1. $H((x, s), 0) = (x, \alpha(s)) = (\mathrm{id}_X \times \alpha)(x, s)$ for all $(x, s) \in X \times I$;

2. $H((x, s), 1) = (x, s) = \mathrm{id}_{X \times I}(x, s)$ for all $(x, s) \in X \times I$;

3. $H((x, 0), t) = (x, 0) \in A$ for all $x \in X$ and all $t \in I$;

4. $H((x, 1), t) = (x, 1) \in A$ for all $x \in X$ and all $t \in I$;

5. $H((x_0, s), t) = (x_0, (1 - t)\alpha(s) + ts) \in A$ for all s, $t \in I$.

The first of the two properties above say that H defines a homotopy $\mathrm{id}_X \times \alpha \simeq \mathrm{id}_{X \times I}$, while the last three properties say that this homotopy is such that, for any fixed $t \in I$, the intermediary map $X \times I \to X \times I$ defined by the homotopy maps the subspace A to itself. Proposition 4.2.8 then implies that passing to the quotient gives us a pointed homotopy $\overline{\mathrm{id}_X \times \alpha} \simeq_* \overline{\mathrm{id}_{X\times I}}$, but we have already remarked that $\overline{\mathrm{id}_X \times \alpha} = f$, and it is clear that $\overline{\mathrm{id}_{X\times I}} = \mathrm{id}_{\Sigma X}$, so the result follows.

2. The proof of this item is very similar to that of the previous one. We now consider the map $H : (X \times I) \times I \to X \times I$ defined as

$$H((x, s), t) := (x, (1 - s)\alpha(s) + tp),$$

where $p := \alpha(0) = \alpha(1)$ is equal to either 0 or 1 according to the hypothesis in the statement of the result. By direct computations we can show that H defines a homotopy $\mathrm{id}_X \times \alpha \simeq \mathrm{id}_X \times \mathrm{ct}_{I,p}$, and moreover, for any fixed $t \in I$, the intermediary map $X \times I \to X \times I$ induced by this homotopy maps the subspace A to itself. Using Proposition 4.2.8 again gives us a pointed homotopy $\overline{\mathrm{id}_X \times \alpha} = f \simeq_* \overline{\mathrm{id}_X \times \mathrm{ct}_{I,p}}$, but since $p \in \{0, 1\}$, the induced map $\overline{\mathrm{id}_X \times \mathrm{ct}_{I,p}}$ is equal to the constant map $\mathrm{ct}_{\Sigma X, *_{\Sigma X}}$, so the result follows. $\square$

With the use of this Lemma, the rest of the proof is straightforward. As we had already remarked, from equation (5.1) we see that the map $\alpha : I \to I$ defined as

$$\alpha(s) := \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

satisfies $q \circ (\mathrm{id}_x \times \alpha) = \langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle \circ \mu \circ q$, and since $\alpha(0) = 0$ and $\alpha(1) = 1$, it follows from Lemma 5.1.3 that

$$\langle \mathrm{id}_{\Sigma X}, \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \rangle \circ \mu \simeq_* \mathrm{id}_{\Sigma X}.$$

The other composition is proved similarly. We have the formula

$$(\langle \mathrm{ct}_{\Sigma X, *_{\Sigma X}}, \mathrm{id}_{\Sigma X} \rangle \circ \mu)([x, s]) = \begin{cases} *_{\Sigma X}, & \text{if } 0 \leq s \leq \frac{1}{2} \\ [x, 2t - 1], & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases} \tag{5.2}$$

so the map $\alpha : I \to I$ defined as

$$\alpha(s) := \begin{cases} 0, & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2s - 1, & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{5.3}$$

is such that

$$q \circ (\mathrm{id}_X \times \alpha) = \langle \mathrm{ct}_{\Sigma X, *_{\Sigma X}}, \mathrm{id}_{\Sigma X} \rangle \circ \mu \circ q;$$

and Lemma 5.1.3 then implies

$$\langle \mathrm{ct}_{\Sigma X, *_{\Sigma X}}, \mathrm{id}_{\Sigma X} \rangle \circ \mu \simeq_* \mathrm{id}_{\Sigma X}.$$

Now we have to prove the coassociativity up to homotopy of μ, thus we need to construct a pointed homotopy

$$(\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu \simeq_* A \circ (\mu \vee \mathrm{id}_{\Sigma X}) \circ \mu,$$

where $A : (\Sigma X \vee \Sigma X) \vee \Sigma X \to \Sigma X \vee (\Sigma X \vee \Sigma X)$ is the associator homeomorphism defined as in Remark A.1.10. Following the notation of said Remark, if we recall that $\mathrm{id}_{\Sigma X} \vee \mu$ by definition satisfies the equalities

$$(\mathrm{id}_{\Sigma X} \vee \mu) \circ j_1 = J_1' \quad \text{and} \quad (\mathrm{id}_{\Sigma X} \vee \mu) \circ j_2 = J_2' \circ \mu,$$

we obtain the formula

$$((\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu)([x,s]) = \begin{cases} J_1'([x,2s]), & \text{if } 0 \leq s \leq \frac{1}{2} \\ J_2'(j_1([x,4s-2])), & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\ J_2'(j_2([x,4s-3])), & \text{if } \frac{3}{4} \leq s \leq 1. \end{cases} \tag{5.4}$$

Now, for the other composition, the wedge sum $\mu \vee \mathrm{id}_{\Sigma X} : \Sigma X \vee \Sigma X \to (\Sigma X \vee \Sigma X) \vee \Sigma X$ satisfies by definition the equalities

$$(\mu \vee \mathrm{id}_{\Sigma X}) \circ j_1 = J_1 \circ \mu \quad \text{and} \quad (\mu \vee \mathrm{id}_{\Sigma X}) \circ j_2 = J_2.$$

Using these we obtain the equality

$$((\mu \vee \mathrm{id}_{\Sigma X}) \circ \mu)([x,s]) = \begin{cases} J_1(j_1([x,4s])), & \text{if } 0 \leq s \leq \frac{1}{4} \\ J_1(j_2([x,4s-1])), & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ J_2([x,2t-1]), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

We recall also that, according to Remark A.1.10, the associator A by definition satisfies the conditions

$$A \circ J_1 = \langle J_1', J_2' \circ j_1 \rangle \quad \text{and} \quad A \circ J_2 = J_2' \circ j_2,$$

from which we deduce the formula

$$(A \circ (\mu \vee \mathrm{id}_{\Sigma X}) \circ \mu)([x,s]) = \begin{cases} J_1'([x,4s]), & \text{if } 0 \leq s \leq \frac{1}{4} \\ J_2'(j_1([x,4s-1])), & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ J_2'(j_2([x,2s-1])), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.5}$$

We want to use Lemma 5.1.3 to compared (5.4) and (5.5). Define $\alpha : I \to I$ by the formula

$$\alpha(s) := \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{4} \\ s + \frac{1}{4}, & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \frac{1}{2}s + \frac{1}{2}, & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

and consider the product map $\mathrm{id}_X \times \alpha : X \times I \to X \times I$. A direct verification shows that the composite

$$q \circ (\mathrm{id}_X \times \alpha) : X \times I \to \Sigma X$$

is such that

$$(q \circ (\mathrm{id}_X \times \alpha))(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I) \subseteq \{*_{\Sigma X}\},$$

therefore it can be factored through the quotient to give a pointed map

$$\beta : (\Sigma X, *_{\Sigma X}) \to (\Sigma X, *_{\Sigma X}).$$

The map α is hand-crafted to ensure that the induced map β satisfies the equation

$$(\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu \circ \beta = A \circ (\mu \vee \mathrm{id}_{\Sigma X}) \circ \mu.$$

Since $\alpha(0) = 0$ and $\alpha(1) = 1$, Lemma 5.1.3 implies $\beta \simeq_* \mathrm{id}_{\Sigma X}$, therefore

$$\begin{aligned} A \circ (\mu \vee \mathrm{id}_{\Sigma X}) \circ \mu &= (\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu \circ \beta \\ &\simeq_* (\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu \circ \mathrm{id}_{\Sigma X} \\ &= (\mathrm{id}_{\Sigma X} \vee \mu) \circ \mu. \end{aligned}$$

We still have to define an H-co-inversion map. Let $r : I \to I$ be defined as $r(s) \coloneqq 1 - s$ for every $sinI$, and consider the product $\mathrm{id}_X \times r : X \times I \to X \times I$. The composition $q \circ (\mathrm{id}_X \times r) : X \times I \to \Sigma X$ satisfies

$$(q \circ (\mathrm{id}_X \times r))(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I) \subseteq \{*_{\Sigma X}\},$$

thus it induces a pointed map

$$\nu : (\Sigma X, *_{\Sigma X}) \to (\Sigma X, *_{\Sigma X})$$

as shown in the diagram below.

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{id}_X \times r"] \arrow[d, "q"'] & X \times I \arrow[d, "q"] \\ \Sigma X \arrow[r, dashed, "\nu"'] & \Sigma X \end{tikzcd}$$

The only thing left is showing that ν really behaves as an H-co-inversion, that is, we need to construct pointed homotopies

$$\langle \nu, \mathrm{id}_{\Sigma X} \rangle \circ \mu \simeq_* \mathrm{ct}_{\Sigma X, *_{\Sigma X}} \simeq_* \langle \mathrm{id}_{\Sigma X}, \nu \rangle.$$

On the one hand we have the formula

$$(\langle \nu, \mathrm{id}_{\Sigma X} \rangle \circ \mu)([x, s]) = \begin{cases} [x, 1 - 2s], & \text{if } 0 \leq s \leq \frac{1}{2} \\ [x, 2s - 1], & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.6}$$

The map $\alpha : I \to I$ given by

$$\alpha(s) := \begin{cases} 1-2s, & \text{if } 0 \le s \le \frac{1}{2} \\ 2s-1, & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

is such that

$$q \circ (\mathrm{id}_X \times \alpha) = \langle \nu, \mathrm{id}_{\Sigma X} \rangle \circ \mu \circ q,$$

so Lemma 5.1.3 implies the desired homotopy

$$\langle \nu, \mathrm{id}_{\Sigma X} \rangle \circ \mu \simeq_* \mathrm{ct}_{\Sigma X, *_{\Sigma X}}.$$

On the other hand, we also have the formula

$$(\langle \mathrm{id}_{\Sigma X}, \nu \rangle \circ \mu)([x,s]) = \begin{cases} [x, 2s], & \text{if } 0 \le s \le \frac{1}{2} \\ [x, 2-2s], & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{5.7}$$

In this case, we consider the map $\alpha : I \to I$ defined as

$$\alpha(s) := \begin{cases} 2s, & \text{if } 0 \le s \le \frac{1}{2} \\ 2-2s, & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

which satisfies the equation

$$q \circ (\mathrm{id}_X \times \alpha) = \langle \mathrm{id}_{\Sigma X}, \nu \rangle \circ \mu \circ q,$$

and use Lemma 5.1.3 one last time to conclude that

$$\langle \mathrm{id}_{\Sigma X}, \nu \rangle \circ \mu \simeq_* \mathrm{ct}_{\Sigma X, *_{\Sigma X}}. \qquad \blacksquare$$

Now that we have a "recipe" for obtaining H-cogroups, we can apply it to some particular examples. The simplest pointed space there is the singleton space $(\{\mathrm{pt}\}, \mathrm{pt})$. The previous theorem implies the reduced suspension $(\Sigma\{\mathrm{pt}\}, *_{\Sigma\{\mathrm{pt}\}})$ admits an H-cogroup structure, but this structure is in fact rather trivial. Indeed, since there is a pointed homeomorphism

$$\Sigma\{\mathrm{pt}\} \cong \{\mathrm{pt}\},$$

the H-comultiplication $\mu_{\Sigma\{\mathrm{pt}\}}$ of $\Sigma\{\mathrm{pt}\}$ corresponds to an H-comultiplication

$$\mu_{\{\mathrm{pt}\}} : \{\mathrm{pt}\} \to \{\mathrm{pt}\} \vee \{\mathrm{pt}\},$$

and if we take into account the existence of a pointed homeomorphism

$$\{\mathrm{pt}\} \vee \{\mathrm{pt}\} \cong \{\mathrm{pt}\},$$

then this H-comultiplication corresponds simply to the identity map of $\{\mathrm{pt}\}$. Similarly, the H-co-inversion $\nu_{\Sigma\{\mathrm{pt}\}} : \Sigma\{\mathrm{pt}\} \to \Sigma\{\mathrm{pt}\}$ corresponds to a map $\{\mathrm{pt}\} \to \{\mathrm{pt}\}$ which is necessarily the identity map of the singleton space.

The next simplest pointed space is the 0-dimensional sphere $S^0 = \{-1, 1\}$. In this case, we already have a non-trivial and extremely important example of H-cogroup structure on the suspension. First, we make precise the way in which we can "transport" H-cogroup structures along pointed homeomorphisms. In fact, since an H-cogroup structure is something homotopical in nature, this transport procedure can actually be done along the weaker notion of a pointed homotopy equivalence.

5.1.4 Proposition (Transport of H-cogroup structure)**.** Let $((X, x_0), \mu, \nu)$ be an H-cogroup. If (Y, y_0) is another pointed space, and $h : (Y, y_0) \to (X, x_0)$ is a pointed homotopy equivalence, then (Y, y_0) admits a structure of H-cogroup such that h becomes an H-cogroup morphism.

Proof. Let $h^{-1} : (X, x_0) \to (Y, y_0)$ be any homotopy inverse to h. Define a map $\mu_Y : Y \to Y \vee Y$ by the diagram below.

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[rrr, dashed, bend right, "\mu_Y"'] & X \arrow[r, "\mu"] & X \vee X \arrow[r, "h^{-1} \vee h^{-1}"] & Y \vee Y \end{tikzcd}$$

We must check that y_0 is a counit up to homotopy for μ_Y, which means that we must construct homotopies

$$\langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ \mu_Y \simeq_* \mathrm{id}_Y \simeq_* \langle \mathrm{ct}_{Y,y_0}, \mathrm{id}_Y \rangle \circ \mu_Y .$$

The idea is to somehow relate the maps above with the corresponding ones for X, and then use the homotopical properties of μ.

The way to relate the structure of X and Y is to write the "obvious" diagrams and check that they commute (sometimes only up to homotopy). In this first part, we check the commutativity in details, but for the later parts, in order to avoid excessive repetition, we just give an outline of the proof.

We claim that the diagram below commutes up to homotopy.

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\mu_Y"'] & X \arrow[d, "\mu"] \\ Y \vee Y \arrow[r, "h \vee h"] \arrow[d, "\langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle"'] & X \vee X \arrow[d, "\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

For the top square, unpacking the definition of μ_Y we get

$$(h \vee h) \circ \mu_Y = (h \vee h) \circ (h^{-1} \vee h^{-1}) \circ \mu \circ h,$$

and if we use the functoriality of the wedge sum, as well as the fact that it preserves pointed homotopies, we obtain

$$\begin{aligned}(h \vee h) \circ (h^{-1} \vee h^{-1}) \circ \mu \circ h &= ((h \circ h^{-1}) \vee (h \circ h^{-1})) \circ \mu \circ h \\ &\simeq_* (\mathrm{id}_X \vee \mathrm{id}_X) \circ \mu \circ h \\ &= \mathrm{id}_{X \vee X} \circ \mu \circ h \\ &= \mu \circ h;\end{aligned}$$

showing the commutativity up to homotopy of the first square.

The bottom square actually commutes strictly, not only up to homotopy. Indeed, if $j_1^X,\ j_2^X : X \to X \vee X$ and $j_1^Y, j_2^Y : Y \to Y \vee Y$ are the canonical injections, on the one hand we have

$$\begin{aligned}\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ (h \vee h) \circ j_1^Y &= \langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle j_1^X \circ h \\ &= \mathrm{id}_X \circ h \\ &= h,\end{aligned}$$

while on the other we have

$$h \circ \langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ j_1^Y = h \circ \mathrm{id}_Y = h$$

showing that there is an equality

$$\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ (h \vee h) \circ j_1^Y = h \circ \langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ j_1^Y.$$

An analogous argument shows that there is also an equality

$$\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ (h \vee h) \circ j_2^Y = h \circ \langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ j_2^Y.$$

Combining these two with the universal property of the coproduct then gives us the desired equality

$$\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ (h \vee h) = h \circ \langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle.$$

Now that we know that the mentioned diagram commutes up to homotopy, we can also state the homotopy

$$h \circ (\langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ \mu_Y) \simeq_* (\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ \mu) \circ h,$$

but since x_0 is a counit up to homotopy for μ, we also have the homotopy

$$\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ \mu \simeq_* \mathrm{id}_X,$$

and substituting this into the previous one gives us

$$h \circ (\langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ \mu_Y) \simeq_* \mathrm{id}_X \circ h = h.$$

Finally, composing both sides of this equality with h^{-1} gives us the desired pointed homotopy

$$\langle \mathrm{id}_Y, \mathrm{ct}_{Y,y_0} \rangle \circ \mu_Y \simeq_* \mathrm{id}_Y.$$

The proof that there exists a pointed homotopy

$$\langle \mathrm{ct}_{Y,y_0}, \mathrm{id}_Y \rangle \circ \mu_Y \simeq_* \mathrm{id}_Y$$

is completely analogous. We just write the corresponding diagram relating $\langle \mathrm{ct}_{Y,y_0}, \mathrm{id}_Y \rangle \circ \mu_Y$ with $\langle \mathrm{ct}_{X,x_0}, \mathrm{id}_X \rangle \circ \mu$, which commutes up to homotopy by an analogous argument, and then use the same reasoning as above to deduce the required pointed homotopy.

Now we prove that μ_Y is also coassociative up to homotopy. We denote the associators of X and Y by A_X and A_Y respectively. We must show that there exists a pointed homotopy

$$A_Y \circ (\mu_Y \vee \mathrm{id}_Y) \circ \mu_Y \simeq_* (\mathrm{id}_Y \vee \mu_Y) \circ \mu_Y.$$

The trick again is to use the "obvious" homotopy commutative diagram to relate both sides of the expression above in terms of the corresponding expressions for X.

First we rewrite the composition on the left-hand side using the homotopy commutative diagram below.

$$\begin{tikzcd}
Y \arrow[r, "h"] \arrow[d, "\mu_Y"'] & X \arrow[d, "\mu"] \\
Y \vee Y \arrow[r, "h \vee h"] \arrow[d, "\mu_Y \vee \mathrm{id}_Y"'] & X \vee X \arrow[d, "\mu \vee \mathrm{id}_X"] \\
(Y \vee Y) \vee Y \arrow[r, "(h \vee h) \vee h"] \arrow[d, "A_Y"'] & (X \vee X) \vee X \arrow[d, "A_X"] \\
Y \vee (Y \vee Y) \arrow[r, "h \vee (h \vee h)"'] & X \vee (X \vee X)
\end{tikzcd}$$

The top and middle rectangles commute up to homotopy due to the definition of μ_Y, the functoriality of the wedge and its compatibility with pointed homotopies. The bottom rectangle actually commutes strictly (not just up to homotopy) as a consequence of the naturality of the associator homeomorphism applied to the pointed map $h : (Y, y_0) \to (X, x_0)$.

This commutativity up to homotopy allows us to write

$$(h \vee (h \vee h)) \circ (A_Y \circ (\mu_Y \vee \mathrm{id}_Y) \circ \mu_Y) \simeq_* (A_X \circ (\mu \vee \mathrm{id}_X) \circ \mu) \circ h,$$

but if we recall that we have the pointed homotopy

$$A_X \circ (\mu \vee \mathrm{id}_X) \circ \mu \simeq_* (\mathrm{id}_X \vee \mu) \circ \mu,$$

then the previous expression can be rewritten as

$$(h \vee (h \vee h)) \circ (A_Y \circ (\mu_Y \vee \mathrm{id}_Y) \circ \mu_Y) \simeq_* ((\mathrm{id}_X \vee \mu) \circ \mu) \circ h. \tag{5.8}$$

Now, one would hope that the right-hand side of (5.8) can be rewritten in terms of the composition $(\mathrm{id}_Y \vee \mu_Y) \circ \mu_Y$. Indeed, we have the homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\mu_Y"'] & X \arrow[d, "\mu"] \\ Y \vee Y \arrow[r, "h\vee h"] \arrow[d, "\mathrm{id}_Y \vee \mu_Y"'] & X \vee X \arrow[d, "\mathrm{id}_X \vee \mu"] \\ Y \vee (Y \vee Y) \arrow[r, "h\vee(h\vee h)"'] & X \vee (X \vee X), \end{tikzcd}$$

which allows us to write

$$(h \vee (h \vee h)) \circ ((\mathrm{id}_Y \vee \mu_Y) \circ \mu_Y) \simeq_* ((\mathrm{id}_X \vee \mu) \circ \mu) \circ h. \tag{5.9}$$

Comparing equations (5.8) and (5.9) give us

$$(h \vee (h \vee h)) \circ (A_Y \circ (\mu_Y \vee \mathrm{id}_Y) \circ \mu_Y) \simeq_* (h \vee (h \vee h)) \circ ((\mathrm{id}_Y \vee \mu_Y) \circ \mu_Y),$$

and if we then compose both sides above with $h^{-1} \vee (h^{-1} \vee h^{-1})$ we obtain the desired pointed homotopy

$$A_Y \circ (\mu_Y \vee \mathrm{id}_Y) \circ \mathrm{id}_Y \simeq_* (\mathrm{id}_Y \vee \mu_Y) \circ \mu_Y.$$

We define a co-inversion map $\nu_Y : (Y, y_0) \to (Y, y_0)$ by using the map h and the co-inversion ν of X.

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[rrr, dashed, bend right, "\nu_Y"'] & X \arrow[r, "\nu"] & X \arrow[r, "h^{-1}"] & Y \end{tikzcd}$$

We must construct the two pointed homotopies indicated below.

$$\langle \mathrm{id}_Y, \nu_Y \rangle \circ \mu_Y \simeq_* \mathrm{ct}_{Y,y_0} \simeq_* \langle \nu_Y, \mathrm{id}_Y \rangle \circ \mu_Y.$$

The strategy behind the proof is the same. For the left-hand side, we use the homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\mu_Y"'] & X \arrow[d, "\mu"] \\ Y \vee Y \arrow[r, "h\vee h"] \arrow[d, "\langle \mathrm{id}_Y, \nu_Y \rangle"'] & X \vee X \arrow[d, "\langle \mathrm{id}_X, \nu \rangle"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

and the homotopical properties of μ to write

$$h \circ (\langle \mathrm{id}_Y, \nu_Y \rangle \circ \mu_Y) \simeq_* (\langle \mathrm{id}_X, \nu \rangle \circ \mu) \circ h \simeq_* \mathrm{ct}_{X,x_0} \circ h = \mathrm{ct}_{Y,x_0}.$$

Composing both sides above with h^{-1} then gives us the desired pointed homotopy

$$\langle \mathrm{id}_Y, \nu_Y \rangle \circ \mu_Y \simeq_* \mathrm{ct}_{Y,y_0}.$$

The other pointed homotopy is obtained by an analogous reasoning. ∎

Let us return to the study of some explicit H-cogroups. Recall that, for every $n \geq 1$, we have the pointed homeomorphisms

$$\Sigma S^{n-1} \cong S^{n-1} \wedge S^1 \cong S^n,$$

The next result follows from this observation by applying Proposition 5.1.4.

5.1.5 Corollary. For every $n \geq 1$, the n-sphere S^n admits an H-cogroup structure.

Let us examine this H-cogroup structure in detail for the particular case of the circle S^1 by following the construction given in the proof of Proposition 5.1.4. We start by giving an explicit description of the pointed homeomorphism $\Sigma S^0 \cong S^1$. Let $\exp : I \to S^1$ be the usual quotient map which identifies the two endpoints of the unit interval. Define $\Phi : S^0 \times I \to S^1$ by the formula

$$\Phi(t, s) := \begin{cases} \exp(s), & \text{if } t = -1 \\ *_{S^1}, & \text{if } t = 1. \end{cases}$$

This map satisfies the condition

$$\Phi(S^0 \times \{0\} \cup S^0 \times \{1\} \cup \{1\} \times I) \subseteq \{*_{S^1}\},$$

so it descends to a pointed map

$$\phi : (\Sigma S^0, *_{\Sigma S^0}) \to (S^1, *_{S^1}).$$

Conversely, the map $\Psi : I \to \Sigma S^0$ defined as

$$\Psi(s) := [-1, s]$$

satisfies $\Psi(0) = \Psi(1) = *_{\Sigma S^0}$, therefore it can be factored through exp to define a pointed map

$$\psi : (S^1, *_{S^1}) \to (\Sigma S^0, *_{\Sigma S^0}).$$

The maps ϕ and ψ defined above are inverse to one another. If $\mu_{\Sigma S^0}$ denotes the H-comultiplication map of ΣS^0, according to Proposition 5.1.4 we obtain an H-comultiplication map $\mu_{S^1} : S^1 \to S^1 \vee S^1$ as the composition

$$\begin{tikzcd} S^1 \arrow[r, "\psi"] \arrow[rrr, dashed, bend right, "\mu_{S^1}"'] & \Sigma S^0 \arrow[r, "\mu"] & \Sigma S^0 \vee \Sigma S^0 \arrow[r, "\phi \vee \phi"] & S^1 \vee S^1 \end{tikzcd}$$

Using the definitions for ϕ and ψ we can show that μ_{S^1} can be described in the following way: given a point in S^1 of the form $\exp(s)$ for some $s \in I$, we have

$$\mu_{S^1}(\exp(s)) = \begin{cases} j_1(\exp(2s)), & \text{if } 0 \leq s \leq \frac{1}{2} \\ j_2(\exp(2s-1)), & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

where $j_1, j_2 : S^1 \to S^1 \vee S^1$ are the canonical injections.

We can also understand μ_{S^1} geometrically. The wedge sum $S^1 \vee S^1$ can be visualized as two tangent circles in the plane. The canonical injections j_1 and j_2 map S^1 to either one of these two tangent circles. The exponential map $\exp : I \to S^1$ wraps the unit interval around the circle, with the restriction $\exp|_{[0,\frac{1}{2}]}$ covering the upper hemisphere, and the restriction $\exp|_{[\frac{1}{2},1]}$ covering the lower one. The map μ then uses the circle S^1 to cover the whole wedge sum $S^1 \vee S^1$, the upper hemisphere is used to cover one of the two circles by applying exp with "double the speed", while the lower hemisphere is used to cover the other circle that makes up the wedge sum, like shown in Figure 6.

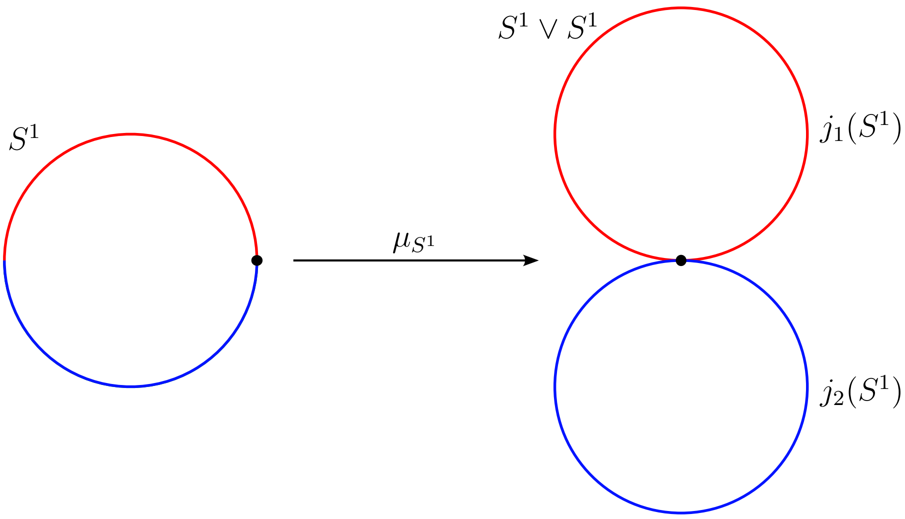

Figure 6 – The H-comultiplication map of the circle.

Sometimes μ_{S^1} is also called the *pinching map*. This is because, since μ_{S^1} maps $*_{S^1}$ and $-*_{S^1}$ both to the basepoint of $S^1 \vee S^1$, we can consider the quotient

$$q : S^1 \to S^1/\{*_{S^1}, -*_{S^1}\}$$

and factor μ_{S^1} through it to obtain a pointed homeomorphism

$$S^1/\{*_{S^1}, -*_{S^1}\} \cong S^1 \vee S^1;$$

and one way to visualize $S^1/\{*_{S^1}, -*_{S^1}\}$ is by pinching the equator of S^1 to a single point.

There is a similar geometric interpretation of the H-comultiplication maps for the higher dimensional spheres. The pointed homeomorphism $\phi : \Sigma S^n \to S^{n+1}$ maps a point $[x, s] \in \Sigma S^n$ to the point

$$\frac{1}{2}(i(x) + *_{S^{n+1}}) + \left(\frac{\cos(t)}{2}\right) \cdot (*_{S^{n+1}} - i(x)) + \left(\frac{\|*_{S^{n+1}} - i(x)\| \sin(t)}{2}\right) \cdot (0, \ldots, 0, 1),$$

where $i : S^n \to S^{n+1}$ is the inclusion at the equator. Recall that geometrically, for a fixed $x \in S^n$, as we vary $t \in I$, the image $\phi([x, t])$ describes a loop on the surface of S^{n+1} that starts at $*_{S^{n+1}}$, moves across the upper hemisphere, passes through $i(x)$ when $t = \frac{1}{2}$, then moves across the lower hemisphere, and then finally comes back to $*_{S^{n+1}}$ when $t = 1$. We see then that the map $\mu_{S^{n+1}} : S^{n+1} \to S^{n+1} \vee S^{n+1}$ uses the upper hemisphere of S^n to cover one copy of S^n inside the wedge sum, and then uses the lower hemisphere to cover the other copy of S^n. In particular, the whole equator gets mapped to the basepoint of wedge sum, so we have the same pinching effect as before.

5.2 Homotopy groups

In this section we finally introduce the homotopy groups of a pointed space. The payoff for using the language of H-cogroups is that the definition is uniform in all dimensions, and it automatically comes with the property of being functorial for pointed spaces and pointed maps. Our approach heavily relies on the concepts of group and cogroup objects developed in Appendix A.1 and on the results proved therein.

The goal of this section is merely to define the homotopy groups, we do not delve to deep into its properties for now. In a later section we will use the notion of H-groups to deduce some interesting basic properties satisfied by the homotopy groups. It is only in the next chapter that we obtain some tools for really computing some homotopy groups coming from the theory of locally trivial bundles.

We first recall some basic categorical properties of HoTop_* that we have already studied. The singleton pointed space $(\{\mathrm{pt}\}, \mathrm{pt})$ is both an initial and terminal object of the category Top_*. Consequently, it is also both initial and terminal in the pointed homotopy category HoTop_*: given a pointed space (X, x_0), $[\mathrm{ct}_{\{\mathrm{pt}\}, x_0}]_*$ is the only pointed homotopy class from $(\{\mathrm{pt}\}, \mathrm{pt})$ to (X, x_0), while $[\mathrm{ct}_{X, \mathrm{pt}}]_*$ is the only pointed homotopy class from (X, x_0) to $(\{\mathrm{pt}\}, \mathrm{pt})$. We have also seen in Corollary 4.2.7 that HoTop_* admits all finite products and coproducts.

These categorical properties ensure that the category HoTop_* supports the algebraic notion of cogroups objects. The next result shows that we have in fact already obtained a bunch of cogroup objects in HoTop_*.

5.2.1 Lemma. If $((X, x_0), \mu, \nu)$ is an H-cogroup, then $((X, x_0), [\mu]_*, [\mathrm{ct}_{X,\mathrm{pt}}]_*, [\nu]_*)$ defines a cogroup object in HoTop_*.

Proof. The idea behind the proof is pretty simple: the homotopical conditions that μ and ν satisfy imply the equalities that the comultiplication and co-inversion morphisms of a cogroup object must satisfy. There are, however, some subtleties that we must take care of. The problem is that, while being a cogroup object in HoTop_* involves morphisms and equalities in this category, being an H-cogroup involves maps and pointed homotopies in Top_*, so in order to prove the result in question, we need to understand how categorical constructions in HoTop_* can be obtained from the analogous constructions in Top_*.

We saw in Corollary 4.2.7 that, given two pointed spaces (X, x_0) and (Y, y_0), the triple $((X \vee Y, *), [j_1]_*, [j_2]_*)$ defines a coproduct for (X, x_0) and (Y, y_0) in the pointed homotopy category HoTop_*, where j_1 and j_2 are the canonical injections into the wedge sum. More succinctly, we can say that the coproduct in HoTop_* can be obtained by the corresponding coproduct in Top_*. In order to prove this, we showed that, given pointed homotopy classes $[f]_* : (X, x_0) \to (Z, z_0)$ and $[g]_* : (Y, y_0) \to (Z, z_0)$, the pointed homotopy class $[\langle f, g\rangle]_* : (X \vee Y, *) \to (Z, z_0)$ is the unique morphism (in HoTop_*) factoring $[f]_*$ and $[g]_*$ through the morphisms $[j_1]_*$ and $[j_2]_*$. In other words, the morphism $\langle [f]_*, [g]_*\rangle$ induced by the universal property of the coproduct in HoTop_* is defined in terms of the analogous induced morphism in Top_*.

There are two formal consequences of this that will be useful for us in the current proof. The first is that, given pointed homotopy classes $[f]_* : (W, w_0) \to (Y, y_0)$ and $[g]_* : (X, x_0) \to (Z, z_0)$, the categorical coproduct morphism $[f]_* \sqcup [g]_* : (W \vee X, *) \to (Y \vee Z, *)$ can be described as the pointed homotopy class $[f \vee g]_*$ of the corresponding coproduct map in Top_*. The other consequence is that the associator isomorphism $(X \vee X) \vee X \cong X \vee (X \vee X)$ in the pointed homotopy category HoTop_*, or in other words, the associator pointed homotopy equivalence

$$(X \vee X) \vee X \simeq_* X \vee (X \vee X)$$

is given simply by the pointed homotopy class $[A]_*$ of the associator isomorphism in Top_*.

Keeping these subtleties in mind, in order to prove that $(X, [\mu]_*, [\mathrm{ct}_{X,\mathrm{pt}}]_*, [\nu]_*)$ defines a cogroup structure in HoTop_*, we need to check the commutativity of the three following diagrams:

1. (Existence of two-sided counit)

$$\begin{tikzcd}
X & X \vee X \arrow[l, "{[\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0}\rangle]_*}"'] \\
X \vee X \arrow[u, "{[\langle \mathrm{ct}_{X,x_0}, \mathrm{id}_X\rangle]_*}"] & X \arrow[l, "{[\mu]_*}"] \arrow[u, "{[\mu]_*}"'] \arrow[ul, "{[\mathrm{id}_X]_*}"]
\end{tikzcd}$$

2. (Existence of co-inverses)

$$\begin{tikzcd}
X & & X \vee X \arrow[ll, "{[\langle \mathrm{id}_X, \nu \rangle]_*}"'] \\
X \vee X \arrow[u, "{[\langle \nu, \mathrm{id}_X \rangle]_*}"] & & X \arrow[ll, "{[\mu]_*}"] \arrow[u, "{[\mu]_*}"'] \arrow[ull, "{[\mathrm{ct}_{X,x_0}]_*}"]
\end{tikzcd}$$

3. (Coassociativity)

$$\begin{tikzcd}
X \vee (X \vee X) & & (X \vee X) \vee X \arrow[ll, "{[A]_*}"'] \\
X \vee X \arrow[u, "{[\mathrm{id}_X \vee \mu]_*}"] & & X \vee X \arrow[u, "{[\mu \vee \mathrm{id}_X]_*}"'] \\
& X \arrow[ul, "{[\mu]_*}"] \arrow[ur, "{[\mu]_*}"'] &
\end{tikzcd}$$

The commutativity of these diagrams actually follows directly from the defining properties of an H-cogroup. For example, the H-comultiplication by definition satisfies the homotopical conditions

$$\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle \circ \mu \simeq_* \mathrm{id}_X \simeq_* \langle \mathrm{ct}_{X,x_0}, \mathrm{id}_X \rangle \circ \mu,$$

so, if we pass to pointed homotopical classes, this becomes the equality

$$[\langle \mathrm{id}_X, \mathrm{ct}_{X,x_0} \rangle]_* \circ [\mu]_* = [\mathrm{id}_X]_* = [\langle \mathrm{ct}_{X,x_0}, \mathrm{id}_X \rangle]_* \circ [\mu]_*;$$

which means precisely that the first of the three diagrams above commute. The required commutativity of the two other diagrams then follows from the two other defining properties of an H-cogroup. ■

Now that we have cogroup objects, applying the tools developed in Appendix A.1, in particular Theorem A.2.2, allows us to obtain many algebraic objects from homotopical ones.

5.2.2 Corollary. If $((X, x_0), \mu, \nu)$ is an H-cogroup, then for any pointed space (Y, y_0), the set of pointed homotopy classes $[X, Y]_*$ admits a group structure such that, if $\alpha : (Y, y_0) \to (Z, z_0)$ is a pointed map, then the pushforward function along the pointed homotopy class $[\alpha]_*$

$$\mathsf{HoTop}(X, [\alpha]_*) : [X, Y]_* \to [X, Z]_*$$

defines a group homomorphism.

Following the discussion after Theorem A.2.2, we can explicitly describe the group structure on $[X, Y]_*$. The binary product

$$\cdot_Y : [X, Y]_* \times [X, Y]_* \to [X, Y]_*$$

is described explicitly as

$$[f]_* \cdot_Y [g]_* \coloneqq ([f]_* \sqcup [g]_*) \circ [\mu]_*,$$

but according to the discussion at the start of the proof of Lemma 5.2.1, the coproduct morphism $[f]_* \sqcup [g]_*$ is given by the pointed homotopy class $[f \vee g]_*$ of the wedge sum, therefore the previous formula can be rewritten as

$$[f]_* \cdot_Y [g]_* \coloneqq [\langle f, g\rangle \circ \mu]_* \tag{5.10}$$

for all $[f]_*$, $[g]_* \in [X, Y]_*$. The unit for such product is given by composing the counit $[\mathrm{ct}_{X,\mathrm{pt}}]$ with the unique pointed homotopy class $[\mathrm{ct}_{\{\mathrm{pt}\},y_0}] : (\{\mathrm{pt}\}, \mathrm{pt}) \to (Y, y_0)$, but this composition is simply equal to $[\mathrm{ct}_{X,y_0}]_*$. Lastly, given a pointed homotopy class $[f]_* \in [X, Y]_*$ its inverse with respect to $\cdot_Y$ is given explicitly in terms of the H-co-inversion as

$$[f]_*^{-1} \coloneqq [f \circ \nu]_*. \tag{5.11}$$

Of course, the result of Corollary 5.2.2 would not be very useful had we not described explicit examples of H-cogroups coming from reduced suspensions in the previous section.

5.2.3 Corollary. For any two pointed spaces (X, x_0) and (Y, y_0), the set $[\Sigma X, Y]_*$ admits a group structure, such that, if $\alpha : (Y, y_0) \to (Z, z_0)$ is a pointed map, then the pushforward along $[\alpha]_*$ defines a group homomorphism $[\Sigma X, Y]_* \to [\Sigma X, Z]_*$.

Since we have an explicit description of the H-comultiplication μ and of the H-co-inversion of ΣX, we also have a more explicit description of the induced product on $[\Sigma X, Y]_*$. Given two pointed homotopy classes $[f_1,]_*$, $[f_2]_* \in [\Sigma X, Y]_*$, according to equation (5.10) their product $[f_1]_* \cdot_Y [f_2]_*$ is given by the pointed homotopy class of the map

$$\langle f_1, f_2\rangle \circ \mu : \Sigma X \to Y.$$

Now recall that the H-comultiplication map $\mu : \Sigma X \to \Sigma X \vee \Sigma X$ constructed in Theorem 5.1.2 is given by the formula

$$\mu([x, s]) = \begin{cases} j_1([x, 2s]), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ j_2([x, 2s - 1]), & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

where j_1, $j_2 : \Sigma X \to \Sigma X \vee \Sigma X$ are the canonical injections. Since the induced morphism $\langle f_1, f_2\rangle$ satisfies $\langle f_1, f_2\rangle \circ j_1 = f_1$ and $\langle f_1, f_2\rangle \circ j_2$, it follows that the pointed map $\langle f_1, f_2\rangle \circ \mu$ *representing* the product $[f_1]_* \cdot_Y [f_2]_*$ can be described explicitly as

$$(\langle f_1, f_2\rangle \circ \mu)([x, s]) = \begin{cases} f_1([x, 2s]), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f_2([x, 2s - 1]), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.12}$$

We can also obtain a more explicit description of the inverse of a pointed homotopy class $[f]_* \in [\Sigma X, Y]_*$. According to Equation (5.11), this inverse $[f]_*^{-1}$ is given by the pointed homotopy class of the map

$$f \circ \nu : \Sigma X \to Y.$$

The H-co-inversion map $\nu : \Sigma X \to \Sigma X$ is described as

$$\nu([x, s]) = [x, 1 - s]$$

for every point $[x, s] \in \Sigma X$, therefore the composite $f \circ \nu$ representing the inverse of $[f]_*$ is given explicitly by

$$(f \circ \nu)([x, s]) = f([x, 1 - s]). \tag{5.13}$$

Even more important than the fact that reduced suspensions are H-cogroups is the fact that all spheres S^n for $n \geq 1$ are homeomorphic to reduced suspensions, and therefore inherit an H-cogroup structure too (Corollary 5.1.5), a result which together with Corollary 5.2.2 implies the next one.

5.2.4 Corollary. For every $n \geq 1$ and for every pointed space (X, x_0), the set of pointed homotopy classes $[S^n, X]_*$ admits a group structure such that, if $f : (X, x_0) \to (Y, y_0)$ is pointed map, then the pushforward along $[f]_*$ defines a group homomorphism $[S^n, X]_* \to [S^n, Y]_*$.

At last, we have the infamous homotopy groups.

5.2.5 Definition. Let (X, x_0) be a pointed space, and consider an integer $n \geq 1$. The group $[S^n, X]_*$ induced by the H-comultiplication map on S^n according to Corollary 5.2.4 is called the **n-th homotopy group of (X, x_0)** or also the **n-th homotopy group of X based at x_0**, and is denoted by $\pi_n(X, x_0)$. In the case where $n = 1$, $\pi_1(X, x_0)$ is traditionally called the **fundamental group of X at x_0**.

If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, we know from Corollary 5.2.4 that the pushforward along $[f]_*$ defines a group homomorphism $\pi_n(X, x_0) \to \pi_n(Y, y_0)$ which we denote by $\pi_n(f)$. Of course, if $g : (X, x_0) \to (Y, y_0)$ is pointed homotopic to f, then the induced homomorphism $\pi_n(g)$ coincides with $\pi_n(f)$, since these only depend on the homotopy classes $[g]_*$ and $[f]_*$ which are equal.

All of this can be rephrased by saying that the construction of the homotopy group defines either a functor of type $\mathsf{Top}_* \to \mathsf{Grp}$, or a functor of type $\mathsf{HoTop}_* \to \mathsf{Grp}$. We denote both of these functors by π_n and refer to them both as the **n-th homotopy group functor**. The two points of view are useful for different things. Notice for example that, since functors always preserve isomorphisms, thinking of π_n as a functor of type

$\mathsf{Top}_* \to \mathsf{Grp}$ shows that, if $f : (X, x_0) \to (Y, y_0)$ is a pointed homeomorphism, then $\pi_n(f)$ is a group isomorphism; while thinking of π_n as a functor of type $\mathsf{HoTop}_* \to \mathsf{Grp}$ implies the following stronger result:

5.2.6 Proposition. If $f : (X, x_0) \to (Y, y_0)$ is a pointed homotopy equivalence, then the induced group homomorphism $\pi_n(f) : (X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for every integer $n \geq 1$.

Even though we cannot do much with the homotopy groups yet, we can state a definition that will be crucial for our study of Obstruction Theory.

5.2.7 Definition. Given an integer $n \geq 1$, we say a space X is n**-connected** if it is path-connected, and if the homotopy groups $\pi_k(X, x_0)$ are trivial for all $1 \leq k \leq n$ and all points $x_0 \in X$. A 1-connected is more often said to be a **simply-connected space**. We also refer to a path-connected space as a 0**-connected** space.

5.2.8 Remark. So far we have only worked with pointed spaces when dealing with the homotopy group. Any map $f : X \to Y$ can be turned into a pointed map by choosing an arbitrary basepoint $x_0 \in X$, and then taking $f(x_0)$ as a basepoint for Y. Consequently, for every $x_0 \in X$ we have the induced homomorphism $\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ between the homotopy groups.

Unfortunately, this construction is not always adequate, the domain X might not have a natural choice of basepoint, or the corresponding basepoint $f(x_0)$ might not be relevant to the problem at hand. Another problem is that this strategy for choosing basepoints does not allow us to compare maps with are homotopic in the unpointed sense. If $g : X \to Y$ is another map homotopic to f, the basepoint $g(x_0)$ in Y induced by g might be different from $f(x_0)$, and then it is not possible to compare the homomorphisms

$$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \quad \text{and} \quad \pi_n(g) : \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$$

with the tools we have so far. Nevertheless, the points $f(x_0)$ and $g(x_0)$ are not entirely unrelated, since the existence of a homotopy $f \simeq g$ allows us to obtain a path connecting $f(x_0)$ to $g(x_0)$ for nay choice of initial basepoint $x_0 \in X$.

We will see later that the existence of paths between two points allows us to compare the homotopy groups based at these points, but since there might exist many paths connecting two points, we might end up with many ways to compare the different homotopy groups.

We end this section with an alternative description of the set $\pi_0(X)$ of path-components which more closely resembles the definition of the homotopy groups. Notice first that, although $\pi_0(X)$ is in general merely a set, the choice of a basepoint $x_0 \in X$

gives rise to a distinguished element in $\pi_0(X)$: the path-component $[x_0]$ of the chosen basepoint. The pair $(\pi_0(X), [x_0])$ therefore defines a *pointed set*, and when we want to talk about this pointed set instead of just $\pi_0(X)$, we employ the notation $\pi_0(X, x_0)$. This of course means that instead of the functor $\pi_0 : \mathsf{HoTop} \to \mathsf{Set}$, we can now consider a functor $\pi_0 : \mathsf{HoTop}_* \to \mathsf{Set}_*$.

The sphere $S^0 \subseteq \mathbb{R}$ consists of the two disjoint points -1 and $+1$. A pointed map $f : (S^0, +1) \to (X, x_0)$ is then uniquely determined by the choice of point $f(-1) \in X$. Given a point $x \in X$, let $\theta_x : (S^0, +1) \to (X, x_0)$ be the induced pointed map. Notice that the map θ_{x_0} induced by the basepoint itself is the constant map $\mathrm{ct}_{S^0, x_0} : (S^0, +1) \to (X, x_0)$.

A path $\gamma : I \to X$ from x to x' induces a pointed homotopy from θ_x to $\theta_{x'}$, therefore we can consider the pointed function $\Theta_{(X,x_0)} : \pi_0(X, x_0) \to [S^0, X]_*$ sending the path-component $[x]$ to the pointed homotopy class $[\theta_x]_*$. It is straightforward to show that $\Theta_{(X,x_0)}$ is a bijection: if two components $[x]$ and $[x']$ are such that $[\theta_x]_* = [\theta_{x'}]_*$, then a pointed homotopy $\theta_x \Rightarrow_* \theta_{x'}$ defines a path from x to x' by restricting it to $\{-1\} \times I$; and any pointed homotopy class $[\alpha]_* \in [S^0, X]_*$ is equal to $'[\theta_{x_\alpha}]$, where $x_\alpha \coloneqq \alpha(-1) \in X$.'

The bijection $\Theta_{(X,x_0)}$ depends naturally on (X, x_0). Given a pointed homotopy class $[f]_* : (X, x_0) \to (Y, y_0)$, the compositions $[S^0, f]_* \circ \Theta_{(X,x_0)}$ and $\Theta_{(Y,y_0)} \circ \pi_0(f)$ both send a path component $[x] \in \pi_0(X, x_0)$ to the pointed homotopy class in $[S^0, X]_*$ represented by the map $S^0 \to X$ given by $-1 \mapsto f(x)$ and $+1 \mapsto y_0$; showing the commutativity of the diagram below.

$$\begin{tikzcd}
\pi_0(X, x_0) \arrow[r, "\pi_0(f)"] \arrow[d, "\Theta_{(X,x_0)}"'] & \pi_0(Y, y_0) \arrow[d, "\Theta_{(Y,y_0)}"] \\
{[S^0, X]_*} \arrow[r, "{[S^0, f]_*}"'] & {[S^0, Y]_*}
\end{tikzcd}$$

We summarize this discussion in the next result for later referencing.

5.2.9 Proposition. The path-components functor $\pi_0 : \mathsf{HoTop}_* \to \mathsf{Set}_*$ is naturally isomorphic to the representable functor $[S^0, -]_* : \mathsf{HoTop}_* \to \mathsf{Set}_*$.

5.3 H-groups

In this section we introduce the concept of H-groups, which are dual to the H-cogroups introduced in Section 5.1. The progression of ideas is similar to that of the aforementioned section, we first define H-groups, and then later we describe a family of examples. In the next section we will then see how the interaction of H-cogroups and H-groups allows us to deduce some properties of the homotopy groups.

5.3.1 Definition. An **H-group** consists of a pointed space (X, x_0) together with the following data:

- A pointed map $m : (X \times X, (x_0, x_0)) \to (X, x_0)$ called the **H-multiplication**;
- A pointed map $\mathrm{inv} : (X, x_0) \to (X, x_0)$ called the **H-inversion**.

These maps are required to satisfy the following homotopical conditions:

1. $m \circ (\mathrm{id}_X, \mathrm{ct}_{X,x_0}) \simeq_* \mathrm{id}_X \simeq_* m \circ (\mathrm{ct}_{X,x_0}, \mathrm{id}_X)$;
2. $m \circ (\mathrm{id}_X, \mathrm{inv}) \simeq_* \mathrm{ct}_{X,x_0} \simeq_* m \circ (\mathrm{inv}, \mathrm{id}_X)$;
3. $m \circ (m \times \mathrm{id}_X) \simeq_* m \circ (\mathrm{id}_X \times m) \circ A$,

where A denotes the product associator pointed homeomorphism

$$A : (X \times X) \times X \xrightarrow{\cong} X \times (X \times X).$$

The geometric interpretation of an H-group is simpler than that of an H-cogroup. The maps m and inv are like the multiplication and inversion maps of an ordinary group, but the usual properties satisfied in a group only hold up to homotopy. In the first of the properties above, the map $m \circ (\mathrm{id}_X, \mathrm{ct}_{X,x_0})$ sends a point $x \in X$ to $m(x, x_0)$. If we had a strict equality $m \circ (\mathrm{id}_X, \mathrm{ct}_{X,x_0}) = \mathrm{id}_X$, then $m(x, x_0)$ would be equal to x itself. In an H-group, even though this equality does not necessarily hold, by virtue of the pointed homotopy $m \circ (\mathrm{id}_X, \mathrm{ct}_{X,x_0}) \simeq_* \mathrm{id}_X$, there is a path connecting $m(x, x_0)$ and x, and this path depends continuously on x. There is a similar interpretation for the other two properties. The point $\mathrm{inv}(x)$ is not necessarily an inverse for x with respect to m, but $m(x, \mathrm{inv}(x))$ and $m(\mathrm{inv}(x), x)$ can be connected to the basepoint x_0 by a path depending continuously on x. Lastly, for any three points x_1, x_2, $x_3 \in X$, the points $m(m(x_1, x_2), x_3)$ and $m(x_1, m(x_2, x_3))$ might be different, but there is a path connecting the two, and this path depends continuously on the three chosen points.

The most important result on H-groups for now is the fact that we have already encountered a family of them.

5.3.2 Theorem. If (X, x_0) is any pointed space, then its loop space $(\Omega X, \mathrm{ct}_{S^1,x_0})$ admits an H-group structure.

Proof. We first define an H-multiplication map

$$m : (\Omega X \times \Omega X, (\omega_{x_0}, \omega_{x_0})) \to (\Omega X, \omega_{x_0}).$$

Consider the map $\psi : (\Omega X \times \Omega X) \times I \to X$ defined by the formula

$$\psi((f,g),s) := \begin{cases} f(\exp(2s)), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ g(\exp(2s-1)), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

This is a well-defined function, since for $s = \frac{1}{2}$ the equalities

$$f(\exp(2 \cdot \frac{1}{2})) = f(\exp(1)) = f(*) = x_0$$

and

$$g(\exp(2 \cdot \frac{1}{2} - 1)) = g(\exp(0)) = g(*) = x_0$$

hold for any pair of loops $(f,g) \in \Omega x \times \Omega X$.

The restriction of ψ to the subspace $(\Omega X \times \Omega X) \times [0, \frac{1}{2}]$ is given by the composition shown below

$$\begin{tikzcd} (\Omega X \times \Omega X) \times [0, \frac{1}{2}] \arrow[r] \arrow[rrrr, bend right, "\psi|_{(\Omega X \times \Omega X)\times[0,\frac{1}{2}]}"'] & \Omega X \times [0, \frac{1}{2}] \arrow[r, "\mathrm{id}_X \times (2\cdot)"] & \Omega X \times I \arrow[r, "\mathrm{id}_X \times \exp"] & \Omega X \times S^1 \arrow[r] & X, \end{tikzcd}$$

where the first map is a suitable combination of canonical projections, and where the last map is the usual evaluation map, therefore this restriction is continuous. Similarly, the restriction of ψ to the subspace $(\Omega X \times \Omega X) \times [\frac{1}{2}, 1]$ is also continuous, thus ψ itself is continuous by virtue of the Pasting Lemma.

Now consider the map

$$\mathrm{id}_{\Omega X \times \Omega X} \times \exp : (\Omega X \times \Omega X) \times I \to (\Omega X \times \Omega X) \times S^1,$$

which is a quotient map due to the fact that exp is a proper map. This map has two types of fibers: given $((f,g),z) \in (\Omega X \times \Omega X) \times S^1$, if $z \neq *$, then the fiber this point consists of a single point, while if $z = *$, then the fiber over $((f,g),z)$ is equal to $\{((f,g),0),((f,g),1)\}$. Since

$$\psi((f,g),0) = f(\exp(2 \cdot 0)) = f(exp(0)) = f(*) = x_0,$$

and also

$$\psi((f,g),1) = g(\exp(2 \cdot 1 - 1)) = g(\exp(1)) = g(*) = x_0,$$

ψ is constant on the fibers of $\mathrm{id}_{\Omega X \times \Omega X} \times \exp$, therefore we have an induced map

$$\overline{\psi} : (\Omega X \times \Omega X) \times S^1 \to X$$

as shown in the diagram below.

$$\begin{tikzcd} (\Omega X \times \Omega X) \times I \arrow[r, "\psi"] \arrow[d, "\mathrm{id}_{\Omega X \times \Omega X} \times \exp"'] & X \\ (\Omega X \times \Omega X) \times S^1 \arrow[ur, dashed, "\overline{\psi}"'] & \end{tikzcd}$$

The computations we have already performed show that

$$\overline{\psi}((\Omega X \times \Omega X) \times \{*\}) \subseteq \{x_0\}.$$

Moreover, a simple computation shows that $\overline{\psi}$ also satisfies

$$\overline{\psi}(\{(\omega_{x_0}, \omega_{x_0})\} \times S^1) \subseteq \{x_0\}.$$

It then follows from Lemma 3.2.4 that the exponential adjoint $m \coloneqq \lambda\overline{\psi}$ defines a pointed map

$$(\Omega X \times \Omega X, (\omega_{x_0}, \omega_{x_0})) \to (\Omega X, \omega_{x_0}).$$

We now prove that the map m obtained above satisfied the required conditions for an H-multiplication. The first thing we need to prove is that the constant loop ω_{x_0} behaves as an H-unit, that is, we need to construct pointed homotopies

$$m \circ (\mathrm{id}_{\Omega X}, \mathrm{ct}_{\Omega X, \omega_{x_0}}) \simeq_* \mathrm{id}_{\Omega X} \simeq_* m \circ (\mathrm{ct}_{\Omega X, \omega_{x_0}}, \mathrm{id}_{\Omega X}).$$

Let us first understand the map $m \circ (\mathrm{id}_{\Omega X}, \mathrm{ct}_{\Omega X, \omega_{x_0}}) : \Omega X \to \Omega X$ on the left. It sends any loop $f \in \Omega X$ to the loop $m(f, \omega_{x_0}) \in \Omega X$ defined as

$$[m(f, \omega_{x_0})](\exp(s)) = \begin{cases} f(\exp(2s)), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ x_0, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.14}$$

The expression above shows that $m(f, \omega_{x_0})$ is obtained from f by a certain reparameterization: we use f on $[0, \frac{1}{2}]$ with double the speed, and then we remain at the basepoint x_0 on $[0, \frac{1}{2}]$. We can express this a real reparameterization of the domain S^1 of the loops. Let $\alpha : I \to I$ be the map defined as

$$\alpha(s) \coloneqq \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.15}$$

Since $\alpha(\partial I) \subseteq \partial I$, we can obtain a pointed quotient map $\overline{\alpha} : (S^1, *) \to (S^1, *)$ that fits in the commutative diagram below.

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "\exp"'] & I \arrow[d, "\exp"] \\ S^1 \arrow[r, dashed, "\overline{\alpha}"'] & S^1 \end{tikzcd}$$

Equation (5.14) then shows that the map $m \circ (\mathrm{id}_{\Omega X}, \mathrm{ct}_{\Omega X, \omega_{x_0}})$ is equal to the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$.

It is clear that we need a reparameterization lemma analogous to Lemma 5.1.3 to understand how the properties of α affect the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$.

5.3.3 Lemma (Reparamaterization II)**.** Let (x, x_0) be an arbitrary pointed space. Suppose $\alpha : I \to I$ is a map such that $\alpha(\partial I) \subseteq \partial I$, and let $\overline{\alpha} : (S^1, *) \to (S^1, *)$ be the pointed map induced by passage to the quotient as shown in the diagram below.

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "\exp"'] & I \arrow[d, "\exp"] \\ S^1 \arrow[r, dashed, "\overline{\alpha}"'] & S^1 \end{tikzcd}$$

1. If $\alpha(0) = \alpha(1)$, then the pullback map $\mathrm{Map}_*(\overline{\alpha}, X) : \Omega X \to \Omega X$ is pointed null homotopic.
2. If $\alpha(0) = 0$ and $\alpha(1) = 1$, then the pullback map $\mathrm{Map}_*(\overline{\alpha}) : (\Omega X, \omega_{x_0}) \to (\Omega X, \omega_{x_0})$ is pointed homotopic to $\mathrm{id}_{\Omega X}$.

Proof of the Lemma. The proof is just a combination of several other results.

1. Let $p := \alpha(0) = \alpha(1) \in \partial I$, and define a map $H : I \times I \to I$ as

$$H(s, t) := (1 - t)\alpha(s) + tp.$$

Direct computations show that H satisfies the following properties:

(i) $H(s, 0) = \alpha(s)$ for every $s \in I$;

(ii) $H(s, 1) = p = \mathrm{ct}_{I,p}(s)$ for every $s \in I$;

(iii) $H(\partial I \times I) \subseteq \partial I$.

It follows from Proposition 4.2.8 that the induced map $\overline{\alpha}$ is pointed homotopic to $\overline{\mathrm{ct}_{I,p}} = \mathrm{ct}_{S^1,*}$. Since S^1 is locally compact Hausdorff, by applying Proposition 4.2.11 we conclude that the pullback maps

$$\mathrm{Map}_*(\overline{\alpha}, X),\ \mathrm{Map}_*(\mathrm{ct}_{S^1,*}, X) : (\Omega X, \omega_{x_0}) \to (\Omega X, \omega_{x_0})$$

are pointed homotopic, but the pullback $\mathrm{Map}_*(\mathrm{ct}_{S^1,*}, X)$ is just the constant map $\mathrm{ct}_{\Omega X, \omega_{x_0}}$; thus $\mathrm{Map}_*(\overline{\alpha}, X)$ is pointed null homotopic.

2. Consider the map $H : I \times I \to I$ defined as

$$H(s, t) := (1 - t)\alpha(s) + ts$$

for all $(s, t) \in I \times I$. Again, direct computations show that H satisfies the following properties:

(i) $H(s, 0) = \alpha(s)$ for every $s \in I$;

(ii) $H(s,1) = s = \mathrm{id}_I(s)$ for every $s \in I$;

(iii) $H(\partial I \times I) \subseteq \partial I$.

Applying Proposition 4.2.8 again, we conclude that $\overline{\alpha}$ is pointed homotopic to $\overline{\mathrm{id}_I}$, but this latter map is just the identity id_{S^1} on the circle. Proposition 4.2.11 then implies that $\mathrm{Map}_*(\overline{\alpha}, X)$ is pointed homotopic to $\mathrm{Map}_*(\mathrm{id}_{S^1}, X)$, but by functoriality we know that this latter pullback is nothing but the identity $\mathrm{id}_{\Omega X}$ on the loop space. $\square$

With Lemma 5.3.3 at our disposal, the rest of the current proof is very similar to the proof of Theorem 5.1.2. As we saw before the statement of the lemma, the map $m \circ (\mathrm{id}_{\Omega X}, \mathrm{ct}_{\Omega X, \omega_{x_0}})$ is equal to the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$, where $\overline{\alpha}$ is induced by the map $\alpha : I \to I$ defined in Equation (5.15). Since $\alpha(0) = 0$ and $\alpha(1) = 1$, it follows from Lemma 5.3.3 that $m \circ (\mathrm{id}_{\Omega X}, \mathrm{ct}_{\Omega X, \omega_{x_0}})$ is pointed homotopic to $\mathrm{id}_{\Omega X}$.

Now for the composition $m \circ (\mathrm{ct}_{\Omega X, \omega_{x_0}}, \mathrm{id}_{\Omega X})$, unpacking the definitions we see that it sends a loop $f \in \Omega X$ to the loop $m(\omega_{x_0}, f)$ given by

$$[m(\omega_{x_0}, f)](\exp(s)) = \begin{cases} x_0, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f(\exp(2s-1)), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.16}$$

This means $m \circ (\mathrm{ct}_{\Omega X, \omega_{x_0}}, \mathrm{id}_{\Omega X})$ is equal to the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$, where $\overline{\alpha}$ is induced by the map $\alpha : I \to I$ defined as

$$\alpha(s) := \begin{cases} 0, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2s-1, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.17}$$

Since $\alpha(0) = 0$ and $\alpha(1) = 1$, we know from Lemma 5.3.3 that $m \circ (\mathrm{ct}_{\Omega X, \omega_{x_0}}, \mathrm{id}_{\Omega X})$ is pointed homotopic to $\mathrm{id}_{\Omega X}$.

We now deal with the H-associativity of m. We must exhibit a pointed homotopy

$$m \circ (m \times \mathrm{id}_{\Omega X}) \simeq_* m \circ (\mathrm{id}_{\Omega X} \times m) \circ A.$$

The idea is to show that one side of the equation above can be obtained from the other using a certain pullback. The map on the left-hand side above sends a triple of loops $((f,g),h) \in \Omega X$ to the loop $m(m(f,g),h)$ described explicitly as

$$[m(m(f,g),h)](\exp(s)) = \begin{cases} f(\exp(4s)), & \text{if } 0 \leq s \leq \frac{1}{4}, \\ g(\exp(4s-1)), & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ h(\exp(2s-1)), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.18}$$

Similarly, the composition on the right-hand side sends the same triple $((f,g),h)$ to the loop $m(f,m(g,h))$ described as

$$[m(f,m(g,h))](\exp(s)) = \begin{cases} f(\exp(2s)), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ g(\exp(4s-2)), & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4}, \\ h(\exp(4s-3)), & \text{if } \frac{3}{4} \leq s \leq 1. \end{cases} \tag{5.19}$$

Now, just like we did in the proof of Theorem 5.1.2, consider the map $\alpha : I \to I$ defined as

$$\alpha(s) \coloneqq \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{4} \\ s+\frac{1}{4}, & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \frac{1}{2}s+\frac{1}{2}, & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases} \tag{5.20}$$

which satisfies $\alpha(\partial I) \subseteq \partial I$. Comparing equations (5.18) and (5.19) we see that α is such that the induced pullback $\mathrm{Map}_*(\overline{\alpha}, X)$ fits in the equality

$$m \circ (m \times \mathrm{id}_{\Omega X}) = m \circ (\mathrm{id}_{\Omega X} \times m) \circ A \circ \mathrm{Map}_*(\overline{\alpha}, X);$$

but since $\alpha(0) = 0$ and $\alpha(1) = 1$, Lemma 5.3.3 implies that the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$ is pointed homotopic to $\mathrm{id}_{\Omega X}$, and by combining this with the previous equality we obtain the desired pointed homotopy relation.

We still have to define an H-inversion map and prove that it satisfies the two required conditions. Consider the map $r : I \to I$ that reverses the interval, that is, $r(s) \coloneqq 1 - s$ for every $s \in I$. Since $r(\partial I) \subseteq \partial I$, we have the corresponding pointed map $\overline{r} : (S^1, *) \to (S^1, *)$ like shown below.

$$\begin{tikzcd} I \arrow[r, "r"] \arrow[d, "\exp"'] & I \arrow[d, "\exp"] \\ S^1 \arrow[r, dashed, "\overline{r}"'] & S^1 \end{tikzcd}$$

We then define an H-inversion map via the pullback $\mathrm{inv} \coloneqq \mathrm{Map}_*(\overline{r}, X) : (\Omega X, \omega_{x_0}) \to (\Omega X, \omega_{x_0})$. This inversion sends a loop $f \in \Omega X$ to the loop $\mathrm{inv}(f)$ defined by the formula

$$[\mathrm{inv}(f)](\exp(s)) = f(\exp(1-s)) \tag{5.21}$$

for every $s \in I$.

In order to show that inv really defines an H-inversion, we need to show that the homotopical relations

$$m \circ (\mathrm{id}_{\Omega X}, \mathrm{inv}) \simeq_* \mathrm{ct}_{\Omega X, \omega_{x_0}} \simeq_* m \circ (\mathrm{inv}, \mathrm{id}_{\Omega X})$$

hold. The map on the left sends an arbitrary loop f to the loop $m(f, \mathrm{inv}(f))$ given by the formula

$$[m(f, \mathrm{inv}(f))](\exp(s)) = \begin{cases} f(\exp(2s)), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f(\exp(2-2s)), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.22}$$

This means that $m \circ (\mathrm{id}_{\Omega X}, \mathrm{inv})$ is nothing but the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$, where $\alpha : I \to I$ is defined as

$$\alpha(s) := \begin{cases} 2s, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2 - 2s, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.23}$$

Since $\alpha(0) = \alpha(1)$, it follows from *Lemma* 5.3.3 that $\mathrm{Map}_*(\overline{\alpha}, X) = m \circ (\mathrm{id}_{\Omega X}, \mathrm{inv})$ is pointed homotopic to $\mathrm{ct}_{\Omega X, \omega_{x_0}}$. Similarly, the other composition $m \circ (\mathrm{inv}, \mathrm{id}_{\Omega X})$ sends a loop f to the loop $m(\mathrm{inv}(f), f)$ defined by the formula

$$[m(\mathrm{inv}(f), f)](\exp(s)) = \begin{cases} f(\exp(1 - 2s)), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f(2s - 1), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.24}$$

It can also be described as the pullback $\mathrm{Map}_*(\overline{\alpha}, X)$, where this time $\alpha : I \to I$ is defined as

$$\alpha(s) := \begin{cases} 1 - 2s, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2s - 1, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{5.25}$$

Lemma 5.3.3 once again implies $\mathrm{Map}_*(\overline{\alpha}, X) = m \circ (\mathrm{inv}, \mathrm{id}_{\Omega X})$ is pointed homotopic to $\mathrm{ct}_{\Omega X, \omega_{x_0}}$ because $\alpha(0) = \alpha(1)$. ■

5.4 Commutativity results

In this section, we combine the notions of H-group from Section 5.3 with the results of Appendix A.1 to deduce some properties of the homotopy groups. The first result says that, dually to how H-cogroups give rise to cogroup objects in HoTop_*, H-groups give rise to group objects in this same category. The proof follows that of Lemma 5.2.1, when we pass to the pointed homotopy category HoTop_*, the properties satisfied up to homotopy by an H-group become exactly the algebraic properties that a group object must satisfy.

5.4.1 Lemma. If $((X, x_0), m, \mathrm{inv})$ is an H-group, then $((X, x_0), [m]_*, [\mathrm{ct}_{\{\mathrm{pt}\}, x_0}]_*, [\mathrm{inv}]_*)$ defines a group object in HoTop_*.

Recall from Theorem A.2.1 that a group object induces an ordinary group structure on any set of morphisms *into* it. When interpreted in the category HoTop_*, this implies the next result.

5.4.2 Corollary. If $((X, x_0), m, \mathrm{inv})$ is an H-group, then for any pointed space (W, w_0), the set of pointed homotopy classes $[W, X]_*$ admits a group structure such that, if $\alpha : (W, w_0) \to (W', w_0')$ is a pointed map, then the pullback function along the pointed homotopy class $[\alpha]_*$

$$\mathsf{HoTop}_*([\alpha]_*, X) : [W', X]_* \to [W, X]_*$$

defines a group homomorphism.

We recall from the proof of Theorem A.2.1 how this group structure is defined. Given two pointed homotopy classes $[f]_*, [g]_* \in [W, X]_*$, their product $[f]_* \cdot_W [g]_*$ is the pointed homotopy class defined as the composition

$$[f]_* \cdot_W [g]_* \coloneqq [m]_* \circ ([f]_*, [g]_*),$$

where $([f]_*, [g]_*)$ is the pointed homotopy class $W \to X \times X$ induced from the universal property of the product. We saw in Corollary 4.2.7 that this pointed homotopy class is precisely $[(f, g)]_*$, where $(f, g) : W \to X \times X$ is the usual induced pointed map in Top_*. This allows us to rewrite the previous expression for the product in the form

$$[f]_* \cdot_W [g]_* \coloneqq [m \circ (f, g)]_*. \tag{5.26}$$

The unit e_W for the product $\cdot_W$ is obtained by composing the pointed homotopy class $[\mathrm{ct}_{W,\mathrm{pt}}]_*$, which is the unique morphism from (W, w_0) to $(\{\mathrm{pt}\}, \mathrm{pt})$ in HoTop_*, with the unit morphism $[\mathrm{ct}_{\{\mathrm{pt}\},x_0}]_*$ from the group structure, therefore we have the equality

$$e_W = [\mathrm{ct}_{\{\mathrm{pt}\},x_0}]_* \circ [\mathrm{ct}_{W,\mathrm{pt}}]_* = [\mathrm{ct}_{W,x_0}]_*. \tag{5.27}$$

Lastly, the inverse of a pointed homotopy class $[f]_* \in [W, X]$ is obtained by composing it with the inversion morphism $[\mathrm{inv}]_* : (X, x_0) \to (X, x_0)$ from the group structure, so that we can write

$$[f]_*^{-1} = [\mathrm{inv} \circ f]_*. \tag{5.28}$$

We can specialize the previous results to the H-groups obtained in Section 5.3 via loop spaces.

5.4.3 Corollary. For any two pointed spaces (W, w_0) and (X, x_0), the set $[W, \Omega X]_*$ admits a group structure such that, if $\alpha : (W, w_0) \to (W', w_0')$ is a pointed map, then the pullback along $[\alpha]_*$ defines a group homomorphism $[W', \Omega X]_* \to [W, \Omega X]_*$.

Just like we gave explicit descriptions for the products and inverses of elements of $[\Sigma X, Y]_*$ coming from the H-cogroup structure on ΣX, we also have explicit formulas for the products and inverses of elements in $[W, \Omega X]_*$ coming from the H-group structure of ΩX.

According to equation (5.26), given two pointed homotopy classes $[g_1]_*, [g_2]_* \in [W, \Omega X]_*$, their product $[g_1]_* \cdot_W [g_2]_*$ is given by the pointed homotopy class of the map $m \circ (f, g) : W \to \Omega X$. Recall that the H-multiplication m constructed in Theorem 5.3.2 sends a pair of loops $(\omega_1, \omega_2) \in \Omega X \times \Omega X$ to the loop $m(\omega_1, \omega_2) : S^1 \to X$ described by the formula

$$(m(\omega_1, \omega_2))(\exp(s)) = \begin{cases} \omega_1(\exp(2s)), & \text{if } 0 \le s \le \frac{1}{2}, \\ \omega_2(\exp(2s-1)), & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Now given two maps g_1, $g_2 : W \to \Omega X$, that is, two families of loops on (X, x_0) parameterized by W, the composite $m \circ (g_1, g_2) : W \to \Omega X$ is the family of loops which sends each point $w \in W$ to the loop $m(g_1(w), g_2(w))$ described explicitly as

$$[m(g_1(w), g_2(w))](\exp(s)) = \begin{cases} [g_1(w)](\exp(2s)), & \text{if } 0 \le s \le \frac{1}{2}, \\ [g_2(w)](\exp(2s-1)), & \text{if } \frac{1}{2} \le s \le 1; \end{cases} \tag{5.29}$$

that is, $m(g_1(w), g_2(w))$ is the loop obtained by first traversing the loop $g_1(w)$, and then traversing the loop $g_2(w)$, both with twice their usual speed.

Now, according to (5.28), the inverse $[g]_*^{-1}$ of a class $[g]_* \in [W, \Omega X]_*$ is given by the pointed homotopy class of the map $\mathrm{inv} \circ g : W \to \Omega X$. The H-inversion map $\mathrm{inv} : \Omega X \to \Omega X$ transforms a loop $\omega \in \Omega X$ into the loop $\mathrm{inv}(\omega) : S^1 \to X$ described as

$$[\mathrm{inv}(\omega)](\exp(s)) = \omega(\exp(1-s))$$

for every $s \in I$. It follows that $\mathrm{inv} \circ g : W \to \Omega X$ sends each point $w \in W$ to the loop $\mathrm{inv}(g(w))$ on (X, x_0) described explicitly as

$$[\mathrm{inv}(g(w))](\exp(s)) = [g(w)](\exp(1-s)) \tag{5.30}$$

for every $s \in I$.

So far we have studied H-groups and H-cogroups separately, but our main interest in this section is on the results obtained by studying the interaction between these two concepts. We know that we have a group structure on sets of morphisms coming *out* of a cogroup object, as well a group structure on sets of morphisms coming *into* a group object. If we then look at the morphisms from a cogroup object to a group object, we have two possible group structures *a priori*, but according to Proposition A.2.3, these two structures coincide and are automatically commutative. Applying this to H-cogroups and H-groups gives us the next result.

5.4.4 Proposition. Let $((X, x_0), \mu, \nu)$ be an H-cogroup, and let $((Y, y_0), m, \mathrm{inv})$ be an H-group. Denote by $\cdot_Y$ the product on $[X, Y]_*$ induced by the cogroup object obtained from (X, x_0), and denote by $\cdot_X$ the product on $[X, Y]_*$ induced by the group object obtained from (Y, y_0). Then these two products coincide, and they define an abelian group structure on the set $[X, Y]_*$.

This has immediate consequences for the homotopy groups of H-groups.

5.4.5 Corollary. If $((X, x_0), m, \mathrm{inv})$ is an H-group, then its homotopy groups $\pi_n(X, x_0)$ are abelian for all integers $n \ge 1$.

Proof. By definition, $\pi_n(X, x_0)$ is the group of pointed homotopy classes $[S^n, X]_*$ obtained by equipping S^n with an H-cogroup structure induced via the homeomorphism $S^n \cong \Sigma S^{n-1}$. The result then follows immediately from Proposition 5.4.4. ■

The simplest examples of H-groups are those where the homotopical conditions are satisfied strictly, also known as topological groups, so the previous result immediately implies the next corollary.

5.4.6 Corollary. The homotopy groups of a topological group based at any point are all abelian. In particular, the homotopy groups of the circle S^1 based at any point are all abelian.

Of course, using other tools, like covering space theory, we can show that $\pi_1(S^1, p)$ is in fact isomorphic to $\mathbb{Z}$ for any choice of basepoint p, but it is still interesting that the fact that $\pi_1(S^1, p)$ is abelian holds for purely formal reasons. As we will see in the next chapter, the higher homotopy groups $\pi_n(S^1, p)$ for $n \geq 2$ are in fact trivial, so their commutativity is not very surprising.

We finish this section by showing how Corollary 5.4.5 can be used to show that the higher homotopy groups of *any* space are commutative. The idea is that, given a pointed homotopy class $[f]_* \in \pi_n(X, x_0)$, if we look at the representing pointed map $f : (S^n, *_{S^n}) \to (X, x_0)$, and we identify S^n with ΣS^{n-1}, then we can use the Eckmann-Hilton Duality of Corollary 3.4.6 to obtain a pointed map of type $(S^{n-1}, *_{S^{-1}}) \to (\Omega X, \omega_{x_0})$, which gives rise to an element of the homotopy group $\pi_{n-1}(\Omega X, \omega_{x_0})$. This suggests that there is a bijection $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, \omega_{x_0})$. This is indeed true, and moreover, if $n \geq 2$, this is not only a bijection but also an isomorphism of groups. In order to show this, however, we need to take a slight detour to understand how the pointed exponential adjoint interacts with pointed homotopies.

5.4.7 Proposition. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed spaces, and suppose Y is locally compact and Hausdorff.

1. If $f_1,\ f_2 : (X \wedge Y, *) \to (Z, z_0)$ are pointed homotopic, then the pointed exponential adjuncts $\lambda^* f_1,\ \lambda^* f_2 : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$ are pointed homotopic.

2. If $g_1,\ g_2 : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$ are pointed homotopic, then the inverse exponential adjuncts $(\lambda^*)^{-1} g_1,\ (\lambda^*)^{-1} g_2 : (X \wedge Y, *) \to (Z, z_0)$ are pointed homotopic.

Proof. 1. Recall that, given a pointed map $f : (X \wedge Y, *) \to (Z, z_0)$, its pointed exponential adjunct can be described as the composition

$$\lambda^* f = \mathrm{Map}_*(Y, f) \circ \iota_X^*,$$

where $\iota_X^* : (X, x_0) \to \mathrm{Map}_*(Y, X \wedge Y)$ is the unit morphism which associates to each point $x \in X$ the pointed map $\iota_X^*(x) : (Y, y_0) \to (X \wedge Y, *)$ defined by $[\iota_X^*(x)](y) := [x, y]$. Since $f_1 \simeq_* f_2$, and Y is locally compact Hausdorff by hypothesis, we know from

Proposition 4.2.11 that there is also a pointed homotopy $\mathrm{Map}_*(Y, f_1) \simeq_* \mathrm{Map}_*(Y, f_2)$. If we then recall that composition preserves pointed homotopies, we see that

$$\begin{aligned} \lambda^* f_1 &= \mathrm{Map}_*(Y, f_1) \circ \iota_X^* \\ &\simeq_* \mathrm{Map}_*(Y, f_2) \circ \iota_X^* \\ &= \lambda^* f_2. \end{aligned}$$

2. Recall that, given a pointed map $g : (X, x_0) \to (\mathrm{Map}_*(Y, Z), \mathrm{ct}_{Y,z_0})$, its inverse pointed exponential adjunct can be described as the composition

$$(\lambda^*)^{-1} g = \mathrm{ev}^*_{Y,Z} \circ (g \wedge \mathrm{id}_Y),$$

where $\mathrm{ev}^*_{Y,Z} : \mathrm{Map}_*(Y, Z) \wedge Y \to Z$ is the counit of the pointed exponential adjunction obtained by factoring the usual evaluation $\mathrm{ev}_{Y,Z}$ through the quotient map defining the smash product in question. Since $g_1 \simeq_* g_2$, by Corollary 4.2.10 we deduce that $g_1 \wedge \mathrm{id}_Y \simeq_* g_2 \wedge \mathrm{id}_Y$ also holds. Combining this with the compatibility of composition with pointed homotopies we see that

$$\begin{aligned} (\lambda^*)^{-1} g_1 &= \mathrm{ev}^*_{Y,Z} \circ (g_1 \wedge \mathrm{id}_Y) \\ &\simeq_* \mathrm{ev}^*_{Y,Z} \circ (g_2 \wedge \mathrm{id}_Y) \\ &= (\lambda^*)^{-1} g_2. \end{aligned}$$ ∎

As a corollary, we deduce that the pointed exponential adjunction descends to a natural bijection between the sets of pointed homotopy classes.

5.4.8 Corollary. Given pointed spaces (X, x_0), (Y, y_0) and (Z, z_0), if Y is locally compact Hausdorff, then the pointed exponential adjunction descends to a bijection

$$[X \wedge Y, Z]_* \cong [X, \mathrm{Map}_*(Y, Z)]_*$$

which depends naturally on (X, x_0) and (Y, y_0).

Proof. Consider the function $\mathrm{Ho}(\lambda^*) : [X \wedge Y, Z]_* \to [X, \mathrm{Map}_*(Y, Z)]_*$ defined as

$$\mathrm{Ho}(\lambda^*)([f]_*) \coloneqq [\lambda^* f]_* \quad \forall [f]_* \in [X \wedge Y, Z]_*.$$

The result of Proposition 5.4.7 guarantees that $\mathrm{Ho}(\lambda^*)$ is a well-defined function.

We can also define a function $\mathrm{Ho}((\lambda^*)^{-1}) : [X, \mathrm{Map}_*(Y, Z)]_* \to [X \wedge Y, Z]_*$ in the opposite direction as

$$\mathrm{Ho}((\lambda^*)^{-1})([g]_*) \coloneqq [(\lambda^*)^{-1} g]_* \quad \forall [g]_* \in [X, \mathrm{Map}_*(Y, Z)]_*.$$

This is well-defined by virtue of Proposition 5.4.7 again.

The fact that $\mathrm{Ho}(\lambda^*)$ and $\mathrm{Ho}((\lambda^*)^{-1})$ are inverse to one another follows from the fact that λ^* and $(\lambda^*)^{-1}$ are inverse to one another, while the naturality of $\mathrm{Ho}(\lambda^*)$ follows from the corresponding naturality of λ^*. ∎

We are particularly interested in applying this result to the case where the locally compact Hausdorff space is the circle S^1. Recall that the Eckmann-Hilton Duality (Corollary 3.4.6) gives us a bijection

$$\mathsf{Top}_*(\Sigma X, Y) \cong \mathsf{Top}_*(X, \Omega Y)$$

depending naturally on both (X, x_0) and (Y, y_0). We briefly recall its construction, since it will be used in the next result. The first step consists of identifying the reduced suspension ΣX with the smash product $X \wedge S^1$ by means of the only pointed homeomorphism $\psi_X : X \wedge S^1 \to \Sigma X$ satisfying the equation

$$\psi_X([x, \exp(s)]) = [x, s]$$

for any $x \in X$ and $s \in I$.

Pulling back along this homeomorphism gives us a bijection

$$\mathsf{Top}_*(\Sigma X, Y) \cong \mathsf{Top}_*(X \wedge S^1, Y),$$

and then composing this with the pointed exponential adjunction gives us the desired bijection, which we denote by EH temporarily[1], as shown below.

$$\begin{tikzcd} \mathsf{Top}_*(\Sigma X, Y) \arrow[r, "\mathsf{Top}_*(\psi_X, Y)"] \arrow[rr, bend right, dashed, "\mathrm{EH}"'] & \mathsf{Top}_*(X \wedge S^1, Y) \arrow[r, "\lambda^*"] & \mathsf{Top}_*(X, \Omega Y) \end{tikzcd}$$

Explicitly, given a pointed map $f : \Sigma X \to Z$, the adjunct $\mathrm{EH}(f) : X \to \Omega Z$ sends a point $x \in X$ to the loop $\mathrm{EH}(f)(x)$ on (Z, z_0) described by the formula

$$[\mathrm{EH}(f)(x)](\exp(s)) = f([x, s])$$

for every $s \in I$.

This bijection continues to hold on the homotopical level, but we want to understand how it interacts with the extra algebraic structure present on this level.

5.4.9 Proposition. Given pointed spaces (X, x_0) and (Y, y_0), the Eckmann-Hilton Duality induces an isomorphism of groups

$$[\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*,$$

where the group structure on the left comes from the H-cogroup structure on ΣX, while the group structure on the right comes from the H-group structure on ΩY.

[1] EH here stands for Eckmann-Hilton.

Proof. The bijection comes from applying the bijection EH of the Eckmann-Hilton Duality at the level of pointed homotopy classes, that is, we consider the function

$$\overline{\mathrm{EH}} : [\Sigma X, Y]_* \to [X, \Omega Y]_*$$

defined as

$$\overline{\mathrm{EH}}([f]_*) := [\mathrm{EH}(f)]_* = [\lambda^*(f \circ \psi_X)]_*.$$

This is well-defined, because if $f \simeq_* f'$, then also $f \circ \psi_X \simeq_* f' \circ \psi_X$, and thus $\lambda^*(f \circ \psi_X) \simeq_* \lambda^*(f' \circ \psi_X)$ by virtue of Proposition 5.4.7. The fact that $\overline{\mathrm{EH}}$ is a bijection follows from the fact that EH itself is a bijection.

The only thing left to show is that $\overline{\mathrm{EH}}$ is a group homomorphism. Given pointed homotopy classes $[f_1]_*$, $[f_2]_* \in [\Sigma X, Y]_*$, we must show the equality

$$\overline{\mathrm{EH}}([f_1]_* \cdot_Y [f_2]_*) = \overline{\mathrm{EH}}([f_1]_*) \cdot_X \overline{\mathrm{EH}}([f_2]_*).$$

Unpacking the definition of $\overline{\mathrm{EH}}$ as well as the definitions of the products $\cdot_Y$ and $\cdot_X$, the expression above can be rewritten as

$$[\mathrm{EH}(\langle f_1, f_2\rangle \circ \mu)]_* = [m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2))]_*,$$

which is equivalent to the following homotopical condition:

$$\mathrm{EH}(\langle f_1, f_2\rangle \circ \mu) \simeq_* m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2)). \tag{5.31}$$

The proof that this homotopical condition holds consists of simply comparing both sides of the expression. ON the left-hand side, it follows from our discussion before the statement of the proposition (or from equation (3.11)), that the adjunct $\mathrm{EH}(\langle f_1, f_2\rangle \circ \mu) : X \to \Omega Y$ sends any point $x \in X$ to a loop on (Y, y_0) given by the formula

$$(\mathrm{EH}(\langle f_1, f_2\rangle \circ \mu)(x))(\exp(s)) = (\langle f_1, f_2\rangle \circ \mu)([x, s]).$$

We can then use equation (5.12) to conclude that the equality

$$[\mathrm{EH}(\langle f_1, f_2\rangle \circ \mu)(x)](\exp(s)) = \begin{cases} f_1([x, 2s]), & \text{if } 0 \le s \le \frac{1}{2}, \\ f_2([x, 2s-1]), & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

holds for every $s \in I$.

Now we analyze the right-hand side of (5.31). We know from (5.29) that $m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2)) : X \to \Omega Y$ sends any point $x \in X$ to the loop on (Y, y_0) given by the formula

$$[(m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2))(x)](\exp(s)) = \begin{cases} [\mathrm{EH}(f_1)(x)](\exp(2s)), & \text{if } 0 \le s \le \frac{1}{2}, \\ [\mathrm{EH}(f_2)(x)](\exp(2s-1)), & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

If we then apply equation (3.11) to rewrite the parts of the expression above involving $\mathrm{EH}(f_1)$ and $\mathrm{EH}(f_2)$ we conclude that the equality

$$[(m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2)))(x)](\exp(s)) = \begin{cases} f_1([x, 2s]), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f_2([x, 2s-1]), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Comparing the end results we see that, for any $x \in X$, we have the equality of loops

$$\mathrm{EH}(\langle f_1, f_2 \rangle \circ \mu)(x) = (m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2)))(x),$$

from which we deduce the equality of maps

$$\mathrm{EH}(\langle f_1, f_2 \rangle \circ \mu) = m \circ (\mathrm{EH}(f_1), \mathrm{EH}(f_2)).$$

This means that the homotopical relation (5.31) we wanted to prove holds strictly in fact, and not simply up to homotopy. ∎

As promised, a corollary of this is that all the homotopy groups of a space are in bijection with a lower dimensional homotopy groups of its loop space.

5.4.10 Corollary. Given a pointed space (X, x_0), for every integer $n \geq 1$, there is an isomorphism of groups

$$\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, \omega_{x_0})$$

depending naturally on (X, x_0). Consequently, the homotopy groups $\pi_n(X, x_0)$ are abelian for all integers $n \geq 2$.

Proof. Let $\Phi : \Sigma S^{n-1} \to S^n$ be the pointed homeomorphism of Proposition 3.4.7. Since the H-cogroup structure on S^n comes from this identification with a reduced suspension, by pulling back along the pointed homotopy class $[\Phi]_*$ we obtain a natural isomorphism of groups

$$\pi_n(X, x_0) = [S^n, X]_* \cong [\Sigma S^{n-1}, X]_*.$$

If we combine this with the natural isomorphism

$$[\Sigma S^{n-1}, X]_* \cong [S^{n-1}, \Omega X]_*$$

of Proposition 5.4.9 we conclude that there exists a natural isomorphism of groups

$$\pi_n(X, x_0) \cong [S^{n-1}, \Omega X]_*.$$

The last step is to identify the group on right with $\pi_{n-1}(\Omega X, \omega_{x_0})$. The group isomorphism $[\Sigma S^{n-1}, X]_* \cong [S^{n-1}, \Omega X]_*$ from Proposition 5.4.9 considers $[S^{n-1}, \Omega X]_*$ equipped with the group structure induced by the H-group structure on ΩX, but the homotopy group $\pi_{n-1}(\Omega X, \omega_{x_0})$ comes from equipping $[S^{n-1}, \Omega X]_*$ with the group structure induced by the H-cogroup structure on S^{n-1}. Luckily, these two group structures coincide by virtue of Proposition 5.4.4, and we really end up with a natural isomorphism of groups

$$\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, \omega_{x_0}).$$

The fact that $\pi_n(X, x_0)$ is abelian for $n \geq 2$ then follows from the fact that the homotopy groups $\pi_{n-1}(\Omega X, \omega_{x_0})$ are all abelian due to ΩX being an H-group (Corollary 5.4.5). ■

5.5 Change of basepoint

This section is concerned with addressing an issue raised in the section where we defined homotopy groups: how do different choices of basepoints affect the homotopy groups. In summary, we show in the present section that the data of the homotopy groups can be "transported along paths", so that a path $\gamma : I \to X$ from a point x_0 to a point x_1 gives rise to a group homomorphism $\pi_n(X, x_0) \to \pi_n(X, x_1)$ for every integer $n \geq 1$. By further exploring the properties of this transport procedure we eventually show that the homotopy groups are constant (up to isomorphism!) on the path-components of a space. This does not mean that we can always forget the basepoints, but it does make our life easier in many cases.

The next result is the main tool used in the construction of this transport procedures.

5.5.1 Proposition. Fix an integer $n \geq 0$, let X be any space, and suppose we are given a homotopy $h : S^n \times I \to X$. If there exists a map $F : D^{n+1} \to X$ such that $F(x) = h(x, 0)$ for every $x \in S^n$, then there exists a homotopy $H : D^{n+1} \times I \to X$ with the following properties:

1. $H(x, 0) = F(x)$ for every $x \in D^{n+1}$,

2. $H(x, t) = h(x, t)$ for every $x \in S^n$ and every $t \in I$.

In other words, the homotopy $h : S^n \times I \to X$ can be extended to a homotopy $H : D^{n+1} \times I \to X$ as soon as its initial stage $h_0 : S^n \to X$ can be extended to a map $F : D^{n+1} \to X$.

Proof. The two numbered conditions above impose restrictions on the behavior of the extension H on the subspace $(D^{n+1} \times \{0\}) \cup (S^n \times I) \subseteq D^{n+1} \times I$. Recall that there is a homeomorphism of pairs

$$u : (D^{n+1} \times I, D^{n+1} \times \{0\}) \to (D^{n+1} \times I, (D^{n+1} \times \{0\}) \cup (S^n \times I)),$$

so we can work with the subspace $D^{n+1} \times \{0\}$ instead of $(D^{n+1} \times \{0\}) \cup (S^n \times I)$.

First, let $\varphi : (D^{n+1} \times \{0\}) \cup (S^n \times I)$ be defined as follows:

$$\varphi(x,t) := \begin{cases} F(x), & \text{if } t = 0, \\ h(x,t), & \text{if } x \in S^n. \end{cases}$$

The fact that $F|_{S^n} = h_0$ ensures that φ is well-defined, and the Pasting Lemma then implies its continuity.

Using φ and the homeomorphism u we define $\psi : D^{n+1} \times \{0\}$ as the composition

$$\psi := \varphi \circ u|_{D^{n+1} \times \{0\}}.$$

Since u maps $D^{n+1} \times \{0\}$ to the subspace $(D^{n+1} \times \{0\}) \cup (S^n \times I)$ where φ is defined, the composition above is well-defined.

It is easy to extend ψ to a homotopy $\Psi : D^{n+1} \times I \to X$, we just project $D^{n+1} \times I$ down to $D^{n+1} \times \{0\}$ and apply ψ, that is, we define $\Psi : D^{n+1} \times I \to X$ by the formula

$$\Psi(x,t) := \psi(x,0) \quad \forall\, (x,t) \in D^{n+1} \times I.$$

Now we undo the effect of u by defining $H : D^{n+1} \times I \to X$ via the composition

$$H := \Psi \circ u^{-1}.$$

The only thing left is checking that H satisfies the required conditions.

1. For any $(x,0) \in D^{n+1} \times \{0\}$, the inverse image $u^{-1}(x,0)$ is of the form $(x',0)$ for some $x' \in D^{n+1}$, therefore

$$\begin{aligned} H(x,0) &= \Psi(u^{-1}(x,0)) \\ &= \Psi(x',0) \\ &= \psi(x',0) \\ &= \varphi(u(x',0)) \\ &= \varphi(x,0) \\ &= F(x). \end{aligned}$$

2. For any $(x,t) \in S^n \times I$, its inverse image $u^{-1}(x,t)$ is of the form $(x'',0)$ for some $x'' \in D^{n+1}$, therefore

$$\begin{aligned} H(x,t) &= \Psi(u^{-1}(x,t)) \\ &= \Psi(x'',0) \\ &= \psi(x'',0) \\ &= \varphi(u(x'',0)) \\ &= \varphi(x,t) \\ &= h(x,t). \end{aligned}$$

■

Arguing cell by cell we can generalize this result to the case where (X, A) is any n-cellular pair.

5.5.2 Corollary. Let (X, A) be an n-cellular pair, Y an arbitrary space, and consider a homotopy $h : A \times I \to Y$. If $F : X \to Y$ is a map such that the equality $F(a) = h(a, 0)$ holds for every $a \in A$, then there exists a homotopy $H : X \times I \to Y$ satisfying the following conditions:

1. $H(x, 0) = F(x)$ for every $x \in X$;
2. $H(a, t) = h(a, t)$ for every $(a, t) \in A \times I$.

This can be further generalized to any relative CW-complex (X, A) by induction over the skeletal filtration.

5.5.3 Corollary. Let (X, A) be a relative CW-complex, Y be an arbitrary space, and consider a homotopy $h : A \times I \to Y$. If $F : X \to Y$ is a map such that the equality $F(a) = h(a, 0)$ holds for every $a \in A$, then there exists a homotopy $H : X \times I \to Y$ satisfying the following conditions:

1. $H(x, 0) = F(x)$ for every $x \in X$;
2. $H(a, t) = h(a, t)$ for every $(a, t) \in A \times I$.

Recall that we proved in Example 1.2.5 that the pair $(S^n, *_{S^n})$ is n-cellular for every integer $n \geq 1$. We can then specialize Corollary 5.5.2 to this particular case keeping in mind that a homotopy $\{*_{S^n}\} \times I \to X$ is equivalent to a map $I \to X$.

5.5.4 Corollary. Let X be any space, and consider a path $\gamma : I \to X$. If $f : S^n \to X$ is a map such that $f(*_{S^n}) = \gamma(0)$, then there exists a homotopy $H : S^n \times I \to X$ satisfying the following conditions:

1. $H(x, 0) = f(x)$ for every $x \in S^n$;
2. $H(*_{S^n}, t) = \gamma(t)$ for every $t \in I$.

This last result is the fundamental tool used for relating the homotopy groups for different choices of basepoints.

5.5.5 Construction. Let X be an arbitrary space, x_0, $x_1 \in X$ two different choices of basepoints, and suppose $\gamma : I \to X$ is a path from x_0 to x_1.

For every integer $n \geq 1$, we define a function $t_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ as follows: given an element $[f]_* \in \pi_n(X, x_0)$ represented by a pointed map $f : (S^n, *_{S^n}) \to (X, x_0)$, since $f(*_{S^n}) = x_0 = \gamma(0)$, by Corollary 5.5.4 there exists a homotopy $h : S^n \times I \to X$ satisfying the following conditions:

1. $h(x,0) = f(x)$ for every $x \in S^n$;

2. $h(*_{S^n}, t) = \gamma(t)$ for every $t \in I$.

Let us say that a homotopy h satisfying these conditions is *adapted to* γ *and* f. If we look at the final stage $h_1 : S^n \to X$ of this homotopy we see that

$$h_1(*_{S^n}) = h(*_{S^n}, 1) = \gamma(1) = x_1,$$

therefore we have a pointed $h_1 : (S^n, *_{S^n}) \to (X, x_1)$, and it is reasonable to define

$$t_\gamma([f]_*) \coloneqq [H_1]_* \in \pi_n(X, x_1).$$

Of course, one needs to check that this construction is well-defined, which means that it must be independent of the chosen homotopy H adapted to f and γ, and also independent of the pointed homotopy class of f. This can be shown by using the extension Corollary 5.5.2 for suitably chosen cellular pairs. For example, if $h' : S^n \times I \to X$ is another homotopy adapted to f and γ, let

$$A \coloneqq (S^n \times \{0\}) \cup (S^n \times \{1\}) \cup (\{*_{S^n}\} \times I) \subseteq S^n \times I,$$

and consider the homotopy $\phi : A \times I \to X$ defined as follows:

$$\phi((x,s),t) \coloneqq \begin{cases} h(x,t), & \text{if } s = 0, \\ h'(x,t), & \text{if } s = 1, \\ \gamma(t), & \text{if } x = *_{S^n}. \end{cases}$$

This is well-defined since both h and h' coincide with γ when restricted to $\{*_{S^n}\} \times I$. Now, since the pair $(S^n \times I, S^n \times \{0\} \cup S^n \times \{1\} \cup \{*_{S^n}\} \times I)$ is $(n+1)$-cellular according to Proposition 1.2.8, by Corollary 5.5.2 we know that ϕ can be extended to a homotopy $\Phi : (S^n \times I) \times I \to X$ if its initial stage ϕ_0 can be extended. Substituting $t = 0$ in the expression for ϕ and using the defining properties of h and h' one can show that

$$\phi((x,s),0) = \begin{cases} f(x), & \text{if } s = 0, \\ f(x), & \text{if } s = 1, \\ x_0, & \text{if } x = *_{S^n}. \end{cases}$$

This means that the map $S^n \times I \to X$ defined as $(x,s) \mapsto f(x)$ is an extension of ϕ_0, therefore we obtain the homotopy $\Phi : (S^n \times I) \times I$ satisfying

1. $\Phi((x,s),0) = f(x)$ for every $(x,s) \in S^n \times I$;

2. $\Phi((x,s),t) = \phi((x,s),t)$ if $((x,s),t) \in A \times I$.

Having such map Φ, a simple computation shows that the rule $(x, s) \mapsto \Phi((x, s), 1)$ defines a pointed homotopy from h_1 to h_1' as desired.

The next result summarizes the main properties of this transport construction.

5.5.6 Proposition. Fix an integer $n \geq 1$, and let X be an arbitrary space.

1. If γ, $\gamma' : I \to X$ are two paths from x_0 to x_1 which are homotopic as paths, then the two transport functions t_γ, $t_{\gamma'} : \pi_n(X, x_0) \to \pi_n(x, x_1)$ are equal.
2. The transport along the constant path $\mathrm{ct}_{I,x_0} : I \to X$ is equal to the identity, i.e., the equality $t_{\mathrm{ct}_{I,x_0}} = \mathrm{id}_{\pi_n(X,x_0)}$ holds.
3. If $\gamma_0 : I \to X$ is a path from x_0 to x_1, $\gamma_1 : I \to X$ is a path from x_1 to x_2, and $\gamma_0 \cdot \gamma_1 : I \to X$ denotes the concatenation of the two paths, then the equality $t_{\gamma_0 \cdot \gamma_1} = t_{\gamma_1} \circ t_{\gamma_0}$ holds.
4. If $\gamma : I \to X$ is a path from x_0 to x_1, then the transport function $t_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ defines a group homomorphism.

An immediate corollary of this is the comparison between homotopy groups based at different points in the same path-component.

5.5.7 Corollary. If X is any space, and x_0, $x_1 \in X$ are two points belonging to the same path-component of X, then there is an isomorphism of groups $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for every integer $n \geq 1$.

Proof. Let $\gamma : I \to X$ be a path from x_0 to x_1, which gives an associated transport map $t_\gamma : \pi_n(x, x_0) \to \pi_n(X, x_1)$. If $\overline{\gamma} : I \to X$ denotes the inverse of the path γ, we also have an associated transport $t_{\overline{\gamma}} : \pi_n(X, x_1) \to \pi_n(x, x_0)$. Since $\gamma \cdot \overline{\gamma}$ is homotopic as a path to ct_{I,x_0}, it follows from the above properties that

$$t_{\overline{\gamma}} \circ t_\gamma = t_{\gamma \cdot \overline{\gamma}} = t_{\mathrm{ct}_{I,x_0}} = \mathrm{id}_{\pi_n(X,x_0)}.$$

Analogously, since $\overline{\gamma} \cdot \gamma$ is homotopic as a path to ct_{I,x_1}, we also have

$$t_\gamma \circ t_{\overline{\gamma}} = t_{\overline{\gamma} \cdot \gamma} = t_{\mathrm{ct}_{I,x_1}} = \mathrm{id}_{\pi_n(X,x_1)}.$$

These two chains of equalities show that t_γ and $t_{\overline{\gamma}}$ define inverse group homomorphism, therefore there is an isomorphism $\pi_n(x, x_0) \cong \pi_n(x, x_1)$. ■

It is important to remark, however, that this isomorphism is not naturally defined, we have to specify a path between $\gamma : I \to X$ from x_0 to x_1 before obtaining an isomorphism $t_\gamma : \pi_n(X, x_0) \cong \pi_n(x, x_1)$, and if $\gamma' : I \to X$ is another path from x_0 to x_1, it is possible for the isomorphism $t_{\gamma'}$ to be different from t_γ. Of course, if γ and γ' are homotopic as paths, then t_γ and $t_{\gamma'}$ coincide by the first item of Proposition 5.5.6. In particular, if X is simply connected, then for any two points x_0, $x_1 \in X$ there is a naturally specified isomorphism $\pi_n(x, x_0) \cong \pi_n(X, x_1)$ obtained by choosing *any* path between these points.

There is a nice way to formalize this discussion in terms of group actions. Given a pointed $x_0 \in X$, let $\mathsf{Top}(I, \partial I; X, x_0)$ denote the set of paths $\gamma : I \to X$ such that $\gamma(\partial I) \subseteq \{x_0\}$, i.e., the set of loops at the point x_0. There is a natural bijection

$$\mathsf{Top}(I, \partial I; x, x_0) \cong \mathsf{Top}_*(S^1, X)$$

assigning to each loop $\gamma : I \to X$ its quotient $\overline{\gamma} : S^1 \to X$ through the exponential map $\exp : I \to S^1$, and assigning to each pointed $f : (S^1, *_{S^1}) \to (X, x_0)$ the loop $f \circ \exp : I \to X$. Moreover, this bijection transforms path homotopies of loops on the left into pointed homotopies on the right, so it induces a bijection

$$[I, \partial I; X, x_0] \cong [S^1, X]_*$$[2]

of appropriate sets of homotopy classes.

The properties of the transport maps allow us to define a function

$$[I, \partial I; X, x_0] \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$

by assigning to each pair $([\gamma], [f]_*)$ the element $t_\gamma([f]_*)$, i.e., the result of transporting the pointed homotopy class $[f]_*$ along the loop γ representing the path-homotopy class. Taking the bijection $[I, \partial I; X, x_0] \cong [S^1, X] = \pi_1(X, x_0)$, this can also be seen as a function of type

$$\pi_1(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$

that assigns to a pair of pointed homotopy classes $([\alpha]_*, [f]_*)$ the element $t_{\alpha\circ\exp}([f]_*) \in \pi_n(x, x_0)$ obtained by transporting $[f]_*$ along the path $\alpha \circ \exp : I \to X$. We will from now on use the notation $[\alpha]_* \cdot [f]_*$ to denote this operation. The next result explains why this is a good choice of notation.

5.5.8 Proposition. The function $\cdot : \pi_1(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$ defines an action of $\pi_n(X, x_0)$ on $\pi_n(X, x_0)$ by automorphisms.

Sketch of proof. We already know from Proposition 5.5.6 that, for a fixed $[\alpha]_* \in \pi_1(X, x_0)$, the transport map $t_{\alpha\circ\exp} : \pi_n(X, x_0) \to \pi_n(x, x_0)$ is a group homomorphism, or in other words, the function $[\alpha]_* \cdot (-) : \pi_n(x, x_0) \to \pi_n(x, x_0)$ mapping $[f]_*$ to $[\alpha]_* \cdot [f]_*$ is a group homomorphism.

[2] It is important to stress that the notation $[I, \partial I; X, x_0]$ on the left-hand side denotes the set of homotopy classes of *paths*, also known as *homotopies relative to* ∂I, so that each stage of the homotopy defines another loop based at x_0.

Moreover, the inverse $[\alpha]_*^{-1}$ is by definition equal to $[\alpha \circ \nu_{S^1}]_*$, and a direct verification shows that the path $(\alpha \circ \nu_{S^1}) \circ \exp : I \to X$ is homotopic as a path to $\overline{\alpha \circ \exp}$, i.e., the reverse path of $\alpha \circ \exp$. This means that the function $[\alpha]_*^{-1} \cdot (-)$ is the inverse of $[\alpha]_* \cdot (-)$, which defines therefore an automorphism of $\pi_n(X, x_0)$.

The only thing left is showing that this action is compatible with the group structure on $\pi_1(X, x_0)$. The identity of the fundamental group is $[\mathrm{ct}_{S^1,x_0}]_*$, and the corresponding path $\mathrm{ct}_{S^1,x_0} \circ \exp : I \to X$ is the constant path ct_{I,x_0}. Since we already know from Proposition 5.5.6 that the transport $t_{\mathrm{ct}_{I,x_0}}$ along the constant path is the identity $\mathrm{id}_{\pi_n(X,x_0)}$, it follows that $[\mathrm{ct}_{S^1,x_0}]_* \cdot (-)$ is equal to this same identity. Lastly, given two elements $[\alpha]_*$, $[\beta]_* \in \pi_1(X, x_0)$, their product $[f]_* \cdot [g]_*$ in $\pi_1(X, x_0)$ is given by the pointed homotopy class $[\langle \alpha, \beta \rangle \circ \mu_{S^1}]_*$. A direct computation the shows that the corresponding path

$$\langle f, g \rangle \circ \mu_{S^1} \circ \exp : I \to X$$

is in fact equal to the concatenation

$$(f \circ \exp) \cdot (g \circ \exp) : I \to X,$$

and since the transport along a concatenation is equal to the composition of the individual transports, it follows that the function $[\langle f, g \rangle \circ \mu_{S^1}]_* \cdot (-)$ is equal to the composition $([g]_* \cdot (-)) \circ ([f]_* \cdot (-))$. ■

We can then state a definition that will be crucial during our study of Obstruction Theory.

5.5.9 Definition. A topological space X is called n**-simple** for some integer $n \geq 1$ if it is path-connected, and if the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is trivial for every choice of basepoint $x_0 \in X$.

In an n-simple space, the homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ at two different basepoints x_0, x_1 are naturally identified. If γ, $\gamma' : I \to X$ are two distinct paths from x_0 to x_1, then the concatenation $\gamma \cdot \overline{\gamma'}$ defines a loop based at x_0. Now, since the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is trivial, the transport map $t_{\gamma \cdot \overline{\gamma'}} : \pi_n(X, x_0) \to \pi_n(X, x_0)$ must be equal to the identity $\mathrm{id}_{\pi_n(x,x_0)}$. If we use the properties of Proposition 5.5.6 we see that

$$t_{\gamma \cdot \overline{\gamma'}} = t_{\overline{\gamma'}} \circ t_\gamma = t_{\gamma'}^{-1} \circ t_\gamma,$$

therefore we have the equality $t_{\gamma'}^{-1} \circ t_\gamma = \mathrm{id}_{\pi_n(x,x_0)}$, which implies that $t_\gamma = t_{\gamma'}$. This shows that we can identify $\pi_n(X, x_0)$ with $\pi_n(X, x_1)$ by choosing *any* path joining the two basepoints.

CHAPTER

6

LOCALLY TRIVIAL BUNDLES

This chapter is devoted to a particular class of maps known as *locally trivial bundles*. There are two main reasons behind our interest in this class of maps:

1. locally trivial bundles have nice homotopical properties which allow us to develop some computational tools for studying homotopy groups;

2. our study of Obstruction Theory will be restricted to the analysis of the possibility or impossibility of constructing and extending sections of a locally trivial bundles.

The two reasons outlined above hopefully make it clear that we are interested in locally trivial bundles primarily for their homotopical properties, and this influences the choice of topics developed in this chapter.

After introducing the basic definitions, we investigate locally trivial bundles arising from group actions. We prove a theorem giving sufficient conditions for the orbit map of an action to be a locally trivial bundle, and use this theorem to obtain several important examples that will be useful later on. After this, we turn to the homotopical study of locally trivial bundles by first proving the crucial result of Feldbau characterizing bundles over cubes and disks, and then applying this result to study the homotopy lifting properties of bundles, as well as some properties of bundles over CW-complexes. The study of the homotopy lifting property leads to a long exact sequence relating the homotopy groups of the various spaces that make up a locally trivial bundle. We finish the chapter with some computations to illustrate the use of the long exact sequence.

Since our interest is mainly in the sections dealing with the Homotopy Theory of locally trivial bundles, the first couple of sections contain mostly either sketches of proofs, or references for complete proofs in the literature. We do however discuss in details some examples that reappear later in the text.

6.1 First definitions and examples

This first section introduces the concept of locally trivial bundles, as well as the more restricted notions of covering maps and vector bundles. We also study important examples of these concepts that will reappear in later sections.

Roughly speaking, a locally trivial bundle is a map that locally looks like the projection out of a product space into one of its factors.

6.1.1 Definition. A **locally trivial bundle** consists of a quadruple (E, B, F, p) where:

- E, B and F are spaces,
- $p : E \to B$ is a map.

These data are required to satisfy the following *local triviality condition*: every point $b \in B$ admits a neighborhood $U \subseteq B$ for which there exists a homeomorphism $\varphi_U : p^{-1}(U) \to U \times F$ that fits in the commutative triangle below.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd} \tag{6.1}$$

Each of the objects appearing in the definition above have particular names: E is called the **total space**, B is called the **base space**, F is called the **typical fiber**, and p is usually called the **projection map**. The neighborhood U appearing in the local triviality condition is called a **trivializing neighborhood**, or sometimes also a **distinguished neighborhood**, and the homeomorphism φ_U comparing $p^{-1}(U)$ with the product $U \times F$ is called a **local trivialization**.

Sometimes, instead of saying that the quadruple (E, B, F, p) is a locally trivial bundle, we also simply say that the map $p : E \to B$ defines a **locally trivial bundle with typical fiber** F.

Let us discuss the geometric meaning of this definition. If $p : E \to B$ is a locally trivial bundle with typical fiber F, then the parts of the total space E that are over sufficiently small parts of the base space B look like a product with the typical fiber. Even though E might not be globally like the product space $B \times F$, if we restrict to the part $p^{-1}(U)$ of E that is over a trivializing neighborhood $U \subseteq B$, then this part looks like a product $U \times F$ by means of the trivialization φ_U.

Notice that the condition $\pi_1 \circ \varphi_U = p$ implies that φ_U maps the fiber $p^{-1}(b)$ over a point $b \in U$ to the subspace $\{b\} \times F \subseteq U \times F$, but since φ_U is a homeomorphism by hypothesis, the restriction $\varphi_U|_{p^{-1}(U)}$ actually defines a homeomorphism between the fiber

$p^{-1}(b)$ and the space $\{b\} \times F$, and this space is itself homeomorphic to the typical fiber F. This means that, in a locally trivial bundle, *every* fiber is topologically the same as the typical fiber. We can then regard the total space E of the bundle as being obtained by gluing together a bunch of copies of the typical fiber F, but this gluing is far from being arbitrary: fibers over nearby points of the base space B are in some sense "parallel", just like the fibers of the canonical projection $B \times F \to B$ are all parallel in our usual depictions of products.

6.1.2 Example. Given any two spaces B and F, the canonical projection $\pi_1 : B \times F \to B$ defines a locally trivial bundle. Notice that, if we take $U \coloneqq B$, then $\pi_1^{-1}(U) = \pi_1^{-1}(B) = B \times F = U \times F$, and then the identity map $\mathrm{id}_{B\times F}$ defines a trivialization, since it trivially fits in the commutative triangle below.

$$\begin{tikzcd}
\pi_1^{-1}(U) = B \times F \arrow[rr, "\mathrm{id}_{B\times F}"] \arrow[dr, "\pi_1"'] & & U \times F = B \times F \arrow[dl, "\pi_1"] \\
& U = B &
\end{tikzcd}$$

This is called the **trivial bundle over B with typical fiber F**.

We have a special terminology for locally trivial bundle $p : E \to B$ that are equivalent to the trivial bundle $\pi_1 : B \times F \to F$.

6.1.3 Definition. A locally trivial bundle $p : E \to B$ with typical fiber F is said to be **trivial** if B itself is a trivializing neighborhood, that is, if there exists a homeomorphism $\varphi : E \to B \times F$ that fits in the commutative triangle below.

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "\pi_1"] \\
& B &
\end{tikzcd}$$

In this case, the homeomorphism φ is called a **global trivialization** of the bundle.

6.1.1 Bundles from group actions

In this subsection we study some useful results relating group actions to locally trivial bundles, and then use these results to deduce some important examples of locally trivial bundles that will reappear later on. The contents presented here is based mainly on (STROM, 2011, Section 15.3) and (DIECK, 2008, Section 14.1). Since our focus for the moment is on the examples, we only sketch the simpler proofs, or instead point to references in the literature when possible.

Let G be a topological group, and suppose we have a map $\rho : G \times X \to X$ defining a (left) action of G on X. There are a number of objects associated with this action. Given any point $x \in X$, the subspace

$$Gx \coloneqq \{g \cdot x \mid g \in G\}$$

is called the **orbit** of the point x. The subgroup $G_x \subseteq G$ defined as

$$G_x \coloneqq \{g \in G \mid g \cdot x = x\}$$

is called the **stabilizer of** x, or also the **isotropy group of** x. The action of G induces an equivalence relation $\sim_G$ defined as $x \sim_G x'$ if and only $x' = g \cdot x$ for some $g \in G$. The equivalence classes of this relation are precisely the orbits of the action, and the quotient space $X/\sim_G$ is aptly called the **orbit space** and is commonly denoted by X/G.

We are interested in obtaining conditions under which the canonical projection $\pi : X \to X/G$ - called the **orbit map** of the action - defines a locally trivial bundle with typical fiber G. In order to obtain nice results, however, we will need to consider a more restricted notion of locally trivial bundle by imposing a certain compatibility condition with the action that we now explain.

If $U \subseteq X/G$ is any subset, then its inverse image $\pi^{-1}(U) \subseteq X$ satisfies a *G-stability condition*: for any $x \in \pi^{-1}(U)$ and any $g \in G$, the point $g \cdot x$ still belongs to $\pi^{-1}(U)$, since $\pi(g \cdot x) = \pi(x) \in U$. This means that the action of G on X restricts to an action of G on $\pi^{-1}(U)$. Given a local trivialization $\varphi : \pi^{-1}(U) \to U \times G$ for π, since G also acts on $U \times G$ by the rule $g \cdot (b, g') \coloneqq (b, gg')$, we can ask for φ to satisfy the condition $\varphi(g \cdot x) = g \cdot \varphi(x)$. When this happens, we say that φ is a **G-equivariant local trivialization**, or more simply a **local G-trivialization**.

We can now state our goal: we want to find conditions under which every point $b \in X/G$ has a neighborhood U over which there exists a local G-trivialization $\varphi_U : \pi^{-1}(U) \to U \times G$. This of course implies that $\pi : X \to X/G$ is a locally trivial bundle with typical fiber G, but local G-trivialiaty is stronger than mere local triviality.

It turns out that this local G-triviality condition is equivalent to a combination of global and local properties of both the action and the orbit map $X \to X/G$. Roughly speaking, the local G-triviality condition is equivalent to the combination of the following conditions:

1. two global conditions on the action which ensure that the orbits of the action are all topologically equivalent to G;
2. a local condition on the projection $X \to X/G$ allowing us to continuously choose points in different orbits of the action.

The next example shows that, in general, the orbits of an action are topologically distinct, so we really need to impose restrictions on the action to avoid this problem.

6.1.4 Example. Regard the multiplicative group $\mathbb{Z}_2 = \{-1, 1\}$ as a discrete topological group, and consider the action $\rho : \mathbb{Z}_2 \times S^1 \to S^1$ defined as

$$\rho(g, (x, y)) := \begin{cases} (x, y), & \text{if } g = 1, \\ (x, -y), & \text{if } g = -1. \end{cases}$$

The orbit of a point $(x_0, y_0) \in S^1$ is the set $\{(x_0, y_0), (x_0, -y_0)\}$. Notice that, if $y_0 \neq 0$, then the orbit contains two points, but if $y_0 = 0$, then the orbit contains a single point. This means that there are topologically distinct orbits, or equivalently, that the projection $\pi : S^1 \to S^1/\mathbb{Z}_2$ has topologically distinct fibers, so there is no hope of it being a locally trivial bundle.

I am not familiar with a general condition that ensures the different orbits of the action are similar in some sense, so I will use one which is sufficient for our purpose. An action $\rho : G \times X \to X$ is said to be **free** if, for any point $x \in X$, the equality $g \cdot x = x$ only holds if $g = e$. This is equivalent to saying that all the stabilizers G_x are trivial. Notice that the action of Example 6.1.4 is not free, because the stabilizers of the points $(1, 0)$ and $(-1, 0)$ are both equal to $\mathbb{Z}_2$.

When dealing with free actions, there is more hope for the quotient $X \to X/G$ to be locally trivial. This is because in this case the orbits are *more similar* than in the case of an arbitrary action. More precisely, for any action $\rho : G \times X \to X$ and for any point $x \in X$, we always have a surjection $\rho_x : G \to Gx$ defined as

$$\rho_x(g) := g \cdot x \quad \forall\, g \in G.$$

If g_1, $g_2 \in G$ are such that $\rho_x(g_1) = \rho_x(g_2)$, then $g_1 \cdot x = g_2 \cdot x$, which can be rewritten as $(g_2^{-1} g_1) \cdot x = x$. If the action is free, this implies $g_2^{-1} g_1 = e$, therefore $g_1 = g_2$. This shows that in a free action there is always a *continuous bijection* $\rho_x : G \to Gx$ between the orbits and the group itself.

In general, there is no reason to expect ρ_x to be a homeomorphism, so we have not solved the problem of topologically distinct fibers yet, but we are closer to a solution. If the action of G on X is free, then there is an obvious inverse *function* $\rho_x^{-1} : Gx \to G$ sending $g \cdot x$ to g, that is, ρ_x^{-1} "extracts" the group elements from the points of the orbit. We want to consider actions where this extraction can be made continuously for all orbits, and we now formalize this idea.

The action $\rho : G \times X \to X$ and the projection $\pi_2 : G \times X \to X$ together determine a map

$$(\pi_2, \rho) : G \times X \to X \times X,$$

called the **shear map** of the action, which sends (g, x) to $(x, g \cdot x)$. The image of this map is the subspace

$$C(X) := \{(x, g \cdot x) \in X \times X \mid x \in X,\, g \in G\}.$$

If the action ρ is free, then its shear map is injective, and we can then consider the **translation function**[1] $t : C(X) \to G$ defined as the composition

$$t := \pi_1 \circ (\pi_2, \rho)^{-1}.$$

Explicitly, t sends a point $(x, g \cdot x)$ to $g \in G$, so it is related to the extraction process mentioned above. In principle, t is just a function, and if t happens to be continuous, we say that the action is **weakly proper**.

The next result shows that, if we want the projection $X \to X/G$ to be a locally trivial bundle, then it is reasonable to work with free and weakly proper actions.

6.1.5 Lemma. Let $\rho : G \times X \to X$ be a continuous action of the topological group G on the space X. Suppose that, for each $b \in X/G$, we can find a neighborhood $U \subseteq X/G$ over which there exists a G-equivariant local trivialization $\varphi : \pi^{-1}(U) \to U \times G$ of the canonical projection $\pi : X \to X/G$. Under these conditions, the action of G on X is both free and weakly proper.

Sketch of proof. For any space Y, notice that the action of G on $Y \times G$ defined as $g \cdot (y, h) := (y, gh)$ is free. Indeed, if $g \cdot (y, h) = (y, h)$, then $gh = h$, which implies $g = e$.

This action is also weakly proper. The translation map $t : C(Y \times G) \to G$ can be described as the composition below,

$$C(Y \times G) \longrightarrow G \times G \xrightarrow{\mathrm{id}_G \times \mathrm{inv}} G \times G \xrightarrow{\quad m \quad} G$$

where the first map is a combination of projections sending $((y, h), (y, gh))$ to (gh, h), and m and inv are the multiplication and inversion maps of G.

The result then follows by using local G-trivializations over neighborhood $U \subseteq X/G$ to compare the action ρ on $\pi^{-1}(U)$ with the action of G on $U \times G$. ■

Free and weakly proper actions solve the problem of topologically distinct fibers. We have already discussed how the map $\rho_x : G \to Gx$ defines a continuous bijection in the case of a free action. Now, if the action is moreover weakly proper, then the continuity of the translation function implies that we can continuously invert the shear map, therefore it defines a homeomorphism

$$s : X \times G \xrightarrow{\cong} C(X).$$

Restricting s gives us a homeomorphism between $\{x\} \times G$ and the subspace

$$s(\{x\} \times G) = \{(x, g \cdot x) \in X \times X \mid g \in G\} = \{x\} \times Gx.$$

Using that there are homeomorphisms $G \xrightarrow{\cong} \{x\} \times G$ and $\{x\} \times Gx \xrightarrow{\cong} Gx$ we deduce that $\rho_x : G \to Gx$ is also a homeomorphism.

[1] Some authors call t the division function of the action.

So far we know that, if the projection $X \to X/G$ is locally G-trivial, then the action is free and weakly proper, and that a free and weakly proper action has all its orbits homeomorphic to the group G itself. These two conditions are almost sufficient to guarantee that the orbit map $X \to X/G$ is *globally* G-trivial! The space X is partitioned into orbits, and each of these orbits is homeomorphic to G, so it seems like we can compare X to $(X/G) \times X$ by means of a map $\varphi : X \to (X/G) \times G$ whose restriction to the orbit Gx sends every point of the form $g \cdot x$ to the pair (Gx, g). Notice that this definition depends on choosing for each orbit $b \in X/G$ a *representing point* in X, that is, a point $x \in X$ such that $b = Gx$. Of course, *any* point on the orbit can serve as a representing point for it, and without further assumptions on the action, it is not possible to guarantee that we can continuously choose a representing point for each orbit. In order to do this, we introduce the concept of sections.

6.1.6 Definition. If $p : X \to Y$ is a map, then a **section** of p is a map $s : Y \to X$ satisfying the equation $p \circ s = \mathrm{id}_Y$. If $U \subseteq Y$ is an open subset, a section of the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is called a **local section of** p **over** U.

At the moment we are interested in constructing local sections of the orbit map $X \to X/G$. Maps out of a quotient can always be constructed by starting with a map on the "over" space, and then factoring it through the quotient. The next result is a useful lemma for defining local sections by a similar factoring process.

6.1.7 Lemma. Let $p : X \to Y$ be a surjective quotient map. Given an open subset $U \subseteq Y$, denote by $p_U : p^{-1}(U) \to U$ the restriction $p|_{p^{-1}(U)}$. The following assertions are equivalent:

1. there exists a local section $s : U \to p^{-1}(U)$ of p over U;
2. there exists a map $S : p^{-1}(U) \to p^{-1}(U)$ which is constant on the fibers of p_U and satisfies the equation $p_U \circ S = p_U$.

Proof. Suppose that $s : U \to p^{-1}(U)$ is a local section of p over U. Define $S : p^{-1}(U) \to p^{-1}(U)$ as the composition $S \coloneqq s \circ p_U$. If x_1, $x_2 \in X$ are such that $p_U(x_1) = p_U(x_2)$, then

$$S(x_1) = s(p_U(x_1)) = s(p_U(x_2)) = S(x_2),$$

showing that S is constant on the fibers of p_U. Now, using that s is a local section we see that

$$\begin{aligned} p_U \circ S &= p_U \circ s \circ p_U \\ &= \mathrm{id}_U \circ p_U \\ &= p_U. \end{aligned}$$

Conversely, suppose S satisfies the conditions in the statement. Since $p^{-1}(U)$ is an open subset of X saturated with respect to p, according to (BROWN, 2006, result 4.3.1), $p_U : p^{-1}(U) \to U$ is still a quotient map. Since S is constant on the fibers of p_U by hypothesis, it can be factored through it to define a map $s : U \to p^{-1}(U)$.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "S"] \arrow[d, "p|_{p^{-1}(U)}"'] & p^{-1}(U) \\ U \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

We claim that s is a local section of p over U. In order to see this, using the equality $p_U \circ S = p_U$ we first notice that s satisfies

$$p_U \circ s \circ p_U = p_U \circ S = p_U.$$

Now, since p_U is surjective, we can cancel it from both sides of the equation above to deduce that $p_U \circ s = \mathrm{id}_U$ as desired. ■

The concept of section solves our problem of representing points for orbits: a section of the orbit map $X \to X/G$ is precisely a way to continuously choose a point in each of the orbits. With this in mind, the next result is expected. See (DIECK, 2008, Propositions 14.1.5 and 14.1.7) for a proof.

6.1.8 Theorem. Let $\rho : G \times X \to X$ be a continuous action of a topological group G on a space X. The following assertions are equivalent:

1. the orbit map $\pi : X \to X/G$ is globally G-trivial, that is, there exists a G-equivariant homeomorphism $\varphi : X \to (X/G) \times G$ such that $\pi_1 \circ \varphi = \pi$;
2. the action is free, weakly proper, and the orbit map admits a section.

Now that we know conditions ensuring the global G-triviality of the orbit map of an action, in order to obtain conditions for local triviality we just need to reformulate the previous result in local terms.

6.1.9 Theorem. Let $\rho : G \times X \to X$ be a continuous action of a topological group G on a space X. The following assertions are equivalent:

1. the orbit map $\pi : X \to X/G$ is locally G-trivial, that is, every $b \in X/G$ has a neighborhood $U \subseteq X/G$ over which there exists a G-equivariant homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times G$ such that $\pi_1 \circ \varphi_U = \pi$;
2. the action is free, weakly proper, and every $b \in X/G$ has a neighborhood $U \subseteq X/G$ over which the orbit map admits a local section.

6.1.10 Remark. One could wonder why we did not make the apparently weaker assumptions that the action $\rho : G \times X \to X$ in Theorem 6.1.9 is only locally free and locally weakly proper. This is because these two assumption imply that ρ is globally free and weakly proper. More precisely, if we suppose that each $b \in X/G$ has a neighborhood U such that the restricted action $\rho_U : G \times \pi^{-1}(U) \to \pi^{-1}(U)$ is free and weakly proper, then it follows that ρ itself also satisfies these two properties.

The examples below show that many important locally trivial bundles in Algebraic Topology can be described as orbit maps of group actions.

6.1.11 Example. Regard the integers $\mathbb{Z}$ as a discrete topological group, and consider the action $\rho : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ defined as

$$\rho(n,t) \coloneqq t + n \quad \forall\,(n,t) \in \mathbb{Z} \times \mathbb{R}.$$

Geometrically, each $n \in \mathbb{Z}$ acts by translating the points of $\mathbb{R}$.

We now show that the orbit map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ defines a locally $\mathbb{Z}$-trivial bundle. First, it is clear that the action in question is free, since the equality $t+n = t$ only happens if $n = 0$. For the weak properness, notice that the translation function $C(\mathbb{R}) \to \mathbb{Z}$ sending $(t, t+n)$ to n can be described by simply subtracting the first coordinate from the second, therefore it is continuous. Now, given a point $[t] \in \mathbb{R}/\mathbb{Z}$ represented by a point $t \in \mathbb{R}$, consider the open neighborhood U of $[t]$ defined as $U \coloneqq p(V)$, where $V \coloneqq \left(t - \frac{1}{2}, t + \frac{1}{2}\right)$ is an open interval around t. The restriction $p|_V : V \to U$ is injective, because it $[t] = [t']$, then $t - t' = n$ for some $n \in \mathbb{Z}$, but since the distance between any two points of V is strictly smaller than 1, this equality can only happen if $n = 0$, therefore $t = t'$. This means that $p|_V$ is a continuous an open bijection from V to U, thus a homeomorphism. A local section of p over U can then be obtained as the inverse of $p|_V$.

The previous paragraph shows that the action ρ in question is free, weakly proper, and that the orbit map p admits local sections. It then follows from Theorem 6.1.9 that $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a locally $\mathbb{Z}$-trivial. And why is this particular bundle important? The answer is that it is merely an alternative description to a famous bundle, as we now show.

Since the complex exponential is $2\pi-$-periodic, the quotient map $E : \mathbb{R} \to S^1$ given by $E(t) \coloneqq e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$ is constant on the orbits of the action ρ, therefore it can be factored through the orbit map to define $\overline{E} : \mathbb{R}/\mathbb{Z} \to S^1$.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[dl, "p"'] \arrow[dr, "E"] & \\
\mathbb{R}/\mathbb{Z} \arrow[rr, dashed, "\overline{E}"'] & & S^1
\end{tikzcd}$$

Conversely, given a point $z \in S^1$ of the form $e^{2\pi i t} = E(t)$, its fiber $E^{-1}(z)$ is equal to $\{t + n \mid n \in \mathbb{Z}\}$, i.e., it is precisely the orbit of t under the action of $\mathbb{Z}$; therefore the orbit map p is constant on the fibers of the quotient map E, so it can also be factored through it to define $\overline{p} : S^1 \to \mathbb{R}/\mathbb{Z}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[dl, "p"'] \arrow[dr, "E"] & \\ \mathbb{R}/\mathbb{Z} & & S^1 \arrow[ll, dashed, "\overline{p}"] \end{tikzcd}$$

Using the defining properties of $\overline{E}$ and $\overline{p}$ it is straightforward to show that they are inverse maps. Of course, E is the famous covering of the circle by the real line, and we have simply given an alternative description of it in terms of orbit maps of group actions.

6.1.12 Example. Recall that, as a mere set, the n-dimensional projective space $\mathbb{R}P^n$ can be defined as

$$\mathbb{R}P^n := \{\ell \subseteq \mathbb{R}^{n+1} \mid \ell \text{ is a 1-dimensional linear real subspace}\}.$$

Since every 1-dimensional linear real subspace of $\mathbb{R}^{n+1}$ is generated by a non-zero vector, we have a surjective function $q_n : S^n \to \mathbb{R}P^n$ which sends each point $x \in S^n$ to the linear real subspace $\langle x \rangle$ spanned by it. We then regard $\mathbb{R}P^n$ as a topological space by equipping it with the quotient topology induced by the function q_n.

Standard results from the theory of quotient spaces imply that there is a homeomorphism $\phi : \mathbb{R}P^n \to S^n/\sim_{q_n}$ that fits in the commutative triangle below.

$$\begin{tikzcd} & S^n \arrow[dl, "q_n"'] \arrow[dr, "\pi"] & \\ \mathbb{R}P^n \arrow[rr, dashed, "\phi"'] & & S^n/\sim_{q_n} \end{tikzcd}$$

Here $\sim_{q_n}$ is the equivalence relation defined as $x \sim_{q_n} y$ if and only if $q_n(x) = q_n(y)$.

In this case, there is a very simple explicit description for $\sim_{q_n}$. If $x \sim_{q_n} y$, then $\langle x \rangle = \langle y \rangle$, so there exists a real number $t > 0$ such that $y = tx$. Taking norms of both sides we conclude that $|t| = 1$, that is, $t = \pm 1$. The equivalence $\sim_{q_n}$ can then be restated as $x \sim_{q_n} y$ if and only if $x = y$ or $y = -x$.

This equivalence relation can in fact be seen as induced by a group action. Regard $\mathbb{Z}_2$ as a discrete group, and define an action $\rho : \mathbb{Z}_2 \times S^n \to S^n$ by the formula

$$\rho(g, x) := \begin{cases} x, & \text{if } g = 1, \\ -x, & \text{if } g = -1. \end{cases}$$

The equivalence relation induced by this action is precisely $\sim_{q_n}$, so we can regard the quotient $S^n/\sim_{q_n}$ as the orbit space $S^n/\mathbb{Z}_2$ and use Theorem 6.1.9 to study the local triviality of the orbit map $\pi : S^n \to S^n/\mathbb{Z}_2$.

Since there is no point $x \in S^n$ satisfying the equation $x = -x$, the action of $-1 \in \mathbb{Z}_2$ does not fix any points of S^n, therefore the action of $\mathbb{Z}_2$ in question is free. The image of the shear map $S^n \times \mathbb{Z}_2 \to S^n \times S^n$ is the subspace

$$C(S^n) = \{(x, x) \mid x \in S^n\} \cup \{(x, -x) \mid x \in S^n\},$$

which can be written as the disjoint union of two closed subspaces. The restriction of the translation function $t : C(S^n) \to \mathbb{Z}_2$ to the first of these subspaces is constant and equal 1, and thus continuous; while the restriction to the second subspace is constant and equal to 1, therefore also continuous. It follows from the Pasting Lemma that t is continuous, so the action in question is weakly proper.

Now we show that the orbit map $\pi : S^n \to S^n/\mathbb{Z}_2$ admits local sections. It is a standard fact from the theory of topological groups that the orbit map is *always* open (see for example (BROWN, 2006, result 11.1.2)). It follows that the subset $U_j \subseteq S^n/\mathbb{Z}_2$ defined as

$$U_j \coloneqq \{[y] \in S^n/\mathbb{Z}_2 \mid y = (y_1, \ldots, y_{n+1}) \in S^n,\ y_j \neq 0\}$$

is open.

The map $S_j : \pi^{-1}(U_j) \to \pi^{-1}(U_j)$ defined as

$$S_j(x_1, \ldots, x_{n+1}) \coloneqq \frac{x_j}{|x_j|}(x_1, \ldots, x_{n+1})$$

is constant on the fibers of $\pi|_{\pi^{-1}(U_j)}$, because $S_j(-x) = S_j(x)$ holds for every $x \in \pi^{-1}(U_j)$, and it also satisfies $\pi|_{\pi^{-1}(U_j)} \circ S = \pi|_{\pi^{-1}(U_j)}$, because $S_j(x)$ is obtained from x by scalar multiplication. It follows from Lemma 6.1.7 that the map $s_j : U_j \to \pi^{-1}(U_j)$ obtained by factoring S_j through the quotient is a local section of π over U_j.

If we vary $j \in \{1, \ldots, n+1\}$, then the open subsets of $S^n/\mathbb{Z}_2$ of the form U_j define an open covering of the orbit space. This means that every orbit lies in an open subset over which we can define a local section. Theorem 6.1.9 then implies that there exists a local trivialization $\psi : U_j \times \mathbb{Z}_2 \to \pi^{-1}(U_j)$. Explicitly, this map is given by

$$\psi([x_1, \ldots, x_{n+1}], g) = \begin{cases} \dfrac{x_j}{|x_j|}(x_1, \ldots, x_{n+1}), & \text{if } g = 1, \\ -\dfrac{x_j}{|x_j|}(x_1, \ldots, x_{n+1}), & \text{if } g = -1. \end{cases}$$

Notice, moreover, that ψ is $\mathbb{Z}_2$-equivariant, because $\psi([x], -1) = -\psi([x], 1)$.

6.1.13 Example. The n-dimensional complex projective space $\mathbb{C}P^n$ has an analogous construction. As a set, it is defined as

$$\mathbb{C}P^n \coloneqq \{\ell \subseteq \mathbb{C}^{n+1} \mid \ell \text{ is a 1-dimensional complex linear subspace of } \mathbb{C}^{n+1}\}.$$

Let S be the set of unit norm vectors in $\mathbb{C}^{n+1}$. Since every 1-dimension complex linear subspace of $\mathbb{C}^{n+1}$ can be generated by a vector in S, the map $q_n : S \to \mathbb{C}P^n$ defined a $q_n(z) := \langle z \rangle$ is a surjection, and we can then equip $\mathbb{C}P^n$ with the quotient topology induced by q_n.

Like in Example 6.1.12, we can also regard $\mathbb{C}P^n$ as a quotient of S by the equivalence relation $\sim_{q_n}$ defined as $z \sim_{q_n} z'$ if and only if $q_n(z) = q_n(z')$. Using that the vectors of S have unit norm, a simple computation then shows that $z \sim_{q_n} z'$ if and only if there exists a complex number $w \in S^1$ such that $z' = w \cdot z$, where the dot denotes the usual scalar multiplication in $\mathbb{C}^{n+1}$. In other words, $\sim_{q_n}$ is the equivalence relation induced by the group action $\rho : S^1 \times S \to S$ defined as

$$\rho(w, (z_1, \ldots, z_{n+1})) := w \cdot (z_1, \ldots, z_{n+1}) = (wz_1, \ldots, wz_{n+1}).$$

We then have a homeomorphism $f : S/S^1 \to \mathbb{C}P^n$ that fits in the commutative triangle below.

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "q_n"] & \\ S/S^1 \arrow[rr, dashed, "f"'] & & \mathbb{C}P^n \end{tikzcd}$$

If $z = (z_1, \ldots, z_{n+1}) \in S$ and $w \in S^1$ are such that $w \cdot z = z$, then $wz_i = z_i$ holds for every $i \in \{1, \ldots, n+1\}$. Since $z \neq \mathbf{0}$, at least one of its coordinates, say z_i, is non-zero, and then the equality $wz_i = z_i$ implies $w = 1$; proving that the action of S^1 on S is free.

In order to prove that the action is also weakly proper, consider first the usual hermitian product $B : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ given by

$$B((w_1, \ldots, w_{n+1}), (z_1, \ldots, z_{n+1})) := \sum_{i=0}^{n+1} w_i \overline{z_i}.$$

For any $z \in S$ and $w \in S^1$ we have the equality

$$B(z, w \cdot z) = \overline{w},$$

therefore the translation function t of the action can be described as the composition of B (suitably restricted) with the complex conjugation map, which means that t is a continuous function.

Now we deal with the construction of local sections. Similar to what we did in Example 6.1.12, we consider the open subset $U_j \subseteq \mathbb{C}P^n$ defined as

$$U_j := \{[z] \in S/S^1 \mid z = (z_1, \ldots, z_{n+1}) \in S,\ z_j \neq 0\}.$$

Now let $S_j : \pi^{-1}(U_j) \to \pi^{-1}(U_j)$ be defined as

$$S_j(z_1, \ldots, z_{n+1}) := \frac{z_j}{\|z_j\|} \cdot (z_1, \ldots, z_{n+1}).$$

Reasoning like in Example 6.1.12 we can show that S_j is constant on the fibers of the orbit map and that it satisfies the equation $\pi|_{\pi^{-1}(U_j)} \circ S_j = \pi|_{\pi^{-1}(U_j)}$. It follows from

Lemma 6.1.7 that by factoring S_j through π we obtain a map $s_j : U_j \to \pi^{-1}(U_j)$ which is a local section of π over U_j.

Theorem 6.1.9 then implies that the map $\psi : U_j \times S^1 \to \pi^{-1}(U_j)$ defined as

$$\psi([z_1, \ldots, z_{n+1}], w) := \frac{wz_j}{\|z_j\|} \cdot (z_1, \ldots, z_{n+1})$$

defines an S^1-equivariant local trivialization of π, and since S/S^1 can be covered by open subsets of the form U_j for varying $j \in \{1, \ldots, n+1\}$, the projection $\pi : S \to S/S^1$ is a locally trivial bundle with typical fiber S^1, the same being true of the projection $q_n : S \to \mathbb{C}P^n$.

We now study some results closely related to Theorem 6.1.9. Suppose $\rho : G \times X \to X$ is a continuous action of a topological group G on a space X, and suppose that this action is free, weakly proper, and admits local sections, so that the orbit map $\pi : X \to X/G$ is locally trivial with typical fiber G. If $H \leq G$ is a subgroup, it becomes a topological group when equipped with the subspace topology, and it gives rise to a restricted action

$$\rho_H := \rho|_{H \times X} : H \times X \to X.$$

The orbit map $\pi_H : X \to X/H$ obtained from this restricted action might not be a locally H-trivial bundle, because it might not admit local sections. A rough explanation for this possible non-existence of local sections is that, even though every orbit of H is contained in an orbit of G, this containment might be strict, and then, even if we are able to continuously choose points in sufficiently close orbits of G, it might happen that the chosen points are outside the orbits of H.

There is, however, a map related to π and π_H that is a locally trivial bundle. The orbit map $\pi : X \to X/G$ is constant on the orbits of H, therefore it can be factored through $\pi_H : X \to X/H$ to define an induced map $p : X/H \to X/G$. Explicitly, p sends an H-orbit of the form Hx to the corresponding G-orbit Gx.

$$\begin{tikzcd} & X \arrow[dl, "\pi_H"'] \arrow[d, "\pi"] \\ X/H \arrow[r, dashed, "p"'] & X/G \end{tikzcd}$$

The next result shows that this induced map p defines locally trivial bundle.

6.1.14 Proposition. Let $\rho : G \times X \to X$ be a continuous, free and weakly proper action of the topological group G on the space X, and suppose that the orbit map $\pi : X \to X/G$ admits local sections. Given a subgroup $H \leq G$, let $\rho_H : H \times X \to X$ denote the restricted action. The map $p : X/H \to X/G$ obtained by factoring π through π_H is a locally trivial bundle with typical fiber G/H, where G/H denotes the orbit space of the action of H on G by multiplication on the right.

Sketch of proof. Let $U \subseteq X/G$ be an open subset together with a G-equivariant local trivialization $\varphi : \pi^{-1}(U) \to U \times G$. We will construct a local trivialization $\overline{\varphi} : p^{-1}(U) \to U \times (G/H)$ of p over this same subset.

Since $\pi_H^{-1}(p^{-1}(U)) = \pi^{-1}(U)$, by (BROWN, 2006, result 4.3.1) we know that the restricted projection $\pi_H|_{\pi^{-1}(U)} : \pi^{-1}(U) \to p^{-1}(U)$ is still a quotient map. If $q : G \to G/H$ denotes the orbit map given by $g \mapsto gH$, then using the G-equivariance of φ one can show that the composite map

$$(\mathrm{id}_U \times q) \circ \varphi : \pi^{-1}(U) \to U \times (G/H)$$

is constant on the fibers of $\pi_H|_{\pi^{-1}(U)}$, therefore it can be factored through this projection to define a map

$$\overline{\varphi} : p^{-1}(U) \to U \times (G/H).$$

Now let $\psi : U \times G \to \pi^{-1}(U)$ be the inverse map of the G-trivialization φ. Since $q : G \to G/H$ is an open map, being the orbit map of an action, the product $\mathrm{id}_U \times q : U \times G \to U \times (G/H)$ is an open surjection, and thus a quotient map. Using the G-equivariance of ψ we can show that $\pi_H \circ \psi : U \times G \to p^{-1}(U)$ is constant on the fibers of $\mathrm{id}_U \times q$, therefore it can be factored to define a map $\overline{\psi} : U \times (G/H) \to p^{-1}(U)$.

A straightforward computation using the defining properties of $\overline{\varphi}$ and $\overline{\psi}$ shows that these two maps are inverse to one another. Moreover, they are compatible with the maps $p : p^{-1}(U) \to U$ and $\pi_1 : U \times (G/H) \to U$, defining therefore local trivializations. ■

The next example is an application of Proposition 6.1.14 which will be relevant later for the construction of characteristic classes.

6.1.15 Example (Stiefel manifolds)**.** Let $\mathbb{F}$ denote either the field of real or complex numbers. Given integers $1 \leq k \leq n$, a **k-frame** in $\mathbb{F}^n$ is a k-tuple $(v_1, \dots, v_k) \in (\mathbb{F}^n)^k$ of linearly independent vectors. The space of all k-tuples in $\mathbb{F}^n$

$$V_k^\circ(\mathbb{F}^n) := \{(v_1, \dots, v_k) \in (\mathbb{F}^n)^k \mid (v_1, \dots, v_k) \text{ is a } k\text{-frame}\},$$

regarded as a subspace of $(\mathbb{F}^n)^k$ is called the **open Stiefel manifold of k-frames in** $\mathbb{F}^n$.

Let $\langle -, - \rangle$ denote the usual inner product on $\mathbb{F}^n$ (euclidean if $\mathbb{F} = \mathbb{R}$, and hermitian if $\mathbb{F} = \mathbb{C}$). If a k-frame $(v_1, \dots, v_k)$ is such that $\|v_i\| = 1$ for every $i \in \{1, \dots, k\}$, and $v_i \perp v_j$ for all $i \neq j$, then we say $(v_1, \dots, v_k)$ is an **orthonormal k-frame**. The space

$$V_k(\mathbb{F}^n) := \{(v_1, \dots, v_k) \in V_k^\circ(\mathbb{F}^n) \mid (v_1, \dots, v_k) \text{ is an orthonormal } k\text{-frame}\}$$

is called the **Stiefel manifold of orthonormal k-frames in** $\mathbb{F}^n$. Notice that $V_k(\mathbb{F}^n)$ is a closed subspace of the product of spheres $(S^{dn-1})^k$, where $d = 1$ if $\mathbb{F} = \mathbb{R}$, and $d = 2$ if $\mathbb{F} = \mathbb{C}$; therefore $V_k(\mathbb{F}^n)$ is compact.

The Gram-Schmidt orthonormalization algorithm defines a map $\mathrm{GS} : V_k^\circ(\mathbb{F}^n) \to V_k(\mathbb{F}^n)$. Since the algorithm does nothing when applied to a k-frame that is already orthonormal, GS is a *retraction*.

Now given another integer $1 \leq j \leq k$, there is a projection map $q_{k,j} : V_k(\mathbb{F}^n) \to V_j(\mathbb{F}^n)$ defined as

$$q_{k,j}(v_1, \ldots, v_k) \coloneqq (v_1, \ldots, v_j).$$

We want to show that $q_{k,j}$ is a locally trivial bundle. In order to do this, we give an alternative description of $V_k(\mathbb{F}^n)$ in terms of group actions.

The group $O(\mathbb{F}^n)$ of orthogonal transformation of $\mathbb{F}^n$ to itself acts on $V_k(\mathbb{F}^n)$ via the map $\rho : O(\mathbb{F}^n) \times V_k(\mathbb{F}^n) \to V_k(\mathbb{F}^n)$ defined as

$$\rho(T, (v_1, \ldots, v_k)) = T \cdot (v_1, \ldots, v_k) \coloneqq (T(v_1), \ldots, T(v_k)).$$

If $e_1, \ldots, e_n$ are the canonical basis vectors of $\mathbb{F}^n$, we let $e \coloneqq (e_1, \ldots, e_k) \in V_k(\mathbb{F}^n)$. If $(v_1, \ldots, v_k)$ is any orthonormal k-frame in $\mathbb{F}^n$, there exists an orthogonal transformation $T \in O(\mathbb{F}^n)$ such that $(v_1, \ldots, v_k) = T \cdot e$. This means that the action ρ is transitive, and that the map $\rho_e : O(\mathbb{F}^n) \to V_k(\mathbb{F}^n)$ defined as

$$\rho_e(T) \coloneqq T \cdot e = (T(e_1), \ldots, T(e_k))$$

is surjective.

The isotropy subgroup $O(\mathbb{F}^n)_e \leq O(\mathbb{F}^n)$ consists of the orthogonal transformations $T \in O(\mathbb{F}^n)$ satisfying $T(e_i) = e_i$ for every $i \in \{1, \ldots, k\}$. If T_1, $T_2 \in O(\mathbb{F}^n)$ are such that $\rho_e(T_1) = \rho(T_2)$, then the orthogonal transformation $T_2^{-1} \circ T_1$ belongs to $O(\mathbb{F}^n)_e$. In other words, if we let $O(\mathbb{F}^n)_e$ act on $O(\mathbb{F}^n)$ by multiplication (= composition) *on the right*, then $\rho_e(T_1) = \rho_e(T_2)$ if and only if T_1 and T_2 are in the same orbit under this action of $O(\mathbb{F}^n)_e$. We can then factor ρ_e through the orbit map $\pi : O(\mathbb{F}^n) \to O(\mathbb{F}^n)/O(\mathbb{F}^n)_e$ to obtain a map $\phi_{n,k} : O(\mathbb{F}^n)/O(\mathbb{F}^n)_e \to V_k(\mathbb{F}^n)$.

$$\begin{tikzcd}
O(\mathbb{F}^n) \arrow[r, "\rho_e"] \arrow[d, "\pi"'] & V_k(\mathbb{R}^n) \\
O(\mathbb{F}^n)/O(\mathbb{F}^n)_e \arrow[ur, dashed, "\phi_{n,k}"'] &
\end{tikzcd}$$

Since ρ_e is surjective, and $O(\mathbb{F}^n)/O(\mathbb{F}^n)_e$ is compact, the induced map $\phi_{n,k}$ is a homeomorphism.

There is a more familiar description of the isotropy group $O(\mathbb{F}^n)_e$. There is an "inclusion map" $i : O(\mathbb{F}^{n-k}) \to O(\mathbb{F}^n)$ mapping an orthogonal transformation $T : \mathbb{F}^{n-k} \to \mathbb{F}^{n-k}$ to the orthogonal transformation $\widehat{T} : \mathbb{F}^n \to \mathbb{F}^n$ defined as

$$\widehat{T}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \coloneqq (x_1, \ldots, x_k, T(x_{k+1}, \ldots, x_n)),$$

where we are making the harmless identification $\mathbb{F}^n \cong \mathbb{F}^k \oplus \mathbb{F}^{n-k}$. Notice that $\widehat{T}$ fixes the first k basis vectors of $\mathbb{F}^n$, that is, $\widehat{T} \in O(\mathbb{F}^n)_e$, and the inclusion $O(\mathbb{F}^{n-k}) \hookrightarrow O(\mathbb{F}^n)$ just described establishes a homeomorphism $O(\mathbb{F}^n)_e \cong O(\mathbb{F}^{n-k})$. For this reason, we usually write the quotient $O(\mathbb{F}^n)/O(\mathbb{F}^n)_e$ as $O(\mathbb{F}^n)/O(\mathbb{F}^{n-k})$, so that we have a homeomorphism

$$\phi_{n,k} : O(\mathbb{F}^n)/O(\mathbb{F}^{n-k}) \xrightarrow{\cong} V_k(\mathbb{F}^n).$$

For an integer $1 \leq j \leq k$, we also have the analogous inclusion $O(\mathbb{F}^{n-j}) \hookrightarrow O(\mathbb{F}^n)$ identifying $O(\mathbb{F}^{n-j})$ with the orthogonal transformations in $O(\mathbb{F}^n)$ that fix the first j canonical basic vectors of $\mathbb{F}^n$. With these identifications in mind, we have the subgroup inclusions $O(\mathbb{F}^{n-k}) \leq O(\mathbb{F}^{n-j}) \leq O(\mathbb{F}^n)$, and therefore we have an induced map

$$p_{k,j} : O(\mathbb{F}^n)/O(\mathbb{F}^{n-k}) \to O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})$$

fitting in the commutative triangle below.

$$\begin{tikzcd}
 & O(\mathbb{F}^n) \arrow[dl, "\pi_{n,k}"'] \arrow[dr, "\pi_{n,j}"] & \\
O(\mathbb{F}^n)/O(\mathbb{F}^{n-k}) \arrow[rr, dashed, "p_{k,j}"'] & & O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})
\end{tikzcd}$$

A direct computation shows that $p_{k,j}$ fits into the commutative square below.

$$\begin{tikzcd}
O(\mathbb{F}^n)/O(\mathbb{F}^{n-k}) \arrow[r, "\phi_{n,k}"] \arrow[d, "p_{k,j}"'] & V_k(\mathbb{F}^n) \arrow[d, "q_{k,j}"] \\
O(\mathbb{F}^n)/O(\mathbb{F}^{n-j}) \arrow[r, "\phi_{n,j}"'] & V_j(\mathbb{F}^n)
\end{tikzcd}$$

This means that we can study the local triviality of $q_{k,j}$ by instead studying the local triviality of $p_{k,j}$, and for this we have Proposition 6.1.14 at our disposal.

The action of a subgroup on the ambient group by multiplication on the left or on the right is always free and weakly proper, therefore the action of $O(\mathbb{F}^{n-j})$ on $O(\mathbb{F}^n)$ is free and weakly proper. We now show that the orbit map $\pi : O(\mathbb{F}^n) \to O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})$ admits local sections. Fix $n-j$ vectors $w_1, \ldots, w_{n-j} \in \mathbb{F}^n$. Using the continuity of the determinant we can show that the subset $V \subseteq V_j(\mathbb{F}^n)$ defined as

$$V := \{(v_1, \ldots, v_j) \in V_j(\mathbb{F}^n) \mid \langle v_1, \ldots, v_j, w_1, \ldots, w_{n-j} \rangle = \mathbb{F}^n\}$$

is open, therefore $U := \phi_{n,j}^{-1}(V)$ is an open subset of the orbit space $O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})$. The inverse image $\pi^{-1}(U)$ consists of those orthogonal transformations $T \in O(\mathbb{F}^n)$ such that

$$\langle T(e_1), \ldots, T(e_j), w_1, \ldots, w_{n-j} \rangle = \mathbb{F}^n.$$

Define a map $\Sigma : \pi^{-1}(U) \to \pi^{-1}(U)$ by setting $\Sigma(T)$ to be the unique orthogonal transformation satisfying

$$(\Sigma(T)(e_1), \ldots, \Sigma(T)(e_n)) = \mathrm{GS}(T(e_1), \ldots, T(e_j), w_1, \ldots, w_{n-j}).$$

Notice that Σ is constant on the orbits of $O(\mathbb{F}^{n-j})$. Indeed, if $S \in O(\mathbb{F}^{n-j})$, that is, if S fixes the first j basic vectors, then

$$\mathrm{GS}((T \circ S)(e_1), \ldots, (T \circ S)(e_j), w_1, \ldots, w_{n-j}) = \mathrm{GS}(T(e_1), \ldots, T(e_j), w_1, \ldots, w_{n-j}),$$

which means that $\Sigma(T \circ S) = \Sigma(T)$. Moreover, since $(T(e_1), \ldots, T(e_j))$ is already an orthonormal j-frame, the Gram-Schmidt algorithm does not change the first j components of the n-frame $(T(e_1), \ldots, T(e_j), w_1, \ldots, w_{n-j})$. This means that the orthogonal transformation $T^{-1} \circ \Sigma(T)$ fixes the first j basic vectors, i.e., $T^{-1} \circ \Sigma(T) \in O(\mathbb{F}^{n-j})$. It follows that $\Sigma(T)$ and T are in the same orbit of $O(\mathbb{F}^{n-j})$, therefore $\pi(\Sigma(T)) = \pi(T)$ holds for every $T \in \pi^{-1}(U)$. Lemma 6.1.7 implies that Σ can be factored through π to define a local section $s : U \to \pi^{-1}(U)$ of the orbit map $\pi : O(\mathbb{F}^n) \to O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})$.

We have shown so far that the action of $O(\mathbb{F}^{n-j})$ on $O(\mathbb{F}^n)$ is free, weakly proper and admits local sections. If we then apply Proposition 6.1.14 to the subgroup $O(\mathbb{F}^{n-k}) \leq O(\mathbb{F}^{n-j})$ we deduce that the induced map

$$p_{k,j} : O(\mathbb{F}^n)/O(\mathbb{F}^{n-k}) \to O(\mathbb{F}^n)/O(\mathbb{F}^{n-j})$$

is a locally trivial bundle whose typical fiber is the quotient space $O(\mathbb{F}^{n-j})/O(\mathbb{F}^{n-k})$, which is homeomorphic to the Stiefel manifold $V_{k-j}(\mathbb{F}^{n-j})$ via the map

$$\phi_{n-j,k-j} : O(\mathbb{F}^{n-j})/O(\mathbb{F}^{n-k}) \to V_{k-j}(\mathbb{F}^{n-j}).$$

Thinking only in terms of Stiefel manifolds, this example shows that the projection $q_{k,j} : V_k(\mathbb{F}^n) \to V_j(\mathbb{F}^n)$ is a locally trivial bundle with typical fiber $V_{k-j}(\mathbb{F}^{n-j})$.

6.1.16 Example. Regard the additive group $\mathbb{Z}$ as a discrete topological group, and consider its action on the infinite horizontal strip $\mathbb{R} \times I$ defined by the formula

$$\rho(n, (s,t)) \coloneqq \left(s + n, \tfrac{1}{2} + (-1)^n \left(t - \tfrac{1}{2}\right)\right).$$

This action is free, since if $n \cdot (s,t) = (s,t)$, then in particular $s + n = s$, which implies $n = 0$. Moreover, the translation function $t : C(\mathbb{R} \times I) \to \mathbb{Z}$ can be described as a combination of projections and a subtraction as shown below,

$$\left((s,t), \left(s + n, \tfrac{1}{2} + (-1)^n \left(t - \tfrac{1}{2}\right)\right)\right) \mapsto (s, s+n) \mapsto s + n - s = n$$

therefore the action ρ is also weakly proper.

Let us describe the orbit space $M \coloneqq \mathbb{R} \times I/\mathbb{Z}$ of this action. Notice that, for any point (s,t), we can find another point (s', t') which belongs to the subspace $I \times I$ and which is in the same orbit of (s,t). Indeed, if we let n be the greatest integer satisfying $n \leq s$, then the point

$$(s', t') \coloneqq \left(s - n, \tfrac{1}{2} + (-1)^n \left(t - \tfrac{1}{2}\right)\right)$$

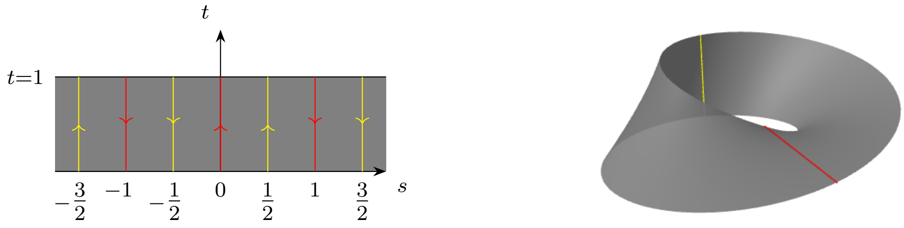

Figure 7 – Line segments on the Möbius strip.

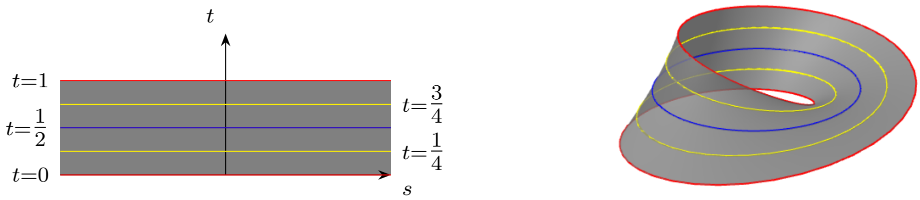

Figure 8 – Topological circles on the Möbius strip.

belongs to $I \times I$ and satisfies $n \cdot (s', t') = (s, t)$. This means that the restriction of the orbit map $\pi : \mathbb{R} \times I \to \mathbb{R} \times I/\mathbb{Z}$ to the unit square $I \times I$ is surjective, thus we can regard the orbit space $\mathbb{R} \times I/\mathbb{Z}$ as a quotient $I \times I$. The only identifications made on $I \times I$ by the action are those of the form $(0, t) \sim 1 \cdot (0, t) = (1, 1 - t)$. This means that $\mathbb{R} \times I/\mathbb{Z}$ can be obtained from $I \times I$ by identifying the left side $\{0\} \times I$ of the square with the left side $\{1\} \times I$, but performing a half-twist before making the identification; therefore the orbit space $\mathbb{R} \times I/\mathbb{Z}$ is a *Möbius strip*.

We can better visualize the orbit map $\pi : \mathbb{R} \times I \to M$ by studying how it affects particular subspaces of $\mathbb{R} \times I$. In Figure 7 we can see how different line segments in $\mathbb{R} \times I$ are identified with alternating orientations, and then how these correspond to line segments on the surface of the Möbius strip.

It is also interesting to see how the orbit map affects lines of the form $\mathbb{R} \times \{t\}$ for some $t \in I$. Their images are topologically equivalent to circles, but due to the way that the points in $\mathbb{R} \times I$ are identified, pairs of lines which are symmetric with respect to the central line $\mathbb{R} \times \left\{\frac{1}{2}\right\}$ have equal images, as shown in Figure 8

Let $E : \mathbb{R} \to S^1$ de the map defined as $E(s) := (\cos(2\pi s), \sin(2\pi s))$, so that $E|_I$ is the exponential map $\exp : I \to S^1$ we have been using so far. Using E, we define a map $P : \mathbb{R} \times I \to S^1$ by the formula $P(s, t) := E(s)$. The periodicity of E implies that $P(n \cdot (s, t)) = P(s, t)$ holds for any $(s, t) \in \mathbb{R} \times I$ and any $n \in \mathbb{Z}$, therefore P can be factored through the orbit map of the action to define a map $p : M \to S^1$.

We want to show that the map p constructed above is a locally trivial bundle with typical fiber the unit interval I. Consider the open subset $U := S^1 \setminus \{(1, 0)\}$. Since $E(s) = (1, 0)$ if and only if $s = n$ for some $n \in \mathbb{Z}$, if we set $U_n := (n, n + 1) \times I$ for each

$n \in \mathbb{Z}$, then we can write

$$P^{-1}(U) = \bigcup_{n \in \mathbb{Z}} U_n.$$

The map p was defined by factoring P through π, so the equality $\pi^{-1}(p^{-1}(U)) = P^{-1}(U)$ holds, and by restriction we have a surjection $\pi|_{P^{-1}(U)} : P^{-1}(U) \to p^{-1}(U)$. Now, for every point $(s,t) \in P^{-1}(U)$ we can find another point $(s',t') \in U_0$ such that $(s,t) \sim (s',t')$, which means that the restriction $\pi|_{U_0} : U_0 \to p^{-1}(U)$ is still surjective. Moreover, this restriction is also an open map, and thus a quotient map, because the orbit map $\pi : \mathbb{R}\times I \to M$ is itself an open map. Notice, however, that $\pi|_{U_0}$ is also injective, because if (s_1,t_1), $(s_2,t_2) \in U_0$ are such that $\pi(s_1,t_1) = \pi(s_2,t_2)$, then in particular $s_2 = s_1 + n$ for some integer $n \in \mathbb{Z}$, but since s_1 and s_2 are both in $(0,1)$, this is only possible if $n = 0$, which then implies $(s_1,t_1) = (s_2,t_2)$.

The reasoning of the previous paragraph implies that $\pi|_{U_0} : U_0 \to p^{-1}(U)$ is a *homeomorphism*. Now, since the restriction $E|_{(0,1)} : (0,1) \to U$ is also a homeomorphism, so is the map $\Phi_U : U_0 \to U \times I$ defined as

$$\Phi_U(s,t) \coloneqq (E(s),t) = ((\cos(2\pi s), \sin(2\pi s)), t)$$

for every $(s,t) \in U_0$. Using this we define a homeomorphism $\varphi_U : p^{-1}(U) \to U \times I$ via the composition $\varphi_U \coloneqq \Phi \circ \pi|_{U_0}^{-1}$. Notice then that, given $[s,t] \in M$ with $(s,t) \in U_0$, we have

$$\pi_1(\varphi_U([s,t])) = \pi_1(E(s),t) = E(s) = p([s,t]); \tag{6.2}$$

showing that the homeomorphism φ commutes with the projections of $p^{-1}(U)$ and $U \times I$, and therefore that it defines a local trivialization.

The open subset U does not cover all of S^1, however. We still need to construct a local trivialization around the point $(1,0) \in S^1$. Consider the open neighborhood of such point defined by $V \coloneqq S^1 \setminus \{(-1,0)\}$. Since $E(s) = -1$ if and only if $s = n + \frac{1}{2}$ for some $n \in \mathbb{Z}$, we can write

$$P^{-1}(V) = \left(\mathbb{R} \setminus \left\{n + \tfrac{1}{2} \mid n \in \mathbb{Z}\right\}\right) \times I$$

Similarly to the previous case, if we consider, for each $n \in \mathbb{Z}$, the subset

$$V_n \coloneqq \left(n + \tfrac{1}{2}, n + 1 + \tfrac{1}{2}\right) \times I,$$

then we can rewrite the previous expression as

$$P^{-1}(V) = \bigcup_{n \in \mathbb{Z}} V_n.$$

The restricted exponential $E|_{\left(\frac{1}{2},\frac{3}{2}\right)} : \left(\frac{1}{2}, \frac{3}{2}\right) \to V$ is again a homeomorphism, and using it we define a homeomorphism $\Phi_V : V_0 \to V \times I$ by the formula

$$\Phi_V(s,t) \coloneqq (E(s),t) \quad \forall\, (s,t) \in V_0.$$

Using the same reasoning as in the previous case we can show that the restriction $\pi|_{V_0} : V_0 \to p^{-1}(V)$ is a homeomorphism, and combining everything we define the required trivial localization $\varphi_V : p^{-1}(V) \to V \times I$ via the composition $\varphi_V \coloneqq \Phi_V \circ \pi|_{V_0}^{-1}$.

The trivializations φ_U and φ_V constructed above *look similar*, and this might suggest that they actually coincide where both are defined, i.e., that $\varphi_V(\varphi_U^{-1}(z,t)) = (z,t)$ for every $(z,t) \in (U \cap V) \times I$, but this is far from being true. Consider for example $z = (0,1)$, so that for any $t \in I$ we can write $((0,1),t) = \left(E\left(\frac{1}{4}\right),t\right)$, and therefore

$$\varphi_U^{-1}((0,1),t) = \left[\tfrac{1}{4}, t\right].$$

Now, if we are careless we might say that

$$\varphi_V\left(\left[\tfrac{1}{4}, t\right]\right) = \left(E\left(\tfrac{1}{4}\right), t\right) = ((0,1),t)$$

and conclude that φ_V and φ_U really coincide. The problem is that before applying φ_V we need to rewrite the input in the form $[s,t]$ with (s,t) belonging to V_0, and $\left(\frac{1}{4}, t\right)$ does not belong to V_0! In order to fix this, we note that $\left(\frac{5}{4}, 1-t\right)$ belongs to V_0, and it is in the same orbit of $\left(\frac{1}{4}, t\right)$ since

$$\left(\tfrac{5}{4}, 1-t\right) = \left(\tfrac{1}{4}+1, \tfrac{1}{2} + (-1)^1\left(t - \tfrac{1}{2}\right)\right) = 1 \cdot \left(\tfrac{1}{4}, t\right).$$

It follows that

$$\varphi_V\left(\left[\tfrac{1}{4}, t\right]\right) = \varphi_V\left(\left[\tfrac{5}{4}, 1-t\right]\right) = \left(E\left(\tfrac{5}{4}\right), 1-t\right) = ((0,1), 1-t);$$

therefore, going back and forth along the trivializations results in a vertical reflection of $(U \cap V) \times I$ along the axis $(U \cap V) \times \left\{\frac{1}{2}\right\}$.

6.2 Feldbau's Theorem

The goal of this short section is to prove Feldbau's Theorem, one of the most important basic results in the theory of locally trivial bundles. It plays a crucial role in the next section where we study the lifting properties of the projection map of a locally trivial bundle, and it is these properties that allow us to develop Obstruction Theory for locally trivial bundles.

The lemma below on locally trivial bundles over cylinders is the main technical result used in the proof of Feldbau's Theorem.

6.2.1 Lemma. Let B be any space, and suppose $\xi = (E, F, B \times [a,b], p)$ is a locally trivial bundle over the cylinder $B \times [a,b]$. If there exists $c \in (a,b)$ such that the restricted bundles $\xi|_{B\times[a,c]}$ and $\xi|_{B\times[c,b]}$ are both trivial, then ξ itself is trivial.

Proof. Consider local trivializations

$$\varphi : (B \times [a,c]) \times F \to p^{-1}(B \times [a,c]),$$
$$\psi : (B \times [c,b]) \times F \to p^{-1}(B \times [c,b]).$$

We want to somehow glue these two trivializations along $(B \times \{c\}) \times F$, but since they do not necessarily agree on this subspace, we need to modify them before performing this gluing.

The restriction $\varphi|_{(B\times\{c\})\times F}$ defines a homeomorphism $(B\times\{c\})\times F \cong p^{-1}(B\times\{c\})$, the same being true of the restriction $\psi|_{(B\times\{c\})\times F}$. We can then find a homeomorphism $\alpha : F \to F$ such that the equality

$$\psi^{-1}(\varphi((x,c),z)) = ((x,c),\alpha(z))$$

holds for every $x \in B$ and $z \in F$. Using this auxiliary map we define $\Phi : (B\times[a,b])\times F \to E$ by the formula

$$\Phi((x,t),z) := \begin{cases} \varphi((x,t),z), & \text{if } a \le t \le c, \\ \psi((x,t),\alpha(z)), & \text{if } c \le t \le b. \end{cases}$$

Notice that, for $t = c$, the two expression above coincide since

$$\psi((x,c),\alpha(z)) = \psi(\psi^{-1}(\varphi((x,c),z))) = \varphi((x,c),z),$$

therefore Φ is well-defined and continuous by the Pasting Lemma.

Using that φ, ψ and α are homeomorphisms, it is straightforward to check that Φ is a bijection. Now let $A \subseteq (B \times [a,b]) \times F$ be a closed subset. The intersections

$$A_1 := A \cap ((B \times [a,c]) \times F) \quad \text{and} \quad A_2 := A \cap ((B \times [c,b]) \times F)$$

are closed in $(B \times [a,c]) \times F$ and $(B \times [c,b]) \times F$, respectively, and we can write

$$\Phi(A) = \Phi(A_1) \cup \Phi(A_2).$$

Since Φ is equal to φ on the subspace $(B \times [a,c]) \times F$, we have the equality

$$\Phi(A_1) = \varphi(A_1),$$

which is a closed subset of $p^{-1}(B\times[a,c])$, since φ is a homeomorphism; but $p^{-1}(B\times[a,c])$ is closed in E, therefore $\Phi(A_1)$ is closed in E. For the other intersection, by definition of Φ we can write

$$\Phi(A_2) = (\psi \circ (\mathrm{id}_{B\times[c,b]} \times \alpha))(A_2),$$

so this image is a closed subset of $p^{-1}(B\times[c,b])$, since $\psi\circ(\mathrm{id}_{B\times[c,b]}\times\alpha)$ is a homeomorphism; and then reasoning like in the previous case we deduce that $\Phi(A_2)$ is closed in E too.

It follows that $\Phi(A)$ is the union of two closed subsets of E, so it is also closed in E. This means that Φ is a closed map, but since we already know that it is a bijection, we deduce that Φ is in fact a homeomorphism, and it defines the desired global trivialization of ξ. ■

6.2.2 Remark. We will need a version of Lemma 6.2.1 which follows from the one we just proved, although it certainly looks stronger. In this version, we consider a finite number of spaces $X_1, \ldots, X_n$, and a locally trivial bundle

$$p : E \to X_1 \times \cdots \times X_{i-1} \times [a, b] \times X_i \times \cdots \times X_n.$$

If there exists a number $c \in (a, b)$ such that restrictions of the bundle above to $X_1 \times \cdots X_{i-1} \times [a, c] \times X_i \times \cdots \times X_n$ and $X_1 \times \cdots \times X_{i-1} \times [c, b] \times X_i \times \cdots X_n$ are both trivial, then the original bundle is also trivial.

This version follows from the one we proved because there is a homeomorphism

$$X_1 \times \cdots \times X_{i-1} \times [a, b] \times X_i \times \cdots \times X_n \cong (X_1 \times \cdots \times X_n) \times [a, b],$$

therefore the bundle in question is isomorphic to a bundle over $(X_1 \times \cdots X_n) \times [a, b]$ to which we can apply Lemma 6.2.1.

With Lemma 6.2.1 we can prove the main result of the section.

6.2.3 Theorem (Feldbau)**.** Every locally trivial bundle over the n-cube I^n is globally trivial, where $n \geq 0$ is an integer.

Proof. Let $\xi = (E, F, I^n, p)$ be a locally trivial bundle over I^n. By the Lebesgue Covering Lemma, we can find a sufficiently big integer $N > 0$ such that, for any integers $i_1, \ldots, i_n \in \{0, \ldots, N-1\}$, the subcube

$$\left[\tfrac{i_1}{N}, \tfrac{i_1+1}{N}\right] \times \cdots \times \left[\tfrac{i_n}{N}, \tfrac{i_n+1}{N}\right] \subseteq I^n$$

is contained in a trivializing neighborhood, therefore the restriction of ξ to this subcube is trivial.

Now, if we fix the first $n-1$ integers $i_1, \ldots, i_{n-1} \in \{0, \ldots, N-1\}$, we claim that ξ is actually trivial when restricted to the strip

$$\left[\tfrac{i_1}{N}, \tfrac{i_1+1}{N}\right] \times \cdots \times \left[\tfrac{i_{n-1}}{N}, \tfrac{i_{n-1}+1}{N}\right] \times I.$$

Indeed, if we let

$$C_n := \left[\tfrac{i_1}{N}, \tfrac{i_1+1}{N}\right] \times \cdots \times \left[\tfrac{i_n}{N}, \tfrac{i_n+1}{N}\right]$$

denote the "horizontal part" of this strip, then we know that ξ is trivial when restricted to the subcubes $C_n \times \left[0, \frac{1}{N}\right]$ and $C_n \times \left[\frac{1}{N}, \frac{2}{N}\right]$, therefore ξ is trivial when restricted to the subcube $C_n \times \left[0, \frac{2}{N}\right]$ by Lemma 6.2.1 But ξ is also trivial when restricted to $C_n \times \left[\frac{2}{N}, \frac{3}{N}\right]$, so using Lemma 6.2.1 again we conclude that ξ is trivial over $C_n \times \left[0, \frac{3}{N}\right]$. Proceeding like this we eventually deduce that ξ is trivial over $C_n \times I$.

Now consider only $n-1$ integers $i_1, \dots, i_{n-1} \in \{0, \dots, N-1\}$, and let

$$C_{n-1} := \left[\frac{i_1}{N}, \frac{i_1+1}{N}\right] \times \cdots \times \left[\frac{i_{n-1}}{N}, \frac{i_{n-1}+1}{N}\right].$$

We claim that ξ is trivial over $C_{n-1} \times I \times I$. Indeed, by the previous paragraph we know that ξ is trivial over $C_{n-1} \times \left[0, \frac{1}{N}\right] \times I$ and $C_{n-1} \times \left[\frac{1}{N}, \frac{2}{N}\right] \times I$, so Lemma 6.2.1 (and also Remark 6.2.2) implies that ξ is trivial over $C_{n-1} \times \left[0, \frac{2}{N}\right] \times I$. Proceeding like in the previous case we eventually deduce the desired triviality condition.

Repeating the reasoning above a bunch of times we eventually show that, for any choice of integer $i_1 \in \{0, \dots, N-1\}$, the bundle ξ is trivial over $\left[\frac{i_1}{N}, \frac{i_1+1}{N}\right] \times (I \times \cdots \times I)$. Varying i_1 and applying Lemma 6.2.1 like in the previous cases we see that ξ is trivial over I^n ■

6.3 Lifting properties

In this section we prove several results characterizing the lifting properties of the projection map of a locally trivial bundle. We first prove that certain simple inclusions can be lifted, and then use this to prove similar lifting results for certain inclusion of CW-complexes and for homotopically simple maps. These results will be crucial for the construction of the long exact sequence of homotopy groups associated to a locally trivial bundle in the next section, which constitutes our main tool for calculating some homotopy groups.

We begin by defining the meaning of a *lifting property*.

6.3.1 Definition. We say that a map $p : E \to B$ has the **right lifting property** with respect to a map $i : X \to Y$, if, given any maps $f : X \to E$ and $g : Y \to B$ such that the square below is commutative,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ Y \arrow[r, "g"'] & B \end{tikzcd}$$

there exists a diagonal map $h : Y \to E$ such that the resulting diagram below is still commutative.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ Y \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

Our first important theorem is that the projection of a locally trivial bundle has the right lifting property with respect to a family of simple inclusions.

6.3.2 Theorem. Let $p : E \to B$ be a locally trivial bundle with typical fiber F. Then p has the right lifting property with respect to the inclusion $i_{I^n,0} : I^n \to I^n \times I$ for every integer $n \geq 0$.

Proof. We first prove that the result holds in the case where the locally trivial bundle is globally trivial. Consider then the commutative square below.

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d, "i_{I^n,0}"'] & B \times F \arrow[d, "\pi_1"] \\ I^n \times I \arrow[r, "g"'] & B \end{tikzcd}$$

Define $h : I^n \times I \to B \times F$ by the formula

$$h(s,t) \coloneqq (g(s,t), \pi_2(f(s)))$$

for any $s = (s_1, \ldots, s_n) \in I^n$ and any $t \in I$, where $\pi_2 : B \times F \to F$ is the canonical projection.

Let us check that h defined as above satisfies the two required conditions. On the one hand

$$\pi_1(h(s,t)) = \pi_1(g(s,t), \pi_2(f(s))) = g(s,t),$$

which means that $\pi_1 \circ h = g$. On the other hand,

$$\pi_1(h(i_{I^n,0}(s))) = \pi_1(h(s,0)) = g(s,0) = \pi_1(f(s)),$$

and also

$$\pi_2(h(i_{I^n,0}(s))) = \pi_2(h(s,0)) = \pi_2(g(s,0), \pi_2(f(s))) = \pi_2(f(s)).$$

These two equalities mean that $\pi_1 \circ h \circ i_{I^n,0} = \pi_1 \circ f$ and $\pi_2 \circ h \circ i_{I^n,0} = \pi_2 \circ f$, and by combining these two equalities we deduce that $h \circ i_{I^n,0} = f$.

Now let $p : E \to B$ be an arbitrary locally trivial bundle with typical fiber F. Consider the pullback bundle $g^*p : g^*E \to I^n \times I$ which fits in the commutative square below.

$$\begin{tikzcd} g^*E \arrow[r, "G"] \arrow[d, "g^*p"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "g"'] & B \end{tikzcd}$$

Using the universal property of the product we obtain a map $\alpha : I^n \to g^*E$ that fits in the commutative diagram below.

$$\begin{tikzcd}
I^n \arrow[r, "\alpha"] \arrow[d, "i_{I^n,0}"'] \arrow[rr, bend left, "f"] & g^*E \arrow[r, "G"] \arrow[d, "g^*p"] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "\mathrm{id}"'] & I^n \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

Since $I^n \times I$ is homeomorphic to the cube I^{n+1}, Feldbau's Theorem (Theorem 6.2.3) implies that the pullback $g^*p : g^*E \to I^n \times I$ is a trivial bundle, therefore we can find a diagonal map $\beta : I^n \times I \to g^*E$ such that the resulting diagram below is still commutative.

$$\begin{tikzcd}
I^n \arrow[r, "\alpha"] \arrow[d, "i_{I^n,0}"'] \arrow[rr, bend left, "f"] & g^*E \arrow[r, "G"] \arrow[d, "g^*p"] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "\beta"] & I^n \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

We then define $h : I^n \times I \to E$ as $h \coloneqq G \circ \beta$. This map satisfies the two required conditions, since on the one hand

$$\begin{aligned}
p \circ h &= p \circ G \circ \beta \\
&= g \circ g^*p \circ \beta \\
&= g \circ \mathrm{id}_{I^n \times I} \\
&= g,
\end{aligned}$$

while on the other

$$\begin{aligned}
h \circ i_{I^n,0} &= G \circ \beta \circ i_{I^n,0} \\
&= G \circ \alpha \\
&= f.
\end{aligned}$$

■

Since the n-dimensional cube I^n is homeomorphic to the n-dimensional disk D^n, Theorem 6.3.2 can also be stated in terms of disks.

6.3.3 Corollary. Let $p : E \to B$ be a locally trivial bundle with typical fiber F. Then p has the right lifting property with respect to the inclusion $i_{D^n,0} : D^n \to D^n \times I$ for every integer $n \geq 0$.

So far we have proved that we can lift maps $g : D^n \times I \to B$ through a locally trivial projection $p : E \to B$ by specifying the initial stage of the lift, that is, by imposing conditions on the values $h(x, 0)$ for a lift $h : D^n \times I \to E$. In many cases, however, we will need a more refined control over the lift obtained by specifying not just its initial stage, but also its behavior along a subspace $A \subseteq X$, that is, by specifying the values $h(x, t)$ for $x \in A$ and $t \in I$.

Naturally, this is not always possible, we need to impose some nice conditions on the subspace. We start by studying the case of the subspace inclusion $S^{n-1} \subseteq D^n$. The problem then is to lift a map $g : D^n \times I \to B$ through the projection $p : E \to B$ subject to predetermined conditions on the subspace $D^n \times \{0\} \cup S^{n-1} \times I \subseteq D^n \times I$. The trick is to compare this inclusion with a more familiar one.

6.3.4 Lemma ("Stacking cups" homeomorphism)**.** For every integer $n \geq 0$, there exists a homeomorphism $u : D^n \times I \to D^n \times I$ such that $u(D^n \times \{0\}) = (D^n \times \{0\}) \cup (S^{n-1} \times I)$.

Sketch of proof. If $n = 0$ the result is trivial, because in this case $S^{-1} = \varnothing$, and then the identity $\mathrm{id}_{D^n \times I}$ satisfies the required conditions, so we suppose from now on that $n \geq 1$.

Recall that the map $q : S^{n-1} \times I \to D^n$ given by $(x, t) \mapsto t \cdot x$ is a quotient map. In particular $q \times \mathrm{id}_I : (S^{n-1} \times I) \times I \to D^n \times I$ is also a quotient map, since I is locally compact Hausdorff.

Define $U : (S^{n-1} \times I) \times I \to D^n \times I$ by the formula

$$U((x, s), t) := \begin{cases} (2s(1-t) \cdot x, t), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ ((1-t) \cdot x, 2\left(s - \frac{1}{2}\right)(1-t) + t), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

This is a well-defined function, because at $s = \frac{1}{2}$ both expressions evaluate to $((1-t) \cdot x, t)$, and the Pasting Lemma then implies its continuity.

The only fibers of $q \times \mathrm{id}_I : (S^{n-1} \times I) \times I \to D^n \times I$ which contain more than a single point are those over points of the form $(\mathbf{0}, t)$, and we can describe them explicitly as

$$(q \times \mathrm{id}_I)^{-1}(\mathbf{0}, t) = (S^{n-1} \times \{0\}) \times \{t\}.$$

Since for any $x \in S^{n-1}$ we have $U((x, 0), t) = (\mathbf{0}, t)$, it follows that U is constant on the fibers of $q \times \mathrm{id}_I$, therefore it can be factored through it to define a map $u : D^n \times I \to D^n \times I$.

Notice that the map U satisfies

$$U\left(\left(S^{n-1} \times \left[0, \tfrac{1}{2}\right]\right) \times \{0\}\right) \subseteq D^n \times \{0\} \quad \text{and} \quad U\left(\left(S^{n-1} \times \left[\tfrac{1}{2}, 1\right]\right) \times \{0\}\right) \subseteq S^{n-1} \times I,$$

therefore the induced map u satisfies $u(D^n \times \{0\}) \subseteq (D^n \times \{0\}) \cup (S^{n-1} \times I)$.

We trust the reader will believe that this map u is a homeomorphism. ∎

With this auxiliary homeomorphism at our disposal, we can prove the existence of lifts satisfying extra conditions.

6.3.5 Proposition. Let $p : E \to B$ be a locally trivial bundle with typical fiber F. Given any maps $f : (D^n \times \{0\}) \cup (S^{n-1} \times I) \to E$ and $g : D^n \times I \to B$ satisfying $p \circ f = g \circ i$, where $i : (D^n \times \{0\}) \cup (S^{n-1} \times I) \hookrightarrow D^n \times I$ is the inclusion, there exists a map $h : D^n \times I \to E$ that makes the diagram below commute.

$$\begin{tikzcd}
(D^n \times \{0\}) \cup (S^{n-1} \times I) \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
D^n \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Proof. Consider the homeomorphism u of Lemma 6.3.4 which fits in the commutative, and form the commutative square below.

$$\begin{tikzcd}
D^n \arrow[r, "f \circ u \circ i_{D^n,0}"] \arrow[d, "i_{D^n,0}"'] & E \arrow[d, "p"] \\
D^n \times I \arrow[r, "g \circ u"'] & B
\end{tikzcd}$$

Since $(u \circ i_{D^n,0})(D^n) \subseteq (D^n \times \{0\}) \cup (S^{n-1} \times I)$, and the restriction of g to this subspace coincides with $p \circ f$ by hypothesis, the diagram above is commutative.

Using Corollary 6.3.3 we obtain a diagonal map $\theta : D^n \times I \to E$ satisfying the equations $p \circ \theta = g \circ u$ and $\theta \circ i_{D^n,0} = f \circ u \circ i_{D^n,0}$. We claim that the map $h : D^n \times I \to E$ defined as $h := \theta \circ u^{-1}$ is the desired lift. Indeed, on the one hand the equality $p \circ \theta = g \circ u$ implies $p \circ \theta \circ u^{-1} = g$, that is, $p \circ h = h$; and on the other, since any point $(y, t) \in (D^n \times \{0\}) \cup (S^{n-1} \times I)$ is of the form $(y, t) = u(x, 0)$ for some $x \in D^n$, we have

$$h(y, t) = h(u(x, 0)) = \theta(u^{-1}(u(x, 0))) = \theta(x, 0) = f(u(x, 0)) = f(y, t). \qquad \blacksquare$$

We now derive some useful corollaries from this enhanced lifting result.

6.3.6 Corollary. Let $p : E \to B$ be a locally trivial bundle with typical fiber F, and let (X, A) be a pair such that X is obtained from A by cell attachments. Suppose we are given maps $f : (X \times \{0\}) \cup (A \times I) \to E$ and $g : X \times I \to B$ such that $p \circ f = g|_{(X \times \{0\}) \cup (A \times I)}$. There exists a map $h : X \times I \to E$ making the diagram below commute.

$$\begin{tikzcd}
(X \times \{0\}) \cup (A \times I) \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Proof. By the definition of cell attachment, there exists a family of maps $\{\Phi_e : D^{d(e)} \to X\}_{e \in \mathcal{E}}$ satisfying the two following conditions:

1. $\Phi_e(S^{d(e)-1}) \subseteq A$ for every $e \in \mathcal{E}$;

2. the diagram below is a pushout square.

$$\begin{tikzcd} \coprod_{e\in\mathcal{E}} S^{d(e)-1} \arrow[r, "\langle\varphi_e\rangle_{e\in\mathcal{E}}"] \arrow[d, tail] & A \arrow[d, tail] \\ \coprod_{e\in\mathcal{E}} D^{d(e)} \arrow[r, "\langle\Phi_e\rangle_{e\in\mathcal{E}}"'] & X \end{tikzcd}$$

Since the functor $-\times I$ is a left adjoint, it commutes with coproducts and pushouts, therefore the diagram below is also a pushout square.

$$\begin{tikzcd} \coprod_{e\in\mathcal{E}} (S^{d(e)-1}\times I) \arrow[r, "\langle\varphi_e\times\mathrm{id}_I\rangle_{e\in\mathcal{E}}"] \arrow[d, tail] & A\times I \arrow[d, tail] \\ \coprod_{e\in\mathcal{E}} (D^{d(e)}\times I) \arrow[r, "\langle\Phi_e\times\mathrm{id}_I\rangle_{e\in\mathcal{E}}"'] & X\times I \end{tikzcd}$$

For every $e\in\mathcal{E}$, the restriction of $\Phi_e\times\mathrm{id}_I$ to $(D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I)$ takes values in $(X\times\{0\})\cup(A\times I)$, therefore we can consider the map $f_e : (D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I)\to E$ defined by the composition

$$f_e := f\circ(\Phi_e\times\mathrm{id}_I)|_{(D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I)}.$$

We also consider the map $g_e : D^{d(e)}\times I\to B$ defined as

$$g_e := g\circ(\Phi_e\times\mathrm{id}_I).$$

Since f is a partial lift of g over $(X\times\{0\})\cup(A\times I)$, it follows that f_e is a partial lift of g_e over $(D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I)$, i.e., the square below is commutative.

$$\begin{tikzcd} (D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I) \arrow[r, "f_e"] \arrow[d, tail] & E \arrow[d, "p"] \\ D^{d(e)}\times I \arrow[r, "g_e"'] & B \end{tikzcd}$$

Applying Proposition 6.3.5 we obtain a map $h_e : D^{d(e)}\times I\to E$ lifting g_e and coinciding with f_e on the subspace $(D^{d(e)}\times\{0\})\cup(S^{d(e)-1}\times I)$. The maps h_e so defined for all $e\in\mathcal{E}$ give rise to an induced map

$$\langle h_e\rangle_{e\in\mathcal{E}} : \coprod_{e\in\mathcal{E}} D^{d(e)}\times I\to E.$$

The defining property of h_e is such that the restriction $h_e|_{S^{d(e)}\times I}$ is equal to $f_e|_{S^{d(e)}\times I}$, and this latter restriction is the same as the composition $f|_{A\times I}\circ(\varphi_e\times\mathrm{id}_I)$. Since this holds for every $e\in\mathcal{E}$, it follows that the "outer square" of the diagram below commutes, therefore the universal property of the pushout implies the existence of the map $h : X \times I \to E$ making the whole diagram commute.

$$\begin{tikzcd}
\coprod_{e\in\mathcal{E}} S^{d(e)-1} \times I \arrow[r, "\langle\varphi_e \times \mathrm{id}_I\rangle_{e\in\mathcal{E}}"] \arrow[d, tail] & A \times I \arrow[d, tail] \arrow[ddr, bend left, "f|_{A\times I}"] & \\
\coprod_{e\in\mathcal{E}} D^{d(e)} \times I \arrow[r, "\langle\Phi_e \times \mathrm{id}_I\rangle_{e\in\mathcal{E}}"'] \arrow[drr, bend right, "\langle h_e\rangle_{e\in\mathcal{E}}"'] & X \times I \arrow[dr, dashed, "h"] & \\
 & & E
\end{tikzcd}$$

We would like to show that h is the desired lift of g. In order to do this, we need to show that h satisfies the equation $p \circ h = g$, and that its restriction to $(X \times \{0\}) \cup (A \times I)$ coincides with f. From the diagram above we see immediately that h coincides with f over $A \times I$. Now, given $(x, 0) \in X \times \{0\}$, if $x \in A$, then we have already shown that $h(x, 0) = f(x, 0)$; but if $x \in X \setminus A$, then according to Definition 1.2.1 there is a point $a \in D^{d(e)}$ such that $x = \Phi_e(a)$ for some $e \in \mathcal{E}$, therefore

$$\begin{aligned}
h(x,0) &= h(\Phi_e \times \mathrm{id}_I(a,0)) \\
&= h_e(a,0) \\
&= f_e(a,0) \\
&= f(\Phi_e \times \mathrm{id}_I(a,0)) \\
&= f(x,0).
\end{aligned}$$

In order to show that h is a lift of g, notice that

$$(p \circ h)|_{A\times I} = p \circ h|_{A\times I} = p \circ f|_{A\times I} = g|_{A\times I},$$

and also that

$$\begin{aligned}
p \circ h \circ \langle\Phi_e \times \mathrm{id}_I\rangle_{e\in\mathcal{E}} &= p \circ \langle h_e\rangle_{e\in\mathcal{E}} \\
&= \langle p \circ h_e\rangle_{e\in\mathcal{E}} \\
&= \langle g_e\rangle_{e\in\mathcal{E}} \\
&= \langle g \circ (\Phi_e \times \mathrm{id}_I)\rangle_{e\in\mathcal{E}} \\
&= g \circ \langle\Phi_e \times \mathrm{id}_I\rangle_{e\ni\mathcal{E}}.
\end{aligned}$$

This means that both g and $p \circ h$ make the diagram below commute, so the uniqueness in the universal property of the pushout implies that $p \circ h = g$.

$$\begin{tikzcd}
\coprod_{e\in\mathcal{E}} S^{d(e)-1} \times I \arrow[r, "\langle \varphi_e \times \mathrm{id}_I \rangle_{e\in\mathcal{E}}"] \arrow[d, tail] & A \times I \arrow[d, tail] \arrow[ddr, bend left, "g|_{A\times I}"] & \\
\coprod_{e\in\mathcal{E}} D^{d(e)} \times I \arrow[r, "\langle \Phi_e \times \mathrm{id}_I \rangle_{e\in\mathcal{E}}"'] \arrow[drr, bend right, "g\circ\langle \Phi_e \times \mathrm{id}_I \rangle_{e\in\mathcal{E}}"'] & X \times I \arrow[dr, shift left, "p\circ h"] \arrow[dr, shift right, "g"'] & \\
& & B
\end{tikzcd}$$

■

The previous result has a particularly simple form for CW-complexes.

6.3.7 Corollary. Let $p : E \to B$ be a locally trivial bundle with typical fiber F, and let X be a CW-complex. Suppose we are given maps $f : X \to E$ and $g : X \times I \to B$ such that $g \circ i_{X,0} = p \circ f$. There exists a map $h : X \times I \to E$ making the diagram below commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_{X,0}"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

In other words, p has the right lifting property with respect to the cylinder inclusion $i_{X,0} : X \to X \times I$ for any CW-complex X.

Proof. Consider the skeletal filtration of X:

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

Using that X is the colimit of its skeletons, it follows that the existence of h is equivalent to the existence of a collection of maps $\{h_n : X_n \times I \to E\}_{n\geq 0}$ satisfying the following conditions:

1. $h_n|_{X_{n-1}\times I} = h_{n-1}$ for every $n \geq 1$;

2. $p \circ h_n = g|_{X_n\times I}$ for every $n\ geq0$;

3. $h_n \circ i_{X_n,0} = f|_{X_n}$ for every $n \geq 0$.

The first of these conditions ensures that the collection $\{h_n\}_{n\geq 0}$ induces a map $h : X\times I \to E$, while the two other conditions ensure that the induced map h will satisfy the required commutativity properties.

We construct the maps h_n by induction. Since X_0 is obtained by attaching points (0-dimensional disks) to the empty set, X_0 consists of a discrete collection of points

$\{\Phi_e(\mathrm{pt})\}_{e\in\mathcal{E}_0}$, where $\Phi_e : D^0 \to X_0$ are the attaching maps. The product $X_0 \times I$ is then a disjoint union of copies of the unit interval, and for each $e \in \mathcal{E}_0$ we define

$$h_0(\Phi_e(\mathrm{pt}), t) \coloneqq f(\Phi_e(\mathrm{pt}))$$

for every $t \in I$, so that h_0 automatically satisfies $p \circ h_0 = g \circ i_{X_0,0}$ and $h_0 \circ i_{X_0,0} = f|_{X_0}$.

Now suppose we have defined $h_n : X_n \times I \to E$ satisfying the required conditions, and let us define h_{n+1}. Let $F : (X_{n+1} \times \{0\}) \cup (X_n \times I) \to E$ be the map defined by the conditions $F(x, 0) = f(x)$ for every $x \in X_{n+1}$, and $F|_{X_n\times I} = h_n$. Notice that this is well-defined because $h_n \circ i_{X_n,0} = f|_{X_n}$ holds by the inductive hypothesis, and the continuity of F then follows from the Pasting Lemma. If we combine the initial hypothesis that $p \circ f = g \circ i_{X,0}$ with the inductive hypothesis saying that $p \circ h_n = g|_{X_n\times I}$, it follows that the outer square below is commutative, therefore from Corollary 6.3.6 we deduce the existence of a map $h_{n+1} : X_{n+1} \times I \to E$ making the whole diagram commute.

$$\begin{tikzcd}
(X_{n+1} \times \{0\}) \cup (X_n \times I) \arrow[r, "F"] \arrow[d, tail] & E \arrow[d, "p"] \\
X_{n+1} \times I \arrow[r, "g|_{X_{n+1}\times I}"'] \arrow[ur, dashed, "h_{n+1}"] & B
\end{tikzcd}$$

The condition $p \circ h_{n+1} = g|_{X_{n+1}\times I}$ is immediate from the commutativity, while the other two conditions that h_{n+1} must satisfy follow from the fact that h_{n+1} coincides with F when restricted to $(X_{n+1} \times \{0\}) \cup (X_n \times I)$. ■

We finish this section by making some general remarks on lifting properties. In Homotopy Theory, a map $p : E \to B$ which has the right lifting property with respect to the inclusion $i_{I^n,0} : I^n \to I^n \times I$ for every integer $n \geq 0$ is called a **Serre fibration**. Theorem 6.3.2 can then be restated as saying that the projection $p : E \to B$ of a locally trivial bundle is a Serre fibration. The proof we gave using Lebesgue's Covering Lemma and Feldbau's Theorem can be adapted to show the following slightly stronger result: *if $p : E \to B$ is a map for which there exists an open cover $\{U_j\}_{j\in J}$ of B such that the restricted projection $p|_{p^{-1}(U_j)} : p^{-1}(U_j) \to U_j$ is a Serre fibration for every $j \in J$, then p itself is a Serre fibration.* Brifely, a local Serre fibration is a Serre fibration. This implies Theorem 6.3.2 because every trivial bundle is a Serre fibration, so a locally trivial bundle is a local Serre fibration.

The results of this section show that a Serre fibration has the right lifting property with respect to the cylinder inclusion $X \to X \times I$ also when X is a CW-complex, and that under certain conditions we can impose additional conditions on the lift.

If we demand that $p : E \to B$ satisfies the stronger condition of having the right lifting property with respect to the inclusion $i_{X,0} : X \to X \times I$ for *any* space, not just cubes or CW-complexes, then we say that p is a **Hurewicz fibration** or simply a **fibration**.

This is a much stronger condition than being a Serre fibration, and it implies some nice properties on p like the homotopy equivalence of fibers, for example.

Over nice spaces there is a local-to-global result for detecting fibrations: *if B is a paracompact Hausdorff space, and $p : E \to B$ is a map for which there exists an open cover $\{U_j\}_{j\in J}$ of B such that the restricted projection $p|_{p^{-1}(U_j)} : p^{-1}(U_j) \to U_j$ is a fibration for every $j \in J$, then p itself is a fibration.* This result has a much more difficult proof than the analogous result for Serre fibrations requiring some serious juggling of partitions of unity. One important consequence of this is that every locally trivial bundle over a paracompact Hausdorff space is a fibration, since trivial bundles are clearly fibrations. We leave this registered here as a theorem, since we will need to refer to it later when talking about Obstruction Theory.

6.3.8 Theorem. If $p : E \to B$ is a locally trivial bundle with typical fiber F, and B is a paracompact Hausdorff space, then p is a fibration.

6.4 A long exact sequence

In this section we show how the lifting properties satisfied by the projection of a locally trivial bundle allow us to construct a long exact sequence relating the homotopy groups of the base space, the total space, and the typical fiber.

We first define a slightly generalized notion of exactness.

6.4.1 Definition. A triple of pointed *sets* and pointed *functions*

$$(X, x_0) \xrightarrow{\ f\ } (Y, y_0) \xrightarrow{\ g\ } (Z, z_0)$$

is said to be **exact** if the equality $\operatorname{Im} f = g^{-1}(z_0)$ holds.

A group G gives rise to the pointed set (G, e_G), where e_G is the identity element of the group, and since group homomorphisms preserve identities, a group homomorphism $f : G \to H$ gives rise to a pointed function $f : (G, e_G) \to (H, e_H)$. Notice that the definition of exact triple of pointed sets is such that a triple of groups and group homomorphisms

$$G \xrightarrow{\ f\ } H \xrightarrow{\ g\ } L$$

is exact if, and only if, the corresponding triple of pointed sets and pointed functions

$$(G, e_G) \xrightarrow{\ f\ } (H, e_H) \xrightarrow{\ g\ } (L, e_L)$$

is exact. During the rest of this chapter, whenever we encounter a group in a context involving pointed sets, we choose its identity element as basepoint, and therefore treat it as a pointed set.

We first construct exact triples of homotopy groups in different dimensions, and then later we study how these different triples can be connected to form a long exact sequence. The proof only uses the lifting properties proved in the previous section, so the results holds in fact for any Serre fibration.

6.4.2 Proposition. Let $p : E \to B$ be a locally trivial bundle with typical fiber F. Given a point $b_0 \in B$, let $i : p^{-1}(b_0) \hookrightarrow E$ be the inclusion of the fiber over b_0 into the total space. For any choice of basepoint $e_0 \in p^{-1}(b_0)$, and for any integer $n\ geq 0$, the sequence of pointed sets and pointed functions below is exact.

$$\pi_n(p^{-1}(b_0), e_0) \xrightarrow{\pi_n(i)} \pi_n(E, e_0) \xrightarrow{\pi_n(p)} \pi_n(B, b_0)$$

Proof. Let $[f] \in \pi_n(p^{-1}(b_0), e_0)$ be a pointed homotopy class represented by the pointed map $f : (S^n, *_{S^n}) \to (p^{-1}(b_0), e_0)$. Since the image of f is contained in the fiber over b_0, the composite $p \circ i \circ f$ is constant and equal to b_0, therefore

$$(\pi_n(p) \circ \pi_n(i))([f]_*) = [p \circ i \circ f]_* = [\mathrm{ct}_{S^n, b_0}]_*;$$

showing that the inclusion $\operatorname{Im} \pi_n(i) \subseteq \pi_n(p)^{-1}([\mathrm{ct}_{S^n, b_0}]_*)$ holds.

Conversely, suppose $[f]_* \in \pi_n(E, e_0)$ belongs to the kernel of p, so there exists a pointed homotopy $g : S^n \times I \to B$ from $p \circ f$ to ct_{S^n, b_0}. Recall from Example 1.2.5 that the pair $(S^n, *_{S^n})$ is n-cellular, therefore Corollary 6.3.6 allows us to lift the homotopy g through p by first lifting it partially over the subspace $(S^n \times \{0\}) \cup (\{*_{S^n}\} \times I)$. With this in mind, consider the map $G : (S^n \times \{0\}) \cup (\{*_{S^n} \times I\}) \to E$ defined as

$$G(x, t) \coloneqq \begin{cases} f(x), & \text{if } t = 0, \\ e_0, & \text{if } x = *_{S^n}. \end{cases}$$

This is well-defined as a function since for $x = *_{S^n}$ and $t = 0$ the first expression for G gives $f(*_{S^n}) = b_0$, which coincides with the value given by the second expression. It then follows from the Pasting Lemma that G is continuous.

We claim that G is a partial lift of g on $(S^n \times \{0\}) \cup (\{*_{S^n}\} \times I)$. On the one hand, for every $x \in S^n$, we have

$$p(G(x, 0)) = p(f(x)) = g(x, 0),$$

since g by hypothesis starts at $p \circ f$; and on the other

$$p(G(*_{S^n}, t)) = p(e_0) = b_0 = g(*_{S^n}, t)$$

for every $t \in I$, where we used the fact that g is a *pointed* homotopy.

Applying Corollary 6.3.6 we obtain a map $h : S^n \times I \to E$ making the diagram below commute.

$$\begin{tikzcd}
(S^n \times \{0\}) \cup (\{*_{S^n}\} \times I) \arrow[r, "G"] \arrow[d, hook] & E \arrow[d, "p"] \\
S^n \times I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Let $h_1 : S^n \to E$ be the "final stage" of the homotopy h, that is, $h_1(x) \coloneqq h(x,1)$ for every $x \in S^n$. Since h is a lift of g through p, we have

$$p(h_1(x)) = p(h(x,1)) = g(x,1) = \mathrm{ct}_{S^n,b_0}(x) = b_0,$$

showing that h_1 takes values in the fiber $p^{-1}(b_0)$, so we may regard it as a map $h_1 : S^n \to p^{-1}(b_0)$. Moreover, since h coincides with G on $\{*_{S^n}\} \times I$, we have

$$h_1(*_{S^n}) = h(*_{S^n}, 1) = G(*_{S^n}, 1) = e_0,$$

so h_1 defines a *pointed* map $(S^n, *_{S^n}) \to (p^{-1}(b_0), e_0)$. Lastly, since h defines a pointed homotopy from f to h_1, since

$$h(x,0) = G(x,0) = f(x)$$

holds for every $x \in S^n$, and

$$h(*_{S^n}, t) = G(*_{S^n}, t) = e_0$$

holds for every $t \in I$. We conclude at last that $\pi_n(i)([h_1]_*) = [i \circ h_1]_* = [f]_*$; proving that the reverse inclusion $\pi_n(p)^{-1}([\mathrm{ct}_{S^n,b_0}]_*) \subseteq \mathrm{Im}\,\pi_n(i)$ also holds.

We remark that the proof given also works for $n = 0$, since for any pointed space (X, x_0), there is a natural pointed bijection between the set of path-components $\pi_0(X, x_0) \coloneqq (\pi_0(X), [x_0])$ and the set of pointed homotopy classes $([S^0, X]_*, [\mathrm{ct}_{S^0,x_0}]_*)$. ■

Even more important than this exactness property is the fact that there are connecting maps relating the various exact sequences above turning them into a long exact sequence. We first describe this construction before analyzing its exactness properties.

6.4.3 Construction. Given a locally trivial bundle $p : E \to B$ with typical fiber F, choose a basepoint $b_0 \in B$, as well as a basepoint $e_0 \in p^{-1}(b_0) \subseteq E$. We will construct, for every integer $n \geq 1$, a function $\partial_n : \pi_n(B, b_0) \to \pi_{n-1}(p^{-1}(b_0))$. The construction depends only on the lifting properties proved in the previous section, so it works more generally for Serre fibrations.

Consider an element $[f]_* \in \pi_n(B, b_0)$ represented by a pointed map $f : (S^n, *_{S^n}) \to (B, b_0)$. Recall from Proposition 3.4.7 that there is a pointed homeomorphism

$$W : (\Sigma S^{n-1}, *) \xrightarrow{\cong} (S^n, *_{S^n},)$$

and this allows us to define a map of type $S^{n-1} \times I \to B$ via the composition shown below,

$$\begin{tikzcd} S^{n-1} \times I \arrow[r, "\pi"] \arrow[rrr, dashed, bend right, "f\circ W\circ\pi"'] & \Sigma S^{n-1} \arrow[r, "W"] & S^n \arrow[r, "f"] & B \end{tikzcd}$$

where $\pi : S^{n-1} \times I \to \Sigma S^{n-1}$ denotes the canonical projection.

Let $\mathrm{ct}_{e_0} : (S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I) \to E$ be the map which is constant and equal to e_0. Since the projection π maps $(S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$ to the basepoint $*$ of the reduced suspension, and W and f are both pointed maps, the composition $f \circ W \circ \pi$ is constant and equal to b_0 on $(S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$, therefore ct_{e_0} is a partial lift of $f \circ W \circ \pi$ over this subspace, or equivalently, the outer square in the diagram below is commutative. Applying Corollary 6.3.6 we then obtain a diagonal map $h^f : S^{n-1} \times I \to E$ making the whole diagram commute.

$$\begin{tikzcd} (S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I) \arrow[r, "\mathrm{ct}_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ S^{n-1} \times I \arrow[r, "f\circ W\circ\pi"'] \arrow[ur, dashed, "h^f"] & B \end{tikzcd}$$

We will say that a map $S^{n-1} \times I \to E$ satisfying the same commutativity properties of h^f is *adapted to* f.

Let $h^f_1 : S^{n-1} \to E$ be the final stage of the homotopy h^f, that is,

$$h^f_1(x) \coloneqq h^f(x, 1) \quad \forall x \in S^{n-1}.$$

The commutativity properties of h^f imply that h^f_1 is pointed, since

$$h^f_1(*_{S^{n-1}}) = h^f(*_{S^{n-1}}, 1) = e_0.$$

Moreover, since h^f lifts $f \circ W \circ \pi$, and π maps the subspace $S^{n-1} \times \{1\}$ to the basepoint $*$ of the reduced suspension, for any $x \in S^{n-1}$ we have the chain of equalities

$$\begin{aligned} p(h^f_1(x)) &= p(h^f(x, 1)) \\ &= f(W(\pi(x, 1))) \\ &= f(W(*)) \\ &= f(*_{S^n}) \\ &= b_0, \end{aligned}$$

which means that the image of h^f_1 is contained entirely in the fiber $p^{-1}(b_0)$; therefore we can regard it as a pointed map $h^f_1 : (S^{n-1}, *_{S^{n-1}}) \to (p^{-1}(b_0), e_0)$.

It is tempting to define a function $\partial_n : \pi_n(B, b_0) \to \pi_{n-1}(p^{-1}(b_0), e_0)$ via the formula

$$\partial_n([f]_*) \coloneqq [h^f_1]_*.$$

By the previous paragraph we know that this defines an element of $\pi_{n-1}(p^{-1}(b_0), e_0)$ indeed, but we need to be careful. Before doing so we need to check that this definition is homotopy invariant, that is, if $f' : (S^n, *_{S^n}) \to (B, b_0)$ is pointed homotopic to f, we must show that applying the above procedure to f' instead of f yields the same element of $\pi_{n-1}(p^{-1}(b_0), e_0)$. There is another more subtle possible problem with this definition. The homotopy h^f was obtained from the lifting properties of p, but Corollary 6.3.6 is only concerned with the *existence* of lifts, it does not say anything about *uniqueness*. If we choose a different homotopy $h : S^{n-1} \times I \to E$ satisfying the same commutativity properties of h^f, do we obtain the same element of $\pi_{n-1}(p^{-1}(b_0), e_0)$?

We first show that the pointed homotopy class $[h^f_1]_*$ is independent of the lift h^f, since this will be crucial for studying the other properties of ∂_n. Suppose $h : S^{n-1} \times I \to E$ is another homotopy adapted to f, so it fits in a commutative diagram analogous to (6.4.3). We would like to construct a pointed homotopy $H : S^{n-1} \times I \to p^{-1}(b_0)$ such that $H(x, 0) = h^f_1(x) = h^f(x, 1)$ and $H(x, 1) = h(x, 1)$. The idea is that this homotopy is given by the final stage of a "homotopy of homotopies" $\widetilde{H} : (S^{n-1} \times I) \times I \to E$, which can be constructed by applying a suitable lifting property. This lifting property comes from the fact that the pair

$$(S^{n-1} \times I,\, S^{n-1} \times \{0\} \cup S^{n-1} \times \{1\} \cup \{*_{S^{n-1}}\} \times I)$$

is n-cellular according to Proposition 1.2.8, since by Example 1.2.5 $(S^{n-1}, \{*_{S^{n-1}}\})$ is $(n-1)$-cellular, while by Example 1.2.3 $(I, \partial I)$ is 1-cellular.

Let $\varphi : (S^{n-1} \times I) \times I \to B$ be defined as

$$\varphi((x, s), t) \coloneqq (f \circ W \circ \pi)(x, t) \quad \forall\, ((x, s), t) \in (S^{n-1} \times I) \times I.$$

According to Corollary 6.3.6, we can lift φ through p by first lifting it partially over the subspace

$$X \coloneqq (S^{n-1} \times I) \times \{0\} \;\cup\; (S^{n-1} \times \{0\} \cup S^{n-1} \times \{1\} \cup \{*_{S^{n-1}}\} \times I) \times I.$$

Consider then the map $\Phi : X \to E$ defined as follows:

$$\Phi((x, s), t) \coloneqq \begin{cases} e_0, & \text{if } t = 0, \\ h^f(x, t), & \text{if } s = 0, \\ h(x, t), & \text{if } s = 1, \\ e_0, & \text{if } x = *_{S^{n-1}}. \end{cases}$$

The domains of definition of the first two expressions for Φ intersect on $(S^{n-1} \times \{0\}) \times \{0\}$, and the two expressions agree on this intersection since $h^f(x, 0) = e_0$ holds for every $x \in S^{n-1}$. Continuing like this we can show that, whenever two of the expressions appearing in the definition of Φ have intersecting domains of definition, the two possible expressions agree on this intersection. Proving this is as tedious as it sounds. In the end, this argument shows that Φ is well-defined, and its continuity then follows from the Pasting Lemma once again.

Using the fact that h^f and h both lift $f \circ W \circ \pi$, and working case by case we can show that Φ is a partial lift of φ over the subspace X, and Corollary 6.3.6 then gives us a map $H : (S^{n-1} \times I) \times I \to E$ as shown below.

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, hook] & E \arrow[d, "p"] \\ (S^{n-1} \times I) \times I \arrow[r, "\varphi"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

Let $H_1 : S^{n-1} \times I \to E$ be the final stage of H, and notice that it takes values in $p^{-1}(b_0)$, since

$$p(H_1(x,s)) = p(H((x,s),1)) = \varphi((x,s),1) = f(W(\pi(x,1))) = b_0.$$

The following are true for every $x \in S^{n-1}$:

1. $H_1(x,0) = H((x,0),1) = h^f(x,1) = h^f_1(x)$,
2. $H_1(x,1) = H((x,1),1) = h(x,1) = h_1(x)$,
3. $H_1(*_{S^{n-1}}, s) = H((*_{S^{n-1}}, s), 1) = e_0$.

This shows that H_1 defines a pointed homotopy $h^f_1 \simeq_* h_1$, therefore the element $\partial_n([f]_*)$ is independent of the choice of homotopy adapted to f.

Now we show that the element $[h^f_1]_* \in \pi_{n-1}(p^{-1}(b_0), e_0)$ only depends on the pointed homotopy class of f. Let $f' : (S^n, *_{S^n}) \to (B, b_0)$ be another pointed map such that $f' \simeq_* f$, and let $g : S^n \times I \to B$ be a pointed homotopy from f' to f. Let h^f, $h^{f'}$: $S^{n-1} \times I \to E$ be homotopies adapted to f and f', respectively. The idea is to use g together with the lifting properties of $p : E \to B$ to construct a pointed homotopy from $h^{f'}_1$ to h^f_1. In order to do this, consider first the map $\varphi : (S^{n-1} \times I) \times I \to B$ defined as

$$\varphi((x,s),t) \coloneqq g(W(\pi(x,t)),s)$$

for every $((x,s),t) \in (S^{n-1} \times I) \times I$. Consider also $\Phi : X \to E$ defined as follows:

$$\Phi((x,s),t) \coloneqq \begin{cases} e_0, & \text{if } t = 0, \\ h^{f'}(x,t), & \text{if } s = 0, \\ h^f(x,t), & \text{if } s = 1, \\ e_0, & \text{if } x = *_{S^{n-1}}. \end{cases}$$

Comparing each of the two expressions and using the commutativity properties defining $h^{f'}$ and h^f one can show that Φ is well-defined, and its continuity follows from the Pasting Lemma. Working case by case we can show that Φ is a partial lift of φ over X. For example, if $s = 0$, then on the one hand

$$p(\Phi((x,0),t)) = p(h^{f'}(x,t)) = f'(W(\pi(x,t))),$$

while on the other

$$\varphi((x,0),t) = g(W(\pi(x,t)),0) = f'(W(\pi(x,t)));$$

so Φ lifts φ over $(S^{n-1} \times \{0\}) \times I \subseteq X$. Reasoning like this for each of the closed subsets that make up X we eventually show that the equality $p \circ \Phi = g|_X$ is indeed true.

Applying Corollary 6.3.6 we obtain the map $H : (S^{n-1} \times I) \times I \to E$ depicted in the commutative diagram below.

$$\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, hook] & E \arrow[d, "p"] \\
(S^{n-1} \times I) \times I \arrow[r, "\varphi"'] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

Like before, let $H_1 : (S^{n-1} \times I) \times I$ be the final stage of H_1. Using the fact that H lifts φ, together with the fact that g is a pointed homotopy, we can show that H takes values in the fiber $p^{-1}(b_0)$. Using the commutativity properties of H, for every $x \in S^{n-1}$ we have the following:

1. $H_1(x,0) = H((x,0),1) = h^{f'}(x,1) = h_1^{f'}(x)$;

2. $H_1(x,1) = H((x,1),1) = h^f(x,1) = h_1^f(x)$;

3. $H_1(*_{S^{n-1}}, s) = H((*_{S^{n-1}}, s), 1) = e_0$.

It follows that H defines a pointed homotopy $h_1^{f'} \simeq_* h_1^f$.

We have shown at last that the function $\partial_n : \pi_n(B, b_0) \to \pi_{n-1}(p^{-1}(b_0), e_0)$ sending $[f]_*$ to $[h_1^f]_*$, where $h^f : (S^{n-1} \times I) \times I \to E$ is *any* homotopy adapted to f, is well-defined. We will refer to it as the **connecting map** associated with the triple $(p^{-1}(b_0), e_0) \rightarrowtail (E, e_0) \to (B, b_0)$. The main result of the present section concerns the algebraic and exactness properties of this connecting map.

6.4.4 Theorem. Let $p : E \to B$ be a locally trivial bundle with typical fiber F, and choose basepoints $b_0 \in B$ and $e_0 \in p^{-1}(b_0) \subseteq E$.

1. The various connecting maps fit together in a long exact sequence of pointed sets.

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
\pi_n(p^{-1}(b_0), e_0) \arrow[r, "\pi_n(i)"] & \pi_n(E, e_0) \arrow[r, "\pi_n(p)"] & \pi_n(B, b_0) \arrow[d, "\partial_n"] \\
\pi_{n-1}(B, b_0) \arrow[d] & \pi_{n-1}(E, e_0) \arrow[l, "\pi_{n-1}(p)"'] & \pi_{n-1}(p^{-1}(b_0), e_0) \arrow[l, "\pi_{n-1}(i)"'] \\
\vdots \arrow[d] & & \\
\pi_1(p^{-1}(b_0), e_0) \arrow[r, "\pi_1(i)"] & \pi_1(E, e_0) \arrow[r, "\pi_1(p)"] & \pi_1(b, b_0) \arrow[d, "\partial_1"] \\
\pi_0(B, b_0) & \pi_0(E, e_0) \arrow[l, "\pi_0(p)"'] & \pi_0(p^{-1}(b_0), e_0) \arrow[l, "\pi_0(i)"']
\end{tikzcd}$$

2. The connecting map $\partial_n : \pi_n(B, b_0) \to \pi_{n-1}(p^{-1}(b_0), e_0)$ is a group homomorphism for every integer $n \geq 2$.

Proof. 1. We first show that the triples of the form

$$\pi_n(b, b_0) \xrightarrow{\partial_n} \pi_{n-1}(p^{-1}(b_0), e_0) \xrightarrow{\pi_{n-1}(i)} \pi_{n-1}(E, e_0)$$

are exact. Given $[f]_* \in \pi_n(B, b_0)$, let $h : S^{n-1} \times I \to E$ be a homotopy adapted to f, so that we have the equality $\partial_n([f]_*) = [h_1]_*$. We need to show that $i \circ h_1 : S^{n-1} \to E$ is pointed null homotopic. This is actually easy, because since h is adapted to f, it satisfies the two following properties:

1. $h(x, 0) = e_0 = \mathrm{ct}_{S^{n-1}, e_0}$ for every $x \in S^{n-1}$,
2. $h(*_{S^{n-1}}, t) = e_0$ for every $t \in I$.

It follows that h defines a pointed homotopy $\mathrm{ct}_{S^{n-1}, e_0} \simeq_* i \circ h_1$, from which we deduce that $\pi_{n-1}(i)([h_1]_*) = [\mathrm{ct}_{S^{n-1}, e_0}]_*$. This proves that the inclusion $\mathrm{Im}\, \partial_n \subseteq \pi_{n-1}(i)^{-1}([\mathrm{ct}_{S^{n-1}}, e_0]_*)$ holds.

Conversely, suppose $[f]_* \in \pi_{n-1}(p^{-1}(b_0), e_0)$ is such that $i \circ f \simeq_* \mathrm{ct}_{S^{n-1}, e_0}$, and let $h : S^{n-1} \times I \to E$ be a pointed homotopy from $\mathrm{ct}_{S^{n-1}, e_0}$ to $i \circ f$. It is reasonable to look for a pointed map $F : (S^n, *_{S^n}) \to (B, b_0)$ such that h itself is adapted to it. In order to do this, notice that h is constant and equal to e_0 on the subspace $(S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$, therefore $p \circ h$ is constant and equal to b_0 on this same subspace. Moreover,

$$p(h(S^{n-1} \times \{1\})) = p(f(S^{n-1})) \subseteq \{b_0\},$$

where we used the fact that f takes values in $p^{-1}(b_0)$ by hypothesis.

In summary, $p \circ h : S^{n-1} \times I \to B$ is constant and equal to b_0 on $(S^{n-1} \times \{0\}) \cup (S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$, therefore it can be factored through $\pi_{n-1} : S^{n-1} \times I \to \Sigma S^{n-1}$ to define a pointed map $\overline{p \circ h} : (\Sigma S^{n-1}, *_{\Sigma S^{n-1}}) \to (B, b_0)$, and using it we define $F : (S^n, *_{S^n}) \to (B, b_0)$ via the composition

$$F \coloneqq \overline{p \circ h} \circ W_{n-1}^{-1}$$

as shown below.

$$\begin{tikzcd} S^{n-1} \times I \arrow[r, "p\circ h"] \arrow[d, "\pi_{n-1}"'] & B \\ \Sigma S^{n-1} \arrow[ur, "\overline{p\circ h}"] & S^n \arrow[l, "W_{n-1}^{-1}"] \arrow[u, dashed, "F"'] \end{tikzcd}$$

The map F really has h as an associated homotopy since

$$\begin{aligned} F \circ W_{n-1} \circ \pi_{n-1} &= \overline{p \circ h} \circ W_{n-1}^{-1} \circ W_{n-1} \circ \pi_{n-1} \\ &= \overline{p \circ h} \circ \pi_{n-1} \\ &= p \circ h. \end{aligned}$$

It follows that $\partial_n([F]_*) = [h_1]_* = [f]_*$, proving that the inclusion $\pi_{n-1}(i)^{-1}([\mathrm{ct}_{S^{n-1},e_0}]_*) \subseteq \operatorname{Im} \partial_n$ also holds.

The only thing left is showing the exactness of the triples of the form

$$\pi_n(E, e_0) \xrightarrow{\pi_n(p)} \pi_n(B, b_0) \xrightarrow{\partial_n} \pi_{n-1}(p^{-1}(b_0), e_0).$$

Consider an arbitrary element $[f]_* \in \pi_n(E, e_0)$, and let $h : S^{n-1} \times I \to E$ be defined via the composition

$$h \coloneqq f \circ W_{n-1} \circ \pi_{n-1},$$

as shown in the diagram below.

$$\begin{tikzcd} S^{n-1} \times I \arrow[r, "\pi_{n-1}"] \arrow[rrr, dashed, bend right, "h"'] & \Sigma S^{n-1} \arrow[r, "W_{n-1}"] & S^n \arrow[r, "f"] & E \end{tikzcd}$$

Since π_{n-1} maps $(S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$ to $*_{\Sigma S^{n-1}}$, and W_{n-1} and f are both pointed, h maps $(S^{n-1} \times \{0\}) \cup (\{*_{S^{n-1}}\} \times I)$ to e_0. Moreover, the definition of h immediately implies the equality

$$p \circ h = (p \circ f) \circ W_{n-1} \circ \pi_{n-1},$$

i.e., h is adapted to $p \circ f$. It follows that $\partial_n([p \circ f]_*) = [h_1]_*$, but $h_1 = \mathrm{ct}_{S^{n-1},e_0}$ since π_{n-1} maps $S^{n-1} \times \{1\}$ to $*_{\Sigma S^{n-1}}$. We have then proved the inclusion $\operatorname{Im} \pi_n(p) \subseteq \partial_n^{-1}([\mathrm{ct}_{S^{n-1}}, e]_*)$.

The reverse inclusion is a bit harder to show. Let $[f]_* \in \pi_n(B, b_0)$ be such that $\partial_n([f]_*) = [\mathrm{ct}_{S^{n-1},e_0}]_*$ in $\pi_{n-1}(p^{-1}(b_0), e_0)$. This means that, if $h : S^{n-1} \times I \to E$ is a homotopy adapted to f, then we have an equality $[h_1]_* = [\mathrm{ct}_{S^{n-1},e_0}]_*$, so there is a pointed homotopy $h_{\mathrm{aux}} : S^{n-1} \times I \to p^{-1}(b_0)$ from h_1 to $\mathrm{ct}_{S^{n-1},e_0}$. The map h_1 is not necessarily constant and equal to e_0, but we claim that by using h_{aux} and the lifting properties of p we can modify h to obtain another homotopy $h' : S^{n-1} \times I \to E$ which is still adapted to f, but which is also constant and equal to e_0 on $S^{n-1} \times \{1\}$.

In order to do this, let $\varphi : (S^{n-1} \times I) \times I \to B$ be defined as

$$\varphi((x,s),t) \coloneqq (f \circ W_{n-1} \circ \pi_{n-1})(x,s).$$

We can lift φ partially over the subspace

$$X \coloneqq (S^{n-1} \times I) \times \{0\} \cup (S^{n-1} \times \{0\}) \times I \cup (S^{n-1} \times \{1\}) \times I \cup (\{*_{S^{n-1}}\} \times I) \times I$$

by defining $\Phi : X \to E$ as follows:

$$\Phi((x,s),t) \coloneqq \begin{cases} h(x,s), & \text{if } t = 0, \\ e_0, & \text{if } s = 0, \\ h_{\mathrm{aux}}(x,t), & \text{if } s = 1, \\ e_0, & \text{if } x = *_{S^{n-1}}. \end{cases}$$

One can show that Φ is well-defined by comparing the expressions above in pairs. Moreover, using that h is adapted to f, and that the image of h_{aux} is contained in the fiber $p^{-1}(b_0)$, one can show that Φ is indeed a lift of φ through p. We thus obtain the diagonal map $\Psi : (S^{n-1} \times I) \times I \to E$ shown in the commutative diagram below.

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, hook] & E \arrow[d, "p"] \\ (S^{n-1} \times I) \times I \arrow[r, "\varphi"'] \arrow[ur, dashed, "\Psi"] & B \end{tikzcd}$$

We now define $h' : S^{n-1} \times I \to E$ as $h'(x,s) \coloneqq \Psi((x,s),1)$ for every $(x,s) \in S^{n-1} \times I$. This map h' satisfies the following properties:

(i) $h'(x,0) = \Psi((x,0),1) = e_0$ for every $x \in S^{n-1}$;

(ii) $p(h'(x,s)) = p(\Psi((x,s),1)) = \varphi((x,s),1) = (f \circ W_{n-1} \circ \pi_{n-1})(x,s)$ for every $(x,s) \in S^{n-1} \times I$;

(iii) $h'(x,1) = \Psi((x,1),1) = h_{\mathrm{aux}}(x,1) = e_0$ for every $x \in S^{n-1}$.

The first two properties say that h' is adapted to f, while the third one says that h' is constant and equal to e_0 on $S^{n-1} \times \{1\}$; thus h' is the homotopy we were looking for.

Equipped with this improved homotopy h', we now construct the desired map $F : (S^n, *_{S^n}) \to (E, e_0)$. Consider the map $\varphi : (S^{n-1} \times I) \times I \to B$ as defined before, but this time let $\Phi' : X \to E$ be the alternative lifting defined as follows:

$$\Phi'((x,s),t) := \begin{cases} h'(x,s), & \text{if } t = 0, \\ e_0, & \text{if } s = 0, \\ e_0, & \text{if } s = 1, \\ e_0, & \text{if } x = *_{S^{n-1}}. \end{cases}$$

Notice that, since $h'(x, 1) = e_0$ for every $x \in S^{n-1}$, Φ' is well-defined, and a case-by-case computation shows that it really lifts φ over X. The lifting properties of p then imply the existence of a diagonal map $H : (S^{n-1} \times I) \times I \to E$ as shown below.

$$\begin{tikzcd} X \arrow[r, "\Phi'"] \arrow[d, hook] & E \arrow[d, "p"] \\ (S^{n-1} \times I) \times I \arrow[r, "\varphi"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

The definition of Φ' ensures that, for every $t \in I$, the inclusion

$$H((S^{n-1} \times \{0\}) \times \{t\} \cup (S^{n-1} \times \{1\}) \times \{t\} \cup (\{*_{S^{n-1}}\} \times I) \times \{t\}) \subseteq \{e_0\}$$

holds, therefore H can be factored through the quotient $\pi_{n-1} \times \mathrm{id}_I : (S^{n-2} \times I) \times I \to \Sigma S^{n-1} \times I$ to define $\overline{H} : \Sigma S^{n-1} \times I \to E$. Using this factored homotopy we define $K : S^n \times I \to E$ via the composition $K := \overline{H} \circ (W_{n-1}^{-1} \times \mathrm{id}_I)$.

We would like to show that the final stage $F := K_1$ of the homotopy K is the desired map. Notice first that F is pointed since

$$F(*_{S^n}) = K(*_{S^n}, 1) = \overline{H}(W_{n-1}^{-1}(*_{S^n}), 1) = \overline{H}(*_{\Sigma S^{n-1}}, 1) = H((*_{S^{n-1}}, 0), 1) = e_0.$$

Moreover, since $F = K_1 \simeq_* K_0$, if we can show the equality $p \circ K_0 = f$, then the homotopy relation $p \circ F \simeq_* f$ will follow. In order to show this, we begin by noting that

$$\begin{aligned} p \circ \overline{H}_0 \circ \pi_{n-1} &= p \circ H_0 \\ &= \varphi_0 \\ &= f \circ W_{n-1} \circ \pi_{n-1}, \end{aligned}$$

and by eliminating π_{n-1} we deduce that

$$p \circ \overline{H}_0 = f \circ W_{n-1}.$$

Using this expression we then see that K_0 satisfies

$$\begin{aligned} p \circ K_0 &= p \circ \overline{H}_0 \circ W_{n-1}^{-1} \\ &= f \circ W_{n-1} \circ W_{n-1}^{-1} \\ &= f. \end{aligned}$$

Overall, we have seen that F defines an element $[F]_* \in \pi_n(E, e_0)$ such that $\pi_n(p)([F]_*) = [p \circ F]_* = [f]_*$; proving that the reverse inclusion $\partial_n^{-1}([\mathrm{ct}_{S^{n-1}, e_0}]_*) \subseteq \operatorname{Im} \pi_n(p)$ also holds. ■

6.4.5 Remark. A personal remark about the proof above. The reader has certainly noticed that we did not prove that the connecting map ∂_n is a group homomorphism. In the literature, the most common proof is actually indirect: one first shows that, for any pair (X, A) and any basepoint $x_0 \in A$, there is an exact sequence of pointed sets and groups of the form

$$\cdots \longrightarrow \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \xrightarrow{\Delta} \pi_{n-1}(A, x_0) \longrightarrow \cdots .$$

Here, $\pi_n(X, A, x_0)$ denotes the *relative* homotopy sets of the pair (X, A) at the basepoint x_0. These sets have a group structure when $n \geq 2$, and they are also abelian when $n \geq 3$. They capture in some sense the possible loss of injectivity of the map $\pi_n(A, x_0) \to \pi_n(X, x_0)$ induced by the inclusion $(A, x_0) \rightarrowtail (X, x_0)$. In the sequence above, $\Delta : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is a naturally defined map which is easily shown to be a group homomorphism in suitable dimensions.

When $p : E \to B$ is a locally trivial bundle (or more generally a Serre fibration), the lifting properties can be used to show that the projection p induces a bijection (isomorphism)

$$p_* : \pi_n(E, p^{-1}(A), e_0) \xrightarrow{\cong} \pi_n(B, A, b_0)$$

of relative homotopy sets (groups), where $b_0 \in A$ is a basepoint, and $e_0 \in p^{-1}(b_0) \subseteq p^{-1}(A)$ is a basepoint lying over b_0. If we consider $A \coloneqq \{b_0\}$ as being the basepoint itself, then the aforementioned isomorphism becomes $\pi_n(E, p^{-1}(b_0), e_0) \cong \pi_n(B, \{b_0\}, b_0)$, and if we take into account that there is also an isomorphism $\pi_n(B, \{b_0\}, b_0) \cong \pi_n(B, b_0)$, we end up with an isomorphism

$$p_* : \pi_n(E, p^{-1}(b_0), e_0) \xrightarrow{\cong} \pi_n(B, b_0).$$

This isomorphism can be used to "divert" the long exact sequence associated with the pointed pair $(E, p^{-1}(b_0), e_0)$ by considering the composite map depicted below.

$$\begin{tikzcd}
\pi_n(E, p^{-1}(b_0), e_0) \arrow[r, "\Delta"] \arrow[d, "p_*"'] & \pi_{n-1}(p^{-1}(b_0), e_0) \\
\pi_n(B, b_0) \arrow[ur, dashed] &
\end{tikzcd}$$

The composition $\Delta \circ (p_*)^{-1} : \pi_n(B, b_0) \to \pi_{n-1}(p^{-1}(b_0), e_0)$ is precisely the connecting map ∂_n that we constructed. Notice that, in this approach, the fact that ∂_n is a group homomorphism in certain dimensions follows from the fact that both Δ and p_* are group homomorphisms.

As I said, this is the most common approach for constructing the long exact sequence of homotopy groups of a Serre fibration. However, due to time and space limitations, I decided to not include a discussion of relative homotopy groups in the text, so that is why we had to construct the sequence from scratch. Using the lifting properties we managed to prove that it is exact, but I couldn't for the life of me prove that ∂_n is a group homomorphism directly, and I also did not find a reference that proves it without appealing to the exact sequence of a pair. The closest proofs that I found in the literature are in (GOERSS; JARDINE, 2009, Section 1.7, Lemma 7.3) and (LURIE, 2022, Subsection 3.2.4, Proposition 3.2.4.4). These two proofs, however, are written in the language of *simplicial sets*, but since the homotopy theory of simplicial sets is equivalent (in a precise sense) to the classical homotopy theory of spaces, they can probably be translated and adapted to our context.

6.4.1 Some computations

In this subsection we apply the long exact sequence constructed above to explicitly compute some homotopy groups that will be useful later on.

We start with one of the simplest non-trivial spaces whose homotopy groups can be fully determined.

6.4.6 Proposition. The homotopy groups of the circle can be described as follows:

$$\pi_j(S^1, *_{S^1}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{if } j > 1. \end{cases}$$

Proof. We showed in Example 6.1.11 that the map $E : \mathbb{R} \to S^1$ defined as $E(t) := e^{2\pi i t}$ is a locally trivial bundle with typical fiber the discrete space $\mathbb{Z}$. The origin $0 \in \mathbb{R}$ is in the fiber $E^{-1}(*_{S^1}) = \mathbb{Z}$, so consider the following fragment of the exact sequence of the bundle in question:

$$\pi_j(\mathbb{Z}, 0) \xrightarrow{\pi_j(i)} \pi_j(\mathbb{R}, 0) \xrightarrow{\pi_j(E)} \pi_j(S^1, *_{S^1}) \xrightarrow{\partial_j} \pi_{j-1}(\mathbb{Z}, 0)$$

If $j > 1$, since the fiber $\mathbb{Z}$ is discrete, and the spheres S^j and S^{j-1} are connected (it is important that $j > 1$ for this!), the only pointed maps $(S^j, *_{S^j}) \to (\mathbb{Z}, 0)$ and $(S^{j-1}, *_{S^{j-1}}) \to (\mathbb{Z}, 0)$ are the constant ones, therefore the groups $\pi_j(\mathbb{Z}, 0)$ and $\pi_{j-1}(\mathbb{Z}, 0)$ appearing at the endpoints of the sequence are both trivial. Exactness then implies that we have an isomorphism

$$\pi_j(E) : \pi_j(\mathbb{R}, 0) \xrightarrow{\cong} \pi_j(S^1, *_{S^1}).$$

Now, the real line $\mathbb{R}$ can be contracted to 0, i.e., $(\mathbb{R}, 0)$ is pointed contractible, therefore $\pi_j(\mathbb{R}, 0)$ is trivial, and the isomorphism above implies that $\pi_j(S^1, *_{S^1})$ is also trivial for $j > 1$.

Now let us deal with the fundamental group. We have the following fragment of the exact sequence:

$$\pi_1(\mathbb{R},0) \xrightarrow{\pi_1(E)} \pi_1(S^1,*_{S^1}) \xrightarrow{\partial_1} \pi_0(\mathbb{Z},0)$$

There is a natural pointed bijection $\pi_0(\mathbb{Z},0) \cong (\mathbb{Z},0)$ mapping a pointed homotopy class $[f : (S^0,+1) \to (\mathbb{Z},0)]_*$ to the integer $f(-1)$. Notice that this is indeed well-defined, since if $f' : (S^0,+1) \to (\mathbb{Z},0)$ is another pointed map, and $h : f \Rightarrow f'$ is a pointed homotopy, the fact $\mathbb{Z}$ is discrete forces the restriction $h|_{\{-1\}\times I}$ to be constant, therefore $f(-1) = f'(-1)$.

Under this identification, the connecting map $\partial_1 : \pi_1(S^1,*_{S^1}) \to (\mathbb{Z},0)$ can be described as follows: given an element $[f]_* \in \pi_1(S^1,*_{S^1})$, if $h^f : S^0 \times I \to \mathbb{R}$ is a homotopy adapted to f, then

$$\partial_1([f]_*) = h_1^f(-1) = h^f(-1,1).$$

Keeping this identification in mind, we claim that the connecting map ∂_1 is in fact a group homomorphism, with $\mathbb{Z}$ regarded as a group via its addition operation. Suppose $[f]_*$, $[g]_* \in \pi_1(S^1,*_{S^1})$ are such that $\partial_1([f]_*) = m$ and $\partial_1([g]_*) = n$. This means that there are homotopies h^f, $h^g : S^0 \times I \to \mathbb{R}$ adapted to f and g, respectively, such that $h^f(-1,1) = m$ and $h^g(-1,1) = n$. Now, the product $[f]_* \cdot [g]_* \in \pi_1(S^1,*_{S^1})$ is equal to $[\langle f,g\rangle \circ \mu_{S^1}]_*$, so in order to evaluate $\partial_1([f]_* \cdot [g]_*)$, we need to find a homotopy $H : S^0 \times I \to \mathbb{R}$ adapted to $\langle f,g\rangle \circ \mu_{S^1}$, which means that it must fit in the commutative diagram below.

$$\begin{tikzcd}
(S^0 \times \{0\}) \cup (\{+1\} \times I) \arrow[r, "\mathrm{ct}_0"] \arrow[d, hook] & \mathbb{R} \arrow[d, "E"] \\
S^0 \times I \arrow[ur, "H"] \arrow[r, "\langle f,g\rangle \circ \mu_{S^1} \circ W_0 \circ p_0"'] & S^1
\end{tikzcd}$$

Unpacking the definition of the H-comultiplication μ_{S^1}, we can show that the map on the bottom can be alternatively described as

$$\langle f,g\rangle \circ \mu_{S^1} \circ W_0 \circ p_0 = \langle f \circ W_0, g \circ W_0\rangle \circ \mu_0 \circ p_0,$$

where μ_0 is the standard H-comultiplication on ΣS^0. More explicitly, for any $(s,t) \in S^0 \times I$ we have

$$(\langle f \circ W_0, g \circ W_0\rangle \circ \mu_0 \circ p_0)(s,t) = \begin{cases} *_{S^1}, & \text{if } s = +1, \\ (f \circ W_0 \circ p_0)(-1,2t), & \text{if } s = -1 \text{ and } 0 \le t \le \frac{1}{2}, \\ (g \circ W_0 \circ p_0)(-1,2t-1), & \text{if } s = 1 \text{ and } \frac{1}{2} \le t \le 1. \end{cases}$$

Using the action of $\mathbb{Z}$ on $\mathbb{R}$ it is easy to combine h^f and h^g to obtain the desired homotopy. We define $H : S^0 \times I \to \mathbb{R}$ as follows:

$$H(s,t) \coloneqq \begin{cases} 0, & \text{if } s = +1, \\ h^f(-1, 2t), & \text{if } s = -1 \text{ and } 0 \leq t \leq \frac{1}{2}, \\ m + h^g(-1, 2t-1), & \text{if } s = -1 \text{ and } \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is well-defined, because $s = -1$ and $t = \frac{1}{2}$ we have

$$m + h^g(-1, 2 \cdot \tfrac{1}{2} - 1) = m + h^g(-1, 0) = m + 0 = m = h^f(-1, 1) = h^f(-1, 2 \cdot \tfrac{1}{2}).$$

It is clear from the definition that H maps the segment $\{+1\} \times I$ to 0, and since h^f maps $S^0 \times \{0\}$ to 0, the same is also true of H. Moreover, since h^f and h^g are adapted to f and g, we have the equalities

$$E \circ h^f = f \circ W_0 \circ p_0 \quad \text{and} \quad E \circ h^g = g \circ W_0 \circ p_0,$$

and using these we can show with a direct computation that

$$E \circ H = \langle f \circ W_0, g \circ W_0 \rangle \circ \mu_0 \circ p_0$$

also holds.

The conclusions of the previous paragraph show that H is adapted to $\langle f, g \rangle \circ \mu_{S^1}$, therefore

$$\begin{aligned} \partial_1([f]_* \cdot [g]_*) &= \partial_1([\langle f, g \rangle \circ \mu_{S^1}]_*) \\ &= H(-1, 1) \\ &= m + h^g(-1, 1) \\ &= m + n \\ &= \partial_1([f]_*) + \partial_1([g]_*); \end{aligned}$$

proving that ∂_1 is a group homomorphism.

The exactness of the sequence implies

$$\partial_1^{-1}(0) = \operatorname{Im} \pi_1(E),$$

but since $(\mathbb{R}, 0)$ is pointed contractible, $\pi_1(\mathbb{R}, 0)$ is the trivial group, thus the image $\operatorname{Im} \pi_1(E)$ above is also trivial. Knowing that ∂_1 is a homomorphism, the equality above means that its kernel is trivial, which implies that it is injective.

The only thing left is proving the surjectivity of ∂_1. Since $\mathbb{Z}$ is generated by 1, it suffices to prove that 1 belongs to the image of ∂_1. We will show that $\partial_1([\mathrm{id}_{S^1}]_*) = 1$. Let $h : S^0 \times I \to \mathbb{R}$ be defined as

$$h(s,t) \coloneqq \begin{cases} 0, & \text{if } s = +1, \\ t, & \text{if } s = -1. \end{cases}$$

According to the discussion after the proof of Proposition 3.4.7, the composition $W_0 \circ p_0 : S^0 \times I \to S^1$ is described explicitly

$$(W_0 \circ p_0)(s,t) = \begin{cases} *_{S^1}, & \text{if } s = +1, \\ e^{2\pi i t}, & \text{if } s = -1. \end{cases}$$

This means that the equality $E \circ h = W_0 \circ p_0$ holds, therefore h is adapted to id_{S^1}, and then by definition

$$\partial_1([\mathrm{id}_{S^1}]_*) = h(-1,1) = 1. \qquad \blacksquare$$

For the next computation, we will have to assume the following results about the homotopy groups of a sphere S^n for $n \geq 1$:

$$\pi_j(S^n, *_{S^n}) = \begin{cases} 0, & \text{if } 0 \leq j \leq n-1, \\ \mathbb{Z}, & \text{if } j = n. \end{cases}$$

6.4.7 Proposition. Given integers $1 \leq k \leq n$, the homotopy groups of the complex Stiefel manifold $V_k(\mathbb{C}^n)$ can be described as follows:

$$\pi_j(V_k(\mathbb{C}^n)) \cong \begin{cases} 0, & \text{if } 0 \leq j \leq 2(n-k); \\ \mathbb{Z}, & \text{if } j = 2(n-k)+1. \end{cases}$$

Proof. The proof is by a double induction on both n and k. If $n = 1$, then necessarily $k = 1$ too, and $V_1(\mathbb{C})$ is simply the set of unit norm complex numbers, that is, $V_1(\mathbb{C}) = S^1$, and from Proposition 6.4.6 we know that

$$\pi_j(V_1(\mathbb{C})) = \pi_j(S^1) \cong \begin{cases} 0, & \text{if } j = 0, \\ \mathbb{Z}, & \text{if } j = 1; \end{cases}$$

therefore the statement is true in this case.

Now suppose n is an integer such that the statement is true for all smaller non-negative integers. We then perform an induction on k. For $k = 1$, $V_1(\mathbb{C}^n)$ is the set of unit norm vectors on $\mathbb{C}^n$, so if we consider the usual identification $\mathbb{C}^n = \mathbb{R}^{2n}$, then $V_1(\mathbb{C}^n) = S^{2n-1}$. If we then use the result about homotopy groups of spheres mentioned above, we deduce that

$$\pi_j(V_1(\mathbb{C}^n)) = \pi_j(S^{2n-1}) \cong \begin{cases} 0, & \text{if } 0 \leq j \leq 2n-2, \\ \mathbb{Z}, & \text{if } j = 2n-1; \end{cases}$$

which is what we had to show.

Now, suppose $2 \leq k \leq n$, and assume that the statement has been proved for integers smaller than k. We then have to prove that the statement holds for k itself.

The trick to use the inductive hypotheses (both of them!) is to recall that the map $q_{k,1} : V_k(\mathbb{C}^n) \to V_1(\mathbb{C}^n)$ mapping a k-frame $(v_1, \dots, v_k)$ to the vector v_1 is a locally trivial bundle with typical fiber $V_{k-1}(\mathbb{C}^{n-1})$, as was showed in Example 6.1.15.

Let us first deal with path-connectedness. Choosing appropriated basepoints $b_0 \in V_1(\mathbb{C}^n)$ and $e_0 \in q_{k,1}^{-1}(b_0)$, if we let $e_0' \in V_{k-1}(\mathbb{C}^{n-1})$ be the image of e_0 with respect to the identification $q_{k,1}^{-1}(b_0) \cong V_{k-1}(\mathbb{C}^{n-1})$, then from the exact sequence of a locally trivial bundle we have the following exact triple of pointed sets:

$$\pi_0(V_{k-1}(\mathbb{C}^{n-1}), e_0') \longrightarrow \pi_0(V_k(\mathbb{C}^n), e_0) \longrightarrow \pi_0(V_1(\mathbb{C}^n), b_0).$$

The inductive hypotheses imply that both pointed sets at the sides contain only a single element, therefore the same is true of $\pi_0(V_k(\mathbb{C}^n), e_0)$; proving the path-connectedness of $V_k(\mathbb{C}^n)$.

Keeping in mind this path-connectedness, we now omit the basepoints from our notation. For $j \geq 1$, we have the following exact sequence of homotopy groups:

$$\pi_j(V_{k-1}(\mathbb{C}^{n-1})) \longrightarrow \pi_j(V_k(\mathbb{C}^n)) \longrightarrow \pi_j(V_1(\mathbb{C}^n)) \longrightarrow \pi_{j-1}(V_{k-1}(\mathbb{C}^{n-1})).$$

Notice that $2((n-1)-(k-1)) = 2(n-k)$, so if $j \leq 2(n-k)$, by the inductive hypotheses we know that the groups $\pi_j(V_{k-1}(\mathbb{C}^{n-1}))$ and $\pi_{j-1}(V_{k-1}(\mathbb{C}^{n-1}))$ are both trivial. Exactness then implies that there is an isomorphism

$$\pi_j(V_k(\mathbb{C}^n)) \cong \pi_j(V_1(\mathbb{C}^n)),$$

and since $2(n-k) \leq 2n-2$ because $k \geq 2$, j is smaller than $2n-2$, so $\pi_j(V_1(\mathbb{C}^n))$ is trivial; therefore $\pi_j(V_k(\mathbb{C}^n))$ is also trivial.

The only case left is when $j = 2(n-k)+1$. We now consider the following fragment of the exact sequence:

$$\pi_{j+1}(V_1(\mathbb{C}^n)) \longrightarrow \pi_j(V_{k-1}(\mathbb{C}^{n-1})) \longrightarrow \pi_j(V_k(\mathbb{C}^n)) \longrightarrow \pi_j(V_1(\mathbb{C}^n)).$$

Since $k \geq 2$, $n-k \leq n-2$, therefore

$$j = 2(n-k)+1 \leq 2(n-2)+1 = 2n-3.$$

It follows that both j and $j+1$ are smaller than $2n-2$, therefore the groups $\pi_{j+1}(V_1(\mathbb{C}^n))$ and $\pi_j(V_1(\mathbb{C}^n))$ appearing at the endpoints are both trivial. We then have an isomorphism

$$\pi_{2(n-k)+1}(V_{k-1}(\mathbb{C}^{n-1}) \cong \pi_{2(n-k)+1}(V_k(\mathbb{C}^n)).$$

Now we just need to notice that $2((n-1)-(k-1))+1 = 2(n-k)+1$, so by the inductive hypothesis

$$\pi_{2(n-k)+1}(V_{k-1}(\mathbb{C}^{n-1})) \cong \mathbb{Z};$$

therefore $\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) \cong \mathbb{Z}$ also holds. ∎

There is a similar result for real Stiefel manifolds, but it is more complicated.

6.4.8 Proposition. Given integers $1 \leq k \leq$, the real Stiefel manifold $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, that is, it is path-connected, and $\pi_j(V_k(\mathbb{R}^n))$ is trivial for every $1 \leq j \leq n-k-1$. Moreover, its $(n-k)$-th homotopy group is given by

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1 \text{ or if } n-k \text{ is even,} \\ \mathbb{Z}_2, & \text{otherwise.} \end{cases}$$

CHAPTER

7

OBSTRUCTION THEORY

We have at last developed all the concepts required for analyzing the main results from Obstruction Theory, which is the goal of the present chapter. We start out by broadly describing the goals of Obstruction Theory in terms of the so-called *extension-lifting property*. This is a big class of problems whose analysis in the most general case is way beyond the scope of this text, so, after giving some examples of such problems, we turn our attention to a particular class of them: constructing and extending maps and section of locally trivial bundles over CW-complexes. We first deal with the case of globally trivial bundles where the problem can be restated in terms of constructing and extending maps. In this simple case we can already discuss some important subtleties that arise, and also get a taste of the different techniques and ingredients required in the proofs. After studying this simple case, we go back to the more general context of only locally trivial bundles where the same results continue to hold, but with an added layer of possible complications.

Many of the proofs in this chapter would be too technical, or would at the very least require too much sidetracking to recall the necessary concepts, so we decided to many of the proofs, and focus instead on the flow of ideas and on the subtleties that must be dealt with when trying to formalize all the results.

7.1 Extension-lifting problems

Many of the problems one encounters when doing Algebraic Topology have the following form: we are given spaces E, B and X, a subspace $A \subseteq X$, and also maps

$f : A \to E$, $p : E \to B$ and $g : X \to B$ that fit in a commutative square.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd} \tag{7.1}$$

We will usually say that a diagram like this *poses an extension-lifting problem*. We are then interested in finding a diagonal $h : X \to E$ such that the resulting diagram is still commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd} \tag{7.2}$$

This commutativity is equivalent to the two following equalities:

1. $p \circ h = g$;
2. $h|_A = f$.

The first equality says that h is a **lift** of g through p, while the second one says that h is an **extension** of f. Diagram (7.1) says that f is a lift of g, but only over the subspace $A \subseteq X$, i.e., f is a *partial lift*. The map h then furnishes an extension of the partial lift f to a global lift defined on the whole space X. We call such map h a *solution* to the extension-lifting problem posed by (7.1).

Let us see some examples of common problem that can be formulated in this framework of extension-lifting problems.

7.1.1 Example (Lifting homotopies)**.** Consider the extension-lifting problem posed by the commutative diagram below.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"'] & B \end{tikzcd}$$

In this case, the map g is a homotopy between the maps $g_0, g_1 : X \to B$, and the commutativity condition says that f (or more precisely the composition $f \circ i_{X,0}$) is a lift of the initial stage g_0 of g through p.

If $h : X \times I \to E$ is a solution to the problem, then it is a homotopy lifting the entire homotopy g through p, and moreover its initial stage h_0 coincides with the given initial lift f. When such a solution *always* exist, it means that, in order to lift a homotopy $X \times I \to B$ to a homotopy $X \times I \to E$, we just need to lift its initial stage. We have already encountered problems of this kind when studying locally trivial bundles (or Serre fibrations more generally). We saw in Corollary 6.3.7 that, if $p : E \to B$ is a locally trivial bundle with typical fiber F, then the extension-lifting problem in question always has a solution if X is a CW-complex.

7.1.2 Example (Extending maps)**.** Consider the extension-lifting problem posed by the commutative diagram below,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & X \times F \arrow[d, "\pi_1"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

where $\pi_1 : X \times F \to X$ denotes the canonical projection to the first coordinate.

The commutativity condition says that f is a **section** of π_1 over the subspace $A \subseteq X$, i.e., a partial section. If $F : X \to X \times F$ is a solution to the problem, then it defines a global section of π_1. This can be reformulated in more familiar terms by noticing that, since F is a map into a product, it is completely determined by the compositions $\pi_1 \circ F$ and $\pi_2 \circ F$, and since $\pi_1 \circ F$ is forced to be equal to id_X by the commutativity conditions, F is completely determined by the map $\pi_2 \circ F : X \to X \times F$. In other words, constructing a section of π_1 extending f is the same thing as constructing a map of type $X \to F$ extending f.

7.1.3 Example (Constructing and extending sections)**.** Let $p : E \to X$ be a locally trivial bundle with typical fiber F, and consider the extension-lifting problem posed by the following diagram.

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, tail] & E \arrow[d, "p"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

The commutativity condition expressed above says that s is **partial section** of the bundle over the subspace $A \subseteq X$. If $S : X \to E$ solves the extension-lifting problem, then it defines a **global section** of p extending the one defined over the subspace A.

Notice that the previous example is a particular example of this one, since the projection onto the first coordinate $\pi_1 : X \times F \to X$ is the worldwide famous trivial bundle with typical fiber F.

This problem of extending and constructing sections of bundles is in general non-trivial. For example, $TS^2 \setminus \{\mathbf{0}\}$ be the space of *non-zero* tangent vectors on the 2-sphere S^2. The usual tangent bundle projection $p : TS^2 \to S^2$ restricts to a locally trivial bundle projection $p' : TS^2 \setminus \{\mathbf{0}\} \to S^2$ whose fiber has the homotopy type of a circle.

This bundle certainly admits local sections over some subspace of S^2. For example, if we regard S^1 as a subspace of S^2 via the embedding at the equator, then we can consider the vector field $v : S^1 \to TS^2 \setminus \mathbf{0}$ running parallel to the equator. We could then search for a solution to the extension-lifting problem

$$\begin{tikzcd} S^1 \arrow[r, "v"] \arrow[d, tail] & TS^2 \setminus \{\mathbf{0}\} \arrow[d, "p'"] \\ S^2 \arrow[r, "\mathrm{id}_{S^2}"'] & S^2 \end{tikzcd}$$

However, one of the most important results of Algebraic Topology says that this particular problem cannot be solved: if $V : S^2 \to TS^2 \setminus \{\mathbf{0}\}$ were such a solution, then it would define a vector field on S^2 without any singularities, but this is impossible by the *hairy ball theorem*.

When we later study the results of Obstruction Theory, we will see that the lack of solution to this problem is simultaneously related to two algebro-topological properties: the non-vanishing of the first homotopy group $\pi_1(S^1)$ of the circle, and also the non-vanishing of the second homology group $H_2(S^2)$ of the sphere.

7.2 Obstruction Theory for maps

In this section we develop Obstruction Theory for analyzing the problem of constructing and extending maps. As we discussed in one of the examples of the previous section, given a pair (X, A), a space Y, and a map $f : A \to Y$, the problem of extending f to a map $F : X \to Y$ is equivalently the extension-lifting problem posed by the diagram below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & X \times Y \arrow[d, "\pi_1"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

We make the simplifying assumption that the pair (X, A) is n-cellular for some integer $n \geq 0$. The advantage of this is that we can then restate the extension problem in homotopical terms by using Corollary 2.3.11.

7.2.1 Theorem. Let Y be any space, and let (X, A) be an n-cellular pair for some integer $n \geq 0$, with $\{\Phi_e : D^n \to X\}_{e \in \mathcal{E}}$ being its family of characteristic maps. Given a map $f : A \to Y$, the following are equivalent:

1. f can be extended to a map $F : X \to Y$;

2. the map $f_e := f \circ \varphi_e : S^{n-1} \to Y$ is null homotopic for every $e \in \mathcal{E}$.

Proof. Suppose first that the extension F exists. Consider, for each $e \in \mathcal{E}$, the composite map $F_e : D^n \to Y$ defined as $F_e := F \circ \Phi_e$. Now, since $\varphi_e = \Phi_e|_{S^{n-1}}$, $F|_A = f$, and

$\varphi_e(S^{n-1}) \subseteq A$, we see that

$$F|_{S^{n-1}} = F \circ \Phi_e|_{S^{n-1}} = F \circ \varphi_e = f,$$

therefore F_e is an extension of f_e, It then follows from Corollary 2.3.11 that each of the maps f_e is null homotopic.

Conversely, suppose each of the maps $f_e : S^{n-1} \to Y$ is null homotopic. Applying Corollary 2.3.11 again implies the existence of maps $F_e : D^n \to Y$ extending f_e for every $e \in \mathcal{E}$.

Recall now that by the definition of cellular pair we have the pushout square below.

$$\begin{tikzcd} \coprod_{e\in\mathcal{E}} S^{n-1} \arrow[r, "\langle\varphi_e\rangle"] \arrow[d, tail] & A \arrow[d, tail] \\ \coprod_{e\in\mathcal{E}} D^n \arrow[r, "\langle\Phi_e\rangle"'] & X \end{tikzcd}$$

The various maps F_e together induce a map $\langle F_e \rangle : \coprod_{e\in\mathcal{E}} D^n \to Y$, and since for each $e \in \mathcal{E}$ the restriction $F_e|_{S^{n-1}}$ equal $f_e = f \circ \varphi_e$, the "outer shell" of the diagram below commutes; therefore by the universal property of the pushout we obtain the diagonal map $F : X \to Y$ depicted.

$$\begin{tikzcd} \coprod_{e\in\mathcal{E}} S^{n-1} \arrow[r, "\langle\varphi_e\rangle"] \arrow[d, tail] & A \arrow[d, tail] \arrow[ddr, bend left, "f"] & \\ \coprod_{e\in\mathcal{E}} D^n \arrow[r, "\langle\Phi_e\rangle"'] \arrow[drr, bend right, "\langle F_e\rangle"'] & X \arrow[dr, dashed, "F"] & \\ & & Y \end{tikzcd}$$

It is then immediate from the commutativity properties of F that it extends the initial map f. ■

Since relative CW-complexes are obtained by successive cell attachments to an initial subspace, it seems reasonable to apply this result to the problem of extension on relative CW-complexes.

Let (X, A) be a relative CW-complex with skeletal filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

Given a map $f : A \to Y$, we can use the previous result to investigate the possibility of extending f to higher stages of the skeletal filtration.

By the definition of relative CW-complex we know that the pair (A, X_0) is 0-cellular, which means that X_0 is given by the disjoint union of A and a disjoint collection of points.

$$X_0 \cong A \sqcup \left(\coprod_{e \in \mathcal{E}_0} \{\mathrm{pt}\} \right).$$

It is then always possible to extend f to X_0: if we choose for each $e \in \mathcal{E}_0$ an arbitrary point $y_e \in Y$, then the collection of maps $\{\mathrm{ct}_{\{\mathrm{pt}\},y_e}\}_{e\in\mathcal{E}_0}$ gives rise to a map $g : \coprod_{e\in\mathcal{E}_0}\{\mathrm{pt}\} \to Y$, and taking into account the description of X_0 as a disjoint union we then have the induced map

$$f_0 \coloneqq \langle f, g\rangle : X_0 \to Y.$$

Okay, so extending over to the 0-skeleton is always possible, but what about extending to the 1-skeleton? According to the definition of relative CW-complex, the pair (X_1, X_0) is 1-cellular, so there exists a collection of characteristic maps $\{\Phi_e : I \to X_1\}_{e\in\mathcal{E}_1}$ that fit in the pushout square below.

$$\begin{tikzcd}
\coprod\limits_{e\in\mathcal{E}_1} \{0,1\} \arrow[r, "\langle\varphi_e\rangle"] \arrow[d, tail] & X_0 \arrow[d, tail] \\
\coprod\limits_{e\in\mathcal{E}_1} I \arrow[r, "\langle\Phi_e\rangle"'] & X_1
\end{tikzcd}$$

We already have the partial extension $f_0 : X_0 \to Y$, so it seems reasonable to look for a map $g : \coprod_{e\in\mathcal{E}_1} I \to Y$ which we can combine with f_0 to use the universal property of the pushout to obtain a map $f_1 : X_1 \to Y$. In order for this to work, the maps g and f_0 must be compatible in some way. Precisely, for every $e \in \mathcal{E}_1$, the maps $(g \circ i_e)|_{\{0,1\}}$, $f_0 \circ \varphi_e : \{0, 1\} \to Y$ must coincide, which means that $g \circ i_e$ is a path in Y from $f_0(\varphi_e(0))$ to $f_0(\varphi_e(1))$. If we want to be sure that such a map exists, we can suppose that the target space Y is *path-connected*. If this is the case, then for every $e \in \mathcal{E}_1$ we can choose a path $g_e : I \to Y$ from $f_0(\varphi_e(0))$ to $f_0(\varphi_e(1))$. All these paths together define a map

$$\langle g_e\rangle : \coprod_{e\in\mathcal{E}_1} I \to Y,$$

and by construction it fits together with $f_0 \circ \langle\varphi_e\rangle$ in the commutative "outer shell" of the diagram below.

$$\begin{tikzcd}
\coprod\limits_{e\in\mathcal{E}_1} \{0,1\} \arrow[r, "\langle\varphi_e\rangle"] \arrow[d, tail] & X_0 \arrow[d, tail] \arrow[ddr, bend left, "f_0"] & \\
\coprod\limits_{e\in\mathcal{E}_1} I \arrow[r, "\langle\Phi_e\rangle"'] \arrow[drr, bend right, "\langle g_e\rangle"'] & X_1 \arrow[dr, dashed, "f_1"] & \\
 & & Y
\end{tikzcd}$$

The diagonal map $f_1 : X_1 \to Y$ is obtained via the universal property of the pushout, and it is clear from the commutativity properties of the diagram that f_1 is an extension of f_0, and therefore also an extension of f.

The crucial part in this reasoning of extending to the 1-skeleton was assuming that Y is path-connected. This simpler case already illustrates the general principle that how far it can extend f up the skeletal fibration depends on the connectivity of the target space. The next result makes this precise.

7.2.2 Theorem. Let (X, A) be a relative CW-complex with skeletal filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

If Y is a $(n-1)$-connected space for some integer $n \geq 1$, then any map $f : A \to Y$ can be extended to a map $f_n : X_n \to Y$ defined on the n-skeleton.

Proof. The discussion above shows that f can always be extended to a map $f_0 : X_0 \to Y$ defined on the 0-skeleton.

Now suppose we have extended f to a map $f_k : X_k \to Y$ on the k-skeleton for some integer $0 < k \leq n-1$. We will show that f_k can then be extended to the next stage $f_{k+1} : X_{k+1} \to Y$ of the filtration. By definition of relative CW-complex we know the pair (X_{k+1}, X_k) is $(k+1)$-cellular, so there exists a collection of characteristic maps $\{\Phi_e : D^{k+1} \to X_{k+1}\}_{e \in \mathcal{E}_{k+1}}$ which together fit in a pushout square.

$$\begin{tikzcd}
\coprod\limits_{e \in \mathcal{E}_{k+1}} S^k \arrow[r, "\langle \varphi_e \rangle"] \arrow[d, tail] & X_k \arrow[d, tail] \\
\coprod\limits_{e \in \mathcal{E}_{k+1}} D^{k+1} \arrow[r, "\langle \Phi_e \rangle"'] & X_{k+1}
\end{tikzcd}$$

For each $e \in \mathcal{E}_{k+1}$, we can regard the composition $f_k \circ \varphi_e : S^k \to Y$ as a pointed map of type $(S^k, *_{S^k}) \to (Y, y_e)$, where $y_e := f_k(\varphi_e(*_{S^k}))$. This map then defines an element $[f_k \circ \varphi_e]_*$ of the k-th homotopy group $\pi_k(Y, y_e)$. Since $k \leq n$, and Y is n-connected, this homotopy group is in fact trivial, therefore $[f_k \circ \varphi_e]_* = [\mathrm{ct}_{S^k, y_e}]_*$, which implies that $f_k \circ \varphi_e$ is null homotopic (even pointed null homotopic).

The reasoning above shows that $f_k \circ \varphi_e$ is null homotopic for every $e \in \mathcal{E}_{k+1}$. Theorem 7.2.1 then implies that f_k can be extended to a map $f_{k+1} : X_{k+1} \to Y$ as claimed. Now, if k was equal to $n-1$, then we have succeeded in extending f to the n-skeleton; if not, then $k+1$ is still strictly smaller than n, and the argument still applies, so we can then extend f_{k+1} to a map $f_{k+2} : X_{k+2} \to Y$. Continuing like this inductively we eventually obtain a map $f_n : X_n \to Y$ extending the initial map f. ∎

7.2.1 The obstruction cocycle

Theorem 7.2.2 tells us that, the more connected the target space Y is, the easier it is to extend a map $f : A \to Y$ through higher stages of the skeletal filtration of a relative CW-complex (X, A).

But what happens when Y has some non-vanishing homotopy groups? Suppose we have a partially extended map $f_n : X_n \to Y$ defined on the n-skeleton, and that Y does not necessarily have trivial homotopy groups in dimension n. By the definition of CW-complex, the pair (X_{n+1}, X_n) is $(n+1)$-cellular, so there is a collection of characteristic maps $\{\Phi_e : D^{n+1} \to X_n\}_{n\in\mathcal{E}_{n+1}}$ that give rise to a certain pushout square. Theorem 7.2.1 still applies, so that f_n can be extended to a map $f_{n+1} : X_{n+1} \to Y$ if and only if the compositions $f_n \circ \varphi_e : S^n \to Y$ are null homotopic for every $e \in \mathcal{E}_{n+1}$. If like before we set $y_e := f_n(\varphi_e(*_{S^n}))$, and we assume at the very least that Y is *path-connected*, then the pointed map $f_n \circ \varphi_e : (S^n, *_{S^n}) \to (Y, y_e)$ is null homotopic if and only if $[f_n \circ \varphi_e]_*$ is the trivial element of $\pi_n(Y, y_e)$.

7.2.3 Remark. Here we have to be a bit careful with basepoints. If $[f_n \circ \varphi_e]_*$ defines the trivial element of $\pi_n(Y, y_e)$, then $f_n \circ \varphi_e$ is certainly null homotopic, even pointed null homotopic. For the converse, if $f_n \circ \varphi_e$ is null homotopic, since this homotopy need not be pointed, its final stage might not define an element of $\pi_n(Y, y_e)$. More precisely, if $h : S^n \times I \to Y$ is the homotopy of $f_n \circ \varphi_e$ to a constant map $\mathrm{ct}_{S^n,y}$ for some point $y \in Y$, then we have the trivial element of $\pi_n(Y, y)$, and not of $\pi_n(Y, y_e)$.

Nonetheless, if we consider the path $\gamma : I \to Y$ defined as $\gamma(t) := h(*_{S^n}, t)$ for every $t \in I$, that is, if we keep track of how the image of the basepoint $*_{S^n}$ moves during the null homotopy, we have the transport map $t_\gamma : \pi_n(Y, y_e) \to \pi_n(Y, y)$. Moreover, h is a homotopy adapted to $f_n \circ \varphi_e$ and γ, so we know that the image $t_\gamma([f_n \circ \varphi_e]_*)$ is exactly the pointed homotopy class $[h_1]_*$, which we know is the trivial element of $\pi_n(Y, y)$. Since the transport map t_γ is an *isomorphism* of homotopy groups according to Proposition 5.5.6, it follows that $[f_n \circ \varphi_e]_*$ must also be equal to the trivial element of $\pi_n(Y, y_e)$.

In our present context, it might very well happen that $[f_n \circ \varphi_e]_*$ does *not* define the trivial element of $\pi_n(Y, y_e)$. In this case, it is then not possible to extend f_n further to the $(n+1)$-skeleton, the non-vanishing homotopy class $[f_n \circ \varphi_e]_*$ represents an *obstruction* to the possibility of extending.

Our goal is to quantify algebraically this extension, and in order to do this we use the *cellular cohomology* of the relative CW-complex (X, A). Let us very briefly recall how this is constructed. If $\{\Phi_e : D^n \to X_{n-1}\}_{e\in\mathcal{E}_n}$ are the characteristic maps for the n-skeleton, we can consider the collection of open cells $\{\Phi_e(D^n \setminus S^{n-1})\}_{e\in\mathcal{E}_n}$. The group of n-dimensional relative cellular chains $\mathcal{C}_n(X, A)$ can be defined as the free abelian group generated by the open n-cells of (X, A). A boundary operator $\partial_n : \mathcal{C}_n(X, A) \to \mathcal{C}_{n-1}(X, A)$

is constructed as follows, if $\Phi_e(D^n \setminus S^{n-1})$ is an open n-cell of (X, A), and $\Phi_f(D^{n-1} \setminus S^{n-2})$ is an open $(n-1)$-cell of (X, A), it turns out that there is a homeomorphism

$$X_{n-1}/(X_{n-1} \setminus \Phi_f(D^{n-1} \setminus S^{n-2})) \cong D^{n-1}/S^{n-2} \cong S^{n-1},$$

so via the composition

$$S^{n-1} \xrightarrow{\varphi_e} X_{n-1} \xrightarrow{p} X_{n-1}/(X_{n-1} \setminus \Phi_f(D^{n-1} \setminus S^{n-2})) \xrightarrow{\cong} D^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$$

we end up with a self-map $S^{n-1} \to S^{n-1}$ whose degree is called the *incidence number* of the n-cell $\Phi_e(D^n \setminus S^{n-1})$ and the $(n-1)$-cell $\Phi_f(D^{n-1} \setminus S^{n-2})$, and is denoted by $[e : f]$. We then define the boundary operator $\partial_n : \mathcal{C}_n(X, A) \to \mathcal{C}_{n-1}(X, A)$ on the n-cell $\Phi_e(D^n \setminus S^{n-1})$ as

$$\partial_n(\Phi_e(D^n \setminus S^{n-1})) := \sum_{f \in \mathcal{E}_{n-1}} [e : f] \cdot \Phi_f(D^{n-1} \setminus S^{n-2}),$$

where the sum is over all the $(n-1)$-cells of (X, A). Since $\mathcal{C}_n(X, A)$ is freely generated by the open n-cells, the definition above can be extended by linearity to a group homomorphism $\partial_n : \mathcal{C}_n(X, A) \to \mathcal{C}_{n-1}(X, A)$.

This procedure then gives us a chain complex $\{\mathcal{C}_n(X, A)\}_{n \geq 0}$ whose homology groups define the *relative cellular homology groups* of the relative CW-complex (X, A). The construction can then be dualized to define cellular cohomology, that is, we consider an abelian group G, define the group of cellular cochains $\mathcal{C}^n(X, A; G)$ as the group of homomorphisms $\mathcal{C}_n(X, A) \to G$, and define a coboundary operator

$$\delta^n : \mathcal{C}^n(X, A; G) \to \mathcal{C}^{n+1}(X, A; G)$$

by dualizing ∂_{n+1}. We end up with a cochain complex $\{\mathcal{C}^n(X, A; G)\}_{n \geq 0}$ whose cohomology groups define the *relative cellular cohomology groups with coefficients in* G of (X, A)

Back to our obstruction problem, we have seen that the possibility of extending $f_n : X_n \to Y$ further to f_{n+1} is controlled by the elements $[f_n \circ \varphi_e]_* \in \pi_n(Y, y_e)$. We can consider this as assigning an element of $\pi_n(Y, y_e)$ to each $(n+1)$-cell in the $(n+1)$-skeleton of (X, A), something that looks a lot like a cellular cochain as described above. There is a problem, however, since the elements $[f_n \circ \varphi_e]_*$ assigned to the $(n+1)$-cells live in different groups for each $e \in \mathcal{E}_{n+1}$. We can make the assumption that Y is path-connected, which allows us to identify the homotopy groups $\pi_n(Y, y_e)$ for different $(n+1)$-cells $e \in \mathcal{E}_{n+1}$, but recall that these identifications are not canonical, they are obtained by transporting along paths between the various basepoints, and we might run into problems if we choose paths arbitrarily. At this point, we make the simplifying assumption that the target space Y is n-simple, which allows us to *canonically* identify the homotopy groups at different basepoints. Keeping this in mind, we from now on denote the homotopy groups of Y simply by $\pi_n(Y)$, omitting the basepoint.

With these considerations in place, we can make the following definition.

7.2.4 Definition. Let (X, A) be a relative CW-complex, and let Y be an n-simple space. Given a map $f : X_n \to Y$ defined on the n-skeleton, the cellular $(n+1)$-cochain obtained by assigning to each $(n+1)$-cell $\Phi_e(D^{n+1} \setminus S^n)$ of (X, A) the pointed homotopy class $[f_n \circ \varphi_e]_* \in \pi_n(Y)$ is called the **obstruction cochain of** f, and is denoted by $\theta^{n+1}(f) \in \mathcal{C}^{n+1}(X, A; \pi_n(Y))$.

Since a relative cellular cochain in $\mathcal{C}^{n+1}(X, A; \pi_n(Y))$ is the same thing as a group homomorphism $\mathcal{C}_{n+1}(X, A) \to \pi_n(Y)$, and $\mathcal{C}_{n+1}(X, A)$ is by definition freely generated by the open $(n+1)$-cells of (X, A), a cellular cochain vanishes if and only if it maps each open $(n+1)$-cell to the trivial element of $\pi_n(Y)$. Specializing this to the case of the obstruction cochain $\theta^{n+1}(f) \in \mathcal{C}^{n+1}(X, A; \pi_n(Y))$ we obtain the following algebraic reformulation of Theorem 7.2.1:

7.2.5 Corollary. Let (X, A) be a relative CW-complex, and let Y be an n-simple space. A map $f : X_n \to Y$ defined on the n-skeleton of (X, A) admits an extension to the $(n+1)$-skeleton if and only if its obstruction cochain $\theta^{n+1}(f) \in \mathcal{C}^{n+1}(X, A; \pi_n(Y))$ vanishes.

We have taken a step towards an algebraic condition for the existence of an extension of $f : X_n \to Y$ to the $(n+1)$-skeleton with Corollary 7.2.5. Unfortunately, it is not of much practical use, since verifying if $\theta^{n+1}(f)$ vanishes is the same thing as verifying if the maps $f \circ \varphi_e : S^n \to Y$ are null homotopic.

Nevertheless, this reformulation suggests that the extension problem has a cohomological nature to it. This is indeed true, and the main result of Obstruction Theory makes precise the connection between extensions and cohomology. The crucial result behind this connection is the following:

7.2.6 Theorem. Let (X, A) be a relative CW-complex, and let Y be an n-simple space. If $f : X_n \to Y$ is a map, then its obstruction cochain $\theta^{n+1}(f) \in \mathcal{C}^{n+1}(X, A; \pi_n(Y))$ is in fact a relative cellular cocycle.

All proofs of this result rely in some way or another on auxiliary results that tell us how the homotopy groups are related to the homology groups. The most famous result of this type is the *Hurewicz Theorem*, which basically states that the homotopy group $\pi_n(X, x_0)$ of a path-connected space is isomorphic to the corresponding homology group $H_n(X)$ if the homotopy groups $\pi_i(X, x_0)$ are trivial for all $1 \leq i \leq n-1$.[1] For proofs, see for example (STEENROD, 1951, Part III, Section 32) and (DAVIS; KIRK, 2001, Theorem 7.6).

[1] There is a subtlety in dimension $n = 1$, since $\pi_1(X, x_0)$ is in general non-abelian, even if X is path-connected (that is, if $\pi_0(X)$ is trivial). In this case, the Hurewicz Theorem states that $H_1(X)$ is isomorphic to the *abelianization* of $\pi_1(X, x_0)$.

7.2.7 Definition. Let (X, A) be a relative CW-complex, and let Y be an n-simple space. Given a map $f : X_n \to Y$, the relative cohomology class $[\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n(Y))$ determined by the obstruction cocycle of f is called the **obstruction class** of f.

7.2.2 The difference cochain

We now take another step towards understanding the connection between extensions and cohomology by understanding how the obstruction cochain of f depends on its values on the previous skeleton X_{n-1}. The main result needed for this comparison is the following:

7.2.8 Lemma. Let (X, A) be a relative CW-complex, and let Y be an n-simple space. If f, $g : X_n \to Y$ are two maps such that their restriction $f|_{X_{n-1}}$, $g|_{X_{n-1}} : X_{n-1} \to Y$ to the previous skeleton are homotopic, then there exists a relative cellular cochain $d(f, g) \in \mathcal{C}^n(X, A; \pi_n(Y))$ satisfying the equality

$$\delta^n(d(f, g)) = \theta^{n+1}(f) - \theta^{n+1}(g).$$

Sketch of proof. Since $(I, \partial I)$ is a relative CW-complex with $X_0 = X_{-1} = \partial I$, and $X_1 = I$, where the only characteristic map is the identity $\mathrm{id}_I : (I, \partial I) \to (I, \partial I)$ (notice that we are making the harmless identification of pairs $(D^1, S^0) \cong (I, \partial I)$), the pair $(X \times I, A \times I)$ is also relative CW-complex whose k-th skeleton $(X \times I)_k$ is given by $(X_k \partial I) \cup (X_{k-1} \times I)$. In this case, the characteristic maps for the k-cells can be identified with the products

$$\Phi_e \times \mathrm{id}_I : D^k \times I \to (X \times I)_k$$

taking into account the existence of a homeomorphism of pairs

$$(D^k \times I, D^k \times \partial I \,\cup\, S^{k-1} \times I) \cong (D^{k+1}, S^k).$$[2]

Let $H : X_{k-1} \times I \to Y$ be a homotopy from $f|_{X_{n-1}}$ to $g|_{X_{n-1}}$. These maps can be combined to define a map $\Gamma : (X \times I)_n \to Y$ as

$$\Gamma(x, t) \coloneqq \begin{cases} f(x), & \text{if } x \in X_k,\ t = 0, \\ g(x), & \text{if } x \in X_k,\ t = 1, \\ H(x, t), & \text{if } x \in X_{k-1}. \end{cases}$$

This map Γ gives rise to an obstruction cochain $\theta^{n+1}(\Gamma) \in \mathcal{C}^{n+1}(X \times I, A \times I, \pi_n(Y))$ measuring the possibility of extending it to the $(n+1)$-skeleton of $X \times I$. Using this we define a cochain $d(f, g) \in \mathcal{C}^n(X, A; \pi_n(Y))$ as follows: given an open n-cell $\Phi^e(D^n \setminus S^{n-1})$,

[2] See the discussion at Example 1.2.7.

the product $\Phi_e \times \mathrm{id}_I : D^n \times \mathrm{id}_I$ can be identified with the characteristic map of an $(n+1)$-cell in $X \times I$, and we then define

$$d(\Phi_e(D^n \setminus S^{n-1})) := (-1)^{n+1} \cdot \theta^{n+1}(\Gamma)(\Phi_e(D^n \setminus S^{n-1}) \times (I \setminus \partial I)).$$

This assigns an element in $\pi_n(Y)$ to each open n-cell of (X, A) by considering the element that is assigned to a corresponding product open $(n+1)$-cell in $(X \times I, A \times I)$ by the obstruction cochain $\theta^{n+1}(\Gamma)$.

The fact that $\theta^n(d(f,g)) = \theta^{n+1}(f) - \theta^{n+1}(g)$ follows by direct comparison cell-by-cell using that $\theta^{n+1}(\Gamma)$ is a cocycle. There are some subtleties since we have to understand how the boundary operator of $(X \times I, A \times I)$ relates to the boundary operators of (X, A) and $(I, \partial I)$. See (DAVIS; KIRK, 2001, Lemma 7.8) for a detailed computation. ■

The cochain $d(f, g)$ constructed above is called the **difference cochain** of f and g. Notice that its definition requires first choosing a homotopy between the restrictions $f|_{X_{n-1}}$ and $g|_{X_{n-1}}$.

Lemma 7.2.8 allows us to prove the first part of the connection between extensions and cohomology classes.

7.2.9 Corollary. Let (X, A) be a relative Cw-complex, and let Y be an n-simple space. Given a map $f : X_n \to Y$, if there exists a map $F : X_{n+1} \to Y$ whose restriction $F|_{X_{n-1}}$ is homotopic to the restriction $f|_{X_{n-1}}$, then the obstruction class of f $[\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n(Y))$ vanishes.

Proof. Let $g := F|_{X_n} : X_n \to Y$, and notice that $g|_{X_{n-1}} = F|_{X_{n-1}}$, therefore $f|_{X_{n-1}}$ and $g|_{X_{n-1}}$ are homotopic by hypothesis. According to Lemma 7.2.8, there exists a difference cochain $d(f, g) \in \mathcal{C}^n(X, A; \pi_n(Y))$ such that

$$\delta^n(d(f,g)) = \theta^{n+1}(f) - \theta^{n+1}(g),$$

and since F is an extension of g, the obstruction cochain on the right vanishes, and we are then left with the equality $\theta^{n+1}(f) = \delta^n(d(f,g))$. This means that $\theta^{n+1}(f)$ is a coboundary in $\mathcal{C}^{n+1}(X, A; \pi_n(Y))$, from which we deduce that its cohomology class $[\theta^{n+1}(f)]$ is zero. ■

7.2.3 The main extension result

At the end of the previous subsection we proved that, if a map $f : X_n \to Y$ can be extended to X_{n+1} after being modified onX_n, in the sense that the restriction $f|_{X_{n-1}}$ is homotopically to a map which does extend to X_{n+1}, then the obstruction class $[\theta^{n+1}(f)]$ vanishes.

The goal of this subsection is to prove that the converse is also true, that is, if the obstruction class vanishes, then f can be extended to X_{n+1} after being modified on X_{n-1}.

In all the next results, (X, A) is a relative CW-complex, Y is an n-simples space.

The proof of the main result depends on the following:

7.2.10 Proposition (Realization)**.** Given a map $f_0 : X_n \to Y$, a homotopy $H : X_{n-1} \times I \to Y$ such that $H_0 = f_0|_{X_{n-1}}$, and a cochain $d \in \mathcal{C}^n(X, A; \pi_n(Y))$, there exists a map $f_1 : X_n \to Y$ such that $H_1 = f_1|_{X_{n-1}}$ and $d(f_0, f_1) = d$. In summary, given f_0 and H, any cochain of $\mathcal{C}^n(X, A; \pi_n(Y))$ can be realized as a difference cochain.

The proof depends on a simple homotopical result.

7.2.11 Lemma. For any map $f : D^n \times \{0\} \cup S^{n-1} \times I \to Y$, and for any homotopy class $\alpha \in [\partial(D^n \times I), Y]$[3], there exists a map $F : \partial(D^n \times I) \to Y$ such that $[F] = [\alpha]$, and whose restriction to $D^n \times \{0\} \cup S^{n-1} \times I$ coincides with f.

Proof. Consider the representing map $\alpha : \partial(D^n \times I) \to Y$. If D denotes the subspace $D^n \times \{0\} \cup S^{n-1} \times I$, then D is contractible, therefore f and $\alpha|_D$ are both null homotopic, therefore homotopic to one another. Consider then a homotopy $h : D \times I \to Y$ starting at f and ending at the restriction $\alpha|_D$, and notice that α itself extends the final stage of h. Since the pair $(\partial(D^n \times I), D)$ is n-cellular - we just need to attach the "cap" $D^n \times \{1\}$ - it satisfies the Homotopy Extension Property[4], therefore we can extend h to a homotopy $H : \partial(D^n \times I) \times I \to Y$ satisfying the following properties:

1. $H(x, 1) = \alpha(x)$ for every $x \in \partial(D^n \times I)$;

2. $H(x, t) = h(x, t)$ for every $(x, t) \in D \times I$.

We then let $F := H_0 : \partial(D^n \times I)$ be the initial stage of this extended homotopy. It is homotopic to α, so the equality $[F] = [\alpha]$ holds, and its restriction $F|_D$ coincides with f by virtue of the second property above. ■

Proof of Proposition 7.2.10. Consider an n-cell $\Phi_e : (D^n, S^{n-1}) \to (X_n, X_{n-1})$ of (X, A). Let $f : D^n \times \{0\} \cup S^{n-1} \times I \to Y$ be defined as

$$\lambda^e(x, t) := \begin{cases} f_0(\Phi_e(x)), & \text{if } x \in D^n,\ t = 0, \\ H(\varphi_e(x), t), & \text{if } x \in S^{n-1},\ t \in I. \end{cases}$$

[3] The notation $\partial(D^n \times I)$ denotes the "outer shell" of the filled cylinder $D^n \times I$, i.e., the subspace $S^{n-1} \times I \cup D^n \times \{0\} \cup D^n \times \{1\}$, which is topologically an n-sphere.

[4] Technically, we are using the fact that, if (X, A) is a cellular pair, then the inclusion $i : A \rightarrowtail X$ is a cofibration

Consider also $[\alpha] := d(\Phi_e(D^n \setminus S^{n-1}))$ the homotopy class assigned to the open cell $\Phi_e(D^n \setminus S^{n-1})$ by the cellular cochain d.[5]

Applying the previous Lemma we obtain a map $F^e : \partial(D^n \times I) \to Y$ such that $[F^e] = [\alpha]$ and whose restriction to D coincides with λ^e. We then let $f_1^e : D^n \to Y$ be defined as $f_1^e := F_1^e$. Doing this for every n-cell of (X, A) we can glue together the various maps f_1^e (formally, we use the universal property of the pushout) to obtain a map $f_1 : X_n \to Y$. Since $\lambda^e(x, 1) = H(\varphi_e(x), 1) = F^e(x, 1)$ holds for every $x \in S^{n-1}$ and every open n-cell, this map f_1 we have defined satisfies the equality $f|_{X_{n-1}} = H_1$.

The only thing left is showing that the difference cochain $d(f_0, f_1)$ has the specified value. If we go over its construction, we see that it assigns to each n-cell $\Phi_e(D^n \setminus S^{n-1})$ an element $\pi_n(Y)$ which can be identified with the class $[\Gamma \circ (\Phi_e \times \mathrm{id}_I)] \in [\partial(D^n \times I), Y]$, where $\Gamma : X_n \times \partial I \cup X_{n-1} \times I \to Y$ is defined as

$$\Phi(x, t) := \begin{cases} f_0(x), & \text{if } x \in x_n,\ t = 0, \\ f_1(x), & \text{if } x \ni X_n,\ t = 1, \\ H(x, t), & \text{if } x \in X_{n-1}. \end{cases}$$

This means that $\Phi \circ (\Phi_e \times \mathrm{id}_I)$ is equal precisely to the map F^e we obtained by applying the previous Lemma. Since this map satisfied $[F] = [\alpha]$, and α was obtained from evaluating the cochain d on the open n-cell $\Phi_e(D^n \setminus S^{n-1})$ via the identification $\partial(D^n \times I) \cong S^n$, if we revert this identification we see that $[F]$ gets mapped precisely to the value $d(\Phi_e(D^n \setminus S^{n-1})) \in \pi_n(Y)$. In summary, the element of $\pi_n(Y)$ assigned to the open cell $\Phi_e(D^n \setminus S^{n-1})$ by the difference cochain $d(f_0, f_1)$ is equal to the element assigned by the cochain d, and since this holds for every n-cell, we deduce that $d(f_0, f_1) = d$ as desired. ■

We finally have all the ingredients to prove the main theorem.

7.2.12 Theorem. Let (X, A) be a CW-complex, and let Y be an n-simple space. Given a map $f : X_n \to Y$, its obstruction class $[\theta^{n-1}(f) \in H^{n+1}(X, A; \pi_n(Y))$ vanishes if and only if there exists a map $F : X_{n+1} \to Y$ whose restriction $F|_{X_{n-1}}$ is homotopic to the restriction $f|_{X_{n-1}}$.

Proof. We have already proved that, if the extension F exists, then $[\theta^{n+1}(f)]$ vanishes. For the converse, suppose this cohomology class vanishes, so that $\theta^{n+1}(f) = \delta^n(d)$ for some cellular cochain $d \in \mathcal{C}^n(X, A; \pi_n(Y))$.

Consider the homotopy $H : X_{n-1} \times I \to Y$ given by $H(x, t) := f|_{X_{n-1}}(x)$. By definition the initial stage H_0 agrees wit $f|_{X_{n-1}}$, and we can then apply Proposition 7.2.10

[5] This cochain actually assigns an element to the open cell an element in the homotopy group $\pi_n(Y)$, but this gives rise to a homotopy class in $[\partial(D^n \times I), Y]$ since $\partial(D^n \times I) \cong S^n$ as we have already remarked.

to obtain a map $f' : X_n \to Y$ such that $H_1 = f'|_{X_{n-1}}$, and such that the difference cochain $d(f, f')$ is equal precisely to d.

We just need to show that f' can be extended to a map $F : X_{n+1} \to Y$. In order to see this, recall that the difference cochain satisfies the equality

$$\delta^n(d(f, f')) = \theta^{n+1}(f) - \theta^{n+1}(f').$$

In our case, this can be rewritten as

$$\delta^n(d) = \theta^{n+1}(f) - \theta^{n+1}(f'),$$

but d was chosen to satisfy $\theta^{n+1}(f) = \delta^n(d)$, which implies that the obstruction cochain $\theta^{n+1}(f')$ vanishes, thus it can be extended to X_{n+1}. ■

It is interesting to note that the extension F constructed above is such that the restriction $F|_{X_{n-1}}$ is not only homotopic to $f|_{X_{n-1}}$, but in fact *equal* to it.

7.3 Obstruction Theory for sections

The theory developed in the previous section can be adapted to study the extension lifting problem for sections of a locally trivial bundle. In this section we briefly go over the main results. We do not include proofs, but mostly point out where are the differences and subtleties in adapting the obstruction theory for maps to the context of sections.

Throughout the section, let X be a CW-complex, and suppose $p : E \to X$ is a locally trivial bundle with typical fiber F. Given a section $s : X_n \to E$ of the bundle, we are interested in investigating the possibility of extending s to higher stages of the skeletal filtration of X such that the extension is still a section of the bundle.

If we suppose that s is defined on the n-skeleton of X_n, and we consider the characteristic map $\Phi_e : (D^{n+1}, S^n) \to (X_n, X_{n-1})$ of an $(n+1)$-cell, we have the composite $s_e := s \circ \varphi_e : S^n \to E$. The projection $p \circ s_e : S^{n-1} \to B$ is null homotopic, because $p \circ s_e = p \circ s \circ \varphi_e = \mathrm{id}_B \circ \varphi_e = \varphi_e$, and Φ_e extends φ_e.

If we then consider a null homotopy $h : S^n \times I \to B$ of $p \circ s_e$ to some constant map $\mathrm{ct}_{S^n,x}$, using the lifting property of locally trivial bundles we can find a lift $\widetilde{h} : S^n \times I \to E$ of h such that its final stage h_1 maps S^n entirely to the fiber $p^{-1}(x)$. This procedure then assigns an element in $\pi_n(p^{-1}(x), *)$ to each $(n+1)$-cell of (X, A). We have the same possible problem of choice of basepoints in the fiber $p^{-1}(x)$, so we better assume that the fibers are n-simple. Since the fibers are all homeomorphic to the typical fiber F, it suffices to assume that F is n-simple.

We then have a procedure that assigns to every open $(n+1)$-cell of X an element in the homotopy group $\pi_n(p^{-1}(x))$ of some fiber. Since all fibers are homeomorphic to F, we can identify $\pi_n(p^{-1}(x))$ with $\pi_n(F)$. It is tempting to say that this whole procedure of assigning homotopy classes in $\pi_n(F)$ to the $(n+1)$-cells of X defines a cellular cochain $\mathcal{C}^{n+1}(X;\pi_n(F))$. Unfortunately, there is still a possible problem. When we identify the fiber $p^{-1}(x)$ with F, this identification is not canonical, it depends on choosing a local trivialization around the point x. Now, different points will give rise to potentially different local trivialization, and we might end up with a mess of different identifications between fibers over points of x and the typical fiber F.

There are two ways to circumvent this problem:

1. we fix once and for all a point x_0, and fix an identification $p^{-1}(x_0) \cong F$. This will serve as the standard identification with the typical fiber. If we can somehow naturally identify other fiber $p^{-1}(x)$ with $p^{-1}(x_0)$, then we can also identify them naturally with F. The situation is similar to the process of identifying homotopy groups based at different points.

2. We can give up on assigning elements of the single group $\pi_n(F)$ to the open $(n+1)$-cells, and instead allow a varying family of groups. The local triviality of the bundle ensures that at least locally it is possible to make a consistent choice of groups. It is possible to develop an alternative cohomology theory based on this idea of varying coefficient groups called *cohomology with local coefficients*. If we use this technology, then the process described above does indeed produce a cochain on this generalized sense, and the theory can be developed in this way.

The second approach is way beyond the scope of this text, so we comment a little on the first. If we recall that every CW-complex is a paracompact space, then it follows that every locally trivial bundle over a CW-complex is in fact a *Hurewicz fibration*. This means that the projection $p : E \to X$ satisfies the right-lifting property with respect to *any* cylinder inclusion $Z \rightarrowtail Z \times I$, not just those defined on other CW-complexes.

This allows us to transport information on fibers along paths on the base space. More precisely, if $\gamma : I \to X$ is a path from x_0 to x_1, and we denote the respective fibers by F_{x_0} and F_{x_1}, then the homotopy $h : F_{x_0} \times I \to B$ given by $h(u,t) := \gamma(t)$ can be lifted to a homotopy $\widetilde{h} : F_{x_0} \times I \to E$, and its final stage $\widehat{h}_1 : F_{x_0} \to E$ actually takes values in the other fiber F_{x_1}.

Let $t_\gamma : F_{x_0} \to F_{x_1}$ be this map obtained from the path γ. This construction satisfies properties similar to those satisfied by the transport map for the homotopy groups, in particular:

1. if $\widehat{h}$ is another lift of h, then the final stages $\widehat{h}_1$ and $\widetilde{h}_1$ are homotopic, therefore the definition of t_γ is independent of the chosen lift;

2. if $\gamma \simeq \gamma'$ as paths, then t_γ and $t_{\gamma'}$ are homotopic maps;
3. if we consider the constant path $\mathrm{ct}_{I,x_0} : I \to X$, then the associated transport map $t_{\mathrm{ct}_{I,x_0}} : F_{x_0} \to F_{x_0}$ is homotopic to the identity;
4. if $\gamma' : I \to X$ is a path from x_1 to x_2, and $\gamma \cdot \gamma'$ denotes the concatenation of the two paths, then $t_{\gamma\cdot\gamma'}$ is homotopic to the composition $t_{\gamma'} \circ t_\gamma$.

This allows us to associate to path-homotopy class $[\gamma]$ between two points x_0 and x_1 a homotopy class $[F_{x_0}, F_{x_1}]$ of maps between the corresponding fibers.

The algebraic properties "up to homotopy" satisfied by the transport construction that were mentioned above allow us to define an action of the fundamental group $\pi_1(X, x_0)$ on the homotopy group $\pi_n(F_{x_0})$: given a class $[\alpha]_* \in \pi_1(X, x_0)$, the properties listed above imply that the associated transport map $t_\alpha : F_{x_0} \to F_{x_0}$ is actually a homotopy equivalence of the fiber, whose inverse up to homotopy can be obtained by transporting along the reverse loop $\overline{\alpha}$, and the homotopy class $[t_\alpha] \in [F_{x_0}, F_{x_0}]$ induces a group homomorphism by the rule $[f] \mapsto [t_\alpha \circ f]$, where $[f] \in \pi_n(F_{x_0})$. Notice that we are not worrying about the basepoints because we supposed that the typical fiber F is n-simple, so the same is true of the fiber F_{x_0}. We can avoid the problem of canonical identification between fibers if we demand that this action be trivial.

7.3.1 Definition. A locally trivial bundle $p : E \to X$ with typical fiber F is said to be n**-simple** if the following conditions are satisfied:

1. X is path connected;
2. the typical fiber F is n-simple;
3. for every point $x_0 \in X$, the action of the fundamental group $\pi_1(X, x_0)$ on the homotopy group $\pi_n(F_{x_0})$ of the corresponding fiber is trivial.

If we restrict ourselves to working with n-simple locally trivial bundles, then the construction described at the beginning of the section goes through. Fix a point $x_0 \in X$, identify F_{x_0}, which we will denote by F, since it is homeomorphic to the typical fiber, and then by transporting along paths every other fiber can be identified with F. Given a section $s : X_n \to E$ over the n-skeleton, we assign to each $(n+1)$-cell $\Phi_e : (D^{n+1}, S^n) \to (X_{n+1}, X_n)$ the element of $\pi_n(F)$ by first lifting a null homotopy $h : S^n \times I \to B$ of the composition $p \circ s \circ \varphi_e$ to a homotopy $\widetilde{h} : S^n \times I \to E$, and then looking at the homotopy class of the last stage of this lift $[\widetilde{h}_1]$. Again, this class lives in the n-th homotopy group of *some* fiber, but the assumption of n-simplicity allows us to identify this with $\pi_n(F)$. We obtain then obtain a cellular cochain $\theta^{n+1}(s) \in \mathcal{C}^{n+1}(X; \pi_n(F))$ called the **obstruction cochain of** s.

We then have results analogous to the ones proved in the previous chapter.

7.3.2 Theorem. Let X be a path-connected CW-complex, and let $p : E \to X$ be an n-simple locally trivial bundle with typical fiber F. Given a section $s : X_n \to E$ of the bundle over the n-skeleton, the following are true:

1. the section s can be extended to a section $S : X_{n+1} \to E$ if and only if its obstruction cochain $\theta^{n+1}(s)$ vanishes.

2. The obstruction cochain $\theta^{n+1}(s)$ is a cellular cocycle.

3. The cohomology class $[\theta^{n+1}(s)] \in H^{n+1}(X; \pi_n(F))$ vanishes if and only if there exists a section $S : X_{n+1} \to E$ whose restriction $S|_{X_{n-1}}$ to the $(n-1)$-skeleton coincides with the restriction $s|_{X_{n-1}}$.

CHAPTER

8

SOME APPLICATIONS

In this final chapter we briefly describe some applications of the results of Obstruction Theory obtained in the previous chapter. It is a short chapter mainly concerned with exposing the potential ways in which Obstruction Theory can be used in other areas of Mathematics.

The first section discusses characteristic classes of vector bundles. These are cohomology classes that can be naturally associated with any vector bundle, and which are connected with its topology. In our obstruction-theoretic approach, these cohomology classes show up as obstruction classes of some locally trivial bundle naturally associated to the initial vector bundle.

In the second and last section, we discuss how the results of Obstruction Theory have been applied in Singularity Theory to define the so-called local Euler obstruction, a numerical invariant which in some sense measures the complexity of a singularity at a point of a space. We then briefly describe how this invariant was first used to construct Chern classes for possibly singular spaces.

This chapter is of a more expository nature, without worrying about the technicalities. Compared to the previous chapters, this one reads more like a mathematical prose than a technical text.

8.1 Characteristic classes

Let X be a path-connected CW-complex, and let $p : E \to X$ be a real vector bundle of rank $n \geq 2$. Suppose also that the bundle is *orientable*, that is, we can cover X by trivializing neighborhoods $\{U_i\}_{i\in I}$ such that, for each i, $j \in I$, over the intersection $U_i \cap U_j$ the change of coordinates map $\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ is of the form $(x, v) \mapsto (x, t_{ij}(x)(v))$, where $t_{ij}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation preserving linear isomorphism which depends continuously on $x \in U_i \cap U_j$.

Every vector bundle comes equipped with a *zero section* $i : X \to E$ which maps each point $x \in X$ to the corresponding zero vector $\mathbf{0}_x \in p^{-1}(x)$ in the fiber above. We can obtain a new bundle $p_0 : E_0 \to X$ by removing the image of this zero section, i.e., by defining $E_0 := E \setminus i(X)$. The resulting projection is no longer a vector bundle, of course, but it is still a locally trivial bundle whose typical fiber is the pointed euclidean space $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

This typical fiber is homotopy equivalent to a sphere S^{n-1}, so some of its homotopy groups can be explicitly described as

$$\pi_j(\mathbb{R}^n \setminus \{\mathbf{0}\}) = \begin{cases} 0, & \text{if } 0 \leq j \leq n-2, \\ \mathbb{Z}, & \text{if } j = n-1. \end{cases}$$

We then see that the typical fiber is $(n-1)$-connected. This is why we restricted to the case $n \geq 2$ by the way, since when $n = 1$, the fiber $\mathbb{R} \setminus \{0\}$ is not path-connected.

This connectedness means that we can find a section $s : X_{n-1} \to E_0$ over the $(n-1)$-skeleton. The orientability hypothesis on the vector bundle ensures that $p_0 : E_0 \to X$ is $(n-1)$-simple, and so by the results of Obstruction Theory we know that there is an obstruction class $[\theta^n(s)] \in H^n(X; \mathbb{Z})$ which measures the possibility of extending s to n-skeleton. It follows from the connectedness of the fiber that this obstruction class is actually uniquely determined, if $s' : X_{n-1} \to E_0$ is another section, then $[\theta^n(s')]$ is equal to $[\theta^n(s)]$. In other words, the cohomology class obtained this way is intrinsic to the bundle, and it is called the **Euler class** of the bundle, and denoted by $e(E)$ or $e(p)$.

The construction of the Euler class can be generalized. Since every CW-complex is paracompact, a vector bundle over a CW-complex can always be equipped with a metric. This allows us to define a new bundle $V_k(p) : V_k(E) \to X$, where $V_k(E)$ is obtained by replacing each fiber $p^{-1}(x)$ with the Stiefel manifold $V_k(p^{-1}(x))$ with respect to the metric mentioned above. As a set, $V_k(E)$ is then given by the disjoint union $\bigsqcup_{x \in X} V_k(p^{-1}(x))$, and by suitably topologizing this the projection $V_k(p) : V_k(E) \to X$ becomes a locally trivial bundle with typical fiber the usual Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$.

We mentioned in Proposition 6.4.8 that $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, so we can always find a section $s : X_{n-k} \to V_k(E)$ defined over the $(n-k)$-skeleton. The possibility of extending this section to X_{n-k+1} is controlled by its obstruction class $[\theta^{n-k+1}(s)] \in H^{n-k+1}(X; \pi_{n-k}(V_k(\mathbb{R}^n)))$. Recall that the homotopy group $\pi_{n-k}(V_k(\mathbb{R}^n))$ is a bit complicated:

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1 \text{ or } n-k \text{ is even}, \\ \mathbb{Z}_2, & \text{otherwise}. \end{cases}$$

This means that, depending on the particular values of n and k, the obstruction class $[\theta^{n-k+1}(s)]$ either lives in $H^{n-k+1}(X;\mathbb{Z})$ or in $H^{n-k+1}(X;\mathbb{Z}_2)$. If we reduce $\mathbb{Z}$ mod 2, then we can always obtain a cohomology class in $H^{n+k-1}(X;\mathbb{Z}_2)$ regardless of the values of n and k. Like in the case of the Euler class, this obstruction class does not depend on the section $s : X_{n-k} \to V_k(E)$ considered initially, being intrinsic to the bundle. The $\mathbb{Z}_2$-cohomology class obtained in this way starting with any section is called the $(n-k+1)$-th **Stiefel-Whitney class** of the bundle, and it is denoted as $w_{n-k+1}(E)$.

Geometrically, the Stiefel-Whitney classes measure how far up the skeletal filtration of X we can find a continuously varying family of k-frames in the fibers above. For example, if X is a smooth manifold, and the initial bundle is the tangent bundle $TM \to M$, then w_{n-k+1} measures in some sense on how much of the manifold M we can define a collection of k linearly independent vector fields.

We end our discussion of characteristic classes with a brief comment on Chern classes. Let $p : E \to B$ be a complex vector bundle over a path-connected CW-complex X. It can be equipped with a continuously varying family of hermitian products on each of its fibers, and we can then perform the previous construction by replacing each fiber $p^{-1}(x)$ with $V_j(p^{-1}(x))$, where we are now considering the Stiefel manifold of complex frames that are orthonormal with respect to the hermitian product. This gives us a locally trivial bundle $\widetilde{p} : V_k(E) \to X$ whose typical fiber is the complex Stiefel manifold $V_k(\mathbb{C}^n)$ of orthonormal frames with respect to the usual hermitian product on $\mathcal{C}^n$.

Luckily, the homotopy of the complex Stiefel manifolds is a bit simpler than that of their real counterpart, and we showed in Proposition 6.4.7 that some of the first homotopy groups can be described as follows:

$$\pi_j(V_k(\mathbb{C}^n)) = \begin{cases} 0, & \text{if } 0 \leq j \leq 2(n-k), \\ \mathbb{Z}, & \text{if } j = 2(n-k)+1. \end{cases}$$

This means that we can always find a section $s : X_{2(n-k)+1} \to V_k(E)$ over the $(2(n-k)+1)$-th skeleton. The possibility of extending it to the next stage of the filtration is controlled by its obstruction class $\theta^{2(n-k+1)}(X;\mathbb{Z})$. Like with the two previous characteristic classes, this cohomology class does not depend on the section initially considered, it is intrinsic to the bundle. It is called the $(n-k+1)$-th **Chern class** of E, and is commonly denoted by $c_{n-k+1}(E)$.

8.2 Local Euler obstruction

In the previous section, we were mainly interested in explaining how some characteristic classes of vector bundles arise from the main results of Obstruction Theory. Part of the importance of these classes, however, is how they admit multiple constructions in different areas, each of them revealing a connection with another topic. Some possible constructions are:

- the classical construction via Obstruction Theory as presented in the previous section;
- a construction via the Leray-Hirsch Theorem and the projectivization of a vector bundle;
- a construction via the Thom Isomorphism and the Steenrod squaring operations in cohomology;
- in Differential Geometry, Chern-Weil Theory allows us to relate Chern classes, which live in singular or cellular cohomology, with classes in de Rham cohomology.

Beyond the various possible constructions, the importance of characteristic classes can also be perceived by looking at some theorems related to these objects.

1. Every manifold has associated to it a vector bundle: its tangent bundle $TM \to M$. The Stiefel-Whitney classes $w_i(TM)$ of the tangent bundle are also called the Stiefel-Whitney bundles of M. Using the obstruction-theoretic construction of these classes one can show the following result: *a manifold M is orientable if and only if its first Stiefel-Whitney class $w_1(M)$ vanishes.*
2. Given a closed manifold M, let $[M]$ denote its fundamental class in homology with $\mathbb{Z}_2$ coefficients. The evaluations $\langle w_i(M), [M]\rangle$ are called the *Stiefel-Whitney numbers* of M. René Thom proved the following result: *two closed manifolds M and N are cobordant if and only if their Stiefel-Whitney numbers agree.*
3. There exists a special $BO(n)$, called the *classifying space of the orthogonal group $O(n)$*, as well as a rank n vector bundle $EO(n) \to BO(n)$, called the *universal rank n vector bundle*: which together satisfy the following properties: if X is a sufficiently nice space (like a CW-complex), and $p : E \to B$ is a rank n vector bundle over X, then there exists a map $f : X \to BO(n)$ such that $p : E \to B$ is the pullback of the universal bundle $EO(n) \to BO(n)$ along the map f, which is called the *classifying map* for the bundle. It turns out that the cohomology ring of the space $BO(n)$ is determined by the Stiefel-Whitney classes of the bundle $EO(n) \to BO(n)$. Using the classifying maps we can then obtain the Stiefel-Whitney classes of any rank n bundle over a nice space from the corresponding classes of the universal rank n bundle.

Historically, characteristic classes have been important tools in the study of the topology and geometry of manifolds. One reason behind this usefulness is the fact that manifolds have many vector bundles associated to them by means of the tangent bundle and modifications of it. When we enter the world of *singular* space, the situation is more complicated. Since these spaces are characterized by having points without a well-defined tangent space, we cannot associate a vector bundle to a general singular space. This leaves us wondering if the theory of characteristic classes can be of any use in Singularity Theory.

The goal of this final section is to give a brief overview of how the techniques of Obstruction Theory can be used to construct characteristic classes for a certain class of singular spaces. We do not give proofs, that is beyond the scope of this work, but at times we offer some pointers to the literature.

Before looking for characteristic classes for singular spaces, we first reformulate the usual characteristic classes in a *functorial* way. This is possible due to the following:

8.2.1 Proposition. The Chern classes enjoy the following properties:

1. If ξ is a rank n complex vector bundle, then $c_i(\xi) = 0$ for every $i > n$.
2. If ξ is a complex vector bundle over the space Y, and $f : X \to Y$ is a map, then the Chern classes of the pullback bundle $f^*\xi$ satisfy the equality $c_i(f^*\xi) = f^*(c_i(\xi))$, where $f^* : H^{2i}(Y;\mathbb{Z}) \to H^{2i}(X;\mathbb{Z})$ denotes the morphism induced in cohomology.
3. If η and ξ are two complex vector bundles over the same base space, then the Chern classes of the Whitney sum $\eta \oplus \eta$ can be described via the formula
$$c_k(\eta \oplus \xi) = \sum_{i+j=k} c_i(\eta) \smile c_j(\xi).$$
4. The first Chern class $c_1(\gamma)$ of the tautological line bundle over the complex projective line $\mathbb{CP}^1$ is a generator of the cohomology group $H^2(\mathbb{CP}^1;\mathbb{Z}) \cong H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.

These properties can actually be proved via the obstruction-theoretic construction of Chern classes, see for example (FOMENKO; FUCHS, 2016, Section 19.5) for a proof in the context of Stiefel-Whitney classes. Other approaches to characteristic provide easier proofs, however. It is also interesting to note that these properties completely characterize the Chern classes, so they can be taken as axioms.

Let us see how these properties allow us to think of Chern classes functorially. Let HoCW be the category whose objects are CW-complexes, and whose morphisms are homotopy classes of maps. Given an integer $i \geq 0$, we define two (contravariant) functors on this category:

1. The first is the cohomology functor $H^{2i}(-;\mathbb{Z}) : \mathsf{HoCW}^{\mathsf{op}} \to \mathsf{Set}$ sending a CW-complex to its singular cohomology group $H^{2i}(X;\mathbb{Z})$, and sending a homotopy class $[f] : X \to Y$ to the induced morphism $f^* : H^{2i}(Y;\mathbb{Z}) \to H^{2i}(X,\mathbb{Z})$.

2. The second is the functor $\mathrm{Vec} : \mathsf{HoCW}^{\mathsf{op}} \to \mathsf{Set}$ sending a CW-complex to the set $\mathrm{Vec}(X)$ of isomorphism classes of complex vector bundles over X, and sending a homotopy class $[f] : X \to Y$ to the function $\mathrm{Vec}([f]) : \mathrm{Vec}(Y) \to \mathrm{Vec}(X)$ given by $[\xi] \mapsto [f^*\xi]$, where $[\xi]$ denotes the set of isomorphism classes of ξ. This is well-defined by virtue of the well-known result that pullbacks of a vector bundle along homotopic maps are isomorphic.

The Chern classes allow us to define natural transformations of type

$$\mathrm{Vec} \Rightarrow H^{2i}(-;\mathbb{Z}).$$

More precisely, we can consider for each CW-complex X the function $c_i^X : \mathrm{Vec}(X) \to H^{2i}(X;\mathbb{Z})$ defined as

$$c_i^X([\xi]) \coloneqq c_i(\xi) \in H^{2i}(X;\mathbb{Z}),$$

that is, c_i^X sends the isomorphism class $[\xi]$ to the i-th Chern class $c_i(\xi)$. This is well-defined by virtue of the naturality property of Chern classes, and it satisfies the commutativity condition characterizing natural transformations.

$$\begin{tikzcd}
\mathrm{Vec}(X) \arrow[d, "c_i^X"'] & \mathrm{Vec}(Y) \arrow[l, "\mathrm{Vec}([f])"'] \arrow[d, "c_i^Y"] \\
H^{2i}(X;\mathbb{Z}) & H^{2i}(Y;\mathbb{Z}) \arrow[l, "f^*"]
\end{tikzcd}$$

This formulation makes it clear that the characteristic classes allows us to naturally translate geometric information about a space (vector bundles over it) into algebraic information (cohomology classes). It is this functorial formulation which represents a bridge to the world of characteristic classes for singular spaces.

Alexander Grothendieck and Pierre Deligne conjectured the following result:

8.2.2 Theorem. Let C denote the category of complex algebraic varieties and algebraic maps. Let $\mathcal{F} : \mathsf{C} \to \mathsf{Ab}$ be the "constructible functions" functor, and let $H_{2*}(-;\mathbb{Z}) : \mathsf{C} \to \mathsf{Ab}$ be the total homology functor.

There exists a unique natural transformation $c : \mathcal{F} \Rightarrow H_{2*}(-;\mathbb{Z})$ satisfying the following normalization condition: if X is smooth, and $[X]$ denotes its fundamental class, then the equality $c_*(X) \frown [X] = c_X(\mathrm{ct}_{X,1})$.

Some explanation of this statement is needed:

- Very roughly speaking, a constructible function on a complex algebraic variety X is a function $X \to \mathbb{Z}$ which is constant on the pieces of a nice decomposition of X into smooth pieces.

- The homology functor $H_{2*}(-;\mathbb{Z})$ sends a space X to the direct sum $\bigoplus_{i\geq 0} H_{2i}(X;\mathbb{Z})$ of all its even dimensional homology groups. Similarly, $c_*(X) \in H_{2*}(X;\mathbb{Z})$ is the total Chern class of X, that is, the formal sum of all its Chern classes.

The idea is that this natural transformation c behaves as the natural transformation induced by the usual Chern classes that we mentioned before. For each $X \in \mathsf{C}$, the homology class $c_X(\mathrm{ct}_{X,1}) \in H_{2*}(X;\mathbb{Z})$ coming from the constant constructible function is called the **Chern-Schwartz-MacPherson class** of X. The normalization condition in the theorem statement says that, if X is in fact smooth, then its Chern-Schwartz-MacPherson class agrees with its usual Chern class up to Poincaré duality.

The reason we have stated a theorem, and not a conjecture, is because it was proved by Robert MacPherson in (MacPherson, 1974). Interestingly, some years before this, the French mathematician Marie-HéLène Schwartz had in fact already constructed Chern classes for some singular spaces (even before the Grothendieck-Deligne conjecture was stated!), and in later work together with Jean-Paul Brasselet they showed that her construction was in fact dual to MacPherson's.

8.2.3 Remark. On a personal remark, one thing that has never been clear for me about this Grothendieck-Deligne conjecture is: why constructible functions? If classical characteristic classes are useful for turning geometric data into algebraic data, why consider constructible functions in the singular case? What is the geometric content of constructible functions? I discussed this with some other students and looked for some information in the literature, but I did not find a precise explanation.

Having explained the general context of part of the historical origins of the theory of characteristic classes for singular spaces, we dedicate the rest of this chapter (and of the text!) to very briefly explain the construction introduced by MacPherson, and how results from Obstruction Theory played a rôle in it.

The first step is to find a substitute for the tangent bundle. Let X be a d-dimensional complex algebraic variety embedded in a smooth complex ambient manifold M of dimension N. Using the tangent bundle TM of M, we define a map $\gamma : X_{\mathrm{reg}} \to G_d(TM)$ over the regular part of X as

$$\gamma(x) \coloneqq (x, T_x X_{\mathrm{reg}}).$$

Here, $G_d(TM)$ denotes the bundle over M obtained by replacing each of the tangent space of M with their corresponding grassmannian of d-planes. The closure of the image of γ in $G_d(TM)$ is called the **Nash modification** of X and is denoted by $\widetilde{X}$. The restriction of the projection $G_d(TM) \to TM$ to $\widetilde{X}$ defines a map $\nu : \widetilde{X} \to X$ called the **Nash blow-up**.

Over $G_d(TM)$ we have the tautological bundle $T \to G_d(TM)$ whose points are the triples (m, V, v), where m is a point of M, $V \leq T_m M$ is a d-plane, and $v \in V$ is a vector. The restriction of this tautological bundle to the Nash modification is called the **Nash bundle** of X, and is denoted by $\widetilde{TX}$.

A first idea for obtaining a class as described in the theorem above is to define

$$c_M(X) \coloneqq \nu_*(c_*(\widetilde{TX}) \frown [\widetilde{X}]),$$

i.e., we consider the total Chern class $c_*(\widetilde{TX})$ of the Nash bundle, dualize to obtain a homology class in $H_{2*}(\widetilde{X};\mathbb{Z})$, and then push it down to $H_{2*}(X;\mathbb{Z})$ via the pushforward ν_* along the Nash blow-up. The class $c_M(X)$ obtained like this is called the **Chern-Mather class** of X.

Unfortunately, even though $c_M(X)$ satisfies the required normalization condition when X is smooth, i.e., it is Poincaré-dual to the usual Chern class, it does not depend naturally on the initial variety X. Nevertheless, these Chern-Mather classes are still part of the solution to the problem. More precisely, using it we can associate to every subvariety $V \subseteq X$ a homology class in $H_{2*}(X;\mathbb{Z})$ via the rule $V \mapsto i^V_*(c_M(V))$, where $i^V_* : H_{2*}(V;\mathbb{Z}) \to H_{2*}(X;\mathbb{Z})$ is the morphism induced by the inclusion map $i^V : V \rightarrowtail X$. This assignment can be extended to a group homomorphism $c_M : \mathcal{A}_*(X) \to H_{2*}(X;\mathbb{Z})$, where $\mathcal{A}_*(X)$ is the group of *algebraic cycles on* X, which is defined as the free abelian group generated by all the subvarieties of X.

The component $c_X : \mathcal{F}(X) \to H_{2*}(X;\mathbb{Z})$ of the natural transformation $c : \mathcal{F} \Rightarrow H_{2*}(-;\mathbb{Z})$ we are trying to define is obtained by composing the Chern-Mather homomorphism $c_M : \mathcal{A}_*(X) \to H_{2*}(X;\mathbb{Z})$ constructed above with a certain isomorphism $\mathcal{F}(X) \overset{\cong}{\to} \mathcal{A}_*(X)$ between the group of constructible functions and the group of algebraic cycles.

This is where Obstruction Theory finally shows up. Suppose V is a v-dimensional complex algebraic variety embedded in a smooth complex algebraic variety N. Given a point $p \in V$, let $z = (z_1, \ldots, z_n)$ be a set of local coordinates for N on a neighborhood U of this point such that $z_i(p) = 0$ for every i. Let $\|z\|^2 : U \to \mathbb{R}$ be the map defined as

$$\|z\|^2(y) \coloneqq \sqrt{z_1(y)\overline{z_1(y)} + \cdots + z_n(y)\overline{z_n(y)}}$$

for each $y \in U$. This map induces a differential form $d\|z\|^2$, which is a section of the cotangent bundle TN^* over this neighborhood U. This section can be pulled-back along the Nash blow-up $\nu : \widetilde{X} \to X$ to define a section r of the dual bundle $\widetilde{TX}^*$ over the neighborhood $\nu^{-1}(U) \subseteq \widetilde{X}$.

The next result, whose proof requires using techniques from the Stratification Theory of algebraic varieties, is crucial for the definition of the local Euler obstruction.

8.2.4 Lemma. Given $\varepsilon \geq 0$, let $B^\times_\varepsilon$ be the set of points in $y \in U$ such that $0 < \|y\| \leq \varepsilon$, i.e., it is a punctured ball in the metric associated with the local coordinates on U.

If ε is sufficiently small, then the section r of $\widetilde{TX}^*$ is non-zero on the subspace $\nu^{-1}(B_\varepsilon^\times)$.

Knowing this, for a sufficiently small ε, let B_ε be the *closed* ε-ball in this coordinate neighborhood, and let S_ε be the corresponding boundary sphere. The Lemma above implies that r is non-zero on the subspace $\nu^{-1}(S_\varepsilon)$ lying over the ε-sphere. The possibility of extending r from $\nu^{-1}(S_\varepsilon)$ to $\nu^{-1}(B_\varepsilon)$ is controlled by its obstruction class

$$\theta \in H^v(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z}).$$

If $\mathcal{O}_\varepsilon \in H_v(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$ is the orientation homology class of the pair of spaces $(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon))$, we have

8.2.5 Definition. The integer

$$\mathrm{Eu}_p(V) \coloneqq \langle \theta, \mathcal{O}_\varepsilon \rangle$$

is called the **local Euler obstruction of V at p.**

Back to the original context, given a subvariety $V \leq X$, using the local Euler obstruction we can define a function $\mathrm{Eu}_V : X \to \mathbb{Z}$ as follows:

$$\mathrm{Eu}_V(p) \coloneqq \begin{cases} \mathrm{Eu}_p(V), & \text{if } p \in V, \\ 0, & \text{if } p \notin V. \end{cases}$$

One of the key properties of the Euler obstruction is that $\mathrm{Eu}_p(V) = 1$ if p is a smooth point of the variety V. By choosing a stratification of X adapted to V, using this property one can then show that $\mathrm{Eu}_V : X \to \mathbb{Z}$ is a constructible function.

The assignment $V \mapsto \mathrm{Eu}_V$ induces a group homomorphism $T : \mathcal{A}_*(X) \to \mathcal{F}(X)$, and MacPherson proves in (MacPherson, 1974) that this homomorphism T is in fact an *isomorphism* of groups. He then goes on to define the desired morphism $c_X : \mathcal{F}(X) \to H_{2*}(X; \mathbb{Z})$ via the composition

$$c_X \coloneqq c_M \circ T^{-1}$$

as shown below.

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "T^{-1}"] \arrow[rr, bend right, dashed, "c_X"'] & \mathcal{A}_*(X) \arrow[r, "c_M"] & H_{2*}(X; \mathbb{Z}) \end{tikzcd}$$

It is straightforward to show that this definition of c_X satisfies the normalization condition. If X is smooth, we want to show that $c_X(\mathrm{ct}_{X,1})$ is Poincaré-dual to the classical Chern class $c_*(X)$. By definition, $c_X(\mathrm{ct}_{X,1}) = c_M(T^{-1}(\mathrm{ct}_{X,1}))$, and in order to evaluate $T^{-1}(\mathrm{ct}_{X,1})$, we need to find an algebraic cycle α of X such that $T(\alpha) = \mathrm{ct}_{X,1}$, but this is easy: consider the algebraic cycle $1 \cdot X \in \mathcal{A}_*(X)$, then $T(1 \cdot X) = \mathrm{Eu}_X$, but since X is smooth, $\mathrm{Eu}_p(X) = 1$ for every $p \in X$, i.e., $\mathrm{Eu}_X = \mathrm{ct}_{X,1}$. We then have

$$c_X(\mathrm{ct}_{X,1}) = c_M(T^{-1}(\mathrm{ct}_{X,1})) = c_M(X),$$

but as we remarked above, when X is smooth, its Chern-Mather class is already Poincaré-dual to its usual Chern class, from which we deduce that

$$c_X(\mathrm{ct}_{X,1}) \frown [X] = c_M(X) \frown [X] = c_*(X).$$

The proof that $c_X : \mathcal{F}(X) \to H_{2*}(X; \mathbb{Z})$ depends naturally on X is much more difficult, however. It involves an intricate construction, called the *graph construction*, and deeper results about algebraic cycles on algebraic varieties. A proof, albeit very succinct, can be found in MacPherson's original article (MacPherson, 1974).

We end this section with some final remarks on the local Euler obstruction. Unfortunately, obstruction-theoretic constructions tend to be difficult work with in practice, so without additional tools it can be difficult to calculate the local Euler obstruction. Fortunately, several results of computational nature have been obtained since the introduction of the concept. One famous such result by Brasselet-Lê-Seade roughly says that, for a variety X in some $\mathbb{C}^N$, the local Euler obstruction at $\mathbf{0} \in X$ can be computed via the weighted sum

$$\mathrm{Eu}_{\mathbf{0}}(X) = \sum_{i=1}^{k} \chi(V_i \cap B_\varepsilon \cap \ell^{-1}(t_0)) \cdot \mathrm{Eu}_{p_i}(V_i).$$

In this formula, $\{V_i\}_{1 \leq i \leq k}$ is a nice stratification of the variety X (what is called a *Whitney stratification*), p_i is an arbitrary point in the stratum V_i, ε and t_0 are sufficiently small real numbers, ℓ is a suitable complex linear functional on $\mathbb{C}^N$, B_ε is the closed ball of radius ε centered at the origin of $\mathbb{C}^N$, and χ denotes the usual Euler characteristic Geometrically, this says that the local Euler obstruction can be calculated by looking at how the hyperplane $\ell^{-1}(t_0)$ intersects the different strata V_i near the origin $\mathbf{0}$. For a proof, see (BRASSELET; TRáNG; SEADE, 2000).

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APPENDIX

A

GROUP AND COGROUP OBJECTS

The goal of this appendix chapter is to analyze the categorical concepts of group and cogroup objects. Since these concepts are more specific than the other ones used in the text so far, and they are not always covered in introductory texts, I felt that an appendix dedicated to them would be good.

We first introduce group objects and analyze some of the more common examples. We then prove a seemingly innocuous result characterizing group objects in the category of groups - the Eckmann-Hilton Argument - which is in fact very useful for studying the commutativity of the higher homotopy groups. After this, we analyze the dual concept of cogroups.

The last section then explains how these categorical notions of group and cogroup can be used to generate whole families of ordinary groups. The results of this section can be seen as a possible explanation behind the existence of group structures on certain sets of pointed homotopy classes.

A.1 Definitions and examples

In this last section we introduce the categorical notions of group and cogroup objects. These are special types of objects of a category which, when they exist, allow us to obtain algebraic objects from the category in question. These concepts will be specially important for us because they will be used to obtain algebraic objects from topological spaces, in particular the infamous homotopy groups.

The definition of a group object comes from rewriting the axioms that define a group in a diagrammatic way that can be stated in any category.

A.1.1 Definition. Let C be a category with binary products and a terminal object $*$. A **group object** in C is a tuple (G, m, e, inv) where

1. m is a morphism of type $G \times G \to G$,
2. e is a morphism of type $* \to G$,
3. inv is a morphism of type $G \to G$.

These morphisms must satisfy the commutativity conditions imposed by the three diagrams below, where $!_G : G \to *$ denotes the *unique* morphism from G to the terminal object $*$, and $A : (G \times G) \times G \to G \times (G \times G)$ denotes the associator isomorphism:

(G1) (Existence of a two-sided unit)

$$\begin{tikzcd} G \arrow[r, "(e\circ !_G, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e\circ !_G)"'] \arrow[dr, "\mathrm{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

(G2) (Associativity)

$$\begin{tikzcd} (G \times G) \times G \arrow[rr, "A"] \arrow[d, "m \times \mathrm{id}_G"'] & & G \times (G \times G) \arrow[d, "\mathrm{id}_G \times m"] \\ G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

(G3) (Existence of inverses)

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id}_G, \mathrm{inv})"] \arrow[d, "(\mathrm{inv}, \mathrm{id}_G)"'] \arrow[dr, "e\circ !_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Each of the morphisms that make up the structure of a group object has a corresponding name: m is the **multiplication morphism**, e is the **unit morphism**, and inv is the **inversion morphism**.

A.1.2 Remark. Let us recall the construction of the associator isomorphism A mentioned above, since we will need to make use of its defining properties later on. Consider the canonical projections below:

$$\begin{aligned} &\pi_1 : G \times G \to G, \\ &\pi_2 : G \times G \to G, \\ &\Pi_1 : (G \times G) \times G \to G \times G, \\ &\Pi_2 : (G \times G) \times G \to G, \\ &\Pi'_1 : G \times (G \times G) \to G, \\ &\Pi'_2 : G \times (G \times G) \to G \times G. \end{aligned}$$

A morphism $A : (G \times G) \times G \to G \times (G \times G)$ is completely determined by the composite morphisms $\Pi_1' \circ A : (G \times G) \times G \to G$ and $\Pi_2' \circ A : (G \times G) \times G \to G \times G$. The second composite morphism $\Pi_2' \circ A$ is itself determined by the morphisms $\pi_1 \circ \Pi_2' \circ A$, $\pi_2 \circ \Pi_2' \circ A : (G \times G) \times G \to G$. Using the universal property of the product twice, we can then define A as the unique morphism $(G\times G)\times G \to G\times(G\times G)$ satisfying the following equations:

$$\begin{cases} \Pi_1' \circ A = \pi_1 \circ \Pi_1, \\ \pi_1 \circ \Pi_2' \circ A = \pi_2 \circ \Pi_1, \\ \pi_2 \circ \Pi_2' \circ A = \Pi_2. \end{cases} \tag{A.1}$$

Let us examine examples of group objects in particular categories to see if this definition really captures the idea of a group.

A.1.3 Example. The category Set has all binary products, which are given by the usual cartesian product construction, and a terminal object is given simply by a singleton set $\{\mathrm{pt}\}$, so the unique morphism $!_X : X \to \{\mathrm{pt}\}$ is just the constant function $\mathrm{ct}_{X,\mathrm{pt}} : X \to \{\mathrm{pt}\}$.

Suppose G is a group object in Set, so that we have functions $e : \{\mathrm{pt}\} \to G$, $m : G \times G \to G$ and $\mathrm{inv} : G \to G$ satisfying the commutativity conditions stated above. The function $e : \{\mathrm{pt}\} \to G$ determines an element $e(\mathrm{pt}) \in G$ which we will denote by e_G. We usually think of a function of type $G \times G \to G$ as a binary operation on G, so we will use the notation $g_1 \cdot g_2 \coloneqq m(g_1, g_2)$. Let us analyze what the commutative diagrams in the definition of a group object mean in terms of explicit elements.

The first diagram imposes the equalities $m \circ (\mathrm{id}_G, e \circ \mathrm{ct}_{G,\mathrm{pt}}) = \mathrm{id}_G$ and $m \circ (e \circ \mathrm{ct}_{G,\mathrm{pt}}, m) = \mathrm{id}_G$. Evaluating the left side of the first equation on an arbitrary element $g \in G$ gives us

$$(m \circ (\mathrm{id}_G, e \circ \mathrm{ct}_{G,\mathrm{pt}}))(g) = m(g, e(\mathrm{pt})) = g \cdot e_G,$$

so the first equation says that $g \cdot e_G = g$ holds for every $g \in G$; and by an analogous computation we see that the second equation says that $e_G \cdot g = g$ holds for every $g \in G$. The commutativity of the first diagram (G1) is then equivalent to saying that e_G is a two side-unit for the binary operation defined by m.

The second diagram states the equality $m \circ (m, \mathrm{id}_G) = m \circ (\mathrm{id}_G, m) \circ A$, where the associator bijection $A : (G \times G) \times G \to G \times (G \times G)$ maps $((g_1, g_2), g_3)$ to $(g_1, (g_2, g_3))$. On the one hand

$$\begin{aligned} (m \circ (m, \mathrm{id}_G))((g_1, g_2), g_3) &= m(g_1 \cdot g_2, g_3) \\ &= (g_1 \cdot g_2) \cdot g_3, \end{aligned}$$

while on the other

$$\begin{aligned}(m \circ (\mathrm{id}_G, m) \circ A)((g_1, g_2), g_3) &= (m \circ (\mathrm{id}_G, m))(g_1, (g_2, g_3))\\ &= m(g_1, g_2 \cdot g_3)\\ &= g_1 \cdot (g_2 \cdot g_3).\end{aligned}$$

The commutativity condition of (G2) then says that the equality

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

holds for every g_1, g_2, $g_3 \in G$, or in other words, that the binary operation defined by m is associative.

Lastly, the third diagram imposes the equalities $m \circ (\mathrm{id}_G, \mathrm{inv}) = e \circ !_G$ and $m \circ (\mathrm{inv}, \mathrm{id}_G) = e \circ !_G$. We already know that the right-hand side of these equations is the constant function ct_{G,e_G}. Now, evaluating the left-hand side of the first of these equations on an element $g \in G$ gives us

$$\begin{aligned}(m \circ (\mathrm{id}_G, \mathrm{inv}))(g) &= m(g, \mathrm{inv}(g))\\ &= g \cdot \mathrm{inv}(g);\end{aligned}$$

while evaluating the left-hand side of the second equations gives us

$$\begin{aligned}(m \circ (\mathrm{inv}, \mathrm{id}))(g) &= m(\mathrm{inv}(g), g)\\ &= \mathrm{inv}(g) \cdot g.\end{aligned}$$

The commutativity of (G3) then says that the equalities

$$g \cdot \mathrm{inv}(g) = e_G = \mathrm{inv}(g) \cdot g$$

hold for any $g \in G$. Since we have already seen that e_G is a unit for the binary product defined by m, these equalities mean that $\mathrm{inv}(g)$ is the inverse of g with respect to this product.

All this reasoning shows that a group object in the category set is nothing but an ordinary group.

A.1.4 Example. The category Top also has binary products given by the cartesian product of sets equipped with the usual product topology, and a terminal object is also given by a singleton set $\{\mathrm{pt}\}$ equipped with the discrete topology. The unique morphism $!_X : X \to \{\mathrm{pt}\}$ in this case is also given by the constant map $\mathrm{ct}_{X,\mathrm{pt}}$, and the associating homeomorphism $A : (X \times Y) \times Z \to (X \times Y) \times Z$ is also given explicitly on elements by $A((x, y), z) \coloneqq (x(, y, z))$.

If G is a group object in Top, then we have *maps* - not just functions - $e : \{\mathrm{pt}\} \to G$, $m : G \times G \to G$ and $\mathrm{inv} : G \to G$. The interpretation of the map e choosing an element of

G also holds in this case, since any function defined on a discrete space is automatically a map. Moreover, since composition of maps is the same as composition of ordinary functions, the explicit interpretation of the commutativity conditions that we gave in Example A.1.3 also holds in this topological case.

In summary, a group object in Top is like an ordinary group, but with the added hypothesis that the multiplication m and inversion functions inv are *continuous*. Not surprisingly, group objects in Top are also called *topological groups*.

A.1.5 Example. Let Mfld be the category of smooth manifolds and smooth maps. It has binary products given by the usual smooth structure on the product space of two manifolds. It also has a terminal object given by the singleton set $\{\mathrm{pt}\}$ regarded as a zero-dimensional manifold, with the constant map $\mathrm{ct}_{M,\mathrm{pt}} : M \to \{\mathrm{pt}\}$ being the unique smooth map. The explicit description of the association isomorphism $A : (M_1 \times M_2) \times M_3 \to M_1 \times (M_2 \times M_3)$ is the same as in the previous examples

Just as before, if G is a group object in Diff, then G is a smooth manifold together with *smooth maps* $m : G \times G \to G$ and $\mathrm{inv} : G \to G$ that together satisfy the axioms of a group. In other words, a group object in Mfld is precisely a Lie group.

Now we want to investigate the following question: what is a group object in the category Grp of groups and morphisms of groups? This may seem weird at first because we are essentially defining a group using an object that is already a group and using maps that are already morphisms of groups. In order to answer this question, we need an auxiliary result that will be very important for Homotopy Theory.

A.1.6 Theorem (Eckmann-Hilton Argument)**.** Let X be a set and consider two binary operations $\odot, \otimes : X \times X \to X$ satisfying the following conditions:

1. both operations are unital, i.e., there are elements $1_\odot, 1_\otimes \in X$ such that $1_\odot \odot x = x \odot 1_\odot = x$ and $1_\otimes \otimes x = x \otimes 1_\otimes = x$ hold for every $x \in X$;

2. the equation $(w \odot x) \otimes (y \odot z) = (w \otimes y) \odot (x \otimes z)$ holds for every $w, x, y, z \in X$.

Under these assumptions, the following properties hold:

1. $1_\odot = 1_\otimes$;

2. $\odot = \otimes$;

3. $\odot$ is commutative and associative.

Proof. 1. Using the compatibility between the two operations we have

$$\begin{aligned} 1_\otimes &= 1_\otimes \otimes 1_\otimes \\ &= (1_\odot \odot 1_\otimes) \otimes (1_\otimes \odot 1_\odot) \\ &= (1_\odot \otimes 1_\otimes) \odot (1_\otimes \otimes 1_\odot) \\ &= 1_\odot \odot 1_\odot \\ &= 1_\odot. \end{aligned}$$

2. Given any x, $y \in X$, using the previous item and the compatibility condition we see that

$$\begin{aligned} x \otimes y &= (x \odot 1_\odot) \otimes (1_\odot \odot y) \\ &= (x \otimes 1_\odot) \odot (1_\odot \otimes y) \\ &= (x \otimes 1_\otimes) \odot (1_\otimes \otimes y) \\ &= x \odot y. \end{aligned}$$

3. Knowing that both operations are identical, we can rewrite the compatibility condition solely in terms of a single operation, so that we are left with the equation

$$(w \odot x) \odot (y \odot z) = (w \odot y) \odot (x \odot z).$$

Given x, $y \in X$, we have

$$\begin{aligned} x \odot y &= (1_\odot \odot x) \odot (y \odot 1_\odot) \\ &= (1 \odot y) \odot (x \odot 1_\odot) \\ &= y \odot x; \end{aligned}$$

showing the commutativity of $\odot$.

Lastly, for the associativity, given x, y, $z \in X$, using the compatibility condition we see that

$$\begin{aligned} (x \odot y) \odot z &= (x \odot y) \odot (1_\odot \odot z) \\ &= (x \odot 1_\odot) \odot (y \odot z) \\ &= x \odot (y \odot z). \end{aligned}$$

■

A.1.7 Remark. In Algebra, a pair $(M, \odot)$, where M is a set and $\odot : M \times M \to M$ is a binary operation on M is called a *magma*. If there exists a double-sided unit $1 \in M$, then the triple $(M, \odot, 1)$ is a *unital magma*. The Eckmann-Hilton Argument can then be restated as saying that, if two unital magma structures on a set satisfy the compatibility condition in the statement of Theorem A.1.6, then the two structures coincide, and in fact define commutative and associative magma.

One important consequence of Theorem A.1.6 is the characterization of group object in the category of groups.

A.1.8 Corollary. A group object in Grp is an abelian group.

Proof. The category Grp satisfies the conditions required in Definition A.1.1. A terminal object in Grp is given by the singleton group $\{\mathrm{pt}\}$ with the only binary product possible. The unique group homomorphism $1_G : G \to \{\mathrm{pt}\}$ is of course the constant function $\mathrm{ct}_{G,\mathrm{pt}}$. Moreover, if G and H are groups, the cartesian product $G \times H$ has the structure of a group with product defined as

$$(g_1, h_1) \cdot (g_2, h_2) \coloneqq (g_1 \cdot g_2, h_1 \cdot h_2),$$

and this is structure is such that $G \times H$ is a categorical product for G and H.

Suppose G is a group object in Grp. Since G is an object of the category Grp, it has the structure of an ordinary group. We will denote its ordinary multiplication by $\cdot : G \times G \to G$, its ordinary identity by 1 and the ordinary inverse of an element $g \in G$ by g^{-1}. Forgetting about the inverses for a moment, the triple $(G, \cdot, 1)$ is a unital magma in the sense of Remark A.1.7.

Now, since G is also a *group object* in Grp, there are *group homomorphisms* $m : G \times G \to G$, $e : \{\mathrm{pt}\} \to G$ and $\mathrm{inv} : G \to G$ satisfying some commutativity conditions, where $G \times G$ has the product group structure described in the first paragraph. We may regard m as defining a second binary product on G, and the commutativity axiom (G1) then says that the triple $(G, m, 1)$ is a unital magma, where we used the equality $e(\mathrm{pt}) = 1$ coming from the fact that a group homomorphism preserves units.

Recall that the group structure on $G \times G$ is defined by the product

$$(g_1, g_2) \cdot (g_3, g_4) \coloneqq (g_1 \cdot g_3, g_2 \cdot g_4).$$

Since m is a group homomorphism by hypothesis, the equality

$$m((g_1, g_2) \cdot (g_3, g_4)) = m(g_1, g_2) \cdot m(g_3, g_4)$$

holds for any g_1, g_2, g_3, $g_4 \in G$. Unpacking the definition of $(g_1, g_2) \cdot (g_3, g_4)$, this equality is equivalent to

$$m(g_1 \cdot g_3, g_2 \cdot g_4) = m(g_1, g_2) \cdot m(g_3, g_4).$$

If we introduce the auxiliary notation

$$g \cdot g' \coloneqq m(g, g')$$

for the binary product defined by m, then the previous equality can be rewritten as

$$(g_1 \cdot g_3) \otimes (g_2 \cdot g_4) = (g_1 \cdot g_2) \otimes (g_3 \otimes g_4).$$

This equality shows that the unital magmas $(G, \cdot, 1)$ and $(G, \otimes, 1)$ satisfy the compatibility condition of Theorem A.1.6, therefore $\cdot$ must be commutative, that is, G must be an abelian group.

As an extra, we can also show that the inversion inv coincides with the inversion $(-)^{-1}$. Given $g \in G$, since m coincides with $\cdot$ by Theorem A.1.6, we have

$$g \cdot \mathrm{inv}(g) = m(g, \mathrm{inv}(g)) = 1 = m(\mathrm{inv}(g), g) = \mathrm{inv}(g) \cdot g.$$

This shows that $\mathrm{inv}(g)$ is an inverse to g with respect to the product $\cdot$, and the uniqueness of inverses in a group then implies that $\mathrm{inv}(g) = g^{-1}$. ∎

Now we introduce the concept dual to group objects: cogroup objects. Whereas in a group two elements can be combined into one, in a cogroup an element can in some sense be separated in two. This may seem weird at first, but over the course of the text we will encounter very natural examples of cogroups in Homotopy Theory. In fact (spoiler!), the most important spaces in Algebraic Topology are cogroups.

The definition of a cogroup object is obtained by dualizing the definition of a group, which informally means that we reverse the directions of arrows in diagrams and dualize all the categorical constructions involved.

A.1.9 Definition. Suppose C is a category with binary coproducts and an initial object 0. A **cogroup object** in C is a tuple $(G, \mu, \varepsilon, \nu)$ where

1. μ is a morphism of type $G \to G \sqcup G$,
2. ε is a morphism of type $G \to 0$,
3. ν is a morphism of type $G \to G$.

These morphisms must satisfy the commutativity conditions imposed by the three diagrams below, where $!_G : 0 \to G$ is the unique morphism from the initial object 0 to G, and $A : (G \sqcup G) \sqcup G \to G \sqcup (G \sqcup G)$ is the associator isomorphism:

(CG1) (Existence of two-sided counit)

$$\begin{tikzcd}
G & G \sqcup G \arrow[l, "\langle !_G \circ \varepsilon, \mathrm{id}_G \rangle"'] \\
G \sqcup G \arrow[u, "\langle \mathrm{id}_G, !_G \circ \varepsilon \rangle"] & G \arrow[l, "\mu"] \arrow[u, "\mu"'] \arrow[ul, "\mathrm{id}_G"]
\end{tikzcd}$$

(CG2) (Coassociativity)

$$\begin{tikzcd}
G \sqcup (G \sqcup G) & & (G \sqcup G) \sqcup G \arrow[ll, "A"'] \\
G \sqcup G \arrow[u, "\mathrm{id}_G \sqcup \mu"] & & G \sqcup G \arrow[u, "\mu \sqcup \mathrm{id}_G"'] \\
& G \arrow[ul, "\mu"] \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(CG3) (Existence of co-inverses)

$$\begin{tikzcd}
G & G \sqcup G \arrow[l, "\langle \mathrm{id}_G, \nu \rangle"'] \\
G \sqcup G \arrow[u, "\langle \nu, \mathrm{id}_G \rangle"] & G \arrow[l, "\mu"] \arrow[u, "\mu"'] \arrow[ul, "!_G \circ \varepsilon" description]
\end{tikzcd}$$

The morphisms in a cogroup structure also have special names: μ is the **comultiplication morphism**, ε is the **counit morphism**, and ν is the **co-inversion morphism**.

A.1.10 Remark. The description of the associator isomorphism for the coproduct is dual to the description of the analogous isomorphism for the product we gave in Remark A.1.2. Consider the canonical injections below:

$$\begin{aligned}
&j_1 : G \to G \sqcup G, \\
&j_2 : G \to G \sqcup G, \\
&J_1 : G \sqcup G \to (G \sqcup G) \sqcup G, \\
&J_2 : G \to (G \sqcup G) \sqcup G, \\
&J_1' : G \to G \sqcup (G \sqcup G), \\
&J_2' : G \sqcup G \to G \sqcup (G \sqcup G).
\end{aligned}$$

The morphism $A : (G \sqcup G) \sqcup G \to G \sqcup (G \sqcup G)$ we are trying to define is uniquely determined by the compositions $A \circ J_1 : G \sqcup G \to G \sqcup (G \sqcup G)$ and $A \circ J_2 : G \to G \sqcup (G \sqcup G)$, and $A \circ J_1$ is itself determined by the compositions $A \circ J_1 \circ j_1$ and $A \circ J_1 \circ j_2$ from G to $G \sqcup (G \sqcup G)$. Using the universal property of the coproduct twice we can then define A as the only morphism of its type satisfying the following equations:

$$\begin{cases} A \circ J_1 \circ j_1 = J_1', \\ A \circ J_1 \circ j_2 = J_2' \circ j_1, \\ A \circ J_2 = J_2' \circ j_2. \end{cases} \tag{A.2}$$

Most of the interesting examples of cogroups we will meet are connected with Homotopy Theory, so they will be studied in a future chapter. Nevertheless, there are some basic examples that we can examine already.

A.1.11 Example. In any category with coproducts and an initial object, the initial object 0 itself admits a cogroup structure, with comultiplication given by the unique morphism $!_{0\sqcup 0} : 0 \to 0 \sqcup 0$, and both counit and co-inversion given by $\mathrm{id}_0 : 0 \to 0$, which is the unique endomorphism of 0.

All the required commutativity conditions follow from the universal property characterizing 0, that is, the fact that for any object $A \in \mathsf{C}$ there is a unique morphism in $\mathsf{C}(0, A)$. For example, condition (CG1) follows because $\langle !_{0\sqcup 0} \circ e, \mathrm{id}_0 \rangle$ and $\langle \mathrm{id}_0, !_{0\sqcup 0} \circ e \rangle$ both belong to $\mathsf{C}(0, 0)$, therefore they must be equal.

In particular, in the category Set, the empty set $\varnothing$ admits such a cogroup structure because it is an initial object. In fact, it is the *only* set admitting a cogroup structure, because if $X \in \mathsf{Set}$ admits one, then in particular there is a map $e : X \to \varnothing$, but this is only possible if $X = \varnothing$. This is a possible explanation for the initial weirdness of cogroups: whereas the study of group objects in Set - also known as Group Theory :) - is extremely rich, the study of cogroup objects in Set is trivial.

A.1.12 Example. Let R be a commutative ring with unit, and consider the category ${}_{\mathsf{R}}\mathsf{Mod}$ of left R-modules. Let us recall some basic facts and constructions of this category. It has an initial object given by the trivial R-module $\{0\}$ containing only the neutral element for addition. Any two R-modules M and N have a coproduct given by the direct sum $M \oplus N$, with canonical injections $i_1 : M \to M \oplus N$ and $i_2 : N \to M \oplus N$ defined as

$$i_1(m) \coloneqq (m, 0) \quad \text{and} \quad i_2(n) \coloneqq (0, n)$$

for every $m \in M$ and $n \in N$. If we consider another R-module P together with morphisms $f : M \to P$ and $g : N \to P$, then the induced morphism $\langle f, g \rangle : M \oplus N \to P$ is given explicitly by

$$\langle f, g \rangle (m, n) \coloneqq f(m) + g(n) \quad \forall\, (m, n) \in M \oplus N.$$

Lastly, the associating isomorphism $A : (M \oplus M) \oplus M \to M \oplus (M \oplus M)$ maps $((m_1, m_2), m_3)$ to $(m_1, (m_2, m_3))$.

We now show that any object of ${}_{\mathsf{R}}\mathsf{Mod}$ admits the structure of a cogroup. Given an R-module M, we define a comultiplication by using the diagonal map $\Delta : M \to M \oplus M$, so that $m \in M$ is mapped to (m, m). The only choice of counit map $\varepsilon : M \to \{0\}$ is the zero map. Lastly, a co-inversion morphism $\nu : M \to M$ is defined as $\nu(m) \coloneqq -m$ for every $m \in M$.

We now check the commutativity conditions. Let $m \in M$ be an arbitrary element.

For the counit condition, we have

$$\begin{aligned}(\langle \mathrm{id}_M, !_M \circ \varepsilon\rangle \circ \Delta)(m) &= \langle \mathrm{id}_M, !_M \circ \varepsilon\rangle(m, m)\\ &= \mathrm{id}_M(m) + (!_M \circ \varepsilon)(m)\\ &= \mathrm{id}_M(m) + 0\\ &= \mathrm{id}_M(m),\end{aligned}$$

and also

$$\begin{aligned}(\langle !_M \circ \varepsilon, \mathrm{id}_M\rangle \circ \Delta)(m) &= \langle !_M \circ \varepsilon, \mathrm{id}_M\rangle(m, m)\\ &= (!_M \circ \varepsilon)(m) + \mathrm{id}_M(m)\\ &= 0 + \mathrm{id}_M(m)\\ &= \mathrm{id}_M(m).\end{aligned}$$

For the coassociativity, on the one hand

$$\begin{aligned}&((\mathrm{id}_M \sqcup \Delta) \circ \Delta)(m)\\ &\qquad = (\mathrm{id}_M \sqcup \Delta)(m, m)\\ &\qquad = (\mathrm{id}_M(m), \Delta(m))\\ &\qquad = (m, (m, m)),\end{aligned}$$

while on the other

$$\begin{aligned}&(A \circ (\Delta \sqcup \mathrm{id}_M) \circ \Delta)(m)\\ &\qquad = (A \circ (\Delta \sqcup \mathrm{id}_M))(m, m)\\ &\qquad = A(\Delta(m), \mathrm{id}(m))\\ &\qquad = A((m, m), m)\\ &\qquad = (m, (m, m)).\end{aligned}$$

Lastly, for the co-inversion condition, we have

$$\begin{aligned}&(\langle \nu, \mathrm{id}_M\rangle \circ \Delta)(m)\\ &\qquad = \langle \nu, \mathrm{id}_M\rangle(m, m)\\ &\qquad = \nu(m) + \mathrm{id}_M(m)\\ &\qquad = -m + m\\ &\qquad = 0\\ &\qquad = (!_M \circ \varepsilon)(m),\end{aligned}$$

and also

$$\begin{aligned}((\langle \mathrm{id}_M, \nu\rangle \circ \Delta)(m) &= \langle \mathrm{id}_M, \nu\rangle(m, m) \\ &= \mathrm{id}_M(m) + \nu(m) \\ &= m + (-m) \\ &= 0 \\ &= (!_M \circ \varepsilon)(m).\end{aligned}$$

A.2 Ordinary groups from (co)group objects

After introducing the notions of group and cogroup objects, we now study how these concepts can be used to obtain a family of ordinary groups. This is a procedure which allows us to extract algebraic information from any category with sufficient structure possessing either group or cogroup objects.

A.2.1 Theorem. Suppose C is a locally small category with terminal object $*$ and with all binary products. Let (G, m, e, inv) be a group object in C. Then, for any other object $X \in \mathsf{C}$, the set of morphisms $\mathsf{C}(X, G)$ admits a group structure such that, for any morphism $\alpha : X \to Y$, the pullback function $\mathsf{C}(\alpha, G) : \mathsf{C}(Y, G) \to \mathsf{C}(X, G)$ defines a group homomorphism.

Proof. We define a binary product $\cdot_X : \mathsf{C}(X, G) \times \mathsf{C}(X, G) \to \mathsf{C}(X, G)$ using the formula

$$f \cdot_X g := m \circ (f, g) \tag{A.3}$$

for every f, $g \in \mathsf{C}(X, G)$. This definition makes sense: given two morphisms f, $g : X \to G$, by the universal property of the product we obtain an induced morphism $(f, g) : X \to G \times G$ which we may then compose with the multiplication morphism $m : G \times G \to G$ to obtain another morphism from X to G.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & G & \rightsquigarrow & X \arrow[r, "{(f,g)}"] \arrow[rr, dashed, bend right, "f \cdot_X g"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

We now need to prove that this binary product $\cdot_X$ really defines a group structure on $\mathsf{C}(X, G)$. We start by exhibiting an identity element for it. Let $!_X : X \to *$ be the unique morphism from X to the terminal object. Combining this with the unit $e : * \to G$ we obtain the morphism

$$e_X := e \circ {!_X} : X \to G. \tag{A.4}$$

Given any $f \in \mathsf{C}(X, G)$, by definition we have $f \cdot e_X = m \circ (f, e_X) = m \circ (f, e \circ !_X)$, but we have the equality $!_X = !_G \circ f$, since $!_G \circ f$ also defines a map from X to $*$. Using this relation leaves us with the equality

$$f \cdot e_X = m \circ (f, e \circ !_G \circ f).$$

We claim that the right-hand side can be rewritten as

$$(f, e \circ !_G \circ f) = (\mathrm{id}_G, e \circ !_G) \circ f. \tag{A.5}$$

Indeed, first recall that, if $\pi_1,\ \pi_2 : G \times G \to G$ are the canonical projects, then $(f, e \circ !_G \circ f)$ is the *only* morphism of its type satisfying the equations

$$\pi_1 \circ (f, e \circ !_G \circ f) = f \quad \text{and} \quad \pi_2 \circ (f, e \circ !_G \circ f) = e \circ !_G \circ f.$$

With this in mind, notice that on the one hand

$$\pi_1 \circ (\mathrm{id}_G, e \circ !_G) \circ f = \mathrm{id}_G \circ f = f,$$

and on the other

$$\pi_2 \circ (\mathrm{id}_G, e \circ !_G) \circ f = e \circ !_G \circ f;$$

and these two equalities together imply (A.5). Using this newly obtained identity we see that

$$\begin{aligned} f \cdot_X e_X &= m \circ (f, e \circ !_G \circ f) \\ &= m \circ (\mathrm{id}_G, e \circ !_G) \circ f \\ &= \mathrm{id}_G \circ f && \text{(by axiom (G1))} \\ &= f; \end{aligned}$$

proving that e_X is a right-identity for the product $\cdot_A$.

The proof that e_X is also a left identity is similar. For any $g \in \mathsf{C}(X, G)$ we have

$$\begin{aligned} e_X \cdot g &= m \circ (e_X, g) \\ &= m \circ (e \circ !_X, g) \\ &= m \circ (e \circ !_G \circ g, g) \\ &= m \circ (e \circ !_G, \mathrm{id}_G) \circ g \\ &= \mathrm{id}_G \circ g \\ &= g. \end{aligned}$$

Now we show the existence of inverses. Given a morphism $f \in \mathsf{C}(X, G)$, we will show that $\mathrm{inv} \circ f : A \to G$ is the inverse of f with respect to the product $\cdot_X$. On the one hand

$$\begin{aligned} f \cdot_X (\mathrm{inv} \circ f) &= m \circ (f, \mathrm{inv} \circ f) \\ &= m \circ (\mathrm{id}_G, \mathrm{inv}) \circ f \\ &= e \circ !_G \circ f \\ &= e \circ !_X \\ &= e_X, \end{aligned}$$

which proves that $\mathrm{inv} \circ f$ is a right-inverse for f; and on the other

$$\begin{aligned} (\mathrm{inv} \circ f) \cdot_X f &= m \circ (\mathrm{inv} \circ f, f) \\ &= m \circ (\mathrm{inv}, \mathrm{id}_G) \circ f \\ &= e \circ !_G \circ f \\ &= e \circ !_X \\ &= e_X, \end{aligned}$$

which proves that $\mathrm{inv} \circ f$ is also a left-inverse for f.

Now we turn to proving the associativity of $\cdot_X$, the hardest part of the proof. Given three morphisms f, g, $h \in \mathsf{C}(X, G)$, in order to show the associativity of $\cdot_X$ we must show the equality

$$m \circ (m \circ (f, g), h) = m \circ (f, m \circ (g, h)).$$

We first claim that the equality

$$A \circ ((f, g), h) = (f, (g, h)) \tag{A.6}$$

holds, which is equivalent to the commutativity of the triangle below.

$$\begin{tikzcd} & X \arrow[dl, "((f,g),h)"'] \arrow[dr, "(f,(g,h))"] & \\ (G \times G) \times G \arrow[rr, "A"'] & & G \times (G \times G) \end{tikzcd}$$

Following the notation of Remark A.1.2, the equality will follow from the universal property of the product if we manage to show the equalities

$$\begin{cases} \Pi'_1 \circ A \circ ((f, g), h) = f, \\ \Pi'_2 \circ A \circ ((f, g), h) = (g, h). \end{cases}$$

Using the relations (A.1) characterizing the associator isomorphism A we see that

$$\begin{aligned} \Pi'_1 \circ A \circ ((g, h), h) &= \pi_1 \circ \Pi_1 \circ ((f, g), h) \\ &= \pi_1 \circ (f, g) \\ &= f, \end{aligned}$$

which shows the first of the required equalities. For the second equality, we first note that

$$\begin{aligned} \pi_1 \circ \Pi'_2 \circ A \circ ((f,g),h) &= \pi_2 \circ \Pi_1 \circ ((f,g),h) && \text{(by (A.1))} \\ &= \pi_2 \circ (f,g) \\ &= g, \end{aligned}$$

and then we also note that

$$\begin{aligned} \pi_2 \circ \Pi'_2 \circ A \circ ((f,g),h) &= \Pi_2 \circ ((f,g),h) && \text{(by (A.1))} \\ &= h; \end{aligned}$$

but these equalities together imply that

$$\Pi'_2 \circ A \circ ((f,g),h) = (g,h)$$

as desired.

Now, we know from axiom (G3) that the equality

$$m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m) \circ A$$

holds. If we precompose both sides with the morphism $((f,g),h)$ and use (A.6) we obtain

$$m \circ (m \times \mathrm{id}_G) \circ ((f,g),h) = m \circ (\mathrm{id}_G \times m) \circ (f,(g,h)). \tag{A.7}$$

This is in fact equivalent to the equality we want to prove. In order to show this, we first show that

$$(m \times \mathrm{id}_G) \circ ((f,g),h) = (m \circ (f,g),h). \tag{A.8}$$

Again, this is shown by looking at the compositions with the canonical projections. The product morphism $m \times \mathrm{id}_G$ by definition satisfies the equalities

$$\pi_1 \circ (m \times \mathrm{id}_G) = m \circ \Pi_1 \quad \text{and} \quad \pi_2 \circ (m \times \mathrm{id}_G) = \mathrm{id}_G \circ \Pi_2 = \Pi_2.$$

Using these relations we see that

$$\begin{aligned} \pi_1 \circ (m \times \mathrm{id}_G) \circ ((f,g),h) &= m \circ \Pi_1 \circ ((f,g),h) \\ &= m \circ (f,g), \end{aligned}$$

and also

$$\begin{aligned} \pi_2 \circ (m \times \mathrm{id}_G) \circ ((f,g),h) &= \Pi_2 \circ ((f,g),h) \\ &= h; \end{aligned}$$

and these two equalities together imply (A.8). An analogous reasoning shows that we also have the equality

$$(\mathrm{id}_G \times m) \circ (f,(g,h)) = (f, m \circ (g,h)). \tag{A.9}$$

Finally, substituting (A.8) and (A.9) into (A.7) yields

$$m \circ (m \circ (f, g), h) = m \circ (f, m \circ (g, h)),$$

which is precisely the equality we wanted to show.

The only thing left is showing that, for a morphism $\alpha : X \to Y$, the pullback $\mathsf{C}(f, G) : \mathsf{C}(Y, G) \to \mathsf{C}(X, G)$ defines a group homomorphism. This follows by a direct computation: given $f,\ g \in \mathsf{C}(Y, G)$, we have

$$\begin{aligned}
\mathsf{C}(\alpha, G)(f \cdot_Y g) &= (f \cdot_Y g) \circ \alpha \\
&= m \circ (f, g) \circ \alpha \\
&= m \circ (f \circ \alpha, g \circ \alpha) \\
&= (f \circ \alpha) \cdot_X (g \circ \alpha) \\
&= \mathsf{C}(\alpha, G)(f) \cdot_X \mathsf{C}(\alpha, G)(g).
\end{aligned}$$

■

All the results proved so far have dual ones concerning cogroups. They can be proved either by direct arguments analogous to the ones we have already given, or by applying the results we have already obtained to the opposite category. We now state these dual results without proof in order to be able to reference them later on.

A.2.2 Theorem. Suppose C is a locally small category with initial object 0 and with all binary coproducts. Let $(G, \mu, \varepsilon, \nu)$ be a cogroup object in C. Then, for any other object $X \in \mathsf{C}$, the set of morphisms $\mathsf{C}(G, X)$ admits a group structure such that, for any morphism $\alpha : X \to Y$, the pushforward function $\mathsf{C}(G, \alpha) : \mathsf{C}(G, X) \to \mathsf{C}(G, Y)$ defines a group homomorphism.

The description of the product on the set $\mathsf{C}(G, X)$ is dual to that of Theorem A.2.1. Given two morphisms $f,\ g : G \to X$, by the universal property of the coproduct we have an induced map $\langle f, g\rangle : G \sqcup G \to X$, and combining this with the comultiplication morphism we define

$$f \cdot_X g := \langle f, g\rangle \circ \mu, \tag{A.10}$$

which is another morphism $G \to X$ as shown below.

$$\begin{tikzcd} G \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} G \arrow[r, "\mu"] \arrow[rr, dashed, bend right, "f \cdot_X g"'] & G \sqcup G \arrow[r, "\langle f, g\rangle"] & X \end{tikzcd}$$

A unit for this product $\cdot_X$ is given by the morphisms

$$e_X := !_X \circ \varepsilon, \tag{A.11}$$

and lastly, an inverse for a morphism $f : G \to X$ with respect to the product $\cdot_X$ is given by the morphism

$$f^{-1} := f \circ \nu. \tag{A.12}$$

After studying how to obtain ordinary group structures from categorical group and cogroup objects, we study a particular case of this construction where we deal simultaneously with both group and cogroup objects. The next result is one of the underlying principles behind the commutativity of the higher dimensional homotopy groups.

A.2.3 Proposition. Let C be a locally small category with an initial object 0, a terminal object $*$, and having all binary products and coproducts. Let $(G, \mu, \varepsilon, \nu)$ be a cogroup object in C, and let (H, m, e, inv) be a group object in C. Denote by $\cdot_H$ the binary product on $\mathsf{C}(G, H)$ coming from the cogroup structure, and denote by $\cdot_G$ the binary product on the same set coming from the group structure. Then $\cdot_H$ and $\cdot_G$ coincide, and they define an abelian group structure on the set of morphisms $\mathsf{C}(G, H)$.

Proof. The setup suggests the use of the Eckmann-Hilton Argument (Theorem A.1.6), so we need to verify the compatibility of the products $\cdot_H$ and $\cdot_G$, that is, given morphisms α, β, γ, $\delta \in \mathsf{C}(G, H)$, we need to prove the equality

$$(\alpha \cdot_H \beta) \cdot_G (\gamma \cdot_H \cdot \delta) = (\alpha \cdot_G \gamma) \cdot_H (\beta \cdot_G \delta).$$

Unpacking the definitions of the products $\cdot_H$ and $\cdot_G$, the expression above can be rewritten as

$$m \circ (\langle \alpha, \beta \rangle \circ \mu, \langle \gamma, \delta \rangle \circ \mu) = \langle m \circ (\alpha, \gamma), m \circ (\beta, \delta) \rangle \circ \mu. \tag{A.13}$$

By studying the compositions of the morphism $(\langle \alpha, \beta \rangle \circ \mu, \langle \alpha, \beta \rangle \circ \mu)$ with the canonical projections π_1, $\pi_2 : H \times H \to H$ we deduce the equality

$$(\langle \alpha, \beta \rangle \circ \mu, \langle \gamma, \delta \rangle \circ \mu) = (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ \mu,$$

and similarly, by studying the composition of $\langle m \circ (\alpha, \gamma), m \circ (\beta, \delta) \rangle$ with the canonical injections j_1, $j_2 : G \to G \sqcup G$ we deduce that

$$\langle m \circ (\alpha, \gamma), m \circ (\beta, \delta) \rangle = m \circ \langle (\alpha, \gamma), (\beta, \delta) \rangle.$$

Substituting these two equalities into (A.13) shows that the equality we are trying to prove can be rewritten as

$$m \circ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ \mu = m \circ \langle (\alpha, \gamma), (\beta, \delta) \rangle \circ \mu. \tag{A.14}$$

Comparing the two sides of this, we see that in fact it suffices to prove the equality

$$(\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) = \langle (\alpha, \gamma), (\beta, \delta) \rangle. \tag{A.15}$$

The right-hand side of the expression above denotes a morphism of type $G \sqcup G \to H \times H$ obtained by applying the universal property of the coproduct, therefore (A.15) is equivalent to the pair of equalities below:

$$\begin{cases} (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_1 = (\alpha, \gamma), \\ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_2 = (\beta, \delta). \end{cases} \tag{A.16}$$

Both morphisms appearing on the right-hand side above are obtained from the universal property of the product, so it makes sense to study the composition of the morphisms on the left-hand side with the canonical projections π_1, $\pi_2 : H \times H \to H$. We have

$$\pi_1 \circ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_1 = \langle \alpha, \beta \rangle \circ j_1 = \alpha,$$

and also

$$\pi_2 \circ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_1 = \langle \gamma, \delta \rangle \circ j_1 = \gamma;$$

implying the first line of (A.16) By completely analogous computations we also have the equalities

$$\pi_1 \circ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_2 = \beta \quad \text{and} \quad \pi_2 \circ (\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle) \circ j_2 = \delta;$$

which together imply the second line of (A.16). ■

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