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On properties about local cohomology modules, finiteness of torsion and extension functors, and integral closure relative to Artinian modules

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Propriedades sobre módulos de cohomologia local, finitude dos funtores torção e extensão, e fecho integral relativo a módulos Artinianos

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutora em Ciências – Matemática. *VERSÃO REVISADA*

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Aos meus pais, sem os quais nada disso teria sido possível.

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A todos vocês o meu carinho e o meu MUITO OBRIGADA!

"First, think.
Second, believe.
Third, dream.
And finally, dare."
(Walt Disney)

RESUMO

MERIGHE, L. C. **Propriedades sobre módulos de cohomologia local, finitude dos funtores torção e extensão, e fecho integral relativo a módulos Artinianos**. 2019. 135 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

Sejam R um anel Noetheriano comutativo com unidade $1 \neq 0$, $\mathfrak{a}$ um ideal de R e M e N módulos sobre R.

Nessa tese, fazemos contribuições ao estudo dos módulos de cohomologia local generalizada, a saber $\mathrm{H}^i_{\mathfrak{a}}(M,N)$, com aplicações ao estudo dos ideais primos anexados de R, funtores torção e extensão, e fecho integral e multiplicidades relativos a módulos artinianos.

Em particular, estabelecemos resultados nos seguintes temas: contar o número de módulos de cohomologia local generalizados no topo não isomorfos; condições para os módulos de cohomologia local e os funtores torção e extensão aplicados a R-módulos terem características finitas (finitamente gerado, finitos primos associados, etc), serem cofinitos, serem artinianos e serem representáveis; e condições para a igualdade entre tipos de fechos integrais e multiplicidades.

Palavras-chave: Cohomologia local generalizada, Álgebra homológica, Primos anexados, Fecho integral, Multiplicidade.

ABSTRACT

MERIGHE, L. C. **On properties about local cohomology modules, finiteness of torsion and extension functors, and integral closure relative to Artinian modules**. 2019. 135 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

Let R be a non-zero commutative Noetherian ring with unit $1 \neq 0$, $\mathfrak{a}$ be an ideal of R, and M and N be R-modules.

This thesis makes a contribution to the study of generalized local cohomology modules, namely $\mathrm{H}^i_{\mathfrak{a}}(M,N)$, with applications for the study of attached primes, torsion product and extension functors, and integral closures and multiplicities relative to Artinian modules.

In particular, we obtained results on the following topics: counting the number of non-isomorphic top generalized local cohomology modules, conditions to finiteness, cofiniteness, artinianess and representability of generalized local cohomology modules, torsion product and extension functors applied to R-modules, and conditions to equality between some types of integral closures and multiplicities.

Keywords: Generalized local cohomology, Homological algebra, Attached primes, Integral closure, Multiplicity.

LIST OF SYMBOLS

$\mathbb{N} = \{1, 2, 3, \ldots\}$ — natural numbers

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ — natural numbers with 0

$|S|$ — number of elements of some set S

$\mathrm{Hom}_R(M, N)$ — set of all R-homomorphisms $h : M \to N$

$\mathfrak{F}$ — functor

$\{M_n\}_{n\in\mathbb{Z}} = \mathbf{M}_\bullet$ — chain complex

$\{M^n\}_{n\in\mathbb{Z}} = \mathbf{M}^\bullet$ — cochain complex

$\mathrm{H}_n(\mathbf{M}_\bullet)$ — homology module

$\mathrm{H}^n(\mathbf{M}^\bullet)$ — cohomology module

$f^\bullet : M^\bullet \to N^\bullet$ — map of cochain complexes

$f_\bullet : M_\bullet \to N_\bullet$ — map of chain complexes

$\mathbf{I}^\bullet$ — injective resolution

$\mathbf{I}^{\bullet,M}$ — deleted injective resolution

$\mathbf{P}_\bullet$ — projective resolution

$\mathbf{P}_{\bullet,M}$ — deleted projective resolution

$\mathfrak{L}_i\mathfrak{F}(-)$ — ith left derived functor

$\mathfrak{R}^i\mathfrak{F}(-)$ — ith right derived functor

$\mathrm{Ext}^i_R(-,-)$ — extension functor

$\mathrm{Tor}^R_i(-,-)$ — torsion functor

$\mathrm{pdim}_R(M)$ — projective dimension of M

$\mathrm{idim}_R(N)$ — injective dimension of N

$\Gamma_\mathfrak{a}(-)$ — $\mathfrak{a}$-torsion functor

$\mathrm{ZD}_R(M)$ — set of zero divisors of M

$\mathrm{NZD}_R(M)$ — set of non-zero divisors of M

$\mathrm{Ass}_R(M)$ — set of associated primes of M

$\mathrm{H}^i_{\mathfrak{a}}(M)$ — ith local cohomology module of M with respect to $\mathfrak{a}$

$\mathrm{H}^i_{\mathfrak{a}}(M,N)$ — ith generalized local cohomology module of M and N with respect to $\mathfrak{a}$

$\dim M or \dim_R M$ — Krull dimension of M

$\mathrm{Supp}_R(M)$ — support of M

$\mathrm{grade}_M(\mathfrak{a})$ — grade of $\mathfrak{a}$

$\mathrm{depth}\, M$ — depth of M

$\mathrm{Ann}_R(M)$ — annihilator of M

$\mathrm{Att}_R(L)$ — set of attached primes of L

$E(R/\mathfrak{m})$ — injective hull of the simple R-module $R/\mathfrak{m}$

$\mathrm{Assh}(M,N)$ — notation for $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N))$

$\mathrm{Assr}(M,N)$ — notation for $\mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{m}}(M,N))$

$\mathscr{S}$ — Serre category

$\bar{I}$ — integral closure of I

$I^{*(H)}$ — ST-closure of I on H

$\bar{I}^{(M)}$ — integral closure of I relative to M

$\lambda_R(M/\mathfrak{a}^n M)$ — Hilbert function

$e(\mathfrak{a};M)$ — multiplicity of $\mathfrak{a}$ on M

$e(\mathfrak{a})$ — multiplicity of $\mathfrak{a}$

$\lambda(0 :_H \mathfrak{a}^n)$ — Hilbert-Samuel polynomial of $\mathfrak{a}$ relative to H

$e'(\mathfrak{a};H)$ — multiplicity of $\mathfrak{a}$ relative to the Artinian module H

$\mathrm{Ndim}_R(M)$ — Noetherian dimension of M

CONTENTS

CHAPTER

1

INTRODUCTION

Homological algebra had its origins in the 19th century, via the work of B. Riemann (1857) and E. Betti (1871) on “homology numbers” as well as the rigorous development of the notion of homology numbers by H. Poincaré (1895). In 1925, E. Noether changed the focus to “homology groups” of a space, and, inspired by this new perspective, in 1929 L. Mayer introduced the purely algebraic notions of chain complex, its subgroup of cycles and the homology groups of a complex. In addition, in the 1930s, algebraic techniques were developed for computational purposes. From 1940-1955, some topologically-motivated techniques for computing homology were applied to define and explore the homology and cohomology of several algebraic systems: derived functors of tensorial product and Hom (namely, Tor and Ext) for abelian groups, homology and cohomology of groups and Lie algebras, and the cohomology of associative algebras. Moreover, sheaves, sheaf cohomology and spectral sequences were introduced.. Slowly the subject became more algebraic.

At this point, H. Cartan and S. Eilenberg’s book (1956) (CARTAN; EILENBERG, 1999) redirected the field completely. Their systematic use of derived functors, defined via projective and injective resolutions of modules, consolidate all the previous diverse theories of homology. The search for a general setting for derived functors led to the notion of abelian categories, and the search for nontrivial examples of projective modules led to the rise of algebraic K-theory.

The approach changed dramatically when J.-P. Serre characterized regular local rings using Homological Algebra (they are the commutative Noetherian local rings of “finite global dimension”). This enabled him to prove that any localization of a regular local ring is itself regular (until then, only special cases of this were known). The importance of regular local rings in algebra grew out of results obtained by homological methods. At the same time, M. Auslander and D. A. Buchsbaum also characterized regular local rings, and they completed the work of M. Nagata by using global dimension to prove that every regular local ring is a unique factorization domain. Furthermore, the study of injective resolutions leads A. Grothendieck to define the theory of sheaf cohomology, the discovery of Gorenstein rings and Local Duality in both Ring

Theory and Algebraic Geometry. In turn, cohomological methods play an important role in Grothendieck's rewriting of the foundations of algebraic geometry, including the development of derived categories.

The reader can see more details about homological algebra history in (WEIBEL, 1999).

Nowadays, Homology Algebra is a great theory and a powerful tool that is being developed and used in the areas of commutative and non-commutative algebra, number theory, group theory and algebraic geometry, algebraic topology, etc. It should be emphasized that a study of modern algebraic geometry, for example, would be practically unthinkable without the sheaf cohomology .

On the other hand, as a particular case of Homological Algebra theory, Local Cohomology was introduced by A. Grothendieck in the early 1960s (GROTHENDIECK, 1967), partly to answer a conjecture of Pierre Samuel about when certain types of commutative rings are unique factorization domains. Specifically: Let R be a Noetherian local ring and $\widehat{R}$ its completion with respect to the maximal ideal. If $\widehat{R}$ is a complete intersection and for each prime ideal $\mathfrak{p}$ of R of height less or equal to 3, $R_{\mathfrak{p}}$ is a UFD (unique factorization domain), then R is a UFD.

The origin of local cohomology is also attributed as a tool to prove Lefschetz-type Theorems in Algebraic Geometry and also for the resolution of several conjectures in Homological Algebra.

Regardless of the motivation, local cohomology, among many other attributes, allows one to answer many difficult questions. A good example of such a problem, where local cohomology provides a partial answer, is the question of how many generators ideals have up to radical. It turns out that local cohomology has since become an indispensable tool and is the subject of much research in the theory of commutative Noetherian rings. Research topics on this subject are, for example: vanishing and non-vanishing, associated primes, attached primes, finiteness, cofiniteness and artinianess.

When R is a commutative Noetherian ring with nonzero identity, $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R, M and N are two R-modules, and $i \in \mathbb{Z}$, Herzog (HERZOG, 1974) introduced a concept that can be seen as an extension of local cohomology modules, called the ith generalized local cohomology module with respect to $\mathfrak{a}$ and M and N, namely,

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

In order to have a better understanding of this structure, several authors investigated different subjects: vanishing, non-vanishing, artinianness, finiteness properties, among others. See for example (DIVAANI-AAZAR; SAZEEDEH; TOUSI, 2005), (GU; CHU, 2009), (MACDONALD; SHARP, 1972a), (DEHGHANI-ZADEH,).

One of our aims in this thesis is to study some of those structures of generalized local cohomology modules, as when they are representable, Artinian and $\mathfrak{b}$-cofinite, and apply those properties to see examples on the study of torsion product and extension functors, and integral closures and multiplicities relative to Artinian modules, as we are going to describe next. It is worth mentioning that although we will see several structures of the generalized local cohomology modules, the main thread during this work will be the fact that, under some conditions, the those modules are examples of Artinian modules that are not finitely generated, which also motivated us to study several properties of Artinian modules.

Our philosophy throughout has been to give a careful and accessible overview of basic ideas and some important results to provide the reader with some background knowledge to understand what we have done in all the Chapter of this thesis.

In Chapter 2, we present an introduction to some fundamental concepts on Homological Algebra, such as: functors from the category of R-modules to itself (where we introduce $\mathrm{Hom}_R(-,-)$ and $(-\otimes_R -)$), exact sequences, chain and cochain complexes, injective and projective modules, resolutions and dimensions, and derived functors (where we introduce $\mathrm{Ext}^i_R(-,-)$ and $\mathrm{Tor}^R_i(-,-)$). Here we focus on a chapter of fundamental results for the subjects that will be used in the rest of the text. Therefore, if the reader is already familiar with all those concepts, he or she can feel free to start reading the next Chapter.

In Chapter 3, we formally define local cohomology modules and generalized local cohomology modules. Here, we see some basic properties involving the functor Gamma, or $\mathfrak{a}$-torsion, and some of the main known results about (generalized) local cohomology modules, such as under which conditions those modules vanish or non-vanish, are finitely generated or not, and are Artinian or not. All the information in this Chapter is going to be important for what comes next. Those modules have some geometric interpretation, but we still focus here just on the algebraic notion.

In Chapter 4, we study a theory that can be thought of as a dual of the theory of primary decomposition and associated primes of a module over a commutative ring: we introduce the definitions and give some properties about secondary R-modules, representable R-modules and the set of attached primes of an R-module. Concerning this last set, we prove that this does not depend on the minimal secondary representation of the R-module (Uniqueness Theorem), and that every Artinian module is representable (Existence Theorem).

Once the fundamentals have been established, at the end of this Chapter we explore some relationships between those concepts and generalized local cohomology modules. Attached primes and secondary representation for local cohomology modules over finitely generated modules is an ongoing topic of investigation (see for example (SHARP, 1975), (MACDONALD; SHARP, 1972a), (DIBAEI; YASSEMI, 2005a), (GU; CHU, 2009), (NHAN; QUY, 2014) and (CHAU; NHAN, 2014)). Thus we have the following natural question.

Question 1.0.1. Under which conditions is it possible to characterize the attached primes of $\mathrm{H}^i_{\mathfrak{a}}(M,N)$? When is the set of attached primes of $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ finite?

In order to answer this question, we show some examples in Section 4.3 of cases where this is already known and offer something new in Theorem 4.4.3.

As a consequence of the study of attached primes of generalized local cohomology modules, we can also investigate the following question.

Question 1.0.2. Let $\operatorname{pdim} M = d < \infty$ and $\dim N = n < \infty$. Is it possible to count the number of non-isomorphic top local cohomology modules $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$?

We finish Chapter 4 by giving an affirmative answer to this question in Theorem 4.5.16 and Corollary 4.5.17.

In Chapter 5, we establish some definitions about finiteness of an R-module: $\mathfrak{a}$-cofinite, minimax, $\mathfrak{a}$-cominimax, weakly Laskerian, and $\mathfrak{a}$-weakly cofinite. Besides, we explore some properties about a Serre subcategory of the category of R-modules and we reached a very interesting result on this subject, Theorem 5.1.16. Furthermore, we define a new class of modules, called $\mathfrak{a}$-weakly finite modules.

All those preparatory results are done in order to provide some answers to interesting open questions in homological algebra and local cohomology theory.

Question 1.0.3. When is $\operatorname{Ext}_R^i(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^j(N))$ finitely generated for all integer i and j?

This question was proposed by Hartshorne (HARTSHORNE, 1970) who, in turn, was motivated by the conjecture made by Grothendieck (GROTHENDIECK, 2005) about the finiteness of $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^j(N))$. Thus, as a generalization of Hartshorne's conjecture, we have another question.

Question 1.0.4. When is $\mathrm{H}_{\mathfrak{a}}^i(M,N)$ $\mathfrak{a}$-cofinite for all i?

Concerning this question, several results were obtained, and in most of them the finiteness assumption on the modules is crucial in the proof (for example (CUONG; GOTO; HOANG, 2015), (DIVAANI-AAZAR; SAZEEDEH, 2004) and (NASBI; VAHIDI; AMOLI, 2017)). In this Chapter, we show Theorem 5.2.4 and Theorem 5.2.7 which provide an answer to Question 1.0.4 and are the most important results in Section 5.2.

In the last section of Chapter 5, we focus on answering question 1.0.3, which can be seen from a different point of view.

Question 1.0.5. When are $\operatorname{Ext}_R^i(M,N)$ and $\operatorname{Tor}_i^R(M,N)$ $\mathfrak{a}$-cofinite (or Artinian, or finite length, or $\mathfrak{a}$-cominimax, or $\mathfrak{a}$-weakly cofinite) for all (or for some) integer i?

This Question has been studied by (KUBIK; LEAMER; SATHER-WAGSTAFF, 2011), (MELKERSSON, 1990), and (MELKERSSON, 2005), for example. Therefore, my contribution in Section 5.3 is to prove Theorem 5.3.4, Theorem 5.3.9 and Theorem 5.3.14.

In Chapter 6, we move on to defining some types of reductions, integral closures, and multiplicities; giving special attention to integral closures and multiplicities relative to Artinian modules.

The concepts of reduction and integral closure have played a very important role in commutative algebra, number theory and algebraic geometry since the mid twentieth century. Recent results on these concepts can be found in (HUNEKE; SWANSON, 2006) and (VASCONCELOS, 2006). This leads to some important questions.

Question 1.0.6. (D. Rees) Fix H an Artinian R-module. Is there a relationship between $\overline{\mathfrak{b}}$, the classical Northcott-Rees integral closure of $\mathfrak{b}$, and $\mathfrak{b}^{*(H)}$,the integral closure of $\mathfrak{b}$ relative to the Artinian R-module H? Are there any Artinian R-module for which they are equal?

Besides, there is an important Theorem from Rees (6.2.8) that provides a relationship between integral closure and multiplicity. Thus, in view of Rees's Theorem, if we give an answer to the previous question, we will be able to find some answers to another question.

Question 1.0.7. What is the relationship between $e(\mathfrak{a})$, the multiplicity of $\mathfrak{a}$, and $e'(\mathfrak{a};H)$, the multiplicity of $\mathfrak{a}$ relative to an Artinian R-module H?

Moreover, it is possible to formulate another question.

Question 1.0.8. Is it possible to formulate a Theorem analogous to Rees Theorem involving integral closures and multiplicities relative to Artinian modules?

In (SHARP; TIRAŞ; YASSI, 1990), they responded Question 1.0.6 in a particular case. Our aim in this Chapter is to answer those questions, which we have done in our main results: Theorem 6.3.6 and Theorem 6.3.7. Furthermore, we end Chapter 6 with examples of Theorem 6.3.6 and applications of it on generalized local cohomology modules.

The answer to Question 1.0.8 is still open and it will be subject to future work.

Before beginning, let us fix some notation that will be used during this thesis: every time we talk about a commutative ring, we mean it to be a commutative ring with non-zero identity; when necessary, we will use the notation $|S|$ to denote the number of elements of some set S; to us, $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$.

We hope this work motivates other people to work on commutative algebra. Enjoy your reading!

CHAPTER

2

HOMOLOGICAL ALGEBRA

In this chapter, we present background material and the relevant tools from homological algebra that will be used during this work. We give some of the proofs here in order to help the reader to understand the results. We refer the reader to (ROTMAN, 2008) and (BLAND, 2011) for a thorough discussion of the subject.

If the reader is familiar with concepts such as exact sequences, functors, injective and projective modules, projective and injective resolutions, projective and injective dimension, homology and cohomology modules, and derived functors, such as Tor and Ext; he or she can skip to Chapter 3, where subjects that are not as elementary as those in Chapter 2 can be found.

Throughout this Chapter, unless otherwise noted, R will be a commutative Noetherian.

2.1 Category of R-modules and Functors

A *category* is an algebraic structure that comprises "objects" that are linked by "arrows". A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. An example that we will use in this work is what we call the *category of R-modules*, whose objects are R-modules and whose arrows are R-homomorphisms.

Furthermore, a *functor* is a type of mapping between two categories; it sends information from one category to another.

Let M and N be two R-modules. Denote by $\mathrm{Hom}_R(M,N)$ the set of all R-homomorphisms $h : M \to N$. This set has an R-module structure: let $h, l \in \mathrm{Hom}_R(M,N)$ and $a \in R$, then, for all $m \in M$,

$$\begin{aligned} (h+l)(m) &:= h(m)+l(m) \\ (ah)(m) &:= ah(m). \end{aligned}$$

In what follows, let R' be another commutative Noetherian ring.

Definition 2.1.1. An *additive covariant functor* from the category of R-modules to the category of R'-modules is the map

$$\mathfrak{F} = \mathfrak{F}(-) : (M \xrightarrow{h} N) \rightsquigarrow (\mathfrak{F}(M) \xrightarrow{\mathfrak{F}(h)} \mathfrak{F}(N)),$$

which associates for each R-module M an R'-module $\mathfrak{F}(M)$, and associates for each R-homomorphism $h : M \to N$ an R'-homomorphism $\mathfrak{F}(h) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ such that:

1. $\mathfrak{F}(\mathrm{id}_M) = \mathrm{id}_{\mathfrak{F}(M)}$, for each R-module M;
2. $\mathfrak{F}(\ell \circ h) = \mathfrak{F}(\ell) \circ \mathfrak{F}(h)$, where $h \in \mathrm{Hom}_R(M,N)$ and $\ell \in \mathrm{Hom}_R(N,P)$;
3. $\mathfrak{F}(\ell + h) = \mathfrak{F}(\ell) + \mathfrak{F}(h)$, where $\ell, h \in \mathrm{Hom}_R(M,N)$.

Definition 2.1.2. An *additive contravariant functor* from the category of R-modules to the category of R'-modules is the map

$$\mathfrak{F} = \mathfrak{F}(-) : (M \xrightarrow{h} N) \rightsquigarrow (\mathfrak{F}(N) \xrightarrow{\mathfrak{F}(h)} \mathfrak{F}(M)),$$

which associates for each R-module M an R'-module $\mathfrak{F}(M)$, and associates for each R-homomorphism $h : M \to N$ an R'-homomorphism $\mathfrak{F}(h) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ such that:

1. $\mathfrak{F}(\mathrm{id}_M) = \mathrm{id}_{\mathfrak{F}(M)}$, for each R-module M;
2. $\mathfrak{F}(\ell \circ h) = \mathfrak{F}(h) \circ \mathfrak{F}(\ell)$, where $h \in \mathrm{Hom}_R(M,N)$ and $\ell \in \mathrm{Hom}_R(N,P)$;
3. $\mathfrak{F}(\ell + h) = \mathfrak{F}(\ell) + \mathfrak{F}(h)$, where $\ell, h \in \mathrm{Hom}_R(M,N)$.

Example 2.1.3. (Contravariant Functor $\mathrm{Hom}_R(-,X)$**)**

Fix X an R-module. Then $\mathrm{Hom}_R(-,X)$ is a functor from the category of R-modules to itself such that:

$$\mathrm{Hom}_R(-,X) : (M \xrightarrow{f} N) \rightsquigarrow (\mathrm{Hom}_R(N,X) \xrightarrow{f^*} \mathrm{Hom}_R(M,X)),$$

which associates for each R-module M an R-module $\mathrm{Hom}_R(M,X)$ and associates for each R-homomorphism $f : M \to N$ an R-homomorphism $f^* = \mathrm{Hom}_R(f,X) : \mathrm{Hom}_R(N,X) \to \mathrm{Hom}_R(M,X)$, where $f^*(h) = \mathrm{Hom}_R(f,X)(h) := h \circ f$, for all $h \in \mathrm{Hom}_R(N,X)$. Note that $\mathrm{Hom}_R(-,X)$ is a contravariant functor.

Example 2.1.4. (Covariant Functor $\mathrm{Hom}_R(X,-)$**)**

Fix X an R-module. Then $\mathrm{Hom}_R(X,-)$ is a functor from the category of R-modules to itself such that:

$$\mathrm{Hom}_R(X,-) : (M \xrightarrow{f} N) \rightsquigarrow (\mathrm{Hom}_R(X,M) \xrightarrow{f_*} \mathrm{Hom}_R(X,N)),$$

which associates for each R-module M an R-module $\mathrm{Hom}_R(X,M)$ and associates for each R-homomorphis $f : M \to N$ an R-homomorphism $f_* = \mathrm{Hom}_R(X,f) : \mathrm{Hom}_R(X,M) \to \mathrm{Hom}_R(X,N)$, where $f_*(h) = \mathrm{Hom}_R(X,f)(h) := f \circ h$, for all $h \in \mathrm{Hom}_R(X,M)$. Note that $\mathrm{Hom}_R(X,-)$ is a covariant functor.

Example 2.1.5. (Covariant Functors $-\otimes_R X$ and $X \otimes_R -$)

Fix X an R-module. Then $-\otimes_R X$ is a functor from the category of R-modules to itself such that:

$$-\otimes_R X : (M \xrightarrow{f} N) \rightsquigarrow (M \otimes_R X \xrightarrow{\overline{f}} N \otimes_R X),$$

which associates for each R-module M an R-module $M \otimes_R X$ and associates for each R-homomorphis $f : M \to N$ an R-homomorphism $\overline{f} = f \otimes_R \mathrm{id}_X : M \otimes_R X \to N \otimes_R X$. Note that $-\otimes_R X$ is a covariant functor.

In the same way, we can define the covariant functor $X \otimes_R -$.

Definition 2.1.6. A *sequence* $M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ of R-modules and R-homomorphisms is said to be *exact* on M if $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Let $n \in \mathbb{Z}$. A *sequence*

$$S : \cdots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to \cdots,$$

is said to be *exact* (or a *long exact sequence*) if it is exact on M_n, for all $n \in \mathbb{Z}$.

A sequence $S : 0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ which is exact on M_1, M and M_2 is said to be a *short exact sequence*.

Definition 2.1.7. Let $\mathfrak{F} = \mathfrak{F}(-)$ be a covariant functor from the category of R-modules to the category of R'-modules and $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ be a short exact sequence of R-modules.

1. $\mathfrak{F}$ is said to be a *left exact functor* if

$$0 \to \mathfrak{F}(M_1) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(M) \xrightarrow{\mathfrak{F}(g)} \mathfrak{F}(M_2)$$

is an exact sequence of R'-modules.

2. $\mathfrak{F}$ is said to be a *right exact functor* if

$$\mathfrak{F}(M_1) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(M) \xrightarrow{\mathfrak{F}(g)} \mathfrak{F}(M_2) \to 0$$

is an exact sequence of R'-modules.

3. If $\mathfrak{F}$ is left exact and right exact, then it is said to be a *covariant exact functor*.

Now let $\mathfrak{F} = \mathfrak{F}(-)$ be a contravariant functor from the category of R-modules to the category of R'-modules.

1. $\mathfrak{F}$ is said to be a *left exact functor* if

$$0 \to \mathfrak{F}(M_2) \stackrel{\mathfrak{F}(f)}{\to} \mathfrak{F}(M) \stackrel{\mathfrak{F}(g)}{\to} \mathfrak{F}(M_1)$$

is an exact sequence of R'-modules.

2. $\mathfrak{F}$ is said to be a *right exact functor* if

$$\mathfrak{F}(M_2) \stackrel{\mathfrak{F}(f)}{\to} \mathfrak{F}(M) \stackrel{\mathfrak{F}(g)}{\to} \mathfrak{F}(M_1) \to 0$$

is an exact sequence of R'-modules.

3. If $\mathfrak{F}$ is left exact and right exact, then it is said to be a *contravariant exact functor*.

Proposition 2.1.8. Let X be an R-module. The functors $\mathrm{Hom}_R(-,X)$ and $\mathrm{Hom}_R(X,-)$ are left exact functors from the category of R-modules to itself.

Proof. Let $0 \to M_1 \stackrel{f}{\to} M \stackrel{g}{\to} M_2$ be an exact sequence of R-modules, we show that

$$0 \to \mathrm{Hom}_R(X,M_1) \stackrel{f_*}{\to} \mathrm{Hom}_R(X,M) \stackrel{g_*}{\to} \mathrm{Hom}_R(X,M_2)$$

is exact. To prove this, it is sufficient to show that f_* is an injective R-homomorphism and $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$. Note that f_* and g_* are homomorphisms.

First we show that f_* is injective. If $h \in \mathrm{Hom}_R(X,M_1)$ is such that $f_*(h) = 0$ then $f \circ h = 0$, that is, $f(h(x)) = 0$ for all $x \in X$. Thus $h(x) = 0$ for all $x \in X$, since f is injective.

Now, let $h \in \mathrm{Im}(f_*)$, then there exists $h' \in \mathrm{Hom}_R(X,M_1)$ such that $h = f_*(h')$. Thus $g_*(h) = g_*(f_*(h')) = g(f(h')) = 0$, since $g \circ f = 0$. Therefore $h \in \mathrm{Ker}(g_*)$ and $\mathrm{Im}(f_*) \subseteq \mathrm{Ker}(g_*)$.

Finally, let us prove the other inclusion. If $h \in \mathrm{Ker}(g_*)$ then $g \circ h = g_*(h) = 0$, that is, for all $x \in X$, $g(h(x)) = 0$ and $h(x) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Since f is injective, for each $x \in X$, there is only one element $y \in M_1$ such that $f(y) = h(x)$. Then let $h'' \in \mathrm{Hom}_R(X,M_1)$ such that $h''(x) = y$ and $f(h''(x)) = f(y) = h(x)$. Thus, $f_*(h'') = h \in \mathrm{Im}(f_*)$ and then $\mathrm{Ker}(g_*) \subseteq \mathrm{Im}(f_*)$. Therefore $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$ and we prove that the functor $\mathrm{Hom}_R(X,-)$ is left exact.

The proof for $\mathrm{Hom}_R(-,X)$ is analogous. □

Proposition 2.1.9. Let X be an R-module. The functors $- \otimes_R X$ and $X \otimes_R -$ are right exact functors from the category of R-modules to itself.

Proof. Let $M_1 \stackrel{f}{\to} M \stackrel{g}{\to} M_2 \to 0$ be an exact sequence of R-modules. We show that the sequence

$$M_1 \otimes_R X \stackrel{f \otimes \mathrm{id}_X}{\longrightarrow} M \otimes_R X \stackrel{g \otimes \mathrm{id}_X}{\longrightarrow} M_2 \otimes_R X \to 0$$

is also exact. To prove this, it is sufficient to show that $\mathrm{Im}(f \otimes \mathrm{id}_X) = \mathrm{Ker}(g \otimes \mathrm{id}_X)$ and $g \otimes \mathrm{id}_X$ is a surjective R-homomorphism.

Since g and id_X are surjective, it follows that $g \otimes \mathrm{id}_X$ is surjective.

Since $g \circ f \equiv 0$, then $\mathrm{Im}(f \otimes \mathrm{id}_X) \subseteq \mathrm{Ker}(g \otimes \mathrm{id}_X)$ and there exists induced map

$$h : (M \otimes_R X)/\mathrm{Im}(f \otimes \mathrm{id}_X) \to M_2 \otimes_R X,$$

such that

$$h\,(y \otimes x + \mathrm{Im}(f \otimes \mathrm{id}_X)) = g(y) \otimes x.$$

Hence we have the following commutative diagram, where the first line is right exact.

$$\begin{array}{ccccccc} M_1 \otimes_R X & \xrightarrow{f \otimes \mathrm{id}_X} & M \otimes_R X & \longrightarrow & (M \otimes_R X)/\mathrm{Im}(f \otimes \mathrm{id}_X) & \longrightarrow & 0 \\ & & {\scriptstyle g \otimes \mathrm{id}_X} \big\downarrow & \swarrow {\scriptstyle h} & & & \\ & & M_2 \otimes_R X & & & & \end{array}$$

Define the R-bilinear map $\rho : M_2 \times_R X \to (M \otimes_R X)/\mathrm{Im}(f \otimes \mathrm{id}_X)$ as $\rho(z,x) = y \otimes x + \mathrm{Im}(f \otimes \mathrm{id}_X)$, where $y \in M$ is such that $g(y) = z$. Let's show that this map is well defined: if $y' \in M$ is another element such that $g(y') = z$, then $y - y' \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Let $u \in M_1$ be such that $f(u) = y - y'$. Then $(f \otimes \mathrm{id}_X)(u \otimes x) = f(u) \otimes x = (y - y') \otimes x$, and $(y - y') \otimes x \in \mathrm{Im}(f \otimes \mathrm{id}_X)$. Therefore $y \otimes x + \mathrm{Im}(f \otimes \mathrm{id}_X) = y' \otimes x + \mathrm{Im}(f \otimes \mathrm{id}_X)$ and then ρ is well defined.

By the definition of ρ, there exists an R-homomorphism

$$\bar{h} : M_2 \otimes_R X \to (M \otimes_R X)/\mathrm{Im}(f \otimes \mathrm{id}_X)$$

such that

$$\bar{h}(z \otimes x) = y \otimes x + \mathrm{Im}(f \otimes \mathrm{id}_X)$$

where $y \in M$ is such that $g(z) = y$, and then we can see that $\bar{h} = h^{-1}$. Thus, h is an isomorphism. Now $(M \otimes_R X)/\mathrm{Ker}(g \otimes \mathrm{id}_X) \cong M_2 \otimes_R X$, by Isomorphism Theorem. Hence $(M \otimes_R X)/\mathrm{Ker}(g \otimes \mathrm{id}_X) \cong (M \otimes_R X)/\mathrm{Im}(f \otimes \mathrm{id}_X)$, therefore $\mathrm{Ker}(f \otimes \mathrm{id}_X) = \mathrm{Im}(f \otimes \mathrm{id}_X)$.

The proof for $X \otimes_R -$ is analogous. □

The next result shows us some relation between those two functors.

Theorem 2.1.10. ((ROTMAN, 2008, Theorem 2.75)) Let R and S be commutative rings with unit, A be an R-module, B be an (R,S)-bimodule and C be a S-module. Then, there exists a natural isomorphism

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)).$$

2.2 Homology and Cohomology Properties

The aim of this section is to define and ensure the existence of long exact sequences of cohomology R-modules and homology R-modules.

Definition 2.2.1. A sequence of R-modules and R-homomorphisms

$$\{M_n\}_{n\in\mathbb{Z}} = \mathbf{M}_\bullet : \cdots \longrightarrow M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \longrightarrow \cdots$$

is said to be a *chain complex* if $\alpha_n \circ \alpha_{n+1} \equiv 0$ for each $n \in \mathbb{Z}$. Each map α_n is said to be a *differential operator*.

A chain complex $\mathbf{M}_\bullet$ is said to be exact on M_n if $\mathrm{Im}(\alpha_{n+1}) = \mathrm{Ker}(\alpha_n)$. $\mathbf{M}_\bullet$ is an *exact chain complex* if it is exact on M_n, for all $n \in \mathbb{Z}$.

A chain complex such that

$$\mathbf{M}_\bullet : \cdots \longrightarrow M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$$

is said to be a *positive chain complex*.

Definition 2.2.2. A sequence of R-modules and R-homomorphisms

$$\{M^n\}_{n\in\mathbb{Z}} = \mathbf{M}^\bullet : \cdots \longrightarrow M^{n-1} \xrightarrow{\alpha^{n-1}} M^n \xrightarrow{\alpha^n} M^{n+1} \longrightarrow \cdots$$

is said to be a *cochain complex* (or co-complex) if $\alpha^n \circ \alpha^{n-1} \equiv 0$ for each $n \in \mathbb{Z}$. Each map α^n is said to be a *differential operator*.

A complex $\mathbf{M}^\bullet$ is said to be exact on M^n if $\mathrm{Im}(\alpha^{n-1}) = \mathrm{Ker}(\alpha^n)$. $\mathbf{M}^\bullet$ is an *exact cochain complex* if it is exact on M^n, for all $n \in \mathbb{Z}$.

A cochain complex such that

$$\mathbf{M}^\bullet : 0 \longrightarrow M^0 \xrightarrow{\alpha^0} M^1 \xrightarrow{\alpha^1} \cdots \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \cdots$$

is said to be a *positive cochain complex*.

Definition 2.2.3. If $\mathbf{M}_\bullet$ is a chain complex of R-modules, then $\mathrm{H}_n(\mathbf{M}_\bullet) = \mathrm{Ker}(\alpha_n)/\mathrm{Im}(\alpha_{n+1})$ is said to be the *nth homology module* of $\mathbf{M}_\bullet$, where $n \in \mathbb{Z}$.

If $\mathbf{M}^\bullet$ is a cochain complex of R-modules, then $\mathrm{H}^n(\mathbf{M}^\bullet) = \mathrm{Ker}(\alpha^n)/\mathrm{Im}(\alpha^{n-1})$ is said to be the *nth cohomology module* of $\mathbf{M}^\bullet$.

Example 2.2.4. Let

$$\mathbf{M}_\bullet : \ 0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} -6 \\ 9 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 2 \end{pmatrix}} \mathbb{Z} \longrightarrow 0$$

be a sequence of $\mathbb{Z}$-modules. To show this is a chain complex, we need to show that the product of pairs of adjacent matrices are zero.

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} -6 \\ 9 \end{pmatrix} = (0).$$

By computing the homology modules in each degree, it follows that:

$$\mathrm{H}_0(\mathbf{M}_\bullet) = \frac{\mathrm{Ker}(\mathbb{Z} \to 0)}{\mathrm{Im}\left(\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 2 \end{pmatrix}} \mathbb{Z}\right)} = \frac{\mathbb{Z}}{(3,2)\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0,$$

$$\mathrm{H}_1(\mathbf{M}_\bullet) = \frac{\mathrm{Ker}\left(\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 2 \end{pmatrix}} \mathbb{Z}\right)}{\mathrm{Im}\left(\mathbb{Z} \xrightarrow{\begin{pmatrix} -6 \\ 9 \end{pmatrix}} \mathbb{Z}^2\right)} = \frac{\begin{pmatrix} -2 \\ 3 \end{pmatrix}\mathbb{Z}}{\begin{pmatrix} -6 \\ 9 \end{pmatrix}\mathbb{Z}} = \frac{\begin{pmatrix} -2 \\ 3 \end{pmatrix}\mathbb{Z}}{3\begin{pmatrix} -2 \\ 3 \end{pmatrix}\mathbb{Z}} = \frac{\mathbb{Z}}{3\mathbb{Z}} = \mathbb{Z}_3,$$

$$\mathrm{H}_2(\mathbf{M}_\bullet) = \frac{\mathrm{Ker}\left(\mathbb{Z} \xrightarrow{\begin{pmatrix} -6 \\ 9 \end{pmatrix}} \mathbb{Z}^2\right)}{\mathrm{Im}(0 \to \mathbb{Z})} = \frac{0\mathbb{Z}}{0Z} = 0.$$

Furthermore, $\mathrm{H}_i(\mathbf{M}_\bullet) = 0$ for every integer $i \neq 0,1,2$, since $M_i = 0$, for all $i \neq 0,1,2$.

In what follows, we define maps between the cochain complex (and see an analogous to chain complex) in order to state two important results in homological algebra: the Snake Lemma and the existence of connecting homomorphism of cohomology R-modules.

We do all the results for cochain complexes but analogous proofs can be done for chain complexes.

Definition 2.2.5. Let $M^\bullet$ and $N^\bullet$ be two cochain complexes. Then, a map of cochain complexes of degree k, also called *map of cochains complexes*, $f^\bullet : M^\bullet \to N^\bullet$ is a family of R-homomorphisms $f^\bullet = \{f^i : M^i \to N^{i+k}\}$ such that the following diagram commutes, for all $i \in \mathbb{Z}$:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\alpha^{i-1}} & M^i & \xrightarrow{\alpha^i} & M^{i+1} & \xrightarrow{\alpha^{i+1}} & \cdots \\ & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \xrightarrow{\beta^{i+k-1}} & N^{i+k} & \xrightarrow{\beta^{i+k}} & N^{i+k+1} & \xrightarrow{\beta^{i+k+1}} & \cdots \end{array}$$

The definition of a *map of chain complexes* of degree k is given in the same way.

Note that $M^\bullet$ is an *exact cochain complex* (respectively *exact chain complex* $M_\bullet$) *on* M^i (respectively on M_i) for all $i \in \mathbb{Z}$ if and only if $\mathrm{H}^i(M^\bullet) = 0$ (respectively $\mathrm{H}_i(M_\bullet) = 0$), for all $i \in \mathbb{Z}$.

Proposition 2.2.6. If $f^\bullet : M^\bullet \to N^\bullet$ is a map of cochain complexes of degree 1, then for each $i \in \mathbb{Z}$ there exists an R-linear map $\mathrm{H}^i(f^\bullet) : \mathrm{H}^i(M^\bullet) \to \mathrm{H}^i(N^\bullet)$ defined by

$$\mathrm{H}^i(f^\bullet)(x + \mathrm{Im}(\alpha^{i-1})) = f^i(x) + \mathrm{Im}(\beta^{i-1}),$$

for all $x + \mathrm{Im}(\alpha^{i-1}) \in \mathrm{H}^i(M^\bullet)$.

Proof. For each $i \in \mathbb{Z}$, there is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\alpha^{i-2}} & M^{i-1} & \xrightarrow{\alpha^{i-1}} & M^i & \xrightarrow{\alpha^i} & M^{i+1} & \xrightarrow{\alpha^{i+1}} \cdots \\
& \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & \\
\cdots \xrightarrow{\beta^{i-2}} & N^{i-1} & \xrightarrow{\beta^{i-1}} & N^i & \xrightarrow{\beta^i} & N^{i+1} & \xrightarrow{\beta^{i+1}} \cdots
\end{array}$$

If $x \in \mathrm{Ker}(\alpha^i)$, then

$$\beta^i(f^i(x)) = f^{i+1}(\alpha^i(x)) = 0$$

and $f^i(x) \in \mathrm{Ker}(\beta^i)$. Thus, we can see that $\mathrm{H}^i(f^\bullet)$ applies $\mathrm{Ker}(\alpha^i)/\mathrm{Im}(\alpha_{i-1})$ in $\mathrm{Ker}(\beta^i)/\mathrm{Im}(\beta^{i-1})$.

Now, let's show that the map $\mathrm{H}^i(f^\bullet)$ is well defined. Let $x, x' \in \mathrm{Ker}(\alpha^i)$ and assume that $x + \mathrm{Im}(\alpha^{i-1}) = x' + \mathrm{Im}(\alpha^{i-1})$. Then $x - x' \in \mathrm{Im}(\alpha^{i-1})$, and there exists $y \in M^{i-1}$ such that $\alpha^{i-1}(y) = x - x'$. Thus,

$$f^i(x) - f^i(x') = f^i(x - x') = f^i(\alpha^{i-1}(y)) = \beta^{i-1}(f^{i-1}(y)),$$

therefore, $f^i(x) - f^i(x') \in \mathrm{Im}(\beta^{i-1})$. This shows that $\mathrm{H}^i(f^\bullet)$ is well defined.

We also need to show that $\mathrm{H}^i(f^\bullet)$ is R-linear, but this follows from the fact that f_i is R-linear. □

By a similar way, we can define a map between homologies.

A sequence of cochain complexes (similarly for chain complexes) $L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet$ is said to be *exact* if $L^i \xrightarrow{f^i} M^i \xrightarrow{g^i} N^i$ is an exact sequence of R-modules, for all $i \in \mathbb{Z}$. A sequence of cochain complexes (similarly for chain complexes) $0 \to L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet \to 0$ is a *short exact* *sequence* if the following diagram commutes

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^{i-1} & \xrightarrow{f^{n-1}} & M^{i-1} & \xrightarrow{g^{n-1}} & N^{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha^{i-1}} & & \downarrow{\scriptstyle \beta^{i-1}} & & \downarrow{\scriptstyle \gamma^{i-1}} & & \\
0 & \longrightarrow & L^{i} & \xrightarrow{f^{i}} & M^{i} & \xrightarrow{g^{i}} & N^{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha^{i}} & & \downarrow{\scriptstyle \beta^{i}} & & \downarrow{\scriptstyle \gamma^{i}} & & \\
0 & \longrightarrow & L^{i+1} & \xrightarrow{f^{i+1}} & M^{i+1} & \xrightarrow{g^{i+1}} & N^{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where the lines are short exact sequences of R-modules and of R-homomorphisms and the rows are cochain complexes $L^\bullet$, $M^\bullet$ and $N^\bullet$, respectively (similar to chain complexes).

For each exact sequence of cochain complexes (respectively chain complexes) $0 \to L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet \to 0$ there exists a long exact sequence of cohomology modules (respectively homology modules):

$$
\cdots \longrightarrow \mathrm{H}^{i-1}(N^\bullet) \xrightarrow{\Phi^{i-1}} \mathrm{H}^{i}(L^\bullet) \xrightarrow{\mathrm{H}^i(f^\bullet)} \mathrm{H}^{i}(M^\bullet) \xrightarrow{\mathrm{H}^i(g^\bullet)} \mathrm{H}^{i}(N^\bullet) \xrightarrow{\Phi^{i}} \mathrm{H}^{i+1}(L^\bullet) \longrightarrow \cdots .
$$

We prove the existence of this long exact sequence using some auxiliary Lemmas, which follows next. Before doing this, note that if this diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\alpha) & \longrightarrow & M_1 & \xrightarrow{\alpha} & M_2 & \longrightarrow & \mathrm{Coker}(\alpha) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} & & \\
0 & \longrightarrow & \mathrm{Ker}(\beta) & \longrightarrow & N_1 & \xrightarrow{\beta} & N_2 & \longrightarrow & \mathrm{Coker}(\beta) & \xrightarrow{i} & 0,
\end{array}
$$

of R-modules and R-homomorphisms, is commutative, then there are induced maps $\overline{f} : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $\overline{g} : \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta)$ defined by $\overline{f}(x) = f(x)$ for all $x \in \mathrm{Ker}(\beta)$ and $\overline{g}(x + \mathrm{Im}(\alpha)) = x + \mathrm{Im}(\beta)$, respectively.

Lemma 2.2.7. (**Snake Lemma**, (ROTMAN, 2008, Corollary 6.12), (BLAND, 2011, Lemma 11.1.9)) Let

$$
\begin{array}{ccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
\end{array}
$$

be a commutative diagram of R-modules and R-homomorphisms with exact lines. Then there is a R-linear map $\Phi : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ such that this sequence of R-modules

$$
\mathrm{Ker}(\alpha) \xrightarrow{\overline{f_1}} \mathrm{Ker}(\beta) \xrightarrow{\overline{g_1}} \mathrm{Ker}(\gamma) \xrightarrow{\Phi} \mathrm{Coker}(\alpha) \xrightarrow{\overline{f_2}} \mathrm{Coker}(\beta) \xrightarrow{\overline{g_2}} \mathrm{Coker}(\gamma)
$$

is an exact sequence. We call this map connecting R-homomorphism.

Lemma 2.2.8. If $M^\bullet$ is a cochain complex of R-modules and R-homomorphisms, then the map $\alpha^i : M^i \to M^{i+1}$ induces an R-homomorphism $\overline{\alpha}^i : \mathrm{Coker}(\alpha^{i-1}) \to \mathrm{Ker}(\alpha^{i+1})$ of R-modules. Moreover $\mathrm{H}^i(M^\bullet) = \mathrm{Ker}(\overline{\alpha}^i)$ and $\mathrm{H}^{i+1}(M^\bullet) = \mathrm{Coker}(\overline{\alpha}^i)$.

Proof. Since $\mathrm{Im}(\alpha^{i-1}) \subseteq \mathrm{Ker}(\alpha^i)$, we can define a surjective R-homomorphism $\overline{\overline{\alpha}}_i : M^i/\mathrm{Im}(\alpha^{i-1}) \to M^i/\mathrm{Ker}(\alpha^i)$ as follows: $x + \mathrm{Im}(\alpha^{i-1}) \mapsto x + \mathrm{Ker}(\alpha^i)$. By the Isomorphism Theorem, $M^i/\mathrm{Ker}(\alpha^i) \cong \mathrm{Im}(\alpha^i) \subseteq \mathrm{Ker}(\alpha^{i+1})$, thus the map $\overline{\alpha}^i : \mathrm{Coker}(\alpha^{i-1}) \to \mathrm{Ker}(\alpha^{i+1})$ follows, where $\overline{\overline{\alpha}}^i$ is the restriction of $\overline{\alpha}^i$ onto the image.

Now, $\mathrm{H}^i(M^\bullet) = \mathrm{Ker}(\alpha^i)/\mathrm{Im}(\alpha^{i-1}) = \mathrm{Ker}(\overline{\overline{\alpha}}^i) = \mathrm{Ker}(\overline{\alpha}^i)$, where the second equality follows since $\overline{\overline{\alpha}}^i$ is surjective and $\mathrm{H}^{i+1}(M^\bullet) = \mathrm{Ker}(\alpha^{i+1})/\mathrm{Im}(\alpha^i) = \mathrm{Coker}(\overline{\overline{\alpha}}^i) = \mathrm{Coker}(\overline{\alpha}^i)$, where the second equality follows from the above isomorphism and by the definition of $\mathrm{Coker}(\overline{\overline{\alpha}}^i)$. $\square$

In the Snake Lemma (Lemma 2.2.7), if f_1 is injective and g_2 is surjective, then $\overline{f}_1$ is injective and $\overline{g}_2$ is surjective.

Consider an exact sequence of cochain complexes $0 \to L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet \to 0$. Then, for every $i \in \mathbb{Z}$, there is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}(\alpha^i) & \to & \mathrm{Ker}(\beta^i) & \to & \mathrm{Ker}(\gamma^i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L^i & \to & M^i & \to & N^i & \to & 0 \\
 & & \downarrow{\scriptstyle \alpha^i} & & \downarrow{\scriptstyle \beta^i} & & \downarrow{\scriptstyle \gamma^i} & & \\
0 & \to & L^{i+1} & \to & M^{i+1} & \to & N^{i+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Coker}(\alpha^i) & \to & \mathrm{Coker}(\beta^i) & \to & \mathrm{Coker}(\gamma^i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the lines are short exact sequence.

By using this diagram and Lemma 2.2.8, we obtain the diagram

$$\begin{array}{ccccccc}
 & & & & \mathrm{H}^i(N^\bullet) = \mathrm{Ker}(\overline{\gamma}^i) & & \\
 & & & & \downarrow & & \\
 & \mathrm{Coker}(\alpha^{i+1}) & \longrightarrow & \mathrm{Coker}(\beta^{i+1}) & \longrightarrow \mathrm{Coker}(\gamma^{i+1}) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & \downarrow & & \\
0 \longrightarrow & \mathrm{Ker}(\alpha^{i+1}) & \longrightarrow & \mathrm{Ker}(\beta^{i+1}) & \longrightarrow \mathrm{Ker}(\gamma^{i+1}) & & \\
 & \downarrow & & & & & \\
 & \mathrm{H}^{i+1}(L^\bullet) = \mathrm{Coker}(\overline{\alpha}^i) & & & & &
\end{array}$$

for each $i \in \mathbb{Z}$. Thus, by Snake Lemma (Lemma 2.2.7), there is a connecting R-homomorphism $\Phi^i : \mathrm{H}^i(N^\bullet) \to \mathrm{H}^{i+1}(L^\bullet)$. Therefore, there are the long exact sequences of cohomology R-modules.

Therefore, we just prove the next Theorem.

Theorem 2.2.9. Let

$$0 \to L^\bullet \xrightarrow{f^\bullet} M^\bullet \xrightarrow{g^\bullet} N^\bullet \to 0$$

be a short exact sequence of cochain complexes. Then, there is a long exact sequence of cohomology modules

$$\cdots \longrightarrow \mathrm{H}^{i-1}(N^\bullet) \xrightarrow{\Phi^{i-1}} \mathrm{H}^i(L^\bullet) \xrightarrow{\mathrm{H}^i(f^\bullet)} \mathrm{H}^i(M^\bullet) \xrightarrow{\mathrm{H}^i(g^\bullet)} \mathrm{H}^i(N^\bullet) \xrightarrow{\Phi^i} \mathrm{H}^{i+1}(L^\bullet) \longrightarrow \cdots.$$

The map Φ^i is a connecting R-homomorphism of cohomology modules for each $i \in \mathbb{Z}$.

In a similar way, it is possible to ensure the existence of a long exact sequence of homology R-modules.

Theorem 2.2.10. Let

$$0 \to L_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} N_\bullet \to 0$$

be a short exact sequence of chain complexes. Then, there is a long exact sequence of homology modules

$$\cdots \longrightarrow \mathrm{H}_{i+1}(N_\bullet) \xrightarrow{\Phi_{i+1}} \mathrm{H}_i(L_\bullet) \xrightarrow{\mathrm{H}_i(f_\bullet)} \mathrm{H}_i(M_\bullet) \xrightarrow{\mathrm{H}_i(g_\bullet)} \mathrm{H}_i(N_\bullet) \xrightarrow{\Phi_i} \mathrm{H}_{i-1}(L_\bullet) \longrightarrow \cdots.$$

The map Φ_i is a connecting R-homomorphism of homology modules for each $i \in \mathbb{Z}$.

2.3 Injective Modules and Injective Resolutions

Definition 2.3.1. An R-module M is said to be *injective* if one of the equivalent conditions holds:

1. For each monomorphism of R-modules $i : X \to N$ and each homomorphism of R-modules $\phi : X \to M$, there exists an homomorphism of R-modules $\tilde{\phi} : N \to M$ such that $\phi = \tilde{\phi} \circ i$. In other words, the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{\ i\ } & N \\ & & {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle\tilde{\phi}} & \\ & & M & & \end{array}$$

2. The functor $\mathrm{Hom}_R(-,M)$ is exact.

Let's prove those two conditions are equivalent:

$(1) \Rightarrow (2)$ Let M be an R-module such that condition (1) holds. Since the contravariant functor $\mathrm{Hom}_R(-,M)$ is left exact, by Proposition 2.1.8, we just need to show that if $0 \longrightarrow N_1 \xrightarrow{g} N_2$ is an exact sequence of R-modules then the sequence

$$\mathrm{Hom}_R(N_2,M) \xrightarrow{g^*} \mathrm{Hom}_R(N_1,M) \longrightarrow 0$$

is exact.

If $f \in \mathrm{Hom}_R(N_1,M)$, then there exists a map $h : N_2 \to M$ such that $f = h \circ g = g^*(h)$, so g^* is surjective. Therefore, $\mathrm{Hom}_R(N_2,M) \xrightarrow{g^*} \mathrm{Hom}_R(N_1,M) \longrightarrow 0$ is exact.

$(2) \Leftarrow (1)$ Let $0 \longrightarrow N_1 \xrightarrow{g} N_2$ be an exact sequence of R-modules. Thus, $\mathrm{Hom}_R(N_2,M) \xrightarrow{g^*} \mathrm{Hom}_R(N_1,M) \longrightarrow 0$ is exact. Let $f \in \mathrm{Hom}_R(N_1,M)$. Since g^* is surjective, there exists $h \in \mathrm{Hom}_R(N_2,M)$ such that $g^*(h) = h \circ g = f$. Therefore, condition (1) holds.

We will see some examples of injective modules later.

Proposition 2.3.2. Let $R \to S$ be a ring homomorphism and let M be an injective R-module. Then, $\mathrm{Hom}_R(S,M)$ is an injective S-module.

Proof. Note that $\mathrm{Hom}_R(S,M)$ is an S-module with the multiplication:

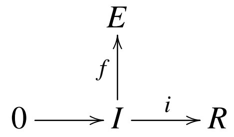
$$\begin{array}{rcl} S \times \mathrm{Hom}_R(S,M) & \longrightarrow & \mathrm{Hom}_R(S,M) \\ (s, f(s')) & \longmapsto & f(ss'), \quad \forall s' \in S. \end{array}$$

By Theorem 2.1.10, there are the following isomorphisms:

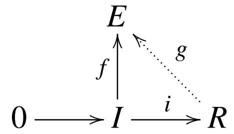
$$\mathrm{Hom}_S(-, \mathrm{Hom}_R(S,M)) \cong \mathrm{Hom}_R(- \otimes_S S, M) \cong \mathrm{Hom}_R(-,M).$$

Since $\mathrm{Hom}_R(-,M)$ is exact, $\mathrm{Hom}_S(-,\mathrm{Hom}_R(S,M))$ is exact. Therefore, $\mathrm{Hom}_R(S,M)$ is an injective S-module. □

Theorem 2.3.3. **(Baer's Criterion**, (ROTMAN, 2008, Theorem 3.30), (BLAND, 2011, Proposition 5.1.3)) Let E be an R-module. Then, E is injective if and only if for all ideal $I \subseteq R$ and each diagram



there exists a map $g : R \to E$ such that the following diagram commutes



Now, we define and give some examples about divisible modules, which are important in the proof of the result that says that every R-module can be embedded on an injective module.

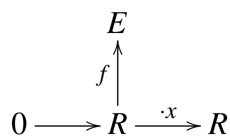
Definition 2.3.4. Let R be a domain (also called integral domain). An R-module M is *divisible* if $rM = M$ for all $r \in R$ non-zero. In other words, if the product map $r\cdot : M \to M$ is surjective for all non-zero $r \in R$.

Note that if R is a principal ideal domain (PID), then an R-module M is divisible if and only if for each non-zero $r \in R$ and $u \in M$, there exists $v \in M$ such that $rv = u$.

Example 2.3.5.

1. $\mathbb{Q}$ in a divisible $\mathbb{Z}$-module. More generally, if R is a domain, its field of fractions $\mathrm{Frac}(R)$ is divisible.
2. If M is divisible and $N \subseteq M$, then $\dfrac{M}{N}$ is divisible.
3. Direct sums and direct products of divisible modules are divisible.
4. Any injective module is divisible.

   Let E be an injective R-module. Let $x \in R$ be a non-zero divisor and $u \in E$. Consider the diagram

   where $f(1) = u$. Then, there exists $g : R \to E$ such that

   $$u = f(1) = g(1 \cdot x) = xg(1).$$

   Therefore, it is enough to take $v = g(1) \in E$.

5. If R is a PID, then an R-module E is injective if and only if E is divisible.

   $(\Rightarrow)$ Is always true by the previous item.

(⇐) Suppose E is divisible. Let $x \in R$ be a non-zero divisor and $u \in E$. Consider the diagram

$$\begin{array}{ccccc} & & E & & \\ & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & (x) & \xrightarrow{i} & R \end{array}$$

such that $f(x) = u$ and i is the inclusion map. Since R is PID, to show that E is injective it is enough to show there exists $g : R \to E$ that makes the diagram commute (by Baer's Criterion), which follows from E being divisible: there exists $v \in E$ such that $f(x) = u = xv$, so just define $g(x) = xv$. Therefore, the result follows.

6. Let $N \subseteq M$ be R-modules and suppose N and M/N are divisible. Then M is divisible.

   Let $x \in R$ be a non-zero divisor and $u \in M$. Since M/N is divisible, $\overline{u} = x\overline{v}$, for some $\overline{v} \in M/N$, that is, $u - xv \in N$. On the other hand, N is divisible, then $u - xv = xw$, for some $w \in N$. Therefore, $u = x(v + w)$.

Example 2.3.6. By Example 2.3.5 items 1 and 5, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Definition 2.3.7. Let L be a set of indexes. An R-module F is said to be *free* if $F = \oplus_{i \in L} R_i$, where $R_i \cong R$. In other words, F is a free R-module if F has a basis.

Definition 2.3.8. Let M be an R-module and L be a set of indexes.

1. We say the set $\{x_i\}_{i \in L}$, $x_i \in M$, generates M if every element $x \in M$ can be written as $x = \sum_i r_i x_i$ where $r_i \in R$ and only a finite number of them is non-zero. An R-module M is said to be *finitely generated* if the set of indexes L is finite, that is, there exists $x_1, \ldots, x_k \in M$ such that $M = \{r_1 x_1 + \ldots + r_k x_k \mid r_i \in R\} = Rx_1 + \ldots + x_k R = (x_1, \ldots, x_k)$.

2. We say that M is *finitely presented* if there exist integers $n, m \in \mathbb{N}$ and an exact sequence

$$R^{\oplus m} \to R^{\oplus n} \to M \to 0.$$

   This exact sequence is called the *presentation* of M.

Note that an R-module M only has a basis if it is free. M can be finitely generated but does not have a basis, as in following example.

Example 2.3.9. Let $R = \mathbb{C}[x, y, z]$ and $M = (xy, xz)$ an R-module. We know that M is finitely generated by $m_1 = xy$ and $m_2 = xz$. But, since

$$zm_1 - ym_2 = z(xy) - y(xz) = 0,$$

$\{m_1, m_2\}$ is not a basis. Even more, M does not have a basis.

Remark 2.3.10. Let L be a set of indexes. Let M be an R-module such that $\{x_i\}_{i\in L}$ is a set of generators. Take $F = \oplus_{i\in L} R_i$, where $R_i \cong R$, and consider the elements $e_i = (0,0,\ldots,1,0,\ldots) \in F$, where 1 is in the i-th coordinate.

Note that there exists a natural homomorphism $F \to M$ such that $e_i \to x_i$, for each $i \in L$. Therefore, the map $\sum_{i\in L} r_i e_i \to \sum_{i\in L} r_i x_i \in M$ is surjective and is well defined, and provides that F is free.

Remark 2.3.11. Every R-module M is a homeomorphic image of a free R-module. Furthermore, if M is finitely generated, then the free R-module can be chosen as a finitely generated R-module.

To show this, it is enough to consider a set $\{m_i\}_{i\in L}$ of generators of M, where L is an index set, and define a surjective homomorphism $f : R^L \to M$, as in Remark 2.3.10.

What follows next is an important result about injective modules.

Proposition 2.3.12. Let M be an R-module. Then, there exists an injective R-module E such that $M \subseteq E$, that is, every module can be embedded on a injective R-module.

Proof. Particular case: Suppose that $R = \mathbb{Z}$. By Remark 2.3.11, every $\mathbb{Z}$-module M is such that

$$M \cong \frac{\oplus \mathbb{Z}}{H} \hookrightarrow \frac{\oplus \mathbb{Q}}{H} := \mathrm{I}_{\mathbb{Z}}.$$

By Example 2.3.5 (1), $\mathbb{Q}$ is divisible, so $\mathrm{I}_{\mathbb{Z}}$ is a divisible $\mathbb{Z}$-module, by Example 2.3.5 (3) and (2). Finally, $\mathrm{I}_{\mathbb{Z}}$ is an injective R-module, by Example 2.3.5 (6), since $\mathbb{Z}$ is PID.

General case: There exists a canonical ring map $\varphi : \mathbb{Z} \to R$ such that 1 goes to 1. Then, by Proposition 2.3.2, $E := \mathrm{Hom}_{\mathbb{Z}}(R, \mathrm{I}_{\mathbb{Z}})$ is an injective R-module. Since M is an R-module and then a $\mathbb{Z}$-module (via φ), there is an injective map $0 \to M \to \mathrm{I}_{\mathbb{Z}}$. Applying in this the left exact functor $\mathrm{Hom}_{\mathbb{Z}}(R, \cdot)$, it follows that

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, \mathrm{I}_{\mathbb{Z}}) = E.$$

Since $M \cong \mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M)$, it follows that $0 \to M \to E$ is exact. Therefore, $M \subseteq E$. $\square$

By using Proposition 2.3.12 we now are able to construct an injective resolution of an R-module.

Definition 2.3.13. Let M be an R-module. An exact sequence

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\alpha^{-1}} I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{\alpha^{i-1}} I^i \longrightarrow \cdots$$

is called *injective resolution* of M if each I^i is an injective R-module, for all $i = 0, 1, 2, \ldots$.

If we remove the R-module M of the sequence $\mathbf{I}^\bullet$, then we obtain a cochain complex

$$\mathbf{I}^{\bullet,M} : 0 \longrightarrow I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{\alpha^{i-1}} I^i \longrightarrow \cdots$$

which is called *deleted injective resolution* of M.

The next result guarantees the existence of an injective resolution of an R-module.

Theorem 2.3.14. Every R-module admits an injective resolution.

Proof. Let M be an R-module. The proof will be done by induction over i.

By Proposition 2.3.12, there exists a map and an injective R-module such that the sequence $0 \longrightarrow M \xrightarrow{\psi_M} I^0$ is exact. Consider a quotient map $p^0 : I^0 \to \mathrm{Coker}(\psi_M)$. Again by Proposition 2.3.12, let I^1 be an injective R-module such that $i^0 : \mathrm{Coker}(\psi_M) \to I^1$ is an injective homomorphism. Let $d^0 = i^0 \circ p^0$, then there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\psi_M} I^0 \xrightarrow{d^0} I^1.$$

Suppose that we have already chosen $I^0, I^1, \ldots, I^i$ and $d^0, d^1, \ldots, d^{i-1}$. As before, consider $p^i : I^i \to \mathrm{Coker}(d^{i-1})$. Let I^{i+1} be an injective R-module such that the homomorphism $i^i : I^i/d^{i-1}(I^{i-1}) \to I^{i+1}$ is injective. Define $d^i = i^i \circ p^i$, that is

$$\begin{array}{ccccc} 0 & \longrightarrow & \dfrac{I^i}{d^{i-1}(I^{i-1})} & \xrightarrow{i^i} & I^{i+1} \\ & & {\scriptstyle p^i}\uparrow & \nearrow{\scriptstyle d^i} & \\ & & I^i & & \end{array}$$

Since i^i is injective, $\mathrm{Ker}(d^i) = \mathrm{Ker}(p^i) = \mathrm{Im}(d^{i-1})$, and so $\mathrm{Ker}(d^i) = \mathrm{Im}(d^{i-1})$. Therefore, there exists an exact sequence

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\psi_M} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{d^{i-1}} I^i \longrightarrow \ldots$$

where each I^i is an injective R-module, for all $i = 0, 1, 2, \ldots$. □

Example 2.3.15. $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha^{-1}} \mathbb{Q} \xrightarrow{\alpha^0} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$ is an injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$, where α^{-1} is the canonical injection and α^0 is the natural mapping.

Let M be an R-module. Note that if

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\alpha^{-1}} I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{\alpha^{i-1}} I^i \longrightarrow \ldots$$

is an injective resolution of M, then

$$\mathbf{I}^{\bullet,M} : 0 \xrightarrow{\alpha^{-1}} I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{\alpha^{i-1}} I^i \longrightarrow \ldots$$

and $H^0(\mathbf{I}^{\bullet,M}) \cong M$. This means that even if we take M away from the resolution and calculate the cohomology, we do note lose any information about M.

2.3.1 Injective Hull

The notations and the results of this subsection follow the ones given in (ROTMAN, 2008, Chapter 3) and (LAM, 2012, §3D). Therefore, we do not do the proofs here.

Definition 2.3.16. Let M and E be R-modules such that $M \subset E$. E is said to be an *essential extension* of M if every nonzero submodule of E intersects M in a non-trivial way. An essential extension $E \supseteq M$ is said to be *maximal* if no module properly containing E can be an essential extension of M.

If $E \supseteq M$ is an essential extension, we shall also say that M is an *essential submodule* of E, and we can denote this as $M \subseteq_e E$.

The notion of an essential extension leads us to a new interpretation of injectivity, as follows.

Lemma 2.3.17. An R-module M is injective if and only if it has no proper essential extensions.

Lemma 2.3.18. Any R-module has a maximal essential extension.

Theorem 2.3.19. For R-modules $M \subseteq E$, the following are equivalent:

(i) E is maximal essential over M.

(ii) E is injective, and is essential over M.

(iii) E is minimal injective over M.

Definition 2.3.20. If the R-modules $M \subseteq E$ satisfy one of the (equivalent) properties of the above theorem, we say that E is an *injective hull* (or injective envelope) of M.

Note that, by Lemma 2.3.18 any R-module M has an injective hull. In view of the following result, one can say “the” injective hull of M and denote it by $E(M)$.

Corollary 2.3.21. Any two injective hulls E, E' of M are isomorphic over M; that is, there exists an isomorphism $g : E' \to E$ which is the identity on M.

Example 2.3.22. The injective hull of an injective module is itself.

Example 2.3.23. ((LAM, 2012, Example 3.35)) The injective hull of an integral domain is its field of fractions.

The next result is an important one about injective hull that will be used latter on this text. Remember that an R-module is said to be simple if it has no non-zero proper submodules, for example the R-module $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R.

Theorem 2.3.24. ((VAMOS, 1968, Theorem 2)) For a commutative ring R, the following conditions are equivalent:

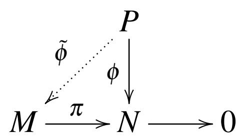
(i) Every injective hull of a simple R-module is Artinian;

(ii) The localization $R_{\mathfrak{m}}$ is Noetherian for every maximal ideal $\mathfrak{m}$ of R.

2.4 Projective Modules and Projective Resolutions

Definition 2.4.1. An R-module P is said to be *projective* if one of the equivalent conditions holds:

1. For each epimorphism of R-modules $\pi : M \to N$ and each homomorphism of R-modules $\phi : P \to N$, there exists an homomorphism of R-modules $\tilde{\phi} : P \to M$ such that $\phi = \pi \circ \tilde{\phi}$. In other words, the following diagram is commutative:



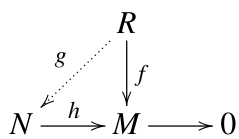
2. The functor $\mathrm{Hom}_R(P, -)$ is exact.

The proof of equivalence of those two conditions follows from an analogous way of what we have done in the injective case.

Remark 2.4.2. Let L be a set of indexes. If $\{M_i\}_{i \in L}$ is a family of R-modules, then $\bigoplus_L M_{i \in L}$ is a projective R-module if and only if each M_i is projective.

Lemma 2.4.3. A ring R is a projective R-module.

Proof. Consider the following diagram of R-modules and R-homomorphisms



which is line exact. We need to find an R-homomorphism $g : R \to N$ such that the diagram turns commutative. Let $f(1) = y$, and let $x \in N$ such that $h(x) = y$. Define $g(r) = rx$. Then g is well defined, is an R-homomorphism, and $f = h \circ g$. □

Proposition 2.4.4. Every free R-module is projective.

Proof. If F is a free R-module, then there exists a set of indexes L such that $R^L \cong F$. The result follows using Lemma 2.4.3 and Remark 2.4.2. □

Corollary 2.4.5. Every R-module is a homeomorphic image of a projective R-module.

Proof. By Remark 2.3.11, every R-module is a homeomorphic image of a free R-module. Therefore, the result follows by Proposition 2.4.4. □

Example 2.4.6. The R-module R^n is projective, for all $n \in \mathbb{N}$.

By using Corollary 2.4.5, we are able to construct a projective resolution of an R-module. A projective resolution can be defined in an analogous way to what we have done to define an injective resolution, as we can see in the following definition.

Definition 2.4.7. Let M be an R-module. An exact sequence

$$\mathbf{P}_{\bullet} : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is called *projective resolution* of M if each P_i is a projective R-module, for all $i = 0, 1, 2, \ldots$.

Note that, as we already have done in the injective case, if we remove the R-module M of the sequence $\mathbf{P}_{\bullet}$, then we obtain a chain complex

$$\mathbf{P}_{\bullet,M} : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \longrightarrow 0$$

which is called *deleted projective resolution* of M.

The next result ensures the existence of a projective resolution of an R-module.

Theorem 2.4.8. Every R-module admits a projective resolution.

Proof. Let M be an R-module. By Corollary 2.4.5, there exists a projective R-module P_0 and an epimosphism $\alpha_0 : P_o \to M$. Thus, we have the exact sequence

$$0 \to K_0 \xrightarrow{p_0} P_0 \xrightarrow{\alpha_0} M \to 0$$

where $K_0 = \mathrm{Ker}(\alpha_0)$ and p_0 is a canonical injection. Repeating this process, there will be some exact sequences of R-modules and R-homomorphisms:

$$0 \to K_0 \xrightarrow{p_0} P_0 \xrightarrow{\alpha_0} M \to 0$$

$$0 \to K_1 \xrightarrow{p_1} P_1 \xrightarrow{\beta_1} K_0 \to 0$$

$$0 \to K_2 \xrightarrow{p_2} P_2 \xrightarrow{\beta_2} K_1 \to 0$$

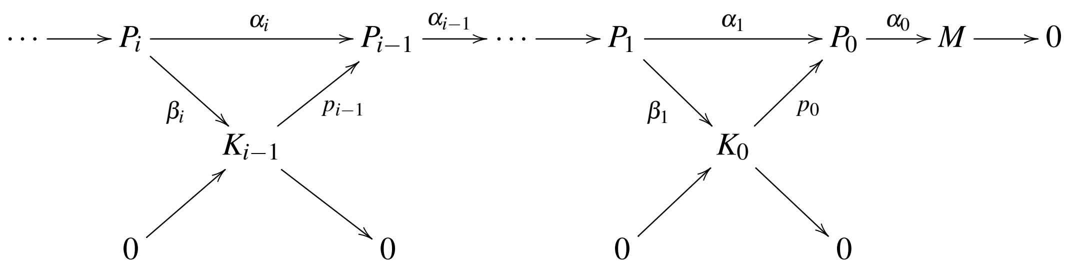
$$\vdots$$

$$0 \to K_i \xrightarrow{p_i} P_i \xrightarrow{\beta_i} K_{i-1} \to 0$$

$$\vdots$$

where P_i is a projective R-module, $K_i = \mathrm{Ker}(\beta_i)$, and p_i is a canonical injection, for $i = 0, 1, 2, \ldots$, with $\beta_0 = \alpha_0$.

Putting all those short exact sequences together, it follows that



where $\alpha_i = p_{i-1} \circ \beta_i$, for $i = 1, 2, \ldots$. Then,

$$\mathrm{Im}(\alpha_i) = \mathrm{Im}(\beta_i) = K_{i-1} = \mathrm{Ker}(\beta_{i-1}) = \mathrm{Ker}(\alpha_{i-1})$$

for $i = 1, 2, \ldots$, and

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is a projective resolution of M. □

Let M be an R-module. Note that if

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is a projective resolution of M, then

$$\mathbf{P}_{\bullet,M} : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \longrightarrow 0$$

and

$$H_0(\mathbf{P}_{\bullet,M}) = \mathrm{Ker}(P_0 \to 0)/\mathrm{Im}(\alpha_1) = P_0/\mathrm{Ker}(\alpha_0) \cong M.$$

This means that even if we take M away from the resolution and calculate the homology, we do note lose any information about M.

Let's see a numerical example that illustrates Theorem 2.4.8.

Example 2.4.9. Let k be a field and $R = k[x,y,z]$. Let $I = (x,y,z)$ be an ideal of R. We are going to construct a projective resolution to the R-module R/I. Since R is a projective R-module, we start with $R \to R/I \to 0$.

Since $I = (x,y,z)$, there exist a surjective map $\alpha_0 : R^3 \to I$ defined by $(a,b,c) \mapsto ax + by + cz$. Therefore, there is an exact sequence

$$0 \to K_0 \xrightarrow{p_0} R^3 \xrightarrow{\alpha_0} M \to 0$$

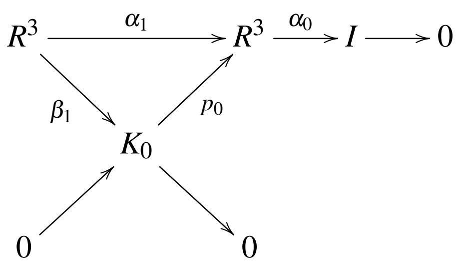
where $K_0 = \text{Ker}(\alpha_0)$. Let's compute K_0. Note that

$$\alpha_0 \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} = \alpha_0 \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} = \alpha_0 \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = 0,$$

and those elements generate K_0. Then,

$$K_0 = \left[\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} \right] = [v_1, v_2, v_3].$$

As before, there is a surjective map $\beta_1 : R^3 \to K_0$ defined by $(a,b,c) \mapsto av_1 + bv_2 + cv_3$. Therefore, there is a commutative diagram



where p_0 is the inclusion and $\alpha_1 = p_0 \circ \beta_1$. Thus, $\text{Ker}(\alpha_0) = \text{Im}(\alpha_1)$.

Let's compute $K_1 = \text{Ker}(\alpha_1) = \text{Ker}(\beta_1)$. Since

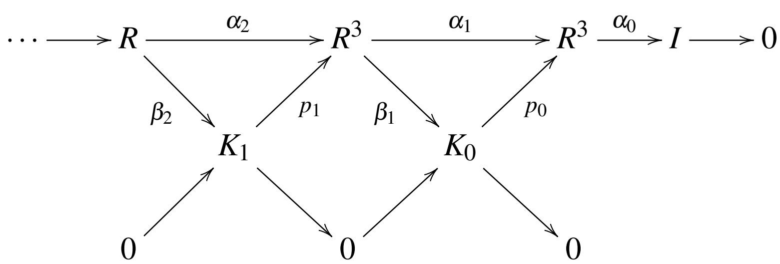
$$\alpha_1 = \begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}$$

and

$$\alpha_1 \begin{pmatrix} z \\ -y \\ x \end{pmatrix} = 0$$

we have $K_1 = \left[\begin{pmatrix} z \\ -y \\ x \end{pmatrix} \right] = [v_1]$.

As before, define the surjective map $\beta_2 : R \to K_1$ such that $\beta_2(a) = av_1$. Then,



where p_1 is the inclusion and $\alpha_2 = p_1 \circ \beta_2$.

Again, let's compute $K_2 = \mathrm{Ker}(\alpha_2) = \mathrm{Ker}(\beta_2)$. Since

$$\alpha_2 = \begin{pmatrix} z \\ -y \\ x \end{pmatrix},$$

$K_2 = 0$.

Therefore, we have the following resolution

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} R^3 \xrightarrow{(x\,y\,z)} R \longrightarrow R/I \to 0.$$

The next example shows that infinite projective resolutions can happen.

Example 2.4.10. Let k be a field and $R = k[x]/(x^2)$. Take $\bar{x}$ the image of x over R, that is, $\bar{x} = x + (x^2)$. Let $M = R/(\bar{x})$ an R-module. A free resolution of M is

$$\mathbf{P}_\bullet : \cdots \longrightarrow R \xrightarrow{\alpha_i} R \longrightarrow \cdots \longrightarrow R \xrightarrow{\alpha_1} R \xrightarrow{\alpha_0} M \longrightarrow 0$$

where each map is the multiplication by x. Let's prove this resolution is exact: the ring R can be identified as the set of all $a + bx$ where $a, b \in k$. Then, $x(a+bx) = xa$. Thus, $\mathrm{Im}(\alpha_i) = \mathrm{Ker}(\alpha_{i-1})$. Since the last map is a projection, it follows the exactness of this complex. Therefore, it is an infinite projective resolution to M.

2.5 Derived Functors

In this section, we define and study left derived functors and right derived functors. After this we define and give some properties of the functors $\mathrm{Ext}^i_R(-,-)$ and $\mathrm{Tor}^R_i(-,-)$ which are examples of right and left derived functors, respectively. Further up, we will define another kind of derived functor: the local cohomology modules.

Next is an important Lemma that will be used in this section.

Lemma 2.5.1. Let M and N be two R-modules and

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a projective resolution of M. Suppose

$$\mathbf{Q}_\bullet : \cdots \longrightarrow N_i \xrightarrow{\beta_i} N_{i-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{\beta_1} N_0 \xrightarrow{\beta_0} N \longrightarrow 0$$

is exact. Then, for each R-homomorphism $f : M \to N$, there exists a map of complexes $f_\bullet : P_{\bullet,M} \to Q_{\bullet,N}$ such that $\beta_i \circ f_i = f_{i-1} \circ \alpha_i$, for all $i = 0, 1, 2, \ldots$.

Proof. The proof is done by using induction over i.

Since P_0 is a project R-module, there is an R-homomorphism $f \circ \alpha_0 : P_0 \to N$, and β_0 is a surjective R-homomorphism, then there exists an R-homomorphism $f_0 : P_0 \to N_0$ such that the diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} & & \\ N_0 & \xrightarrow{\beta_0} & N & \longrightarrow & 0 \end{array}$$

commutes.

Suppose that there are R-homomorphisms $f_0, f_1, \dots, f_{i-1}$ such that the diagram

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & P_i & \xrightarrow{\alpha_i} & P_{i-1} & \xrightarrow{\alpha_{i-1}} & P_{i-2} & \xrightarrow{\alpha_{i-2}} \cdots \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\ & {\scriptstyle f_i}\downarrow & & {\scriptstyle f_{i-1}}\downarrow & & {\scriptstyle f_{i-2}}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \\ \cdots \longrightarrow & N_i & \xrightarrow{\beta_i} & N_{i-1} & \xrightarrow{\beta_{i-1}} & N_{i-2} & \xrightarrow{\beta_{i-2}} \cdots \xrightarrow{\beta_1} & N_0 & \xrightarrow{\beta_0} & N & \longrightarrow & 0 \end{array}$$

commutes. Our goal is to find $f_i : P_i \to N_n$ such that $\beta_i \circ f_i = f_{i-1} \circ \alpha_i$. By induction and using the previous diagram, $\beta_{i-1} \circ f_{i-1} = f_{i-2} \circ \alpha_{i-1}$, then $\beta_{i-1} \circ f_{i-1} \circ \alpha_i = f_{i-2} \circ \alpha_{i-2} \circ \alpha_i = 0$ therefore $\mathrm{Im}(f_{i-1} \circ \alpha_i) \subseteq \mathrm{Ker}(\beta_{i-1}) = \mathrm{Im}(\beta_i)$. Since P_i is a projective R-module, the diagram

$$\begin{array}{ccccc} & {\scriptstyle f_i}\swarrow & P_n & & \\ & & \downarrow{\scriptstyle f_{i-1}\circ\alpha_i} & & \\ N_n & \xrightarrow{\beta_i} & \mathrm{Im}(\beta_i) & \longrightarrow & 0 \end{array}$$

can be completed commutatively by the R-homomorphism $f_i : P_i \to N_i$. □

There is an analogous result for injective modules. The proof is also analogous, so we omit it.

Lemma 2.5.2. Let M and N be two R-modules and

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\alpha^{-1}} I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \xrightarrow{\alpha^{i-1}} I^i \longrightarrow \cdots$$

be an injective resolution of M. Suppose

$$\mathbf{Q}^\bullet : 0 \longrightarrow N \xrightarrow{\beta^{-1}} N^0 \xrightarrow{\beta^0} N^1 \longrightarrow \cdots \longrightarrow N^{i-1} \xrightarrow{\beta^{i-1}} N^i \longrightarrow \cdots$$

is exact. Then, for each R-homomorphism $f : M \to N$, there exists a map of complexes $f^\bullet : I^{\bullet,M} \to Q^{\bullet,N}$ such that $f^{i-1} \circ \beta^{i-1} = f^i \circ \alpha^{i-1}$, for all $i = 0, 1, 2, \dots$.

Let $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ be two projective resolutions of two R-modules, M and N, respectively, and let $f : M \to N$ be an R-homomorphism. By Lemma 2.5.1, there exists a commutative diagram

$$\begin{array}{ccccccccccccc} \mathbf{P}_\bullet : \cdots \longrightarrow & P_i & \xrightarrow{\alpha_i} & P_{i-1} & \xrightarrow{\alpha_{i-1}} & P_{i-2} & \xrightarrow{\alpha_{i-2}} \cdots \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\ & {\scriptstyle f_i}\downarrow & & {\scriptstyle f_{i-1}}\downarrow & & {\scriptstyle f_{i-2}}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \\ \mathbf{Q}_\bullet : \cdots \longrightarrow & Q_i & \xrightarrow{\beta_i} & Q_{i-1} & \xrightarrow{\beta_{i-1}} & Q_{i-2} & \xrightarrow{\beta_{i-2}} \cdots \xrightarrow{\beta_1} & Q_0 & \xrightarrow{\beta_0} & N & \longrightarrow & 0 \end{array}$$

where $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ is a family of R-homomorphisms induced by f.

Let $\mathfrak{F}$ be an additive covariant functor from the category of R-modules to the category of S-modules. Applying $\mathfrak{F}$ to the commutative diagram below, it follows that

$$\begin{array}{ccccccccccccc}
\mathfrak{F}(\mathbf{P}_{\bullet,M}) : \cdots \longrightarrow & \mathfrak{F}(P_i) & \xrightarrow{\mathfrak{F}(\alpha_i)} & \mathfrak{F}(P_{i-1}) & \xrightarrow{\mathfrak{F}(\alpha_{i-1})} & \mathfrak{F}(P_{i-2}) & \xrightarrow{\mathfrak{F}(\alpha_{i-2})} & \cdots & \xrightarrow{\mathfrak{F}(\alpha_1)} & \mathfrak{F}(P_0) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \mathfrak{F}(f_i)} & & \downarrow{\scriptstyle \mathfrak{F}(f_{i-1})} & & \downarrow{\scriptstyle \mathfrak{F}(f_{i-2})} & & & & \downarrow{\scriptstyle \mathfrak{F}(f_0)} & \\
\mathfrak{F}(\mathbf{Q}_{\bullet,N}) : \cdots \longrightarrow & \mathfrak{F}(Q_i) & \xrightarrow{\mathfrak{F}(\beta_i)} & \mathfrak{F}(N_{i-1}) & \xrightarrow{\mathfrak{F}(\beta_{i-1})} & \mathfrak{F}(Q_{i-2}) & \xrightarrow{\mathfrak{F}(\beta_{i-2})} & \cdots & \xrightarrow{\mathfrak{F}(\beta_1)} & \mathfrak{F}(Q_0) & \longrightarrow 0
\end{array}$$

which is also a commutative diagram and the lines are chain complexes.

By a result analogous of Proposition 2.2.6 for homologies, there exists a family of maps $H_i(\mathfrak{F}(f_\bullet))$ from the ith homology of $\mathfrak{F}(\mathbf{P}_{\bullet,M})$ to the ith homology of $\mathfrak{F}(\mathbf{Q}_{\bullet,N})$. Furthermore, if $\mathfrak{F}$ is an additive contravariant functor, then there will be a family of maps from cohomology to cohomology.

The next result ensures that the homology and cohomology modules are unique up to isomorphism.

Proposition 2.5.3. (BLAND, 2011, Proposition 11.3.1) Let $\mathfrak{F}$ be an additive covariant functor from the category of R-modules to the category of S-modules. Let M and N be two R-modules.

1. If $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ are two projective resolutions of the R-module M, then $\mathrm{H}_i(\mathfrak{F}(\mathbf{P}_{\bullet,M})) \cong \mathrm{H}_i(\mathfrak{F}(\mathbf{Q}_{\bullet,M}))$, for all $i \geq 0$. Furthermore, if $f : M \to N$ is an R-homomorphism, and $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ are projective resolutions of M and N, respectively, then the map $\mathrm{H}_i(\mathfrak{F}(f_\bullet)) : \mathrm{H}_i(\mathfrak{F}(\mathbf{P}_{\bullet,M})) \to \mathrm{H}_i(\mathfrak{F}(\mathbf{P}_{\bullet,N}))$, for all $i \geq 0$, does not depend on the choice of the map of complexes $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ which is induced by f.

2. If $\mathbf{I}^\bullet$ and $\mathbf{J}^\bullet$ are two injective resolutions of the R-module M, then $\mathrm{H}^i(\mathfrak{F}(\mathbf{I}^{\bullet,M})) \cong \mathrm{H}^i(\mathfrak{F}(\mathbf{J}^{\bullet,M}))$, for all $i \geq 0$. Furthermore, if $f : M \to N$ is an R-homomorphism, and $\mathbf{I}^\bullet$ and $\mathbf{J}^\bullet$ are injective resolutions of M and N, respectively, then $\mathrm{H}^i(\mathfrak{F}(f^\bullet)) : \mathrm{H}^i(\mathfrak{F}(\mathbf{I}^{\bullet,M})) \to \mathrm{H}^i(\mathfrak{F}(\mathbf{I}^{\bullet,N}))$, for all $i \geq 0$, does not depend on the choice of the map of co-complexes $f^\bullet : \mathbf{I}^{\bullet,M} \to \mathbf{J}^{\bullet,N}$ which is induced by f.

Let $\mathfrak{F}$ be an additive covariant functor from the category of R-modules (here denoted by M_R) to the category of S-modules (here denoted by M_S). Using Proposition 2.5.3, it is possible to define right and left derived functors related to $\mathfrak{F}$.

1. **The Functor $\mathfrak{L}_i\mathfrak{F}(-) : M_R \to M_S$.**

   Let M be an R-module. Choose a projective resolution to M

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0,$$

and apply the functor $\mathfrak{F}$ on $\mathbf{P}_{\bullet,M}$:

$$\mathfrak{F}(\mathbf{P}_{\bullet,M}) : \cdots \longrightarrow \mathfrak{F}(P_i) \xrightarrow{\mathfrak{F}(\alpha_i)} \mathfrak{F}(P_{i-1}) \longrightarrow \cdots \longrightarrow \mathfrak{F}(P_1) \xrightarrow{\mathfrak{F}(\alpha_1)} \mathfrak{F}(P_0) \xrightarrow{\mathfrak{F}(\alpha_0)} 0.$$

This is a chain complex of S-modules. Thus, define

$$\mathfrak{L}_i\mathfrak{F}(M) := \mathrm{H}_i(\mathfrak{F}(\mathbf{P}_{\bullet,M})),$$

which is called the *ith left derived functor related to* $\mathfrak{F}$.

Let N be another R-module. For each R-homomorphism $f : M \to N$, there is a map of homology S-modules on level i:

$$\mathrm{H}_i(\mathfrak{F}(f_\bullet)) : \mathrm{H}_i(\mathfrak{F}(\mathbf{P}_{\bullet,M})) \to \mathrm{H}_i(\mathfrak{F}(\mathbf{Q}_{\bullet,N}))$$

where $\mathbf{Q}_\bullet$ is a projective resolution of N and $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ is a complex map induced by f. By Proposition 2.5.3 (1), the module $\mathrm{H}_i(\mathfrak{F}(f_\bullet))$ depends only on f and does not depend on $f_\bullet$. If we denote $\mathfrak{L}_i\mathfrak{F}(f) := \mathrm{H}_i(\mathfrak{F}(f_\bullet))$, then

$$\mathfrak{L}_i\mathfrak{F}(f) : \mathfrak{L}_i\mathfrak{F}(M) \to \mathfrak{L}_i\mathfrak{F}(N)$$

is an additive covariant functor from the category of R-modules to the category of S-modules, for each integer $i \geq 0$.

2. **The Functor $\mathfrak{R}^i\mathfrak{F}(-) : M_R \to M_S$.**

Let M be an R-module. Choose an injective resolution to M

$$\mathbf{I}^\bullet : 0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{i-1} \longrightarrow I^i \longrightarrow \cdots,$$

and apply the functor $\mathfrak{F}$ on $\mathbf{I}^{\bullet,M}$:

$$\mathfrak{F}(\mathbf{I}^\bullet) : 0 \longrightarrow \mathfrak{F}(I^0) \longrightarrow \mathfrak{F}(I^1) \longrightarrow \cdots \longrightarrow \mathfrak{F}(I^{i-1}) \longrightarrow \mathfrak{F}(I^i) \longrightarrow \cdots.$$

This is a cochain complex of S-modules. Thus, define

$$\mathfrak{R}^i\mathfrak{F}(M) := \mathrm{H}^i(\mathfrak{F}(\mathbf{I}^{\bullet,M})),$$

which is called the *ith right derived functor related to* $\mathfrak{F}$.

Let N be another R-module. For each R-homomorphism $f : M \to N$, there is a map of cohomology S-modules on level i:

$$\mathrm{H}^i(\mathfrak{F}(f^\bullet)) : \mathrm{H}^i(\mathfrak{F}(\mathbf{I}^{\bullet,M})) \to \mathrm{H}^i(\mathfrak{F}(\mathbf{J}^{\bullet,N}))$$

where $\mathbf{J}^\bullet$ is an injective resolution of N and $f^\bullet : \mathbf{I}^{\bullet,M} \to \mathbf{J}^{\bullet,N}$ is a co-complex map induced by f. By Proposition 2.5.3 (2), the module $\mathrm{H}^i(\mathfrak{F}(f^\bullet))$ depends only on f and does not depend on $f^\bullet$. If we denote $\mathfrak{R}^i\mathfrak{F}(f) := \mathrm{H}^i(\mathfrak{F}(f^\bullet))$, then

$$\mathfrak{R}^i\mathfrak{F}(f) : \mathfrak{R}^i\mathfrak{F}(M) \to \mathfrak{R}^i\mathfrak{F}(N)$$

is an additive covariant functor from the category of R-modules to the category of S-modules, for each integer $i \geq 0$.

Note that if $\mathfrak{F}$ is an additive contravariant functor from the category of R-modules to the category of S-modules, then it is possible to define functors $\mathfrak{L}_i\mathfrak{F}$ and $\mathfrak{R}^i\mathfrak{F}$. In this case, to define $\mathfrak{L}_i\mathfrak{F}$ it is enough to choose an injective resolution to the module; and, to define $\mathfrak{R}^i\mathfrak{F}$, it is enough to choose a projective resolution to the module.

Definition 2.5.4. Let $\mathfrak{F}$ and $\mathfrak{G} : M_R \to M_S$ be two additive covariant functors. Let M and N be two R-modules and suppose that there exists a S-homomorphism $\eta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ such that, for each R-homomorphism $f : M \to N$ the diagram

$$\begin{array}{ccc} \mathfrak{F}(M) & \xrightarrow{\eta_M} & \mathfrak{G}(M) \\ {\scriptstyle \mathfrak{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathfrak{G}(f)} \\ \mathfrak{F}(N) & \xrightarrow{\eta_N} & \mathfrak{G}(N) \end{array}$$

commutes. The class of S-homomorphisms $\eta = \{\eta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)\}$ is called a *natural transformation* from $\mathfrak{F}$ to $\mathfrak{G}$. If η_M is an S-isomorphism for each $M \in M_R$, then η is called a natural isomorphism, and $\mathfrak{F}$ and $\mathfrak{G}$ are said to be *naturally equivalent functors*, denoted by $\mathfrak{F} \cong \mathfrak{G}$

There is an analogous to this definition to additive contravariant functors.

Proposition 2.5.5. Let $\mathfrak{F}$ be an additive covariant functor from the category of R-modules to the category of S-modules. Then:

1. If $\mathfrak{F}$ is right exact, then $\mathfrak{L}_0\mathfrak{F}(M) \cong \mathfrak{F}(M)$, for each R-module M. Furthermore, $\mathfrak{L}_0\mathfrak{F}$ and $\mathfrak{F}$ are naturally equivalent functors. If M is a projective R-module, then $\mathfrak{L}_i\mathfrak{F}(M) = 0$, for all integer $i \geq 0$.

2. If $\mathfrak{F}$ is left exact, then $\mathfrak{R}^0\mathfrak{F}(M) \cong \mathfrak{F}(M)$, for each R-module M. Furthermore, $\mathfrak{R}^0\mathfrak{F}$ and $\mathfrak{F}$ are naturally equivalent functors. If M is an injective R-module, then $\mathfrak{R}^i\mathfrak{F}(M) = 0$, for all integer $i \geq 0$.

Proof. 1. Let M and N be two R-modules and $\mathbf{P}_\bullet$ be a projective resolution of M. Then,

$$P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0,$$

is exact. Since $\mathfrak{F}$ is right exact, it follows that

$$\mathfrak{F}(P_1) \xrightarrow{\mathfrak{F}(\alpha_1)} \mathfrak{F}(P_0) \xrightarrow{\mathfrak{F}(\alpha_0)} \mathfrak{F}(M) \longrightarrow 0$$

is an exact sequence.

Consider the sequence

$$\mathfrak{F}(P_1) \xrightarrow{\mathfrak{F}(\alpha_1)} \mathfrak{F}(P_0) \longrightarrow 0$$

and

$$\begin{aligned} \mathfrak{L}_0\mathfrak{F}(M) &= \mathrm{H}_0(\mathfrak{F}(\mathbf{P}_{\bullet,M})) = \mathrm{Ker}(\mathfrak{F}(P_0) \to 0)/\mathrm{Im}(\mathfrak{F}(\alpha_1)) \\ &= \mathfrak{F}(P_0)/\mathrm{Ker}(\mathfrak{F}(\alpha_0)) \cong \mathfrak{F}(M) \end{aligned}$$

Then, there exists an isomorphism of S-modules $\eta_M : \mathfrak{L}_0\mathfrak{F}(M) \to \mathfrak{F}(M)$.

We claim the family $\{\eta_M : \mathfrak{L}_0\mathfrak{F}(M) \to \mathfrak{F}(M)\}$ of those natural S-isomorphisms induces a natural isomorphism $\eta : \mathfrak{L}_0\mathfrak{F} \to \mathfrak{F}$. Indeed, let $f : M \to N$ be an R-homomorphism and $\mathbf{Q}_\bullet$ be a projective resolution of N. If $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ is a map between chain complexes induced by f, then there exists a commutative diagram:

$$\begin{array}{ccc} \mathfrak{L}_0\mathfrak{F}(M) & \xrightarrow{\eta_M} & \mathfrak{F}(M) \\ {\scriptstyle \mathfrak{L}_0\mathfrak{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}(f)} \\ \mathfrak{L}_0\mathfrak{F}(N) & \xrightarrow{\eta_N} & \mathfrak{F}(N) \end{array}$$

where η_M and η_N are S-isomorphisms. Thus, $\mathfrak{L}_0\mathfrak{F}$ and $\mathfrak{F}$ are naturally equivalent functors.

Now, if M is a projective R-module, then

$$\mathbf{P}_\bullet : \cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{Id}_M} M \longrightarrow 0,$$

is a projective resolution of M. Therefore,

$$\mathfrak{F}(\mathbf{P}_{\bullet,M}) : \cdots \longrightarrow 0 \longrightarrow \mathfrak{F}(M) \longrightarrow 0,$$

is a chain complex and $\mathfrak{L}_i\mathfrak{F}(M) = 0$, for all $i = 1, 2, \ldots$.

2. Follows in an analogous way of what has been done in item (1) before. □

The next Proposition is similar to Proposition 2.5.5 and the proof is analogous to what we have done.

Proposition 2.5.6. Let $\mathfrak{F}$ be an additive contravariant functor from the category of R-modules to the category of S-modules. Then:

1. If $\mathfrak{F}$ is right exact, then $\mathfrak{L}_0\mathfrak{F}(M) \cong \mathfrak{F}(M)$, for each R-module M. Furthermore, $\mathfrak{L}_0\mathfrak{F}$ and $\mathfrak{F}$ are naturally equivalent functors. If M is an injective R-module, then $\mathfrak{L}_i\mathfrak{F}(M) = 0$, for all integer $i \geq 0$.

2. If $\mathfrak{F}$ is left exact, then $\mathfrak{R}^0\mathfrak{F}(M) \cong \mathfrak{F}(M)$, for each R-module M. Furthermore, $\mathfrak{R}^0\mathfrak{F}$ and $\mathfrak{F}$ are naturally equivalent functors. If M is a projective R-module, then $\mathfrak{R}^i\mathfrak{F}(M) = 0$, for all integer $i \geq 0$.

2.6 The Functor $\mathrm{Ext}^i_R(-,-)$

Let X be a fixed R-module. By Examples 2.1.3 and 2.1.4, respectively, we already know that $\mathrm{Hom}_R(-,X)$ is an additive contravariant functor and $\mathrm{Hom}_R(X,-)$ is an additive covariant functor, both from the category of R-modules to itself.

2.6.1 The right derived functor related to $\mathrm{Hom}_R(-,X)$

Let M and N be two R-modules and

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a projective resolution of M. Apply the functor $\mathrm{Hom}_R(-,X)$ on $\mathbf{P}_{\bullet,M}$, then if follows the cochain complex

$$\mathrm{Hom}_R(\mathbf{P}_{\bullet,M},X) : 0 \longrightarrow \mathrm{Hom}_R(P_0,X) \xrightarrow{\alpha_1^*} \cdots \xrightarrow{\alpha_i^*} \mathrm{Hom}_R(P_i,X) \xrightarrow{\alpha_{i+1}^*} \cdots .$$

Define

$$\mathrm{Ext}^i_R(-,X) := \mathfrak{R}^i\mathrm{Hom}_R(-,X) = \mathrm{H}^i(\mathrm{Hom}_R(\mathbf{P}_{\bullet,M},X)),$$

for all integer $i \geq 0$.

Now, consider the R-homomorphism $f : M \to N$ and projective resolutions $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ of M and N, respectively. Then,

$$\mathrm{Ext}^i_R(f,X) := \mathrm{H}^i(\mathrm{Hom}_R(f_\bullet,X))$$

where $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ is a map between complexes induced by f. Thus

$$\mathrm{Ext}^i_R(f,X) : \ \mathrm{Ext}^i_R(N,X) \to \mathrm{Ext}^i_R(M,X).$$

Therefore, for each integer $i \geq 0$

$$\mathrm{Ext}^i_R(-,X) : \ M_R \to M_R$$

is an additive contravariant functor, called the *ith extension functor* of $\mathrm{Hom}_R(-,X)$.

2.6.2 The right derived functor related to $\mathrm{Hom}_R(X,-)$

Instead of using projective resolutions, it is possible to use an injective resolution to the R-module M and proceed in a similar way to what has been done in Section 2.6.1 to define another functor from the category of R-modules to itself: $\mathrm{Ext}^i_R(X,-)$, for all integer $i \geq 0$.

Therefore, for each integer $i \geq 0$

$$\mathrm{Ext}^i_R(X,-) : \ M_R \to M_R$$

is an additive covariant functor, called the *ith extension functor* of $\mathrm{Hom}_R(X,-)$.

2.6.3 Properties

Let X be an R-module. What follows is intended to show that for each short exact sequence $0 \to L \to M \to N \to 0$ of R-modules and R-homomorphisms, there exists a long exact sequence in cohomology corresponding to the contravariant functor $\mathrm{Ext}^i_R(-,X)$ and a long exact sequence in cohomology corresponding to the covariant functor $\mathrm{Ext}^i_R(X,-)$.

Lemma 2.6.1. (**Horse Shoe Lemma for Projectives**, (BLAND, 2011, Lemma 11.4.2)) Consider the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \gamma_2} & & \\
 & & P_1 & & & & R_1 & & \\
 & & \downarrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \gamma_1} & & \\
 & & P_0 & & & & R_0 & & \\
 & & \downarrow{\scriptstyle \alpha_0} & & & & \downarrow{\scriptstyle \gamma_0} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow{\scriptstyle \gamma} & & \\
 & & 0 & & & & 0 & &
\end{array}$$

where the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact and $\mathbf{P}_\bullet$ and $\mathbf{R}_\bullet$ are projective resolutions of L and N, respectively. Then, there is a projective resolution $\mathbf{Q}_\bullet$ of M and chain maps $f_\bullet : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$ and $g_\bullet : \mathbf{Q}_\bullet \to \mathbf{R}_\bullet$ such that

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_2} & & \\
0 & \longrightarrow & P_1 & \xrightarrow{f_1} & Q_1 & \xrightarrow{g_1} & R_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma_1} & & \\
0 & \longrightarrow & P_0 & \xrightarrow{f_0} & Q_0 & \xrightarrow{g_0} & R_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \gamma_0} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is a commutative row exact diagram. Furthermore, $Q_i = P_i \oplus R_i$, for all integer $i \geq 0$.

There exists a similar result for injective resolutions.

Lemma 2.6.2. (**Horse Shoe Lemma for Injectives**, (BLAND, 2011, Proposition 11.4.8)) If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules and R-module homomorphisms and $\mathbf{D}^\bullet$ and $\mathbf{F}^\bullet$ are injective resolutions of L and N respectively, then there is an injective resolution $\mathbf{E}^\bullet$ of M and cochain maps $f^\bullet : \mathbf{D}^\bullet \to \mathbf{E}^\bullet$ and $g^\bullet : \mathbf{E}^\bullet \to \mathbf{F}^\bullet$ such that $0 \to D^i \xrightarrow{f^i} E^i \xrightarrow{g^i} F^i \to 0$ is a short exact sequence of cochain complexes, where $E^i = D^i \oplus F^i$, for all integer $i \geq 0$.

Now, it is possible to construct both exact sequences. We prove just the first one; the second one is analogous.

Proposition 2.6.3. If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence of R-modules and R-homomorphisms, then for each R-module X, there is a long exact cohomology sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(N,X) \xrightarrow{g^*} \mathrm{Hom}_R(M,X) \xrightarrow{f^*} \mathrm{Hom}_R(L,X) \xrightarrow{\Phi^0} \\ \xrightarrow{\Phi^0} \mathrm{Ext}^1_R(N,X) \xrightarrow{\mathrm{Ext}^1_R(g,X)} \mathrm{Ext}^1_R(M,X) \xrightarrow{\mathrm{Ext}^1_R(f,X)} \mathrm{Ext}^1_R(L,X) \xrightarrow{\Phi^1} \cdots \\ \cdots \xrightarrow{\Phi^{i-1}} \mathrm{Ext}^i_R(N,X) \xrightarrow{\mathrm{Ext}^i_R(g,X)} \mathrm{Ext}^i_R(M,X) \xrightarrow{\mathrm{Ext}^i_R(f,X)} \mathrm{Ext}^i_R(L,X) \xrightarrow{\Phi^i} \ldots \end{array}$$

where Φ^i is a connecting R-homomorphism for each $i \geq 0$.

Proof. If $\mathbf{P}_\bullet$ and $\mathbf{R}_\bullet$ are projective resolutions of the R-modules L and N, respectively, then the Horse Shoe Lemma for Projectives (2.6.1) shows that there is a projective resolution $\mathbf{Q}_\bullet$ of the R-module M and chain maps $f_\bullet : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$ and $g_\bullet : \mathbf{Q}_\bullet \to \mathbf{R}_\bullet$ such that $0 \to \mathbf{P}_\bullet \xrightarrow{f_\bullet} \mathbf{Q}_\bullet \xrightarrow{g_\bullet} \mathbf{R}_\bullet \to 0$ is a short exact sequence of chain complexes. Note that $0 \to P_i \xrightarrow{f_i} Q_i \xrightarrow{g_i} R_i \to 0$ is a short exact sequence with $Q_i = P_i \oplus R_i$, for all $i \geq 0$. Then, there is a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(R_0,X) & \xrightarrow{g_0^*} & \mathrm{Hom}_R(R_1,X) & \xrightarrow{\gamma_2^*} & \mathrm{Hom}_R(R_2,X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \gamma_1^*} & & \downarrow{\scriptstyle g_1^*} & & \downarrow{\scriptstyle g_2^*} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(P_0 \oplus R_0,X) & \xrightarrow{\beta_1^*} & \mathrm{Hom}_R(P_1 \oplus R_1,X) & \xrightarrow{\beta_2^*} & \mathrm{Hom}_R(P_1 \oplus R_2,X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_0^*} & & \downarrow{\scriptstyle f_1^*} & & \downarrow{\scriptstyle f_2^*} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(P_0,X) & \xrightarrow{\alpha_1^*} & \mathrm{Hom}_R(P_1,X) & \xrightarrow{\alpha_2^*} & \mathrm{Hom}_R(P_2,X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the columns are exact and the rows give $\mathrm{Ext}^i_R(N,X)$, $\mathrm{Ext}^i_R(M,X)$, and $\mathrm{Ext}^i_R(L,X)$, for all $i \geq 0$. Thus,

$$\mathbf{0} \longrightarrow \mathrm{Hom}_R(\mathbf{R}_{\bullet,N},X) \xrightarrow{g_\bullet^*} \mathrm{Hom}_R(\mathbf{Q}_{\bullet,M},X) \xrightarrow{f_\bullet^*} \mathrm{Hom}_R(\mathbf{P}_{\bullet,L},X) \longrightarrow \mathbf{0}$$

is an exact sequence of cochain complexes. Since the contravariant functor $\mathrm{Hom}_R(-,X)$ is left exact and additive, by Proposition 2.5.6 item (2), it follows that

$$\mathrm{Ext}^0_R(N,X) = H^0(\mathrm{Hom}_R(\mathbf{P}_{\bullet,N},X)) = \mathrm{Hom}_R(N,X),$$

$$\mathrm{Ext}^0_R(M,X) = H^0(\mathrm{Hom}_R(\mathbf{P}_{\bullet,M},X)) = \mathrm{Hom}_R(M,X) \quad \text{and}$$

$$\mathrm{Ext}^0_R(L,X) = H^0(\mathrm{Hom}_R(\mathbf{P}_{\bullet,L},X)) = \mathrm{Hom}_R(L,X).$$

Therefore, by Proposition 2.2.9, we conclude the result. □

Proposition 2.6.4. If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules and R-module homomorphisms, then for each R-module X, there is a long exact cohomology sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(X,L) \xrightarrow{f_*} \mathrm{Hom}_R(X,M) \xrightarrow{g_*} \mathrm{Hom}_R(X,N) \xrightarrow{\Phi^0} \\ \xrightarrow{\Phi^0} \mathrm{Ext}^1_R(X,L) \xrightarrow{\mathrm{Ext}^1_R(X,f)} \mathrm{Ext}^1_R(X,M) \xrightarrow{\mathrm{Ext}^1_R(X,g)} \mathrm{Ext}^1_R(L,N) \xrightarrow{\Phi^1} \cdots \\ \cdots \xrightarrow{\Phi^{i-1}} \mathrm{Ext}^i_R(X,L) \xrightarrow{\mathrm{Ext}^i_R(X,f)} \mathrm{Ext}^i_R(X,M) \xrightarrow{\mathrm{Ext}^i_R(X,g)} \mathrm{Ext}^i_R(X,N) \xrightarrow{\Phi^i} \cdots \end{array}$$

where Φ^i is a connecting R-homomorphism for each $i \geq 0$.

Next two Propositions give some properties about Ext functor, related to projective and injective modules.

Proposition 2.6.5. Let M be an R-module. The following are equivalent:

(i) M is projective;

(ii) $\mathrm{Ext}^n_R(M,X) = 0$, for each R-module X and for all $n \geq 1$;

(iii) $\mathrm{Ext}^1_R(M,X) = 0$, for every R-module X.

Proof. $(i) \Rightarrow (ii)$ Let M be a projective R-module and X be an R-module. Since $\mathrm{Hom}_R(-,X)$ is a left exact additive contravariant functor, it follows from Proposition 2.5.6 item (2) that $\mathrm{Ext}^n_R(M,X) = 0$ for every integer $n \geq 1$.

$(ii) \Rightarrow (iii)$ Immediate.

$(iii) \Rightarrow (i)$ Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of R-modules. By Proposition 2.6.4, there exists a long exact sequence

$$0 \to \mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'') \to \mathrm{Ext}^1_R(M,N') \to \cdots .$$

Since $\mathrm{Ext}^1_R(M,X) = 0$ for each R-module X, it follows that

$$0 \to \mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'') \to 0$$

is exact. Therefore, $\mathrm{Hom}_R(M,-)$ is an exact functor and then M is a projective R-module. □

Proposition 2.6.6. Let M be an R-module. The following are equivalent:

(i) M is injective;

(ii) $\mathrm{Ext}^n_R(X,M) = 0$, for each R-module X and for all $n \geq 1$;

(iii) $\mathrm{Ext}^1_R(X,M) = 0$, for every R-module X.

The proof of this result is analogous of the proof of Proposition 2.6.5, then we omit it.

2.7 The Functor $\operatorname{Tor}_i^R(-,-)$

Let X be a fixed R-module. By Example 2.1.5, we already know that $-\otimes_R X$ and $X\otimes_R -$ are additive covariant functors from the category of R-modules to itself.

2.7.1 The left derived functor related to $-\otimes_R X$ and $X\otimes_R -$

Let M and N be two R-modules and

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_i \xrightarrow{\alpha_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be
cochain complex

$$\mathbf{P}_{\bullet,M}\otimes_R X : \cdots \to P_i\otimes_R X \xrightarrow{\alpha_i\otimes \mathrm{Id}_X} P_{i-1}\otimes_R X \to \cdots \to P_1\otimes_R X \xrightarrow{\alpha_1\otimes \mathrm{Id}_X} P_0\otimes_R X \to 0.$$

Define

$$\operatorname{Tor}_i^R(-,X) := \mathfrak{L}_i(-\otimes_R X) = \mathrm{H}_i(\mathbf{P}_{\bullet,M}\otimes_R X)$$

for all integer $i \geq 0$.

Now, consider the R-homomorphism $f : M \to N$ and a projective resolution $\mathbf{Q}_\bullet$ of N. Then,

$$\operatorname{Tor}_i^R(f,X) := \mathrm{H}_i(f_\bullet \otimes \mathrm{Id}_X)$$

where $f_\bullet : \mathbf{P}_{\bullet,M} \to \mathbf{Q}_{\bullet,N}$ is a map between complexes induced by f. Thus

$$\operatorname{Tor}_i^R(f,X) : \operatorname{Tor}_i^R(M,X) \to \operatorname{Tor}_i^R(N,X).$$

Therefore, for each integer $i \geq 0$,

$$\operatorname{Tor}_i^R(-,X) : M_R \to M_R$$

is a left exact additive covariant functor, called the *ith torsion product functor* of $-\otimes_R X$.

Using a similar construction, it is possible to define another left exact additive covariant functor:

$$\operatorname{Tor}_i^R(X,-) : M_R \to M_R$$

for all integer $i \geq 0$.

2.7.2 Properties

Let X be an R-module and $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence of R-modules and R-homomorphisms. Then, there exists a short exact sequence of chain complex:

$$0 \longrightarrow \mathbf{P}_{\bullet,L} \xrightarrow{f_\bullet} \mathbf{Q}_{\bullet,M} \xrightarrow{g_\bullet} \mathbf{R}_{\bullet,N} \longrightarrow 0,$$

where $\mathbf{P}_\bullet$,$\mathbf{Q}_\bullet$ and $\mathbf{R}_\bullet$ are projective resolutions of L, M and N, respectively. By the Horse Shoe Lemma for Projectives (2.6.1), $Q_i = P_i \oplus R_i$ for all $i \geq 0$. Then,

$$0 \to \mathbf{P}_{\bullet,L} \otimes_R X \stackrel{\mathbf{f}\otimes \mathrm{Id}_X}{\longrightarrow} \mathbf{Q}_{\bullet,M} \otimes_R X \stackrel{\mathbf{g}\otimes \mathrm{id}_X}{\longrightarrow} \mathbf{R}_{\bullet,N} \otimes_R X \to 0.$$

is an exact sequence of chains.

Taking cohomology modules over R, there exists a long exact sequence

$$\begin{array}{l}\cdots \longrightarrow \mathrm{Tor}_i^R(L,X) \stackrel{\mathrm{Tor}_i^R(f,X)}{\longrightarrow} \mathrm{Tor}_i^R(M,X) \stackrel{\mathrm{Tor}_i^R(g,X)}{\longrightarrow} \mathrm{Tor}_i^R(N,X) \stackrel{\Phi_i}{\longrightarrow} \\ \cdots \stackrel{\Phi_2}{\longrightarrow} \mathrm{Tor}_1^R(L,X) \stackrel{\mathrm{Tor}_1^R(f,X)}{\longrightarrow} \mathrm{Tor}_1^R(M,X) \stackrel{\mathrm{Tor}_1^R(g,X)}{\longrightarrow} \mathrm{Tor}_1^R(N,X) \stackrel{\Phi_1}{\longrightarrow} \\ L \otimes_R X \stackrel{f\otimes \mathrm{Id}_X}{\longrightarrow} M \otimes_R X \stackrel{g\otimes \mathrm{Id}_X}{\longrightarrow} N \otimes_R X \longrightarrow 0.\end{array}$$

Note that, since R is a commutative ring, $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$. Thus, there also exists a long exact sequence

$$\begin{array}{l}\cdots \longrightarrow \mathrm{Tor}_i^R(X,L) \longrightarrow \mathrm{Tor}_i^R(X,M) \longrightarrow \mathrm{Tor}_i^R(X,N) \longrightarrow \\ \cdots \stackrel{\Phi_2}{\longrightarrow} \mathrm{Tor}_1^R(X,L) \longrightarrow \mathrm{Tor}_1^R(X,M) \longrightarrow \mathrm{Tor}_1^R(X,N) \stackrel{\Phi_1}{\longrightarrow} \\ X \otimes_R L \longrightarrow X \otimes_R M \longrightarrow X \otimes_R N \longrightarrow 0.\end{array}$$

Next there are some properties about Tor functor, related to some kinds of R-modules.

Lemma 2.7.1. If M is a projective R-module, then $\mathrm{Tor}_i^R(X,M) = 0$, for each R-module X and for all integer $i \geq 1$.

Proof. Let X be an R-module. Since Tor_i is a left derived functor of $N \otimes_R -$, it follows from Proposition 2.5.5 item 1 that $\mathrm{Tor}_i^R(X,M) = 0$ for all integer $i \geq 1$. □

Definition 2.7.2. An R-module X is said to be *flat* if the sequence

$$0 \to M_1 \otimes_R X \to M \otimes_R X \to M_2 \otimes_R X$$

is exact, whenever $0 \to M_1 \to M \to M_2$ is an exact sequence of R-modules. In other words, X is a flat R-module if the functor $- \otimes_R X$ is left exact.

Theorem 2.7.3. Let M be an R-module. The following are equivalent:

(i) M is flat;

(ii) $\mathrm{Tor}_i^R(M,X) = 0$, for each R-module X and for all integer $i \geq 1$;

(iii) $\mathrm{Tor}_1^R(M,X) = 0$, for each finitely generated R-module X;

(iv) $\mathrm{Tor}_1^R(M,R/\mathfrak{a}) = 0$, for all ideal $\mathfrak{a} \subset R$.

Proof. $(i) \Rightarrow (ii)$ Follows from Lemma 2.7.1.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ Immediate.

$(iv) \Rightarrow (iii)$ Let X be a finitely generated R-module generated by the set $\{x_i \mid x_i \in N, i = 1, \ldots, s\}$. Let's prove this, using induction over s, that $\mathrm{Tor}_1^R(M,X) = 0$.

If $s = 1$, then there exists $x \in X$ such that $X = xR$. Thus, the map $R \xrightarrow{\cdot x} X$ is surjective, and define $\mathrm{Ker}(\cdot x) = \mathfrak{a}$. Therefore $R/\mathfrak{a} \cong X$ and $\mathrm{Tor}_1^R(M,X) = \mathrm{Tor}_1^R(M,R/\mathfrak{a}) = 0$, since $\mathfrak{a}$ is finitely generated (R is Noetherian).

Now, suppose $s > 1$ and $X = \sum_{j=1}^{s} Rx_j$. Let $X' = \sum_{j=1}^{s-1} Rx_j$ and the exact sequence

$$0 \to X' \to X \to X'' \to 0,$$

where $X'' = X/X'$. By induction, $\mathrm{Tor}_1^R(M,X') = 0$, and by the case $s = 1$, $\mathrm{Tor}_1^R(M,X'') = 0$. Therefore, using the long exact sequence of Tor, it follows that $\mathrm{Tor}_1^R(M,X) = 0$.

$(iii) \Rightarrow (iv)$ Let $X' \to X$ be an injective homomorphism of R-modules and

$$0 \to X' \to X \to X'' \to 0$$

be an exact sequence where $X'' = X/X'$. To show that M is flat, it is enough to show that $M \otimes_R X' \to M \otimes_R X$ is an injective R-homomorphism.

If X is a finitely generated R-module, then X'' is also finitely generated. Therefore, $\mathrm{Tor}_1^R(M,X'') = 0$ by hypothesis. Thus, the long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(M,X'') \longrightarrow M \otimes_R X' \longrightarrow M \otimes_R X \longrightarrow M \otimes_R X'' \longrightarrow 0$$

shows

$$0 \longrightarrow M \otimes_R X' \longrightarrow M \otimes_R X \longrightarrow M \otimes_R X'' \longrightarrow 0$$

is exact. Therefore, $M \otimes_R X' \to M \otimes_R X$ is injective.

If X is not finitely generated, let's show that $\mathrm{Ker}\,(M \otimes_R X' \to M \otimes_R X) = 0$. Let $z = \sum_{j=i}^{r} m_j \otimes n_i \in \mathrm{Ker}\,(M \otimes_R X' \to M \otimes_R X)$. We claim that $z = 0$. To do this, one can replace X' by the submodule generated by $n_1, \ldots, n_r$. Besides, there exists a finitely generated submodule $\tilde{X} \subset X$ containing X' such that the image of z is trivial in $M \otimes_R \tilde{X}$. So $z \in \mathrm{Ker}\left(M \otimes_R X' \to M \otimes_R \tilde{X}\right) = 0$, by the previous case. Therefore, $M \otimes_R X' \to M \otimes_R X$ is injective. □

2.8 Dimensions

2.8.1 Projective Dimension

Let M be an R-module and consider a projective resolution of M:

$$\mathbf{P}_\bullet : \quad 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

In this case, $\mathbf{P}_\bullet$ is said to have length n.

Definition 2.8.1. Let M be an R-module. The *projective dimension* of M is defined by

$$\mathrm{pdim}_R(M) := \inf\{n \mid n \text{ is the length of a projective resolution of } M\}.$$

The next result gives a characterization to projective R-modules using its projective dimension.

Proposition 2.8.2. M is a projective R-module if and only if $\mathrm{pdim}_R(M) = 0$.

Proof. If M is a projective R-module, then

$$0 \longrightarrow P_0 = M \xrightarrow{Id_M} M \longrightarrow 0$$

is a projective resolution to M. Therefore, $\mathrm{pdim}_R(M) = 0$.

Reciprocally, if $\mathrm{pdim}_R(M) = 0$, then there exists a projective resolution of M

$$0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Thus, $M \cong P_0$. Therefore, M is projective. □

Next we describe the results involving the projective dimension of an R-module and functors Ext and Tor.

Lemma 2.8.3. (**Dimension Shifting**) Let $m,n \in \mathbb{Z}$ be positives and $0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to M \to 0$ be an exact sequence of R-modules. If $\mathrm{pdim}(L_i) < m$ for each $i = 0,\ldots,n-1$, then for all R-module N,

$$\mathrm{Ext}_R^m(L_n,N) = \mathrm{Ext}_R^{m+n}(M,N) \quad \text{and} \quad \mathrm{Tor}_m^R(L_n,N) = \mathrm{Tor}_{n+m}^R(M,N).$$

Proof. Let $Z_i = \mathrm{Im}(L_i \to L_{i-1})$, for $i \geq 1$, and $Z_0 = M$. For each $i \geq 0$, there is a short exact sequence

$$0 \longrightarrow Z_{i+1} \longrightarrow L_i \longrightarrow Z_i \longrightarrow 0,$$

and two long exact sequences

$$\cdots \longrightarrow \mathrm{Ext}_R^j(L_i,N) \longrightarrow \mathrm{Ext}_R^j(Z_{i+1},N) \longrightarrow \mathrm{Ext}_R^{j+1}(Z_i,N) \longrightarrow \mathrm{Ext}_R^{j+1}(L_i,N) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Tor}_{j+1}^R(L_i,N) \longrightarrow \mathrm{Tor}_{j+1}^R(Z_i,N) \longrightarrow \mathrm{Tor}_j^R(Z_{j+1},N) \longrightarrow \mathrm{Tor}_j^R(L_i,N) \longrightarrow \cdots.$$

When $j > m$, $\mathrm{Ext}_R^j(L_i,N) = \mathrm{Ext}_R^{j+1}(L_i,N) = 0$ and $\mathrm{Tor}_{j+1}^R(L_i,N) = \mathrm{Tor}_j^R(L_i,N) = 0$. Therefore,

$$\mathrm{Ext}_R^j(Z_{i+1},N) \cong \mathrm{Ext}_R^{j+1}(Z_i,N) \quad \text{and} \quad \mathrm{Tor}_{j+1}^R(Z_i,N) \cong \mathrm{Tor}_j^R(Z_{j+1},N).$$

Since $Z_n = L_n$ and $Z_0 = M$, then

$$\mathrm{Ext}_R^m(L_n,N) = \mathrm{Ext}_R^m(Z_n,N) \cong \mathrm{Ext}_R^{m+1} \cong \cdots \cong \mathrm{Ext}_R^{m+n}(Z_0,N) = \mathrm{Ext}_R^{m+n}(M,N)$$

and

$$\mathrm{Tor}_m^R(L_n,N) = \mathrm{Tor}_m^R(Z_n,N) \cong \mathrm{Tor}_{m+1}^R \cong \cdots \cong \mathrm{Tor}_{m+n}^R(Z_0,N) = \mathrm{Tor}_{m+n}^R(M,N).$$

□

Theorem 2.8.4. Let M be an R-module and $n \in \mathbb{N}$. The following are equivalent:

(i) $\mathrm{pdim}_R(M) \leq n$;

(ii) $\mathrm{Ext}_R^i(M,X) = 0$, for each R-module X and for all $i > n$;

(iii) $\mathrm{Ext}_R^{n+1}(M,X) = 0$, for each R-module X;

(iv) If

$$0 \longrightarrow K_{n-1} \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a long exact sequence of R-modules such that P_i is a projective R-module for each $i = 0,\ldots,n-1$, then K_{n-1} is a projective R-module.

Proof. $(i) \Rightarrow (ii)$ Since $\mathrm{pdim}_R(M) \leq n$, there exists a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Applying functor $\mathrm{Hom}_R(-,X)$, which is a left exact additive contravariant functor, on this projective resolution, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P_0,X) \xrightarrow{\alpha_1^*} \mathrm{Hom}_R(P_1,X) \xrightarrow{\alpha_2^*} \cdots \xrightarrow{\alpha_n^*} \mathrm{Hom}_R(P_n,X),$$

then

$$\mathrm{Ext}_R^i(M,X) = \frac{\ker(\alpha_{i+1}^*)}{\mathrm{Im}(\alpha_i^*)} = 0, \qquad \forall i > n.$$

$(ii) \Rightarrow (iii)$ Immediate.

$(iii) \Rightarrow (iv)$ Let

$$C: \quad 0 \longrightarrow K_{n-1} \longrightarrow P_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

an exact sequence such that each P_i projective, for all $i = 0,\ldots,n-1$.

We claim that $\mathrm{Ext}_R^1(K_{n-1},X) = 0$, for each R-module X. Indeed, by Lemma 2.8.3, $\mathrm{Ext}_R^1(K_{n-1},X) = \mathrm{Ext}_R^{n+1}(M,X) = 0$. Therefore, by Proposition 2.6.5, it follows that K_{n-1} is projective.

$(iv) \Rightarrow (i)$ Since we are assuming (iv), consider the following projective resolution of the R-module M

$$0 \longrightarrow K_{n-1} \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Therefore, $\mathrm{pdim}_R(M) \leqslant n$. □

The next Corollary follows immediately.

Corollary 2.8.5. Let M and N be two R-modules, then

$$\mathrm{pdim}_R(M) = \sup\{n \mid \mathrm{Ext}_R^n(M,N) \neq 0\}.$$

Next, we have an important result involving projective dimension and the proof can be done by using Horse Shoe Lemma for Projectives (2.6.1).

Proposition 2.8.6. Let $0 \to M \to N \to T \to 0$ be a short exact sequence of R-modules. Then

$$\mathrm{pdim}_R(N) \leqslant \max\{\mathrm{pdim}_R(M), \mathrm{pdim}_R(T)\}.$$

2.8.2 Injective Dimension

The definition of injective dimension is dual to what has been done to projective dimension.

Let N be an R-module and consider an injective resolution of N:

$$\mathbf{I}^\bullet : 0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow 0.$$

In this case, $\mathbf{I}^\bullet$ is said to have length n.

Definition 2.8.7. Let N be an R-module. The *injective dimension* of N is defined by

$$\mathrm{idim}_R(N) := \inf\{n \mid n \text{ is the length of a injective resolution of } N\}.$$

The proof of the next results are analogous to what has been done before to projective dimension, using injective resolutions.

Lemma 2.8.8. Let $m,n \in \mathbb{Z}$ be positives and $0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0$ be an exact sequence of R-modules. If $\mathrm{idim}(I^i) < m$ for each $i = 0,\ldots,n-1$, then for all R-module M,

$$\mathrm{Ext}_R^m(M,I^n) = \mathrm{Ext}_R^{m+n}(M,N) \quad \text{and} \quad \mathrm{Tor}_m^R(M,I^n) = \mathrm{Tor}_{n+m}^R(M,N).$$

Theorem 2.8.9. Let N be an R-module and $n \in \mathbb{N}$. The following are equivalent:

(i) $\mathrm{idim}_R(N) \leqslant n$;

(ii) $\operatorname{Ext}_R^i(X,N) = 0$, for each R-module X and for all $i > n$;

(iii) $\operatorname{Ext}_R^{n+1}(X,N) = 0$, for each R-module X;

(v) If

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow C^{n-1} \longrightarrow 0$$

is a long exact sequence of R-modules such that E^i is a injective module for each $i = 0,\ldots,n-1$, then C^{n-1} is an injective R-module.

Corollary 2.8.10. Let M and N be two R-modules, then

$$\operatorname{idim}_R(N) = \sup\{n \mid \operatorname{Ext}_R^n(M,N) \neq 0\}.$$

CHAPTER

3

LOCAL COHOMOLOGY MODULES

In this Chapter, we aim to establish the definition of local cohomology modules and generalized local cohomology modules. We start by defining the functor $\mathfrak{a}$-torsion (or Gamma functor). Besides, we give some properties and main results about them (such as when they vanish, when they are finitely generated, and when they are Artinian). All the information in this Chapter is going to be used in the following chapters.

To see some results mentioned in this Chapter and more results about this subject, the reader can look at (BRODMANN; SHARP, 2012) and (GROTHENDIECK, 1967).

During this Chapter, R will be a non-zero commutative Noetherian ring with identity.

3.1 Gamma Functor and $\mathfrak{a}$-Torsion Modules

In this section, we define the Gamma functor, which is an important tool in the definition of local cohomology modules.

Let $\mathfrak{a}$ be an ideal of R, M be an R-module and N be a submodule of M. The set

$$(N :_M \mathfrak{a}) := \{m \in M \mid m\mathfrak{a} \subseteq N\}$$

is a submodule of M.

Definition 3.1.1. Let M be an R-module. The $\mathfrak{a}$-*torsion module* of M is defined by

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n).$$

Note that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. Furthermore, there are some more properties of this module.

Properties 3.1.2. Let M and N be two R-modules and $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R.

1. $\Gamma_0(M) = M$ and $\Gamma_R(M) = 0$.

2. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\Gamma_{\mathfrak{b}}(M) \subseteq \Gamma_{\mathfrak{a}}(M)$.

3. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M)$.

4. $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M) \cap \Gamma_{\mathfrak{b}}(M)$.

5. If $f : M \to N$ is an R-homomorphism, then $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$.

6. $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$.

7. If M is finitely generated, the there exists $n_0 \in \mathbb{N}$ such that

$$\Gamma_{\mathfrak{a}}(M) = (0 :_M \mathfrak{a}^{n_0}).$$

Proof. Some of these Properties follow by the definition of $\Gamma_{\mathfrak{a}}(M)$. Let's prove the other ones.

5. Let $y \in f(\Gamma_{\mathfrak{a}}(M))$. Thus, there is $x \in \Gamma_{\mathfrak{a}}(M)$ such that $f(x) = y$. Let $r \in R$. Since $r^n y = r^n f(x) = f(r^n x) = 0$, follows that $y \in \Gamma_{\mathfrak{a}}(N)$.

7. Since M is finitely generated and R is Noetherian, M is a Noetherian module. Consider the sequence of submodules of M, $(0 :_M \mathfrak{a}^1) \subseteq (0 :_M \mathfrak{a}^2) \subseteq \cdots$. Thus, there exists $n_0 \in \mathbb{N}$ such that $(0 :_M \mathfrak{a}^{n_0}) = (0 :_M \mathfrak{a}^{n_0+k})$, for $k \leq 0$. Therefore,

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}} (0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^{n_0}).$$

□

Definition 3.1.3. Let $\mathfrak{a}$ be an ideal of R. An R-module M is said to be $\mathfrak{a}$-*torsion* if $\Gamma_{\mathfrak{a}}(M) = M$. On the other hand, an R-module M is said to be $\mathfrak{a}$-*torsion free*, if $\Gamma_{\mathfrak{a}}(M) = 0$.

Note that $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$, therefore $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.

The next result tells us that $\Gamma_{\mathfrak{a}}(-)$ is a functor from the category of R-modules to itself, called $\mathfrak{a}$-*torsion functor*, or, *Gamma functor*.

Proposition 3.1.4. $\Gamma_{\mathfrak{a}}(-)$ is a left exact additive covariant functor from the category of R-modules to itself.

Proof. Let $f : M \to N$ an homomorphism of R-modules. By Property 3.1.2 item (5), $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$, so there is an R-homomorphism

$$\begin{aligned} \Gamma_{\mathfrak{a}}(h) : \Gamma_{\mathfrak{a}}(M) &\to \Gamma_{\mathfrak{a}}(N) \\ m &\mapsto f(m). \end{aligned}$$

Therefore, $\Gamma_{\mathfrak{a}}(-)$ is an additive covariant functor from the category of R-modules to itself.

Furthermore, let $0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0$ a short exact sequence of R-modules. We claim that

$$0 \to \Gamma_{\mathfrak{a}}(N) \xrightarrow{\Gamma_{\mathfrak{a}}(f)} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(g)} \Gamma_{\mathfrak{a}}(P)$$

is exact. Indeed, since f is injective, if $\Gamma_{\mathfrak{a}}(f)(m) := f(m) = 0$, then $m = 0$ So, $\Gamma_{\mathfrak{a}}(f)(m)$ is injective.

Now, since $g \circ f = 0$, then

$$\Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f) = \Gamma_{\mathfrak{a}}(g \circ f) = \Gamma_{\mathfrak{a}}(0) = 0.$$

Therefore, $\mathrm{Im}(\Gamma_{\mathfrak{a}}(f)) \subseteq \mathrm{Ker}(\Gamma_{\mathfrak{a}}(g))$.

Let $m \in \mathrm{Ker}(\Gamma_{\mathfrak{a}}(g))$, so $g(m) = 0$ and, since $m \in \Gamma_{\mathfrak{a}}(M)$, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$. Let $y \in N$ such that $f(y) = m$. We finish the proof by showing $y \in \Gamma_{\mathfrak{a}}(N)$. To do this, note that for each $r \in \mathfrak{a}^n$, $f(ry) = rf(y) = rm = 0$. Since f is injective, $ry = 0$, then $\mathfrak{a}^n y = 0$. Thus, $y \in \Gamma_{\mathfrak{a}}(N)$. Therefore, $\mathrm{Im}(\Gamma_{\mathfrak{a}}(f)) = \mathrm{Ker}(\Gamma_{\mathfrak{a}}(g))$ and the sequence is exact. □

Theorem 3.1.5. $\Gamma_{\mathfrak{a}}(M) = \varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, M)$, for an R-module M.

Proof. For each R-module M and integer $t \geq 0$,

$$\begin{array}{ccc} \mathrm{Hom}_R(R/\mathfrak{a}^t, M) & \xrightarrow{\cong} & \{x \in M \mid \mathfrak{a}^t x = 0\} \\ f & \mapsto & f(1) \end{array}$$

With this identification, one has a directed system

$$\mathrm{Hom}_R(R/\mathfrak{a}, M) \subseteq \cdots \subseteq \mathrm{Hom}_R(R/\mathfrak{a}^t, M) \subseteq \mathrm{Hom}_R(R/\mathfrak{a}^{t+1}, M) \subseteq \cdots$$

of submodules of $\Gamma_{\mathfrak{a}}(M)$. Therefore,

$$\Gamma_{\mathfrak{a}}(M) = \varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, M).$$

□

Moreover, there is an important property involving injective R-modules and this functor, which will be used in Section 3.3.

Proposition 3.1.6. ((BRODMANN; SHARP, 2012, Theorem 2.1.4)) Let I be an injective R-module. Then $\Gamma_{\mathfrak{a}}(I)$ is also an injective R-module.

Define

$$\mathrm{ZD}_R(M) := \{x \in R \mid \exists m \in M \setminus \{0\} \text{ such that } xm = 0\}$$

the *set of zero divisors* of M over R. And, analogously, define

$$\mathrm{NZD}_R(M) := R \setminus \mathrm{ZD}_R(M)$$

the *set of non-zero divisors* of M over R.

Note that $\mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)} \mathfrak{p}$, where $\mathrm{Ass}_R(M)$ denotes the set of associated primes of M.

Lemma 3.1.7. Let M be an R-module.

1. If $\Gamma_{\mathfrak{a}}(M) \neq 0$, then $\mathfrak{a} \subseteq \mathrm{ZD}_R(M)$.
2. Suppose $|\mathrm{Ass}_R(M)| < \infty$. If $\mathfrak{a} \subseteq \mathrm{ZD}_R(M)$, then $\Gamma_{\mathfrak{a}}(M) \neq 0$.

Proof. 1. Let's prove: if $\mathfrak{a} \cap \mathrm{NZD}_R(M) \neq \emptyset$, then $\Gamma_{\mathfrak{a}}(M) = 0$.

Indeed, let $r \in \mathfrak{a}$ be a non-zero divisor of M and $m \in \Gamma_{\mathfrak{a}}(M)$. Thus, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$, in particular, $r^n m = 0$. Since r^n is a non-zero divisor of M, it follows that $m = 0$. Therefore, $\Gamma_{\mathfrak{a}}(M) = 0$.

2. By hypothesis $|\mathrm{Ass}_R(M)| < \infty$. Therefore we can write $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Since $\mathfrak{a} \in \mathrm{ZD}_R(M)$ and $\mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)} \mathfrak{p}$, $\mathfrak{a} \subseteq \bigcup_{i=1}^{r} \mathfrak{p}_i$. Thus, by Prime Avoidance Lemma there exists some $i \in \{1, \dots, r\}$, such that $\mathfrak{a} \subseteq \mathfrak{p}_i$. Since $\mathfrak{p}_i \in \mathrm{Ass}_R(M)$, there exists $v \in M$, $v \neq 0$, such that $\mathfrak{p}_i = (0 :_R v)$. So $\mathfrak{a}v \subseteq \mathfrak{p}_i v = 0$ and then $v \in \Gamma_{\mathfrak{a}}(M)$. Therefore, $\Gamma_{\mathfrak{a}}(M) \neq 0$. □

Lemma 3.1.8. Let M and N be two R-modules, and $\mathfrak{a}$ be an ideal of R. Then, $\mathfrak{a} \cap \mathrm{NZD}_R(N) \neq \emptyset$ if and only if $\mathfrak{a} \cap \mathrm{NZD}_R(\mathrm{Hom}_R(M,N)) \neq \emptyset$.

Proof. Suppose that $\mathfrak{a} \cap \mathrm{NZD}_R(N) \neq \emptyset$ and let $x \in \mathfrak{a} \cap \mathrm{NZD}_R(N)$. Our claim is: $x \in \mathrm{NZD}_R(\mathrm{Hom}_R(M,N)) = \{r \in R \mid r \cdot f \neq 0,\ \forall f \in \mathrm{Hom}_R(M,N)\}$. Indeed,

$$r \cdot f \neq 0 \Leftrightarrow \exists m \in M \text{ such that } rf(m) \neq 0, \quad \forall f \in \mathrm{Hom}(M,N).$$

But, $f(m) \in N$ and $x \cdot n \neq 0$, for all $n \in N$, since $x \in \mathrm{NZD}_R(N)$. Hence, $x \in \mathfrak{a} \cap \mathrm{NZD}_R(\mathrm{Hom}_R(M,N))$. Therefore, $\mathfrak{a} \cap \mathrm{NZD}_R(\mathrm{Hom}_R(M,N)) \neq \emptyset$.

Conversely, suppose $\mathfrak{a} \cap \mathrm{NZD}_R(N) = \emptyset$. We show $\mathfrak{a} \cap \mathrm{NZD}_R(\mathrm{Hom}_R(M,N)) = \emptyset$. Since M is an R-module, by Remark 2.3.11 there exists a surjection $R^J \twoheadrightarrow M \to 0$, where J is a set of indexes. By applying functor $\mathrm{Hom}_R(-,N)$ on the previous exact sequence, we have the following exact sequence

$$0 \to \mathrm{Hom}_R(M,N) \hookrightarrow \mathrm{Hom}_R(R^J,N),$$

where $\mathrm{Hom}_R(R^J,N) \cong \bigoplus_{j\in J} N$. Hence,

$$\mathrm{NZD}(\mathrm{Hom}_R(M,N)) \hookrightarrow \mathrm{NZD}_R(\bigoplus N) = \mathrm{NZD}_R(N)$$

and

$$\mathfrak{a} \cap \mathrm{NZD}(\mathrm{Hom}_R(M,N)) \hookrightarrow \mathfrak{a} \cap \mathrm{NZD}_R(N) = \emptyset.$$

Therefore, $\mathfrak{a} \cap \mathrm{NZD}(\mathrm{Hom}_R(M,N)) = \emptyset$. □

Proposition 3.1.9. Let M be a finitely generated R-module. Then,

1. $\mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \cap V(\mathfrak{a})$.

2. $\mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \setminus V(\mathfrak{a})$.

Proof. 1. Let $\mathfrak{p} \in \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$. We already know $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M, thus, $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Since M is finitely generated, by Property 3.1.2 item (7), there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$, that is, $\mathfrak{a}^n \subseteq (0 :_R \Gamma_{\mathfrak{a}}(M))$. On the other hand, $\mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M)) \subseteq V((0 :_R \Gamma_{\mathfrak{a}}(M)))$, so $\mathfrak{a}^n \subseteq (0 :_R \Gamma_{\mathfrak{a}}(M)) \subseteq \mathfrak{p}$. Therefore, $\mathfrak{p} \in V(\mathfrak{a})$.

Now, let $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap V(\mathfrak{a})$. Therefore, $\mathfrak{a} \subseteq \mathfrak{p}$ and there exists $v \in M$ such that $\mathfrak{p} = (0 :_R v)$. Thus, $\mathfrak{a}v = 0$ and then $v \in \Gamma_{\mathfrak{a}}(M)$. Therefore, $\mathfrak{p} = (0 :_R v) \in \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$.

2. Let $\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus V(\mathfrak{a})$. Note that $\mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M)) \cup \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$. Since $\mathfrak{p} \notin V(\mathfrak{a})$, by item (1), $\mathfrak{p} \notin \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(M))$. Therefore, $\mathfrak{p} \in \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$.

Now, let $\mathfrak{p} \in \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$. Since M is finitely generated, by Property 3.1.2 item (6), $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$. Note that $M/\Gamma_{\mathfrak{a}}(M)$ is also finitely generated, therefore, by Lemma 3.1.7 item (2), there exists $x \in \mathrm{NZD}_R(M/\Gamma_{\mathfrak{a}}(M)) \cap \mathfrak{a}$. Furthermore, $\mathfrak{p} \in \mathrm{ZD}_R(M/\Gamma_{\mathfrak{a}}(M))$, since $\mathrm{ZD}_R(M/\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$. Thus, $x \notin \mathfrak{p}$.

By the choice of $\mathfrak{p}$, there exists an element $\overline{v} \in M/\Gamma_{\mathfrak{a}}(M)$ such that $\mathfrak{p} = (0 :_R \overline{v})$. Therefore, $\mathfrak{p}v \subseteq \Gamma_{\mathfrak{a}}(M)$. Since M is finitely generated, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$, by Property 3.1.2 item (7). Thus $\mathfrak{p}(x^n v) = x^n\mathfrak{p}v \subseteq \mathfrak{a}^n\Gamma_{\mathfrak{a}}(M) = 0$ and then $\mathfrak{p} \subseteq (0 :_R x^n v)$.

On the other hand, let $s \in (0 :_R x^n v)$. Thus, $s(x^n v) = 0 \in \Gamma_{\mathfrak{a}}(M)$, and $(sx^n)\overline{v} = \overline{0}$ on the quotient $M/\Gamma_{\mathfrak{a}}(M)$, which implies $sx^n \in (0 :_R \overline{v}) = \mathfrak{p}$. Since $x \notin \mathfrak{p}$, follows that $s \in \mathfrak{p}$. Furthermore, $\mathfrak{p} \notin V(\mathfrak{a})$, since $x \in \mathfrak{a}$. Therefore, $\mathfrak{p} = (0 :_R x^n v)$, with $x^n v \in M$ and $\mathfrak{p} \notin V(\mathfrak{a})$, as we wish. □

3.2 Local Cohomology

In this section, we define a local cohomology module and local cohomology functor. These concepts were introduced by Grothendieck and can be found in (BRODMANN; SHARP, 2012). The properties about these modules are going to be given in a further section.

Let M be an R-module, $\mathfrak{a}$ be an ideal of R and let

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{\alpha^0} I^1 \xrightarrow{\alpha^1} \cdots \longrightarrow I^i \xrightarrow{\alpha^i} I^{i+1} \longrightarrow \cdots$$

be an injective resolution of M (which exists by Theorem 2.3.14). Apply the left exact additive covariant functor $\Gamma_{\mathfrak{a}}(-)$ on $\mathbf{I}^{\bullet,M}$, then it follows that the cochain complex

$$\Gamma_{\mathfrak{a}}(\mathbf{I}^{\bullet,M}) : 0 \longrightarrow \Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^0)} \Gamma_{\mathfrak{a}}(I^1) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^1)} \cdots \longrightarrow \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^i)} \Gamma_{\mathfrak{a}}(I^{i+1}) \longrightarrow \cdots.$$

Define the *ith local cohomology module* $\mathrm{H}^i_{\mathfrak{a}}(M)$ of M with respect to $\mathfrak{a}$ as the ith right derived functor of the $\mathfrak{a}$-torsion functor. That is,

$$\mathrm{H}^i_{\mathfrak{a}}(M) := \mathfrak{R}^i\Gamma_{\mathfrak{a}}(M) = \ker(\Gamma_{\mathfrak{a}}(\alpha^i))/\operatorname{Im}(\Gamma_{\mathfrak{a}}(\alpha^{i-1})),$$

for all integer $i \geq 0$.

Now, let $f : M \to N$ be an R-homomorphism, and $\mathbf{I}^\bullet$ and $\mathbf{J}^\bullet$ be injective resolutions of M and N, respectively. Define an homomorphism on the ith local cohomology module with respect to $\mathfrak{a}$ by

$$\begin{array}{rcl} \mathrm{H}^i_{\mathfrak{a}}(f) : \mathrm{H}^i_{\mathfrak{a}}(M) & \to & \mathrm{H}^i_{\mathfrak{a}}(N) \\ m + \operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{i-1}_M)) & \mapsto & f^i(m) + \operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{i-1}_N)). \end{array}$$

where $f^\bullet : \mathbf{I}^{\bullet,M} \to \mathbf{J}^{\bullet,N}$ is a map between co-complexes induced by f. Note that $\mathrm{H}^i_{\mathfrak{a}}(f)$ is well defined since $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$.

Therefore, for each integer $i \geq 0$,

$$\mathrm{H}^i_{\mathfrak{a}}(-) : M_R \to M_R$$

is an additive covariant functor, called the *ith local cohomology functor* of M with respect to $\mathfrak{a}$.

By Proposition 2.5.5 item (2), the next Proposition follows.

Proposition 3.2.1. $\mathrm{H}^0_{\mathfrak{a}}(-) = \Gamma_{\mathfrak{a}}(-)$,where $\mathfrak{a}$ is an ideal of R.

Furthermore, it is possible to define the local cohomology functor via the Ext functor, as we can see in the next Theorem.

Theorem 3.2.2. ((BRODMANN; SHARP, 2012, Theorem 1.3.8)) There is a unique isomorphism of functors from the category of R-modules to itself

$$(\varphi^i_{\mathfrak{a}})_{i\in\mathbb{N}_0} : \left(\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, -) \right) \xrightarrow{\cong} \left(\mathrm{H}^i_{\mathfrak{a}}(-)\right)_{i\in\mathbb{N}_0}.$$

Consequently, for each R-module M, there is an isomorphism

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M) \cong \mathrm{H}^i_{\mathfrak{a}}(M).$$

3.3 Generalized Local Cohomology Modules

We now explore a generalization to what we have seen before: we define the generalized local cohomology module of two R-modules with respect to an ideal.

Let M and N be two R-modules, and $\mathfrak{a}$ be an ideal of R. Define

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_{n} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N),$$

for all integer $i \geq 0$, that we call the *ith generalized local cohomology module* of M and N with respect to $\mathfrak{a}$, introduced by Herzog in (HERZOG, 1974). If $M = R$ the definition of generalized local cohomology modules reduces to the case of Theorem 3.2.2 and we have the local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(N)$.

Furthermore, there is another way to define generalized local cohomology modules, which is similar to what has been done to local cohomology modules, as we will see next.

Consider an injective resolution of the R-module N

$$\mathbf{E}^{\bullet} : 0 \longrightarrow N \xrightarrow{\alpha} E^0 \xrightarrow{\alpha^0} E^1 \xrightarrow{\alpha^1} \cdots \longrightarrow E^i \xrightarrow{\alpha^i} E^{i+1} \longrightarrow \cdots.$$

Apply the functor $\Gamma_{\mathfrak{a}}(-)$ to $\mathbf{E}^{\bullet,N}$, then we have the following cochain complex

$$\Gamma_{\mathfrak{a}}(\mathbf{E}^{\bullet,N}) : 0 \longrightarrow \Gamma_{\mathfrak{a}}(E^0) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^0)} \Gamma_{\mathfrak{a}}(E^1) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^1)} \cdots \longrightarrow \Gamma_{\mathfrak{a}}(E^i) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha^i)} \Gamma_{\mathfrak{a}}(E^{i+1}) \longrightarrow \cdots,$$

where each $\Gamma_{\mathfrak{a}}(E^i)$, $i \in \mathbb{N}_0$, is injective, by Proposition 3.1.6. Again, applying the functor $\mathrm{Hom}_R(M,-)$, it follows that

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^0)) \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^1)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^i)) \longrightarrow \\ &\longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{i+1})) \longrightarrow \cdots \end{aligned}$$

Then, we can take the cohomology of this cochain complex: $\mathrm{H}^i(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(\mathbf{E}^{\bullet})))$, which is isomorphic to $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ in the hypothesis of the following Theorem.

Theorem 3.3.1. Let M and N be two R-modules, and $\mathfrak{a}$ be an ideal of R. The following hold.

1. Let $0 \to N \to E^{\bullet}$ be an injective resolution of N. Then

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \overset{(a)}{\cong} \mathrm{H}^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^{\bullet}))) \overset{(b)}{\cong} \mathrm{H}^i(\mathrm{H}^0_{\mathfrak{a}}(M,E^{\bullet})).$$

2. Moreover, if M is finitely generated, then

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \mathrm{H}^i(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{\bullet}))).$$

With the goal of establishing Theorem 3.3.1, we first prove the following preliminary result.

Lemma 3.3.2. Let M and N two R-modules such that M is finitely generated. Let $0 \to N \to E^{\bullet}$ be an injective resolution of N. Then

$$\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^{\bullet})) = \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{\bullet})).$$

Proof.

$$\begin{aligned}
\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^\bullet)) &= \varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(M,E^\bullet)) \\
&= \varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t \otimes_R M, E^\bullet) \\
&= \varinjlim_t \mathrm{Hom}_R(M, \mathrm{Hom}_R(R/\mathfrak{a}^t, E^\bullet)) \\
&= \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet)). \qquad (3.1)
\end{aligned}$$

Note that we are using the fact that M is finitely generated at equality (3.1), by (BROWN, 1975, Corollary of Theorem 1). □

Now, it is possible to prove Theorem 3.3.1:

Proof. 1. (a)

$$\begin{aligned}
\mathrm{H}^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^\bullet))) &\cong \mathrm{H}^i(\varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(M,E^\bullet))) \\
&= \mathrm{H}^i(\varinjlim_t \mathrm{Hom}_R(M/\mathfrak{a}^t M, E^\bullet)) \\
&= \varinjlim_t \mathrm{H}^i(\mathrm{Hom}_R(M/\mathfrak{a}^t M, E^\bullet)) \\
&= \varinjlim_t \mathrm{Ext}^i_R(M/\mathfrak{a}^t M, E^\bullet) \\
&= \mathrm{H}^i_{\mathfrak{a}}(M,N).
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{H}^i(\mathrm{H}^0_{\mathfrak{a}}(M,E^\bullet)) &= \mathrm{H}^i(\varinjlim_t \mathrm{Hom}_R(M/\mathfrak{a}^t M, E^\bullet)) \\
&= \mathrm{H}^i(\varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, \mathrm{Hom}_R(M,E^\bullet))) \\
&\cong \mathrm{H}^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^\bullet))).
\end{aligned}$$

2.

$$\begin{aligned}
\mathrm{H}^i(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^\bullet))) &\cong \mathrm{H}^i(\mathrm{Hom}_R(M, \varinjlim_t \mathrm{Hom}_R(R/\mathfrak{a}^t, E^\bullet))) \\
&= \mathrm{H}^i(\varinjlim_t \mathrm{Hom}_R(M, \mathrm{Hom}_R(R/\mathfrak{a}^t, E^\bullet))) \qquad (3.2) \\
&= \varinjlim_t \mathrm{H}^i(\mathrm{Hom}_R(M, \mathrm{Hom}_R(R/\mathfrak{a}^t, E^\bullet))) \\
&= \varinjlim_t \mathrm{H}^i(\mathrm{Hom}_R(M/\mathfrak{a}^t M, E^\bullet)) \\
&= \varinjlim_t \mathrm{Ext}^i_R(M/\mathfrak{a}^t M, E^\bullet) \\
&= \mathrm{H}^i_{\mathfrak{a}}(M,N).
\end{aligned}$$

Note that we are using the fact that M is finitely generated at equality 3.2, by (BROWN, 1975, Corollary of Theorem 1).

□

From the Theorem 3.3.1, it follows that:

Corollary 3.3.3. Let M and N be two R-modules, and $\mathfrak{a}$ be an ideal of R. Then $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-torsion.

Corollary 3.3.4. Let M and N be two R-modules such that M is finitely generated, and $\mathfrak{a}$ be an ideal of R. Then

$$\mathrm{H}^0_{\mathfrak{a}}(M,N) \cong \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(N)).$$

Corollary 3.3.5. Let M and N be two R-modules such that M is a finitely generated, then

$$\mathrm{H}^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)) \cong \mathrm{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(N)), \quad \text{for all } i \in \mathbb{N}_0.$$

3.4 Some results

Throughout this section, let $\mathfrak{a}$ be an ideal of R, where R is a commutative Noetherian ring.

When Groethendiek first defines local cohomology modules, it was used in a geometric point of view. His aim was to use local cohomology modules with respect to an R-module M to obtain information about the module M itself, as we can see, for example in Theorem 3.4.16 and Corollary 3.4.17. On the other hand, to investigate properties regarding local cohomology modules and generalized local cohomology modules is another ongoing topic. Some questions that have been attempted to be answered are as follows.

- When are $\mathrm{H}^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(M,N)$ zero or non-zero?
- When are $\mathrm{H}^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(M,N)$ finitely generated?
- When are $\mathrm{H}^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(M,N)$ Artinian?

There are many important results concerning those questions of local cohomology modules and generalized local cohomology modules. In this section, we will see some of those results that answer those questions in some cases. More results will be shown in Chapters 4 (Section 4.3) and 5 (Section 5.2).

The notations and most of the results of this sections can be found in (BRODMANN; SHARP, 2012) essentially in Chapters 6 and 7, although the results themselves are somewhat older and can be found in their original versions in (GROTHENDIECK, 1967). The results that do not have references in front of them are our contribution to the theory.

Before starting, let's remember a definition of Krull dimension.

Definition 3.4.1. The *Krull dimension* of an R-module M, denoted by $\dim M$ or $\dim_R M$, is the supremum of lengths of chains of prime ideals in the support of M if this supremum exists, and ∞ otherwise.

When M is finitely generated, $\dim M = \dim R/(0 :_R M)$, but this is not the case if M is not finitely generated. To the general case, it is possible to define

$$\dim M := \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}.$$

Note that the two definitions coincide when M is a finitely generated R-module. We adopt the convention that the dimension of the zero R-module is -1.

The following three results concern the vanishing of local cohomology modules.

Theorem 3.4.2. (BRODMANN; SHARP, 2012, 3.3.1) Let M be an R-module. Suppose that $\mathfrak{a}$ can be generated by t elements. Then, $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$, for all $i > t$.

Theorem 3.4.3. (Grothendieck's Vanishing Theorem) (BRODMANN; SHARP, 2012, 6.1.2) Let M be an R-module. Then $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$, for all $i > \dim M$.

Theorem 3.4.4. ((SUZUKI, 1978, Lemma 3.1)) Assume that $(R,\mathfrak{m})$ is local, let M and N be two R-modules such that $\operatorname{pdim} M = d < \infty$ and $\dim N = n < \infty$. Then $\mathrm{H}^i_{\mathfrak{m}}(M,N) = 0$, for all $i > d+n$.

Theorem 3.4.4 is the "best" result concerning vanishing of generalized local cohomology modules related to the dimensions of the R-modules. Furthermore, there is no vanishing or non-vanishing result to what we call the "top" generalized local cohomology modules, that is, we do not know if $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) \neq 0$ or $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) = 0$.

On the other hand, for local cohomology modules, we have the next Theorem, which is about non-vanishing of local cohomology modules. It can be regarded as a companion to Grothendieck's Vanishing Theorem, because it shows that, in some circumstances, this Vanishing Theorem is the best possible. Furthermore, it tells us when local cohomology modules are not finitely generated.

Theorem 3.4.5. (BRODMANN; SHARP, 2012, 6.1.4 and 6.1.7) Assume that $(R,\mathfrak{m})$ is local, and let M be a non-zero, finitely generated R-module such that $\dim M = n$. Then $H^n_{\mathfrak{m}}(M) \neq 0$. Furthermore, if $n > 0$, $H^n_{\mathfrak{m}}(M)$ is not finitely generated.

In Section 5.2, Proposition 5.2.10 also gives us a non finitely generated condition, but for generalized local cohomology modules.

Now it is time to explore connections between regular sequences and local cohomology modules. Some of the following results are already true for local cohomology modules and here we give proof for generalized local cohomology modules. They are also about the vanishing of those modules.

Proposition 3.4.6. Let M and N be two R-modules. Let $(x_1,\dots,x_r)$ be an N-sequence in $\mathfrak{a}$. Then, $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$, for all $i<r$.

Proof. We do the proof by induction on r.

Let $r=1$. Then, $x_1\in\mathfrak{a}\cap\mathrm{NZD}_R(N)$, hence $\mathfrak{a}\cap\mathrm{NZD}_R(N)\neq\emptyset$ and, by Lemma 3.1.8, $\mathfrak{a}\cap\mathrm{NZD}_R(\mathrm{Hom}_R(M,N))\neq\emptyset$. Then, $\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N))=0$. Therefore, by Theorem 3.3.1,

$$\mathrm{H}^0_{\mathfrak{a}}(M,N)=\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N))=0.$$

Now, let $r>1$. Then $(x_1,\dots,x_{r-1})$ is an N-sequence in $\mathfrak{a}$. Hence, by induction, $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$, for all $i<r-1$. It remains to be shown that $\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N)=0$.

We know $x_1\in\mathrm{NZD}_R(N)$ and $(x_2,\dots,x_r)$ is an N/x_1N-sequence in $\mathfrak{a}$. Applying the functor $\mathrm{H}^i_{\mathfrak{a}}(M,-)$ on the exact sequence

$$0\longrightarrow N\xrightarrow{\cdot x_1}N\longrightarrow N/x_1N\longrightarrow 0$$

we have

$$\mathrm{H}^{r-2}_{\mathfrak{a}}(M,N/x_1N)\longrightarrow\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N)\xrightarrow{\cdot x_1}\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N).$$

Since $(x_2,\dots,x_r)$ is an N/x_1N-sequence in $\mathfrak{a}$ we obtain by induction that, $\mathrm{H}^{r-2}_{\mathfrak{a}}(M,N/x_1N)=0$. Therefore, the multiplication homomorphism

$$x_1\cdot:\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N)\to\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N)$$

is injective. As $H^{r-1}_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-torsion (by Corollary 3.3.3) and $x_1\in\mathfrak{a}$, it follows $\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N)=0$. □

Corollary 3.4.7. Let M be an R-module and $(x_1,\dots,x_r)$ be an M-sequence in $\mathfrak{a}$. Then, $\mathrm{H}^i_{\mathfrak{a}}(M)=0$, for all $i<r$.

Proposition 3.4.8. Let M and N be two R-modules such that $|\mathrm{Ass}_R(N)|<\infty$, and $r\in\mathbb{N}$. Then, the following statements are equivalent:

(i) There is a N-sequence of length r in $\mathfrak{a}$;

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$ for all $i<r$.

Proof. $(i)\Rightarrow(ii)$ Clear by Proposition 3.4.6 (does not need $|\mathrm{Ass}_R(N)|<\infty$).

$(ii)\Rightarrow(i)$ Let $\mathrm{H}^i_{\mathfrak{a}}(M,N)=0$ for $i\in\{0,\dots,r-1\}$. We construct an R-sequence $(x_1,\dots,x_r)$ in $\mathfrak{a}$ by induction in r.

Let $r = 1$. Then $\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) = \mathrm{H}^0_{\mathfrak{a}}(M,N) = 0$, by hypothesis. By Lemma 3.1.7 (b) it follows $\mathfrak{a} \nsubseteq \mathrm{ZD}_R(\mathrm{Hom}(M,N))$ and hence $\mathfrak{a} \cap \mathrm{NZD}_R(\mathrm{Hom}(M,N)) \neq \emptyset$. Then, by Lemma 3.1.8, $\mathfrak{a} \cap \mathrm{NZD}_R(N) \neq \emptyset$. Therefore, we can find an element $x \in \mathfrak{a} \cap \mathrm{NZD}_R(N)$. This proves the case $r = 1$.

Now, let $r > 1$. By the case $r = 1$ there is some $x_1 \in \mathfrak{a} \cap \mathrm{NZD}_R(N)$. Thus, we have the exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot x_1} N \longrightarrow N/x_1N \longrightarrow 0,$$

and it gives us the exact sequence of cohomology modules

$$\mathrm{H}^j_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^j_{\mathfrak{a}}(M,N/x_1N) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{a}}(M,N),$$

for all $j \in \mathbb{N}$. It shows that $\mathrm{H}^j_{\mathfrak{a}}(M,N/x_1N) = 0$ for all $j < r-1$. So, by induction, there is an N/x_1N-sequence $(x_2,\ldots,x_r)$ in $\mathfrak{a}$. Therefore, $(x_1,x_2,\ldots,x_r)$ is an N-sequence in $\mathfrak{a}$. □

Corollary 3.4.9. Let M be a R-module such that $|\mathrm{Ass}_R(M)| < \infty$, and $r \in \mathbb{N}$. Then, the following statements are equivalent:

(i) There is an M-sequence of length r in $\mathfrak{a}$;

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i < r$.

Definition 3.4.10. Let M be a finitely generated R-module. We call the *grade* of $\mathfrak{a}$ with respect to M, denoted by $\mathrm{grade}_M \mathfrak{a}$, the supremum of the length of all M-sequences contained in $\mathfrak{a}$. In symbols,

$$\mathrm{grade}_M(\mathfrak{a}) := \sup\{r \in \mathbb{N} \mid \exists\ M\text{-sequence of length } r \text{ in } \mathfrak{a}\}.$$

Proposition 3.4.11. (BRODMANN; SHARP, 2012, 6.2.4) Let M be a finitely generated R-module such that $\mathfrak{a}M \neq M$. Then all maximal M-sequences contained in $\mathfrak{a}$ have the same length, namely $\mathrm{grade}_M(\mathfrak{a})$.

Furthermore,

$$\mathrm{grade}_M(\mathfrak{a}) = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_R(R/\mathfrak{a},M) \neq 0\}.$$

Definition 3.4.12. Suppose $(R,\mathfrak{m})$ is a commutative Noetherian local ring and M is a non-zero finitely generated R-module. The *depth* of M is defined as $\mathrm{depth}\, M := \mathrm{grade}_M(\mathfrak{m})$.

Theorem 3.4.13. Let M and N be two finitely generated R-modules. Then $\mathrm{grade}_N(\mathfrak{a})$ is the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0$.

Proof. Define $\rho := \inf\{i \in \mathbb{N} : \mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0\} \in \mathbb{N} \cup \{\infty\}$.

Let $g \in \mathbb{N}$ be such that $g \leq \mathrm{grade}_N(\mathfrak{a})$. Then, there is an N-sequence $(x_1,\ldots,x_r)$ in $\mathfrak{a}$ with $r \geq g$. By Proposition 3.4.8 it follows $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i < r$, so $g \leq r \leq \rho$. This proves $\mathrm{grade}_N(\mathfrak{a}) \leq \rho$.

Now, let $r \leq \rho$. Then $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i < r$. By Proposition 3.4.8 there is an N-sequence $(x_1,\dots,x_r)$ in $\mathfrak{a}$, so $\mathrm{grade}_N(\mathfrak{a}) \geq r$. This proves $\mathrm{grade}_N(\mathfrak{a}) \geq \rho$. □

In other words, the previous Theorem says

$$\mathrm{grade}_N(\mathfrak{a}) = \inf\{i \in \mathbb{N} : \mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0\}.$$

The proofs of the following Corollaries are immediate.

Corollary 3.4.14. Let M and N be two finitely generated R-modules. Let $\rho = \mathrm{grade}_N(\mathfrak{a})$. Then $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i < \rho$.

Corollary 3.4.15. (SUZUKI, 1978, Theorem 2.3) Assume $(R,\mathfrak{m})$ is local, and let M and N be two non-zero finitely generated R-modules and $t = \operatorname{depth} N$. Then $\mathrm{H}^t_{\mathfrak{m}}(M,N) \neq 0$ and $\mathrm{H}^i_{\mathfrak{m}}(M,N) = 0$, for all $i < t$.

Theorem 3.4.16. (BRODMANN; SHARP, 2012, 6.2.8) Assume that $(R,\mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module. Then any integer i for which $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$ must satisfy

$$\operatorname{depth} M \leq i \leq \dim M,$$

while for i at either extremity of this range we have $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$.

Corollary 3.4.17. Assume that $(R,\mathfrak{m})$ is local, and let M be a non-zero, finitely generated R-module. Then there is exactly one integer i for which $H^i_{\mathfrak{m}}(M) \neq 0$ if and only if $\operatorname{depth} M = \dim M$, that is, if and only if M is a Cohen-Macaulay R-module.

Note that we do not have a result such as Corollary 3.4.17 to generalized local cohomology modules, since we do not know when the top generalized local cohomology modules vanish or not.

Next, we establish another vanishing theorem for local cohomology modules, namely the local Lichtenbaum-Hartshorne Vanishing Theorem. While Grothendieck's Vanishing Theorem can be regarded as "algebraic" in nature, the Lichtenbaum-Hartshorne Theorem is of "analytic" nature, in the sense that it is intimately related with "formal" methods and techniques, that is, with passage to completions of local rings and with the structure theory for complete local rings.

Theorem 3.4.18. (Lichtenbaum–Hartshorne Vanishing Theorem, (BRODMANN; SHARP, 2012, 8.2.1)) Suppose that $(R,\mathfrak{m})$ is a commutative Noetherian local ring of dimension n, and also that $\mathfrak{a}$ is a proper ideal of R. Then the following statements are equivalent:

(i) $\mathrm{H}^n_{\mathfrak{a}}(R) = 0$;

(ii) For each (necessarily minimal) prime ideal $\mathfrak{p}$ of $\widehat{R}$, the completion of R, satisfying $\dim \widehat{R}/\mathfrak{p} = n$, we have $\dim \widehat{R}/(\mathfrak{a}\widehat{R} + \mathfrak{p}) > 0$.

Moving on to Artinian modules, the next two results are about the artinianess of local cohomology modules. To generalized local cohomology modules, see Section 5.2.

Theorem 3.4.19. (MACDONALD; SHARP, 1972b) Assume that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module. Then, the R-module $H^i_{\mathfrak{m}}(M)$ is Artinian for all $i \in \mathbb{N}_0$.

Theorem 3.4.20. (SHARP, 1981) Let M be a non-zero, finitely generated R-module such that $\dim M = n$. Then, the R-module $H^n_{\mathfrak{a}}(M)$ is Artinian.

CHAPTER

4

SECONDARY REPRESENTATION AND ATTACHED PRIMES

In this Chapter, we study a theory that can be thought of as a dual of the theory of primary decomposition and associated primes of a module over a commutative ring: secondary representation and attached primes of a module over a commutative ring, respectively.

This is a little known subject which does not have a vast body of literature. Therefore, this led us to compile a very detailed collection of results in Sections 4.1 and 4.2, starting with the definition (Uniqueness Theorem) and proving their existence (Existence Theorem), so that the reader can have a sufficient background to continue. Furthermore, in Section 4.3 we exhibit some examples. Our presentation and treatment of those topics closely follows the one in (MACDONALD, 1973).

Our contribution in this Chapter are Section 4.4, where we prove that some generalized local cohomology modules have secondary representation and we can explicitly calculate the set of those attached primes, and Section 4.5, where we count the number of non-isomorphic top generalized local cohomology modules using the set of those attached primes.

4.1 Uniqueness Theorem and Some Properties

Definition 4.1.1. Let R be a commutative Noetherian ring. Let L be an R-module. A prime ideal $\mathfrak{p}$ of R is said to be *attached* to L if $\mathfrak{p} = (K :_R L) = \operatorname{Ann}_R(L/K)$ for some submodule K of L.

Denote by $\operatorname{Att}_R(L)$ the set of attached prime ideals of the R-module L.

Now, assume R is a commutative ring (not necessarily Noetherian).

Definition 4.1.2. A non-zero R-module L is called *secondary* if for each element $x \in R$, the endomorphism $\varphi_{x,L} : L \to L$ defined by multiplication by x, is either surjective or nilpotent.

Proposition 4.1.3. If an R-module L is secondary, then $\sqrt{\mathrm{Ann}_R(L)}$ is a prime ideal $\mathfrak{p}$.

Proof. Let $a,b \in R$ such that $ab \in \sqrt{\mathrm{Ann}_R(L)}$. Then $(ab)^n L = 0$, for some integer $n > 0$. If $b \notin \sqrt{\mathrm{Ann}_R(L)}$, then $\varphi_{b,L}$ is surjective, that is $b^n L = L$, since L is secondary. Thus $a^n L = a^n(b^n L) = (ab)^n L = 0$. Therefore, $a \in \sqrt{\mathrm{Ann}_R(L)}$ and thus, $\sqrt{\mathrm{Ann}_R(L)}$ is prime. □

In this case, L is said to be $\mathfrak{p}$-*secondary*.

The following Lemmas highlight some properties about $\mathfrak{p}$-secondary modules.

Lemma 4.1.4. Finite direct sums and non-zero quotients of $\mathfrak{p}$-secondary modules are $\mathfrak{p}$-secondary.

Proof. Let $L = L_1 \oplus L_2$ be a non-zero finite direct sum of two $\mathfrak{p}$-secondary modules, that is $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(L_1)} = \sqrt{\mathrm{Ann}_R(L_2)}$. Let $x \in R$ and assume x is not surjective, that is $x(L_1 \oplus L_2) \neq (L_1 \oplus L_2)$ which implies that either $xL_1 \neq L_1$ or $xL_2 \neq L_2$.

Suppose $xL_1 \neq L_1$, then there exists an integer $n > 0$ such that $x^n L_1 = 0$, since L_1 is $\mathfrak{p}$-secondary. Therefore, $x \in \mathfrak{p} = \sqrt{\mathrm{Ann}_R(L_1)}$. But, since $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(L_1)} = \sqrt{\mathrm{Ann}_R(L_2)}$, there exists another integer $m > 0$ such that $x^m \in \mathrm{Ann}_R(L_2)$, that is $x^m L_2 = 0$. Take $k = \max\{n,m\}$. Then $x^k(L_1 \oplus L_2) = 0$. Thus, L is secondary.

Let's show $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(L)}$. From what we have done before, $\mathfrak{p} \subseteq \sqrt{\mathrm{Ann}_R(L)}$. Assume that there exists $x \in \sqrt{\mathrm{Ann}_R(L)}$ such that $x \notin \mathfrak{p}$. Therefore, φ_{x,L_1} and φ_{x,L_2} are always surjective, that is $x^n L_1 = L_1$ and $x^m L_2 = L_2$ for every integer $n,m > 0$. Thus, $x^k L = L$ for every integer $k > 0$, which is a contradiction, since $x \in \sqrt{\mathrm{Ann}_R(L)}$ and $L \neq 0$.

Next, let L be $\mathfrak{p}$-secondary and $\varphi : L \to L' = L/K$ be a natural projection from L to a non-zero quotient of L. Let $x \in R$ such that $\varphi_{x,L}$ is surjective, that is, $xL = L$. Therefore, $xL' = L'$ as $xL + K = L$. On the other hand, assume $\varphi_{x,L}$ is nilpotent, that is, there exists an integer $n > 0$ such that $x^n L = 0_L$. So $x^n L' = 0_{L'}$, since $x^n L + K = K$. Thus, $x \in \mathrm{Ann}_R(L')$. Therefore, we just prove L' is secondary. As before, it is possible to show $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(L')}$. Therefore, L' is $\mathfrak{p}$-secondary. □

Lemma 4.1.5. Let L be an R-module, $\mathfrak{p}$ a prime ideal of R and let $L_1,\ldots,L_r$ be $\mathfrak{p}$-secondary submodules of L. Then, $S = L_1 + \ldots + L_r$ is $\mathfrak{p}$-secondary.

Proof. Since each L_i is non-zero, for $i = 1,\ldots,r$, then S is non-zero. There exists a natural surjection $L_1 \oplus \cdots \oplus L_r \to S$ given by $(l_1,\ldots,l_r) \mapsto l_1 + \cdots + l_r$, so S is a quotient of $L_1 \oplus \cdots \oplus L_r$. Therefore, S is $\mathfrak{p}$-secondary. □

Definition 4.1.6. A *secondary representation* for an R-module L is a finite expression $L = L_1 + L_2 + \ldots + L_s$, where L_i is secondary for $1 \leq i \leq s$. By Lemma 4.1.5, we can assume that $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(L_i)}$ are all distinct and then, the representation is minimal.

We say that the R-module L is *representable* if there exists such an expression.

Definition 4.1.7. If L admits a minimal secondary representation $L = L_1 + L_2 + \ldots + L_s$, then we define the *set of the attached prime ideals* of R as $\text{Att}_R(L) = \{\mathfrak{p}_i = \sqrt{\text{Ann}_R(L_i)} \mid 1 \leq i \leq s\}$.

The minimal elements of $\text{Att}_R(L)$ are said to be *isolated*, and the others *embedded*.

The Uniqueness Theorem that follows shows that the set of the attached prime ideals of R is well defined when the R-module is representable. In other words, it ensures that the set of prime ideals does not depend on the minimal secondary representation of the R-module. Moreover, the equivalence $(i) \Leftrightarrow (v)$ shows that definitions 4.1.1 and 4.1.7 are equivalent, if L is representable.

Theorem 4.1.8. **(Uniqueness Theorem)** Let L be a representable R-module. The set of attached prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, as before, depends only on L and not on the minimal secondary representation. More precisely, the following conditions on a prime ideal $\mathfrak{p}$ are equivalent:

(i) $\mathfrak{p}$ is one of $\mathfrak{p}_i$, for $i = 1, \ldots, n$;

(ii) L has a $\mathfrak{p}$-secondary quotient module;

(iii) L has a quotient Q such that $\sqrt{\text{Ann}_R(Q)} = \mathfrak{p}$;

(iv) L has a quotient Q such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $\text{Ann}_R(Q)$;

(v) Suppose that R is a Noetherian ring. There exists a quotient module L' of L whose annihilator is exactly $\mathfrak{p}$.

With the aim of proving Theorem 4.1.8, we first prove some preliminary results.

First, let $x, y \in R$. Remember that an R-ideal I is primary if $xy \in I$, then either $x \in I$ or $y^n \in I$ for some $n \geq 1$.

Lemma 4.1.9. The annihilator of a $\mathfrak{p}$-secondary module is a $\mathfrak{p}$-primary ideal.

Proof. Let L be a $\mathfrak{p}$-secondary module and $I = \text{Ann}_R(L)$. Let $ab \in I$ and assume $b^n \notin I$, for all $n \in \mathbb{N}$. Since L is secondary and $\varphi_{b,L}$ is not nilpotent, we must have $\varphi_{b,L}$ surjective, that is, $bL = L$. Thus, since $ab \in I = \text{Ann}_R(L)$, $0 = abL = aL$. So $a \in I$. Therefore, I is primary and then $\mathfrak{p}$-primary, since $\mathfrak{p} = \sqrt{\text{Ann}_R(L)} = \sqrt{I}$. □

Lemma 4.1.10. Let L be a representable R-module. The annihilator $\mathfrak{a}$ of L is a decomposable ideal in R, and $\text{Ass}_R(R/\mathfrak{a}) \subset \text{Att}_R(L)$.

Proof. Let $L = L_1 + \cdots + L_s$ be a minimal secondary representation, $\mathfrak{p}_i = \sqrt{\text{Ann}_R(L_i)}$, and $\mathfrak{q}_i = \text{Ann}_R(L_i)$, for $i = 1, \ldots, s$. Then, by Lemma 4.1.9, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal and $\mathfrak{a} = \cap_{i=1}^{s} \mathfrak{q}_i$. Since $\mathfrak{a}$ is the zero ideal in $R/\mathfrak{a}$, the inclusion follows. □

Lemma 4.1.11. Assume L is a representable R-module, and let Q be a quotient of L. Then Q is representable and $\mathrm{Att}_R(Q) \subseteq \mathrm{Att}_R(L)$.

Proof. Let $Q = L/K$, $L = L_1 + \cdots + L_s$ and $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(L_i)}$. Then $Q = \sum_{i=1}^{s}(L_i + K)/K$ and $(L_i + K)/K \cong L_i/(L_i \cap K)$ is either $\mathfrak{p}_i$-secondary or zero, by Lemma 4.1.4. □

Now it is possible to do the proof of Theorem 4.1.8.

Proof. (Uniqueness Theorem)

During the proof, assume $L = L_1 + \cdots + L_s$ is a minimal representation of L, and $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(L_i)}$, for $i = 1, \ldots, s$.

$(i) \Rightarrow (ii)$ Let $K_i = \sum_{j \neq i} L_j$. Then $L/K_i \neq 0$, since the representation of L is minimal, and

$$L/K_i = (K_i + L_i)/K_i \cong L_i/(L_i \cap K_i)$$

is $\mathfrak{p}_i$-secondary, by Lemma 4.1.4.

$(ii) \Rightarrow (iii)$ Immediate.

$(iii) \Rightarrow (i)$ Let $Q = L/K$. Rearrange the L_i such that $L_i \nsubseteq K$ for $1 \leq i \leq r$ and $L_i \subseteq K$ for $r + 1 \leq i \leq s$. Then

$$L/K = \sum_{i=1}^{s}(L_i + K)/K = \sum_{i=1}^{r}(L_i + K)/K,$$

and $(L_i + K)/K \cong L_i/(L_i \cap K)$ is $\mathfrak{p}_i$-secondary for $1 \leq i \leq r$, by Lemma 4.1.4. Hence

$$\mathfrak{p} = \sqrt{\mathrm{Ann}_R(L/K)} = \bigcap_{i=1}^{r} \sqrt{\mathrm{Ann}_R((L_i + K)/K)} = \bigcap_{i=1}^{r} \mathfrak{p}_i,$$

therefore $\mathfrak{p}$ is one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$.

$(iii) \Rightarrow (iv)$ If Q is a quotient module of L such that $\sqrt{\mathrm{Ann}_R(Q)} = \mathfrak{p}$, then $\mathfrak{p}$ is the unique minimal element of the set of prime ideals containing $\mathrm{Ann}_R(Q)$.

$(iv) \Rightarrow (i)$ By Lemma 4.1.11, Q is representable. Hence $\mathfrak{a} = \mathrm{Ann}_R(Q)$ is decomposable, by Lemma 4.1.10, and

$$\mathfrak{p} \in \mathrm{Ass}_R(R/\mathfrak{a}) \subseteq \mathrm{Att}_R(Q) \subseteq \mathrm{Att}_R(L),$$

by Lemmas 4.1.10 and 4.1.11.

$(ii) \Rightarrow (v)$ Let Q be a $\mathfrak{p}$-secondary quotient module of L. Then, by Lemma 4.1.9, $\mathrm{Ann}_R(Q)$ is a $\mathfrak{p}$-primary ideal, hence it contains a power of $\mathfrak{p}$ (since R is Noetherian, by hypothesis on (v)). So $\mathfrak{p}^k Q = 0$, for some integer $k > 0$. Since $Q \neq 0$, it follows that $Q \neq \mathfrak{p}Q$. Hence $L' = Q/\mathfrak{p}Q$ is $\mathfrak{p}$-secondary, by Lemma 4.1.4, and $\mathrm{Ann}_R(L') = \mathfrak{p}$.

$(v) \Rightarrow (iii)$ Since $\mathrm{Ann}_R(L') = \mathfrak{p}$ is a prime ideal, it follows that $\sqrt{\mathrm{Ann}_R(L')} = \mathfrak{p}$. □

The next result gives us information about the elements that belong to the set of attached primes of a representable R-module. This result also provides an Artinian analogue to the fact that if M is a finitely generated module over a commutative Noetherian ring R, and $r \in R$, r is a non-zero divisor on M if and only if r lies outside all the associated prime ideals of M.

Theorem 4.1.12. Let L be a representable R-module and $\mathrm{Att}_R(L) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$, as before. Let $x \in R$. Then

1. $\varphi_{x,L}$ is surjective if and only if $x \notin \bigcup_{i=1}^{s} \mathfrak{p}_i$.
2. $\varphi_{x,L}$ is nilpotent if and only if $x \in \bigcap_{i=1}^{s} \mathfrak{p}_i$.

Proof. 1. If $x \notin \cup_{i=1}^{s} \mathfrak{p}_i$, then $xL_i = L_i$, for all $1 \le i \le s$. Hence $xL = L$.

Conversely, if $x \in \mathfrak{p}_i$, for some $i = 1, \ldots, s$, then $x^r L_i$ for some integer $r > 0$. Thus,

$$x^r L = \sum_{j=1}^{s} x^r L_j \subseteq \sum_{j \neq i} L_j \neq L,$$

and therefore $xL \neq L$, i.e., $\varphi_{x,L}$ is not surjective.

2. $\varphi_{x,L}$ is nilpotent if and only if each φ_{x,L_i} is nilpotent, for $i = 1, \ldots, s$. That is if and only if $x \in \mathfrak{p}_i$, for $i = 1, \ldots, s$. □

In what follows, there will be some properties about the set of the attached primes of an R-module.

Theorem 4.1.13. Let L be a representable R-module and N be a representable submodule of L. Then

$$\mathrm{Att}_R(L/N) \subseteq \mathrm{Att}_R(L) \subseteq \mathrm{Att}_R(N) \cup \mathrm{Att}_R(L/N).$$

Proof. The left-hand inclusion was proved in Lemma 4.1.11. To prove the other inclusion, let $\mathfrak{p} \in \mathrm{Att}_R(L)$ and let L/K be a $\mathfrak{p}$-secondary quotient of L. Consider $Q = K + N$.

If $Q = L$, then

$$L/K = (K+N)/K \cong N/(N \cap K),$$

hence N has a $\mathfrak{p}$-secondary quotient module, and therefore $\mathfrak{p} \in \mathrm{Att}_R(N)$ by the First Uniqueness Theorem.

On the other hand, if $Q \neq L$, then L/Q is a quotient of L/K, and thus it is $\mathfrak{p}$-secondary, by Lemma 4.1.4; but it is also a quotient of L/N, therefore $\mathfrak{p} \in \mathrm{Att}_R(L/N)$. □

Corollary 4.1.14. Let $L_1, \ldots, L_r$ be representable R-modules. Then $L_1 \oplus \cdots \oplus L_r$ is representable, and

$$\mathrm{Att}_R(L_1 \oplus \cdots \oplus L_r) = \bigcup_{i=1}^{r} \mathrm{Att}_R(L_i).$$

Proof. It is clear that the direct sum is representable. The second assertion follows from Theorem 4.1.13, by induction on r. □

Proposition 4.1.15 ((RUSH, 1980, Lemma 1.2))**.** If M is an R-module with $\mathrm{Att}_R(M) = \{\mathfrak{p}\}$ where $\mathfrak{p}$ is a minimal prime of R, then M is secondary.

Proposition 4.1.16. Let L be a representable R-module such that $L = L_1 + \cdots + L_s$ and $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(L_i)}$. For $i = 1, \ldots, s$,

$$\mathrm{Att}_R(L/L_i) = \mathrm{Att}_R(L) - \{\mathfrak{p}_i\}.$$

Proof. Note that

$$L/L_i = \sum_{j \neq i} (L_j + L_i)/L_i$$

and this is a minimal secondary representation, since $(N_j + N_i)/N_i \cong N_j/(N_j \cap N_i)$ is $\mathfrak{p}_i$-secondary. Therefore, the result follows. □

Theorem 4.1.17. Let L be a representable R-module such that $L = L_1 + \cdots + L_s$ and $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R(L_i)}$, for $i = 1, \ldots, s$.

1. L has a composition series

$$L = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = 0$$

in which each N_{i-1}/N_i is secondary.

2. In any such composition series, if $\mathfrak{q}_i = \sqrt{\mathrm{Ann}_R(N_{i-1}/N_i)}$ for $1 \leq i \leq r$, then

$$\mathrm{Att}_R(L) \subseteq \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}.$$

Proof. 1. We may take $N_i = \sum_{j>i} L_j$ (and $r = s$), then

$$N_{i-1}/N_i = (N_i + L_i)/N_i \cong L_i/(L_i \cap N_i)$$

is a non-zero quotient module of L_i, hence it is $\mathfrak{p}_i$-secondary.

2. Applying Theorem 4.1.13 again and again, it follows that

$$\mathrm{Att}_R(L) \subseteq \bigcup_{i=1}^{r} \mathrm{Att}_R(N_{i-1}/N_i) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}.$$

□

4.2 Existence Theorem

Definition 4.2.1. An R-module M is said to be *sum-irreducible* if $M \neq 0$ and the sum of any two proper submodules of M is always a proper submodule.

Lemma 4.2.2. If M is an Artinian R-module and is sum-irreducible, then M is secondary.

Proof. Suppose M is not secondary, then there exists $x \in R$ such that $M \neq xM$ and $x^n M \neq 0$ for all integer $n > 0$.

Consider the sequence $(x^n M)_{n \geq 0}$ of submodules of M. Since M is Artinian, this sequence is stationary, that is, there exists an integer $p > 0$ such that $x^p M = x^{p+1} M = \cdots$. Let $M_1 = \operatorname{Ker}(\varphi_{x^p,M})$ and $M_2 = x^p M$. Then M_1 and M_2 are proper submodules of M, since we are assuming that M is not secondary.

We claim that $M = M_1 + M_2$. Let $u \in M$. Then $x^p u = x^{2p} v$ for some $v \in M$, hence $x^p(u - x^p v) = x^p u - x^{2p} v = 0$, thus $u - x^p v \in M_1$ and therefore $u \in M_1 + M_2$. Therefore, M is not sum-irreducible, which is a contradiction. □

Theorem 4.2.3. **(Existence Theorem)** Every Artinian R-module has a secondary representation.

Proof. Suppose M is an Artinian R-module which is not representable and consider the set Δ of all non-zero submodules of M which are not representable. Note that $\Delta \neq \emptyset$, since $M \in \Delta$.

Since M is Artinian, Δ has a minimal element, say N. Certainly N is not secondary and $N \neq 0$, hence, by Lemma 4.2.2, N is the sum of two strictly smaller submodules N_1 and N_2. By the minimality of N, each one of N_1 and N_2 is representable and, therefore, so is N, which is a contradiction. □

Corollary 4.2.4. If H is an Artinian R-module, then $\operatorname{Att}_R(H)$ is a finite set.

The proof in fact provides us with a representation of an Artinian module M as a sum of sum-irreducible submodules: $M = \sum_{i=1}^{m} S_i$, where each S_i is sum-irreducible submodules of M. If none of the summands S_i redundant for all $i = 1, \ldots, m$, then the representation is said to be minimal.

One question that comes to mind is the following.

Question 4.2.5. What happens if the R-module is Noetherian? Can it be representable?

The next two results answer this question and give us kind of reciprocal of the Existence Theorem. They are important to this theory, but we could not find them in the bibliography. Thus, together with Roger Wiegand, we prove it here.

Theorem 4.2.6. Let M be a Noetherian R-module. If M is secondary, then $\dim M = 0$, i.e., M is an Artinian module.

Proof. Suppose $\dim M > 0$. Therefore there exists two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$ with $M_\mathfrak{p} \neq 0 \neq M_\mathfrak{q}$. Choose $x \in \mathfrak{q} \setminus \mathfrak{p}$. Since M is a secondary R-module, then $xM = M$ or $x^n M = 0$ for some integer $n > 0$.

If $xM = M$, then there exists $r \in R$ such that $(1+rx)M = 0$, since M is a Noetherian module. Note that $1 + rx \notin \mathfrak{q}$ (otherwise $1 \in \mathfrak{q}$, which is not true), thus $M_\mathfrak{q} = 0$ which is a contradiction.

Otherwise, if $xM \neq M$, we must have $x^n M = 0$ for some integer $n > 0$, that is $x^n \in (0 :_R M)$. On the other hand, $\mathfrak{p} \in \mathrm{Supp}_R(M) = \mathrm{V}((0 :_R M))$, since $M_\mathfrak{p} \neq 0$. Hence $x^n \in \mathfrak{p}$ and so $x \in \mathfrak{p}$, which is a contradiction. Therefore, $\dim M = 0$. □

Corollary 4.2.7. Let M be a Noetherian R-module. If M is representable, then $\dim M = 0$.

Proof. Let $M = M_1 + \cdots + M_s$ be a secondary representation of M. By Theorem 4.2.6, $\dim M_i = 0$ for $i = 1, \ldots, s$.

Suppose $\dim M > 0$. Again, as in the previous proof, there exists two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Supp}_R(M)$. Thus, there exists $i, j \in \{1, \ldots, s\}$ such that $(M_i)_\mathfrak{p} \neq 0$ and $(M_j)_\mathfrak{q} \neq 0$. So $(M_i)_\mathfrak{q} \neq 0$, since $(M_i)_\mathfrak{p} \neq 0$ and $\mathfrak{p} \subsetneq \mathfrak{q}$. Therefore, $\dim M_i > 0$, which is a contradiction. □

4.3 Properties and Examples about Attached Primes

In this section, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring.

Let $E := E(R/\mathfrak{m})$ be the injective hull of the simple R-module $R/\mathfrak{m}$. Denote by $D(-) := \mathrm{Hom}_R(-, E)$ the additive, exact, contravariant functor from the category R-modules to itself. The next Theorem, establishes a correspondence between Noetherian modules and Artinian modules via the functor $D(-)$.

Theorem 4.3.1. (**Matlis' Duality**, (OOISHI, 1976, Theorem 1.6))

1. If M is a finitely generated R-module, then $D(M)$ is an Artinian R-module.
2. Assume R is complete (with respect to the $\mathfrak{m}$-adic topology). If H is an Artinian R-module, then $D(H)$ is a finitely generated R-module.
3. If M is a finitely generated R-module, then $D(D(M))$ is isomorphic to $\widehat{M} = M \otimes_R \widehat{R}$.
4. If H is an Artinian R-module, then $D(D(H))$ is isomorphic to H.

The next result is a tool to switch between attached primes of Artinian modules and associated primes of Noetherian modules, using Matlis' Duality.

Proposition 4.3.2. ((OOISHI, 1976, Proposition 2.7)) If M is a finitely generated R-module, then

$$\mathrm{Ass}_R(M) = \mathrm{Att}_R(D(M)).$$

Corollary 4.3.3. If R is complete and H is an Artinian R-module, then

$$\mathrm{Att}_R(H) = \mathrm{Ass}_R(D(H)).$$

Since it is easier to calculate the associated primes of a finitely generated module, using this Corollary 4.3.3 it is possible to know which are the attached primes of an Artinian module H, as in the following example.

Example 4.3.4. Let $R = \mathbb{C}[[x,y]]$ be the formal power series ring in two variables, which is a complete Noetherian local ring with maximal ideal $\mathfrak{m} = (x,y)$. Note that the injective hull of $R/\mathfrak{m}$ is $E = E(R/\mathfrak{m}) = E(\mathbb{C}) = \mathbb{C}$, by Example 2.3.23.

We claim that $\mathbb{C}$ is an Artinian R-module. In fact,

$$\mathbb{C} = E \cong \mathrm{Hom}_R(R,E) = D(R).$$

Since R is a finitely generated R-module, the artinianess of $\mathbb{C}$ follows from Theorem 4.3.1 item 1.

On the other hand, $D(\mathbb{C}) = D(D(R)) \cong R$, since R is complete.

Therefore, by Corollary 4.3.3,

$$\mathrm{Att}_R(\mathbb{C}) = \mathrm{Ass}_R(\mathbb{C}[[x,y]]) = \{(0)\}.$$

Now, lets see some examples about the set of attached primes of the top local cohomology modules.

Example 4.3.5. ((DIBAEI; YASSEMI, 2005a, Theorem A)) Let $\mathfrak{a}$ be an ideal of R and N be a finitely generated R-module such that $\dim N = n$. We already know that $\mathrm{H}^n_{\mathfrak{a}}(N)$ is an Artinian R-module. Furthermore,

$$\mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(N)) = \{\mathfrak{p} \in \mathrm{Ass}_R(N) \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\},$$

where $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0\}$.

Example 4.3.6. ((DIBAEI; YASSEMI, 2005a, Theorem B)) Let $\mathfrak{a}$ be an ideal of R and N be an R-module such that $\dim N = \dim R = n$, then $\mathrm{H}^n_{\mathfrak{a}}(N)$ has a secondary representation and

$$\mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(N)) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R(N) \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\},$$

where $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0\}$.

We will generalize Example 4.3.6 in the next Section 4.4, in Theorem 4.4.3, to generalized local cohomology modules.

The next example is a generalization for Example 4.3.5.

Example 4.3.7. ((GU; CHU, 2009, Proposition 2.2 (i) and Theorem 2.3)) Let $\mathfrak{a}$ be an ideal R and let M and N be two finitely generated R-modules such that $\operatorname{pdim}(M) = d < \infty$, and $\dim N = n < \infty$. Then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{a}$-cofinite, and

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p}) = d+n\},$$

where $\operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p}) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{p}) \neq 0\}$.

In particular, if $\mathfrak{a} = \mathfrak{m}$, then

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \dim R/\mathfrak{p} = n\}.$$

The next Proposition relates the set of attached primes of local cohomology modules and generalized local cohomology modules.

Proposition 4.3.8. (GU; CHU, 2009, Proposition 2.2 (ii)) Let $\mathfrak{a}$ be an ideal R and let M and N be two finitely generated R-modules such that $\operatorname{pdim}(M) = d < \infty$, and $\dim N = n < \infty$. Then,

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Att}_R(\mathrm{H}^{n}_{\mathfrak{a}}(N)).$$

The equality is not true in general, as we can see in (FATHI; TEHRANIAN; ZAKERI, 2015, Example 5.7).

In the next section, there will be more results about the attached primes of the generalized local cohomology modules.

4.4 Results on Attached primes of the top generalized local cohomology modules

In this section, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$.

We have already seen some conditions for generalized local cohomology modules to be representable modules and, even more, Artinian modules. Thus, we can think about its attached primes and try to characterize this set. The main purpose of this section is to prove (DIBAEI; YASSEMI, 2005a, Theorem B), stated in Example 4.3.6, for generalized local cohomology modules. In order to do this, we give some preliminary results.

Lemma 4.4.1. Let M be a finitely generated R-module and $N = \varinjlim_{j\in J} N_j$ another R-module, where $\{N_j \mid j \in J\}$ is a family of finitely generated R-modules. Then

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) = \varinjlim_{j\in J} \mathrm{H}^i_{\mathfrak{a}}(M,N_j), \quad \text{for all } i \in \mathbb{N}_0.$$

Proof. Since M is finitely generated, $M/\mathfrak{a}^nM$ is finitely generated. Then, by (BROWN, 1975, Corollary of Theorem 1), it is possible to commute the direct limit and the functor $\operatorname{Ext}^i_R(M/\mathfrak{a}^nM,-)$. Therefore,

$$\begin{aligned} \mathrm{H}^i_{\mathfrak{a}}(M,N) &= \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^nM,N) \\ &= \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^nM,\varinjlim_{j\in J} N_j) \\ &= \varinjlim_{j\in J}(\varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^nM,N_j)) \\ &= \varinjlim_{j\in J} \mathrm{H}^i_{\mathfrak{a}}(M,N_j). \end{aligned}$$

□

The next Proposition is very technical. We decided to enunciate it separately to clear the proof of Theorem 4.4.3.

Proposition 4.4.2. Let M and N be two R-modules such that M is finitely generated, $\operatorname{pdim} M = d < \infty$, and $\dim N = n < \infty$. Suppose $\operatorname{Ass}_R(L)$ is a finite set and $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L)) \subseteq \operatorname{Ass}_R(L)$ for every submodule L of N (in particular for $L = N$). Moreover, suppose $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \neq \emptyset$. Then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L)$ has a secondary representation, for all R-submodule $L \subseteq N$.

Proof. Let $\operatorname{Ass}_R(N) = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_l\}$. We proceed by induction on l.

If $l = 1$, then $\emptyset \neq \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{p}_1\}$. Therefore, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is a $\mathfrak{p}_1$-secondary R-module, by Proposition 4.1.15.

Now assume $l > 1$. By (BOURBAKI, 1972, Proposition 2 - p.263), for each i, $1 \leq i \leq l$, there exist $L_i \subseteq N$ submodule such that $\operatorname{Ass}_R(L_i) = \{\mathfrak{p}_i\}$ and $\operatorname{Ass}_R(N/L_i) = \operatorname{Ass}_R(N) \setminus \{\mathfrak{p}_i\}$. By hypothesis,

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i)) \subseteq \operatorname{Ass}_R(L_i) = \{\mathfrak{p}_i\} \quad \text{and} \quad \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N/L_i)) \subseteq \operatorname{Ass}_R(N) \setminus \{\mathfrak{p}_i\}.$$

Thus, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i)$ is $\mathfrak{p}_i$-secondary or zero and $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N/L_i)$ has secondary representation by induction.

By the exact sequence

$$\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i) \xrightarrow{\varphi_i} \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N/L_i) \longrightarrow 0,$$

we obtain that $\varphi_i(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i))$ is $\mathfrak{p}_i$-secondary or zero. If $\varphi_i(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i)) = 0$ for some i, then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N/L_i)$ which has secondary representation. Therefore, we may assume $\varphi_i(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i)) \neq 0$ for all $i = 1,\ldots,l$. Hence

$$\begin{aligned} \operatorname{Att}_R\left(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)/\sum_{i=1}^{l}\varphi_i(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i))\right) &\subseteq \bigcap_{i=1}^{l} \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)/\varphi_i(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i))) \\ &= \bigcap_{i=1}^{l} \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L_i)) = \emptyset. \end{aligned}$$

Therefore,

$$\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)=\sum_{i=1}^{l}\varphi_i(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_i))$$

which is a secondary representation for $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$. □

The next Theorem is the main result of this section.

Theorem 4.4.3. Let M and N be two R-modules, where M is finitely generated such that $\operatorname{pdim} M=d<\infty$. Suppose $\dim N=\dim R=n$. Then, $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ has a secondary representation and

$$\operatorname{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))\subseteq\{\mathfrak{p}\in\operatorname{Ass}_R(N)\mid \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p})=d+n\}.$$

Proof. First we show the inclusion of the set of attached primes. We can assume $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)\neq 0$. Let $X=\{\mathfrak{p}\in\operatorname{Ass}_R(N)\mid \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p})=d+n\}$. By (BOURBAKI, 1972, Proposition 2 - p.263), there exists a submodule $L\subseteq N$ such that $\operatorname{Ass}_R(N/L)=X$ and $\operatorname{Ass}_R(L)=\operatorname{Ass}_R(N)\setminus X$.

Consider the exact sequence

$$\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L)\longrightarrow\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)\longrightarrow\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N/L)\longrightarrow\mathrm{H}_{\mathfrak{a}}^{d+n+1}(M,L).$$

Then, $\mathrm{H}_{\mathfrak{a}}^{d+n+1}(M,L)=0$, since $n+1>\dim L$. On the other hand, by Lemma 4.4.1, $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L)=\varinjlim_{j\in I}\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_j)$, where $\{L_j\mid j\in I\}$ is a family of finite submodules of L such that $L=\varinjlim_{j\in I}L_j$.

We claim that $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_j)=0$, for all $j\in I$. If $\dim L_j=l_j<n$, then $d+n>d+l_j$ and therefore $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_j)=0$. Now, if $\dim L_j=n$, then

$$\operatorname{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_j))=\{\mathfrak{p}\in\operatorname{Ass}_R(L_j)\mid \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p})=d+n\}.$$

On the other hand, as L_j is a submodule of L and $\operatorname{Ass}_R(L)=\operatorname{Ass}_R(N)\setminus X$, we have $\operatorname{Ass}_R(L_j)\cap X=\emptyset$. Then, $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L_j)=0$ and we can conclude $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,L)=0$. Therefore, $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)\cong\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N/L)$ and hence we may assume that $L=0$ and $\operatorname{Ass}_R(N)=X$.

Now, we claim $\operatorname{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))\subseteq\operatorname{Ass}_R(N)$. If $r\notin\bigcup_{\mathfrak{p}\in\operatorname{Ass}(N)}\mathfrak{p}$, then the short exact sequence

$$0\longrightarrow N\xrightarrow{\cdot r}N\longrightarrow N/rN\longrightarrow 0$$

induces the exact sequence

$$\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)\xrightarrow{\cdot r}\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)\longrightarrow\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N/rN).$$

Since $\operatorname{cd}(\mathfrak{a},M,N/rN)<n+d$, we have $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N/rN)=0$. Then $r\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)=\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ and $r\notin\bigcup_{\mathfrak{p}\in\operatorname{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))}\mathfrak{p}$. Therefore, $\bigcup_{\mathfrak{p}\in\operatorname{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))}\mathfrak{p}\subseteq\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R(N)}\mathfrak{p}$.

We claim that $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(R)$, and therefore $|\mathrm{Ass}_R(N)| < \infty$. To see this, let $\mathfrak{p} \in \mathrm{Ass}_R(N)$ then

$$n+d = \mathrm{cd}(\mathfrak{a}, M, R/\mathfrak{p}) \leq \mathrm{pdim}(M) + \dim R/\mathfrak{p} \leq \mathrm{pdim}(M) + \dim R = n+d,$$

which implies $\dim R/\mathfrak{p} = n$. Thus $\mathfrak{p}$ is minimal over the ideal 0 and $\mathfrak{p} \in \mathrm{Ass}_R(R)$, which is a finite set, since R is Noetherian.

If now $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N))$, then $\mathfrak{p} \subseteq \mathfrak{q}$, for some $\mathfrak{q} \in \mathrm{Ass}_R(N)$. On the other hand, since

$$\mathrm{cd}(\mathfrak{a}, M, R/\mathfrak{p}) \leq \mathrm{pdim}(M) + \dim R/\mathfrak{p} \leq \mathrm{pdim}(M) + \dim R = d+n$$

and

$$n+d = \mathrm{cd}(\mathfrak{a}, M, R/\mathfrak{q}) \leq \mathrm{cd}(\mathfrak{a}, M, R/\mathfrak{p}) \leq n+d$$

we have $\mathfrak{p} = \mathfrak{q}$. Therefore, $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \mathrm{Ass}_R(N)$.

The secondary representation of $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ follows immediately from Proposition 4.4.2 and the first part of this proof. □

The inclusion shown in the previous Theorem is strict in general, see the following example. But, if N is finitely generated the equality holds, it was shown in (GU; CHU, 2009, Theorem 2.3).

Example 4.4.4. ((DIBAEI; YASSEMI, 2005a, Example 6)) Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring with $\dim R = n > 0$. Choose a prime ideal $\mathfrak{p}$ of R such that $\dim R/\mathfrak{p} = d > 0$. We have $\mathrm{cd}(\mathfrak{m}, R/\mathfrak{p}) = d$, so $\mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Ass}_R(N) \mid \mathrm{cd}(\mathfrak{a}, M, R/\mathfrak{p}) = d+n\}$. But, since $E(R/\mathfrak{p})$ is Artinian (by Theorem 2.3.24), $\mathrm{H}^d_{\mathfrak{m}}(E(R/\mathfrak{p})) = 0$ and $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(E(R/\mathfrak{p}))) = \emptyset$.

4.5 Application: counting the number of non-isomorphic top generalized local cohomology modules

In this part, we also assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring, and M and N are two finitely generated R-modules. Suppose $\dim R = r$, $\dim N = n < \infty$, and $\mathrm{pdim}\, M = d < \infty$.

The next result can be seen as an extension of (DIBAEI; YASSEMI, 2005b, Theorem 1.5).

Proposition 4.5.1. Let L be an R-module such that $\dim L = n$. Suppose $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,L) \neq 0$ and $\mathfrak{a} \subseteq \mathfrak{b}$. Then there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{b}}(M,L) \to \mathrm{H}^{d+n}_{\mathfrak{a}}(M,L)$.

Proof. We may assume that $\mathfrak{a} \neq \mathfrak{b}$ and choose $x \in \mathfrak{b} \setminus \mathfrak{a}$. By (DIVAANI-AAZAR; HAJIKARIMI, 2011, Lemma 3.1), there is an exact sequence

$$\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M,L) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,L) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x,L_x),$$

where L_x is the localization of L at $\{x^i \mid i \geq 0\}$. Note that $\dim_{R_x}(L_x) < n$ and so $\mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x, N_x) = 0$. Thus, there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M,N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$. Now the assertion follows by assuming $\mathfrak{b} = \mathfrak{a} + (x_1, \ldots, x_r)$ and applying the argument for finite steps. □

Based on this elementary Proposition, one can ask Question 1.0.2, which we enunciate again.

Question 4.5.2. Let $\operatorname{pdim} M = d < \infty$ and $\dim N = n < \infty$. Is it possible to count the number of non-isomorphic top local cohomology modules $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$?

The purpose of this section is to study this question. The results that we obtained here extends some of the results in (DIBAEI; JAFARI, 2008).

Define the following notation to the sets

$$\operatorname{Assh}(M,N) := \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) \text{ and } \operatorname{Assr}(M,N) := \operatorname{Att}_R(\mathrm{H}^{r}_{\mathfrak{m}}(M,N)).$$

Theorem 4.5.3 ((GU; CHU, 2009, Theorem 3.3))**.** If $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) = \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{b}}(M,N))$, then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{b}}(M,N)$.

The following Proposition is analogous to Theorem 4.5.3 and the proof is similar, so we omit it here.

Proposition 4.5.4. Assume $\operatorname{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Ass}_R(N)$. If $\operatorname{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N)) = \operatorname{Att}_R(\mathrm{H}^{r}_{\mathfrak{b}}(M,N))$, then $\mathrm{H}^{r}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{r}_{\mathfrak{b}}(M,N)$.

The following result is an easy consequence of Theorem 4.5.3, just noticing that $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Assh}(M,N)$ (by Proposition 4.5.1).

Proposition 4.5.5. The number of non-isomorphic top generalized local cohomology modules $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is less than or equal to $2^{|\operatorname{Assh}(M,N)|}$, where $\mathfrak{a}$ is an ideal of R.

This Proposition leads us to ask the following:

Question 4.5.6. When does the equality hold?

When $M = R$ and $(R, \mathfrak{m})$ is a commutative Noetherian local ring complete with respect to the $\mathfrak{m}$-adic topology, it has been proved in (DIBAEI; JAFARI, 2008, Corollary 2.9).

Our aim from now on is to answer this question when M is not necessarily equal to R. The answer from Question 4.5.6 when M is not necessarily equal to R can be found in the Corollary 4.5.17.

Corollary 4.5.7. Assume that $|\operatorname{Assh}(M,N)| = 1$. For any ideal $\mathfrak{a}$ of R, if $\operatorname{cd}(\mathfrak{a}, M, N) = d+n$, then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$, in particular $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is an Artinian R-module.

Proof. Note that $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \neq 0$, since $\mathrm{cd}(\mathfrak{a},M,N) = d+n$. Moreover, $\mathrm{H}_{\mathfrak{m}}^{d+n}(M,N) \neq 0$. As the number of non-isomorphic top generalized local cohomology modules $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ is not greater than 1, we have the result. □

The next result will be a key point of the main result of this section.

Proposition 4.5.8. Assume that $n \geq 1$ and $T \subset \mathrm{Assh}(M,N)$ is a non-empty set. Set $\mathrm{Assh}(M,N) \setminus T = \{\mathfrak{q}_1,\ldots,\mathfrak{q}_r\}$. If

(i) there exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) = T$,

then

(ii) for each $1 \leq i \leq r$, there exists $Q_i \in \mathrm{Supp}_R(N)$ such that $\dim(R/Q_i) = 1$ with $\bigcap_{\mathfrak{p}\in T}\mathfrak{p} \nsubseteq Q_i$ and $\mathfrak{q}_i \subseteq Q_i$.

Proof. As $T = \mathrm{Att}_R((\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) \subseteq \mathrm{Att}_R((\mathrm{H}_{\mathfrak{a}}^{n}(N))$, by Proposition 4.3.8, then $\mathrm{H}_{\mathfrak{a}}^{n}(R/\mathfrak{p}) \neq 0$, for all $\mathfrak{p} \in T$. By the Lichtenbaum-Hartshorne Theorem (3.4.18), this is equivalent to say that $\mathfrak{a}+\mathfrak{p}$ is a $\mathfrak{m}$-primary ideal, for all $\mathfrak{p} \in T$.

On the other hand, for each $1 \leq i \leq r$, $\mathfrak{q}_i \notin T$, which is equivalent to say that $\mathfrak{a}+\mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal (again by the Lichtenbaum-Hartshorne Theorem, 3.4.18). Then, there exists a prime ideal $Q_i \in \mathrm{Supp}_R(N)$ such that $\dim(R/Q_i) = 1$ and $\mathfrak{a}+\mathfrak{q}_i \subseteq Q_i$. In this case, we have $\bigcap_{\mathfrak{p}\in T}\mathfrak{p} \nsubseteq Q_i$. □

Now, consider the following sentence:

(i') There exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) \subseteq T$.

In the previous hypothesis, we can show $(ii) \Rightarrow (i')$, where $\mathfrak{a} = \bigcap_{i=1}^{r} Q_i$.

Proof. Set $\mathfrak{a} = \bigcap_{i=1}^{r} Q_i$. For each $1 \leq i \leq r$, $\mathfrak{a}+\mathfrak{q}_i \subseteq Q_i$ is equivalent to say that $\mathfrak{a}+\mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal, hence by the Lichtenbaum-Hartshorne Theorem (3.4.18), $\mathrm{H}_{\mathfrak{a}}^{n}(R/Q_i) = 0$. Then $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{n}(N)) \subseteq T$. Therefore, $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) \subseteq T$. □

Corollary 4.5.9. If $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \neq 0$, then there is an ideal $\mathfrak{b} \subseteq R$ such that $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \cong \mathrm{H}_{\mathfrak{b}}^{d+n}(M,N)$ and $\dim R/\mathfrak{b} \leq 1$.

Proof. If $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) = \mathrm{Assh}(M,N)$, then $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \cong \mathrm{H}_{\mathfrak{m}}^{d+n}(M,N)$ by Theorem 4.5.3.

Otherwise, $d+n \geq 1$ and $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))$ is a proper subset of $\mathrm{Assh}(M,N)$. Set $\mathrm{Assh}(M,N) \setminus \mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) := \{\mathfrak{q}_1,\ldots,\mathfrak{q}_r\}$. By the proof of Proposition 4.5.8, for each $1 \leq i \leq r$, there exists $Q_i \in \mathrm{Supp}_R(M)$ with $\dim(R/Q_i) = 1$ such that $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) =$

$\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{b}}(M,N))$, where $\mathfrak{b} = \bigcap_{i=1}^{r} Q_i$. Again, by Theorem 4.5.3, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{b}}(M,N)$. Since $\dim R/\mathfrak{m} = 0$ and $\dim R/\mathfrak{b} = 1$, we have the result. □

4.5.1 The Cohen-Macaulay case

Definition 4.5.10. Let R be a Noetherian local ring. A finitely generated R-module M is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring* . A *maximal Cohen-Macaulay module* is a Cohen-Macaulay module M such that $\dim M = \dim R$.

In this part, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian Cohen-Macaulay local ring, complete with respect to the $\mathfrak{m}$-adic topology, such that $\dim R = r$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. We also assume that M and N are two finitely generated R-modules such that $\operatorname{pdim} M = d < \infty$ and $\dim N = n < \infty$.

Note that $\mathrm{H}^{r}_{\mathfrak{a}}(M,N)$ is an Artinian R-module for all R-ideal $\mathfrak{a}$, by (DIVAANI-AAZAR; HAJIKARIMI, 2011, Theorem 3.8), and $\mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \mathrm{Supp}_R(N) \cap \mathrm{Ass}_R(M) \mid \dim(R/\mathfrak{a}+\mathfrak{p}) = 0\}$, by Theorem 3.9 of the same paper.

Proposition 4.5.11. Assume $r \geq 1$ and T is a proper non-empty subset of $\mathrm{Assr}(M,N)$. Set $\mathrm{Assr}(M,N) \setminus T = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$. The following statements are equivalent:

(i) There exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N)) = T$;

(ii) For each $1 \leq i \leq s$, there exists $Q_i \in \mathrm{Supp}_R(N)$ such that $\dim(R/Q_i) = 1$ with $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$ and $\mathfrak{q}_i \subseteq Q_i$.

Where $\mathfrak{a} = \bigcap_{i=1}^{s} Q_i$.

Proof. $(i) \Rightarrow (ii)$ Since $T = \{\mathfrak{p} \in \mathrm{Supp}_R(N) \cap \mathrm{Ass}_R(M) \mid \dim(R/\mathfrak{a}+\mathfrak{p}) = 0\} \neq \emptyset$, $\mathfrak{a}+\mathfrak{p}$ is a $\mathfrak{m}$-primary ideal, for all $\mathfrak{p} \in T$.

On the other hand, by the same argument, $\mathfrak{a}+\mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal, since $\mathfrak{q}_i \notin T$, for all $1 \leq i \leq s$. Then there exists $Q_i \in \mathrm{Supp}_R(N)$ such that $\dim(R/Q_i) = 1$ and $\mathfrak{a}+\mathfrak{q}_i \subseteq Q_i$. Therefore, $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$.

$(ii) \Rightarrow (i)$ Set $\mathfrak{a} = \bigcap_{i=1}^{s} Q_i$. For all $1 \leq i \leq s$, $\mathfrak{a}+\mathfrak{q}_i \subseteq Q_i$, then $\mathfrak{a}+\mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal. Thus, $\mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N)) \subseteq T$.

On the other hand, take $\mathfrak{p} \in T \subseteq \mathrm{Assr}(M,N) = \mathrm{Supp}_R(N) \cap \mathrm{Ass}_R(M)$ and let $Q \in \mathrm{Supp}_R(N)$ such that $\mathfrak{a}+\mathfrak{p} \subseteq Q$. Then there exists $i \in \{1,\ldots,s\}$ such that $Q_i \subseteq Q$. Notice that $\mathfrak{p} \nsubseteq Q_i$, since $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$, then $Q_i \neq Q$. Thus, $Q = \mathfrak{m}$. So $\mathfrak{a}+\mathfrak{p}$ is a $\mathfrak{m}$-primary ideal then $\dim(R/\mathfrak{a}+\mathfrak{p}) = 0$. Therefore, $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{a}}(M,N))$. □

Corollary 4.5.12. If $r=1$, then any subset of $\mathrm{Assr}(M,N)$ is equal to $\mathrm{Att}_R(\mathrm{H}^1_{\mathfrak{a}}(M,N))$, for some ideal $\mathfrak{a}$ of R.

Proof. In the notation of the Proposition 4.5.11, it is enough to take $Q_i=\mathfrak{q}_i$, for $i=1,\ldots,s$. □

Lemma 4.5.13. Assume that $r\geq 2$. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be ideals of R. Then there exists an ideal $\mathfrak{b}$ of R such that $\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{b}}(M,N))=\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{a}_1}(M,N))\cap\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{a}_2}(M,N))$.

Proof. Consider $T_1=\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{a}_1}(M,N))$ and $T_2=\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{a}_2}(M,N))$. Suppose $T_1\cap T_2$ is a non-empty proper subset of $\mathrm{Assr}(M,N)$. Let $\mathfrak{q}\in\mathrm{Assr}(M,N)\setminus(T_1\cap T_2)=(\mathrm{Assr}(M,N)\setminus T_1)\cup(\mathrm{Assr}(M,N)\setminus T_2)$. By Proposition 4.5.11, there exists $Q\in\mathrm{Supp}_R(N)$ such that $\dim(R/Q)=1$ with $\mathfrak{q}\subseteq Q$ and $\bigcap_{\mathfrak{p}\in T_1}\mathfrak{p}\nsubseteq Q$ or $\bigcap_{\mathfrak{p}\in T_2}\mathfrak{p}\nsubseteq Q$. Thus, there exists Q satisfying the above conditions such that $\bigcap_{\mathfrak{p}\in T_1\cap T_2}\mathfrak{p}\nsubseteq Q$. Therefore, again by Proposition 4.5.11, there exists an ideal $\mathfrak{b}$ of R such that $\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{b}}(M,N))=T_1\cap T_2$. □

Lemma 4.5.14. Assume that $r=d+n\geq 2$. Let T be a non-empty subset of $\mathrm{Assr}(M,N)=\mathrm{Assh}(M,N)$ and $T'=\mathrm{Assh}(M,N)\setminus T=\{\mathfrak{q}\}$. If $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq\bigcap_{\mathfrak{b}\in\mathrm{Assh}(M,R/\mathfrak{p})}\mathfrak{b}$, then there exists a prime ideal $Q\in\mathrm{Supp}_R(N)$ such that $\dim(R/Q)=1$ and $\mathrm{Att}_R(\mathrm{H}^{d+n}_Q(M,N))=T$.

Proof. First note that $\dim R/\mathfrak{q}=d+n=r$, since $\mathfrak{q}\in T'\subset\mathrm{Assh}(M,N)$. We prove by induction in j $(0\leq j\leq r-1)$ that there exists a chain of prime ideals $Q_0\subset Q_1\subset\cdots\subset Q_j\subset\mathfrak{m}$ such that $Q_0\in\mathrm{Assh}(M,R/\mathfrak{q})$, $\dim(R/Q_j)=r-j$ and $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq Q_j$.

Now, note that there exists $Q_0\in\mathrm{Assh}(M,R/\mathfrak{q})$ such that $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq Q_0$, since $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq\bigcap_{\mathfrak{b}\in\mathrm{Assh}(M,R/\mathfrak{q})}\mathfrak{b}$, and $\dim(R/Q_0)=\dim R/\mathfrak{q}=r$.

Let $0<j\leq r-1$ and suppose we already prove the existence of the chain of prime ideals $Q_0\subset Q_1\subset\cdots\subset Q_{j-1}$ such that $Q_0\in\mathrm{Assh}(M,R/\mathfrak{q})$, $\dim(R/Q_j)=r-(j-1)$ and $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq Q_{j-1}$. Note that $r-j+1\geq 2$, since $r-j-1\geq 0$. Thus, the set

$$V=\{\mathfrak{q}\in\mathrm{Supp}_R(N)\mid Q_{j-1}\subset\mathfrak{q}\subset\mathfrak{q}'\subset\mathfrak{m},\ \dim R/\mathfrak{q}=r-j,$$

$$\mathfrak{q}'\in\mathrm{Spec}(R),\ \dim R/\mathfrak{q}'=r-j-1\}$$

is non-empty and therefore, by Ratliff's weak existence Theorem ((MATSUMURA, 1989, Theorem 31.2)), is infinite.

Since $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq Q_{j-1}$, we have that $Q_{j-1}\subset Q_{j-1}+\bigcap_{\mathfrak{p}\in T}\mathfrak{p}$. If $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\subseteq\mathfrak{q}$ for $\mathfrak{q}\in V$, then $\mathfrak{q}$ is a minimal prime ideal of $Q_{j-1}+\bigcap_{\mathfrak{p}\in T}\mathfrak{p}$. Thus, there is $Q_j\in V$ such that $\bigcap_{\mathfrak{p}\in T}\mathfrak{p}\nsubseteq Q_j$, since V is an infinite set and the amount of minimal primes of $Q_{j-1}+\bigcap_{\mathfrak{p}\in T}\mathfrak{p}$ is finite. Therefore, taking $Q:=Q_{r-1}$, by Proposition 4.5.11, the result is as follows. □

Corollary 4.5.15. Assume that $r=d+n\geq 2$. If $T\subseteq\mathrm{Assh}(M,N)$ is non-empty and $T'=\mathrm{Assh}(M,N)\setminus T=\{\mathfrak{q}\}$, then there is an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N))=T$.

Proof. Define, for any prime ideal $\mathfrak{b} \in \operatorname{Spec}(R)$,

$$\operatorname{ht}_N(\mathfrak{b}) = \inf\{s \mid P_0 \subseteq P_1 \subseteq \cdots \subseteq P_s \subseteq \mathfrak{b}, \text{ with } P_i \in \operatorname{Supp}_R(N)\}.$$

Note that $\operatorname{ht}_N(\mathfrak{q}) = 0$, since $\mathfrak{q} \in \operatorname{Ass}_R(N) \subseteq \operatorname{Supp}_R(N)$. Thus $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq \mathfrak{q} \subseteq \bigcap_{\mathfrak{b}\in \operatorname{Assh}(M,R/\mathfrak{p})} \mathfrak{b}$. Therefore, by Lemma 4.5.14, the result follows. □

Now we are able to show the main result of this section.

Theorem 4.5.16. Assume that $r = d+n$. If $T \subseteq \operatorname{Assh}(M,N)$, then there exists an ideal $\mathfrak{a}$ of R such that $T = \operatorname{Att}_R(\operatorname{H}^{d+n}_{\mathfrak{a}}(M,N))$.

Proof. By Corollary 4.5.12, we may assume $r \geq 2$ and T is a non-empty proper subset of $\operatorname{Assh}(M,N)$. Let $T = \{\mathfrak{p}_1,\dots,\mathfrak{p}_t\}$ and $\operatorname{Assh}(M,N)\setminus T = \{\mathfrak{p}_{t+1},\dots,\mathfrak{p}_{t+s}\}$. We use induction over s. For $s = 1$, the result holds by Corollary 4.5.15.

Suppose that $s > 1$ and the result holds for the case $s-1$. Consider $T_1 = \{\mathfrak{p}_1,\dots,\mathfrak{p}_t,\mathfrak{p}_{t+1}\}$ and $T_2 = \{\mathfrak{p}_1,\dots,\mathfrak{p}_t,\mathfrak{p}_{t+2}\}$. By induction, there exists ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of R such that $T_1 = \operatorname{Att}_R(\operatorname{H}^{d+n}_{\mathfrak{a}_1}(M,N))$ and $T_2 = \operatorname{Att}_R(\operatorname{H}^{d+n}_{\mathfrak{a}_2}(M,N))$. Therefore, by Lemma 4.5.13, there exists an ideal $\mathfrak{a}$ of R such that $T = T_1 \cap T_2 = \operatorname{Att}_R(\operatorname{H}^{d+n}_{\mathfrak{a}}(M,N))$. □

As an immediate consequence of Theorem 4.5.16, we have the main result of this section which counts the number of non-isomorphic top generalized local cohomology modules and answers Question 1.0.2.

Corollary 4.5.17. The number of non-isomorphic top local cohomology modules $\operatorname{H}^{d+n}_{\mathfrak{a}}(M,N)$ is equal to $2^{|\operatorname{Assh}(M,N)|}$, when $r = d+n$.

CHAPTER

5

ON FINITENESS PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY MODULES AND OF TORSION AND EXTENSION FUNCTORS

Let R be a commutative Noetherian ring, $\mathfrak{a}$ be an ideal of R, and M and N be two R-modules.

In this Chapter, we provide some answers on the extension of Hartshorne's conjecture about the cofiniteness of torsion product and extension functors, Questions 1.0.3 and 1.0.4 in the Introduction.

For this purpose, in Section 5.1 we establish some definitions about finiteness of an R-module: $\mathfrak{a}$-cofinite, minimax, $\mathfrak{a}$-cominimax, weakly Laskerian, and $\mathfrak{a}$-weakly cofinite, showing a connection between them. Besides, we explore some properties about a Serre category and we reached Theorem 5.1.16, which is a very interesting result when it comes to this subject. Furthermore, we define a new class of modules, called $\mathfrak{a}$-weakly finite modules.

In Sections 5.2 and 5.3, we study the artinianess and $\mathfrak{a}$-cofiniteness of (generalized) local cohomology modules, in the local and non-local case, and the behavior of extension and torsion product functors when it comes to finiteness situation. The most important results are Theorem 5.2.4 and Theorem 5.2.7 in Section 5.2, and Theorem 5.3.4, Theorem 5.3.9 and Theorem 5.3.14, in Section 5.3.

For conventions of notation, basic results, and terminology not given here, the reader should consult the books (MATSUMURA, 1989) and (BRODMANN; SHARP, 2012).

The results presented in this Chapter can be found in (FREITAS; JORGE PÉREZ; MERIGHE, 2018).

5.1 Some definitions

In this section, we fix our notation and remember some definitions, for the convenience of the reader.

Definition 5.1.1. ((HARTSHORNE, 1970)) An R-module N is said to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is a finitely generated module for all $i \in \mathbb{N}_0$.

Note that every finitely generated R-module N such that $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ is $\mathfrak{a}$-cofinite.

Definition 5.1.2. ((ZÖSCHINGER, 1986)) An R-module M is called *minimax*, if there is a finitely generated submodule N of M such that M/N is Artinian.

Note that Noetherian modules are minimax, as are Artinian modules.

Proposition 5.1.3. Let M be a minimax R-module. If there is $s \in \mathbb{N}$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ maximal ideals of R such that $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_s M = 0$, then M has finite length.

Proof. Consider the short exact sequence $0 \to L \to M \to K \to 0$, where L is finitely generated and K is an Artinian R-module. Note that $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_s L = 0$, then L is Artinian, since we already know that L is finitely generated. Thus, M is an Artinian module and $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_s M = 0$. Therefore, M has finite length. □

Remark 5.1.4. The set of associated primes of any minimax R-module is finite.

To prove Remark 5.1.4, we will need the following Lemma.

Lemma 5.1.5. Let H be an Artinian R-module.

1. $\mathrm{Supp}_R(H) \subseteq \text{m-Spec}(R)$;
2. $\mathrm{Min}_R(H) = \mathrm{Ass}_R(H) = \mathrm{Supp}_R(H)$;
3. The support of H is a finite set.

Proof. 1. Assume there exists $\mathfrak{p} \in \mathrm{Supp}_R(H) \setminus \text{m-Spec}(R)$. Since $H_\mathfrak{p} \neq 0$, let $a \in H$ be an element such that its image under the natural map $H \to H_\mathfrak{p}$ is non-zero. Then $\mathrm{Ann}(a) \subseteq \mathfrak{p}$. Thus, there is a surjective map $aR \twoheadrightarrow R/\mathfrak{p}$, and then $R/\mathfrak{p}$ is an homomorphic image of a submodule of an Artinian module. Therefore, $R/\mathfrak{p}$ is Artinian.

However, since $\mathfrak{p} \notin \text{m-Spec}(R)$, $R/\mathfrak{p}$ is a ring of positive dimension. Thus, it cannot be Artinian. Therefore, from this contradiction, it follows that $\mathrm{Supp}_R(H) \subseteq \text{m-Spec}(R)$.

2. By item 1, each $\mathfrak{m} \in \operatorname{Supp}_R(H)$ is both maximal and minimal in $\operatorname{Supp}_R(H)$. Thus, $\operatorname{Supp}_R(H) \subseteq \operatorname{Min}_R(H)$.

On the other hand, the inclusions $\operatorname{Min}_R(H) \subseteq \operatorname{Ass}_R(H) \subseteq \operatorname{Supp}_R(H)$ hold for all modules, see (MATSUMURA, 1989, Theorem 6.5 (ii) and (iii)) (the proof of which only uses that the module is finitely generated for part (i)).

Therefore, $\operatorname{Min}_R(H) = \operatorname{Ass}_R(H) = \operatorname{Supp}_R(H)$.

3. By item 2, for each $\mathfrak{m}_i \in \operatorname{Supp}_R(H)$, we have $\mathfrak{m}_i \in \operatorname{Ass}_R(H)$. Hence, there is a submodule $H_i \subseteq H$ such that $H_i \cong R/\mathfrak{m}_i$. Let

$$H' := \sum_{\mathfrak{m}_i \in \operatorname{Supp}_R(H)} H_i \cong \bigoplus_{\mathfrak{m}_i \in \operatorname{Supp}_R(H)} R/\mathfrak{m}_i.$$

Since H is an Artinian R-module and H' is a submodule of H, then H' is an Artinian R-module and then this direct sum must be finite. Therefore, $\operatorname{Supp}_R(H)$ is a finite set. □

Now, let's prove Remark 5.1.4.

Proof. Let L be a minimax R-module. Then, there is an exact sequence $0 \to M \to L \to H \to 0$ such that M is a finitely generated R-module and H is an Artinian R-module. Hence, by the exact sequence, $\operatorname{Ass}_R(L) \subseteq \operatorname{Ass}_R(M) \cup \operatorname{Ass}_R(H)$. Therefore, since $\operatorname{Ass}_R(M)$ and $\operatorname{Ass}_R(H)$ are finite sets, it follows that $\operatorname{Ass}_R(L)$ is finite. □

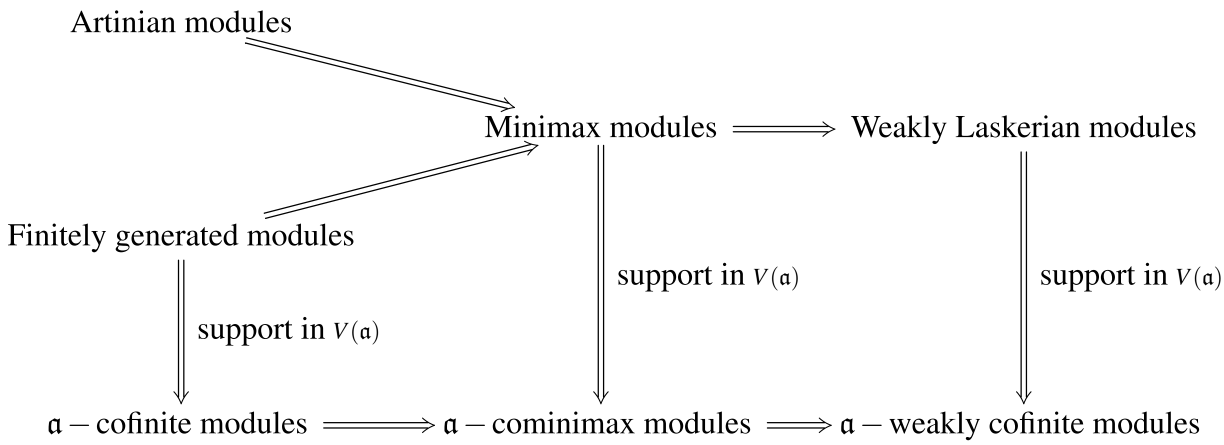
Definition 5.1.6. ((AZAMI; NAGHIPOUR; VAKILI, 2009)) An R-module M is said to be $\mathfrak{a}$-*cominimax* if $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is minimax for all $j \in \mathbb{N}_0$.

Definition 5.1.7. ((DIVAANI-AAZAR; MAFI, 2005)) An R-module M is said to be *weakly Laskerian*, if the set of associated primes of any quotient module of M is finite.

In particular, by Remark 5.1.4, minimax R-modules are weakly Laskerian R-modules.

Definition 5.1.8. ((DIVAANI-AAZAR; MAFI, 2006)) An R-module M is said to be $\mathfrak{a}$-*weakly cofinite* if $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $j \in \mathbb{N}_0$.

The next diagram shows the correspondence between all those definitions.



Where the downward arrows follow from Proposition 5.1.10.

5.1.1 Serre subcategory of the category of R-modules

A subcategory $\mathscr{S}$ of the category of R-modules is said to be a *Serre category* if it is a class of R-modules closed under taking submodules, quotients and extensions. In other words, if in any short exact sequence of R-modules and R-homomorphisms, the middle module is in $\mathscr{S}$ if and only if the two other modules are in $\mathscr{S}$. The classes of Noetherian modules, of Artinian modules, of minimax modules, of weakly Laskerian, and of Matlis reflexive are examples of Serre subcategories.

On the other hand, the classes of free modules, of $\mathfrak{a}$-cofinite modules, of projective modules, and of representable modules are not examples of Serre subcategories. In all of these examples, their submodules do not preserve the property.

Here, we will give some results about Serre's categories that will be used in the following sections.

Lemma 5.1.9 ((MELKERSSON, 2005, Corollary 4.4))**.** The class of Artinian $\mathfrak{a}$-cofinite is a Serre subcategory of the category of R-modules.

Proposition 5.1.10. Let $\mathscr{S}$ denote a Serre subcategory of the category of R-modules. Let M be a finitely generated R-module and $N \in \mathscr{S}$ any R-module. Then for all integer $i \geq 0$, $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}^R_i(M,N)$ are elements of $\mathscr{S}$.

Proof. The proof follows from the definition of extension and torsion product functors. □

Lemma 5.1.11. Let M be an R-module such that $M/\mathfrak{a}M \in \mathscr{S}$. Then $M/\mathfrak{a}^n M \in \mathscr{S}$ for all $n \in \mathbb{N}$.

Proof. We use induction on n.

If $n = 1$, then it is true by hypothesis. Now let $n > 1$ and suppose the result is true for $n-1$. Since $M/\mathfrak{a}^{n-1}M \in \mathscr{S}$, $(M/\mathfrak{a}^{n-1}M)^k \in \mathscr{S}$, for all $k \in \mathbb{N}_0$. There is an exact sequence

$$(M/\mathfrak{a}^{n-1}M)^t \xrightarrow{f} M/\mathfrak{a}^n M \xrightarrow{g} M/\mathfrak{a}M \longrightarrow 0$$

where $\mathfrak{a} = (x_1, \ldots, x_t)$ and

$$f(m_1 + \mathfrak{a}^{n-1}M, \ldots, m_t + \mathfrak{a}^{n-1}M) = x_1 m_1 + \ldots + x_t m_t + \mathfrak{a}^n M.$$

Therefore, $M/\mathfrak{a}^n M \in \mathscr{S}$. □

Let $\underline{x} = x_1, \ldots, x_t$. In the next two Theorems we will use Koszul complexes, $K_\bullet(\underline{x})$; Koszul homology, $\mathrm{H}_q(\underline{x}; M) = \mathrm{H}_q(K_\bullet(\underline{x}) \otimes_R M)$; and Koszul cohomology, $\mathrm{H}^q(\underline{x}; M) = \mathrm{H}^q(\mathrm{Hom}_R(K_\bullet(\underline{x}), M))$. The reader can refer to (WEIBEL, 1995) for more details.

Theorem 5.1.12. Let M be an R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) \in \mathscr{S}$ for all $i \in \mathbb{N}_0$. Then $M/\mathfrak{a}^n M \in \mathscr{S}$ for all $n \in \mathbb{N}$.

Proof. In view of Lemma 5.1.11, it is enough to prove that $M/\mathfrak{a}M \in \mathscr{S}$.

Let $\mathfrak{a} = (x_1, \ldots, x_t)$ and $\underline{x} = x_1, \ldots, x_t$. Then $M/\mathfrak{a}M \cong \mathrm{H}^t(\underline{x}; M)$ and $\mathrm{H}^j(\underline{x}; M) = Z^j/B^j$, where B^j and Z^j are the modules of coboundaries and cocycles of the complex $\mathrm{Hom}_R(K_\bullet(\underline{x}), M)$, respectively, and where $K_\bullet(\underline{x})$ is the Koszul complex on $\underline{x}$.

Put

$$\mathscr{C} = \{N \mid \mathrm{Ext}^i_R(R/\mathfrak{a}, N) \in \mathscr{S} \text{ for all } i \in \mathbb{N}_0\}.$$

Our claim is that $B^j \in \mathscr{C}$ for all $j = 0, 1, \ldots, t$. We prove this by induction on j.

If $j = 0$, $B^0 = 0 \in \mathscr{C}$.

Now, assume that $B^l \in \mathscr{C}$. Put $C^j = \mathrm{Hom}_R(K_j(\underline{x}), M)/B^j$. Since $K_l(\underline{x})$ is a finite free R-module, it follows that $\mathrm{Hom}_R(K_l(\underline{x}), M) \in \mathscr{C}$. Now, since $B^l \in \mathscr{C}$, we have that $C^l \in \mathscr{C}$. Hence $(0 :_{C^l} \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, C^l) \in \mathscr{S}$.

Because of $\mathfrak{a}\mathrm{H}^l(\underline{x}; M) = 0$, it follows that $\mathrm{H}^l(\underline{x}; M) \subseteq (0 :_{C^l} \mathfrak{a})$, and therefore $\mathrm{H}^l(\underline{x}; M) \in \mathscr{S}$. Consequently, from the short exact sequence

$$0 \longrightarrow \mathrm{H}^l(\underline{x}; M) \longrightarrow C^l \longrightarrow B^{l+1} \longrightarrow 0$$

we deduce that $B^{l+1} \in \mathscr{C}$. Hence by induction we have proved that $B^j \in \mathscr{C}$, for all $j \in \mathbb{N}_0$.

Now, since $B^t \in \mathscr{C}$ and $\mathrm{Hom}_R(K_t(\underline{x}), M) \in \mathscr{C}$, we obtain $C^t \in \mathscr{C}$. Hence $(0 :_{C^t} \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, C^t) \in \mathscr{S}$. Thus $\mathrm{H}^t(\underline{x}; M) \subseteq (0 :_{C^t} \mathfrak{a})$ is also in $\mathscr{S}$. Therefore, $M/\mathfrak{a}M \in \mathscr{S}$. □

The next Theorem is a generalization of (MELKERSSON, 2005, Theorem 2.1).

Theorem 5.1.13. Let M be an R-module and $\mathfrak{a} = (x_1, \ldots, x_t)$ be an ideal of R. Then the following conditions are equivalent:

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$;

(ii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$;

(iii) The Koszul cohomology modules $\mathrm{H}^i(x_1, \ldots, x_t; M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$.

Proof. $(i) \Rightarrow (ii)$ Let

$$\mathbb{F}_\bullet : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. Consider the complex $\mathbb{F}_\bullet \otimes_R M$; it follows that $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of this new complex, respectively.

Put

$$\mathscr{C} = \{N \mid \mathrm{Ext}^j_R(R/\mathfrak{a}, N) \in \mathscr{S} \text{ for all } j \in \mathbb{N}_0\}.$$

Our claim is that $Z_i \in \mathscr{C}$ for all $i \in \mathbb{N}_0$. We prove this by induction on i.

If $i = 0$, $Z_0 = F_0 \otimes_R M \in \mathscr{C}$, since F_0 is a finitely generated free R-module.

Now, assume that $Z_l \in \mathscr{C}$. Consider the short exact sequence

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \longrightarrow 0, \tag{5.1}$$

where we can see $B_i \cong (F_{i+1} \otimes_R M)/Z_{i+1}$. Hence, we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \longrightarrow \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \longrightarrow 0.$$

Therefore, $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ is homomorphic image of $Z_i/\mathfrak{a}Z_i$, for all $i \in \mathbb{N}_0$.

Now, since $Z_l \in \mathscr{C}$, $\mathrm{Ext}^j_R(R/\mathfrak{a}, Z_l) \in \mathscr{S}$, for all $j \in \mathbb{N}_0$; then $Z_l/\mathfrak{a}Z_l \in \mathscr{S}$, by Theorem 5.1.12. Thus $\mathrm{Tor}^R_l(R/\mathfrak{a}, M) \in \mathscr{S}$. Therefore, we deduce from (5.1) that $B_l \in \mathscr{C}$ and so $Z_{l+1} \in \mathscr{C}$.

Hence by induction we have proved that $Z_i \in \mathscr{C}$, for all $i \in \mathbb{N}_0$. It follows by Theorem 5.1.12 that $Z_i/\mathfrak{a}Z_i \in \mathscr{S}$, for all $i \in \mathbb{N}_0$, and therefore $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$.

$(ii) \Rightarrow (iii)$ Let $\underline{x} = x_1, \ldots, x_t$. As $\mathrm{H}^i(\underline{x}; M) \cong \mathrm{H}_{t-i}(\underline{x}; M)$, it is sufficient to show that $\mathrm{H}_i(\underline{x}; M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$.

Consider the Koszul complex

$$K_\bullet(\underline{x}) : 0 \longrightarrow K_t(\underline{x}) \longrightarrow K_{t-1}(\underline{x}) \longrightarrow \cdots \longrightarrow K_1(\underline{x}) \longrightarrow K_0(\underline{x}) \longrightarrow 0.$$

Then $\mathrm{H}_i(\underline{x}; M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $K_\bullet(\underline{x}) \otimes_R M$, respectively.

Put

$$\mathscr{C} = \{N \mid \mathrm{Tor}_j^R(R/\mathfrak{a}, N) \in \mathscr{S}, \text{ for all } j \in \mathbb{N}_0\}.$$

Consider the short exact sequence

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow \mathrm{H}_i(\underline{x}; M) \longrightarrow 0.$$

Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \longrightarrow \mathrm{H}_i(\underline{x}; M) \longrightarrow 0,$$

thus $\mathrm{H}_i(\underline{x}; M)$ is a homomorphic image of $Z_i/\mathfrak{a}Z_i$, for all $i \in \mathbb{N}_0$.

Now, analogous to the proof of the implication (i) $\Rightarrow$ (ii), we can show that $Z_i \in \mathscr{C}$, for all $i \in \mathbb{N}_0$. Since $Z_i/\mathfrak{a}Z_i = \mathrm{Tor}_0^R(R/\mathfrak{a}, Z_i) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$, we have $\mathrm{H}_i(\underline{x}; M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$.

$(iii) \Rightarrow (i)$ Let

$$\mathbb{F}_\bullet : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. Consider the complex $\mathrm{Hom}_R(\mathbb{F}_\bullet, M)$; it follows that $\mathrm{Ext}_R^i(R/\mathfrak{a}, M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of this new complex, respectively.

Let $\underline{x} = x_1, \ldots, x_t$. Put

$$\mathscr{C} = \{N \mid \mathrm{H}^j(\underline{x}; N) \in \mathscr{S}, \text{ for all } j \in \mathbb{N}_0\}.$$

Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^i(R/\mathfrak{a}, M) \longrightarrow C^i \longrightarrow B^{i+1} \longrightarrow 0,$$

where $C^i = \mathrm{Hom}_R(F_i, M)/B^i$. Then $B^i \in \mathscr{C}$ (as in the proof of Theorem 5.1.12), for all $i \in \mathbb{N}_0$. Thus, $C_i \in \mathscr{C}$, for all $i \in \mathbb{N}_0$.

Therefore, since

$$\mathrm{Ext}_R^i(R/\mathfrak{a}, M) \subseteq (0 :_{C^i} \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, C^i) \cong \mathrm{H}^0(\underline{x}; C^i)$$

and $\mathrm{H}^0(\underline{x}; C^i) \in \mathscr{S}$, we can see that $\mathrm{Ext}_R^i(R/\mathfrak{a}, M) \in \mathscr{S}$, for all $i \in \mathbb{N}_0$. □

The next two Lemmas are going to be used on the proof of the most important result of this section.

Lemma 5.1.14. Let M be a finitely generated R-module and N be an arbitrary R-module. Let t be a non-negative integer such that $\mathrm{Tor}_i^R(M, N) \in \mathscr{S}$ for all $i \leq t$. Then $\mathrm{Tor}_i^R(L, N) \in \mathscr{S}$ for all $i \leq t$, whenever L is a finitely generated R-module such that $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(M)$.

Proof. Since $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(M)$, there exists a chain of R-modules

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of M (by Gruson's Theorem, (VASCONCELOS, 1974, Theorem 4.1)).

Consider the exact sequences

$$0 \to K \to M^n \to L_1 \to 0$$

$$0 \to L_1 \to L_2 \to L_2/L_1 \to 0$$

$$\vdots$$

$$0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0$$

for some positive integer n and some finitely generated R-module K.

Let $i \leq t$ and $1 \leq j \leq k$. From the long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}^R(L_j/L_{j-1}, N) \to \mathrm{Tor}_i^R(L_{j-1}, N) \to \mathrm{Tor}_i^R(L_j, N) \to \mathrm{Tor}_i^R(L_j/L_{j-1}, N) \to \cdots$$

and properties of Serre subcategories, $\mathrm{Tor}_i^R(L_j, N) \in \mathscr{S}$ if and only if $\mathrm{Tor}_i^R(L_{j-1}, N) \in \mathscr{S}$. Using an easy induction on k, it suffices to prove the case when $k = 1$.

Therefore, consider the exact sequence mentioned above

$$0 \to K \to M^n \to L \to 0. \tag{5.2}$$

We now use induction on t.

If $t = 0$, we have that $L \otimes_R N$ is a homomorphic image of $M^n \otimes_R N$ which belongs to $\mathscr{S}$. Then, $L \otimes_R N \in \mathscr{S}$.

Now, let's assume $t > 0$ and $\mathrm{Tor}_i^R(L', N) \in \mathscr{S}$ for every finitely generated R-module L' with $\mathrm{Supp}_R(L') \subseteq \mathrm{Supp}_R(M)$ and all $i < t$. The exact sequence (5.2) induces the long exact sequence

$$\cdots \to \mathrm{Tor}_i^R(M^n, N) \to \mathrm{Tor}_i^R(L, N) \to \mathrm{Tor}_{i-1}^R(K, N) \to \cdots$$

so that, by the inductive hypothesis, $\mathrm{Tor}_{i-1}^R(K, N) \in \mathscr{S}$ for all $i \leq t$. On the other hand, $\mathrm{Tor}_i^R(M^n, N) \cong \bigoplus^n \mathrm{Tor}_i^R(M, N) \in \mathscr{S}$. Therefore, $\mathrm{Tor}_i^R(L, N) \in \mathscr{S}$ for all $i \leq t$, and the proof is complete. □

Lemma 5.1.15. Let M be a finitely generated R-module and N be an arbitrary R-module. Let t be a non-negative integer such that $\mathrm{Ext}_R^i(M, N) \in \mathscr{S}$ for all $i \leq t$. Then $\mathrm{Ext}_R^i(L, N) \in \mathscr{S}$ for all $i \leq t$, whenever L is a finitely generated R-module such that $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(M)$.

Proof. The proof follows by a similar way to what has been done in Lemma 5.1.14. □

The next Theorem is the most important result of this section and it is a key point for the rest of the chapter.

Theorem 5.1.16. Let t be a non-negative integer. Then, for an R-module N, the following conditions are equivalent:

(i) $\operatorname{Tor}_i^R(R/\mathfrak{a},N) \in \mathscr{S}$, for all $i \leq t$;

(ii) For any finitely generated R-module M with $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$, $\operatorname{Tor}_i^R(M,N) \in \mathscr{S}$ for all $i \leq t$;

(iii) For any R-ideal $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, $\operatorname{Tor}_i^R(R/\mathfrak{b},N) \in \mathscr{S}$, for all $i \leq t$;

(iv) For any minimal prime $\mathfrak{p}$ over $\mathfrak{a}$, $\operatorname{Tor}_i^R(R/\mathfrak{p},N) \in \mathscr{S}$, for all $i \leq t$;

(v) $\operatorname{Ext}_R^i(R/\mathfrak{a},N) \in \mathscr{S}$, for all $i \leq t$;

(vi) For any finitely generated R-module M with $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$, $\operatorname{Ext}_R^i(M,N) \in \mathscr{S}$ for all $i \leq t$;

(vii) For any R-ideal $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, $\operatorname{Ext}_R^i(R/\mathfrak{b},N) \in \mathscr{S}$, for all $i \leq t$;

(viii) For any minimal prime $\mathfrak{p}$ over $\mathfrak{a}$, $\operatorname{Ext}_R^i(R/\mathfrak{p},N) \in \mathscr{S}$ for all $i \leq t$.

Proof. $(i) \Rightarrow (ii)$ It follows from Lemma 5.1.14, since $\operatorname{Supp}_R(R/\mathfrak{a}) = V(\mathfrak{a})$.

$(ii) \Rightarrow (iii)$ Take $M = R/\mathfrak{b}$ and observe that $\operatorname{Supp}_R(R/\mathfrak{b}) = V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

$(iii) \Rightarrow (iv)$ Immediate.

$(iv) \Rightarrow (i)$ Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the minimal primes of $\mathfrak{a}$. Then, by assumption, $\operatorname{Tor}_i^R(R/\mathfrak{p}_j,N) \in \mathscr{S}$ for all $j = 1,\ldots,n$. Hence $\bigoplus_{j=1}^n \operatorname{Tor}_i^R(R/\mathfrak{p}_j,N) \cong \operatorname{Tor}_i^R(\bigoplus_{j=1}^n R/\mathfrak{p}_j,N) \in \mathscr{S}$. Since $\operatorname{Supp}_R(R/\mathfrak{a}) = \operatorname{Supp}_R(\bigoplus_{j=1}^n R/\mathfrak{p}_j)$, it follows by Lemma 5.1.14 that $\operatorname{Tor}_i^R(R/\mathfrak{a},N) \in \mathscr{S}$ for all $i \leq t$, as required.

$(v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (v)$ It follows in a similar way to what has been done previously, using Lemma 5.1.15.

$(i) \Leftrightarrow (v)$ Follows by Theorem 5.1.13. □

5.1.2 The class of $\mathfrak{a}$-weakly finite modules over M

In this section, we define a new class of R-modules, the $\mathfrak{a}$-weakly finite modules over M. This definition is a generalization of the definition of weakly finite R-modules (BAGHERI, 2014, Definition 2.1).

Definition 5.1.17. Let M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Let $\mathscr{W}$ be the largest class of R-modules, i.e., the union of all such classes, satisfying the following four properties:

1. If $N \in \mathscr{W}$, then $\operatorname{Hom}_R(R/\mathfrak{a}, N)$ and $\operatorname{Hom}_R(M/\mathfrak{a}M, N)$ are finitely generated.

2. If N is a non-zero element of $\mathscr{W}$ and x is a regular element of R, then $N \neq xN$, $N/xN \in \mathscr{W}$ and $\dim N/xN = \dim N - 1$.

3. If $N \in \mathscr{W}$, then $|\operatorname{Ass}_R(N)| < \infty$.

4. If $N \in \mathscr{W}$, then $N/\Gamma_{\mathfrak{a}}(N) \in \mathscr{W}$.

We say an R-module N is $\mathfrak{a}$-*weakly finite over* M, if it belongs to $\mathscr{W}$.

If $M = R$, we say that the R-module is $\mathfrak{a}$-*weakly finite* . If $M = R$, $(R, \mathfrak{m})$ is a local ring and $\mathfrak{a} = \mathfrak{m}$ we just say that the R-module is *weakly finite*.

Note that if M is a finitely generated R-module, then M is a weakly finite R-module. ((BAGHERI, 2014, Lemma 2.1)) has shown, in the local case, that the class of $\mathfrak{a}$-cofinite modules is contained in the class of weakly finite modules. Our next proposition shows a similar behaviour for $\mathfrak{a}$-weakly finite modules, and the proof is analogous to what was done in (BAGHERI, 2014, Lemma 2.1).

Proposition 5.1.18. Let $(R, \mathfrak{m})$ be a local ring and let M be a non-zero R-module of dimension $n > 0$. If N is $\mathfrak{a}$-cofinite, then N is $\mathfrak{a}$-weakly finite.

Proof. We show that $\mathfrak{a}$-cofinite modules satisfy the axiom of $\mathfrak{a}$-weakly finite modules.

The first axiom is true, since $\operatorname{Ext}^0_R(R/\mathfrak{a}, N) = \operatorname{Hom}_R(R/\mathfrak{a}, N)$ is finitely generated by hypothesis.

To the second axiom, let $r \in \mathfrak{m}$ a regular element of R. By applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to the short exact sequence $0 \to N \xrightarrow{\cdot} N \to N/rN \to 0$, we have the long exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, N) \to \operatorname{Hom}_R(R/\mathfrak{a}, N) \to \operatorname{Hom}_R(R/\mathfrak{a}, N/rN). \tag{5.3}$$

If $N = rN$, then $\operatorname{Hom}_R(R/\mathfrak{a}, N) \cong r\operatorname{Hom}_R(R/\mathfrak{a}, N)$. By Nakayama's Lemma, $\operatorname{Hom}_R(R/\mathfrak{a}, N) = 0$ and this is a contradiction because

$$\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, N)) = \operatorname{Ass}_R(N) \cap \operatorname{V}(\mathfrak{a}) \neq \emptyset.$$

On the other hand,

$$\begin{aligned} n-1 &= \dim(\operatorname{Hom}_R(R/\mathfrak{a}, N)/r\operatorname{Hom}_R(R/\mathfrak{a}, N)) \\ &\leq \dim(\operatorname{Hom}_R(R/\mathfrak{a}, N/rN)) \\ &\leq \dim(N/rN) \leq n-1, \end{aligned}$$

follows $\dim(N/rN) = n-1$. Clearly, $\operatorname{Supp}_R(N/rN) \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, N/rN)$ is finitely generated for all integer $i \geq 0$.

To the third axiom, note that

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a},N)) = \mathrm{Ass}_R(N) \cap \mathrm{V}(\mathfrak{a}).$$

Since $\mathrm{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a})$, $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a},N)) = \mathrm{Ass}_R(N)$. By assumption, $\mathrm{Hom}_R(R/\mathfrak{a},N)$ is finitely generated and so has finitely many associated primes. Therefore, $\mathrm{Ass}_R(N)$ is a finite set.

To the last axiom, by using the exact sequence (5.3) together with the fact that $\mathrm{Ext}^i_R(R/\mathfrak{a},N)$ is finitely generated, we obtain that N/rN is weakly finite. Respectively, using the short exact sequence $0 \to \Gamma_{\mathfrak{m}}(N) \to N \to N/\Gamma_{\mathfrak{m}}(N) \to 0$, we conclude that $N/\Gamma_{\mathfrak{m}}(N)$ is also weakly finite. □

5.2 Cofiniteness and Artinianness of Generalized Local Cohomology Modules

During this section, let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R.

Bagheri has shown in the local case that the local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(N)$ is Artinian when N is a weakly finite R-module, (BAGHERI, 2014, Theorem 2.1). Our main aim in this section is to generalize this result to generalized local cohomology modules and in the non-local case (Theorem 5.2.4). To do this, we will need some previous results.

Lemma 5.2.1 ((BRODMANN; SHARP, 2012, Lemma 2.1.1))**.** If T is an R-module such that $|\mathrm{Ass}_R(T)| < \infty$, then T is $\mathfrak{a}$-torsion free if and only if there is an element $r \in \mathfrak{a}$ such that r is T-regular.

Proposition 5.2.2 ((MELKERSSON, 2005, Proposition 4.1))**.** Let M be an R-module with support in $V(\mathfrak{a})$. Then M is Artinian and $\mathfrak{a}$-cofinite if and only if $(0 :_M \mathfrak{a})$ has finite length. If there is an element $x \in \mathfrak{a}$ such that $(0 :_M x)$ is Artinian and $\mathfrak{a}$-cofinite, then M is Artinian and $\mathfrak{a}$-cofinite.

The next result provides us a property about the artinianess and cofinniteness of the Gamma functor.

Lemma 5.2.3. Suppose that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\dim R/\mathfrak{b} = 0$. Let N be an $\mathfrak{a}$-weakly finite R-module. Then $\Gamma_{\mathfrak{b}}(N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite.

Proof. Note that $(0 :_N \mathfrak{a}) = \mathrm{Hom}_R(R/\mathfrak{a},N)$ is finitely generated, since N is $\mathfrak{a}$-weakly finite. Since $\mathfrak{a} \subseteq \mathfrak{b}$, $\mathrm{Hom}_R(R/\mathfrak{b},N) = (0 :_N \mathfrak{b}) \subseteq (0 :_N \mathfrak{a})$ and, since R is a Noetherian ring, $\mathrm{Hom}_R(R/\mathfrak{b},N)$ is finitely generated. Moreover, applying functors $\mathrm{Hom}_R(R/\mathfrak{a},-)$ and $\mathrm{Hom}_R(R/\mathfrak{b},-)$ to the exact sequence $0 \to \Gamma_{\mathfrak{b}}(N) \to N$ and using the same argument, we conclude $\mathrm{Hom}_R(R/\mathfrak{a},\Gamma_{\mathfrak{b}}(N))$ and $\mathrm{Hom}_R(R/\mathfrak{b},\Gamma_{\mathfrak{b}}(N))$ are finitely generated R-modules.

Since $\dim R/\mathfrak{b} = 0$, it follows that $\operatorname{Hom}_R(R/\mathfrak{b}, \Gamma_\mathfrak{b}(N)) = (0 :_{\Gamma_\mathfrak{b}(N)} \mathfrak{b})$ has finite length. Thus, by Proposition 5.2.2, $\Gamma_\mathfrak{b}(N)$ is Artinian and $\mathfrak{b}$-cofinite. Hence, $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_\mathfrak{b}(N)) = (0 :_{\Gamma_\mathfrak{b}(N)} \mathfrak{a})$ is Artinian and therefore, has finite length. Then, again by Proposition 5.2.2, $\Gamma_\mathfrak{b}(N)$ is also $\mathfrak{a}$-cofinite. □

Now we are able to show the first main result of this section.

Theorem 5.2.4. Suppose that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\dim R/\mathfrak{b} = 0$. Let M be a finitely generated R-module and let N be a $\mathfrak{a}$-weakly finite R-module over M. Then $\operatorname{H}^i_\mathfrak{b}(M,N)$ is an Artinian R-module and is $\mathfrak{a}$- and $\mathfrak{b}$-cofinite, for all $i \in \mathbb{N}_0$.

Proof. We use induction on i.

Assume that $i = 0$. We claim $\operatorname{H}^0_\mathfrak{b}(M,N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Since, by Corollary 3.3.4, $\operatorname{H}^0_\mathfrak{b}(M,N) \cong \operatorname{Hom}_R(M, \Gamma_\mathfrak{b}(N))$ the assertion follows from Lemma 5.2.3, Lemma 5.1.9 and Proposition 5.1.10.

Now, assume $i > 0$ and that $\operatorname{H}^j_\mathfrak{b}(M,N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite, for $j < i$. Consider the exact sequence

$$\operatorname{H}^i_\mathfrak{b}(M, \Gamma_\mathfrak{b}(N)) \to \operatorname{H}^i_\mathfrak{b}(M,N) \to \operatorname{H}^i_\mathfrak{b}(M, N/\Gamma_\mathfrak{b}(N)).$$

By Lemma 5.2.3, $\Gamma_\mathfrak{b}(N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Since M is finitely generated, Lemma 3.3.5 and Proposition 5.1.10 imply that $\operatorname{H}^i_\mathfrak{b}(M, \Gamma_\mathfrak{b}(N))$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Therefore, $\operatorname{H}^i_\mathfrak{b}(M,N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite if and only if $\operatorname{H}^i_\mathfrak{b}(M, N/\Gamma_\mathfrak{b}(N))$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Hence we may assume that $\Gamma_\mathfrak{b}(N) = 0$ and so, by Lemma 5.2.1, there is an element $r \in \mathfrak{b}$ which is N-regular.

The short exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot r} N \longrightarrow N/rN \longrightarrow 0$$

induces the following exact sequence

$$\operatorname{H}^{i-1}_\mathfrak{b}(M, N/rN) \longrightarrow \operatorname{H}^i_\mathfrak{b}(M,N) \xrightarrow{\cdot r} \operatorname{H}^i_\mathfrak{b}(M,N).$$

Since N/rN is $\mathfrak{a}$-weakly finite, by induction $\operatorname{H}^{i-1}_\mathfrak{b}(M, N/rN)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Thus, by Lemma 5.1.9, $(0 :_{\operatorname{H}^i_\mathfrak{b}(M,N)} r)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. Therefore, by Proposition 5.2.2, $\operatorname{H}^i_\mathfrak{b}(M,N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. □

The next two Corollaries follow immediately.

Corollary 5.2.5. Let $(R, \mathfrak{m})$ be a local ring. Let M be a finitely generated R-module and N be a weakly finite R-module over M. Then $\operatorname{H}^i_\mathfrak{m}(M,N)$ is Artinian, for all $i \in \mathbb{N}_0$.

Corollary 5.2.6 ((BAGHERI, 2014, Theorem 2.1))**.** Let $(R, \mathfrak{m})$ be a local ring and let N be a weakly finite R-module. Then $\operatorname{H}^i_\mathfrak{m}(N)$ is Artinian, for all $i \in \mathbb{N}_0$.

5.2.1 The local case

From now on, in this section, we assume R is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$.

The next result give us artinianess and cofiniteness conditions to the top generalized local cohomology modules.

Theorem 5.2.7. Let M be a finitely generated R-module such that $\operatorname{pdim} M = d < \infty$ and let N be a $\mathfrak{b}$-weakly finite R-module over M such that $\dim N = n < \infty$. Suppose $\mathfrak{a} \subseteq \mathfrak{b}$. Then $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{b}$-cofinite.

In particular, if $\mathfrak{a} = \mathfrak{b}$, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{a}$-cofinite.

Proof. If $\mathfrak{a} = \mathfrak{m}$ just use Theorem 5.2.4.

Now assume $\mathfrak{a} \neq \mathfrak{m}$ and choose $x \in \mathfrak{m} \setminus \mathfrak{a}$. By (DIVAANI-AAZAR; HAJIKARIMI, 2011, Lemma 3.1), there is an exact sequence

$$\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M,N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x,N_x),$$

where N_x is the localization of N at $\{x^i \mid i \geq 0\}$. Note that $\dim_{R_x}(N_x) < n$ (since $\mathfrak{m}_x = 0$), therefore $\mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x,N_x) = 0$. Thus, there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M,N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$. Now assuming $\mathfrak{m} = \mathfrak{a} + (x_1,\ldots,x_r)$ and repeating this argument, we obtain the surjection

$$\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) \twoheadrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N).$$

Therefore, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{b}$-cofinite, since $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$ is Artinian and $\mathfrak{b}$-cofinite from Theorem 5.2.4. □

Corollary 5.2.8. Let N be a $\mathfrak{b}$-weakly finite R-module such that $\dim N = n < \infty$. Suppose $\mathfrak{a} \subseteq \mathfrak{b}$. Then $\mathrm{H}^{n}_{\mathfrak{a}}(N)$ is Artinian and $\mathfrak{b}$-cofinite.

In particular, if $\mathfrak{a} = \mathfrak{b}$, $\mathrm{H}^{n}_{\mathfrak{a}}(N)$ is Artinian and $\mathfrak{a}$-cofinite.

The next result gives us an important isomorphism using a $\mathfrak{a}$-weakly finite module. By Corollary 5.2.8, the proof follows analogously to Proposition 4.3.8.

Proposition 5.2.9. Let M and N be two R-modules such that M is finitely generated and N is $\mathfrak{a}$-weakly finite over M. Assume $\operatorname{pdim} M = d < \infty$ and $\dim N = n < \infty$. Moreover, assume $n \in \mathbb{N}$ and that there exists an $\mathfrak{a}$-filter regular sequence $x_1,\ldots,x_n$ on N. Then,

1. $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \operatorname{Ext}^{d}_{R}(M,\mathrm{H}^{n}_{\mathfrak{a}}(N))$.
2. $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq \operatorname{Att}_R(\mathrm{H}^{n}_{\mathfrak{a}}(N))$.

Proposition 5.2.10. Let M and N be two R-modules such that M is finitely generated and N is weakly finite. Assume $\operatorname{pdim} M = d < \infty$, $\dim R = \dim N = n$ and $0 < n < \infty$. Moreover, assume $n \in \mathbb{N}$ and that there exists an $\mathfrak{m}$-filter regular sequence $x_1, \ldots, x_n$ on N. If $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) \neq 0$, then it is not finitely generated.

Proof. As $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N) \neq 0$, then $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) \neq \emptyset$. By item 2 of Proposition 5.2.9 and Example 4.3.6,

$$\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) \subseteq \operatorname{Att}_R(\mathrm{H}^{n}_{\mathfrak{m}}(N)) \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \dim R/\mathfrak{p} = n\}.$$

Since $n > 0$, we have $\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)) \nsubseteq \{\mathfrak{m}\}$. Since $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$ is Artinian, it follows that $\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$ is not finitely generated by (BRODMANN; SHARP, 2012, Corollary 7.2.12). □

5.3 Cofiniteness of Torsion and Extension functors

During this section, $\widehat{R}$ will denote the completion of R with respect to the maximal ideal $\mathfrak{m}$, when $(R,\mathfrak{m})$ is a commutative Noetherian local ring. Furthermore, $\mathfrak{a}$ and $\mathfrak{b}$ will always denote two ideals of any ring R.

Our purpose in this section is to give some answers to Question 1.0.5 in the Introduction, which we enunciate again.

Question 5.3.1. When are $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}^R_i(M,N)$ $\mathfrak{a}$-cofinite (or Artinian, or finite length, or $\mathfrak{a}$-cominimax, or $\mathfrak{a}$-weakly cofinite) for all (or for some) integer i?

In view of what was done in Section 5.2, we have the following results.

Proposition 5.3.2. Suppose that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\dim R/\mathfrak{b} = 0$. Let M be a finitely generated R-module and N be an $\mathfrak{a}$-weakly finite R-module over M. Then $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ and $\operatorname{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ have finite length, for all i and j in $\mathbb{N}_0$.

Proof. By Theorem 5.2.4, $\mathrm{H}^i_{\mathfrak{b}}(M,N)$ is Artinian and $\mathfrak{a}$- and $\mathfrak{b}$-cofinite. So, $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ and $\operatorname{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ are finitely generated and, since $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are finitely generated $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ and $\operatorname{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ are Artinian by Proposition 5.1.10. Therefore, $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ and $\operatorname{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ have finite length. □

As a consequence of this Proposition, we have the following result:

Corollary 5.3.3. Let $(R,\mathfrak{m})$ be a local ring. Suppose $\mathfrak{a} \subseteq \mathfrak{b}$ such that $\dim R/\mathfrak{b} = 0$. Let M be a finitely generated R-module and N be a $\mathfrak{a}$-weakly finite R-module over M. Then,,

$$\operatorname{Att}_R(\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) = \{\mathfrak{m}\} = \operatorname{Att}_R(\operatorname{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{b}}(M,N))),$$

for all i and j in $\mathbb{N}_0$.

Proof. We will show that $\mathrm{Att}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) = \{\mathfrak{m}\}$; the other one is analogous.

By Proposition 5.3.2, $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))$ has finite length, therefore $\mathrm{Supp}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) = \{\mathfrak{m}\}$. Now, by (OOISHI, 1976, Proposition 2.9 (2) and (3)), $\mathrm{Att}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) \subseteq \{\mathfrak{m}\}$. Therefore, $\mathrm{Att}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) = \{\mathfrak{m}\}$, since $\mathrm{Att}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{b}}(M,N))) \neq \emptyset$. □

The reader can compare the next result with (KUBIK; LEAMER; SATHER-WAGSTAFF, 2011, Theorem 2.3).

Theorem 5.3.4. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring. Let H and N be two R-modules such that H is Artinian and $\mathfrak{a}$-cofinite and N is minimax. Then, for each $i \geq 0$, the module $\mathrm{Ext}^i_R(N,H)$ is minimax and $\widehat{\mathfrak{a}}$-cofinite over $\widehat{R}$.

Proof. Since N is minimax, there exists a submodule $L \subseteq N$ such that L is finitely generated and N/L is Artinian. We obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(N/L,H) \to \mathrm{Ext}^i_R(N,H) \to \mathrm{Ext}^i_R(L,H) \to \cdots.$$

By Proposition 5.1.10, since Artinian and $\mathfrak{a}$-cofinite modules form a Serre subcategory of the category of R-modules, we obtain that $\mathrm{Ext}^i_R(L,H)$ is Artinian and $\mathfrak{a}$-cofinite. Then, it is also Artinian and $\widehat{\mathfrak{a}}$-cofinite over $\widehat{R}$. Thus, $\mathrm{Ext}^i_R(L,H)$ is minimax and $\widehat{\mathfrak{a}}$-cofinite over $\widehat{R}$.

On the other hand, by (KUBIK; LEAMER; SATHER-WAGSTAFF, 2011, Corollary 2.3), $\mathrm{Ext}^i_R(N/L,H)$ is Noetherian over $\widehat{R}$. Thus $\mathrm{Ext}^i_R(N/L,H)$ is minimax and $\widehat{\mathfrak{a}}$-cofinite over $\widehat{R}$.

Therefore, since the class of $\widehat{\mathfrak{a}}$-cofinite minimax modules is a Serre subcategory (MELKERSSON, 2005, Corollary 4.4), $\mathrm{Ext}^i_R(N,H)$ is minimax and $\widehat{\mathfrak{a}}$-cofinite over $\widehat{R}$. □

The next two results partially answer a generalization of Hartshorne's conjecture.

Corollary 5.3.5. Let $(R,\mathfrak{m})$ be a complete local ring and $\mathfrak{a}$ an R-ideal such that $\dim R/\mathfrak{a} = 0$. Let M be a finitely generated R-module, and let N be an $\mathfrak{a}$-weakly finite R-module over M. Then, for any minimal R-module L and for each $i \geq 0$ and $j \geq 0$, $\mathrm{Ext}^i_R(L, \mathrm{H}^j_{\mathfrak{a}}(M,N))$ is minimax and $\mathfrak{a}$-cofinite R-module.

Proof. The proof follows by Theorem 5.2.4 and Theorem 5.3.4. □

Corollary 5.3.6. Let $(R,\mathfrak{m})$ be a complete local ring. Let M be a finitely generated R-module such that $\mathrm{pdim}\, M = d < \infty$ and let N be an $\mathfrak{a}$-weakly finite R-module over M such that $\dim N = n < \infty$. Then, for each $i \geq 0$, $\mathrm{Ext}^i_R(L, \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N))$ is minimax and $\mathfrak{a}$-cofinite R-module, for any minimax module L.

Proof. The proof follows by Theorem 5.2.7 and Theorem 5.3.4. □

Lemma 5.3.7. Let N be a non-zero $\mathfrak{a}$-cominimax R-module. Then, for any non-zero R-module M of finite length, the R-module $\mathrm{Tor}_i^R(M,N)$ is minimax and has finite length, for all $i \in \mathbb{N}_0$.

Proof. First, note that $\mathrm{Supp}_R(M)$ is a finite non-empty subset of the set of all maximal ideals of R. Let $\mathrm{Supp}_R(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$ and $\mathfrak{b} = \mathfrak{m}_1\mathfrak{m}_2 \ldots \mathfrak{m}_r$. As $\mathrm{Supp}_R(M) = V(\mathfrak{b})$, by Lemma 5.1.14, it is sufficient to show that $\mathrm{Tor}_i^R(R/\mathfrak{b},N)$ has finite length for all $i \geq 0$. By the isomorphism $\mathrm{Tor}_i^R(R/\mathfrak{b},N) \cong \bigoplus_{j=1}^r \mathrm{Tor}_i^R(R/\mathfrak{m}_j,N)$, it is enough to show that $\mathrm{Tor}_i^R(R/\mathfrak{m}_j,N)$ has finite length for all $i \in \mathbb{N}_0$ and $j = 1, \ldots, r$.

Fix j and let $i \geq 0$ be an integer such that $\mathrm{Tor}_i^R(R/\mathfrak{m}_j,N) \neq 0$. Note that $\mathfrak{m}_j \in \mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}_R^i(R/\mathfrak{a},N)$ is minimax for all $i \geq 0$. Hence, $\mathrm{Ext}_R^i(R/\mathfrak{m}_j,N)$ is minimax and has finite length by Theorem 5.1.16 and Proposition 5.1.3. Therefore, applying Theorem 5.1.16 again, we obtain that $\mathrm{Tor}_i^R(R/\mathfrak{m}_j,N)$ is minimax and has finite length for all $i \in \mathbb{N}_0$ and $j = 1, \ldots, r$. $\square$

Corollary 5.3.8. Let N be a non-zero minimax R-module. Then, for any non-zero R-module M of finite length, the R-module $M \otimes_R N$ is minimax and has finite length, for all $i \in \mathbb{N}_0$.

Proof. Take the ideal $\mathfrak{a} = 0$ in Lemma 5.3.7. $\square$

Now we are able to show one of the main results of this section.

Theorem 5.3.9. Let N be a non-zero $\mathfrak{a}$-cominimax R-module and M be a finitely generated R-module.

(i) If $\dim M = 1$, then the R-module $\mathrm{Tor}_i^R(M,N)$ is Artinian and $\mathfrak{a}$-cofinite, for all $i \in \mathbb{N}_0$.

(ii) If $\dim M = 2$, then the R-module $\mathrm{Tor}_i^R(M,N)$ is $\mathfrak{a}$-cofinite, for all $i \in \mathbb{N}_0$.

Proof. (i) Since N is $\mathfrak{a}$-cominimax and $\mathrm{Supp}_R(\Gamma_\mathfrak{a}(M)) \subseteq V(\mathfrak{a})$, we obtain that $\mathrm{Tor}_i^R(\Gamma_\mathfrak{a}(M),N)$ is minimax for all $i \geq 0$ by Theorem 5.1.16. Now, by the short exact sequence

$$0 \to \Gamma_\mathfrak{a}(M) \to M \to M/\Gamma_\mathfrak{a}(M) \to 0,$$

we can deduce the following long exact sequence, for all $i \geq 0$

$$\cdots \to \mathrm{Tor}_i^R(\Gamma_\mathfrak{a}(M),N) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M/\Gamma_\mathfrak{a}(M),N) \to \mathrm{Tor}_{i-1}^R(\Gamma_\mathfrak{a}(M),N) \to \cdots.$$

Therefore, it is sufficient to show that, for all $i \geq 0$, $\mathrm{Tor}_i^R(M/\Gamma_\mathfrak{a}(M),N)$ is Artinian and $\mathfrak{a}$-cofinite. Hence, we may assume that $\Gamma_\mathfrak{a}(M) = 0$ and therefore ensure the existence of $x \in \mathfrak{a} \setminus \cup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$ by Lemma 5.2.1. From the short exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0,$$

we obtain the following long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_i^R(M/xM,N) \longrightarrow \mathrm{Tor}_{i-1}^R(M,N) \xrightarrow{\cdot x} \mathrm{Tor}_{i-1}^R(M,N) \longrightarrow \mathrm{Tor}_{i-1}^R(M/xM,N) \longrightarrow \cdots.$$

By Lemma 5.3.7, the R-module $\operatorname{Tor}_i^R(M/xM,N)$ is of finite length, for all $i \in \mathbb{N}_0$, because M/xM has finite length. Thus, by the long exact sequence, $(0 :_{\operatorname{Tor}_i^R(M,N)} x)$ has finite length for all $i \geq 0$, and therefore $(0 :_{\operatorname{Tor}_i^R(M,N)} \mathfrak{a})$ also has finite length. Finally, since $\operatorname{Supp}_R(\operatorname{Tor}_i^R(M,N)) \subseteq \operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, we can conclude that $\operatorname{Tor}_i^R(M,N)$ is $\mathfrak{a}$-torsion. Therefore for all $i \in \mathbb{N}_0$, $\operatorname{Tor}_i^R(M,N)$ is an Artinian R-module by (MELKERSSON, 1990, Theorem 1.3). The $\mathfrak{a}$-cofiniteness of $\operatorname{Tor}_i^R(M,N)$ follows by (MELKERSSON, 2005, Theorem 4.3).

(ii) Proceeding similarly to the proof of item (i), we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$, and so take $x \in \mathfrak{a} \setminus \cup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$. The short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

induces following long exact sequence

$$\cdots \to \operatorname{Tor}_i^R(M/xM,N) \to \operatorname{Tor}_{i-1}^R(M,N) \xrightarrow{\cdot x} \operatorname{Tor}_{i-1}^R(M,N) \to \operatorname{Tor}_{i-1}^R(M/xM,N) \to \cdots.$$

Therefore $(0 :_{\operatorname{Tor}_i^R(M,N)} \mathfrak{a})$ and $\operatorname{Tor}_i^R(M,N)/x\operatorname{Tor}_i^R(M,N)$ are Artinian R-modules, by item (i) and Lemma 5.3.7, and $\mathfrak{a}$-cofinite, by (MELKERSSON, 2005, Corollary 4.4), for all $i \geq 0$. Therefore $\operatorname{Tor}_i^R(M,N)$ is $\mathfrak{a}$-cofinite for all $i \in \mathbb{N}_0$, by (MELKERSSON, 2005, Corollary 3.4). □

Corollary 5.3.10. Let $\mathfrak{a}$ be an ideal of R such that $\dim R/\mathfrak{a} = 0$. Let L and M be two finitely generated R-modules and let N be an $\mathfrak{a}$-weakly finite R-module over M. Then:

(i) If $\dim L = 1$, then $\operatorname{Tor}_i^R(L, \operatorname{H}_{\mathfrak{a}}^j(M,N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i, j \geq 0$.

(ii) If $\dim L = 2$, then $\operatorname{Tor}_i^R(L, \operatorname{H}_{\mathfrak{a}}^j(M,N))$ is $\mathfrak{a}$-cofinite for all $i, j \geq 0$.

Proof. The result follows by Theorem 5.2.4 and Theorem 5.3.9. □

Proposition 5.3.11. Let N be a non-zero $\mathfrak{a}$-cominimax R-module and M be a finitely generated R-module. If $\dim N \leq 1$, then the R-module $\operatorname{Tor}_i^R(M,N)$ is $\mathfrak{a}$-cominimax, for all $i \in \mathbb{N}_0$.

Proof. Let $\mathbb{F}_\bullet$ be a resolution of M consisting of finite free R-modules. Since $\operatorname{Tor}_i^R(M,N) = \operatorname{H}_i(\mathbb{F}_\bullet \otimes_R N)$ is a subquotient of a finite direct sum of copies of N, the result follows by the fact that the category of $\mathfrak{a}$-cominimax modules with dimension less than or equal to 1 is an Abelian category (IRANI, 2017, Theorem 2.5). □

Corollary 5.3.12. Let N be a non-zero $\mathfrak{a}$-cofinite R-module and M be a finitely generated R-module. If $\dim N \leq 1$, then the R-module $\operatorname{Tor}_i^R(M,N)$ is $\mathfrak{a}$-cofinite, for all $i \in \mathbb{N}_0$.

Now we investigate the behavior of torsion product functors for larger dimensions. Our development of this includes weakly Laskerian modules, which are key points.

Remark 5.3.13. Note that if R is a Noetherian ring, $\mathfrak{a}$ is an ideal of R and L an R-module, then $\operatorname{Tor}_i^R(R/\mathfrak{a},L)$ is a weakly Laskerian R-module for all $i \geq 0$ if and only if $\operatorname{Ext}_R^i(R/\mathfrak{a},L)$ is weakly Laskerian R-module for all $i \geq 0$.

The proof of this Remark follows by Theorem 5.1.16 and the fact that the class of weakly Laskerian modules is a Serre subcategory of the category of R-modules.

Theorem 5.3.14. Let $(R,\mathfrak{m})$ be a local ring and $\mathfrak{a}$ be an R-ideal. Let N be a non-zero $\mathfrak{a}$-cominimax R-module and M be a finitely generated R-module. Then the R-module $\mathrm{Tor}_i^R(M,N)$ is $\mathfrak{a}$-weakly cofinite for all $i \in \mathbb{N}_0$ when one of the following cases holds:

(i) $\dim N \leq 2$.

(ii) $\dim M = 3$.

Proof. Note that, in the view of Remark 5.3.13, take $L = \mathrm{Tor}_i^R(M,N)$. It is sufficient to show that the R-modules $\mathrm{Tor}_j^R(R/\mathfrak{a}, \mathrm{Tor}_i^R(M,N))$ are weakly Laskerian for all $i \geq 0$ and $j \geq 0$.

For this purpose, consider the set $\Lambda = \{\mathrm{Tor}_j^R(R/\mathfrak{a}, \mathrm{Tor}_i^R(M,N)) \mid i \geq 0 \text{ and } j \geq 0\}$. Let $K \in \Lambda$ and let K' be a submodule of K. The proof is complete if we show that the set $\mathrm{Ass}_R(K/K')$ is finite. Note that we may assume that R is complete by (MATSUMURA, 1989, Ex 7.7) and (MARLEY, 2001, Lemma 2.1).

Suppose that $\mathrm{Ass}_R(K/K')$ is an infinite set. Hence, we can consider $\{\mathfrak{p}_s\}_{s=1}^{\infty}$ a countably infinite subset of non-maximal elements of $\mathrm{Ass}_R(K/K')$. Furthermore $\mathfrak{m} \not\subseteq \cup_{s=1}^{\infty} \mathfrak{p}_s$ by (MARLEY; VASSILEV, , Lemma 3.2). Define $S := R \backslash \cup_{s=1}^{\infty} \mathfrak{p}_s$. Now, we will analyze the cases (i) and (ii).

If $\dim N \leq 2$, then we can conclude that $S^{-1}N$ is a $S^{-1}\mathfrak{a}$-cominimax $S^{-1}R$-module of dimension at most one by (MATSUMURA, 1989, Ex 7.7) and (ROSHAN-SHEKALGOURABI; HASSANZADEH-LELEKAAMI, 2016, Lemma 3.4). Therefore, $\mathrm{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N)$ is $S^{-1}\mathfrak{a}$-cominimax for all $i \in \mathbb{N}_0$, by Proposition 5.3.11.

Therefore $S^{-1}K/S^{-1}K'$ is a minimax $S^{-1}R$-module and so, $\mathrm{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}K')$ is finite by Remark 5.1.4. However, for each s, we have that $S^{-1}\mathfrak{p}_s \in \mathrm{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}K')$, and so we obtain a contradiction. This completes the proof.

In case $\dim M = 3$, we obtain that $\mathrm{Tor}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ is $S^{-1}\mathfrak{a}$-cominimax, by Lemma 5.3.7 and Theorem 5.3.9. Now the proof follows similarly to the one previously made. □

Now we give some applications of the results shown in this section.

Corollary 5.3.15. Let N be a non-zero minimax R-module and M be a finitely generated R-module.

1. If $\dim \mathrm{H}_{\mathfrak{a}}^i(N) \leq 1$ (e.g. $\dim N \leq 1$ or $\dim R/\mathfrak{a} = 1$), then the R-module $\mathrm{Tor}_j^R(M, \mathrm{H}_{\mathfrak{a}}^i(N))$ is $\mathfrak{a}$-cominimax for all $i \geq 0$ and $j \geq 0$. Moreover, for for all $i \geq 1$ and $j \geq 0$, the R-module $\mathrm{Tor}_j^R(M, \mathrm{H}_{\mathfrak{a}}^i(N))$ is $\mathfrak{a}$-cofinite.

2. If $(R,\mathfrak{m})$ is a local ring and $\dim \mathrm{H}_{\mathfrak{a}}^i(M) \leq 2$ (e.g. $\dim R/I \leq 2$) , then the R-module $\mathrm{Tor}_j^R(M, \mathrm{H}_{\mathfrak{a}}^i(N))$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$ and $j \geq 0$.

3. Let L and M be two finitely generated R-modules such that $\operatorname{pdim} M = d < \infty$ and $\dim L = 3$, and let N be an $\mathfrak{a}$-weakly finite R-module over M such that $\dim N = n < \infty$. Then, for each $i \geq 0$, $\operatorname{Tor}_i^R(L, \mathrm{H}_{\mathfrak{a}}^{d+n}(M,N))$ is $\mathfrak{a}$-weakly cofinite R-module.

Proof. 1. First note that $\mathrm{H}_{\mathfrak{a}}^i(N)$ is an $\mathfrak{a}$-cominimax R-module for all $i \geq 0$, by (ABBASI; ROSHAN-SHEKALGOURABI; HASSANZADEH-LELEKAAMI, 2014, Theorem 2.2). Now the first statement follows by Proposition 5.3.11. The cofiniteness of $\operatorname{Tor}_j^R(M, \mathrm{H}_{\mathfrak{a}}^i(N))$ follows by the fact that $\operatorname{Ext}_R^j(M, \mathrm{H}_{\mathfrak{a}}^i(N))$ is finite for all $i \geq 1$ and $j \geq 0$ by (ABBASI; ROSHAN-SHEKALGOURABI; HASSANZADEH-LELEKAAMI, 2014, Theorem 2.2) and Corollary 5.3.12.

2. The proof follows analogously to Theorem 5.3.14 (i), using the previous item.

3. Apply Theorem 5.2.7 and Theorem 5.3.14 (ii). □

CHAPTER

6

ON INTEGRAL CLOSURES AND MULTIPLICITIES RELATIVE TO AN ARTINIAN MODULE

The theory about Noetherian modules is quite extensive and complete, and involves concepts such as primary decomposition, associated primes, Krull dimension, regular sequences, systems of parameters, integral closure, reduction, multiplicities of ideals $\mathfrak{m}$-primary, among others. Details about this theory can be seen, for example, in the book (HUNEKE; SWANSON, 2006). On the other hand, if we look at the case where M is an Artinian R-module but not necessarily Noetherian, there are natural questions of how the above-mentioned concepts could be defined in this dual situation. In this sense, some definitions appeared and are highly important: attached primes, secondary decomposition, Noetherian dimension, co-regular sequences, integral closure and reduction relative to Artinian modules, and multiplicities of an ideal relative to Artinian modules, for example. As an example of this duality we have the following: if M is a Noetherian R-module, an element $x \in M$ is said to be *regular* if the multiplication map $M \xrightarrow{\cdot x} M$ is injective. When M is an Artinian R-module, not necessarily Noetherian, if $M \xrightarrow{\cdot x} M$ is a surjective map, $x \in M$ is said to be a *co-regular* element.

In this chapter, we explore the relationship between $\overline{\mathfrak{b}}$, the classical Northcott-Rees integral closure of $\mathfrak{b}$, and $\mathfrak{b}^{*(H)}$, the integral closure of $\mathfrak{b}$ relative to an Artinian R-module H (also called here by ST-closure of $\mathfrak{b}$ on H), in order to study the relation between $e(\mathfrak{a})$, the multiplicity of $\mathfrak{a}$, and $e'(\mathfrak{a};H)$, multiplicity of $\mathfrak{a}$ relative to H. The main results that give the relationship are Theorem 6.3.6 and Theorem 6.3.7.

Our development of this theory includes the following: in Section 6.1, we present the definitions of reduction and integral closure, which are defined for commutative Noetherian rings. Moreover, we exhibit the definitions of the concepts of reduction and integral closure of an ideal in a commutative ring A (not necessarily Noetherian) relative to an Artinian A-module H

(not necessarily finitely generated), and relative to a Noetherian A-module M. In Section 6.2, we exhibit the concepts of multiplicities and we see how they behave in relation to integral closures. This behavior motivates some question that we discuss and prove in Section 6.3. Furthermore, in Section 6.4, we give some examples and applications.

The results presented here can be found in (JORGE PÉREZ; MERIGHE, 2020).

6.1 Integral Closures and its Properties

Throughout this section, A will be a commutative ring (not necessarily Noetherian), B will be a commutative Noetherian ring, and $(R, \mathfrak{m})$ will be a commutative Noetherian local ring.

Definition 6.1.1. ((NORTHCOTT; REES, 1954)) Given I and J two ideals of B, we say that I is a *reduction* of J if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $IJ^s = J^{s+1}$.

An element x of B is said to be *integrally dependent on I* if there exists elements $c_1, \ldots, c_n \in B$, $n \in \mathbb{N}$, with $c_i \in I^i$, for $i = 1, \ldots, n$, such that

$$x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n = 0.$$

Moreover,

$$\bar{I} := \{y \in B \mid y \text{ is integrally dependent on } I\}$$

is an ideal of B, called the *integral closure* of I.

How should a reduction and integral closure be defined if the ring is not Noetherian?

We shall see this next.

Let H be an Artinian A-module, note that H can be finitely generated or not.

Definition 6.1.2. ((SHARP; TAHERIZADEH, 1988)) Let I and J be two ideals of A, we say that I is a *reduction of J relative to H* if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $(0 :_H IJ^s) = (0 :_H J^{s+1})$.

An element x of A is said to be *integrally dependent on I relative to H* if there exists $n \in \mathbb{N}$ such that

$$\left(0 :_H \sum_{i=1}^{n} x^{n-i} I^i\right) \subseteq (0 :_H x^n).$$

Moreover,

$$I^{*(H)} := \{y \in A \mid y \text{ is integrally dependent on } I \text{ relative to } H\}$$

is an ideal of A, called the *integral closure of I relative to H*. Here, we will call the integral closure of I relative to H by the *ST-closure of I on H*, referring this integral closure to its authors: Sharp and Taherizadeh (ST)

Remark 6.1.3. In particular, $0^{*(H)} = \sqrt{(0:_A H)}$.

R. Y. Sharp and A.-J.Taherizadeh showed that these definitions have properties which reflect some of the classical concepts of reduction and integral closure. For example, the next two results, which will be useful in what follows.

Lemma 6.1.4. ((SHARP; TAHERIZADEH, 1988, Theorem 2.4 (iv))) Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Then $\mathfrak{a}^{*(H)} = \mathfrak{b}^{*(H)}$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are both reductions of $\mathfrak{a}+\mathfrak{b}$ relative to H.

Corollary 6.1.5. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. Then $\mathfrak{a}^{*(H)} = \mathfrak{b}^{*(H)}$ if and only if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ relative to H.

To see more known facts on integral closure relative to an Artinian module, one refers to (SHARP; TAHERIZADEH, 1988).

Note that, if I is an ideal of B, $\overline{I} \subseteq I^{*(H)}$. One of the questions that we tried to answer during this work is: when $\overline{I} = I^{*(H)}$?

Let's see another definition that will help us, which is a dual concept to what was previously defined.

Definition 6.1.6. ((SHARP; TIRAŞ; YASSI, 1990)) Let M be a Noetherian A-module. Let I and J be two ideals of A, we say that I is a *reduction of J relative to M* if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $IJ^sM = J^{s+1}M$.

An element x of A is said to be *integrally dependent on I relative to M* if there exists $n \in \mathbb{N}$ such that

$$x^nM \subseteq \sum_{i=1}^{n} x^{n-i}I^iM.$$

Moreover,

$$\overline{I}^{(M)} := \{x \in A \mid x \text{ is integrally dependent on } I \text{ relative to } M\}$$

is an ideal of A ((SHARP; TIRAŞ; YASSI, 1990), Corollary 1.5(vi)), called the *integral closure of I relative to M*.

Remark 6.1.7. ((SHARP; TIRAŞ; YASSI, 1990, Remark 1.6)) Let I be an ideal of A and let M be a Noetherian A-module. If we write $\widetilde{I} := (I+(0:_A M))/(0:_A M)$ an ideal of $\widetilde{A} := A/(0:_A M)$, then we have

$$\overline{I}^{(M)} \supseteq (0:_A M) \qquad \text{and} \qquad \overline{\widetilde{I}} = \overline{I}^{(M)}/(0:_A M).$$

Let $E := E(R/\mathfrak{m})$ be the injective hull of the R-module $R/\mathfrak{m}$. Denote by $D(-) := \operatorname{Hom}_R(-,E)$ the additive, exact, contravariant functor from the category R-modules to itself, as in Section 4.3.

Next Theorem is an important tool to switch between ST-closures on Artinians modules and integral closures relative to Noetherian modules. It was due to (SHARP; TIRAŞ; YASSI, 1990, Theorem 2.1), but here we have a simpler version which is what we need.

Theorem 6.1.8. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R and M be a finitely generated R-module. Then,

1. $D(M)$ is an Artinian R-module.
2. $\mathfrak{a}M \subseteq \mathfrak{b}M$ if and only if $(0 :_{D(M)} \mathfrak{b}) \subseteq (0 :_{D(M)} \mathfrak{a})$.
3. $\overline{\mathfrak{a}}^{(M)} = \mathfrak{a}^{*(D(M))}$; that is, the integral closure of $\mathfrak{a}$ relative to the Noetherian module M is equal to the ST-closure of $\mathfrak{a}$ on the Artinian module $D(M)$.

Proof. 1. Due to Matlis Duality Theorem, 4.3.1.

2. The standard equivalence between the functors

$$\mathrm{Hom}_R(-\otimes_R M, E) \quad \text{and} \quad \mathrm{Hom}_R(-, \mathrm{Hom}_R(M, E)),$$

by Theorem 2.1.10, provides us with an R-isomorphism

$$\psi_{\mathfrak{a}} : D(M/\mathfrak{a}M) = \mathrm{Hom}_R(M/\mathfrak{a}M, E) \to (0 :_{D(M)} \mathfrak{a})$$

such that for each $f \in D(M/\mathfrak{a}M)$, we have $(\psi_{\mathfrak{a}}(f))(m) = f(m + \mathfrak{a}M)$, for all $m \in M$.

($\Rightarrow$) Suppose $\mathfrak{a}M \subseteq \mathfrak{b}M$. Then, there is a natural induced R-epimorphism $\pi : M/\mathfrak{a}M \to M/\mathfrak{b}M$. Therefore, to prove that $(0 :_{D(M)} \mathfrak{b}) \subseteq (0 :_{D(M)} \mathfrak{a})$, it is enough to consider the injective homomorphism

$$\psi_{\mathfrak{a}} \circ D(\pi) \circ (\psi_{\mathfrak{b}})^{-1} : (0 :_{D(M)} \mathfrak{b}) \to (0 :_{D(M)} \mathfrak{a}).$$

($\Leftarrow$) Conversely, assume $(0 :_{D(M)} \mathfrak{b}) \subseteq (0 :_{D(M)} \mathfrak{a})$. By contradiction, suppose $\mathfrak{a}M \nsubseteq \mathfrak{b}M$, then there exists $y \in \mathfrak{a}M \setminus \mathfrak{b}M$. Since E is the injective hull of the R-module $R/\mathfrak{m}$, it is an injective cogenerator for R, therefore there exists $f \in D(M/\mathfrak{b}M)$ such that $f(y + \mathfrak{b}M) \neq 0$. Now

$$\psi_{\mathfrak{b}}(f) \in (0 :_{D(M)} \mathfrak{b}) \subseteq (0 :_{D(M)} \mathfrak{a}),$$

and hence $\mathfrak{a}(\psi_{\mathfrak{b}}(f)) = 0$. Since $y \in \mathfrak{a}M$, it follows that $(\psi_{\mathfrak{b}}(f))(y) = 0$, that is, $f(y + \mathfrak{b}M) = 0$, which is a contradiction. Hence $\mathfrak{a}M \subseteq \mathfrak{b}M$.

3. Let $x \in \overline{\mathfrak{a}}^{(M)}$. Hence, there exists $n \in \mathbb{N}$ such that $x^n M \subseteq \sum_{i=1}^{n} x^{n-i}\mathfrak{a}^i M$, then it follows from part 2 that

$$\left(0 :_{D(M)} \sum_{i=1}^{n} x^{n-i}\mathfrak{a}^i\right) \subseteq (0 :_{D(M)} x^n).$$

Thus, $x \in \mathfrak{a}^{*(D(M))}$.

The other inclusion can be proved following the reverse argument. □

6.2 Multiplicities

This section is devoted to the theory of multiplicity and its relationship to integral closure. We use (HUNEKE; SWANSON, 2006) as our primary reference, where more results can be found. We begin with the traditional concept of multiplicity of an ideal on a finitely generated module and some properties. After that, we give the definition of the multiplicity of an ideal relative to an Artinian module, we give more properties and we try to find a relation between them.

During this section, $(R,\mathfrak{m})$ will be a commutative Noetherian local ring such that $\dim R = r$. Furthermore, M will be a non-zero finitely generated R-module.

The next results gives us the definition of an important polynomial.

Theorem 6.2.1. **(The Hilbert–Samuel Polynomia**l, (HUNEKE; SWANSON, 2006, Theorem 11.1.3)) Let $\mathfrak{a}$ be a $\mathfrak{m}$-primary ideal. There exists a polynomial $P(n)$, with rational coefficients, such that for all $n >> 0$, $P(n) = \lambda_R(M/\mathfrak{a}^n M)$. Furthermore, the degree in n of $P(n)$ is $\dim M$, which is at most r.

Definition 6.2.2. The $\lambda_R(M/\mathfrak{a}^n M)$, in Theorem 6.2.1, is called the *Hilbert function* and $P(n)$ is called the *Hilbert–Samuel polynomial* of $\mathfrak{a}$ with respect to M. If the dependence of $P(n)$ on $\mathfrak{a}$ and M needs to be specified, we write $P_{\mathfrak{a},M}(n)$.

Definition 6.2.3. Let $\mathfrak{a}$ be a $\mathfrak{m}$-primary ideal. One can define the *multiplicity* of $\mathfrak{a}$ *on* M as follows

$$e(\mathfrak{a};M) = \lim_{n\to\infty} \frac{r!}{n^r}\lambda(M/\mathfrak{a}^n M).$$

In other words, $e(\mathfrak{a};M)$ is the leading coefficient of the Hilbert–Samuel polynomial of $\mathfrak{a}$ with respect to M.

If $M = R$, then denote $e(\mathfrak{a};R)$ by $e(\mathfrak{a})$ and call it the *multiplicity* of $\mathfrak{a}$.

The next result gives us a relation between $e(\mathfrak{a};M)$ and $e(\mathfrak{a})$. Here $\mathrm{rk}_R(M)$ denotes the rank of the R-module M.

Proposition 6.2.4. (HUNEKE; SWANSON, 2006, Corollary 11.2.6) Let $(R,\mathfrak{m})$ be a Noetherian local domain, $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, and M be a finitely generated R-module. Then $e(\mathfrak{a};M) = e(\mathfrak{a})\cdot \mathrm{rk}_R(M)$.

The next result is the first step to give a relation between the integral closure and multiplicity of an ideal on a finitely generated R-module.

Proposition 6.2.5. ((HUNEKE; SWANSON, 2006, Proposition 11.2.1)) Let $(R,\mathfrak{m})$ be a Noetherian local ring, and assume that $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{m}$-primary ideals such that $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$. Let M be a finitely generated R-module. Then $e(\mathfrak{a};M) = e(\mathfrak{b};M)$.

The final step is due to Rees. To enunciate it, we will need two definitions.

Definition 6.2.6. If A is a ring with finite Krull dimension, A is said to be *equidimensional* if $\dim A/\mathfrak{p} = \dim A$ for every minimal prime $\mathfrak{p}$ of A.

Definition 6.2.7. Let $(R,\mathfrak{m})$ be a commutative Noetehrian local ring. R is said to be *formally equidimensional* (or *quasi-unmixed*) if its completion $\overline{R}$ is equidimensional.

One way to think about formally equidimensional rings is that in such rings parameters are "true" parameters, i.e., their images stay parameters after completing and going modulo the minimal prime ideals.

Now, we have this important result from Rees.

Theorem 6.2.8. (**Rees's Theorem**, (REES, 1961)) Let $(R,\mathfrak{m})$ be a commutative formally equidimensional Noetherian local ring, and $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R. Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $e(\mathfrak{a}) = e(\mathfrak{b})$.

From now on, let H be a non-zero Artinian R-module (note that H does not need to be finitely generated). Lets see what kind of definitions we can do using Artinian R-modules.

Let $\mathfrak{a} \subsetneq \mathfrak{m}$ be an ideal of R such that $\lambda(0 :_H \mathfrak{a}) < \infty$. (KIRBY, 1973, Theorem 2) and (SHARP; TAHERIZADEH, 1988, Theorem 4.1) showed that for large $n \in \mathbb{N}$ the length $\lambda(0 :_H \mathfrak{a}^n)$ is a polynomial function, called *Hilbert-Samuel polynomial of* $\mathfrak{a}$ *relative to* H. (KIRBY, 1990, Proposition 2.7) has shown that this polynomial function has degree $d = \operatorname{Ndim}_R(H)$ (Noetherian dimension, as we define in Subsection 6.2.1).

Definition 6.2.9. Let $\mathfrak{a} \subsetneq \mathfrak{m}$ be an ideal of R such that $\lambda(0 :_H \mathfrak{a}) < \infty$. One can define the *multiplicity of* $\mathfrak{a}$ *relative to the Artinian module* H as follows

$$e'(\mathfrak{a};H) = \lim_{n\to\infty} \frac{d!}{n^d}\lambda(0 :_H \mathfrak{a}^n).$$

In other words, $e'(\mathfrak{a};H)$ is the leading coefficient of the Hilbert–Samuel polynomial of $\mathfrak{a}$ with respect to H.

The next Proposition is the same as Proposition 6.2.5 but for Artinian modules.

Proposition 6.2.10. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and $\lambda(0 :_H \mathfrak{b}) < \infty$. If $\mathfrak{a}^{*(H)} = \mathfrak{b}^{*(H)}$, then $e'(\mathfrak{a};H) = e'(\mathfrak{b};H)$.

Proof. By Corollary 6.1.5, $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ relative to H. Therefore, there exists $s \in \mathbb{N}_0$ such that $(0 :_H \mathfrak{a}^{s+1}) = (0 :_H \mathfrak{b}\mathfrak{a}^s)$. Hence, for all $n \geq s+1$, $(0 :_H \mathfrak{a}^n) = (0 :_H \mathfrak{b}^{n-s}\mathfrak{a}^s)$.

Since $(0 :_H \mathfrak{a}^n) \subseteq (0 :_H \mathfrak{b}^n)$ and $(0 :_H \mathfrak{b}^{n-s}) \subseteq (0 :_H \mathfrak{b}^{n-s}\mathfrak{a}^s)$, we have that

$$\lambda(0 :_H \mathfrak{b}^{n-s}) \leq \lambda(0 :_H \mathfrak{b}^{n-s}\mathfrak{a}^s) = \lambda(0 :_H \mathfrak{a}^n) \leq (0 :_H \mathfrak{b}^n) < \infty,$$

for all $n \geq s$.

We know that, for large n, $\lambda(0 :_H \mathfrak{b}^{n-s})$, $\lambda(0 :_H \mathfrak{b}^n)$ and $\lambda(0 :_H \mathfrak{a}^n)$ are polynomials in n of degree $d = \mathrm{Ndim}_R(H)$. Therefore, after dividing by n^d and taking limits as $n \to \infty$, we see that $e'(\mathfrak{a}; H) = e'(\mathfrak{b}; H)$. □

There is no version of Rees' Theorem to integral closure relative to an Artinian module and multiplicity of an ideal relative to an Artinian module. This is an open question and we are still working on it to see what can be done.

6.2.1 Noetherian Dimension

Assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring. Let M and H be two R-modules such that H is Artinian.

In this section, we are going to see the concept of the Noetherian dimension, denoted by $\mathrm{Ndim}_R(H)$, of an Artinian R-module, which was introduced by (ROBERTS, 1975) (who called it by the name of "Krull dimension") and had the nomenclature changed by (KIRBY, 1990).

Definition 6.2.11. The *Noetherian dimension*, $\mathrm{Ndim}_R(M)$, of an R-module M is defined inductively as follows:

When $M = 0$ we put $\mathrm{Ndim}_R(M) = -1$.

Let $r \geq 0$ be an integer. Assume that those R-modules which have Noetherian dimension less than r have been specified.

We put $\mathrm{Ndim}_R(M) = r$ when

(i) $\mathrm{Ndim}_R(M) < r$ is false; and

(ii) for every ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of submodules of M there exists an integer $m_0 \geq 0$ such that $\mathrm{Ndim}_R(M_{n+1}/M_n) < r$ for all $n \geq m_0$.

If M is an R-module such that, for all integers $r \geq -1$, M does not have Noetherian dimension r, we say M has infinite Noetherian dimension.

Note that those R-modules M with $\mathrm{Ndim}_R(M) = 0$ are precisely the non-zero Noetherian modules.

In what follows, let's see some properties about Noetherian dimension.

Proposition 6.2.12. ((ROBERTS, 1975, Proposition 1)) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then $\mathrm{Ndim}_R(M) = \max\{\mathrm{Ndim}_R(L), \mathrm{Ndim}(N)\}$.

Theorem 6.2.13. ((KIRBY, 1990, Theorem 2.6)) An Artinian R-module H has finite Noetherian dimension.

The next result gives us an easier way to calculate the Noetherian dimension.

Theorem 6.2.14. ((ROBERTS, 1975, Theorem 6)) Let H be an Artinian R-module. Then $\mathrm{Ndim}_R(H) = -1$ if and only if $H = 0$. If $H \neq 0$,

$$\mathrm{Ndim}_R(H) = \inf\{k \mid \exists\, x_1,\ldots,x_k \in \mathfrak{m} \text{ such that } \lambda(0 :_H x_1,\ldots,x_k) < \infty\},$$

where $(0 :_H x_1,\ldots,x_k) := \{h \in H \mid h(x_1,\ldots,x_k) = 0\}$.

6.3 Relating those concepts

In this section, let $(R,\mathfrak{m})$ be a commutative Noetherian local ring. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R. Let M be a finitely generated R-module and H be an Artinian R-module (which can be finitely generated or not).

In view of what we have done in Section 6.1, a natural question is Question 1.0.6 in the Introduction, which we enunciate here again.

Question 6.3.1. (D. Rees) Fix H an Artinian R-module. Is there a relationship between $\overline{\mathfrak{b}}$, the classical Northcott-Rees integral closure of $\mathfrak{b}$, and $\mathfrak{b}^{*(H)}$, the ST-closure of $\mathfrak{b}$ on the Artinian R-module H? Are there any Artinian R-module for which they are equal?

Using Rees's Theorem, if we give an answer to the previous question, we will be able to find some answer to Question 1.0.7:

Question 6.3.2. What is the relationship between $e(\mathfrak{a})$, the multiplicity of $\mathfrak{a}$, and $e'(\mathfrak{a};H)$, the multiplicity of $\mathfrak{a}$ relative to H?

(SHARP; TIRAŞ; YASSI, 1990) responded to Rees' question in a particular case: when $(R,\mathfrak{m})$ is a quasi-unmixed local ring of dimension r and $H = H^r_{\mathfrak{m}}(R)$, the rth local cohomology module of R with respect to $\mathfrak{m}$. They proved, under those conditions, that $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$.

However, it is not true in general. R. Y. Sharp, Y. Tiras and M. Yassi showed an example where the equality does not hold, which we will see next.

Example 6.3.3. Consider the local ring $B := k[[X_1,X_2,X_3]]$, where k is a field, and let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals of B given by $\mathfrak{p} = (X_1)$ and $\mathfrak{q} = (X_2,X_3)$; let $A = B/\mathfrak{p}\cap\mathfrak{q}$ and note that $\dim A = 2$. Take $H = \mathrm{H}^2_{\mathfrak{m}}(A)$.

Then the classical integral closure $\overline{0}$ of the zero ideal 0 of A is just the nilradical $\sqrt{0}$ of A, and this is zero because A is a reduced ring.

However, $0^{*(H)}$, the ST-closure of 0 on H, is not zero, as we now show. Since $(0 :_A H) \subseteq 0^{*(H)}$, because of Remark 6.1.3, it is sufficient, in order to show that $0^{*(H)} \neq 0$, to establish that $(0 :_A H) \neq 0$. Lets prove this last statement: by Theorem 3.4.5, $H \neq 0$, and by Example 4.3.5

$$\emptyset \neq \mathrm{Att}_A(H) = \{\mathfrak{p} \in \mathrm{Ass}_A(A) \mid \dim A/\mathfrak{p} = 2\}.$$

Suppose $(0:_A H)=0$, so

$$\bigcap_{\mathfrak{p}\in \operatorname{Att}_A(H)} \mathfrak{p} = \sqrt{(0:_A H)} = \sqrt{0} = \bigcap_{\mathfrak{p}\in \operatorname{Spec}(A)} \mathfrak{p}$$

thus, every $\mathfrak{p}\in \operatorname{Spec}(A)$ must be in $\operatorname{Att}_A(H)$, that is, $\mathfrak{p}\in \operatorname{Ass}_A(A)$ and $\dim A/\mathfrak{p}=2$, for all $\mathfrak{p}\in \operatorname{Spec}(A)$. But $\overline{\mathfrak{q}}=\mathfrak{q}/\mathfrak{p}\cap\mathfrak{q}\in \operatorname{Spec}(A)\cap \operatorname{Ass}_A(A)$ and $\dim A/\overline{\mathfrak{q}}=1$, which is a contradiction.

However, it is still an interesting question to find classes of R-modules for which the equality holds.

The answer of R. Y. Sharp, Y. Tiras and M. Yassi is restricted to local cohomology. Thus, the question asked by D. Rees can be rephrased in the following way:

Question 6.3.4. What is the largest class of modules where the equality in Rees' question is true?

Therefore, our aim in this section is to generalize the results given in (SHARP; TIRAŞ; YASSI, 1990) for an Artinian R-module H. To do this, first consider the next Lemma.

Lemma 6.3.5. Every minimal prime ideal of R is in $\operatorname{Att}_R(H)$ if and only if the R-ideal $(0:_R H)$ is nilpotent.

Proof. Since H is an Artinian R-module, it has a reduced secondary representation, say $H = S_1+S_2+\ldots+S_l$, where $\sqrt{(0:_R S_i)}=\mathfrak{p}_i$, for all $i=1,\ldots,l$. Thus $\operatorname{Att}_R(H)=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_l\}$ and

$$\sqrt{(0:_R H)} = \bigcap_{i=1}^{l} \mathfrak{p}_i. \tag{6.1}$$

Therefore, the desired result follows easily. □

Now we return to Question 6.3.4 and we have an important result of this section.

Theorem 6.3.6. Assume $(R,\mathfrak{m})$ is a commutative Noetherian complete local ring. Let H be an Artinian R-module. The following conditions are equivalent:

(i) $\overline{\mathfrak{a}}=\mathfrak{a}^{*(H)}$ for every ideal $\mathfrak{a}$ of R;

(ii) $\overline{0}=0^{*(H)}$;

(iii) every minimal prime ideal of R belongs to $\operatorname{Att}_R(H)$;

(iv) $(0:_R H)$ is nilpotent.

Proof. $(iii) \Leftrightarrow (iv)$ By Lemma 6.3.5.

$(i) \Rightarrow (ii)$ Just take $\mathfrak{a} = 0$.

$(ii) \Rightarrow (iv)$ Assume $\overline{0} = 0^{*(H)}$. Since $\overline{0} = \sqrt{0}$ is nilpotent and $(0 :_R H) \subseteq 0^{*(H)}$ by (6.1.3), it follows that $(0 :_R H)$ is nilpotent.

$(iii) \Rightarrow (i)$ Let $\mathfrak{a}$ an arbitrary ideal of R. By Theorem 6.1.8 (iii) it follows that $\overline{\mathfrak{a}}^{(D(H))} = \mathfrak{a}^{*(H)}$, since $D(H)$ is a Noetherian R-module. To finish the proof it is sufficient to show that $\overline{\mathfrak{a}}^{(D(H))} = \overline{\mathfrak{a}}$.

By Remark 6.1.7, we have

$$(0 :_R D(H)) \subseteq \overline{\mathfrak{a}}^{(D(H))} \text{ and } \overline{\overline{\mathfrak{a}}} = \overline{\mathfrak{a}}^{(D(H))}/(0 :_R D(H)).$$

Note that $(0 :_R D(H)) = (0 :_R H)$ is nilpotent by Lemma 6.3.5, then

$$(0 :_R D(H)) \subseteq \overline{\mathfrak{a}} \text{ and } \overline{\overline{\mathfrak{a}}} = \overline{\mathfrak{a}}/(0 :_R D(H)).$$

Therefore, $\overline{\mathfrak{a}}^{(D(H))} = \overline{\mathfrak{a}}$. □

Now, we can prove the following Theorem, which gives a relation between multiplicity of an ideal and multiplicity of an ideal over an Artinian R-module.

Theorem 6.3.7. Let $(R, \mathfrak{m})$ be a commutative complete formally equidimensional Noetherian local ring, and $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R. Let H be an Artinian R-module such that every minimal prime ideal of R belongs to $\mathrm{Att}_R(H)$. Suppose $\lambda(0 :_H \mathfrak{b}) < \infty$ and $e(\mathfrak{a}) = e(\mathfrak{b})$. Then $e'(\mathfrak{a}; H) = e'(\mathfrak{b}; H)$.

Proof. Use Rees's Theorem, Theorem 6.3.6 and Proposition 6.2.10. □

Let M be a finitely generated R-module. In view of Theorem 6.3.7, one could improve Question 6.3.2 in the following way:

Question 6.3.8. What is the relationship between $e(\mathfrak{a}; M)$, the multiplicity of $\mathfrak{a}$ on M, and $e'(\mathfrak{a}; H)$, multiplicity of $\mathfrak{a}$ relative to H?

To give an answer this question, we shall assume $(R, \mathfrak{m})$ is a Noetherian local domain. In this case, there is a result similar to Rees's Theorem, for finitely generated R-modules M such that $\mathrm{rk}_R(M) \neq 0$.

Theorem 6.3.9. Let $(R, \mathfrak{m})$ be a formally equidimensional Noetherian local domain, $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module such that $\mathrm{rk}_R(M) \neq 0$. Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $e(\mathfrak{a}; M) = e(\mathfrak{b}; M)$.

Proof. Use Rees's Theorem and Proposition 6.2.4. □

So, we can rewrite an analogous to Theorem 6.3.7 in the following way.

Theorem 6.3.10. Let $(R,\mathfrak{m})$ be a commutative complete formally equidimensional Noetherian local domain, and $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R. Let M be a finitely generated R-module such that $\mathrm{rk}_R(M) \neq 0$, and let H be an Artinian R-module such that every minimal prime ideal of R belongs to $\mathrm{Att}_R(H)$. Suppose $\lambda(0:_H \mathfrak{b}) < \infty$ and $e(\mathfrak{a};M) = e(\mathfrak{b};M)$. Then $e'(\mathfrak{a};H) = e'(\mathfrak{b};H)$.

Proof. Use Theorem 6.3.9, Theorem 6.3.6 and Proposition 6.2.10. □

Corollary 6.3.11. Let $(R,\mathfrak{m})$ be a commutative complete formally equidimensional Noetherian local domain, and $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R. Let H be an Artinian R-module such that every minimal prime ideal of R belongs to $\mathrm{Att}_R(H)$ and $\mathrm{rk}_R(D(H)) \neq 0$. Suppose $\lambda(0:_H \mathfrak{b}) < \infty$ and $e(\mathfrak{a};D(H)) = e(\mathfrak{b};D(H))$. Then $e'(\mathfrak{a};H) = e'(\mathfrak{b};H)$.

Corollary 6.3.12. Let $(R,\mathfrak{m})$ be a commutative complete formally equidimensional Noetherian local domain, and $\mathfrak{b} \subseteq \mathfrak{a}$ be two $\mathfrak{m}$-primary ideals of R. Let M be a finitely generated R-module such that $\mathrm{rk}_R(M) \neq 0$ and every minimal prime ideal of R belongs to $\mathrm{Ass}_R(M)$. Suppose $\lambda(0:_H \mathfrak{b}) < \infty$ and $e(\mathfrak{a};M) = e(\mathfrak{b};M)$. Then $e'(\mathfrak{a};D(M)) = e'(\mathfrak{b};D(M))$.

6.4 Examples and Applications

As a consequence of Theorem 6.3.6, we have some examples and applications involving Artinian modules, local cohomology modules and generalized local cohomology modules, as follows.

Example 6.4.1. Let $(R,\mathfrak{m})$ be a commutative Artinian complete local ring. Let H be an Artinian R-module, thus there is a reduced secondary representation for H, $H = S_1 + \cdots + S_n$. Notice that $P_i = \sqrt{(0:_R S_i)} \in \mathrm{Spec}(R) = \{\mathfrak{m}\}$ for all $i = 1,\ldots,s$. Then, $\mathrm{Att}_R(H) = \{\mathfrak{m}\}$. Therefore, for all ideal $\mathfrak{b}$ of R, $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$.

Example 6.4.2. Let $(R,\mathfrak{m})$ be a commutative Noetherian complete local ring. Let M be a finitely generated R-module such that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(R)$.

Take $H = D(M)$, which is Artinian. Then, by Proposition 4.3.2, $\mathrm{Att}_R(H) = \mathrm{Ass}_R(M)$. Moreover, every minimal prime ideal of R is in $\mathrm{Ass}_R(M)$. Thus, every minimal prime ideal of R belongs to $\mathrm{Att}_R(H)$. Therefore, $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$ for every ideal $\mathfrak{b}$ of R.

The following Lemma gives us some conditions to verify when the hypothesis $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(R)$ in Example 6.4.2 is satisfied, and it is an important tool in what follows. First, recall a definition.

Definition 6.4.3. Let R be a commutative Noetherian ring. An R-module M is called a *syzygy* R-module if there exists an injective R-homomorphism $\iota : M \to F$, where F is a free R-module.

Definition 6.4.4. Let R be a commutative Noetherian ring. An R-module M is called *faithful* module M the annihilator of M is the zero ideal.

Lemma 6.4.5. Let R be a commutative Noetherian ring. Let M be a finitely generated syzygy R-module. Then,

$$\operatorname{Ass}_R(R/\operatorname{Ann}_R(M)) \subseteq \operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(R).$$

In particular, if M is a faithful R-module, then $\operatorname{Ass}_R(M) = \operatorname{Ass}_R(R)$.

Proof. Set $I = \operatorname{Ann}_R(M)$. There is an inclusion $R/I \hookrightarrow \operatorname{Hom}_R(M,M)$, and hence $\operatorname{Ass}_R(R/I) \subseteq \operatorname{Ass}_R(M)$. Now, as M is a syzygy R-module, $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(R)$. □

Now, there is another consequence of Theorem 6.3.6.

Proposition 6.4.6. Let $(R,\mathfrak{m})$ be a commutative Noetherian complete local ring and let M be a faithful finitely generated syzygy module. Then $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$, for every ideal $\mathfrak{b}$ of R, where $H = D(M)$.

Proof. Let $H = D(M)$, which is an Artinian R-module. By Proposition 4.3.2, $\operatorname{Att}_R(H) = \operatorname{Ass}_R(M)$. Also, by Lemma 6.4.5, $\operatorname{Ass}_R(R) = \operatorname{Ass}_R(M)$, then every minimal prime of R is in $\operatorname{Ass}_R(M)$.

Therefore, every minimal prime ideal of R belong to $\operatorname{Att}_R(H)$. The result now follows immediately from Theorem 6.3.6. □

Assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring (note that here we are not assuming R is complete). Let M and N be two R-modules. Take $H = \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$. The next Lemma shows that $\mathfrak{b}^{*(H)}$, the ST-closure of $\mathfrak{b}$ on the Artinian R-module H, is invariant under completion.

Lemma 6.4.7. Let $H = \mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)$ be as above. Then, H has a natural structure as a module over $(\widehat{R},\widehat{\mathfrak{m}})$, the $\mathfrak{m}$-adic completion of R, and $\widehat{H} \cong H^{d+n}_{\widehat{a}}(\widehat{M},\widehat{N})$. Furthermore, for any ideal $\mathfrak{b}$ of R,

$$(\mathfrak{b}\widehat{R})^{*(H)} \cap R = \mathfrak{b}^{*(H)}.$$

Proof. Since H is Artinian, there is a natural $\widehat{R}$-module structure on H. Besides, $\widehat{H} = H \otimes_R \widehat{R} \cong H^{d+n}_{\mathfrak{a}\widehat{R}}(M \otimes_R \widehat{R}, N \otimes_R \widehat{R}) = H^{d+n}_{\widehat{\mathfrak{a}}}(\widehat{M},\widehat{N})$, since $\widehat{R}$ is R-flat and $\mathfrak{a}\widehat{R} = \widehat{\mathfrak{a}}$.

For the second part, let $x \in R$ and $n \in \mathbb{N}$. The proof follows from the equality

$$\left(0 :_H \sum_{i=1}^{n} x^{n-i}\mathfrak{b}^i\right) = \left(0 :_H \sum_{i=1}^{n} x^{n-i}(\mathfrak{b}\widehat{R})^i\right).$$

□

Using these facts, we have the following application.

Proposition 6.4.8. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring. Let M, N be two finitely generated R-modules such that $d = \operatorname{pdim}(M) < \infty$ and $n = \dim N < \infty$. Let $\mathfrak{a}$ be an ideal of R. Set $H := \mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ and suppose $\operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p}) = d+n$ for all minimal prime ideals $\mathfrak{p}$ of R. Then, for all ideal $\mathfrak{b}$ of R, $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$.

Proof. We already know H is an Artinian R-module and

$$\operatorname{Att}_R(H) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p}) = d+n\}.$$

By using (6.1),

$$\sqrt{(0:_R H)} = \bigcap_{\substack{\mathfrak{p}\in \operatorname{Spec}(R)\\ \operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p})=d+n}} \mathfrak{p}.$$

then we can conclude that $(0:_R H)$ is nilpotent if and only if every prime ideal $\mathfrak{p}$ of R is such that $\operatorname{cd}(\mathfrak{a},M,R/\mathfrak{p}) = d+n$.

Note that by Lemma 6.4.7 and the fact that $\overline{(\mathfrak{a}\widehat{R})} \cap R = \overline{\mathfrak{a}}$ for every ideal $\mathfrak{a}$ of R (JR, 1981, Lemma 2.5), we may assume R is complete. Therefore, we conclude the proof using Theorem 6.3.6. □

As another application of our main result we obtain the next Corollary, which is one of the main results obtained by (SHARP; TIRAŞ; YASSI, 1990, Corollary 3.5), but without the hypothesis that $(R,\mathfrak{m})$ is a homomorphic image of a Gorenstein local ring. Taking $M = R$ in Proposition 6.4.8, we have what follows.

Corollary 6.4.9. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring. Let N be a finitely generated R-module such that $n = \dim N$. If $H := \mathrm{H}_{\mathfrak{a}}^{n}(N)$ and $\operatorname{cd}(\mathfrak{a},R/\mathfrak{p}) = n$ for all minimal prime ideals $\mathfrak{p}$ of R, then $\overline{\mathfrak{b}} = \mathfrak{b}^{*(H)}$, for every ideal $\mathfrak{b}$ of R.

Example 6.4.10. Let k be a field of characteristic 0 and $S = k[x_1,x_2,x_3]$ denote the ring of polynomials over k. Set $\mathfrak{n} = (x_1,x_2,x_3)$ a maximal ideal, $\mathfrak{a} = (x_2^2 - x_1^2 - x_1^3)$, $\mathfrak{b} = (x_2)$ and $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$. Let y_i denote the image of x_i in $S/\mathfrak{c}$. Let $R = (S/\mathfrak{c})_{\mathfrak{n}/\mathfrak{c}}$ and

$$I = (y_1 + y_2 - y_2y_3)R + ((y_3-1)^2(y_1+1)-1)R$$

According to (NHAN; CHAU, 2012, Example 3.9), $\operatorname{Ass}_R(R) = \{\mathfrak{a}R, \mathfrak{b}R\}$, $\dim R = 2$ and $\operatorname{Att}_R(H_I^2(R)) = \{\mathfrak{a}R, \mathfrak{b}R\}$. Therefore $\operatorname{Ass}_R(R) = \operatorname{Att}_R(H_I^2(R))$, and every minimal prime ideal of R belongs to $\operatorname{Att}_R(H_I^2(R))$. Hence $\overline{J} = J^{*(H_I^2(R))}$ for every ideal J of R.

INDEX

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