# Equilibrium states and their local product structure for partially hyperbolic diffeomorphisms

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SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

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# Estados de equilíbrio e sua estrutura de produto local para difeomorfismos parcialmente hiperbólicos

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# Abstract

We address the problem of existence and uniqueness (or finiteness) of ergodic equilibrium states for a natural class of partially hyperbolic diffeomorphisms homotopic to Anosov. We propose to study the disintegration of equilibrium states along the central foliation as a tool to develop the theory of equilibrium states for partially hyperbolic dynamics. For a large class of low variational potentials we obtain existence and uniqueness of the equilibrium state and we also obtain a dichotomy between finiteness of ergodic equilibrium states and hyperbolicity of such measures.

We also prove that the measure of maximal entropy for accessible partially hyperbolic diffeomorphisms of 3-manifold having compact center leaves can be written locally as the product of three measures defined on the local stable, central and unstable foliations provided that such measure is unique. We verify that the local product structure does not hold when the number of measures of maximal entropy is larger than one.

**Keywords:** Equilibrium states, disintegration of measures, local product structure, partially hyperbolic diffeomorphisms, measure of maximal entropy.

# Resumo

Abordamos o problema de existência e unicidade (ou finitude) dos estados de equilíbrio ergódicos para uma classe natural de difeomorfismos parcialmente hiperbólicos homotópicos a um Anosov. Propomos estudar a desintegração dos estados de equilíbrio ao longo da folheação central como uma ferramenta para desenvolver a teoria de estados de equilíbrio para sistemas parcialmente hiperbólicos. Para uma classe de potenciais com variação pequena obtemos existência e unicidade de estados de equilíbrio e também obtemos uma dicotomia entre finitude dos estados de equilíbrio ergódicos e hiperbolicidade de tais medidas.

Obtemos também que as medidas de máxima entropia para difeomorfismos parcialmente hiperbólicos acessíveis definidos numa 3-variedade tendo folhas centrais compactas podem ser escritas localmente como o produto de três medidas definidas nas folheações stável, central e instável locais sempre que tal medida é única. Verificamos que a estrutura de produto local não é valida quando o número de medidas de máxima entropia é maior que um.

**Palavras-chaves:** Estados de equilíbrio, desintegração de medidas, estrutura de produto local, difeomorfismos parcialmente hiperbólicos, medidas de máxima entropia.

# Contents

Chapter 1

# Introduction

An equilibrium state for a continuous map $f : M \to M$ with respect to a potential $\phi : M \to \mathbb{R}$ is an invariant Borel probability measure $\mu$ that maximizes the quantity $h_\mu(f) + \int \phi d\mu$ among all invariant measures. In the particular case $\phi \equiv 0$, such maximizer is called measure of maximal entropy (if it exists). It is an old and very important problem to know about existence and uniqueness of equilibrium states. For hyperbolic dynamics and expanding endomorphisms this problem was extensively studied by Sinai [52], Ruelle [50], and Bowen [5], [6].

In the other hand, considerable research was done for non-uniformly hyperbolic systems. For one dimensional systems we mention the important contributions made by Keller [28], Hofbauer [24], Buzzi [13], Buzzi and Sarig [15], Pesin and Senti [40], Bruin [9], Bruin and Keller [10], Bruin and Todd [11], Iommi and Todd [27], among others. In dimension bigger than one, Oliveira [36], Oliveira and Viana [38], and Varandas and Viana [58], considered non-uniformyly expanding maps, Arbieto and Prudente [1], Rios and Siquiera [45], Leplaideur, Oliveira and Rios [33], considered partially hyperbolic horseshoes, Climenhaga, Fisher and Thompson [16], considered the robustly transitive diffeomorphisms of Mañé and Bonatti-Viana, Pesin, Senti and Zhang [41], [42], considered the Katok map and towers of hyperbolic type; for a more complete picture of equilibrium states for non-uniformly hyperbolic see the survey [18].

We also mention that R. Spatzier and D. Visscher [53] proved uniqueness of equilibrium state for frame flows on closed, oriented, negatively curved $n-$manifold, $n$ odd and ($n \neq 7$) and potentials induced by potentials defined on unit tangent bundles, i.e constant on the fibers of the bundle $FM \to SM$ where $FM$ and $SM$

are respectively frame bundle and unit tangent bundle.

## 1.1 Equilibrium states for partially hyperbolic diffeomorphisms in $\mathbb{T}^3$

We consider partially hyperbolic systems, i.e., diffeomorphisms of compact manifolds $f : M \to M$ with an invariant spliting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that vectors in $E^s$ are exponentially contracted under iteration, vectors in $E^u$ are exponentially expanded, while vectors in $E^c$ are neither contracted as strongly as any vector in $E^s$ nor expanded as strongly as any vector in $E^u$. For those class of diffeomorphisms, it is well known that there are foliations $\mathcal{F}^\sigma$ tangent to the sub-bundles $E^\sigma$ for $\sigma = s, u$. In general, it is not true that there is a foliation tangent to the central sub-bundle $E^c$ (see for instance [47]). However, by Brin, Burago, Ivanov [7] all absolutely partially hyperbolic diffeomorphism on $\mathbb{T}^3$ admit a foliation tangent to $E^c$, the center foliation.

It is well known that all entropy-expansive maps have equilibrium states with respect to any continuos potential. By the work of Diaz, Fisher, Pacifico and Vieitez [20] all partially hyperbolic diffeomorphisms with 1-dimensional center direction are entropy-expansive. Therefore, all partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ have equilibrium states with respect to any continuous potential. However, we can ask the following.

**Question 1.1.1.** *Do all absolutely partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ have a unique equilibrium state associated to Hölder continuous potentials?*

J. Buzzi, T. Fisher, M. Sambarino and C. Vásquez [14] showed that the Mañé's example has a unique measure of maximal entropy, and R. Ures in [57] proved the same property for any absolutely partially hyperbolic diffeomorphisms homotopic to a linear Anosov diffeomorphism of $\mathbb{T}^3$. We call this class of diffeomorphisms as Derived from Anosov diffeomorphisms (see Definition 2.2.7).

F. Rodriguez Hertz, A. Rodriguez Hertz, A. Tahzibi and R. Ures in [49] showed that partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ with compact central leaves have finitely many ergodic measures of maximal entropy. In fact they show that "typically" ([49], Theorem 1), there is more than one measure of maximal entropy.

By a work of Hammerlindl [23], the absolutely partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ are classified into two groups of diffeomorphisms mentioned above

(Derived from Anosov and diffeomorphims with compact central leaves). Then, we can ask the next.

**Question 1.1.2.** *Do all Derived from Anosov diffeomorphisms have a unique equilibrium state associated to Hölder continuous potentials?*

By a recent work of Climenhaga, Fisher and Thompson [16], the Mañé's example has a unique equilibrium states with respect to Hölder continuous potential satisfying some technical conditions.

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a derived from Anosov diffeomorphism, it is well known that $f$ is semi-conjugated to a linear Anosov diffeomorphism $A$ on $\mathbb{T}^3$ by a semiconjugacy $H$. Let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a potential Hölder continuous and considered the potential $\phi$ definded by $\phi = \psi \circ H$.

In this thesis we study equilibrium states for Derived from Anosov diffeomorphisms on $\mathbb{T}^3$ associated to potential $\phi = \psi \circ H$, for which we make to use of disintegration of measures along the central foliation of these diffeomorphisms.

Our first result gives a partial answer to the question 1.1.2. We denote by $C$ the set of points where $H$ fails to be injective.

**Theorem A.** *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a Derived from Anosov diffeomorphism and let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a Hölder continuous potential. Define $\phi = \psi \circ H$ and let $\mu$ be an ergodic equilibrium state for $f$ with respect to $\phi$:*

*1) If $\mu(C) = 0$, then $\mu$ is the unique equilibrium state.*

*2) If $\mu(C) = 1$, then $\mu$ is virtually hyperbolic (see 2.3.9) and there exists necessarily another equilibrium state $\eta$ for $(f, \phi)$.*

The proof of the above theorem enables us to conclude a dichotomy between finiteness of ergodic equilibrium states and hyperbolicity of such measures.

**Theorem B.** *Let $f$ and $\phi$ be as in Theorem A. Then either there is an ergodic non-hyperbolic equilibrium state or the number of ergodic equilibrium states is finite.*

Recall that $f$ has a unique measure of maximal entropy. Moreover, we can show that under small variation hypothesis of the potential, the equilibrium state is unique.

Let $\lambda_1, \lambda_2$ and $\lambda_3$ be the Lyapunov exponents of $A$ such that $\lambda_3 < 0 < \lambda_2 < \lambda_1$.

**Theorem C.** *Let $f$ and $\phi$ be as in Theorem A. If the potential satisfies* $\sup_{\mathbb{T}^3} \phi - \inf_{\mathbb{T}^3} \phi < \lambda_2$*, then there exists a unique equilibrium state for* $(f, \phi)$.

The "small" variational condition in the above theorem is common in the literature to achieve uniqueness of equilibrium states and it has been considered by K. Oliveira and M. Viana [38] for non-uniformly expanding maps on compact manifolds, by I. Rios and J. Siqueira [45] for partially hyperbolic horsehoes, by F. Hofbauer and G. Keller [25] for piece wise monotomic maps, by H. Bruin and M. Todd [12] for interval maps, and by M. Denker, M. Urbánski [19] for rational maps on the Riemann sphere.

Our approach to study the uniqueness (or finiteness) of the equilibrium states for Derived from Anosov diffeomorphisms is based on using the disintegration of equilibrium state along the central foliation. To prove Theorem A, we prove in Lemma 3.1.1 that the conditional measure for equilibrium states along central foliation are monoatomic, that is, it consist of a unique atom per leaf. Similar results were studied by Ponce, Tahzibi and Varão in [43], [44]. Since that each center leaf has a unique atom, we can construct another equilibrium state which is also virtually hyperbolic. To show finiteness of ergodic equilibrium states when such measures are hyperbolic (see Theorem B) we supposse that there exist an infinite number of such measures, since such measures are virtually hyperbolic (see Lemma 3.1.1) is possible to construct an hyperbolic ergodic equilibrium states. To prove the uniqueness of equilibrium state for Derived from Anosov associated to potential with small variation (see Theorem C), we prove in Proposition 3.3.1 that the metric entropy of $f$ with respect to equilibrium states is less or equal than $\lambda_1$.

## 1.2 Local product structure

The local product structure of measures of maximal entropy for uniformly hyperbolic diffeomorphisms was showed by the work of Ruelle and Sullivan [51], and Leplaideur in [32] showed that the unique equilibrium states associated for Hölder continuous potential for such diffeomorphisms has local product structure. Roughly speaking such property means that locally the measure can be written as the product of two measures defined in the stable and unstable manifold. For non-uniformly hyperbolic diffeomophisms it was showed by Barreira, Pesin and Schemling [4] that hyperbolic measures have "almost" local product structure and using this property of hyperbolic measures they prove the long-standing Eckmann-Ruelle conjecture in dimension theory of smooth dynamical systems.

Since absolutely partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ have measures of maximal entropy and there exist stable, central and unstable foliations, we can ask:

**Question 1.2.1.** *Can the measures of maximal entropy for absolutely partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ be written locally as the product of three measures defined on the local stable, central and unstable foliations?*

The previous question was enunciated by F. Rodriguez Hertz in a context more general (see [46], Section 4).

Our next result answer the above question for such diffeomorfisms with compact central leaves, more specifically we prove:

**Theorem D.** *Let $f : M \to M$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism of a three dimensional closed manifold $M$. Assume that $f$ is dynamically coherent with compact one-dimensional central leaves and has the accessibility property. Then*

1) *If $\mu$ is the unique measure of maximal entropy for $f$ (see item 1, Theorem 4.0.4), then for each $x \in M$, there are measures $\mu_x^s$, $\mu_x^u$ and $\mu_x^c$ defined in $\mathcal{F}^s_{loc}(x)$ (local stable manifold of $x$), $\mathcal{F}^u_{loc}(x)$ and $\mathcal{F}^c_{loc}(x)$ respectively, such that*

$$\mu = \varphi_* \mu_x^s \times \mu_x^c \times \mu_x^u \tag{1.1}$$

*on $\varphi(\mathcal{F}^s_{loc}(x) \times \mathcal{F}^c_{loc}(x) \times \mathcal{F}^u_{loc}(x))$ where $\varphi : \mathcal{F}^s_{loc}(x) \times \mathcal{F}^c_{loc}(x) \times \mathcal{F}^u_{loc}(x) \to M$ is defined by*

$$\varphi(z,t,w) = \mathcal{F}^u_{loc}(\mathcal{F}^c_{loc}(z) \cap \mathcal{F}^s_{loc}(t)) \cap \mathcal{F}^{cs}_{loc}(w)$$

2) *If $\mu$ is a hyperbolic ergodic measure of maximal entropy for $f$ (see item 2, theorem 4.0.4), then do not exist measures defined in the local stable, unstable and central manifold such that (1.1) is true for $\mu$.*

We also verified that the local product structure does not hold for equilibrium states of derived from Anosov diffeomorphisms which are virtually hyperbolics (see Theorem 4.1.6).

## 1.3 Structure of the thesis

This work is organized as follows:

In Chapter 2 we state some fundamental background concepts and results in equilibrium states, partially hyperbolic dynamics, ergodic theory, measure disintegration theory and local product structure of measures.

In Chapter 3 we study equilibrium states for Derived from Anosov diffeomorphisms. In the Section 3.1 we prove Theorem A showing ergodic equilibrium states are virtually hyperbolic and that is used to construct twin measures. The construction of the above result is used to prove in the Section 3.2 Theorem B. In the Section 3.3 we prove Theorem C showing that equilibrium states associated to potentials with small variation are not virtually hyperbolic. In the Section 3.4 we study the center Lyapunov exponent of equilibrium states, showing an inequality between such exponent and the center Lyapunov exponent of the linear part.

In Chapter 4 we study local product structure of measures of maximal entropy for partially hyperbolic diffeomorphisms in 3-dimensional manifold, dynamically coherent with compact one-dimensional central leaves and with accessibility property.

Chapter 2

# Preliminaries

## 2.1 Equilibrium states

Let $(M, \mu, \mathcal{B})$ be a probability space, where $(M, d)$ is a compact metric space, $\mu$ a Borel probability measure and $\mathcal{B}$ is the Borel $\sigma$-algebra. Let $f : M \to M$ be a measurable transformation. The measure $\mu$ is said to be $f$-invariant if

$$f_*\mu(B) := \mu(f^{-1}(B)) = \mu(B), \text{ for all } B \in \mathcal{B}.$$

We denoted by $\mathcal{M}(f)$ the set of $f-$invariant Borel probability measures. If $f : M \to M$ is a continuous transformation, then $\mathcal{M}(f) \neq \emptyset$.

An $f$-invariant measure is called ergodic if $f^{-1}(B) = B$, implies $\mu(B) = 0$ or 1. The ergodic measure set is denoted by $\mathcal{M}_e(f)$.

Let $\mathcal{P}$ be a finite partition of $M$. The entropy of the partition $\mathcal{P}$ is defined by

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

We denoted by $f^{-1}(\mathcal{P}) = \{f^{-1}(P) : P \in \mathcal{P}\}$, $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ and $\mathcal{P}^n = \vee_{j=0}^{n-1} f^{-j}(\mathcal{P})$.

**Definition 2.1.1.** *Let $f : M \to M$ be a measurable transformation preserving a probability measure $\mu$ in $M$. The metric entropy of $f$ with respect to $\mu$ and a measurable partition $\mathcal{P}$ of $M$ is defined by*

$$h_\mu(f, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

*And the metric entropy of $f$ with respect to $\mu$ is defined by*

$$h_\mu(f) := \sup\{h_\mu(f, \mathcal{P}) : \mathcal{P} \textit{ is a partition of } M\}.$$

Let $f : M \to M$ be a continuous transformation of a metric space $(M, d)$ and let $K \subseteq M$ be an any compact subset. Let $n \in \mathbb{N}$ and $\epsilon > 0$. We say that a subset $E \subseteq K$ is $(n, \epsilon)$-separated if for $x, y \in E$, $x \neq y$, there exist $i \in \{0, 1, \cdots, n\}$ such that $d(f^i(x), f^i(y)) \geq \epsilon$. Defined

$$h_n(f, K, \epsilon) := \sup\{\#E : E \subseteq K \text{ is } (n, \epsilon) - \text{separated}\}$$

and

$$h(f, K) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log h_n(f, K, \epsilon).$$

**Definition 2.1.2.** *Let $f : M \to M$ be a continuous transformation of a compact metric space $M$. The topological entropy of $f$ is defined by*

$$h_{top}(f) := h(f, M).$$

**Theorem 2.1.3** (Variational Principle)**.** *If $f : M \to M$ is a continuous transformation of a compact metric space, then*

$$h_{top}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}(f)\}.$$

**Definition 2.1.4.** *An $f$-invariant probability measure $\mu$ is called a measure of maximal entropy for $f$ if*

$$h_\mu(f) = h_{top}(f).$$

**Theorem 2.1.5** (Ledrappier-Walters Variational principle, [30])**.** *Let $M$ and $N$ be compact metric spaces and $f : M \to M$, $g : N \to N$, $\pi : M \to N$ be a continuous maps such that $\pi$ is surjective and $\pi \circ f = g \circ \pi$. Then*

$$\sup_{\mu : \pi_* \mu = \nu} h_\mu(f) = h_\nu(g) + \int_N h(f, \pi^{-1}(y)) d\nu(y).$$

**Definition 2.1.6.** *Consider a continuous map $f : M \to M$ on a compact manifold $M$. We say that an $f-$invariant Borel probability measure $\mu$ is an equilibrium state for $f$ with respect to a potential $\phi \in C^0(M, \mathbb{R})$ if it satisfies*

$$h_\mu(f) + \int \phi d\mu = \sup\{h_\eta(f) + \int \phi d\eta : \eta \in \mathcal{M}(f)\}.$$

*Remark* 2.1.7. In the above definition we can change $\mathcal{M}(f)$ by $\mathcal{M}_e(f)$. This is a consequence of the affine property of the metric entropy (that is $h_{(1-t)\mu + t\nu}(f) = (1-t)h_\mu(f) + t h_\nu(f)$, for $0 < t < 1$) which was generalized by Jacobs for ergodic decomposition (see 2.3.11). Let $\mu \in \mathcal{M}(f)$ and $\{\mu_P : P \in \mathcal{P}\}$ its ergodic decomposition. By theorems 2.3.10 and 2.3.11, we have

$$h_\mu(f) + \int \phi d\mu = \int \left( h_{\mu_P}(f) + \int \phi d\mu_P \right) d\tilde{\mu}(P).$$

So,

$$\sup\{h_\mu(f) + \int \phi d\mu : \mu \in \mathcal{M}(f)\} \leq \sup\{h_\mu(f) + \int \phi d\mu : \mu \in \mathcal{M}_e(f)\}.$$

**Lemma 2.1.8.** *Let $X$ be a compact metric space and let $\psi : X \to [-\infty, \infty)$ a map. The following are equivalent:*

1. *$\psi$ is upper semicontinuous.*
2. *$\{x : \psi(x) < c\}$ is closet set for each $c \in \mathbb{R}$.*
3. *If $x, x_n \in X$ and $\lim_{n\to\infty} x_n = x$, then $\limsup_{n\to\infty} \psi(x_n) \leq \psi(x)$.*

*Proof.* See [[29], Lemma 4.1.5]. □

We define the entropy function that is denoted by $h : \mathcal{M}(f) \to [0, \infty)$ and defined by

$$h(\mu) := h_\mu(f).$$

**Proposition 2.1.9.** *If the entropy function is upper semicontinuous, then $f$ has an equilibrium states with respect to any potential continuous $\phi : M \to \mathbb{R}$. Moreover, the equilibrium states set for $(f, \phi)$ is compact and convex subset of $\mathcal{M}(f)$.*

*Proof.* Defined $P_\phi : \mathcal{M}(f) \to \mathbb{R}$ by $P_\phi(\mu) = h_\mu(f) + \int \phi d\mu$. Since that the entropy function is upper semicontinuous, we have $P_\phi$ and as $\mathcal{M}(f)$ is compact then, $P_\phi$ has maximum, i.e., there exist $\mu \in \mathcal{M}(f)$ such that

$$h_\mu(f) + \int \phi d\mu \geq \sup\{h_\eta(f) + \int \phi d\eta : \eta \in \mathcal{M}(f)\}.$$

Then $\mu$ is an equilibrium states for $(f, \phi)$. On the other hand, for each $k \in \mathbb{N}$ we define the sets

$$F_k := \{\mu : P_\phi(\mu) \geq P(\phi) - \frac{1}{k}\}$$

where $P(\phi) = \sup\{h_\eta(f) + \int \phi d\eta : \eta \in \mathcal{M}(f)\}$. Since that $P_\phi$ is upper semicontinuous, we have that each $F_k$ is closed (compact) set in $\mathcal{M}(f)$ (see 2.1.8). Hence,

$$\bigcap_{k\in\mathbb{N}} F_k = \{\mu : h_\mu(f) + \int \phi d\mu = P(\phi)\}$$

the equilibrium states set for $(f, \phi)$ is compact. □

Let $f : M \to M$ be a continuous transformation of a metric space $(M, d)$. Given $\epsilon > 0$ let

$$\Gamma_\epsilon^+(x) := \{y \in M : d(f^k(x), f^k(y)) < \epsilon \text{ for all } k \geq 0\}.$$

If $f$ is invertible we define

$$\Gamma_\epsilon(x) := \{y \in M : d(f^k(x), f^k(y)) < \epsilon \text{ for all } k \in \mathbb{Z}\}.$$

The map $f$ is called $\epsilon-$expansive if $\Gamma_\epsilon(x) = \{x\}$. Define

$$h_f^*(\epsilon) := \sup_{x \in M} h(f, \Gamma_\epsilon^+(x)).$$

**Definition 2.1.10.** *Let $f : M \to M$ be a continuous transformation of a metric space $(M, d)$. $f$ is called entropy expansive (or h-expansive) if exist some $\epsilon > 0$ such that*

$$h_f^*(\epsilon) = 0.$$

*And $f$ is called asymptotically h-expansive if*

$$\lim_{\epsilon \to 0} h_f^*(\epsilon) = 0.$$

*Remark* 2.1.11. $\epsilon-$expansive $\Rightarrow$ entropy expansive (or h-expansive)$\Rightarrow$ asymptotically h-expansive.

The next theorem gives a relation to the entropy function and asymptotically h-expansive.

**Theorem 2.1.12** (Misiurewicz, [35])**.** *If $f : M \to M$ is asymptotically h-expansive. Then the entropy function is uppersemicontinuous.*

**Corollary 2.1.13.** *If $f : M \to M$ is asymptotically h-expansive. Then $f$ has an equilibrium states with respect to any continuous potential$\phi : M \to \mathbb{R}$.*

## 2.2 Partially hyperbolic diffeomorphisms

**Definition 2.2.1.** *Let $M$ be a closed manifold. A diffeomorphism $f : M \to M$ is called partially hyperbolic if the tangent bundle $TM$ admits a $Df$-invariant descomposition $TM = E^s \oplus E^c \oplus E^u$ such that all unit vectors $v^\sigma \in E_x^\sigma (\sigma = s, c, u)$ for all $x \in M$ satisfy:*

$$\| Df(x)v^s \| < \| Df(x)v^c \| < \| Df(x)v^u \|$$

*and moreover* $\| Df|_{E^s}\| < 1$ *and* $\| Df^{-1}|_{E^u}\| < 1$.

*We called* $f$ *absolutely partially hyperbolic, if it is partially hyperbolic and for any* $x, y, z \in M$

$$\| Df(x)v^s \| < \| Df(y)v^c \| < \| Df(z)v^u \|$$

*where* $v^s \in E^s_x$, $v^c \in E^c_y$ *and* $v^u \in E^u_z$.

The set of absolutely partially hyperbolic diffeomorphism is $C^1$ open inside the set of all diffeomorphisms of $M$.

For partially hyperbolic diffeomorphisms, it is well known that there are foliations $\mathcal{F}^\sigma$ tangent to the sub-bundles $E^\sigma$ for $\sigma = s, u$. The leaf of $\mathcal{F}^\sigma$ containing $x$ will be called $\mathcal{F}^\sigma(x)$, for $\sigma = s, u$. In general, it is not true that there is a foliation tangent to the central sub-bundle $E^c$.

**Definition 2.2.2.** *A diffeomorphism* $f : M \to M$ *partially hyperbolic is called dinamically coherent if there exist invariant foliations* $\mathcal{F}^{c\sigma}$ *tangent to* $E^{c\sigma}$ *for* $\sigma = s, u$.

Note that by taking the intersection of these foliations, we obtain an invariant foliation $\mathcal{F}^c$ tangent to $E^c$ that subfoliates $\mathcal{F}^{c\sigma}$ for $\sigma = s, u$. Hertz-Hertz-Ures [47] gave an example of partially hyperbolic diffeomorphism on $\mathbb{T}^3$ which is not dinamically coherent. However, Brin, Burago, Ivanov [7] showed that all absolutely partially hyperbolic diffeomorphism on $\mathbb{T}^3$ admit central foliation tangent to $E^c$.

**Theorem 2.2.3** (Brin, Burago, Ivanov [7] )**.** *All absolutely partially hyperbolic diffeomorphisms on* $\mathbb{T}^3$ *are dynamically coherent.*

**Definition 2.2.4.** *A partially hyperbolic diffeomorphism is called accessible if one can join any two points in the manifold by a path which is piecewise tangent to either* $E^s$ *or* $E^u$.

**Theorem 2.2.5** (Hertz-Hertz-Ures [48])**.** *For all* $1 \le r \le \infty$, *accessibility is open and dense in the of* $C^r$ *partially hyperbolic diffeomorphisms on a compact Riemannian manifold* $M$, *preserving a smooth probability measure* $m$, *with one dimensional center bundle.*

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism. Consider $f_* : \mathbb{Z}^3 \to \mathbb{Z}^3$ the action of $f$ on the fundamental group of $\mathbb{Z}^3$ . $f_*$ can be extended to $\mathbb{R}^3$ and the extension is the lift of a unique linear automorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$.

**Definition 2.2.6.** *Given $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism, the unique linear automorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$ with lift $f_* : \mathbb{R}^3 \to \mathbb{R}^3$ as constructed in the previous paragraph, is called the linearization of $f$.*

It can be proved that the linearization $A$ of an absolutely partially hyperbolic diffeomorphism $f$, is a partially hyperbolic automorphism of torus (see [8]).

A. Hammerlindl in [23], proves that any absolutely partially hyperbolic diffeomorphism $f$ on $\mathbb{T}^3$ is leaf conjugated to its linearization. This means that there exist an homeomorphism $G : \mathbb{T}^3 \to \mathbb{T}^3$ such that $G$ sends the central leaves of $f$ to central leaves of $f_*$ and conjugates the dynamics of the leaf spaces. In particular the central leaves of $f$ are all homeomorphic. As a consequence of Hammerlindl's result, we have that absolutely partially hyperbolic diffeomorphism $f$ on $\mathbb{T}^3$ fall into distinct groups:

1. If $f$ is homotopic to a linear Anosov, then every center leaf is dense in $\mathbb{T}^3$ and is homeomorphic to a line.

2. If $f$ is not homotopic to a linear Anosov, then every center leaf is homeomorphic to a circle $\mathbb{S}^1$.

**Definition 2.2.7.** *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism. $f$ is called Derived from Anosov (DA) diffeomorphism if is homotopic to a linear Anosov $A : \mathbb{T}^3 \to \mathbb{T}^3$.*

**Definition 2.2.8.** *A foliation $\mathcal{F}$ defined on a manifold $M$ is quasi-isometric if the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal cover of $M$ has the following property: There exist positive constants $Q, Q'$ such that for all $x, y$ in a common leaf of $\tilde{\mathcal{F}}$ we have*

$$d_{\tilde{\mathcal{F}}}(x, y) \leq Q \parallel x - y \parallel + Q'$$

*where $d_{\tilde{\mathcal{F}}}$ denotes the Riemannian metric on $\tilde{\mathcal{F}}$ and $\parallel x - y \parallel$ is the distance on the universal cover.*

For absolutely partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ the stable, unstable and central foliations are quasi-isometric in the universal covering $\mathbb{R}^3$ (see [23], [7]).

### 2.2.1 Geometry of the center foliation

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a Derived from Anosov diffeomorphism. By a well-known result of Franks [21] $f$ is semiconjugated to $A$. More specifically, there exists $H : \mathbb{T}^3 \to \mathbb{T}^3$ homotopic to the identity such that

$$H \circ f = A \circ H.$$

Moreover, this semi conjugacy has the property that there exists a constant $K > 0$ such that if $\tilde{H} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the lift of $H$ to $\mathbb{R}^3$ we have $\|\tilde{H}(x) - x\| \leq K$ for all $x \in \mathbb{R}^3$, and given two points $a, b \in \mathbb{R}^3$, there exists a constant $\Omega > 0$ with

$$\tilde{H}(a) = \tilde{H}(b) \Leftrightarrow \| \tilde{f}^n(a) - \tilde{f}^n(a) \| < \Omega \text{ for all } n \in \mathbb{Z}.$$

Using this characterization of the semi conjugacy and the quasi-isometry property on the center foliation proved by A. Hammerlindl [23], R. Ures proved the following result.

**Proposition 2.2.9** (Ures [57]). *For all $z \in \mathbb{T}^3$, the pre-image $H^{-1}(z)$ is a compact connected subset (i.e. an arc or point) of the center manifold with uniformly bounded length.*

**Proposition 2.2.10** (Ures [57]). *The $H$-image of a center manifold of $f$ is a center manifold of $A$, that is,*

$$H(\mathcal{F}^c_f(x)) = \mathcal{F}^c_A(H(x)).$$

Conversely, observe that Proposition 2.2.9 implies that $H^{-1}(\mathcal{F}^c_A(x))$ is contained in $\mathcal{F}^c_f(z)$ for any $z \in H^{-1}(x)$. Moreover, we have that $H^{-1}(\mathcal{F}^c_A(x)) = \mathcal{F}^c_f(z)$.

**Lemma 2.2.11** (Ures [57]). *If $\nu$ is an $A-$invariant measure and $\mu$ $f$-invariant measure such that $\nu = H_*\mu$. Then*

$$h_\mu(f) = h_\nu(A).$$

*Proof.* By Ledrappier-Walters variational principle [30], we have

$$\sup_{\mu : H_*\mu = \nu} h_\mu(f) = h_\nu(A) + \int_N h(f, H^{-1}(y)) d\nu(y). \tag{2.1}$$

Since that $H^{-1}(y)$ is a compact connected interval (including the case of just a point) of the center manifold with uniformly bounded length, we have $h(f, H^{-1}(y)) = 0$, for all $y \in \mathbb{T}^3$. Therefore of 2.1

$$h_\mu(f) \leq \sup_{\mu : H_*\mu = \nu} h_\mu(f) = h_\nu(A).$$

The other inequality is immediate since that $f$ is semiconjugated to $A$. $\square$

## 2.3 Disintegration of measures

In order to prove uniqueness (or finiteness) of equilibrium states, we propose to study the conditional measures of equilibrium states on the leaves of central foliation. In what follows we review some basic properties of disintegration of measures.

Let $(M, \mu, \mathcal{B})$ be a probability space, where $M$ is a compact metric space, $\mu$ a probability measure and $\mathcal{B}$ the borelian $\sigma$-algebra. Given a partition $\mathcal{P}$ of $M$ by measurable sets, we associate the probability space $(\tilde{M} := M/\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ by the following way. Let $\pi : M \to \tilde{M}$ be the canonical projection, that is, $\pi$ associates a point $x$ of $M$ to the partition element of $\mathcal{P}$ that contains it. Then we define $\widetilde{\mu} := \pi_*\mu$ and $\widetilde{\mathcal{B}} := \pi_*\mathcal{B}$.

**Definition 2.3.1.** *Given a partition $\mathcal{P}$. A family $\{\mu_P\}_{P\in\mathcal{P}}$ is a system of conditional measures for $\mu$ (with respect to $\mathcal{P}$) if*

*i) given $\phi \in C^0(M)$, then $P \mapsto \int \phi d\mu_P$ is measurable.*

*ii) $\mu_P(P) = 1$ $\widetilde{\mu}$-a.e.*

*iii) $\mu = \int_{\tilde{M}} \mu_P d\tilde{\mu}$, i.e if $\phi \in C^0(M)$, then $\int_M \phi d\mu = \int_{\tilde{M}} \int_P \phi d\mu_P d\tilde{\mu}$.*

When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ *disintegrates* the measure $\mu$. There exists an equivalent form of writing the disintegration formula above:

$$\mu = \int_M \mu_x d\mu$$

by considering the conditional measures $\mu_x, x \in M$ where $\mu_x = \mu_y$ if $y \in \mathcal{P}(x)$. In this work we use both formulation to simplify the notations whenever it is necessary.

**Proposition 2.3.2** ( [22], [55])**.** *If $\{\mu_P\}$ and $\{\nu_P\}$ are conditional measures that disintegrate $\mu$, then $\mu_P = \nu_P$ $\widetilde{\mu}$-a.e.*

**Corollary 2.3.3.** *If $T : M \to M$ preserves a probability $\mu$ and the partition $\mathcal{P}$, then $T_*\mu_P = \mu_{T(P)}, \widetilde{\mu}$-a.e.*

*Proof.* It follows from the fact that $\{T_*\mu_P\}_{P\in\mathcal{P}}$ is also a disintegration of $\mu$ and essential uniqueness of system of disintegration. □

**Definition 2.3.4.** *We say that a partition $\mathcal{P}$ is measurable (or countably generated) with respect to $\mu$ if there exist a measurable family $\{A_i\}_{i\in\mathbb{N}}$ and a measurable set*

*$F$ of full measure such that if $B \in \mathcal{P}$, then there exists a sequence $\{B_i\}$, where $B_i \in \{A_i, A_i^c\}$ such that $B \cap F = \bigcap_i B_i \cap F$.*

**Theorem 2.3.5** (Rokhlin's disintegration [55])**.** *Let $\mathcal{P}$ be a measurable partition of a compact metric space $M$ and $\mu$ a Borel probability. Then there exists a disintegration by conditional measures for $\mu$.*

Let us state a simple but usefull remark which comes from essential uniqueness of disintegration.

*Remark* 2.3.6. Let $(M, \mathcal{B}, \mu)$ be a probability space, $\mathcal{P}$ a measurable partition of $M$ and $X \subset M$ a measurable subset of positive measure. Then $X$ is called $\mathcal{P}-$saturated if for any $x \in X$ then $\mathcal{P}(x)$, the atom of partition containing $x$, is contained in $X$. Let $\mu|_X$ be the normalized (probability) restriction of $\mu$ on $X$. For any $P \in \mathcal{P}$ such that $P \subset X$, the conditional measures of $\mu$ and $\mu|_X$ coincide, that is $\mu_P = (\mu|_X)_P$.

More generally, if $X \subset M$ is a measurable subset with positive measure then $\mathcal{P}$ induces a measurable partition on $X$. Namely,

$$\mathcal{P}_X := \{P_X | P_X := P \cap X; P \in \mathcal{P}\}$$

is a measurable partition of $X$. So, by Rokhlin theorem we consider the conditional measures $(\mu|_X)_{P_X}\}$ obtaining by disintegration of the probability $\mu|X$ on the atoms of partition $\mathcal{P}_X$. We will use later in the thesis the following fact which can be verified using the essential uniqueness of conditional measures: $(\mu|_X)_{P_X} = (\mu_P)|_{P_X}$ and consequently $\mu_P \leq (\mu|_X)_{P_X}$ on $P_X \subseteq P$.

### 2.3.1 Atomic disintegration along foliations

In general the partition by the leaves of a foliation may be non-measurable. It is for instance the case for the stable and unstable foliations of Anosov diffeomorphisms with respect to measures of non vanishing metric entropy (see [17], Theorem 3.1). Therefore, by disintegration of a measure along the leaves of a foliation we mean the disintegration on compact foliated boxes. In principle, the conditional measures depend on the foliated boxes, however, two different foliated boxes induce proportional conditional measures. See [3] for a discussion on this issue.

**Definition 2.3.7.** *We say that a foliation $\mathcal{F}$ has atomic disintegration with respect to a measure $\mu$ if the conditional measures on any foliated box are sum of Dirac measures.*

Equivalently we could define atomic disintegration as follows: there exist a full measurable subset $Z$ such that $Z$ intersects all leaves in at most a countable set.

Although the disintegration of a measure along a general foliation is defined in compact foliated boxes, it makes sense to say that a foliation $\mathcal{F}$ has a quantity $k_0 \in \mathbb{N}$ atoms per leaf. The meaning of "per leaf" should always be understood as a generic leaf, i.e. almost every leaf. That means that there is a set $A$ of $\mu$-full measure which intersects a generic leaf on exactly $k_0$ points.

In the atomic disintegration case, it may happen that almost all leaves intersect a full meaure set in a non finite but countable number of points.

Let us state a recent result of Yang-Viana[1] [59].

**Theorem 2.3.8.** *Let $f$ be a DA diffeomorphism and $\mu$ an invariant measure with $h_\mu > \lambda_1$ then the disintegration of $\mu$ along central foliation can not be atomic.*

Let $f$ be a derived from Anosov (or more generally any partially hyperbolic diffeomorphism) diffeomorphism.

**Definition 2.3.9.** *An $f$-invariant measure $\mu$ is called virtually hyperbolic if there exists a full measurable invariant subset $Z$ such that $Z$ intersects each center leaf in at most one point.*

The above definition was given in [34] in the context of algebraic automorphisms and the existence of such measures in partially hyperbolic diffeomorphism also had been noticed by (see for instance [54], [43]). If $\mu$ is virtually hyperbolic, then the central foliation is measurable with respect to $\mu$ and conditional measures along center leaves are dirac. Indeed the partition into central leaves is equivalent to the partition into points.

We can deduce from Rokhlin's disintegration theorem the ergodic decomposition theorem (see [37], Theorem 5.1.3).

**Theorem 2.3.10** (Ergodic decomposition)**.** *Let $M$ be a complete separable metric space, $f : M \to M$ be a mesurable transformation and $\mu$ be an invariant probability measure. Then exist a measurable set $M_0$ of $M$ with $\mu(M_0) = 1$, a partition $\mathcal{P}$ of $M_0$ into measurable subsets and a family $\{\mu_P : P \in \mathcal{P}\}$ of probability measure on $M$, satisfying.*

[1]We thank J. Yang for awaring us on the existence of this result when we were working on this project on the same time.

*i)* $\mu_P(P) = 1$ *for* $\tilde{\mu}-$*almost every* $P \in \mathcal{P}$.

*ii)* $P \to \mu_P(E)$ *is measurable, for every measurable set* $E \subset M$.

*iii)* $\mu_P(P)$ *is ergodic and* $f$*-invariant for* $\tilde{\mu}-$*almost every* $P \in \mathcal{P}$.

*iv)* $\mu(E) = \int_{\mathcal{P}} \mu_P(E) d\tilde{\mu}$, *for every measurable set* $E \subset M$.

The next result due to Jacobs is the generalized the property afim the metric entropy for ergodic decomposition, for a proof see for instance ([37], Theorem 9.6.2).

**Theorem 2.3.11** (Jacobs). *Let* $M$ *be a complete separable metric space,* $f : M \to M$ *be a mesurable transformation and* $\mu$ *be an invarint probability measure. If* $\{\mu_P : P \in \mathcal{P}\}$ *is the ergodic decomposition of* $\mu$*, then*

$$h_\mu(f) = \int h_{\mu_P}(f) d\tilde{\mu}.$$

### 2.3.2 Lyapunov exponent

Let $M$ be a compact manifold and $f : M \to M$ be a diffeomorphism. Given $x \in M$ and $v \in T_x M$, define the Lyapunov exponent

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \| Df^n v \|$$

For every $x \in M$ the function $\lambda(x, .)$ takes on finitely many values $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$ where $d = \dim M$. If $\mu$ is a $f$-invariant measure, then by Osedelec's theorem (see for instance [39]), there exist $\Lambda$ with $\mu(\Lambda) = 1$ such that these numbers exist in $\Lambda$ and are called the Lyapunov exponent of $(f, \mu)$. The functions $\lambda_i(x)$ are Borel and are invariant under $f$; in particular, if $\mu$ is an ergodic measure, then $\lambda_i(x) = \lambda_i(\mu)$ is constant almost everywhere for each $i = 1, \cdots, d$.

**Definition 2.3.12.** *Let* $M$ *be a compact manifold,* $f : M \to M$ *be a diffeomorphism and* $\mu$ *be an* $f-$*invariant measure. If none of the Lyapunov exponent of* $(f, \mu)$ *is equal to zero, then* $\mu$ *is called a hyperbolic measure.*

Let $A : \mathbb{T}^3 \to \mathbb{T}^3$ be a Anosov linear diffeomorphism with three invariant subbundle $T(\mathbb{T}^3) = E^u \oplus E^c \oplus E^s$ and $\lambda_1 > \lambda_2 > 0 > \lambda_3$ (or $\lambda_1 > 0 > \lambda_2 > \lambda_3$) are three Lyapunov exponents of $A$. Let $\nu$ be an $A$-invariant measure and $\xi$ be a measurable partition of $\mathbb{T}^3$ with respect to $\nu$. We say that $\xi$ is $\nu$-subordinate to the foliation $\mathcal{F}$ ($=\mathcal{F}^{cu}_A$ or $\mathcal{F}^u_A$.) if for $\nu$-almost every $x$, we have

1) $\xi(x) \subset \mathcal{F}(x)$;

2) $\xi(x)$ contains an open neighborhood of $x$ inside the leaf $\mathcal{F}(x)$.

Let $B_{\mathcal{F}}(x, \epsilon)$ denote the open ball in $\mathcal{F}(x)$ centered at $x$ of radius $\epsilon$. Let $\xi$ be a measurable partition subordinated to $\mathcal{F}$ with conditional measures $\{\nu_x\}$. For $x \in \Lambda$ define

$$\delta^{\mathcal{F}}(x) = \lim_{\epsilon \to 0} \frac{\log \nu_x B_{\mathcal{F}}(x, \epsilon)}{\log \epsilon}.$$

This number is well defined independent of $\xi$ (see [31], Proposition 7.3.1). $\delta^{\mathcal{F}}$ is called the dimension (or pointwise dimension) of $\nu$ on $\mathcal{F}-$manifold.

The next result is due to Ledrappier and Young and holds in much more generality than the stated below. See [31].

**Theorem 2.3.13.** *Let $A : \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov linear diffeomorphism with three invariant subbundle $T(\mathbb{T}^3) = E^u \oplus E^c \oplus E^s$ and $\lambda_1 > \lambda_2 > 0 > \lambda_3$ are three Lyapunov exponents of $A$. Let $\nu$ be an $A$-invariant measure, then*

$$h_\nu(A) = \lambda_1 \delta^u + \lambda_2(\delta^{cu} - \delta^u)$$

*where $\delta^{cu} = \delta^{\mathcal{F}^{cu}}$ and $\delta^u = \delta^{\mathcal{F}^u}$.*

Chapter 3

# Equilibrium states for derived from Anosov diffeomorphisms

The existence of equilibrium states for partially hyperbolic diffeomorphisms with 1-dimensional center direction associated to any continuous potential is guaranteed as a consequence of the work of Diaz, Fisher, Pacifico and Vieitez. In fact, they proved that these systems are entropy expansive (see 3.0.1) and Misiurewicz proved that for entropy-expansive systems the entropy function is upper semicontinuous (see 2.1.12). By Proposition 2.1.9 we have the existence of equilibrium states for these systems with respect to any continuous potential.

**Theorem 3.0.1** (L. Diaz, T. Fisher M. Pacifico J. Vieitez, [20]). *Let $f : M \to M$ be a partially hyperbolic diffeomorphisms with $Df$-invariant descomposition $TM = E^s \oplus E^c \oplus E^u$ such that $dimE^c = 1$. Then $f$ is entropy expansive.*

In particular we have the existence of equilibrium states for partially hyperbolic diffeomorphisms on 3-torus associated to any continuous potential.

**Corollary 3.0.2.** *All partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ have at least one equilibrium states with respect to any continuous potential.*

**Question 3.0.3.** *Is it true that any Hölder continuous potential admits a unique equilibrium state for a derived from Anosov diffeomorphism?*

R. Ures proved that when the potential is constant all derived from Anosov diffeomorphisms have a unique equilibrium state with respect to this potential.

**Theorem 3.0.4** (Ures, [57]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a DA diffeomorphism. Then, $f$ has a unique measure of maximal entropy.*

Climenhaga, Fisher and Thompson in a recent work answered the above question positively for a especial class of derived from Anosov diffeomorphisms. More specifically, they proved uniqueness of equilibrium states for natural class of potentials in the setting of Mañé and Bonatti-Viana class of robustly transitive diffeomorphisms (see 3.0.5). We observe that in one hand their result is more general, as it treats non partially hyperbolic setting. On the other hand, the class under consideration in their result is the special type of systems which are localized perturbations of uniformly hyperbolic dynamics. In fact their result "give a quantitative criterion for existence and uniqueness of equilibrium state involving the topological pressure, the norm and variation of the potential, the tail entropy, and the $C^0$ size of the perturbation from the original Anosov map for the Mañé and Bonatti type examples". If the potential is far from being constant, then the localized perturbation should be small.

**Theorem 3.0.5** (V. Climenhaga, T. Fisher and J. Thompson [16]). *Let $f_A : \mathbb{T}^3 \to \mathbb{T}^3$ be a Mañé example diffeomorphism and let $\phi : \mathbb{T}^3 \to \mathbb{R}$ be Hölder continuous potential. Then in any $C^0$-neighborhood of $f_A$ there exist a $C^1$-open set $\mathcal{U} \subset Diff(\mathbb{T}^3)$ which contains diffeomorphisms from the Mañé family of examples, and for every $g \in \mathcal{U}$ we have:*

- *$g$ is partially hyperbolic diffeomorphism and not Anosov.*
- *The system $(\mathbb{T}^3, g, \phi)$ has a unique equilibrium state.*

We give a partial answer to the question 3.0.3. In the setting of derived from Anosov diffeomorphisms we study the uniqueness (or finiteness) of ergodic equilibrium states associated to potentials defined for the Anosov (action on homotopy) model. More specifically, let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a derived from Anosov diffeomorphism. By a well-known result of Franks [21] $f$ is semiconjugated to a linear Anosov $A$. More specifically, there exists $H : \mathbb{T}^3 \to \mathbb{T}^3$ homotopic to the identity such that

$$H \circ f = A \circ H \tag{3.1}$$

Let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a potential Hölder continuous and considered the potential $\phi$ defined by $\phi = \psi \circ H$.

$$\begin{tikzcd}
\mathbb{T}^3 \arrow[r, "f"] \arrow[d, "H"] & \mathbb{T}^3 \arrow[d, "H"] \arrow[dr, "\phi=\psi\circ H"] & \\
\mathbb{T}^3 \arrow[r, "A"] & \mathbb{T}^3 \arrow[r, "\psi"] & \mathbb{R}
\end{tikzcd}$$

We study uniqueness (or finiteness) of ergodic equilibrium states for DA diffeomorphisms with respect to the potential $\phi = \psi \circ H$.

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a Derived from Anosov diffeomorphism. By corollary 3.0.2 there exists equilibrium state for $f$ with respect to any potential continuous. In our case, when the potential is $\phi = \psi \circ H$ the existence of equilibrium states for $(f, \phi)$ follow directly the existence of equilibrium states of $(A, \psi)$.

**Lemma 3.0.6.** *If $\nu$ is an equilibrium state for $(A, \phi)$, then any $\mu \in \mathcal{M}(f)$ such that $H_*\mu = \nu$ is an equilibrium state for $(f, \phi = \psi \circ H)$.*

*Proof.* We claim that $\{\mu \in \mathcal{M}(f) : H_*\mu = \nu\}$ is not empty. In fact,

$$T(\varphi \circ H) := \int \varphi d\nu$$

defines a positive linear funtional in $\{\varphi \circ H : \varphi \in C^0(\mathbb{T}^3)\} \subset C^0(\mathbb{T}^3)$ and $T$ can be extended to $C^0(\mathbb{T}^3)$, still positive. Since that $T(1) = T(1 \circ H) = 1$, by Riezs theorem there exists $\eta$ probability Borel measure in $\mathbb{T}^3$ such that $T$ is identified with $\eta$. In particular $H_*\eta = \nu$. We considere the sequence $\mu_n = \dfrac{1}{n} \sum_{i=0}^{n-1} f_*^i \eta$. By compactness let $\mu$ such that $\lim_{n_k \to \infty} \mu_{n_k} = \mu$, then $\mu$ is a $f$-invariant and $H_*\mu = \nu$.

On other hand, by Lemma 2.2.11, we have

$$\begin{aligned} \sup\{h_\eta(f) + \int \phi d\eta : \eta \in \mathcal{M}(f)\} &= \sup\{h_{H_*\eta}(A) + \int \psi dH_*\eta : \eta \in \mathcal{M}(f)\} \\ &\leq \sup\{h_\xi(A) + \int \psi d\xi : \xi \in \mathcal{M}(A)\} \\ &\leq h_\nu(A) + \int \psi d\nu \end{aligned}$$

By the last inequality, we have that any measure $\mu \in \mathcal{M}(f)$ such that $H_*\mu = \nu$ is an equilibrium state for $(f, \phi = \psi \circ H)$. □

*Remark* 3.0.7. Reciprocaly, if $\mu$ is an equilibrium state for $(f, \phi = \psi \circ H)$, then $H_*\mu$ is an equilibrium state for $(A, \psi)$ and it is the unique one as $\psi$ is Hölder continuous (see [5]).

Let

$$C = \{x \in \mathbb{T}^3 : \#H^{-1}H(x) > 1\}.$$

We denoted by $\mathcal{N}(x) := H^{-1}H(x)$. By the discussion in the section 2.2.1 each $\mathcal{N}(x)$ is a compact connected interval (including the case of just a point) in $\mathcal{F}^c(x)$. We say that $\mathcal{N}(x)$ is a collapse interval if $\#\mathcal{N}(x) > 1$. Then $C$ is union of these intervals

$$C = \bigcup_{\#\mathcal{N}(x)>1} \mathcal{N}(x) \tag{3.2}$$

We call $C$, the set of collapse intervals.

**Lemma 3.0.8.** *$C$ is an $f$-invariant set and $H^{-1}H(C) = C$.*

*Proof.* Let $y \in C$, then $y \in \mathcal{N}(x)$ for any $x$. Since that $\#\mathcal{N}(x) > 1$ we can suppose $x \neq y$. $H(x) = H(y)$, by (3.1) we have

$$H(f(x)) = A(H(x)) = A(H(y)) = H(f(y))$$

$f(y) \in \mathcal{N}(f(x))$. As, $x \neq y$ then $\#\mathcal{N}(f(x)) > 1$. Therefore

$$f(C) = C.$$

On other hand, suppose that $H^{-1}H(C) \nsubseteq C$. There exist $y \in H^{-1}H(C)$ such that $y \notin C$. then, $H(x) = H(y)$, with $x \in C$ and $H(f(x)) = A(H(x)) = A(H(y)) = H(f(y))$. As $y \notin C$ and $f(C) = C$, we have $y = x \in C$. This is a contradiction and proves that $H^{-1}H(C) = C$. $\square$

Let $\mu \in \mathcal{M}(f)$ be an equilibrium states for $f$ with respect to $\psi \circ H$ and by remark 2.1.7, we can suppose that $\mu$ is ergodic. The next result gives a partial answer for the question 3.0.3, more specifically we showed the following result.

**Theorem A.** *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a DA diffeomorphism and let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a Hölder continuous potential. Define $\phi = \psi \circ H$ and let $\mu$ be an ergodic equilibrium states for $f$ with respect to $\phi$:*

*1) If $\mu(C) = 0$, then $\mu$ is the unique equilibrium state.*

*2) If $\mu(C) = 1$, then $\mu$ is virtually hyperbolic (see 2.3.9) and there exists necessarily another equilibrium state $\eta$ for $(f, \phi)$.*

*Remark* 3.0.9. Observe that clearly the first item of the above theorem implies that in the second case (if it occurs) any other ergodic equilibrium state give total mass to the union of collapse intervals $C$.

## 3.1 Equilibrium states and virtual hyperbolicity

Observe that the partition into central leaves is not necessarily a measurable partition and we are not allowed apriori to apply Rokhlin disintegration result to this partition. However, the preimage by $h$ of the partition into points is a measurable partition (see [43]).

**Lemma 3.1.1.** *If $\mu(C) = 1$, then $\mu$ is virtually hyperbolic.*

*Proof.* The similar arguments appear in [44] and here we repeat for completeness. Under the hypothesis of the lemma we just consider the partition into collapse intervals:

$$\mathcal{N} := \{\mathcal{N}(x) : \mathcal{N}(x) \quad \text{is a non trivial closed interval}\}$$

So we can speak about disintegration of $\mu$ along collapse intervals. We denote by $\mu_{\mathcal{N}(x)}$ the conditional measure supported on the collapse interval containing $x$. Of course, if $\mathcal{N}(x) = \mathcal{N}(y)$ then $\mu_{\mathcal{N}(x)} = \mu_{\mathcal{N}(y)}$.

We claim that the disintegration is atomic. In fact, suppose that this is not the case. Fix an orientation for the central leaves and for each collapse interval (any element of the partition $\mathcal{N}$) and consider the left extreme point of them. It can be proved that the left extreme point of collapse intervals form a measurable set (see [44]). We call these sets as point zero, that is if $x \in \mathcal{C}$ then $0_x$ means the left extreme point associated to the segment $\mathcal{N}(x)$ where $\mathcal{N}(x) \in \mathcal{N}$ which contains $x$. If $y \in \mathcal{N}(x)$ then $[0_x, y]$ stands for the segment inside the center leaf which contains $0_x$ and $y$.

We now consider the set

$$H_\alpha = \{y : [0_x, y] \subset \mathcal{N}(x) \mid \mu_{\mathcal{N}(x)}([0_x, y]) \leq \alpha\}$$

observe that $H_\alpha$ is an invariant set. This comes from the fact that $f(\mathcal{N}(x)) = \mathcal{N}(f(x))$ and $f_*\mu_{\mathcal{N}(x)} = \mu_{\mathcal{N}(f(x))}$. Hence $H_\alpha$ is an invariant set. From the definition of disintegration and $H_\alpha$ notice that $\mu(H_\alpha) \leq \alpha$. By ergodicity we have $\mu(H_\alpha) = 0$ for all $\alpha < 1$. On the other hand, if $\alpha_n = 1 - \dfrac{1}{n}$, then $H_{\alpha_n} \subset H_{\alpha_{n+1}}$. Since we are assuming that we are not in the atomic case, we have

$$C \setminus \{1_x : x \in C\} \subset \cup_n H_{\alpha_n}$$

where $1_x$ denote the right extreme point associated to the segment $\mathcal{N}(x)$. Therefore $\mu(\{1_x : x \in C\}) = 1$, and this is a contradiction since $\mu_{\mathcal{N}(x)}(1_x) = 0$.

Thus, the disintegration is indeed atomic. In fact there exist at most one atom in each collapse interval. Indeed, let $A_n$ be the set of atoms with weight belonging to the interval $[\frac{1}{n+1}, \frac{1}{n})$. As disintegration is unique and $\mu$ is invariant, $A_n$ is invariant and by ergodicity and usual measure theory argument we get that all of the atoms have full weight. Consequently there is at most one atom in each collapse interval. So, we get a full measurable subset $\mathcal{M} \subset \mathbb{T}^3$ such that intersects each center leaf in at most a countable number of points. Observe that $H$ restricted to $\mathcal{M}$ is injective.

Now the idea is to use theorem $B$ of [43] and conclude that we have exactly one atom per (global) leaf. Although their result is for volume measure, it applies also in our setting. Let us review the main arguments. First of all we show that the number of atoms on central leaves is bounded. By this we mean that there exist a full measurable subset which intersects all center leaves in a finite (uniformly bounded) number of points.

By contradiction, suppose that this is not the case. So every full measurable subset of $\mathcal{M}$ intersects any typical center leaves in infinitely many points. Define $\nu := H_*\mu$ and observe that it is an invariant measure by the linear hyperbolic automorphism. Any measurable subset of $H(\mathcal{M})$ of $\nu-$full measure intersects almost all center leaves in an infinite (countable) number of points.

Let $\{R_i\}$ be a Markov partition for $A$ and consider the partition $\mathcal{P} := \{\mathcal{F}^c_R(x), x \in R_i \quad \text{for some } i\}$ where $\mathcal{F}^c_R(x)$ denotes the connected component of $\mathcal{F}^c(x) \cap R_i$ and contains $x$ in its interior. The partition $\mathcal{P}$ is a measurable partition and by Rokhlin theorem we can disintegrate $\nu$ along the elements of this partition. As $\nu$ is an equilibrium state for Anosov automorphism, it gives zero mass to the boundary of Markov partition. Let $\nu_x$ be the conditional measure supported on $\mathcal{F}^c_R(x)$. Observe that, as $H(\mathcal{M})$ intersects typical leaves in a countable number of points, the conditional measures $\nu_x$ should be atomic.

**Proposition 3.1.2.** *There is a natural number $\alpha_0 \in \mathbb{N}$ such that for $\nu-$almost every $x$, $\nu_x$ contains exactly $\alpha_0$ atoms.*

*Proof.* Firstly we observe that:

**Lemma 3.1.3.** *$A_*\nu_x \leq \nu_{A(x)}$ restricted to the subsets of $\mathcal{F}^c_R(A(x))$.*

Observe that $A_*\nu_x$ and $\nu_{A(x)}$ are probability measures defined respectively on $A(\mathcal{F}^c_R(x))$ and $\mathcal{F}^c_R(A(x))$. Fix an element of Markov partition $R_i$. By Remark 2.3.6, $\nu_x, x \in R_i$ coincides with the disintegration of the normalized restriction of $\nu$ on $R_i$

which we denote by $\nu|_{R_i}$. As $\nu$ is invariant $A_*(\nu|_{R_i}) = \nu|_{A(R_i)}$, by essential uniqueness of disintegration, $A_*\nu_x$ coincides with the disintegration of $\nu|_{A(R_i)}$ along the partition $A(\mathcal{F}^c_R(x)), x \in R_i$.

For any $j$ such that $A(R_i) \cap R_j \neq \emptyset$, by Markov property $A(R_i)$ crosses $R_j$ completely in the center-unstable direction and so for all $x \in R_i$,

$$\mathcal{F}^c_R(A(x)) \subset A(\mathcal{F}^c_R(x)).$$

Again by remark 2.3.6 we conclude that $A_*\nu_x \leq \nu_{A(x)}$ on $\mathcal{F}^c_R(A(x))$.

Given any $\delta \geq 0$ consider the set $K_\delta = \{x \in \mathbb{T}^3 \mid \nu_x(\{x\}) > \delta\}$, that is, the set of atoms with weight at least $\delta$. If $x \in K_\delta$ then

$$\delta < \nu_x(\{x\}) = A_*\nu_x(\{A(x)\}) \leq \nu_{A(x)}(\{A(x)\}).$$

Thus $A(K_\delta) \subset K_\delta$, and by the ergodicity of $A$ we have that $\nu(K_\delta)$ is zero or one, for each $\delta \geq 0$. Note that $\nu(A_0) = 1$ and $\nu(A_1) = 0$. Let $\delta_0$ be the critical point for which $\nu(A_\delta)$ changes value, i.e, $\delta_0 = \sup\{\delta : \nu(K_\delta) = 1\}$. This means that all the atoms have weight $\delta_0$. Due to the atomicity of disintegration, the value of $\delta_0$ has to be a strictly positive number. Since $\nu_x$ is a probability we have an $\alpha_0 := 1/\delta_0$ number of atoms as claimed. □

In particular the above lemma shows that given a fixed length $L \in \mathbb{R}$ there exist $N \in \mathbb{N}$ such that the number of atoms in any typical center plaque of size $L$ is at most $N$. Recall that we had supposed that $H(\mathcal{M})$ intrinsically intersects center leaves in infinitely many points. So, take a center plaque $D \subset \mathcal{F}^c_x$ with more than $N$ atoms. By backward contraction along central leaves by $A$ we get a large $n > 0$ such that the length of $A^{-n}(D)$ is less than $L$. As $\nu$ is invariant and disintegration is unique we get a center plaque with length less than $L$ containing more than $N$ atoms which is absurd. The proof of lemma is complete.

□

### 3.1.1 Proof of Theorem A

*1.* If $\eta$ is another equilibrium state for $(f, \psi \circ H)$, then $H_*\mu = H_*\eta$ (see remark 3.0.7). Let $\varphi : \mathbb{T}^3 \to \mathbb{R}$ be any continuous map. Since $H^{-1}H(C) = C$, implies $\eta(C) = 0$. Hence,

$$\begin{aligned}\int \varphi d\mu &= \int_{\mathbb{T}^3\backslash C} \varphi d\mu = \int_{\mathbb{T}^3\backslash C} \varphi \circ H^{-1} \circ H d\mu = \int_{\mathbb{T}^3\backslash C} \varphi \circ H^{-1} dH_*\mu \\ &= \int_{\mathbb{T}^3\backslash C} \varphi \circ H^{-1} dH_*\eta = \int_{\mathbb{T}^3\backslash C} \varphi d\eta \\ &= \int \varphi d\eta\end{aligned}$$

This implies, $\mu = \eta$.

2. We have seen that if $\mu(C) = 1$ then central foliation is measure theoretically equivalent to the partition of $\mathbb{T}^3$ into points and consequently measurable. We denote by $(\tilde{M}, \tilde{\mu})$ the quotient space $\mathbb{T}^3/\mathcal{F}^c$ equipped with the quotient measure. Observe that by virtual hyperbolicity proved above, any element $\tilde{x} \in \tilde{M}$ can be considered as a unique collapse interval inside the center leaf $\mathcal{F}^c(x)$. From now on we denote this collapse interval by $\mathcal{N}(\tilde{x})$. We denote by $\tilde{f} : \tilde{M} \to \tilde{M}$ the induced map on the quotient space. Clearly as $\mu$ is invariant by $f$ then $\tilde{\mu}$ obtained by natural quotient is invariant by $\tilde{f}$.

Now, by Lemma 3.1.1, we have that

$$\mu = \int \delta_{a(\tilde{x})} d\tilde{\mu}(\tilde{x})$$

where $a(\tilde{x}) \in \mathcal{N}(\tilde{x})$ and $\mathcal{N}(\tilde{x})$ is the collapse interval corresponding to $\tilde{x}$. Let $b(\tilde{x}) \neq a(\tilde{x})$ be the left extreme point of $\mathcal{N}(\tilde{x})$. We define

$$\eta = \int \delta_{b(\tilde{x})} d\tilde{\mu}(\tilde{x})$$

$\eta$ is well defined because $\{b(\tilde{x})\}$ is a measurable set. We claim that $\eta$ is $f-$invariant measure and $H_*\mu = H_*\eta$, and ergodicity of $\mu$ implies ergodicity of $\eta$. To show that $\eta$ is invariant take any continuous $\xi$ and observe that:

$$\begin{aligned}\int \xi \circ f d\eta &= \int \int \xi \circ f d\delta_{b(\tilde{x})} d\tilde{\mu} \\ &= \int \xi(f(b(\tilde{x}))) d\tilde{\mu} = \int \xi(b(\tilde{f}(\tilde{x}))) d\tilde{\mu} \\ &= \int \xi(b(\tilde{x})) d\tilde{\mu} = \int \xi d\eta.\end{aligned}$$

where the third equality comes from the invariance of collapse intervals and that $f$ preserves orientation on center foliation. The fourth equality is consequence of invariance of $\tilde{\mu}$ by $\tilde{f}$.

To prove the ergodicity of $\eta$, consider any invariant subset $D$ with $\eta(D) > 0$. Observe that $\tilde{\mu}$ is ergodic as an invariant measure of $\tilde{f}$. As $f(b(\tilde{x})) = b(\tilde{f}(\tilde{x}))$ and $D$ is invariant we have that $\{\tilde{x} : \chi_D(b(\tilde{x})) = 1\}$ is an $\tilde{f}$ invariant subset of $\tilde{M}$. So, ergodicity of $\tilde{\mu}$ implies that it has full measure. This implies that $\eta(D) = 1$.

By essential uniqueness of disintegration $\eta \neq \mu$.

On the other hand, as $H(a(\tilde{x})) = H(b(\tilde{x}))$ and $\phi = \psi \circ H$ we have:

$$\int \phi d\eta = \int \phi(b(\tilde{x}))d\tilde{\mu} = \int \phi(a(\tilde{x}))d\tilde{\mu} = \int \phi d\mu.$$

This implies that $\eta$ is an equilibrium state for $(f, \phi)$.

□

**Question 3.1.4.** *Is there any $\phi = \psi \circ H$ with $\psi : \mathbb{T}^3 \to \mathbb{R}$ Hölder continuous such that its equilibrium states satisfied the case 2 of the above theorem ?*

Currently we do not know $\phi = \psi \circ H$ with $\psi$ Hölder continuous that satisfies item 2 of the above theorem, although continuous examples exist. Indeed, let $\nu$ be a ergodic measure $A-$invariant such that $\nu(H(C)) = 1$. By a result of Ruelle [50], there exist a continuous map $\psi : \mathbb{T}^3 \to \mathbb{R}$ such that $\nu$ is an equilibrium state for $(A, \psi)$. Hence, if $\mu$ is $f-$invariant such that $H_*\mu = \nu$, then $\mu(C) = 1$ and $\mu$ is an equilibrium state for $(f, \phi = \psi \circ H)$.

*Remark* 3.1.5. Let $f$ be a $C^2$ DA diffeomorphism and $m$ be the measure of maximal entropy for $f$. The center Lyapunov exponent $\lambda^c(m)$ of $m$ is positive (see [57], Theorem 5.1). Considered the potential $\phi = \log \|Df(x)|_{E^c(x)}\|$. Note the $\phi$ is Hölder continuous because $f$ is $C^2$ and the distribution $E^c$ is Hölder. We claim that any ergodic equilibrium state for $(f, \phi)$ is hyperbolic with positive center Lyapunov exponent. In fact, let $\mu$ an ergodic equilibrium state for $(f, \phi)$, we have

$$h_\mu(f) + \lambda^c(\mu) = h_\mu(f) + \int \phi d\mu \geq h_m(f) + \int \phi dm = h_m(f) + \lambda^c(m).$$

If $\lambda^c(\mu) \leq 0$, then $h_\mu(f) \geq h_m(f) + \lambda^c(m) > h_{top}(f)$. This is a contradiction and proves the claim.

**Question 3.1.6.** *If there is a potential that satisfied the item 2 of the above theorem. Are there a finite number of ergodic equilibrium states associated to such potential?*

The next theorem gives a answer positive for the above question. The proof of the above theorem enables us conclude a dichotomy between finiteness of ergodic equilibrium states and hyperbolicity of such measures, more especifically we showed the next result.

**Theorem B.** *Let $f$ and $\phi$ be as in Theorem A. Then either there is an ergodic non-hyperbolic equilibrium state or the number of ergodic equilibrium states is finite.*

## 3.2 Proof of Theorem B

Let $\mu, \nu$ be an ergodic equilibrium states for $(f, \phi)$ such that $\mu(C) = \nu(C) = 1$. By Lemma 3.1.1 we have that

$$\mu = \int \delta_{a(\tilde{x})} d\tilde{\mu}(\tilde{x}), \nu = \int \delta_{b(\tilde{x})} d\tilde{\mu}(\tilde{x})$$

the next lemma prove that the atoms $a(\tilde{x}), b(\tilde{x})$ can not be in the same Pesin stable manifold if $\mu \neq \nu$, more specifically we prove.

**Lemma 3.2.1.** *Let $\mu = \int \delta_{a(\tilde{x})} d\tilde{\mu}(\tilde{x})$ and $\nu = \int \delta_{b(\tilde{x})} d\tilde{\mu}(\tilde{x})$ be $f-$invariant. If $\lim_{n\to\infty} d(f^n(a(\tilde{x})), f^n(b(\tilde{x})) = 0$, then $\mu = \nu$.*

*Proof.* Since $f_*\mu = \mu$ and $f_*\nu = \nu$, we have $\mu = f_*^n \mu = \int \delta_{f^n(a(\tilde{x}))} d\tilde{\mu}$ and $\nu = f_*^n \nu = \int \delta_{f^n(b(\tilde{x}))} d\tilde{\mu}$.

Let $\varphi : \mathbb{T}^3 \to \mathbb{R}$ be any Lipschitz map. Hence,

$$\begin{aligned} \mid \int \varphi d\mu - \int \varphi d\nu \mid &= \mid \int \varphi(f^n(a(\tilde{x}))d\tilde{\mu} - \int \varphi(f^n(b(\tilde{x}))d\tilde{\mu} \mid \\ &\leq \int \mid \varphi(f^n(a(\tilde{x})) - \varphi(f^n(b(\tilde{x})) \mid d\tilde{\mu} \\ &\leq \int kd(f^n(a(\tilde{x})), f^n(b(\tilde{x})))d\tilde{\mu} \end{aligned}$$

This implies that $\int \varphi d\mu = \int \varphi d\nu$, since that Lipschitz map set is dense in $C(\mathbb{T}^3)$, we have that last equality is holds for $\varphi \in C(\mathbb{T}^3)$. $\square$

***Proof of Theorem B***. To prove the dichotomy in the statement of the theorem, suppose that there does not exist any ergodic equilibrium state with zero central Lyapunov exponent. Now, by contradiction suppose that there is a sequence of ergodic equilibrium states $\{\mu_n\}$ for $(f, \phi)$ with negative center Lyapunov exponent.

By theorem A we have that $\mu_n(C) = 1$. By Lemma 3.1.1 all $\mu_n$ are virtually hyperbolic. Observe that for all $n$, $H_*\mu_n = \nu$ where $\nu$ is the unique equilibrium state for $(A, \psi)$.

By uniqueness of disintegration we conclude that the dirac conditional measures of $\mu_n$ are push forwarded to dirac disintegration of $\nu$ along central leaves of $A$. This implies that $\tilde{\mu}_m = \tilde{\mu}_n$ for all $n, m$ where $\tilde{\mu}_n$ is the quotient measure obtained from the disintegration of $\mu_n$ along central foliation (see section 2.3). In other words, all $\mu_n$ are virtually hyperbolic and for any two $m, n$ there exist full measurable subsets $Z_m, Z_n$, $\mu_m(Z_m) = \mu_n(Z_n) = 1$ such that $Z_m$ and $Z_n$ intersect almost all center leaves in a unique point and the intersection point belong to the same collapse interval. So,

$$\mu_n = \int \delta_{a_n(\tilde{x})} d\tilde{\mu}.$$

where $\tilde{\mu}$ stands for the quotient measure for all $\mu_n$. We emphasize that the dirac masses $a_n(\tilde{x})$ are in the same collapse interval for all $n$. Now by compactness of collapse intervals, let $a(\tilde{x}) \in \mathcal{N}(\tilde{x})$ such that $\lim_{n\to\infty} a_n(\tilde{x}) = a(\tilde{x})$. Since for any $n$, $\{a_n(\tilde{x})\}$ is a measurable invariant set it comes out that $\{a(\tilde{x})\}$ is measurable and invariant. Define,

$$\eta = \int \delta_{a(\tilde{x})} d\tilde{\mu}(\tilde{x})$$

Thus,

$$\lim_{n\to\infty} \mu_n = \eta$$

Since $f$ is entropy expansive (see Theorem 3.0.1), we have

$$\limsup_{n\to\infty} h_{\mu_n}(f) \leq h_\eta(f)$$

Hence,

$$\limsup_{n\to\infty} h_{\mu_n}(f) + \int \phi d\mu_n \leq h_\eta(f) + \int \phi d\eta$$

This implies that $\eta$ is an equilibrium state for $(f, \phi)$. Then, $H_*\mu_n = H_*\eta$ and thus $\eta(C) = 1$.

Since,

$$\lambda^c(\mu_n) = \int \log \| Df|_{E^c}\| \, d\mu_n$$

we have,

$$0 \geq \lim_{n\to\infty} \lambda^c(\mu_n) = \lim_{n\to\infty} \int \log \| Df|_{E^c}\| \, d\mu_n = \int \log \| Df|_{E^c}\| \, d\eta = \lambda^c(\eta)$$

We claim that $\lambda^c(\eta) = 0$. If not, $\lambda^c(\eta) < 0$ and by $\lim_{n\to\infty} a_n(\tilde{x}) = a(\tilde{x})$ there exist $n_0$ such that $a_n(\tilde{x})$ belong to local stable manifold of $a(\tilde{x})$, for $n \geq n_0$. By Lemma 3.2.1, we have that $\eta = \mu_n$, which it's a contradiction. Then $\lambda^c(\eta) = 0$.

We have that $\eta$ is an equilibrium state for $(f, \phi)$ with $\lambda^c(\eta) = 0$. Using similar argument in the proof of the second item of Theorem A. it is easy to see that $\eta$ is an ergodic measure. This yields a contradiction to our assumption. This concludes the proof of the dichotomy. $\square$

Recall that $f$ has a unique measure of maximal entropy and our first result says that if the equilibrium state gives zero mass to the collapse intervals set, then the equilibrium state is unique. We can ask the next.

**Question 3.2.2.** *Are there potentials whose equilibrium state give zero mass to the collapse intervals set?*

The next theorem answer the above theorem. Indeed, we can show that under small variation hypothesis of the potential, the equilibrium state is unique.

Let $\lambda_1, \lambda_2$ and $\lambda_3$ be the Lyapunov exponents of $A$ such that $\lambda_3 < 0 < \lambda_2 < \lambda_1$.

**Theorem C.** *Let $f$ and $\phi$ be as in Theorem A. If the potential satisfies $\sup_{\mathbb{T}^3} \phi - \inf_{\mathbb{T}^3} \phi < \lambda_2$, then there exists a unique equilibrium state for $(f, \phi)$.*

# 3.3 Uniqueness of equilibrium states

Theorem C is a consequence of Theorem A and 2.3.8. However, we include a proof which is interesting by itself. We prove in Proposition 3.3.1 that the metric entropy of $f$ with respect to equilibrium states with total mass to the collapse intervals set is less or equal than $\lambda_1$, and such result we using to prove uniqueness of equilibrium state for Derived from Anosov associated to potential with small variation.

## 3.3.1 Equilibrium states and nonatomic desintegration

Let $\mu$ be an equilibrium state for $(f, \phi = \psi \circ H)$.

**Proposition 3.3.1.** *If $\mu(C) = 1$, then $h_\mu(f) \leq \lambda_1$.*

**Lemma 3.3.2.** *Let $m$ be a probability measure on $\mathbb{R}^p \times \mathbb{R}^q$, $\pi$ projection onto $\mathbb{R}^p$, $m_t$ conditional measures of $m$ along the fibers of $\pi$. Define*

$$\gamma(t) = \liminf_{\epsilon\to 0} \frac{\log m \circ \pi^{-1} B^p(t, \epsilon)}{\log \epsilon}$$

*and let $\delta \geq 0$ be such that at $m-$a.e. $(s,t)$*

$$\delta \leq \liminf_{\epsilon \to 0} \frac{\log m_t B^q(s,\epsilon)}{\log \epsilon}$$

*Then, at $m-$a.e. $(s,t)$*

$$\delta + \gamma(t) \leq \liminf_{\epsilon \to 0} \frac{\log m B^{p+q}((s,t),\epsilon)}{\log \epsilon}$$

*Proof.* The proof can be find in [31], but here we write for completeness.

Fix $\sigma > 0$, we can find $N_1$ and a set $A_1$ with $m(A_1) \geq 1 - \sigma$ such that for all $(s,t) \in A_1$ and $n \geq N_1$,

$$m_t B^q(s, 2e^{-n}) \leq e^{-n\delta} e^{n\sigma}.$$

By the Lebesgue density theorem we can find $N_2$ and a set $A_2$ with $m(A_2) \geq 1 - 2\sigma$ that for all $(s,t) \in A_2$ and $n \geq N_2$,

$$m(A_1 \cap B^{p+q}((s,t), e^{-n})) \geq \frac{1}{2} m B^{p+q}((s,t), e^{-n}).$$

If $(s_0, t_0) \in A_2$ and $n \geq N_2$, we have

$$\begin{aligned} mB^{p+q}((s_0,t_0),e^{-n}) &\leq 2\int_{B^p(t_0,e^{-n})} m_t(A_1 \cap B^q(s_0,e^{-n})) m \circ \pi^{-1}(dt) \\ &\leq 2e^{-n\delta} e^{n\sigma} m \circ \pi^{-1} B^p(t_0, e^{-n}) \end{aligned}$$

because for each $t$, there exist some $u(t)$ with $(t, u(t)) \in A_1 \cap B^q(s_0, e^{-n})$ and thus $A_1 \cap B^q(s_0, e^{-n}) \cap \pi^{-1}\{t\} \subset B^q(u(t), 2e^{-n}) \cap \pi^{-1}\{t\}$. The lemma follows when $n \longrightarrow \infty$ and $\sigma \longrightarrow 0$.

□

By Lemma 3.1.1, we have that if $\mu(C) = 1$, then $\mu$ is virtually hyperbolic. Let $\nu = H_* \mu$ and $R$ be a Markov's rectangle of $A$. We normalize the restriction of $\nu$ on $A$. Let $\mathcal{F}^{cu}$ be a typical unstable leaf of $A$. Consider $R^{cu} = R \cap \mathcal{F}^{cu}$. Observe that $R^{cu}$ is foliated by strong unstable plaques and also by central (weak unstable) plaques. Denote by $\nu^{cu}$ the conditional measure of $\nu$ (normalized and restricted on $R$) on $R^{cu}$.

Since disintegration of $\nu$ along central foliation is mono-atomic, we have

$$\nu^{cu} = \int \delta_{a(t)} d\nu^{uu}(t)$$

where $a(t)$ is the unique atom on the central leaf of $t$ and $\nu^{uu}$ is the quotient measure on the quotient of $R^{cu}$ by central plaques. This quotient space can be identified by a strong unstable plaque.

**Lemma 3.3.3.** *If $\delta^{cu}$ denote the dimension of $\nu^{cu}$, then*

$$\delta^{uu} := \liminf_{\epsilon \to 0} \frac{\log \nu^{uu}(B^{uu}(x,\epsilon))}{\log \epsilon} = \delta^{cu}$$

*where $B^{uu}(x,\epsilon)$ denote the open ball with center $x$ and radius $\epsilon$ on the strong unstable leaf of $x$.*

*Proof.* The inequality

$$\delta^{cu} \geq \liminf_{\epsilon \to 0} \frac{\log \nu^{uu}(B^{uu}(x,\epsilon))}{\log \epsilon} \tag{3.3}$$

is immediate by Lemma 3.3.2.

On the other hand, we define:

$$D = \{x \in R^{cu} : \exists \alpha > 0 | \nu^{cu}(B^{uu}(x,\epsilon) \times B^{c}(x,\epsilon)) \geq \alpha \nu^{cu}(B^{uu}(x,\epsilon) \times \mathcal{F}^{c}(x)), \forall \epsilon > 0\}$$

where $B^{c}(x,\epsilon)$ denote the open ball with center $x$ and radius $\epsilon$ on the central leaf $x$.

We claim that $\nu^{cu}(D) = 1$. In fact, we prove that all atom $a(x)$ are in $D$. By definition of conditional measure

$$1 = \delta_{a(x)}(B^{c}(a(x),\gamma))) = \lim_{\epsilon \to 0} \frac{\nu^{cu}(B^{uu}(a(x),\epsilon)) \times B^{c}(a(x),\gamma))}{\nu^{cu}(B^{uu}(a(x),\epsilon) \times \mathcal{F}^{c}(x))}$$

since $a(x)$ is the unique atom on the central leaf of $x$, we have that the last equality is hold for all $\gamma > 0$.

Hence,

$$1 = \lim_{\epsilon \to 0} \frac{\nu^{cu}(B^{uu}(a(x),\epsilon)) \times B^{c}(a(x),\epsilon))}{\nu^{cu}(B^{uu}(a(x),\epsilon) \times \mathcal{F}^{c}(x))}$$

we take a large enough $n$, then there exist $\epsilon_0 > 0$ such that

$$\frac{n-1}{n} \nu^{cu}(B^{uu}(a(x),\epsilon) \times \mathcal{F}^{c}(x)) < \nu^{cu}(B^{uu}(a(x),\epsilon)) \times B^{c}(a(x),\epsilon)), \forall \epsilon < \epsilon_0$$

this proves the claim.

If $x \in D$ and since that $h^{c}_{*}\nu^{cu} = \nu^{uu}$ ($h^{c}$ is the central holonomy in $R^{cu}$), then

$$\begin{array}{rcl} \nu^{cu}(B^{uu}(x,\epsilon) \times B^{c}(x,\epsilon)) & \geq & \alpha \nu^{cu}(B^{uu}(x,\epsilon) \times \mathcal{F}^{c}(x)) \\ & \geq & \alpha \nu^{uu}(B^{uu}(x,\epsilon)) \end{array}$$

so,

$$\delta^{cu} \leq \liminf_{\epsilon \to 0} \frac{\log \nu^{uu}(B^{uu}(x,\epsilon)) + \log \alpha}{\log \epsilon}. \tag{3.4}$$

By (3.3) and (3.4) the lemma is proved.

$\square$

***Proof of the Proposition 3.3.1.*** By Ledrappier and Young's formula (see 2.3.13) and $h_\nu(A) = h_\mu(f)$, it comes out that

$$h_\mu(f) = \lambda_1\delta_1 + \lambda_2(\delta^{cu} - \delta_1)$$

where $\delta_1$ is the dimension the measure on the strong unstable leaf. By Lemma 3.3.2, we have $\delta_1 \le \delta^{cu}$ and by Lemma 3.3.3, we have $\delta_1 \le \delta^{uu}$.

So,

$$\delta^{uu}(\lambda_1 - \lambda_2) \ge \delta_1(\lambda_1 - \lambda_2)$$

by Lemma 3.3.3,

$$\delta^{uu}\lambda_1 \ge \lambda_1\delta_1 + \lambda_2(\delta^{cu} - \delta_1) = h_\mu(f)$$

since $\delta^{uu} \le 1$, we have $h_\mu(f) \le \lambda_1$. □

### 3.3.2 Proof of Theorem C

*Proof of Theorem C.* We claim that if the potential $\phi$ satisfies the low variational hypothesis of the theorem then the entropy of any equilibrium state of $\phi$ is larger than $\lambda_1$. To see this it is enough to take $\mu$ as any equilibrium state of $\phi$ and $\eta$ the measure of maximal entropy.

$$h_\mu(f) + \int \phi d\mu \ge h_\eta(f) + \int \phi d\eta = \lambda_1 + \lambda_2 + \int \phi d\eta$$

So,

$$h_\mu(f) \ge \lambda_1 + \lambda_2 + (\int \phi d\eta - \int \phi d\mu) \ge \lambda_1 + \lambda_2 - (sup\phi - inf\phi) > \lambda_1.$$

By Proposition 3.3.1, we have that all ergodic equilibrium state that satisfies the low variational hypothesis give zero mass to the union of collapse intervals $C$. Hence, by item 1 of Theorem A, if the potential $\phi$ satisfies the low variational hypothesis, then $(f, \phi)$ has a unique equilibrium state. □

## 3.4 Center Lyapunov exponent and equilibrium state

In this section we study the center Lyapunov exponent of the equilibrium states. Ures in [57], showed that the center Lyapunov exponent of the measure of maximal entropy is greater or equal to the center Lyapunov exponent of the linear part. The Theorem 3.4.1 generalizes the previous result for equilibrium states and the corollary 3.4.3 showes that the center Lyapunov exponent of the unique equilibrium state associated to potential with small variation is positive.

**Theorem 3.4.1.** *Let $\mu$ be an equilibrium state for $f$ w.r.t. a potential $\phi = \psi \circ H$. If $\lambda^c(\mu) > 0$, then*

$$\lambda_2 \leq \lambda^c(\mu) + \sup \phi - \inf \phi.$$

The proof of this result is similar to arguments of Ures ([57], Theorem 5.1) and it is based in a Pesin-Ruelle-like inequality proved by Y. Hua, R. Saghin and Z.Xia in [26]. We repeat for completeness. Let $\mathcal{W}$ be a foliation. Let $W_r(x)$ be the ball of the leaf $W(x)$ with radius $r$ and centered at $x$. Let

$$\chi_{\mathcal{W}}(x, f) = \limsup_{n\to\infty} \frac{1}{n} \log(vol(f^n(W_r(x)))$$

$\chi_{\mathcal{W}}(x, f)$ is the volume growth rate of the foliation at $x$. Let

$$\chi_{\mathcal{W}}(f) = \sup_{x\in M} \chi_{\mathcal{W}}(x, f)$$

Then, $\chi_{\mathcal{W}}(f)$ is the maximum volume growth rate of $\mathcal{W}$ under $f$. Let us denote $\chi_u(f) = \chi_{\mathcal{W}^u}(f)$ when $f$ is a partially hyperbolic diffeomorphism.

**Theorem 3.4.2** ([26])**.** *Let $f$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism. Let $\mu$ be an ergodic measure and $\lambda_i^c(\mu)$ the center Lyapunov exponent of $\mu$. Then,*

$$h_\mu(f) = \chi_u(f) + \sum_{\lambda_i^c>0} \lambda_i^c(\mu).$$

***proof of Theorem*** 3.4.1. We claim that

$$\chi_u(f) \leq \lambda_1. \tag{3.5}$$

In fact, since $\mathcal{W}^u$ is 1-dimensional the volume is the length. Then, consider

$$\frac{1}{n} \log(vol(f^n(W_r(x))).$$

Observe that $\chi_u(f) = \chi_u(\tilde{f})$ where $\tilde{f}$ is any lift of $f$ to universal cover. On the one hand, since $\mathcal{W}^u$ is quasi-isometric, we have that

$$\frac{1}{n}\log(vol(\tilde{f}^n(W_r^u(\tilde{x}))) \leq \frac{1}{n}\log(Q\text{diam}(\tilde{f}^n(W_r^u(\tilde{x})))$$

for some constant $Q$. On the other hand, $\tilde{H}(\tilde{f}^n(W_r^u(\tilde{x}))) = \tilde{A}^n(\tilde{H}(W_r^u(\tilde{x})))$. Let $C = \text{diam}(\tilde{H}(W_r^u(\tilde{x})))$. Then, $\text{diam}(\tilde{A}^n(\tilde{H}(W_r^u(\tilde{x})))) \leq C\exp(n\lambda_1)$. Since $\tilde{H}$ is at bounded distance from the identity we have that there exists a constant $K$ such that $\text{diam}(\tilde{f}^n(W_r^u(\tilde{x}))) \leq C\exp(n\lambda_1) + K$. Thus,

$$\frac{1}{n}\log(vol(\tilde{f}^n(W_r^u(\tilde{x}))) \leq \frac{1}{n}(Q(C\exp(n\lambda_1) + K))$$

Then, $\chi_u(f) \leq \lambda_1$.

On the other hand, We have that

$$\lambda_1 + \lambda_2 + \int \phi d\eta = h_\eta(f) + \int \phi d\eta \leq h_\mu(f) + \int \phi d\mu$$

where $\eta$ is such that $h_*\eta = vol$. Hence,

$$\lambda_1 + \lambda_2 + \int \phi d\eta \leq \chi_u(f) + \lambda^c(\mu) + \int \phi d\mu \leq \lambda_1 + \lambda^c(\mu) + \int \phi d\mu$$

and then,

$$\lambda_2 \leq \lambda^c(\mu) + \int \phi d\mu - \int \phi d\eta \leq \lambda^c(\mu) + \sup\phi - \inf\phi.$$

Therefore the theorem is proved. □

The proof the above theorem, we have the next corollary.

**Corollary 3.4.3.** *If $\mu$ is the unique equilibrium state for $f$ with respect to a potential $\phi = \psi \circ H$ with $\sup\phi - \inf\phi < \lambda_2$. Then, the center Lyapunov exponent of $\mu$ is positive.*

*Proof.* Suppose $\lambda^c(\mu) \leq 0$. By Theorem 3.4.2 and (3.5)

$$\lambda_1 + \lambda_2 + \int \phi d\eta = h_\eta(f) + \int \phi d\eta \leq h_\mu(f) + \int \phi d\mu \leq \lambda_1 + \int \phi d\mu$$

where $\eta$ is such that $h_*\eta = vol$. Hence,

$$\lambda_2 \leq \int \phi d\mu - \int \phi d\eta \leq \sup\phi - \inf\phi.$$

This is a contradiction and proves that $\lambda^c(\mu) > 0$.

□

Chapter 4

# Local product structure

Ruelle and Sullivan in [51] showed that the unique measure of maximal entropy for uniformly hyperbolic diffeomorphisms possesses local product structure, i.e., such measure can be written as the product of two measures defined on the stable and unstable manifold, more specifically they prove the next result.

**Theorem 4.0.1** (Ruelle-Sullivan [51]). *Let $f : M \longrightarrow M$ be an uniformly hyperbolic diffeomorphism, $\mu$ be the unique measure of maximal entropy of $f$ and* $\log \lambda$ *be the topological entropy of $f$. Then for each $x \in M$ there is a measure $\mu_x^s$ on $\mathcal{F}_\epsilon^s(x)$ (local stable manifold of $x$) and a measure $\mu_x^u$ on $\mathcal{F}_\epsilon^u(x)$ such that*

*1) $supp(\mu_x^s) \subset \mathcal{F}_\epsilon^s(x)$, $supp(\mu_x^u) \subset \mathcal{F}_\epsilon^u(x)$.*

*2) If $d(x, x') < \delta$, then*

$$(P^s_{x,x'})_* \mu^s_{x'} | \mathcal{F}^s_\delta(x') = \mu^s_x | P^s_{x,x'}(\mathcal{F}^s_\delta(x'))$$

*where $P^s_{x,x'}$ is the unstable holonomy. Analogously we have invariance of $\mu^u$ for the stable holonomy.*

*3) $f_* \mu_x^s = \lambda^{-1} \mu^s_{f(x)} | \mathcal{F}^s_\epsilon(f(x))$.*

*4) $f_* \mu_x^u = \lambda \mu^u_{f(x)} | \mathcal{F}^u_\epsilon(f(x))$.*

*5) $[.,.]_*(\mu_x^u \times \mu_x^s) = \mu$ on $\mathcal{F}^u_\epsilon(x) \times \mathcal{F}^s_\epsilon(x)$.*

*Where $[.,.] : \mathcal{F}^u_\epsilon(x) \times \mathcal{F}^s_\epsilon(x) \to M$ is defined by $[y, z] = \mathcal{F}^s_\epsilon(y) \cap \mathcal{F}^u_\epsilon(z)$.*

In [32], Leplaideur showed that the unique equilibrium states for uniformly hyperbolic diffeomorphisms associated to Hölder continuous potential possesses local product structure, more specifically he proves the next result.

**Theorem 4.0.2** (Leplaideur [32]). *Let $f : M \longrightarrow M$ be an uniformly hyperbolic diffeomorphism and $\mu$ be the unique equilibrium state for $f$ associated to Hölder continuous potential $\phi$. Then $\mu$ has local product structure.*

$$d\mu([y,z]) = \varphi_x(y,z)d\mu_x^u(y) \times d\mu_x^s(z)$$

*where $\mu_x^u$ and $\mu_x^s$ denote the contitional measures of $\mu$ with respect to any measurable partition subordinate to the unstable and the stable foliation; $y$ is any point in $\mathcal{F}_{loc}^u$ and $z$ is any point in $\mathcal{F}_{loc}^s$, and $\varphi_x$ is non-negative Borel function.*

*Moreover $\mu$, $\mu_x^u$ and $\mu_x^s$ have pointwise dimensions, $\delta$, $\delta^u$ and $\delta^s$, $\mu$ almost everywhere constant, and*

$$\delta = \delta^u + \delta^s \tag{4.1}$$

This last equality 4.1 is a particular case of a general fact in the non-uniform hyperbolic case that was prove by Barreira, Pesin and Schemling [4] using that hyperbolic measures for such systems have "almost" local product structure, more specifically they proved:

**Theorem 4.0.3** (Barreira-Pesin-Schmeling [4]). *Let $f$ be a $C^{1+\alpha}$ diffeomorphism on a smooth Riemannian manifold $M$ without boundary, and $\mu$ an $f$-invariant compactly supported ergodic Borel probability measure. If $\mu$ is hyperbolic then the following properties hold:*

*1) for every $\delta > 0$ there exist a set $\Lambda \subset M$ with $\mu(\Lambda) > 1 - \delta$ and a constant $k \geq 1$ such that for every $x \in \Lambda$ and every sufficiently small $r$ (depending on $x$), we have*

$$r^{\delta}\mu_x^s(B^s(x,\frac{r}{k}))\mu_x^u(B^u(x,\frac{r}{k})) \leq \mu(B(x,r)) \leq r^{-\delta}\mu_x^s(B^s(x,kr))\mu_x^u(B^u(x,kr)).$$

*2) The pointwise dimension of $\mu$ exists almost everywhere and it is equal to the sum of the stable and unstable pointwise dimensions, i.e.,*

$$\delta = \delta^s + \delta^u.$$

We are interested in studying local product structure for measure of maximal entropy of partially hyperbolic diffeomorphisms defined in 3-manifolds. F. Hertz, A. Hertz, A. Tahzibi and R. Ures in [49], studied measure of maximal entropy for such systems, more specifically they prove the next result.

**Theorem 4.0.4** (Hertz-Hertz-Tahzibi-Ures [49]). *Let $f : M \to M$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism of a three dimensional closed manifold $M$. Assume that $f$ is dynamically coherent with compact one-dimensional central leaves and has the accessibility property. Then $f$ has finitely many ergodic measures of maximal entropy. There are two possibilities.*

*1) $f$ has a unique measure of maximal entropy $\mu$. The central Lyapunov exponent $\lambda_c(\mu)$ vanishes and $(f, \mu)$ is isomorphic to a Bernoulli shift.*

*2) $f$ has more than one ergodic measure of maximal entropy, all of which have a non-vanishing central Lyapunov exponent. The central Lyapunov exponent $\lambda_c(\mu)$ in non-zero and $(f, \mu)$ is a finite extension of a Bernoulli shift for any such measure $\mu$. Some of these measures have a positive central exponent and some have a negative central exponent.*

We can ask the following:

**Question 4.0.5.** *Do all measure of maximal entropy for the previous systems have local product structure? More specifically. Can the measures of maximal entropy be written as the product of three measures defined on the local stable, central and unstable foliations?*

Our next result gives an answer to the previous question showing that measure of maximal entropy for accessible partially hyperbolic diffeomorphisms of 3-manifold having compact center leaves can be written locally as the product of three measures defined on the local stable, central and unstable foliations when such measure is unique and we verify that local product structure does not hold when the number of measures of maximal entropy is larger than one, more specifically we prove the following.

**Theorem D.** *Let $f : M \to M$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism of a three dimensional closed manifold $M$. Assume that $f$ is dynamically coherent with compact one-dimensional central leaves and has the accessibility property. Then*

*1) If $\mu$ is the unique measure of maximal entropy for $f$ (see item 1, theorem 4.0.4), then for each $x \in M$, there are measures $\mu_x^s$, $\mu_x^u$ and $\mu_x^c$ defined in $\mathcal{F}_{loc}^s(x)$ (local stable manifold of $x$), $\mathcal{F}_{loc}^u(x)$ and $\mathcal{F}_{loc}^c(x)$ respectively, such that*

$$\mu = \varphi_* \mu_x^s \times \mu_x^c \times \mu_x^u \tag{4.2}$$

*on $\varphi(\mathcal{F}_{loc}^s(x) \times \mathcal{F}_{loc}^c(x) \times \mathcal{F}_{loc}^u(x))$ where $\varphi : \mathcal{F}_{loc}^s(x) \times \mathcal{F}_{loc}^c(x) \times \mathcal{F}_{loc}^u(x) \to M$ is defined by $\varphi(z,t,w) = \mathcal{F}_{loc}^u(\mathcal{F}_{loc}^c(z) \cap \mathcal{F}_{loc}^s(t)) \cap \mathcal{F}_{loc}^{cs}(w)$ (see Figure 4.1).*

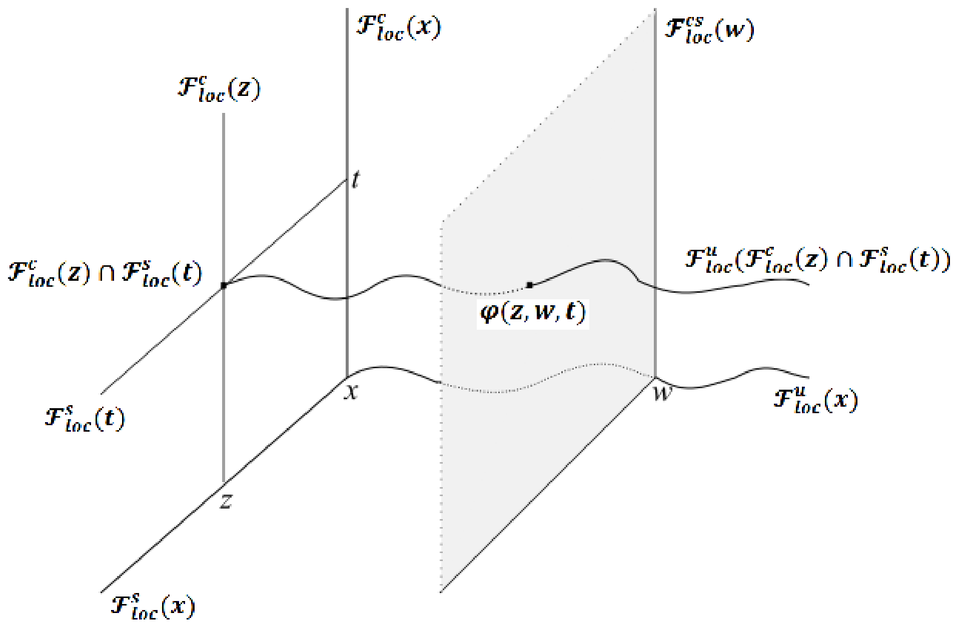


Figure 4.1:

*2) If $\mu$ is a hyperbolic ergodic measure of maximal entropy for $f$ (see item 2, Theorem 4.0.4), then do not exist measures defined in the local stable, unstable and central manifold such that (4.2) is true for $\mu$.*

## 4.1 Proof of Theorem D

*Remark* 4.1.1. F. Rodriguez Herz, A. Rodriguez Hertz, A. Tahzibi and R. Ures in [49] showed the followings: Let $\hat{M} = M/\mathcal{F}^c$ be the quotient space equipped with the quotient topology, $F$ be the dynamic induced on $\hat{M}$ and $\pi : M \to \hat{M}$ be the quotient map.

1) $(\hat{M}, F)$ is conjugated to an Anosov homeomorphism of $\mathbb{T}^2$.

2) If $m$ is the unique measure of maximal entropy for $(\hat{M}, F)$, then $\mu$ such that $\pi_*\mu = m$ is a measure of maximal entropy for $(M, f)$.

3) If $\mu$ is the unique measure of maximal entropy for $f$, then the conditional measures of $\mu$ along the center foliation are atom free.

4) If $\mu$ is a hyperbolic ergodic measure of maximal entropy for $f$ (see item 2, theorem 4.0.4), then the conditional measures of $\mu$ along the center foliation are atomic.

The Anosov homeomorphism $F$ admits two topological foliations $W^s$ and $W^u$ with similar dynamical properties as in the diffeomorphism case. If $\pi(x) = \hat{x}$ with $x \in M$, then the leaves are topological manifolds and

$$W^s(\hat{x}) = \bigcup_n F^{-n} W^s_\epsilon(F^n(\hat{x})), W^u(\hat{x}) = \bigcup_n F^n W^u_\epsilon(F^{-n}(\hat{x}))$$

Where

$$W^s_\epsilon(\hat{x}) = \{\hat{y} \in \hat{M} : d(F^n(\hat{x}), F^n(\hat{y})) \leq \epsilon\},$$

$$W^u_\epsilon(\hat{x}) = \{\hat{y} \in \hat{M} : d(F^{-n}(\hat{x}), F^{-n}(\hat{y})) \leq \epsilon\}.$$

By remark 4.1.1 we have that $\pi_*\mu = m$ and since that $m$ is the measure of maximal entropy for $F$ we have by Theorem 4.0.1 that for each $\pi(x) = \hat{x}$, with $x \in M$ there are measures $m^s_{\hat{x}}$ and $m^u_{\hat{x}}$ defined in $W^s_{loc}(\hat{x})$ (local stable set of $\hat{x}$) and $W^u_{loc}(\hat{x})$ respectively, such that

$$m = [.,.]_* m^s_{\hat{x}} \times m^u_{\hat{x}} \tag{4.3}$$

on $[.,.](W^s_{loc}(\hat{x}) \times W^u_{loc}(\hat{x}))$, where $[\hat{z}, \hat{w}] = W^u_{loc}(\hat{z}) \cap W^s_{loc}(\hat{w})$.

Since $\pi \mid_{\mathcal{F}^\sigma_{loc}(x)}: \mathcal{F}^\sigma_{loc}(x) \to W^\sigma_{loc}(\hat{x})$ $(\sigma = s, u)$ is a homeomorphism, we can define the measures

$$\mu^\sigma_x = ((\pi \mid_{\mathcal{F}^\sigma_{loc}(x)})^{-1})_* m^\sigma_{\hat{x}} \tag{4.4}$$

in $\mathcal{F}^\sigma_{loc}(x)$ with $\sigma = s, u$.

**Proposition 4.1.2.** *$\mu^s_x$ is c-invariant on $\mathcal{F}^{cs}$ and $\mu^u_x$ is c-invariant on $\mathcal{F}^{cu}$.*

*Proof.* We denote by $h^c_{xy}$ the center holonomy on $\mathcal{F}^{cs}$ between $\mathcal{F}^s_{loc}(x)$ and $\mathcal{F}^s_{loc}(y)$.

Since

$$h^c_{xy} \circ (\pi \mid_{\mathcal{F}^s_{loc}(x)})^{-1} = (\pi \mid_{\mathcal{F}^s_{loc}(y)})^{-1}$$

we have that $(h^c_{xy})_* \mu^s_x = \mu^s_y$.

$\square$

***Proof of Theorem D***. Firt we consider the case when there exit a unique measure of maximal entropy. Since $\pi_* \mu = m$ and $m$ has local product structure and the central Lyapunov exponent of $\mu$ is zero, we have by Invariance Principle of Avila and Viana [2], that $\mu$ admits a disintegration along of the central foliation $\{\mu_{\hat{x}} : \hat{x} \in \hat{M}\}$ which is $s$-invariant and $u$-invariant, and varies continuously with $\hat{x}$ on supp$\pi_*\mu = \hat{M}$. Moreover, we have that $supp(\mu_{\hat{x}}) = \mathcal{F}^c(x)$ and the measures $\mu_{\hat{x}}$ are atom free.

Hence,

$$\mu(A) = \int_{\hat{M}} \mu_{\hat{x}}(A) dm, A \subset M \tag{4.5}$$

and we can define $\mu^c_x$ in $\mathcal{F}^c_{loc}(x)$ by $\mu_{\hat{x}} \mid_{\mathcal{F}^c_{loc}(x)}$.

On the other hand. Let $z \in \mathcal{F}^s_{loc}(x)$ and $w \in \mathcal{F}^u_{loc}(x)$ fixed. We define

$$\psi : \mathcal{F}^c_{loc}(x) \to \mathcal{F}^c_{loc}(\underbrace{\varphi(z, x, w)}_{y})$$

by $\psi(t) = \varphi(z, t, w)$. Since $\{\mu_{\hat{x}} : \hat{x} \in \hat{M}\}$ is $s$-invariant and $u$-invariant, we have that

$$\psi_* \mu^c_x = \mu^c_y \tag{4.6}$$

If $A \subset \varphi(\mathcal{F}^s_{loc}(x) \times \mathcal{F}^c_{loc}(x) \times \mathcal{F}^u_{loc}(x))$, then of (4.3) and (4.5), we have

$$\mu(A) = \int_{\hat{M}} \mu_{\hat{x}}(A) d([,]_* m^s_{\hat{x}} \times m^u_{\hat{x}})$$

By (4.4), we have

$$\begin{aligned} \mu(A) &= \int_{\hat{M}} \mu_{\hat{y}}(A) d([,]_* \circ (\pi \mid_{\mathcal{F}^s_{loc}(x)} \times \pi \mid_{\mathcal{F}^u_{loc}(x)})_* \mu^s_x \times \mu^u_x)(\hat{y}) \\ &= \int_{\mathcal{F}^s_{loc}(x) \times \mathcal{F}^u_{loc}(x)} \mu_{[\pi|_{\mathcal{F}^s_{loc}(x)}(z), \pi|_{\mathcal{F}^u_{loc}(x)}(w)]}(A) d(\mu^s_x \times \mu^u_x)(z, w). \end{aligned} \tag{4.7}$$

On the other hand, by (4.6), we have

$$
\begin{aligned}
\mu_{[\pi|_{\mathcal{F}^s_{loc}(x)}(z),\pi|_{\mathcal{F}^u_{loc}(x)}(w)]}(A) &= \mu^c_{\varphi(z,x,w)}(A)\\
&= \int_{\mathcal{F}^c_{loc}(\varphi(z,x,w))} 1_A d\mu^c_{\varphi(z,x,w)}\\
&= \int_{\mathcal{F}^c_{loc}(x)} 1_A \circ \psi(t) d\mu^c_x(t)\\
&= \int_{\mathcal{F}^c_{loc}(x)} 1_A \circ \varphi(z,t,w) d\mu^c_x(t).
\end{aligned}
\tag{4.8}
$$

So, by (4.8) in (4.7), we have

$$
\begin{aligned}
\mu(A) &= \int_{\mathcal{F}^s_{loc}(x)\times\mathcal{F}^u_{loc}(x)} (\int_{\mathcal{F}^c_{loc}(x)} 1_A \circ \varphi(z,t,w) d\mu^c_x(t)) d(\mu^s_x \times \mu^u_x)(z,w)\\
&= \int_{\mathcal{F}^s_{loc}(x)\times\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)} (1_A \circ \varphi(z,t,w)) d(\mu^s_x \times \mu^c_x \times \mu^u_x)(z,t,w)\\
&= \int_M 1_A d(\varphi_*\mu^s_x \times \mu^c_x \times \mu^u_x)\\
&= \varphi_*\mu^s_x \times \mu^c_x \times \mu^u_x(A).
\end{aligned}
$$

Now we prove the second item of the theorem. Let us begin to prove the next lemma.

**Lemma 4.1.3.** *If $\mu$ such that*

$$\mu = \varphi_*\mu^s_x \times \mu^c_x \times \mu^u_x$$

*with $\mu^s_x$, $\mu^u_x$ and $\mu^c_x$ defined in $\mathcal{F}^s_{loc}(x)$, $\mathcal{F}^u_{loc}(x)$, $\mathcal{F}^c_{loc}(x)$ respectively. Where $\varphi : \mathcal{F}^s_{loc}(x) \times \mathcal{F}^c_{loc}(x) \times \mathcal{F}^u_{loc}(x) \to M$ is defined by*

$$\varphi(z,t,w) = \mathcal{F}^u_{loc}(\mathcal{F}^c_{loc}(z) \cap \mathcal{F}^s_{loc}(t)) \cap \mathcal{F}^{cs}_{loc}(w).$$

*Then $\{\mu^c_x\}$ is s-invariant and u-invariant.*

*Proof.* Since $\mu = \varphi_*\mu^s_x \times \mu^c_x \times \mu^u_x$, we have

$$\mu = \int_{\mathcal{F}^u_{loc}(x)} \varphi(\cdot,\cdot,w)_*(\mu^s_x \times \mu^c_x) d\mu^u_x(w).$$

We denote by $\mu^{cs}_x$ the conditional measure of $\mu$ on $\mathcal{F}^s_{loc}(x) \times \mathcal{F}^c_{loc}(x)$, then

$$\mu^{cs}_x = \varphi(\cdot,\cdot,x)_*(\mu^s_x \times \mu^c_x) \tag{4.9}$$

On the other hand, we denote by $\mu_z$ the conditional measure of $\mu^{cs}_x$ along the center foliation. Let $I$ be an interval in $\mathcal{F}^c_{loc}(z)$ with $z \in \mathcal{F}^s_{loc}(x)$, by definition of conditional measure, we have that

$$\mu_z(I) = \lim_{\delta \to 0} \frac{\mu_x^{cs}(\varphi(\cdot,\cdot,x)(I_\delta \times h_{zx}^s(I)))}{\mu_x^{cs}(\varphi(\cdot,\cdot,x)(I_\delta \times \mathcal{F}_{loc}^c(x)))}$$

where $h_{zx}^s$ is the stable holonomy ($h_{zx}^s : \mathcal{F}_{loc}^c(z) \to \mathcal{F}_{loc}^c(x)$, $h_{zx}^s(t) = \mathcal{F}_{loc}^c(x) \cap \mathcal{F}_{loc}^s(t)$) and $I_\delta \subset \mathcal{F}_{loc}^s(x)$.

So, by (4.9)

$$\mu_z(I) = \lim_{\delta \to 0} \frac{\mu_x^s(I_\delta) \cdot \mu_x^c(h_{zx}^s(I))}{\mu_x^s(I_\delta) \cdot \mu_x^c(\mathcal{F}_{loc}^c(x))} = \mu_x^c(h_{zx}^s(I)).$$

In particular, $\mu_z = \mu_z^c$. Hence,

$$(h_{zx}^s)_* \mu_z^c = \mu_x^c.$$

By the same arguments $\{\mu_x^c\}$ is $u$-invariant.

□

Now we complete the proof of the second item of theorem. Suppose that $\mu = \varphi_* \mu_x^s \times \mu_x^c \times \mu_x^u$ as in the above lemma. By Lemma 4.1.3 we have that $\{\mu_x^c\}$ is $s$-invariant and $u$-invariant. By item 4 of the remark 4.1.1 the conditional measures $\{\mu_x\}$ of $\mu$ along the center foliation are atomic with the same finite number of atoms.

On the other hand, let $z, w \in \mathcal{F}_{loc}^c(x)$ such that $z$ is an atom of $\mu_x$ and $w$ is not an atom. By the accessibility property of $f$ there exist a path $\lambda$ which is piecewise tangent to $E^s$ or to $E^u$ that joins $z$ and $w$. So, $\lambda$ can be covered by a finite number of cubes $\mathcal{F}_{loc}^s(y) \times \mathcal{F}_{loc}^c(y) \times \mathcal{F}_{loc}^u(y)$. Since that $\{\mu_x = \mu_x^c\}$ is $s, u$-invariant, we have that $w$ is a atom of $\mu_x$, which it's a contradiction. This concludes the proof of the item 2 of the theorem. □

For absolutely partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ with compact center leaves the previous theorem shows that the measure of maximal entropy can be written as the product of three measures defined on the stable, central and unstable manifold when such measure is unique. When such measure is hyperbolic this measure have not the above property. We can ask:

**Question 4.1.4.** *Can the measures of maximal entropy for absolutely partially hyperbolic diffeomorphisms homotopic to a linear Anosov diffeomorphism of $\mathbb{T}^3$ ("Derived from Anosov") be written as the product of three measures defined on the local stable, central and unstable manifold?*

Currently the above question is open, but since that such measures are hyperbolic (see [57]) have "almost" local product structure by Theorem 4.0.3.

**Question 4.1.5.** *Do all equilibrium states of Derived from Anosov diffeomorphism associated to potential that was studied in the chapter 3 can be written as the product of three measures defined on the local stable, central and unstable manifold?*

The next result answers negatively the above question for ergodic equilibrium states with total mass to the collapse intervals set. Once equilibrium states associated to potential with small variation are hyperbolic (see Corollary 3.4.3), we have that such measures have "almost" local product structure (see Theorem 4.0.3).

**Theorem 4.1.6.** *Let $f$ and $\phi$ be as in Theorem A. Let $\mu$ be an ergodic equilibrium state for $(f,\phi)$. If $\mu$ is virtually hyperbolic (see item 2, Theorem A), then $\mu$ can not be written as the product of three measures defined on the local stable, central and unstable manifolds.*

*Proof.* Let $\mu$ be an ergodic equilibrium state for Derived from Anosov diffeomorphism associated to any potential as was studied in the previous chapter. Once $\mu(C)=1$, by Proposition 3.1.1 we have that $\mu$ is virtually hyperbolic. Denote by $\mu^{cu}$ the conditional measure of $\mu$ (normalized and restricted on $\mathcal{F}^s_{loc}(x)\times\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)$ ) on $\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)$. Since disintegration of $\mu$ along central foliation is mono-atomic, we have

$$\mu^{cu}=\int\delta_{a(t)}d\mu^{uu}(t)$$

where $a(t)$ is the unique atom on the central leaf of $t$ and $\mu^{uu}$ is the quotient measure on the quotient of $\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)$ by central plaques.

Let us assume that

$$\mu=\varphi_*\mu^s_x\times\mu^c_x\times\mu^u_x$$

with $\mu^s_x$, $\mu^u_x$ and $\mu^c_x$ defined in $\mathcal{F}^s_{loc}(x)$, $\mathcal{F}^u_{loc}(x)$, $\mathcal{F}^c_{loc}(x)$ respectively. By Lemma 4.1.3, $\mu^c_x$ is $s,u$-invariant. Therefore, there exist $z\in\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)$ such that

$$\mu^{cu}(B\cap\mathcal{F}^u_{loc}(z))=1 \tag{4.10}$$

where $B$ is the set of unique atom in the center leaf of the conditional measure of $\mu^{cs}$ along of the center foliation.

We claim that there exist a set $D\subset\mathcal{F}^c_{loc}(x)\times\mathcal{F}^u_{loc}(x)$ such that $D\cap\mathcal{F}^u_{loc}(z)=\emptyset$ and $\mu^{cu}(D)>0$. In fact, let $\nu=H_*\mu$ and $R$ be a Markov's rectangle of $A$. We normalize the restriction of $\nu$ on $R$. Let $\mathcal{F}^{cu}$ be a typical unstable leaf of $A$. Consider $R^{cu} = R \cap \mathcal{F}^{cu}$ and denote by $\nu^{cu}$ the conditional measure of $\nu$ (normalized and restricted on $R$) on $R^{cu}$. We can suppose that $\mathcal{F}^c_{loc}(x) \times \mathcal{F}^u_{loc}(x) \subset H^{-1}(R^{cu})$.

We can take $n > 0$ such that

$$A_{cu}^{-n}(R^{cu}) \cap H(\mathcal{F}^u_{loc}(z)) = \emptyset$$

where $A_{cu} : R^{cu} \to R^{cu}$ is defined by $A_{cu}(w) = R^{cu} \cap \mathcal{F}^s_A(A^{r(w)}(w))$, with $r(w) := \min\{n : A^n(w) \in R\}$ (for more details see [32]). Since $\nu^{cu}$ gives positive measure for all set open in $R^{cu}$ (see [32], Lemma 5.10). In particular $\nu^{cu}(A_{cu}^{-n}(R^{cu})) > 0$. Hence, $D = H^{-1}A_{cu}^{-n}(R^{cu})$ satisfies the claim.

By the claim and (4.10), we have a contradiction and this ending the proof.

$\square$

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