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Centers and isochronicity of some polynomial differential systems

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Centros e isocronicidade de alguns sistemas diferenciais polinomiais

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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To my family, teachers and friends.

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"You must be the change you wish to see in the world."
(Mahatma Gandhi)

ABSTRACT

FERNANDES, W. **Centers and isochronicity of some polynomial differential systems**. 2017. 179 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

The center-focus and isochronicity problems are two classic problem in the qualitative theory of ordinary differential equations (ODEs). Although such problems have been studied during more than hundred years a complete understanding of them is far from be reached. Recently the computational algebra tools have been contributing significantly with the development of such problems. The aim of this thesis is to contribute with the studies of the center-focus and isochronicity problem. Using computational algebra tools we find conditions for the existence of two simultaneous centers for a family of quintic systems possessing symmetry. The studies of the simultaneous existence of two centers in differential systems is known as the bi-center problem. We investigate conditions for the isochronicity of centers for families of cubic and quintic systems and we study its global behaviour in the Poincaré disk. Finally, we study the existence of invariant surfaces and first integrals in a family of 3-dimensional systems. Such family is known as the May-Leonard asymmetric system and it appears in modelling, for instance it is a model for the competition of three species.

Key-words: Isochronous centers, Darboux integrability, invariant surfaces and curves, differential systems with symmetry, primary decompositions of ideals.

RESUMO

FERNANDES, W. **Centers and isochronicity of some polynomial differential systems**. 2017. 179 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

Os problemas do foco-centro e da isocronicidade são dois problemas clássicos da teoria qualitativa das equações diferenciais ordinárias (EDOs). Apesar de tais problemas serem investigados a mais de cem anos ainda pouco se sabe sobre eles. Recentemente o uso e desenvolvimento de ferramentas algebro-computacionais tem contribuído significativamente em seu avanço. O objetivo desta tese é colaborar com o estudo do problema do foco-centro e da isocronicidade. Utilizando ferramentas algebro-computacionais encontramos condições para a existência simultânea de dois centros em famílias de sistemas diferenciais quínticos com simetria. O estudo sobre a existência simultânea de dois centros é também conhecido como problema do bi-centro. Investigamos condições para a isocronicidade de centros para famílias de sistemas cubicos e quínticos e estudamos o comportamento global de suas órbitas no disco de Poincaré. Finalmente, tratamos da existência de superfícies invariantes e integrais primeiras para uma familia de sistemas 3-dimensionais encontrado entre outras situações na modelagem da competição entre três espécies e conhecido como sistema de May-Leonard.

Palavras-chave: Centros isócronos, integrabilidade Darbouxiana, curvas e superfícies invariantes, sistemas diferenciais com simetria, decomposição primária de ideais.

CONTENTS

INTRODUCTION

The history of the theory of ordinary differential equations goes back the 17th century when Newton and Leibniz proposed, independently, the differential calculus. In that time, this new tool became easy the possibility of solving many questions asked by ancient mathematicians, acquiring the appreciation of many researches. Through the years the development of the theory of ordinary differential equations have been contributing with the investigation of a big amount of problems in many different areas, for instance, physical phenomena, interaction of species, business transactions, and several situations in engineering. However, together with the evolution of such theory the difficult of finding and estimate their solutions growth significantly. It has given rise to the qualitative theory of the differential equations, introduced by Poincaré [97]. This theory consists of deducing the behaviour of the solutions of an ODE without computing them explicitly, just knowing certain properties of the vector fields that define them.

With the development of the qualitative theory of differential systems a large amount of problems were arising. Among them is, for instance, the most investigated problem in the qualitative theory of dynamical systems in the plane, the Hilbert's 16th problem [60, 61]. In short, this problem discusses on the number of limit cycles in polynomial systems in the plane. Although many researchers have been working in this problem it has no a complete answer and it remains difficult to solve.

In this thesis we study two classical problems of qualitative theory of ordinary differential equations, the *center-focus* and *isochronicity* problems. The history of both problems go back to Poincaré and Lyapunov. In short, the center-focus problem consists in to decide when a singular point of an ordinary differential equation is a center (the orbit of every point in a neighbourhood of the singular point is a closed orbit) or a focus (the orbits of every point in a neighbourhood of the singular point spirals towards or away from the singular point). Similarly to the 16th Hilbert problem, the center-focus has been studied since the end of the 19th century and it remains unresolved even for cubic polynomial systems. A general method introduced by Poincaré and Lyapunov reduces the center-focus problem to the problem of finding a first integral for the system, that is the integrability problem. Such characterization lead us to solve an infinite system of polynomial equations whose variables are parameters of the system of differential equations. That is, the center-focus problem is reduced to the problem of finding the variety of the ideal generated by a collection of polynomials, called the focus quantities of the system.

The isochronicity problem consists in to decide when all periodic solutions in a neighbourhood of a center have the same period. Such problem has attracted studies from the time of Huygens and the Bernoullis (17th century) and as the two problem mentioned above it is completely solved only for few families of polynomial systems. The studies of the isochronicity problem for polynomial systems became more intensively when it was proved that the existence of an isochronous center is direct connect with the linearizability of the system. The reason is that, similarly to the center-focus problem, the linearizability problem consists in to compute a collection of polynomials whose variables are parameters of the system, called linearizability quantities and then finding the variety of the ideal generated by them. Such variety lies in the complex field, where the computational methods are more efficient.

In view of the difficulty to obtain results for these three problem, the researchers have been improving and giving new statements for them restricting their analysis on families of differential systems with specific properties. We dare say that the complete study of the huge family of generic differential systems is impossible.

It is worth mentioning that in the recent decades have seen a surge of interest in such problems. Certainly an important reason for this is that their resolutions involves extremely laborious computations, which nowadays can be carried out using powerful computational facilities. Applications of concepts that could not be utilized even 30 years ago are now feasible, often even on a personal computer, because of advances in the mathematical theory, in the computer software of computational algebra, and in computer technology.

The main objective of this thesis is to increase the knowledge on the center-focus and isochronicity problems. We look for conditions for the existence of centers and isochronous centers on different families of differential systems. As the history on researching these problems says, the investigations carried out are very difficult and we have to look for different tools and results to solve them. Moreover, in some cases we have to restrict the studies of a family of systems to some subfamilies and then gather the results obtained for such subfamilies to obtain a conclusion for the whole family.

In the literature the existence of two simultaneous centers in planar differential systems was investigated only for very few particular families of systems. Kirnitskaya and Sibirskii in [68] and Li in [70] studied this problem for quadratic systems. Conti [32] and Chen, Lu and Wang [23] investigated particular families of cubic systems and Du [39] a particular family of polynomial systems of degree seven. Recently, Liu and Li [79] studied a family of $\mathbb{Z}_2$-equivariant cubic systems, obtaining conditions for the existence of two simultaneous centers in such system (called here as bi-center). In the second chapter of this thesis we study the isochronicity of the centers found by Liu and Li [79] and we investigate the existence of bi-centers and isochronous bi-centers for a family of $\mathbb{Z}_2$-equivariant quintic systems. Moreover we study the global behaviour in the Poicaré disk of the $\mathbb{Z}_2$-equivariant quintic systems when it possess an isochronous bi-center. Part of the results on this investigation are in:

V. G. Romanovski, W. Fernandes, R. Oliveira. Bi-center problem for some classes of $\mathbb{Z}_2$-equivariant systems. *J. Comput. Appl. Math* **320** (2017), 61–75.

Studying the isochronocity of a family of cubic systems with the property that the infinity is fulfilled of singular points we have faced with some results on the literature which are not compatible. In the paper [85] published in 1997 the authors obtained four series of conditions for the existence of an isochronous center for such system, however in the more recent paper [20] published in 1999 the authors gave five conditions for existence of isochronous center for the same system. In Chapter 3 of this thesis we clarify the conditions for isochronicity for such system. Moreover, we study the coexistence of isochronous centers in the system. The results on this investigation are in:

W. Fernandes, V. G. Romanovski, M. Sultanova, Y. Tang. Isochronicity and linearizability of a planar cubic system. *J. Math. Anal. Appl.* **450** (2017), 795–813.

Our more recent contribution investigating the isochronicity of centers for planar differential systems was done in a family of system known as the generalized Riccati system. In Chapter 4 we obtain four families of such system in the complex field being linearizable where two of then represents systems with an isochronous center at the origin. In addition we study the global structures in the Poicaré disk of such system when it possess an isochronous center at the origin. The results of this investigation are contained in the preprint:

V. G. Romanovski, W. Fernandes, Y. Tang, Y. Tian. Linearizability and critical period bifurcations of a generalized Riccati system. Preprint available at https://www.researchgate.net/publication/313844845_Linearizability_and_critical_period_bifurcations_of_a_generalized_Riccati_system.

Generalizing the concept of center we study persistent centers for complex differential systems. Such kind of centers were introduced by Cima, Gasull and Medrado in [31] for real differential systems. In [27], Chen, Romanovski and Zhang generalized the notion of persistent centers for complex systems. In Chapter 5 we present the investigation of the linearizability of persistent centers for complex systems. Firstly we introduce the definition of linearizable persistent centers and then we describe an computational approach to perform our investigation. The method consists in to compute some polynomials which we call by persistent linearizability quantities. The results are described in the preprint:

M. Mencinger, W. Fernandes, B. Ferčec R. Oliveira. On linearizability of persistent and weakly persistent cubic centers. Preprint available at http://conteudo.icmc.usp.br/CMS/Arquivos/arquivos_enviados/BIBLIOTECA_158_Nota%20Serie%20Mat%20423.pdf.

Extending the studies of differential systems from the plane to the space, we study a three-dimensional differential system known as the May-Laonard asymmetric model. Such system belongs to the well known family of Lotka-Volterra systems and it is a generalization of the model introduced by May and Leonard in [91] describing the competition of three species. The dynamics of the May-Leonard asymmetric system was studied in [6, 28, 63, 117]. In Chapter 6 our aim is to study the integrability of May-Leonard asymmetric model. Using algorithms of the elimination theory we first find families of such systems admitting invariant planes and invariant surfaces defined by quadratic polynomials. Then we look for first integrals of the Darboux type constructed using these invariant surfaces and find subfamilies of the system admitting one or two independent first integrals. It is important to mention that the approach proposed in this investigation can be used to study integrability of many other mathematical models described by polynomial systems of differential equations. The results are described in the preprint:

V. Antonov, W. Fernandes, V. G. Romanovski, N. L. Shceglova. First integrals of the May-Leonard assymetric system. Preprint available at Notas da Série Matemática ICMC-USP, number 436, see http://conteudo.icmc.usp.br/Portal/conteudoDinamicoSemVinculo.php?id_conteudos=719.

The first chapter describes the basic definitions, notions and results used in this thesis. A short introduction on the qualitative theory of ordinary differential equations is presented. The concepts of Gröber basis and decomposition of afine varieties which are the basis of the algorithms used in computational algebra methods are introduced and different methods for investigating the center-focus problem and the isochronicity problem are described.

We finish this thesis presenting our final considerations and giving some directions for further investigations.

I take this opportunity to thank the committee members Claudia Valls, Claudio Pessoa and Joan Torregrosa for being present on the day of the defense (either in person or by video-conference) and also for the valuable comments and corrections which have enriched this thesis further.

Have a good reading!

CHAPTER 1

PRELIMINARIES AND BASIC RESULTS

In this chapter we introduce some basic definitions, notions and results used in this thesis. In the first section a short introduction on the qualitative theory of ordinary differential equations is presented. For more details the reader can see e.g. [9, 43, 62, 110, 112]. The second section contains the concepts of Gröber basis and decomposition of afine varieties which are the basis of the algorithms used in computational algebra methods and are close related to the investigation proposed in this thesis, for more details we refer [33, 103]. The last two sections are devoted to present two very known problems of the qualitative theory of ordinary differential equations, the center problem and the isochronicity problem. The center problem consists of distinguishing between a center and a focus for a given polynomial differential system. The isochronicity problem is the problem to decide when all period solutions in a neighbourhood of a center have the same period. For investigating such problems we describe different approaches and computational methods, see also [103].

1.1 Singular points and global phase portraits of ODEs

Let $f : U \to \mathbb{R}^n$ be a map defined in an open set $U \subseteq \mathbb{R} \times \mathbb{R}^n$. Consider the system of ordinary differential equations

$$\dot{x} = f(t,x), \tag{1.1}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\dot{x}$ denotes the derivative of x with respect to the variable t. A function $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is called a *solution* of system (1.1) if $(t, x(t)) \in U$ for all $t \in I$ and $\dot{x}(t) = f(t, x(t))$ for all $t \in I$. If I is the largest interval for which $x(t)$ satisfies (1.1) then $x(t)$ is called the *maximal solution*.

The solution of (1.1) may not exist or if it there exists can be not unique. The existence and uniqueness of a solution are determined by certain conditions imposed on the function f, as described in the next results, see for instance [62, 110, 112] for details.

Proposition 1.1.1. *If f is a continuous map defined in the open set $U \subseteq \mathbb{R}^{n+1}$ then given any point $(t_0, x_0) \in U$ there exists a solution $x : I \to \mathbb{R}^n$ of (1.1) such that $t_0 \in I$ and $x(t_0) = x_0$.*

Proposition 1.1.2. *If f and $\partial f/\partial x_i$, $1 \le i \le n$, are continuous in the open set $U \subseteq \mathbb{R}^{n+1}$, then given any point $(t_0, x_0) \in U$ there exists a unique solution $x : I \to \mathbb{R}^n$ of (1.1) such that $t_0 \in I$ and $x(t_0) = x_0$.*

In this thesis we focus on autonomous systems, i.e. systems of the form

$$\dot{x} = f(x). \tag{1.2}$$

Each point x_0 for which $f(x_0) = 0$ is called a *singular point* of (1.2) (or *fixed point* or *equilibrium point* or *critical point*). Moreover, if x_0 is a singular point of (1.2) then the solution $x(t)$ of system (1.2) is constant for all t. A non singular point of (1.2) is called a *regular point* (or *ordinary point*). The qualitative behaviour of the solutions in this case is represented by a family of curves, endowed with the orientation of $x(t)$ for t increasing. These curves are called *orbits* or *trajectories* of system (1.2). The geometrical representation of the qualitative behaviour of the orbits of (1.2) is called its *phase portrait*.

The next theorem describe the qualitative behaviour of the solutions around a regular point. A proof can be found for instance in [43, 62].

Theorem 1.1.3 (Flow box theorem)**.** *In a sufficiently small neighbourhood of a regular point x_0 of system (1.2) there is a differentiable change of coordinates $y = y(x)$ which transforms (1.2) into the system $\dot{y} = \overline{f}(y)$, where $\overline{f}(y) = (0, 0, \ldots, 1)$.*

The Flow box theorem assures that the orbits near regular points behaves as the orbits of the constant system $\overline{f}(y) = (0, 0, \ldots, 1)$. For singular points the local behaviour of the orbits is not so simple to be determined.

As in this thesis we are interested to study planar systems of ordinary differential equations, from now on we consider system (1.2) where $f : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. systems of the form

$$\dot{x} = f_1(x, y), \qquad \dot{y} = f_2(x, y). \tag{1.3}$$

An arbitrary singular point (x_0, y_0) of system (1.3) can be always moved to the origin by the transformation $x_1 = x - x_0$, $y_1 = y - y_0$. Hence, when interested in the behaviour of trajectories near a singular point, without loss of generality we can assume that system (1.3) has the singular point at the origin.

Before explaining the behaviour of trajectories near a singular point at the origin of the general planar system (1.3), we shall discuss the local behaviour at the origin of planar linear systems, i.e. systems of the form

$$\dot{x} = Ax, \tag{1.4}$$

where A is a matrix 2x2. Let B be the Jordan canonical form of A such as $B = P^{-1}AP$ for some matrix P. The linear change of variables $x = Py$, transforms system (1.4) into $\dot{y} = By$. When A has nonzero eigenvalues the matrix B has one of the following forms

$$(i)\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad (ii)\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad (iii)\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

According to the signs of λ_1, λ_2, λ, α and β of the matrices above one can determine the behaviour of the trajectories of system (1.4) near the singular point $(0,0)$. It can occur one of the following four types:

- *node* - it can occur in case (i) when λ_1 and λ_2 are real having the same sign (see Figure 1.(a)) or in case (ii) when $\lambda \neq 0$ is real (see Figure 1.(b)). Moreover there are two different types:
  (1) *stable* if $\lambda_1 < 0$ and $\lambda_2 < 0$ in (i) or $\lambda < 0$ in (ii);
  (2) *unstable* if $\lambda_1 > 0$ and $\lambda_2 > 0$ in (i) or $\lambda > 0$ in (ii).
- *saddle* (see Figure 1.(c)) - it can occur in case (i) when λ_1 and λ_2 are real and have different signs.
- *focus* (see Figure 1.(d)) - it can occur in case (iii) when α and β are real non zero. In this case there are also two types:
  (1) *stable* if $\alpha < 0$;
  (2) *unstable* if $\alpha > 0$.
- *center* (see Figure 1.(e)) - it can occur in case (iii) when $\alpha = 0$ and $\beta \neq 0$, i.e. when the eigenvalues of B are pure imaginary.

A singular point that is known to be either a node, a focus, or a center is termed an *antisaddle*.

The matrix A is said to be *hyperbolic* if the real part of their eigenvalues is not zero. Thus, by the analysis above, if A is a hyperbolic matrix the singular point at the origin of system (1.4) can be a node, a saddle or a focus.

Now we return to the study of the local behaviour of the orbits at a singular point (x_0, y_0) of the general planar system (1.3). We say that the system $\dot{x} = Dfx$, where Df is the matrix

$$Df = \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(x_0,y_0) & \dfrac{\partial f_1}{\partial y}(x_0,y_0) \\ \dfrac{\partial f_2}{\partial x}(x_0,y_0) & \dfrac{\partial f_2}{\partial y}(x_0,y_0) \end{bmatrix},$$

is the *linearization* of system (1.3) at the point (x_0, y_0). A singular point (x_0, y_0) of (1.3) is called *nondegenerate* if the determinant of the matrix Df is nonzero. If the determinant of Df is equal to zero the singular point (x_0, y_0) is called *degenerate*.

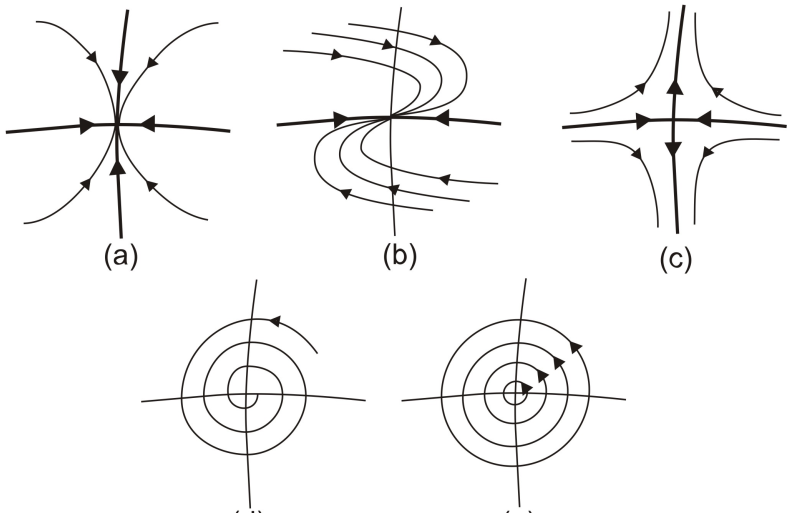

Figure 1 – Behaviour of the orbits of system (1.4) near singular points: (a) and (b) stable node, (c) saddle, (d) stable focus, (e) center.

A nondegenerate singular point (x_0, y_0) of (1.3) is said to be *hyperbolic* if the matrix Df is hyperbolic. The following theorem characterizes the behaviour of the orbits of a system of the form (1.3) in a neighbourhood of a hyperbolic singular point, a proof can be found in [59].

Theorem 1.1.4 (Hartman–Grobman Theorem)**.** *Let (x_0, y_0) be a hyperbolic singular point of (1.3). Then in a neighbourhood of (x_0, y_0) the orbits of (1.3) are topologically equivalent to the orbits of its linearization $\dot{x} = Dfx$.*

In the case of a nondegenerate singular point of (1.3) which is not hyperbolic, that is, the eigenvalues of Df are purely imaginary, it is well known that it can be either a center or a focus. However, it is still an open problem to find a complete characterization to distinguish them and it is the main propose of our studies (see Section 1.3).

For degenerate singular points the local structure of the orbits can be more complicated to be determined. They are split in three classes: semi-hyperbolic, nilpotent and linearly zero. Next we briefly present some details for each one of them. We do not present here the theorems which characterize the local behaviour of the orbits near such points, since it is not used in this thesis. They can be found on the given references.

- *semi-hyperbolic*: when the trace of Df is nonzero. The characterization of the local behaviour of the orbits for this case is done by Andronov, Leontovich, Gordon and Maier in [5].

- *nilpotent*: when the trace of Df is equal to zero, but Df is not identically zero. Andreev in [4] provide a complete characterization for the local behaviour of the orbits for such singular points.

- *linearly zero*: when Df is equal to zero matrix. The local behaviour of the orbits for these singular points can be determined using blow-up (see [13, 14, 74]). We also refer [43] and [2] for more details about blow-up.

In addition to singular and regular points another important element composing the phase portrait of a system of ODE is the so called limit cycle. By definition a *limit cycle* of system (1.3) is an isolated closed (periodic) orbit which corresponds to a periodic solution of (1.3). In Figure 2 we present different types of limit cycles.

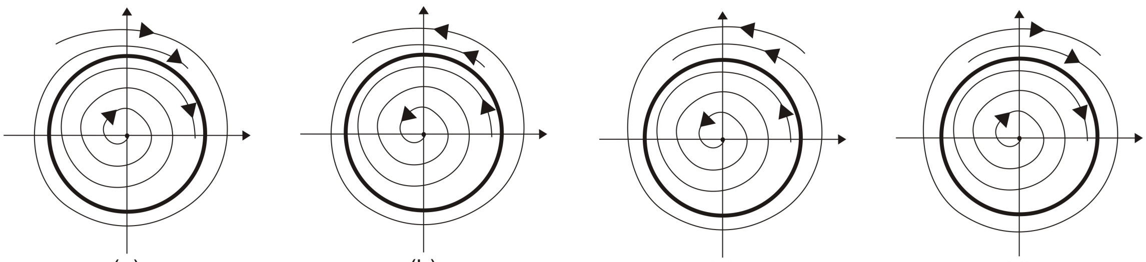

Figure 2 – (a) stable limit cycle, (b) unstable limit cycle, (c) and (d) semistable limit cycles.

Let us define the α- and ω-*limit set* of a point p in the phase space, denoted by $\alpha(p)$ and $\omega(p)$ respectively. A point q is α-*limit point* of p if there exists a sequence $\{t_n\}$, such that $t_n \to -\infty$ and $\lim_{n\to\infty} \varphi(t_n) = q$, where $\varphi(t)$ denotes the trajectory going through the point p. Similarly, a point q is ω-*limit point* of p, if there exists the sequence $\{t_n\}$, such that $t_n \to \infty$ and $\lim_{n\to\infty} \varphi(t_n) = q$, where $\varphi(t)$ denotes the trajectory going through the point p.

The next theorem, which holds only for planar systems, gives a characterization of the α- and ω- limit sets of a point p. A proof can be found in [43, 62, 110, 112].

Theorem 1.1.5 (Poincaré-Bendixson Theorem)**.** *Let $\varphi(t) = \varphi(t, p)$ the trajectory of (1.3) passing through the point p and defined for all $t \geq 0$ such that $\varphi(t) \subset K$ for all $t \geq 0$, where $K \subset \mathbb{R}^2$ is a compact set. Assume that (1.3) has only finitely many singular points in K. Then, one of the following cases holds:*

(i) If $\omega(p)$ contains only regular points then $\omega(p)$ is a period orbit;

(ii) If $\omega(p)$ does not contains regular points then $\omega(p)$ is a singular point;

(iii) If $\omega(p)$ contains both regular ad singular points then $\omega(p)$ is composed by a set of regular trajectories and singularities such that each regular trajectory goes to a singularity of $\omega(p)$ when $t \to \pm\infty$.

Remark 1.1.6. An analogue result is obtained from Theorem 1.1.5 for $\alpha(p)$ if we suppose that $\varphi(t) \subset K$ for all $t \leq 0$, where $K \subset \mathbb{R}^2$ is a compact set.

It follows from Theorem 1.1.5 that if a closed trajectory γ is a subset of $\alpha(p)$ or $\omega(p)$ for some p that does not lie in γ, then γ is a limit cycle.

The informations about the types of all singular points, the existence of limit cycles and trajectories connecting singular point are essential to determine the phase portrait of a differential system. However we do not obtain information about the behaviour of the orbits close to the infinity. We now discuss a method used to study the behaviour of the trajectories of polynomial differential systems at the infinity by compactifying the whole plane $\mathbb{R}^2$.

We note that there exist different approaches used to compactify the plane $\mathbb{R}^2$, for example using the stereographic projection of the sphere onto the plane. In which case the infinity of $\mathbb{R}^2$ is identified with a single point (see [10]). However, Poincaré [96] introduce a technique for studying the behaviour of trajectories at the infinity by using the so called Poincaré sphere. The advantage of such technique is that the singular points at the infinity are displayed along the equator of the sphere, where the study of such singular points become simpler than using the Bendixson's technique (but even so, some of them are still very complicated). Nevertheless, for our purpose, the Poincaré compactification is very useful.

In order to present the Poincaré compactification we have to introduce some notations. We refer to the reader [43] for more details.

Let system (1.3) be a polynomial system, that is, $f_1(x,y) = P(x,y)$ and $f_2(x,y) = Q(x,y)$, where P and Q are polynomials. We associate to such system the planar vector field

$$\mathscr{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We recall that the degree of $\mathscr{X}$ is $d = \max\{\deg(P), \deg(G)\}$.

We consider $\mathbb{R}^2$ as the plane $(x_1,x_2,1)$ in $\mathbb{R}^3$. Let $\mathbb{S}^2 = \{y = (y_1,y_2,y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ be the sphere in $\mathbb{R}^3$, which we shall call as *Poincaré sphere*, $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ be the equator of $\mathbb{S}^2$, $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$ be the northern hemisphere of $\mathbb{S}^2$ and $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$ be the southern hemisphere of $\mathbb{S}^2$.

We consider the projection of the vector field $\mathscr{X}$ from $\mathbb{R}^2$ to $\mathbb{S}^2$ given by the central projections

$$f^+ : \mathbb{R}^2 \to \mathbb{S}^2 \quad \text{and} \quad f^- : \mathbb{R}^2 \to \mathbb{S}^2,$$

where $f^+(x)$ (respectively, $f^-(x)$) is the intersection of the straight line passing through the point y and the origin with the northern (respectively, southern) hemisphere of $\mathbb{S}^2$,

$$f^+(x) = \left(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\right), \quad f^-(x) = \left(-\frac{x_1}{\Delta(x)}, -\frac{x_2}{\Delta(x)}, -\frac{1}{\Delta(x)}\right),$$

where $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$.

We obtain the induced vector fields $\overline{\mathscr{X}}(y) = Df^+(x)\mathscr{X}(x)$, where $y = f^+(x)$ on H_+ and, $\overline{\mathscr{X}}(y) = Df^-(x)\mathscr{X}(x)$, where $y = f^-(x)$ on H_-. We note that $\overline{\mathscr{X}}$ is a vector field on $\mathbb{S}^2 \setminus \mathbb{S}^1$

which is tangent to $\mathbb{S}^2$. The points at infinity of $\mathbb{R}^2$ (two for each direction) are in bijective correspondence to the points on $\mathbb{S}^1$.

The natural procedure now is to try to extend the induced vector field $\overline{\mathscr{X}}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$. Unfortunately, in general it does not stay bounded as we get close to $\mathbb{S}^1$, obstructing the extension. However, by multiplying the vector field by the factor $\rho(x) = y_3^{d-1}$, the extension becomes possible. The extended vector field on $\mathbb{S}^2$ is called *Poincaré compactification* of the vector field $\mathscr{X}$ on $\mathbb{R}^2$ and it is denoted by $p(\mathscr{X})$.

If we know the behaviour of $p(\mathscr{X})$ near $\mathbb{S}^1$, we know the behaviour of $\mathscr{X}$ in a neighbourhood of the infinity. Moreover, the Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathscr{X})$.

Since $\mathbb{S}^2$ is a differentiable manifold we consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1,2,3$ for computing the expression for $p(\mathscr{X})$. The diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1,2,3$ are the inverses of the central projections from the planes tangent at the points $(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1)$, and $(0,0,-1)$ respectively. We denote by (u,v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1,2,3$.

The expression for $p(\mathscr{X})$ in the local chart (U_1,F_1) is given by

$$\dot{u} = v^n \left[-u\tilde{P}\left(\frac{1}{v},\frac{u}{v}\right) + \tilde{Q}\left(\frac{1}{v},\frac{u}{v}\right)\right], \quad \dot{v} = -v^{n+1}\tilde{P}\left(\frac{1}{v},\frac{u}{v}\right),$$

for (U_2,F_2) is

$$\dot{u} = v^n \left[\tilde{P}\left(\frac{u}{v},\frac{1}{v}\right) - u\tilde{Q}\left(\frac{u}{v},\frac{1}{v}\right)\right], \quad \dot{v} = -v^{n+1}\tilde{Q}\left(\frac{u}{v},\frac{1}{v}\right),$$

and for (U_3,F_3) is

$$\dot{u} = \tilde{P}(u,v), \quad \dot{v} = \tilde{Q}(u,v).$$

The expressions for V_i's are the same as that for U_i's but multiplied by the factor $(-1)^{n-1}$. In these coordinates $v = 0$ always denotes the points of $\mathbb{S}^1$.

We say that a singular point of $\mathscr{X}$ or $p(\mathscr{X})$ is *finite* (respectively, *infinite*) if it is a singular point of $p(\mathscr{X})$ which lies in $\mathbb{S}^2 \setminus \mathbb{S}^1$ (respectively, $\mathbb{S}^1$).

Due to the fact that infinite singular points appear in pairs of diametrally opposite points, it is enough to study half of them, and using the degree of the vector field, we can determine the other half, that is, it suffices to study only the local charts (U_k,F_k), $k = 1,2,3$. singular points.

Finally, it is said that two polynomial vector fields $\mathscr{X}$ and $\mathscr{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathscr{X})$ into orbits of the flow induced by $p(\mathscr{Y})$, preserving or not the sense of all orbits.

1.2 Polynomial ideals and affine varieties

In this section we recall some basic concepts of commutative algebra and algebraic geometry and describe few main algorithms related to polynomial ideals, for details see for instance [33]. The results and notions present in this section are very important for this thesis since the main problem considered here can be, as explained in next chapters, reduced to the studies of polynomial ideals and their varieties, see [103] for more details.

A *polynomial* in variables $x_1, x_2, ..., x_n$ with coefficients in a field k is a formal expression of the form

$$f = \sum_{\alpha \in S} a_\alpha \mathbf{x}^\alpha \tag{1.5}$$

where:

- S is a finite subset of $\mathbb{N}_0^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
- $a_\alpha \in k$,
- for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\mathbf{x}^\alpha$ denotes the *monomial* $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

The product $a_\alpha \mathbf{x}^\alpha$ is called a *term* of the polynomial f. The set of all polynomials in the variables $x_1, x_2, ..., x_n$ with coefficients in k is denoted by $k[x_1, x_2, ..., x_n]$. With the natural operations of addition and multiplication, $k[x_1, x_2, ..., x_n]$ is a commutative ring. The degree of a monomial $\mathbf{x}^\alpha$ is the number $\mid \alpha \mid = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The degree of a polynomial f as in (1.5), denoted by $deg(f)$, is the maximum of $\mid \alpha \mid$ among all monomials (with nonzero coefficients a_α) of f.

Denoting by k a field and by n a natural number, the space

$$k^n = \{(a_1, a_2, ..., a_n) : a_1, a_2, ..., a_n \in k\}$$

is called *n-dimensional affine space*.

Definition 1.2.1. Let $f_1, ..., f_s$ be polynomials of $k[x_1, x_2, ..., x_n]$. The *affine variety* defined by the polynomials $f_1, ..., f_s$ is the set

$$\mathbf{V}(f_1, ..., f_s) = \{(a_1, a_2, ..., a_n) \in k_n : f_j(a_1, a_2, ..., a_n) = 0 \text{ for } 1 \leq j \leq s\}.$$

A *subvariety* of a variety V is a subset of V that is itself an affine variety.

It is obvious from the definition of $\mathbf{V}(f_1, ..., f_m)$ that the affine variety is the set of all solutions of the system

$$f_1 = 0, f_2 = 0, ..., f_m = 0.$$

So, there are many collections of polynomials defining the same variety. To understand better the concept of variety we have to present the notion of ideal.

Definition 1.2.2. A subset I of the polynomial ring $k[x_1,x_2,\ldots,x_n]$ is an *ideal* of $k[x_1,x_2,\ldots,x_n]$ if it satisfies

(i) $0 \in I$,

(ii) if $f,g \in I$, then $f+g \in I$,

(iii) if $f \in I$ and $h \in k[x_1,x_2,\ldots,x_n]$, then $fh \in I$.

Let $f_1,f_2,\ldots,f_s$ be polynomials of $k[x_1,x_2,\ldots,x_n]$. We denote

$$\langle f_1,f_2,\ldots,f_s\rangle = \left\{ \sum_{j=1}^{s} h_j f_j : h_1,\ldots,h_s \in k[x_1,x_2,\ldots,x_n] \right\}.$$

It is easy see that $\langle f_1,f_2,\ldots,f_s\rangle$ is an ideal in $k[x_1,x_2,\ldots,x_n]$. The polynomials $f_1,\ldots,f_s$, are called *generators* of this ideal. An ideal $I \subset k[x_1,x_2,\ldots,x_n]$ is said to be *finitely generated* if there exist polynomials $f_1,f_2,\ldots,f_m \in k[x_1,x_2,\ldots,x_n]$ such that $I = \langle f_1,f_2,\ldots,f_m\rangle$ and the set $\{f_1,f_2,\ldots,f_s\}$ is called a basis of I. For the proof of the next theorem see [33].

Theorem 1.2.3 (Hilbert Basis Theorem)**.** *Let k be a field and let I be an ideal in the polynomial ring $k[x_1,x_2,\ldots,x_n]$. Then the ideal I is finitely generated.*

It follows from Theorem 1.2.3 that each ascending chain of ideals in $k[x_1,x_2,\ldots,x_n]$, $I_1 \subset I_2 \subset \ldots$, stabilizes at some m. That is, there exists $m \geq 1$, such that for every $j > m$, $I_j = I_m$.

Theorem 1.2.4 ([33])**.** *Let $f_1,\ldots,f_m$ and $g_1,\ldots,g_s$ be two bases of the ideal $I \subset k[x_1,\ldots,x_n]$, i.e, $I = \langle f_1,\ldots,f_m\rangle = \langle g_1,\ldots,g_s\rangle$. Then $\mathbf{V}(f_1,\ldots,f_m) = \mathbf{V}(g_1,\ldots,g_s)$.*

It follows from Theorem 1.2.4, we see that two different finite collections of polynomials can generate the same ideal. Conversely, given a variety V there is a natural association between V and an ideal called *ideal of the variety* V. This ideal is defined by

$$\mathbf{I}(V) = \{f \in k[x_1,\ldots,x_n] : f(a_1,\ldots,a_n) = 0 \text{ for all } (a_1,\ldots,a_n) \in V\}.$$

Let $f_1,\ldots,f_m \in k[x_1,\ldots,x_n]$ and $I = \langle f_1,\ldots,f_m\rangle$. Then, $\langle f_1,\ldots,f_m\rangle \subset \mathbf{I}(\mathbf{V}(f_1,\ldots,f_m))$. Moreover, given any affine varieties $V,W \subset k^n$, it holds that

(i) $V \subset W$ if and only if $\mathbf{I}(W) \subset \mathbf{I}(V)$,

(ii) $V = W$ if and only if $\mathbf{I}(W) = \mathbf{I}(V)$.

One of the main objects of computational algebra is the so called Gröbner basis. Introduced by Bruno Buchberger (see [12]) in the second half of the last century.

Definition 1.2.5. Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ be elements of $\mathbb{N}_0^n$. We define the following orders of elements in $\mathbb{N}_0^n$

(a) LEXICOGRAPHIC ORDER. $\alpha >_{lex} \beta$ if and only if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the leftmost nonzero entry is positive.

(b) DEGREE LEXICOGRAPHIC ORDER. $\alpha >_{deglex} \beta$ if and only if

$$| \alpha | = \sum_{j=1}^{n} \alpha_j > | \beta | = \sum_{j=1}^{n} \beta_j, \quad \text{or} \quad | \alpha | = | \beta | \text{ and } \alpha >_{lex} \beta.$$

(c) REVERSE LEXICOGRAPHIC ORDER. $\alpha >_{rev} \beta$ if and only if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the rightmost nonzero entry is negative.

(d) DEGREE REVERSE LEXICOGRAPHIC ORDER. $\alpha >_{degrev} \beta$ if and only if

$$| \alpha | = \sum_{j=1}^{n} \alpha_j > | \beta | = \sum_{j=1}^{n} \beta_j, \quad \text{or} \quad | \alpha | = | \beta | \text{ and } \alpha >_{rev} \beta.$$

For example, if $\alpha = (3,2,5,2)$ and $\beta = (1,6,0,5)$ then α is greater than β with respect to all four orders.

From such n-tuples orders arise the monomials orders in the natural way: $a_\alpha \mathbf{x}^\alpha > a_\beta \mathbf{x}^\beta$ if $\alpha > \beta$, where the order of variables is $x_1 > ... > x_n$. For example, for x^3y, $xy^4 \in k[x,y]$ and choosing $x > y$, we have

- $x^3y >_{lex} xy^4$ (since $(3,1) >_{lex} (1,4)$),
- $x^3y <_{deglex} xy^4$ (since $3+1 < 1+4$ yields $(3,1) <_{deglex} (1,4)$),
- $x^3y >_{rev} xy^4$ (since $(3,1) >_{rev} (1,4)$),
- $x^3y <_{degrev} xy^4$ (since $(3,1) <_{degrev} (1,4)$).

With a fixed variables order $(>)$ on $k[x_1,...,x_n]$ and a specific monomial order $\alpha_1 > ... > \alpha_s$, each polynomial $f \in k[x_1,...,x_n]$ can be written in the standard form, with respect to such order,

$$f = a_1\mathbf{x}^{\alpha_1} + a_2\mathbf{x}^{\alpha_2} + \cdots + a_s\mathbf{x}^{\alpha_s}, \tag{1.6}$$

where $a_j \neq 0$ for $j = 1,...,s$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and $1 \leq i,j \leq s$.

The term $a_1\mathbf{x}^{\alpha_1}$ of (1.6) is called the *leading term* and it is denoted by $LT(f)$. The monomial $\mathbf{x}^{\alpha_1}$ of (1.6) is called *leading monomial* and it is denoted by $LM(f)$. The coefficient a_1 of (1.6) is called *leading coefficient* and it is denoted by $LC(f)$.

Given two monomials $\mathbf{x}^\alpha$, $\mathbf{x}^\beta \in k[x_1,...,x_n]$ the *least common multiple* of $\mathbf{x}^\alpha$ and $\mathbf{x}^\beta$ is the monomial $\mathbf{x}^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ denoted by $LCM(\mathbf{x}^\alpha, \mathbf{x}^\beta)$ such that $\lambda_j = max(\alpha_j, \beta_j)$, $1 \leq j \leq n$.

The monomial $\mathbf{x}^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is *divisible* by the monomial $\mathbf{x}^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n}$, that is $\mathbf{x}^{\alpha} \mid \mathbf{x}^{\beta}$, if and only $\alpha_i \geq \beta_i$ for all $1 \leq i \leq n$.

The division of n-variable polynomial is a generalization of division of single-variable polynomial. In order to present the Multivariable Division Algorithm it is necessary to introduce some notions.

Definition 1.2.6. (a) Let $f, g, h \in k[x_1, ..., x_n]$, where $g \neq 0$. We say that the polynomial f reduces to polynomial h modulo g in one step,

$$f \xrightarrow{g} h,$$

if and only if $LM(g)$ divides a nonzero term X of f and $h = f - \frac{X}{LT(g)} g$.

(b) Let $f, f_1, ..., f_m, h \in k[x_1, ..., x_n]$, where $f_i \neq 0$ for $1 \leq i \leq n$, and let F denote the set $\{f_1, ..., f_m\}$. We say that the polynomial f reduces to polynomial h modulo F,

$$f \xrightarrow{F} h,$$

if and only if there exists a sequence of indices $j_1, ..., j_s \in \{1, ..., m\}$ and a sequence of polynomials $h_1, ..., h_s$ such that

$$f \xrightarrow{f_{j_1}} h_1 \xrightarrow{f_{j_2}} h_2 \xrightarrow{f_{j_3}} \cdots \xrightarrow{f_{j_{s-1}}} h_s \xrightarrow{f_{j_s}} h.$$

Definition 1.2.7. Let $f, f_1, ..., f_m, h \in k[x_1, ..., x_n]$, where $f_i \neq 0$ for $1 \leq i \leq n$, and let F denote the set $\{f_1, ..., f_m\}$.

(a) A polynomial $r \in k[x_1, ..., x_n]$ is reduced with respect to F if either $r = 0$ or no monomial appearing in the polynomial r is divisible by any element of the set $\{LM(f_1), ..., LM(f_m)\}$.

(b) A polynomial $r \in k[x_1, ..., x_n]$ is a remainder for f with respect to F if $f \xrightarrow{F} r$ and r is reduced with respect to F.

Next we present the Multivariable Division Algorithm [1], which is analogue to the division algorithm for polynomials of one variable.

Multivariable Division Algorithm

INPUT: $f, f_1, ..., f_m, h \in k[x_1, ..., x_n]$ and let $F = \{f_1, ..., f_m\}$.

OUTPUT: The polynomials $u_1, ..., u_m, r \in k[x_1, ..., x_n]$ such that $f = u_1 f_1 + \cdots + u_m f_m + r$, where r is reduced with respect to F and $max(LM(f_1), ..., LM(f_m), LM(r)) = LM(f)$.

Procedure

STEP 1: $u_i := 0$ for all $1 \leq i \leq m$, $r := 0$ and $h := f$.

STEP 2: WHILE $h \neq 0$ DO

IF there exist f_j such that $LM(f_j)$ divides $LM(h)$
THEN for the least f_j such that $LM(f_j)$ divides $LM(h)$

$$u_j := u_j - \frac{LT(h)}{LT(f_j)},$$
$$h := h - \frac{LT(h)}{LT(f_j)} f_j,$$

ELSE

$$r := r + LT(h),$$
$$h := h - LT(h).$$

A difference between the multivariable division and the single-variable algorithms is that the quotient and remainder are not unique. They depend on the order of divisors in the set F as well as on the term order chosen for the polynomial ring. However there always exists a set of polynomials with a special property that defines such ideal. This property is the following: after the reduction of the polynomial $f \in I$ by this set, the obtained remainder is unique. This set is the so-called *Gröbner basis*.

Definition 1.2.8. A *Gröbner basis* of an ideal $I \in k[x_1,...,x_n]$ is a finite nonempty subset $G = \{g_1,...,g_m\}$ of $I \setminus \{0\}$ with the following property: for every nonzero $f \in I$, there exists $g_j \in G$ such that $LT(g_j) \mid LT(f)$.

The following theorem provides four equivalent statements on Gröbner basis. The second one is very important. It answer the question: given a polynomial f and an ideal I in $k[x_1,...,x_n]$, how to determine if $f \in I$? This problem is called the Ideal Membership Problem.

Theorem 1.2.9. *Let $I \in k[x_1,...,x_n]$ be a nonzero ideal, $f \in k[x_1,...,x_n]$ an arbitrary polynomial and $G = \{g_1,...,g_m\}$ a finite set of nonzero elements of I. Then the following statements are equivalent:*

(1) G is Gröbner basis for the ideal I;

(2) $f \in I \Leftrightarrow f \xrightarrow{G} 0$;

(3) $f \in I \Leftrightarrow f = \sum_{j=1}^{s} u_j g_j$ and $LM(f) = \max_{1 \leq j \leq s}(LM(u_j)LM(g_j))$;

(4) $\langle LT(G) \rangle = \langle LT(I) \rangle$.

Now the problem is how to compute a Gröeber basis of an ideal. The method used to solve this problem is the so-called *Buchberger Algorithm* which is based on the so called *S-polynomials* and the *Buchberger's Criterion*.

Definition 1.2.10. Let f and g be nonzero elements of $k[x_1,...,x_n]$ with $LM(f) = \mathbf{x}^\alpha$ and $LM(g) = \mathbf{x}^\beta$. The *S-polynomial* of f and g is the polynomial

$$S(f,g) = \frac{LCM(LM(f),LM(g))}{LT(f)} f - \frac{LCM(LM(f),LM(g))}{LT(g)} g.$$

The Buchberger's Criterion [12] says that $G = \{g_1,...,g_m\} \subset I$ is a Gröbner basis of a nonzero ideal I with respect to a order $<$ if and only if $S(g_1,g_j) \xrightarrow{G} 0$ for all $i \neq j$.

The Buchberger Algorithm to compute a Gröbner basis G of an ideal $I = \langle f_1,...,f_s\rangle \subset k[x_1,...,x_n]$ is the following.

Buchberger's Algorithm

INPUT: $G := \{f_1,...,f_s\}$.

OUTPUT: Gröbner basis of $I = \langle f_1,...,f_s\rangle$.

Procedure

STEP 1: Compute for each pair $g_i, g_j \in G$ the S-polynomial $S(g_i,g_j)$ and reduce it by the set G, $S(g_i,g_j) \xrightarrow{G} r_{ij}$.

STEP 2: If all r_{ij} are equal to zero, then G is Gröbner basis, else add r_{ij} to G and go to STEP 1.

Besides the Buchberger's Algorithm provides a Gröebner basis of an ideal, this base is not unique. It happens because the division algorithm can produce different remainders for different orderings of polynomials in the set of divisors.

Definition 1.2.11. A Gröbner basis $G = \{g_1,...,g_m\}$ is called reduced if for all i, $1 \leq i \leq m$, $LC(g_i) = 1$ and no term of g_i is divisible by any $LT(g_j)$ when $j \neq i$.

If a term order is fixed then every ideal has a unique reduced Gröbner basis with respect to this order. Nowadays all major computer algebra systems such as MATHEMATICA, MAPLE, SINGULAR and many others have routines to compute Gröbner bases.

Now we introduce some basic properties of polynomial ideals and its connection with affine varieties. Moreover, we present the concept of decomposition of an affine variety and the main results and algorithms related to that.

Recall that to solve a system of linear equations, an effective method is to reduce the system to an equivalent one in which an initial string of variables are missing from some of the equations. The next definition and theorem provide a way to eliminate a group of variables from a system of nonlinear polynomials. Moreover, it provides a way to find all solutions of a polynomial system in the case that the solution set is finite, or in other words, to find the variety of a polynomial ideal in the case that the variety is zero-dimensional.

Definition 1.2.12. Given an ideal $I = \langle f_1, ..., f_m \rangle \subset k[x_1, ..., x_n]$, where $x_1 > x_2 > \cdots > x_n$ and a fixed $\ell \in \{0, 1, ..., n-1\}$, the *$\ell$th elimination ideal* of I is the ideal

$$I_\ell = I \cap k[x_{\ell+1}, ..., x_n].$$

Any point $(a_{\ell+1}, ..., a_n) \in V(I_\ell)$ is called a *partial solution* of the system $f_1 = 0, ..., f_m = 0$.

Theorem 1.2.13 (Elimination Theorem [33])**.** *Let I be an ideal of $k[x_1, ..., x_n]$ with the fixed lexicographic term order, $x_1 > \cdots > x_n$, and let G be Gröbner basis of I. Then for every ℓ, $0 \leq \ell \leq n-1$, the set*

$$G_\ell := G \cap k[x_{\ell+1}, ..., x_n]$$

is a Gröbner basis for the ℓth elimination ideal I_ℓ.

From the geometric point of view, the elimination of the variables from the system is in fact the projection of the variety $\mathbf{V}(I) \subset k^n$ onto a lower dimensional subspace $k^{n-\ell}$.

Next we present some properties and results regarding affine varieties which are essential to construct the algorithms and computational procedures used in this thesis.

Theorem 1.2.14 (Weak Hilbert Nullstelensatz [33])**.** *If I is an ideal in $\mathbb{C}[x_1, ..., x_n]$ such that $\mathbf{V}(I) = \emptyset$, then $I = C[x_1, ..., x_n]$.*

The Weak Hilbert Nullstellensatz provides a way for checking whether a given system of polynomial equations $f_1 = 0, ..., f_m = 0$, has a solution over $\mathbb{C}$. It follows from the theorem that the system has a solution if and only if the reduced Gröbner basis of ideal $I = \langle f_1, ..., f_m \rangle$ with respect to any chosen term ordering is different from $\langle 1 \rangle$.

Definition 1.2.15. Given an ideal $I \in k[x_1, ..., x_n]$ the *radical* of I, denoted by $\sqrt{I}$, is the set

$$\sqrt{I} = \{f \in k[x_1, ..., x_n] : \text{ there exists } p \in \mathbb{N} \text{ such that } f^p \in I\}.$$

If for a given ideal I, $\sqrt{I} = I$, then I is called *radical ideal*.

If I is an ideal, then $\sqrt{I}$ is also an ideal and determines the same affine variety as I, that is $\mathbf{V}(\sqrt{I}) = \mathbf{V}(I)$.

The next theorem states that if a polynomial f vanishes in all points of a variety $\mathbf{V}(I) \subset \mathbb{C}^n$, then some power of f must belongs to I, that is, f belongs to the radical of I.

Theorem 1.2.16 (Strong Hilbert Nullstellensatz [33])**.** *Let $f, f_1, ..., f_s \in \mathbb{C}[x_1, ..., x_n]$. Then $f \in \boldsymbol{I}(\boldsymbol{V}(f_1, ..., f_s))$ if and only if there exists an integer $m \geq 1$ such that $f^m \in \langle f_1, ..., f_s \rangle$. In other words, for any ideal $I \subset \mathbb{C}[x_1, ..., x_n]$,*

$$\sqrt{I} = \boldsymbol{I}(\boldsymbol{V}(I)).$$

The following result characterizes whether two ideals provides the same affine variety.

Proposition 1.2.17 ([33]). *Let I and J be ideals in $\mathbb{C}[x_1,...,x_n]$. The variety of I is the same as the variety of J, $\mathbf{V}(I) = \mathbf{V}(J)$, if and only if $\sqrt{I} = \sqrt{J}$.*

The computation of a radical of an ideal can be very difficult but it is relatively easy to check whether some polynomial belongs to the radical of a given ideal. The following theorem provides a method for checking whether a polynomial is an element of the radical of a given ideal.

Theorem 1.2.18 ([103]). *Given an arbitrary field k and an ideal $I = \langle f_1,...,f_m\rangle \in k[x_1,...,x_n]$, the polynomial $f \in k[x_1,...,x_n]$ is an element of $\sqrt{I}$ if and only if $1 \in J := \langle f_1,...,f_m, 1-wf\rangle \subset k[x_1,...,x_n,w]$.*

Theorem 1.2.18 provides the so-called *Radical membership test*. The test says that for a polynomial f and an ideal $I = \langle f_1,...,f_m\rangle \in k[x_1,...,x_n]$, $f \in \sqrt{I}$ if and only if the reduced Gröbner basis of the ideal $\langle 1-wf, f_1,...,f_m\rangle$ is equal to $\{1\}$. Thus, to check whether the varieties of two ideals $I = \langle f_1,...,f_m\rangle$ and $J = \langle g_1,...,g_s\rangle$ are the same in $\mathbb{C}[x_1,...,x_n]$, $\mathbf{V}(I) = \mathbf{V}(J)$, it is necessary to check that all polynomials $f_j, 1 \leq j \leq m$, are in $\sqrt{J}$, and vice versa.

Consider now the ideals $I, J \in k[x_1,...,x_n]$. Define the operations

(1) the *intersection* of I and J, $I \cap J$:

$$I \cap J := \{f \in k[x_1,...,x_n] : f \in I \text{ and } f \in J\},$$

(2) the *sum* of I and J, $I+J$:

$$I+J := \{f+g \in k[x_1,...,x_n] : f \in I \text{ and } g \in J\},$$

(3) the *quotient* $I : J$:

$$I : J := \{f \in k[x_1,...,x_n] : fg \in I \text{ for all } g \in J\}.$$

It is easy to see that three sets defined above are ideals. Moreover, in [33] it is proven that the intersection of two given ideals $I = \langle f_1,...,f_m\rangle$ and $J = \langle g_1,...,g_s\rangle$, is equal to the ideal

$$\langle tf_1,...,tf_m, (1-t)g_1,...,(1-t)g_s\rangle \cap k[x_1,...,x_n].$$

Following we present algorithms to compute Gröbner basis of ideal $I \cap J$ and a basis of ideal $I : J$. The proof can be found in [103].

Algorithm for computing $I \cap J$

INPUT: Ideals $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ in $k[x_1, \ldots, x_n]$.

OUTPUT: Gröbner basis of $I \cap J$.

Procedure

STEP 1: Compute Gröbner basis G' of ideal $\langle t f_1, \ldots, t f_m, (1-t)g_1, \ldots, (1-t)g_s \rangle$ in $k[t, x_1, \ldots, x_n]$ with respect to lexicographic order and with $t > x_1 > \cdots > x_n$.

STEP 2: Compute $G = G' \cap k[x_1, \ldots, x_n]$.

Algorithm for computing $I : J$

INPUT: Ideals $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ in $k[x_1, \ldots, x_n]$.

OUTPUT: A basis $\{f_{s_1}, \ldots, f_{s_{p_s}}\}$ of $I : J$.

Procedure

STEP 1: FOR $j = 1, \ldots, s$: compute $I \cap \langle g_j \rangle = \langle h_{j_1}, \ldots, h_{j_{m_j}} \rangle$.

STEP 2: FOR $j = 1, \ldots, s$: compute $I : \langle g_j \rangle = \langle h_{j_1}/g_j, \ldots, h_{j_{m_j}}/g_j \rangle$.

STEP 3: $K := I : \langle g_1 \rangle$.

STEP 4: FOR $j = 2, \ldots, s$: compute $K := K \cap (I : \langle g_j \rangle) = \langle f_{j_1}, \ldots, f_{j_{p_j}} \rangle$.

Following is presented the definition of the Zariski closure. This concept is necessary in the next theorem which connect the tree operations on ideals defined above with affine varieties.

Definition 1.2.19. The *Zariski closure* of a set $S \subset k^n$, denoted by $\overline{S}$, is the smallest variety containing S.

Theorem 1.2.20 ([33, 103])**.** *Given ideals I and J in $k[x_1, \ldots, x_n]$ it holds*

(1) $\boldsymbol{V}(I \cap J) = \boldsymbol{V}(I) \cup \boldsymbol{V}(J)$,

(2) $\boldsymbol{V}(I + J) = \boldsymbol{V}(I) \cap \boldsymbol{V}(J)$,

(3) $\overline{\boldsymbol{V}(I) \setminus \boldsymbol{V}(J)} \subset \boldsymbol{V}(I : J)$ *and,*

(4) if $k = \mathbb{C}$ *and* I *is radical ideal, then* $\overline{\boldsymbol{V}(I) \setminus \boldsymbol{V}(J)} = \boldsymbol{V}(I : J)$.

An affine variety V can have a very complicated structure. So, it is profitable to write V in a less complicated way.

Definition 1.2.21. A nonempty affine variety $V \subset k^n$ is said to be *irreducible* if $V = V_1 \cup V_2$, for affine varieties V_1 and V_2, only if either $V_1 = V$ or $V_2 = V$.

Irreducible affine varieties are close related to a class of ideals called prime ideals.

Definition 1.2.22. Let I be a proper ideal in $k[x_1,\dots,x_n]$. We say that I is a *prime* ideal if $fg \in I$ implies that either $f \in I$ or $g \in I$. And I is called a *primary* ideal if $fg \in I$ implies that either $f \in I$ or $g^p \in I$ for some $p \in \mathbb{N}$.

Prime ideals and primary ideals have the following properties.

Proposition 1.2.23. *a) Every prime ideal is a radical ideal;*

b) Suppose $P_1,\dots,P_s$ are prime ideals in $k[x_1,\dots,x_n]$. Then the ideal $I = \cap_{j=1}^{s} P_j$ is a radical ideal.

c) I is a primary ideal if and only if $\sqrt{I}$ is prime. In this case $\sqrt{I}$ is called associated prime ideal of I.

A *primary decomposition* of an ideal $I \subset k[x_1,\dots,x_n]$ is a representation of I as a finite intersection of primary ideals Q_j, $I = \cap_{j=1}^{s} Q_j$. The decomposition is called a *minimal primary decomposition* if the associated prime ideals $\sqrt{Q_j}$ are all distinct and $\cap_{i\neq j} Q_j \not\subseteq Q_j$ for any j. A minimal primary decomposition of a polynomial ideal always exists, but it is not necessarily unique (see e.g [33]).

Theorem 1.2.24 ([33])**.** *A nonempty affine variety $V \subset k^n$ is irreducible if and only if $\boldsymbol{I}(V)$ is a prime ideal.*

Each affine variety $V \subset k^n$ can be written as a union of a finite number of irreducible varieties, $V = V_1 \cup \cdots \cup V_n$. This decomposition is called a *minimal decomposition* if $V_i \not\subseteq V_j$ for $i \neq j$.

Moreover, every variety $V \subset k^n$ has a minimal decomposition $V = V_1 \cup \cdots \cup V_n$, and this decomposition is unique up to the order of the V_i's.

The minimal decomposition of an affine variety can be computed using different algorithms implemented in some computer algebra systems, as SINGULAR, MAPLE etc..

In the following example using the routine `minAssGTZ` [36] (which is based on the algorithm of [50]) of SINGULAR [35] we find the minimal decomposition of the variety of an ideal I.

Example 1.2.25. The code for carrying out the decomposition of the variety of the ideal $I = \langle yz + z^3, xz - z, xy + z \rangle$, using the computer algebra system SINGULAR is as follows.

```
LIB"primdec.lib";
ring r=0,(x,y,z),dp;
poly f1=y*z+z^2;
poly f2=x*z-z;
poly f3=x*y+z;

ideal i=f1,f2,f3;
minAssGTZ(i);
```

and the output is:

```
[1]:
_[1]=x*y+z
_[2]=y
[2]:
_[1]=x*y+z
_[2]=x
[3]:
_[1]=x*y+z
_[2]=x-1
```

Hence, the minimal decomposition of the variety $\mathbf{V}(I)$ is

$$\mathbf{V}(I) = \mathbf{V}(\langle xy+z, y\rangle) \cup \mathbf{V}(\langle xy+z, x\rangle) \cup \mathbf{V}(\langle xy+z, x-1\rangle).$$

Unfortunately, very often the computation of the decomposition of a variety is very difficult and it consumes a lot of memory and time. In order to simplify the procedure to find a irreducible decomposition of a variety an approach using modular computations, that is computations over a field with characteristic p, $\mathbb{Z}_p = \mathbb{Z}/\langle p\rangle$, where p is a prime number can be used. The modular calculations still keep essential information on the original system. So, it is often possible to extract this information to reconstruct the exact solution of the polynomial system over the field of rational numbers. It can be done using the following rational reconstruction algorithm of [114].

Rational Reconstruction Algorithm

INPUT: The prime number $p \in \mathbb{Z}$ and $c \in \mathbb{Z}_p$.
OUTPUT: The integers v_2 and v_3 such that $v_3/v_2 \equiv c(mod p)$, hence such that $v_3 = v_2c + kp$, for some $k \in \mathbb{Z}$.

Procedure

STEP 1: Define $u = (u_1, u_2, u_3) := (1, 0, p)$, $v = (v_1, v_2, v_3) := (1, 0, c)$.
STEP 2: WHILE $\sqrt{\frac{p}{2}} \leq v_3$, DO $\{q := \lfloor u_3/v_3 \rfloor, r := u - qv, u := v, v := r\}$.
STEP 3: IF $| v_2 | \geq \sqrt{\frac{p}{2}}$, THEN error().
STEP 4: Return v_3, v_2.
The function $\lfloor . \rfloor$ stands for the floor function.

The Rational Reconstruction Algorithm is used in the following Decomposition Algorithm with Modular Arithmetics, suggested in [101], to compute the irreducible decomposition of a variety.

Modular Arithmetics Decomposition Algorithm

INPUT: The ideal $I := \langle f_1, \ldots, f_m \rangle$.
OUTPUT: The decomposition of $I := \langle f_1, \ldots, f_m \rangle$ over the field of rational numbers.

Procedure

STEP 1: Compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ in $\mathbb{Z}_p$, where p is a chosen prime number.

STEP 2: Reconstruct the ideals $\tilde{Q}_1, \ldots, \tilde{Q}_s$ to ideals Q_i, $i = 1, \ldots, s$, in $\mathbb{Q}$ using the rational reconstruction algorithm.

STEP 3: For each i, using the radical membership test, check whether the polynomials $f_1, \ldots, f_m$ are in the radicals of the ideals Q_i. That is, whether the reduced Gröbner basis of the ideal $\langle 1 - wf_j, Q_i \rangle$, for $1 \leq j \leq m$, is equal to $\{1\}$.
If YES, then go to step 4, otherwise take another prime p and go to STEP 1.

STEP 4: Compute the intersection of Q_i over the rational numbers $Q = \cap_{i=1}^{s} Q_i \subset \mathbb{Q}[x_1, \ldots, x_n]$.

STEP 5: Check that $\sqrt{Q} = \sqrt{I}$, that is:
(a) for any $q_i \in Q$, the reduced Gröbner basis of the ideal $\langle 1 - wq_i, I \rangle$ is equal to $\{1\}$,
(b) for any $f_j \in I$, the reduced Gröbner basis of the ideal $\langle 1 - wf_j, Q \rangle$ is equal to $\{1\}$.
If it is the case, then $\mathbf{V}(I) = \cup_{i=1}^{s} \mathbf{V}(Q_i)$. If NOT, then go to STEP 1 and choose another prime p.

1.3 The center and integrability problems

Let us consider a planar system of ordinary differential equations $\dot{u} = f(u)$, where $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is analytic on a neighbourhood U of 0, where $f(0) = 0$ and the eigenvalues of the linear part of f at 0 are $\alpha \pm ib$ with $b \neq 0$. In Section 1.1 we have seen that if the system is linear then the trajectory of every point spirals towards or away from 0 when $\alpha \neq 0$, i.e, 0 is a focus, and the trajectory of every point except 0 is a closed curve when $\alpha = 0$, i.e, 0 is a center. For nonlinear systems Theorem 1.1.4 says that, in the first case, $\alpha \neq 0$, the behaviour of the trajectories in a neighbourhood of the origin follow the behaviour of the linear system determined by the linear part of f at 0, i.e, they spiral towards or away from the origin. However, the second case, $\alpha = 0$, the linear approximation does not necessarily determine the geometric behaviour of the trajectories of the nonlinear system in a neighbourhood of the origin. For example, the origin of the system

$$\begin{aligned} \dot{u} &= -v - u(u^2 + v^2), \\ \dot{v} &= u - v(u^2 + v^2), \end{aligned} \tag{1.7}$$

is a center for the corresponding linear system. However, in polar coordinates system (1.7) is $\dot{r} = -r^3$, $\dot{\varphi} = 1$, which means that every trajectory of (1.7) spirals towards the origin. So it is a stable focus. But there exist examples where the addition of higher-order terms does not destroy the center.

The problem of distinguishing a singular point between a center and a focus is well known as *the Poincaré center problem* or *the center-focus problem* or only *the center problem*.

Although the center problem have been studied during more than hundred years by many authors it is unresolved even for planar systems with cubic nonlinearities, i.e. systems of the form

$$\dot{u} = -v + f_1(u, v), \qquad \dot{v} = u + f_2(u, v), \tag{1.8}$$

where f_1 and f_2 are polynomials of degree three. For polynomial quadratic systems, i.e. systems of the form (1.8) where f_1 and f_2 are polynomials of two, it is solved in [42, 66, 67, 109], where the authors presented necessary and sufficient conditions for the existence of a center. For cubic systems only particular families were investigated, see e.g. [3, 24, 41, 78, 86, 103, 106, 107, 111] and references therein. There are also some works on the center problem for families of polynomial systems of higher degrees, see e.g. [17, 18, 45, 46, 54, 56] and references given there.

Next we describe a general approach for studying the center problem using polar coordinates.

It is well known that by a nonsingular linear change of coordinates any real analytic system $\dot{u} = f(u)$ defined on a neighbourhood of 0 in $\mathbb{R}^2$ with $f(0) = 0$ such that the eigenvalues of the linear part of f at 0 are $\alpha \pm ib$ with $b \neq 0$ can be written in the form

$$\begin{aligned} \dot{u} &= \alpha u - \beta v + F(u, v), \\ \dot{v} &= \beta u + \alpha v + G(u, v), \end{aligned} \tag{1.9}$$

where $F(u,v) = \sum_{k=2}^{\infty} F^{(k)}(u,v)$, $G(u,v) = \sum_{k=2}^{\infty} G^{(k)}(u,v)$, and $F^{(k)}(u,v)$ and $G^{(k)}(u,v)$ (if nonzero) are homogeneous polynomials of degree k.

Introducing polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, we obtain

$$\begin{aligned}
\dot r &= \alpha r + F(r\cos\varphi, r\sin\varphi)\cos\varphi + G(r\cos\varphi, r\sin\varphi)\sin\varphi,\\
\dot\varphi &= \beta - r^{-1}\left[F(r\cos\varphi, r\sin\varphi)\sin\varphi - G(r\cos\varphi, r\sin\varphi)\cos\varphi\right].
\end{aligned} \tag{1.10}$$

Dividing the first equation of (1.10) by the second one, we obtain

$$\frac{dr}{d\varphi} = \frac{\alpha r + r^2 F(r\cos\varphi, r\sin\varphi)}{\beta + rG(r\cos\varphi, r\sin\varphi)} = R(r,\varphi). \tag{1.11}$$

The function $R(r,\varphi)$ is a 2π-periodic function of φ and it is analytic for all φ and for $\mid r \mid < r^*$, for some sufficiently small r^*. The fact that the origin is a singularity for (1.9) corresponds to the fact that $R(0,\varphi) \equiv 0$, so that $r = 0$ is a solution of (1.11). Expanding $R(r,\varphi)$ in power series in r we obtain

$$\frac{dr}{d\varphi} = R(r,\varphi) = rR_1(\varphi) + r^2 R_2(\varphi) + \cdots = \frac{\alpha}{\beta} r + \cdots, \tag{1.12}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ. The series is convergent for all φ and for all sufficiently small r.

Denote by $r = f(\varphi, \varphi_0, r_0)$ the solution of (1.12) with initial conditions $r = r_0$ and $\varphi = \varphi_0$. The function $f(\varphi, \varphi_0, r_0)$ is an analytic function of all three variables φ, φ_0, r_0 and it has the property that

$$f(\varphi, \varphi_0, r_0) \equiv 0, \tag{1.13}$$

since $r = 0$ is a solution of (1.12). From equation (1.13) using continuous dependence of solutions on parameters we conclude that every trajectory of system (1.10) in a sufficiently small neighbourhood of the origin crosses every ray $\varphi = c$, $0 \le c < 2\pi$. This implies that in order to investigate the trajectories in a sufficiently small neighbourhood of the origin it is sufficient to consider all trajectories passing through a segment $\Sigma = \{(u,v) : v = 0, 0 \le u \le r^*\}$ for r^* sufficiently small, that is, all solutions $r = f(\varphi, 0, r_0)$. The function $r = f(\varphi, 0, r_0)$ can be expanded in a power series in r_0,

$$r = f(\varphi, 0, r_0) = w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots,$$

which is convergent for all $0 \le \varphi \le 2\pi$ and for $\mid r_0 \mid < r^*$. This function is a solution of (1.12), hence

$$\begin{aligned}
&w_1' r_0 + w_2' r_0^2 + \cdots\\
&\equiv R_1(\varphi)\left(w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots\right) + R_2(\varphi)\left(w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots\right) + \cdots,
\end{aligned}$$

where the derivatives are taken with respect to the variable φ. Equating the coefficients of like powers of r_0 in this identity, we obtain the recurrence differential equations

$$\begin{aligned}
w_1' &= R_1(\varphi) w_1,\\
w_2' &= R_1(\varphi) w_2 + R_2(\varphi) w_1^2,\\
w_3' &= R_1(\varphi) w_3 + 2R_2(\varphi) w_1 w_2 + R_3(\varphi) w_1^3,\\
&\vdots
\end{aligned} \tag{1.14}$$

Taking into account the initial condition $r = f(0,0,r_0) = r_0$ we obtain

$$w_1(0) = 1, \quad w_j(0) = 0 \quad for \quad j > 1.$$

Using these conditions we can consequently find the functions $w_j(\varphi)$ by integrating equations (1.14). In particular,

$$w_1(\varphi) = e^{\frac{\alpha}{\beta}\varphi}. \tag{1.15}$$

Setting $\varphi = 2\pi$ in the solution $r = f(\varphi,0,r_0)$ we obtain the value $r = f(2\pi,0,r_0)$, corresponding to the point of Σ where the trajectory $r = f(\varphi,0,r_0)$ first intersects Σ again.

Definition 1.3.1. Fix the system of the form (1.9).

(a) The function

$$\mathscr{R}(r_0) = f(2\pi,0,r_0) = \tilde{\eta}_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots$$

(defined for $\mid r_0 \mid < r^*$), where $\tilde{\eta}_1 = w_1(2\pi)$ and $\eta_j = w_j(2\pi)$ for $j \geq 2$, is called the *Poincaré first return map* (or simply the *return map*).

(b) The function

$$\mathscr{P}(r_0) = \mathscr{R}(r_0) = \eta_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{1.16}$$

is called the *difference function*.

(c) The coefficient η_j, $j \in \mathbb{N}$, is called the *jth Lyapunov number*.

Zeros of (1.16) correspond to *cycles* (closed orbits, that is, orbits that are ovals) of system (1.10), isolated zeros correspond to *limit cycles* (isolated closed orbits). It is not difficult to prove that the first nonzero coefficient of the expansion (1.16) is the coefficient of an odd power of r_0.

The next theorem states that the Lyapunov numbers completely determine the behaviour of the trajectories of system (1.10) near the origin. A proof can be found in [103].

Theorem 1.3.2. *System (1.10) has a center at the origin if and only if all Lyapunov numbers are zero. Moreover, if $\eta_1 = 0$, or if for some $k \in \mathbb{N}$*

$$\eta_1 = \eta_2 = \cdots = \eta_{2k} = 0, \quad \eta_{2k+1} \neq 0, \tag{1.17}$$

then all trajectories in a neighbourhood of the origin are spirals and the origin is a focus, which is stable if $\eta_1 < 0$ or (1.17) holds for $\eta_{2k+1} < 0$ and it is unstable if $\eta_1 > 0$ or (1.17) holds for $\eta_{2k+1} > 0$.

From (1.15) and Theorem 1.3.2 we see that if $\alpha < 0$, then the origin of (1.10) is a stable focus and if $\alpha > 0$ then the origin is an unstable focus of (1.10).

Another approach can be used to characterize when a system of the form (1.9) posses a center at the origin. It is based on the well known Poincaré Lyapunov Theorem. To states this theorem we first remind some concepts.

We denote by $\mathfrak{X}$ the vector field associated to system (1.9)

$$\mathfrak{X} = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v}.$$

A local first integral of system (1.9) is a nonconstant differentiable function $\Phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, from a neighbourhood Ω of the origin, which is constant on trajectories of (1.9), equivalently,

$$\mathfrak{X}\Phi = \dot{u}\frac{\partial \Phi}{\partial u} + \dot{v}\frac{\partial \Phi}{\partial v} \equiv 0.$$

A formal first integral of system (1.9) is a formal power series Φ in the variables u and v satisfying $\mathfrak{X}\Phi \equiv 0$.

Theorem 1.3.3 (Poincaré-Lyapunov Theorem [73, 97])**.** *System* (1.9) *has a center at the origin if and only if* $\alpha = 0$ *and the system admits a local analytic first integral of the form*

$$\Phi(u,v) = u^2 + v^2 + \sum_{k+l\geq 3} w_{k,l}u^k v^l. \tag{1.18}$$

Moreover, the existence of a formal first integral Φ *of the form* (1.18) *implies the existence of a local analytic first integral of the same form.*

We now come to our central object of study, planar polynomial systems having a singularity at the origin at which the eigenvalues of the linear part are purely imaginary, and whose coefficients depend on parameters. Up to a time rescaling any such system can be written in the form

$$\dot{u} = -v + \sum_{p+q=2}^{n} a_{p,q}u^p v^q = P(u,v), \qquad \dot{v} = u + \sum_{p+q=2}^{n} b_{p,q}u^p v^q = Q(u,v). \tag{1.19}$$

To determine if a system of the form (1.19) has a center at the origin, by Theorem 1.3.3 we must look for a formal first integral of the form (1.18), which is equivalent to

$$\mathfrak{X}\Phi = P(u,v)\frac{\partial \Phi}{\partial u} + Q(u,v)\frac{\partial \Phi}{\partial v} \equiv 0. \tag{1.20}$$

To find necessary conditions for the existence of a first integral of the form (1.18) for system (1.19) we look for a formal series (1.18) satisfying (1.20). To start the computational process for finding the first N conditions for integrability we write down the initial string of (1.18) up to order $2N+2$

$$\Phi_{2N+2}(u,v) = u^2 + v^2 + \sum_{k+l=3}^{2N+2} w_{j,k}u^j v^k.$$

Then for each $i = 3, \ldots, 2N+2$ we equate the coefficients of terms of order i in the expression

$$\frac{\partial \Phi_{2N+2}}{\partial u}P(u,v) + \frac{\partial \Phi_{2N+2}}{\partial v}Q(u,v), \tag{1.21}$$

to zero obtaining $2N$ systems of linear equations of unknown variables $w_{j,k}$. Then, we look for solutions of the linear systems obtained starting from system that corresponds to $i = 3$. Linear systems corresponding to odd $i = 2\ell - 1$ always have unique solutions. After solving the system we substitute the obtained values of $w_{j,k}$ into the linear systems corresponding to $i > 2\ell - 1$. For systems that correspond to even $i = 2\ell$, there is one equation more than the number of variables. After dropping a suitable equation we obtain the system with the unique solution. After solving the system we assign the value 0 for the undefined $w_{j,k}$ (with $j + k = 2\ell$) and substitute the obtained values of $w_{j,k}$ into the linear systems corresponding to $i > 2\ell$. Next, we evaluate (1.21) with the found $w_{j,k}$ ($j + k \leq 2\ell$) and find the coefficient of $u^{2\ell-2}v^2$ which we denote by $\upsilon_{\ell-1}$. Computing in this way we obtain a list of polynomials $\upsilon_1, \upsilon_2, \upsilon_3, \ldots$, in the parameters $a_{p,q}, b_{q,p}$ of system (1.19). This construction process always yields a series of the form (1.18) for which

$$\mathfrak{X}\Phi = \upsilon_1 (uv)^2 + \upsilon_2 (u^2 v)^2 + \upsilon_3 (u^3 v)^2 + \cdots + \upsilon_k (u^k v)^2 + \cdots . \tag{1.22}$$

Definition 1.3.4. We call the polynomial υ_k on the right side of (1.22) by the kth *real focus quantity* for the singularity at the origin of system (1.19). The ideal defined by the real focus quantities, $\mathscr{B}^{\mathbb{R}} = \langle \upsilon_1, \upsilon_2, \ldots \rangle \subset \mathbb{C}[a,b]$, where a and b represents all the parameters $a_{p,q}$ and $b_{p,q}$, respectively, of system (1.19) is called the *real Bautin ideal*. The affine variety $V_{\mathscr{C}}^{\mathbb{R}} = \mathbf{V}(\mathscr{B}^{\mathbb{R}})$, is called the *real center variety* of system (1.19).

Remark 1.3.5. The standard nomenclature in the literature for the polynomial υ_k is only "focus quantity". However in the next subsection we present another method for solving the center problem using the complexification of a real system. This method yields polynomials called focus quantities as well. Hence, we decided to change smoothly the names in order to clarify the object we are using. The same changes happen with the nomenclatures and notations of the "Bautin ideal" and the "center variety".

The polynomials υ_k represent obstacles for the existence of the first integral of the form (1.18), that is, system (1.19) admits a first integral of the form (1.18) if and only if $\upsilon_k = 0$, for all $k \geq 1$. Thus, the simultaneous vanishing of all focus quantities provide conditions which characterize when a system of the form (1.19) has a center at the origin.

By the Hilbert Basis Theorem there exists a positive integer k such that $\mathscr{B}^{\mathbb{R}} = \mathscr{B}_k^{\mathbb{R}} = \langle \upsilon_1, \ldots, \upsilon_k \rangle$. Note that the inclusion $V_{\mathscr{C}}^{\mathbb{R}} = \mathbf{V}(\mathscr{B}^{\mathbb{R}}) \subset \mathbf{V}(\mathscr{B}_k^{\mathbb{R}})$ holds for any $k \geq 1$. The opposite inclusion is verified finding the irreducible decomposition of $\mathbf{V}(\mathscr{B}_k^{\mathbb{R}})$ (see Section 1.2) and then checking that any point of each component of the decomposition corresponds to a system having a center at the origin. To find the irreducible decomposition of $\mathbf{V}(\mathscr{B}_k^{\mathbb{R}})$ we performed computations with the routine `minAssGTZ` of the computer algebra system SINGULAR.

1.3.1 Complexification of real systems

Since we are investigating the behaviour of solutions of a real analytic system in the case that the eigenvalues of the linear part at a singularity are complex, i.e, when the system has the form (1.19), it is convenient to associate to it a two-dimensional complex system that can be profitably studied to gain information about the original real system. To this end, we introduce the complex variable $x = u + iv$, obtaining from (1.19) the system of the form

$$\dot{x} = i(x - X(x,\bar{x})), \tag{1.23}$$

where $i = \sqrt{-1}$, $X = (P + iQ)/i$ and P and Q are evaluated at $((x+\bar{x})/2, (x-\bar{x})/2i)$. Adjoining to equation (1.23) its complex conjugate, $\dot{\bar{x}} = -i(x - \overline{X(x,\bar{x})})$, and replacing $\bar{x}$ by y, we obtain the pair of equations which can be written in the form

$$\begin{aligned} \dot{x} &= i\left(x - \sum_{p+q=1}^{n-1} A_{p,q} x^{p+1} y^q\right) = i(x - X(x,y)), \\ \dot{y} &= -i\left(y - \sum_{p+q=1}^{n-1} B_{q,p} x^q y^{p+1}\right) = -i(y - Y(x,y)), \end{aligned} \tag{1.24}$$

where $p \geq -1$, $q \geq 0$ and $B_{q,p} = \overline{A}_{p,q}$. Systems (1.23) and (1.24) are called the *complexification* of the real system (1.19).

Example 1.3.6. Consider the real system

$$\dot{u} = -v + u^2, \qquad \dot{v} = u + 2uv. \tag{1.25}$$

Introducing $x = u + iv$ we obtain

$$\dot{x} = ix + \left(\frac{x+\bar{x}}{2}\right)^2 + i\left(\frac{x+\bar{x}}{2}\right)\left(\frac{x-\bar{x}}{2i}\right) = i\left(x - i\frac{x^2}{2} - i\frac{xy}{2}\right).$$

Adjoining the complex conjugate of the equation above and replacing $\bar{x}$ by y, we obtain

$$\dot{x} = i\left(x - i\frac{x^2}{2} - i\frac{xy}{2}\right) = ix + \frac{x^2}{2} + \frac{xy}{2}, \qquad \dot{y} = -i\left(y + i\frac{y^2}{2} + i\frac{xy}{2}\right) = -iy + \frac{y^2}{2} + \frac{xy}{2},$$

which is the complexification of system (1.25).

Throughout the text, unless otherwise mentioned, we consider a more general class of systems of the form (1.24), where $A_{p,q}$, $B_{q,p}$ and x,y are not necessarily complex conjugated, i.e, $B_{q,p}$ is not necessarily equal to $\overline{A}_{p,q}$ and y is not necessarily equal to $\bar{x}$.

The next theorem is a version of the Poincaré Lyapunov Theorem for the complexification (1.24) of system (1.19). A proof can be found in [103].

Theorem 1.3.7. *System (1.19) has a center at the origin if and only if its complexification (1.24) has a formal first integral of the form $\Psi = xy + \cdots$, whre "$\cdots$" denote hight order terms.*

Following Dulac [42], it is common to extend the concept of a center to certain complex systems.

Definition 1.3.8. Consider the system

$$\dot{x} = x - \tilde{P}(x,y), \qquad \dot{y} = -y + \tilde{Q}(x,y), \tag{1.26}$$

where x and y are complex variables and $\tilde{P}$ and $\tilde{Q}$ are complex series without constant or linear terms that are convergent in a neighbourhood of the origin. We say that system (1.26) has a *center* at the origin if it has a formal first integral of the form

$$\Psi(x,y) = xy + \sum_{j+k\geq 3} w_{j-1,k-1} x^j y^k, \tag{1.27}$$

equivalently,

$$\mathfrak{X}\Psi = \dot{x}\frac{\partial \Psi}{\partial x} + \dot{y}\frac{\partial \Psi}{\partial y} \equiv 0.$$

Remark 1.3.9. If in (1.26) $\tilde{P}$ and $\tilde{Q}$ satisfy the condition $\tilde{Q}(x,\bar{x}) = \overline{\tilde{P}(x,\bar{x})}$, then the system

$$\dot{x} = i(x - \tilde{P}(x,y)), \qquad \dot{y} = -i(y - \tilde{Q}(x,y)), \tag{1.28}$$

is the complexification of a real system. If there exists a formal first integral of the form (1.27) for system (1.28), then by Definition 1.3.8 this system has a center at the origin of $\mathbb{C}^2$ and, by Theorem 1.3.7, the corresponding real system has a center at $(0,0) \in \mathbb{R}^2$.

To find necessary conditions for the existence of a first integral of the form (1.27) for system (1.24) one can proceed similarly to the previous subsection. The analogue construction process always yields a series of the form (1.27) for which $\mathfrak{X}\Psi = \frac{\partial \Psi}{\partial x}\dot{x} + \frac{\partial \Psi}{\partial y}\dot{y}$ reduces to

$$\mathfrak{X}\Psi = g_{11}(xy)^2 + g_{22}(xy)^3 + g_{33}(xy)^4 + \cdots. \tag{1.29}$$

Definition 1.3.10. We call the polynomial g_{kk} on the right-hand side of (1.29) by the kth *complex focus quantity* for the singularity at the origin of system (1.24). The ideal defined by the complex focus quantities, $\mathscr{B}^{\mathbb{C}} = \langle g_{11}, g_{22}, \ldots \rangle \subset \mathbb{C}[A,B]$, where A and B denotes the all parameters $A_{p,q}$ and $B_{p,q}$ of (1.24), respectively, is called the *complex Bautin ideal*. The affine variety $V_{\mathscr{C}}^{\mathbb{C}} = \mathbf{V}(\mathscr{B}^{\mathbb{C}})$, is called the *complex center variety* of system (1.24).

Remark 1.3.11. As explained in Remark 1.3.5 the standard nomenclature of the polynomial g_{kk} in literature is only "focus quantity". However to avoid mistakes we have chance smoothly the name. The same happens again for the "Bautin ideal" and the "center variety".

Finally, the process for finding the variety $V_{\mathscr{C}}^{\mathbb{C}}$ is analogue to the process for finding the variety $V_{\mathscr{C}}^{\mathbb{R}}$.

1.3.2 Mechanisms to study the integrability of differential systems

In this subsection we briefly present three mechanisms to study the integrability of polynomial systems. The time-reversible and Hamiltonian systems and the Darboux first integrals. See more details in [103].

Consider a polynomial system of the form

$$\dot{x} = U(x,y), \qquad \dot{y} = V(x,y), \tag{1.30}$$

where $x, y \in \mathbb{R}$ or $\mathbb{C}$ and $U(x,y)$ and $V(x,y)$ are polynomials of degree m in $\mathbb{R}[x,y]$ or $\mathbb{C}[x,y]$ without constant terms and nonconstant common factor. We say that a line L is an axis of symmetry of system (1.30) if as point sets the orbits of the system are symmetric with respect to L. When direction of flow is taken into account, there are two types of symmetry of a real system with respect to a line L: *mirror symmetry*, meaning that when the phase portrait is reflected in the line L it is unchanged; and *time-reversible symmetry*, meaning that when the phase portrait is reflected in the line L and then the sense of every orbit is reversed (corresponding to a reversal of time), the original phase portrait is obtained. Time-reversible symmetry is of interest in connection with the center problem because for any antisaddle on L, presence of the symmetry prevent the possibility that the singularity be a focus, hence forces it to be of center type.

System (1.30) is said to be a *Hamiltonian system* if there is a function $H : \mathbb{C}^2 \to \mathbb{C}$, called the *Hamiltonian* of the system, such that $U = -\frac{\partial H}{\partial y}$ and $V = \frac{\partial H}{\partial x}$. It follows immediately that

$$\mathfrak{X}H = \frac{\partial H}{\partial x}U + \frac{\partial H}{\partial y}V \equiv 0,$$

so that H is a first integral of the system. Thus an antisaddle of any real Hamiltonian system, and the singularity at the origin of any complex Hamiltonian system of the form (1.24), is known to be a center.

A *Darboux factor* of system (1.30) is a polynomial $f(x,y)$ satisfying

$$\frac{\partial f}{\partial x}U + \frac{\partial f}{\partial y}V = Kf,$$

where $K(x,y)$ is a polynomial of degree at most $m-1$, called the *cofactor of f*.

A *Darboux first integral* of system (1.30) is a first integral of the form $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, where $f_1, f_2, \ldots, f_k$ are Darboux factors of system (1.30) and $\alpha_i \in \mathbb{R}$, $1 \le i \le k$. If a first integral of system (1.30) cannot be found, then we turn our attention to possible existence of an integrating factor. A *Darboux integrating factor* of system (1.30) is an integrating factor of the form $\mu = f_1^{\beta_1} \cdots f_k^{\beta_k}$, where $f_1, f_2, \ldots, f_k$ are Darboux factors of system (1.30) and $\beta_i \in \mathbb{R}$, $1 \le i \le k$.

Suppose that $f_1, f_2, \ldots, f_k$ are Darboux factors of system (1.30) with respective cofactors $K_1, K_2, \ldots, K_k$. Then it is not difficult to prove that if there exist constants $\alpha_i \in \mathbb{R}$, $1 \le i \le k$, satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \tag{1.31}$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a Darboux first integral of (1.30), and if there exist constants $\beta_i \in \mathbb{R}$, $1 \le i \le k$, satisfying

$$\sum_{i=1}^{k} \beta_i K_i + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{1.32}$$

then $\mu = f_1^{\beta_1} \cdots f_k^{\beta_k}$ is a Darboux integrating factor of system (1.30).

1.4 The isochronicity and linearizability problems

In the previous section we presented methods for determining whether the antisaddle at the origin of the real polynomial system (1.19) is a center or a focus, and more generally if the singularity at the origin of the complex polynomial system (1.24) is a center. In this section we assume that the singularity in question is known to be a center and present methods for determining whether or not it is *isochronous*, that is, whether or not every periodic orbit in a neighbourhood of the origin has the same period.

Isochronicity was already studied in the 17th century when Christian Huygens [64] observed that a pendulum clock has a monotone period function and oscillates with a shorter oscillation period when it has a smaller energy, that is, when the clock's spring unwinds. He wanted to make a clock which would oscillate isochronously and would be more precise. It appears that his solution, the cycloidal pendulum, is probably the first example of a nonlinear isochronous system.

The studies of the isochronicity problem for polynomial systems become more intensively when Poincaré and Lyapunov proved that the existence of an isochronous center is direct connect with the linearizability of the system. This result has become the studies of the isochronicity problem much simpler, since the linearizability problem can be extended to the complex field, where the computational methods are more efficient. In 1964 Loud [87] found the necessary and sufficient conditions for isochronicity of the system

$$\dot{x} = -y + P'(x,y), \qquad \dot{y} = x + Q'(x,y), \tag{1.33}$$

with P' and Q' being quadratic homegeneous polynomials. Latter on, the isochronicity problem was solved for system (1.33) when P' and Q' are homogeneous polynomials of degree three [95] (see also [72]) and degree five [98]. However in the case of the linear center perturbed by homogeneous polynomials of degree four the problem is still unsolved, although some partial results were obtained [21, 53]. The reason is that linearizability quantities (defined below) have more complicate expressions in the case of homogeneous perturbations of degree four, than in the case of homogeneous perturbations of degree five. There are also many works devoted to the investigation of particular families of some other polynomial systems, see e.g. [3, 16, 26, 30, 89, 103] and references therein. Many works also deal with investigation of isochronicity of Hamiltonian systems, see e.g. [34, 29, 57, 65, 81] and references given there.

Next we present an approach for computing the so called *Period function*, which computes the period of every periodic solution in a neighbourhood of the origin.

Consider the polynomial system (1.19), namely

$$\dot{u} = -v + \sum_{p+q=2}^{n} a_{p,q} u^p v^q = P(u,v), \qquad \dot{v} = u + \sum_{p+q=2}^{n} b_{p,q} u^p v^q = Q(u,v).$$

Introducing the polar coordinates $u = r\cos\varphi$, $v = r\sin\varphi$ yields the equation of the trajectories

$$\frac{dr}{d\varphi} = \frac{r^2 F(r\cos\varphi, r\sin\varphi)}{1 + rG(r\cos\varphi, r\sin\varphi)} = R(r,\varphi). \tag{1.34}$$

As in Section 1.3 we can choose the line segment $\Sigma = \{(u,v); v = 0, 0 \le u \le r^*\}$, where r^* is chosen to be small enough. Now we consider the solution of (1.34) with the initial condition $r = r_0$, $\varphi_0 = 0$, and expand it in a power series in r_0 to obtain

$$r(\varphi, r_0 = w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + w_3(\varphi) r_0^3 + \cdots, \tag{1.35}$$

which is convergent for all $\varphi \in [0, 2\pi]$ and all $\mid r_0 \mid \le r^*$. The $r(\varphi)$ from (1.35) is a solution of (1.12) and inserting $r(\varphi, r_0)$ into (1.12) yields recurrence differential equations (1.14) for the functions $w_j(\varphi)$. We consider one revolution of $r = r(\varphi, r_0)$ beginning on $r_0 \in \Sigma$ where φ is assumed to be 0 and study the return to Σ (which happens at $\varphi = 2\pi$, that is, after one revolution). Thus the Poncaré return map $\mathscr{R}(r)$ is defined by

$$\mathscr{R}(r_0) = r(2\pi, r_0) = r_0 + w_2(2\pi) r_0^2 + w_3(2\pi) r_0^3 + + \cdots, \tag{1.36}$$

where the coefficients $\eta_j := w_j(2\pi)$ for $j > 1$ of (1.36) are the Lyapunov numbers. Suppose now that the origin of (1.19) is a center. Then for small enough $r_0 < r^*$ the trajectory of (1.19) is an oval (i.e. a simple closed curve) and we can consider the so-called *period function* $T(r)$. To obtain $T(r)$ we note that

$$\dot{\varphi} = 1 + rG(r, \cos\varphi, \sin\varphi) = 1 + \sum_{k=1}^{\infty} \zeta_k(\varphi) r^k.$$

Since $w_1 \equiv 0$ substituting instead of r expression (1.35) with $r_0 = r$ into this equation we obtain

$$\dot{\varphi} = 1 + \sum_{k=1}^{\infty} \zeta_k(\varphi) \left(r + \sum_{k=2}^{\infty} w_k(\varphi) r^k \right)^k.$$

Therefore

$$\frac{dt}{d\varphi} = \frac{1}{1 + \sum_{k=1}^{\infty} F_k(\varphi) r^k} = 1 + \sum_{k=1}^{\infty} \Psi_k(\varphi) r^k,$$

where $\sum_{k=1}^{\infty} \Psi_k(\varphi) r^k$ is an analytic function. After integrating we obtain

$$t = \varphi + \sum_{k=1}^{\infty} \int_0^{\varphi} \Psi_k(s) ds \cdot r^k = \varphi + \sum_{k=1}^{\infty} \theta_k(\varphi) r^k$$

(note that r^* is small enough and actually fixed, so the series above converges for $\varphi \in [0, 2\pi]$) and setting $\varphi = 2\pi$ (and $t = T$) one can obtain the "one revolution time" or the period

$$T(r) = 2\pi\left(1 + \sum_{k=1}^{\infty} T_k r^k\right),$$

where $T_k = \frac{1}{2\pi}\theta_k(2\pi)$ for $k \geq 1$. A center is isochronous if all solutions in its neighbourhood have the same period. Thus the center at the origin of system (1.19) is isochronous if and only if $T_k = 0$ for $k \geq 1$.

The process to obtain the period function demands integration of trigonometric functions, which can be a very difficult computational problem. Another approach based on the linearizability of the system is described below.

The following theorem which goes back to Poincaré and Lyapunov shows that there is an intimate relation between linearizability and isochronicity. A proof can be found e.g. in [103].

Theorem 1.4.1. *The origin of system* (1.19) *is an isochronous center if and only if the system is linearizable, i.e, if and only if there is an analytic change of coordinates*

$$u_1 = u + \sum_{m+n\geq 2} c_{m,n}u^m v^n, \qquad v_1 = v + \sum_{m+n\geq 2} d_{m,n}u^m v^n, \tag{1.37}$$

that reduces (1.19) *to the linear center*

$$\dot{u}_1 = -v_1, \qquad \dot{v}_1 = u_1.$$

It follows from Theorem 1.4.1 that solving the isochronicity problem is equivalent to solving the linearizability problem, but the investigation of the latter problem is computationally much simpler.

Differentiating with respect to t both sides of each equation of (1.37) we obtain

$$\begin{aligned}
\dot{u}_1 =& \dot{u} + \left(\sum_{m+n\geq 2} mc_{m,n}u^{m-1}v^n\right)\dot{u} + \left(\sum_{m+n\geq 2} nc_{m,n}u^m v^{n-1}\right)\dot{v},\\
\dot{v}_1 =& \dot{v} + \left(\sum_{m+n\geq 2} md_{m,n}u^{m-1}v^n\right)\dot{u} + \left(\sum_{m+n\geq 2} nd_{m,n}u^m v^{n-1}\right)\dot{v}.
\end{aligned}$$

Thus, substitution (1.37) linearizes system (1.19) if it holds that

$$\begin{aligned}
&\sum_{m+n\geq 2} d_{m,n}u^m v^n + \sum_{p+q=2}^{n} a_{p,q}u^p v^q + \left(\sum_{m+n\geq 2} mc_{m,n}u^{m-1}v^n\right)\left(-v + \sum_{p+q=2}^{n} a_{p,q}u^p v^q\right)\\
&+ \left(\sum_{m+n\geq 2} nc_{m,n}u^m v^{n-1}\right)\left(u + \sum_{p+q=2}^{n} b_{p,q}u^p v^q\right) \equiv 0,\\
&- \sum_{m+n\geq 2} c_{m,n}u^m v^n + \sum_{p+q=2}^{n} b_{p,q}u^p v^q + \left(\sum_{m+n\geq 2} md_{m,n}u^{m-1}v^n\right)\left(-v + \sum_{p+q=2}^{n} a_{p,q}u^p v^q\right)\\
&+ \left(\sum_{m+n\geq 2} nd_{m,n}u^m v^{n-1}\right)\left(u + \sum_{p+q=2}^{n} b_{p,q}u^p v^q\right) \equiv 0.
\end{aligned} \tag{1.38}$$

Obstacles for the fulfilment of equations in (1.38) give us necessary conditions for existence of a linearizing change of coordinates (1.37) for system (1.19). Thus, a computational procedure to find necessary conditions for linearizability can be described as follows.

(1) Write the left hand sides of two equations in (1.38) in the form $\sum_{k,l\geq 2} h_1^{(k,l)} u^k v^l$, and $\sum_{k,l\geq 2} h_2^{(k,l)} u^k v^l$, respectively, where $h_1^{(k,l)}$ and $h_2^{(k,l)}$ are polynomials in the parameters $a_{p,q}, b_{p,q}$ $(p+q=2,\dots,n)$ of system (1.19) and $c_{m,n}$, $d_{m,n}$ $(m+n\geq 2)$ of (1.37).

(2) Solve the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=2)$ for the coefficients $c_{m,n}$, $d_{m,n}$ $(m+n=2)$ of (1.37).

(3) Solve the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=3)$ for the coefficients $c_{m,n}$, $d_{m,n}$ $(m+n=3)$ of (1.37). In general case the system cannot be solved. However dropping from it two suitable equations we obtain a system that has a solution. We denote the two dropped polynomials on the left hand sides of these two equations by i_1 and j_1.

(4) Proceed step-by-step solving the polynomial systems $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=r,\ r>3)$. Generally speaking, at all steps when $r=k+l$ is an odd number the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=r)$ cannot be solved. Dropping on each such step two suitable equations (and denoting by $i_{(r-1)/2}$ and $j_{(r-1)/2}$ the corresponding polynomials), we obtain a system that has a solution.

This procedure yields the polynomials i_k and j_k which are polynomials in the parameters $a_{p,q}$ and $b_{p,q}$ of system (1.19) which we call by *the real linearizability quantities*. It is clear that system (1.19) admits a linearizing change of coordinates (1.37) if and only if $i_k=j_k=0$ for all $k\geq 1$. Thus, the simultaneous vanishing of all real linearizability quantities provide conditions which characterize when system (1.19) is linearizable (equivalently it has an isochronous center at the origin). The ideal $\mathscr{L}^{\mathbb{R}}=\langle i_1,j_1,i_2,j_2,\dots\rangle\subset\mathbb{C}[a,b]$ defined by the real linearizability quantities is called the *real linearizability ideal* and its affine variety, $V_{\mathscr{L}}^{\mathbb{R}}=\mathbf{V}(\mathscr{L}^{\mathbb{R}})$, is called the *real linearizability variety*. Therefore, the linearizability problem will be solved finding the variety $V_{\mathscr{L}}$.

Remark 1.4.2. The standard nomenclature in the literature for the polynomials i_k and j_k is only "linearizability quantities". However, next we present another method for solving the linearizability problem using the complexification of a real system. This method yields polynomials called linearizability quantities as well. Hence, we decided to change smoothly the names in order to clarify which object we are using. The same changes happen with the nomenclatures and notations of the "linearizability ideal" and the "linearizability variety".

By the Hilbert Basis Theorem there exists a positive integer k such that $\mathscr{L}^{\mathbb{R}}=\mathscr{L}_k^{\mathbb{R}}=\langle i_1,j_1,\dots,i_k,j_k\rangle$. Note that the inclusion $V_{\mathscr{L}}^{\mathbb{R}}\subset\mathbf{V}(\mathscr{L}_k^{\mathbb{R}})$ holds for any $k\geq 1$. The opposite inclusion is verified finding the irreducible decomposition of the variety $\mathbf{V}(\mathscr{L}_k^{\mathbb{R}})$ and then checking that any point of each component of the decomposition corresponds to a linearizable system. The irreducible decomposition can be found using the routine `minAssGTZ` of the computer algebra system SINGULAR, however it involves extremely laborious calculations.

We can also investigate the linearizability problem for the complexification of the real system (1.19), system (1.24). For system (1.24) the linearizability problem is to decide whether the system can be transformed in to the linear system $\dot{x} = x$, $\dot{y} = -y$, by an analytic change of coordinates

$$x_1 = x + \sum_{m+n\geq 2} c_{m,n} x^m y^n, \quad y_1 = y + \sum_{m+n\geq 2} d_{m,n} x^m y^n.$$

By a process similar to the explained above (see [103] for more details) we obtain polynomials I_{kk}, J_{kk}, $k \geq 1$, in parameters $A_{p,q}$ and $B_{p,q}$ of system (1.24) which we call by *complex linearizability quantities*. As in the real case, the simultaneous vanishing of all complex linearizability quantities provide conditions which characterize when system (1.24) is linearizable. The ideal $\mathscr{L}^{\mathbb{C}} = \langle I_{11}, J_{11}, I_{22}, J_{22}, \ldots \rangle \subset \mathbb{C}[A,B]$ defined by the complex linearizability quantities is called the *complex linearizability ideal* and its affine variety, $V_{\mathscr{L}}^{\mathbb{C}} = \mathbf{V}(\mathscr{L}^{\mathbb{C}})$, is called the *complex linearizability variety*.

Remark 1.4.3. As explained in Remark 1.4.2 the standard nomenclature of the polynomials I_{kk} and J_{kk} in literature is only "linearizability quantity". However to avoid mistakes we have chance smoothly the name. The same happens again for the "linearizability ideal" and the "linearizability variety".

Finally, the process for finding the variety $V_{\mathscr{L}}^{\mathbb{C}}$ is analogue to the process for finding the variety $V_{\mathscr{L}}^{\mathbb{R}}$.

1.4.1 Darboux linearizability

In Subsection 1.3.2 we present the Darboux method to construct first integrals by using Darboux factors. In this section we describe the most used tool to contruct linearizable change of coordinates, the Darboux linearization method witch is also based on the existence of Darboux factors. To apply the method we have to change the system of interest into a system of the form (1.24).

A *Darboux linearization* of system (1.24) is an analytic change of coordinates, $x_1 = X_1(x,y)$, $y_1 = Y_1(x,y)$, such that

$$X_1(x,y) = \prod_{j=0}^{m} f_j^{\alpha_j}(x,y) = x + X_1'(x,y),$$
$$Y_1(x,y) = \prod_{j=0}^{n} g_j^{\beta_j}(x,y) = y + Y_1'(x,y),$$

where $f_j, g_j \in \mathbb{C}[x,y]$, $\alpha_j, \beta_j \in \mathbb{C}$, and X_1' and Y_1' have neither constant nor linear terms.

A system is said to be *Darboux linearizable* if it admits a Darboux linearization. The next theorem provides a way to construct a Darboux linearization using Darboux factors (see e.g. [103] for a proof).

Theorem 1.4.4. *System* (1.24) *is Darboux linearizable if and only if there exits $s+1\geq 1$ Darboux factors $f_0,\dots,f_s$ with corresponding cofactors $K_0,\dots,K_s$ and $t+1\geq 1$ Darboux factors $g_0,\dots,g_t$ with corresponding cofactors $L_0,\dots,L_t$ with the following properties:*

(i) $f_0(x,y)=x+\cdots$ *but* $f_j(0,0)=1$ *for* $j\geq 1$*;*

(ii) $g_0(x,y)=y+\cdots$ *but* $g_j(0,0)=1$ *for* $j\geq 1$*; and*

(iii) there are $s+t$ constants $\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_t\in\mathbb{C}$ such that

$$K_0+\alpha_1K_1+\cdots+\alpha_sK_s=1 \quad \textit{and} \quad L_0+\beta_1L_1+\cdots+\beta_tL_t=-1. \tag{1.39}$$

The Darboux linearization is then given by

$$x_1=H_1(x,y)=f_0f_1^{\alpha_1}\cdots f_s^{\alpha_s}, \qquad y_1=H_2(x,y)=g_0g_1^{\beta_1}\cdots g_t^{\beta_t}.$$

Sometimes we cannot find enough Darboux factors to construct both Darboux linearizable transformations. In such case if we can find or at least prove the existence of a first integral of system (1.24) then it can be possible to construct linearizing transformations, as stated in the following two theorems. For the proofs see [103] and [38], respectively.

Theorem 1.4.5. *Suppose that system* (1.24) *has a first integral Ψ of the form* (1.27) *and that there exist Darboux factors $f_1,\dots,f_s$ satisfying condition (i) of Theorem 1.4.4 and the first equation of* (1.39)*. Then system* (1.24) *is linearizable by the transformation*

$$x_1=H_1(x,y)=f_0f_1^{\alpha_1}\cdots f_s^{\alpha_s}, \qquad y_1=H_2(x,y)=\frac{\Psi}{H_1(x,y)}.$$

If in Theorem 1.4.5 instead of condition (i) of Theorem 1.4.4 and the first equation of (1.39) we assume that condition (ii) of Theorem 1.4.4 and the second equation of (1.39) hold, then system (1.24) is linearizable by the transformation

$$x_1=H_1(x,y)=\frac{\Psi}{H_2(x,y)}, \qquad y_1=H_2(x,y)=f_0f_1^{\alpha_1}\cdots f_s^{\alpha_s}.$$

Theorem 1.4.6. *Assume that system* (1.24) *has a first integral Ψ of the form* (1.27) *and Darboux factors $f_1,\dots,f_s$ of the form $1+$h.o.t. with corresponding cofactors $K_1,\dots,K_s$. In such case, if*

$$(1-c)\frac{\dot{x}}{x}-c\frac{\dot{y}}{y}+\sum_{j=1}^{s}\alpha_jK_j=1, \quad \textit{and} \quad -c\frac{\dot{x}}{x}+(1-c)\frac{\dot{y}}{y}+\sum_{j=1}^{s}\alpha_jK_j=-1,$$

for some $c,\alpha_1,\dots,\alpha_s\in\mathbb{C}$, then the first and second equations of (1.24) *are linearized by the substitutions*

$$x_1=x^{1-c}y^{-c}\Psi^cf_1^{\alpha_1}\cdots f_s^{\alpha_s}, \qquad y_1=x^{-c}y^{1-c}\Psi^cf_1^{\alpha_1}\cdots f_s^{\alpha_s},$$

respectively.

CHAPTER

2

BI-CENTER PROBLEM

In this chapter we investigate the so called bi-center problem for some families of cubic and quintic systems. Part of the results presented here are published in Journal of Computational and Applied Mathematics [99]. This chapter is organized as follows. In Section 2.1 we introduce the bi-center problem and we give the motivation for such investigation. Section 2.2 is devoted to the study of the isochronicity of bi-centers of a cubic system. In Section 2.3 we present the results on the investigation of the existence of a bi-center and its isochronicity for a quintic system.

2.1 Motivation for the study

The existence of two simultaneous centers in planar differential systems was investigated only for very few particular families of systems. Kirnitskaya and Sibirskii in [68] and Li in [70] studied this problem for quadratic systems. They presented necessary and sufficient conditions for a planar quadratic differential system to have two centers simultaneously. Conti [32] investigated a particular family of cubic systems obtaining the first cubic system possessing two centers. Chen, Lu and Wang [23] studied other particular family of cubic systems called the Kukles system and found conditions for the existence of two centers in the family and Du [39] investigated the problem of existing two centers and their isochronicity in a particular family of polynomial systems of degree seven.

A differential system

$$\dot{x} = X(x,y), \qquad \dot{y} = Y(x,y) \tag{2.1}$$

is said to be $\mathbb{Z}_2$-*equivariant* with respect to a point P if its phase portrait is unchanged after a rotation on angle π around P (see for instance Li [71] for more details about $\mathbb{Z}_q$-equivariant systems and their importance for studies on Hilbert's 16th problem). The problem of characterizing the existence of two simultaneous centers in a $\mathbb{Z}_2$-equivariant system is called here as the *bi-center problem*. More precisely, assume that A and B are singular points of system (2.1) and that such system is $\mathbb{Z}_2$-equivariant with respect to the middle point of the line segment AB. In this case, we say that system (2.1) has a *bi-center* at points A and B if A and B are singular points of the center type.

Recently Liu and Li [79] studied the $\mathbb{Z}_2$-equivariant cubic system of the form

$$\dot{x} = X_1(x,y) + X_3(x,y), \quad \dot{y} = Y_1(x,y) + Y_3(x,y), \tag{2.2}$$

where $X_i(x,y), Y_i(x,y)$ $(i = 1,3)$ are homogeneous polynomials of degree i in variables x and y having two weak foci or centers at the points $(-1,0)$ and $(1,0)$. They presented the necessary and sufficient conditions for system (2.2) to have a bi-center at these points. System (2.2) was also investigated by Yu and Han [115, 116] and Liu and Huang [77] who studied limit cycle bifurcations in the system. Note that (2.2) is a rather general family of $\mathbb{Z}_2$-equivariant cubic systems, since any cubic system with two singular points A and B which is $\mathbb{Z}_2$-equivariant with respect to the middle point of the line segment AB, by a linear change of coordinates can be transformed to system (2.2).

The main purpose of our investigation is to find conditions for the existence of isochronous bi-centers at the points $(-1,0)$ and $(1,0)$ for system (2.2) and the existence of bi-centers and isochronous bi-centers for the $\mathbb{Z}_2$-equivariant quintic system of the form

$$\dot{x} = X_1(x,y) + X_5(x,y) = X(x,y), \quad \dot{y} = Y_1(x,y) + Y_5(x,y) = Y(x,y), \tag{2.3}$$

where $X_i(x,y), Y_i(x,y)$ $(i = 1,5)$ are homogeneous polynomials of degree i in the variables x and y and assuming that system (2.3) has two weak foci or centers at the points $(-1,0)$ and $(1,0)$.

It is worth to mention that the reason for choosing homogeneous polynomials of degree five in (2.3) (rather than polynomials of degree four) is to assure the existence of $\mathbb{Z}_2$-equivariant symmetry with respect to origin, which can appear only if the polynomials defining the system have just odd degree monomials. So, if we replace in (2.3) $X_5(x,y)$ and $Y_5(x,y)$ with homogeneous polynomials of degree four, then the system cannot have a bi-center at $(-1,0)$ and $(1,0)$, and if we add monomials of degree three to the perturbation of the linear part in system (2.3) then the study becomes computationally unfeasible.

Through this chapter we apply the procedures described in Chapter 1 to compute the real focus quantities and real linearizability quantities. However, to simplify notations and nomenclatures we use here the classical terminology and notation - focus quantities, linearizability quantities and the same for the ideals and affine varieties - for representing the real ones.

2.2 Isochronicity of a $\mathbb{Z}_2$-equivariant cubic system

In [79] Liu and Li studied the bi-center problem for a $\mathbb{Z}_2$-equivariant cubic system of the form (2.2) and found the necessary and sufficient conditions for existence of a bi-center at the points $(-1,0)$ and $(1,0)$. After a change of coordinates, they obtained from (2.2) the following standard form of the system (see Theorem 7 of [79]):

$$\begin{aligned}\dot{x} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3,\\ \dot{y} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3,\end{aligned} \tag{2.4}$$

where $a_1,\dots,a_6$ are real parameters.

The following eleven necessary and sufficient conditions for the existence of a bi-center at the points $(-1,0)$ and $(1,0)$ for system (2.4) are given in Theorem 11 of [79]:

(1) $a_4 = 0$, $a_1 = -a_5$, $a_6 = -a_2/3$;

(2) $a_4 = 0$, $a_1 + a_5 \neq 0$, $a_2 = a_6 = 0$;

(3) $a_1 + a_5 \neq 0$, $a_6 = -(a_2 + 2a_1a_2 - 2a_4 + 2a_2a_5 - 2a_4a_5)/3$, $2(1+a_1)(a_1+a_5)^2 - a_4^2(1+2a_1+2a_5) = 0$, $3(a_1+a_5)(-a_3+2(1+a_1)(1+a_5)) - 2a_4(2a_4(1+a_5)+a_2(2+a_1+a_5)) = 0$;

(4) $a_3 = 2(1+a_1)(1+a_5)$, $a_6 = -(a_2 + 2a_1a_2 - 2a_4 + 2a_2a_5 - 2a_4a_5)$, $2(1+a_5)a_4 + (2+a_1+a_5)a_2 = 0$;

(5) $a_4 \neq 0$, $a_1 = -(2-3a_4^2)/3$, $a_2 = a_4$, $a_3 = a_24(1-a_4^2+a_5)$, $a_6 = a_4(1-a_4^2)$;

(6) $a_4 \neq 0$, $a_1 = -(8-5a_4^2)/8$, $a_2 = a_4/2$, $a_5 = -(8+a_4^2)/8$, $a_3 = -5a_4^4/32$, $a_6 = a_4(2-a_4^2)/4$;

(7) $a_4 \neq 0$, $a_1 = -(32+15a_4^2)/32$, $a_2 = a_4/4$, $a_3 = a_4^2(64+15a_4^2)/512$, $a_5 = -(96+17a_4^2)/32$, $a_6 = -3a_4(4+a_4^2)/16$;

(8) $a_4 \neq 0$, $a_1 = -(50+21a_4^2)/50$, $a_2 = a_4/5$, $a_3 = a_4^2/1250(250+63a_4^2)$, $a_5 = -(200+39a_4^2)/50$, $a_6 = -a_4/25(35+9a_4^2)$;

(9) $a_4 \neq 0$, $a_1 = -(9+4a_4^2)/9$, $a_2 = 0$, $a_3 = 0$, $a_6 = 2a_4/3(1+a_5)$;

(10) $a_4 \neq 0$, $a_1 = -(8+3a_4^2)/8$, $a_2 = -a_4/2$, $a_3 = 3a_4^2(4+a_4^2+4a_5)/16$, $a_6 = a_4(4-a_4^2+8a_5)/8$;

(11) $a_4 \neq 0$, $a_1 = -(32+15a_4^2)/32$, $a_2 = -a_4/4$, $a_3 = a_4^2/512(832+495a_4^2)$, $a_5 = (160+111a_4^2)/32$, $a_6 = a_4(76+45a_4^2)/16$.

We investigate the isochronicity of the bi-centers found by Liu and Li in [79]. We obtain the following result.

Theorem 2.2.1. *System* (2.4) *has an isochronous bi-center at the points* $(1,0)$ *and* $(-1,0)$ *if and only if one of the following conditions is satisfied:*

(i) $a_1 = -3/2,\ a_2 = 0,\ a_3 = 1/2,\ a_4 = 0,\ a_5 = -3/2,\ a_6 = 0$;

(ii) $a_1 = -3,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = -9,\ a_6 = 0$.

Proof. To compute the linearizability quantities we move the singular point (1,0) to the origin. Applying the translation $u = x-1$, $v = y$, we obtain from (2.4) the system

$$\begin{aligned} \dot{u} &= -v + a_1(2+u)uv + a_2(1+u)v^2 + a_3v^3, \\ \dot{v} &= u + \frac{3}{2}u^2 + \frac{1}{2}u^3 + a_4(2+u)uv + a_5(1+u)v^2 + a_6v^3. \end{aligned} \tag{2.5}$$

Using the computer algebra system MATHEMATICA we computed the first seven pairs of the linearizability quantities for system (2.5) using the procedure described in Section 1.4. Their expressions are very large, so we present here only the first two pairs:

$$\begin{aligned}
i_1 &= 18 + 4a_1^2 + 10a_2^2 + 9a_3 - 2a_2a_4 + 4a_4^2 + 12a_5 - 10a_1a_5 + 4a_5^2, \\
j_1 &= a_2 + 2a_1a_2 - 2a_4 + 2a_2a_5 - 2a_4a_5 + 3a_6, \\
i_2 &= \frac{1}{90}(486 + 144a_1 - 132a_1^2 - 48a_1^3 - 32a_1^4 - 240a_2^2 - 24a_1a_2^2 - 272a_1^2a_2^2 - 189a_3 - 108a_1a_3 \\
&\quad + 168a_2^2a_3 - 1026a_2a_4 + 396a_1a_2a_4 - 40a_1^2a_2a_4 - 224a_2^3a_4 - 84a_2a_3a_4 + 276a_4^2 - 264a_1a_4^2 \\
&\quad - 48a_1^2a_4^2 - 24a_2^2a_4^2 - 84a_3a_4^2 - 168a_2a_4^3 - 16a_4^4 + 288a_5 - 30a_1a_5 - 48a_1^2a_5 + 16a_1^3a_5 \\
&\quad - 372a_2^2a_5 - 880a_1a_2^2a_5 - 198a_3a_5 - 144a_1a_3a_5 - 612a_2a_4a_5 + 1216a_1a_2a_4a_5 + 192a_4^2a_5 + \\
&\quad - 204a_5^2 + 12a_1a5^2 + 96a_1^2a_5^2 - 272a_2^2a_5^2 - 72a_3a_5^2 + 80a_2a_4a_5^2 - 72a_4^2a_5^2 - 168a_5^3 + 16a_1a_5^3 + \\
&\quad - 32a_5^4 + 342a_2a_6 - 684a_1a_2a_6 - 216a_4a_6 + 276a_1a_4a_6 - 180a_2a_5a_6 - 240a_1a_4^2a_5 \\
&\quad + 132a_4a_5a_6 - 72a_1^2a_3), \\
j_2 &= \frac{1}{90}(-279a_2 + 342a_1a_2 - 288a_1^2a_2 - 16a_1^3a_2 - 168a_2^3 + 224a_1a_2^3 - 162a_2a_3 + 240a_1a_2a_3 \\
&\quad + 24a_1a_4 + 144a_1^2a_4 - 16a_1^3a_4 - 60a_2^2a_4 - 344a_1a_2^2a_4 + 36a_3a_4 + 12a_1a_3a_4 + 300a_2a_4^2 \\
&\quad - 72a_4^3 - 16a_1a_4^3 + 144a_2a_5 - 48a_1a_2a_5 - 528a_1^2a_2a_5 + 224a_2^3a_5 + 240a_2a_3a_5 - 102a_4a_5 \\
&\quad + 168a_1^2a_4a_5 - 368a_2^2a_4a_5 + 12a_3a_4a_5 + 496a_2a_4^2a_5 - 88a_4^3a_5 - 84a_2a_5^2 - 528a_1a_2a_5^2 \\
&\quad + 264a_1a_4a_5^2 - 16a_2a_5^3 + 80a_4a_5^3 + 135a_6 - 252a_1a_6 - 216a_1^2a_6 + 36a_2^2a_6 - 456a_2a_4a_6 \\
&\quad - 198a_5a_6 - 360a_1a_5a_6 - 144a_5^2a_6 - 162a_4 + 112a_1a_2a_4^2 + 288a_1a_4a_5 + 180a_4a_5^2 + 60a_4^2a_6).
\end{aligned}$$

The reader can easily compute the others quantities using any available computer algebra system.

As explained in Section 1.4 the next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathscr{L}_7) = \mathbf{V}(\langle i_1, j_1, ..., i_7, j_7\rangle)$. We use the routine `minAssGTZ` of SINGULAR. Since an isochronous center must be a center, to make the computations easier we investigated the existence of an isochronous center using the eleven cases found by Liu and Li in [79] and given above. Thus, our proof is split in eleven cases corresponding to these eleven conditions.

Case (1). Computation with `minAssGTZ` of SINGULAR [1] shows that

$$I = \langle \mathscr{L}_7, a_4, a_1 + a_5, a_6 + a_2/3 \rangle,$$

is a primary ideal whose minimal associate prime is

$$I_1 = \langle a_4,\, a_6^2 + 1,\, 2a_5 - 3,\, a_1 + a_5,\, -2a_4a_5 + a_2 - 2a_4 + 3a_6,\, f \rangle,$$

where $f = \frac{40}{9}a_4^2a_5^2 + \frac{76}{9}a_4^2a_5 - \frac{40}{3}a_4a_5a_6 + \frac{40}{9}a_4^2 + 2a_5^2 - \frac{38}{3}a_4a_6 + 10a_6^2 + a_3 + \frac{4}{3}a_5 + 2.$

Analyzing the variety of this ideal we see that the corresponding polynomial system has only complex solutions, however all parameters of system (2.4) are real, so $\mathbf{V}(I)$ is the empty set in $\mathbb{R}^6$. Thus system (2.4) can not have an isochronous center if condition (1) of Theorem 11 of [79] holds.

Case (2). The irreducible decomposition of the variety of the ideal

$$\langle \mathscr{L}_7, a_2, a_4, a_6 \rangle$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the varieties of the following two ideals:

$$I_1 = \langle a_6,\, 2a_5 + 3,\, a_4,\, \frac{4}{9}a_1^2 + \frac{4}{9}a_4^2 - \frac{10}{9}a_1a_5 + \frac{4}{9}a_5^2 + a_3 + \frac{4}{3}a_5 + 2, a_2,\, 2a_1 + 3 \rangle,$$
$$I_2 = \langle a_6,\, a_5 + 9,\, a_4,\, \frac{4}{9}a_1^2 + \frac{4}{9}a_4^2 - \frac{10}{9}a_1a_5 + \frac{4}{9}a_5^2 + a_3 + \frac{4}{3}a_5 + 2, a_2,\, a_1 + 3 \rangle.$$

The varieties of the ideals I_1 and I_2 give conditions (i) and (ii) of this theorem. Thus, (i) and (ii) are the necessary conditions for existence of an isochronous bi-center at the points $(-1,0)$ and $(1,0)$ for system (2.4). Now we need to show that these two conditions are also sufficient for existence of an isochronous bi-center. To do so, we look for Darboux linearizations of the corresponding systems.

Condition (i). In this case system (2.4) becomes

$$\dot{x} = y(\frac{1}{2} - \frac{3}{2}x^2 + \frac{1}{2}y^2), \quad \dot{y} = x(-\frac{1}{2} + \frac{1}{2}x^2 - \frac{3}{2}y^2). \tag{2.6}$$

Translating the point (1,0) to the origin using the substitution $u = x - 1$, $v = y$, and then performing the complexification $z = u + iv$, $w = u - iv$, we obtain the system

$$\dot{z} = z(1 + \frac{3}{2}z + \frac{1}{2}z^2), \quad \dot{w} = w(-1 - \frac{3}{2}w - \frac{1}{2}w^2). \tag{2.7}$$

This system is a particular case of the system studied in [55] (namely, it satisfies condition (1) of Theorem 4 of [55]) where it was shown that this system is linearizable. Thus, system (2.7) is linearizable at the origin. Consequently, system (2.6) has an isochronous bi-center at the points $(1,0)$ and $(-1,0)$.

[1] The code used in this computation is present in Appendix A.

Condition (ii). The conditions of this case yield the system

$$\dot{x} = y(2-3x^2), \quad \dot{y} = x(-\frac{1}{2}+\frac{1}{2}x^2-9y^2). \tag{2.8}$$

As in the previous case applying the translation $u = x-1$, $v = y$ and the complexification, $z = u+iv$, $w = u-iv$ to system (2.8) we obtain the system

$$\begin{aligned}
\dot{z} &= z+\frac{33z^2}{8}-\frac{15zw}{4}+\frac{9w^2}{8}+\frac{25z^3}{16}-\frac{9z^2w}{16}-\frac{21zw^2}{16}+\frac{13w^3}{16},\\
\dot{w} &= -w-\frac{9z^2}{8}+\frac{15zw}{4}-\frac{33w^2}{8}-\frac{13z^3}{16}+\frac{21z^2w}{16}+\frac{9zw^2}{16}-\frac{25w^3}{16}.
\end{aligned} \tag{2.9}$$

This system has the Darboux factors

$$\begin{aligned}
l_1 &= z+\frac{3z^2}{8}-\frac{3zw}{4}+\frac{3w^2}{8}+\frac{z^3}{16}+\frac{3z^2w}{16}+\frac{3zw^2}{16}+\frac{w^3}{16},\\
l_2 &= w+\frac{3z^2}{8}+\frac{3zw}{4}+\frac{3w^2}{8}+\frac{z^3}{16}+\frac{3z^2w}{16}+\frac{3zw^2}{16}+\frac{w^3}{16},\\
l_3 &= 1+3z+3w+\frac{3z^2}{4}+\frac{3zw}{2}+\frac{3w^2}{4},
\end{aligned}$$

with respective cofactors

$$\begin{aligned}
k_1 &= \frac{1}{4}(4+18z-18w+9z^2-9w^2),\\
k_2 &= \frac{1}{4}(-4+18z-18w+9z^2-9w^2),\\
k_3 &= \frac{3}{2}(2z-2w+z^2-w^2).
\end{aligned}$$

It is easy to verify that conditions (1.39) are satisfied with $f_0 = l_1$, $f_1 = l_3$, $g_0 = l_2$, $g_1 = l_3$ and constants $\alpha_1 = \beta_1 = -\dfrac{3}{2}$. Thus, the Darboux linearization of system (2.9) is given by the following analytic change of coordinates:

$$\begin{aligned}
z_1 &= \frac{16z+6z^2+z^3+12zw+3z^2w+6w^2+3zw^2+w^3}{2(4+12z+3z^2+12w+6zw+3w^2)^{3/2}},\\
w_1 &= \frac{6z^2+z^3+16w+12zw+3z^2w+6w^2+3zw^2+w^3}{2(4+12z+3z^2+12w+6zw+3w^2)^{3/2}}.
\end{aligned}$$

Thus system (2.9) is linearizable and therefore system (2.8) has an isochronous bi-center at the points $(1,0)$ and $(-1,0)$.

Case (3). The irreducible decomposition of the variety of the ideal

$$\langle \mathscr{L}_7, f_1, f_2, f_3 \rangle, \tag{2.10}$$

where

$$\begin{aligned}
f_1 =& 3a_6+(a_2+2a_1a_2-2a_4+2a_2a_5-2a_4a_5),\\
f_2 =& 2(1+a_1)(a_1+a_5)^2-a_4^2(1+2a_1+2a_5),\\
f_3 =& 3(a_1+a_5)(-a_3+2(1+a_1)(1+a_5))-2a_4(2a_4(1+a_5)+a_2(2+a_1+a_5)),
\end{aligned}$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the seven components presented in Appendix C. Analyzing the components we see that the corresponding polynomial systems have only complex solutions, so the varieties are empty sets in $\mathbb{R}^6$. Thus, system (2.4) cannot have an isochronous center if condition (3) of Theorem 11 of [79] holds.

The remaining eight cases (*Cases* (4),(5),...,(11)) are analogous to *Case* (3). All the varieties are empty sets in $\mathbb{R}^6$. Therefore system (2.4) can not have an isochronous center if conditions (4), (5),..., (11) of Theorem 11 of [79] holds. □

In [79] the authors gave examples of Hamiltonian systems (2.2) having more than two centers. Systems (2.6) and (2.8) are examples of non-Hamiltonian systems with three centers. Indeed, since the vector field associated to systems (2.6) and (2.8) is symmetric with respect to the y-axis, for both cases the origin is a center. Simple computations show that both systems are non-Hamiltonian. Moreover, all three centers of systems (2.6) and (2.8) are isochronous.

Proposition 2.2.2. *If condition (i) or (ii) of Theorem 2.2.1 holds, then system (2.4) has an isochronous center at the origin.*

Proof. By a linear change of coordinates system (2.6) is transformed to the system

$$\dot{x_1} = -y_1 - 3x_1^2y_1 + y_1^3, \quad \dot{y_1} = x_1 + x_1^3 - 3x_1y_1^2. \tag{2.11}$$

and system (2.8) is transformed to the system

$$\dot{x_1} = -y_1 + 3x_1^2y_1, \quad \dot{y_1} = x_1 - 2x_1^3 + 9x_1y_1^2. \tag{2.12}$$

System (2.11) corresponds to system (S_1^*) in Table II of [89], and it was shown in [89] that it possess an isochronous center at the origin. Thus, system (2.6) also has an isochronous center at origin. System (2.12) corresponds to system (S_3^*) in Table II of [89] so it has an isochronous center at the origin. Hence, system (2.8) has an isochronous center at origin as well. □

By Proposition 2.2.2 under the conditions of Theorem 2.2.1 system (2.4) has three isochronous centers at points $(-1,0)$, $(0,0)$ and $(1,0)$. We are unaware about other examples of cubic systems with 3 isochronous centers. In Figures 3 and 4 we present the local phase portraits of systems (2.6) and (2.8), respectively.

2.3 Centers and isochronicity of a $\mathbb{Z}_2$-equivariant quintic system

In this section we investigate the existence of a bi-center and an isochronous bi-center for a class of $\mathbb{Z}_2$-invariant quintic system.

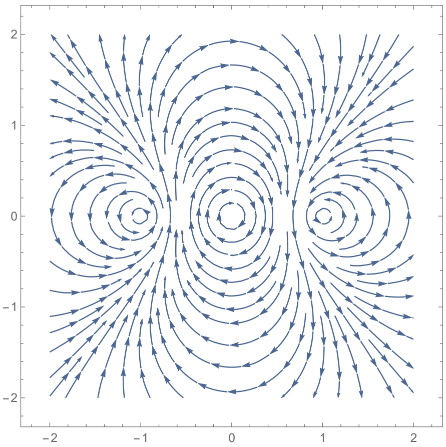

Figure 3 – Local phase portrait of system (2.6).

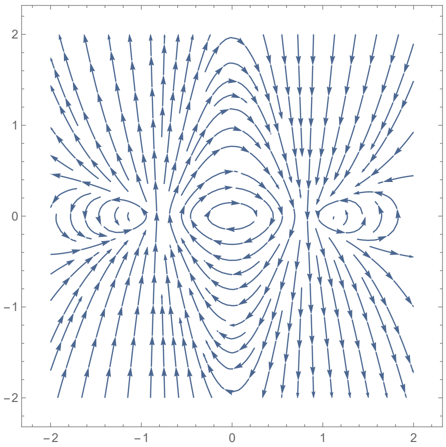

Figure 4 – Local phase portrait of system (2.8).

Before applying the methods described in Chapter 1 to study system (2.3) we look for a canonical form of the system for which the computations are simpler.

Suppose that system (2.3) satisfies the following conditions

$$X(1,0)=Y(1,0)=0, \quad \frac{\partial X(1,0)}{\partial x}=\frac{\partial Y(1,0)}{\partial y}=0, \quad \frac{\partial X(1,0)}{\partial y}=-1, \quad \frac{\partial Y(1,0)}{\partial x}=1, \tag{2.13}$$

which mean that the point $(1,0)$ is a singular point of (2.3), and its linearization at the point $(1,0)$ is

$$\frac{dx}{dt}=-y, \quad \frac{dy}{dt}=x-1. \tag{2.14}$$

According to Lemma 6 in [79], if system (2.3) has two weak foci or centers at $(-1,0)$ and $(1,0)$ then the transformation

$$x_1=x+\left(\frac{\partial Y(1,0)}{\partial y}\Big/\frac{\partial Y(1,0)}{\partial x}\right)y, \quad y_1=\left(r\Big/\frac{\partial Y(1,0)}{\partial x}\right)y, \quad t_1=rt,$$

where

$$r=\sqrt{\frac{\partial X(1,0)}{\partial x}\frac{\partial Y(1,0)}{\partial y}-\frac{\partial X(1,0)}{\partial y}\frac{\partial Y(1,0)}{\partial x}},$$

carries system (2.3) into a system whose linear part at the point $(1,0)$ is given by (2.14) and the conditions (2.13) hold. Moreover, this linear transformation preserves the distance between the origin and the two singular points. Thus, the assumption that the center are at the points $(-1,0)$ and $(1,0)$ is not a restriction, as shown in the next proposition.

Proposition 2.3.1. *Suppose that system (2.3) has two weak foci or centers that could be arranged at the points $(-1,0)$ and $(1,0)$. Then there is a change of coordinates that maps the orbits of system (2.3) to the orbits of the system*

$$\begin{aligned}\dot{x}&=-(a_1+1)y+a_1x^4y+a_2x^3y^2+a_3x^2y^3+a_4xy^4+a_5y^5,\\ \dot{y}&=-\frac{1}{4}x-a_6y+\frac{1}{4}x^5+a_6x^4y+a_7x^3y^2+a_8x^2y^3+a_9xy^4+a_{10}y^5,\end{aligned} \tag{2.15}$$

where $a_i\in\mathbb{R}$, $i=1,\ldots,10$.

System (2.15) is a $\mathbb{Z}_2$-equivariant quintic system, so the existence of a bi-center or an isochronous bi-center at the points $(1,0)$ and $(-1,0)$ for such system follows from the existence of a center or an isochronous center at the point $(1,0)$.

To compute the focus and linearizability quantities we have to move the singular point (1,0) to the origin. Applying the transformation $u = x-1$, $v = y$, to system (2.15) we obtain

$$\begin{aligned}\dot{u} =& -v+4a_1uv+a_2v^2+6a_1u^2v+3a_2uv^2+a_3v^3+4a_1u^3v+3a_2u^2v^2+2a_3uv^3+\\&+a_4v^4+a_1u^4v+a_2u^3v^2+a_3u^2v^3+a_4uv^4+a_5v^5,\\ \dot{v} =& u+\frac{5u^2}{2}+4a_6uv+a_7v^2+\frac{5u^3}{2}+6a_6u^2v+3a_7uv^2+a_8v^3+\frac{5u^4}{4}+4a_6u^3v+\\&+3a_7u^2v^2+2a_8uv^3+a_9v^4+\frac{u^5}{4}+a_6u^4v+a_7u^3v^2+a_8u^2v^3+a_9uv^4+a_{10}v^5.\end{aligned} \tag{2.16}$$

Our goal is to find conditions on the parameters of system (2.15) for the existence of bi-centers and isochronous bi-centers at the points $(1,0)$ and $(-1,0)$. Unfortunately, because this system has ten parameters the computational approaches described in Chapter 1 become unfeasible for the whole family (2.15). So we have studied some subfamilies of (2.15) fixing different choices of parameters. Then gathering the obtained results for such subfamilies, we conjecture the conditions for the existence of bi-centers and isochronous bi-centers for the whole family (2.15).

2.3.1 Bi-center conditions

In this subsection we investigate the existence of a bi-center for some subfamilies of system (2.15).

We first assume that system (2.15) posses the y-axis as an invariant curve, i.e, we fix two parameters of the original system (2.15), $a_1 = -1$ and $a_5 = 0$. Thus, we look for necessary and sufficient conditions for the system

$$\begin{aligned}\dot{x} &= -x^4y+a_2x^3y^2+a_3x^2y^3+a_4xy^4,\\ \dot{y} &= -\frac{1}{4}x-a_6y+\frac{1}{4}x^5+a_6x^4y+a_7x^3y^2+a_8x^2y^3+a_9xy^4+a_{10}y^5,\end{aligned} \tag{2.17}$$

to have a bi-center at the points $(-1,0)$ and $(1,0)$, or, equivalently, for the system

$$\begin{aligned}\dot{u} =& -v-4uv+a_2v^2-6u^2v+3a_2uv^2+a_3v^3-4u^3v+3a_2u^2v^2+2a_3uv^3+a_4v^4+\\&-u^4v+a_2u^3v^2+a_3u^2v^3+a_4uv^4,\\ \dot{v} =& u+\frac{5u^2}{2}+4a_6uv+a_7v^2+\frac{5u^3}{2}+6a_6u^2v+3a_7uv^2+a_8v^3+\frac{5u^4}{4}+4a_6u^3v+\\&+3a_7u^2v^2+2a_8uv^3+a_9v^4+\frac{u^5}{4}+a_6u^4v+a_7u^3v^2+a_8u^2v^3+a_9uv^4+a_{10}v^5,\end{aligned} \tag{2.18}$$

to have a center at the origin. We obtain the following result.

Theorem 2.3.2. *System (2.17) has a bi-center at the points $(-1,0)$ and $(1,0)$ if one of the following conditions holds:*

(1) $a_6 = 0,\ a_8 = \dfrac{1}{3}(a_2 - 2a_2a_7),\ a_9 = \dfrac{1}{2}(a_3 - a_3a_7),\ a_{10} = \dfrac{1}{5}(3a_4 - 2a_4a_7);$

(2) $a_2 = -4a_6,\ a_4 = 4a_3a_6,\ a_8 = 4a_6a_7,\ a_{10} = 4a_6a_9;$

(3) $a_4 = 4(a_3a_6 - 4a_2a_6^2 - 16a_6^3),\ a_8 = \dfrac{1}{3}(a_2 + 4a_6 - 2a_2a_7 + 4a_6a_7),$
$a_9 = \dfrac{1}{6}(3a_3 - 4a_2a_6 - 16a_6^2 - 3a_3a_7 - 4a_2a_6a_7 - 16a_6^2a_7),$
$a_{10} = 2a_6(-a_3 + 4a_2a_6 + 16a_6^2)(-1 + a_7);$

(4) $a_7 = -1,\ a_8 = a_2,\ a_9 = a_3,\ a_{10} = a_4.$

Proof. Using the computer algebra system MATHEMATICA we computed the first nine nonzero focus quantities for system (2.18) using the procedure described in Section 1.3. Their expressions are very large, so we present here only the first two:

$$
\begin{aligned}
\upsilon_1 =& -a_2 - 4a_6 + 2a_2a_7 - 4a_6a_7 + 3a_8,\\
\upsilon_2 =& 60a_{10} + 75a_2 - 40a_2a_3 - 36a_4 + 300a_6 + 80a_2^2a_6 - 16a_3a_6 + 256a_2a_6^2 - 256a_6^3\\
&- 154a_2a_7 + 40a_2a_3a_7 + 24a_4a_7 + 284a_6a_7 + 64a_3a_6a_7 + 320a_2a_6^2a_7 - 256a_6^3a_7\\
&+ 148a_2a_7^2 + 544a_6a_7^{-}280a_2a_7^3 + 560a_6a_7^3 - 225a_8 - 80a_2a_6a_8 + 64a_6^2a_8 + 12a_7a_8\\
&- 420a_7^2a_8 + 80a_2a_9 + 80a_6a_9.
\end{aligned}
$$

The reader can easily compute the others quantities using any available computer algebra system.

As explained in Section 1.3 the next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathscr{B}_9) = \mathbf{V}(\langle \upsilon_1, \ldots, \upsilon_9 \rangle)$. We use the routine `minAssGTZ` SINGULAR. Due to the complexity of the focus quantities the computations become unfeasible over the field of rational numbers[2]. To be able to complete our computations we computed in the field of finite characteristic 32003 and then lifted the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm of [114] and the computational procedure of [101] (see Section 1.2). The irreducible decomposition of the variety of the ideal $\mathscr{B}_9$ computed using the routine `minAssGTZ` of SINGULAR over the field of characteristic 32003[3] consists of 4 irreducible components. After the rational reconstruction we obtain the four conditions given in the statement of this theorem, which are necessary conditions for existence of a bi-center at the points $(-1,0)$ and $(1,0)$ for system (2.17).

Now we show that they are also sufficient.

[2] The computations done in this thesis were performed in a machine with the following characteristics: *memory: 32GB; processor: Intel(R) Core(TM) i7-3770 CPU @ 3.40GHz; disk: 2TB.*

[3] The code used to compute the irreducible decomposition of $\mathscr{B}_9$ is present in Appendix B.

Case (1). Under condition (1) of Theorem 2.3.2 system (2.17) becomes

$$\begin{aligned}\dot{x} &= -xy(x^3 - a_2x^2y - a_3xy^2 - a_4y^3),\\ \dot{y} &= -\frac{x}{4} + \frac{x^5}{4} + a_7x^3y^2 + \left(\frac{a_2 - 2a_2a_7}{3}\right)x^2y^3 + \left(\frac{a_3 - a_3a_7}{2}\right)xy^4 + \left(\frac{3a_4 - 2a_4a_7}{5}\right)y^5.\end{aligned} \tag{2.19}$$

System (2.19) has the Darboux factor $l_1 = x$ with the cofactor $k_1 = -y(x^3 - a_2x^2y - a_3xy^2 - a_4y^3)$. It is easy to verify that $\beta_1 = 2a_7 - 4$ is a solution of equation (1.32). Thus, $\mu = x^{2a_7-4}$ is an integrating factor of system (2.19) and computing we obtain the first integral

$$\begin{aligned}H(x,y) =& x^{2a_7-3}(-(15a_7+15)x + (15a_7-15)x^5 + (60a_7^2-60)x^3y^2\\ &+ (40a_2 - 40a_2a_7^2)x^2y^3 + (30a_3 - 30a_3a_7^2)xy^4 + (24a_4 - 24a_4a_7^2)y^5),\end{aligned}$$

which is analytic in a neighborhood of the point (1,0). Thus, the points $(-1,0)$ and $(1,0)$ are centers of (2.19).

Case (2). Under condition (2) of Theorem 2.3.2 system (2.17) has the form

$$\begin{aligned}\dot{x} =& -xy(x+4a_6y)(x^2 - a_3y^2),\\ \dot{y} =& \frac{1}{4}(x+4a_6y)(-1 + x^4 + 4a_7x^2y^2 + 4a_9y^4).\end{aligned} \tag{2.20}$$

Note that polynomials on the right hand sides of equation (2.20) have the common factor $x + 4a_6y$, so after the reparametrization of time we obtain the system

$$\begin{aligned}\dot{x} =& -xy(x^2 - a_3y^2) = P(x,y),\\ \dot{y} =& \frac{1}{4}(-1 + x^4 + 4a_7x^2y^2 + 4a_9y^4) = Q(x,y).\end{aligned} \tag{2.21}$$

Obviously, system (2.21) posses time-reversible symmetry with respect to the x-axis. Thus, the singular points $(-1,0)$ and $(1,0)$ are centers for (2.20).

Case (3). In this case the corresponding system (2.17) is written as

$$\begin{aligned}\dot{x} =& -xy(x+4a_6y)(x^2 - a_2xy - 4a_6xy - a_3y^2 + 4a_2a_6y^2 + 16a_6^2y^2),\\ \dot{y} =& \frac{1}{12}(x+4a_6y)(-3 + 3x^4 + 12a_7x^2y^2 + 4a_2xy^3 + 16a_6xy^3 - 8a_2a_7xy^3 - 32a_6a_7xy^3 +\\ &+ 6a_3y^4 - 24a_2a_6y^4 - 96a_6^2y^4 - 6a_3a_7y^4 + 24a_2a_6a_7y^4 + 96a_6^2a_7y^4).\end{aligned} \tag{2.22}$$

Applying the reparametrization of time we obtain the system

$$\begin{aligned}\dot{x} =& -xy(x^2 - (a_2+4a_6)xy + (-a_3 + 4a_2a_6 + 16a_6^2)y^2),\\ \dot{y} =& \frac{1}{12}(-3 + 3x^4 + 12a_7x^2y^2 + (4a_2 + 16a_6 - 8a_2a_7 - 32a_6a_7)xy^3\\ &+ (6a_3 - 24a_2a_6 - 96a_6^2 - 6a_3a_7 + 24a_2a_6a_7 + 96a_6^2a_7)y^4).\end{aligned} \tag{2.23}$$

System (2.23) has the Darboux factor $l_1 = x$ with cofactor $k_1 = -y(x^2 - a_2xy - 4a_6xy - a_3y^2 + 4a_2a_6y^2 + 16a_6^2y^2)$. It is easy to verify that $\beta_1 = 2a_7 - 3$ is a solution of equation (1.32). Thus,

$\mu = x^{2a_7-3}$ is an integrating factor of system (2.23) and computing we obtain the first integral

$$\begin{aligned} H(x,y) =& x^{2a_7-2}(-3-3a7+(3a_7-3)x^4+(12a_7^2-12)x^2y^2 \\ &+(8a_2+32a_6-8a_2a_7^2-32a_6a_7^2)xy^3 \\ &+(6a_3-24a_2a_6-96a_6^2-6a_3a_7^2+24a_2a_6a_7^2+96a_6^2a_7^2)y^4), \end{aligned}$$

which is analytic in a neighborhood of the point (1,0). Thus the points $(-1,0)$ and $(1,0)$ are centers of (2.22).

Case (4). In this case system (2.18) becomes

$$\begin{aligned} \dot{u} =& -v-4uv+a_2v^2-6u^2v+3a_2uv^2+a_3v^3-4u^3v+3a_2u^2v^2+2a_3uv^3+a_4v^4+ \\ &-u^4v+a_2u^3v^2+a_3u^2v^3+a_4uv^4, \\ \dot{v} =& u+\frac{5u^2}{2}+4a_6uv-v^2+\frac{5u^3}{2}+6a_6u^2v-3uv^2+a_2v^3+\frac{5u^4}{4}+4a_6u^3v \\ &-3u^2v^2+2a_2uv^3+a_3v^4+\frac{u^5}{4}+a_6u^4v-u^3v^2+a_2u^2v^3+a_3uv^4+a_4v^5. \end{aligned} \tag{2.24}$$

System (2.24) has the Darboux factors $l_1 = 1+u$ and $l_2 = 1+u+a_6v$, with the respective cofactors

$$\begin{aligned} k_1 =& v(-1-3u-3u^2-u^3+a_2v+2a_2uv+a_2u^2v+a_3v^2+a_3uv^2+a_4v^3), \\ k_2 =& 4a_6u+6a_6u^2+4a_6u^3+a_6u^4-v-3uv-3u^2v-u^3v+a_2v^2+ \\ & 2a_2uv^2+a_2u^2v^2+a_3v^3+a_3uv^3+a_4v^4. \end{aligned}$$

It is easy to verify that $\beta_1 = -5$ and $\beta_2 = -1$ is a solution of equation (1.32). Thus, $\mu = (1+u)^{-5}(1+u+a_6v)^{-1}$ is an integrating factor of system (2.24) of the form $\mu = 1+\cdots$. It implies the existence of an analytic first integral for (2.24). Hence the origin of (2.24) is a center and, therefore, system (2.17) has a bi-center at the points $(-1,0)$ and $(1,0)$. □

As it is mentioned in the proof of Theorem 2.3.2, to be able to compute the irreducible decomposition of the variety $\mathbf{V}(\mathscr{B}_9)$ it was necessary to use modular arithmetic. Since we were not able to complete the last step of the computational procedure of [101] (described in Section 1.2), we do not know if a component of the variety $\mathbf{V}(\mathscr{B}_9)$ was lost in the computations, that is, if conditions in Theorem 2.3.2 are all necessary conditions [4]. So it is still an open problem to prove that the conditions in the statement of Theorem 2.3.2 are necessary and sufficient for the existence of a bi-center for system (2.17).

We also investigate the existence of bi-centers for others subfamilies of system (2.15). However we were not able to perform the computations using our computational facilities for such subfamilies due to complexity of the ideal $\mathscr{B}_9$.

[4] In this case we say that the list of conditions is complete with high probability.

2.3.2 Isochronous bi-center conditions

In this subsection we study the isochronicity problem for some subfamilies of system (2.15).

Firstly, we consider system (2.17), i.e, system (2.15) possessing y-axis as an invariant curve. We obtain the following result.

Theorem 2.3.3. *Any system* (2.17) *satisfying one of conditions (1)–(4) of Theorem* 2.3.2 *has non-isochronous centers at the points* $(-1,0)$ *and* $(1,0)$.

Proof. Using MATHEMATICA and the procedure described in Section 1.4 we have computed the first nine pairs of the linearizability quantities for system (2.18). Their expressions are very large, so we present here only the first two pairs.

$$
\begin{aligned}
i_1 &= \frac{1}{9}(48+10a_2^2+9a_3-4a_2a_6+16a_6^2+36a_7+4a_7^2),\\
j_1 &= \frac{1}{3}(-a_2-4a_6+2a_2a_7-4a_6a_7+3a_8).\\
i_2 &= \frac{1}{90}(720-1008a_2^2-351a_3+168a_2^2a_3+252a_2a_4-5988a_2a_6-448a_2^3a_6-168a_2a_3a_6\\
&\quad +3120a_6^2-96a_2^2a_6^2-336a_3a_6^2-1344a_2a_6^3-256a_6^4-156a_7+740a_2^2a_7-174a_3a_7\\
&\quad +2720a_6^2a_7-852a_7^2-272a_2^2a_7^2-72a_3a_7^2+160a_2a_6a_7^2-288a_6^2a_7^2-376a_7^3-32a_7^4\\
&\quad +1782a_2a_8-1656a_6a_8-180a_2a_7a_8+264a_6a7a_8+360a_9+96a_7a_9+216a_4a_6\\
&\quad -6248a_2a_6a_7),\\
j_2 &= \frac{1}{90}(90a_{10}-3459a_2-712a_2^3-690a_2a_3-54a_4-12a_6+696a_2^2a_6+48a_3a_6+240a_2a_6^2\\
&\quad -1416a_2a_7+224a_2^3a_7+240a_2a_3a_7+36a_4a_7-708a_6a_7-736a_2^2a_6a_7+24a_3a_6a_7\\
&\quad -704a_6^3a_7+596a_2a_7^2-280a_6a_7^2-16a_2a_7^3+160a_6a_7^3+297a_8+36a_2^2a_8-912a_2a_6a_8\\
&\quad +240a_6^2a_8+162a_7a_8-144a_7^2a_8+120a_2a_9+120a_6a_9-448a_6^3+1984a_2a_6^2a_7).
\end{aligned}
$$

Following the method explained in Section 1.4 and using the routine `minAssGTZ` of SINGULAR we compute the irreducible decomposition of the variety $\mathbf{V}(\mathscr{L}_9)=\mathbf{V}(\langle i_1,j_1,...,i_9,j_9\rangle)$. Since an isochronous center must be a center, to make the computations easier we investigate the existence of an isochronous center using the four conditions described in Theorem 2.3.2. Thus, our proof is split in four cases corresponding to these four conditions.

Case (1). Computing the reduced Gröbner basis we see that

$$\langle \mathscr{L}_9, a_6, 3a_8-a_2+2a_2a_7, 2a_9-a_3+a_3a_7, 5a_{10}-3a_4+2a_4a_7\rangle = \langle 1\rangle.$$

Thus, system (2.17) does not have isochronous centers if condition (1) of Theorem 2.3.2 holds.

Case (2). The irreducible decomposition of the variety of the ideal

$$\langle \mathscr{L}_9, a_2+4a_6, a_4-4a_3a_6, a_8-4a_6a_7, a_{10}-4a_6a_9\rangle \tag{2.25}$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the six components in Appendix D. Analyzing the components we see that the corresponding polynomial systems have only complex solutions, so the varieties are empty sets in $\mathbb{R}^8$. Thus system (2.17) cannot have an isochronous center if condition (2) of Theorem 2.3.2 holds.

The remaining two cases (*Cases* (3) and (4)) are analogous to *Case* (2). All the varieties are empty sets in $\mathbb{R}^8$. Therefore system (2.17) cannot have an isochronous center if conditions (3) and (4) of Theorem 2.3.2 holds. □

Remark 2.3.4. If the conditions in the statement of Theorem 2.3.2 are necessary and sufficient for the existence of a bi-center for system (2.17) then such system cannot have isochronous centers at the points $(-1,0)$ and $(1,0)$. Another way to prove that system (2.17) cannot have an isochronous bi-center is analysing the variety $\mathbf{V}(\mathscr{L}_9)$ without using the conditions presented in Theorem 2.3.2, however we were not able to perform such analysis due to complexity of the ideal $\mathscr{L}_9$.

Summarizing, in the subfamily (2.17) of the quintic system (2.15) we did not find any isochronous bi-centers, whereas we found two isochronous bi-centers in the family of cubic systems (2.2). We note in the case of isochronous centers for the cubic system (2.2) there exist a third center at origin. For system (2.17) such additional center is not possible since the y-axis is an invariant curve of the system.

Such analysis lead us to investigate the existence of isochronous bi-centers for subfamilies of system (2.15) fixing different parameters,

$$a_1 = 0; \quad a_2 = 0; \quad a_3 = 0; \quad a_4 = 0; \quad a_5 = 0; \quad a_7 = 0; \quad a_8 = 0; \quad a_9 = 0; \quad a_{10} = 0.$$

Doing $a_2 = 0$ in system (2.15) we obtain the following system

$$\begin{aligned} \dot{x} &= -(a_1+1)y + a_1x^4y + a_3x^2y^3 + a_4xy^4 + a_5y^5, \\ \dot{y} &= -\frac{1}{4}x - a_6y + \frac{1}{4}x^5 + a_6x^4y + a_7x^3y^2 + a_8x^2y^3 + a_9xy^4 + a_{10}y^5. \end{aligned} \tag{2.26}$$

For system (2.26) the following result holds.

Theorem 2.3.5. *System* (2.26) *has an isochronous bi-center at the points* $(-1,0)$ *and* $(1,0)$ *if one of the following conditions holds:*

(1) $a_2 = a_4 = a_6 = a_8 = a_{10} = 0, a_1 = -2, a_3 = -4, a_5 = 16, a_7 = -1, a_9 = -8;$

(2) $a_2 = a_4 = a_6 = a_8 = a_{10} = 0, a_1 = -5, a_3 = 48, a_5 = 0, a_7 = -13, a_9 = -144;$

(3) $a_2 = a_4 = a_6 = a_8 = a_{10} = 0, a_1 = -2, a_3 = 8, a_5 = 0, a_7 = -4, a_9 = -4;$

(4) $a_2 = a_4 = a_6 = a_8 = a_{10} = 0, a_1 = -5, a_3 = 0, a_5 = 0, a_7 = -25, a_9 = 0;$

(5) $a_2 = a_4 = a_6 = a_8 = a_{10} = 0, a_1 = -\frac{5}{4}, a_3 = \frac{5}{2}, a_5 = -\frac{1}{4}, a_7 = -\frac{5}{2}, a_9 = \frac{5}{4}.$

Proof. Using the computer algebra system MATHEMATICA we computed the first eight pairs of the linearizability quantities, $i_1, j_1, \ldots, i_8, j_8$, for the whole family (2.16). The first pair is

$$i_1 = 40+8a_1+16a_1^2+10a_2^2+9a_3-4a_2a_6+16a_6^2+16a_7-20a_1a_7+4a_7^2,$$
$$j_1 = 3a_2+4a_1a_2-4a_6+2a_2a_7-4a_6a_7+3a_8.$$

The irreducible decomposition of the variety of the ideal $J=\langle \mathscr{L}_8, a_2\rangle$ computed using the routine `minAssGTZ` of SINGULAR over a field of characteristic 32452843 [5] is composed by the varieties of 16 ideals. After lifting the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm of [114] and the computational procedure of [101] (see Section 1.2), we obtain the following ideals

$$\begin{aligned}
J_1 =&\langle a_{10}, 8+a_9, a_8, 1+a_7, a_6, -16+a_5, a_4, 4+a_3, a_2, 2+a_1\rangle,\\
J_2 =&\langle a_{10}, 144+a_9, a_8, 13+a_7, a_6, a_5, a_4, -48+a_3, a_2, 5+a_1\rangle,\\
J_3 =&\langle a_{10}, 4+a_9, a_8, 4+a_7, a_6, a_5, a_4, -8+a_3, a_2, 2+a_1\rangle,\\
J_4 =&\langle a_{10}, a_9, a_8, 25+a_7, a_6, a_5, a_4, a_3, a_2, 5+a_1\rangle,\\
J_5 =&\langle a_{10}, -5+4a_9, a_8, 5+2a_7, a_6, 1+4a_5, a_4, -5+2a_3, a_2, 5+4a_1\rangle,\\
J_6 =&\langle 1+16a_{10}^2, -5+4a_9, 10a_{10}+a_8, 5+2a_7, -5a_{10}+a_6, a_5, a_4, a_3, a_2, a_1\rangle,\\
J_7 =&\langle a_{10}, a_9, a_8, 1+a_7, 36+25a_6^2, a_5, a_4, a_3, a_2, 1+5a_1\rangle,\\
J_8 =&\langle a_{10}, a_9, a_8, 1+a_7, 9+4a_6^2, a_5, a_4, a_3, a_2, -1+4a_1\rangle,\\
J_9 =&\langle a_{10}, a_9, a_8, 1+a_7, 1+a_6^2, a_5, a_4, a_3, a_2, 1+a_1\rangle,\\
J_{10} =&\langle 256+a_{10}^2, 4+a_9, -a_{10}+8a_8, -2+a_7, -a_{10}+32a_6, a_5, -3a_{10}+a_4, 36+a_3, a_2, 3+a_1\rangle,\\
J_{11} =&\langle 16384+a_{10}^2, -80+a_9, -a_{10}+8a_8, -11+a_7, -a_{10}+128a_6, a_5, -9a_{10}+a_4, 432+a_3,\\
&\quad a_2, 9+a_1\rangle,\\
J_{12} =&\langle 289+1024a_{10}^2, 103+64a_9, 8a_{10}+17a_8, 11+8a_7, -16a_{10}+17a_6, 45+256a_5,\\
&\quad 12a_{10}+17a_4, 9+32a_3, a_2, 21+16a_1\rangle,\\
J_{13} =&\langle 121+a_{10}^2, 67+4a_9, 2a_{10}+11a_8, 5+2a_7, -a_{10}+11a_6, 45+4a_5, 18a_{10}+11a_4,\\
&\quad -9+2a_3, a_2, 9+4a_1\rangle,\\
J_{14} =&\langle 5607424+9765625a_{10}^2, 448+625a_9, -25a_{10}+74a_8, 49+25a_7, 625a_{10}+2368a_6,\\
&\quad -6144+15625a_5, -57a_{10}+37a_4, -768+625a_3, a_2, 33+25a_1\rangle,\\
J_{15} =&\langle 49+16a_{10}^2, 7+4a_9, -2a_{10}+7a_8, 5+2a_7, a_{10}+7a_6, -3+2a_5, -12a_{10}+7a_4, -3+a_3,\\
&\quad a_2, 3+2a_1\rangle,\\
J_{16} =&\langle a_{10}, a_9, 9216+a_8^2, -23+a_7, 32a_6-a_8, a_5, a_4, a_3, a_2, -7+a_1\rangle.
\end{aligned}$$

The varieties of J_1, J_2, J_3, J_4, and J_5 provide conditions (1), (2), (3), (4), and (5), respectively, of the Theorem. Analysing the remaining 11 varieties we see that they are the empty set in $\mathbb{R}^9$. Thus,

[5] As it is mentioned in the proof of Theorem 2.3.2, we were not able to complete the last step of the computational procedure of [101]. So, we say that the conditions the statement of Theorem 2.3.5 holds with high probability.

conditions $(1)-(5)$ are necessary conditions for the existence of an isochronous bi-center at the points $(-1,0)$ and $(1,0)$ for system (2.26). Now we need to prove that they are also sufficient.

Case (1) : Under conditions (1) system (2.15) becomes

$$\dot{x}=y-2x^4y-4x^2y^3+16y^5, \qquad \dot{y}=-\frac{x}{4}+\frac{x^5}{4}-x^3y^2-8xy^4. \tag{2.27}$$

Translating the point $(1,0)$ to the origin, using the substitution $u=x-1, v=y$, and then the complexification $z=u+iv, w=u-iv$, we obtain the system

$$\begin{aligned}
\dot{z}=&z+\frac{1}{128}(-144w^2-40w^3-30w^4+45w^5+96wz-312w^2z\\
&+40w^3z-255w^4z+368z^2+456wz^2-372w^2z^2+530w^3z^2\\
&+216z^3+552wz^3-654w^2z^3-30z^4+465wz^4-99z^5),\\
\dot{w}=&-w+\frac{1}{128}(-368w^2-216w^3+30w^4+99w^5-96wz-456w^2z\\
&-552w^3z-465w^4z+144z^2+312wz^2+372w^2z^2+654w^3z^2\\
&+40z^3-40wz^3-530w^2z^3+30z^4+255wz^4-45z^5).
\end{aligned} \tag{2.28}$$

System (2.28) has the Darboux factors

$$\begin{aligned}
l_1=&z+\frac{1}{64}(-24w^2+16w^3-3w^4+48wz-96w^2z+28w^3z+72z^2\\
&+144wz^2-90w^2z^2+108wz^3-27z^4),\\
l_2=&w+\frac{1}{64}(72w^2-27w^4+48wz+144w^2z+108w^3z-24z^2-96wz^2\\
&-90w^2z^2+16z^3+28wz^3-3z^4),\\
l_3=&1+\frac{3}{2}z-\frac{1}{2}w,\\
l_4=&1-\frac{1}{2}z+\frac{3}{2}w,
\end{aligned}$$

with respective cofactors

$$\begin{aligned}
k_1=&-\frac{1}{16}(-2+w-3z)^3(2+3w-z),\\
k_2=&\frac{1}{16}(-2+w-3z)(2+3w-z)^3,\\
k_3=&\frac{1}{32}(16w+12w^3-9w^4+48z+48wz-36w^2z+48w^3z+48z^2\\
&+84wz^2-90w^2z^2+4z^3+88wz^3-21z^4),\\
k_4=&\frac{1}{32}(-48w-48w^2-4w^3+21w^4-16z-48wz-84w^2z-88w^3z+36wz^2\\
&+90w^2z^2-12z^3-48wz^3+9z^4).
\end{aligned}$$

It is easy to verify that conditions (1.39) are satisfied with $f_0=l_1, f_1=l_3, f_2=l_4, g_0=l_2, g_1=l_3, g_2=l_4$ and constants $\alpha_1=-3$, $\alpha_2=-1$, $\beta_1=-1$, $\beta_2=-3$. Thus a Darboux linearization for system (2.28) is given by

$$z_1=l_1l_3^{-3}l_4^{-1}, \qquad w_1=l_2l_3^{-1}l_4^{-3}.$$

Then system (2.28) is linearizable and therefore system (2.27) has an isochronous bi-center at the points $(-1,0)$ and $(1,0)$.

Case (2) : Under conditions (2) system (2.15) becomes

$$\dot{x} = 4y - 5x^4y + 48x^2y^3, \qquad \dot{y} = -\frac{x}{4} + \frac{x^5}{4} - 13x^3y^2 - 144xy^4. \tag{2.29}$$

As in the previous case, after apply the translation $u = x-1, v = y$, and the complexification $x = u+iv, y = u-iv$, we obtain the system

$$\begin{aligned}
\dot{z} =& z + \frac{1}{128}(-144w^2 - 584w^3 - 1758w^4 - 735w^5 - 672wz + 1320w^2z \\
&+ 5864w^3z + 1917w^4z + 1136z^2 - 2328wz^2 - 7476w^2z^2 - 902w^3z^2 \\
&+ 1912z^3 + 3432wz^3 - 1590w^2z^3 + 98z^4 + 1653wz^4 - 311z^5), \\
\dot{w} =& -w + \frac{1}{128}(-1136w^2 - 1912w^3 - 98w^4 + 311w^5 + 672wz + 2328w^2z \\
&- 3432w^3z - 1653w^4z + 144z^2 - 1320wz^2 + 7476w^2z^2 + 1590w^3z^2 \\
&+ 584z^3 - 5864wz^3 + 902w^2z^3 + 1758z^4 - 1917wz^4 + 735z^5).
\end{aligned} \tag{2.30}$$

System (2.30) has the Darboux factors

$$\begin{aligned}
l_1 =& z + \frac{1}{64}(-24w^2 - 16w^3 - 3w^4 + 48wz - 4w^3z + 72z^2 \\
&+ 48wz^2 + 6w^2z^2 + 32z^3 + 12wz^3 + 5z^4), \\
l_2 =& w + \frac{1}{64}(72w^2 + 32w^3 + 5w^4 + 48wz + 48w^2z + 12w^3z \\
&- 24z^2 + 6w^2z^2 - 16z^3 - 4wz^3 - 3z^4), \\
l_3 =& 1 + \frac{5}{2}x - \frac{3}{2}y, \\
l_4 =& 1 - \frac{3}{2}x + \frac{5}{2}y, \\
l_5 =& 1 + \frac{1}{16}(16 + 96w - 120w^2 - 168w^3 - 45w^4 + 96z + 528wz + 264w^2z \\
&+ 12w^3z - 120z^2 + 264wz^2 + 114w^2z^2 - 168z^3 + 12wz^3 - 45z^4),
\end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{1/2} l_4^{3/2} l_5^{-3/2}, \qquad w_1 = l_2 l_3^{3/2} l_4^{1/2} l_5^{-3/2}.$$

Then system (2.30) is linearizable and therefore system (2.29) has an isochronous bi-center at the points $(-1,0)$ and $(1,0)$.

Case (3) : Under conditions (3) system (2.15) becomes

$$\dot{x} = y - 2x^4y + 8x^2y^3, \qquad \dot{y} = -\frac{x}{4} + \frac{x^5}{4} - 4x^3y^2 - 4xy^4. \tag{2.31}$$

As the previous cases, after apply the translation and the complexification, we obtain the system

$$
\begin{aligned}
\dot{z} =& z + \frac{1}{128}(464z^2 + 552z^3 + 266z^4 + 41z^5 - 96zw - 264z^2w \\
& + 40z^3w + 61z^4w - 48w^2 + 120zw^2 - 324z^2w^2 - 102z^3w^2 \\
& - 88w^3 + 296zw^3 - 6z^2w^3 - 118w^4 + 77zw^4 - 39w^5), \\
\dot{w} =& - w + \frac{1}{128}(48z^2 + 88z^3 + 118z^4 + 39z^5 + 96zw - 120z^2w \\
& - 296z^3w - 77z^4w - 464w^2 + 264zw^2 + 324z^2w^2 + 6z^3w^2 \\
& - 552w^3 - 40zw^3 + 102z^2w^3 - 266w^4 - 61zw^4 - 41w^5).
\end{aligned}
\tag{2.32}
$$

We do not find an enough number of invariant curves to construct a Darboux linearization for system (2.32). Thus, this case remains open. We believe that the linearizability of such system can be proved using another method which is not considered in this thesis.

Case (4) : Under conditions (4) system (2.15) becomes

$$
\dot{x} = 4y - 5x^4y, \qquad \dot{y} = -\frac{x}{4} + \frac{x^5}{4} - 25x^3y^2. \tag{2.33}
$$

Applying the translation and the complexification we obtain the system

$$
\begin{aligned}
\dot{z} =& z + \frac{1}{128}(240w^2 + 760w^3 + 450w^4 + 81w^5 - 1440wz - 1560w^2z \\
& - 280w^3z + 45w^4z + 1520z^2 - 600wz^2 - 1140w^2z^2 - 230w^3z^2 \\
& + 1720z^3 + 360wz^3 - 150w^2z^3 + 770z^4 + 165wz^4 + 121z^5), \\
\dot{w} =& - w + \frac{1}{128}(-1520w^2 - 1720w^3 - 770w^4 - 121w^5 + 1440wz + 600w^2z \\
& - 360w^3z - 165w^4z - 240z^2 + 1560wz^2 + 1140w^2z^2 + 150w^3z^2 \\
& - 760z^3 + 280wz^3 + 230w^2z^3 - 450z^4 - 45wz^4 - 81z^5).
\end{aligned}
\tag{2.34}
$$

System (2.34) has the Darboux factors

$$
\begin{aligned}
l_1 =& z + \frac{1}{128}(80w^2 + 40w^3 + 10w^4 + w^5 + 160wz + 120w^2z + 40w^3z + 5w^4z + 80z^2 \\
& + 120wz^2 + 60w^2z^2 + 10w^3z^2 + 40z^3 + 40wz^3 + 10w^2z^3 + 10z^4 + 5wz^4 + z^5), \\
l_2 =& w + \frac{1}{128}(80w^2 + 40w^3 + 10w^4 + w^5 + 160wz + 120w^2z + 40w^3z + 5w^4z + 80z^2 \\
& + 120wz^2 + 60w^2z^2 + 10w^3z^2 + 40z^3 + 40wz^3 + 10w^2z^3 + 10z^4 + 5wz^4 + z^5), \\
l_3 =& 1 + (5 - 2\sqrt{5})z + \frac{1}{4}(5 - 2\sqrt{5})z^2 + (5 - 2\sqrt{5})w + \frac{1}{2}(5 - 2\sqrt{5})zw + \frac{1}{4}(5 - 2\sqrt{5})w^2, \\
l_4 =& 1 + (5 + 2\sqrt{5})z + \frac{1}{4}(5 + 2\sqrt{5})z^2 + (5 + 2\sqrt{5})w + \frac{1}{2}(5 + 2\sqrt{5})zw + \frac{1}{4}(5 + 2\sqrt{5})w^2,
\end{aligned}
$$

which allow to construct the Darboux linearization

$$
z_1 = l_1 l_3^{-5/4} l_4^{-5/4}, \qquad w_1 = l_2 l_3^{-5/4} l_4^{-5/4}.
$$

Then system (2.34) is linearizable and therefore system (2.33) has an isochronous bi-center at the points $(-1,0)$ and $(1,0)$.

Case (5) : Under conditions (5) system (2.15) becomes

$$\dot{x} = \frac{y}{4} - \frac{5}{4}x^4y + \frac{5}{2}x^2y^3 - \frac{y^5}{4}, \qquad \dot{y} = -\frac{x}{4} + \frac{x^5}{4} - \frac{5}{2}x^3y^2 + \frac{5}{4}xy^4. \tag{2.35}$$

Applying the translation and the complexification we obtain the system

$$\dot{z} = \frac{1}{4}z(1+z)(2+z)(2+2z+z^2), \qquad \dot{w} = -\frac{1}{4}w(1+w)(2+w)(2+2w+w^2). \tag{2.36}$$

System (2.36) has the Darboux factors

$$l_1 = z, \qquad l_2 = w, \qquad l_3 = 1+z, \qquad l_4 = 1+w,$$
$$l_5 = 1+z+\frac{z^2}{2}, \qquad l_6 = 1+w+\frac{w^2}{2}, \qquad l_7 = 1+\frac{z}{2}, \qquad l_8 = 1+\frac{w}{2},$$

which allow to construct the Darboux linearization

$$z_1 = \frac{z(2+z)(2+2z+z^2)}{4(1+z)^4}, \qquad w_1 = \frac{w(2+w)(2+2w+w^2)}{4(1+w)^4}.$$

Then system (2.36) is linearizable and therefore system (2.35) has an isochronous bi-center at the points $(-1,0)$ and $(1,0)$. □

Unfortunately, for the remaining proposed conditions

$$a_1 = 0; \quad a_3 = 0; \quad a_4 = 0; \quad a_5 = 0; \quad a_7 = 0; \quad a_8 = 0; \quad a_9 = 0; \quad a_{10} = 0,$$

we are not able to perform the computations for system (2.15) due to complexity of the ideal $\mathscr{L}_8$. However, looking in the space of parameters of system (2.15), i.e., $\mathbb{R}^{10}$, we see that each condition of Theorem 2.3.5 lies in the set $X = \{(a_1, a_2, ..., a_{10}); a_1 \neq 0 \text{ and } a_7 \neq 0\}$. So, if there exist another subfamily of system (2.15) with an isochronous center at the points $(1,0)$ and $(-1,0)$ its parameters should lies in the complement set of X, i.e, the set $Y = \{(a_1, ..., a_{10}); a_1 = 0 \text{ or } a_7 = 0\}$. It lead us to investigate subfamilies of (2.15) obtained doing two parameters equal to zero, where one of them is a_1 or a_7. In other words we consider systems of the form (2.15) with one of the conditions below

(1) $a_1 = a_3 = 0$;	(2) $a_1 = a_4 = 0$;	(3) $a_1 = a_5 = 0$;
(4) $a_1 = a_6 = 0$;	(5) $a_1 = a_7 = 0$;	(6) $a_1 = a_8 = 0$;
(7) $a_1 = a_9 = 0$;	(8) $a_1 = a_{10} = 0$;	(9) $a_7 = a_3 = 0$;
(10) $a_7 = a_4 = 0$;	(11) $a_7 = a_5 = 0$;	(12) $a_7 = a_6 = 0$;
(13) $a_7 = a_8 = 0$;	(14) $a_7 = a_9 = 0$;	(15) $a_7 = a_{10} = 0$.

We obtain the following result.

Theorem 2.3.6. *Any system of the form* (2.15) *satisfying one of conditions (1)-(15) above has non-isochronous center at the points* $(-1,0)$ *and* $(1,0)$.

Proof. When $a_1 = a_3 = 0$ we have computed using the routine `minAssGTZ` over a field of characteristic 32452843 the variety of the ideal $J_{1,3} = \langle \mathscr{L}_8, a_1, a_3 \rangle$. After lifting the result to the ring of polynomials with rational coefficients, we obtain that the variety of $J_{1,3}$ is equal to the variety of the ideal $\tilde{J}_{1,3} = \langle a_1, a_2, a_3, a_4, a_5, -5+4a_9, 25+4a_8^2, 5+2a_7, 2a_6+a_8, 10a_{10}+a_8 \rangle$. It is clear its variety is the empty set in $\mathbb{R}^8$. So there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_1 = a_3 = 0$.

When $a_1 = a_4 = 0$, using an analogue process we obtain the variety of the ideal $J_{1,4} = \langle \mathscr{L}_8, a_1, a_4 \rangle$ is equal to the variety of the ideal $\tilde{J}_{1,4} = \langle a_1, a_2, a_3, a_4, a_5, 1+16a_{10}^2, -5+4a_9, 10a_{10}+a_8, 5+2a_7, -5a_{10}+a_6 \rangle$. It is clear its variety is the empty set in $\mathbb{R}^8$. So there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_1 = a_4 = 0$.

When $a_1 = a_5 = 0$, we obtain that $J_{1,5} = \langle \mathscr{L}_8, a_1, a_5 \rangle = J_{1,4}$. So, the variety of $J_{1,5}$ is the empty set in $\mathbb{R}^8$. Thus there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_1 = a_5 = 0$.

When $a_1 = a_6 = 0$, $a_1 = a_7 = 0$, $a_1 = a_8 = 0$, $a_1 = a_9 = 0$ and $a_1 = a_{10} = 0$ we obtain that $J_{1,6} = J_{1,7} = J_{1,8} = J_{1,9} = J_{1,10} = \langle 1 \rangle$, where $J_{1,6} = \langle \mathscr{L}_8, a_1, a_6 \rangle$, $J_{1,7} = \langle \mathscr{L}_8, a_1, a_7 \rangle$, $J_{1,8} = \langle \mathscr{L}_8, a_1, a_8 \rangle$, $J_{1,9} = \langle \mathscr{L}_8, a_1, a_9 \rangle$, and $J_{1,10} = \langle \mathscr{L}_8, a_1, a_{10} \rangle$. Thus there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_1 = a_6 = 0$, $a_1 = a_7 = 0$, $a_1 = a_8 = 0$, $a_1 = a_9 = 0$, and $a_1 = a_{10} = 0$.

When $a_7 = a_3 = 0$, $a_7 = a_4 = 0$, $a_7 = a_8 = 0$, $a_7 = a_9 = 0$, and $a_7 = a_{10} = 0$ we obtain that the varieties of the ideals $J_{7,3} = \langle \mathscr{L}_8, a_7, a_3 \rangle$, $J_{7,4} = \langle \mathscr{L}_8, a_7, a_4 \rangle$, $J_{7,8} = \langle \mathscr{L}_8, a_7, a_8 \rangle$, $J_{7,9} = \langle \mathscr{L}_8, a_7, a_9 \rangle$, and $J_{7,10} = \langle \mathscr{L}_8, a_7, a_{10} \rangle$ are equal to the variety of the ideal $\tilde{J}_7 = \langle a_{10}, a_9, a_8, a_7, 1+4a_6^2, a_5, a_4, a_3, a_2+4a_6, 1+a_1 \rangle$. It is easy to see that its variety is the empty set in $\mathbb{R}^8$. Thus there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_7 = a_3 = 0$, $a_7 = a_4 = 0$, $a_7 = a_8 = 0$, $a_7 = a_9 = 0$, and $a_7 = a_{10} = 0$.

When $a_7 = a_5 = 0$, we obtain that the variety of the ideal $J_{7,5} = \langle \mathscr{L}_8, a_7, a_5 \rangle$, is equal to the union of the varieties of the ideals

$$
\begin{aligned}
J_{7,5}^1 =& \tilde{J}_7,\\
J_{7,5}^2 =& \langle 1+a_{10}^2, 1+a_9, -2a_{10}+a_8, a_7, -a_{10}+a_6, a_5, a_{10}+a_4, 1+a_3, a_{10}+a_2, 1+4a_1 \rangle,\\
J_{7,5}^3 =& \langle 729+a_{10}^2, 21+a_9, -2a_{10}+9a_8, a_7, -a_{10}+54a_6, a_5, -a_{10}+3a_4, 39+a_3,\\
& -a_{10}+27a_2, 19+4a_1 \rangle,\\
J_{7,5}^4 =& \langle 1+16a_{10}^2, -3+4a_9, 2a_{10}+a_8, a_7, -a_{10}+a_6, a_5, -2a_{10}+a_4, -2+a_3,\\
& 10a_{10}+a_2, 1+a_1 \rangle.
\end{aligned}
$$

It is easy to see that the varieties generated by the ideals above are the empty set in $\mathbb{R}^8$. Thus there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_7 = a_5 = 0$.

When $a_7 = a_6 = 0$, we obtain that $J_{7,6} = \langle \mathscr{L}_8, a_7, a_6 \rangle = \langle 1 \rangle$. Thus there is no real isochronous center for system (2.15) at the points $(-1,0)$ and $(1,0)$ when $a_7 = a_6 = 0$. □

After consider so many cases we statement two conjectures.

Conjecture 2.3.7. *If* $a_1 = 0$ *or* $a_7 = 0$ *system* (2.15) *does not have an isochronous center at the points* $(-1,0)$ *and* $(1,0)$.

From the studies carry out till now we believe that the next result can hold.

Conjecture 2.3.8. *System* (2.15) *has an isochronous bi-center at the points* $(1,0)$ *and* $(-1,0)$ *if and only if one of the 5 conditions of Theorem 2.3.5 holds.*

Remark 2.3.9. Note that in all conditions $(1)-(5)$ of Theorem 2.3.5 the coefficients with even index are equal to zero. Moreover, if $a_2 = a_4 = a_6 = a_8 = a_{10} = 0$, system (2.15) is reversible, since in this case $P(x,-y) = -P(x,y)$ and $Q(x,-y) = Q(x,y)$. So, it is natural to ask if the existence of isochronous bi-centers at the points $(-1,0)$ and $(1,0)$ for system (2.15) implies in the reversibility of such system. We are not able to answer such question till now.

2.3.3 Global dynamics

In this section we study the global dynamics of system (2.15) having an isochronous bi-center at the points $(1,0)$ and $(-1,0)$ in the Poincaré disk. We obtain an interesting structure in its dynamics.

Theorem 2.3.10. *The global phase portrait of system* (2.15) *under conditions (1)-(5) of Theorem 2.3.5 is topologically equivalent to one of the 3 phase portraits in Figure 5.*

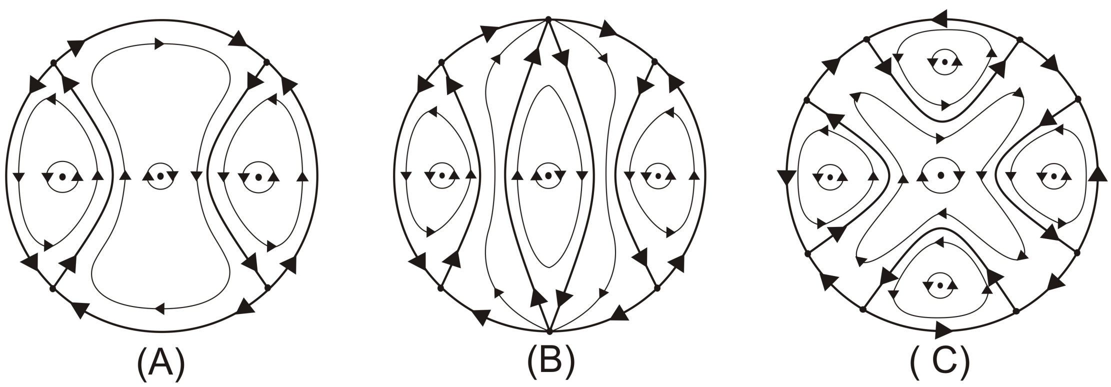

Figure 5 – Global phase portraits of system (2.15) under conditions (1)-(5) of Theorem 2.3.5.

Proof. By Theorem 2.3.5 system (2.15) with $a_2 = 0$ has an isochronous bi-center at the points $(1,0)$ and $(-1,0)$ if one of the 5 conditions holds. We note that for the 5 conditions the correspondent systems do not have free parameters. So, we investigate separately the phase portraits for the five cases. Thus our proof is split in these 5 cases. Firstly we remind that $(-1,0)$ and $(1,0)$ are isochronous centers for (2.15) for all five cases and, as mentioned before, the correspondent systems are reversible.

Case (1): For this case system (2.15) becomes system (2.27). It is easy to verify that only $(-1,0)$, $(0,0)$ and $(1,0)$ are finite singularities of (2.27). The linear part of (2.27) in $(0,0)$ has purely imaginary eigenvalues. So $(0,0)$ is a focus or a center. As (2.27) has time-reversible symmetry with respect to x-axis, the singular point $(0,0)$ must be a center.

Now we study the infinite singular points. In the local chart U_1 system (2.27) becomes

$$\dot{u} = -\frac{1}{4}(1+4u^2)(-1+16u^4+v^4), \qquad \dot{v} = -uv(-2-4u^2+16u^4+v^4). \tag{2.37}$$

This system has two infinite singular points in the equator line $v=0$, they are $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$. The linear part of (2.37) in $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$ are $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Thus, $(-\frac{1}{2},0)$ and $(-\frac{1}{2},0)$ are saddle points on the local chart U_1.

In the local chart U_2 we only need to study the origin. On this chart, system (2.27) becomes

$$\dot{u} = -\frac{1}{4}(4+u^2)(-16+u^4-v^4), \qquad \dot{v} = -\frac{1}{4}uv(-32-4u^2+u^4-v^4),$$

so $(0,0)$ is not a singular point.

Therefore, considering the information obtained from the finite singularities, infinity singularities, the symmetry of the system and the continuity of the orbits we obtain that the global phase portrait of system (2.27) is topologically equivalent to Figure 5.A.

Case (2): For this case system (2.15) becomes system (2.29). The finite singularities of (2.29) are $(-1,0)$, $(0,0)$ and $(1,0)$. The linear part of (2.29) in $(0,0)$ has purely imaginary eigenvalues. So $(0,0)$ is a focus or a center. As (2.29) has time-reversible symmetry with respect to x-axis, the singular point $(0,0)$ must be a center.

Now we analyse the infinite singular points. In the local chart U_1 system (2.29) becomes

$$\dot{u} = -\frac{1}{4}(1+16u^2)(-1+48u^2+v^4), \qquad \dot{v} = -uv(-5+48u^2+4v^4). \tag{2.38}$$

This system has two infinite singular points in the equator line $v=0$, they are $(-\frac{1}{4\sqrt{3}},0)$ and $(\frac{1}{4\sqrt{3}},0)$. The linear part of (2.38) in $(-\frac{1}{4\sqrt{3}},0)$ and $(\frac{1}{4\sqrt{3}},0)$ are $\begin{pmatrix} \frac{8}{\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{8}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$, respectively. Thus, $(-\frac{1}{4\sqrt{3}},0)$ and $(\frac{1}{4\sqrt{3}},0)$ are saddle points on the local chart U_1.

In the local chart U_2 system (2.29) becomes

$$\dot{u} = -\frac{1}{4}(16+u^2)(-48u^2+u^4-v^4), \qquad \dot{v} = -\frac{1}{4}uv((-576-52u^2+u^4-v^4). \tag{2.39}$$

The linear part of (2.39) in $(0,0)$ is the null matrix, i.e, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so we need to apply blow up to investigate the local behaviour of the orbits around this point. Applying the directional blow up in the u-axis, i.e, doing the change of coordinates $u = vz, v = v$, and a time rescaling $dT = vdt$, we obtain the system

$$\dot{z} = 4v^2 + 48z^2 - 5v^2z^4, \qquad \dot{v} = -\frac{1}{4}vz(-576 - v^4 - 52v^2z^2 + v^4z^4). \tag{2.40}$$

This system has only $(0,0)$ as singular point and the linear part of (2.40) in $(0,0)$ is again the null matrix. Applying the directional blow up in the z-axis, i.e, doing the change of coordinates $z = vw, v = v$, and a time rescaling $ds = vdT$, we obtain the system

$$\dot{w} = \frac{1}{4}(16 - 384w^2 - v^4w^2 - 72v^4w^4 + v^8w^6), \qquad \dot{v} = -\frac{1}{4}vzvw(-576 - v^4 - 52v^4w^2 + v^8w^4). \tag{2.41}$$

This system has the singular points $(-\frac{1}{2\sqrt{6}}, 0)$ and $(\frac{1}{2\sqrt{6}}, 0)$. The linear part of (2.41) in $(-\frac{1}{2\sqrt{6}}, 0)$ and $(\frac{1}{2\sqrt{6}}, 0)$ are $\begin{pmatrix} 16\sqrt{6} & 0 \\ 0 & -12\sqrt{6} \end{pmatrix}$ and $\begin{pmatrix} -16\sqrt{6} & 0 \\ 0 & 12\sqrt{6} \end{pmatrix}$, respectively. Thus, $(-\frac{1}{2\sqrt{6}}, 0)$ and $(\frac{1}{2\sqrt{6}}, 0)$ are saddle points, see Figure 6.a. The blow down process is described in Figure 6.

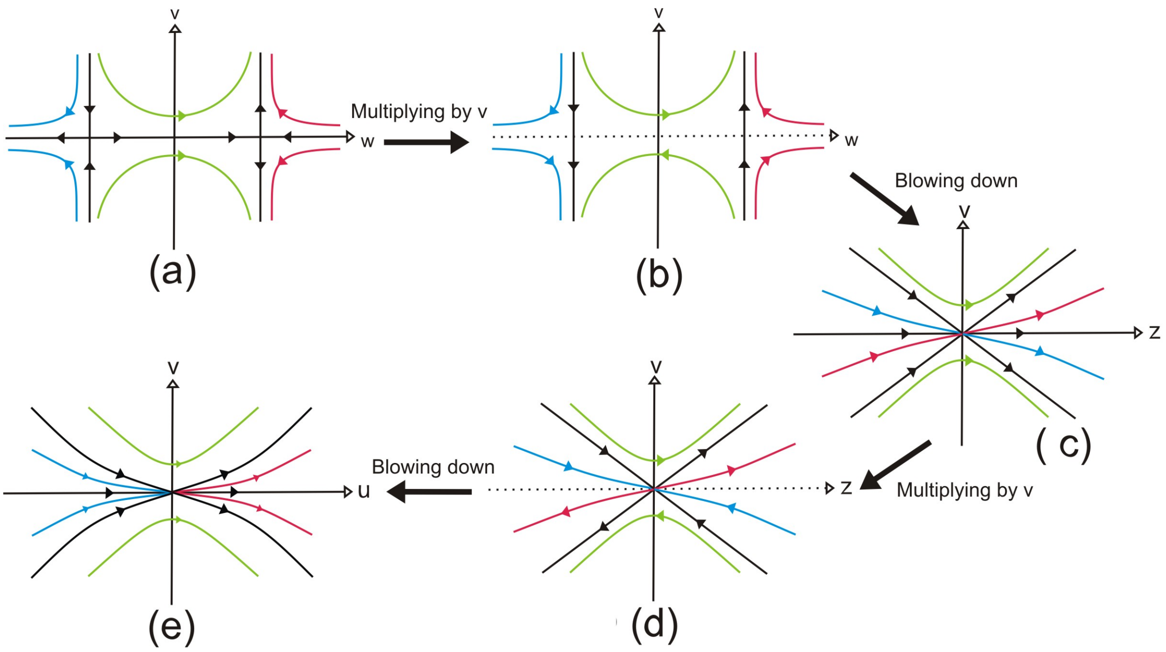

Figure 6 – Blowing down process relative to the origin of system (2.39).

To return from system (2.41) to system (2.40) we first multiply by v (undoing the time rescaling), it becomes w-axis a singularities axis, moreover the sense of all orbits in the half-plan when $v < 0$ are changed, see Figure 6.b. The next stage is the first blowing down, when the third and fourth quadrants of Figure 6.b are changed, see Figure 6.c. Note that the z-axis of system (2.40) is an invariant line. Next we multiply by v (undoing the time rescaling), so the sense of the orbits in the half-plan when $v < 0$ are changed again, and the z-axis becomes a singularities axis, see Figure 6.d. Finally, doing the last blowing down from system (2.40) to system (2.39), the third and fourth quadrants of Figure 6.d are changed. Thus the correct configuration of the orbits close to the origin of U_2 is according to Figure 6.e.

Therefore, considering the information obtained from the finite singularities, infinity singularities, the symmetry of the system and the continuity of the orbits we obtain that the global phase portrait of system (2.29) is topologically equivalent to Figure 5.B.

Case (3): For this case system (2.15) becomes system (2.31). The finite singularities of (2.31) are $(-1,0)$, $(0,0)$ and $(1,0)$. The linear part of (2.31) in $(0,0)$ has purely imaginary eigenvalues. So $(0,0)$ is a focus or a center. As (2.31) has time-reversible symmetry with respect to x-axis, the singular point $(0,0)$ must be a center.

Now we analyse the infinite singular points. In the local chart U_1 system (2.31) becomes

$$\dot{u} = -\frac{1}{4}(1+4u^2)(-1+12u^2+v^4), \qquad \dot{v} = -uv(-2+8u^2+v^4). \tag{2.42}$$

This system has two infinite singular points in the equator line $v=0$, they are $(-\frac{1}{2\sqrt{3}},0)$ and $(\frac{1}{2\sqrt{3}},0)$. The linear part of (2.42) in $(-\frac{1}{2\sqrt{3}},0)$ and $(\frac{1}{2\sqrt{3}},0)$ are $\begin{pmatrix} \frac{4}{\sqrt{3}} & 0 \\ 0 & -\frac{2}{3\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{4}{\sqrt{3}} & 0 \\ 0 & \frac{2}{3\sqrt{3}} \end{pmatrix}$, respectively. Thus, $(-\frac{1}{2\sqrt{3}},0)$ and $(\frac{1}{2\sqrt{3}},0)$ are saddle points on the local chart U_1.

In the local chart U_2 system (2.31) becomes

$$\dot{u} = -\frac{1}{4}(4+u^2)(-12u^2+u^4-v^4), \qquad \dot{v} = -\frac{1}{4}uv(-16-16u^2+u^4-v^4). \tag{2.43}$$

The linear part of (2.43) in $(0,0)$ is the null matrix. Applying the directional blow up in the v-axis, i.e, doing the change of coordinates $u=u, v=uz$, and a time rescaling $dT=udt$, we obtain the system

$$\dot{z} = \frac{1}{4}u(4+u^2)(12-u^2+u^2z^4), \qquad \dot{v} = -z(8-2u^2+u^2z^4). \tag{2.44}$$

This system has only $(0,0)$ as singular point, and the linear part of (2.44) in $(0,0)$ is $\begin{pmatrix} 12 & 0 \\ 0 & -8 \end{pmatrix}$. Thus, $(0,0)$ is a saddle point, see Figure 7.a. The blow down process is described in Figure 7.

To return from system (2.44) to system (2.43) we first multiply by u (undoing the time rescaling), it becomes z-axis a singularities axis, moreover the sense of all orbits in the half-plan when $u<0$ are changed, see Figure 7.b. The next stage is the blowing down, when the second and third quadrants of Figure 7.b are changed. Moreover we note that the vectors field of (2.43) in a point of the form $(0,v)$ is $(v^4,0)$. It shows that the correct configuration of the orbits close to the origin of U_2 is according to Figure 7.c.

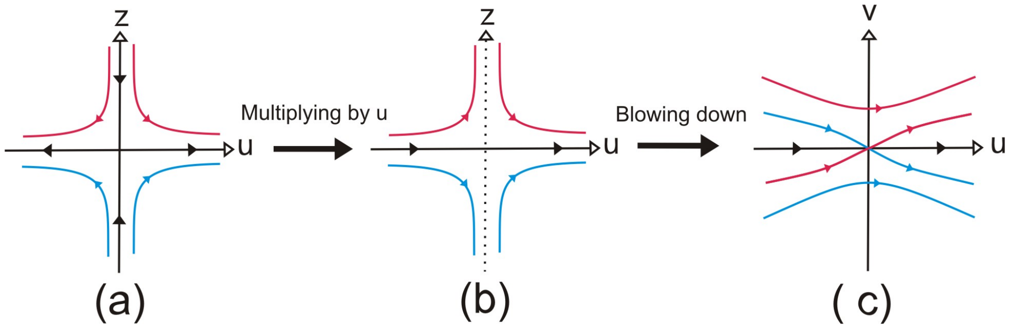

Figure 7 – Blowing down process relative to the origin of system (2.43).

Therefore, considering the information obtained from the finite singularities, infinity singularities, the symmetry of the system and the continuity of the orbits we obtain that the global phase portrait of system (2.31) is topologically equivalent to Figure 5.B.

Case (4): For this case system (2.15) becomes system (2.33). The finite singularities of (2.33) are $(-1,0)$, $(0,0)$ and $(1,0)$. The linear part of (2.33) in $(0,0)$ has purely imaginary eigenvalues. So $(0,0)$ is a focus or a center. As (2.33) has time-reversible symmetry with respect to x-axis, the singular point $(0,0)$ must be a center.

Now we analyse the infinite singular points. In the local chart U_1 system (2.33) becomes

$$\dot{u} = \frac{1}{4}(1-80u^2-v^4-16u^2v^4), \qquad \dot{v} = -uv(-5+4v^4). \tag{2.45}$$

This system has two infinite singular points in the equator line $v=0$, $(-\frac{1}{4\sqrt{5}},0)$ and $(\frac{1}{4\sqrt{5}},0)$. The linear part of (2.45) in $(-\frac{1}{4\sqrt{5}},0)$ and $(\frac{1}{4\sqrt{5}},0)$ are $\begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & -\frac{\sqrt{5}}{4} \end{pmatrix}$ and $\begin{pmatrix} -2\sqrt{5} & 0 \\ 0 & \frac{\sqrt{5}}{4} \end{pmatrix}$, respectively. Thus, $(-\frac{1}{4\sqrt{5}},0)$ and $(\frac{1}{4\sqrt{5}},0)$ are saddle points on the local chart U_1.

In the local chart U_2 system (2.33) becomes

$$\dot{u} = \frac{1}{4}(80u^4-u^6+16v^4+u^2v^4), \qquad \dot{v} = -\frac{1}{4}uv(-100u^2+u^4-v^4). \tag{2.46}$$

The linear part of (2.46) in $(0,0)$ is the null matrix. Applying the directional blow up in the v-axis, i.e, doing the change of coordinates $u=u, v=uz$, and a time rescaling $dT=u^3dt$, we obtain the system

$$\dot{z} = \frac{1}{4}u(80-u^2+16z^4+u^2z^4), \qquad \dot{v} = -z(-5+4z^4). \tag{2.47}$$

This system has $(0,-\frac{\sqrt[4]{5}}{\sqrt{2}},0)$, $(0,0)$ and $(0,\frac{\sqrt[4]{5}}{\sqrt{2}},0)$ as singular point, with respective linear parts of (2.44), $\begin{pmatrix} 25 & 0 \\ 0 & -20 \end{pmatrix}$, $\begin{pmatrix} 20 & 0 \\ 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 25 & 0 \\ 0 & -20 \end{pmatrix}$. Thus, $(0,-\frac{\sqrt[4]{5}}{\sqrt{2}},0)$ and $(0,\frac{\sqrt[4]{5}}{\sqrt{2}},0)$ are saddle points and $(0,0)$ is a unstable node, see Figure 8.a. The blow down process is described in Figure 8.

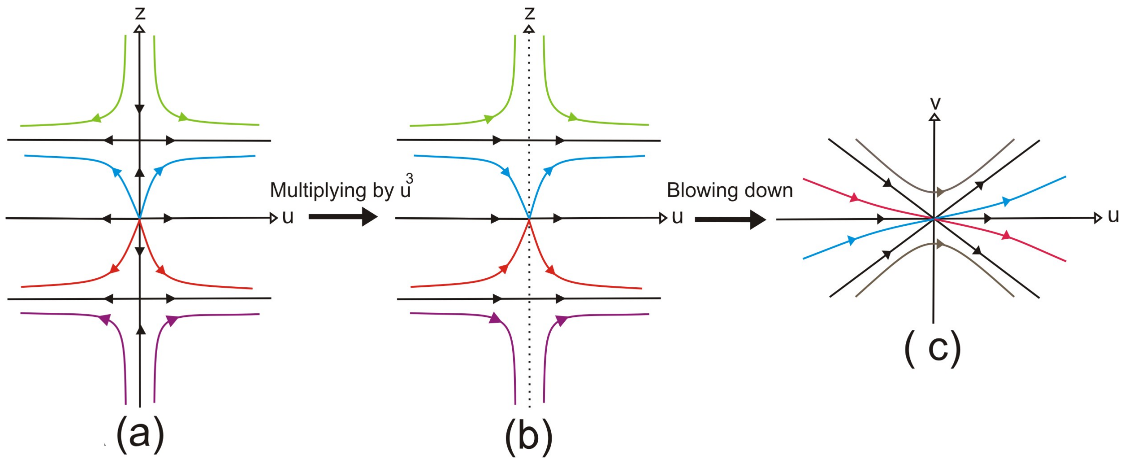

Figure 8 – Blowing down process relative to the origin of system (2.46).

To return from system (2.47) to system (2.46) we first multiply by u^2 (undoing the time rescaling), it becomes z-axis a singularities axis, see Figure 8.b. The next stage is the blowing down, when the second and third quadrants of Figure 8.b are changed. Moreover we note that the vectors field of (2.46) in a point of the form $(0,v)$ is $(4v^4,0)$. It shows that the correct configuration of the orbits close to the origin of U_2 is according to Figure 8.c.

Therefore, considering the information obtained from the finite singularities, infinity singularities, the symmetry of the system and the continuity of the orbits we obtain that the global phase portrait of system (2.33) is topologically equivalent to Figure 5.B.

Case (5): For this case system (2.15) becomes system (2.35). It is easy to verify that only $(-1,0)$, $(0,0)$, $(1,0)$, $(0,-1)$ and $(0,1)$ are finite singularities of (2.35). The linear part of (2.35) in $(0,0)$, $(0,-1)$ and $(0,1)$ has purely imaginary eigenvalues. So $(0,0)$, $(0,-1)$ and $(0,1)$ are foci or centers. As (2.35) has time-reversible symmetry with respect to y-axis, these three singular points must be a centers.

Now we study the infinite singular points. In the loca chart U_1 system (2.35) becomes

$$\dot{u} = \frac{1}{4}(1+u^2)(1-6u^2+u^4-v^4), \qquad \dot{v} = \frac{1}{4}uv(5-10u^2+u^4-v^4).$$

This system has four infinite singular points in the equator line $v=0$, $(-1-\sqrt{2},0)$, $(1-\sqrt{2},0)$, $(-1+\sqrt{2},0)$ and $(1+\sqrt{2},0)$ where their respective linear parts are

$$\begin{pmatrix} -8(3+2\sqrt{2}) & 0 \\ 0 & 2(3+2\sqrt{2}) \end{pmatrix}, \qquad \begin{pmatrix} -8(-3+2\sqrt{2}) & 0 \\ 0 & 2(-3+2\sqrt{2}) \end{pmatrix},$$

$$\begin{pmatrix} 8(-3+2\sqrt{2}) & 0 \\ 0 & -2(-3+2\sqrt{2}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 8(3+2\sqrt{2}) & 0 \\ 0 & -2(3+2\sqrt{2}) \end{pmatrix},$$

respectively. Thus, each point is a saddle point.

In the local chart U_2 system (2.27) becomes

$$\dot{u} = -\frac{1}{4}(1+u^2)(1-6u^2+u^4-v^4), \qquad \dot{v} = -\frac{1}{4}uv(5-10u^2+u^4-v^4),$$

so $(0,0)$ is not a singular point.

Therefore, considering the information obtained from the finite singularities, infinity singularities, the symmetry of the system and the continuity of the orbits we obtain that the global phase portrait of system (2.35) is topologically equivalent to Figure 5.C.

□

2.3.4 Coexistence of isochronous centers

The study of the global dynamics of system (2.15) under conditions for existence of isochronous centers shows the existence of more centers in all five cases. We investigate if such new centers are isochronous as the ones in $(1,0)$ and $(-1,0)$.

Theorem 2.3.11. *If system (2.15) under one of the five conditions in Theorem 2.3.5 has a center besides $(1,0)$ and $(-1,0)$, then this center is also isochronous.*

Proof. We split our proof in 5 cases corresponding to the cases of Theorem 2.3.5.

Case (1) : For this case system (2.15) becomes (2.27). In the proof of Theorem 2.3.10 we see that, besides $(1,0)$ and $(-1,0)$, only the origin is a center. This system is not in the standard form, so by a linear change of coordinates we can rewrite it in the standard form

$$\dot{x} = -y + \frac{x^4y}{8} + \frac{x^2y^3}{16} - \frac{y^5}{16}, \qquad \dot{y} = x - \frac{x^5}{16} + \frac{x^3y^2}{16} + \frac{xy^4}{8},$$

and its complexification is

$$\dot{z} = z - \frac{3z^4w}{64} - \frac{z^2w^3}{64}, \qquad \dot{w} = -w + \frac{z^3w^2}{64} + \frac{3zw^4}{64}. \tag{2.48}$$

System (2.48) has the Darboux factors

$$l_1 = z, \qquad l_2 = w, \qquad l_3 = 1 - \frac{z^3w}{16}, \qquad l_4 = 1 - \frac{z^3w}{16} - \frac{zw^3}{16},$$

which allow to construct the Darboux linearization

$$z_1 = z\left(1 - \frac{z^3w}{16}\right)^{-\frac{1}{2}}\left(1 - \frac{z^3w}{16} - \frac{zw^3}{16}\right)^{\frac{1}{8}}, \qquad w_1 = w\left(1 - \frac{z^3w}{16}\right)^{\frac{1}{2}}\left(1 - \frac{z^3w}{16} - \frac{zw^3}{16}\right)^{-\frac{3}{8}}.$$

Thus system (2.48) is linearizable and therefore the origin of system (2.27) is an isochronous center.

Case (2) : For this case system (2.15) becomes (2.29). In the proof of Theorem 2.3.10 we see that, besides $(1,0)$ and $(-1,0)$, only the origin is a center. By a linear change of coordinates system (2.29) is written in the standard form

$$\dot{x} = -y + \frac{5x^4y}{4} - \frac{3x^2y^3}{4}, \qquad \dot{y} = x - x^5 + \frac{13x^3y^2}{4} + \frac{9xy^4}{4},$$

and its complexification is

$$\dot{z}=z-\frac{z^5}{8}-\frac{9z^4w}{16}+\frac{z^2w^3}{16}-\frac{3zw^4}{8},\qquad \dot{w}=-w+\frac{3z^4w}{8}-\frac{z^3w^2}{16}+\frac{9zw^4}{16}+\frac{w^5}{8}. \tag{2.49}$$

System (2.49) has the Darboux factors

$$l_1=z,\qquad l_2=w,\qquad l_3=1-\frac{z^3w}{8}-\frac{3z^2w^2}{8}-\frac{3zw^3}{8}-\frac{w^4}{8},$$
$$l_4=1-\frac{9z^3w}{8}+\frac{9z^2w^2}{8}-\frac{9zw^3}{8},\qquad l_5=1-\frac{z^4}{8}-\frac{3z^3w}{8}-\frac{3z^2w^2}{8}-\frac{zw^3}{8},$$

which allow to construct the Darboux linearization

$$z_1=l_1l_3^{\frac{3}{4}}l_4^{-\frac{1}{4}}l_5^{-\frac{1}{4}},\qquad w_1=l_2l_3^{-\frac{1}{4}}l_4^{-\frac{1}{4}}l_5^{\frac{3}{4}}.$$

Thus system (2.49) is linearizable and therefore the origin of system (2.29) is an isochronous center.

Case (3) : For this case system (2.15) becomes (2.31). In the proof of Theorem 2.3.10 we see that, besides $(1,0)$ and $(-1,0)$, only the origin is a center. By a linear change of coordinates system (2.31) is written in the standard form

$$\dot{x}=-y+\frac{x^4y}{8}-\frac{x^2y^3}{8},\qquad \dot{y}=x-\frac{x^5}{16}+\frac{x^3y^2}{4}+\frac{xy^4}{16}, \tag{2.50}$$

and its complexification is

$$\begin{aligned}\dot{z}=&z-\frac{z^5}{64}-\frac{z^4w}{32}-\frac{zw^4}{64},\\ \dot{w}=&-w+\frac{z^4w}{64}+\frac{zw^4}{32}+\frac{w^5}{64}.\end{aligned} \tag{2.51}$$

Since the origin of (2.31) is a center, the origin of system (2.50) is a center as well. Thus its complexification, that is system (2.51), has a first integral Ψ of the form (1.27) (see [103] Theorem 3.3.1). Moreover, system (2.51) has the Darboux factors

$$l_1=1-\frac{z^2}{8}-\frac{zw}{8},\qquad l_2=1+\frac{z^2}{8}+\frac{zw}{8},$$
$$l_3=1-\frac{zw}{8}-\frac{w^2}{8},\qquad l_4=1+\frac{zw}{8}+\frac{w^2}{8},$$

which by Theorem 1.4.6 allow to construct the Darboux linearization

$$z_1=z^{\frac{1}{2}}w^{-\frac{1}{2}}\Psi^{\frac{1}{2}}f_1^{-\frac{1}{4}}f_2^{\frac{1}{4}}f_3^{-\frac{1}{4}}f_4^{\frac{1}{4}},\qquad w_1=z^{-\frac{1}{2}}w^{\frac{1}{2}}\Psi^{\frac{1}{2}}f_1^{\frac{1}{4}}f_2^{-\frac{1}{4}}f_3^{\frac{1}{4}}f_4^{-\frac{1}{4}}.$$

Hence, system (2.51) is linearizable and therefore the origin of system (2.31) is an isochronous center.

Case (4) : For this case system (2.15) becomes (2.33). In the proof of Theorem 2.3.10 we see that, besides $(1,0)$ and $(-1,0)$, only the origin is a center. By a linear change of coordinates system (2.33) is written in the standard form

$$\dot{x}=-y+320x^4y,\qquad \dot{y}=x-256x^5+1600x^3y^2,$$

and its complexification is

$$\begin{aligned} \dot{z} =& z - 68z^5 - 120z^4w + 40z^2w^3 - 60zw^4 - 48w^5, \\ \dot{w} =& -w + 48z^5 + 60z^4w - 40z^3w^2 + 120zw^4 + 68w^5. \end{aligned} \tag{2.52}$$

System (2.52) has the Darboux factors

$$\begin{aligned} l_1 &= z - 8z^5 - 40z^4w - 80z^3w^2 - 80z^2w^3 - 40zw^4 - 8w^5, \\ l_2 &= w - \left(8z^5 + 40z^4w + 80z^3w^2 + 80z^2w^3 + 40zw^4 + 8w^5\right), \\ l_3 &= -1 + 20z^4 + 80z^3w + 120z^2w^2 + 80zw^3 + 20w^4, \end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{-\frac{5}{4}}, \qquad w_1 = l_2 l_3^{-\frac{5}{4}}.$$

Thus system (2.52) is linearizable and therefore the origin of system (2.33) is an isochronous center.

Case (5) : For this case system (2.15) becomes (2.35). In the proof of Theorem 2.3.10 we see that, besides $(1,0)$ and $(-1,0)$, $(0,-1)$, $(0,0)$, and $(0,1)$ are centers. To study the origin we change system (2.35) (by linear change of coordinates) to the standard form

$$\dot{x} = -y + 5x^4y - 10x^2y^3 + y^5, \qquad \dot{y} = x - x^5 + 10x^3y^2 - 5xy^4. \tag{2.53}$$

The complexification of system (2.53) is

$$\dot{z} = z - z^5, \qquad \dot{w} = -w + w^5. \tag{2.54}$$

System (2.54) has the Darboux factors

$$\begin{array}{lllll} l_1 = z, & l_2 = 1+z, & l_3 = 1-z, & l_4 = 1+iz, & l_5 = 1-iz, \\ l_6 = w, & l_7 = 1+w, & l_8 = 1-w, & l_9 = 1+iw, & l_{10} = 1-iw, \end{array}$$

which allow to construct the Darboux linearization

$$z_1 = \frac{z}{(1-z^4)^{\frac{1}{4}}}, \qquad w_1 = \frac{w}{(1-w^4)^{\frac{1}{4}}}.$$

Thus system (2.54) is linearizable and therefore the origin of system (2.35) is an isochronous center.

To study the isochronicity of the centers in $(0,-1)$, and $(0,1)$ is enough to study the isochronicity of $(0,1)$, since system (2.35) symmetric with respect to x-axis. Moving the point $(0,1)$ to the origin, i.e, doing the change of coordinates $u = x$, $v = y - 1$, we obtain system

$$\begin{aligned} \dot{u} &= -v + \frac{1}{4}(10u^2 - 5u^4 + 30u^2v - 5u^4v - 10v^2 + 30u^2v^2 - 10v^3 + 10u^2v^3 - 5v^4 - v^5), \\ \dot{v} &= u + \frac{u}{4}(-10u^2 + u^4 + 20v - 20u^2v + 30v^2 - 10u^2v^2 + 20v^3 + 5v^4), \end{aligned}$$

and its complexification is

$$\dot{z} = z - \frac{5i}{2}z^2 - \frac{5}{2}z^3 + \frac{5i}{4}z^4 + \frac{z^5}{4}, \qquad \dot{w} = -w - \frac{5i}{2}w^2 + \frac{5}{2}w^3 + \frac{5i}{4}w^4 - \frac{w^5}{4}. \tag{2.55}$$

System (2.55) has the Darboux factors

$$\begin{array}{lllll} l_1 = z, & l_2 = 1 - iz, & l_3 = 1 - \frac{i}{2}z, & l_4 = 1 - \frac{1+i}{2}z, & l_5 = 1 + \frac{1-i}{2}z, \\ l_6 = w, & l_7 = 1 + iw, & l_8 = 1 + \frac{i}{2}w, & l_9 = 1 - \frac{1-i}{2}w, & l_{10} = 1 + \frac{1+i}{2}w, \end{array}$$

which allow to construct the Darboux linearization

$$z_1 = \frac{z(4 - 6iz - 4z^2 + iz^3)}{4(i+z)^4}, \qquad w_1 = \frac{w(4 + 6iw - 4w^2 - iw^3)}{4(-i+w)^4}.$$

Thus system (2.55) is linearizable and therefore the centers in $(0,-1)$, and $(0,1)$ of system (2.35) are isochronous.

□

Remark 2.3.12. In [98, Theorem 2, case 5] the authors present a Darboux linearization for the following system

$$\dot{x} = x - b_5x^5 - \frac{4b_1b_5x^4y}{b_2} - b_1xy^4, \qquad \dot{y} = -y + b_5x^4y + b_2xy^4 + b_1y^5. \tag{2.56}$$

System (2.51) in the proof of Theorem 2.3.11 is a subcase of (2.56). Thus, the linearizability of (2.51) should follows directly from the linearizability of (2.56). However there exists a misprint in the proof of the linearizability of such system. The invariant curve of (2.56) presented in [98] is not correct. The correct ones should be

$$l_1 = 1 - \frac{b_2^2x^2y^2}{4b_1} - b_2xy^3 - b_1y^4, \quad \text{and} \quad l_2 = 1 - b_5x^4 - \frac{4b_1b_5x^3y}{b_2} - \frac{4b_1^2b_5x^2y^2}{b_2^2},$$

which, by Theorem 1.4.6, allow to construct a Darboux linearization for (2.56).

To conclude, from the studies carried out till now everything indicates that the existence of an isochronous bi-center is connected with the existence of three or more isochronous centers for cubic and quintic $\mathbb{Z}_2$-equivariant systems. In Section 2.2 we present examples of cubic $\mathbb{Z}_2$-equivariant systems with 3 isochronous centers. As far as we know, the systems in the proof of Theorem 2.3.11 are the first examples of quintic system with 3 and 5 isochronous centers.

CHAPTER

3

ISOCHRONICITY AND LINEARIZABILITY OF A CUBIC SYSTEM

In this chapter we investigate the linearizability problem for a family of cubic complex planar systems with the property that its infinity is filled up of singular points. In Section 3.1 we recall the obtained results up to now on the investigation of system 3.1 and our motivation for the study. In Section 3.2 we present our main result in this line, Theorem 3.2.1, which gives conditions for linearizability of system (3.1). Section 3.3 is devoted to present the relation among Theorem 3.2.1 in this chapter and the results in [85] and [20]. Finally, in Section 3.4 we discuss the coexistence of isochronous centers in system (3.1). The results present in this chapter are published in Journal of Mathematical Analysis and Applications [49].

3.1 Motivation for the study

Consider the following family of planar cubic systems

$$\begin{aligned} \dot{x} &= -y + p_2(x,y) + xr_2(x,y) = P(x,y), \\ \dot{y} &= x + q_2(x,y) + yr_2(x,y) = Q(x,y), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} p_2 &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ q_2 &= b_{20}x^2 + b_{11}xy + b_{02}y^2, \\ r_2 &= r_{20}x^2 + r_{11}xy + r_{02}y^2. \end{aligned}$$

System (3.1) has been studied in [19, 20, 58, 85] for the case when all parameters are real. In [19] the authors have shown that real system (3.1) has a center at the origin if and only if in polar coordinates after some transformations it can be written in one of the four forms described in the paper. The same characterization was done in [20] for isochronous centers where five forms were presented. However from their results it is difficult to determine the conditions on parameters of polynomials p_2, q_2, r_2 for the existence of centers and isochronous centers. Conditions on parameters of p_2, q_2, r_2 for the existence of a center were obtained in [85] and latter on using another approach in [58]. In the paper [85] published in 1997 the authors obtained the necessary and sufficient conditions for existence of isochronous center of system (3.1) represented by four series of condition on coefficients of the system, however in the more recent paper [20] published in 1999 the authors gave five conditions for existence of isochronous center of system (3.1).

One of the aims of this investigation is to clarify the conditions for isochronicity of system (3.1). For this purpose we use an approach different from the ones of [85] and [20]. We consider system (3.1) with complex coefficients and find conditions for linearization of such system. Since in real systems the linearizability of the system is equivalent to its isochronicity, the groups of conditions we obtain contains all the conditions corresponding to isochronous systems of the form (3.1). We obtain five groups of conditions on the parameters of the system for linearizability of system (3.1) and we show that each linearizable systems is Darboux linearizable.

3.2 Linearizability conditions

In this section we obtain conditions for linearizability of system (3.1) with complex parameters.

Without loss of generality, we can assume $b_{02} = -b_{20}$ in system (3.1). Indeed, if $a_{02} + a_{20} \neq 0$, we apply the transformation $\tilde{x} = x + (b_{02} + b_{20})y/(a_{02} + a_{20})$, $\tilde{y} = y - (b_{02} + b_{20})x/(a_{02} + a_{20})$ to system (3.1) obtaining a system with such property. If $a_{02} + a_{20} = 0$, we only make the change $(x, y) \to (y, x)$ together with the time scaling $dt = -d\tau$ to obtain the same effect.

Theorem 3.2.1. *Complex system (3.1) with $b_{02} = -b_{20}$ is linearizable at the origin if one of the following conditions holds:*

(1) $4a_{20}^2 + a_{11}^2 + 4a_{11}b_{20} + 4b_{20}^2 - 4a_{20}b_{11} + b_{11}^2 = r_{20} + r_{02} = a_{02} + a_{20} = 0,$

(2) $a_{02} = r_{02} = a_{11} + 2b_{20} = b_{11} - 4a_{20} = r_{11} + b_{20}^2 = r_{20} - a_{20}b_{20} = 0,$

(3) $4a_{02} + a_{20} = a_{11} + 2b_{20} = 2b_{11} - a_{20} = 4r_{02} + a_{20}b_{20} = r_{11} + b_{20}^2 = r_{20} - a_{20}b_{20} = 0,$

(4) $a_{02} = r_{02} = a_{11} + 2b_{20} = b_{11} - a_{20} = r_{20} - a_{20}b_{20} = 0,$

(5) $9a_{11}^2 - 12a_{11}b_{20} + 4b_{20}^2 + 4b_{11}^2 = -6a_{11}b_{20} + 4b_{20}^2 + 2a_{20}b_{11} - b_{11}^2 = 6a_{20}a_{11} - 4a_{20}b_{20} - 3a_{11}b_{11} + 10b_{20}b_{11} = 4a_{20}^2 - 12a_{11}b_{20} + 24b_{20}^2 - b_{11}^2 = -\frac{4}{3}b_{20}^2 - \frac{b_{11}^2}{3} + r_{11} = \frac{4}{9}a_{20}b_{20} + \frac{a_{11}b_{11}}{6} - \frac{b_{20}b_{11}}{9} + r_{02} = \frac{a_{20}a_{11}}{6} - \frac{a_{20}b_{20}}{3} + \frac{a_{11}b_{11}}{12} - \frac{b_{20}b_{11}}{6} + r_{20} + r_{02} = a_{02} + \frac{a_{20}}{3} - \frac{b_{11}}{3} = 0.$

Proof. Following the method explained in Section 1.4 we computed the first eight pairs of the linearizability quantities for system (3.1). The first pair is

$$\begin{aligned}
i_1 =&\frac{1}{9}(10a_{02}^2+a_{11}^2+10a_{02}a_{20}+4a_{20}^2-a_{02}b_{11}-5a_{20}b_{11}+b_{11}^2+4a_{11}b_{20}+4b_{20}^2),\\
j_1 =&\frac{1}{3}(a_{02}a_{11}+a_{11}a_{20}-2a_{02}b_{20}-2a_{20}b_{20}+4r_{02}+4r_{20}),
\end{aligned}$$

and the second pair reduced by the Groebner basis of $\langle i_1, j_1\rangle$ is

$$\begin{aligned}
\tilde{i}_2 =&\frac{1}{750}(-10a_{11}^2a_{20}^2+200a02a_{20}^3+160a_{20}^4+10a_{11}^2a_{20}b_{11}-600a_{02}a_{20}^2b_{11}-520a_{20}^3b_{11}\\
&+6a_{11}^2b_{11}^2+490a_{02}a_{20}b_{11}^2+464a_{20}^2b_{11}^2-96a_{02}b_{11}^3-110a_{20}b_{11}^3+6b_{11}^4-170a_{11}^3b_{20}\\
&-720a_{11}a_{20}^2b_{20}+720a_{11}a_{20}b_{11}b_{20}-146a_{11}b_{11}^2b_{20}-550a_{11}^2b_{20}^2+2600a_{02}a_{20}b_{20}^2\\
&+3080a_{20}^2b_{20}^2-1580a_{02}b_{11}b_{20}^2-2060a_{20}b_{11}b_{20}^2+154b_{11}^2b_{20}^2-160a_{11}b_{20}^3+520b_{20}^4\\
&-100a_{11}a_{20}r_{02}+560a_{11}b_{11}r_{02}-6800a_{20}b_{20}r_{02}+2180b_{11}b_{20}r_{02}+5250r_{02}^2\\
&-55a_{11}^2r_{11}+50a_{02}a_{20}r_{11}-170a_{20}^2r_{11}+5a_{02}b_{11}r_{11}+225a_{20}b_{11}r_{11}-55b_{11}^2r_{11}\\
&-220a_{11}b_{20}r_{11}-220b_{20}^2r_{11}-100a_{11}a_{20}r_{20}+560a_{11}b_{11}r_{20}+800a_{02}b_{20}r_{20}\\
&-6000a_{20}b_{20}r_{20}+2180b_{11}b_{20}r_{20}+8500r_{02}r_{20}+3250r_{20}^2),\\
\tilde{j}_2 =&\frac{1}{120}(2a_{11}^3a_{20}+8a_{11}a_{20}^3-a_{11}^3b_{11}-12a_{11}a_{20}^2b_{11}+6a_{11}a_{20}b_{11}^2-a_{11}b_{11}^3-4a_{11}^2a_{20}b_{20}\\
&+48a_{02}a_{20}^2b_{20}-6a_{11}^2b_{11}b_{20}+16a_{02}a_{20}b_{11}b_{20}+56a_{20}^2b_{11}b_{20}-4a_{02}b_{11}^2b_{20}\\
&-8a_{20}b_{11}^2b_{20}-2b_{11}^3b_{20}-40a_{11}a_{20}b_{20}^2-12a_{11}b_{11}b_{20}^2-48a_{20}b_{20}^3-8b_{11}b_{20}^3\\
&-24a_{11}^2r_{02}-16a_{20}^2r_{02}+64a_{20}b_{11}r_{02}+4b_{11}^2r_{02}-64a_{11}b_{20}r_{02}-32b_{20}^2r_{02}\\
&+128a_{02}b_{20}r_{11}+128a_{20}b_{20}r_{11}-128r_{02}r_{11}-8a_{11}^2r_{20}-32a_{02}a_{20}r_{20}+16a_{20}^2r_{20}\\
&-80a_{02}b_{11}r_{20}-80a_{20}b_{11}r_{20}+20b_{11}^2r_{20}+32b_{20}^2r_{20}-128r_{11}r_{20}).
\end{aligned}$$

The other polynomials have very long expressions, so we do not present them here, however, the reader can easily compute them using any available computer algebra system.

To find conditions for linearizability we have to solve the system $i_1 = \cdots = i_8 = j_1 = \cdots = j_8 = 0$, or, more precisely, to find the irreducible decomposition of the variety $\mathbf{V}(\mathscr{L}_8)$ of the ideal $\mathscr{L}_8 = \langle i_1, j_1, ..., i_8, j_8\rangle$. We have performed the decomposition of the variety $\mathbf{V}(\mathscr{L}_8)$ using the routine `minAssGTZ` of SINGULAR, however we have not succeeded to complete computations neither over $\mathbb{Q}$ nor over the fields $\mathbb{Z}_{32003}$ and $\mathbb{Z}_{32452843}$.

To find the decomposition we proceed as follows. First, we use the four conditions for isochronicity of real system (3.1) obtained in [85], which are conditions (2), (3) and (4) of the statement of Theorem 3.2.1 and the condition

$$a_{02}+a_{20} = a_{11}+2b_{20} = b_{11}-2a_{20} = r_{02}+r_{20} = 0. \tag{3.2}$$

It is clear that under condition (3.2) and conditions (2), (3) and (4) of Theorem 3.2.1 complex system (3.1) is linearizable, since it is proven in [85] that under such conditions system (3.1) has an isochronous center at the origin.

Denote by J_1 the ideal

$$J_1 = \langle a_{02} + a_{20}, a_{11} + 2b_{20}, b_{11} - 2a_{20}, r_{02} + r_{20} \rangle,$$

and by J_2, J_3, J_4 ideals generated by polynomials of conditions (2), (3) and (4) of Theorem (3.2.1).

Using the ideals $J_1 - J_4$ is is possible find the decomposition of the variety $\mathbf{V}(\mathscr{L}_8)$. The idea is to subtract from $\mathbf{V}(\mathscr{L}_8)$ the components defined by the ideals $J_1 - J_4$ and then find the decomposition of the remaining variety. For this aim we use the Theorem 1.2.20, which says that given two ideals I and H in $k[x_1, ..., x_n]$,

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(H)} \subset \mathbf{V}(I : H),$$

where the overline indicates the Zariski closure. Moreover, if $k = \mathbb{C}$ and I is a radical ideal, then

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(H)} = \mathbf{V}(I : H).$$

Thus, to remove the components $\mathbf{V}(J_1), ..., \mathbf{V}(J_4)$ from $\mathbf{V}(\mathscr{L}_8)$, we compute over the field $\mathbb{Z}_{32003}$ with the `intersect` of SINGULAR the intersection $J = J_1 \cap J_2 \cap J_3 \cap J_4$ (clearly, $\mathbf{V}(J) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_3) \cup \mathbf{V}(J_4)$), then with the `radical` we compute $R = \sqrt{\mathscr{L}_8}$, then with `quotient` we compute the ideal $G = R : J$ and, finally, with `minAssGTZ` we compute the minimal associate primes of G, obtaining that $G = G_1 \cap G_2$, where

$$\begin{aligned}
G_1 =& \langle r_{20} + r_{02}, a_{02} + a_{20}a_{20}^2 + 8001a_{11}^2 + a_{11}b_{20} + b_{20}^2 - a_{20}b_{11} + 8001b_{11}^2 \rangle, \\
G_2 =& \langle a_{02} + 10668a_{20} - 10668b_{11}, a_{11}r_{02} - 10667b_{20}r_{02} + 14224a_{20}r_{11} - 14224b_{11}r_{11}, a_{20}r_{02} + \\
& 16000b_{11}r_{02} + 16001a_{11}r_{11} - b_{20}r_{11}, r_{20}^2 + 6r_{20}r_{02} + 9r_{02}^2 + r_{11}^2, b_{11}r_{20} + 3b_{11}r_{02} - \\
& 16000a_{11}r_{11} - b_{20}r_{11}, b_{20}r_{20} + 3b_{20}r_{02} - 16001a_{20}r_{11} - 8001b_{11}r_{11}, a_{11}r_{20} - 2b_{20}r_{02} - \\
& a_{20}r_{11} - 16001b_{11}r_{11}, a_{20}r_{20} - 15997b_{11}r_{02} + 8003a_{11}r_{11} - 16001b_{20}r_{11}, a_{11}b_{11} - \\
& 2b_{20}b_{11} + 4r_{20} + 6r_{02}, b_{20}^2 + 8001b_{11}^2 + 8000r_{11}, a_{11}b_{20} - 10668a_{20}b_{11} + 10668b_{11}^2 + \\
& 16001r_{11}, a_{20}b_{20} - 16001b_{20}b_{11} + 16000r_{20}, a_{11}^2 - 14224a_{20}b_{11} - 7111b_{11}^2 - 10668r_{11}, \\
& a_{20}a_{11} + b_{20}b_{11} + r_{20} + 3r_{02}, a_{20}^2 - a_{20}b_{11} + 8000b_{11}^2 + 3r_{11}, a_{20}b_{11}r_{11} - 16001b_{11}^2r_{11} - \\
& 9r_{20}r_{02} - 27r_{02}^2 - 6r_{11}^2, b_{20}b_{11}r_{02} - 8001b_{11}^2r_{11} + 8003r_{02}^2 + r_{11}^2, a_{20}b_{11}^2 - 16001b_{11}^3 - \\
& 18b_{20}r_{02} - 6a_{20}r_{11}, b_{11}^2r_{02}r_{11} + b_{20}b_{11}r_{11}^2 + 15997r_{20}r_{02}^2 + 15988r_{02}^3 - r_{20}r_{11}^2 + \\
& 15997r_{02}r_{11}^2, b_{11}^3r_{02} + b_{20}b_{11}^2r_{11} - 9b_{20}r_{02}^2 - 3b_{11}r_{02}r_{11} - 4b_{20}r_{11}^2 \rangle.
\end{aligned}$$

Since $8001 \equiv \frac{1}{4} \mod 32003$, lifting the ideal G_1 from the ring of polynomials over the field $\mathbb{Z}_{32003}$ to the ring of polynomials over the field $\mathbb{Q}$ we obtain polynomials given in condition (1) of Theorem 3.2.1. Similarly, lifting the ideal G_2 we obtain the ideal which we denote by J_5 Simple computations show that $\mathbf{V}(J_5)$ is the same set as the set given by conditions (5) of Theorem 3.2.1.

To check the correctness of the obtained conditions we use the procedure described in [101]. First, we compute the ideal $\tilde{J} = J_1 \cap J_2 \cap J_3 \cap J_4 \cap J_5$ and then we check that the Groebner base of each ideal $\langle \tilde{J}, 1 - wi_k \rangle$, $\langle \tilde{J}, 1 - wj_k \rangle$, for $k = 1, \ldots, 8$ and w being a new variable, computed over $\mathbb{Q}$ is $\{1\}$. By the Radical Membership Test (see Section 1.2) it means that

$$\mathbf{V}(\mathscr{L}_8) \subset \mathbf{V}(\tilde{J}).$$

To check the opposite inclusion it is sufficient to check that

$$\langle \mathscr{L}_8, 1 - wf \rangle = \langle 1 \rangle \tag{3.3}$$

for all polynomials f from a basis of $\tilde{J}$. Unfortunately, we were not able to perform the check over $\mathbb{Q}$, however we have checked that (3.3) holds over few fields of finite characteristic. It yields that (3.3) holds with high probability [8] [1].

We now prove that under each of conditions (1)-(5) of Theorem 3.2.1 system (3.1) is linearizable.

Case (1) : In this case $a_{11} = -2b_{20} \pm (2a_{20} - b_{11})i$. We consider only the case $a_{11} = -2b_{20} + (2a_{20} - b_{11})i$, since when $a_{11} = -2b_{20} - (2a_{20} - b_{11})i$ the consideration is analogous. In this case system (3.1) becomes

$$\begin{aligned}\dot{x} =& -y + a_{20}x^2 + (-2b_{20} + (2a_{20} - b_{11})i)xy - a_{20}y^2 + r_{20}x^3 + r_{11}x^2y - r_{20}xy^2,\\ \dot{y} =& x + b_{20}x^2 + b_{11}xy - b_{20}y^2 + r_{20}x^2y + r_{11}xy^2 - r_{20}y^3.\end{aligned} \tag{3.4}$$

By the change of coordinates $z = x + iy$, $w = x - iy$, we obtain from (3.4) the system

$$\begin{aligned}\dot{z} =& z - (ia_{20} - b_{20})z^2 - \frac{1}{4}(r_{11} + 2ir_{20})z^3 + \frac{1}{4}(r_{11} - 2ir_{20})zw^2,\\ \dot{w} =& -w + \frac{1}{2}(ib_{11} - 2ia_{20})z^2 - \frac{1}{2}(ib_{11} + 2b_{20})w^2 - \frac{1}{4}(r_{11} + 2ir_{20})z^2w + \frac{1}{4}(r_{11} - 2ir_{20})w^3.\end{aligned} \tag{3.5}$$

System (3.5) has Darboux factors

$$\begin{aligned}l_1 =& z,\\ l_3 =& 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} - 4\sqrt{2}\eta_-)z + \frac{1}{4}(ib_{11} + 2b_{20} - i\xi)w,\\ l_4 =& 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} + 4\sqrt{2}\eta_-)z + \frac{1}{4}(ib_{11} + 2b_{20} - i\xi)w,\\ l_5 =& 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} - 4\sqrt{2}\eta_+)z + \frac{1}{4}(ib_{11} + 2b_{20} + i\xi)w,\\ l_6 =& 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} + 4\sqrt{2}\eta_+)z + \frac{1}{4}(ib_{11} + 2b_{20} + i\xi)w,\end{aligned}$$

where $\xi = \sqrt{b_{11}^2 - 4ib_{11}b_{20} - 4b_{20}^2 - 4r_{11} + 8ir_{20}}$ and

$$\eta_{\pm} = \sqrt{-2a_{20}^2 + 2a_{20}b_{11} - b_{11}^2 - 8ia_{20}b_{20} + 2ib_{11}b_{20} + 2b_{20}^2 + 2r_{11} + 4ir_{20} \pm 2a_{20}\xi \mp b_{11}\xi}.$$

[1] For this reason we say in the statement of Theorem 3.2.1 that conditions (1)-(5) are only necessary, but not necessary and sufficient conditions for linearizability of system (3.1).

It is easy to verify that the first condition of (1.39) is satisfied for $f_0 = l_1$, $f_1 = l_4$, $f_2 = l_5$, $f_3 = l_6$, and

$$\begin{aligned}
\alpha_1 &= -\frac{b_{11} - 2ib_{20} + \xi}{2\xi},\\
\alpha_2 &= \frac{b_{11}\eta_+ - 2ib_{20}\eta_+ - b_{11}\eta_- + 2ib_{20}\eta_- - 2i\sqrt{2}a_{20}\xi + 2\sqrt{2}b_{20}\xi - \eta_+\xi - \eta_-\xi}{4\eta_+\xi},\\
\alpha_3 &= \frac{b_{11}(\eta_+ + \eta_-) + (2i\sqrt{2}a_{20} - \eta_+ + \eta_-)\xi - 2ib_{20}(\eta_+ + \eta_- - i\sqrt{2}\xi)}{4\eta_+\xi}.
\end{aligned}$$

Moreover, system (3.5) has the Darboux first integral

$$\Psi(z,w) = l_3^{s_1} l_4^{s_2} l_5^{s_3} l_6^{s_4} = 1 - \frac{i}{2\sqrt{2}}\eta_-\xi zw + o(||(z,w)||^3),$$

where $s_1 = 1,\ s_2 = -1,\ s_3 = -\frac{\eta_-}{\eta_+},\ s_4 = \frac{\eta_-}{\eta_+}$, $f_1 = l_3$, $f_2 = l_4$, $f_3 = l_5$, and $f_4 = l_6$.

Therefore, system (3.5) is linearizable by the substitution

$$z_1 = l_1 l_4^{\alpha_1} l_5^{\alpha_2} l_6^{\alpha_3}, \quad w_1 = \frac{2\sqrt{2}(\Psi(z,w) - 1)i}{\eta_-\xi z_1}.$$

Case (2) : In this case system (3.1) becomes

$$\begin{aligned}
\dot{x} &= -y + a_{20}x^2 + a_{20}b_{20}x^3 - 2b_{20}xy - b_{20}^2x^2y = (b_{20}x+1)(a_{20}x^2 - b_{20}xy - y),\\
\dot{y} &= x + b_{20}x^2 + 4a_{20}xy + a_{20}b_{20}x^2y - b_{20}y^2 - b_{20}^2xy^2,
\end{aligned} \tag{3.6}$$

and performing the change of coordinates $z = x+iy$, $w = x-iy$, we obtain the system

$$\begin{aligned}
\dot{z} &= z + \left(b_{20} - i\frac{5}{4}a_{20}\right)z^2 - \frac{i}{2}a_{20}zw + i\frac{3}{4}a_{20}w^2 + \left(\frac{b_{20}^2}{4} - \frac{i}{4}a_{20}b_{20}\right)z^3\\
&\quad - \frac{i}{2}a_{20}b_{20}z^2w - \left(\frac{b_{20}^2}{4} + \frac{i}{4}a_{20}b_{20}\right)zw^2,\\
\dot{w} &= -w + i\frac{3}{4}a_{20}z^2 - \frac{i}{2}a_{20}zw - \left(b_{20} + i\frac{5}{4}a_{20}\right)w^2 + \left(\frac{b_{20}^2}{4} - \frac{i}{4}a_{20}b_{20}\right)z^2w\\
&\quad - \frac{i}{2}a_{20}b_{20}zw^2 - \left(\frac{b_{20}^2}{4} + \frac{i}{4}a_{20}b_{20}\right)w^3.
\end{aligned} \tag{3.7}$$

System (3.7) has the Darboux factors

$$\begin{aligned}
l_1 &= z + \left(\frac{b_{20}}{2} + \frac{i}{4}a_{20}\right)z^2 + \left(\frac{b_{20}}{2} + \frac{i}{2}a_{20}\right)zw + \frac{i}{4}a_{20}w^2,\\
l_2 &= w - \frac{i}{4}a_{20}z^2 + \left(\frac{b_{20}}{2} - \frac{i}{2}a_{20}\right)zw + \left(\frac{b_{20}}{2} - \frac{i}{4}a_{20}\right)w^2,\\
l_3 &= 1 + \frac{b_{20}}{2}z + \frac{b_{20}}{2}w,\\
l_4 &= 1 - \frac{i}{2}(4a_{20} + ib_{20})z + \frac{1}{2}(b_{20} + 4ia_{20})w,
\end{aligned}$$

which for $a_{20} \neq 0$ allow us to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where

$$\begin{aligned}
\alpha_1 &= -\frac{6a_{20} - ib_{20}}{4a_{20}}, \quad \alpha_2 = -\frac{2a_{20} + ib_{20}}{4a_{20}}, \\
\beta_1 &= -\frac{6a_{20} + ib_{20}}{4a_{20}}, \quad \beta_2 = -\frac{2a_{20} - ib_{20}}{4a_{20}}.
\end{aligned}$$

Since the set of linearizable systems is an affine variety, it is a closed set in the Zariski topology. Thus, system (3.7) is also linearizable when $a_{20} = 0$.

Case (3) : In this case system (3.1) becomes

$$\begin{aligned}
\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy - \frac{a_{20}}{4}y^2 + x\left(a_{20}b_{20}x^2 - b_{20}^2 xy - \frac{a_{20}b_{20}}{4}y^2\right), \\
\dot{y} &= x + b_{20}x^2 + \frac{a_{20}}{2}xy - b_{20}y^2 + y\left(a_{20}b_{20}x^2 - b_{20}^2 xy - \frac{a_{20}b_{20}}{4}y^2\right).
\end{aligned} \tag{3.8}$$

Performing the substitution $z = x + iy$, $w = x - iy$ we obtain

$$\begin{aligned}
\dot{z} =& z + \left(b_{20} - i\frac{7}{16}a_{20}\right)z^2 - i\frac{3}{8}a_{20}zw - i\frac{3}{16}a_{20}w^2 + \left(\frac{b_{20}^2}{4} - i\frac{5}{16}a_{20}b_{20}\right)z^3 \\
&- i\frac{3}{8}a_{20}b_{20}z^2 w - \left(\frac{b_{20}^2}{4} + i\frac{5}{16}a_{20}b_{20}\right)zw^2, \\
\dot{w} =& -w - i\frac{3}{16}a_{20}z^2 - i\frac{3}{8}a_{20}zw - \left(b_{20} + i\frac{7}{16}a_{20}\right)w^2 + \left(\frac{b_{20}^2}{4} - i\frac{5}{16}a_{20}b_{20}\right)z^2 w \\
&- i\frac{3}{8}a_{20}b_{20}zw^2 - \left(\frac{b_{20}^2}{4} + i\frac{5}{16}a_{20}b_{20}\right)w^3.
\end{aligned} \tag{3.9}$$

System (3.9) has the following Darboux factors

$$\begin{aligned}
l_1 =& z + \left(\frac{b_{20}}{2} - \frac{i}{16}a_{20}\right)z^2 + \left(\frac{b_{20}}{2} + \frac{i}{8}a_{20}\right)zw - \frac{i}{16}a_{20}w^2, \\
l_2 =& w + \frac{i}{16}a_{20}z^2 + \left(\frac{b_{20}}{2} - \frac{i}{8}a_{20}\right)zw + \left(\frac{b_{20}}{2} + \frac{i}{16}a_{20}\right)w^2, \\
l_3 =& 1 + \frac{b_{20}}{2}z + \frac{b_{20}}{2}w, \\
l_4 =& 1 - \frac{i}{4}(a_{20} + i2b_{20})z + \frac{i}{4}(a_{20} - i2b_{20})w,
\end{aligned}$$

which for $a_{20} \neq 0$ allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where

$$\begin{aligned}
\alpha_1 &= \frac{i2b_{20}}{a_{20}}, \quad \alpha_2 = -\frac{2a_{20} + i2b_{20}}{a_{20}}, \\
\beta_1 &= -\frac{i2b_{20}}{a_{20}}, \quad \beta_2 = -\frac{2a_{20} - i2b_{20}}{a_{20}}.
\end{aligned}$$

If $a_{20}=0$, case (3) is equivalent to case (2). Thus system (3.9) is linearizable.

Case (4) : In this case system (3.1) becomes

$$\begin{aligned}\dot{x} =& -y+a_{20}x^2-2b_{20}xy+a_{20}b_{20}x^3+r_{11}x^2y,\\ \dot{y} =& x+b_{20}x^2+a_{20}xy-b_{20}y^2+a_{20}b_{20}x^2y+r_{11}xy^2.\end{aligned} \tag{3.10}$$

By doing the change of coordinates $z=x+iy$, $w=x-iy$ we obtain from (3.10) the system

$$\begin{aligned}\dot{z} =& z+(b_{20}-i/2a_{20})z^2-\frac{i}{2}a_{20}zw-\left(\frac{r_{11}}{4}+\frac{i}{4}a_{20}b_{20}\right)z^3-\frac{i}{2}a_{20}b_{20}z^2w\\ &+\left(\frac{r_{11}}{4}-\frac{i}{4}a_{20}b_{20}\right)zw^2,\\ \dot{w} =& -w-\frac{i}{2}a_{20}zw-\left(b_{20}+\frac{i}{2}a_{20}\right)w^2-\left(\frac{r_{11}}{4}+\frac{i}{4}a_{20}b_{20}\right)z^2w-\frac{i}{2}a_{20}b_{20}zw^2\\ &+\left(\frac{r_{11}}{4}-\frac{i}{4}a_{20}b_{20}\right)w^3,\end{aligned}$$

which admits the Darboux factors

$$\begin{aligned}l_1 =& z,\\ l_2 =& w,\\ l_3 =& 1+\frac{1}{4}(-ia_{20}+2b_{20}+iC)z-\frac{i}{4}(-a_{20}+i2b_{20}+C)w,\\ l_4 =& 1-\frac{i}{2}(a_{20}+i2b_{20})z+\frac{i}{2}(a_{20}-i2ib_{20})w-\frac{i}{4}(a_{20}b_{20}-ir_{11})z^2\\ &+\frac{1}{2}(2b_{20}^2+r_{11})zw+\frac{i}{4}(a_{20}b_{20}+ir_{11})w^2,\end{aligned}$$

where $C=\sqrt{a_{20}^2-4b_{20}^2-4r_{11}}$. When $C\neq 0$ we obtain the Darboux linearization

$$z_1=l_1l_3^{\alpha_1}l_4^{\alpha_2},\quad w_1=l_2l_3^{\beta_1}l_4^{\beta_2},$$

where

$$\begin{aligned}\alpha_1 =& \frac{a_{20}+i2b_{20}}{C},\quad \alpha_2=-\frac{a_{20}+i2b_{20}+C}{2C},\\ \beta_1 =& \frac{a_{20}-i2b_{20}}{C},\quad \beta_2=-\frac{a_{20}-i2b_{20}+C}{2C}.\end{aligned}$$

Using the same argument as in case (2) we conclude that the system is linearizable also when $C=0$.

Case (5) : If $b_{20}\neq 0$, we can rewrite the condition of this case as

$$\begin{aligned}r_{11} =& 3a_{02}^2+2a_{20}a_{02}+\frac{a_{20}^2}{3}+\frac{4b_{20}^2}{3},\quad r_{02}=\frac{27a_{02}^3+9a_{02}^2a_{20}-3a_{02}a_{20}^2-a_{20}^3-16a_{20}b_{20}^2}{36b_{20}},\\ a_{11} =& -\frac{9a_{02}^2-a_{20}^2-4b_{20}^2}{6b_{20}},\ r_{20}=a_{02}b_{20}+a_{20}b_{20},\ b_{11}=a_{20}+3a_{02},\ a_{20}=3a_{02}\pm 4b_{20}i.\end{aligned}$$

We only consider the case $a_{20} = 3a_{02} + 4b_{20}i$, since when $a_{20} = 3a_{02} - 4b_{20}i$, the consideration is analogous. Under such condition after performing the substitution $z = x + iy$, $w = x - iy$ system (3.1) becomes

$$\begin{aligned}\dot{z} =& z + (3b_{20} - 3ia_{02})z^2 + (2b_{20} - 2ia_{02})zw + 2ia_{02}w^2 + (2b_{20}^2 - 2a_{02}^2 - 4ia_{02}b_{20})z^3\\ &- (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)z^2w + (4a_{02}^2 + 4ia_{02}b_{20})zw^2,\\ \dot{w} =& - w\Big(1 + (2ia_{02} - 2b_{20})z + (ia_{02} - b_{20})w + (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)z^2\\ &+ (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)zw - (4a_{02}^2 + 4ia_{02}b_{20})w^2\Big).\end{aligned} \tag{3.11}$$

System (3.11) has the Darboux factors

$$\begin{aligned}&l_1 = z - i(a_{02} + ib_{20})z^2 + \frac{2ia_{02}}{3}w^2,\\ &l_2 = w,\\ &l_3 = 1 - 2i(a_{02} + ib_{20})z + i(a_{02} + ib_{20})w,\\ &l_4 = 1 - 4i(a_{02} + ib_{20})z - 4(a_{02} + ib_{20})^2z^2 + 2i(a_{02} + ib_{20})w + 4(a_{02} + ib_{20})^2zw\\ &\qquad + (-a_{02}^2 - 2ia_{02}b_{20} + b_{20}^2)w^2,\end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1l_4^{-1}, \qquad w_1 = l_2l_4^{-\frac{1}{2}}.$$

Similarly as above, using the Zariski closure argument we conclude that the system is linearizable also when $b_{20} = 0$. □

3.3 Relation between isochronicity conditions

In [20] the authors presented conditions for isochronicity of systems of the form (3.1) when all parameters of such system are real. Here we investigate the relation among these conditions and the conditions obtained in Theorem 3.2.1 and in [85]. The following result can be found in [20].

Theorem 3.3.1 (Theorem 1 of [20])**.** *The origin of system* (3.1) *is an isochronous center if and only if system* (3.1) *has one of the following expressions in polar coordinates:*

(a) $\dot{r} = r^2(\cos 3\theta - \frac{7}{3}\cos\theta - k_1\sin\theta) + r^3(-\frac{2k_1}{3} - \frac{2k_1}{3}\cos 2\theta - \frac{k_1^2}{2}\sin 2\theta)$,
$\dot{\theta} = 1 + r(-\sin 3\theta + k_1\cos\theta - \sin\theta)$,

(b) $\dot{r} = r^2(\cos 3\theta + \frac{13}{3}\cos\theta - k_1\sin\theta) + r^3(2k_1 + \frac{10k_1}{3}\cos 2\theta - \frac{k_1^2}{2}\sin 2\theta)$,
$\dot{\theta} = 1 + r(-\sin 3\theta + k_1\cos\theta + \frac{1}{3}\sin\theta)$,

(c) $\dot{r} = r^2k_1\cos\theta + r^3(k_2\cos 2\theta + k_3\sin 2\theta)$, $\quad\dot{\theta} = 1 + rk_1\sin\theta$,

(d) $\dot{r} = r^2(k_1\cos\theta + k_2\sin\theta) + r^3(\frac{k_1k_2}{2} - \frac{k_1k_2}{2}\cos 2\theta + k_3\sin 2\theta)$,
$\dot{\theta} = 1 + rk_1\sin\theta$ *and*

(e) $\dot{r} = r^2(k_1\cos\theta + k_2\sin\theta) + r^3(k_3 + k_4\cos 2\theta + k_5\sin 2\theta), \quad \dot{\theta} = 1,$

where the k_j's in each system are independent and they are functions in the parameters of system (3.1).

We remark that in [20], the second equation of (c) is written as $\dot{\theta} = 1 + rk_1\cos\theta$, however it is a misprint which was corrected in [22].

As it is mentioned in the previous section, from [85] each real system of the form (3.1) is linearizable (equivalently, it has isochronous center at the origin) if and only if condition (3.2) or one of conditions (2), (3) and (4) of Theorem 3.2.1 hold. The next theorem presents the relation between the results obtaned in [85] (and Theorem 3.2.1) and [20].

Theorem 3.3.2. *System* (3.1) *under conditions* (3.2)*,* (2)*,* (3) *and* (4) *of Theorem 3.2.1 can be changed into system (c), (a), (b) and (d) of Theorem 3.3.1, respectively.*

Proof. System (3.1) under condition (3.2) becomes

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy - a_{20}y^2 + x(r_{20}x^2 + r_{11}xy - r_{20}y^2) = P_1(x,y),\\ \dot{y} &= x + b_{20}x^2 + 2a_{20}xy - b_{20}y^2 + y(r_{20}x^2 + r_{11}xy - r_{20}y^2) = Q_1(x,y).\end{aligned} \tag{3.12}$$

Applying the linear transformation

$$x = -a_{20}\tilde{x} + b_{20}\tilde{y}, \quad y = b_{20}\tilde{x} + a_{20}\tilde{y}$$

and a time scaling $dt = -d\tilde{t}$, we change system (3.12) to

$$\begin{aligned}\dot{x} &= -y + k_1(x^2 - y^2) + x\left(k_2x^2 + 2k_3xy - k_2y^2\right),\\ \dot{y} &= x + 2k_1xy + y\left(k_2x^2 + 2k_3xy - k_2y^2\right),\end{aligned} \tag{3.13}$$

where $k_1 = a_{20}^2 + b_{20}^2$, $k_2 = a_{20}b_{20}r_{11} - a_{20}^2r_{20} + b_{20}^2r_{20}$, $k_3 = (a_{20}^2r_{11} - b_{20}^2r_{11} + 4a_{20}b_{20}r_{20})/2$, and below we write x and y instead of $\tilde{x}$ and $\tilde{y}$. Note that system (3.13) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (c).

System (3.1) under condition (2) of Theorem 3.2.1 becomes system (3.6). The transformation $x = \frac{4}{3a_{20}}\tilde{x}, \;\; y = -\frac{4}{3a_{20}}\tilde{y}$ and the time scaling $dt = -d\tilde{t}$ take system (3.6) to

$$\begin{aligned}\dot{x} &= -y - \frac{4}{3}x^2 - 2k_1xy - \frac{x}{3}\left(4k_1x^2 + 3k_1^2xy\right),\\ \dot{y} &= x + k_1x^2 - \frac{16}{3}xy - k_1y^2 - \frac{y}{3}\left(4k_1x^2 + 3k_1^2xy\right),\end{aligned} \tag{3.14}$$

where we write x and y instead of $\tilde{x}$ and $\tilde{y}$, and $k_1 = \frac{4b_{20}}{3a_{20}}$. System (3.14) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (a).

System (3.1) under condition (3) becomes system (3.8). Applying the transformation $x = \frac{16}{3a_{20}}\tilde{x}, \;\; y = \frac{16}{3a_{20}}\tilde{y}$, we transform (3.8) to the system

$$\begin{aligned} \dot{x} =& -y - \frac{16}{3}x^2 - 2k_1xy - \frac{4}{3}y^2 + \frac{k_1}{3}x\left(16x^2 - 3k_1xy - 4y^2\right), \\ \dot{y} =& x + k_1x^2 + \frac{8}{3}xy - k_1y^2 + \frac{k_1}{3}y\left(16x^2 - 3k_1xy - 4y^2\right), \end{aligned} \tag{3.15}$$

where we write x and y instead of $\tilde{x}$ and $\tilde{y}$, and $k_1 = \frac{16b_{20}}{3a_{20}}$. System (3.15) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (b).

System (3.1) under condition (4) becomes system (3.10). The transformation $x = \tilde{y}$, $y = \tilde{x}$ and a time scaling $dt = -d\tilde{t}$ change system (3.10) to

$$\begin{aligned} \dot{x} =& -y + k_1x^2 + k_2xy - k_1y^2 + x\left(2k_3xy + k_1k_2y^2\right), \\ \dot{y} =& x + 2k_1xy + k_2y^2 + y\left(2k_3xy + k_1k_2y^2\right), \end{aligned} \tag{3.16}$$

where $k_1 = b_{20}$, $k_2 = -a_{20}$, $k_3 = -\frac{r_{11}}{2}$, and we write x and y instead of $\tilde{x}$ and $\tilde{y}$. System (3.16) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (d). □

However system (e) from Theorem 3.3.1 does not have an isochronous center at the origin, since, generally speaking, the origin of the system is not a center, but a focus. Indeed, system (e) in Cartesian coordinates takes the form

$$\begin{aligned} \dot{x} &= -y + k_1x^2 + k_2xy + x\left((k_3 + k_4)x^2 + (k_3 - k_4)y^2 + 2k_5xy\right), \\ \dot{y} &= x + k_1xy + k_2y^2 + y\left((k_3 + k_4)x^2 + (k_3 - k_4)y^2 + 2k_5xy\right). \end{aligned} \tag{3.17}$$

We compute the first two focus quantities for system (3.17) using the method explained in Section 1.3. They are $\upsilon_1 = k_3$ and $\upsilon_2 = 2k_1k_2k_5 + k_4(k_1^2 - k_2^2)$. Thus, the origin of system (e) is a focus, which is stable if $k_3 < 0$ or $k_3 = 0$ and $\upsilon_2 < 0$, and unstable if $k_3 > 0$ or $k_3 = 0$ and $\upsilon_2 > 0$. So, the necessary condition for existence of a center and a isochronous center at the origin of system (e) is $k_3 = \upsilon_2 = 0$.

When $k_3 = \upsilon_2 = 0$, by the linear transformation $x_1 = x + \frac{k_2}{k_1}y$, $y_1 = y - \frac{k_2}{k_1}x$, system (3.17) is taken into

$$\begin{aligned} \dot{x} &= -y + k_1x^2 - \tfrac{k_1k_4}{k_2}x^2y, \\ \dot{y} &= x + k_1xy - \tfrac{k_1k_4}{k_2}xy^2. \end{aligned} \tag{3.18}$$

System (3.18) is a special case of system (3.10) when $b_{20} = 0$, which is system (3.1) under condition (4) of Theorem 3.2.1 with $b_{20} = 0$. Therefore, only when $k_3 = \upsilon_2 = 0$, the origin of system (3.17), and thus of system (e), is an isochronous center.

It appears that in [20] the authors made the following mistake in their reasoning. They obtained system (e) from the condition of vanishing of two focus constants (computed using an approach explained in [20], Section 3). Then observing that the second equation of such system is $\dot{\theta} = 1$, they concluded that the system has an isochronous center at the origin. However, as we have shown, unless $k_3 = \upsilon_2 = 0$, the origin of the system is an isochronous focus (see e.g. [3, 52] for definitions) but not a center.

We note that the conditions for isochronicity of system (3.1) are also given in the survey paper [22]. According to Theorem 14.2 of [22] system (3.1) has an isochronous center at the origin if and only if by a change of coordinates and rescaling of time it can be brought to one of systems (a), (b), (d) of Theorem 3.3.1 or to one of systems

$$\begin{aligned} \dot{r} &= r^2\cos\theta + r^3(k_2\cos 2\theta + k_3\sin 2\theta), \\ \dot{\theta} &= 1 + r\sin\theta \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \dot{r} &= r^3\cos 2\theta, \\ \dot{\theta} &= 1. \end{aligned} \tag{3.20}$$

However instead of (3.19) and (3.20) we can use just system (c) of the statement of Theorem 3.3.1, since system (3.19) is a particular case of system (c) if we set in (c) $k_1 = 1$ and system (3.20) is a particular case of system (c) if we set in (c) $k_1 = k_3 = 0$, $k_2 = 1$.

To summarize, in [85] the authors presented four necessary and sufficient conditions for isochronicity of the center at the origin of system (3.1), in the case when all parameters of system (3.1) are *real*. Their conditions are correct and coincide with condition (3.2) and conditions (2)–(4) of Theorem 3.2.1.

In [20] the authors presented five systems, which are systems (a)–(e) of Theorem 3.3.1 and stated that any *real* system with an isochronous center at the origin can be transformed to one of systems (a)–(e). However, as it is shown above, generally speaking system (e) has not a center, but an isochronous focus at the origin (see e.g. [3, 52] for definitions). So system (e) should not be presented in the statements of Theorem 3.3.1.

The five systems presented in [22] are correct, however, as explained above, two of systems of [22] can be combined to give system (c) of Theorem 3.3.1. So there are only four necessary and sufficient conditions for isochronicity of the center at the origin of *real* system (3.1).

In our Theorem 3.2.1 we presented five conditions for *linearizability* of *complex* (3.1). In the case when the parameters of (3.1) are real our conditions coincide with those obtained in [85] since for real parameters condition (1) of Theorem 3.2.1 is equivalent to condition (3.2) and condition (5) is equivalent to the condition that all parameters in (3.1) are equal to zero. Thus, Theorem 3.2.1 contains all conditions for isochronicity of *real* system (3.1) obtained in [20] and [85], but additionally it gives also conditions for linearizability of *complex* system (3.1).

3.4 Coexistence of isochronous centers

In this section we present the study on the existence of more than one isochronous centers for real systems of the form (3.1).

Theorem 3.4.1. *System* (3.1) *has at most two isochronous centers including the origin when all parameters are real. More precisely, under condition* (3.2)*, conditions (3) and (4) of Theorem 3.2.1, system* (3.1) *has at most two isochronous centers. Under condition (2) of Theorem 3.2.1, system* (3.1) *has exactly one isochronous center.*

Proof. We first consider the simplest case, case (2) of Theorem 3.2.1. In this situation, system (3.1) has the form (3.6). From the first equation of (3.6) we see that the coordinates of a singular point must satisfy

$$b_{20}x+1=0 \quad \text{or} \quad a_{20}x^2-b_{20}xy-y=0.$$

Substituting $y=a_{20}x^2/(1+b_{20}x)$ into the right hand side of the second equation of (3.6) we obtain $4a_{20}^2x^2+(b_{20}x+1)^2=0$. Then, we get $x=0$ or $x=-1/b_{20}$. On the other hand, substituting $x=-1/b_{20}$ into the right hand of the second equation of (3.6), we have $-3a_{20}y/b_{20}=0$. Thus, besides the origin $O:(0,0)$ we obtain the singular point $A:(-1/b_{20},0)$ when $a_{20}b_{20}\neq 0$. When $b_{20}=0$ and $a_{20}\neq 0$ there are no singular points and when $a_{20}=0$ and $b_{20}\neq 0$ the line $x=-1/b_{20}$ is filled by singular points.

Computing the determinant of the linear matrix for system (3.6) at the singular point $A:(-1/b_{20},0)$, we find that it is equal to $-3a_{20}^2/b_{20}^2<0$, indicating that A is a saddle. Clearly, any singular point on the line $x=-1/b_{20}$ cannot be an isochronous center when $a_{20}=0$. Therefore, in the case (2) of Theorem 3.2.1, system (3.1) has only the isochronous center at the origin.

Consider now case (3) of Theorem 3.2.1. In this case system (3.1) can be written as

$$\begin{aligned}\dot{x}=&(b_{20}x+1)(4b_{11}x^2-b_{11}y^2-2b_{20}xy-2y)/2:=P_3(x,y),\\ \dot{y}=&x+b_{20}x^2+b_{11}xy-b_{20}y^2-(b_{11}b_{20}/2)y^3-b_{20}^2xy^2+2yb_{11}b_{20}x^2:=Q_3(x,y).\end{aligned}\tag{3.21}$$

From the first equation of (3.21) we see that the coordinates of a singular point must satisfy

$$b_{20}x+1=0 \quad \text{or} \quad g_3(x,y):=4b_{11}x^2-b_{11}y^2-2b_{20}xy-2y=0.$$

Substituting $x=-1/b_{20}$ into the right hand side of the second equation of (3.21), we have $-yb_{11}(b_{20}^2y^2-2)/b_{20}=0$. Thus, if $b_{20}\neq 0$ we find three singular points $A:(-1/b_{20},0)$ and $A_{\pm}:(-1/b_{20},\pm\sqrt{2}/b_{20})$.

If we solve $g_3(x,y)=0$ and substitute the solution into the right hand side of the second equation of (3.21) a very complicated expression arises. However, we only need to find the coordinates of centers of system (3.21) and for such singular point of the center type the trace of the linear matrix is zero. So we calculate

$$
\begin{aligned}
T_3(x,y) :=&\frac{\partial P_3}{\partial x}+\frac{\partial Q_3}{\partial y}\\
=&b_{20}(4b_{11}x^2-b_{11}y^2-2b_{20}xy-2y)/2+(b_{20}x+1)(8b_{11}x-2b_{20}y)/2\\
&+b_{11}x-2b_{20}y-(3/2)b_{11}b_{20}y^2-2b_{20}^2xy+2b_{11}b_{20}x^2,\\
D_3(x,y) :=&\frac{\partial P_3}{\partial x}\frac{\partial Q_3}{\partial y}-\frac{\partial P_3}{\partial y}\frac{\partial Q_3}{\partial x}\\
=&(b_{20}(4b_{11}x^2-b_{11}y^2-2b_{20}xy-2y)/2+(b_{20}x+1))(8b_{11}x-2b_{20}y)/2)\\
&(b_{11}x-2b_{20}y-(3/2)b_{11}b_{20}y^2-2b_{20}^2xy+2b_{11}b_{20}x^2)\\
&-(b_{20}x+1)(-2b_{11}y-2b_{20}x-2)(4b_{11}b_{20}xy-b_{20}^2y^2+b_{11}y+2b_{20}x+1)/2.
\end{aligned}
$$

Computing a Groebner basis of the ideal $\langle g_3, Q_3, T_3\rangle$ we obtain the basis

$$
\mathscr{G}_3 := \{b_{20}x^2+x, b_{11}y^2+2b_{20}xy+2y, b_{11}x\}.
$$

Then if $b_{11}=0$, $O:(0,0)$ is a center or the line $b_{20}x+1=0$ is filled of singular points. If $b_{11}\neq 0$, $B:(0,-2/b_{11})$ is a second singular point.

Notice that all singular points on the line $b_{20}x+1=0$ are degenerate when $b_{11}=0$, since the determinant of the linear matrix at each singular point is zero. Thus, a singular point on the line $b_{20}x+1=0$ cannot be an isochronous center if $b_{11}=0$. By calculations, among all singular points $A:(-1/b_{20},0)$, $A_\pm:(-1/b_{20},\pm\sqrt{2}/b_{20})$ and $B:(0,-2/b_{11})$, only at the point $B:(0,-2/b_{11})$ the trace of the linear part is zero and its determinant is positive simultaneously. So only need to check the isochronicity of the singular point $B:(0,-2/b_{11})$. Moving the singular point $B:(0,-2/b_{11})$ to the origin and making the change

$$
u=\sqrt{2}(-2b_{20}/b_{11}))x-\sqrt{2}y, \quad v=\sqrt{2}x
$$

together with the time scaling $dt=-d\tau$, we obtain from (3.21) the system

$$
\begin{aligned}
\dot{x} =&-y-\frac{\sqrt{2}b_{11}}{2}xy+\frac{\sqrt{2}b_{20}}{2}x^2-\frac{\sqrt{2}b_{20}}{2}y^2+\frac{b_{11}b_{20}}{4}x^3-xb_{11}b_{20}y^2+\frac{b_{20}^2}{2}x^2y,\\
\dot{y} =&x+\frac{\sqrt{2}b_{11}}{4}x^2+\sqrt{2}b_{20}xy-\sqrt{2}b_{11}y^2+\frac{b_{11}b_{20}}{4}x^2y+\frac{b_{20}^2}{2}xy^2-b_{11}b_{20}y^3,
\end{aligned}
\tag{3.22}
$$

where we still write x,y instead of u,v. Performing the change of coordinates, $z=x+iy$, $w=x-iy$, we obtain from system (3.22) the system

$$
\begin{aligned}
\dot{z} =&z+\frac{1}{16}\Big((7\sqrt{2}b_{11}-8i\sqrt{2}b_{20})z^2-6\sqrt{2}b_{11}zw+3\sqrt{2}b_{11}w^2\\
&-(5ib_{11}b_{20}+)2b_{20}^2z^3+6ib_{11}b_{20}z^2w-5ib_{11}b_{20}zw^2+2b_{20}^2zw^2\Big),\\
\dot{w} =&-w+\frac{1}{16}\Big(-3\sqrt{2}b_{11}z^2+6\sqrt{2}b_{11}zw-(7\sqrt{2}b_{11}+8i\sqrt{2}b_{20})w^2\\
&-(5ib_{11}b_{20}+2b_{20}^2)z^2w+6ib_{11}b_{20}zw^2+(2b_{20}^2-5ib_{11}b_{20})w^3\Big).
\end{aligned}
\tag{3.23}
$$

System (3.23) has the Darboux factors

$$
\begin{aligned}
l_1 =& z + \frac{b_{11} - 4ib_{20}}{8\sqrt{2}} z^2 + \frac{b_{11} + 2ib_{20}}{4\sqrt{2}} zw + \frac{b_{11}}{8\sqrt{2}} w^2, \\
l_2 =& w + \frac{b_{11}}{8\sqrt{2}} z^2 + \frac{b_{11} - 2ib_{20}}{4\sqrt{2}} zw + \frac{b_{11} + 4ib_{20}}{8\sqrt{2}} w^2, \\
l_3 =& 1 + \frac{b_{11} - ib_{20}}{2\sqrt{2}} z + \frac{\sqrt{2}b_{11} + i\sqrt{2}b_{20}}{4} w, \\
l_4 =& 1 - \frac{ib_{20}}{2\sqrt{2}} z + \frac{ib_{20}}{2\sqrt{2}} w,
\end{aligned}
$$

which allow to construct the Darboux linearization

$$
z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \qquad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},
$$

where

$$
\alpha_1 = -\frac{2b_{11} - ib_{20}}{b_{11}}, \quad \alpha_2 = -\frac{ib_{20}}{b_{11}}, \quad \beta_1 = \quad -\frac{2b_{11} + ib_{20}}{b_{11}}, \quad \beta_2 = \frac{ib_{20}}{b_{11}}.
$$

Thus, the origin of system (3.22) is an isochronous center. Therefore, system (3.21) has isochronous centers at the origin and at the point $B : (0, -2/b_{11})$ when $b_{11} \neq 0$.

Now consider case (4) of Theorem 3.2.1. In this case system (3.1) has the form

$$
\begin{aligned}
\dot{x} =& -y + a_{20}x^2 - 2b_{20}xy + a_{20}b_{20}x^3 + r_{11}x^2y := P_4(x,y), \\
\dot{y} =& x + a_{20}xy + b_{20}x^2 - b_{20}y^2 + a_{20}b_{20}x^2y + r_{11}xy^2 := Q_4(x,y).
\end{aligned}
\tag{3.24}
$$

It is difficult to find the coordinates of a singular point of system (3.24) explicitly. However, we calculate

$$
\begin{aligned}
T_4(x,y) :=& \frac{\partial P_4}{\partial x} + \frac{\partial Q_4}{\partial y}, \\
D_4(x,y) :=& \frac{\partial P_4}{\partial x}\frac{\partial Q_4}{\partial y} - \frac{\partial P_4}{\partial y}\frac{\partial Q_4}{\partial x}
\end{aligned}
$$

to find only coordinates of centers. Computing a Groebner basis of $\langle P_4, Q_4, T_4 \rangle$ we obtained

$$
\begin{aligned}
\mathscr{G}_4 :=& \{a_{20}xy + 4b_{20}x^2 + 4x,\ a_{20}y^2 + 4b_{20}xy + 4y,\ -3a_{20}^3x + 16a_{20}b_{20}^2x + 16a_{20}r_{11}x, \\
& a_{20}x^2 - 4b_{20}xy - 4y,\ -3a_{20}^2y + 16b_{20}^2y + 16r_{11}y,\ b_{20}x^3 + b_{20}xy^2 + x^2 + y^2, \\
& 64b_{20}^3x^2 + 16b_{20}r_{11}x^2 - 3a_{20}^2x - 12a_{20}b_{20}y + 64b_{20}^2x + 16r_{11}x\}.
\end{aligned}
$$

Doing the first and the second polynomials in $\mathscr{G}_4$ equal to zero, we obtain $y = -4(b_{20}x+1)/a_{20}$ when $a_{20} \neq 0$ or $x = y = 0$. Substituting $y = -4(b_{20}x+1)/a_{20}$ into $\mathscr{G}_4$, we have

$$
\begin{aligned}
& \{4(b_{20}x+1)(3a_{20}^2 - 16b_{20}^2 - 16r_{11})/a_{20}, (16 + (a_{20}^2 + 16b_{20}^2)x^2 + 32b_{20}x)/a_{20}, \\
& (16 + (a_{20}^2 + 16b_{20}^2)x^2 + 32b_{20}x)(b_{20}x+1)/a_{20}^2, -a_{20}x(3a_{20}^2 - 16b_{20}^2 - 16r_{11}), \\
& (64b_{20}^3 + 16b_{20}r_{11})x^2 + (-3a_{20}^2 + 112b_{20}^2 + 16r_{11})x + 48b_{20}\}.
\end{aligned}
\tag{3.25}
$$

Doing the first polynomial of (3.25) equal to zero, we obtain $x = -1/b_{20}$ if $b_{20} \neq 0$ or $3a_{20}^2 - 16b_{20}^2 - 16r_{11} = 0$. Substituting them in (3.25), we obtain

$$\{a_{20}(3a_{20}^2 - 16b_{20}^2 - 16r_{11})/b_{20}, a_{20}/b_{20}^2, 3a_{20}^2/b_{20}\}$$

and

$$\begin{gathered}\{(a_{20}^2x^2 + (4b_{20}x+4)^2)/a_{20},\ (b_{20}x+1)(a_{20}^2x^2 + 16b_{20}^2x^2 + 32b_{20}x + 16)/a_{20}^2, \\ 3b_{20}(a_{20}^2x^2 + 16b_{20}^2x^2 + 32b_{20}x + 16)\},\end{gathered}$$

respectively. If $b_{20} = 0$, (3.25) becomes

$$\begin{gathered}\{-a_{20}(3a_{20}^2 - 16r_{11})x, (16 + a_{20}^2x^2)/a_{20}, 4(3a_{20}^2 - 16r_{11})/a_{20}, \\ (16 + a_{20}^2x^2)/a_{20}^2, -(3a_{20}^2 - 16r_{11})x\}.\end{gathered}$$

From the first and the second polynomials in the above three sets, we see that on the line $y = -4(b_{20}x+1)/a_{20}$ there is no singular points of center type when $a_{20} \neq 0$.

When $a_{20} = 0$, the basis $\mathscr{G}_4$ becomes $\{b_{20}x^2 + x, b_{20}xy + y, b_{20}^2y + r_{11}y\}$ and we obtain that $b_{20}x + 1 = 0, (b_{20}^2 + r_{11})y = 0$ or $x = 0, y = 0$. If $a_{20} = 0$, $b_{20}x + 1 = 0$ and $b_{20}^2 + r_{11} = 0$, the line $b_{20}x + 1 = 0$ is full of singular points, none of which can be an isochronous center of system (3.24). Hence, we only get the unique possible center $A : (-1/b_{20}, 0)$ if $a_{20} = 0$, at which the trace of the linear matrix for system (3.24) is zero and its determinant is $r_{11}/b_{20}^2 + 1$. If $a_{20} = 0$, after moving the point $(-1/(2b_{20}), 0)$ to the origin, system (3.24) is taking into system

$$\begin{aligned}\dot{x} &= \frac{r_{11}}{4b_{20}^2}y - \frac{2b_{20}^2 + r_{11}}{b_{20}}xy + r_{11}x^2y, \\ \dot{y} &= -\frac{1}{4b_{20}} + b_{20}x^2 - \frac{2b_{20}^2 + r_{11}}{2b_{20}}y^2 + r_{11}xy^2,\end{aligned} \tag{3.26}$$

which is symmetric with respect to the x-axis. Moreover, the singular points $(\pm 1/(2b_{20}), 0)$ of system (3.26) correspond to the singular points A and O of system (3.24) respectively. Thus, except of the origin $O : (0,0)$ we get another isochronous center at the singular point $A : (-1/b_{20}, 0)$ when $a_{20} = 0$, $r_{11}/b_{20}^2 + 1 > 0$ and $b_{20} \neq 0$. Therefore, in case (4) system (3.1) has at most two isochronous centers.

At last, we study the case when condition (3.2) is fulfilled. Under such condition system (3.1) becomes system (3.12), namely

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy - a_{20}y^2 + x(r_{20}x^2 + r_{11}xy - r_{20}y^2) = P_1(x,y), \\ \dot{y} &= x + b_{20}x^2 + 2a_{20}xy - b_{20}y^2 + y(r_{20}x^2 + r_{11}xy - r_{20}y^2) = Q_1(x,y).\end{aligned}$$

Similarly to case (4), we only consider singular points of center type avoiding complicated calculations with the coordinates of all singular points. We calculate

$$\begin{aligned}T_1(x,y) &:= \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y}, \\ D_1(x,y) &:= \frac{\partial P_1}{\partial x}\frac{\partial Q_1}{\partial y} - \frac{\partial P_1}{\partial y}\frac{\partial Q_1}{\partial x},\end{aligned}$$

to find coordinates of centers. The Groebner basis of $\langle P_1, Q_1, T_1\rangle$ is

$$\mathscr{G}_1 := \{a_{20}y + b_{20}x + 1, r_{11}xy + r_{20}x^2 - r_{20}y^2 + a_{20}x - b_{20}y, a_{20}r_{11}y^2 + a_{20}r_{20}xy + b_{20}r_{20}y^2 + a_{20}^2y + b_{20}^2y + r_{11}y + r_{20}x + a_{20}\}.$$

If $a_{20} = b_{20} = 0$, system (3.12) cannot have other centers except of the origin. Without loss of generality we suppose $b_{20} \neq 0$. If $a_{20} \neq 0$ the discussion is similar and we only need to make the change $(x,y) \to (y,x)$ with the time rescaling $dt = -d\tau$. From the first polynomial in $\mathscr{G}_1$, we get $x = -(a_{20}y+1)/b_{20}$. Substituting it into $\mathscr{G}_1$, we have

$$g_1 := a_0 + a_1 y + a_2 y^2 = 0, \tag{3.27}$$

where $a_0 = a_{20}b_{20} - r_{20}$, $a_1 = a_{20}^2 b_{20} + b_{20}^3 - 2a_{20}r_{20} + b_{20}r_{11}$ and $a_2 = -a_{20}^2 r_{20} + a_{20}b_{20}r_{11} + b_{20}^2 r_{20}$. Thus, from (3.27) we find two roots $y_\pm = (-a_1 \pm \sqrt{a_1^2 - 4a_2a_0})/(2a_2)$ and then we obtain two singular points $C_\pm : (-(a_{20}y_\pm + 1)/b_{20}, y_\pm)$ when $d_0 := a_1^2 - 4a_2a_0 > 0$ and $a_2 \neq 0$. At $C_\pm$ the trace of the linear matrix of system (3.12) is zero and its determinant is

$$\tilde{D}_\pm := \frac{d_0(\mp(a_{20}^2 + b_{20}^2)\sqrt{d_0} - b_{20}(d_0/b_{20}^2 - b_{20}^2 r_{11} + 4a_{20}b_{20}r_{20} + a_{20}^2 r_{11} - r_{11}^2 - 4r_{20}^2))}{2b_{20}^3(a_{20}^2 r_{20} - a_{20}b_{20}r_{11} - b_{20}^2 r_{20})^2}.$$

Moreover,

$$\tilde{D}_+\tilde{D}_- = -\frac{d_0^2}{b_{20}^4(a_{20}^2 r_{20} - a_{20}b_{20}r_{11} - b_{20}^2 r_{20})^2} < 0,$$

implying that at most one of C_+ and C_- is a center. Actually, when $r_{20} = a_{20}b_{20}$, we find that the singular point $B : (0, -2/b_{11})$ is an isochronous center, since it is possible to show that system (3.12) is Darboux linearizable at this point. Therefore, if (3.2) holds, system (3.1) has at most two isochronous centers. □

CHAPTER 4

LINEARIZABILITY FOR A GENERALIZED RICCATI SYSTEM

In this chapter we investigate the isochronicity and linearizability problems for a cubic polynomial differential system which can be considered as a generalization of the Riccati system. The chapter is organized as follows. In Section 4.1 we introduce the problem giving the motivation for the investigation. Section 4.2 and Section 4.3 contain the results on the linearizability of system (4.2) and the results on the study of the global dynamics of system (4.2) when the origin is an isochronous center, respectively. The results present in this chapter are part of the preprint [100] available at https://www.researchgate.net/publication/313844845_Linearizability_and_critical_period_bifurcations_of_a_generalized_Riccati_system.

4.1 Motivation for the study

The classic Riccati equation is a system written in the form

$$\dot{x} = 1, \quad \dot{y} = g_2(x)y^2 + g_1(x)y + g_0(x), \tag{4.1}$$

where each $g_j(x)$ is a $\mathscr{C}^1$ function with respect to x and $g_2(x)g_0(x) \not\equiv 0$. System (4.1) becomes a special case of Bernoulli system if $g_0(x) \equiv 0$, and it obviously is a linear differential system if $g_2(x) \equiv 0$.

The Riccati equation has been investigated by many authors, see for example [83, 84] and references therein. They are important since they can be used to solve second-order ordinary differential equations and can be applied in studying the third-order Schwarzian differential equation [93]. It also has many applications in both physics and mathematics. For instance, renormalization group equations for running coupling constants in quantum field theories [15], nonlinear physics [90], Newton's laws of motion [94], thermodynamics [105] and variational calculus [118].

Recently, Llibre and Valls [83, 84] investigated the planar differential system

$$\dot{x} = f(y), \qquad \dot{y} = g_2(x)y^2 + g_1(x)y + g_0(x),$$

where $g_2(x)$, $g_1(x)$ and $g_0(x)$ are polynomials, which is called the *generalized Riccati system* (it becomes the classic Riccati system when $f(y) \equiv 1$). Here we study a subfamily of the generalized Riccati system, the cubic systems of the form

$$\begin{aligned} \dot{x} &= -y + a_{02}y^2 + a_{03}y^3, \\ \dot{y} &= (b_{02} + b_{12}x)y^2 + (b_{11}x + b_{21}x^2)y + (x + b_{20}x^2 + b_{30}x^3) \\ &= x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2, \end{aligned} \tag{4.2}$$

where $x, y \in \mathbb{R}^2$ and all parameters a_{ij}, b_{ij} are in $\mathbb{C}$ or $\mathbb{R}$.

The main goal of this investigation is to obtain conditions on parameters a_{ij} and b_{ij} for the linearizability of system (4.2), and to study the global structures of the trajectories when such system has an isochronous center at the origin. The main results of this chapter are Theorem 4.2.1, which gives conditions for the linearizability of system (4.2) and Theorem 4.3.1 which gives the global dynamics of system (4.2) when the origin is an isochronous center.

4.2 Linearizability conditions

Regarding to the linearizability of system (4.2) we obtain the following result.

Theorem 4.2.1. *System (4.2) is linearizable at the origin if one of the following conditions holds:*

(1) $b_{12} = a_{02} = b_{30} = b_{21} = a_{03} = b_{02} + b_{20} = b_{11}^2 + 4b_{20}^2 = 0,$

(2) $b_{12} = a_{02} = b_{20} = b_{02} = b_{21} = a_{03} = 9b_{30} - b_{11}^2 = 0,$

(3) $b_{12} = a_{02} = b_{11} = b_{20} = b_{30} = b_{21} = 9a_{03} + 4b_{02}^2 = 0,$

(4) $b_{12} = b_{30} = b_{21} = a_{03} = 2b_{02} + 5b_{20} = 10a_{02} - 3b_{11} = 4b_{11}^2 + 25b_{20}^2 = 0.$

Proof. Using the computer algebra system MATHEMATICA and the procedure described in Section 1.4 we have computed the first nine pairs of the linearizability quantities $i_1, j_1, \ldots, i_9, j_9$ for system (4.2). Their expressions are very large, so we only present the first pair

$$\begin{aligned} i_1 &= 10a_{02}^2 + 9a_{03} + 4b_{02}^2 - a_{02}b_{11} + b_{11}^2 - 3b_{12} + 10b_{02}b_{20} + 10b_{20}^2 - 9b_{30}, \\ j_1 &= 2a_{02}b_{02} - b_{02}b_{11} - b_{11}b_{20} + b_{21}. \end{aligned}$$

The reader can easily compute the other quantities using any available computer algebra system.

Following the process explained in Section 1.4 we compute the irreducible decomposition of the variety $\mathbf{V}(\mathscr{L}_9) = \mathbf{V}(\langle i_1, j_1, ..., i_9, j_9\rangle)$. We were not able to complete the computations using the routine `minAssGTZ` of SINGULAR over the field of the rational numbers with our computational facilities. Performing the computations over the field of characteristic 32452843 we obtain that $\mathbf{V}(\mathscr{L}_9)$ is equal to the union of the varieties of four ideals. After lifting these four ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm we obtain the ideals

$$
\begin{aligned}
J_1 =& \langle b_{12}, a_{02}, b_{30}, b_{21}, a_{03}, b_{02} + b_{20}, b_{11}^2 + 4b_{20}^2\rangle,\\
J_2 =& \langle b_{12}, a_{02}, b_{20}, b_{02}, b_{21}, a_{03}, 9b_{30} - b_{11}^2\rangle,\\
J_3 =& \langle b_{12}, a_{02}, b_{11}, b_{20}, b_{30}, b_{21}, 9a_{03} + 4b_{02}^2\rangle,\\
J_4 =& \langle b_{12}, b_{30}, b_{21}, a_{03}, 2b_{02} + 5b_{20}, 10a_{02} - 3b_{11}, 4b_{11}^2 + 25b_{20}^2\rangle.
\end{aligned}
$$

The varieties of J_1, J_2, J_3 and J_4 provide conditions (1), (2), (3) and (4) of the theorem, respectively.

To check the correctness of the obtained conditions we use the procedure described in [101]. First, we computed the ideal $J = J_1 \cap J_2 \cap J_3 \cap J_4$, which defines the union of all four sets given in the statement of the theorem. Then we check that $\mathbf{V}(J) = \mathbf{V}(\mathscr{L}_9)$. According to the Radical Membership Test (see Section 1.2), to verify the inclusion $\mathbf{V}(J) \supset \mathbf{V}(\mathscr{L}_9)$ it is sufficient to check that the Groebner bases of all ideals $\langle J, 1 - wi_k\rangle$, $\langle J, 1 - wj_k\rangle$ for $k = 1, \ldots, 9$ and w being a new variable, computed over $\mathbb{Q}$ are $\{1\}$. The computations show that this is the case. To check the opposite inclusion, $\mathbf{V}(J) \subset \mathbf{V}(\mathscr{L}_9)$, it is sufficient to check that Groebner bases of the ideals $\langle \mathscr{L}_9, 1 - wf_i\rangle$ (where the polynomials f_i's are the polynomials of a basis of J) computed over $\mathbb{Q}$ are equal to $\{1\}$. Unfortunately, we were not able to perform these computations over $\mathbb{Q}$ however we have checked that all the bases are $\{1\}$ over few fields of finite characteristic. It yields that the list of conditions in Theorem 4.2.1 is the complete list of linearizability conditions for system (4.2) with high probability [8].

We now prove that under each of conditions (1)–(4) of Theorem 4.2.1 system (4.2) is linearizable.

Condition (1). In this case $b_{11} = \pm 2b_{20}i$. Here we consider only the case $b_{11} = 2b_{20}i$, since when $b_{11} = -2b_{20}i$ the proof is analogous. Performing the change of variables $z = x + iy$, $w = x - iy$, system (4.2) becomes

$$
\begin{aligned}
\dot{z} =& z + b_{20}z^2,\\
\dot{w} =& -w - b_{20}z^2,
\end{aligned}
\tag{4.3}
$$

which is a quadratic system. By Theorem 4.5.1 in [103] system (4.3) is Darboux linearizable and, therefore, system (4.2) is linearizable if condition (1) holds.

Condition (2). After substitution $z = x+iy$, $w = x-iy$ system (4.2) becomes

$$\begin{aligned}\dot{z} =& z+\frac{1}{72}(-18ib_{11}z^2+18ib_{11}w^2+b_{11}^2z^3+3b_{11}^2z^2w+3b_{11}^2zw^2+b_{11}^2w^3),\\ \dot{w} =& -w+\frac{1}{72}(18ib_{11}z^2-18ib_{11}w^2-b_{11}^2z^3-3b_{11}^2z^2w-3b_{11}^2zw^2-b_{11}^2w^3).\end{aligned} \tag{4.4}$$

It has the Darboux factors

$$\begin{aligned} l_1 =& z+\frac{ib_{11}}{12}z^2+\frac{ib_{11}}{6}zw+\frac{ib_{11}}{12}w^2,\\ l_2 =& w-\frac{ib_{11}}{12}z^2-\frac{ib_{11}}{6}zw-\frac{ib_{11}}{12}w^2,\\ l_3 =& 1-\frac{ib_{11}}{6}z+\frac{b_{11}^2}{36}z^2+\frac{ib_{11}}{6}w+\frac{b_{11}^2}{18}zw+\frac{b_{11}^2}{36}w^2,\\ l_4 =& 1-\frac{ib_{11}}{3}z+\frac{b_{11}^2}{36}z^2+\frac{ib_{11}}{3}w+\frac{b_{11}^2}{18}zw+\frac{b_{11}^2}{36}w^2\end{aligned}$$

with the respective cofactors

$$\begin{aligned} k_1 =& 1-\frac{ib_{11}}{6}z-\frac{ib_{11}}{6}w, \quad & k_2 =& -1-\frac{ib_{11}}{6}z-\frac{ib_{11}}{6}w,\\ k_3 =& -\frac{ib_{11}}{6}z-\frac{ib_{11}}{6}w, & k_4 =& -\frac{ib_{11}}{3}z-\frac{ib_{11}}{3}w.\end{aligned}$$

It is easy to verify that the equations (1.39) are satisfied using $f_0 = l_1$, $f_1 = l_3$, $f_2 = l_4$, $g_0 = l_2$, $g_1 = l_3$ and $g_2 = l_4$ with $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1 = 1$ and $\beta_2 = -1$. Hence the Darboux linearization for system (4.4) is given by the analytic change of coordinates

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2}.$$

Thus, system (4.4) is linearizable and therefore the corresponding system (4.2) is linearizable as well.

Condition (3). In this case performing the substitution $z = x+iy$, $w = x-iy$ the corresponding system (4.2) is changed to

$$\begin{aligned}\dot{z} =& z+\frac{1}{36}(-9b_{02}z^2+18b_{02}zw-9b_{02}w^2-2b_{02}^2z^3+6b_{02}^2z^2w-6b_{02}^2zw^2+2b_{02}^2w^3),\\ \dot{w} =& -w+\frac{1}{36}(9b_{02}z^2-18b_{02}zw+9b_{02}w^2-2b_{02}^2z^3+6b_{02}^2z^2w-6b_{02}^2zw^2+2b_{02}^2w^3).\end{aligned} \tag{4.5}$$

System (4.5) has the Darboux factors

$$\begin{aligned} l_1 =& z-\frac{b_{02}}{12}z^2+\frac{b_{02}}{6}zw-\frac{b_{02}}{12}w^2,\\ l_2 =& w-\frac{b_{02}}{12}z^2+\frac{b_{02}}{6}zw-\frac{b_{02}}{12}w^2,\\ l_3 =& 1-\frac{2b_{02}}{3}z+\frac{2b_{02}^2}{9}z^2-\frac{2b_{02}}{3}w-\frac{4b_{02}^2}{9}zw+\frac{2b_{02}^2}{9}w^2,\\ l_4 =& 1-\frac{b_{02}}{3}z+\frac{b_{02}^2}{18}z^2-\frac{b_{02}}{3}w-\frac{b_{02}^2}{9}zw+\frac{b_{02}^2}{18}w^2,\end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where $\alpha_1 = 1,\ \alpha_2 = -3,\ \beta_1 = 1$ and $\beta_2 = -3$.

Condition (4). For this condition it is easy to see that $b_{11} = \pm 5b_{20}i/2$. Here we consider only the case $b_{11} = 5b_{20}i/2$, since when $b_{11} = -5b_{20}i/2$ the proof is analogous. By transformation $z = x+iy$, $w = x-iy$ system (4.2) becomes

$$\begin{aligned} \dot{z} =& z + \frac{1}{16}(21b_{20}z^2 - 6b_{20}zw + b_{20}w^2), \\ \dot{w} =& -w + \frac{1}{16}(-27b_{20}z^2 + 18b_{20}zw - 7b_{20}w^2). \end{aligned} \tag{4.6}$$

System (4.6) has the Darboux factors

$$\begin{aligned} l_1 =& z + \frac{1}{16}3b_{20}z^2 + \frac{1}{8}b_{20}zw + \frac{1}{48}b_{20}w^2, \\ l_2 =& w + \frac{1}{16}9b_{20}z^2 + \frac{3b_{20}}{8}zw + \frac{b_{20}}{16}w^2, \\ l_3 =& 1 + 3b_{20}z + \frac{27b_{20}^2}{8}z^2 + b_{20}w - \frac{3b_{20}^2}{4}zw + \frac{3b_{20}^2}{8}w^2, \\ l_4 =& 1 + \frac{3b_{20}}{2}z + \frac{9b_{20}^2}{16}z^2 + \frac{b_{20}}{2}w + \frac{3b_{20}^2}{8}zw + \frac{b_{20}^2}{16}w^2, \end{aligned}$$

yielding the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where $\alpha_1 = 1,\ \alpha_2 = -3,\ \beta_1 = 1$ and $\beta_2 = -3$. □

4.3 Global dynamics

In this section, we study the global structures of system (4.2) in the Poincaré disc for the case when it has an isochronous center at the origin and all parameters of the system are real.

Theorem 4.3.1. *The global phase portrait of system* (4.2) *possessing an isochronous center listed in Theorem 3.2.1 is topologically equivalent to one of phase portraits in Fig. 9.*

Proof. From Theorem 4.2.1 system (4.2) is linearizable if one of the conditions (1)–(4) holds. Under conditions (1) and (4) real systems (4.2) becomes the linear system $\dot{x} = -y,\ \dot{y} = x$ and its phase portrait is presented in Figure 9.A.

Under conditions (2) and (3) system (4.2) becomes

$$\dot{x} = -y, \qquad \dot{y} = x + b_{11}xy + \frac{b_{11}^2}{9}x^3, \tag{4.7}$$

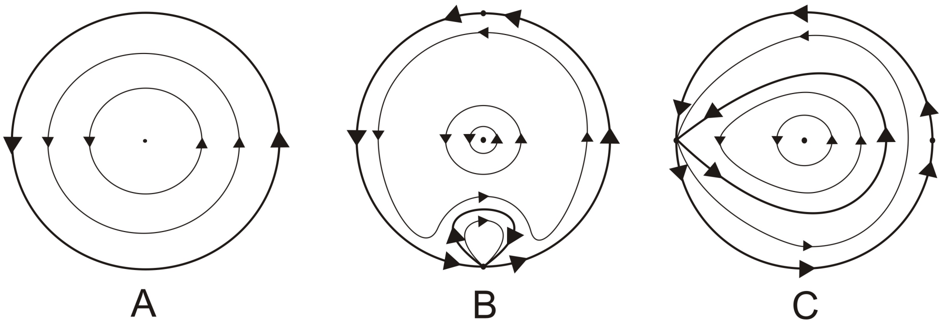

Figure 9 – Global phase portraits of system (4.2) possessing an isochronous center at the origin.

and

$$\dot{x} = -y - \frac{4}{9}b_{02}^2 y^3, \qquad \dot{y} = x + b_{02}y^2, \tag{4.8}$$

respectively.

Note that if $b_{11} = 0$ in (4.7) and $b_{02} = 0$ in (4.8), then both systems are the canonic linear systems and have a global center shown in Figure 9.A. Thus, we consider the cases when $b_{11} \neq 0$ and $b_{02} \neq 0$. In both cases by a linear change of coordinates we can reduce systems (4.7) and (4.8) to systems

$$\dot{x} = -y, \qquad \dot{y} = x + xy + \frac{x^3}{9}, \tag{4.9}$$

and

$$\dot{x} = -y - \frac{4}{9}y^3, \qquad \dot{y} = x + y^2, \tag{4.10}$$

respectively.

System (4.9) has only the isochronous center at $(0,0)$ as a finite singular point. Now we analyze its singular points at infinity. In the local chart U_1 system (4.9) becomes

$$\dot{u} = \frac{1}{9}(1 + 9uv + 9v^2 + 9u^2v^2), \qquad \dot{v} = uv^3.$$

This system has no real singular points. So the unique possible infinite singular point is the origin of the local chart U_2. In the local chart U_2 system (4.9) becomes

$$\dot{u} = \frac{1}{9}(-u^4 - 9u^2v - 9v^2 - 9u^2v^2), \qquad \dot{v} = -\frac{1}{9}uv(u^2 + 9v + 9v^2). \tag{4.11}$$

It is clear that $(0,0)$ is a singular point of (4.11) and the linear part of (4.11) at $(0,0)$ is the null matrix, i.e, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Applying the directional blow-up in the v-axis, i.e, doing the change of coordinates $(u,v) \mapsto (u,uz)$, and a time rescaling $dT = udt$, we obtain the system

$$\dot{u} = -\frac{1}{9}u(u^2 + 9uz + 9z^2 + 9u^2z^2), \quad \dot{z} = z^3. \tag{4.12}$$

This system has only $(0,0)$ as singular point, and the linear part of (4.12) in $(0,0)$ is again the null matrix. Applying once more directional blow-up in the z-axis, i.e, doing the change of coordinates $(u,v) \mapsto (u,uw)$, and a time rescaling $ds = u^2 dT$, we obtain the system

$$\dot{u} = -\frac{1}{9}u(1+9w+9w^2+9u^2w^2), \quad \dot{w} = \frac{1}{9}w(1+9w+18w^2+9u^2w^2). \tag{4.13}$$

This system has the singular points $(0,-\frac{1}{3})$, $(0,-\frac{1}{6})$ and $(0,0)$ in the w-axis. The linear parts of (4.13) in $(0,-\frac{1}{3})$, $(0,-\frac{1}{6})$ and $(0,0)$ are $\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{36} & 0 \\ 0 & -\frac{1}{18} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$, respectively. Thus, $(0,-\frac{1}{3})$ is a noddle, and $(0,-\frac{1}{6})$ and $(0,0)$ are saddle points, see Figure 10.a.

The complete blowing down process is described in Figure 10. To return from system

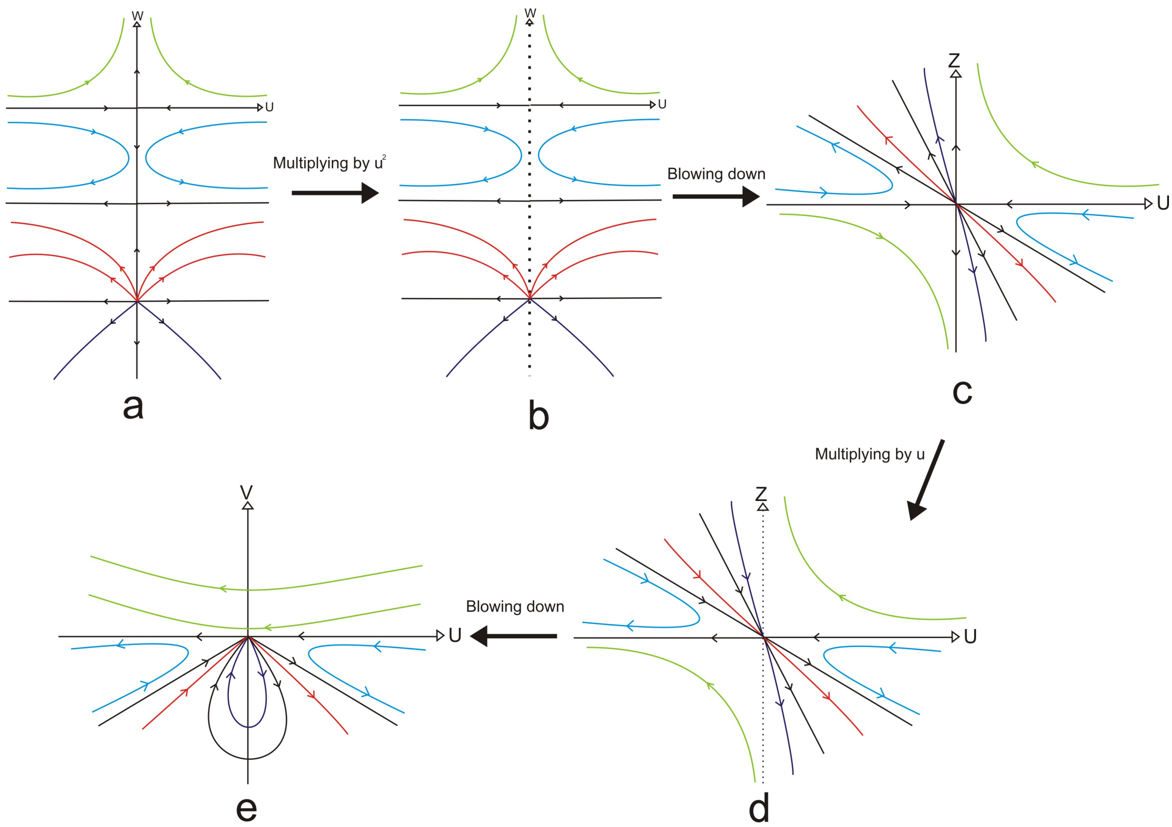

Figure 10 – Blowing down process relative to the origin of system (4.11).

(4.13) to system (4.12) we first multiply by u^2 (undoing the time rescaling), it becomes w-axis a singularities axis, see Figure 10.b. The next stage is the first blowing down, when the second and third quadrants of Figure 10.b are changed, see Figure 10.c. Note that the z-axis of system (4.12) is an invariant line. Next we multiply by u (undoing the time rescaling), in this stage the sense of the orbits in the half-plan when $u<0$ are changed, moreover the z-axis becomes a singularities axis, see Figure 10.d. Finally, doing the last blowing down from system (4.12) to system (4.11), the second and third quadrants of Figure 10.d are changed. Moreover we note that the vectors field of (4.11) in a point of the form $(0,v)$ is $(-v^2,0)$. It shows that the correct configuration of the orbits close to the origin of U_2 is according to Figure 10.e.

Therefore, considering the information obtained from the finite singularities, infinity singularities and the continuity of the orbits we have that the global phase portrait of system

(4.9) is topologically equivalent to the one in Figure 9.B.

Now we study system (4.10). This system has only the isochronous center at $(0,0)$ as a finite singular point. For the infinite singular points, in the local chart U_1 system (4.10) becomes

$$\dot{u} = \frac{1}{9}(4u^4 + 9u^2v + 9v^2 + 9u^2v^2), \qquad \dot{v} = \frac{1}{9}uv(4u^2 + 9v^2). \tag{4.14}$$

This system has only $(0,0)$ as a singular point, and the linear part of (4.14) at $(0,0)$ is the null matrix. Applying the directional blow-up in the v-axis, i.e, doing the change of coordinates $(u,v) \mapsto (u,uz)$, and a time rescaling $dT = udt$, we obtain the system

$$\dot{u} = \frac{1}{9}u(4u^2 + 9uz + 9z^2 + 9u^2z^2), \quad \dot{z} = -z^2(u+z). \tag{4.15}$$

This system has only $(0,0)$ as singular point, and the linear part of (4.15) in $(0,0)$ is again the null matrix. Applying the directional blow-up in the z-axis, i.e, doing the change of coordinates $(u,v) \mapsto (u,uw)$, and a time rescaling $ds = u^2dT$, we obtain the system

$$\dot{u} = \frac{1}{9}u(4 + 9w + 9w^2 + 9u^2w^2), \quad \dot{w} = -\frac{1}{9}w(4 + 18w + 18w^2 + 9u^2w^2). \tag{4.16}$$

This system has the singular points $(0,-\frac{2}{3})$, $(0,-\frac{1}{3})$ and $(0,0)$ in the w-axis. The linear parts of (4.16) in $(0,-\frac{2}{3})$, $(0,-\frac{1}{3})$ and $(0,0)$ are $\begin{pmatrix} \frac{2}{9} & 0 \\ 0 & -\frac{4}{9} \end{pmatrix}$, $\begin{pmatrix} \frac{2}{9} & 0 \\ 0 & \frac{2}{9} \end{pmatrix}$ and $\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & -\frac{4}{9} \end{pmatrix}$, respectively. Thus, $(0,-\frac{1}{3})$ is a noddle, and $(0,-\frac{2}{3})$ and $(0,0)$ are saddle points, see Figure 11.a.

The complete blowing down process is described in Figure 11. To return from system (4.16) to system (4.15) we first multiply by u^2 (undoing the time rescaling), it becomes w-axis a singularities axis, see Figure 11.b. The next stage is the first blowing down, when the second and third quadrants of Figure 11.b are changed, see Figure 11.c. Note that the z-axis of system (4.15) is an invariant line. Next we multiply by u (undoing the time rescaling), in this stage the sense of the orbits in the half-plan when $u<0$ are changed, moreover the z-axis becomes a singularities axis, see Figure 11.d. Finally, doing the last blowing down from system (4.15) to system (4.14), the second and third quadrants of Figure 11.d are changed. Moreover we note that the vector field of (4.14) in a point of the form $(0,v)$ is $(v^2,0)$. It shows that the correct configuration of the orbits close to the origin of U_1 is according to Figure 11.e.

In the local chart U_2 system (4.10) becomes

$$\dot{u} = \frac{1}{9}(-4 - 9uv - 9v^2 - 9u^2v^2), \qquad \dot{v} = -v^2(1+uv). \tag{4.17}$$

As it is mentioned above, we need to study only the origin of this chart, but $(0,0)$ is not a singular point for system (4.17). Thus, considering the information obtained from the finite singularities, infinity singularities and the continuity of the orbits the global phase portrait of system (4.10) is topologically equivalent to the portrait in Figure 9.C. □

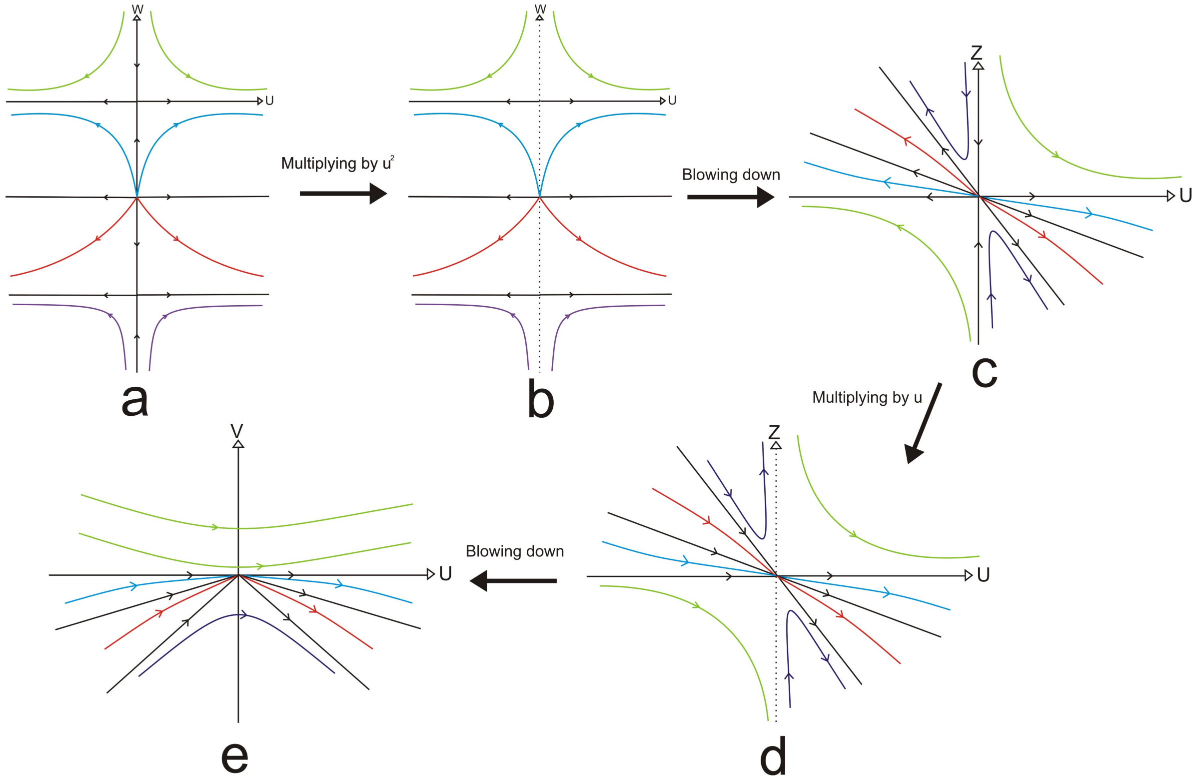

Figure 11 – Blowing down process relative to the origin of system (4.14).

Summarizing, we have found four families of system (4.2) which are linearizable. However, only two families represent systems with an isochronous center at the origin. It is because we have considered system (4.2) with complex parameters and when restricted to the real field only two families appears. To determine the global phase portraits we have used the blow up technique to desingularize the singular points at the infinity.

CHAPTER

5

LINEARIZABILITY OF PERSISTENT CENTERS

This chapter is devoted to investigate persistent centers for complex differential systems. Such kind of centers were introduced by Cima, Gasull and Medrado in [31] for real differential systems. In [27], Chen, Romanovski and Zhang generalized the notion of persistent centers for complex systems. In this chapter we investigate the linearizability of persistent centers. The results here discussed are described in [92], manuscript available in http://conteudo.icmc.usp.br/CMS/Arquivos/arquivos_enviados/BIBLIOTECA_158_Nota%20Serie%20Mat%20423.pdf. This chapter is organized as follows. In Section 5.1 we give the motivation for this investigation. In Section 5.2 we describe the computational method used on the investigation. Finally, Section 5.3 contains our main results.

5.1 Motivation for the study

There are several generalizations of the classical center problem. One of these notions is the following. The singular point $O = (0,0)$ in a complex system of the form

$$\dot{x} = px + P(x,y), \quad \dot{y} = -qy + Q(x,y),$$

where $x, y \in \mathbb{C}$, $p, q \in \mathbb{N}$, and $P, Q \in \mathbb{C}[x,y]$ is called a $p : -q$ *resonant center* if there exists a local analytic first integral of the form

$$\Phi(x,y) = x^q y^p + \sum_{j+k>p+q+1} \phi_{j-q,k-p} x^j y^k.$$

Some results for $p : -q$ resonant centers can be found in [25, 37, 40, 47, 51, 75, 76, 104, 119]. Some other generalizations are discussed also in [44] and [80].

Before to state the definitions of persistent center and a weakly persistent center we remind that the complexification of the real system (1.19) is system (1.23), namely $\dot{x} = i(x - X(x,\bar{x}))$, and more generally system (1.24), namely $\dot{x} = i(x - X(x,y))$, $\dot{y} = -i(y - Y(x,y))$.

Definition 5.1.1 ([31]). The origin of (1.23) is a *persistent center* (respectively, *weakly persistent center*) if it is a center for

$$\dot{x} = ix + \lambda iX(x,\bar{x}), \quad x \in \mathbb{C}$$

for all $\lambda \in \mathbb{C}$ (respectively, $\lambda \in \mathbb{R}$).

Definition 5.1.2 ([27]). The origin of (1.24) is a *persistent center* (respectively, *weakly persistent center*) if it is a center for

$$\dot{x} = ix + \lambda iX(x,y), \quad \dot{y} = -iy + \mu iY(x,y), \quad x,y \in \mathbb{C} \tag{5.1}$$

for all $\lambda, \mu \in \mathbb{C}$ (respectively, $\lambda = \mu \in \mathbb{C}$).

In [31] authors found some general systems of type (1.23) where the origin is a persistent center: a) $\dot{x} = ix + Ax^2 + C\bar{x}^2$ (quadratic), b) $\dot{x} = ix + X(x)$, with $X(0) = X'(0) = 0$ (holomorphic), c) $\dot{x} = ix + X(\bar{x})$, with $X(0) = X'(0) = 0$ (hamiltonian), d) $\dot{x} = ix + x\bar{x}X(\bar{x})$ (separated), and e) $\dot{x} = ix + Bx^k\bar{x}^l\Psi(x\bar{x})$, with $k \neq l+1$ (reversible), where $A,B,C \in \mathbb{C}$ and Ψ a real analytic function such that the series expansion for $x^k\bar{x}^l\Psi(x\bar{x})$ starts with second order terms. Furthermore, for some special cases of (1.23), where

$$X(x,\bar{x}) = Ax^2 + Bx\bar{x} + C\bar{x}^2 + Dx^3 + Ex^2\bar{x} + Fx\bar{x}^2 + G\bar{x}^3,$$

the origin is proven to be a persistent center (see [31, Theorem 1.2]). This result was generalized in [27] for systems (1.24), see [27, Theorem 2.2].

Generalizing the concepts of linearizable systems (see Section 1.4) we introduce some new definitions concerning linearizability of persistent and weakly persistent centers of complex system (1.24).

Definition 5.1.3. The origin O is called a *linearizable persistent center* of system (1.24) if it is a persistent center and there exists an analytic change of coordinates of the form

$$x = x_1 + \sum_{j+k\geq 2} c_{j,k}x_1^j y_1^k, \quad y = y_1 + \sum_{j+k\geq 2} d_{j,k}x_1^j y_1^k,$$

that reduces (5.1) to the system

$$\dot{x}_1 = ix_1, \quad \dot{y}_1 = -iy_1$$

for all $\lambda, \mu \in \mathbb{C}$.

Definition 5.1.4. The origin O is called a *linearizable weakly persistent center* of system (1.24) if it is a weakly persistent center and system (5.1) is linearizable for all $\lambda, \mu \in \mathbb{C}$.

Our main goal is to study the existence of linearizable persistent centers and linearizable weakly persistent centers for the family of cubic differential systems

$$\begin{aligned} \dot{x} =& ix + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}y^2x + a_{03}y^3, \\ \dot{y} =& -iy + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}y^2x + b_{03}y^3. \end{aligned} \tag{5.2}$$

We obtain the necessary and sufficient conditions for the existence of a linearizable persistent center for system (5.2). Unfortunately, we were not able to characterize the existence of a linearizable weakly persistent center for system (5.2) due to the complexity of the ideal $\mathscr{L}^{wp}$ which is defined in (5.5). Thus, we investigate the existence of linearizable weakly persistent centers for a subfamily of system (5.2) which define a kind of a generalization of the Kolmogorov systems (which we call "semi-Kolmogorov" systems), i.e. systems of the form

$$\begin{aligned} \dot{x} =& ix + x\left(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2\right), \\ \dot{y} =& -iy + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{5.3}$$

We use in this chapter the procedures explained in Section 1.4 to compute the complex linearizability quantities. However, to simplify notations and nomenclatures we use here the classical terminology and notation - linearizability quantities and the same for the ideals and affine varieties - for representing the complex ones.

5.2 Persistent and weakly persistent linearizability quantities

In this section an approach for studying the linearizability problem for persistent centers and weakly persistent centers for system (5.2) is described. The first step is the calculation of the linearizability quantities for system

$$\begin{aligned} \dot{x} &= ix + \lambda\left(a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}y^2x + a_{03}y^3\right), \\ \dot{y} &= -iy + \mu\left(b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}y^2x + b_{03}y^3\right), \end{aligned} \tag{5.4}$$

$\lambda, \mu \in \mathbb{C}$. Using the method explained in Section 1.4 we obtain that

$$i_k = \sum_{m,n} i_k^{(m,n)}(a,b)\lambda^m\mu^n, \quad \text{and} \quad j_k = \sum_{m,n} j_k^{(m,n)}(a,b)\lambda^m\mu^n.$$

We look at these linearizability quantities as polynomials in λ and μ and denote the coefficient of $\lambda^m\mu^n$ as $i_k^{(m,n)}$ and $j_k^{(m,n)}$ for $i_k(\lambda,\mu,a,b)$ and $j_k(\lambda,\mu,a,b)$, respectively (where $a = (a_{20},a_{11},a_{02},a_{30},a_{21},a_{12},a_{03})$ and $b = (b_{20},b_{11},b_{02},b_{30},b_{21},b_{12},b_{03})$), and call it the $k_{(m,n)}-$th *persistent linearizability quantity* (according to i_k and j_k). If the origin is a linearizable center of (5.4) for all $\lambda,\mu \in \mathbb{C}$, then it is by Definition 5.1.3 a linearizable persistent center of (5.2).

Note, that by Theorem 2.1 of [27] one can always assume that $\lambda\mu \neq 0$ in (5.4). This gives rise to define the following sets of polynomials

$$L_k = \left\{ i_k^{(m,n)}, j_k^{(m,n)}; \quad m,n \in \mathbb{N}_0, m+n \neq 0 \right\}, \quad k = 1,2,3,\ldots$$

and the ideals

$$\begin{aligned} \mathscr{L}^p &:= \langle L_1, L_2, \ldots, L_k, \ldots \rangle, \\ \mathscr{L}_k^p &:= \langle L_1, L_2, \ldots, L_k \rangle. \end{aligned}$$

According to Definition 5.1.2 setting $\lambda=\mu$ to (5.1) one can consider the linearizability of a possibly weakly persistent center. Now, one can consider $i_k(\lambda,a,b)$ and $j_k(\lambda,a,b)$ and define the coefficients, $i_k^{(m)}$, $j_k^{(m)}$, corresponding to λ^m in the series expansion of the linearizabilizy quantities

$$i_k=\sum_m i_k^{(m)}(a,b)\lambda^m, \quad j_k=\sum_m j_k^{(m)}(a,b)\lambda^m,$$

and call them $k_{(m)}-$th *weakly persistent linearizability quantities*. Next, defining the following sets of polynomials

$$L_k^w=\left\{i_k^{(m)}, j_k^{(m)}; \quad m\in\mathbb{N}\right\}, \quad k=1,2,3,\dots$$

one can define the ideals

$$\begin{aligned}\mathscr{L}^{wp} &:= \langle L_1^w, L_2^w, \dots, L_k^w, \dots\rangle, \\ \mathscr{L}_k^{wp} &:= \langle L_1^w, L_2^w, \dots, L_k^w\rangle.\end{aligned} \tag{5.5}$$

Note that the ideals $\mathscr{L}^{wp}$, $\mathscr{L}_k^{wp}$, $\mathscr{L}^p$ and $\mathscr{L}_k^p$ are ideals in the polynomial ring $\mathbb{C}[a,b]$. It follows from the Hilbert Basis Theorem that the ideals $\mathscr{L}^p$ and $\mathscr{L}^{wp}$ are finitely generated, that is, there exists $N_1,N_2\geq 1$ such that for every $k>N_1$, $\mathscr{L}_k^p=\mathscr{L}_{N_1}^p$ and for every $k>N_2$, $\mathscr{L}_k^{wp}=\mathscr{L}_{N_2}^{wp}$. We define the corresponding persistent and weakly persistent linearizability variety for systems (5.1) in a natural way

$$V_{\mathscr{L}^p}=\mathbf{V}(\mathscr{L}^p), \quad V_{\mathscr{L}^{wp}}=\mathbf{V}(\mathscr{L}^{wp}).$$

In the rest of the work we will search for $V_{\mathscr{L}^p}$ for systems (5.4) and $V_{\mathscr{L}^{wp}}$ for systems (5.4) with $a_{02}=a_{03}=0$. Note that the persistent center variety and the persistent linearizability variety is much easier to obtain than the (regular) center variety and (regular) linearizability variety for a cubic system (5.2) since the corresponding persistent linearizability quantities, $i_k^{(m,n)}$, $j_k^{(m,n)}$, are splited compared to (regular) linearizability quantities $i_k(\lambda,\mu,a,b)$ and $j_k(\lambda,\mu,a,b)$.

In Figure 12 the relation between the varieties of (regular) centers $V_{\mathscr{C}}$, weakly persistent centers V_{wpc}, persistent centers V_{pc}, (regular) linearizable centers $V_{\mathscr{L}}$, linearizable weakly persistent centers, and linearizable persistent centers for system (5.2) is presented. Moreover, we obtain examples of systems in each one of them:

- system $\dot{x}=ix+x\left(a_{11}y+a_{12}y^2\right), \quad \dot{y}=-iy+y\left(b_{11}xy+b_{21}x^2y\right)$, has a persistent center at the origin (see [27], Corollary 3.1, case (2)) which is not linearizable, that is, it belongs to $V_{pc}\setminus V_{\mathscr{L}^p}$;
- system

$$\begin{aligned}\dot{x}=&ix+x\left(a_{20}x+a_{11}y+a_{30}x^2+a_{21}xy+\frac{b_{03}(2a_{11}+b_{02})}{b_{02}}y^2\right),\\ \dot{y}=&-iy+y\left(\frac{a_{11}a_{20}}{b_{02}}x+b_{02}y+\frac{a_{30}(2a_{11}+b_{02})}{b_{02}}x^2-a_{21}xy+b_{03}y^2\right),\end{aligned}$$

has a weakly persistent center at the origin (see [27], Theorem 3.1, case (2)) which is neither linearizable nor a persistent center, that is, it belongs to $V_{wpc} \setminus (V_{pc} \cup V_{\mathscr{L}^{wp}})$;

- system $\dot{x} = ix + x\left(a_{20}x + a_{30}x^2 + a_{12}y^2\right), \quad \dot{y} = -iy + y\left(b_{02}y + a_{30}x^2 + a_{12}y^2\right)$, has a linearizable weakly persistent center at the origin (see Theorem 5.3.2, case (12)) which is not a persistent center, that is, it belongs to $V_{\mathscr{L}^{wp}} \setminus V_{pc}$;
- system $\dot{x} = ix + x\left(a_{20}x + a_{30}x^2\right), \quad \dot{y} = -iy + y\left(b_{11}x + b_{02}y + b_{03}y^2\right)$, has a linearizable center at the origin (see [102], Theorem 3, case VIII (3)) which is not a weakly persistent center, that is, it belongs to $V_{\mathscr{L}} \setminus V_{wpc}$;
- system
$$\begin{aligned}\dot{x} &= ix + x\left(a_{20}x - b_{21}x^2 + b_{12}xy - b_{03}y^2\right),\\ \dot{y} &= -iy + y\left(b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2\right),\end{aligned}$$
has a center at the origin (see [48], Theorem 6, case J_2) which is neither a weakly persistent center nor linearizable, that is, it belongs to $V_{\mathscr{C}} \setminus (V_{wpc} \cup V_{\mathscr{L}})$.

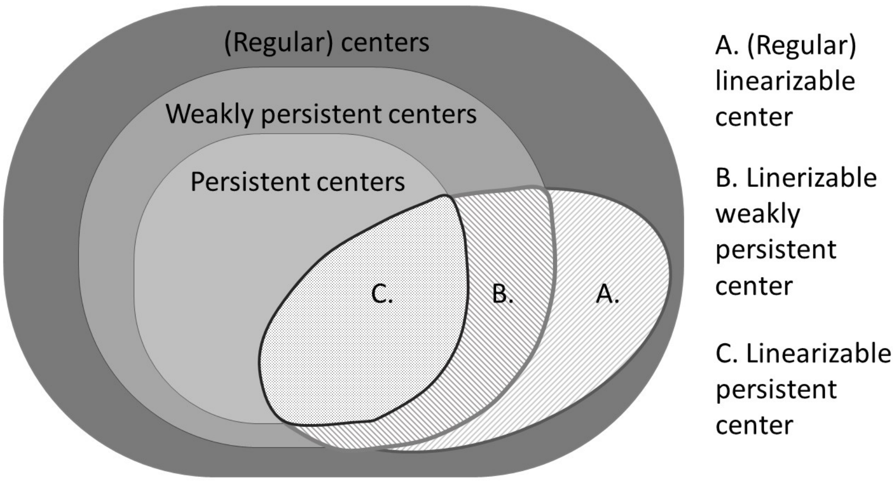

Figure 12 – The relation between the varieties $V_{\mathscr{C}}$, V_{pc}, V_{wpc}, $V_{\mathscr{L}}$, $V_{\mathscr{L}^p}$, $V_{\mathscr{L}^{wp}} \subset \mathbb{C}^{14}$ of the cubic system (5.2).

In the next two propositions we consider systems (1.24) and (5.1) as quasi-homogeneous systems of degree n, i.e. there exist $n \geq 2$ such that $X(tx,ty) = t^n X(x,y)$ and $Y(tx,ty) = t^n Y(x,y)$, $\forall x,y,t$.

Proposition 5.2.1. *If system (1.24) is quasi-homogeneous of degree n, $n \geq 2$, then the origin is a weakly persistent center if and only if it is a center.*

See [27, p. 114 – 115], for the proof.

Proposition 5.2.2. *If system (1.24) is quasi-homogeneous of degree n, $n \geq 2$, then the origin is a linearizable weakly persistent center if and only if it is a linearizable center, i.e, $V_{\mathscr{L}^{wp}} = V_{\mathscr{L}}$, where $V_{\mathscr{L}^{wp}}$ and $V_{\mathscr{L}}$ are the weak persistent linearizability variety and the linearizability variety respectively, of system (1.24).*

Proof. Under the assumption of the proposition we write systems (1.24) and (5.1) for $\lambda = \mu$ in the form

$$\dot{x} = ix + X_n(x,y), \quad \dot{y} = -iy + Y_n(x,y), \tag{5.6}$$

and

$$\dot{x} = ix + \lambda i X_n(x,y), \quad \dot{y} = -iy + \lambda i Y_n(x,y), \tag{5.7}$$

respectively, where $n \geq 2$, $X_n(x,y)$ and $Y_n(x,y)$ are homogeneous polynomials of degree n in x and y.

We shall prove that system (5.6) is equivalent to system (5.7), up to a linear change of variables, so both systems must have the same linearizability varieties. In fact, for any $\gamma \neq 0$ consider the linear change of variables

$$x_1 = \gamma x, \quad y_1 = \gamma y.$$

Applying this change to system (5.6) and using the homogeneity of X_n and Y_n we obtain the system

$$\dot{x}_1 = ix_1 + \left(\frac{1}{\gamma}\right)^n iX_n(x_1,y_1), \quad \dot{y}_1 = -iy_1 + \left(\frac{1}{\gamma}\right)^n iY_n(x_1,y_1).$$

Setting $\lambda = \left(\frac{1}{\gamma}\right)^n$ we arrive to (5.7), which completes the proof. $\square$

Now let us consider the relation between systems

$$\dot{x} = Ax + \mathbf{X}(x), \tag{5.8}$$

and

$$\dot{x} = Ax + \lambda \mathbf{X}(x), \tag{5.9}$$

in terms of the linearizing transformation, where $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, A is a complex matrix and $\mathbf{X} : \mathbb{C} \to \mathbb{C}$ is an analytic map starting with quadratic terms. Suppose that (5.8) can be linearized to

$$\dot{y} = Ay, \tag{5.10}$$

by the linearizing transformation

$$x = y + \mathbf{h}(y).$$

Suppose that $\sigma_A = \{\kappa_1, \kappa_2, ..., \kappa_n\}$ and $A = diag(\kappa_1, \kappa_2, ..., \kappa_n)$ (i.e. A is in the Jordan form). We follow the text in ([103, Sec 2.2]) and denote

$$\mathbf{h}(y) = \mathbf{h}^{(2)}(y) + \mathbf{h}^{(3)}(y) + \cdots + \mathbf{h}^{(k)}(y) + \cdots$$
$$\mathbf{X}(x) = \mathbf{X}^{(2)}(x) + \mathbf{X}^{(3)}(x) + \cdots + \mathbf{X}^{(k)}(x) + \cdots$$

and write $\mathscr{H}_k$ for the linear space of all possible (vectorial) monomials. For example for $x = (x_1, x_2) \in \mathbb{C}^2$ we have

$$\mathscr{H}_2 = \left\{ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} \right\}.$$

The basis elements of $\mathscr{H}_k$ are usually written as $\left(x_1^{\alpha_1}x_2^{\alpha_2},0\right)^T$ and $\left(0,x_1^{\alpha_1}x_2^{\alpha_2}\right)^T$. Denote

$$\kappa = (\kappa_1, \kappa_2, ..., \kappa_n),$$
$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

Now we define the homological operator $\mathscr{L} : \mathscr{H}_k \to \mathscr{H}_k$ as

$$\mathscr{L} : \mathbf{p}(y) \longmapsto d\mathbf{p}(y)Ay - A\mathbf{p}(y),$$

where $d\mathbf{p}$ is the Jacobian matrix of $\mathbf{p}$. In linear space $\mathscr{H}_k$ the eigenvalues, λ_k, of $\mathscr{L}$ are defined by

$$\lambda_m = \alpha \cdot \kappa - \kappa_m.$$

Any monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdot\dots\cdot x_n^{\alpha_n}$ in the $m-$th component on the right hand side of the systems (5.8) and (5.9) for which $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies the equation $\alpha \cdot \kappa - \kappa_m = 0$ is said to be *resonant*.

Denote by $\mathscr{K}_k$ the complement to Image($\mathscr{L}$) in $\mathscr{H}_k$. Note that resonant terms can not be removed from the normal form. They are elements of $\mathscr{K}_k$. Suppose that $\mathscr{K}_k = \varnothing$ for any $k \geq 2$ for system (5.8). According to Theorem 2.2.3 of [103, Sec 2.2] if for a system (5.8) for all $k \geq 2$ one has $\mathscr{K}_k = \varnothing$, then this system can be linearized to (5.10).

In the following lemma we find the relation between the linearizing transformation of (5.9) and the linearizing transformation of (5.8). By a let us denote all the parameters in $\mathbf{X}(x)$ from the right hand side of (5.8) and (5.9). Note that $\mathbf{h} = \mathbf{h}(a,y)$.

Lemma 5.2.3. *Suppose that $\mathbf{h} = \mathbf{h}(a,y)$ is a linearizing transformation of (5.8). Then $\widetilde{\mathbf{h}} = \mathbf{h}(\lambda a, y)$ is the linearizing transformation of (5.9).*

Proof. Suppose that for all $\mathscr{H}_k$ the sets $\mathscr{K}_k$ are empty for (5.8) and that

$$d\mathbf{h}^{(k)}(a,y)Ay - A\mathbf{h}^{(k)}(a,y) = \mathbf{X}^{(k)}(y), \tag{5.11}$$

for all y, then

$$\mathbf{h}(a,y) = \mathbf{h}^{(2)}(a,y) + \mathbf{h}^{(3)}(a,y) + \dots + \mathbf{h}^{(k)}(a,y) + \cdots.$$

It is obvious that systems (5.8) and (5.9) have the same resonant terms. Therefore, there exist also the linearizing transformation for (5.9) which we denote by $\widetilde{\mathbf{h}}$. For $\widetilde{\mathbf{h}}$ in $\mathscr{H}_k$ the set $\mathscr{K}_k$ is empty and $d\widetilde{\mathbf{h}}^{(k)}(y)Ay - A\widetilde{\mathbf{h}}^{(k)}(y) = \lambda\mathbf{X}^{(k)}(y)$ for all y.

Next we prove that $\widetilde{\mathbf{h}}$ can be set to the form $\widetilde{\mathbf{h}} = \mathbf{h}(\lambda a, y)$. First note that for all $k \geq 2$ and for all y and $\lambda \neq 0$ we have

$$\mathbf{h}^{(k)}(\lambda a, y) = \frac{1}{\lambda^{k-1}}\mathbf{h}^{(k)}(a, \lambda y) \quad \text{and} \quad \mathbf{X}^{(k)}(y) = \frac{1}{\lambda^k}\mathbf{X}^{(k)}(\lambda y). \tag{5.12}$$

Substituting y by λy in equation (5.11) and dividing it by λ^{k-1} yields

$$\frac{1}{\lambda^{k-2}}d\mathbf{h}^{(k)}(a,\lambda y)\cdot\frac{1}{\lambda}A\lambda y-\frac{1}{\lambda^{k-1}}A\mathbf{h}^{(k)}(a,\lambda y)=\frac{\lambda}{\lambda^{k}}\mathbf{X}^{(k)}(\lambda y).$$

Using equations (5.12) we rewrite the above equation in the form

$$d\mathbf{h}^{(k)}(\lambda a,y)Ay-A\mathbf{h}^{(k)}(\lambda a,y)=\lambda\mathbf{X}^{(k)}(y),$$

which proves that $\mathbf{h}(\lambda a,y)$ is a linearizing transformation for system (5.9). □

5.3 Main results

In the investigation of conditions for the linearizability of a persistent center of system (5.2) we obtain the following result.

Theorem 5.3.1. *System (5.2) has a linearizable persistent center at the origin if and only if one of the following conditions holds:*

(1) $b_{30}=b_{21}=b_{20}=b_{12}=b_{11}=a_{30}=a_{21}=a_{20}=0,$

(2) $b_{30}=b_{21}=b_{20}=b_{12}=b_{11}=a_{30}=a_{21}=a_{11}=0,$

(3) $b_{30}=b_{21}=b_{20}=b_{12}=b_{11}=a_{21}=a_{12}=a_{11}=a_{03}=a_{02}=0,$

(4) $b_{12}=b_{11}=b_{03}=a_{21}=a_{12}=a_{11}=a_{03}=a_{02}=0,$

(5) $b_{12}=b_{03}=b_{02}=a_{21}=a_{12}=a_{11}=a_{03}=a_{02}=0.$

Proof. The linearizability quantities of complex cubic system (5.4) are polynomials in λ and μ with coefficients $i_k^{(m,n)}(a,b)$, $j_k^{(m,n)}(a,b)$ being polynomials in a and b. So (5.2) has a linearizable center at the origin for all $\lambda,\mu\in\mathbb{C}$ if and only if all polynomials $i_k^{(m,n)}(a,b)$'s and $j_k^{(m,n)}(a,b)$'s vanish (i.e. $(a,b)\in V_{\mathscr{L}^p}$). We compute the first seven pairs, $i_1,\ j_1,\dots,i_7,\ j_7$, of linearizability quantities for system (5.4). As the quantities are very large we present below only the first pair

$$\begin{aligned}
i_1 =&a_{21}\lambda+ia_{11}a_{20}\lambda^2-ia_{11}b_{11}\lambda\mu-\frac{2}{3}ia_{02}b_{20}\lambda\mu,\\
j_1 =&b_{12}\mu+ia_{11}b_{11}\lambda\mu+\frac{2}{3}ia_{02}b_{20}\lambda\mu-ib_{02}b_{11}\mu^2.
\end{aligned}$$

As explained in the previous section, the next computational step is to compute the irreducible decomposition of the variety of ideal $\mathscr{L}^p=\langle L_1,L_2,\dots\rangle$. We use the routine `minAssGTZ` of SINGULAR. Performing computations over the field of the rational numbers the same decomposition is obtained for $\mathbf{V}(\mathscr{L}_5^p)$, $\mathbf{V}(\mathscr{L}_6^p)$ and $\mathbf{V}(\mathscr{L}_7^p)$, but for $\mathbf{V}(\mathscr{L}_4^p)$ a different decomposition is obtained. This lead us to expect that $\mathbf{V}(\mathscr{L}^p)=\mathbf{V}(\mathscr{L}_5^p)$.

The decomposition of $\mathbf{V}(\mathscr{L}_5^p)$ consists of five components listed in Theorem 5.3.1 and these are necessary conditions for a linearizable persistent center at the origin of system (5.2).

Now we need to prove that each of these five conditions is also sufficient.

Case (1). Systems (5.2) and (5.4) are

$$\dot{x} = ix + a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{5.13}$$

and

$$\dot{x} = ix + \lambda(a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{5.14}$$

respectively. System (5.14) has the Darboux factors

$$\begin{aligned}
g_0 =& y,\\
g_1 =& 1 + \frac{1}{2}i\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y,\\
g_2 =& 1 + \frac{1}{2}i\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y,
\end{aligned}$$

with respective cofactors

$$\begin{aligned}
l_0 =& -i + \mu b_{02}y + \mu b_{03}y^2,\\
l_1 =& \frac{1}{2}\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2,\\
l_2 =& \frac{1}{2}\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2.
\end{aligned}$$

It is easy to verify that the second equation on (1.39) is satisfied for

$$\beta_1 = \frac{\sqrt{\mu}b_{02} - \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}} \quad \text{and} \quad \beta_2 = -\frac{\sqrt{\mu}b_{02} + \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}},$$

so according to Theorem 1.4.4 we obtain that the second equation of (5.14) is linearizable by the change of coordinates

$$y_1 = y g_1^{\beta_1} g_2^{\beta_2}.$$

In order to find a change of coordinates for the first equation of (5.14) we look for the existence of a first integral. By Theorem 3.2 of [27], (5.14) possess a first integral $\Psi(x,y)$. Thus, as explained in Subsection 1.4.1, the first equation of (5.14) is linearizable by the change of coordinates $x_1 = \frac{\Psi}{y_1}$. Therefore, O is a linearizable persistent center of (5.13).

Case (2). Systems (5.2) and (5.4) are

$$\dot{x} = ix + a_{20}x^2 + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{5.15}$$

and

$$\dot{x} = ix + \lambda(a_{20}x^2 + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{5.16}$$

respectively. System (5.16) has the Darboux factors

$$\begin{aligned} g_0 &= y, \\ g_1 &= 1 + \frac{1}{2}i\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \\ g_2 &= 1 + \frac{1}{2}i\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \end{aligned}$$

with respective cofactors

$$\begin{aligned} l_0 &= -i + \mu b_{02}y + \mu b_{03}y^2, \\ l_1 &= \frac{1}{2}\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2, \\ l_2 &= \frac{1}{2}\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2. \end{aligned}$$

It is easy to verify that the second equation on (1.39) is satisfied for

$$\beta_1 = \frac{\sqrt{\mu}b_{02} - \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}} \quad \text{and} \quad \beta_2 = -\frac{\sqrt{\mu}b_{02} + \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}},$$

so according to Theorem 1.4.4 we obtain that the second equation of (5.14) is linearizable by the change of coordinates

$$y_1 = y g_1^{\beta_1} g_2^{\beta_2}.$$

In order to find a change of coordinates for the first equation of (5.14) we look for the existence of a first integral. By Theorem 3.2 (condition 7) of [27], (5.16) possess a first integral $\Psi(x,y)$. Thus, the first equation of (5.16) is linearizable by the change of coordinates $x_1 = \frac{\Psi}{y_1}$. Therefore, O is a linearizable persistent center of (5.15).

Case (3). Systems (5.2) and (5.4) are

$$\dot{x} = ix + a_{20}x^2 + a_{30}x^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{5.17}$$

and

$$\dot{x} = ix + \lambda(a_{20}x^2 + a_{30}x^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{5.18}$$

respectively. System (5.18) has the Darboux factors

$$\begin{aligned}
l_1 =& x,\\
l_2 =& 1-\frac{1}{2}i\left(\lambda a_{20}-\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x,\\
l_3 =& 1-\frac{1}{2}i\left(\lambda a_{20}+\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x,\\
l_4 =& y,\\
l_5 =& 1+\frac{1}{2}i\left(\mu b_{02}-\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y,\\
l_6 =& 1+\frac{1}{2}i\left(\mu b_{02}+\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y,
\end{aligned}$$

with respective cofactors

$$\begin{aligned}
k_1 =& i+\lambda a_{20}x+\lambda a_{30}x^2,\\
k_2 =& \frac{1}{2}\left(\lambda a_{20}-\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x+\lambda a_{30}x^2,\\
k_3 =& \frac{1}{2}\left(\lambda a_{20}+\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x+\lambda a_{30}x^2,\\
k_4 =& -i+\mu b_{02}y+\mu b_{03}y^2,\\
k_5 =& \frac{1}{2}\left(\mu b_{02}-\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y+\mu b_{03}y^2,\\
k_6 =& \frac{1}{2}\left(\mu b_{02}+\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y+\mu b_{03}y^2.
\end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$, $f_1 = l_2$, $f_2 = l_3$, $g_0 = l_4$, $g_1 = l_5$ and $g_2 = l_6$, the equations on (1.39) are satisfied for

$$\alpha_1 = \frac{\sqrt{\lambda}a_{20}-\sqrt{-4ia_{30}+\lambda a_{20}^2}}{2\sqrt{-4ia_{30}+\lambda a_{20}^2}},\quad \alpha_2 = -\frac{\sqrt{\lambda}a_{20}+\sqrt{-4ia_{30}+\lambda a_{20}^2}}{2\sqrt{-4ia_{30}+\lambda a_{20}^2}},$$

$$\beta_1 = \frac{\sqrt{\mu}b_{02}-\sqrt{4ib_{03}+\mu b_{02}^2}}{2\sqrt{4ib_{03}+\mu b_{02}^2}}\quad\text{and}\quad \beta_2 = -\frac{\sqrt{\mu}b_{02}+\sqrt{4ib_{03}+\mu b_{02}^2}}{2\sqrt{4ib_{03}+\mu b_{02}^2}}.$$

So we obtain that (5.18) is linearizable by the change of coordinates

$$x_1 = xf_1^{\alpha_1}f_2^{\alpha_2},\quad y_1 = yg_1^{\beta_1}g_2^{\beta_2}.$$

Therefore, O is a linearizable persistent center of (5.17).

Case (4). System (5.2) satisfying conditions (4) of this theorem can be transformed into system (5.15), where

$$(a_{20},a_{30},b_{02},b_{20},b_{21},b_{30}) = -(b_{02},b_{03},a_{20},a_{02},a_{12},a_{03}),$$

by the change of coordinates $(x,y,t) \to (y,x,-t)$. Therefore, O is a linearizable persistent center for this case.

Case (5). System (5.2) satisfying conditions (5) of this theorem can be transformed into system (5.13), where

$$(a_{20}, a_{30}, b_{11}, b_{20}, b_{21}, b_{30}) = -(b_{02}, b_{03}, a_{11}, a_{02}, a_{12}, a_{03}),$$

by the change of coordinates $(x,y,t) \to (y,x,-t)$. Therefore, O is a linearizable persistent center for this case. $\square$

In the investigation of conditions for the linearizability of a weakly persistent center of system (5.3) we obtain the following result. Note that the Lotka-Volterra systems considered in [27], [38] and [102] are all subcases of (5.3).

Theorem 5.3.2. *System (5.3) has a linearizable weakly persistent center at the origin if and only if one of the following conditions holds*

(1) $b_{21} = b_{20} = b_{12} = b_{11} = b_{03} = a_{30} = a_{21} = a_{20} = a_{12} = b_{02} - 3a_{11} = 0$,

(2) $b_{30} = b_{21} = b_{12} = b_{11} = b_{02} = a_{30} = a_{21} = a_{20} = a_{11} = 2a_{12} - b_{03} = 0$,

(3) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{20} = 0$,

(4) $b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{20} = 3a_{12} - b_{03} = 3a_{11} + b_{02} = 0$,

(5) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{11} = 0$,

(6) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{21} = a_{20} = a_{12} = 2a_{11} + b_{02} = 0$,

(7) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{21} = a_{12} = a_{11} = 0$,

(8) $b_{30} = b_{21} = b_{20} = b_{12} = b_{02} = a_{30} = a_{21} = a_{11} = a_{20} - 2b_{11} = a_{12} - 2b_{03} = 0$,

(9) $b_{30} = b_{21} = b_{20} = b_{12} = b_{02} = a_{21} = a_{12} = a_{11} = a_{20} + 2b_{11} = 0$,

(10) $b_{30} = b_{12} = b_{11} = b_{03} = a_{21} = a_{20} = a_{12} = 2a_{30} - b_{21} = 2a_{11} - b_{02} = 0$,

(11) $b_{12} = b_{11} = b_{03} = a_{21} = a_{12} = a_{11} = 0$,

(12) $b_{30} = b_{12} = b_{11} = a_{21} = a_{11} = a_{30} - b_{21} = a_{12} - b_{03} = 0$,

(13) $b_{12} = b_{03} = b_{02} = a_{21} = a_{12} = a_{11} = 0$,

(14) $b_{30} = b_{20} = b_{12} = a_{21} = a_{30} - b_{21} = a_{20} - b_{11} = a_{12} - b_{03} = a_{11} - b_{02} = b_{03}b_{11}^2 + b_{02}^2 b_{21} = 0$.

Proof. The linearizability quantities of system

$$\begin{aligned}\dot{x} =& ix+\lambda x\left(a_{20}x+a_{11}y+a_{30}x^2+a_{21}xy+a_{12}y^2\right),\\ \dot{y} =& -iy+\lambda\left(b_{20}x^2+b_{11}xy+b_{02}y^2+b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3\right),\end{aligned} \tag{5.19}$$

are polynomials in λ with coefficients $i_k^{(m)}(a,b)$, $j_k^{(m)}(a,b)$ being polynomials in a and b. So (5.3) has a linearizable center at the origin for all $\lambda \in \mathbb{C}$ if and only if all polynomials $i_k^{(m)}(a,b)$'s and $j_k^{(m)}(a,b)$'s vanish (i.e. $(a,b) \in \mathbf{V}(\mathscr{L}^{wp})$).

We compute the first six pairs of linearizability quantities, i_1, j_1, ..., i_6, j_6 of system (5.19). As the quantities are very large we present only the first pair:

$$\begin{aligned}i_1 =& a_{21}\lambda+i\left(a_{11}a_{20}-a_{11}b_{11}\right)\lambda^2,\\ j_1 =& b_{12}\lambda+i\left(a_{11}b_{11}-b_{02}b_{11}\right)\lambda^2.\end{aligned}$$

Performing computations using routine `minAssGTZ` of SINGULAR over the field of the rational numbers the same decomposition is obtained for $\mathbf{V}(\mathscr{L}_5^{wp})$ and $\mathbf{V}(\mathscr{L}_6^{wp})$ but for $\mathbf{V}(\mathscr{L}_4^{wp})$ a different decomposition is obtained. It lead us to expect that $\mathbf{V}(\mathscr{L}^{wp}) = \mathbf{V}(\mathscr{L}_5^{wp})$.

The decomposition of $\mathbf{V}(\mathscr{L}_5^{wp})$ consists of 14 components listed in the statement of the theorem. Thus, the necessary conditions for existence of a linearizable weakly persistent center at the origin for (5.3) are obtained.

Now we prove the sufficiency of each of these 14 conditions.

Case (1). Systems (5.3) and (5.19) are

$$\dot{x} = ix+a_{11}xy, \quad \dot{y} = -iy+3a_{11}y^2+b_{30}x^3, \tag{5.20}$$

and

$$\dot{x} = ix+\lambda(a_{11}xy), \quad \dot{y} = -iy+\lambda(3a_{11}y^2+b_{30}x^3), \tag{5.21}$$

respectively. System (5.21) has the Darboux factors

$$\begin{aligned}l_1 =& x,\\ l_2 =& 1+3i\lambda a_{11}y-\lambda^2 a_{11}b_{30}x^3,\\ l_3 =& y+\frac{1}{4}i\lambda b_{30}x^3,\end{aligned}$$

with respective cofactors

$$\begin{aligned}k_1 =& i+\lambda a_{11}y,\\ k_2 =& 3\lambda a_{11}y,\\ k_3 =& -i+3\lambda a_{11}y.\end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$, $f_1 = l_2$, $g_0 = l_3$ and $g_1 = l_2$ the equations on (1.39) are satisfied for $\alpha_1 = -\dfrac{1}{3}$ and $\beta_1 = -1$. So we obtain that (5.21) is linearizable by the change of coordinates

$$
\begin{aligned}
x_1 =& x\left(1+3i\lambda a_{11}y-\lambda^2 a_{11}b_{30}x^3\right)^{-1/3},\\
y_1 =& \left(y+\frac{1}{4}i\lambda b_{30}x^3\right)\left(1+3i\lambda a_{11}y-\lambda^2 a_{11}b_{30}x^3\right)^{-1}.
\end{aligned}
$$

Therefore, O is a linearizable weakly persistent center of (5.20).

Case (2). Systems (5.3) and (5.19) are

$$
\dot{x}=ix+a_{12}xy^2,\quad \dot{y}=-iy+b_{20}x^2+2a_{12}y^3, \tag{5.22}
$$

and

$$
\dot{x}=ix+\lambda(a_{12}xy^2),\quad \dot{y}=-iy+\lambda(b_{20}x^2+2a_{12}y^3), \tag{5.23}
$$

respectively. System (5.23) has the Darboux factors

$$
\begin{aligned}
l_1 =& x,\\
l_2 =& 1+2i\lambda a_{12}y^2-4\lambda^2 a_{12}b_{20}x^2y-i\lambda^3 a_{12}b_{20}^2x^4,\\
l_3 =& y+\frac{1}{3}i\lambda b_{20}x^2,
\end{aligned}
$$

with respective cofactors

$$
\begin{aligned}
k_1 =& i+\lambda a_{12}y^2,\\
k_2 =& 4\lambda a_{12}y^2,\\
k_3 =& -i+2\lambda a_{12}y^2.
\end{aligned}
$$

Choosing $f_0=l_1$, $f_1=l_2$, $g_0=l_3$ and $g_1=l_2$ the equations on (1.39) are satisfied for $\alpha_1=-\dfrac{1}{4}$ and $\beta_1=-\dfrac{1}{2}$. So we obtain that (5.23) is linearizable by the change of coordinates

$$
\begin{aligned}
x_1 =& x\left(1+2i\lambda a_{12}y^2-4\lambda^2 a_{12}b_{20}x^2y-i\lambda^3 a_{12}b_{20}^2x^4\right)^{-1/4},\\
y_1 =& \left(y+\frac{1}{3}i\lambda b_{20}x^2\right)\left(1+2i\lambda a_{12}y^2-4\lambda^2 a_{12}b_{20}x^2y-i\lambda^3 a_{12}b_{20}^2x^4\right)^{-1/2}.
\end{aligned}
$$

Therefore, O is a linearizable weakly persistent center of (5.22).

Case (3). Systems (5.3) and (5.19) are

$$
\dot{x}=ix+a_{11}xy+a_{12}xy^2,\quad \dot{y}=-iy+b_{02}y^2+b_{03}y^3, \tag{5.24}
$$

and

$$
\dot{x}=ix+\lambda(a_{11}xy+a_{12}xy^2),\quad \dot{y}=-iy+\lambda(b_{02}y^2+b_{03}y^3), \tag{5.25}
$$

respectively. System (5.24) is considered in [38], Theorem 2 (case 4), where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.25) is also linearizable. Therefore, O is a linearizable weakly persistent center of (5.24).

Case (4). Systems (5.3) and (5.19) are

$$\begin{aligned}\dot{x} &= x(i+a_{11}y+a_{12}y^2),\\ \dot{y} &= -iy+b_{30}x^3-3a_{11}y^2+3a_{12}y^3,\end{aligned}$$

and

$$\begin{aligned}\dot{x} &= x(i+a_{11}\lambda y+a_{12}\lambda y^2)=P(x,y),\\ \dot{y} &= -iy+b_{30}\lambda x^3-3a_{11}\lambda y^2+3a_{12}\lambda y^3=Q(x,y).\end{aligned} \tag{5.26}$$

For system (5.26) we find only one Darboux factor $f_0 = x$, with the corresponding cofactor $K_0 = i + x\,(a_{20} + a_{30}x)\,\lambda$, which is not enough to construct neither a Darboux first integral nor an integrating factor. To prove that there exists a first integral of the form (1.27) we follow the approach in [55] and first make the substitution $z = x/y$. In the new coordinates system (5.26) takes the form

$$\begin{aligned}\dot{z} =& z(2i+4a_{11}\lambda y-2a_{12}\lambda y^2-b_{30}\lambda y^2z^3),\\ \dot{y} =& -iy-3a_{11}\lambda y^2+3a_{12}\lambda y^3+b_{30}\lambda y^3z^3.\end{aligned} \tag{5.27}$$

We look for a first integral for (5.27) of the form

$$F(z,y)=zy^2\sum_{k=2}^{\infty}f_k(z)y^k.$$

The functions f_k are determined recursively by the differential equation

$$\begin{aligned}f_k'(z)-\frac{k}{2z}f_k(z)=&\frac{-\lambda((3k-2)a_{12}-(k-1)b_{30}z^3)f_{k-2}(z)}{2iz}-\frac{4\lambda a_{11}f_{k-1}'(z)}{2i}+\\ &+\frac{(3n-1)\lambda a_{11}f_{k-1}(z)}{2iz}+\frac{\lambda(2a_{12}+b_{30}z^3)f_{k-2}'(z)}{2i}.\end{aligned} \tag{5.28}$$

For $k = 2,3,4,5,6,7$ we find

$$\begin{aligned}f_2(z)=&z,\quad f_3(z)=4i\lambda a_{11}z,\quad f_4(z)=\frac{1}{2}i\lambda z(b_{30}z^3+28i\lambda a_{11}^2-8a_{12}),\\ f_5(z)=&\frac{1}{3}\lambda^2a_{11}z(84a_{12}-13b_{30}z^3-140i\lambda a_{11}^2),\\ f_6(z)=&\frac{1}{48}\lambda^2z(-3(224a_{12}^2-96a_{12}b_{30}z^3+b_{30}^2z^6)+40ia_{11}^2(168a_{12}-31b_{30}z^3)\lambda+7280a_{11}^4\lambda^2),\\ f_7(z)=&\frac{1}{42}z(-5880ia_{11}a_{12}^2\lambda^3+2870ia_{11}a_{12}b_{30}z^3\lambda^3-55ia_{11}b_{30}^2z^6\lambda^3-25480a_{11}^3a_{12}\lambda^4\\ &+5460a_{11}^3b_{30}z^3\lambda^4+20384ia_{11}^5\lambda^5).\end{aligned}$$

We claim that $f_k(z) = zp_{k-a}$ if $k \equiv a(mod\ 3)$, where $a \in \{0,1,2\}$ and p_{k-a} is a polynomial of degree $k-a$. Thus, we have three cases:

(i) $k \equiv 0(mod\ 3)$;

(ii) $k \equiv 1(mod\ 3)$;

(iii) $k \equiv 2(mod\ 3)$.

Assume that $f_k(z) = zp_{k-a}(z)$ for $k = 1,\dots,n-1$ where $k \equiv a(mod\ 3)$ and $a \in \{0,1,2\}$. We compute $f_k(z)$ for $k = n$. To this end we solve (5.28) for $k = n$. Let us first consider case (i). If $k = n \equiv 0(mod\ 3)$ then $n-1 \equiv 2(mod\ 3)$ and $n-2 \equiv 1(mod\ 3)$ and $f_{n-1}(z) = zq_{n-3}(z)$, $f_{n-2}(z) = zr_{n-3}(z)$, where $q_{n-3}(z)$ and $r_{n-3}(z)$ are polynomials of degree $n-3$. We want to show that $f_n(z) = zp_n(z)$, where $p_n(z)$ is polynomial of degree n. Using the induction assumption about f_{n-1} and f_{n-2} differential equation (5.28) becomes

$$f_n'(z) - \frac{n}{2z}f_n(z) = P_n(z) + zQ_{n-1}(z) = C_n(z),$$

where $P_n(z)$, $C_n(z)$ and $Q_{n-1}(z)$ are polynomials of degree n, n and $n-1$, respectively and, if $P_n(z) = A_0 + A_1 z + \cdots + A_n z^n$ and $Q_n(z) = B_0 + B_1 z + \cdots + B_n z^n$, then $C_n(z) = c_0 + c_1 z + \cdots + c_n z^n$, where $c_0 = A_0$, $c_1 = A_1 + B_0$, ..., $c_n = A_n + B_{n-1}$.

As the general solution of linear differential equation of the form $f'(z) + p(z)f(z) = q(z)$ is

$$f(z) = Ce^{-\int p(z)dz} + e^{-\int p(z)dz} \int e^{\int p(z)dz} q(z)dz,$$

and in our case we have $p(z) = -\frac{n}{2z}$ and $q(z) = C_n(z)$, it follows that $e^{-\int p(z)dz} = z^{\frac{n}{2}}$ and the solution is

$$\begin{aligned}
f_n(z) =& Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{C_n(z)}{z^{\frac{n}{2}}} dz \\
=& Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int (c_0 z^{-\frac{n}{2}} + (c_1)z^{1-\frac{n}{2}} + \cdots + (c_n)z^{\frac{n}{2}})dz \\
=& Cz^{\frac{n}{2}} + z^{\frac{n}{2}}(c_0 z^{1-\frac{n}{2}} + c_1 z^{2-\frac{n}{2}} + \cdots + c_n z^{\frac{n}{2}+1}) \\
=& Cz^{\frac{n}{2}} + c_0 z + c_1 z^2 + \cdots + c_n z^{n+1} \\
=& Cz^{\frac{n}{2}} + z(c_0 + c_1 z + \cdots + c_n z^n).
\end{aligned}$$

Choosing $C = 0$ we obtain $f_n(z) = zp_n(z)$.

Now we shortly consider case (ii). If $n-1 \equiv 0\ (mod\ 3)$ then $n-2 \equiv 2\ (mod\ 3)$ and $n \equiv 1\ (mod\ 3)$. Assuming that $f_{n-1}(z) = zq_{n-1}(z)$ and $f_{n-2}(z) = zr_{n-4}(z)$, we want to show that $f_n = zp_{n-1}$, where q_{n-1} and p_{k-1} are polynomials of degree $n-1$ and r_{n-4} is polynomial of degree $n-4$. Using the induction assumption about f_{n-1} and f_{n-2} differential equation (5.28) becomes

$$f_n'(z) - \frac{n}{2z}f_n(z) = P_{n-1}(z) + zQ_{n-2}(z) = C_{n-1}(z),$$

where $P_{n-1}(z), C_{n-1}(z)$ and $Q_{n-2}(z)$ are polynomials of degree $n-1, n-1$ and $n-2$, respectively. Then the solution to (5.28) is

$$\begin{aligned}
f_n(z) =& Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{C_{n-1}(z)}{z^{\frac{n}{2}}} dz \\
=& Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{c_0 + c_1 z + \cdots c_{n-1} z^{n-1}}{z^{\frac{n}{2}}} dz \\
=& Cz^{\frac{n}{2}} + z^{\frac{n}{2}}(c_0 z^{1-\frac{n}{2}} + c_1 z^{2-\frac{n}{2}} + \cdots + c_{n-1} z^{\frac{n}{2}}) \\
=& Cz^{\frac{n}{2}} + z(c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}).
\end{aligned}$$

Choosing again $C=0$ we have shown that $f_n(z)=zp_{n-1}(z)$.

In the case (iii) when $k-2=n-2\equiv 0$ $(mod\ 3)$ the proof is similar as in the previous two cases. We assume that $f_{n-2}=zr_{n-2}(z)$ and $f_{n-1}=zq_{n+1}(z)$ and we prove that $f_n=zp_{n+1}(z)$.

Thus, we have shown that $f_2(z)=z$, $f_3(z)=4i\lambda a_{11}z$ and for $k>3$ we have proven that $f_k=zp_{k-a}$, if $k\equiv a$ $(mod\ 3)$. Therefore, (formal) first integral of system (5.27) is $F(z,y)=z^2y^4+\sum_{i+j=7}^{\infty}\phi_{ij}z^iy^j$ (and $\Psi(x,y)=\sqrt{F(x/y,y)}$ is a first integral of system (5.26) of the form (1.27)).

Now we look for a linearization $Y=Y(x,y)$ of the second equation of system (5.26). The function Y should satisfy the equation

$$\frac{\partial Y}{\partial x}P(x,y)+\frac{\partial Y}{\partial y}Q(x,y)+iY=0.$$

After the substitution $z=x/y$, this equation is written as

$$\frac{1}{y}\frac{\partial Y}{\partial z}P(x,y)+(\frac{\partial Y}{\partial y}-\frac{z}{y}\frac{\partial Y}{\partial z})Q(x,y)+iY=0.$$

We look for a transformation of the form

$$Y(z,y)=\sum_{k=1}^{\infty}g_k(z)y^k,$$

where $g_k(z)$ are polynomials satisfying a first-order linear differential equation,

$$\begin{aligned}&(k-2)\lambda(3a_{12}+b_{30}z^3)g_{k-2}(z)-3(k-1)\lambda a_{11}g_{k-1}(z)-(k-1)ig_k(z)\\&+z(-\lambda(2a_{12}+b_{30}z^3)g'_{k-2}(z)+4\lambda a_{11}g'_{k-1}(z)+2ig'_k(z))=0.\end{aligned}\tag{5.29}$$

Solving the equation we obtain

$$\begin{aligned}g_1(z)=&1,\quad g_2(z)=3i\lambda a_{11},\quad g_3(z)=-\frac{3}{2}i\lambda a_{12}+\frac{1}{4}i\lambda b_{30}z^3-9\lambda^2a_{11}^2,\\g_4(z)=&\frac{3}{4}\lambda^2a_{11}(14a_{12}-3b_{30}z^3-36i\lambda a_{11}^2).\end{aligned}$$

We assume that $g_k(z)=p_{k-a}(z)$ where $k\equiv a$ $(mod\ 3)$ and $a\in\{0,1,2\}$. We prove this again by induction. We assume that equation $g_k(z)=p_{k-a}(z)$ holds for $k=1,2,\dots,n-1$ and we want to show that this is the case also for $k=n$. To end this task one must consider three cases: $a=0$, $a=1$ and $a=2$. The proof is very similar as the proof of integrability for system (5.26), thus we only briefly consider the case $a=0$, i.e. $k=n\equiv 0$ $(mod\ 3)$. By the induction assumption we have $g_{n-2}(z)=r_{n-3}(z)$ and $g_{n-1}(z)=q_{n-3}(z)$, and we want to show that $g_n(z)=r_n(z)$. Differential equation (5.29) becomes

$$g'_n(z)-\frac{n-1}{2z}g_n(z)=\frac{C_n(z)}{z},$$

where $C_n(z)$ is polynomial of degree n. The solution of the latter equation is

$$\begin{aligned}g_n(z)=&Cz^{\frac{n-1}{2}}+z^{\frac{n-1}{2}}\int\frac{C_n(z)}{z^{\frac{n+1}{2}}}dz\\=&Cz^{\frac{n-1}{2}}+c_0+c_1z+\cdots+c_nz^n.\end{aligned}$$

Choosing $C = 0$ we obtain $g_n(z) = p_n(z)$, where $p_n(z)$ is polynomial of degree n. In a similar way one can prove that if $n-2 \equiv 0$ $(mod\ 3)$ $(n-1 \equiv 0$ $(mod\ 3)$ $)$ then $g_n = p_{n-2}$ $(g_n = p_{n-1})$.

Then $Y(\frac{x}{y}, y)$ is a series in x and y of the form $Y(x,y) = y + \sum_{i+j=2}^{\infty} Y_{ij}x^i y^j$ and the second equation of (5.26) can be linearized by the transformation $Y = Y(x,y)$. The first equation of (5.26) can be linearized by the transformation $X = X(x,y) = \frac{\Psi(x,y)}{Y(x,y)}$ [1] .

Case (5). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(a_{20}x + a_{12}y^2), \quad \dot{y} = -iy + y(b_{02}y + b_{03}y^2), \tag{5.30}$$

and

$$\dot{x} = ix + \lambda x(a_{20}x + a_{12}y^2), \quad \dot{y} = -iy + \lambda y(b_{02}y + b_{03}y^2), \tag{5.31}$$

respectively. System (5.30) is considered in [38, Theorem 2, case 3], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.31) is linearizable. Therefore, O is a linearizable weakly persistent center of (5.30).

Case (6). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(a_{30}x^2 + a_{11}y), \quad \dot{y} = -iy + y(-2a_{11}y + b_{03}y^2), \tag{5.32}$$

and

$$\dot{x} = ix + \lambda x(a_{30}x^2 + a_{11}y), \quad \dot{y} = -iy + \lambda y(-2a_{11}y + b_{03}y^2), \tag{5.33}$$

respectively. System (5.32) is considered in [38, Theorem 2, case 5], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.33) is linearizable. Therefore, O is a linearizable weakly persistent center of (5.32).

Case (7). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(a_{20}x + a_{30}x^2), \quad \dot{y} = -iy + y(b_{02}y + b_{03}y^2), \tag{5.34}$$

and

$$\dot{x} = ix + \lambda x(a_{20}x + a_{30}x^2), \quad \dot{y} = -iy + \lambda y(b_{02}y + b_{03}y^2), \tag{5.35}$$

respectively. System (5.34) is a subcase of the case considered in [102, Theorem 3, case VIII (3)], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.35) is linearizable. Therefore, O is a linearizable weakly persistent center of (5.34).

Case (8). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(2b_{11}x + 2b_{03}y^2), \quad \dot{y} = -iy + y(b_{11}x + b_{03}y^2), \tag{5.36}$$

and

$$\dot{x} = ix + \lambda x(2b_{11}x + 2b_{03}y^2), \quad \dot{y} = -iy + \lambda y(b_{11}x + b_{03}y^2), \tag{5.37}$$

[1] We thank Prof. B. Ferčec for the proof of *Case (4)* of Theorem 5.3.2.

respectively. System (5.36) is considered in [38, Theorem 2, case 6], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.37) is linearizable. Therefore, O is a linearizable weakly persistent center of (5.36).

Case (9). System (5.3) satisfying conditions (9) of this theorem can be transformed into system (5.3) satisfying conditions (6) of this theorem , where

$$(a_{30}, b_{03}, b_{11}) = -(b_{03}, a_{30}, a_{11}),$$

by the change of coordinates $(x, y, t) \to (y, x, -t)$. Therefore, O is a linearizable weakly persistent center for this case

Case(10). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + b_{20}x^2 + 2a_{11}y^2 + 2a_{30}x^2y, \tag{5.38}$$

and

$$\dot{x} = ix + \lambda x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + \lambda(b_{20}x^2 + 2a_{11}y^2 + 2a_{30}x^2y), \tag{5.39}$$

respectively. System (5.39) has the Darboux factors

$$\begin{aligned} l_1 =& x, \\ l_2 =& 1 + 2i\lambda a_{11}y - \left(i\lambda a_{30} + \lambda^2 a_{11}b_{20}\right)x^2, \\ l_3 =& y + \frac{1}{3}i\lambda b_{20}x^2, \end{aligned}$$

with respective cofactors

$$\begin{aligned} k_1 =& i + \lambda a_{11}y + \lambda a_{30}x^2, \\ k_2 =& 2\lambda a_{11}y + 2\lambda a_{30}x^2, \\ k_3 =& -i + 2\lambda a_{11}y + 2\lambda a_{30}x^2. \end{aligned}$$

Choosing $f_0 = l_1$, $f_1 = l_2$, $g_0 = l_3$ and $g_1 = l_2$ the equations on (1.39) are satisfied for $\alpha_1 = -\dfrac{1}{2}$ and $\beta_1 = -1$. So we obtain that (5.39) is linearizable by the change of coordinates

$$\begin{aligned} x_1 =& x\left(1 + 2i\lambda a_{11}y - \left(i\lambda a_{30} + \lambda^2 a_{11}b_{20}\right)x^2\right)^{-1/2}, \\ y_1 =& \left(y + \frac{1}{3}i\lambda b_{20}x^2\right)\left(1 + 2i\lambda a_{11}y - \left(i\lambda a_{30} + \lambda^2 a_{11}b_{20}\right)x^2\right)^{-1}. \end{aligned}$$

Therefore, O is a linearizable weakly persistent center of (5.38).

Case (11). Systems (5.3) and (5.19) are

$$\begin{aligned} \dot{x} &= ix + x\left(a_{20}x + a_{30}x^2\right), \\ \dot{y} &= -iy + b_{20}x^2 + b_{30}x^3 + b_{21}x^2y + b_{02}y^2, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= ix + \lambda x\left(a_{20}x + a_{30}x^2\right) = \bar{P}(x,y), \\ \dot{y} &= -iy + \lambda(b_{20}x^2 + b_{30}x^3 + b_{21}x^2y + b_{02}y^2) = \bar{Q}(x,y). \end{aligned} \tag{5.40}$$

We find the Darboux factors

$$
\begin{aligned}
f_0 &= x,\\
f_1 &= 1+\frac{1}{2}x\left(-ia_{20}\lambda-\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}\right),\\
f_2 &= 1+\frac{1}{2}x\left(-ia_{20}\lambda+\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}\right),
\end{aligned}
$$

with the corresponding cofactors

$$
\begin{aligned}
k_0 &= i+x\left(a_{20}+a_{30}x\right)\lambda,\\
k_1 &= \frac{1}{2}x\left(a_{20}\lambda+2a_{30}x\lambda-i\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}\right),\\
k_2 &= \frac{1}{2}x\left(a_{20}\lambda+2a_{30}x\lambda+i\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}\right).
\end{aligned}
$$

We found the solution for the first equation on (1.39) for k_1 and k_2 to be

$$
\alpha_1=-\frac{1}{2}-\frac{a_{20}\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}}{8a_{30}+2ia_{20}^2\lambda},\quad \alpha_2=-\frac{1}{2}+\frac{a_{20}\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}}{8a_{30}+2ia_{20}^2\lambda}.
$$

Therefore the first equation of (5.40) is linearizable by the change of coordinates

$$
x_1 = x f_1^{\alpha_1} f_2^{\alpha_2}.
$$

Using the Darboux factors $p_1=f_1$ and $p_2=f_2$ with corresponding cofactors $K_1=k_1$ and $K_2=k_2$ we can construct an integrating factor $\mu=p_1^{\beta_1}p_2^{\beta_2}$ by solving equation (1.32) for β_1, β_2 and obtain

$$
\begin{aligned}
\beta_1 =& -\frac{12a_{30}+\left(4b_{21}+a_{20}\left(\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}+3ia_{20}\lambda\right)\right)}{2\left(4a_{30}+ia_{20}^2\lambda\right)}+\\
&+\frac{a_{20}b_{21}\left(\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}-ia_{20}\lambda\right)}{2a_{30}\left(4a_{30}+ia_{20}^2\lambda\right)},\\
\beta_2 =& -\frac{12a_{30}+\left(4b_{21}+a_{20}\left(3ia_{20}\lambda-\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}\right)\right)}{2\left(4a_{30}+ia_{20}^2\lambda\right)}+\\
&+\frac{a_{20}b_{21}\left(\sqrt{\lambda\left(4ia_{30}-a_{20}^2\lambda\right)}+ia_{20}\lambda\right)}{2a_{30}\left(4a_{30}+ia_{20}^2\lambda\right)},
\end{aligned}
$$

which implies that system (5.40) has an analytic first integral of the form (1.27) according to [3]. So the second equation of (5.40) is linearizable by the change of coordinates $y_1=\frac{\Psi}{x_1}$.

Case (12). Systems (5.3) and (5.19) are

$$
\begin{aligned}
\dot{x} &= ix+x\left(a_{20}x+a_{30}x^2+a_{12}y^2\right),\\
\dot{y} &= -iy+b_{20}x^2+a_{30}x^2y+b_{02}y^2+a_{12}y^3,
\end{aligned}
$$

and

$$\begin{aligned}\dot{x} &= ix + \lambda x\left(a_{20}x + a_{30}x^2 + a_{12}y^2\right),\\ \dot{y} &= -iy + \lambda\left(b_{20}x^2 + a_{30}x^2y + b_{02}y^2 + a_{12}y^3\right).\end{aligned} \tag{5.41}$$

We find the Darboux factors

$$\begin{aligned}
f_0 &= x,\\
f_1 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda - \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) - x(a_{20}\lambda + A_+)\Big),\\
f_2 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda - \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) + x(-a_{20}\lambda + A_+)\Big),\\
f_3 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda + \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) - x(a_{20}\lambda + A_-)\Big),\\
f_4 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda + \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) + x(-a_{20}\lambda + A_-)\Big),
\end{aligned}$$

with corresponding cofactors

$$\begin{aligned}
k_0 &= i + (x(a_{20} + a_{30}x) + a_{12}y^2)\lambda,\\
k_1 &= \frac{1}{2}\Big(\lambda\left(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\right) - y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} + xA_+\Big),\\
k_2 &= \frac{1}{2}\Big(\lambda\left(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\right) - y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} - xA_+\Big),\\
k_3 &= \frac{1}{2}\Big(\lambda\left(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\right) + y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} + xA_-\Big),\\
k_4 &= \frac{1}{2}\Big(\lambda\left(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\right) + y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} - xA_-\Big),
\end{aligned}$$

where

$$A_\pm = \sqrt{\lambda\Big(-4ia_{30} + a_{20}^2\lambda - 2b_{02}b_{20}\lambda \pm 2b_{20}\sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}\Big)}.$$

We find the solution for the first equation on (1.39) for α_1, α_2, α_3, and α_4 being

$$\begin{aligned}
\alpha_1 &= -\frac{b_{02}\sqrt{\lambda}}{2\sqrt{4ia_{12} + b_{02}^2\lambda}},\\
\alpha_2 &= -\frac{1}{2},\\
\alpha_3 &= \frac{-2a_{20}\lambda\sqrt{4ia_{12} + b_{02}^2\lambda} + \Big(b_{02}\sqrt{\lambda} - \sqrt{4ia_{12} + b_{02}^2\lambda}\Big)\Big(A_- + A_+\Big)}{4A_-\sqrt{4ia_{12} + b_{02}^2\lambda}},\\
\alpha_4 &= \frac{2a_{20}\lambda\sqrt{4ia_{12} + b_{02}^2\lambda} + \Big(b_{02}\sqrt{\lambda} - \sqrt{4ia_{12} + b_{02}^2\lambda}\Big)\Big(A_- - A_+\Big)}{4A_-\sqrt{4ia_{12} + b_{02}^2\lambda}}.
\end{aligned}$$

Thus the first equation of (5.41) is linearizable by the change of coordinates

$$x_1 = x f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_4}.$$

Now, using the Darboux factors f_1,..., f_4 and the corresponding cofactors we construct a Darboux first integral for system (5.41). First, we solve equation (1.31) for $\alpha_1, \ldots, \alpha_4$ using $k_1, \ldots, k_4$ and obtain $\alpha_1 = A_-$, $\alpha_2 = -A_-$, $\alpha_3 = -A_+$, and $\alpha_4 = A_+$. Therefore, the Darboux first integral for system (5.41) is

$$\Psi = f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_4}.$$

Actually, $\Psi_1(x,y) = \frac{\Psi - 1}{A_- A_+ \sqrt{\lambda}\sqrt{4ia_{12}+b_{02}^2\lambda}}$, is the first integral whose series expansion is of the form (1.27) and, therefore, the second equation of system (5.41) can be linearized by the change of coordinates $y_1 = \frac{\Psi_1}{x_1}$.

Case (13). Systems (5.3) and (5.19) are

$$\begin{aligned} \dot{x} &= x(i + a_{20}x + a_{30}x^2), \\ \dot{y} &= -iy + b_{20}x^2 + b_{30}x^3 + b_{11}xy + b_{21}x^2y, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= x(i + a_{20}\lambda x + a_{30}\lambda x^2) = P_1(x,y), \\ \dot{y} &= -iy + b_{20}\lambda x^2 + b_{30}\lambda x^3 + b_{11}\lambda xy + b_{21}\lambda x^2y = Q_1(x,y). \end{aligned} \tag{5.42}$$

We find the Darboux factors

$$\begin{aligned} f_0 &= x, \\ f_1 &= \frac{1}{2}\left(2 + 2ia_{30}\lambda x^2 + x\left(\sqrt{\lambda\left(a_{20}^2\lambda - 4ia_{30}\right)} - \lambda a_{20}\right)(x\lambda a_{20} + 2i)\right), \\ f_2 &= \frac{1}{2}\left(2 + 2ia_{30}\lambda x^2 - x\left(\sqrt{\lambda\left(a_{20}^2\lambda - 4ia_{30}\right)} + \lambda a_{20}\right)(x\lambda a_{20} + 2i)\right), \end{aligned}$$

with the corresponding cofactors

$$\begin{aligned} K_0 &= i + x\lambda(a_{20} + a_{30}x), \\ K_1 &= x\left(a_{20}\lambda + 2a_{30}\lambda x + \sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}\right), \\ K_2 &= x\left(a_{20}\lambda + 2a_{30}\lambda x - \sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}\right). \end{aligned}$$

We find the solution for the first equation on (1.39) for α_1 and α_2 being

$$\alpha_1 = \frac{1}{4}\Big(-1 - \frac{a_{20}\lambda}{\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}}\Big), \quad \alpha_2 = \frac{1}{4}\Big(-1 + \frac{a_{20}\lambda}{\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}}\Big).$$

Therefore the first equation of (5.42) is linearizable by the change of coordinates

$$x_1 = x f_1^{\alpha_1} f_2^{\alpha_2}.$$

Using cofactors K_1 and K_2 we solve equation (1.32) obtaining

$$\begin{aligned} \beta_1 &= \frac{\lambda(a_{20}b_{21} - 2a_{30}b_{11} - a_{20}a_{30})}{4a_{30}\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}} - \frac{3}{4} - \frac{b_{21}}{4a_{30}}, \\ \beta_2 &= -\frac{\lambda(a_{20}b_{21} - 2a_{30}b_{11} - a_{20}a_{30})}{4a_{30}\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}} - \frac{3}{4} - \frac{b_{21}}{4a_{30}}. \end{aligned}$$

Thus, an integrating factor of system (5.42) is

$$\mu = f_1^{\beta_1} f_2^{\beta_2},$$

which implies that system (5.42) has an analytic first integral of the form (1.27). So, the second equation of (5.42) is linearizable by the change of coordinates $y_1 = \frac{\Psi}{x_1}$.

Case (14). Systems (5.3) and (5.19) are

$$\dot{x} = ix + x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + y(-2a_{11}x + b_{03}y^2), \tag{5.43}$$

and

$$\dot{x} = ix + \lambda x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + \lambda y(-2a_{11}x + b_{03}y^2), \tag{5.44}$$

respectively. System (5.43) is considered in [38, Theorem 2, case 1], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 5.2.3 system (5.44) is linearizable. Therefore, O is a linearizable weakly persistent center of (5.43). □

Remark 5.3.3. We remark that system (5.3) is system (5.2) with $a_{02} = a_{03} = 0$. Notice that if we apply to the conditions of Theorem 5.3.2 the involution $a_{ij} \leftrightarrow b_{ji}$, then we obtain conditions for linearizable weakly persistent center at the origin of the dual system of (5.3), i.e. system (5.2) with $b_{20} = b_{30} = 0$:

$$\begin{aligned}
\dot{x} =& ix + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,\\
\dot{y} =& -iy + y(b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2).
\end{aligned}$$

CHAPTER

6

ON THE MAY-LEONARD ASYMMETRIC SYSTEM

The main goal of this chapter is to study some aspects of the May-Leonard asymmetric system which is a tree dimensional quadratic Lotka-Voltera system. The study done here was obtained using the Darboux theory for 3-dimensional systems. We characterize the existence of irreducible invariant surfaces of degree two for the May-Leonard asymmetric system and we give conditions to the existence of Darboux first integrals for such system. The results present here are in [7], manuscript available at http://conteudo.icmc.usp.br/Portal/conteudoDinamicoSemVinculo.php?id_conteudos=719, n$^{\text{O}}$ 436.

6.1 Motivation for the study

An important class of mathematical models describing different phenomena in biology, ecology and chemistry are the so-called Lotka-Volterra systems which are written in the form

$$\dot{x}_i = x_i\Big(\sum_{j=1}^{n} a_{ij}x_j + b_i\Big) \qquad (i = 1, \ldots, n). \tag{6.1}$$

They were introduced independently by Lotka and Volterra in the 1920s to model the interaction among species, see [88, 113], and have been, and continue to be, intensively studied. For the class of systems (6.1) the case $n = 3$ is the most studied. One of the simplest models of such type describing a competition of three species was introduced by May and Leonard in [91]. It is a model depending on two parameters and written as the differential system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha y - \beta z),\\ \dot{y} &= y(1 - \beta x - y - \alpha z),\\ \dot{z} &= z(1 - \alpha x - \beta y - z). \end{aligned} \tag{6.2}$$

where $x,y,z \geq 0$, $0 < \alpha < 1 < \beta$, and

$$\alpha + \beta > 2.$$

It was showed in [91] that system (6.2) has four singular points in $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3, x,y,z \geq 0\}$. Three of them are on the boundary of $\mathbb{R}^3_+$,

$$E_1 = (1,0,0),\ E_2 = (0,1,0),\ E_3 = (0,0,1),$$

and the forth one is in the interior point

$$C = ((1+\alpha+\beta)^{-1}, (1+\alpha+\beta)^{-1}, (1+\alpha+\beta)^{-1}).$$

There is a separatrix cycle F formed by orbits connecting E_1, E_2 and E_3 on the boundary of $\mathbb{R}^3_+$, and every orbit in $\mathbb{R}^3_+$, except the equilibrium point C has F as the ω-limit. It was showed [91] that in the degenerate case $\alpha + \beta = 2$ the cycle F becomes a triangle on the invariant plane $x+y+z = 1$, all orbits inside the triangle are closed and every orbit in the interior of $\mathbb{R}^3_+$ has one of these closed orbits as ω-limit. Latter on, the dynamics of (6.2) was studied in more details in [11, 108, 117] and some other works.

A generalization of model (6.2) is the model described by the differential system

$$\begin{aligned}
\dot{x} =& x(1 - x - \alpha_1 y - \beta_1 z) = X(x,y,z),\\
\dot{y} =& y(1 - \beta_2 x - y - \alpha_2 z) = Y(x,y,z),\\
\dot{z} =& z(1 - \alpha_3 x - \beta_3 y - z) = Z(x,y,z),
\end{aligned} \tag{6.3}$$

where $x,y,z \geq 0$ and α_i, β_i $(1 \leq i \leq 3)$ are real parameters, which is called the *asymmetric May-Leonard model*. The dynamics of (6.3) was studied in [6, 28, 63, 117]. In particular, Chi, Hsu and Wu [28] studied (6.3) under the assumption

$$0 < \alpha_i < 1 < \beta_i\ (1 \leq i \leq 3) \tag{6.4}$$

and showed that under this assumption the system has a unique interior equilibrium P and if

$$A_1 A_2 A_3 \neq B_1 B_2 B_3,$$

(where $A_i = 1 - \alpha_i, \quad B_i = \beta_i - 1, \ (1 \leq i \leq 3)$), then the system does not have periodic solutions, and if

$$A_1 A_2 A_3 = B_1 B_2 B_3,$$

then there is a family of periodic solutions. It was showed in [6] that even if assumption (6.4) is dropped system (6.3) still can have a family of periodic solutions.

First integrals of May-Leonard system (6.2) where studied by Leach and Miritzis [69] (see also [82]), who obtained the following first integrals:

(i) $H_1 = \frac{xyz}{(x+y+z)^3}$ if $\alpha + \beta = 2$ and $\alpha \neq 1$

(ii) $H_2 = \frac{y(x-z)}{x(y-z)}$ if $\alpha = \beta \neq 1$,

(iii) $H_3 = x/z$ and $H_4 = y/z$ (two independent first integrals) if $\alpha = \beta = 1$.

It was shown in [11] that system (6.2) is completely integrable, that is, it admits two independent first integrals, if either $\alpha + \beta = 2$, or $\beta = \alpha$.

Our aim is to study the integrability of asymmetric May-Leonard model (6.3). Using algorithms of the elimination theory we first find systems of the form (6.3) admitting invariant planes and invariant surfaces defined by quadratic polynomials. Then we look for first integrals of the Darboux type constructed using these invariant surfaces and find subfamilies of (6.3) admitting one or two independent first integrals. As we show the set of systems with first integrals is much larger for system (6.3) than for classical May-Leonard system (6.2).

The proposed approach can be used to study integrability of many other mathematical models described by polynomial systems of differential equations.

6.2 Darboux theory for 3-dimensional systems

In this section we remind some general results on the elimination theory and the Darboux theory of integrability that we shall use in our study.

Considerer system of differential equations

$$\begin{aligned} \dot{x} &= P(x,y,z), \\ \dot{y} &= Q(x,y,z), \\ \dot{z} &= R(x,y,z), \end{aligned} \tag{6.5}$$

where P, Q and R are polynomials of degree at most m, and let $\mathfrak{X}$ be the corresponding vector field,

$$\mathfrak{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

A C^1 function $H : U \to \mathbb{R}$ non-constant in any open subset of $U \subset \mathbb{R}^2$ is a first integral of the differential system (6.5) if and only if $\mathfrak{X}H \equiv 0$ in U. Let $H_1 : U_1 \to \mathbb{R}$ and $H_2 : U_2 \to \mathbb{R}$ be two first integrals of system (6.5). It is said that H_1 and H_2 are independent in $U_1 \cap U_2$ if their gradients are independent in all the points of $U_1 \cap U_2$ except perhaps in a zero Lebesgue measure set. System (6.5) is completely integrable on $U_1 \cap U_2$ if it has two independent first integrals on $U_1 \cap U_2$.

A *Darboux polynomial* of system (6.5) is a polynomial $f(x,y,z)$ such that

$$\mathfrak{X}f := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = Kf,$$

where $K(x,y,z)$ is a polynomial of degree at most $m-1$. The polynomial $K(x,y,z)$ is called the *cofactor of* f. It is easy to see that if f is a Darboux polynomial of (6.5) then the equation $f = 0$

defines an algebraic surface which is invariant under the flow of system (6.5). For this reason f often is referred as invariant algebraic surface of (6.5).

A simple computation shows that if there are Darboux polynomials $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \lambda_i K_i = 0, \tag{6.6}$$

then

$$H = f_1^{\lambda_1} \cdots f_k^{\lambda_k}, \tag{6.7}$$

is a first integral of (6.5). An integral of the form (6.7) is called *a Darboux integral* of system (6.5).

6.3 Darboux polynomials of degree two

Invariant planes of system (6.3) and Darboux integrals constructed from such planes were found in [6]. In this section using the Elimination Theorem (Theorem 1.2.13) we look for Darboux polynomials of system (6.3) of degree two. A general form of a polynomial of degree two is

$$\begin{aligned} f(x,y,z) = {} & h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy \\ & + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2. \end{aligned} \tag{6.8}$$

A cofactor of any Darboux polynomials of system (6.5) is a polynomial of degree one which we write in the form

$$K(x,y,z) = c_0 + c_1x + c_2y + c_3z. \tag{6.9}$$

Polynomial (6.8) will be a Darboux polynomial of system (6.3) with cofactor (6.9) if

$$\mathfrak{X} f = K f, \tag{6.10}$$

where now

$$\mathfrak{X} f = \frac{\partial f}{\partial x} X + \frac{\partial f}{\partial y} Y + \frac{\partial f}{\partial z} Z,$$

with X, Y and Z defined in (6.3).

Comparing the coefficients of similar terms in (6.10), we obtain the polynomial system

$$g_1 = g_2 = ... = g_{19} = g_{20} = 0,$$

where

$$\begin{aligned} g_1 &= -c_0h_{000}, \\ g_2 &= -c_3h_{000} + h_{001} - c_0h_{001}, \\ g_3 &= -h_{001} - c_3h_{001} + 2h_{002} - c_0h_{002}, \\ g_4 &= -2h_{002} - c_3h_{002}, \\ g_5 &= -c_2h_{000} + h_{010} - c_0h_{010}, \\ g_6 &= -\beta_3h_{001} - c_2h_{001} - \alpha_2h_{010} - c_3h_{010} + 2h_{011} - c_0h_{011}, \end{aligned} \tag{6.11}$$

$$
\begin{aligned}
g_7 &= -2\beta_3 h_{002} - c_2 h_{002} - h_{011} - \alpha_2 h_{011} - c_3 h_{011},\\
g_8 &= -h_{010} - c_2 h_{010} + 2h_{020} - c_0 h_{020},\\
g_9 &= -2h_{020} - c_2 h_{020},\\
g_{10} &= -h_{011} - \beta_3 h_{011} - c_2 h_{011} - 2\alpha_2 h_{020} - c_3 h_{020},\\
g_{11} &= -c_1 h_{000} + h_{100} - c_0 h_{100},\\
g_{12} &= -\alpha_3 h_{001} - c_1 h_{001} - \beta_1 h_{100} - c_3 h_{100} + 2h_{101} - c_0 h_{101},\\
g_{13} &= -2\alpha_3 h_{002} - c_1 h_{002} - h_{101} - \beta_1 h_{101} - c_3 h_{101},\\
g_{14} &= -\beta_2 h_{010} - c_1 h_{010} - \alpha_1 h_{100} - c_2 h_{100} + 2h_{110} - c_0 h_{110},\\
g_{15} &= -2\beta_2 h_{020} - c_1 h_{020} - h_{110} - \alpha_1 h_{110} - c_2 h_{110},\\
g_{16} &= -\alpha_3 h_{011} - \beta_2 h_{011} - c_1 h_{011} - \alpha_1 h_{101} - \beta_3 h_{101}\\
&\quad - c_2 h_{101} - \alpha_2 h_{110} - \beta_1 h_{110} - c_3 h_{110},\\
g_{17} &= -h_{100} - c_1 h_{100} + 2h_{200} - c_0 h_{200},\\
g_{18} &= -2h_{200} - c_1 h_{200},\\
g_{19} &= -h_{110} - \beta_2 h_{110} - c_1 h_{110} - 2\alpha_1 h_{200} - c_2 h_{200},\\
g_{20} &= -h_{101} - \alpha_3 h_{101} - c_1 h_{101} - 2\beta_1 h_{200} - c_3 h_{200}.
\end{aligned}
$$

We denote by $I = \langle g_1, g_2, ..., g_{19}, g_{20}\rangle$ the ideal generated by polynomials in (6.11). Since computations based on the Elimination Theorem (Theorem 1.2.13) are very laborious, to simplify them we consider separately the cases $h_{000} = 1$ and $h_{000} = 0$, that is, we look separately for invariant curves $f = 0$ not passing and passing through the origin, so *from now on in this section we assume that* $h_{000} = 1$.

To find Darboux polynomials of system (6.3) of degree two, we have to determine for which values of parameters α_i, β_i ($i = 1,2,3$) system (6.11) has a solution with at least one of coefficient $h_{200}, h_{002}, h_{011}, h_{020}, h_{101}, h_{110}$ different from zero. To satisfy this condition we have six options that can be written in polynomial form as

$$
\begin{aligned}
&1 - wh_{200} = 0,\ \ 1 - wh_{110} = 0,\ \ 1 - wh_{101} = 0,\\
&1 - wh_{020} = 0,\ \ 1 - wh_{011} = 0,\ \ 1 - wh_{002} = 0,
\end{aligned} \tag{6.12}
$$

with w being a new variable. For instance, to find systems of form (6.3) which have surfaces with $h_{200} \neq 0$, we can compute (for example, with the routine `eliminate` of the computer algebra system SINGULAR) the 13-th elimination ideal (see Section 1.2) of the ideal $I^{(1)} = \langle I, 1 - wh_{200}\rangle$, in the ring

$$\mathbb{Q}[w, c_0, c_1, c_2, c_3, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3].$$

Denote this elimination ideal by $I_{13}^{(1)}$ and its variety by V_1 (that is, $V_1 = \mathbf{V}\,(I_{13}^{(1)})$). Proceeding analogously, we can find the other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$ corresponding to the other cases of (6.12). Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, \mathbf{V}(I_{13}^{(6)})$. It is clear that the union $V = V_1 \cup \ldots \cup V_6$ of these six varieties contains the set of all systems (6.3) having invariant surfaces of the form (6.8) not passing through the origin. To compute the irreducible decomposition of the variety V it is sufficient to compute the ideal $J = I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)}$, which defines the variety $V = V_1 \cup \ldots \cup V_6$ and then to find the irreducible decomposition of V. The intersection of ideals can be computed with the routine `intersect` of SINGULAR, and the irreducible decomposition of V can be found with the routine `minAssGTZ`. Theoretically, such computations should give all systems in the family (6.3) having invariant surfaces of degree two. However all the routines `eliminate`, `intersect` and `minAssGTZ` rely on computations of many Groebner bases, and such computations can be rarely completed when computing over the field $\mathbb{Q}$ of rational numbers for polynomials in many variables. To be able to complete our computations we computed in the field of finite characteristic 32003 and then lifted the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm (see Section 1.2).

The primary decomposition of the radical of the ideal

$$J = \bigcap_{i=1}^{6} I_{13}^{(i)} \tag{6.13}$$

computed using the routine `minAssGTZ` in the field of characteristic 32003 consists of 88 ideals, that is, we have 88 irreducible components of the variety $\mathbf{V}(J)$ given in Appendix E. It means there are 88 conditions on the parameters α_i, β_i of system (6.3) for existence of an invariant surface of degree two not passing through the origin.

However some of these conditions give systems with the same dynamics in the phase space, since system (6.3) has a symmetry with respect to simple linear transformations. Namely, it is easily seen that the transformations

$$x \to z,\ y \to x,\ z \to y, \tag{6.14}$$

$$x \to y,\ y \to z,\ z \to x, \tag{6.15}$$

$$x \to y,\ y \to x,\ z \to z, \tag{6.16}$$

$$x \to z,\ y \to y,\ z \to z, \tag{6.17}$$

$$x \to x,\ y \to z,\ z \to y, \tag{6.18}$$

which correspond to relabelling of the coordinate axes, do not change the shape of the system. For instance, under transformation (6.14) system (6.3) is changed into the system

$$\begin{aligned}
\dot{x} =& x(1 - x - \alpha_2 y - \beta_2 z),\\
\dot{y} =& y(1 - \beta_3 x - y - \alpha_3 z),\\
\dot{z} =& z(1 - \alpha_1 x - \beta_1 y - z),
\end{aligned}$$

which can be obtained from system (6.3) by the change of parameters

$$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2. \tag{6.19}$$

Thus, if we have a condition on the parameters of (6.3) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (6.19). For example, as we will see below, system (6.3) has the invariant surface

$$f = 2 - 4x + 2x^2 - 2y + yz$$

if condition (4) of Theorem 6.3.1 is fulfilled, that is, if

$$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0.$$

Applying to (6.3) transformation (6.14) we obtain that system (6.3) has the invariant surface

$$f = 2 - 4z + 2z^2 - 2x + xy$$

if the condition

$$\beta_2 = \beta_3 = \alpha_2 + 1 = \beta_1 - 3 = \alpha_1 + 1 = \alpha_3 - 1/2 = 0$$

holds, that is, condition (4) is changed according to (6.19). Similarly, after substitutions (6.15)–(6.18) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \tag{6.20}$$

$$\alpha_1 \to \beta_2,\ \beta_1 \to \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \tag{6.21}$$

$$\alpha_1 \to \beta_3,\ \beta_1 \to \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \tag{6.22}$$

$$\alpha_1 \to \beta_1,\ \beta_1 \to \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2, \tag{6.23}$$

respectively.

We say, that two conditions for existence of invariant surfaces are *conjugate* if one can be obtained from another by means of one of transformations (6.19)–(6.23). For instance, condition (4) (which is the same as condition (7) from Appendix E) and conditions (10), (19), (25), (33), (47) from Appendix E can be obtained from each other by one of transformations (6.19)–(6.23), so all these conditions are conjugate.

Note that some of the 88 obtained conditions give Darboux polynomials of degree two which are not irreducible, but they are products of two polynomials of degree one. Namely, if
(i) $\alpha_1 = \beta_1 = 0$ (condition (1) of Appendix E), then system (6.3) has the Darboux polynomial $(-1+x)^2$ (and the conjugate conditions are (22) and (36) from Appendix E);
(ii) $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ (condition (5) of Appendix E), then system (6.3) has the Darboux polynomial $(-1+x+z)^2$ (and the conjugate conditions are (44) and (78) from Appendix E);
(iii) $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_2 - 2 = 0$ (condition (88) of Appendix E), then system (6.3) has the Darboux polynomial $(-1+x+y+z)^2$.

From the analysis of the 88 obtained conditions we reach the following result.

Theorem 6.3.1. *System (6.3) has an irreducible invariant surface of degree two not passing through the origin if one of the following conditions or conjugated to it holds:*

1. $\alpha_2 = \beta_1 = \beta_2 - 1/2 = \alpha_1 - 3 = 0,$
2. $\alpha_2 = \beta_1 = \beta_2 - 3 = \alpha_1 - 3 = 0,$
3. $\beta_3 = \beta_1 = \alpha_3 + \beta_2 - 1 = \alpha_2 + 1 = \alpha_1 - \alpha_3 - 1 = 0,$
4. $\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0,$
5. $\beta_3 = \beta_1 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 + 1 = 0,$
6. $\beta_1 = \beta_3 - 3 = \alpha_3 - 3 = \beta_2 - 1 = \alpha_2 - 1/2 = \alpha_1 - 1 = 0,$
7. $\beta_1 = \beta_3 - 3 = \alpha_3 - 1/2 = \beta_2 - 1/2 = \alpha_2 + 1 = \alpha_1 - 3 = 0,$
8. $\beta_1 = \beta_3 - 1/2 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 - 1/2 = 0,$
9. $\beta_1 = \beta_3 - 3 = \alpha_3 + 3 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 3 = 0,$
10. $\beta_1 = \beta_3 - 1/2 = \alpha_3 - 2 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 - 1/2 = 0,$
11. $\beta_1 = \alpha_3 = \beta_2 - \beta_3 - 1 = \alpha_2 + \beta_3 - 2 = \alpha_1 + \beta_3 - 1 = 0,$
12. $\beta_1 = \beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 + 1 = \alpha_1 - \alpha_3 + 2 = 0,$
13. $\beta_3 - 1/2 = \alpha_3 - 1/2 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0,$
14. $\beta_3 - 1/2 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 - 1/2 = 0,$
15. $\beta_3 - 1/2 = \beta_2 - 3 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - 1/2 = 0,$
16. $\beta_3 - 3 = \alpha_3 - 3 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0,$
17. $\beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - \alpha_3 + 2 = 0.$

Proof. For each case of the theorem we give below the irreducible Darboux polynomial f of degree two which defines quadratic invariant surface $f = 0$ not passing through the origin and the corresponding cofactor:

1. $f = 1 - x - 2y + y^2; \quad K = -x - 2y;$
2. $f = 1 - 2x + x^2 - 2y - 2xy + y^2; \quad K = -2(x + y);$
3. $f = 2 - 2x - 2y + yz; \quad K = -x - y;$
4. $f = 2 - 4x + 2x^2 - 2y + yz; \quad K = -2x - y;$

5. $f = 2-4x+2x^2+2xy-2z+xz; \quad K = -2x-z;$

6. $f = 2-4x+2x^2-4y+4xy+2y^2-2z+xz; \quad K = -2x-2y-z;$

7. $f = 1-x-2y+y^2+yz; \quad K = -x-2y;$

8. $f = 2-4x+2x^2-2y-2z+xz; \quad K = -2x-y-z;$

9. $f = 1-2x+x^2-2y-2xy+y^2+yz; \quad K = -2(x+y);$

10. $f = 1-2x+x^2-y-z+xz; \quad K = -2x-y-z;$

11. $f = 1-x-y-z+xz; \quad K = -x-y-z;$

12. $f = 1-2x+x^2-2y+2xy+y^2+yz; \quad K = -2(x+y);$

13. $f = 1-x-y-2z+z^2; \quad K = -x-y-2z;$

14. $f = 1-2x+x^2-y-2z-2xz+z^2; \quad K = -2x-y-2z;$

15. $f = 1-2x+x^2-y-2z+2xz+z^2; \quad K = -2x-y-2z;$

16. $f = 1-2x+x^2-2y+2xy+y^2-2z-2xz-2yz+z^2; \quad K = -2(x+y+z);$

17. $f = 1-2x+x^2-2y+2xy+y^2-2z+2xz-2yz+z^2; \quad K = -2(x+y+z).$

□

6.4 First integrals

In this section we look for Darboux first integrals of system (6.3), which can be constructed using the invariant surfaces obtained in the previous section.

Theorem 6.4.1. *a) If one of conditions 1-3, 11, 12, 17 of Theorem 6.3.1 holds, then the corresponding system (6.3) admits at least one Darboux first integral.*
b) If one of conditions 4-10, 13-16 of Theorem 6.3.1 holds, then the corresponding system (6.3) is completely integrable on $\mathbb{R}^2$ (it admits two independent Darboux first integrals).

Proof. First note that system (6.3) always has the following three invariant surfaces of degree one, with the respective cofactors:

$$\begin{aligned} f_1 &= x; \;\; K_1 = 1-x-\alpha_1 y-\beta_1 z;\\ f_2 &= y; \;\; K_2 = 1-\beta_2 x-y-\alpha_2 z;\\ f_3 &= z; \;\; K_3 = 1-\alpha_3 x-\beta_3 y-z. \end{aligned} \tag{6.24}$$

Case a). To prove statement a) of the theorem we present the Darboux first integrals for each case mentioned in the statement.

If condition 1) of Theorem 6.3.1 is satisfied the system has the form

$$\dot{x} = x(1-x-3y), \quad \dot{y} = y(1-x/2-y)y, \quad \dot{z} = z(1-\alpha_3 x-\beta_3 y-z). \tag{6.25}$$

Besides the invariant surface f_1, f_2 and f_3 given above and the invariant surface $f = 1-x-2y+y^2$, system (6.25) has the following surfaces f_4, f_5 (with cofactors K_4, K_5, respectively),

$$f_4 = x+4y; \;\; K_4 = 1-x-y;$$
$$f_5 = x+2y-2y^2; \;\; K_5 = 1-x-2y.$$

From the corresponding equation (6.6) we find that $\lambda_1 = \lambda_3/2, \lambda_2 = \lambda_4, \lambda_5 = -\lambda_3-2\lambda_4, \lambda_6 = 0$. Thus, for arbitrary λ_3, λ_4 not both equal to zero system (6.25) has a Darboux first integral

$$\widetilde{H} = x^{\lambda_4} y^{\lambda_3} (x+4y)^{\lambda_4} \left(x-2y^2+2y\right)^{-\lambda_3-2\lambda_4} \left(-x+y^2-2y+1\right)^{\frac{\lambda_3}{2}}.$$

In particular, taking $\lambda_4 = 1$ and $= \lambda_3 = 0$ we have the Darboux first integral

$$H = \frac{x(x+4y)}{(x+2y-2y^2)^2}.$$

Since first two equations of (6.25) are independent of z we cannot construct another independent first integral $H_2(x,y)$ of (6.25) using only the surfaces given above, since if such integral would exist then the two-dimensional system

$$\dot{x} = x(1-x-3y), \quad \dot{y} = y(1-x/2-y)y,$$

would have two independent first integrals, which is impossible.

Similarly as above we find families of Darboux integrals for other cases. We list below representatives of the families for the corresponding cases:

2) $H = \dfrac{xy}{(-x+x^2-y-2xy+y^2)^2};$

3) $H = \dfrac{(x+y-yz)^2}{x^2+2xy+y^2-2yz};$

11) $H = \dfrac{xz(1-x-y-z+xz)}{(-x-y-z+2xz)^2};$

12) $H = \dfrac{yz}{(-x+x^2-y+2xy+y^2+yz)^2};$

17) $H = \dfrac{yz}{(-x+x^2-y+2xy+y^2-z+2xz-2yz+z^2)^2}.$

Case b). For each system of this case we present two independent Darboux first integrals.

Condition 4). Besides the invariant surface f_1, f_2, f_3 above and f of the previous theorem, we have the invariant surfaces $f_4 = 4x + y - 2z$ with the cofactor $K_4 = 1 - x - y - z$. Using these polynomials we can find the following two Darboux first integrals:

$$H_1 = \frac{z(2 - 4x + 2x^2 - 2y + yz)}{(4x + y - 2z)},$$
$$H_2 = \frac{yz}{x^2}.$$

To check if these first integrals are independent we compute their gradients and obtain, that are

$$\begin{aligned} G_1 =& \frac{2}{(4x+y-2z)^2}(2z(-2+2x+y)(1+x-z),\ \ -z(1+x-z)^2, \\ & 4x - 8x^2 + 4x^3 + y - 6xy + x^2y - y^2 + 4xyz + y^2z - yz^2), \\ G_2 =& \left(-\frac{2yz}{x^3},\ \frac{z}{x^2},\ \frac{y}{x^2}\right), \end{aligned}$$

respectively. Computations show that the linear combination $aG_1 + bG_2$, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

Therefore the Darboux first integrals H_1 and H_2 are independents.

Condition 5). Besides the invariant surface f_1, f_2, f_3 given in (6.24) and f of the previous theorem, we have the following invariant surfaces passing through the origin with respective cofactors:

$$\begin{aligned} &f_4 = -4xy + 2xz + z^2;\ \ K_4 = -2(-1 + 2x + z); \\ &f_5 = 2y + z;\ \ K_5 = 1 - 3x - y - z; \\ &f_6 = 2x + 2y + z;\ \ K_6 = 1 - x - y - z; \\ &f_7 = -2x + 2x^2 + 2xy - z + xz;\ \ K_7 = 1 - 2x - z. \end{aligned}$$

Using these polynomials we can find the following two Darboux first integrals:

$$H_1 = \frac{xy^2}{z^2(2x + 2y + z)},$$
$$H_2 = \frac{xy^2}{(2y + z)(-2x + 2x^2 + 2xy - z + xz)^2}.$$

The gradients of them are

$$\begin{aligned} G_1 =& \left(\frac{y^2(2y+z)}{z^2(2x+2y+z)^2},\ \frac{2xy(2x+y+z)}{z^2(2x+2y+z)^2},\ -\frac{xy^2(4x+4y+3z)}{z^3(2x+2y+z)^2}\right), \\ G_2 =& \frac{y}{(2y+z)^2(-2x+2x^2+2xy-z+xz)^3}(-y(2y+z)(-2x+6x^2+2xy+z+xz), \\ & 2x(-2xy + 2x^2y - 2xy^2 - 2xz + 2x^2z - yz + xyz - z^2 + xz^2), \\ & -xy(-2x + 2x^2 - 4y + 6xy - 3z + 3xz)), \end{aligned}$$

respectively. Analogously to the previous case to show that these two Darboux first integrals are independent we verify that a linear combination $aG_1 + bG_2$, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

Therefore the Darboux first integrals H_1 and H_2 are independents.

Using the similar computations we get the following pairs of independents Darboux first integrals for the remaining cases:

6. $H_1 = \dfrac{z}{x(2-4x+2x^2-4y+4xy+2y^2-2z+xz)},$

   $H_2 = \dfrac{z(2x+4y+z)}{y^2(2-4x+2x^2-4y+4xy+2y^2-2z+xz)};$

7. $H_1 = -\dfrac{x^2}{yz(-1+x+2y-y^2-yz)},$

   $H_2 = -\dfrac{(x-2z)^2}{z(y+z)(-1+x+2y-y^2-yz)};$

8. $H_1 = \dfrac{xz(2-4x+2x^2-2y-2z+xz)}{(y+z)^2},$

   $H_2 = \dfrac{(2x+z)(y+z)^2}{xy^2};$

9. $H_1 = \dfrac{yz(1-2x+x^2-2y-2xy+y^2+yz)}{x^2},$

   $H_2 = \dfrac{y^2z(4x-z)}{x^2(x^2-2xy+y^2+yz)};$

10. $H_1 = -\dfrac{x(y-z+xz+z^2)}{z(-2x+2x^2-y-z+2xz)},$

    $H_2 = \dfrac{y^2(x+z)}{z(2x-2x^2+y+z-2xz)(y-z+xz+z^2)};$

13. $H_1 = \dfrac{(-x-y-2z+2z^2)^2}{z^2(1-x-y-2z+z^2)},$

    $H_2 = \dfrac{xz^{2\alpha_1-2}(x+y+4z)^{2-2\alpha_1}}{y};$

14. $H_1 = \dfrac{(-2x+2x^2-y-2z-4xz+2z^2)^2}{(x-z)^2(1-2x+x^2-y-2z-2xz+z^2)},$

    $H_2 = \dfrac{x(y+4z)^2(-2x+2x^2-y-2z-4xz+2z^2)^2}{z(x-z)^4(1-2x+x^2-y-2z-2xz+z^2)^2};$

15. $H_1 = \dfrac{y(4x+y+4z)}{(2x-2x^2+y+2z-4xz-2z^2)^2},$

    $H_2 = \dfrac{x(4x+y+4z)^{1-\alpha_3}(-2x+2x^2-y-2z+4xz+2z^2)^{1+\alpha_3}}{z(x+z)^2(1-2x+x^2-y-2z+2xz+z^2)};$

16. $H_1 = \dfrac{(x+y)z}{(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^2},$

$$
\begin{aligned}
H_2 =&\frac{1}{y^4}x^4z^{2-2\alpha_1}(1-2x+x^2-2y+2xy+y^2-2z-2xz-2yz+z^2)^{1-\alpha_1}\\
&(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^{2\alpha_1+2}.
\end{aligned}
$$

□

To summarize, we have found some Darboux first integral of May-Leonard asymetric system (6.3) which are constructed using Darboux polynomials of degree one and two. We do not know if we found all independent first integrals of system (6.3) which can be constructed from Darboux polynomials of degree one and two. To verify if the list is complete, we have to find Darboux polynomials of (6.3), which define invariant algebraic surfaces passing through the origin, that is, polynomials (6.8) with $h_{000} = 0$. A naïve expectation is that this case should be simpler, than the case $h_{000} = 1$, which we have successfully investigated in this paper. However it turns out that the case $h_{000} = 0$ is computationally much more difficult and we were not able to complete computations for this case using our computational facilities. We believe that a reason for this difficulty is that since the origin is a singular point there are many invariant surfaces passing through the origin and it implies a complicate structure of the elimination ideals which we have to compute using our approach.

FINAL CONSIDERATIONS

In this thesis we essentially consider the problems of characterize the existence of centers and isochronous centers for some families of differential systems. The investigation of such problems require some laborious computational algebra methods which must be improved as computational technology is developed.

The studies carry out here have led us to use different techniques of the qualitative theory of ordinary differential equations, for instance, blow up, normal forms and the Darboux theory for integrability and linearizability. Moreover, we have to keep a close relation with the computer algebra systems MATHEMATICA and SINGULAR. We can say that the contact with all these tools have been expanding our horizons and have been providing opportunities to investigate different problem in the qualitative theory of differential systems.

It is important to mention that we have had opportunity to develop projects with excellent researches from different countries, Matej Mencinger and Brigita Ferčec from Slovenia, Yilei Tang and Yun Tian from China, Valery Antonov from Russia, Natalie Shcheglova from Belarus and Marzhan Sultanova from Kazakhstan. The contact with them expanded our network and made possible the development of further works with new collaborators.

In the problem of the existence of bi-centers we believe that there is still lot to be done. As far as we know there are very few studies in such problem in the literature. For a $\mathbb{Z}_2$-equivariant cubic system we obtained two subfamilies possessing an isochronous bi-center. For a $\mathbb{Z}_2$-equivariant quintic system we obtained four and five subfamilies possessing a bi-center and an isochronous bi-center, respectively. These subfamilies possessing isochronous bi-centers provide examples of cubic and quintic systems possessing 3 and 5 isochronous centers, we are unaware about other examples of such fact. In this line we intend to find concrete proofs for the conjectures and open questions remarked through the text. Moreover, we intend to analyse other families of such systems, families possessing homogeneous non-linearities of degree three and five, families with homogeneous non-linearities of degree seven and characterize the global behaviour of such systems in the case of existence isochronous centers. We have a felling that in all these cases the global structures posses a very similar behaviour (as we can see for the presented cases).

For planar systems we have plans to investigate the *cyclicity problem* which consists in estimating the number of limit cycles in planar polynomial system and the *problem of critical period bifurcations* which consists on estimating of the number of critical periods that can arise near the center under small perturbations. In addition we intend to study the so called *uniform isochronous centers* which are centers with the property that all periodic orbits in its neighbourhood have the same constant angular velocity.

Studying three dimensional systems we could see that there is a lot to be done. The characterizations of integrable systems appear very seldom in the literature as well as the existence of limit cycles. Moreover, the algorithms and computational procedures for investigating such problem are still scarce. In this line we intend to improve the computational tools and contribute with new results on different families of such systems.

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APPENDIX A

The SINGULAR code for computing irreducible decomposition of the variety of the ideal $I = \langle \mathscr{L}_7, a_4, a_1 + a_5, a_6 + a_2/3 \rangle$ in Subsection 2.2.

```
LIB"primdec.lib";

ring r=0,(a1, a2, a3, a4, a5, a6),dp;

option(redSB);

poly i1=-18-4*a1^2-10*a2^2-9*a3+2*a2*a4-4*a4^2-12*a5+10*a1*a5-4*a5^2;
poly j1=-3*a2-6*a1*a2+6*a4-6*a2*a5+6*a4*a5-9*a6;
poly i2=...;
...
poly j7=...;

ideal i=i1,j1,i2,j2,i3,j3,i4,j4,i5,j5,i6,j6,i7,j7,a4 a1+a5,3*a6+a2;

minAssGTZ(i);
```

APPENDIX B

.

The SINGULAR code for computing irreducible decomposition of the variety $\mathscr{B}_9$ obtained from system (2.18).

```
LIB"primdec.lib";

ring r=32003,(a1, a2, a3, a4, a5, a6, a7, a8, a9, a10),dp;

option(redSB);

poly v1=-a2-4*a6+2*a2*a7-4*a6*a7+3*a8;
poly v2=60*a10+75*a2-40*a2*a3-36*a4+300*a6+80*a2^2*a6-16*a3*a6+256*a2*a6^2
-256*a6^3-154*a2*a7+40*a2*a3*a7+24*a4*a7+284*a6*a7+64*a3*a6*a7
+320*a2*a6^2*a7-256*a6^3*a7+148*a2*a7^2+544*a6*a7^2-280*a2*a7^3+560*a6*a7^3
-225*a8-80*a2*a6*a8+64*a6^2*a8+12*a7*a8-420*a7^2*a8+80*a2*a9+80*a6*a9;
poly v3=...;
...
poly v9=...;

ideal i=v1,v2,v3,v4,v5,v6,v7,v8,v9;

minAssGTZ(i);
```

APPENDIX C

.

Here is presented the seven prime ideals, $I_1, I_2, \ldots, I_7$, defining the irreducible components of the the variety of the ideal (2.10), which give conditions for the existence of an isochronous bi-center for system (2.4) under condition (3) of Theorem 11 in [79].

$I_1 = \langle f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)}, f_7^{(1)} \rangle$, where

$$
\begin{aligned}
f_1^{(1)} =& 16a_4^2 + 4a_5^2 - 12a_5 + 9, \\
f_2^{(1)} =& 2a_2a_5 - 4a_4a_5 - 3a_2 - 6a_4, \\
f_3^{(1)} =& 8a_2a_4 + 4a_5^2 - 9, \\
f_4^{(1)} =& 4a_2^2 + 4a_5^2 + 12a_5 + 9, \\
f_5^{(1)} =& 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6, \\
f_6^{(1)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2, \\
f_7^{(1)} =& 1767113484457806191819394908 16a_4^2a_5^{13} + 441778371114451547954848727 04a_5^{15} \\
& + 3802937789975548287443966230528a_4^2a_5^{12} + 8182009361595516074745369395 20a_5^{14} \\
& + 35343630268187272246740460929024a_4^2a_5^{11} + 6083104358065908444391981522944a_5^{13} \\
& + 186124816855970198178551925194752a_4^2a_5^{10} + 22162634019713341271269866606592a_5^{12} \\
& + 611958345752720144505463533920256a_4^2a_5^9 + 33276765822057728131243448856576a_5^{11} \\
& + 1302493291690448907138352781529088a_4^2a_5^8 - 28650226910444645119073997135872a_5^{10} \\
& + 1791802131580824336245097947664384a_4^2a_5^7 - 184692866386725515008166861400576a_5^9 \\
& + 1518210963847726688743552391800832a_4^2a_5^6 - 231646381147809069732611923187968a_5^8 \\
& - 478160056922050033090560000000a_4^4a_5^3 + 655722361345603412492915306966016a_4^2a_5^5 \\
& + 331640549651752885161389444520 96a_5^7 - 373501753169380009574400000 00a_4^6 \\
& - 1805842070141330535874560000 00a_4^4a_5^2 - 297076086251379994801693216952 32a_4^2a_5^4 \\
& + 354767765913173682461862769739648a_5^6 + 4202867383971070368153600000 0a_4^4a_5 \\
& - 193222358437787547499641121768320a_4^2a_5^3 + 342794196369769992826332834997728a_5^5
\end{aligned}
$$

$$-2696702111099234574336000000a_4^4 - 1004374597277070825847224974512 00a_4^2a_5^2$$
$$+1032066470246336790749245095098 72a_5^4 - 2315865971608680223100564605 2000a_4^2a_5$$
$$-3929256695955330460039408541928 0a_5^3 - 2018997423058351950276986070000a_4^2$$
$$-3956078943457777996265290479480 0a_5^2 + 1992009350236693384396800000 00a_1$$
$$-1142235899959275482224855235175 0a_5 - 984400343940413185636244664375.$$

$I_2 = \langle f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, f_4^{(2)}, f_5^{(2)}, f_6^{(2)} \rangle$, where

$$f_1^{(2)} = 4a_5 + 3,$$
$$f_2^{(2)} = 16a_4^2 + 9,$$
$$f_3^{(2)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(2)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(2)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(2)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_3 = \langle f_1^{(3)}, f_2^{(3)}, f_3^{(3)}, f_4^{(3)}, f_5^{(3)}, f_6^{(3)} \rangle$, where

$$f_1^{(3)} = 2a_5 + 9,$$
$$f_2^{(3)} = 9a_4^2 + 25,$$
$$f_3^{(3)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(3)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(3)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(3)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_4 = \langle f_1^{(4)}, f_2^{(4)}, f_3^{(4)}, f_4^{(4)}, f_5^{(4)}, f_6^{(4)} \rangle$, where

$$f_1^{(4)} = 6a_5 + 7,$$
$$f_2^{(4)} = a_4^2 + 4,$$
$$f_3^{(4)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(4)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(4)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(4)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_5 = \langle f_1^{(5)}, f_2^{(5)}, f_3^{(5)}, f_4^{(5)}, f_5^{(5)}, f_6^{(5)} \rangle$, where

$$\begin{aligned}
f_1^{(5)} =& 8a_5 + 31,\\
f_2^{(5)} =& a_4^2 + 4,\\
f_3^{(5)} =& 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,\\
f_4^{(5)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,\\
f_5^{(5)} =& 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,\\
f_6^{(5)} =& 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.
\end{aligned}$$

$I_6 = \langle f_1^{(6)}, f_2^{(6)}, f_3^{(6)}, f_4^{(6)}, f_5^{(6)}, f_6^{(6)} \rangle$, where

$$\begin{aligned}
f_1^{(6)} =& 2a_5 - 1,\\
f_2^{(6)} =& a_4^2 + 1,\\
f_3^{(6)} =& 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,\\
f_4^{(6)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,\\
f_5^{(6)} =& 2a_4a_5 + 3a_2 + 2a_4,\\
f_6^{(6)} =& 4a_1 + 2a_5 + 3.
\end{aligned}$$

$I_7 = \langle f_1^{(7)}, f_2^{(7)}, f_3^{(7)}, f_4^{(7)}, f_5^{(7)}, f_6^{(7)} \rangle$, where

$$\begin{aligned}
f_1^{(7)} =& 2a_5 - 7,\\
f_2^{(7)} =& a_4^2 + 1,\\
f_3^{(7)} =& 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,\\
f_4^{(7)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,\\
f_5^{(7)} =& 2a_4a_5 + 3a_2 + 2a_4,\\
f_6^{(7)} =& 4a_1 + 2a_5 + 3.
\end{aligned}$$

APPENDIX D

Here is presented the six prime ideals, $I_1, I_2, \ldots, I_6$, defining the irreducible components of the variety of the ideal (2.25), which give conditions for the existence of an isochronous bi-center for system (2.17) under condition (2) of Theorem 2.3.2.

$I_1 = \langle f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)}, f_7^{(1)}, f_8^{(1)} \rangle$, where

$$
\begin{aligned}
f_1^{(1)} &= a_9,\\
f_2^{(1)} &= a_7,\\
f_3^{(1)} &= 4a_6^2 + 1,\\
f_4^{(1)} &= -4a_6a_9 + a_{10},\\
f_5^{(1)} &= -4a_6a_7 + a_8,\\
f_6^{(1)} &= 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,\\
f_7^{(1)} &= 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3,\\
f_8^{(1)} &= a_2 + 4a_6.
\end{aligned}
$$

$I_2 = \langle f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, f_4^{(2)}, f_5^{(2)}, f_6^{(2)}, f_7^{(2)}, f_8^{(2)} \rangle$, where

$$
\begin{aligned}
f_1^{(2)} &= 81a_9 + 16,\\
f_2^{(2)} &= a_7 + 1,\\
f_3^{(2)} &= 9a_6^2 + 1,\\
f_4^{(2)} &= -4a_6a_9 + a_{10},\\
f_5^{(2)} &= -4a_6a_7 + a_8,\\
f_6^{(2)} &= 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,\\
f_7^{(2)} &= 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3,\\
f_8^{(2)} &= a_2 + 4a_6.
\end{aligned}
$$

$I_3 = \langle f_1^{(3)}, f_2^{(3)}, f_3^{(3)}, f_4^{(3)}, f_5^{(3)}, f_6^{(3)}, f_7^{(3)}, f_8^{(3)} \rangle$, where

$$\begin{aligned}
f_1^{(3)} =& a_9 + 4,\\
f_2^{(3)} =& a_7 - 3,\\
f_3^{(3)} =& 4a_6^2 + 1,\\
f_4^{(3)} =& -4a_6a_9 + a_{10},\\
f_5^{(3)} =& -4a_6a_7 + a_8,\\
f_6^{(3)} =& 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,\\
f_7^{(3)} =& 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3,\\
f_8^{(3)} =& a_2 + 4a_6.
\end{aligned}$$

$I_4 = \langle f_1^{(4)}, f_2^{(4)}, f_3^{(4)}, f_4^{(4)}, f_5^{(4)}, f_6^{(4)}, f_7^{(4)}, f_8^{(4)} \rangle$, where

$$\begin{aligned}
f_1^{(4)} =& a_9 - 48,\\
f_2^{(4)} =& a_7 - 15,\\
f_3^{(4)} =& a_6^2 + 1,\\
f_4^{(4)} =& -4a_6a_9 + a_{10},\\
f_5^{(4)} =& \\
f_6^{(4)} =& 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,\\
f_7^{(4)} =& 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3,\\
f_8^{(4)} =& a_2 + 4a_6.
\end{aligned}$$

$I_5 = \langle f_1^{(5)}, f_2^{(5)}, f_3^{(5)}, f_4^{(5)}, f_5^{(5)}, f_6^{(5)}, f_7^{(5)}, f_8^{(5)} \rangle$, where

$$\begin{aligned}
f_1^{(5)} =& 4a_9 - 1,\\
f_2^{(5)} =& 2a_7 + 3,\\
f_3^{(5)} =& 16a_6^2 + 1,\\
f_4^{(5)} =& -4a_6a_9 + a_{10},\\
f_5^{(5)} =& -4a_6a_7 + a_8,\\
f_6^{(5)} =& 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,\\
f_7^{(5)} =& 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3,\\
f_8^{(5)} =& a_2 + 4a_6.
\end{aligned}$$

$I_6 = \langle f_1^{(6)}, f_2^{(6)}, f_3^{(6)}, f_4^{(6)}, f_5^{(6)}, f_6^{(6)}, f_7^{(6)}, f_8^{(6)} \rangle$, where

$$
\begin{aligned}
f_1^{(6)} =& a_9, \\
f_2^{(6)} =& a_7 - 3, \\
f_3^{(6)} =& a_6^2 + 1, \\
f_4^{(6)} =& -4a_6a_9 + a_{10}, \\
f_5^{(6)} =& -4a_6a_7 + a_8, \\
f_6^{(6)} =& 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6, \\
f_7^{(6)} =& 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \\
f_8^{(6)} =& a_2 + 4a_6.
\end{aligned}
$$

APPENDIX E

Here is listed the irreducible components of the variety of ideal (6.13), which give conditions for existence in system (6.3) invariant surfaces of degree two not passing through the origin of the system.

1. $\alpha_1 = \beta_1 = 0$
2. $-(1/2)+\beta_2 = \alpha_2 = \beta_1 = -3+\alpha_1 = 0$
3. $-3+\beta_2 = \alpha_2 = \beta_1 = -(1/2)+\alpha_1 = 0$
4. $-3+\beta_2 = \alpha_2 = \beta_1 = -3+\alpha_1 = 0$
5. $\alpha_2 = \beta_1 = \beta_2+\alpha_1-2 = 0$
6. $\beta_3 = -1+\alpha_3+\beta_2 = 1+\alpha_2 = \beta_1 = -1+\alpha_1-\alpha_3 = 0$
7. $\beta_3 = 1+\alpha_3 = -3+\beta_2 = 1+\alpha_2 = \beta_1 = -(1/2)+\alpha_1 = 0$
8. $\beta_3 = -3+\alpha_3 = -3+\beta_2 = -(3/2)+\alpha_2 = \beta_1 = 1+\alpha_1 = 0$
9. $-3+\beta_3 = -(3/2)+\alpha_3 = \beta_2 = 1+\alpha_2 = \beta_1 = -3+\alpha_1 = 0$
10. $-3+\beta_3 = 1+\alpha_3 = \beta_2 = -(1/2)+\alpha_2 = \beta_1 = 1+\alpha_1 = 0$
11. $-3+\beta_3 = -3+\alpha_3 = -1+\beta_2 = -(1/2)+\alpha_2 = \beta_1 = -1+\alpha_1 = 0$
12. $-3+\beta_3 = -(1/2)+\alpha_3 = -(1/2)+\beta_2 = 1+\alpha_2 = \beta_1 = -3+\alpha_1 = 0$
13. $1+\alpha_3 = \beta_2 = -2+\alpha_2+\beta_3 = \beta_1 = -1+\alpha_1+\beta_3 = 0$
14. $-(1/2)+\beta_3 = -3+\alpha_3 = -3+\beta_2 = -(3/2)+\alpha_2 = \beta_1 = -(1/2)+\alpha_1 = 0$
15. $-3+\beta_3 = 3+\alpha_3 = -3+\beta_2 = 1+\alpha_2 = \beta_1 = -3+\alpha_1 = 0$

16. $-(1/2)+\beta_3=-2+\alpha_3=-3+\beta_2=-(3/2)+\alpha_2=\beta_1=-(1/2)+\alpha_1=0$

17. $\alpha_3=-1+\beta_2-\beta_3=-2+\alpha_2+\beta_3=\beta_1=-1+\alpha_1+\beta_3=0$

18. $-3+\beta_3=-4+\alpha_3+\beta_2=1+\alpha_2=\beta_1=2+\alpha_1-\alpha_3=0$

19. $1+\beta_3=-3+\alpha_3=1+\beta_2=\alpha_2=-(1/2)+\beta_1=\alpha_1=0$

20. $1+\beta_3=-1+\alpha_3+\beta_2=\alpha_2=-2+\alpha_3+\beta_1=\alpha_1=0$

21. $-(3/2)+\beta_3=-3+\alpha_3=-3+\beta_2=\alpha_2=1+\beta_1=\alpha_1=0$

22. $\alpha_2=\beta_2=0$

23. $-2+\beta_3=-(1/2)+\alpha_3=-(1/2)+\beta_2=\alpha_2=-(3/2)+\beta_1=-3+\alpha_1=0$

24. $-3+\beta_3=-(1/2)+\alpha_3=-(1/2)+\beta_2=\alpha_2=-(3/2)+\beta_1=-3+\alpha_1=0$

25. $1+\beta_3=\alpha_3=-(1/2)+\beta_2=\alpha_2=1+\beta_1=-3+\alpha_1=0$

26. $3+\beta_3=-3+\alpha_3=-3+\beta_2=\alpha_2=1+\beta_1=-3+\alpha_1=0$

27. $-(1/2)+\beta_3=-3+\alpha_3=-3+\beta_2=\alpha_2=1+\beta_1=-(1/2)+\alpha_1=0$

28. $\alpha_3=-1+\beta_2-\beta_3=\alpha_2=1+\beta_1=-1+\alpha_1+\beta_3=0$

29. $-3+\beta_3=\alpha_3=1+\beta_2=\alpha_2=-(3/2)+\beta_1=-3+\alpha_1=0$

30. $-3+\alpha_3=2+\beta_2-\beta_3=\alpha_2=1+\beta_1=-4+\alpha_1+\beta_3=0$

31. $-3+\beta_3=-3+\alpha_3=-1+\beta_2=\alpha_2=-(1/2)+\beta_1=-1+\alpha_1=0$

32. $\beta_3=-1+\alpha_3+\beta_2=\alpha_2=-2+\alpha_3+\beta_1=-1+\alpha_1-\alpha_3=0$

33. $\beta_3=-(1/2)+\alpha_3=\beta_2=1+\alpha_2=-3+\beta_1=1+\alpha_1=0$

34. $\beta_3=\beta_2=1+\alpha_2-\alpha_3=-2+\alpha_3+\beta_1=1+\alpha_1=0$

35. $\beta_3=1+\alpha_3=\beta_2=-3+\alpha_2=-3+\beta_1=-(3/2)+\alpha_1=0$

36. $\alpha_3=\beta_3=0$

37. $\beta_3=-(1/2)+\alpha_3=-3+\beta_1=\alpha_1=0$

38. $\beta_3=-(1/2)+\alpha_3=-(1/2)+\beta_2=-2+\alpha_2=-3+\beta_1=-(3/2)+\alpha_1=0$

39. $\beta_3=-(1/2)+\alpha_3=-(1/2)+\beta_2=-3+\alpha_2=-3+\beta_1=-(3/2)+\alpha_1=0$

40. $\beta_3=-3+\alpha_3=-(1/2)+\beta_1=\alpha_1=0$

41. $\beta_3=-3+\alpha_3=-3+\beta_1=\alpha_1=0$

42. $\beta_3 = -3+\alpha_3 = -3+\beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = 1+\alpha_1 = 0$

43. $\beta_3 = -3+\alpha_3 = -3+\beta_2 = 3+\alpha_2 = -3+\beta_1 = 1+\alpha_1 = 0$

44. $\alpha_1 = \beta_1+\alpha_3-2 = \beta_3 = 0$

45. $\beta_3 = -1+\alpha_3 = -3+\beta_2 = -3+\alpha_2 = -1+\beta_1 = -(1/2)+\alpha_1 = 0$

46. $\beta_3 = -3+\beta_2 = -2+\alpha_2-\alpha_3 = -2+\alpha_3+\beta_1 = 1+\alpha_1 = 0$

47. $-(1/2)+\beta_3 = \alpha_3 = 1+\beta_2 = -3+\alpha_2 = 1+\beta_1 = \alpha_1 = 0$

48. $-(1/2)+\beta_3 = -1+\alpha_3 = -3+\beta_2 = -3+\alpha_2 = -1+\beta_1 = \alpha_1 = 0$

49. $-(1/2)+\beta_3 = \alpha_3 = \beta_2 = -3+\alpha_2 = 0$

50. $-(1/2)+\beta_3 = 1+\alpha_3 = \beta_2 = -3+\alpha_2 = -3+\beta_1 = -(1/2)+\alpha_1 = 0$

51. $-(1/2)+\beta_3 = \alpha_3 = -(3/2)+\beta_2 = -3+\alpha_2 = -2+\beta_1 = -(1/2)+\alpha_1 = 0$

52. $-(1/2)+\beta_3 = \alpha_3 = -(3/2)+\beta_2 = -3+\alpha_2 = -3+\beta_1 = -(1/2)+\alpha_1 = 0$

53. $-(1/2)+\beta_3 = -(1/2)+\alpha_3 = -3+\alpha_2 = -3+\beta_1 = -2+\alpha_1+\beta_2 = 0$

54. $-(1/2)+\beta_3 = -3+\alpha_3 = -3+\beta_2 = -3+\alpha_2 = -3+\beta_1 = -(1/2)+\alpha_1 = 0$

55. $-(1/2)+\beta_3 = -3+\beta_2 = -3+\alpha_2 = -2+\alpha_3+\beta_1 = -(1/2)+\alpha_1 = 0$

56. $-3+\beta_3 = \alpha_3 = \beta_2 = -(1/2)+\alpha_2 = 0$

57. $-3+\beta_3 = \alpha_3 = \beta_2 = -3+\alpha_2 = 0$

58. $-3+\beta_3 = -(3/2)+\alpha_3 = \beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = -2+\alpha_1 = 0$

59. $-3+\beta_3 = -(3/2)+\alpha_3 = \beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = -3+\alpha_1 = 0$

60. $-3+\beta_3 = 1+\alpha_3 = \beta_2 = -3+\alpha_2 = -3+\beta_1 = 3+\alpha_1 = 0$

61. $-3+\beta_3 = \alpha_3 = 1+\beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = -3+\alpha_1 = 0$

62. $-3+\beta_3 = \alpha_3 = 1+\beta_2 = -3+\alpha_2 = 3+\beta_1 = -3+\alpha_1 = 0$

63. $-3+\beta_3 = -(1/2)+\alpha_3 = -(1/2)+\beta_2 = -3+\alpha_2 = -3+\beta_1 = -3+\alpha_1 = 0$

64. $-3+\beta_3 = -3+\alpha_3 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = -2+\alpha_1+\beta_2 = 0$

65. $-3+\beta_3 = -3+\alpha_3 = -3+\alpha_2 = -3+\beta_1 = -2+\alpha_1+\beta_2 = 0$

66. $-3+\beta_3 = -3+\alpha_3 = -3+\beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = -3+\alpha_1 = 0$

67. $-3+\beta_3 = -4+\alpha_3+\beta_2 = -3+\alpha_2 = -2+\alpha_3+\beta_1 = 2+\alpha_1-\alpha_3 = 0$

68. $-3+\beta_3 = -(1/2)+\beta_2 = -(1/2)+\alpha_2 = -2+\alpha_3+\beta_1 = -3+\alpha_1 = 0$

69. $-3+\beta_3 = -3+\beta_2 = -3+\alpha_2 = -2+\alpha_3+\beta_1 = -3+\alpha_1 = 0$

70. $-3+\alpha_3+\beta_3 = \beta_2 = -2+\alpha_2+\beta_3 = 1+\beta_1-\beta_3 = \alpha_1 = 0$

71. $\alpha_3 = 1+\beta_2 = -2+\alpha_2+\beta_3 = 1+\beta_1-\beta_3 = \alpha_1 = 0$

72. $1+\beta_3 = \alpha_3 = -(3/2)+\beta_2 = -3+\alpha_2 = -3+\beta_1 = \alpha_1 = 0$

73. $1+\beta_3 = -(1/2)+\alpha_3 = -(1/2)+\beta_2 = -3+\alpha_2 = -3+\beta_1 = \alpha_1 = 0$

74. $-(3/2)+\beta_3 = -3+\alpha_3 = -2+\beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = \alpha_1 = 0$

75. $-(3/2)+\beta_3 = -3+\alpha_3 = -3+\beta_2 = -(1/2)+\alpha_2 = -(1/2)+\beta_1 = \alpha_1 = 0$

76. $1+\beta_3 = -3+\alpha_3 = 3+\beta_2 = -3+\alpha_2 = -3+\beta_1 = \alpha_1 = 0$

77. $1+\beta_3 = -4+\alpha_3+\beta_2 = -3+\alpha_2 = -2+\alpha_3+\beta_1 = \alpha_1 = 0$

78. $\alpha_3 = \beta_2 = \beta_3+\alpha_1-2 = 0$

79. $1+\alpha_3 = \beta_2 = -2+\alpha_2+\beta_3 = -3+\beta_1 = -4+\alpha_1+\beta_3 = 0$

80. $-1+\beta_3 = -(1/2)+\alpha_3 = \beta_2 = -1+\alpha_2 = -3+\beta_1 = -3+\alpha_1 = 0$

81. $-1+\beta_3 = \alpha_3 = -(1/2)+\beta_2 = -1+\alpha_2 = -3+\beta_1 = -3+\alpha_1 = 0$

82. $-(1/2)+\alpha_3 = -(1/2)+\beta_2 = -2+\alpha_2+\beta_3 = -3+\beta_1 = -3+\alpha_1 = 0$

83. $-3+\alpha_3 = -3+\beta_2 = -2+\alpha_2+\beta_3 = -(1/2)+\beta_1 = -(1/2)+\alpha_1 = 0$

84. $-3+\alpha_3 = -3+\beta_2 = -2+\alpha_2+\beta_3 = -3+\beta_1 = -3+\alpha_1 = 0$

85. $\alpha_3+\beta_3 = -3+\beta_2 = -2+\alpha_2+\beta_3 = -2+\beta_1-\beta_3 = -3+\alpha_1 = 0$

86. $\alpha_3 = 1+\beta_2 = -2+\alpha_2+\beta_3 = -2+\beta_1-\beta_3 = -3+\alpha_1 = 0$

87. $-3+\alpha_3 = 2+\beta_2-\beta_3 = -2+\alpha_2+\beta_3 = -3+\beta_1 = -4+\alpha_1+\beta_3 = 0$

88. $\beta_1+\alpha_3-2 = \beta_2+\alpha_1-2 = \beta_3+\alpha_1-2 = 0.$