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Stability for nonlinear generalized ODEs and for retarded Volterra-Stieltjes integral equations and control theory for these equations and for dynamic equations on time scale

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Estabilidade para EDOs generalizadas não lineares e para equações integrais de Volterra-Stieltjes retardadas e teoria de controle para estas equações e para equações dinâmicas em escala temporal

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutora em Ciências – Matemática. *VERSÃO REVISADA*

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Fiel à sua lei de cada instante

Desassombrado, doido, delirante

Numa paixão de tudo e de si mesmo.

(Vinicius de Moraes)

RESUMO

ANDRADE DA SILVA. F. **Estabilidade para EDOs generalizadas não lineares e para equações integrais de Volterra-Stieltjes retardadas e teoria de controle para estas equações e para equações dinâmicas em escala temporal**. 2021. 205 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Este trabalho têm dois objetivos principais. O primeiro é provar teoremas do tipo Lyapunov e teoremas inversos de Lyapunov a respeito de limitação de soluções, estabilidade regular e estabilidade uniforme para equações diferenciais ordinárias generalizadas e para equações integrais de Volterra-Stieltjes retardadas. Como uma aplicação, estabelecemos condições necessárias e suficientes para um sistema de equações diferenciais ordinárias generalizadas perturbadas e para um sistema de equações integrais de Volterra-Stieltjes retardadas perturbadas, definidas em um espaço de Banach, sejam assimptoticamente controláveis.

O segundo objetivo é investigar a existência e unicidade de uma solução para uma equação integral de Volterra-Stieltjes de segunda ordem, assim como, para uma equação dinâmica em escala temporal, cujas formas integrais contêm Δ-integrais de Perron definidas em espaços de Banach. Também fornecemos uma fórmula da variação-das-contantes para equações dinânicas lineares não homogêneas em escala temporal e estabelecemos resultados sobre controlabilidade para estas equações.

Os resultados inétidos apresentados neste trabalho estão contidos em 3 artigos (veja [4–6]) e em 2 capítulos do livro [13].

Palavras-chave: Teoremas de Lyapunov para estabilidade e limitação de solução, Teoria de controle, Equações diferenciais ordinárias generalizadas, Equações integrais de Volterra-Stieltjes retardadas.

ABSTRACT

ANDRADE DA SILVA. F. **Stability for nonlinear generalized ODEs and for retarded Volterra-Stieltjes integral equations and control theory for these equations and for dynamic equations on time scale**. 2021. 205 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

This work has two main purposes. The first one is to prove Lyapunov-type theorems and converse Lyapunov theorems on boundedness of solutions, regular stability and uniform stability for generalized ODEs and retarded Volterra-Stieltjes integral equations. As an application, we establish necessary and sufficient conditions for a system of perturbed generalized ODEs and for a system of perturbed retarded Volterra-Stieltjes integral equations, defined in a Banach space, to be asymptotically controllable.

The second purpose is to investigate the existence and uniqueness of a solution for a linear Volterra-Stieltjes integral equation of the second kind, as well as for a homogeneous and a nonhomogeneous linear dynamic equations on time scales, whose integral forms contain Perron Δ-integrals defined in Banach spaces. We also provide a variation-of-constant formula for a nonhomogeneous linear dynamic equations on time scales and we establish results on controllability for these equations.

The new results presented in this work are contained in 3 papers (see [4–6]) and in two chapters of the book [13].

Keywords: Lyapunov theorems on stability and boundedness of solutions, Control theory, Generalized ODEs, Retarded Volterra-Stieljes integral equations, Dynamic equations on time scales.

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CHAPTER 1

INTRODUCTION

The notion of derivatives were already known before Isaac Newton and Gottfried Leibniz. These two mathematicians are credited with the invention of calculus around 1670. For Newton, the process of integration was largely seen as an inverse to the operation of differentiation, and the integral was synonymous with the anti-derivative, which is known as the Newton integral. Around 1850 a new approach came out in the work of Augustin-Louis Cauchy and soon after in the work of Georg Riemann. They believed that the definite integral could be interpreted as the area under continuous curves, and could be obtained by summing an infinite series of areas corresponding to approximating rectangles of infinitely small width, which makes this definition of integral is independent of the derivative.

The theory of the Riemann integral was not fully satisfactory. Firstly, the class of integrable curves was limited not only to continuous curves, but also to those with elementary anti-derivatives. Therefore, many important functions do not have a Riemann integral, even after when one extends them to a larger class of integrable functions by allowing "improper" Riemann integrals in. Moreover, the Riemann integral does not yield good convergence theorems. For instance, a pointwise, bounded limit of Riemann integrable functions is not necessarily Riemann integrable. To overcome these deficiencies, Henri Lebesgue proposed a new notion of integration, known as the Lebesgue integral. The Lebesgue integral is strictly more general than the proper Riemann integral, that is, it can integrate a larger class of functions. However, in comparing the improper Riemann integral with the Lebesgue integral, we find that neither is strictly more general than the other. Furthermore, Lebesgue's method is a complex one and a considerably amount of measure theory is required even to define the integral.

Neither the improper Riemann integral nor the Lebesgue integral generated a fully satisfactory construction of anti-derivatives. Slightly more general notions of integral were given by Arnaud Denjoy (1912) and Oskar Perron (1914). Denjoy's and Perron's definitions turned out to be equivalent.

Decades later, independently, Ralph Henstock (1955) and Jaroslav Kurzweil (1957) came up with a much simpler formulation of the Denjoy-Perron integral. In their definition the intuitive approach of the Riemann integral is preserved, but unlike the Riemann integral which considers tagged partitions of an interval with subintervals whose lengths are limited by a fixed constant, Henstock and Kurzweil used a strictly positive function δ (called gauge) to measure the length of each subintervals, that is, the maximum length of the subintervals are allowed to vary. By making this small adjustment, it turned out that their integral also surpassed limitations of the Riemann integral, in particular, every derivative function is integrable. Moreover, the integral of Kurzweil and Henstock allows us to deal with integrands which are highly oscillating and have many discontinuities.

The integral of Kurzweil and Henstock is also known by various names: the Henstock-Kurzweil integral, the generalized Riemann integral, the Perron integral and, because of its definition, it is also called as a gauge integral. However, in the original paper [38] by Kurzweil we find a more general definition of this integral which we refer to as the Kurzweil integral. See Definition B.0.11.

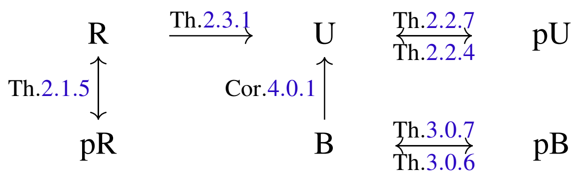
In order to generalize certain classical results on the existence and continuous dependence on a parameter of solutions of ordinary differential equations, J. Kurzweil introduced, in [38], a special class of generalized differential equation, currently known as generalized ordinary differential equations or generalized ODEs, for short. Moreover, he applied his results to classical differential equations with distributing terms which approximate the Dirac function. See [39].

Generalized ODEs are described by integral equations involving the Kurzweil integral and are known to encompass several other types of equations as Volterra-Stieltljes integral equations, measure functional differential equations, among others. Therefore, the theory of generalized ODEs has been shown to act as a unifying theory for many differential equations.

In the present thesis, one of our purposes is to establish necessary and sufficient conditions for a perturbed nonlinear generalized ODE to be asymptotically controllable. Usually, in the definition of asymptotic controllability, some type of stability is required (see Definition 5.0.1). For this reason, we introduced a new concept of uniform stability with respect to perturbations for a trivial solution of a homogeneous nonlinear generalized ODE and studied stability for generalized ODEs via Lyapunov functionals. In particular, we obtained converse Lyapunov theorems on regular and uniform stability for the trivial solution of a homogeneous generalized ODE. Since up to now the theory of Lyapunov functionals for generalized ODEs deals only with homogeneous equations, we needed to establish a relation between uniform stability and uniform stability with respect to perturbations. On the other hand, while we were investigating the theory of Lyapunov functionals for generalized ODEs, we came across the theory of uniform boundedness of solutions of generalized ODEs and we were intrigued to find out whether there exists a relation between this concept and the concept of uniform stability. Therefore, the main goals of this work in the framework of generalized ODEs are:

- to establish converse theorems on regular and uniform stability for a homogeneous generalized ODE;
- to introduce a concept of uniform stability with respect to perturbations for a trivial solution of a homogeneous generalized ODE;
- to provide relations between the concept of uniform stability and uniform stability with respect to perturbations;
- to investigate converse theorems on uniform boundedness of solutions of homogeneous generalized ODEs;
- to set relations between boundedness of solutions of homogeneous and perturbed generalized ODEs;
- to relate boundedness of solutions to uniform stability for homogeneous generalized ODE;
- to introduce the definition of asymptotic controllability for a perturbed generalized ODEs and to apply the Lyapunov theorems to provide a criteria for asymptotic controllability.

Thus, under some hypotheses, we obtained a few relations which are illustrated, in a simplified way, in the following diagram.

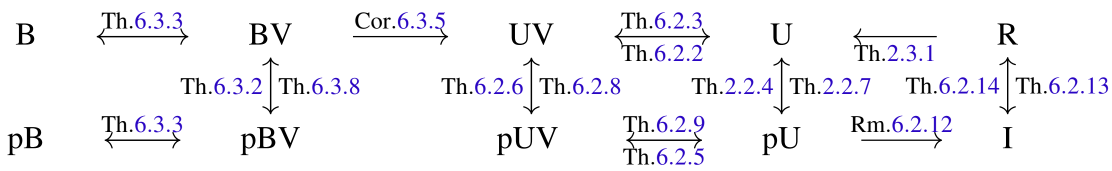


- R = regular stability;
- pR = regular stability with respect to perturbations;
- U = uniform stability;
- pU = uniform stability with respect to perturbations;
- B = uniform boundedness of solutions for homogeneous generalized ODEs;
- pB = uniform boundedness of solutions for perturbed generalized ODEs.

Retarded Volterra-Stieljes integral equations (we write retarded VS integral equations, for short) play an important role not only from a theoretical point of view but also in applications, since they are used in many types of well known mathematical models. Therefore, VS integral equations have been attracting the attention of several researchers (see, for instance, [8,14,15,36]). Motivated by these features, we are interested in obtaining Lyapunov theorems on stability and boundedness of solutions as well as asymptotic controllability results for these equations. To this end, we proved that a retarded VS integral equation can be regarded as a generalized ODE and used this fact to obtain

- a result on the existence and uniqueness of a solution of a retarded VS integral equation;
- Lyapunov-type and converse theorems on integral stability, uniform stability and boundedness of solutions for homogeneous retarded VS integral equations;
- relations between the concepts of integral stability, uniform stability and uniform stability with respect to perturbations;
- relations between boundedness of solutions for homogeneous and perturbed retarded VS integral equations;
- a characterization of asymptotic controllability.

In the sequel, we exhibit a digram which contains the relations above.



- BV = boundedness of the solutions of a homogeneous retarded VS integral equation;
- pBV = boundedness of the solutions of a perturbed retarded VS integral equation;
- UV = uniform stability for the trivial solution of a homogeneous retarded VS equation;
- pUV = uniform stability with respect to pertubations for the trivial solution of a homogeneous retarded VS integral equation (6.2);
- I = integral stability .

Now, we turn our attention to calculus on time scales, introduced in 1988 by Stefan Hilger. This theory allows us to describe continuous, discrete and hybrid systems which have several applications (see [7, 41]). One of the main concepts of the time scale theory is the delta derivative, which is a generalization of the classical time derivative in the continuous time and the finite forward difference in the discrete time. As a consequence, differential equations as well as difference equations are naturally accommodated in this theory (see [10, 11]). In this work, we are also interested in obtaining necessary and sufficient conditions for a nonhomogeneous linear dynamic equations on time scales to be approximately controllable/strictly controllable. To this end, we proved that there exists a relation between the solution of a linear dynamic equation and the solution of a Volterra-Stieltjes integral equation. Moreover, we obtained a variation-of-constant formula for nonhomogeneous dynamic equations on time scales.

The present work is divided in seven chapters and four appendixes which are organized as follows. The second chapter is devoted to the study of stability for the trivial solution of a generalized ODE, where in Section 2.1, we recall the basic concepts and results, presented in [26], on regular, regular attracting and regular asymptotic stability. Not only that, but we weaken the Lipschitzian condition with respect to the second variable on the Lyapunov functional, required in [26, Theorems 6.3 and 6.4], to a condition which allows jumps and, yet, we are able to get converse Lyapunov theorems concerning regular, regular attracting and regular asymptotic stability for generalized ODEs. In Section 2.2, we bring up the Lyapunov-type theorem on uniform stability for generalized ODEs, contained in [13, 23], and we also establish a converse Lyapunov theorem. Moreover, we introduce a concept of uniform stability with respect to perturbations and give relations between this concept and the notion of uniform stability.

The third chapter contains our contributions on uniform boundedness of solutions for generalized ODEs which are two converse Lyapunov theorems and relations between boundedness of solutions for homogeneous and perturbed generalized ODEs.

In Chapter 4, we use the Lyapunov theorems, described in the previous chapters, to obtain relations between uniform stability and boundedness of solutions for generalized ODEs.

Chapter 5 gethers results on control theory in the framework of generalized ODEs. In this chapter, we introduce a definition of asymptotic controllability and we proved that this concept is related to the existence of a Lyapunov functional.

In order to apply our results to a retarded VS equation, we describe, in Chapter 6, a relation between retarded VS equations and generalized ODEs. With this relation at hand, we are able to establish some interesting results for retarded VS equations, such as existence and uniqueness of solutions, Lyapunov-type and converse Lyapunov theorems on several types of stability and boundedness of solutions (namely integral, integral attracting, asymptotic integral, uniform, asymptotic uniform, uniform with respect to perturbations, asymptotic uniform with respect to perturbations). Moreover, we relate these concepts to each other, to boundedness of solutions for homogeneous and perturbed generalized ODEs, and to each type of stability for generalized ODEs (see the last diagram). In Section 6.4, we translate the result on asymptotic controllability for perturbed generalized ODEs to perturbed retarded VS integral equations and, in the end of the section, we present an example.

The goal of Chapter 7 is to present a control theory for dynamic equations on time scales. In Subsection 7.1.1, we prove the existence and uniqueness of a solution of a Volterra-Stieltjes integral equation of the second kind. In Subsection 7.1.2, we consider homogeneous and nonhomogeneous linear dynamic equations on time scales, whose functions are Perron Δ-integrable and we prove the existence and uniqueness of their solutions. Still in Subsection 7.1.2, we give a relation between the solutions of our dynamic equations and the solutions of a Volterra-Stieltjes integral equation. A variation-of-constant formula for nonhomogeneous dynamic equations on time scales is obtained in Section 7.2. The main propose of Section 7.3 is to investigate necessary and sufficient conditions for a nonhomogeneous dynamic equation on time scales, defined in a Banach space with Perron Δ-integrable functions, to be approximately controllable/strictly controllable. Yet in this section, we prove that our theorem on controllability generalizes some other results for the literature for dynamic equations on time scales defined in $\mathbb{R}^n$ with rd-continuous and regressive functions. We also provide examples about strictly controllable dynamic equations on time scales taking values in $\mathbb{R}^2$ and in an arbitrary Banach space.

Appendix A, B, C and D provide, respectively, the basic background on regulated functions, vector integrals, generalized ODEs and the time scales calculus.

CHAPTER 2

STABILITY

This chapter is devoted to the study of regular and uniform stability for the trivial solution of a generalized ODE and its main references are [4, 6, 23, 26].

In the theory of ordinary differential equations, Aleksandr Mikhailovich Lyapunov described, in his doctoral thesis "The General Problem of Stability of Motion" (1892), a method which shows that the existence of a functional guarantees stability of solutions of ordinary differential equations near to an equilibrium point. This method is now referred to as Direct Method of Lyapunov.

Results on stability of the trivial solution in the framework of generalized ODEs started in 1984, when Štefan Schwabik introduced the concept of variational stability (see [48]). Many years later, the authors of [1, 23, 26, 27] developed results on regular, uniform, variational and exponential stability for generalized ODEs.

In the present work, we provide two Lyapunov functional techniques for the study of stability for generalized ODEs. The first one, known as Direct Method of Lyapunov, makes use of a Lyapunov functional to prove stability of the trivial solution of a generalized ODE and it has an important application to control theory (see Chapter 5). The second technique deals with the inverse of the Direct Method of Lyapunov, that is, assuming that the trivial solution of a generalized ODE is stable, we are able to construct a Lyapunov functional. The results which guarantee that "the existence of a Lyapunov functional implies stability" are known as Lyapunov–type theorems. On the other hand, the results which show that "stability implies the existence of a Lyapunov functional" are called converse Lyapunov theorems.

At first, we introduce the basic background on a Lyapunov functional for generalized ODEs.

Let X be a Banach space, equipped with the norm $\|\cdot\|$, $\mathscr{O}$ be an open subset of X such that $0 \in \mathscr{O}$, where 0 is the neutral element of X, and $t_0 \in \mathbb{R}$ with $t_0 \geq 0$. Consider $\Omega = \mathscr{O} \times [t_0, +\infty)$

and the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t), \tag{2.1}$$

where $F : \Omega \to X$ belongs to the class $\mathscr{F}(\Omega, h)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). We also assume that $x \equiv 0$ is a solution of the generalized ODE (2.1). In this case, we say that $x \equiv 0$ is the *trivial solution* of the generalized ODE (2.1).

In what follows, we present a sufficient condition for the existence of the trivial of the generalized ODE (2.1).

Remark 2.0.1. Assume that $F(0,t_2) - F(0,t_1) = 0$ for all $t_1, t_2 \in [t_0, +\infty)$. Then, by the definition of the Kurzweil integral (Definition B.0.11), we have

$$\int_{t_1}^{t_2} DF(0,t) = F(0,t_2) - F(0,t_1) = 0, \quad \text{for all } t_1, t_2 \in [t_0, +\infty)$$

which implies that $x \equiv 0$ is a solution of the generalized ODE (2.1) on $[t_0, +\infty)$.

Notice that, if we assume that $F(x,t)$ is a function which depends only on the variable x, then we can obtain a normalized representation F_1 of F by

$$F_1(z,t) = F(z,t) - F(z,t_0) = 0,$$

for every $z \in \mathscr{O}$. This fact shows us that, in order to get stability results for any solution of the generalized ODE (2.1), it is sufficient to obtain the same results for the trivial solution of (2.1).

In the sequel, we introduce a definition of a Lyapunov functional for the generalized ODE (2.1). We use the symbol $\mathbb{R}^+$ to denote the set of non-negative real numbers, that is, $t \in \mathbb{R}^+$ whenever $t \geq 0$.

Definition 2.0.2. Let $B \subset \mathscr{O}$ be any subset. We say that $V : [t_0, +\infty) \times B \to \mathbb{R}$ is a *Lyapunov functional with respect to the generalized ODE* (2.1), if the following conditions are satisfied:

(L1) $V(\cdot, y) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$, for all $y \in B$;

(L2) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that

$$V(t,y) \geq b(\|y\|),$$

for all $(t,y) \in [t_0, +\infty) \times B$;

(L3) for every maximal solution $x : [s_0, \omega) \to B$ of the generalized ODE (2.1) (see Definition C.0.15), the derivative

$$D^+V(t,x(t)) = \limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \leq 0$$

holds for all $t \in [s_0, \omega) \subset [t_0, +\infty)$, that is, the upper right derivative of V is non-positive along every maximal solution of the generalized ODE (2.1).

We point out that, in some references, the authors replace condition (L3), in Definition 2.0.2, by the following condition

(L'3) the function $[s_0,\omega) \ni t \mapsto V(t,x(t))$ is nonincreasing along every maximal solution, $x : [s_0,\omega) \to B$, of the generalized ODE (2.1).

See, e.g., [22, 23].

In the present work, we chose to use condition (L3). However, in the following lines, we will show that conditions (L3) and (L'3) are related. This fact allows us to work with the most convenient of these two conditions depending on the type of stability we are dealing with. In addition, our results are still valid if we consider condition (L'3) instead of (L3) in Definition 2.0.2. See Remarks 2.1.8, 2.1.20, 2.1.28 and 2.2.17 .

It is clear that, if (L'3) holds, then $V : B\times[t_0,+\infty) \to \mathbb{R}^+$ satisfies condition (L3) as well. In the sequel, we show that with an additional hypothesis on the Lyapunov functional, condition (L3) implies (L'3).

Proposition 2.0.3. Let $B \subset \mathscr{O}$, $V : [t_0,+\infty)\times B \to \mathbb{R}$ be a functional and $f : [s_0,\omega) \to \mathbb{R}$ be a function defined by

$$f(t) = V(t,x(t)), \quad t \in [s_0,\omega),$$

where $x : [s_0,\omega) \to B$ is any maximal solution of the generalized ODE (2.1) on $[s_0,\omega) \subset [t_0,+\infty)$. If f is left-continuous on (s_0,ω) and

$$D^+f(t) = \limsup_{\eta\to 0^+} \frac{f(t+\eta)-f(t)}{\eta} \leq 0, \quad \text{for all } t \in [s_0,\omega), \tag{2.2}$$

then f is nonincreasing, that is, $f(t_1) \geq f(t_2)$, whenever $t_1,t_2 \in [s_0,\omega)$ and $t_1 < t_2$.

Proof. Suppose, by contradiction, there exist $t_1,t_2 \in [s_0,\omega)$ for which

$$t_1 < t_2 \quad \text{and} \quad f(t_1) < f(t_2).$$

Let $\gamma \in \mathbb{R}$ be such that $f(t_1) < \gamma < f(t_2)$ and consider the set $A = \{t \in [t_1,t_2] : f(t) < \gamma\}$. Then, A is upper bounded and $A \neq \emptyset$, once $A \subset [t_1,t_2]$ and $t_1 \in A$. Let $c = \sup A$ and assume that $f(c) \geq \gamma$. Since f is left-continuous, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\gamma - \varepsilon \leq f(c) - \varepsilon < f(t), \quad \text{for all } t \in (c-\delta,c). \tag{2.3}$$

Moreover, by the definition of the supremum, there exists $t_\delta \in A$ such that $t_\delta \in (c-\delta,c)$. Thus, by (2.3), we obtain

$$\gamma - \varepsilon < f(t_\delta). \tag{2.4}$$

Taking $\varepsilon \to 0$ in (2.4), we conclude that $\gamma < f(t_\delta)$ which contradicts the fact that $t_\delta \in A$. Therefore, $f(c) < \gamma$ and, hence, $c \in A$.

On the other hand, since $c = \sup A$, $f(t) \geq \gamma$ for all $t > c$, $t \in [t_1, t_2]$. Then, for $\eta > 0$ sufficiently small, we have

$$f(c+\eta) - f(c) > \gamma - \gamma = 0. \tag{2.5}$$

By (2.5), $D^+ f(c) > 0$, contradicting (2.2). Therefore, the statement follows. □

2.1 Regular stability

In this section, we apply Lyapunov functional techniques to establish necessary and sufficient conditions for the trivial solution of the generalized ODE (2.1) to be regularly stable, regularly attracting or regularly asymptotically stable.

As mentioned in Remark A.0.8 and Lemma C.0.9, for every $[\alpha, \beta] \subset [t_0, +\infty)$, the solution $x : [\alpha, \beta] \to X$ of the generalized ODE (2.1) is a regulated function (see Definition A.0.7), where X is a Banach space. Therefore, it is natural to measure the distance between two solutions of the generalized ODE (2.1) using the usual supremum norm (see Theorem A.0.16). For this reason, the authors of [26], introduced a new concept of stability for generalized ODEs, namely regular stability, which concerns the local behavior of a regulated function initially close to the trivial solution of the generalized ODE (2.1).

Throughout this section, we suppose $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition for the existence of such a solution) and $F : \Omega \to X$ belongs to $\mathscr{F}(\Omega, h)$, where $\Omega = \mathscr{O} \times [t_0, +\infty)$, $\mathscr{O} \subseteq X$ is an open set containing the neutral element of X and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2).

At first, we present the concept of regular stability for generalized ODEs, introduced in [26], and some auxiliary results.

Definition 2.1.1. Let $[\alpha, \beta] \subset [t_0, +\infty)$. The trivial solution of the generalized ODE (2.1) is said to be

(i) *regularly stable*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\bar{x} \in G^-([\alpha, \beta], \mathscr{O})$ satisfies

$$\|\bar{x}(\alpha)\| < \delta \quad \text{and} \quad \sup_{s \in [\alpha, \beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau), t) \right\| < \delta,$$

then

$$\|\bar{x}(t)\| < \varepsilon,$$

for all $t \in [\alpha, \beta]$;

(ii) *regularly attracting*, if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\bar{x} \in G^-([\alpha, \beta], \mathscr{O})$ satisfies

$$\|\bar{x}(\alpha)\| < \delta_0 \quad \text{and} \quad \sup_{s \in [\alpha, \beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau), t) \right\| < \rho,$$

then

$$\|\bar{x}(t)\| < \varepsilon,$$

for all $t \in [\alpha,\beta] \cap [\alpha+T,+\infty)$;

(iii) *regularly asymptotically stable*, if it is both regularly stable and regularly attracting.

Remark 2.1.2. The existence of the Kurzweil integrals, in Definition 2.1.1, are guaranteed by Corollary C.0.4.

Besides the generalized ODE (2.1), we consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t)+P(t)], \tag{2.6}$$

where $F : \Omega \to X$ and $P : [t_0,+\infty) \to X$ is such that $P \in G([t_0,+\infty),X)$.

Remark 2.1.3. Considering that $F \in \mathscr{F}(\Omega,h)$, if $G : \Omega \to X$ is given by

$$G(x,t) = F(x,t)+P(t),$$

for all $(x,t) \in \Omega$, then for all $(x,s_2),(x,s_1),(y,s_1),(y,s_2) \in \Omega$, we have

$$\|G(x,s_2)-G(x,s_1)\| \leq \left| h(s_2)-h(s_1)+\sup_{s\in[t_0,s_2]}\|P(s)-P(t_0)\| - \sup_{s\in[t_0,s_1]}\|P(s)-P(t_0)\| \right|$$

and

$$\|G(x,s_2)-G(x,s_1)-G(y,s_2)+G(y,s_1)\| \leq \|x-y\||h(s_2)-h(s_1)|.$$

By Proposition A.0.20, if $P : [t_0,+\infty) \to X$ is left-continuous on $(t_0,+\infty)$, then $g : [t_0,+\infty) \to \mathbb{R}$, defined by $g(t) = \sup_{s\in[t_0,t]}\|P(s)-P(t_0)\|$, is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Therefore, $G \in \mathscr{F}(\Omega,\widetilde{h})$, where $\widetilde{h}(t) = h(t)+g(t)$, and all results on existence, uniqueness and other properties of a solution of a generalized ODE, presented in Appendix C, hold for the perturbed generalized ODE (2.6).

Based on Definitions B.0.11 and C.0.1, the function $\bar{x} : [\alpha,\beta] \to X$ is a *solution of the perturbed generalized ODE* (2.6) on $[\alpha,\beta] \subset [t_0,+\infty)$, if $(\bar{x}(s),s) \in \Omega$, for all $s \in [\alpha,\beta]$, and

$$\begin{aligned} \bar{x}(t) &= \bar{x}(\alpha) + \int_{\alpha}^{\beta} D[F(\bar{x}(\tau),s)+P(s)] \\ &= \bar{x}(\alpha) + \int_{\alpha}^{\beta} DF(\bar{x}(\tau),s) + P(t) - P(\alpha) \end{aligned} \tag{2.7}$$

holds for all $t \in [\alpha,\beta]$.

From now on, until the end of this subsection, we assume that for all $[\alpha,\beta] \subset [t_0,+\infty)$ and all $x_0 \in \mathscr{O}$, there exists a solution $\bar{x} : [\alpha,\beta] \to X$ of the perturbed generalized ODE (2.6) with initial condition $\bar{x}(\alpha) = x_0$.

The next definition was borrowed from [26].

Definition 2.1.4. Let $[\alpha,\beta] \subset [t_0,+\infty)$ and $x_0 \in \mathscr{O}$. The trivial solution of the generalized ODE (2.1) is called

(i) *regularly stable with respect to perturbations*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\|x_0\| < \delta$ and $P \in G^-([\alpha,\beta],X)$ with

$$\sup_{s\in[\alpha,\beta]} \|P(s) - P(\alpha)\| < \delta,$$

then

$$\|\bar{x}(t,\alpha,x_0)\| = \|\bar{x}(t)\| < \varepsilon, \quad \text{for every } t \in [\alpha,\beta],$$

where $\bar{x} : [\alpha,\beta] \to X$ is a solution of the perturbed generalized ODE (2.6) with initial condition $\bar{x}(\alpha) = x_0$;

(ii) *regularly attracting with respect to perturbations*, if there exists $\widetilde{\delta} > 0$ for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that, if $\|x_0\| < \widetilde{\delta}$ and $P \in G^-([\alpha,\beta],X)$ with

$$\sup_{s\in[\alpha,\beta]} \|P(s) - P(\alpha)\| < \rho,$$

then

$$\|\bar{x}(t,\alpha,x_0)\| = \|\bar{x}(t)\| < \varepsilon, \quad \text{for every } t \in [\alpha,\beta] \cap [\alpha+T,+\infty),$$

where $\bar{x} : [\alpha,\beta] \to X$ is a solution of the perturbed generalized ODE (2.6) with initial condition $\bar{x}(\alpha) = x_0$;

(iii) *regularly asymptotically stable with respect to perturbations*, if it is regularly stable with respect to perturbations and regularly attracting with respect to perturbations.

The next result establishes a relation between regular stability and regular stability with respect perturbations. For a proof of it, see [13, Theorem 8.31] or [26, Theorem 4.7].

Theorem 2.1.5. The following statements hold.

(i) The trivial solution of the generalized ODE (2.1) is regularly stable if and only if it is regularly stable with respect to perturbations.

(ii) The trivial solution of the generalized ODE (2.1) is regularly attracting if and only if it is regularly attracting with respect to perturbations.

(iii) The trivial solution of the generalized ODE (2.1) is regularly asymptotically stable if and only if it is regularly asymptotically stable with respect to perturbations.

2.1.1 Direct method of Lyapunov

In this subsection, we present Lyapunov-type theorems for generalized ODEs, settled down in [26]. Here, we weaken the Lipschitzian condition on the second variable of the Lyapunov functional used in [26, Theorems 6.3 and 6.4] to a condition which allows jumps (see Theorems 2.1.7, 2.1.9 and 2.1.10 in the present work). Almost all results presented in this subsection are new and appear in the article [4] and the book [13].

Throughout this subsection, we suppose X is a Banach space, $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition for the existence of such a solution), $F : \Omega \to X$ belongs to $\mathscr{F}(\Omega, h)$, where $\Omega = X \times [t_0, +\infty)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). By Corollary C.0.20, for all $x_0 \in X$ and all $s_0 \in [t_0, +\infty)$, there exists unique global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$ (See Definition C.0.15 and Remark C.0.16 for the concept of global forward solution).

The proof of the next lemma was inspired by [22, Lemma 3.4].

Lemma 2.1.6. Let $[\alpha, \beta] \subset [t_0, +\infty)$ and $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Suppose $W : [t_0, +\infty) \times X \to \mathbb{R}$ satisfies the following conditions:

(H1) for each $z \in G^-([\alpha, \beta], X)$, the function $[\alpha, \beta] \ni t \mapsto W(t, z(t))$ is left-continuous on $(\alpha, \beta]$;

(H2) if $y, z \in G^-([\alpha, \beta], X)$, then

$$|W(s_2, z(s_2)) - W(s_2, y(s_2)) - W(s_1, z(s_1)) + W(s_1, y(s_1))| \leq \sup_{s \in [s_1, s_2]} \|z(s) - y(s)\| \quad (2.8)$$

holds for every $\alpha \leq s_1 < s_2 \leq \beta$;

(H3) there exists a continuous function $\Phi : X \to \mathbb{R}$ such that, for all local solution of the generalized ODE (2.1), $x : [s_0, \delta(s_0) + s_0] \to X$, with $s_0 \geq t_0$, we have

$$D^+W(t, x(t)) = \limsup_{\eta \to 0^+} \frac{W(t + \eta, x(t + \eta)) - W(t, x(t))}{\eta} < \Phi(x(t)),$$

for all $t \in [s_0, \delta(s_0) + s_0)$.

If $\bar{x} \in G^-([\alpha, \beta], X)$, then, for all $t \in [\alpha, \beta]$, we have

$$W(t, \bar{x}(t)) - W(\alpha, \bar{x}(\alpha)) \leq \sup_{s \in [\alpha, \beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau), t) \right\| + M(t - \alpha), \quad (2.9)$$

where $M = \sup_{s \in [\alpha, \beta]} \Phi(\bar{x}(s))$.

Proof. At first, notice that, by Propositions A.0.13 and A.0.14, we have $\sup_{t\in[\alpha,\beta]}\Phi(\overline{x}(t))<\infty$ for all $\overline{x}\in G^{-}([\alpha,\beta],X)$. Moreover, for all $\overline{x}\in G^{-}([\alpha,\beta],X)$, the existence of the Kurzweil integral $\int_{\alpha}^{\beta}DF(\overline{x}(\tau),s)$ is guaranteed by Corollary C.0.4 and, if $\sigma\in[\alpha,\beta]$ is fixed, then $(\overline{x}(\sigma),\sigma)\in\Omega$, once $\Omega=X\times[t_0,+\infty)$. On the other hand, it is clear that $\Omega=\Omega_F$, where Ω_F is given by (C.6). Therefore, by Theorem C.0.11, there exists a unique local solution $x:[\sigma,\sigma+\eta_1(\sigma)]\to X$ of the generalized ODE (2.1) with initial condition $x(\sigma)=\overline{x}(\sigma)$.

Let $\eta_2>0$ be such that $\eta_2\leq\eta_1(\sigma)$ and $\sigma+\eta_2\leq\beta$. Then, the integral $\int_{\sigma}^{\sigma+\eta_2}DF(x(\tau),t)$ exists because x is a solution of the generalized ODE (2.1) and, by the property of integrability on subintervals, the integral $\int_{\sigma}^{\sigma+\eta_2}D[F(\overline{x}(\tau),t)-F(x(\tau),t)]$ also exists. The existence of this integral ensures that, given $\varepsilon>0$, there exists a gauge δ on $[\sigma,\sigma+\eta]$ (see Definition B.0.1) and we may assume, without loss of generality, that $\eta_2<\delta(\sigma)$. By hypothesis (H3), we can take $\eta\leq\eta_2$ such that the inequality

$$W(\sigma+\eta,x(\sigma+\eta))-W(\sigma,x(\sigma))\leq\eta\,(\Phi(x(\sigma))+\varepsilon) \tag{2.10}$$

holds. Moreover, by Corollary B.0.14, for every $s\in[\sigma,\sigma+\eta]$, we have

$$\left\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-\int_{\sigma}^{s}DF(\overline{x}(\tau),t)\right\|<\frac{\eta\varepsilon}{2}\quad\text{and} \tag{2.11}$$

$$\left\|F(x(\sigma),s)-F(x(\sigma),\sigma)-\int_{\sigma}^{s}DF(x(\tau),t)\right\|<\frac{\eta\varepsilon}{2}. \tag{2.12}$$

By equations (2.11) and (2.12), we get

$$\begin{aligned}
&\sup_{s\in[\sigma,\sigma+\eta]}\left\|\int_{\sigma}^{s}D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\right\|\\
&-\sup_{s\in[\sigma,\sigma+\eta]}\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma)\|\\
\leq\;&\sup_{s\in[\sigma,\sigma+\eta]}\bigg(\bigg\|\int_{\sigma}^{s}D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\\
&-(F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma))\bigg\|\bigg)\\
\leq\;&\sup_{s\in[\sigma,\sigma+\eta]}\left\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-\int_{\sigma}^{s}DF(\overline{x}(\tau),t)\right\|\\
&+\sup_{s\in[\sigma,\sigma+\eta]}\left\|F(x(\sigma),s)-F(x(\sigma),\sigma)-\int_{\sigma}^{s}DF(x(\tau),t)\right\|<\eta\varepsilon.
\end{aligned} \tag{2.13}$$

Taking into account that $F\in\mathscr{F}(\Omega,h)$ and $\overline{x}(\sigma)=x(\sigma)$, we have

$$\begin{aligned}
&\sup_{s\in[\sigma,\sigma+\eta]}\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma)\|\\
\leq\;&\|\overline{x}(\sigma)-x(\sigma)\|\sup_{s\in[\sigma,\sigma+\eta]}|h(s)-h(\sigma)|=0
\end{aligned} \tag{2.14}$$

(see Definition C.0.2). Replacing (2.14) in (2.13), we obtain

$$\sup_{s\in[\sigma,\sigma+\eta]}\left\|\int_{\sigma}^{s}D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\right\|<\varepsilon\eta. \tag{2.15}$$

From hypothesis (2.8) and the relation $x(\sigma)=\bar{x}(\sigma)$, we conclude

$$\begin{aligned}
& W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta)) \\
\overset{\bar{x}(\sigma)=x(\sigma)}{=} \; & W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta))-W(\sigma,\bar{x}(\sigma))+W(\sigma,x(\sigma)) \\
\leq \; & |W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta))-W(\sigma,\bar{x}(\sigma))+W(\sigma,x(\sigma))| \\
\overset{(2.8)}{\leq} \; & \sup_{s\in[\sigma,\sigma+\eta]} \|\bar{x}(s)-x(s)\|
\end{aligned}$$

which implies

$$W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta)) \leq \sup_{s\in[\sigma,\sigma+\eta]} \|\bar{x}(s)-x(s)\|. \tag{2.16}$$

Owing to the fact that x is a local solution of the generalized ODE (2.1), we have

$$x(\sigma)-x(s) = -\int_{\sigma}^{s} DF(x(\tau),t), \quad \text{for all } s\in[\sigma,\sigma+\eta]. \tag{2.17}$$

Thus, by (2.10), (2.15), (2.16), (2.17) and the relation $\bar{x}(\sigma)=x(\sigma)$, we obtain

$$\begin{aligned}
& W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma,\bar{x}(\sigma)) \\
\overset{\bar{x}(\sigma)=x(\sigma)}{=} \; & W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta))+W(\sigma+\eta,x(\sigma+\eta))-W(\sigma,x(\sigma)) \\
\overset{(2.10)}{\leq} \; & W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma+\eta,x(\sigma+\eta))+\eta\,(\Phi(x(\sigma))+\varepsilon) \\
\overset{(2.16)}{\leq} \; & \sup_{s\in[\sigma,\sigma+\eta]} \|\bar{x}(s)-x(s)\|+\eta\,(\Phi(x(\sigma))+\varepsilon) \\
\overset{\bar{x}(\sigma)=x(\sigma)}{=} \; & \sup_{s\in[\sigma,\sigma+\eta]} \|\bar{x}(s)-\bar{x}(\sigma)+x(\sigma)-x(s)\|+\eta\,(\Phi(\bar{x}(\sigma))+\varepsilon) \\
\overset{(2.17)}{=} \; & \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s)-\bar{x}(\sigma)-\int_{\sigma}^{s} DF(x(\tau),t)\right\|+\eta\,(\Phi(\bar{x}(\sigma))+\varepsilon) \\
= \; & \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s)-\bar{x}(\sigma)-\int_{\sigma}^{s} DF(\bar{x}(\tau),t)+\int_{\sigma}^{s} D[F(\bar{x}(\tau),t)-F(x(\tau),t)]\right\| \\
& +\eta\,(\Phi(\bar{x}(\sigma))+\varepsilon) \\
\leq \; & \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s)-\bar{x}(\sigma)-\int_{\sigma}^{s} DF(\bar{x}(\tau),t)\right\| \\
& + \sup_{s\in[\sigma,\sigma+\eta]} \left\|\int_{\sigma}^{s} D[F(\bar{x}(\tau),t)-F(x(\tau),t)]\right\|+\eta\,(\Phi(\bar{x}(\sigma))+\varepsilon) \\
\overset{(2.15)}{\leq} \; & \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s)-\bar{x}(\sigma)-\int_{\sigma}^{s} DF(\bar{x}(\tau),t)\right\|+\varepsilon\eta+\eta\,(\Phi(\bar{x}(\sigma))+\varepsilon).
\end{aligned}$$

Taking $\varepsilon\to 0$ and $M=\sup_{s\in[\alpha,\beta]}\Phi(\bar{x}(t))$, we validate

$$\begin{aligned}
& W(\sigma+\eta,\bar{x}(\sigma+\eta))-W(\sigma,\bar{x}(\sigma)) \\
\leq \; & \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s)-\bar{x}(\sigma)-\int_{\sigma}^{s} DF(\bar{x}(\tau),t)\right\|+\eta M.
\end{aligned} \tag{2.18}$$

In order to proceed with the proof, we define $P:[\alpha,\beta]\to X$ by

$$P(s)=\bar{x}(s)-\int_{\alpha}^{s} DF(\bar{x}(\tau),t), \text{ for all } s\in[\alpha,\beta].$$

Since $\bar{x} \in G^-([\alpha,\beta],X)$, by Lemma C.0.7, $P \in G^-([\alpha,\beta],X)$. Furthermore, if $s \in [\sigma,\beta]$, then

$$\begin{aligned}
&\|P(s)-P(\sigma)\| \\
= &\left\|\bar{x}(s) - \int_\alpha^s DF(\bar{x}(\tau),t) - \bar{x}(\sigma) + \int_\alpha^\sigma DF(\bar{x}(\tau),t)\right\| \\
= &\left\|\bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) - \left(\bar{x}(\sigma) - \bar{x}(\alpha) - \int_\alpha^\sigma DF(\bar{x}(\tau),t)\right)\right\| \\
\leq &\left\|\bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t)\right\| + \left\|\bar{x}(\sigma) - \bar{x}(\alpha) - \int_\alpha^\sigma DF(\bar{x}(\tau),t)\right\| \\
= &\left\|\bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t)\right\| + \|P(\sigma)-P(\alpha)\|
\end{aligned} \tag{2.19}$$

Define a function $f : [\alpha,\beta] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \sup_{s\in[t,\sigma]} \|P(s)-P(\sigma)\| + M(t-\sigma), & t \in [\alpha,\sigma], \\ \sup_{s\in[\sigma,t]} \|P(s)-P(\sigma)\| + M(t-\sigma), & t \in [\sigma,\beta]. \end{cases}$$

Clearly, f is well-defined and, by the left continuity of P, f is left-continuous on $(\alpha,\beta]$ (see Propositions A.0.20 and A.0.21). Moreover,

$$f(\sigma+\eta) - f(\sigma) = \sup_{s\in[\sigma,\sigma+\eta]} \left\|\bar{x}(s) - \bar{x}(\sigma) - \int_\sigma^s DF(\bar{x}(\tau),t)\right\| + M\eta.$$

Using this fact together with (2.18), we get

$$W(\sigma+\eta,\bar{x}(\sigma+\eta)) - W(\sigma,\bar{x}(\sigma)) \leq f(\sigma+\eta) - f(\sigma). \tag{2.20}$$

Then, the functions $[a,b] \ni t \mapsto W(t,\bar{x}(t))$ and $[a,b] \ni t \mapsto f(t)$ satisfy all the hypotheses of Proposition A.0.19 and, hence,

$$W(t,\bar{x}(t)) - W(\alpha,\bar{x}(\alpha)) \leq f(t) - f(\alpha),$$

for all $t \in [\alpha,\beta]$.

We conclude the proof by showing

$$f(t) - f(\alpha) \leq \sup_{s\in[\alpha,t]} \left\|\bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),v)\right\| + M(t-\alpha),$$

for all $t \in [\alpha,\beta]$. To this end, we consider two cases.

Case 1: $\alpha \leq \sigma < t$.

In this case, we have

$$
\begin{aligned}
f(t)-f(\alpha) &= \sup_{s\in[\sigma,t]}\|P(s)-P(\sigma)\|+M(t-\sigma)-\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|-M(\alpha-\sigma)\\
&\overset{(2.19)}{\leq} \sup_{s\in[\sigma,t]}\left(\left\|\overline{x}(s)-\overline{x}(\alpha)-\int_{\alpha}^{s}DF(\overline{x}(\tau),v)\right\|+\|P(\sigma)-P(\alpha)\|\right)\\
&\quad -\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|+M(t-\alpha)\\
&\leq \sup_{s\in[\sigma,t]}\left\|\overline{x}(s)-\overline{x}(\alpha)-\int_{\alpha}^{s}DF(\overline{x}(\tau),v)\right\|\\
&\quad +\|P(\sigma)-P(\alpha)\|-\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|+M(t-\alpha)\\
&\leq \sup_{s\in[\alpha,t]}\left\|\overline{x}(s)-\overline{x}(\alpha)-\int_{\alpha}^{s}DF(\overline{x}(\tau),v)\right\|\\
&\quad +\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|-\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|+M(t-\alpha)\\
&= \sup_{s\in[\alpha,t]}\left\|\overline{x}(s)-\overline{x}(\alpha)-\int_{\alpha}^{s}DF(\overline{x}(\tau),v)\right\|+M(t-\alpha).
\end{aligned}
$$

Case 2: $\alpha<t\leq\sigma$.

In this case, we have

$$
\begin{aligned}
f(t)-f(\alpha) &= \sup_{s\in[t,\sigma]}\|P(s)-P(\sigma)\|+M(t-\sigma)-\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|-M(\alpha-\sigma)\\
&\leq \sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|-\sup_{s\in[\alpha,\sigma]}\|P(s)-P(\sigma)\|+M(t-\alpha)\\
&= M(t-\alpha)\leq \sup_{s\in[\alpha,t]}\left\|\overline{x}(s)-\overline{x}(\alpha)-\int_{\alpha}^{s}DF(\overline{x}(\tau),v)\right\|+M(t-\alpha)
\end{aligned}
$$

and the result follows. □

It is worth mentioning that, when $\Phi\equiv 0$ in the proof of Lemma 2.1.6, we obtain [4, Lemma 3.7] and [13, Lemma 8.32].

Based on Lemma 2.1.6, the following result gives sufficient conditions for regular stability of the trivial solution of the generalized ODE (2.1) and weakens the Lipschitzian condition on the second variable of the Lyapunov functional used in [26, Theorem 6.3].

Theorem 2.1.7. Let $V:[t_0,+\infty)\times X\to\mathbb{R}$ be a Lyapunov functional with respect to the generalized ODE (2.1) and assume that $F\in\mathscr{F}(\Omega,h)$, where $h:[t_0,+\infty)\to\mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Furthermore, suppose V satisfies conditions (H1) and (H2) from Lemma 2.1.6 and

(LR1) there exists an increasing continuous function $a:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $a(0)=0$ such that

$$V(t,z)\leq a(\|z\|),$$

for all $z\in X$ and all $t\in[t_0,+\infty)$.

Then, the trivial solution of the generalized ODE (2.1) is regularly stable.

Proof. From the definition of a Lyapunov functional, there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(0) = 0$ and

$$V(t,y) \geq b(\|y\|),$$

for every $(t,y) \in [t_0,+\infty) \times X$. Let $s_0 \geq t_0$ and $\varepsilon > 0$. Consider the set $A = \{a(t)+t; t \in \mathbb{R}^+\}$. It is clear that $\inf A = 0$ and, by the property of the infimum, for $b(\varepsilon) > 0$, there exists $\delta > 0$, such that $0 < a(\delta)+\delta < b(\varepsilon)$.

Let $[\alpha,\beta] \subset [t_0,+\infty)$ and $\bar{x} \in G^-([\alpha,\beta],X)$ be such that

$$\|\bar{x}(\alpha)\| < \delta \quad \text{and} \quad \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| < \delta.$$

Since $V : [t_0,+\infty) \times X \to \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (2.1), for all solution $x : [s_0, \delta(s_0)+s_0] \to X$ of the generalized ODE (2.1) with $s_0 \geq t_0$, we have

$$D^+V(t,x(t)) = \limsup_{\eta\to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \leq 0,$$

for all $t \in [s_0, \delta(s_0)+s_0)$. Thus, $V : [t_0,+\infty) \times X \to \mathbb{R}$ satisfies all the hypotheses of Lemma 2.1.6 with $\Phi \equiv 0$. By (2.9), for all $t \in [\alpha,\beta]$, we get

$$\begin{aligned} V(t,\bar{x}(t)) &\leq V(\alpha,\bar{x}(\alpha)) + \sup_{s\in[\alpha,t]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| \\ &\leq V(\alpha,\bar{x}(\alpha)) + \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| \\ &\leq a(\|\bar{x}(\alpha)\|) + \delta \\ &\leq a(\delta) + \delta \\ &< b(\varepsilon). \end{aligned}$$

Moreover, by the definition of b, for every $t \in [\alpha,\beta]$, we have

$$b(\|\bar{x}(t)\|) \leq V(t,\bar{x}(t)) < b(\varepsilon).$$

From this fact and since b is an increasing function, we conclude

$$\|\bar{x}(t)\| < \varepsilon, \text{ for all } t \in [\alpha,\beta],$$

which proves that the trivial solution of the generalized ODE (2.1) is regularly stable. □

Remark 2.1.8. If we consider condition (L'3) instead of condition (L3) in Definition 2.0.2, the proof of Theorem 2.1.7 holds, once condition (L'3) implies condition (L3).

The next result is a Lyapunov-type theorem on regular asymptotic stability. A version of such a result when the Lyapunov functional with respect to the generalized ODE satisfies the following Lipschitz condition

$$|V(t,z)-V(t,y)| \leq K\|z-y\|, \quad \text{for all } t \in [t_0,+\infty) \text{ and all } z,y \in \overline{B}_c = \{y \in X; \|y\| \leq c\}$$

can be found in [26, Theorem 6.4].

Theorem 2.1.9. Let $V : [t_0,+\infty) \times \overline{B}_c \to \mathbb{R}$ be a Lyapunov functional with respect to the generalized ODE, where $\overline{B}_c = \{y \in X; \|y\| \leq c\}$ and $c > 0$. Suppose V satisfies the conditions from Theorem 2.1.7 and

(LRA1) there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$ such that for every solution $x : [\alpha,\beta] \subset [t_0,+\infty) \to X$ of (2.1), we have

$$D^+V(t,x(t)) \leq -\Phi(x(t)),$$

for all $t \in [\alpha,\beta)$.

Then, the trivial solution of the generalized ODE (2.1) is regularly asymptotically stable.

Proof. By Theorem 2.1.7, the trivial solution of the generalized ODE (2.1) is regularly stable. Then,

(I) there exists $\delta_0 \in (0,c)$ such that, if $\bar{x} \in G^-([\alpha,\beta],X)$ satisfies

$$\|\bar{x}(\alpha)\| < \delta_0 \quad \text{and} \quad \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| < \delta_0,$$

then

$$\|\bar{x}(t)\| < c, \quad \text{for all } t \in [\alpha,\beta].$$

(II) for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\alpha \leq \gamma < \theta \leq \beta$ and $\bar{x} \in G^-([\gamma,\theta],X)$ for which

$$\|\bar{x}(\gamma)\| < \delta \quad \text{and} \quad \sup_{s\in[\gamma,\theta]} \left\| \bar{x}(s) - \bar{x}(\gamma) - \int_\gamma^s DF(\bar{x}(\tau),t) \right\| < \delta,$$

we have

$$\|\bar{x}(t)\| < \varepsilon,$$

for all $t \in [\gamma,\theta]$.

Define $\rho(\varepsilon) = \frac{\min\{\delta_0,\delta\}}{2}$ and

$$T = T(\varepsilon) = -\left(\frac{a(\delta_0)+\rho(\varepsilon)}{N}\right), \tag{2.21}$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is given by Theorem 2.1.7-(LR1) and

$$N = \sup\{-\Phi(y) : \delta \leq \|y\| < c\} = -\inf\{\Phi(y) : \delta \leq \|y\| < c\} < 0.$$

Assume that $\bar{x} \in G^-([\alpha,\beta],X)$ is such that

$$\|\bar{x}(\alpha)\| < \delta_0 \quad \text{and} \quad \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| < \rho(\varepsilon) < \frac{\delta}{2}. \tag{2.22}$$

We aim to prove that $\|\bar{x}(t)\| < \varepsilon$, for all $t \in [\alpha,\beta] \cap [T+\alpha,+\infty)$.

If $T > \beta - \alpha$, there is nothing to prove, once $[\alpha,\beta] \cap [T+\alpha,+\infty) = \emptyset$. Assume that $T < \beta - \alpha$. We assert that there exists $\bar{t} \in [\alpha,\beta]$ such that $\|\bar{x}(t)\| < \delta$. Suppose to the contrary, that is,

$$\|\bar{x}(t)\| \geq \delta \quad \text{for all } t \in [\alpha,\beta]. \tag{2.23}$$

In particular, $\|\bar{x}(T+\alpha)\| \geq \delta$. Owing to the fact that $V : [t_0,+\infty) \times \overline{B}_c \to \mathbb{R}$ is a Lyapunov functional with respect the generalized ODE (2.1), there exists an incresing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(0) = 0$ and

$$V(T+\alpha,\bar{x}(T+\alpha)) \geq b(\|\bar{x}(T+\alpha)\|) \geq b(\delta) > 0. \tag{2.24}$$

On the other hand, by (2.22), item (I) and (2.23), we conclude that $\delta \leq \|\bar{x}(t)\| < c$, for all $t \in [\alpha,\beta]$. Consequently,

$$\sup_{s\in[\alpha,\beta]} [-\Phi(\bar{x}(s))] \leq N. \tag{2.25}$$

By (2.22) and by the definition of the function $a : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$V(\alpha,\bar{x}(\alpha)) \leq a(\|\bar{x}(\alpha)\|) < a(\delta_0). \tag{2.26}$$

By Lemma 2.1.6, we get

$$\begin{aligned} & V(T+\alpha,\bar{x}(T+\alpha)) \\ \leq \; & V(\alpha,\bar{x}(\alpha)) + \sup_{s\in[\alpha,T+\alpha]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| \\ & + \sup_{s\in[\alpha,T+\alpha]} (-\Phi(\bar{x}(s)))\,(T+\alpha-\alpha) \\ \overset{(2.25),\,(2.26)}{\leq} \; & a(\delta_0) + \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_\alpha^s DF(\bar{x}(\tau),t) \right\| + NT \\ \overset{(2.21),\,(2.22)}{\leq} \; & a(\delta_0) + \rho(\varepsilon) - a(\delta_0) - \rho(\varepsilon) = 0 \end{aligned} \tag{2.27}$$

which contradicts (2.24). Therefore, there exists $\bar{t} \in [\alpha,\beta]$ such that $\|\bar{x}(\bar{t})\| < \delta$.

Let t^* be the smallest point $t \in [\alpha,\beta]$ such that $\|\bar{x}(t)\| < \delta$. Then,

$$\begin{aligned}
& \sup_{s\in[t^*,\beta]} \left\| \bar{x}(s) - \bar{x}(t^*) - \int_{t^*}^{s} DF(\bar{x}(\tau),t) \right\| \\
\leq & \sup_{s\in[t^*,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{t^*} DF(\bar{x}(\tau),t) - \int_{t^*}^{s} DF(\bar{x}(\tau),t) \right. \\
& \left. - \left(\bar{x}(t^*) - \bar{x}(\alpha) - \int_{\alpha}^{t^*} DF(\bar{x}(\tau),t) \right) \right\| \\
\leq & \sup_{s\in[t^*,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| + \left\| \bar{x}(t^*) - \bar{x}(\alpha) - \int_{\alpha}^{t^*} DF(\bar{x}(\tau),t) \right\| \\
\leq & \ 2 \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| \\
\overset{(2.22)}{\leq} & \ 2\rho(\varepsilon) < 2\frac{\delta}{2} = \delta.
\end{aligned}$$

By item (II), $\|\bar{x}(t)\| < \varepsilon$ for all $t \in [t^*,\beta]$. Moreover, by (2.24) and (2.27), $T+\alpha \geq t^*$ and, hence, $[T+\alpha,\beta] \subset [t^*,\beta]$ and $\|\bar{x}(t)\| < \varepsilon$ for all $t \in [T+\alpha,\beta] = [\alpha,\beta] \cap [T+\alpha,\beta]$. □

We point out that the regular asymptotic stability in Theorem 2.1.9 is guaranteed only for regulated functions whose their range is a subset of the open ball $B_c = \{y \in X; \|y\| < c\}$ (see the statement (I) in the proof of Thereom 2.1.9). In order to generalize this result for every regulated function taking values in X, we obtained, in the next result, the same conclusion as in Theorem 2.1.9, but we assume that the Lyapunov functional is defined in $[t_0,+\infty) \times X$ instead of $[t_0,+\infty) \times \overline{B}_c$. Thereby, regular asymptotic stability holds for regulated functions with range outside $\overline{B}_c$. We also highlight that the proof of the next theorem follows similar ideas to the proof of Theorem 2.1.9.

Theorem 2.1.10. Let $V : [t_0,+\infty) \times X \to \mathbb{R}$ be a Lyapunov functional with respect to the generalized ODE. Suppose V satisfies the conditions from Theorem 2.1.7 and

(LRA'1) there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$ such that for every solution $x : [\alpha,\beta] \subset [t_0,+\infty) \to X$ of (2.1), we have

$$D^+V(t,x(t)) \leq -\Phi(x(t)),$$

for all $t \in [\alpha,\beta)$.

Then, the trivial solution of the generalized ODE (2.1) is regularly asymptotically stable.

Proof. Let $\bar{x} \in G^-([\alpha,\beta],X)$ and $\delta_0 > 0$ be given. Since $V : [t_0,+\infty) \times X \to \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (2.1), there exist increasing continuous functions, $a,b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$b(\|y\|) \leq V(t,y) \leq a(\|y\|), \quad \text{for all } (t,y) \in [t_0,+\infty) \times X. \tag{2.28}$$

Moreover, since all the hypotheses of Theorem 2.1.7 are satisfied, the trivial solution of the generalized ODE (2.1) is regularly stable. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha \leq \gamma < \theta \leq \beta$ and

$$\|\bar{x}(\gamma)\| < \delta \quad \text{and} \quad \sup_{s\in[\gamma,\theta]} \left\| \bar{x}(s) - \bar{x}(\gamma) - \int_{\gamma}^{s} DF(\bar{x}(\tau),t) \right\| < \delta, \tag{2.29}$$

then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for all, } t \in [\gamma,\theta]. \tag{2.30}$$

We target to prove that there exist $\rho(\varepsilon) > 0$ and $T(\varepsilon) > 0$ such that if $\bar{x} \in G^{-}([\alpha,\beta],X)$ is such that $\|\bar{x}(\alpha)\| < \delta_0$ and $\sup_{s\in[\alpha,\beta]} \|\bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),s)\| < \rho(\varepsilon)$, then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for all } t \in [\alpha,\beta] \cap [\alpha + T(\varepsilon), +\infty). \tag{2.31}$$

At first, notice that, $M = \sup_{s\in[\alpha,\beta]} -\Phi(\bar{x}(t)) < \infty$ (see Propositions A.0.13 and A.0.14) and $M < 0$, once $\Phi(y) > 0$, whenever $y \neq 0$. Take $\rho(\varepsilon) = \frac{\delta}{2}$ and

$$T = T(\varepsilon) = \frac{a(\delta_0) + \rho(\varepsilon)}{-M} > 0. \tag{2.32}$$

Assume that $\|\bar{x}(\alpha)\| < \delta_0$ and

$$\sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),s) \right\| < \rho(\varepsilon) = \frac{\delta}{2}. \tag{2.33}$$

In order to show that (2.30) holds we consider two cases.

Case 1: $T + \alpha > \beta$.

In this case, $[\alpha,\beta] \cap [\alpha + T, +\infty) = \emptyset$ and (2.31) is trivially satisfied.

Case 2: $T + \alpha \leq \beta$.

Let us prove that $\|\bar{x}(T+\alpha)\| < \delta$. Indeed, suppose $\|\bar{x}(T+\alpha)\| \geq \delta$. Since b is increasing, we have

$$b(\|\bar{x}(T+\alpha)\|) \geq b(\delta) > 0. \tag{2.34}$$

By the fact that a is increasing, $\|x(\alpha)\| < \delta_0$ and, by Lemma 2.1.6, we obtain

$$\begin{aligned}
& V(T+\alpha, \bar{x}(T+\alpha)) \\
\leq \; & V(\alpha, \bar{x}(\alpha)) + \sup_{s\in[\alpha,T+\alpha]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| + M(T + \alpha - \alpha) \\
\overset{(2.28)}{\leq} \; & a(\|\bar{x}(\alpha)\|) + \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| - a(\delta_0) - \rho(\varepsilon) \\
\leq \; & a(\delta_0) + \rho(\varepsilon) - a(\delta_0) - \rho(\varepsilon) = 0
\end{aligned}$$

which contradicts the fact that

$$V(T+\alpha, \bar{x}(T+\alpha)) \overset{(2.28)}{\geq} b(\|\bar{x}(T+\alpha)\|) \overset{(2.34)}{\geq} b(\delta) > 0.$$

Therefore $\|\bar{x}(T+\alpha)\| < \delta$ and,

$$\begin{aligned}
& \sup_{s\in[T+\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(T+\alpha) - \int_{T+\alpha}^{s} DF(\bar{x}(\tau),t) \right\| \\
\leq & \sup_{s\in[T+\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{T+\alpha} DF(\bar{x}(\tau),t) - \int_{T+\alpha}^{s} DF(\bar{x}(\tau),t) \right. \\
& \left. - \left(\bar{x}(T+\alpha) - \bar{x}(\alpha) - \int_{\alpha}^{T+\alpha} DF(\bar{x}(\tau),t) \right) \right\| \\
\leq & \sup_{s\in[T+\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| + \left\| \bar{x}(T+\alpha) - \bar{x}(\alpha) - \int_{\alpha}^{T+\alpha} DF(\bar{x}(\tau),t) \right\| \\
\leq & \ 2 \sup_{s\in[\alpha,\beta]} \left\| \bar{x}(s) - \bar{x}(\alpha) - \int_{\alpha}^{s} DF(\bar{x}(\tau),t) \right\| \\
\overset{(2.33)}{\leq} & \ 2\rho(\varepsilon) = 2\frac{\delta}{2} = \delta.
\end{aligned}$$

These facts together with (2.29) and (2.30) lead to (2.31). □

2.1.2 Converse Lyapunov theorems

Our goal in this subsection is to obtain converse Lyapunov Theorems on regular stability and on regular attractivity with some properties described in Theorems 2.1.7 and 2.1.10. We were motivated by [48, 49], when Š. Schwabik proved a converse Lyapunov Theorem concerning variational stability. The results presented here are new and they can be found in [4] (they also can be found in [13]).

Suppose X is a Banach space, $\mathscr{O} \subseteq X$ is an open subset containing the neutral element of X, $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition for the existence of such a solution), $F : \Omega \to X$ belongs to $\mathscr{F}(\Omega, h)$, where $\Omega = \mathscr{O} \times [t_0, +\infty)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). Moreover, we assume that, for every $(x_0, s_0) \in \Omega$, there exists unique maximal solution $x : [s_0, \omega(s_0, x_0)) \to X$ of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. See Definition C.0.15 for the concept of maximal solution and Theorem C.0.17 and Corollaries C.0.20 and C.0.21 for sufficient conditions for the existence of such a solution.

Let $s \geq t_0$, $y \in \mathscr{O}$ and consider

$$A(s,y) = \{\varphi \in G([t_0,+\infty),\mathscr{O}) : \varphi(t_0) = 0,\ \varphi(s) = y,\ \varphi \text{ is left-continuos on } (t_0,+\infty)\}, \quad (2.35)$$

where the set $G([t_0,+\infty),\mathscr{O})$ is described in Definition A.0.22-(ii).

For $s \geq t_0$ and $y \in \mathscr{O}$, define $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ by

$$V(s,y) = \begin{cases} \inf_{\varphi\in A(s,y)} \left\{ \sup_{\sigma\in[t_0,s]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \right\}, & \text{if } s > t_0, \\ \|y\|, \text{ if } s = t_0. \end{cases} \quad (2.36)$$

By Corollary C.0.4, we know that, for all $\varphi \in A(s,y)$ and all $\sigma \in [t_0,s]$, the Kurzweil integral $\int_{t_0}^{\sigma} DF(\varphi(\tau),t)$ exists. Moreover, by Lemma C.0.5, the function

$$[t_0,s] \ni \sigma \mapsto f(\sigma) := \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t)$$

is also regulated which guarantees that $\sup_{\sigma\in[t_0,s]} \|f(\sigma)\| < \infty$ (see Proposition A.0.13). Therefore, V is well-defined for all $(s,y) \in [t_0,+\infty) \times \mathscr{O}$.

Remark 2.1.11. Let $\sigma > s \geq t_0$, $y \in X$ and $\varphi \in A(s,y)$ be given. If $\widetilde{\varphi} \in G^{-}([s,\sigma],\mathscr{O})$ is such that $\widetilde{\varphi}(s) = \varphi(s)$, then the function $\phi : [t_0,+\infty) \to \mathscr{O}$, defined by

$$\phi(t) = \begin{cases} \varphi(t), & \text{if } t \in [t_0,s], \\ \widetilde{\varphi}(t), & \text{if } t \in [s,\sigma], \\ 0, & \text{if } t \in (\sigma,+\infty), \end{cases}$$

belongs to $A(s,y)$. Indeed, ϕ is well-defined, once $\widetilde{\varphi}(s) = \varphi(s)$. Moreover, since $\varphi \in A(s,x)$, we have $\varphi(t_0) = 0$, $\varphi(s) = y$, φ is regulated on $[t_0,s]$ and φ is left-continuous on $(t_0,+\infty)$. Then, it is clear that $\phi(t_0) = \varphi(t_0) = 0$, $\phi(s) = \varphi(s) = y$, ϕ is left-continuous on $(t_0,+\infty)$ and $\phi \in G([t_0,+\infty),\mathscr{O})$. From these facts, $\phi \in A(s,y)$.

Furthermore, by Lemma C.0.9, if $F \in \mathscr{F}(\Omega,h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$, then every solution $x : [s,\sigma] \to X$ of the generalized ODE (2.1) with $x(s) = y$ is such that $x \in G^{-}([s,\sigma],\mathscr{O})$ and, hence, the function $\widetilde{\phi} : [t_0,+\infty) \to \mathscr{O}$, defined by

$$\widetilde{\phi} = \begin{cases} \varphi(t), & \text{if } t \in [t_0,s], \\ x(t), & \text{if } t \in [s,\sigma], \\ 0, & \text{if } t \in (\sigma,+\infty), \end{cases}$$

belongs to $A(s,y)$.

In the sequel, we present an useful property of the set $A(s,y)$ given by (2.35).

Lemma 2.1.12. For all $s \geq t_0$ and all $y \in \mathscr{O}$, the set $A(s,y)$ is closed, that is, if $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence in $A(s,y)$ which converges to a function φ in $G([t_0,+\infty),\mathscr{O})$ with the topology of locally uniform convergence, then $\varphi \in A(s,y)$.

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $A(s,y)$ which converges to φ in $G([t_0,+\infty),\mathscr{O})$, that is,

$$\lim_{n\to\infty} \sup_{\sigma\in[\alpha,\beta]} \|\varphi_n(\sigma) - \varphi(\sigma)\| = 0, \quad \text{for all } [\alpha,\beta] \subset [t_0,+\infty). \tag{2.37}$$

We need to prove the following assertions:

(i) $\varphi \in G([t_0,+\infty),\mathscr{O})$;

(ii) $\varphi(t_0) = 0$;

(iii) $\varphi(s) = y$;

(iv) φ is left-continuous on $(t_0, +\infty)$.

Item (i) follows directly from (2.37) and the fact that $\varphi_n \in G([\alpha,\beta],\mathscr{O})$ for all $n \in \mathbb{N}$ and all $[\alpha,\beta] \subset [t_0,+\infty)$ (see Theorem A.0.11). Moreover, since $[t_0,s] \subset [t_0,+\infty)$, we have

$$\begin{aligned} \varphi(t_0) &= \lim_{n\to\infty} \varphi_n(t_0) &= \lim_{n\to\infty} 0 &= 0 \quad \text{and} \\ \varphi(s) &= \lim_{n\to\infty} \varphi_n(s) &= \lim_{n\to\infty} y &= y, \end{aligned}$$

which prove items (ii) and (iii). In order to prove (iv), we show that $\varphi \in G^-([\alpha,\beta],\mathscr{O})$ for each $[\alpha,\beta] \subset [t_0,+\infty)$. Indeed, let $[\alpha,\beta] \subset [t_0,+\infty)$ be given and, applying the Moore-Osgood Theorem (see Theorem A.0.12), we obtain

$$\lim_{\sigma\to t^-} \varphi(\sigma) = \lim_{\sigma\to t^-} \lim_{n\to\infty} \varphi_n(\sigma) = \lim_{n\to\infty} \lim_{\sigma\to t^-} \varphi_n(\sigma) = \lim_{n\to\infty} \varphi_n(t^-) = \lim_{n\to\infty} \varphi_n(t) = \varphi(t),$$

for all $t \in (\alpha,\beta]$. Therefore, $\varphi \in A(s,y)$. □

In what follows, we show that the function $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$, defined by (2.36), satisfies some special properties.

Lemma 2.1.13. Let $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.36). Then V satisfies the following conditions:

(i) $V(s,0) = 0$ for all $s \geq t_0$;

(ii) $V(s,y) \geq 0$, for all $y \in \mathscr{O}$ and all $s \geq t_0$.

Proof. Item (i) follows immediately from the fact that $\varphi \equiv 0 \in A(s,0)$.

In order to prove item (ii), notice that

$$\sup_{\sigma\in[t_0,s]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \geq 0,$$

for all $\varphi \in A(s,y)$. Therefore, $V(s,y) \geq 0$, for all $y \in \mathscr{O}$ and all $s \geq t_0$. □

Lemma 2.1.14. Let $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.36) and assume that $F \in \mathscr{F}(\Omega,h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Then,

$$V(s,y) - V(s,z) \leq \|y - z\|,$$

for all $y,z \in \mathscr{O}$ and all $s \in [t_0,+\infty)$.

Proof. At first, assume that $s = t_0$. Then, by the definition of V, we have

$$V(t_0,y) - V(t_0,z) = \|y\| - \|z\| \leq \|y - z\|.$$

Let $s > t_0$, $\varphi \in A(s,z)$ and take $0 < \eta < s - t_0$. Define a function $\varphi_\eta : [t_0,+\infty) \to \mathscr{O}$ by

$$\varphi_\eta(\sigma) = \begin{cases} \varphi(\sigma), & \text{if } \sigma \in [t_0, s-\eta], \\ \varphi(\sigma) + \dfrac{\sigma - s + \eta}{\eta}(y - z), & \text{if } \sigma \in [s-\eta, s], \\ 0, & \text{if } \sigma \in (s, +\infty). \end{cases}$$

As we mentioned in Remark 2.1.11, $\varphi_\eta \in A(s,y)$ and, by the definition of the function V, we have

$$\begin{aligned}
& V(s,y) - \sup_{\sigma\in[t_0,s]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
\leq\ & \sup_{\sigma\in[t_0,s]} \left\| \varphi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_\eta(\tau),t) \right\| - \sup_{\sigma\in[t_0,s]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
\leq\ & \sup_{\sigma\in[t_0,s]} \left\| \varphi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_\eta(\tau),t) - \varphi(\sigma) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
\leq\ & \sup_{\sigma\in[t_0,s-\eta]} \left\| \varphi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_\eta(\tau),t) - \varphi(\sigma) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
& + \sup_{\sigma\in[s-\eta,s]} \left\| \varphi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_\eta(\tau),t) - \varphi(\sigma) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
=\ & \sup_{\sigma\in[t_0,s-\eta]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) - \varphi(\sigma) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
& + \sup_{\sigma\in[s-\eta,s]} \left\| \varphi(\sigma) + \frac{\sigma - s + \eta}{\eta}(y-z) - \int_{t_0}^{\sigma} DF(\varphi_\eta(\tau),t) - \varphi(\sigma) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| \\
=\ & \sup_{\sigma\in[s-\eta,s]} \left\| \frac{\sigma - s + \eta}{\eta}(y-z) - \int_{t_0}^{s-\eta} DF(\varphi_\eta(\tau),t) - \int_{s-\eta}^{\sigma} DF(\varphi_\eta(\tau),t) \right. \\
& \left. + \int_{t_0}^{s-\eta} DF(\varphi(\tau),t) + \int_{s-\eta}^{\sigma} DF(\varphi(\tau),t) \right\| \\
=\ & \sup_{\sigma\in[s-\eta,s]} \left\| \frac{\sigma - s + \eta}{\eta}(y-z) - \int_{t_0}^{s-\eta} DF(\varphi(\tau),t) - \int_{s-\eta}^{\sigma} DF(\varphi_\eta(\tau),t) \right. \\
& \left. + \int_{t_0}^{s-\eta} DF(\varphi(\tau),t) + \int_{s-\eta}^{\sigma} DF(\varphi(\tau),t) \right\| \\
=\ & \sup_{\sigma\in[s-\eta,s]} \left\| \frac{\sigma - s + \eta}{\eta}(y-z) - \int_{s-\eta}^{\sigma} DF(\varphi_\eta(\tau),t) + \int_{s-\eta}^{\sigma} DF(\varphi(\tau),t) \right\| \\
\leq\ & \sup_{\sigma\in[s-\eta,s]} \left\| \frac{\sigma - s + \eta}{\eta}(y-z) \right\| + \sup_{\sigma\in[s-\eta,s]} \left\| \int_{s-\eta}^{\sigma} DF(\varphi_\eta(\tau),t) \right\| \\
& + \sup_{\sigma\in[s-\eta,s]} \left\| \int_{s-\eta}^{\sigma} DF(\varphi(\tau),t) \right\| \\
\leq\ & \|y-z\| + 2 \sup_{\sigma\in[s-\eta,s]} |h(\sigma) - h(s-\eta)| \\
=\ & \|y-z\| + 2|h(s) - h(s-\eta)|,
\end{aligned}$$

where the inequality

$$\left\|\int_{s-\eta}^{\sigma} DF(\varphi_\eta(\tau),t)\right\| + \sup_{\sigma\in[s-\eta,s]}\left\|\int_{s-\eta}^{\sigma} DF(\varphi(\tau),t)\right\| \leq 2|h(\sigma)-h(s-\eta)|$$

follows from Lemma C.0.5. Therefore,

$$V(s,y) - \sup_{\sigma\in[t_0,s]}\left\|\varphi(\sigma)-\int_{t_0}^{\sigma} DF(\varphi(\tau),t)\right\| \leq \|y-z\|+2|h(s)-h(s-\eta)|.$$

Taking $\eta\to 0^+$, we obtain

$$V(s,y) \leq \|y-z\| + \sup_{\sigma\in[t_0,s]}\left\|\varphi(\sigma)-\int_{t_0}^{\sigma} DF(\varphi(\tau),t)\right\|, \tag{2.38}$$

once h is left-continuous on $(t_0,+\infty)$ and, hence,

$$\lim_{\eta\to 0^+}|h(s)-h(s-\eta)| = \lim_{t\to s^-}|h(s)-h(t)| = 0.$$

Taking the infimum over all $\inf \varphi\in A(s,z)$ in (2.38), we conclude

$$V(s,y)\leq \|y-z\|+V(s,z),$$

and the proof is complete. □

As a consequence of Lemmas 2.1.13 and 2.1.14, we have the following result.

Corollary 2.1.15. Let $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$ be defined by (2.36) and assume that $F\in\mathscr{F}(\Omega,h)$, where h is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Then, for all $z\in\mathscr{O}$ and all $s\in[t_0,+\infty)$, we have

$$V(s,z)\leq\|z\|.$$

Proof. By Lemma 2.1.13, $V(s,0)=0$ for all $s\geq t_0$ and, by Lemma 2.1.14, we conclude

$$V(s,z)=V(s,z)-V(s,0)\leq\|z\|,$$

for all $z\in\mathscr{O}$ and all $s\in[t_0,+\infty)$. □

The next result shows that the function $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$, given by (2.36), satisfies condition (L'3) from Definition 2.0.2.

Lemma 2.1.16. Let $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$ be defined by (2.36) and assume that $F\in\mathscr{F}(\Omega,h)$, where h is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Then, for every $s_0\geq t_0$, the function $[s_0,\omega)\ni t\mapsto V(t,x(t))$ is nonincreasing along every maximal solution $x:[s_0,\omega)\to X$ of the generalized ODE (2.1).

Proof. Let $x : [s_0, \omega) \to X$ be a maximal solution of the generalized ODE (2.1), $t_1, t_2 \in [s_0, \omega)$ be such that $t_2 > t_1$, and $\varphi \in A(t_1, x(t_1))$. Define $\phi : [t_0, +\infty) \to \mathscr{O}$ as

$$\phi(\sigma) := \begin{cases} \varphi(\sigma), & \text{if } \sigma \in [t_0, t_1], \\ x(\sigma), & \text{if } \sigma \in [t_1, t_2], \\ 0, & \text{if } \sigma \in (t_2, +\infty). \end{cases}$$

Notice that, by Remark 2.1.11, $\phi \in A(t_2, x(t_2))$. Thus, by the definition of V, we have

$$V(t_2, x(t_2)) \leq \sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\|. \tag{2.39}$$

Owing to the fact that $\phi \in A(t_2, x(t_2))$, Lemma C.0.5 ensures that the function

$$[t_0, t_2] \ni \sigma \mapsto \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s)$$

belongs to $G^-([t_0, t_2], X)$ and, according to Lemma A.0.15, we may consider two cases with respect to

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\|.$$

Case 1: Assume that, for some $v \in [t_0, t_2]$, we have

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| = \left\| \phi(v) - \int_{t_0}^{v} DF(\phi(\tau), s) \right\|.$$

In this case, either $v \in [t_0, t_1]$ or $v \in [t_1, t_2]$.

If $v \in [t_0, t_1]$, then

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| = \sup_{\sigma \in [t_0, t_1]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\|.$$

Since $\phi|_{[t_0, t_1]} = \varphi$, we get

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| = \sup_{\sigma \in [t_0, t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\|.$$

From this and (2.39), we obtain

$$V(t_2, x(t_2)) \leq \sup_{\sigma \in [t_0, t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\|. \tag{2.40}$$

On the other hand, if $v \in [t_1, t_2]$, then $\phi|_{[t_1, t_2]} = x$ and

$$\begin{aligned} \left\| \phi(v) - \int_{t_0}^{v} DF(\phi(\tau), s) \right\| &= \left\| \phi(v) - \int_{t_0}^{t_1} DF(\phi(\tau), s) - \int_{t_1}^{v} DF(\phi(\tau), s) \right\| \\ &= \left\| x(v) - \int_{t_0}^{t_1} DF(\varphi(\tau), s) - \int_{t_1}^{v} DF(x(\tau), s) \right\|. \end{aligned} \tag{2.41}$$

The fact that x is a solution of the generalized ODE (2.1) and $x(t_1) = \varphi(t_1)$ ensure

$$x(v) - \int_{t_1}^{v} DF(x(\tau),s) = x(t_1) = \varphi(t_1). \tag{2.42}$$

Replacing (2.42) in (2.41), we gain

$$\begin{aligned} \left\| \phi(v) - \int_{t_0}^{v} DF(\phi(\tau),s) \right\| &= \left\| \varphi(t_1) - \int_{t_0}^{t_1} DF(\varphi(\tau),s) \right\| \\ &\leq \sup_{\sigma \in [t_0,t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),s) \right\|. \end{aligned} \tag{2.43}$$

Therefore, by equations (2.39) and (2.43), we conclude

$$V(t_2, x(t_2)) \leq \sup_{\sigma \in [t_0,t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),s) \right\|. \tag{2.44}$$

Case 2: Assume that, for some $v \in [t_0, t_2)$, we have

$$\sup_{\sigma \in [t_0,t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\| = \left\| \phi(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\|.$$

Thus, either $v \in [t_0, t_1)$ or $v \in [t_1, t_2)$.

If $v \in [t_0, t_1)$, then $\phi|_{[t_0,t_1]} = \varphi$ and, by (2.39), we get

$$\begin{aligned} V(t_2, x(t_2)) &\leq \sup_{\sigma \in [t_0,t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\| \\ &= \left\| \varphi(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\varphi(\tau),s) \right\| \\ &\leq \sup_{\sigma \in [t_0,t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),s) \right\|. \end{aligned} \tag{2.45}$$

On the other hand, if $v \in [t_1, t_2)$, then $\phi|_{[t_1,t_2]} = x$ and

$$\begin{aligned} &\sup_{\sigma \in [t_0,t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\| \\ &= \left\| \phi(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\| \\ &= \left\| x(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\phi(\tau),s) \right\| \\ &= \left\| x(v^+) - \int_{t_0}^{t_1} DF(\varphi(\tau),s) - \lim_{\sigma \to v^+} \int_{t_1}^{\sigma} DF(x(\tau),s) \right\|. \end{aligned} \tag{2.46}$$

From Remark C.0.10 and equation (2.42), we conclude

$$\begin{aligned} &\left\| x(v^+) - \int_{t_0}^{t_1} DF(\varphi(\tau),s) - \lim_{\sigma \to v^+} \int_{t_1}^{\sigma} DF(x(\tau),s) \right\| \\ &\overset{C.0.10}{=} \left\| x(v) - \int_{t_0}^{t_1} DF(\varphi(\tau),s) - \int_{t_1}^{v} DF(x(\tau),s) \right\| \\ &\overset{(2.42)}{=} \left\| \varphi(t_1) - \int_{t_0}^{t_1} DF(\varphi(\tau),s) \right\| \\ &\leq \sup_{\sigma \in [t_0,t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),s) \right\|. \end{aligned} \tag{2.47}$$

By (2.45), (2.46) and (2.47), we gain

$$\sup_{\sigma\in[t_0,t_2]}\left\|\phi(\sigma)-\int_{t_0}^{\sigma}DF(\phi(\tau),s)\right\| \leq \sup_{\sigma\in[t_0,t_1]}\left\|\varphi(\sigma)-\int_{t_0}^{\sigma}DF(\varphi(\tau),s)\right\|.$$

Using this fact together with (2.39), we obtain

$$V(t_2,x(t_2)) \leq \sup_{\sigma\in[t_0,t_1]}\left\|\varphi(\sigma)-\int_{t_0}^{\sigma}DF(\varphi(\tau),s)\right\|. \tag{2.48}$$

Taking inf over $\varphi\in A(t_1,x(t_1))$ in (2.40), (2.44), (2.45) and (2.48), we conclude

$$V(t_2,x(t_2)) \leq V(t_1,x(t_1)),$$

obtaining the desired result. □

The next result deals with condition (L1) from Definition 2.0.2.

Lemma 2.1.17. Let $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$ be defined by (2.36) and assume that $F\in\mathscr{F}(\Omega,h)$, where $h:[t_0,+\infty)\to\mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Then, $V(\cdot,y):[t_0,+\infty)\to\mathbb{R}$ is left-continuous on $(t_0,+\infty)$ for all $\mathscr{O}\in X$.

Proof. Let $y\in\mathscr{O}$, $\sigma_0\in(t_0,+\infty)$ and $\varepsilon>0$. By Lemma 2.1.12, there exists $\psi\in A(\sigma_0,y)$ such that

$$V(\sigma_0,y)=\sup_{\sigma\in[t_0,\sigma_0]}\left\|\psi(\sigma)-\int_{t_0}^{\sigma}DF(\psi(\tau),t)\right\|. \tag{2.49}$$

Since h and ψ are left-continuous on $(t_0,+\infty)$, there exists $\delta>0$ such that

$$|h(t)-h(\sigma_0)|<\varepsilon \qquad \text{and} \qquad \|\psi(t)-\psi(\sigma_0)\|<\varepsilon, \tag{2.50}$$

for all $t\in(\sigma_0-\delta,\sigma_0)$. Moreover, by the usual properties of the supremum and, by the definition of V, we have

$$V(\sigma_0,y)\geq\sup_{\sigma\in[t_0,t]}\left\|\psi(\sigma)-\int_{t_0}^{\sigma}DF(\psi(\tau),s)\right\|\geq V(t,\psi(t)), \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0). \tag{2.51}$$

Combining (2.51) with the fact that $\psi\in A(\sigma_0,y)$, we get

$$V(t,y)-V(t,\psi(t))\leq\|y-\psi(t)\|=\|\psi(\sigma_0)-\psi(t)\|, \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0). \tag{2.52}$$

By (2.50), (2.51) and (2.52), we conclude that the following inequality

$$V(t,y)-V(\sigma_0,y)\overset{(2.51)}{\leq}V(t,y)-V(t,\psi(t))\overset{(2.52)}{\leq}\|\psi(\sigma_0)-\psi(t)\|\overset{(2.50)}{<}\varepsilon \tag{2.53}$$

holds for all $t\in(\sigma_0-\delta,\sigma_0)$.

On the other hand, for each $t\in(\sigma_0-\delta,\sigma_0)$, let $x_t:[t,\omega(t,y))\to X$ be the maximal solution of the generalized ODE (2.1) with initial condition $x_t(t)=y$ (the existence of such solutions are ensured by the hypotheses in the beginning of this subsection). By Lemma 2.1.16, we have $V(\sigma_0,x_t(\sigma_0))-V(t,x_t(t))\leq 0$, for all $t\in(\sigma_0-\delta,\sigma_0)$ and, consequently,

$$\begin{aligned} V(\sigma_0,y)-V(t,y) &= V(\sigma_0,y)-V(\sigma_0,x_t(\sigma_0))+V(\sigma_0,x_t(\sigma_0))-V(t,x_t(t)) \\ &\leq V(\sigma_0,y)-V(\sigma_0,x_t(\sigma_0)), \end{aligned} \tag{2.54}$$

for all $t\in(\sigma_0-\delta,\sigma_0)$.

Once x_t is a solution of the generalized ODE (2.1) with $x_t(t)=y$, we have

$$x_t(\sigma_0)-y=\int_t^{\sigma_0} DF(x_t(\tau),s), \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0). \tag{2.55}$$

By Lemma 2.1.14, we obtain

$$V(\sigma_0,y)-V(\sigma_0,x_t(\sigma_0))\leq \|x_t(\sigma_0)-y\|, \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0). \tag{2.56}$$

From (2.54), (2.55) and (2.56), we get

$$V(\sigma_0,y)-V(t,y)\leq \left\|\int_t^{\sigma_0} DF(x_t(\tau),s)\right\|, \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0).$$

Using this fact together with Lemma C.0.5 and (2.50), we obtain

$$V(\sigma_0,y)-V(t,y)\leq |h(\sigma_0)-h(t)|<\varepsilon, \quad \text{for all } t\in(\sigma_0-\delta,\sigma_0). \tag{2.57}$$

According to (2.53) and (2.57), $|V(\sigma_0,y)-V(t,y)|<\varepsilon$, for all $t\in(\sigma_0-\delta,\sigma_0)$ which proves that $V(\cdot,y):[t_0,+\infty)\to\mathbb{R}$ is left-continuous at σ_0 for all $y\in\mathscr{O}$. Since σ_0 has be taken arbitrarily in $(t_0,+\infty)$, we conclude that $V(\cdot,y):[t_0,+\infty)\to\mathbb{R}$ is left-continuous at $(t_0,+\infty)$ for all $y\in\mathscr{O}$. □

In the sequel, we prove that if the trivial solution of the generalized ODE is regularly stable, then the function $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$, defined by (2.36), satisfies condition (L2) from Definition 2.0.2.

Lemma 2.1.18. Let $V:[t_0,+\infty)\times\mathscr{O}\to\mathbb{R}$ be defined by (2.36) and assume that $F\in\mathscr{F}(\Omega,h)$, where $h:[t_0,+\infty)\to\mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. If the trivial solution of the generalized ODE (2.1) is regularly stable, then

(1) there exists an increasing continuous function $b:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $b(0)=0$ such that,

$$V(t,x)\geq b(\|x\|),$$

for every $(t,x)\in[t_0,+\infty)\times\mathscr{O}$.

Proof. We assume that (1) does not hold, that is, there are $\varepsilon>0$ and a sequence of pairs $(t_k,x_k)\in[t_0,+\infty)\times\mathscr{O}$, $k\in\mathbb{N}$, such that

$$\varepsilon\leq\|x_k\|, \tag{2.58}$$

$$t_k \to \infty \text{ as } k \to \infty$$

and

$$V(t_k, x_k) \to 0 \text{ as } k \to \infty. \tag{2.59}$$

By the hypotheses and Theorem 2.1.5, the trivial solution of the generalized ODE (2.1) is regularly stable with respect to perturbations. Take $\delta = \delta(\varepsilon) > 0$ as in Definition 2.1.4-(i). By (2.59), there exists $k_0 \in \mathbb{N}$ such that

$$V(t_k, x_k) < \delta, \text{ for all } k > k_0.$$

From this and the fact that $A(t_k, x_k)$ is a closed set (see Lemma 2.37), there exists $\varphi_k \in A(t_k, x_k)$ (with $k > k_0$) such that

$$\sup_{\sigma \in [t_0, t_k]} \left\| \varphi_k(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t) \right\| < \delta. \tag{2.60}$$

We proceed this proof by constructing a perturbed generalized ODE whose solution has initial condition 0. Define $P_k : [t_0, t_k] \to X$ by

$$P_k(\sigma) = \varphi_k(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t), \quad \text{for all } \sigma \in [t_0, t_k].$$

Since $\varphi_k \in A(t_k, x_k)$, we have $\varphi_k \in G^-([t_0, t_k], X)$ and $\varphi_k(t_0) = 0$ which, in turn, imply that $P \in G^-([t_0, t_k], X)$ (see Lemma C.0.7), $P_k(t_0) = 0$ and

$$\begin{aligned}
\sup_{\sigma \in [t_0, t_k]} \|P_k(\sigma) - P_k(t_0)\| &= \sup_{\sigma \in [t_0, t_k]} \|P_k(\sigma)\| \\
&= \sup_{\sigma \in [t_0, t_k]} \left\| \varphi_k(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t) \right\| \\
&\overset{(2.60)}{<} \delta.
\end{aligned}$$

Moreover, for $\sigma \in [t_0, t_k]$, we have

$$\begin{aligned}
\varphi_k(\sigma) &\overset{\varphi_k(t_0)=0}{=} \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t) + \varphi_k(\sigma) - \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t) + \varphi_k(t_0) \\
&\overset{P_k(t_0)=0}{=} \varphi_k(t_0) + \int_{t_0}^{\sigma} DF(\varphi_k(\tau), t) + P_k(\sigma) - P_k(t_0) \\
&= \varphi_k(t_0) + \int_{t_0}^{\sigma} D[F(\varphi_k(\tau), t) + P_k(t)].
\end{aligned}$$

Consequently, $\varphi_k : [t_0, t_k] \to X$ is a solution of the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x, t) + P_k(t)]$$

with initial condition $\varphi_k(t_0) = 0$. Since the trivial solution of the generalized ODE (2.1) is regularly stable with respect to perturbations, $P \in G^-([t_0, t_k], X)$, $\sup_{\sigma \in [t_0, t_k]} \|P_k(\sigma) - P_k(t_0)\| < \delta$ and $\|\varphi_k(t_0)\| = 0 < \delta$, we have $\|\varphi_k(t)\| < \varepsilon$, for all $t \in [t_0, t_k]$. In particular, $\|\varphi_k(t_k)\| = \|x_k\| < \varepsilon$ which contradicts (2.58). $\square$

In what follows, we present a converse Lyapunov theorem on regular stability for the trivial solution of the generalized ODE (2.1).

Theorem 2.1.19. If the trivial solution of the generalized ODE (2.1) is regularly stable, then there exists a function $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ satisfying:

(CLR1) for all $s_0 \geq t_0$, the function
$$[s_0,+\infty) \ni t \mapsto V(t,x(t))$$
is nonincreasing, where $x : [s_0,\omega) \to \mathscr{O}$ is a maximal solution of the generalized ODE (2.1);

(CLR2) $V(\cdot,y) : [t_0,+\infty) \to \mathbb{R}$ is left-continuous on $(t_0,+\infty)$, for all $y \in \mathscr{O}$;

(CLR3) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that
$$V(t,y) \geq b(\|y\|),$$
for every $(t,y) \in [t_0,+\infty) \times \mathscr{O}$;

(CLR4) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, such that
$$V(t,y) \leq a(\|y\|),$$
for all $y \in \mathscr{O}$ and all $t \in [t_0,+\infty)$;

(CLR5) $V(t,0) = 0$ for all $t \in [t_0,+\infty)$;

(CLR6) for all $s_0 \geq t_0$ and all maximal solution $x : [s_0,\omega) \to X$ of the generalized ODE (2.1), the derivative
$$D^+V(t,x(t)) = \limsup_{\eta\to 0^+} \frac{V(t+\eta,x(t+\eta)) - V(t,x(t))}{\eta} \leq 0$$
holds for all $t \in [s_0,\omega)$, that is, the right derivative of V is non-positive along every solution of the generalized ODE (2.1).

Proof. Let $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.36). Properties (CLR1), (CLR2) and (CLR3) are direct consequences of Lemmas 2.1.16, 2.1.17 and 2.1.18 respectively. Property (CLR4) is proved in Corollary 2.1.15 with a as the identity function. Although condition (CLR5) is proved in Lemma 2.1.13, it is also a consequence of conditions (CLR3) and (CLR4), since
$$0 = b(0) \leq V(t,0) \leq a(0) = 0, \quad \text{for all } t \in [t_0,+\infty).$$
Finally, item (CLR6) is a straightforward consequence of item (CLR1) (see the comment before Proposition 2.0.3). □

Remark 2.1.20. Conditions (CLR2), (CLR3) and (CLR6) from Theorem 2.1.19 ensure that the function $V : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$, defined by (2.36), is a Lyapunov functional with respect to the generalized ODE (2.1) in the framework of Definition 2.0.2. On the other hand, conditions

(CLR1), (CLR2) and (CLR3), given in Theorem 2.1.19, shows that V satisfies all the conditions of the definition of a Lyapunov functional presented in [22, 23]. Therefore, no matter which definition of Lyapunov functional we are using, Theorem 2.1.19 ensures that regular stability for generalized ODEs implies in the existence of a Lyapunov functional. Moreover, condition (CLR4) shows that V satisfies the hypothesis (LR1) from the Lyapunov-type Theorem 2.1.7 on regular stability.

Our next goal is to prove the converse Lyapunov theorem on regular attractivity. In order to do this, for $s \geq t_0$ and $y \in \mathscr{O}$, we define $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ as

$$\widetilde{V}(s,y) := \begin{cases} \inf\limits_{\varphi \in A(s,y)} \left\{ \sup\limits_{\sigma \in [t_0,s]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \right\| e^{-s} \right\}, \text{ if } s > t_0, \\ \|y\|, \text{ if } s = t_0, \end{cases} \tag{2.61}$$

where $A(s,y)$ is given by (2.35).

By Corollary C.0.4, we know that, for all $\varphi \in A(s,y)$ and all $\sigma \in [t_0, s]$, the Kurzweil integral $\int_{t_0}^{\sigma} DF(\varphi(\tau),t)$ exists. Moreover, by Lemma C.0.5, the function

$$[t_0,s] \ni \sigma \mapsto f(\sigma) := \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t)$$

is also regulated and, by Proposition A.0.13, $\sup_{\sigma \in [t_0,s]} \|f(\sigma)\| < \infty$. Therefore, $\widetilde{V}$ is well-defined for all $(s,y) \in [t_0,+\infty) \times \mathscr{O}$.

The proofs of the next two lemmas are analogous to the proofs of Lemmas 2.1.14 and 2.1.17 respectively. Therefore, we omit them here.

Lemma 2.1.21. Let $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing functions which is left-continuous on $(t_0, +\infty)$. Then, for all $y, z \in \mathscr{O}$ and all $s \in [t_0, +\infty)$, we have

$$\widetilde{V}(s,y) - \widetilde{V}(s,z) \leq \|y - z\|.$$

Lemma 2.1.22. Let $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then, $\widetilde{V}(\cdot, y) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ for all $y \in \mathscr{O}$.

As a consequence of the definition of the function $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ and Lemma 2.1.21, we have the following result.

Corollary 2.1.23. Let $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega, h)$, where h is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then, for all $z \in \mathscr{O}$ and all $s \in [t_0, +\infty)$,

$$\widetilde{V}(s,z) \leq \|z\|.$$

Proof. Since $\varphi \equiv 0 \in A(s,0)$ for all $s \in [t_0,+\infty)$, we have

$$\widetilde{V}(s,0) = 0, \quad \text{for all } s \in [t_0,+\infty). \tag{2.62}$$

Therefore, the result follows from Lemma 2.1.21 and (2.62). □

Although the proof of the next lemma is similar to the proof of Lemma 2.1.16, it shows the importance of the exponential function in the definition of the function $\widetilde{V} : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ and, hence, we include it here.

Lemma 2.1.24. Let $\widetilde{V} : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega,h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Then, for every maximal solution, $x : [s_0,\omega) \subset [t_0,+\infty) \to X$, of the generalized ODE (2.1), we have

$$D^+\widetilde{V}(t,x(t)) := \limsup_{\eta \to 0^+} \frac{\widetilde{V}(t+\eta,x(t+\eta)) - \widetilde{V}(t,x(t))}{\eta} \leq -\widetilde{V}(t,x(t)),$$

for all $t \in [s_0,\omega)$.

Proof. Let $x : [s_0,\omega) \subset [t_0,+\infty) \to X$ be a maximal solution of the generalized ODE (2.1) and $t \in [s_0,\omega)$. Let $\varphi \in A(t,x(t))$ and $\eta > 0$. Define $\phi_\eta : [t_0,+\infty) \to \mathscr{O}$ by

$$\phi_\eta(\sigma) := \begin{cases} \varphi(\sigma), & \text{if } \sigma \in [t_0,t], \\ x(\sigma), & \text{if } \sigma \in [t,t+\eta], \\ 0, & \text{if } \sigma \in (t+\eta,+\infty). \end{cases}$$

By Remark 2.1.11, $\phi_\eta \in A(t+\eta,x(t+\eta))$ and, by the definition of $\widetilde{V}$, we have

$$\widetilde{V}(t+\eta,x(t+\eta)) \leq \sup_{\sigma \in [t_0,t+\eta]} \left\| \phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau),s) \right\| e^{-(t+\eta)}.$$

Owing to the fact that $F \in \mathscr{F}(\Omega,h)$, we can apply Lemma C.0.7 to guarantee that the function $\phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau),s)$, with $\sigma \in [t_0,t+\eta]$, is regulated and left-continuous on $(t_0,t+\eta]$. According to Lemma A.0.15, we may consider two cases with respect to

$$\sup_{\sigma \in [t_0,t+\eta]} \left\| \phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau),s) \right\| e^{-(t+\eta)}.$$

Case 1: Assume that, for some $v \in [t_0,t+\eta]$, we have

$$\sup_{\sigma \in [t_0,t+\eta]} \left\| \phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau),s) \right\| e^{-(t+\eta)} = \left\| \phi_\eta(v) - \int_{t_0}^{v} DF(\phi_\eta(\tau),s) \right\| e^{-(t+\eta)}.$$

Taking into account that $\phi_\eta|_{[t_0,t]} = \varphi$ and $\phi_\eta|_{[t,t+\eta]} = x$, we conclude

$$
\begin{aligned}
&\widetilde{V}(t+\eta, x(t+\eta)) \leq \sup_{\sigma\in[t_0,t+\eta]} \left\| \phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau), s) \right\| e^{-(t+\eta)} \\
&= \begin{cases} \left\| \phi_\eta(v) - \int_{t_0}^{v} DF(\phi_\eta(\tau), s) \right\| e^{-(t+\eta)}, \text{for some } v \in [t_0, t] \\ \text{or} \\ \left\| \phi_\eta(v) - \int_{t_0}^{v} DF(\phi_\eta(\tau), s) \right\| e^{-(t+\eta)}, \text{for some } v \in [t, t+\eta] \end{cases} \\
&= \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-(t+\eta)}, \text{for some } v \in [t_0, t] \\ \text{or} \\ \left\| x(v) - \int_{t_0}^{t} DF(\varphi(\tau), s) - \int_{t}^{v} DF(x(\tau), s) \right\| e^{-(t+\eta)}, \text{for some } v \in [t, t+\eta]. \end{cases}
\end{aligned}
$$

Since $x(v) - \int_t^v DF(x(\tau), s) = x(t)$ and $\varphi(t) = x(t)$, we get

$$
\begin{aligned}
&\widetilde{V}(t+\eta, x(t+\eta)) \\
&\leq \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-(t+\eta)}, \text{ for some } v \in [t_0, t] \\ \text{or} \\ \left\| x(t) - \int_{t_0}^{t} DF(\varphi(\tau), s) \right\| e^{-(t+\eta)}, \text{ for some } v \in [t, t+\eta] \end{cases} \\
&= \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-(t+\eta)}, \text{ for some } v \in [t_0, t] \\ \text{or} \\ \left\| \varphi(t) - \int_{t_0}^{t} DF(\varphi(\tau), s) \right\| e^{-(t+\eta)}, \text{ for some } v \in [t, t+\eta] \end{cases} \\
&\leq \begin{cases} \sup_{\sigma\in[t_0,t]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\| e^{-t} e^{-\eta}, \\ \text{or} \\ \sup_{\sigma\in[t_0,t]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\| e^{-t} e^{-\eta}. \end{cases}
\end{aligned}
$$

Taking the infimum over all $\varphi \in A(t, x(t))$, we obtain

$$\widetilde{V}(t+\eta, x(t+\eta)) \leq \widetilde{V}(t, x(t)) e^{-\eta}$$

which implies

$$\widetilde{V}(t+\eta, x(t+\eta)) - \widetilde{V}(t, x(t)) \leq \widetilde{V}(t, x(t))(e^{-\eta} - 1).$$

Therefore,

$$
\begin{aligned}
\limsup_{\eta\to 0^+} \frac{\widetilde{V}(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} &\leq \limsup_{\eta\to 0^+} \frac{\widetilde{V}(t, x(t))(e^{-\eta} - 1)}{\eta} \\
&= \widetilde{V}(t, x(t)) \limsup_{\eta\to 0^+} \frac{(e^{-\eta} - 1)}{\eta} \\
&= -\widetilde{V}(t, x(t))
\end{aligned}
$$

obtaining the desired result.

Case 2: Assume that, for some $v \in [t_0, t+\eta]$, we have

$$\sup_{\sigma \in [t_0, t+\eta]} \left\| \phi_\eta(\sigma) - \int_{t_0}^{\sigma} DF(\phi_\eta(\tau), s) \right\| e^{-(t+\eta)} = \left\| \phi_\eta(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\phi_\eta(\tau), s) \right\| e^{-(t+\eta)}.$$

Since the proof of Case 2 is similar to the proof of Case 2 from Lemma 2.1.16, with the same adaptations that we did here, we will omit it. □

The next result ensure that $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$, given by (2.61), satisfies condition (L'3).

Lemma 2.1.25. Let $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then, for every $s_0 \geq t_0$, the function $[s_0, \omega) \ni t \mapsto \widetilde{V}(t, x(t))$ is nonincreasing along every maximal solution $x : [s_0, \omega) \to X$ of the generalized ODE (2.1)

Proof. Let $s_0 \geq t_0$, $x : [s_0, \omega) \to X$ be a maximal solution of the generalized ODE (2.1) and $t_1, t_2 \in [s_0, \omega)$ be such that $t_2 > t_1$. Since the function $\mathbb{R} \ni x \mapsto e^{-x}$ is decreasing, we have

$$e^{-t_2} < e^{-t_1}. \tag{2.63}$$

Let $\varphi \in A(t_1, x(t_1))$ and define $\phi : [t_0, +\infty) \to \mathscr{O}$ by

$$\phi(\sigma) := \begin{cases} \varphi(\sigma), & \text{if } \sigma \in [t_0, t_1], \\ x(\sigma), & \text{if } \sigma \in [t_1, t_2], \\ 0, & \text{if } \sigma \in (t_2, +\infty). \end{cases}$$

Notice that $\phi \in A(t_2, x(t_2))$ (see Remark 2.1.11). By the definition of $\widetilde{V}$, we have

$$\widetilde{V}(t_2, x(t_2)) \leq \sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| e^{-t_2}.$$

By the fact that $F \in \mathscr{F}(\Omega, h)$ and Lemma C.0.7, the function

$$[t_0, t_2] \ni \sigma \mapsto \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s)$$

belongs to $G^-([t_0, t_2], X)$ and, by Lemma A.0.15, we must consider two cases with respect to

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| e^{-t_2}.$$

Case 1: For some $v \in [t_0, t_2]$, we have

$$\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| e^{-t_2} = \left\| \phi(v) - \int_{t_0}^{v} DF(\phi(\tau), s) \right\| e^{-t_2}.$$

In this case, either $v \in [t_0, t_1]$ or $v \in [t_1, t_2]$. Then,

$$
\begin{aligned}
& \widetilde{V}(t_2, x(t_2)) \\
\leq \; & \left\| \phi(v) - \int_{t_0}^{v} DF(\phi(\tau), s) \right\| e^{-t_2}, \text{for some } v \in [t_0, t_2] \\
= \; & \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-t_2}, \text{for some } v \in [t_0, t_1] \\ \text{or} \\ \left\| x(v) - \int_{t_0}^{t_1} DF(\varphi(\tau), s) - \int_{t_1}^{v} DF(x(\tau), t) \right\| e^{-t_2}, \text{for some } v \in [t_1, t_2] \end{cases} \\
\overset{(2.63)}{\leq} \; & \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-t_1}, \text{for some }, v \in [t_0, t_1] \\ \text{or} \\ \left\| x(v) - \int_{t_0}^{t_1} DF(\varphi(\tau), s) - \int_{t_1}^{v} DF(x(\tau), t) \right\| e^{-t_1}, \text{for some } v \in [t_1, t_2]. \end{cases}
\end{aligned}
$$

By the definition of a solution of generalized ODEs and since $\varphi \in A(t_1, x(t_1))$, we get

$$
x(v) - \int_{t_1}^{v} DF(x(\tau), t) = x(t_1) = \varphi(t_1).
$$

Thus,

$$
\begin{aligned}
\widetilde{V}(t_2, x(t_2)) \; & \leq \begin{cases} \left\| \varphi(v) - \int_{t_0}^{v} DF(\varphi(\tau), s) \right\| e^{-t_1}, \; \text{for some } \; v \in [t_0, t_1] \\ \text{or} \\ \left\| \varphi(t_1) - \int_{t_0}^{t_1} DF(\varphi(\tau), t) \right\| e^{-t_1}, \; \text{for some } \; v \in [t_1, t_2] \end{cases} \\
& \leq \begin{cases} \sup_{\sigma \in [t_0, t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\| e^{-t_1} \\ \text{or} \\ \sup_{\sigma \in [t_0, t_1]} \left\| \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau), s) \right\| e^{-t_1}. \end{cases}
\end{aligned}
$$

Taking the infimum over all $\varphi \in A(t_1, x(t_1))$, we obtain

$$
\widetilde{V}(t_2, x(t_2)) \leq \widetilde{V}(t_1, x(t_1)).
$$

Case 2: For some $v \in [t_0, t_2]$,

$$
\sup_{\sigma \in [t_0, t_2]} \left\| \phi(\sigma) - \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| e^{-t_2} = \left\| \phi(v^+) - \lim_{\sigma \to v^+} \int_{t_0}^{\sigma} DF(\phi(\tau), s) \right\| e^{-t_2}
$$

Since the proof of Case 2 is similar to the proof of Case 2 from Lemma 2.1.16 with the same adaptations that we did here, we will omit it. □

In the sequel, we present a result which shows that the function $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$, defined by (2.61), satisfies condition (L2) from Definition 2.0.2. Although its proof is follows the same ideas of the proof of Lemma 2.1.18, we exhibit the details here.

Lemma 2.1.26. Let $\widetilde{V} : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) and assume that $F \in \mathscr{F}(\Omega,h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. If the trivial solution of the generalized ODE (2.1) is regularly attracting, then

(1) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that, for every $(t,x) \in [t_0,+\infty) \times \mathscr{O}$,

$$\widetilde{V}(t,x) \geq b(\|x\|).$$

Proof. Let $\widetilde{V} : [t_0,+\infty) \times \mathscr{O} \to \mathbb{R}$ be given by (2.61). Assume that $\widetilde{V}$ is not positive definite. Then, for $\widetilde{\delta}$ as in Definition 2.1.4-(ii), there exist $\varepsilon > 0$ and a sequence of pairs $(t_k,x_k)_{k\in\mathbb{N}}$ in $[t_0,+\infty) \times \mathscr{O}$ such that

$$\varepsilon \leq \|x_k\|, \tag{2.64}$$

$$t_k \to \infty \text{ as } k \to \infty \tag{2.65}$$

and

$$\widetilde{V}(t_k,x_k) \to 0 \text{ as } k \to \infty. \tag{2.66}$$

Using the hypotheses and Theorem 2.1.5, the trivial solution of the generalized ODE (2.1) is regularly attracting with respect to perturbations. We consider $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ as in Definition 2.1.4-(ii). By (2.65), there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ we have $t_k > T + t_0$ and, by (2.66), we may assume

$$\widetilde{V}(t_k,x_k) < \rho e^{-t_k},$$

for every $k > k_0$. According to the definition of $\widetilde{V}$ and since $A(t_k,x_k)$ is a closed set (see Lemma 2.1.12), fix $k > k_0$ and choose $\varphi \in A(t_k,x_k)$ such that

$$\widetilde{V}(t_k,x_k) \sup_{s\in[t_0,t_k]} \left\|\varphi(s) - \int_{t_0}^{s} DF(\varphi(\tau),t)\right\| e^{-t_k} \leq \rho e^{-t_k}$$

which implies

$$\sup_{s\in[t_0,t_k]} \left\|\varphi(s) - \int_{t_0}^{s} DF(\varphi(\tau),t)\right\| \leq \rho.$$

For $\sigma \in [t_0,t_k]$, define $P : [t_0,t_k] \to X$ by

$$P(\sigma) = \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t).$$

Since $\varphi(t_0) = 0$, we have

$$\sup_{\sigma\in[t_0,t_k]} \|P(\sigma) - P(t_0)\| = \sup_{\sigma\in[t_0,t_k]} \left\|\varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t)\right\| \leq \rho.$$

Moreover,

$$\begin{aligned}
\varphi(\sigma) &= \int_{t_0}^{\sigma} DF(\varphi(\tau),t) + \varphi(\sigma) - \int_{t_0}^{\sigma} DF(\varphi(\tau),t) \\
&= \int_{t_0}^{\sigma} DF(\varphi(\tau),t) + P(\sigma) \\
&\overset{\varphi(t_0)=0=P(t_0)}{=} \varphi(t_0) + \int_{t_0}^{\sigma} DF(\varphi(\tau),t) + P(\sigma) - P(t_0) \\
&= \varphi(t_0) + \int_{t_0}^{\sigma} D[F(\varphi(\tau),t) + P(t)].
\end{aligned}$$

Therefore, $\varphi : [t_0, t_k] \to X$ is a solution of the following perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t) + P(t)]$$

with $\|\varphi(t_0)\| = 0 \leq \widetilde{\delta}$. Since the trivial solution of the generalized ODE (2.1) is regularly attracting, the inequality $\|\varphi(t)\| \leq \varepsilon$ holds for every $t > t_0 + T(\varepsilon)$. In particular,

$$\|\varphi(t_k)\| = \|x_k\| < \varepsilon$$

which contradicts (2.64). □

In what follows, we present a converse Lyapunov theorem on regular attracting for the trivial solution of the generalized ODE (2.1).

Theorem 2.1.27. If the trivial solution $x \equiv 0$ of the generalized ODE (2.1) is regularly attracting, then there exists a function $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ satisfying:

(CLRA1) $\widetilde{V}(\cdot, y) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$, for all $y \in \mathscr{O}$;

(CLRA2) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, such that

$$\widetilde{V}(t,y) \leq a(\|y\|),$$

for all $y \in \mathscr{O}$ and all $t \in [t_0, +\infty)$;

(CLRA3) for every $s_0 \geq t_0$, the function $[s_0, +\omega) \ni t \mapsto \widetilde{V}(t, x(t))$ is nonincreasing along every maximal solution $x : [s_0, +\omega) \to X$ of the generalized ODE (2.1);

(CLRA4) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that

$$\widetilde{V}(t,y) \geq b(\|y\|),$$

for every $(t,y) \in [t_0, +\infty) \times \mathscr{O}$;

(CLRA5) $\widetilde{V}(t, 0) = 0$ for all $t \in [t_0, +\infty)$;

(CLRA6) there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$ such that for every $s_0 \geq t_0$ and every maximal solution $x : [s_0, \omega) \to X$ of the generalized (2.1), we have

$$D^+\widetilde{V}(t, x(t)) \leq -\Phi(x(t)),$$

for all $t \in [s_0, \omega)$.

Proof. Let $\widetilde{V} : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be defined by (2.61) . Items (CLRA1) and (CLRA3) follow directly from Lemmas 2.1.22 and Lemma 2.1.25 respectively. By Lemma 2.1.21, for all $y \in \mathscr{O}$, we have

$$\widetilde{V}(t,y) \leq \|y\|, \quad \text{for all } t \in [t_0, +\infty).$$

Therefore, item (CLRA2) holds for a as the identity function.

Property (CLRA4) is proved in Lemma 2.1.26.

As a consequence of items (CLRA2) and (CLRA4), we obtain property (CLRA5) once

$$0 = b(0) \leq \widetilde{V}(t,0) \leq a(0) = 0, \quad \text{for all } t \in [t_0, +\infty).$$

Let us prove item (CLRA6). By Lemma 2.1.24, for every $s_0 \geq t_0$ and every maximal solution $x : [s_0, \omega) \to X$ of the generalized ODE (2.1), we have

$$D^+\widetilde{V}(t,x(t)) \leq -\widetilde{V}(t,x(t)), \quad \text{for all } t \in [s_0, \omega).$$

On the other hand, by item (CLRA4), we obtain

$$D^+\widetilde{V}(t,x(t)) \leq -b(\|x(t)\|), \quad \text{for all } t \in [s_0, \omega),$$

once $x(t) \in \mathscr{O}$ by the definition of a solution of a generalized ODE. Define $\Phi : X \to \mathbb{R}$ by $\Phi(y) = b(\|y\|)$ for all $y \in X$. Since b is continuous, $b(0) = 0$ and $b(t) > 0$ for all $t > 0$, we have Φ is clearly continuous, $\Phi(0) = b(0) = 0$ and $\Phi(y) = b(\|y\|) > 0$ whenever $y \neq 0$. □

Remark 2.1.28. Conditions (CLRA1), (CLRA4) and (CLRA6) from Theorem 2.1.27 ensure that the function $\widetilde{V} : [t_0, +\infty) \times X \to \mathbb{R}$, defined by (2.61), is a Lyapunov functional with respect to the generalized ODE (2.1) in the framework of Definition 2.0.2. On the other hand, conditions (CLRA1), (CLRA3) and (CLRA4) from Theorem 2.1.27 show that $\widetilde{V}$ satisfies all the conditions of the definition of a Lyapunov functional presented in [22, 23]. Therefore, no matter which definition of Lyapunov functional we are using, Theorem 2.1.27 ensures that if the trivial solution of the generalized ODE (2.1) is regularly attracting, then there exists a Lyapunov functional with respect to (2.1). Moreover, condition (CLRA6) shows that $\widetilde{V}$ satisfies hypothesis (LRA'1) from the Lyapunov-type theorem 2.1.10 on regular attractivity.

The next result is a consequence of Theorems 2.1.19 and 2.1.27 and it is a version of a converse Lyapunov theorem on asymptotic stability.

Corollary 2.1.29. If the trivial solution of the generalized ODE (2.1) is regularly asymptotically stable, then there exists a Lyapunov functional, $V : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$, with respect to the generalized ODE (2.1) satisfying:

(CLRS1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, such that

$$V(t,y) \leq a(\|y\|),$$

for all $y \in \mathscr{O}$ and all $t \in [t_0, +\infty)$;

(CLRS2) for every $s_0 \geq t_0$, the function $[s_0, \omega \ni t \mapsto V(t, x(t))$ is nonincreasing along every maximal solution $x : [s_0, +\infty) \to X$ of the generalized ODE (2.1);

(CLRS3) $V(t, 0) = 0$ for all $t \in [t_0, +\infty)$;

(CLRS4) there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$ such that for every $s_0 \geq t_0$ and every maximal solution $x : [s_0, \omega) \to X$ of the generalized (2.1), we have

$$D^+V(t, x(t)) \leq -\Phi(x(t)),$$

for all $t \in [s_0, \omega)$.

2.2 Uniform stability

In this section, we present the Lyapunov-type theorem on uniform stability of the trivial solution of the generalized ODE (2.1), presented in [13,23], and we establish a converse Lyapunov theorem. Moreover, we introduce a concept of uniform stability with respect to perturbations and give relations between this concept and the notion of uniform stability. The main references for this section are [6, 13, 23]

Assume that X is a Banach space, $\mathscr{O} \subseteq X$ is an open subset such that $0 \in \mathscr{O}$, where 0 represents the neutral element of X and $F : \Omega \to X$ belongs to $\mathscr{F}(\Omega, h)$, where $\Omega = \mathscr{O} \times [t_0, +\infty)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). Moreover, we suppose $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition of such a solution) and, for all $s_0 \in [t_0, +\infty)$ and all $x_0 \in \mathscr{O}$, there exists a unique maximal solution $x : [s_0, \omega(s_0, x_0)) \to X$ of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. In this case, we denote $x(\cdot)$ by $x(\cdot, s_0, x_0)$. We recall that the concept of a maximal solution is given in Definition C.0.15 and sufficient conditions for its existence can be found in Theorem C.0.17 and Corollaries C.0.20 and C.0.21.

In the sequel, we present a definition of uniform stability for generalized ODEs introduced in [23].

Definition 2.2.1. The trivial solution of the generalized ODE (2.1) is said to be

(i) *stable*, if for every $s_0 \geq t_0$, $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $x_0 \in \mathscr{O}$ satisfies

$$\|x_0\| < \delta,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, x_0)),$$

where $x : [s_0, \omega(s_0, x_0)) \to X$ is a maximal solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$;

(ii) *uniformly stable*, if it is stable with δ independent of s_0;

(iii) *uniformly asymptotically stable*, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta_0) \geq 0$ such that if $(x_0, s_0) \in \Omega$ and

$$\|x_0\| < \delta,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, x_0)) \cap [s_0 + T, +\infty),$$

where $x : [s_0, \omega(s_0, x_0)) \to X$ is a maximal solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$.

Now, consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) + P(t), \tag{2.67}$$

where $P : [t_0, +\infty) \to X$ is left-continuous on $(t_0, +\infty)$ and

$$\sup_{s \in [t_0, +\infty)} \|P(s) - P(t_0)\| < \infty.$$

Notice that, $\sup_{s \in [t_0, t]} \|P(s) - P(t_0)\| < \infty$ for all $t \in [t_0, +\infty)$ and, by Proposition A.0.20, the function $g : [t_0, +\infty) \to \mathbb{R}$, defined by $g(t) = \sup_{s \in [t_0, t]} \|P(s) - P(t_0)\|$, is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Therefore, as we mentioned in Remark 2.1.3, we may assume that, for every $(x_0, s_0) \in \Omega$, there exists a unique maximal solution $\bar{x} : [s_0, \omega(s_0, x_0)) \to X$ of the perturbed generalized ODE (2.67) with initial condition $\bar{x}(s_0) = x_0$.

In the sequel, we present a definition of uniform stability with respect to perturbations.

Definition 2.2.2. The trivial solution of the generalized ODE (2.1) is said to be

(i) *stable with respect to perturbations* if, for all $s_0 \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $x_0 \in \mathscr{O}$ with

$$\|x_0\| < \delta \quad \text{and} \quad \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| < \delta,$$

then

$$\|\bar{x}(t, s_0, x_0)\| = \|\bar{x}(t)\| < \varepsilon, \quad \text{for all } s \in [s_0, \omega(s_0, x_0)),$$

where $\bar{x} : [s_0, \omega(s_0, x_0)) \to X$ is a maximal solution of the perturbed generalized ODE (2.67) with initial condition $\bar{x}(s_0) = x_0$;

(ii) *uniformly stable with respect to perturbations*, if it is stable with respect to perturbations with δ independent of s_0.

(iii) *uniformly asymptotically stable with respect to perturbations*, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta_0) \geq 0$ such that if $x_0 \in \mathscr{O}$ and

$$\|x_0\| < \delta \quad \text{and} \quad \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| < \delta,$$

then

$$\|\bar{x}(t, s_0, x_0)\| = \|\bar{x}(t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, x_0)) \cap [s_0 + T, +\infty),$$

where $\bar{x} : [s_0, \omega(s_0, x_0)) \to X$ is a maximal solution of the perturbed generalized ODE (2.67) with initial condition $x(s_0) = x_0$.

In what follows, we assume the existence of global forward solutions of the generalized ODE (2.1) and of the perturbed generalized ODE (2.6).

Theorem 2.2.3. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. If the trivial solution of generalized ODE (2.1) is stable with respect to perturbations, then it is stable.

Proof. For all $s_0 \geq t_0$ and all $x_0 \in X$, let $x : [s_0, +\infty) \to X$ be the global forward solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$, $\varepsilon > 0$, $P : [s_0, +\infty) \to X$ be a function defined by $P(t) = x_0$, for all $t \geq s_0$, and let $\bar{x} : [s_0, +\infty) \to X$ be the global forward solution of the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t) + P(t)],$$

with initial condition $\bar{x}(s_0) = x_0$. Then,

$$\bar{x}(t) = x_0 + \int_{s_0}^{t} DF(\bar{x}(\tau), s) + P(t) - P(s_0) = x_0 + \int_{s_0}^{t} DF(\bar{x}(\tau), s).$$

Therefore, $\bar{x}$ is the global forward solution of the generalized ODE (2.1) with initial condition $\bar{x}(s_0) = x_0$ and, by the uniqueness of a solution, $\bar{x}(t) = x(t)$ for all $t \geq s_0$. Since the trivial solution of the generalized ODE (2.1) is stable with respect to perturbations, there exists $\delta > 0$ such that if $\|x_0\| < \delta$ and $\sup_{s \in [s_0, +\infty)} \|P(s) - P(t_0)\| = 0 < \delta$, then $\|x(t)\| = \|\bar{x}(t)\| < \varepsilon$ for all $t \geq s_0$. This fact leads to the uniform stability for the trivial solution of the generalized ODE (2.1) □

The proof of Theorem 2.2.4, in the sequel, follows as in Theorem 2.2.3 and, therefore, we omit it here.

Theorem 2.2.4. Assume that $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then, the following statements hold.

(i) If the trivial solution of the generalized ODE (2.1) is uniformly stable with respect to perturbations, then it is uniformly stable.

(ii) If the trivial solution of the generalized ODE (2.1) is uniformly asymptotically stable with respect to perturbations, then it is uniformly asymptotically stable.

In order to prove that uniform stability implies uniform stability with respect to perturbations, we need the following auxiliary result.

Lemma 2.2.5. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Furthermore, assume that h is bounded. For every global forward solution x of the generalized ODE (2.1) and for every global forward solution $\bar{x}$ of the perturbed generalized ODE (2.67) on $[s_0, +\infty)$, with $x(s_0) = x_0 = \bar{x}(s_0)$ and $s_0 \geq t_0$, there exists a constant $M > 0$ such that

$$\|x(t) - \bar{x}(t)\| \leq M,$$

for all $t \geq s_0$.

Proof. Since h is bounded, there exists $H > 0$ such that $\sup_{t\in[t_0,+\infty)} |h(t) - h(t_0)| = H$ and, once $\sup_{s\in[s_0,+\infty)} \|P(s) - P(s_0)\| < \infty$, there exists $P > 0$ such that $\sup_{s\in[s_0,+\infty)} \|P(s) - P(s_0)\| = P$. Using these facts together with Lemma C.0.5, we obtain

$$\begin{aligned}
\|x(t) - \bar{x}(t)\| &= \left\| x(s_0) + \int_{s_0}^{t} D[F(x(\tau), s) - F(\bar{x}(\tau), s)] - \bar{x}(s_0) - P(s) + P(s_0) \right\| \\
&\overset{x(s_0)=\bar{x}(s_0)}{\leq} \left\| \int_{s_0}^{t} D[F(x(\tau), s) - F(\bar{x}(\tau), s)] \right\| + \sup_{s\in[s_0,+\infty)} \|P(s) - P(s_0)\| \\
&\overset{Lem. C.0.5}{\leq} 2|h(t) - h(s_0)| + P \\
&\leq (2|h(t) - h(t_0)| + |h(s_0) - h(t_0)|) + P \\
&\leq 2(H + H) + P = 4H + P = M,
\end{aligned}$$

for all $t \geq s_0$ and the statement holds. □

The next result relates the concept of stability to the notion of stability with respect to perturbations.

Theorem 2.2.6. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Moreover, suppose h is bounded. If the trivial solution of the generalized ODE (2.1) is stable, then it is stable with respect to perturbations.

Proof. For all $s_0 \geq t_0$ and all $x_0 \in X$, let $\bar{x} : [s_0, +\infty) \to X$ be a global forward solution of the perturbed generalized ODE (2.67) with initial condition $\bar{x}(s_0) = x_0$ and let $\varepsilon > 0$. Take δ as in Definition 2.2.1-(i) and assume that $\|x_0\| < \frac{\delta}{2}$ and $\sup_{s\in[s_0,+\infty)} \|P(s) - P(t_0)\| < \frac{\delta}{2e^H}$, where $H = \sup_{t\in[t_0,+\infty)} |h(t) - h(t_0)|$. Let $x : [s_0, +\infty) \to X$ be a global forward solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. Then for all $t \geq s_0$, we have

$$\begin{aligned}\|\bar{x}(t) - x(t)\| &\leq \left\| \int_{s_0}^{t} D[F(\bar{x}(\tau), s) - F(x(\tau), s)] \right\| + \|P(t) - P(s_0)\| \\ &\leq \left\| \int_{s_0}^{t} D[F(\bar{x}(\tau), s) - F(x(\tau), s)] \right\| + \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\|.\end{aligned}$$

By Lemma C.0.6, we get

$$\|\bar{x}(t) - x(t)\| \leq \int_{s_0}^{t} \|\bar{x}(s) - x(s)\| dh(s) + \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\|, \quad \text{for all } t \in [s_0, +\infty).$$

Thus, by hypotheses, $\sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| < \infty$, h is nondecreasing and it is also left-continuous on $(t_0, +\infty)$ and, by Lemma 2.2.5, the function $[s_0, +\infty) \ni t \mapsto \|\bar{x}(t) - x(t)\|$ is bounded. Therefore, we are under the hypotheses of the Gronwall-type inequality (see Theorem B.0.10) and, hence,

$$\|\bar{x}(t) - x(t)\| \leq \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| e^{|h(t) - h(s_0)|} \leq \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| e^{H},$$

where $H = \sup_{t \in [t_0, +\infty)} |h(t) - h(t_0)|$. Notice that, $\|x(t)\| < \frac{\varepsilon}{2}$ for all $t \geq s_0$, since the trivial solution of the generalized ODE (2.1) is stable. Therefore, for all $t \in [s_0, +\infty)$, we conclude

$$\|\bar{x}(t)\| \leq \|x(t)\| + \|\bar{x}(t) - x(t)\| < \frac{\varepsilon}{2} + \frac{\delta}{2e^H} e^H < \varepsilon,$$

once $\delta < \varepsilon$. □

The proof of the next result is analogous to the proof of Theorem 2.2.6 and, therefore, we omit it here.

Theorem 2.2.7. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing, left-continuous on $(t_0, +\infty)$ and it is bounded. Then, the following statements hold.

(i) If the trivial solution of the generalized ODE (2.1) is uniformly stable, then it is uniformly stable with respect to perturbations.

(ii) If the trivial solution of the generalized ODE (2.1) is uniformly asymptotically stable, then it is uniformly asymptotically stable with respect to perturbations.

2.2.1 Direct method of Lyapunov

In this subsection, we present Lyapunov-type theorems on uniform stability and uniform asymptotic stability borrowed from [13, 23].

We recall that we are considering $F : \Omega \to X$ belonging to $\mathscr{F}(\Omega, h)$, where $\mathscr{O} \subseteq X$ is an open set containing the neutral element of X, X is a Banach space, $\Omega = \mathscr{O} \times [t_0, +\infty)$ and

$h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). Moreover, we suppose that $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition for the existence of such a solution) and, for all $(x_0, s_0) \in \Omega$, there exists a unique maximal solution $x : [s_0, \omega(s_0, x_0)) \to X$ of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. We point out that the concept of a maximal solution is given in Definition C.0.15 and sufficient conditions for its existence can be found in Theorem C.0.17 and Corollaries C.0.20 and C.0.21.

In what follows, we present a result which ensures that the trivial solution of the generalized ODE (2.1) is uniformly stable. A version of such a result, when the Lyapunov functional with respect to the generalized ODE (2.1) is defined in $[t_0, +\infty) \times \overline{B}_\rho$, $\overline{B}_\rho = \{y \in X; \|y\| \leq \rho\}$, can be found in [13, Theorem 8.18] or [23, Theorem 3.4] and, for a Lyapunov functional defined in $[t_0, +\infty) \times \mathscr{O}$, its proofs follows analogously.

Theorem 2.2.8. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$, and let $V : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ be a Lyapunov functional with respect to the generalized ODE (2.1). Suppose V satisfies the following conditions:

(LU1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$ and

$$V(t, y) \leq a(\|y\|),$$

for all $t \in [t_0, +\infty)$ and all $y \in \mathscr{O}$;

(LU2) for every $s_0 \geq t_0$ and every maximal solution $x : [s_0, \omega) \to \mathscr{O}$ of the generalized ODE (2.1), the function $[s_0, \omega) \ni t \mapsto V(t, x(t))$ is nonincreasing.

Then, the trivial solution of the generalized ODE (2.1) is uniformly stable.

We end this subsection by presenting a Lyapunov-type theorem on uniform asymptotic stability. The reader may consult [13, Theorem 8.20] or [23, Theorem 3.6] for a proof.

Theorem 2.2.9. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$, and let $V : [t_0, +\infty) \times \mathscr{O}$ be a Lyapunov functional with respect to the generalized ODE (2.1). Suppose V satisfies following conditions:

(LA1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$ and

$$V(t, y) \leq a(\|y\|),$$

for all $t \in [t_0, +\infty)$ and all $y \in \mathscr{O}$;

(LA2) there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(y) > 0$, whenever $y \neq 0$ such that, for all maximal solution $x : [s_0, \omega) \to X$ of the generalized ODE (2.1) with $s_0 \geq t_0$, we have

$$V(s, x(s)) - V(t, x(t)) \leq (s - t)\left(-\Phi(x(t))\right),$$

for every $s,t \in [s_0,\omega)$, with $t \leq s$.

Then, the trivial solution of the generalized ODE (2.1) is uniformly asymptotically stable.

2.2.2 Converse Lyapunov theorems

This subsection is devoted to the investigation of a converse Lyapunov theorem on uniform stability. The results presented here are new and can be found in [6].

Throughout this subsection, we assume that X is a Banach space and $\mathscr{O} \subseteq X$ is an open subset such that $0 \in \mathscr{O}$, where 0 represents the neutral element of X. Consider $F \in \mathscr{F}(\Omega,h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$ and $\Omega = \mathscr{O} \times [t_0,+\infty)$ (see Definitions A.0.17 and C.0.2). Furthermore, we suppose $x \equiv 0$ is a solution of the generalized ODE (2.1) (see Remark 2.0.1 for a sufficient condition for the existence of such a solution) and, for all $s_0 \in [t_0,+\infty)$ and all $x_0 \in \mathscr{O}$, there exists a unique maximal solution $x : [s_0,\omega(s_0,x_0)) \to X$ of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. In this case, we denote $x(\cdot)$ by $x(\cdot,s_0,x_0)$. We point out that the concept of a maximal solution is given in Definition C.0.15 and sufficient conditions for its existence can be found in Theorem C.0.17 and Corollaries C.0.20 and C.0.21.

The next result gives a characterization of uniform stability of the trivial solution of the generalized ODE (2.1). Its proof follows as the proof of [30, Lemma 4.1].

Lemma 2.2.10. The trivial solution of the generalized ODE (2.1) is uniformly stable if and only if there exists a function $a : \mathbb{R}^+ \to \mathbb{R}^+$ fufilling the following properties:

(i) a is increasing and continuous;

(ii) $a(0) = 0$;

(iii) for any $x_0 \in \mathscr{O}$ and any $s_0 \geq t_0$, the solution $x(\cdot,s_0,x_0)$ of the generalized ODE (2.1) satisfies

$$\|x(t,s_0,x_0)\| \leq a(\|x_0\|), \quad \text{for all } t \in [s_0,\omega(s_0,x_0)).$$

Proof. We begin by proving the existence of the function $a : \mathbb{R}^+ \to \mathbb{R}^+$. For all $\varepsilon > 0$, let $\widetilde{\delta}(\varepsilon)$ be the least upper bound of all numbers $\delta(\varepsilon)$ occurring in Definition 2.2.1-(ii), that is,

$$\widetilde{\delta}(\varepsilon) = \sup\{\|x(t,s_0,x_0)\|; \|x_0\| < \delta(\varepsilon), s_0 \geq t_0, t \geq s_0\},$$

where $x(\cdot,s_0,x_0)$ is the maximal solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. Since the trivial solution of the generalized ODE (2.1) is uniformly stable, for all $s_0 \geq t_0$ and all $x_0 \in \mathscr{O}$, with $\|x_0\| < \delta(\varepsilon)$, we have $\|x(t,s_0,x_0)\| < \varepsilon$ for all $t \geq s_0$ and, therefore, $\widetilde{\delta}$ is well-defined and it is positive for $\varepsilon > 0$. Moreover, for every $\varepsilon_2 > \varepsilon_1 > 0$, we have

$\widetilde{\delta}(\varepsilon_1) \leq \widetilde{\delta}(\varepsilon_2)$. Indeed, suppose $\widetilde{\delta}(\varepsilon_2) < \widetilde{\delta}(\varepsilon_1)$. Then, there exist $\delta_0 > 0$ and $\widehat{x}_0 \in \mathscr{O}$ such that $\widetilde{\delta}(\varepsilon_2) < \delta_0 < \widetilde{\delta}(\varepsilon_1)$, $\|\widehat{x}_0\| < \delta_0 < \widetilde{\delta}(\varepsilon_1)$ and

$$\|x(t,s_0,\widehat{x}_0)\| \geq \varepsilon_2, \quad \text{for some } t \geq s_0. \tag{2.68}$$

On the other hand, by the uniform stability, we have $\|x(t,s_0,\widehat{x}_0)\| < \varepsilon_1 < \varepsilon_2$ for all $t \geq s_0$ which contradicts (2.68). Therefore, $\widetilde{\delta}(\varepsilon)$ is nondecreasing. In addition, since $\bar{x} \equiv 0$ is a solution of the generalized ODE (2.1), we conclude that $\widetilde{\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Owing to the fact that $\widetilde{\delta}$ is nondecreasing, positive for $\varepsilon > 0$ and tends to zero as ε tends to zero, we may choose an increasing continuous function $\widehat{\delta} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\widetilde{\delta}(\varepsilon) < \widehat{\delta}(\varepsilon)$ for all $\varepsilon > 0$.

Let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be the inverse function of $\widehat{\delta}$. Thus, for any $x_0 \in \mathscr{O}$ with $\|x_0\| < \widehat{\delta}(\varepsilon)$, there exists $\varepsilon_1 > 0$ such that $\|x_0\| = \widehat{\delta}(\varepsilon_1)$ and, since the trivial solution of the generalized ODE (2.1) is uniformly stable, we have $\|x(t,s_0,x_0)\| < \varepsilon_1 = a(\|x_0\|)$ for all $t \geq s_0$.

Reciprocally, assume the existence of the function $a : \mathbb{R}^+ \to \mathbb{R}^+$. By items (i) and (ii), for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a(\delta) < \varepsilon$. Since a is increasing, item (iii) guarantees that if $x \in \mathscr{O}$ is such that $\|x_0\| < \delta$, then for any $s_0 \geq t_0$ and any solution, $x(\cdot,s_0,x_0)$, of the generalized ODE (2.1), we have

$$\|x(t,s_0,x_0)\| < a(\|x_0\|) < a(\delta) < \varepsilon$$

which completes the proof. □

By the fact that $\mathscr{O}$ is an open set, there exists $c > 0$ such that $B_c = \{y \in X; \|y\| < c\} \subset \mathscr{O}$. Let $\rho < c$ and $\overline{B}_\rho = \{y \in X; \|y\| \leq \rho\}$. For all $y \in \overline{B}_\rho$ and all $t \in [t_0,+\infty)$ define

$$V(t,y) = \sup_{\tau \in [t,\omega(t,y))} \|x(\tau,t,y)\|, \tag{2.69}$$

where $x : [t,\omega(t,y)) \to X$ is the maximal solution of the generalized ODE (2.1) with initial condition $x(t) = y$. Notice that, by Lemma 2.2.10, for all $y \in \overline{B}_\rho$ and $t \in [t_0,+\infty)$, we have

$$V(t,y) \leq a(\|y\|) \leq a(\rho).$$

Therefore, V is well-defined for all $t \geq t_0$ and all $y \in \overline{B}_\rho$. Moreover, V satisfies

$$V(t,y) \leq a(\|y\|), \quad \text{for all } (t,y) \in [t_0,+\infty) \times \overline{B}_\rho. \tag{2.70}$$

In what follows, we prove that the function $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$, defined by (2.69), satisfies condition (L2) from Definition 2.0.2.

Lemma 2.2.11. Let $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$ be defined by (2.69). Then, there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that

$$V(t,y) \geq b(\|y\|),$$

for every $(t,y) \in [t_0,+\infty) \times \overline{B}_\rho$.

Proof. Let $y \in \overline{B}_\rho$ and $t \in [t_0, +\infty)$. Let $x : [t, \omega(t,y)) \to X$ be the maximal solution of the generalized ODE (2.1) with initial condition $x(t) = y$. It is clear that the following inequality

$$\|y\| \leq \sup_{\tau \in [t,\omega(t,y))} \|x(\tau,t,y)\| = V(t,y)$$

holds and, hence, the result follows by considering b as the identity function. □

As a consequence of Lemmas 2.2.10 and 2.2.11, we obtain the next result.

Lemma 2.2.12. Let $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ be defined by (2.69). Then, $V(t,0) = 0$ for all $t \in [t_0, +\infty)$.

The next lemma give us condition (L1) from Definition 2.0.2.

Lemma 2.2.13. Let $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ be defined by (2.69) and assume that $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then,

$$V(\cdot, y) : [t_0, +\infty) \to \mathbb{R}$$

is left-continuous on $(t_0, +\infty)$, for all $y \in \overline{B}_\rho$.

Proof. Let $\sigma_0 \in (t_0, +\infty)$, $y \in \overline{B}_\rho$ and $\varepsilon > 0$ be given. Since $F \in \mathscr{F}(\Omega, h)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$, there exists $\delta > 0$ such that

$$\|F(y,s) - F(y,\sigma_0)\| \leq |h(s) - h(s_0)| < \varepsilon, \quad \text{for all } s \in (\sigma_0 - \delta, \sigma_0). \tag{2.71}$$

Let $x : [\sigma_0, \omega) \to X$ be the maximal solution of the generalized ODE (2.1) with initial condition $x(\sigma_0) = y$ and, for all $s \in (\sigma_0 - \delta, \sigma_0)$, define $z_s : [s, \omega) \to X$ by

$$z_s(t) = \begin{cases} y, & \text{if } t \in [s, \sigma_0], \\ x(t), & \text{if } t \in [\sigma_0, \omega) \end{cases}$$

By Remark A.0.8 and Lemma C.0.9, z_s is regulated and, by Corollary C.0.4, the integral $\int_s^t DF(z_s(\tau), v)$ exists for all $t \in [s, \omega)$. Moreover, since x is a solution of the generalized ODE (2.1), for all $t \in [\sigma_0, \omega)$, we have

$$z_s(\sigma_0) + \int_{\sigma_0}^t DF(z_s(\tau), v) = x(\sigma_0) + \int_{\sigma_0}^t DF(x(\tau), v) = x(t) = z_s(t). \tag{2.72}$$

On the other hand, if $t \in [s, \sigma_0]$, then

$$z_s(s) + \int_s^t DF(z_s(\tau), v) = y + \int_s^t DF(y, v) = y + F(y,t) - F(y,s).$$

This fact together with (2.71) reveals

$$z_s(t) = z_s(s) + \int_s^t DF(z_s(\tau), v), \quad \text{for all } t \in [s, \sigma_0]. \tag{2.73}$$

Then, equations (2.72) and (2.73) show that z_s is a solution of the generalized ODE (2.1) with $z_s(s) = y$. By the definition of V, we obtain

$$|V(s,y) - V(\sigma_0,y)| = \left| \sup_{\tau\in[s,\omega)} \|z_s(\tau)\| - \sup_{\tau\in[\sigma_0,\omega)} \|x(\tau)\| \right| = 0, \quad \text{for all } s \in (\sigma_0 - \delta, \sigma_0)$$

which proves that $V(\cdot,y) : [t_0,+\infty) \to \mathbb{R}$ is left-continuous at σ_0. □

The following lemma guarantees that $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$, defined by (2.69), satisfies hypothesis (LU2) from Theorem 2.2.8.

Lemma 2.2.14. The function $[s_0,\omega) \ni t \mapsto V(t,x(t))$ is nonincreasing along every maximal solution $x : [s_0,\omega) \to \overline{B}_\rho$ of the generalized ODE (2.1), where $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$ is defined by (2.69) and $s_0 \geq t_0$.

Proof. Let $x : [s_0,\omega) \to \overline{B}_\rho$ be a maximal solution of the generalized ODE (2.1) and $t_1,t_2 \in [s_0,\omega)$ be such that $t_1 < t_2$. Notice that, since x is a maximal solution of the generalized ODE (2.1), for all $t \in [t_1,\omega)$, we have

$$\begin{aligned} x(t,s_0,x(s_0)) &= x(s_0) + \int_{s_0}^{t} DF(x(\tau),s) = x(s_0) + \int_{s_0}^{t_1} DF(x(\tau),s) + \int_{t_1}^{t} DF(x(\tau),s) \\ &= x(t_1) + \int_{t_1}^{t} DF(x(\tau),s) = x(t,t_1,x(t_1)). \end{aligned}$$

Analogously, $x(t,t_2,x(t_2)) = x(t,s_0,x(s_0))$ for all $t \in [t_2,\omega)$. Then,

$$\begin{aligned} V(t_2,x(t_2)) &= \sup_{\tau\in[t_2,\omega)} \|x(\tau,t_2,x(t_2))\| \\ &= \sup_{\tau\in[t_2,\omega)} \|x(\tau,s_0,x(s_0))\| \\ &\leq \sup_{\tau\in[t_1,\omega)} \|x(\tau,s_0,x(s_0))\| \\ &= \sup_{\tau\in[t_1,\omega)} \|x(\tau,t_1,x(t_1))\| \\ &= V(t_1,x(t_1)) \end{aligned}$$

and the proof is complete. □

The next result shows that the right-derivative of V is non-positive along the solutions of the generalized ODE (2.1), where $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$ is defined by (2.69). Its proof is similar to the proof of Lemma 2.2.14 .

Lemma 2.2.15. Let $V : [t_0,+\infty) \times \overline{B}_\rho \to \mathbb{R}$ be defined by (2.69). For every $s_0 \geq t_0$ and every maximal solution $x : [s_0,\omega) \subset [t_0,+\infty) \to \overline{B}_\rho$ of the generalized ODE (2.1), we have

$$D^+V(t,x(t)) := \limsup_{\eta\to 0^+} \frac{V(t+\eta,x(t+\eta)) - V(t,x(t))}{\eta} \leq 0, \quad t \in [s_0,\omega),$$

that is, the right-derivative of V is non-positive along the solutions of the generalized ODE (2.1).

Proof. Let $s_0 \geq t_0$ and $x : [s_0, \omega) \subset [t_0, +\infty) \to \overline{B}_\rho$ be a maximal solution of the generalized ODE (2.1). Then, for all $t \in [s_0, \omega)$ and all $\tau \in [t+\eta, \omega)$, with $\eta \geq 0$, we have

$$\begin{aligned} x(\tau, t, x(t)) &= x(t) + \int_t^\tau DF(x(s), \sigma) = x(t) + \int_t^{t+\eta} DF(x(s), \sigma) + \int_{t+\eta}^\tau DF(x(s), \sigma) \\ &= x(t+\eta) + \int_{t+\eta}^\tau DF(x(s), \sigma) = x(\tau, t+\eta, x(t+\eta)). \end{aligned}$$

Thus,

$$\begin{aligned} V(t+\eta, x(t+\eta)) - V(t, x(t)) &= \sup_{\tau \in [t+\eta, \omega)} \|x(\tau, t+\eta, x(t+\eta))\| - \sup_{\tau \in [t, \omega)} \|x(\tau, t, x(t))\| \\ &= \sup_{\tau \in [t+\eta, \omega)} \|x(\tau, t, x(t))\| - \sup_{\tau \in [t, \omega)} \|x(\tau, t, x(t))\| \\ &\leq \sup_{\tau \in [t, \omega)} \|x(\tau, t, x(t))\| - \sup_{\tau \in [t, \omega)} \|x(\tau, t, x(t))\| = 0. \end{aligned}$$

Therefore,

$$D^+V(t, x(t)) := \limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \leq 0, \quad t \in [s_0, \omega)$$

which completes the proof. □

In the sequel, we present a converse Lyapunov theorem on uniform stability.

Theorem 2.2.16. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. If the trivial solution $x \equiv 0$ of the generalized ODE (2.1) is uniformly stable, then there exists a functional $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$, $0 < \rho < c$ satisfying:

(CLU1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, such that

$$V(t, y) \leq a(\|y\|),$$

for all $t \in [t_0, +\infty)$ and $y \in \overline{B}_\rho$;

(CLU2) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$, with $b(0) = 0$, such that

$$V(t, y) \geq b(\|y\|),$$

for every $(t, y) \in [t_0, +\infty) \times \overline{B}_\rho$;

(CLU3) $V(t, 0) = 0$ for all $t \in [t_0, +\infty)$;

(CLU4) $V(\cdot, y) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$, for all $y \in \overline{B}_\rho$;

(CLU5) for every $s_0 \geq t_0$, the function $[s_0, \omega) \ni t \mapsto V(t, x(t))$ is nonincreasing along every maximal solution $x : [s_0, \omega) \to \overline{B}_\rho$, of the generalized ODE (2.1).

(CLU6) for every $s_0 \geq t_0$ and every maximal solution $x : [s_0, \omega) \subset [t_0, +\infty) \to \overline{B}_\rho$ of the generalized ODE (2.1), we have

$$D^+V(t, x(t)) := \limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \leq 0, \quad t \in [s_0, \omega),$$

that is, the right-derivative of V is non-positive along the solutions of the generalized ODE (2.1);

Proof. Let $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}$ be defined by (2.69). Then, items (CLU1), (CLU2), (CLU3), (CLU4), (CLU5) and (CLU6) are straightforward consequence of Lemmas 2.2.10, 2.2.11, 2.2.12, 2.2.13, 2.2.14 and 2.2.15 respectively. □

Remark 2.2.17. Conditions (CLU2), (CLU4) and (CLU6) from Theorem 2.2.16 guarantee that the function $V : [t_0, +\infty) \times X \to \mathbb{R}$, defined by (2.69), is a Lyapunov functional with respect to the generalized ODE (2.1) in the framework of Definition 2.0.2. On the other hand, conditions (CLU2), (CLU4) and (CLU5) given in Theorem 2.2.16, show that V satisfies all the conditions of the definition of a Lyapunov functional presented in [22, 23]. Therefore, no matter which definition of Lyapunov functional we are using, Theorem 2.1.19 ensures that uniform stability for generalized ODEs implies in the existence of a Lyapunov functional. Moreover, conditions (CLU1) and (CLU5) in Theorem 2.2.16 show that V satisfies the hypotheses (LU1) and (LU2) from the Lyapunov-type theorem 2.1.7 on uniform stability.

2.3 Relations

In this section, we use the Lyapunov theorems, described in Sections 2.1 and 2.2, to obtain relations between regular stability and uniform stability for the trivial solution of the generalized ODE (2.1). At first, we recall that F in the right-hand side of the generalized ODE (2.1) belongs to the class $\mathscr{F}(\Omega, h)$, where $\Omega = \mathscr{O} \times [t_0, +\infty)$, $\mathscr{O}$ is an open set containing the neutral element of the Banach space X and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$.

The next result shows regular stability implies uniform stability.

Theorem 2.3.1. If the trivial solution of the generalized ODE (2.1) is regularly stable, then it is uniformly stable.

Proof. By Theorem 2.1.19, there exists a Lyapunov functional $V : [t_0, +\infty) \times \mathscr{O} \to \mathbb{R}$ with respect to the generalized ODE (2.1). Moreover, V satisfies all conditions from Theorem 2.2.8 which, in turn, implies that the trivial solution of the generalized ODE (2.1) is uniformly stable. □

The next diagram illustrates the relations between the concepts of regular stability and uniform stability, where, by R we mean regular stability, U represents uniform stability for generalized ODEs, pR denotes regular stability with respect to perturbations and pU means uniform stability with respect to perturbations.

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{\text{Th.2.3.1}} & \mathrm{U} \\ {\scriptstyle\text{Th.2.1.5}}\Big\updownarrow & & \Big\uparrow{\scriptstyle\text{Th.2.2.4}} \\ \mathrm{pR} & & \mathrm{pU} \end{array} \tag{2.74}$$

Moreover, if $h : [t_0, +\infty) \to \mathbb{R}$ is bounded, then we have the following diagram.

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{\text{Th.2.3.1}} & \mathrm{U} \\ {\scriptstyle\text{Th.2.1.5}}\Big\updownarrow & & {\scriptstyle\text{Th.2.2.7}}\Big\updownarrow{\scriptstyle\text{Th.2.2.4}} \\ \mathrm{pR} & & \mathrm{pU} \end{array} \tag{2.75}$$

In what follows, we relate regular asymptotic stability to uniform asymptotic stability.

Using Theorem 2.1.29 instead of 2.1.19 and Theorem 2.2.9 instead of 2.2.8 in the proof of Theorem 2.3.1, we obtain the following result.

Theorem 2.3.2. If the trivial solution of the generalized ODE (2.1) is regular asymptotically stable, then it is uniformly asymptotically stable.

Therefore, if RA, pRA, UA and pUA denotes, respectively, regular asymptotic stability, regular asymptotic stability with respect to perturbations, uniform asymptotic stability and uniform asymptotic stability with respect to perturbations, then we have the following diagrams

$$\begin{array}{ccc} \mathrm{RA} & \xrightarrow{\text{Th.2.3.2}} & \mathrm{UA} \\ {\scriptstyle\text{Th.2.1.5}}\Big\updownarrow & & \Big\uparrow{\scriptstyle\text{Th.2.2.4}} \\ \mathrm{pRA} & & \mathrm{pUA} \end{array} \tag{2.76}$$

and, if in addition $h : [t_0, +\infty) \to \mathbb{R}$ is bounded, then

$$\begin{array}{ccc} \mathrm{RA} & \xrightarrow{\text{Th.2.3.2}} & \mathrm{UA} \\ {\scriptstyle\text{Th.2.1.5}}\Big\updownarrow & & {\scriptstyle\text{Th.2.2.7}}\Big\updownarrow{\scriptstyle\text{Th.2.2.4}} \\ \mathrm{pRA} & & \mathrm{pUA} \end{array} \tag{2.77}$$

We point out that, in [48], Štefan Schwabik proved a Lypunov-type theorem and a converse Lyapunov theorem on a different type of stability for the generalized ODE (2.1) called variational stability. Therefore, using the same arguments of this section, one can prove that the following implications hold:

$$\begin{array}{ccc} \mathrm{V} & \longrightarrow & \mathrm{UA} \\ \Big\updownarrow & & \Big\uparrow{\scriptstyle\text{Th.2.2.4}} \\ \mathrm{pV} & & \mathrm{pUA} \end{array} \tag{2.78}$$

and

$$
\begin{array}{ccc}
\mathrm{VA} & \longrightarrow & \mathrm{UA} \\
\updownarrow & & \Big\uparrow \text{Th.2.2.4} \\
\mathrm{pVA} & & \mathrm{pUA}
\end{array}
\tag{2.79}
$$

Moreover, if $h : [t_0, +\infty) \to \mathbb{R}$ is bounded, then

$$
\begin{array}{ccc}
\mathrm{V} & \longrightarrow & \mathrm{UA} \\
\updownarrow & & \text{Th.2.2.7} \Big\updownarrow \text{Th.2.2.4} \\
\mathrm{pV} & & \mathrm{pUA}
\end{array}
\tag{2.80}
$$

and

$$
\begin{array}{ccc}
\mathrm{V}A & \longrightarrow & \mathrm{UA} \\
\updownarrow & & \text{Th.2.2.7} \Big\updownarrow \text{Th.2.2.4} \\
\mathrm{pVA} & & \mathrm{pUA}
\end{array}
\tag{2.81}
$$

where V, VA, pV and pVA denote, respectively, variational stability, variational asymptotically stability, variational stability with respect to perturbations and variational asymptotically stability with respect to perturbations.

CHAPTER

3

BOUNDEDNESS OF SOLUTIONS

Concepts of boundedness of solution in the setting of generalized ODEs were firstly introduced in [2] and were inspired by the definitions of boundedness of solutions of impulsive functional differential equations presented in [20, 29, 42, 54]. Motived by [2], the authors of [22] proved several criteria, via Direct Method of Lyapunov, for the boundedness of solutions of generalized ODEs and, using the correspondence between the solutions of these equations and the solutions of measure differential equations, they were able to introduce new concepts of boundedness of solutions in the framework of measure differential equations and to prove Lyapunov-type theorems on boundedness of solutions of these equations.

Our goal in this chapter is to present Lyapunov-type theorems, borrowed from [13, 22] and to prove that uniform boundedness of solutions of generalized ODEs implies the existence of a Lyapunov functional (see Definition 2.0.2). Furthermore, we introduce concepts of uniform boundedness of solutions of a perturbed generalized ODE, we give sufficient conditions for the uniform boundedness of solutions of a perturbed generalized ODE to imply uniform boundedness of solutions of a homogeneous generalized ODE and vice-versa. Almost all results presented in this chapter are new and can be found in [6].

We consider X a Banach space, $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, and $F \in \mathscr{F}(X \times [t_0, +\infty), h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). Furthermore, consider the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t). \tag{3.1}$$

By Corollary C.0.20, for all $t \in [t_0, +\infty)$ and all $y \in X$, there exists a unique global forward solution $x : [t, +\infty) \to X$ of the generalized ODE (3.1) with initial condition $x(t) = y$ and we denote $x(\cdot)$ by $x(\cdot, y, t)$.

In the sequel, we recall concepts of boundedness of solutions in the setting of generalized ODEs described in [2].

Definition 3.0.1. We say that the generalized ODE (3.1) is

(i) *uniformly bounded*, if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and all $x_0 \in X$, with $\|x_0\| < \alpha$, we have

$$\|x(s, s_0, x_0)\| = \|x(s)\| < M, \text{ for all } s \geq s_0,$$

where $x : [s_0, +\infty) \to X$ is a global forward solution of the generalized ODE (3.1) with initial condition $x(s_0) = x_0$;

(ii) *quasi-uniformly ultimately bounded*, if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and all $x_0 \in X$, with $\|x_0\| < \alpha$, we have

$$\|x(s, s_0, x_0)\| = \|x(s)\| < B, \text{ for all } s \geq s_0 + T,$$

where $x : [s_0, +\infty) \to X$ is a global forward solution of the generalized ODE (3.1) with initial condition $x(s_0) = x_0$;

(iii) *uniform ultimately bounded*, if it is uniformly bounded and quasi-uniformly ultimately bounded.

In [22], the authors proved the following result which gives a sufficient condition for the uniform boundedness of a solution of the generalized ODE (3.1). For a proof of it, the reader may consult [13, Theorem 11.3] or [22, Theorem 3.5].

Theorem 3.0.2. Let $V : [t_0, +\infty) \times X \to \mathbb{R}$ satisfy the following conditions:

(LB1) for each $z \in G^-([\alpha, \beta], X)$, the function $[\alpha, \beta] \ni t \to V(t, z(t))$ is left-continuous on $(\alpha, \beta]$;

(LB2) there are two increasing functions $p, b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(0) = b(0) = 0$,

$$\lim_{s \to +\infty} b(s) = +\infty$$

and

$$b(\|y\|) \leq V(t, y) \leq p(\|y\|),$$

for every $(t, y) \in [t_0, +\infty) \times X$;

(LB3) for every $s_0 \geq t_0$ and every global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (3.1), we have

$$V(s, x(s)) - V(t, x(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$.

Then, the generalized ODE (3.1) is uniformly bounded.

In what follows, we target to prove a converse Lyapunov theorem.

Assume that the generalized ODE (3.1) is uniformly bounded. For all $t \in [t_0, +\infty)$ and all $y \in X$ define

$$V(t,y) = \sup_{\tau \in [t,+\infty)} \|x(\tau,t,y)\|, \tag{3.2}$$

where $x : [t, +\infty) \to X$ is the global forward solution of the generalized ODE (3.1) with initial condition $x(t) = y$.

Let $\alpha = \|y\| > 0$. Since the generalized ODE (3.1) is uniformly bounded, there exists $M = M(2\alpha) > 0$ such that

$$\|x(\tau,t,y)\| \leq M \text{ for all } \tau \geq t.$$

Then, V is well-defined for all $(t,y) \in [t_0, +\infty) \times X$.

The next result ensures that the boundedness of solutions of the generalized ODE (3.1) implies the existence of a Lyapunov functional satisfying some conditions of Theorem 3.0.2. The reader may check Definition 2.0.2 for the concept of a Lyapunov functional.

Theorem 3.0.3. If the generalized ODE (3.1) is uniformly bounded, then there exists a function $V : [t_0, +\infty) \times X \to \mathbb{R}$ satisfying:

(CB1) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(0) = 0$,

$$\lim_{s \to +\infty} b(s) = +\infty,$$

and

$$b(\|y\|) \leq V(t,y),$$

for every $(t,y) \in [t_0, +\infty) \times X$;

(CB2) $V(\cdot, y)$ is left-continuous on $(t_0, +\infty)$ for every $y \in X$;

(CB3) for every $s_0 \geq t_0$ and every global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (3.1), we have

$$V(s,x(s)) - V(t,x(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$;

(CB4) for all $s_0 \geq t_0$ and all global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (3.1), we have

$$D^+V(t,x(t)) := \limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \leq 0, \quad t \in [s_0, +\infty),$$

that is, the right-derivative of V is non-positive along the solutions of the generalized ODE (3.1);

(CB5) there exists an increasing function $p:\mathbb{R}^+\to\mathbb{R}^+$ such that $p(0)=0$ and

$$p(\|y\|)\geq V(t,y),$$

for every $(t,y)\in[t_0,+\infty)\times X$;

(CB6) $V(t,0)=0$ for all $t\in[t_0,+\infty)$.

Proof. Let $V:[t_0,+\infty)\times X\to\mathbb{R}$ be defined by (3.2). The proofs of items (CB1), (CB2), (CB3) and (CB4) are analogous to the proofs of Lemmas 2.2.11, 2.2.13, 2.2.14 and 2.2.15 respectively. Therefore, we omit them here.

Let us prove condition (CB5). Let $\alpha>0$. Then, by Definition 3.0.1-(i), for all $s_0\geq t_0$ and $x_0\in X$ with $\|x_0\|<\alpha$, there exists $M(\alpha)>0$ such that $\|x(s,s_0,x_0)\|<M(\alpha)$ for all $s\geq s_0$. Therefore, the set

$$\{\|x(s,s_0,x_0)\|;\|x_0\|<\alpha,s_0\geq t_0,s\geq s_0\}$$

is upper bounded for all $s\geq s_0$. Define $\overline{M}:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\overline{M}(\alpha)=\begin{cases}\sup\{\|x(s,s_0,x_0)\|;\|x_0\|<\alpha,s_0\geq t_0,s\geq s_0\}, & \text{if } \alpha>0,\\ 0, & \text{if } \alpha=0.\end{cases}$$

At first, we prove that $\overline{M}$ is nondecreasing. Indeed, let $0<\alpha_1<\alpha_2$ and consider

$$A=\{\|x(s,s_0,x_0)\|;\|x_0\|<\alpha_1,s_0\geq t_0,s\geq s_0\}\quad\text{and}$$
$$B=\{\|x(s,s_0,x_0)\|;\|x_0\|<\alpha_2,s_0\geq t_0,s\geq s_0\}.$$

Then, it is clear that $A\subset B$ which implies that $\overline{M}(\alpha_1)\leq\overline{M}(\alpha_2)$. Since $\overline{M}$ may not be increasing, we can choose a function $\widehat{M}:[0,+\infty)\to[0,+\infty)$ which is non-negative, increasing such that $\widehat{M}(0)=0$ and $\overline{M}(\alpha)\leq\widehat{M}(\alpha)$ for all $\alpha>0$. Define $p:\mathbb{R}^+\to\mathbb{R}^+$ by

$$p(t)=\begin{cases}0, & t=0,\\ \lim\limits_{\varepsilon\to0^+}\widehat{M}(t+\varepsilon), & t>0.\end{cases}$$

Then, if $0\leq t_1<t_2$, we have

$$p(t_1)=\widehat{M}(t_1^+)<\widehat{M}(t_2)\leq\widehat{M}(t_2^+)=p(t_2)$$

which shows that p is an increasing function.

It remains to demonstrate that p satisfies the following inequality

$$p(\|y\|)\geq V(t,y),\quad \text{for all } (t,y)\in[t_0,+\infty)\times X.$$

Let $y\in X$ and $t\in[t_0,+\infty)$ be given and take $\alpha=\|y\|+\varepsilon$, where ε is sufficiently small. Let $x(\cdot,t,y)$ be the global forward solution of the generalized ODE (3.1) with initial condition $x(t)=y$. Then, by the definition of $\overline{M}(\cdot)$ and $\widehat{M}(\cdot)$, we have

$$\|x(\tau,t,y)\|\leq\overline{M}(\alpha)\leq\widehat{M}(\alpha)=\widehat{M}(\|y\|+\varepsilon),$$

for all $\tau \in [t, +\infty)$, which implies

$$V(t,y) = \sup_{\tau \in [t,+\infty)} \|x(\tau,t,y)\| = \lim_{\varepsilon \to 0^+} \sup_{\tau \in [t,+\infty)} \|x(\tau,t,y)\| \leq \lim_{\varepsilon \to 0^+} \widehat{M}(\|y\| + \varepsilon) = p(\|y\|),$$

for all $\tau \in [t, +\infty)$. Therefore, condition (CB5) is proved.

To finish this proof, we notice that (CB6) is a consequence of conditions (CB1) and (CB5), once

$$0 = b(0) \leq V(t,0) \leq p(0) = 0,$$

for all $t \in [t_0, +\infty)$. □

Although the next result is similar to Theorem 3.0.3, it shows that, with the additional hypothesis that $x \equiv 0$ is a solution of the generalized ODE (3.1), the boundedness of solutions of the generalized ODE (3.1) implies the existence of a Lyapunov functional satisfying all conditions of the Lyapunov-type Theorem on uniform stability 2.2.8.

Theorem 3.0.4. If the generalized ODE (3.1) is uniformly bounded and $x \equiv 0$ is a solution of the generalized ODE (3.1), then there exists a function $V : [t_0, +\infty) \times X \to \mathbb{R}$ satisfying:

(CB1) there exists an increasing continuous functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(0) = 0$,

$$\lim_{s \to +\infty} b(s) = +\infty,$$

and

$$b(\|y\|) \leq V(t,y),$$

for all $(t,x) \in [t_0, +\infty) \times X$;

(CB2) for every $s_0 \geq t_0$ and every global forward solution $x : [s_0, +\infty) \to X$, $s_0 \geq t_0$, of the generalized ODE (3.1), we have

$$V(s,x(s)) - V(t,x(t)) \leq 0,$$

for all $s_0 \leq t < s < +\infty$;

(CB3) $V(\cdot, y)$ is left-continuous on $(t_0, +\infty)$ for every $y \in X$;

(CB4) for all $s_0 \geq t_0$ and all global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (3.1), we have

$$D^+V(t,x(t)) := \limsup_{\eta \to 0^+} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \leq 0, \quad t \in [s_0 + \infty),$$

that is, the right-derivative of V is non-positive along the solutions of (3.1);

(CB5) there exist an increasing continuous functions $a : \mathbb{R}^+ \to \mathbb{R}^+$ such that $a(0) = 0$ and

$$V(t,y) \leq a(\|y\|),$$

for every $(t,y) \in [t_0, +\infty) \times X$;

(CB6) $V(t,0)=0$ for all $t\in[t_0,+\infty)$.

Proof. Let $V:[t_0,+\infty)\times X\to\mathbb{R}$ be defined by (3.2). By Theorem 3.0.3, items (CB1) to (CB4) hold. Therefore, it remains to prove items (CB5) and (CB6).

In order to prove item (CB5), we take $\alpha>0$ and define

$$\widetilde{M}(\alpha)=\sup\{\|x(s,s_0,x_0)\|;\|x_0\|<\alpha,s_0\geq t_0,s\geq s_0\}.$$

Since the generalize ODE (3.1) is uniformly bounded, for all $s_0\in[t_0,+\infty)$ and all $x_0\in X$ with $\|x_0\|<\alpha$, there exists $M(\alpha)>0$ such that $\|x(s,s_0,x_0)\|<M(\alpha)$ for all $s\geq s_0$ and, therefore, $\widetilde{M}(\alpha)$ is well-defined for all $\alpha>0$. It is also clear that $\widetilde{M}(\alpha)$ is nondecreasing (see the proof of Theorem 3.0.3). Let $\overline{M}:\mathbb{R}^+\to\mathbb{R}^+$ be defined by

$$\overline{M}(\alpha)=\inf\{\widetilde{M}(r);\alpha<r\}\quad\text{for all }\alpha\in\mathbb{R}^+.$$

We target to prove that $\overline{M}(\alpha)$ is nondecreasing. Let $0<\alpha_1<\alpha_2$, then $\overline{M}(\alpha_1)\leq\widetilde{M}(r)$ for all $r>\alpha_1$. In particular, $\overline{M}(\alpha_1)\leq\widetilde{M}(r)$ for all $r>\alpha_2>\alpha_1$ which enables $\overline{M}(\alpha_1)\leq\overline{M}(\alpha_2)$.

Moreover, we claim that $\overline{M}$ is right-continuous on $[0,+\infty)$. Indeed, let $\alpha_0\in[0,+\infty)$ be fixed and $\varepsilon>0$. By the property of the infimum, there exists $c\in\{\widetilde{M}(r);\alpha_0<r\}$ such that $\overline{M}(\alpha_0)\leq c<\overline{M}(\alpha_0)+\varepsilon$. Since $c\in\{\widetilde{M}(r);\alpha_0<r\}$, it is clear that there exists $\delta>0$ such that $c=\widetilde{M}(\alpha_0+\delta)$ and, hence,

$$\overline{M}(\alpha_0)\leq\widetilde{M}(\alpha_0+\delta)<\overline{M}(\alpha_0)+\varepsilon. \tag{3.3}$$

Despite this fact, $\widetilde{M}(\alpha_0+\delta)\in\{\widetilde{M}(r);\alpha<r\}$ for all $\alpha\in(\alpha_0,\alpha_0+\delta)$ and, by the definition of $\overline{M}$, we have $\overline{M}(\alpha)\leq\widetilde{M}(\alpha_0+\delta)$. Thus, by (3.3), we get

$$\overline{M}(\alpha)\leq\widetilde{M}(\alpha_0+\delta)\leq\overline{M}(\alpha_0)+\varepsilon \tag{3.4}$$

which proves that $\overline{M}$ is right-continuous on $[0,+\infty)$.

Furthermore, we point out that $\overline{M}(0)=0$. In fact, let $\alpha_k=\frac{1}{k}$ for $k\in\mathbb{N}$ and x_k be the solution of the generalized ODE (2.1) with initial condition $x_k(s_0)=x_{k,0}$ such that $\|x_{k,0}\|<\alpha_k$ and $s_0\geq t_0$, where the existence of such a solution is guaranteed by Corollary C.0.20. Then, $x_{k,0}$ goes to zero as $k\to\infty$. By Proposition C.0.23, with $F_k=F$ for all $k=0,1,2,\ldots$, we have $\lim_{k\to\infty}x_k(s)=\bar{x}(s)$ for all $s\in[s_0,+\infty)$, where $\bar{x}$ is the solution of the generalized ODE (2.1) with initial condition $\bar{x}(s_0)=0$. Once $x\equiv0$ is a solution of the generalized ODE (3.1) and, by the uniqueness of a solution, we infer that $\bar{x}\equiv0$ and $\lim_{k\to\infty}x_k(s)=0$ for all $s\in[s_0,+\infty)$. Consequently, $\overline{M}(\alpha)$ goes to zero as α goes to zero and, since $\overline{M}$ is right-continuous on $[0,+\infty)$, we conclude that $\overline{M}(0)=0$.

From the previous paragraphs, $\overline{M}$ is nondecreasing, positive, right-continuous on $[0,+\infty)$ and $\overline{M}(0)=0$. Therefore, there exists an increasing continuous function $a:\mathbb{R}^+\to\mathbb{R}^+$ such that $a(\alpha)\geq\overline{M}(\alpha)$ for all $\alpha\in[0,+\infty)$ and $a(0)=0$.

Finally, for all $t \in [t_0, +\infty)$ and all $y \in X$, we have

$$V(t,y) = \sup_{\tau \in [t,+\infty)} \|x(\tau,t,y)\| < \widetilde{M}(\alpha), \quad \text{for all } \alpha > \|y\|$$

which implies

$$V(t,y) \leq \inf\{\widetilde{M}(\alpha), \|y\| < \alpha\} = \overline{M}(\|y\|) \leq a(\|y\|)$$

and condition (CB5) is proved.

Moreover, condition (CB6) follows from (CB1) and (CB5), since

$$0 = b(0) \leq V(t,0) \leq a(0) = 0, \quad \text{for all } t \in [t_0, +\infty).$$

Therefore, the proof of the theorem is complete. □

Consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t) + P(t)], \tag{3.5}$$

where $F : \Omega \to X$ and $P : [t_0, +\infty) \to X$ is a left-continuous function on $(t_0, +\infty)$ such that

$$\sup_{s \in [t_0, +\infty)} \|P(s) - P(t_0)\| < \infty.$$

Notice that, $\sup_{s \in [t_0,t]} \|P(s) - P(t_0)\| < \infty$ for all $t \in [t_0, +\infty)$ and, by Proposition A.0.20, the function $g : [t_0, +\infty) \to \mathbb{R}$, defined by $g(t) = \sup_{s \in [t_0,t]} \|P(s) - P(t_0)\|$, is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Therefore, as we mentioned in Remark 2.1.3, the function $\bar{x} : [\alpha, \beta] \to X$ is a solution of the perturbed generalized ODE (3.5) on $[\alpha, \beta] \subset [t_0, +\infty)$, if $(\bar{x}(s), s) \in \Omega$, for all $s \in [\alpha, \beta]$, and the following equality

$$\bar{x}(t) = \bar{x}(\alpha) + \int_{\alpha}^{b} DF(\bar{x}(\tau), s) + P(t) - P(\alpha)$$

holds for all $t \in [\alpha, \beta]$. Moreover, since we are considering $\Omega = X \times [t_0, +\infty)$, Corollary C.0.20 ensures the existence of a unique global forward solution $\bar{x} : [s_0, +\infty) \to X$ of the perturbed generalized ODE (3.5) with initial condition $\bar{x}(s_0) = x_0$, for all $s_0 \in [t_0, +\infty)$ and all $x_0 \in X$.

In what follows, we introduce a concept of uniform boundedness of solutions of the perturbed generalized ODE (3.5).

Definition 3.0.5. We say that the perturbed generalized ODE (3.5) is

(i) *uniformly bounded*, if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and for all $x_0 \in X$, with $\|x_0\| < \alpha$, we have

$$\|\bar{x}(s, s_0, x_0)\| < M, \text{ for all } s \geq s_0,$$

where $\bar{x} : [s_0, +\infty) \to X$ is a global forward solution of the perturbed generalized ODE (3.5) with initial condition $\bar{x}(s_0) = x_0$;

(ii) *quasi-uniformly ultimately bounded*, if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and for all $x_0 \in X$, with $\|x_0\| < \alpha$, we have

$$\|\bar{x}(s, s_0, x_0)\| < B, \text{ for all } s \geq s_0 + T,$$

where $\bar{x} : [s_0, +\infty) \to X$ is a global forward solution of the perturbed generalized ODE (3.5) with initial condition $\bar{x}(s_0) = x_0$;

(iii) *uniform ultimately bounded*, if it is uniformly bounded and quasi-uniformly ultimately bounded.

The next results relate the concepts of Definitions 3.0.1 and 3.0.5. The proof of first one is analogous to that of Theorem 2.2.3 and, therefore, we omit it here.

Theorem 3.0.6. The following statements hold.

(i) If the perturbed generalized ODE (3.5) is uniformly bounded, then the generalized ODE (3.1) is uniformly bounded.

(ii) If the perturbed generalized ODE (3.5) is quasi-uniformly ultimately bounded, then the generalized ODE (3.1) is quasi-uniformly ultimately bounded.

(iii) If the perturbed generalized ODE (3.5) is uniform ultimately bounded, then the generalized ODE (3.1) is uniform ultimately bounded.

Theorem 3.0.7. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Furthermore, assume that h is bounded. If the generalized ODE (3.1) is uniformly bounded, then the perturbed generalized ODE (3.5) is uniformly bounded.

Proof. Let $\bar{x} : [s_0, +\infty) \to X$ be the global forward solution of the perturbed generalized ODE (2.6) with initial condition $\bar{x}(s_0) = x_0$, $\alpha > 0$ and $x : [s_0, +\infty) \to X$ be the global forward solution of the generalized ODE (2.1) with initial condition $x(s_0) = x_0$. For all $t \geq s_0$, we have

$$\begin{aligned} \|\bar{x}(t) - x(t)\| &\leq \left\| \int_{s_0}^{t} D[F(\bar{x}(\tau), s) - F(x(\tau), s)] \right\| + \|P(t) - P(s_0)\| \\ &\leq \left\| \int_{s_0}^{t} D[F(\bar{x}(\tau), s) - F(x(\tau), s)] \right\| + \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\|. \end{aligned}$$

Then, by Lemma C.0.6, we have

$$\|\bar{x}(t) - x(t)\| \leq \int_{s_0}^{t} \|\bar{x}(s) - x(s)\| dh(s) + \sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\|,$$

for all $t \geq s_0$. Thus, by hypotheses, $\sup_{s \in [s_0, +\infty)} \|P(s) - P(s_0)\| < \infty$, h is nondecreasing and it is left-continuous on $(t_0, +\infty)$ and, by Lemma 2.2.5, the function $[s_0, +\infty) \ni t \mapsto \|\bar{x}(t) - x(t)\|$ is bounded. Therefore, we are under the hypotheses of the Gronwall-type inequality (see Theorem B.0.10) and, hence,

$$\|\bar{x}(t)-x(t)\| \leq \sup_{s\in[s_0,+\infty)} \|P(s)-P(s_0)\| e^{|h(t)-h(s_0)|} \leq \sup_{s\in[s_0,+\infty)} \|P(s)-P(s_0)\| e^{H},$$

where $H = \sup_{t\in[t_0,+\infty)} |h(t)-h(t_0)|$.

On the other hand, since the generalized ODE (3.1) is uniformly bounded, there exists $M = M(\alpha) > 0$ such that if $\|x_0\| < \alpha$, then

$$\|x(t)\| < M, \text{ for all } t \geq s_0.$$

Taking $\widetilde{M} = M + \sup_{s\in[s_0,+\infty)} \|P(s)-P(s_0)\| e^{H}$, we obtain

$$\|x_0\| < \alpha \Rightarrow \|\bar{x}(t)\| < \widetilde{M}, \text{ for all } t \geq t_0,$$

yielding that the perturbed generalized ODE (3.5) is uniformly bounded. □

The proof of the following theorem is analogous to that of Theorem 3.0.7 and, therefore, we omit it here.

Theorem 3.0.8. Let $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$. Furthermore, assume that h is bounded. Then, the next assertions are true.

(i) If the generalized ODE (3.1) is quasi-uniformly ultimately bounded, then the perturbed generalized ODE (3.5) is quasi-uniformly ultimately bounded.

(ii) If the generalized ODE (3.1) is uniform ultimately bounded, then the perturbed generalized ODE (3.5) is uniform ultimately bounded.

CHAPTER

4

STABILITY X BOUNDEDNESS OF SOLUTIONS

In this chapter, we relate the concepts of stability and boundedness of solutions of generalized ODEs, presented in Chapters 2 and 3.

Consider the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t), \tag{4.1}$$

where X is a Banach space, $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, and $F : X \times [t_0, +\infty) \to X$ is an X-valued function. Moreover, we assume that $x \equiv 0$ is a solution of the generalized ODE (4.1) (see Remark 2.0.1 for a sufficient condition for the existence of such solution) and $F \in \mathscr{F}(X \times [t_0, +\infty), h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$ (see Definitions A.0.17 and C.0.2). Under these conditions, for all $(x_0, s_0) \in X \times [t_0, +\infty)$, there exists a unique global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (4.1) with initial condition $x(s_0) = x_0$. See Corollary C.0.20.

The next result is a consequence of Theorems 2.2.8 and 3.0.4.

Corollary 4.0.1. If the generalized ODE (4.1) is uniformly bounded, then the trivial solution of the generalized ODE (4.1) is uniformly stable.

Proof. By Theorem 3.0.4, there exists a Lyapunov functional $V : [t_0, +\infty) \times X \to \mathbb{R}$ with respect to the generalized ODE (4.1) which satisfies all the hypotheses of Theorem 2.2.8 , which in turns, implies that the trivial solution of the generalized ODE (4.1) is uniformly stable. □

Now, consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) + P(t), \tag{4.2}$$

where $P : [t_0, +\infty) \to X$ is left-continuous on $(t_0, +\infty)$ and

$$\sup_{s \in [t_0, +\infty)} \|P(s) - P(t_0)\| < \infty.$$

Notice that, $\sup_{s \in [t_0,t]} \|P(s) - P(t_0)\| < \infty$ for all $t \in [t_0, +\infty)$ and, by Proposition A.0.20, the function $g : [t_0, +\infty) \to \mathbb{R}$, defined by $g(t) = \sup_{s \in [t_0,t]} \|P(s) - P(t_0)\|$, is a nondecreasing function and it is left-continuous on $(t_0, +\infty)$. Therefore, as we mentioned in Remark 2.1.3, Corollary C.0.20 guarantees that, for all $(x_0, s_0) \in \Omega$, the exists a unique global forward solution $\bar{x} : [s_0, +\infty) \to X$ of the perturbed generalized ODE (4.2) with initial condition $\bar{x}(s_0) = x_0$. Moreover, by the definition of solution, we have

$$\bar{x}(t) = x_0 + \int_{s_0}^{t} DF(\bar{x}(\tau), t) + P(t) - P(s_0), \quad \text{for all } t \in [s_0, +\infty).$$

In the sequel, we establish a relation between boundedness of solutions of the perturbed generalized ODE (4.2) and uniform stability for the generalized ODE (4.1).

Corollary 4.0.2. If the perturbed generalized ODE (4.2) is uniformly bounded, then the trivial solution of the generalized ODE (4.1) is uniformly stable.

Proof. By Theorem 3.0.6, the generalized ODE (4.1) is uniformly bounded. Then, by Corollary 4.0.1, the trivial solution of the generalized ODE (2.1) is uniformly stable. □

The next result is a consequence of the converse Lyapunov theorems for generalized ODEs presented in Chapters 2 and 3 and it will be crucial to prove the asymptotic controllability results for generalized ODEs in Chapter 5.

Corollary 4.0.3. If the function h from the class $\mathscr{F}(X \times [t_0, +\infty), h)$ is bounded, then the trivial solution of the generalized ODE (4.1) is uniformly stable with respect to perturbations.

Proof. Let $\alpha > 0$, $s_0 \geq t_0$, $x_0 \in X$, with $\|x_0\| < \alpha$, and $x : [s_0, +\infty) \to X$ be the global forward solution of the generalized ODE (4.1) with initial condition $x(s_0) = x_0$. Then, by Lemma C.0.9, for all $t \in [s_0, +\infty)$, we have

$$\|x(t) - x_0\| \leq |h(t) - h(s_0)| \leq |h(t) - h(t_0)| + |h(s_0) - h(t_0)| < 2H,$$

where $H = \sup_{s \in [s_0, +\infty)} |h(s) - h(t_0)|$. Taking $M(\alpha) = \alpha + 2H$, we obtain $\|x(t)\| < M(\alpha)$ for all $t \in [s_0, +\infty)$ which implies that the generalized ODE (4.1) is uniformly bounded. Finally, by Corollary 4.0.1, the trivial solution of the generalized ODE (4.1) is uniformly stable and, by Theorem 2.2.7, it is uniformly stable with respect to perturbations. □

We end this chapter by presenting a diagram which illustrates the relations above, where U means uniform stability for the trivial solution of the generalized ODE (4.1), B denotes uniform boundedness of solutions of the generalized ODE (4.1), pU uniform stability with respect to perturbations for the trivial solution of the generalized ODE (4.1), and pB stands for the uniform boundedness of solutions of the perturbed generalized ODE (4.2)

$$\begin{array}{ccc} \mathrm{U} & \xleftrightarrow[\text{Th.2.2.4}]{\text{Th.2.2.7}} & \mathrm{pU} \\ {\scriptstyle \text{Cor.4.0.1}}\updownarrow & & \\ \mathrm{B} & \xleftrightarrow[\text{Th.3.0.6}]{\text{Th.3.0.7}} & \mathrm{pB} \end{array} \tag{4.3}$$

It can be inferred from the diagram that generalized ODEs are robust with respect to perturbations, when this "robustness" concerns the uniform boundedness of solutions or the uniform stability of the trivial solution, and not only this, but also that boundedness of solutions implies uniform stability (see Corollary 4.0.1). Moreover, by diagram (2.75), we have

$$\begin{array}{ccccc} \mathrm{R} & \xrightarrow{\text{Th.2.3.1}} & \mathrm{U} & \xleftrightarrow[\text{Th.2.2.4}]{\text{Th.2.2.7}} & \mathrm{pU} \\ {\scriptstyle \text{Th.2.1.5}}\updownarrow & & {\scriptstyle \text{Cor.4.0.1}}\updownarrow & & \\ \mathrm{pR} & & \mathrm{B} & \xleftrightarrow[\text{Th.3.0.6}]{\text{Th.3.0.7}} & \mathrm{pB} \end{array} \tag{4.4}$$

where, R and pR denote regular stability and regular stability with respect to perturbations respectively.

CHAPTER

5

ASYMPTOTIC CONTROLLABILITY

In this chapter, we use Lyapunov techniques to give necessary and sufficient conditions for a nonlinear perturbed generalized ODEs to be asymptotic controllable. The results presented here are new and can be found in [6].

In the following lines, we specify our contributions concerning control theory.

In [53], the author proved that an ordinary differential equation of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{5.1}$$

is asymptotic controllable if and only if there exists a continuous Lyapunov functional $V : \mathbb{R}^n \to \mathbb{R}$ (satisfying some conditions) with respect to (5.1), where f is locally Lipschitz on the second variable, $x(t) \in \mathbb{R}^n$, $u(t) \in U$ and U is a locally compact metric space (see [53, Theorem 2.5]). The continuity of the Lyapunov functional is strongly used in the proof of [53, Theorem 2.5]. In the present work, our Lyapunov functional $V : X \times [t_0, +\infty) \to \mathbb{R}$ with respect to the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) \tag{5.2}$$

is a two-variable functional which does not need to be continuous on the first variable and is left-continuous on the second variable (see Definition 2.0.2). Thus, discontinuities are allowed. Moreover, our results do not require any Lipschitz-type condition on the right-hand side F. Our requirements are compatible with applications to retarded Volterra–Stieljes integral equations whose solutions may undergo jumps (see Chapter 6-Section 6.4). The interested reader may notice that the results from [35, 45, 53, 55] require continuity of the Lyapunov functional.

In addition, we consider a perturbed generalized ODE of the form

$$\frac{dx}{d\tau} = D[F(x,t) + u(t)] \tag{5.3}$$

where $F : X \times [t_0, +\infty) \to X$ and $u : [t_0, +\infty) \to X$ satisfy some properties and X is a Banach space. Having in mind that the Lyapunov functional with respect to (5.2) does not need to be continuous, we used the converse Lyapunov theorems (see Theorems 2.2.16, 3.0.3 and 3.0.4 presented in the previous chapters) and the relations described previously in order to give a characterization of the asymptotic controllability of equation (5.3). See Theorem 5.0.2 and Corollary 5.0.3. In particular, our results generalize those presented in [53].

Let X be a Banach space and $\Omega = X \times [t_0, +\infty)$, where $t_0 \geq 0$. Consider the perturbed generalized ODE given by

$$\frac{dx}{d\tau} = D[F(x,t) + u(t)], \tag{5.4}$$

where $F : \Omega \to X$ and $u : [t_0, +\infty) \to X$ is a *control function* and assume following conditions:

(AC1) $F(0,t) - F(0,s) = 0$ for $t, s \geq t_0$;

(AC2) $F \in \mathscr{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is nondecreasing, left-continuous on $(t_0, +\infty)$ and bounded. See Definitions A.0.17 and C.0.2

By condition (AC1) and Remark 2.0.1, the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) \tag{5.5}$$

admits trivial solution and, by condition (AC2), for all $(x_0, s_0) \in X \times [t_0, +\infty)$, there exists a unique global forward solution $x : [s_0, +\infty) \to X$ of the generalized ODE (5.5) with initial condition $x(s_0) = x_0$. See Corollary C.0.20. Moreover, for all $\xi \in X$ and all control function $u : [t_0, +\infty) \to X$, we denote by $\bar{x}(\cdot, \xi, u)$ the global forward solution of the perturbed generalized ODE (5.4) with initial condition $\bar{x}(t_0) = \xi \in X$ and control u, provided it exists.

In the sequel, we give a definition of asymptotic controllability for generalized ODEs.

Definition 5.0.1. The perturbed generalized ODE (5.4) is *asymptotically controllable*, if the following properties hold:

(i) (*global part*) for each ξ in X, there exists a control u such that $\bar{x}(t) = \bar{x}(t, \xi, u)$ is defined for all $t \geq t_0$ and, moreover, $\bar{x}(t)$ goes to 0 as t goes to ∞;

(ii) (*stability*) for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any state $\xi \in X$, with $\|\xi\| \leq \delta$, there is a control u as in (i) such that $\|\bar{x}(t)\| \leq \varepsilon$ for all $t \geq t_0$;

(iii) (*bounded controls*) there exist positive numbers η, k such that, if ξ given in (ii) satisfies $\|\xi\| < \eta$, then the control u satisfies $\|u\| \leq k$, where $\|u\| = \sup_{s \in [t_0, +\infty)} \|u(s)\|$.

The next result gives sufficient conditions for the perturbed generalized ODE (5.4) to be asymptotically controllable.

Theorem 5.0.2. If conditions (AC1) and (AC2) hold, then the perturbed generalized ODE (5.4) is asymptotically controllable.

Proof. If $\xi = 0$, then $\bar{x} \equiv 0$ is a solution of the perturbed generalized ODE (5.4) with control $u \equiv 0$ and it is clear that conditions (i) to (iii) of Definition 5.0.1 are fulfilled.

Let $\xi \in X$, $\xi \neq 0$. Consider $u : [t_0, +\infty) \to X$ given by

$$u(t) = \begin{cases} 0, & t = t_0 \\ -\xi, & t > t_0. \end{cases}$$

and $\bar{x} : [t_0, +\infty) \to X$ given by

$$\bar{x}(t) = \begin{cases} \xi, & t = t_0 \\ 0, & t > t_0. \end{cases}$$

Then, for all $t > 0$, we have

$$\bar{x}(t_0) + \int_{t_0}^{t} DF(\bar{x}(\tau), s) + u(t) - u(t_0) = \xi + 0 - \xi = 0 = \bar{x}(t)$$

and, therefore, $\bar{x}$ is a solution of the perturbed generalized ODE (5.4) with initial condition $\bar{x}(t_0) = \xi$. Besides, $\bar{x}$ is defined for all $t \geq t_0$ and $\bar{x}(t)$ goes to 0 as t goes to ∞. This gives the first part of the definition of asymptotic controllability.

In order to prove item (iii) from Definition 5.0.1, we notice that

$$\|u\| = \sup_{s \in [t_0, +\infty)} \|u(s)\| = \|\xi\|.$$

Therefore, the statement holds by taking $k = \eta$.

Finally, since u is bounded and (AC2) is fulfilled, by Corollary 4.0.3, the trivial solution of the generalized ODE (5.5) is uniformly stable with respect to perturbations. This gives the second part of the Definition 5.0.1. □

We point out that the proof of Theorem 5.0.2 uses the fact that boundedness of solutions of the generalized ODE (5.5) implies the existence of a Lyapunov functional and, in turn, the existence of this functional implies the uniform stability for the generalized ODE (5.5) (see Theorem 3.0.4 and Corollary 4.0.1 respectively). On the other hand, if the perturbed generalized ODE (5.4) is asymptotically controllable, then by condition (ii) of Definition 5.0.1 and Theorem 2.2.3, the trivial solution of the generalized ODE (5.5) is uniformly stable and, by Theorem 2.2.16, there exists a Lyapunov functional with respect to the generalized ODE (5.5). Hence, we have the following result.

Corollary 5.0.3. The perturbed generalized ODE (5.4) is asymptotically controllable if and only if there is a Lyapunov functional $V : [t_0, +\infty) \times X \to \mathbb{R}$ with respect to the generalized ODE (5.5) satisfying

(AC1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$ and

$$V(t, y) \leq a(\|y\|),$$

for all $y \in X$ and $t \in [t_0, +\infty)$.

CHAPTER

6

RETARDED VOLTERRA-STIELJTES INTEGRAL EQUATIONS

In this chapter, we apply the results, described in Chapters 2, 3, 4 and 5, to obtain Lyapunov theorems on stability and boundedness of solutions for retarded Volterra-Stieltjes integral equations (we write retarded VS integral equations, for short), as well as, results on asymptotically controllable for a perturbed retarded VS integral equation. Almost all the results in this chapter are new and can be found in [6].

We organized this chapter so that its first section contains a result on the existence of solutions of a retarded VS integral equation, where the given initial condition is evaluated at a point in the interval $[t_0, +\infty)$. We also establish a relation between the solutions of a retarded VS integral equation (perturbed retarded VS integral equation) and the solutions of a certain generalized ODE (perturbed generalized ODE).

The main goal of the second section is to study stability. In Subsection 6.2.1, we present some concepts of stability of the trivial solution of a retarded VS integral equation and relations between these concepts. Moreover, we give relations between types of stability for the trivial solution of a retarded VS integral equation and types of stability for the trivial solution of a generalized ODE. Subsection 6.2.2 is devoted to the proof Lyapunov-type theorems and, in subsection 6.2.3, we use the fact that retarded VS integral equations can be regarded as a generalized ODEs to obtain converse Lyapunov theorems.

In Section 6.3, we recall the definition of boundedness of solutions in the setting of retarded VS integral equations and we obtain converse Lyapunov Theorems. Furthermore, we relate stability and boundedness of solutions in the setting of retarded VS integral equations.

In the last section, we introduce a concept of asymptotic controllability for retarded VS integral equations and we present necessary and sufficient conditions to obtain asymptotic controllability for such equations. In the end of Section 6.4, we include an example.

Throughout this chapter, we consider as X being a Banach space, equipped with the norm $\|\cdot\|$, $[t_0,+\infty)\subset\mathbb{R}$, $r>0$ and the sets $BG([t_0-r,+\infty),X)$ and $G([-r,0],X)$ equipped with the usual supremum norm, that is,

$$\|y\|_\infty = \sup_{\theta\in[t_0-r,+\infty)} \|y(\theta)\|, \quad \text{for all } y\in BG([t_0-r,+\infty),X)$$

and

$$\|\psi\|_\infty = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|, \quad \text{for all } y\in G([-r,0],X).$$

Moreover, we recall that the sets $G([-r,0],X)$ and $BG([t_0-r,+\infty),X)$ are described in Definitions A.0.7 and A.0.22 respectively.

In the sequel, we define a special set of functions in $BG([t_0-r,+\infty),X)$.

Definition 6.0.1. Let $O\subset BG([t_0-r,+\infty),X)$ be an open set. We say that O has the *prolongation property*, if for every $y\in O$, $s_0\in[t_0,+\infty)$ and every $\widetilde{t}\in[s_0,+\infty)$, the function $\widetilde{y}:[t_0-r,+\infty)\to X$, defined by

$$\widetilde{y}(\theta)=\begin{cases} y(s_0-r), & t_0-r\le\theta\le s_0-r,\\ y(\theta), & s_0-r\le\theta\le\widetilde{t},\\ y(\widetilde{t}), & \widetilde{t}\le\theta<+\infty,\end{cases}$$

is also an element of O.

Let $O\subset BG([t_0-r,+\infty),X)$ be an open set with the prolongation property and define

$$S=\{y_t; y\in O, t\in[t_0,+\infty)\}\subset G([-r,0],X). \tag{6.1}$$

Consider the retarded VS integral equation

$$y(t)=y(s_0)+\int_{s_0}^{t} f(y_s,s)dg(s), \quad t\ge s_0, \tag{6.2}$$

where $s_0\ge t_0$, $f:S\times[t_0,+\infty)\to X$, with S given by (6.1), $g:[t_0,+\infty)\to\mathbb{R}$, the integral on the right-hand side is in the sense of Perron-Stieltjes and, for all $s\ge t_0$, the *memory function*, $y_s:[-r,0]\to X$ is given by $y_s(\theta)=y(s+\theta)$ for all $\theta\in[-r,0]$ with $r>0$. We recall that the concept of Perron-Stieltjes integral can be found in Definition B.0.4.

Furthermore, we assume the additional conditions:

(A1) the function $g:[t_0,+\infty)\to\mathbb{R}$ is left-continuous on $(t_0,+\infty)$ and nondecreasing;

(A2) for all $y\in O$ and all $s_1,s_2\in[t_0,+\infty)$, the Perron-Stieltjes integral

$$\int_{s_1}^{s_2} f(y_s,s)dg(s)$$

exists;

(A3) there exists a locally Perron-Stieltjes integrable function $M:[t_0,+\infty)\to\mathbb{R}$ with respect to g such that

$$\left\|\int_{s_1}^{s_2} f(y_s,s)dg(s)\right\| \leq \int_{s_1}^{s_2} M(s)dg(s),$$

for all $y\in O$ and all $s_1,s_2\in[t_0,+\infty)$;

(A4) there exists a locally Perron-Stieltjes integrable function $L:[t_0,+\infty)\to\mathbb{R}$ with respect to g such that

$$\left\|\int_{s_1}^{s_2}[f(y_s,s)-f(z_s,s)]dg(s)\right\| \leq \int_{s_1}^{s_2} L(s)\|y_s-z_s\|_\infty dg(s),$$

for all $y,z\in O$ and all $s_1,s_2\in[t_0,+\infty)$.

We also refer the retarded VS integral equation (6.2) as *homogeneous retarded VS integral equation*. In what follows, we show important properties of the functions M and L from conditions (A3) and (A4).

Remark 6.0.2. Let $y\in BG([t_0-r,+\infty),X)$ and $s_1,s_2\in[t_0,+\infty)$ be such that $s_1\leq s_2$. Then,

$$0\leq\left\|\int_{s_1}^{s_2} f(y_s,s)dg(s)\right\| \leq \int_{s_1}^{s_2} M(s)dg(s) = \int_{t_0}^{s_2} M(s)dg(s) - \int_{t_0}^{s_1} M(s)dg(s),$$

which implies

$$\int_{t_0}^{s_2} M(s)dg(s) \geq \int_{t_0}^{s_1} M(s)dg(s)$$

and, hence, the function $t\mapsto\int_{t_0}^{t} M(s)dg(s)$ is nondecreasing. Moreover, for all $\theta\in[s_1,s_2]$, we have

$$\int_{\theta}^{s_2} M(s)dg(s) \geq \left\|\int_{\theta}^{s_2} f(y_s,s)dg(s)\right\| \geq 0 \quad \text{and}$$

$$\int_{s_1}^{s_2} M(s)dg(s) = \int_{s_1}^{\theta} M(s)dg(s) + \int_{\theta}^{s_2} M(s)dg(s) \geq \int_{s_1}^{\theta} M(s)dg(s).$$

which implies

$$\int_{s_1}^{s_2} M(s)dg(s) \geq \sup_{\theta\in[s_1,s_2]}\int_{s_1}^{\theta} M(s)dg(s). \tag{6.3}$$

Similarly, $t\mapsto\int_{t_0}^{t} L(s)dg(s)$ is a nondecreasing function and

$$\int_{s_1}^{s_2} L(s)dg(s) \geq \sup_{\theta\in[s_1,s_2]}\int_{s_1}^{\theta} L(s)dg(s) \tag{6.4}$$

holds for all $y\in BG([t_0-r,+\infty),X)$ and all $s_1,s_2\in[t_0,+\infty)$ with $s_1\leq s_2$.

In what follows, we present a definition of a solution of the retarded VS integral equation (6.2).

Definition 6.0.3. We say that a function $y\colon[s_0-r,\omega(s_0,\phi))\to X$, $\omega(s_0,\phi)\leq+\infty$, is a *maximal solution of the retarded VS integral equation* (6.2) with initial condition $y_{s_0}=\phi$, $s_0\geq t_0$, if it satisfies the following conditions:

(i) $y(t) = \phi(t - s_0)$ for all $t \in [s_0 - r, s_0]$;

(ii) $(y_t, t) \in S \times [s_0, \omega(s_0, \phi))$;

(iii) the Perron-Stieltjes integral $\int_{s_0}^{t} f(y_s, s)dg(s)$ exists, for all $t \geq s_0$;

(iv) the equality

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s)$$

holds for all $t \in [s_0, \omega(s_0, \phi))$.

When $\omega(s_0, \phi) = +\infty$, y is also known as a *global forward solution*.

Consider the perturbed retarded VS integral equation

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s) + \int_{s_0}^{t} p(s)dv(s), \quad t \geq s_0 \geq t_0, \tag{6.5}$$

where the integrals on the right-hand side are in the sense of Perron-Stieltjes, S is given by (6.1), $f : S \times [t_0, +\infty) \to X$, $p : [t_0, +\infty) \to X$, and $g, v : [t_0, +\infty) \to \mathbb{R}$.

Assume that f satisfies conditions (A2), (A3) and (A4), g satisfies condition (A1) and the functions p and v satisfy the following conditions:

(A5) the function $v : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ and nondecreasing;

(A6) the Perron-Stieltjes integral

$$\int_{t_0}^{t} p(s)dv(s)$$

exists, for all $t \in [t_0, +\infty)$;

(A7) there exists a locally Perron-Stieltjes integrable function $K : [t_0, +\infty) \to \mathbb{R}$ with respect to v such that

$$\left\| \int_{s_1}^{s_2} p(s)dv(s) \right\| \leq \int_{s_1}^{s_2} K(s)dv(s),$$

for all $s_1, s_2 \in [t_0, +\infty)$.

Analogous to Definition 6.0.3 , we say that a function y: $[s_0 - r, \omega(s_0, \phi)) \to X$, with $\omega(s_0, \phi) \leq +\infty$, is a *maximal solution of the perturbed retarded VS integral equation* (6.5) with initial condition $y_{s_0} = \phi$, $s_0 \geq t_0$, if it satisfies the following conditions:

(i) $y(t) = \phi(t - s_0)$ for all $t \in [s_0 - r, s_0]$;

(ii) $(y_t, t) \in S \times [s_0, \omega(s_0, \phi))$;

(iii) the Perron-Stieltjes integrals $\int_{s_0}^{t} f(y_s, s)dg(s)$ and $\int_{s_0}^{t} p(s)dv(s)$ exist, for all $t \geq s_0$;

(iv) the equality

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s) + \int_{s_0}^{t} p(s)dv(s)$$

holds for all $t \in [s_0, \omega(s_0, \phi))$.

When $\omega(s_0, \phi) = +\infty$, y is also known as a *global forward solution*.

6.1 Existence and uniqueness of a solution

As our interest in the next section is to study stability for the trivial solution of the retarded VS integral equation (6.2), it is preferable that the point at which we consider the initial condition is loose within the interval $[t_0, +\infty)$. With this in mind, in this section, we adapted [26, Theorem 3.1], where the initial condition is taken at t_0 to an initial condition taken at some $s_0 \in [t_0, +\infty)$. Moreover, we introduce a relation between a solution of the retarded VS integral equation (6.2) (perturbed retarded VS integral equation (6.5)) and a solution of a generalized ODE (perturbed generalized ODE). Furthermore, we give a sufficient condition for the existence of a maximal solution of the retarded VS integral equation (6.2). The result present here are contained in [6].

At first, we define a special set $\mathbb{O}$. Let $O \subset BG([t_0 - r, +\infty), X)$ be an open set with the prolongation property, $y \in O$ and $s_0 \in [t_0, +\infty)$ be given. For all $t \in [s_0, +\infty)$, define the function $x_{y,s_0}(t) : [t_0 - r, +\infty) \to X$ by

$$x_{y,s_0}(t)(\theta) = \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty \end{cases} \tag{6.6}$$

and

$$\mathbb{O}_{t,s_0} = \{x_{y,s_0}(t) : y \in O,\ t \in [s_0, +\infty), \text{ where } x_{y,s_0}(t) \text{ is given by (6.6)}\}.$$

Consider $\mathbb{O}_{s_0} = \bigcup_{t \in [s_0, +\infty)} \mathbb{O}_{t,s_0}$ and set

$$\mathbb{O} = \bigcup_{s_0 \in [t_0, +\infty)} \mathbb{O}_{s_0}. \tag{6.7}$$

Then, $\mathbb{O} \subset O \subset BG([t_0 - r, +\infty), X)$, once O is a set with the prolongation property.

In what follows, we present an interesting property of the set $\mathbb{O}$.

Remark 6.1.1. For all $\psi \in S$, there exist $y \in O$ and $s_0 \in [t_0, +\infty)$ such that $\psi(\theta) = y_{s_0}(\theta)$ for all $\theta \in [-r, 0]$. Then, the function $x_0 : [t_0 - r, +\infty) \to X$, given by

$$x_0(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \psi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \psi(0), & s_0 \leq \theta < +\infty, \end{cases}$$

belongs to $\mathbb{O}$. Indeed, by the definition of ψ, we have

$$\begin{aligned} x_0(\theta) &= \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \psi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \psi(0), & s_0 \leq \theta < +\infty \end{cases} \\ &= \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq s_0, \\ y(s_0), & s_0 \leq \theta < +\infty, \end{cases} \\ &= x_{y,s_0}(s_0)(\theta). \end{aligned}$$

Define $F : \mathbb{O} \times [t_0, +\infty) \to BG([t_0 - r, +\infty), X)$ by

$$F(y,t)(\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq t_0, \\ \displaystyle\int_{t_0}^{\theta} f(y_s, s) dg(s), & t_0 \leq \theta \leq t, \\ \displaystyle\int_{t_0}^{t} f(y_s, s) dg(s), & t \leq \theta < +\infty. \end{cases} \tag{6.8}$$

Notice that, by condition (A2), the integrals appearing in (6.8) exist and, by condition (A3) and Remark 6.0.2, we have

$$\|F(y,t)\|_\infty = \sup_{\theta \in [t_0,t]} \left\| \int_{t_0}^{\theta} f(y_s, s) dg(s) \right\| \leq \sup_{\theta \in [t_0,t]} \int_{t_0}^{\theta} M(s) dg(s) = \int_{t_0}^{t} M(s) dg(s) < \infty.$$

Therefore, F is well-defined.

Consider the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t). \tag{6.9}$$

We target to prove that every solution of the retarded VS integral equation (6.2) is a solution of the generalized ODE (6.9) and vice-versa.

We show, in the sequel, that there exists a nondrecreasing function $h : [t_0, +\infty) \to \mathbb{R}$ such that F, defined by (6.8), belongs to the class $\mathscr{F}(\Omega, h)$ (see Definition C.0.2). Its proof follows the same ideas from [13, Lemma 4.16].

Lemma 6.1.2. Let $\mathbb{O}$ be given by (6.7), S by (6.1) and $\phi \in S$. Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3) and (A4). Then, F given by (6.8) belongs to the class $\mathscr{F}(\Omega, h)$, where $\Omega = \mathbb{O} \times [t_0, +\infty)$, and $h : [t_0, +\infty) \to \mathbb{R}$, given by

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] dg(s), \quad t \in [t_0, +\infty), \tag{6.10}$$

is a nondecreasing function which is left-continuous on $(t_0, +\infty)$.

Proof. At first, we notice that, by Remark 6.0.2, h is nondecreasing. Moreover, by Theorem B.0.17 and condition (A1), h is left-continuous on $(t_0, +\infty)$.

Let us show that $F \in \mathscr{F}(\Omega, h)$. Let $y \in \mathbb{O}$ and $s_1, s_2 \in [t_0, +\infty)$. Assume, without loss of generality, that $s_1 < s_2$. Then, by the definition of F, we have

$$[F(y,s_2) - F(y,s_1)](\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq s_1, \\ \displaystyle\int_{s_1}^{\theta} f(y_s,s)dg(s), & s_1 \leq \theta \leq s_2, \\ \displaystyle\int_{s_1}^{s_2} f(y_s,s)dg(s), & s_2 \leq \theta < +\infty. \end{cases} \tag{6.11}$$

By (6.11), condition (A3) and Remark 6.0.2 (see equation (6.3)), we conclude

$$\begin{aligned} \|F(y,s_2) - F(y,s_1)\|_\infty &= \sup_{\theta \in [t_0,+\infty)} \|F(y,s_2)(\theta) - F(y,s_1)(\theta)\| \\ &= \sup_{\theta \in [s_1,s_2]} \|F(y,s_2)(\theta) - F(y,s_1)(\theta)\| \\ &\overset{(6.11)}{=} \sup_{\theta \in [s_1,s_2]} \left\| \int_{s_1}^{\theta} f(y_s,s)dg(s) \right\| \\ &\overset{\text{Cond. }(A3)}{\leq} \sup_{\theta \in [s_1,s_2]} \int_{s_1}^{\theta} M(s)dg(s) \\ &\overset{(6.3)}{\leq} \int_{s_1}^{s_2} M(s)dg(s) \\ &\leq \int_{s_1}^{s_2} [M(s) + L(s)]dg(s) = h(s_2) - h(s_1). \end{aligned}$$

Similarly, if $y, z \in \mathbb{O}$ and $s_1, s_2 \in [t_0, +\infty)$, with $s_1 < s_2$, then condition (A4) and Remark 6.0.2 (see equation (6.4)) imply

$$\begin{aligned} \|F(y,s_2) - F(y,s_1) - F(z,s_2) + F(z,s_1)\|_\infty &= \sup_{\theta \in [s_1,s_2]} \left\| \int_{s_1}^{\theta} [f(y_s,s) - f(z_s,s)]dg(s) \right\| \\ &\overset{\text{Cond. }(A4)}{\leq} \sup_{\theta \in [s_1,s_2]} \int_{s_1}^{\theta} L(s)\|y_s - z_s\|_\infty dg(s) \\ &\leq \|y - z\|_\infty \int_{s_1}^{s_2} L(s)dg(s) \\ &\leq \|y - z\|_\infty \int_{s_1}^{s_2} [M(s) + L(s)]dg(s) \\ &= \|y - z\|_\infty h(s_2) - h(s_1) \end{aligned}$$

and the proof is complete. □

The next result gives sufficient conditions for the existence of a maximal solution of the generalized ODE (6.9) (see Definition C.0.15).

Theorem 6.1.3. Let $\mathbb{O}$ be given by (6.7) and F be defined by (6.8). If $s_0 \in [t_0, +\infty)$ is such that $\Delta^+ g(s_0) = g(s_0^+) - g(s_0) = 0$, then for all $x_0 \in \mathbb{O}$, there exists a unique maximal solution $x : [s_0, \omega(s_0, x_0)) \to \mathbb{O}$ of the generalized ODE (6.9) with initial condition $x(s_0) = x_0$.

Proof. By Lemma 6.1.2, $F \in \mathscr{F}(\Omega, h)$, where $\Omega = \mathbb{O} \times [t_0, +\infty)$ and $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$. Then, by Theorem C.0.17, it remains to show that $(x_0,s_0)\in\Omega_F$, that is,

$$x_0+\lim_{s\to s_0^+}F(x_0,s)-F(x_0,s_0)\in\mathbb{O}.$$

Notice that, by the definition of F, Theorem B.0.17 and the fact that $\Delta^+g(s_0)=0$, we have

$$\lim_{s\to s_0^+}F(x_0,s)(\theta)=\begin{cases}0, & t_0-r\leq\theta\leq t_0,\\ \displaystyle\int_{t_0}^{\theta}f((x_0)_s,s)dg(s), & t_0\leq\theta\leq s_0,\\ \displaystyle\int_{t_0}^{s_0}f((x_0)_s,s)dg(s), & s_0\leq\theta<+\infty\end{cases}=F(x_0,s_0)(\theta),$$

which implies $\lim_{s\to s_0^+}F(x_0,s)=F(x_0,s_0)$. Then, since $x_0\in\mathbb{O}$, we conclude

$$x_0+\lim_{s\to s_0^+}F(x_0,s)-F(x_0,s_0)=x_0$$

belongs to $\mathbb{O}$. □

In the sequel, we present an import property concerning the maximal solution of the generalized ODE (6.9). Its proof follows similar ideas to the proof of [24, Lemma 3.7].

Lemma 6.1.4. Let $\mathbb{O}$ be given by (6.7), F by (6.8), $s_0\in[t_0,+\infty)$ and $\psi\in S$. Assume that $x:[s_0,\omega(s_0,x_0))\to BG([t_0-r,+\infty),X)$ is the maximal solution of the generalized ODE (6.9) with initial condition

$$x(s_0)(\theta)=x_0(\theta)=\begin{cases}\psi(-r), & t_0-r\leq\theta\leq s_0-r,\\ \psi(\theta-s_0), & s_0-r\leq\theta\leq s_0,\\ \psi(0), & s_0\leq\theta<+\infty.\end{cases}$$

If $v,\theta\in[s_0,\omega(s_0,x_0))$, then

$$x(v)(\theta)=\begin{cases}x(v)(v), & \theta\geq v,\\ x(\theta)(\theta), & v\geq\theta.\end{cases}\tag{6.12}$$

Moreover, if $\theta\in[s_0,\omega(s_0,x_0))$ and $v\in[t_0-r,s_0]$, then

$$x(\theta)(v)=x(s_0)(v).\tag{6.13}$$

Proof. Let $v,\theta\in[s_0,\omega(s_0,x_0))$. At first, assume that $\theta\geq v$. By the definition of a solution of the generalized ODE (6.9), $x(t)\in\mathbb{O}$ for all $t\in[s_0,\omega(s_0,x_0))$ (see Definition C.0.1). Moreover,

$$x(v)(v)=x(s_0)(v)+\int_{s_0}^{v}DF(x(\tau),s)(v)\quad\text{and}\tag{6.14}$$

$$x(v)(\theta)=x(s_0)(\theta)+\int_{s_0}^{v}DF(x(\tau),s)(\theta).\tag{6.15}$$

Combining equations (6.14) and (6.15) with the fact that $x(s_0)(v) = x(s_0)(\theta)$, we obtain

$$x(v)(\theta) - x(v)(v) = \int_{s_0}^{v} DF(x(\tau),s)(\theta) - \int_{s_0}^{v} DF(x(\tau),s)(v). \tag{6.16}$$

By the definition of the Kurzweil integral, given ε, there exists a gauge δ on $[s_0, v]$ such that

$$\left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})] - \int_{s_0}^{v} DF(x(\tau),s) \right\|_{\infty} < \varepsilon, \tag{6.17}$$

provided $d = (\tau_i, [s_{i-1}, s_i])$ is a δ-fine tagged division of $[s_0, v]$ (see Definitions B.0.1 and B.0.11). By (6.16) and (6.17), we get

$$\begin{aligned}
&\|x(v)(\theta) - x(v)(v)\| = \left\| \int_{s_0}^{v} DF(x(\tau),s)(\theta) - \int_{s_0}^{v} DF(x(\tau),s)(v) \right\| \\
&\leq \Bigg\| \int_{s_0}^{v} DF(x(\tau),s)(\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) \\
&\quad - \int_{s_0}^{v} DF(x(\tau),s)(v) + \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) \Bigg\| \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) \right\| \\
&\leq \left\| \int_{s_0}^{v} DF(x(\tau),s)(\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) \right\| \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) - \int_{s_0}^{v} DF(x(\tau),s)(v) \right\| \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) \right\| \\
&\leq 2 \left\| \int_{s_0}^{v} DF(x(\tau),s) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})] \right\|_{\infty} \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) \right\| \\
&\leq 2\varepsilon + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) - \sum_{i=1}^{|d|} [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v) \right\|.
\end{aligned}$$

In addition, by the definition of F, we have

$$[F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](\theta) = [F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})](v)$$

and, hence,

$$\|x(v)(\theta) - x(v)(v)\| \leq 2\varepsilon.$$

Since ε can be made arbitrarily small, we conclude

$$x(v)(\theta) = x(v)(v).$$

Similarly, if $\theta \leq v$, then

$$x(v)(\theta) = x(s_0)(\theta) + \int_{s_0}^{v} DF(x(\tau),s)(\theta) \quad \text{and}$$

$$x(\theta)(\theta) = x(s_0)(\theta) + \int_{s_0}^{\theta} DF(x(\tau),s)(\theta).$$

Consequently,

$$x(v)(\theta) - x(\theta)(\theta) = \int_{\theta}^{v} DF(x(\tau),s)(\theta).$$

Moreover, if $d = (\tau_i, [s_{i-1}, s_i])$ is a δ-fine tagged division of $[\theta, v]$, then, by the definition of F, we have

$$F(x(\tau_i),s_i)(\theta) - F(x(\tau_i),s_{i-1})(\theta) = 0, \quad \text{for all } i = 1,\ldots,|d|$$

which implies that $\int_{\theta}^{v} DF(x(\tau),s)(\theta) = 0$ and the proof of (6.12) is complete.

On the other hand, if $\theta \in [s_0, \omega(s_0,x_0))$ and $v \in [t_0 - r, s_0]$, then

$$x(\theta)(v) = x(s_0)(v) + \int_{s_0}^{\theta} DF(x(\tau),s)(v). \tag{6.18}$$

Moreover, by the definition of F, if $d = (\tau_i, [s_{i-1}, s_i])$ is a δ-fine tagged division of $[s_0, \theta]$, then

$$F(x(\tau_i),s_i)(v) - F(x(\tau_i),s_{i-1})(v) = 0, \quad \text{for all } i = 1,\ldots,|d|,$$

once $v \leq s_0 < s_{i-1} < s_i$. Thus,

$$\int_{s_0}^{\theta} DF(x(\tau),s)(v) = 0. \tag{6.19}$$

Replacing (6.19) into (6.18), we have $x(\theta)(v) = x(s_0)(v)$ which yields (6.13). □

In the following result, we assume the existence of maximal solutions of the retarded VS integral equation (6.2) and of the generalized ODE (6.9). Moreover, we present a relation between a solution of the generalized ODE (6.9) and a solution of the retarded VS integral equation (6.2). A version of such a result, when the initial condition occurs at time t_0 and the solutions of the retarded VS integral equation (6.2) and the generalized ODE (6.9) are defined in a closed interval $[t_0, t_0 + \sigma]$, with $\sigma > 0$, can be found in [13, Theorems 3.8 and 3.9]. Although the proof, presented in the sequel, is analogous to the proof of [13, Theorems 3.8 and 3.9], it shows the reason for choosing the set $\mathbb{O}$.

Theorem 6.1.5. Let $\mathbb{O}$ be given by (6.7) and S by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). For all $s_0 \geq t_0$, the following statements hold.

(i) Let $y : [s_0 - r, \omega(s_0, \psi)) \to X$ be a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi \in S$. For every $t \in [s_0, \omega(s_0, \psi))$, let

$$x(t)(\theta) = \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty. \end{cases} \tag{6.20}$$

Then, the function $x : [s_0, \omega(s_0, \psi)) \to \mathbb{O}$ is a maximal solution of the generalized ODE (6.9) with initial condition $x(s_0) = x_0$, where

$$x_0(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \psi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \psi(0), & s_0 \leq \theta < +\infty. \end{cases} \tag{6.21}$$

(ii) Conversely, let $\psi \in S$ and x_0 be given by (6.21). If $x : [s_0, \omega(s_0,x_0)) \to \mathbb{O}$ is a maximal solution of the generalized ODE (6.9) with initial condition $x(s_0) = x_0$, then the function $y : [s_0 - r, \omega(s_0,x_0)) \to X$, defined by

$$y(\theta) = \begin{cases} x(s_0)(\theta), & s_0 - r \leq \theta \leq s_0, \\ x(\theta)(\theta), & s_0 \leq \theta < \omega(s_0,x_0), \end{cases} \tag{6.22}$$

is a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$.

Proof. At first, we notice that, by the definition of S, $y_{s_0} = \psi \in S$. Then, $x : [s_0, \omega(s_0, \psi)) \to \mathbb{O}$, given by (6.20), can be rewritten as

$$x(t)(\theta) = \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty \end{cases} = x_{y,s_0}(t)(\theta)$$

Therefore, $x(t) \in \mathbb{O}$ for all $t \in [s_0, \omega(s_0, \psi))$ and, hence, $x : [s_0, \omega(s_0, \psi)) \to \mathbb{O}$ is well-defined.

Let us prove item (i). Let $s_0 \geq t_0$ and $y : [s_0 - r, \omega(s_0, \psi)) \to X$ be a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$. We target to show that, for every $t \in [s_0, \omega(s_0, \psi))$, the integral $\int_{s_0}^t DF(x(\tau), s)$ exists and

$$x(t) - x(s_0) = \int_{s_0}^t DF(x(\tau), s),$$

where $x : [s_0, \omega(s_0, \psi)) \to \mathbb{O}$ is given by (6.20).

Let $\varepsilon > 0$ be given and $h : [t_0, +\infty) \to \mathbb{R}$ be given by (6.10). Once h is nondecreasing (see Remark 6.0.2), Corollary A.0.10 guarantees that there exist at most finitely many points $s \in [s_0, t]$ such that

$$\Delta^+ h(s) = h(s^+) - h(s) \geq \varepsilon. \tag{6.23}$$

Let us denote the points in $[s_0,t]$ for which (6.23) holds by $s_1,\dots,s_m$. Consider a gauge δ on $[s_0,t]$ such that

$$\delta(\tau) < \min\left\{\frac{s_k - s_{k-1}}{2} : k = 2,\dots,m\right\}, \quad \text{for } \tau \in [s_0,t], \quad \text{and}$$

$$\delta(\tau) < \min\{|\tau - s_k| : k = 2,\dots,m\} \quad \text{for } \tau \in [s_0,t] \setminus \{s_k\}_{k=1}^m.$$

Notice that, if a point-interval par $(\tau,[c,d])$, with $[c,d] \subset [s_0,t]$, is δ-fine, then $[c,d]$ contains at most one of the points $s_1,\dots,s_m$ and, if $s_k \in [c,d]$, then

$$\tau = s_k. \tag{6.24}$$

Moreover, by the definition of x (see (6.20)), we have

$$x(s_k)_{s_k} = y_{s_k}, \quad \text{for every } k = 1,\dots,m. \tag{6.25}$$

By (6.25) and Theorem B.0.17, we obtain

$$\lim_{\theta \to s_k^+} \int_{s_k}^{\theta} L(s)\|y_s - x(s_k)_s\|_\infty dg(s) = L(s_k)\|y_{s_k} - x(s_k)_{s_k}\|_\infty \Delta^+ g(t_k) = 0,$$

for every $k = 1,\dots,m$. Then, we can choose a gauge δ on $[s_0,t]$ such that

$$\int_{s_k}^{s_k+\delta(s_k)} L(s)\|y_s - x(s_k)_s\|_\infty dg(s) < \frac{\varepsilon}{2m+1}, \quad \text{for every } k = 1,\dots,m. \tag{6.26}$$

Furthermore, by condition (A3), for all $\tau \in [s_0,t] \setminus \{s_k\}_{k=1}^m$, we have

$$\|y(\tau^+) - y(\tau)\| = \left\| \lim_{\theta \to \tau^+} \int_{\tau}^{\theta} f(y_s,s)dg(s) \right\| \le \lim_{\theta \to \tau^+} \int_{\tau}^{\theta} M(s)dg(s) \le h(\tau^+) - h(\tau)$$

and, by (6.23), we conclude

$$\|y(\tau^+) - y(\tau)\| \le \varepsilon.$$

Therefore, we choose a gauge δ on $[s_0,t]$ such that

$$\|y(\theta) - y(\tau)\| \le \varepsilon, \quad \text{for every } \tau \in [s_0,t] \setminus \{s_k\}_{k=1}^m \text{ and } \theta \in [\tau, \tau + \delta(\tau)). \tag{6.27}$$

Assume that $d = (\tau_i, [t_{i-1},t_i])$, $i = 1,\dots,|d|$, is a δ-fine tagged division of $[t_0,t]$. By (6.20) and the fact that $y : [s_0 - r, \omega(s_0,\psi)) \to X$ is a solution of the retarded VS integral equation (6.2), we have

$$[x(t_i) - x(t_{i-1})](\theta) = \begin{cases} 0, & t_0 - r \le \theta \le t_{i-1}, \\ y(\theta) - y(t_{i-1}) = \displaystyle\int_{t_{i-1}}^{\theta} f(y_s,s)dg(s), & t_{i-1} \le \theta \le t_i, \\ y(t_i) - y(t_{i-1}) = \displaystyle\int_{t_{i-1}}^{t_i} f(y_s,s)dg(s), & t_i \le \theta < +\infty. \end{cases}$$

Moreover, by the definition of F (see (6.8)), for all $i = 1,\dots,|d|$, we conclude

$$[F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq t_{i-1}, \\ \displaystyle\int_{t_{i-1}}^{\theta} f(x(\tau_i)_s,s)dg(s), & t_{i-1} \leq \theta \leq t_i, \\ \displaystyle\int_{t_{i-1}}^{t_i} f(x(\tau_i)_s,s)dg(s), & t_i \leq \theta < +\infty. \end{cases}$$

Then,

$$\begin{aligned} &[x(t_i) - x(t_{i-1})](\theta) - [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1}](\theta) \\ &= \begin{cases} 0, & t_0 - r \leq \theta \leq t_{i-1}, \\ \displaystyle\int_{t_{i-1}}^{\theta} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s), & t_{i-1} \leq \theta \leq t_i, \\ \displaystyle\int_{t_{i-1}}^{t_i} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s), & t_i \leq \theta < +\infty \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\|x(t_i) - x(t_{i-1}) - [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1}]\|_\infty \\ &= \sup_{\theta \in [t_0,+\infty)} \|[x(t_i) - x(t_{i-1})](\theta) - [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1}](\theta)\| \\ &= \sup_{\theta \in [t_{i-1},t_i]} \left\| \int_{t_{i-1}}^{\theta} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s) \right\|. \end{aligned}$$

Notice that, if $s < \tau_i$, then by equation (6.20), we get

$$x(\tau_i)_s(\theta) = x(\tau_i)(s+\theta) = y(s+\theta) = y_s(\theta), \quad \text{for all } \theta \in [-r,0]$$

which implies that $x(\tau_i)_s = y_s$ and

$$\int_{t_{i-1}}^{\theta} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s) = \begin{cases} 0, & t_{i-1} \leq \theta \leq \tau_i, \\ \displaystyle\int_{\tau_i}^{\theta} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s), & \tau_i \leq \theta \leq t_i. \end{cases}$$

Furthermore, by condition (A4) and Remark 6.0.2, we have

$$\begin{aligned} \left\| \int_{t_{i-1}}^{\theta} [f(y_s,s) - f(x(\tau_i)_s,s)]dg(s) \right\| &\leq \left\| \int_{t_{i-1}}^{\theta} L(s)\|y_s - x(\tau_i)_s\|_\infty dg(s) \right\| \\ &\leq \left\| \int_{t_{i-1}}^{t_i} L(s)\|y_s - x(\tau_i)_s\|_\infty dg(s) \right\|. \end{aligned} \tag{6.28}$$

By the definition of the gauge δ, we have two cases.

Case 1: $[t_{i-1},t_i] \cap \{s_1,\dots,s_m\} = \{s_k\}$, for some $k \in \{1,\dots,m\}$.

In this case, we have $\tau_i = s_k$ (see (6.24)) and from (6.26), we obtain

$$\int_{\tau_i}^{t_i} L(s)\|y_s - x(\tau_i)_s\|_\infty \leq \frac{\varepsilon}{2m+1}.$$

Therefore,

$$\|x(t_i) - x(t_{i-1}) - [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1}]\|_\infty \leq \frac{\varepsilon}{2m+1}. \tag{6.29}$$

Case 2: $[t_{i-1},t_i]\cap\{s_1,\ldots,s_m\}=\emptyset$.

In this case, let $s\in[\tau_i,t_i]$. Then,

$$\begin{aligned}\|y_s-x(\tau_i)_s\|_\infty &= \sup_{\theta\in[-r,0]}\|y(s+\theta)-x(\tau_i)(s+\theta)\|\\ &= \sup_{\theta\in[s-r,s]}\|y(\theta)-x(\tau_i)(\theta)\|\end{aligned}$$

By the definition of x (see (6.20)), if $s-r\leq\tau_i$, then

$$\begin{aligned}\|y_s-x(\tau_i)_s\|_\infty &\leq \sup_{\theta\in[s-r,\tau_i]}\|y(\theta)-x(\tau_i)(\theta)\|+\sup_{\theta\in[\tau_i,s]}\|y(\theta)-x(\tau_i)(\theta)\|\\ &= \sup_{\theta\in[s-r,\tau_i]}\|y(\theta)-y(\theta)\|+\sup_{\theta\in[\tau_i,s]}\|y(\theta)-y(\tau_i)\|\\ &= \sup_{\theta\in[\tau_i,s]}\|y(\theta)-y(\tau_i)\|.\end{aligned}\tag{6.30}$$

On the other hand, if $s-r\geq\tau_i$, then

$$\|y_s-x(\tau_i)_s\|_\infty\leq\sup_{\theta\in[s-r,s]}\|y(\theta)-y(\tau_i)\|.\tag{6.31}$$

Since $d=(\tau_i,[t_{i-1},t_i])$ is δ-fine, we have $s\leq t_i<\delta(\tau_i)+\tau_i$ and, hence, $[\tau_i,s]\subset[\tau_i,\delta(\tau_i)+\tau_i)$ and $[s-r,s]\subset[\tau_i,\delta(\tau_i)+\tau_i)$, whenever $s-r\geq\tau_i$. Therefore, by (6.27), (6.30) and (6.31), for all $s\in[\tau_i,t_i]$, we have

$$\|y_s-x(\tau_i)_s\|_\infty\leq\varepsilon.\tag{6.32}$$

By (6.28) and (6.32), we get

$$\|x(t_i)-x(t_{i-1})-[F(x(\tau_i),t_i)-F(x(\tau_i),t_{i-1}]\|_\infty\leq\varepsilon\int_{\tau_i}^{t_i}L(s)dg(s).\tag{6.33}$$

Then, from Cases 1 and 2 (see (6.29) and (6.33)) and the fact that Case 1 occurs at most $2m$ times, we conclude

$$\left\|x(t)-x(s_0)-\sum_{i=1}^{|d|}[F(x(\tau_i),t_i)-F(x(\tau_i),t_{i-1}]\right\|_\infty\leq\varepsilon\int_{s_0}^{t}L(s)dg(s)+\frac{2m\varepsilon}{2m+1}.$$

Since ε and $t\in[s_0,\omega(s_0,\psi))$ are arbitrary, we have

$$x(t)-x(s_0)=\int_{s_0}^{t}DF(x(\tau),s),$$

for all $t\in[s_0,\omega(s_0,\psi))$ which completes the proof of item (i).

Our next goal is to prove item (ii). Let $\psi\in S$ and x_0 be given by (6.21). Assume that $x:[s_0,\omega(s_0,x_0))\to\mathbb{O}$ is a maximal solution of the generalized ODE (6.9) with initial condition $x(s_0)=x_0$ and let $y:[s_0,\omega(s_0,x_0))\to X$ be given by (6.22).

At first, notice that, by (6.22), for all $\theta\in[-r,0]$, we have

$$y_{s_0}(\theta)=y(s_0+\theta)=x(s_0)(s_0+\theta)=\psi(s_0+\theta-s_0)=\psi(\theta)\quad\text{and}$$

$$y(t) = x(s_0)(\theta) = \psi(t - s_0),$$

for all $t \in [s_0 - r, s_0]$. Therefore, condition (i) from Definition 6.0.3 is satisfied.

Moreover, by the definition of y and Lemmas 6.1.2 and C.0.9, $y \in BG([s_0 - r, \omega(s_0, x_0)), X)$. By condition (A2), the Perron-Stieltjes integral $\int_{s_1}^{s_2} f(y_s, s)dg(s)$ exists, for all $s_1, s_2 \in [s_0, \omega(s_0, x_0))$ and, consequently, condition (iii) from Definition 6.0.3 holds.

It remains to show that y satisfies

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s),$$

for all $t \in [s_0, \omega(s_0, x_0))$. Indeed, by the definition of y and Lemma 6.1.4, we have

$$\begin{aligned} y(t) - y(s_0) - \int_{s_0}^{t} f(y_s, s)dg(s) &= x(t)(t) - x(s_0)(s_0) - \int_{s_0}^{t} f(y_s, s)dg(s) \\ &= x(t)(t) - x(s_0)(t) - \int_{s_0}^{t} f(y_s, s)dg(s) \\ &= \int_{s_0}^{t} DF(x(\tau), s)(t) - \int_{s_0}^{t} f(y_s, s)dg(s). \end{aligned} \tag{6.34}$$

Similarly to what we did in item (i), for a fixed $t \in [s_0, \omega(s_0, x_0))$, we may consider a gauge δ on $[s_0, t]$ such that if s_k, $k = 1, \ldots, m$ are points in $[s_0, t]$ for which (6.23) holds, then δ satisfies the following properties:

(I) $\delta(\tau) < \min\left\{\frac{s_k - s_{k-1}}{2} : k = 2, \ldots, m\right\}$, for $\tau \in [s_0, t]$;

(II) $\delta(\tau) < \min\{|\tau - s_k| : k = 2, \ldots, m\}$ for $\tau \in [s_0, t] \setminus \{s_k\}_{k=1}^{m}$;

(III) $\int_{s_k}^{s_k + \delta(s_k)} L(s)\|y_s - x(s_k)_s\|_\infty dg(s) < \frac{\varepsilon}{2m+1}$, $k \in \{1, \ldots, m\}$;

(IV) $\|h(\theta) - h(\tau)\|$, for all $\tau \in [s_0, t] \setminus \{s_k\}_{k=1}^{m}$ and all $\theta \in [\tau, \tau + \delta(\tau))$.

By the definition of the Kurzweil integral $\int_{s_0}^{t} DF(x(\tau), s)$, the gauge δ may be chosen in such a way that, for every $\varepsilon > 0$ and every δ-fine tagged division $d = (\tau_i, [t_{i-1}, t_i])$ of $[s_0, t]$, we have

$$\left\| \int_{s_0}^{t} DF(x(\tau), s) - \sum_{i=1}^{|d|} [F(x(\tau_i), t_i) - F(x(\tau_i), t_{i-1})] \right\|_\infty < \varepsilon. \tag{6.35}$$

By (6.34) and (6.35), we obtain

$$
\begin{aligned}
&\left\| y(t) - y(s_0) - \int_{s_0}^{t} f(y_s,s)dg(s) \right\| \\
&= \left\| \int_{s_0}^{t} DF(x(\tau),s)(t) - \int_{s_0}^{t} f(y_s,s)dg(s) \right\| \\
&\leq \left\| \int_{s_0}^{t} DF(x(\tau),s)(t) - \sum_{i=1}^{|d|} [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) \right\| \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) - \int_{s_0}^{t} f(y_s,s)dg(s) \right\| \\
&\leq \left\| \int_{s_0}^{t} DF(x(\tau),s) - \sum_{i=1}^{|d|} [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})] \right\|_{\infty} \\
&\quad + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) - \int_{s_0}^{t} f(y_s,s)dg(s) \right\| \\
&\leq \varepsilon + \left\| \sum_{i=1}^{|d|} [F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) - \int_{s_0}^{t} f(y_s,s)dg(s) \right\|.
\end{aligned}
$$

From the definition of F, we conclude

$$
[F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) = \int_{t_{i-1}}^{t_i} f(x(\tau_i)_s,s)dg(s)
$$

which implies

$$
[F(x(\tau_i),t_i) - F(x(\tau_i),t_{i-1})](t) - \int_{t_{i-1}}^{t_i} f(y_s,s)dg(s) = \int_{t_{i-1}}^{t_i} [f(x(\tau_i)_s,s) - f(y_s,s)]dg(s).
$$

On the other hand, by the definition of y and Lemma 6.1.4, if $s \in [t_{i-1}, \tau_i]$ and $s - r \geq s_0$, then

$$
x(\tau_i)_s(\theta) = x(\tau_i)(s+\theta) = x(s+\theta)(s+\theta) = y(s+\theta) = y_s(\theta), \quad \text{for all } \theta \in [-r,0]
$$

and, if $s \in [t_{i-1}, \tau_i]$ and $s - r \leq s_0$, thus

$$
\begin{aligned}
x(\tau_i)(s+\theta) &= \begin{cases} x(\tau_i)(s+\theta), & \theta \in [-r, s_0 - s], \\ x(s+\theta)(s+\theta), & \theta \in [s_0 - s, 0]. \end{cases} \\
&= \begin{cases} x(s_0)(s+\theta), & v \in [s-r, s_0], \\ x(s+\theta)(s+\theta), & v \in [s_0, s]. \end{cases} \\
&= y(s+\theta),
\end{aligned}
$$

for all $\theta \in [-r,0]$. Therefore,

$$
x(\tau_i)_s = y_s, \quad \text{for all } s \in [t_{i-1}, \tau_i]. \tag{6.36}
$$

Moreover, if $s \in [\tau_i, t_i]$ and $s - r \geq s_0$, then, for all for all $\theta \in [-r,0]$, we have

$$
y_s(\theta) = y(s+\theta) = x(s+\theta)(s+\theta) = x(t_i)(s+\theta) = x(t_i)_s(\theta), \quad \text{for all } \theta \in [-r,0] \tag{6.37}
$$

and, if $s \in [\tau_i, t_i]$ and $s - r \leq s_0$, then for all $\theta \in [-r, 0]$, we obtain

$$\begin{aligned} x(t_i)(s+\theta) &= \begin{cases} x(s_0)(s+\theta), & \theta \in [-r, s_0 - s], \\ x(s+\theta)(s+\theta), & \theta \in [s_0 - s, 0]. \end{cases} \\ &= y(s+\theta) \end{aligned} \tag{6.38}$$

By (6.37) and (6.38), we conclude

$$x(s_i)_s = y_s, \quad \text{for all } s \in [\tau_i, t_i]. \tag{6.39}$$

From (6.36), (6.39) and condition (A4), we get

$$\begin{aligned} \left\| \int_{t_{i-1}}^{t_i} [f(x(\tau_i)_s, s) - f(y_s, s)] dg(s) \right\| &= \left\| \int_{\tau_i}^{t_i} [f(x(\tau_i)_s, s) - f(x(t_i)_s, s)] dg(s) \right\| \\ &\leq \int_{\tau_i}^{t_i} L(s) \| x(\tau_i)_s - x(t_i)_s \|_\infty dg(s). \end{aligned}$$

As in the proof of item (i), let us consider two cases.

Case 1: $[t_{i-1}, t_i] \cap \{s_1, \ldots, s_m\} = \{s_k\}$, for some $k \in \{1, \ldots, m\}$.

Since x is a solution of the generalized ODE (6.9), by Lemma C.0.9 and by the definition of the gauge δ (see item (III)), we obtain

$$\int_{\tau_i}^{t_i} L(s) \| x(\tau_i)_s - x(t_i)_s \|_\infty dg(s) \leq \frac{\varepsilon}{2m+1},$$

which, in turn, implies

$$\left\| [F(x(\tau_i), t_i) - F(x(\tau_i), t_{i-1})](t) - \int_{t_{i-1}}^{t_i} f(y_s, s) dg(s) \right\| \leq \frac{\varepsilon}{m+1}$$

Case 2: $[t_{i-1}, t_i] \cap \{s_1, \ldots, s_m\} = \emptyset$.

By item (IV), for all $s \in [\tau_i, t_i]$, we have

$$\| x(t_i)_s - x(\tau_i)_s \|_\infty \leq \| x(t_i) - x(\tau_i) \|_\infty \leq |h(t_i) - h(\tau_i)| \leq \varepsilon$$

and, hence,

$$\left\| [F(x(\tau_i), t_i) - F(x(\tau_i), t_{i-1})](t) - \int_{t_{i-1}}^{t_i} f(y_s, s) dg(s) \right\| \leq \varepsilon \int_{\tau_i}^{t_i} L(s) dg(s).$$

From Cases 1 and 2 and the fact that Case 1 occurs at most $2m$ times, we conclude

$$\begin{aligned} &\sum_{i=1}^{|d|} \left\| [F(x(\tau_i), t_i) - F(x(\tau_i), t_{i-1})](t) - \int_{t_{i-1}}^{t_i} f(y_s, s) dg(s) \right\| \\ &\leq \varepsilon \int_{s_0}^{t} L(s) dg(s) + \frac{2m\varepsilon}{2m+1} < \varepsilon \left(2 + \int_{s_0}^{t} L(s) dg(s) \right). \end{aligned}$$

Consequently,

$$\left\| y(t) - y(s_0) - \int_{s_0}^{t} f(y_s, s) dg(s) \right\| < \varepsilon \left(2 + \int_{s_0}^{t} L(s) dg(s) \right).$$

and the statement follows, once ε is arbitrary. □

The next result gives sufficient conditions for the existence and uniqueness of a maximal solution of the retarded VS integral equation (6.2).

Theorem 6.1.6. Let S be given by (6.1). Assume that $g:[t_0,+\infty)\to\mathbb{R}$ satisfies (A1) and $f:S\times[t_0,+\infty)\to X$ satisfies (A2), (A3) and (A4). Let $s_0\in[t_0,+\infty)$ and suppose $\Delta^+g(s_0)=0$. Then, for all $\psi\in S$, there exists a unique maximal solution $y:[s_0,\omega)\to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0}=\psi$.

Proof. Let $x_0:[t_0-r,+\infty)\to X$ be given by

$$x_0(\theta)=\begin{cases}\psi(-r), & t_0-r\le\theta\le s_0-r,\\ \psi(\theta-s_0), & s_0-r\le\theta\le s_0,\\ \psi(0), & s_0\le\theta<+\infty.\end{cases}$$

By Remark 6.1.1, $x_0\in\mathbb{O}$. Moreover, by Theorem 6.1.3, there exists a unique maximal solution $x:[s_0,\omega(s_0,x_0))\to\mathbb{O}$ of the generalized ODE (6.9) with initial condition $x(s_0)=x_0$ and, by Theorem 6.1.5-(ii), the function $y:[s_0-r,\omega(s_0,x_0))\to X$, defined by

$$y(\theta)=\begin{cases}x(s_0)(\theta), & s_0-r\le\theta\le s_0,\\ x(\theta)(\theta), & s_0\le\theta<\omega(s_0,x_0),\end{cases}$$

is a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0}=\psi$.

In order to prove the uniqueness, we assume that $\bar{y}:[s_0-r,\omega(s_0,x_0))\to X$ is another solution of the retarded VS integral equation with initial condition $\bar{y}_{s_0}=\psi$. By Theorem 6.1.5-(i), the function $\bar{x}:[s_0,\omega(s_0,x_0))\to\mathbb{O}$, given by

$$\bar{x}(t)(\theta)=\begin{cases}\bar{y}(s_0-r), & t_0-r\le\theta\le s_0-r,\\ \bar{y}(\theta), & s_0-r\le\theta\le t,\\ \bar{y}(t), & t\le\theta<+\infty,\end{cases}$$

is a solution of the generalized ODE (6.9) with initial condition $\bar{x}(s_0)=x_0$. By the uniqueness of a solution of the generalized ODE (6.9), $\bar{x}(t)=x(t)$ for all $t\in[s_0,\omega(s_0,x_0))$. Thus, for all $\theta\in[s_0-r,s_0]$, we have

$$y(\theta)=x(s_0)(\theta)=\bar{x}(s_0)(\theta)=\bar{y}(\theta)$$

and, for all $\theta\in[s_0,\omega(s_0,x_0))$, we have

$$y(\theta)=x(\theta)(\theta)=\bar{x}(\theta)(\theta)=\bar{y}(\theta).$$

Consequently, $y(\theta)=\bar{y}(\theta)$ for all $\theta\in[s_0-r,\omega(s_0,x_0))$ and the maximal solution of the retarded VS integral equation is unique. □

Our next goal is to relate the solutions of the perturbed retarded VS integral equation (6.5) to the solutions of a perturbed generalized ODE.

Let $F : \mathbb{O} \times [t_0, +\infty) \to BG([t_0 - r, +\infty), X)$ be defined by (6.8), where $\mathbb{O}$ is given by (6.7). Define $P : [t_0, +\infty) \to BG([t_0 - r, +\infty), X)$ by

$$P(t)(\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq t_0, \\ \displaystyle\int_{t_0}^{\theta} p(s)dv(s), & t_0 \leq \theta \leq t, \\ \displaystyle\int_{t_0}^{t} p(s)dv(s), & t \leq \theta < +\infty, \end{cases} \tag{6.40}$$

and consider the following perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t) + P(t)]. \tag{6.41}$$

Notice that, by condition (A6), P is well-defined and, by condition (A5) and Theorem B.0.17, P is left-continuous on $(t_0, +\infty)$. As we mentioned in Remark 2.1.3, all results on existence, uniqueness and other properties of a solution of a generalized ODE, presented in Appendix C, hold for the perturbed generalized ODE (6.41).

The next result relates a solution of the perturbed generalized ODE (6.41) to a solution of the perturbed retarded VS integral equation (6.5). Its proof is analogous to that of Theorem 6.1.7. Therefore, we omit it here.

Theorem 6.1.7. Consider $\mathbb{O} \subset BG([t_0 - r, +\infty), X)$ given by (6.7) and S by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). For all $s_0 \geq t_0$, the following statement hold.

(i) Let $\bar{y} : [s_0 - r, \omega(s_0, \phi)) \to X$ be a maximal solution of the perturbed retarded VS integral equation (6.2) with initial condition $\bar{y}_{s_0} = \phi \in S$. For every $t \in [s_0, \omega(s_0, \phi))$, let

$$\bar{x}(t)(\theta) = \begin{cases} \bar{y}(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ \bar{y}(\theta), & s_0 - r \leq \theta \leq t, \\ \bar{y}(t), & t \leq \theta < +\infty. \end{cases}$$

Then, $\bar{x} : [s_0, \omega(s_0, \phi)) \to \mathbb{O}$ is a maximal solution of the perturbed generalized ODE (6.41) with initial condition $\bar{x}(s_0) = x_0$, where

$$x_0(\theta) = \begin{cases} \phi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \phi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \phi(0), & s_0 \leq \theta < +\infty. \end{cases} \tag{6.42}$$

(ii) Conversely, let $\phi \in S$ and x_0 be given by (6.42). If $\bar{x} : [s_0, \omega(s_0, x_0)) \to \mathbb{O}$ is a maximal solution of the perturbed generalized ODE (6.41) with initial condition $\bar{x}(s_0) = x_0$, then the function $\bar{y} : [s_0 - r, \omega(s_0, x_0)) \to X$, defined by

$$\bar{y}(\theta) = \begin{cases} \bar{x}(t_0)(\theta), & s_0 - r \leq \theta \leq s_0, \\ \bar{x}(\theta)(\theta), & s_0 \leq \theta < \omega(s_0, x_0), \end{cases}$$

is a maximal solution of the perturbed retarded VS integral equation (6.2) with initial condition $\bar{y}_{s_0} = \phi$.

6.2 Stability

This section is devoted to the study of uniform stability, uniform stability with respect to perturbations and integral stability for the trivial solution of the retarded VS integral equation (6.2) and the relations between these concepts. Furthermore, in Sections 6.2.2 and 6.2.3, we applied the results described in Chapter 2 to establish Lyapunov theorems on uniform stability and integral stability for the trivial solution of the retarded VS integral equation (6.2).

Throughout this section, we consider the set S, given by (6.1), and $\mathscr{O}$ by (6.7). We suppose $g : [t_0, +\infty) \to \mathbb{R}$ satisfies(A1) and $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3) and (A4). Moreover, we assume that, for all $s_0 \geq t_0$ and all $\psi \in S$, there exists a unique maximal solution $y : [s_0 - r, \omega(s_0, \psi)) \to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$. In this case, for all $t \in [s_0 - r, \omega(s_0, \psi))$, we denote the memory function y_t by $y_t(s_0, \psi)$. In addition, we suppose the function $f : S \times [t_0, +\infty) \to X$ is such that

$$f(0,t) = 0, \quad \text{for every } t \in [t_0, +\infty). \tag{6.43}$$

Therefore, $y \equiv 0$, is a solution of (6.2). In this case, we say that $y \equiv 0$ is the *trivial solution* of the retarded VS integral equation (6.2). Moreover, notice that if f satisfies (6.43), then F, defined by (6.8), satisfies

$$F(0,t) - F(0,s) = 0, \quad \text{for all } t, s \in [t_0, +\infty)$$

which implies that $x \equiv 0$ is a solution of the generalized ODE (6.9). See Remark 2.0.1.

6.2.1 Basic results

In this subsection, we introduce the concepts of uniform stability, uniform stability with respect to perturbations and integral stability for the trivial solution of the retarded VS integral equation (6.2). Furthermore, we give relations between these concepts and relations between:

- uniform stability for the retarded VS integral equation (6.2) and uniform stability for the generalized ODE (6.9);
- uniform stability with respect to perturbations for the retarded VS integral equation (6.2) and uniform stability with respect to perturbations for the generalized ODE (6.9);
- integral stability for the retarded VS integral equation (6.2) and regular stability for the generalized ODE (6.9).

Almost all the results presented here are new and can be found in [4, 6].

In the following, we present some concepts of stability for the retarded VS integral equation (6.2).

Definition 6.2.1. The trivial solution of (6.2) is said to be

(i) *stable*, if for all $s_0 \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $\psi \in S$ satisfies

$$\|\psi\|_\infty = \sup_{\theta \in [-r,0]} \|\psi(\theta)\| < \delta,$$

then

$$\|y_t(s_0, \psi)\|_\infty = \sup_{\theta \in [-r,0]} \|y(\theta + t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, \psi)),$$

where $y : [s_0 - r, \omega(s_0, \psi)) \to X$ is a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$;

(ii) *uniformly stable*, if it is stable with δ independent of s_0.

(iii) *uniformly asymptotically stable*, if there exists $\delta_0 > 0$ such that for all $\varepsilon > 0$ and all $s_0 \geq t_0$, there exists $T = T(\varepsilon) \geq 0$ such that if $\psi \in S$ satisfies

$$\|\psi\|_\infty = \sup_{\theta \in [-r,0]} \|\psi(\theta)\| < \delta,$$

then

$$\|y_t(s_0, \psi)\|_\infty = \sup_{\theta \in [-r,0]} \|y(\theta + t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, \psi)) \cap [s_0 + T, +\infty),$$

where $y : [s_0 - r, \omega(s_0, \psi)) \to X$ is a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$.

The next two results relate the concepts above to the corresponding concepts of stability for the trivial solution of the generalized ODE (6.9), presented in Definition 2.2.1.

Theorem 6.2.2. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). Then, the following statements hold.

(i) If the trivial solution of the generalized ODE (6.9) is stable, then the trivial solution of the retarded VS integral equation (6.2) is stable.

(ii) If the trivial solution of the generalized ODE (6.9) is uniformly stable, then the trivial solution of the retarded VS integral equation (6.2) is uniformly stable.

(iii) If the trivial solution of the generalized ODE (6.9) is uniformly asymptotically stable, then the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable.

Proof. Due to the similarity of the proofs items (i), (ii) and (iii), we only prove item (ii).

Given $s_0 \geq t_0$ and $\varepsilon > 0$, let $y : [s_0 - r, \omega(s_0, \psi)) \to X$ be the maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$. By Theorem 6.1.5-(i), the function $x : [s_0, \omega(s_0, \psi)) \to \mathbb{O}$, given by

$$x(t)(\theta) = \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty, \end{cases}$$

is the maximal solution of the generalized ODE (6.9) with initial condition $x(s_0) = \widetilde{x}$, where

$$\widetilde{x}(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \psi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \psi(0), & \theta \geq s_0. \end{cases}$$

Moreover, for all $t \in [s_0, \omega(s_0, \psi))$, we have

$$\begin{aligned} \|y_t\|_\infty &= \sup_{\theta \in [-r,0]} \|y(\theta + t)\| = \sup_{\theta \in [t-r,t]} \|y(\theta)\| \\ &= \sup_{\theta \in [t-r,t]} \|x(t)(\theta)\| \leq \sup_{\theta \in [t_0 - r, +\infty)} \|x(t)(\theta)\| = \|x(t)\|_\infty. \end{aligned} \tag{6.44}$$

Since the trivial solution of the generalized ODE (6.9) is uniformly stable, there exists $\delta > 0$ such that, if

$$\|x(s_0)\|_\infty < \delta, \tag{6.45}$$

then

$$\|x(t)\|_\infty < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, \psi)). \tag{6.46}$$

Therefore, by equations (6.44), (6.45) and (6.46), we have

$$\|y_{s_0}\|_\infty = \|x(s_0)\|_\infty < \delta \Rightarrow \|y_t\|_\infty \leq \|x(t)\|_\infty < \varepsilon, \text{ for all } t \geq s_0$$

which shows that the trivial solution of the retarded VS integral equation (6.2) is uniformly stable. □

Theorem 6.2.3. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). Then, the following statements hold.

(i) If the trivial solution of the retarded VS integral equation (6.2) is uniformly stable, then the trivial solution of the generalized ODE (6.9) is uniformly stable.

(ii) If the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable, then the trivial solution of the generalized ODE (6.9) is uniformly asymptotically stable.

Proof. We start by proving item (i). Let $s_0 \geq t_0$, $\varepsilon > 0$ and $x_0 \in \mathbb{O}$. By the definition of the set $\mathbb{O}$, there exist $y \in O$ and $s, \tau_0 \in [t_0, +\infty)$, with $s \geq \tau_0$, such that

$$x_0(\theta) = \begin{cases} y(\tau_0 - r), & t_0 - r \leq \theta \leq \tau_0 - r, \\ y(\theta), & \tau_0 - r \leq \theta \leq s, \\ y(s), & s \leq \theta < +\infty. \end{cases}$$

Let $\psi = y_{\tau_0}$ and define $\bar{x}_0 : [t_0 - r, +\infty) \to X$ by

$$\bar{x}_0(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq \tau_0 - r, \\ \psi(\theta - \tau_0), & \tau_0 - r \leq \theta \leq \tau_0, \\ \psi(0), & \tau_0 \leq \theta < +\infty. \end{cases}$$

By Theorem 6.1.5-(i), for all $t \in [\tau_0, \omega(\tau_0, \psi))$ the function $\bar{x}(t) \in \mathbb{O}$, given by

$$\bar{x}(t)(\theta) = \begin{cases} y(\tau_0 - r), & t_0 - r \leq \theta \leq \tau_0 - r, \\ y(\theta), & \tau_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty, \end{cases}$$

is the maximal solution of the generalized ODE (6.9) with initial condition $\bar{x}(\tau_0) = \bar{x}_0$, where $y : [\tau_0 - r, \omega(\tau_0, \psi)) \to X$ is the maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{\tau_0} = \psi$.

By the fact that $\bar{x}(s) = x_0$, if $x : [s_0, \omega(s_0, x_0)) \to \mathbb{O}$ is the solution of the generalized ODE (6.9) with initial condition $x(s_0) = x_0$, then the function $z : [\tau_0, \omega(s_0, x_0)) \to \mathbb{O}$, defined by

$$z(t) = \begin{cases} \bar{x}(t), & t \in [\tau_0, s], \\ x(t - (s - s_0)), & t \in [s, \omega(s_0, x_0)), \end{cases}$$

is well-defined. Moreover, z is a maximal solution of the generalized ODE (6.9) with initial condition $z(\tau_0) = \bar{x}(\tau_0) = \bar{x}_0$, once $\bar{x}$ and x are solutions of the generalized ODE (6.9) (see Theorem C.0.14). By the uniqueness of solution, we have $z(t) = \bar{x}(t)$ for all $t \in [s_0, \omega(s_0, x_0))$ and, by the fact that $\bar{x}$ is a maximal solution, $\omega(s_0, x_0) \leq \omega(\tau_0, \psi)$. Consequently,

$$x(t - (s - s_0)) = \bar{x}(t), \quad \text{for all } t \in [s, \omega(s_0, x_0)). \tag{6.47}$$

Now, for all $t \geq s_0$, we have $t + s - s_0 \geq t + s - t = s$. Then, by (6.47), we obtain

$$x(t) = x(t + (s - s_0) - (s - s_0)) = \bar{x}(t + (s - s_0)), \quad \text{for all } t \in [s_0, \omega(s_0, x_0)). \tag{6.48}$$

Notice that, since $t \geq s_0$ and $s \geq \tau_0$, we have $t + s \geq s_0 + \tau_0$ and, hence, $t + (s - s_0) \geq \tau_0$. Then, $\bar{x}(t + (s - s_0))$ is well-defined for all $t \in [s_0, \omega(s_0, x_0))$.

On the other hand, since the trivial solution of the retarded VS integral equation (6.2) is uniformly stable, there exists $\delta = \delta(\varepsilon) > 0$, such that, if $\|\psi\|_\infty < \delta$, then

$$\|y_t\|_\infty < \frac{\varepsilon}{3}, \quad \text{for all } t \in [\tau_0, \omega(\tau_0, \psi)). \tag{6.49}$$

Furthermore, it is not difficult to see that $\|\psi\|_\infty = \|\bar{x}_0\|_\infty \leq \|x_0\|_\infty$.

Let $t \in [\tau_0, \omega(\tau_0, \psi))$ and consider the following two cases.

Case 1: $t - r < \tau_0$ and $\|x_0\|_\infty < \delta$.

Then,

$$\begin{aligned} \|\bar{x}(t)\|_\infty &= \sup_{\theta\in[\tau_0-r,t]} \|y(\theta)\| \leq \sup_{\theta\in[\tau_0-r,\tau_0]} \|y(\theta)\| + \sup_{\theta\in[\tau_0,t]} \|y(\theta)\| \\ &\leq \sup_{\theta\in[-r,0]} \|y_{\tau_0}\|_\infty + \sup_{\theta\in[t-r,t]} \|y(\theta)\| \\ &\leq \sup_{\theta\in[-r,0]} \|y_{\tau_0}\|_\infty + \sup_{\theta\in[-r,0]} \|y_t\|_\infty \\ &\overset{(6.49)}{<} 2\frac{\varepsilon}{3} < \varepsilon. \end{aligned} \tag{6.50}$$

Case 2: $\tau_0 \leq t - r$ and $\|x_0\|_\infty < \delta$.

In this case, for all $\theta \in [\tau_0, t-r]$, we get

$$\|y(\theta)\| \leq \|y_\theta\|_\infty < \frac{\varepsilon}{3},$$

which implies

$$\sup_{\theta\in[\tau_0,t-r]} \|y(\theta)\| < \frac{\varepsilon}{3}. \tag{6.51}$$

Consequently,

$$\begin{aligned} \|\bar{x}(t)\|_\infty &= \sup_{\theta\in[\tau_0-r,t]} \|y(\theta)\| \\ &\leq \sup_{\theta\in[\tau_0-r,\tau_0]} \|y(\theta)\| + \sup_{\theta\in[\tau_0,t-r]} \|y(\theta)\| + \sup_{\theta\in[t-r,t]} \|y(\theta)\| \\ &= \|y_{\tau_0}\|_\infty + \sup_{\theta\in[\tau_0,t-r]} \|y(\theta)\| + \|y_t\|_\infty \\ &\overset{(6.49),\,(6.51)}{<} 3\frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{6.52}$$

Therefore, equations (6.50) and (6.52) yield

$$\|\bar{x}(t)\|_\infty \leq \varepsilon, \quad \text{for all } t \in [\tau_0, \omega(\tau_0, \psi)),$$

whenever $\|x_0\|_\infty < \delta$ and, by (6.48), we conclude

$$\|x(t)\|_\infty = \|\bar{x}(t + (s - s_0))\|_\infty \leq \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0, x_0)),$$

whenever $\|x_0\|_\infty < \delta$. Therefore, the trivial solution of the generalized ODE (6.9) is uniformly stable.

The proof of item (ii) is analogous to the proof of item (i) and, hence, we omit it here. □

Henceforward, we assume that, for all $s_0 \geq t_0$ and all $\psi \in S$, there exists a unique maximal solution $\bar{y} : [s_0 - r, \omega(s_0, \psi)) \to X$ of the perturbed retarded VS integral equation (6.5) with initial condition $\bar{y}_{s_0} = \psi$.

In what follows, we describe when the trivial of the retarded VS integral equation (6.2) is stable with respect to perturbations.

Definition 6.2.4. The trivial solution of the retarded VS integral equation (6.2) is said to be

(i) *stable with respect to perturbations* if, for all $s_0 \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $\psi \in S$ satisfies

$$\|\psi\|_\infty = \sup_{\theta\in[-r,0]} \|\psi(\theta)\| < \delta \quad \text{and} \quad \sup_{s\in[s_0,+\infty)} \left\| \int_{s_0}^{s} p(s)dv(s) \right\| < \delta,$$

then

$$\|\bar{y}_t\|_\infty = \sup_{\theta\in[-r,0]} \|\bar{y}(\theta+t)\| < \varepsilon, \quad \text{for all } t \geq s_0,$$

where $\bar{y} : [s_0 - r, \omega(s_0, \psi)) \to X$ is a maximal solution of the perturbed retarded VS integral equation (6.5) with initial condition $\bar{y}_{s_0} = \psi$;

(ii) *uniformly stable with respect to perturbations*, if it is stable with respect to perturbations with δ independent of s_0;

(iii) *uniformly asymptotically stable with respect to perturbations*, if there exists $\delta_0 > 0$ such that for all $\varepsilon > 0$ and all $s_0 \geq t_0$, there exists $T = T(\varepsilon) \geq 0$ such that if $\psi \in S$ satisfies

$$\|\psi\|_\infty = \sup_{\theta\in[-r,0]} \|\psi(\theta)\| < \delta,$$

then

$$\|\bar{y}_t(s_0,\psi)\|_\infty = \sup_{\theta\in[-r,0]} \|\bar{y}(\theta+t)\| < \varepsilon, \quad \text{for all } t \in [s_0, \omega(s_0,\psi)) \cap [s_0+T, +\infty),$$

where $\bar{y} : [s_0 - r, \omega(s_0, \psi)) \to X$ is a maximal solution of the perturbed retarded VS integral equation (6.5) with initial condition $\bar{y}_{s_0} = \psi$.

The next result relates the concepts above to the corresponding concepts for the trivial solution of the generalized ODE (6.9), presented in Definition 2.1.4. Its proof follows by using Theorem 6.1.7 instead of Theorem 6.1.5 in Theorem 6.2.2 and noticing that, if P is defined by (6.41), then

$$\sup_{s\in[s_0,+\infty)} \|P(s) - P(s_0)\|_\infty = \sup_{s\in[s_0,+\infty)} \left\| \int_{s_0}^{s} p(s)dv(s) \right\|.$$

Therefore, we omit it here.

Theorem 6.2.5. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). The following statements hold.

(i) If the trivial solution of the generalized ODE (6.9) is stable with respect to perturbations, then the trivial solution of the retarded VS integral equation (6.2) is stable with respect to perturbations.

(ii) The trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations, then the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations.

(iii) The trivial solution of the generalized ODE (6.9) is uniformly asymptotically stable with respect to perturbations, then the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable with respect to perturbations.

In the next result, we are assuming that, for all $s_0 \geq t_0$ and $\psi \in S$, there exist global forward solutions of the retarded VS integral equation (6.2) and of the perturbed retarded VS integral equation (6.5) with same initial condition ψ evaluated at s_0. Its proof is analogous to the proof of Theorem 2.2.3. Therefore, we omit it here.

Theorem 6.2.6. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). Then, the following assertions hold.

(i) If the trivial solution of the retarded VS integral equation (6.2) is stable with respect to perturbations, then it is stable.

(ii) If the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations, then it is uniformly stable.

(iii) If the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable with respect to perturbations, then it is uniformly asymptotically stable.

In order to show that uniform stability for the trivial solution of the retarded VS integral equation (6.2) implies uniform stability with respect to perturbations, we introduce the following conditions:

($\mathscr{A}3$) there exists a Perron-Stieltjes integrable function $\mathscr{M} : [t_0, +\infty) \to \mathbb{R}$ with respect to g such that
$$\left\| \int_{s_1}^{s_2} f(y_s, s) dg(s) \right\| \leq \int_{s_1}^{s_2} \mathscr{M}(s) dg(s),$$
for all $y \in O$ and all $s_1, s_2 \in [t_0, +\infty)$;

($\mathscr{A}4$) there exists a Perron-Stieltjes integrable function $\mathscr{L} : [t_0, +\infty) \to \mathbb{R}$ with respect to g such that
$$\left\| \int_{s_1}^{s_2} [f(y_s, s) - f(z_s, s)] dg(s) \right\| \leq \int_{s_1}^{s_2} \mathscr{L}(s) \|y_s - z_s\|_\infty dg(s),$$
for all $y, z \in O$ and all $s_1, s_2 \in [t_0, +\infty)$.

In the sequel, we point out that if $f : S\times[t_0,+\infty) \to X$ satisfies (A2), ($\mathscr{A}3$) and ($\mathscr{A}4$), then F, defined by (6.8), satisfies an useful property.

Remark 6.2.7. Assume that $f : S\times[t_0,+\infty) \to X$ satisfies (A2), ($\mathscr{A}3$) and ($\mathscr{A}4$) and define $\widetilde{h} : [t_0,+\infty) \to \mathbb{R}$ by

$$\widetilde{h}(t) = \int_{s_1}^{s_2} [\mathscr{M}(s)+\mathscr{L}(s)]dg(s), \quad t \in [t_0,+\infty). \tag{6.53}$$

Then, $\widetilde{h}$ is a nondecreasing function which is left-continuous on $(t_0,+\infty)$ and F, defined by (6.8), belongs to the class $\mathscr{F}(\Omega,\widetilde{h})$ (the proof of these facts are analogous to the proof of Lemma 6.1.2). Furthermore, since the integrals in conditions ($\mathscr{A}3$) and ($\mathscr{A}4$) are in the sense of Perron-Stieltjes, $\widetilde{h}$ is bounded.

The next theorem shows that with conditions ($\mathscr{A}3$) and ($\mathscr{A}4$), uniform stability with respect to perturbations implies uniform stability.

Theorem 6.2.8. Let S be given by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S\times[t_0,+\infty) \to X$ satisfies (A2), ($\mathscr{A}3$), ($\mathscr{A}4$) and the functions $p : [t_0,+\infty) \to X$ and $v : [t_0,+\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). The following statements hold.

(i) If the trivial solution of the retarded VS integral equation (6.2) is uniformly stable, then it is uniformly stable with respect to perturbations.

(ii) If the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable, then it is uniformly asymptotically stable with respect to perturbations.

Proof. Once the proof of item (ii) is similar to the proof of item (i), we only prove item (i).

Assume that the trivial solution of the retarded VS integral equation (6.2) is uniformly stable. By Theorem 6.2.13, the trivial solution of the generalized ODE (6.9) is uniformly stable.

On the other hand, since $f : S\times[t_0,+\infty) \to X$ satisfies conditions ($\mathscr{A}3$) and ($\mathscr{A}4$), F defined by (6.8), belongs to the class $\mathscr{F}(\Omega,\widetilde{h})$, where $\widetilde{h}$ is bounded (see Remark 6.2.7). Then, by Theorem 2.2.7, the trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations and, by Theorem 6.2.5-(ii), the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations. □

As a consequence of Theorems 6.2.6 and 6.2.8, we have the following result.

Theorem 6.2.9. Let S be given by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S\times[t_0,+\infty) \to X$ satisfies (A2), ($\mathscr{A}3$), ($\mathscr{A}4$) and the functions $p : [t_0,+\infty) \to X$ and $v : [t_0,+\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). The following statements hold.

(i) If the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations, then the trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations.

(ii) If the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable with respect to perturbations, then the trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations.

Proof. Let us start by proving item (i). By Theorem 6.2.6-(ii), the trivial solution of the retarded VS integral equation (6.2) is uniformly stable. On the other hand, it is clear that if a function $f: S\times[t_0,+\infty)\to X$ satisfies conditions ($\mathscr{A}3$) and ($\mathscr{A}4$), then it satisfies conditions (A3) and (A4). Therefore, by Theorem 6.2.6-(ii), the trivial solution of the generalized ODE (6.9) is uniformly stable and, by Theorem 2.2.4-(i), it is uniformly stable with respect to perturbations, once $\widetilde{h}$ from the class $\mathscr{F}(\Omega,\widetilde{h})$ is bounded (see Remark 6.2.7).

The proof of item (ii) is analogous and, therefore, we omit it here. □

In the sequel, we present sufficient conditions for the retarded VS integral equation (6.2) to be uniformly stable with respect to perturbations.

Theorem 6.2.10. Let S be given by (6.1). Assume that $g:[t_0,+\infty)\to\mathbb{R}$ satisfies condition (A1), $f: S\times[t_0,+\infty)\to X$ satisfies (A2), ($\mathscr{A}3$), ($\mathscr{A}4$) and the functions $p:[t_0,+\infty)\to X$ and $v:[t_0,+\infty)\to\mathbb{R}$ satisfy (A5), (A6) and (A7). Then, trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations.

Proof. By Remark 6.2.13, the function $\widetilde{h}$ from the class $\mathscr{F}(\Omega,\widetilde{h})$ is bounded and, by Corollary 4.0.3, the trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations. Then, by Theorem 6.2.5-(ii), the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations. □

We end this subsection by presenting the definition of integral stability for the retarded VS integral equation (6.2), contained in [26], and a relation between this stability and the concept of regular stability for the generalized ODE (6.9) (see Definitions 2.1.4 and 2.1.1).

Definition 6.2.11. Let $[\alpha,\beta]\subset[t_0,+\infty)$ and $\psi\in S$. The trivial solution $y\equiv 0$ of (6.2) is called

(i) *integrally stable*, if for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that if

$$\|\psi\|_\infty=\sup_{\theta\in[-r,0]}\|\psi(\theta)\|<\delta \quad\text{and}\quad \sup_{\theta\in[\alpha,\beta]}\left\|\int_\alpha^\theta p(s)dv(s)\right\|<\delta,$$

then

$$\|\bar{y}_t(\alpha,\psi)\|_\infty=\sup_{\theta\in[-r,0]}\|\bar{y}(\theta+t)\|<\varepsilon,\ \text{for every } t\in[\alpha,\beta],$$

where $\bar{y}(\cdot,\alpha,\psi)$ is a solution of the perturbed retarded VS integral equation (6.5) with $\bar{y}_\alpha = \psi$;

(ii) *integrally attracting*, if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if

$$\|\psi\|_\infty = \sup_{\theta\in[-r,0]} \|\psi(\theta)\| < \delta \quad \text{and} \quad \sup_{\theta\in[\alpha,\beta]} \left\| \int_\alpha^\theta p(s)dv(s) \right\| < \delta,$$

then

$$\|\bar{y}_t(\alpha,\psi)\|_\infty = \sup_{\theta\in[-r,0]} \|\bar{y}(\theta+t)\| < \varepsilon, \text{ for all } t \geq \alpha + T, t \in [\alpha,\beta],$$

where $\bar{y}(\cdot,\alpha,\psi)$ is a solution of the perturbed retarded VS integral equation (6.5) with $\bar{y}_\alpha = \psi$;

(iii) *integrally asymptotically stable*, if it is integrally stable and integrally attracting.

Remark 6.2.12. It is clear that if the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations, then it is integrally stable. Moreover, if the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable with respect to perturbations, then it is integrally asymptotically stable.

The proof of the next result can be found in [26, Theorem 5.1].

Theorem 6.2.13. Let S be given by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S\times[t_0,+\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0,+\infty) \to X$ and $v : [t_0,+\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). Then, the following statements hold.

(i) If the trivial solution of the generalized ODE (6.9) is regularly stable, then the trivial solution of the retarded VS integral equation (6.2) is integrally stable.

(ii) If the trivial solution of the generalized ODE (6.9) is regularly attracting, then the trivial solution of the retarded VS integral equation (6.2) is integrally attracting.

(iii) If the trivial solution of the generalized ODE (6.9) is regularly asymptotically stable, then the trivial solution of the retarded VS integral equation (6.2) is integrally asymptotically stable.

In what follows, we prove the inverse implications of Theorem 6.2.13.

Theorem 6.2.14. Let S be given by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S\times[t_0,+\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0,+\infty) \to X$ and $v : [t_0,+\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). Then, the following statements hold.

(i) If the trivial solution of the retarded VS integral equation (6.2) is integrally stable, then the trivial solution of the generalized ODE (6.9) is regularly stable.

(ii) If the trivial solution of the retarded VS integral equation (6.2) is integrally attracting, then the trivial solution of the generalized ODE (6.9) is regularly attracting.

(iii) If the trivial solution of the retarded VS integral equation (6.2) is integrally asymptotically stable, then the trivial solution of the generalized ODE (6.9) is regularly asymptotically stable.

Proof. Since the proof of items (ii) and (iii) are similar to the proof of item (i), we omit them here.

Let $\varepsilon > 0$, $[\alpha,\beta] \subset [t_0,+\infty)$, $P \in G^{-}([\alpha,\beta], BG([t_0-r,+\infty),X))$ and $x_0 \in \mathbb{O}$. By the definition of $\mathbb{O}$, there exist $y \in O$ and $s, s_0 \in [t_0,+\infty)$, with $s \geq s_0$, such that

$$x_0(\theta) = \begin{cases} y(s_0-r), & t_0-r \leq \theta \leq s_0-r,\\ y(\theta), & s_0-r \leq \theta \leq s,\\ y(s), & s \leq \theta < +\infty. \end{cases}$$

Define $\overline{P} : [t_0,+\infty) \to BG([t_0-r,+\infty),X)$ by

$$\overline{P}(t)(\theta) = \begin{cases} 0, & t_0-r \leq \theta \leq t_0,\\ \displaystyle\int_{t_0}^{\theta} p(s)dv(s), & t_0 \leq \theta \leq t,\\ \displaystyle\int_{t_0}^{t} p(s)dv(s), & t \leq \theta < +\infty. \end{cases}$$

Since $v : [t_0,+\infty) \to \mathbb{R}$ is left-continuous, by Theorem B.0.17, P is a regulated function which is left-continuous on $(t_0,+\infty)$.

Using the ideas from the proof of Theorem 6.2.3, the function $\widetilde{x} : [s_0,\omega) \to \mathbb{O}$, given by

$$\widetilde{x}(t)(\theta) = \begin{cases} y(s_0-r), & t_0-r \leq \theta \leq s_0-r,\\ y(\theta), & s_0-r \leq \theta \leq t,\\ y(t), & t \leq \theta < +\infty, \end{cases}$$

is the solution of the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t)+\overline{P}(t)] \tag{6.54}$$

with initial condition $\widetilde{x}(s_0) = \widetilde{x}_0$, where $y : [s_0-r,\omega) \to X$ is the solution of the perturbed retarded VS equation (6.5) with initial condition $y_{s_0} = \psi$ and $\widetilde{x}_0$ is given by

$$\widetilde{x}_0(\theta) = \begin{cases} \psi(-r), & t_0-r \leq \theta \leq s_0-r,\\ \psi(\theta-s_0), & s_0-r \leq \theta \leq s_0,\\ \psi(0), & s_0 \leq \theta < +\infty, \end{cases}$$

Furthermore, if $\overline{x} : [\alpha,\beta] \to \mathbb{O}$ is the solution of the perturbed generalized ODE (6.54) with initial condition $\overline{x}(\alpha) = x_0$, then $\beta < \omega$, $\|\widetilde{x}_0\|_\infty = \|\psi\|_\infty \leq \|x_0\|_\infty$ and

$$\|\overline{x}(t)\|_\infty = \|\widetilde{x}(t+(s-\alpha))\|_\infty, \quad \text{for all } t \in [\alpha,\beta]. \tag{6.55}$$

Since the retarded VS integral equation (6.2) is integrally stable, there exists $\delta > 0$ such that, if

$$\|\psi\|_\infty < \delta \quad \text{and} \quad \sup_{\theta\in[\alpha,\beta]} \left\| \int_\alpha^\theta p(s)dv(s) \right\| < \delta, \tag{6.56}$$

then,

$$\|y_t\|_\infty < \frac{\varepsilon}{6}, \quad \text{for all } t \in [\alpha,\beta]. \tag{6.57}$$

Define $G : [\alpha,\beta] \to BG([t_0 - r, +\infty), X)$ by $G(t) = P(t) - \overline{P}(t)$, for all $t \in [\alpha,\beta]$ and assume that

$$\|x_0\|_\infty < \frac{\delta}{4e^H}, \quad \sup_{\theta\in[\alpha,\beta]} \|P(\theta) - P(\alpha)\|_\infty < \frac{\delta}{4e^H} \quad \text{and} \quad \sup_{\theta\in[\alpha,\beta]} \left\| \int_\alpha^\theta p(s)dv(s) \right\| < \frac{\delta}{4e^H},$$

where $H = \sup_{s\in[\alpha,\beta]} |h(s) - h(\alpha)|$.

Notice that, for all $t \in [\alpha,\beta]$, we have

$$[\overline{P}(t) - \overline{P}(\alpha)](\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq \alpha, \\ \displaystyle\int_\alpha^\theta p(s)dv(s), & \alpha \leq \theta \leq t, \\ \displaystyle\int_\alpha^t p(s)dv(s), & t \leq \theta < +\infty, \end{cases}$$

and, hence,

$$\sup_{\theta\in[\alpha,\beta]} \|\overline{P}(\theta) - \overline{P}(\alpha)\|_\infty = \sup_{\theta\in[\alpha,\beta]} \left\| \int_\alpha^\theta p(s)dv(s) \right\|, \tag{6.58}$$

$$\|G(t) - G(\alpha)\|_\infty \leq \|P(t) - P(\alpha)\|_\infty + \|\overline{P}(t) - \overline{P}(\alpha)\|_\infty < \frac{\delta}{2e^H} < \infty$$

and, consequently,

$$\sup_{\theta\in[\alpha,\beta]} \|G(\theta) - G(\alpha)\|_\infty < \frac{\delta}{2e^H} < \infty. \tag{6.59}$$

Let $x : [\alpha,\beta] \to \mathbb{O}$ be the solution of the perturbed generalized ODE (6.41) with initial condition $x(\alpha) = x_0$. Then, for all $t \in [\alpha,\beta]$, we have

$$\begin{aligned} \|x(t) - \overline{x}(t)\|_\infty &\leq \left\| \int_\alpha^t DF(x(\tau),s) - \int_\alpha^t DF(\overline{x}(\tau),s) \right\|_\infty + \|G(t) - G(\alpha)\|_\infty \\ &\leq 2|h(t) - h(\alpha)| + \frac{\delta}{2e^H} < 2H + \frac{\delta}{2e^H} < \infty. \end{aligned}$$

Therefore, the function $[\alpha,\beta] \ni t \to \|x(t) - \overline{x}(t)\|_\infty$ is bounded and, by Lemma C.0.6, for all $t \in [\alpha,\beta]$, we get

$$\|\overline{x}(t) - x(t)\|_\infty \leq \int_{s_0}^t \|\overline{x}(s) - x(s)\|_\infty dh(s) + \sup_{\theta\in[\alpha,\beta]} \|G(\theta) - G(\alpha)\|_\infty, \quad \text{for all } t \in [\alpha,\beta].$$

Furthermore, by the Gronwall-type inequality (see Theorem B.0.10), we have

$$\begin{aligned} \|x(t)\|_\infty &\leq \|\overline{x}(t)\|_\infty + \|x(t) - \overline{x}(t)\|_\infty \leq \|\overline{x}(t)\|_\infty + \sup_{\theta\in[\alpha,\beta]} \|G(\theta) - G(\alpha)\|_\infty e^H \\ &< \|\overline{x}(t)\|_\infty + \frac{\delta}{2}. \end{aligned} \tag{6.60}$$

On the other hand, by (6.56) and (6.57) (see Cases 1 and 2 in the proof of Theorem 6.2.3), we conclude

$$\|\widetilde{x}(t)\|_\infty < \frac{\varepsilon}{2}, \quad \text{for all } t \in [s_0, \omega) \tag{6.61}$$

since $\|\psi\|_\infty = \|\widetilde{x}_0\|_\infty \leq \|x_0\|_\infty < \frac{\delta}{4e^H}$. By equations (6.55) and (6.61), we get

$$\|\overline{x}(t)\|_\infty = \|\widetilde{x}(t+(s-\alpha))\|_\infty < \frac{\varepsilon}{2}, \quad \text{for all } t \in [\alpha, \beta]. \tag{6.62}$$

By equations (6.60) and (6.62) and the fact that $\delta < \varepsilon$, we obtain

$$\|x(t)\|_\infty \leq \varepsilon, \quad \text{for all } t \in [\alpha, \beta],$$

which shows that the trivial solution of the generalized ODE (6.9) is regular stable with respect to perturbations and, by Theorem 2.1.5, it is regularly stable. □

6.2.2 Direct method of Lyapunov

This subsection is devoted to proving Lyapunov-type theorems on uniform stability and integral stability for the retarded VS integral equation (6.2).

At first, we introduce a concept of Lyapunov functional with respect to the retarded VS integral equation (6.2) (see also [26, Definition 7.1]). To this end, we recall that $\mathbb{R}^+$ denotes the set of non-negative real numbers.

Definition 6.2.15. Let $E \subset S$, where S is given by (6.1). We say that $U : [t_0, +\infty) \times E \to \mathbb{R}$ is a *Lyapunov functional with respect to the retarded VS integral equation* (6.2), if the following conditions are satisfied:

(i) $U(\cdot, \psi) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$, for all $\psi \in E$;

(ii) there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ such that

$$U(t, \psi) \geq b(\|\psi\|),$$

for every $(t, \psi) \in [t_0, +\infty) \times E$;

(iii) for every $t \geq t_0$ and $\psi \in E$,

$$\dot{U}(t, \psi) := \limsup_{\eta \to 0^+} \frac{U(t+\eta, y_{t+\eta}(t, \psi)) - U(t, y_t(t, \psi))}{\eta} \leq 0$$

holds, where $y(t, \psi)$ denotes the maximal solution of the retarded VS integral equation (6.2) with initial condition $y_t = \psi$ and the functions $y_{t+\eta}, y_t : [-r, 0] \to X$ are defined by $y_{t+\eta}(\theta) = y(t+\eta+\theta)$ and $y_t(\theta) = y(t+\theta)$ for all $\theta \in [-r, 0]$.

In the sequel, we present a Lyapunov-type theorem on uniform stability for the retarded VS integral equation (6.2). Such a result weakens the Lipschitzian condition on the Lyapunov functional found in [26, Theorem 7.3] and its proof follows the same ideas of the proof of [23, Theorem 3.4].

Theorem 6.2.16. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). Let $U : [t_0, +\infty) \times S \to \mathbb{R}$ be a Lyapunov functional with respect to the retarded VS integral equation (6.2). Suppose U satisfies the following conditions:

(LUV1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, and

$$U(t, \psi) \leq a(\|\psi\|_\infty),$$

for all $(t, \psi) \in [t_0, +\infty) \times S$;

(LUV2) for all $s_0 \geq t_0$, the function $[s_0, \omega(s_0, \psi)) \ni t \mapsto U(t, y_t(s_0, \psi))$ is nonincreasing along every maximal solution $y : [s_0 - r, \omega(s_0, \psi)) \to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi \in S$.

Then, the trivial solution of the retarded VS integral equation (6.2) is uniformly stable.

Proof. By the definition of Lyapunov functional, there exists an increasing continuous function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ and

$$b(\|\psi\|_\infty) \leq U(t, \psi), \quad \text{for all } (t, \psi) \in [t_0, +\infty) \times S. \tag{6.63}$$

Let $s_0 \geq t_0$ and $\varepsilon > 0$. Consider $A = \{a(t); t \in \mathbb{R}^+\}$. Since $a(0) = 0$ and a is increasing, $\inf A = 0$. Then, by the property of the infimum, for $b(\varepsilon) > 0$, there exists $\delta > 0$ such that

$$0 < a(\delta) < b(\varepsilon). \tag{6.64}$$

Let $\psi \in S$ and $y : [s_0 - r, \omega(s_0, \psi)) \to X$ be a maximal solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$. Assume that $\|\psi\|_\infty < \delta$. By the fact that a is increasing, we have

$$a(\|\psi\|_\infty) < a(\delta). \tag{6.65}$$

Then, by condition (LUV2) and equations (6.63), (6.64) and (6.65), we obtain

$$b(\|y_t\|_\infty) \leq U(t, y_t) \leq U(t, \psi) \leq a(\|\psi\|_\infty) < a(\delta) < b(\varepsilon),$$

for all $t \in [s_0, \omega(s_0, \psi))$. Once b is increasing, we conclude $\|y_t\|_\infty < \varepsilon$, for all $t \in [s_0, \omega(s_0, \psi))$ and the proof is complete. □

In the next lines, we establish a Lyapunov-type theorem on uniform asymptotic stability for the retarded VS integral equation (6.2). Its proof follows the same ideas as those in Theorem 2.1.9, and therefore, we omit it here.

Theorem 6.2.17. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). Let $U : [t_0, +\infty) \times S \to \mathbb{R}$ be a Lyapunov functional with respect to the retarded VS integral equation (6.2). Suppose U satisfies conditions (LUV1) and (LUV2) from Theorem 6.2.16. Furthermore, assume that there exists a continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ for which $\Phi(0) = 0$, $\Phi(t) > 0$ whenever $t \neq 0$ and

$$D^+U(t, \psi) \leq -\Phi(\|\psi\|_\infty),$$

for all $(t, \psi) \in [t_0, +\infty) \times S$. Then, the trivial solution of the retarded VS integral equation (6.2) is uniformly asymptotically stable.

In what follows, we obtain a Lyapunov-type theorem on integral stability for the retarded VS integral equation (6.2).

Theorem 6.2.18. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), ($\mathscr{A}3$) and ($\mathscr{A}4$). Let $U : [t_0, +\infty) \times S \to \mathbb{R}$ be a Lyapunov functional with respect to the retarded VS integral equation (6.2). Suppose U satisfies conditions (LUV1) and (LUV2) from Theorem 6.2.16. Then, the trivial solution of the retarded VS integral equation (6.2) is integrally stable.

Proof. Once $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), ($\mathscr{A}3$) and ($\mathscr{A}4$), it satisfies conditions (A2), (A3) and (A4). Then, by Theorem 6.2.16, the trivial solution of the retarded VS integral equation (6.2) is uniformly stable. Moreover, by Theorem 6.2.8-(i), the trivial solution of the retarded VS integral equation (6.2) is uniformly stable with respect to perturbations and, by Remark 6.2.12, it is integrally stable. □

We end this section by presenting a Lyapunov-type theorem on integral asymptotic stability for the retarded VS integral equation (6.2). Its proofs is analogous to that of Theorem 6.2.18.

Theorem 6.2.19. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), ($\mathscr{A}3$) and ($\mathscr{A}4$). Let $U : [t_0, +\infty) \times S \to \mathbb{R}$ be a Lyapunov functional with respect to the retarded VS integral equation (6.2). Suppose U satisfies all conditions from Theorem 6.2.17. Then, the trivial solution of the retarded VS integral equation (6.2) is integrally asymptotically stable.

6.2.3 Converse Lyapunov theorems

In this subsection, we prove converse Lyapunov theorems on uniform stability and integral stability for the trivial solution of the retarded VS integral equation (6.2).

At first, we show that the existence of a Lyapunov functional with respect to the generalized ODE (6.9) (see Definition 2.0.2) implies the existence of a Lyapunov functional with respect to the retarded VS integral equation (6.2) (see Definition 6.2.15). This result plays an important role in the proof of the converse Lyapunov theorems for the retarded VS integral equation (6.2).

Theorem 6.2.20. Consider $\mathbb{O} \subset BG([t_0 - r, +\infty), X)$ given by (6.7) and S by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). If there exists a Lyapunov functional $V : [t_0, +\infty) \times \mathbb{O} \to \mathbb{R}$ with respect to the generalized ODE (6.9), then there exists a Lyapunov functional $U : [t_0, +\infty) \times S \to \mathbb{R}$ with respect to the retarded VS integral equation (6.2).

Proof. Given $t \geq t_0$ and $\psi \in S$, let $y\colon [t - r, \omega(t, \psi)) \to X$ be a solution of the retarded VS integral equation (6.2) with initial condition $y_t = \psi$. Then, by Theorem 6.1.5-(i), the function $x\colon [t, \omega(t, \psi)) \to \mathbb{O}$, defined by

$$x(\tau)(\theta) = \begin{cases} y(t-r), & t_0 - r \leq \theta \leq t - r, \\ y(\theta), & t - r \leq \theta \leq \tau, \\ y(\tau), & \tau \leq \theta < +\infty \end{cases} \tag{6.66}$$

is the solution of a generalized ODE (6.9) with initial condition $x(t) = \widetilde{x}$, where

$$\widetilde{x}(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq t - r, \\ \psi(\theta - t), & t - r \leq \theta \leq t, \\ \psi(0), & t \leq \theta < +\infty \end{cases} \tag{6.67}$$

In this case, we write $x_\psi(\tau)$ instead of $x(\tau)$ for all $\tau \in [t, \omega(t, \psi))$. Besides,

$$\|x_\psi(t)\|_\infty = \sup_{\theta \in [t_0 - r, +\infty)} \|x_\psi(t)(\theta)\| = \sup_{\theta \in [t-r, t]} \|y(\theta)\| = \sup_{\theta \in [-r, 0]} \|y_t(\theta)\| = \|\psi\|_\infty \tag{6.68}$$

and, for all $\tau \in [t, \omega(t, \psi))$ and $\theta \in [-r, 0]$, we have

$$(x(\tau))_\tau(\theta) = x(\tau)(\tau + \theta) = y(\tau + \theta) = y_\tau(\theta). \tag{6.69}$$

Define $U : [t_0, +\infty) \times S \to \mathbb{R}$ by

$$U(t, \psi) = V(t, x_\psi(t)), \quad \text{for all } t \in [t_0, +\infty),\ \psi \in S,$$

where $V : [t_0, +\infty) \times \mathbb{O} \to \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (6.9).

Owing to the fact that there exists a one-to-one correspondence between $\psi \in S$ and $x_\psi \in \mathbb{O}$, U is well-defined in $[t_0, +\infty) \times S$ and, by the left-continuity of $V(\cdot, x_\psi(t)) : [t_0, +\infty) \to \mathbb{R}$ on $(t_0, +\infty)$, $U(\cdot, \psi) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ for all $\psi \in S$, which implies condition (i) from Definition 6.2.15. Moreover, by condition (L2) from Definition 2.0.2, we have

$$U(t, \psi) = V(t, x_\psi(t)) \geq b(\|x_\psi(t)\|_\infty) \overset{(6.68)}{=} b(\|\psi\|_\infty), \quad \text{for all } (t, \psi) \in [t_0, +\infty) \times S,$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function such that $b(0) = 0$.

We target to prove that U satisfies condition (iii) from Definition 6.2.15. Consider the solution $y\colon [t-r,\omega(t,\psi)) \to X$ of the retarded VS integral equation (6.2) with initial condition $y_t = \psi$ and let $x\colon [t,\omega(t,\psi)) \to \mathbb{O}$ be the solution of the generalized ODE (6.9) given by (6.66). By (6.69), for all $\eta > 0$ for which $t+\eta < \omega(t,\psi)$, we have $y_{t+\eta}(t,\psi) = (x_\psi(t+\eta))_{t+\eta}$. By the definition of U and the fact that V satisfies condition (L3) of Definition 2.0.2, we have

$$\begin{aligned} &\limsup_{\eta\to 0^+} \frac{U(t+\eta, y_{t+\eta}(t,\psi)) - U(t, y_t(t,\psi))}{\eta} \\ = &\limsup_{\eta\to 0^+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta} \leq 0, \end{aligned}$$

which concludes the proof. □

The next result is a converse Lyapunov theorem on integral stability for the retarded VS integral equation (6.2).

Theorem 6.2.21. Let S be given by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies (A1) and $f : S \times [t_0,+\infty) \to X$ satisfies (A2), (A3) and (A4). If the trivial solution of the retarded VS integral equation (6.2) is integrally stable, then there exists a Lyapunov functional $U : [t_0,+\infty) \times S \to \mathbb{R}$ with respect to the retarded VS integral equation (6.2) satisfying

(CLIV1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, and

$$U(t,\psi) \leq a(\|\psi\|_\infty),$$

for all $(t,\psi) \in [t_0,+\infty) \times S$;

(CLIV2) for all $s_0 \geq t_0$, the function $[s_0,\omega(s_0,\psi)) \ni t \mapsto U(t, y_t(s_0,\psi))$ is nonincreasing along every maximal solution $y : [s_0 - r, \omega(s_0,\psi)) \to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi \in S$.

Proof. By Theorem 6.2.14-(i), the trivial solution of the generalized ODE (6.9) is regularly stable and, by Theorem 2.1.19, there exists a Lyapunov functional $V : [t_0,+\infty) \times \mathbb{O} \to \mathbb{R}$ with respect to the generalized ODE (6.9). By Theorem 6.2.20, the function $U : [t_0,+\infty) \times S \to \mathbb{R}$ by

$$U(t,\psi) = V(t,x(t)), \quad \text{for all } t \in [t_0,+\infty),\ \psi \in S,$$

is a Lyapunov functional with respect to the retarded VS integral equation (6.2), where $x(t)_t = \psi$ is given by

$$x_\psi(t)(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq t - r, \\ \psi(\theta - t), & t - r \leq \theta \leq t, \\ \psi(0), & t \leq \theta < +\infty. \end{cases}$$

Furthermore, by Theorem 2.1.19-(CLR4), there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, and

$$U(t,\psi) = V(t,x_\psi) \leq a(\|\psi\|_\infty),$$

since $\|x_\psi\|_\infty = \|\psi\|_\infty$. Therefore, the proof of item (CLIV1) is complete.

In order to prove item (CLIV2), let $y : [s_0 - r, \omega(s_0,\psi)) \to X$ be a solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi \in S$. Then, by Theorem 6.1.5-(i), the function $x : [s_0, \omega(s_0,\psi)) \to \mathbb{O}$, given by

$$x(t)(\theta) = \begin{cases} y(s_0 - r), & t_0 - r \leq \theta \leq s_0 - r, \\ y(\theta) & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < +\infty, \end{cases}$$

is a solution of the generalized ODE (6.9) with initial condition $x(s_0) = x_0$, where

$$x_0(\theta) = \begin{cases} \psi(-r), & t_0 - r \leq \theta \leq s_0 - r, \\ \psi(\theta - s_0), & s_0 - r \leq \theta \leq s_0, \\ \psi(s_0), & s_0 \leq \theta < +\infty. \end{cases}$$

Notice that, for all $t \in [s_0, \omega(s_0,\psi))$ and $\theta \in [-r, 0]$, we have

$$x(t)_t(\theta) = x(t)(t+\theta) = y(t+\theta) = y_t(\theta)$$

and, by the definition of U, $U(t, y_t(s_0,\psi)) = V(t, x(t))$. Then, by Theorem 2.1.19-(CLR1), for all $t, s \in [s_0, \omega(s_0,\psi))$, $t > s$, we have

$$U(t, y_t(s_0,\psi)) = V(t, x(t)) < V(s, x(s)) = U(s, y_s(s_0,\psi))$$

which proves item (CLIV2). □

In the sequel, we exhibit a converse Lyapunov theorem on uniform stability for the retarded VS integral equation (6.2).

Theorem 6.2.22. Let S be given by (6.1) and $\overline{E}_\rho = \{\psi \in S; \|\psi\|_\infty \leq \rho\}$, $0 < \rho$. Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0, +\infty) \to X$ satisfies conditions (A2), (A3) and (A4). If the trivial solution of the retarded VS integral equation (6.2) is uniformly stable, then there exists a Lyapunov functional $U : [t_0, +\infty) \times \overline{E}_\rho \to \mathbb{R}$ with respect to the retarded VS integral equation (6.2) satisfying

(CLUV1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$, and

$$U(t,\psi) \leq a(\|\psi\|_\infty),$$

for all $(t,\psi) \in [t_0, +\infty) \times \overline{E}_\rho$;

(CLUV2) for all $s_0 \geq t_0$, the function $[s_0, \omega(s_0,\psi)) \ni t \mapsto U(t, y_t(s_0,\psi))$ is nonincreasing along every maximal solution $y : [s_0 - r, \omega(s_0,\psi)) \to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi \in \overline{E}_\rho$.

Proof. By Theorem 6.2.3-(i), the trivial solution of the generalized ODE (6.9) is uniformly stable and, by the Lyapunov-type theorem on uniform stability for generalized ODEs (see Theorem 2.2.8), there exists a Lyapunov functional $V : [t_0,+\infty)\times\overline{G}_\rho \to \mathbb{R}$ with respect to the generalized ODE, where $\overline{G}_\rho = \{x \in \mathbb{O}; \|x\|_\infty \leq \rho\}$.

Let $\psi \in \overline{E}_\rho$ and $t \in [t_0,+\infty)$. Define $x_\psi(t) : [t_0-r,+\infty) \to X$ by

$$x_\psi(t)(\theta) = \begin{cases} \psi(-r), & t_0-r \leq \theta \leq t-r, \\ \psi(\theta-t), & t-r \leq \theta \leq t, \\ \psi(0), & t \leq \theta < +\infty. \end{cases}$$

Then, $x_\psi(t) \in \mathbb{O}$ and

$$\|x\|_\infty = \sup_{\theta\in[t_0-r,+\infty)} \|x(\theta)\| = \sup_{\theta\in[t-r,t]} \|\psi(\theta-t)\| = \sup_{\theta\in[-r,0]} \|\psi(\theta)\| = \|\psi\|_\infty, \tag{6.70}$$

which implies $x_\psi(t) \in \overline{G}_\rho$.

By Theorem 6.2.20, $U : [t_0,+\infty)\times\overline{E}_\rho \to \mathbb{R}$, defined by

$$U(t,\psi) = V(t,x_\psi(t)), \quad \text{for all } t \in [t_0,+\infty),\ \psi \in \overline{E}_\rho,$$

is a Lyapunov functional with respect to the retarded VS integral equation (6.2).

By conditions (LU1) and (LU2) from Theorem 2.2.8, it is not difficult to see that U satisfies conditions (CLUV1) and (CLUV2). □

6.3 Boundedness of solutions

In this section, we introduce concepts of uniform boundedness of solutions of the retarded VS integral equation (6.2) and of the perturbed retarded VS integral equation (6.5). We also relate these concepts to the concepts of stability, described in Section 6.2.1, and to the concept of boundedness of solutions, as well as stability, for the generalized ODEs (6.9) and (6.41). Moreover, we obtain a Lyapunov-type theorem and applied the results from Chapter 3 to prove a converse Lyapunov theorem on uniform boundedness of solutions of the retarded VS integral equation (6.2).

Throughout this section, we consider S be given by (6.1), $g : [t_0,+\infty) \to \mathbb{R}$ satisfying (A1) and $f : S\times[t_0,+\infty) \to X$ satisfying (A2), (A3) and (A4). Moreover, we assume that, for all $s_0 \geq t_0$ and all $\psi \in S$, there exists a unique maximal solution $y : [s_0-r,\omega(s_0,\psi)) \to X$ of the retarded VS integral equation (6.2) and a unique maximal solution $\overline{y}[s_0-r,\omega(s_0,\psi)) \to X$ of the perturbed retarded VS integral equation (6.2) with $y_{s_0} = \psi = \overline{y}_{s_0}$. In addition, we suppose the function $f : S\times[t_0,+\infty) \to X$ satisfies (6.43).

In the next lines, we give a definition of uniform boundedness of solutions of the retarded VS integral equation (6.2) and of the perturbed retarded VS integral equation (6.5).

Definition 6.3.1. We say that

(i) the retarded VS integral equation (6.2) is *uniformly bounded* if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and for all $\psi \in S$, with

$$\|\psi\|_\infty = \sup_{\theta \in [-r,0]} \|\psi(\theta)\| < \alpha,$$

we have

$$\|y_s\|_\infty = \sup_{\theta \in [-r,0]} \|y(\theta + s)\| < M, \text{ for all } s \geq s_0,$$

where $y : [s_0, \omega(s_0, \psi)) \to X$ is a solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$;

(ii) the perturbed retarded VS integral equation (6.5) is *uniformly bounded* if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and for all $\psi \in S$, with

$$\|\psi\|_\infty = \sup_{\theta \in [-r,0]} \|\psi(\theta)\| < \alpha,$$

we have

$$\|\bar{y}_s\|_\infty = \sup_{\theta \in [-r,0]} \|\bar{y}(\theta + s)\| < M, \text{ for all } s \geq s_0,$$

where $\bar{y} : [s_0, \omega(s_0, \psi)) \to X$ is a solution of the perturbed retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$.

In the sequel, we establish a relation between the concepts above. Its proof follows similar ideas to the proof of Theorem 3.0.6-(i).

Theorem 6.3.2. If the perturbed retarded VS integral equation (6.5) is uniformly bounded, then the retarded VS integral equation (6.2) is uniformly bounded.

The proof of the next result is analogous to the proofs of Theorems 6.2.2 and 6.2.3. Therefore, we omit it here.

Theorem 6.3.3. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), (A3), (A4) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). Then, the following statements hold.

(i) The retarded VS integral equation (6.2) is uniformly bounded if and only if the generalized ODE (6.9) is uniformly bounded.

(ii) The perturbed retarded VS integral equation (6.5) is uniformly bounded if and only if the perturbed generalized ODE (6.41) is uniformly bounded.

The next result relates the existence of a Lyapunov function with respect to the retarded VS integral equation (6.2) to the concept of uniform boundedness of solutions of the retarded VS integral equation (6.2). This result is a version of the Lyapunov theorems on uniform boundedness of solutions.

Theorem 6.3.4. Let S be given by (6.1). Assume that $g:[t_0,+\infty)\to\mathbb{R}$ satisfies condition (A1) and $f:S\times[t_0,+\infty)\to X$ satisfies conditions (A2), (A3) and (A4). Then, the trivial solution of the retarded VS integral equation (6.2) is uniformly bounded if and only if there exists a Lyapunov functional with respect with respect to the retarded VS integral equation (6.2) $U:[t_0,+\infty)\times S\to\mathbb{R}$ satisfying the following conditions:

(CBV1) there exist two increasing continuous functions $p,b:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $p(0)=b(0)=0$,

$$\lim_{s\to+\infty} b(s)=+\infty$$

and

$$b(\|\psi\|_\infty)\leq U(t,\psi)\leq p(\|\psi\|_\infty),$$

for all $(t,\psi)\in[t_0,+\infty)\times S$;

(CBV2) for all $s_0\geq t_0$, the function $[s_0,+\infty)\ni t\mapsto U(t,y_t(s_0,\psi))$ is nonincreasing along every global forward solution $y:[s_0-r,+\infty)\to X$ of the retarded VS integral equation (6.2) with initial condition $y_{s_0}=\psi\in S$.

Proof. Assume that the retarded VS integral equation (6.2) is uniformly bounded. Then, by Theorem 6.3.3-(ii) the generalized ODE (6.9) is uniformly bounded. By Theorem 3.0.3, there exist a function $V:[t_0,+\infty)\times\mathbb{O}\to\mathbb{R}$ with respect to generalized ODE (6.9) and two increasing functions $p,b:\mathbb{R}^+\to\mathbb{R}^+$ such that $p(0)=b(0)=0$,

$$\lim_{s\to+\infty} b(s)=+\infty \tag{6.71}$$

and

$$b(\|x\|_\infty)\leq V(t,x)\leq p(\|x\|_\infty),\quad \text{for all } (t,x)\in[t_0,+\infty)\times\mathbb{O}. \tag{6.72}$$

By Theorem 6.2.20, the function $U:[t_0,+\infty)\times S\to\mathbb{R}$, defined by

$$U(t,\psi)=V(t,x_\psi(t)),\quad \text{for all } t\in[t_0,+\infty),\ \psi\in S, \tag{6.73}$$

is a Lyapunov functional with respect to the retarded VS integral equation (6.2), where $x_\psi(t)$ is given by

$$x_\psi(t)(\theta)=\begin{cases}\psi(-r), & t_0-r\leq\theta\leq t-r,\\ \psi(\theta-t), & t-r\leq\theta\leq t,\\ \psi(0), & t\leq\theta<+\infty\end{cases}$$

Then, it is clear that $\|\psi\|_\infty = \|x_\psi(t)\|_\infty$ and, by equations (6.72) and (6.73), we conclude

$$b(\|\psi\|_\infty) \leq U(t,\psi) \leq p(\|\psi\|_\infty), \quad \text{for all } t \in [t_0,+\infty),\ \psi \in S.$$

Moreover, as in the proof of Theorem 6.2.20, U satisfies condition (CBV2).

Reciprocally, suppose there exists a Lyapunov functional with respect to the retarded VS integral equation (6.2). Let $\alpha > 0$. By condition (CBV1), there exists $M = M(\alpha)$ such that

$$p(\alpha) < b(s), \quad \text{for all } s \geq M. \tag{6.74}$$

Let $s_0 \in [t_0,+\infty)$ and $y : [s_0 - r,+\infty) \to X$ be the global forward solution of the retarded VS integral equation (6.2) with initial condition $y_{s_0} = \psi$. Assume that $\|\psi\|_\infty < \alpha$. Then, by condition (CBV2) and the fact p is increasing, we have

$$b(\|y_t(s_0,\psi)\|_\infty) \leq U(t, y_t(s_0,\psi)) \leq U(s_0,\psi) \leq p(\|\psi\|_\infty) < p(\alpha) \overset{(6.74)}{<} b(M),$$

for all $t \geq s_0$. Finally, since b is increasing, we have

$$\|y_t(s_0,\psi)\|_\infty < M,$$

for all $t \geq s_0$ and, consequently, the retarded VS integral equation (6.2) is uniformly bounded. □

The following result is consequence of Theorems 6.2.22 and 6.3.4.

Corollary 6.3.5. If the retarded VS integral equation (6.2) is uniformly bounded, then the trivial solution of the retarded VS integral equation (6.2) is uniformly stable.

The next result gives a relation between the uniform boundedness of solutions of the perturbed retarded VS integral equation (6.5) and uniform stability.

Theorem 6.3.6. If the perturbed retarded VS integral equation (6.5) is uniformly bounded, then the trivial solution of the retarded VS integral equation (6.2) is uniformly stable.

Proof. It is sufficient to notice that, by Theorem 6.3.2, the retarded VS integral equation (6.2) is uniformly bounded and, by Corollary 6.3.5, it is also uniformly stable. □

In what follows, we prove a result which guarantees that the existence of a Lyapunov functional with respect to the retarded VS integral equation (6.2) implies the existence of a Lyapunov functional with respect to the generalized ODE (6.9).

Theorem 6.3.7. Consider $\mathbb{O}$ given by (6.7) and S by (6.1). Assume that $g : [t_0,+\infty) \to \mathbb{R}$ satisfies condition (A1) and $f : S \times [t_0,+\infty) \to X$ satisfies conditions (A2), (A3) and (A4). If there exists a Lyapunov functional $U : [t_0,+\infty) \times S \to \mathbb{R}$ with respect to the retarded VS integral equation (6.2) satisfying conditions (LUV1) and (LUV2) from Theorem 6.2.16, then there exists a Lyapunov functional $V : [t_0,+\infty) \times \mathbb{O} \to \mathbb{R}$ with respect to the generalized ODE (6.9).

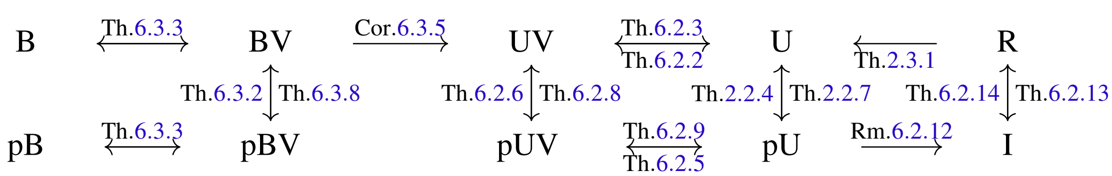
Proof. By Theorem 6.3.4, the solution of the retarded VS integral equation (6.2) is uniformly bounded and, by Theorem 6.3.3-(i), the solution of the generalized ODE (6.9) is uniformly bounded. Moreover, Theorem 3.0.4 ensures the existence of the Lyapunov functional with respect to the generalized ODE (6.9). □

Theorem 6.3.8. Let S be given by (6.1). Assume that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), ($\mathscr{A}3$), ($\mathscr{A}4$) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). If the retarded VS integral equation (6.2) is uniformly bounded, then the perturbed retarded VS integral equation (6.5) is uniformly bounded.

Proof. By Theorem 6.3.3-(i), the generalized ODE (6.9) is uniformly bounded. By Remark 6.2.13, the function $\widetilde{h}$ from the class $\mathscr{F}(\Omega, \widetilde{h})$ is bounded and, therefore, we can apply Theorem 3.0.7 to conclude that the perturbed generalized ODE (6.41) is uniformly bounded. Finally, by Theorem 6.3.3-(ii), the perturbed retarded VS integral equation (6.5) is uniformly bounded. □

We end this section by presenting a diagram which illustrates the relations between the types of stability, contained in Section 6.2.1, and the concepts of uniform boundedness of solutions. We are assuming that $g : [t_0, +\infty) \to \mathbb{R}$ satisfies (A1), $f : S \times [t_0, +\infty) \to X$ satisfies (A2), ($\mathscr{A}3$), ($\mathscr{A}4$) and the functions $p : [t_0, +\infty) \to X$ and $v : [t_0, +\infty) \to \mathbb{R}$ satisfy (A5), (A6) and (A7). Moreover, we consider the following symbols:

- B = boundedness of the solutions of the generalized ODE (6.9);
- pB = boundedness of the solutions of perturbed generalized ODE (6.41);
- BV = boundedness of the solutions of the retarded VS integral equation (6.2);
- pBV = boundedness of the solutions of the perturbed retarded VS integral equation (6.5);
- UV = uniform stability for the trivial solution of the retarded VS equation (6.2);
- pUV = uniform stability with respect to pertubations for the trivial solution of the retarded VS integral equation (6.2);
- U = uniform stability for the trivial solution of the of the generalized ODE (6.9);
- pU = uniform stability with respect to perturbations for the trivial solution of the of the generalized ODE (6.9);
- I = integral stability for the trivial solution of the retarded VS equation (6.2);
- R = regular stability for the trivial solution of the of the generalized ODE (6.9).



6.4 Asymptotic controllability

In this section, we present a definition of asymptotic controllability for perturbed retarded VS integral equations. Moreover, we applied the results, presented in Chapter 5 and in Sections 6.2 and 6.3, to establish necessary and sufficient conditions for a perturbed retarded VS integral equation to be asymptotically controllable. We also include an example to illustrate our main results.

At first, we recall that $O \subset BG([t_0 - r, +\infty), X)$ is an open set with the prolongation property (see Definition 6.0.1) and $S = \{y_t; y \in O, t \in [t_0, +\infty)\} \subset G([-r, 0], X)$.

Consider a perturbed retarded VS integral equation given by

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s) + \int_{s_0}^{t} p(s)dv(s),\ t \geq s_0 \geq t_0, \tag{6.75}$$

where $f : S \times [t_0, +\infty) \to X$, $p : [t_0, +\infty) \to X$ is a *control function* and $g, v : [t_0, +\infty) \to \mathbb{R}$. Moreover, assume that the following conditions are fulfilled:

(C1) the functions $g, v : [t_0, +\infty) \to \mathbb{R}$ are left-continuous on $(t_0, +\infty)$ and nondecreasing;

(C2) $f(0, t) - f(0, s) = 0$ for $t, s \geq t_0$;

(C3) for all $y \in O$ and all $s_1, s_2 \in [t_0, +\infty)$, the Perron-Stieltjes integral
$$\int_{s_1}^{s_2} f(y_s, s)dg(s)$$
exists;

(C4) there exists a Perron-Stieltjes integrable function $\mathscr{M} : [t_0, +\infty) \to \mathbb{R}$ with respect to g such that
$$\left\| \int_{s_1}^{s_2} f(y_s, s)dg(s) \right\| \leq \int_{s_1}^{s_2} \mathscr{M}(s)dg(s),$$
for all $y \in O$ and all $s_1, s_2 \in [t_0, +\infty)$;

(C5) there exists a Perron-Stieltjes integrable function $\mathscr{L} : [t_0, +\infty) \to \mathbb{R}$ with respect to g such that
$$\left\| \int_{s_1}^{s_2} [f(y_s, s) - f(z_s, s)]dg(s) \right\| \leq \int_{s_1}^{s_2} \mathscr{L}(s)\|y_s - z_s\|_\infty dg(s),$$
for all $y, z \in O$ and all $s_1, s_2 \in [t_0, +\infty)$;

(C6) the Perron-Stieltjes integral
$$\int_{t_0}^{t} p(s)dv(s)$$
exists, for all $t \in [t_0, +\infty)$;

(C7) there exists a locally Perron-Stieltjes integrable function $K : [t_0, +\infty) \to \mathbb{R}$ with respect to v such that
$$\left\| \int_{s_1}^{s_2} p(s)dv(s) \right\| \leq \int_{s_1}^{s_2} K(s)dv(s),$$
for all $s_1, s_2 \in [t_0, +\infty)$.

We denote by $\mathbb{P}$ the set of all functions $p : [t_0, +\infty) \to X$ which satisfy (C6) and (C7). In the sequel, we define asymptotic controllability for the perturbed retarded VS integral equation (6.75).

Definition 6.4.1. The perturbed retarded VS integral equation (6.75) is *asymptotically controllable*, if the following properties hold:

(i) (*global part*) for each ϕ in S, there exists a control function $p \in \mathbb{P}$ such that the solution $\bar{y}(t) = \bar{y}(t, \phi, p)$ of the the perturbed retarded VS integral equation (6.75) is defined for all $t \geq t_0$ and, moreover, $\bar{y}(t)$ goes to 0 as t goes to ∞;

(ii) (*Uniform stability*) for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any state $\phi \in S$, with $\|\phi\| \leq \delta$, there is a control p as in (i) such that $\|\bar{y}(t)\| \leq \varepsilon$ for all $t \geq t_0$;

(iii) (*bounded controls*) there exist positive numbers η, k such that if ϕ, given in (ii), satisfies $\|\phi\| < \eta$, then the control p is such that $\|\widetilde{p}\|_\infty \leq k$, where

$$\|\widetilde{p}\|_\infty = \sup_{t \in [t_0, +\infty)} \left\| \int_{t_0}^{t} p(s) dv(s) \right\|.$$

In order to prove asymptotic controllability for the perturbed retarded VS integral equation (6.75), we denote by $\mathbb{U}$ the set of all functions $u : [t_0, +\infty) \to BG([t_0, +\infty), X)$ given by

$$u(t)(\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq t_0, \\ \displaystyle\int_{t_0}^{\theta} p(s) dv(s), & t_0 \leq \theta \leq t, \\ \displaystyle\int_{t_0}^{t} p(s) dv(s), & t \leq \theta < +\infty, \end{cases}$$

for some $p \in \mathbb{P}$.

Theorem 6.4.2. Assume that $g, v : [t_0, +\infty) \to \mathbb{R}$ satisfy condition (C1), $f : S \times [t_0, +\infty) \to X$ satisfies conditions (C2), (C3), (C4) and (C5) and $p \in \mathbb{P}$. Then, the perturbed retarded VS integral equation (6.75) is asymptotically controllable.

Proof. Consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x,t) + u(t)], \tag{6.76}$$

where F is given by (6.8) and $u \in \mathbb{U}$. By Remark 6.2.7, the function $\widetilde{h}$ from the class $\mathscr{F}(\Omega, \widetilde{h})$ is bounded. On the other hand, by condition (C2), $F(0,t) - F(0,s) = 0$ for $t, s \geq t_0$. Therefore, the perturbed generalized ODE (6.76) satisfies all conditions of Theorem 5.0.2 and, consequently, it is asymptotically controllable.

Let $\phi \in S$ and $x_0 : [t_0, +\infty) \to X$ be given by

$$x_0(\theta) = \begin{cases} \phi(\theta - t_0), & t_0 - r \leq \theta \leq t_0, \\ \phi(0), & t_0 \leq \theta < +\infty. \end{cases}$$

Since the perturbed generalized ODE (6.76) is asymptotically controllable, there exists $u \in \mathbb{U}$ such that the solution $\overline{x}(t, x_0, u)$ of the perturbed generalized ODE (6.76) is defined for all $t \geq t_0$ and $\overline{x}(t)$ goes to zero as t goes to $+\infty$.

By the definition of $\mathbb{U}$, there exists $p \in \mathbb{P}$ such that $u : [t_0, +\infty) \to BG([t_0 - r, +\infty), X)$ is given by

$$u(t)(\theta) = \begin{cases} 0, & t_0 - r \leq \theta \leq t_0, \\ \displaystyle\int_{t_0}^{\theta} p(s)dv(s), & t_0 \leq \theta \leq t, \\ \displaystyle\int_{t_0}^{t} p(s)dv(s), & t \leq \theta < +\infty. \end{cases}$$

By Theorem 6.1.7-(ii), $\overline{y} : [t_0 - r, +\infty) \to X$ given by

$$\overline{y}(\theta) = \begin{cases} \overline{x}(t_0)(\theta), & t_0 - r \leq \theta \leq t_0, \\ \overline{x}(\theta)(\theta), & t_0 \leq \theta < +\infty \end{cases}$$

is a solution of the perturbed retarded VS integral equation (6.75) with initial condition $y_{t_0} = \phi$ and control p. Since $\overline{x}(t)$ goes to 0 as t goes to ∞, we conclude that $\overline{y}(t)$ goes to 0 as t goes to ∞ and condition (i) from Definition 6.4.1 is satisfied.

By Corollary 4.0.3, the trivial solution of the generalized ODE (6.9) is uniformly stable with respect to perturbations and, by Theorem 6.2.5-(ii), the trivial solution of the perturbed retarded VS integral equation is uniformly stable with respect to perturbations which yields condition (ii) of Definition 6.4.1.

Let us prove condition (iii) from Definition 6.4.1. At first, notice that

$$\|x_0\|_\infty = \sup_{\theta \in [t_0 - r, \infty)} \|x_0(\theta)\| = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\| = \|\phi\|_\infty$$

and

$$\begin{aligned} \sup_{t \in [t_0, +\infty)} \|u(t)\|_\infty &= \sup_{t \in [t_0, +\infty)} \left(\sup_{\theta \in [t_0 - r, +\infty)} \|u(t)(\theta)\| \right) \\ &= \sup_{t \in [t_0, +\infty)} \left(\sup_{\theta \in [t_0, t]} \left\| \int_{t_0}^{\theta} p(s)dv(s) \right\| \right) = \sup_{\theta \in [t_0, +\infty)} \left\| \int_{t_0}^{\theta} p(s)dv(s) \right\| = \|\widetilde{p}\|_\infty. \end{aligned}$$

By condition (iii) from Definition 5.0.1 and by the previous equations, we conclude that there exist positive numbers η, k such that if $\|\phi\|_\infty = \|x_0\|_\infty < \eta$, then $\|\widetilde{p}\|_\infty = \sup_{t \in [t_0, +\infty)} \|u(t)\|_\infty \leq k$ and the proof is complete. □

The existence of a Lyapunov functional with respect to retarded VS integral equation (6.2) is related to stability and boundedness of solutions of the retarded VS integral equation (6.2) and these concepts are also related to stability with respect to perturbations and to boundedness of solutions of the perturbed retarded VS integral equation (6.75). Therefore, Theorem 6.4.2 implies the next result.

Corollary 6.4.3. Assume that $g, v : [t_0, +\infty) \to \mathbb{R}$ satisfy condition (C1), $f : S \times [t_0, +\infty) \to X$ satisfies conditions (C2)(C3), (C4) and (C5) and $p \in \mathbb{P}$. Then, the perturbed retarded VS integral equation (6.75) is asymptotically controllable if and only if there is a Lyapunov functional $U : [t_0, +\infty) \times S \to \mathbb{R}$ with respect to the retarded VS integral equation

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s)dg(s),\ t \geq s_0 \geq t_0, \tag{6.77}$$

satisfying

(H1) there exists an increasing continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(0) = 0$ and

$$U(t, \psi) \leq a(\|\psi\|_\infty),$$

for all $\psi \in S$ and $t \in [t_0, +\infty)$.

In the sequel, we present an example to illustrate how our results can be applied to perturbed retarded VS integral equations.

Example 6.4.4. Let $\xi : S \to \mathbb{R}^n$ be a function fulfilling:

(i) for all $y \in O$, the mapping $t \mapsto \xi(y_t)$ is Perron-Stieltjes integrable with respect to g, where $g : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function which is left-continuous on $(t_0, +\infty)$;

(ii) $\xi(\psi + \phi) = \xi(\psi) + \xi(\phi)$ for all $\psi, \phi \in S$;

(iii) $\xi(\psi) = 0$, whenever $\psi \equiv 0$;

(iv) $0 < \|\xi(\psi)\| < \|\psi\|_\infty$ for all $\psi \in S$.

Consider functions $f : S \times [t_0, +\infty) \to \mathbb{R}^n$ and $l : [t_0, +\infty) \to \mathbb{R}$ defined by

$$f(\psi, s) = l(s)\xi(\psi), \quad (\psi, s) \in S \times [t_0, +\infty), \tag{6.78}$$

and

$$l(s) = \begin{cases} 1, & \text{if } s \in \mathbb{I} \cap [t_0, +\infty), \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbb{I}$ denotes the set of irrational numbers.

Notice that, for all $s \in [t_0, +\infty)$, we have

$$f(0, s) = l(s)\xi(0) = 0,$$

since $\xi(0) = 0$.

Claim: f, defined by (6.78), satisfies (C3), (C4) and (C5).

Indeed, since the mapping $t \mapsto \xi(y_t)$ is Perron–Stieltjes integrable with respect to g, for all $\varepsilon > 0$, there exists a gauge δ on $[t_0, t]$ such that, for all δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[t_0, t]$, we have

$$\left\| \sum_{i=1}^{|d|} \xi(y_{\tau_i})(g(s_i) - g(s_{i-1})) - \int_{t_0}^{t} \xi(y_s) dg(s) \right\| < \varepsilon. \tag{6.79}$$

Once $l(\tau_i) = 0$ whenever τ_i is rational, we can ignore these rational tags and consider a subsequence $(\tau_i)_{i=1}^n$, $n \leq |d|$, consisting of irrational numbers. Therefore, for all $y \in O$, we have

$$\sum_{i=1}^{|d|} f(y_{\tau_i}, \tau_i)(g(s_i) - g(s_{i-1})) = \sum_{i=1}^{n} \xi(y_{\tau_i})(g(s_i) - g(s_{i-1})). \tag{6.80}$$

By (6.79) and (6.80), the mapping $t \mapsto f(y_t, t)$ is Perron-Stieltjes integrable with respect to g for all $y \in O$, and

$$\int_{t_0}^{t} f(y_s, s) dg(s) = \int_{t_0}^{t} \xi(y_s) dg(s), \quad \text{for all } t \in [t_0, +\infty) \text{ and all } y \in O,$$

which proves condition (C3). On the other hand, since $\|\xi(\psi)\| > 0$ for all $\psi \in S$ and g is nondecreasing, we have

$$\left\| \int_{s_1}^{s_2} f(y_s, s) dg(s) \right\| = \int_{s_1}^{s_2} \xi(y_s) dg(s),$$

for all $s_1, s_2 \in [t_0, +\infty)$ and all $y \in O$. Then, condition (C4) holds. By condition (iv), we have

$$\left\| \int_{s_1}^{s_2} [f(y_s, s) - f(z_s, s)] dg(s) \right\| \leq \int_{s_1}^{s_2} \|y_s - z_s\|_\infty dg(s),$$

for all $s_1, s_2 \in [t_0, +\infty)$ and all $y, z \in O$ (see [24, Theorem 2.1]). Therefore, condition (C5) is satisfied and the *Claim* is proved. Now, consider the perturbed retarded VS integral equation

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s, s) ds + \int_{s_0}^{t} p(s) dv(s),\ t \geq s_0 \geq t_0, \tag{6.81}$$

where f is given by (6.78), $p : [t_0, +\infty) \to \mathbb{R}^n$ satisfies (C6) and (C7) and $v : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ and nondecreasing. We assert that the perturbed retarded VS integral equation (6.81) is asymptotically controllable. Indeed, define $U : [t_0, +\infty) \times S \to \mathbb{R}$ by $U(t, \psi) = \|\psi\|_\infty$ for all $(t, \psi) \in [t_0, +\infty) \times S$. Then, it is clear that $U(\cdot, \psi) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$. Moreover, if $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ are defined by $a(s) = s$ and $b(s) = \frac{s}{2}$ for all $s \in \mathbb{R}^+$, then a and b are increasing continuous functions and

$$b(\|\psi\|_\infty) \leq V(t, \psi) \leq a(\|\psi\|_\infty)$$

holds for all $(t,\psi) \in [t_0,+\infty) \times S$. On the other hand,

$$\begin{aligned}
\dot{U}(t,\psi) &= \limsup_{\eta\to 0^+} \frac{\|y_{t+\eta}(t,\psi)\|_\infty - \|y_t(t,\psi)\|_\infty}{\eta} \\
&= \limsup_{\eta\to 0^+} \frac{\sup_{\theta\in[-r,0]} \|y(t+\eta+\theta)\| - \sup_{\theta\in[-r,0]} \|y(t+\theta)\|}{\eta} \\
&= \limsup_{\eta\to 0^+} \frac{\sup_{\theta\in[-r+\eta,\eta]} \|y(t+\theta)\| - \sup_{\theta\in[-r,0]} \|y(t+\theta)\|}{\eta} \\
&\leq \limsup_{\eta\to 0^+} \frac{\sup_{\theta\in[-r,0]} \|y(t+\theta)\| - \sup_{\theta\in[-r,0]} \|y(t+\theta)\|}{\eta} = 0.
\end{aligned}$$

Therefore, U is a Lyapunov functional with respect to the retarded VS integral equation

$$y(t) = y(s_0) + \int_{s_0}^{t} f(y_s,s)dg(s),\ t \geq s_0 \geq t_0$$

and U satisfies hypothesis (H1) from Corollary 6.4.3. By Corollary 6.4.3, the perturbed retarded VS integral equation (6.81) is asymptotically controllable.

CHAPTER

7

DYNAMIC EQUATIONS ON TIME SCALES

In this chapter, we investigate the existence and uniqueness of a solution for a linear Volterra-Stieltjes integral equation (linear VS integral equation, for short) of the second kind, as well as for a homogeneous and a nonhomogeneous linear dynamic equations on time scales, whose integral forms contain Perron Δ-integrals defined in Banach spaces. We also provide a variation-of-constant formula for a nonhomogeneous linear dynamic equations on time scales and we establish results on controllability for linear dynamic equations. The results presented here are contained in [5].

In the following lines, we specify our contributions concerning the existence and uniqueness of a solution for homogeneous and a nonhomogeneous linear dynamic equation.

The best known results on the existence and uniqueness of a solution for a nonhomogeneous linear dynamic equation of the form

$$x^{\Delta} = a(t)x + f(t) \tag{7.1}$$

and for its corresponding homogeneous equation

$$x^{\Delta} = a(t)x \tag{7.2}$$

on a time scale $\mathbb{T}$, take into account that a is a regressive $n \times n$-matrix-valued function (see Definition 7.3.5) which is rd-continuous and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous (see Appendix D for more details). Moreover, the integrals appearing in the solutions of the dynamic equations (7.1) and (7.2) are in the sense of the Riemann Δ-integral (see [3, 10, 11, 37], for instance).

Furthermore, [13, Theorem 5.32] ensures that the nonlinear dynamic integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} h(y^*(s), s)\Delta s$$

on a time scale $\mathbb{T}$ has a unique solution, where $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, the integral on the right-hand side is in the sense of Perron Δ-integral (see Definition D.0.7), X is a Banach space and the function $h : X \times \mathbb{T}_0 \to X$ satisfies the following conditions:

(H1) for all $t_1, t_2 \in \mathbb{T}_0$ and all regulated function $y : \mathbb{T}_0 \to X$, the Perron Δ-integral $\int_{t_1}^{t_1} h(y(s), s)\Delta s$ exists;

(H2) there exists a locally Perron Δ-integrable function $M : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_1} h(y(s), s)\Delta s \right\| \leq \int_{t_1}^{t_2} M(s)\Delta s$$

holds for every $t_1, t_2 \in \mathbb{T}_0$ and every $y \in G_0(\mathbb{T}_0, X)$ (see Definition D.0.3);

(H3) there exists a locally Perron Δ-integrable function $L : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_1} [h(y(s), s) - h(z(s), s)]\Delta s \right\| \leq \|y - z\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s$$

holds for every $t_1, t_2 \in \mathbb{T}_0$ and every $y, z \in G_0(\mathbb{T}_0, X)$.

Take $h : X \times \mathbb{T}_0 \to X$ as $h(x, s) = a(s)x$ for all $(x, s) \in X \times \mathbb{T}_0$, where $a : \mathbb{T} \to L(X)$ is a given function and $L(X)$ is the Banach space of continuous linear mappings $T : X \to X$. Then, by [13, Theorem 5.32], there exists a unique solution of the linear dynamic integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} a(s)y(s)\Delta s.$$

Since the integral on the right-hand side of the above equation is in the sense of Perron Δ, a and y may be discontinuous functions with highly oscillating behaviour.

We point out that the theorems on the existence and uniqueness of solutions presented in [3, 10, 11, 37] are not a particular case of [13, Theorem 5.32]. Indeed, we notice that there exist a special time scale $\mathbb{T}$ and an rd-continuous function $a : \mathbb{T} \to L(\mathbb{R})$ such that condition (H2) is not fulfilled and condition (H1) holds. Consider the time scale $\mathbb{T} = \mathbb{Z}$ and define $a : \mathbb{T} \to L(\mathbb{R})$ by

$$a(t)s = \left(\frac{(-1)^t}{(1+t)} \right) s, \quad (t, s) \in \mathbb{T} \times \mathbb{R}. \tag{7.3}$$

Then, since $\mathbb{Z}$ does not contain any right-dense point (see Definition D.0.1), a is rd-continuous and $\mathbb{T}_0 = \mathbb{Z}^+ = \{t \in \mathbb{Z}; t \geq 0\}$. Moreover, if we consider $y : \mathbb{Z}^+ \to \mathbb{R}$ being the constant function equal to 1, we have

$$\int_{s}^{t} a(s)y(s)\Delta s = \sum_{k=s}^{t-1} a(k),$$

for all $s, t \in \mathbb{Z}^+$ (see [10, Table 1.3]). Notice that, for all $t \in \mathbb{T}$, we get

$$\int_{t-1}^{t} a(s)y(s)\Delta s = \frac{(-1)^{t-1}}{t}$$

Therefore, for all $T \in \mathbb{Z}^+$, we obtain the finite alternating harmonic series

$$\int_{0}^{T} a(s)y(s)\Delta s = \sum_{t=1}^{T} \int_{t-1}^{t} a(s)y(s)\Delta s = \sum_{t=1}^{T} \frac{(-1)^{t-1}}{t}.$$

Owing the fact that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges, the Perron Δ-integral $\int_0^N a(s)y(s)\Delta s$ exists for a large $N\in\mathbb{Z}^+$. Assume that condition (H1) holds. Then, for all $t_1,t_2\in\mathbb{Z}^+$,

$$\left|\int_{t_1}^{t_2} a(s)y(s)\Delta s\right| \leq \int_{t_1}^{t_2} M(s)\Delta s,$$

and for a large $N\in\mathbb{Z}^+$,

$$\int_0^N M(s)\Delta s=\sum_{t=1}^{N}\int_{t-1}^{t} M(s)\Delta s\geq\sum_{t=1}^{N}\left|\int_{t-1}^{t} a(s)y(s)\Delta s\right|=\sum_{t=1}^{N}\left|\frac{(-1)^{t-1}}{t}\right|=\sum_{t=1}^{N}\frac{1}{t}$$

which leads to a contraction, since the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

It is well-known that the study of the convergence of infinite series is widely used in physics and the position of the source charges are given as functions of time, as in the problem of determining of electrical potential and electrical force exerted by infinite point electrical charges. As an example, if we consider a distribution, where the positive and the negative electrical charges are placed in $1,-3,-5,\ldots$ and $-2,-4,-6,\ldots$ respectively, then the electric potential, in the origin, is given by the series $\frac{q}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$, where ε_0 is a constant and q is the modulo of the electrical changes. See [18, Chapter 2]. Therefore, the function a defined by (7.3) has an application in physics. So, it is interesting that a good theory encompasses this situation as well. This is one reason why we sought to suppress condition (H2) from [13, Theorem 5.32] (in the linear case), and we assume that the functions a and f satisfy conditions (H1) and (H3) among others, instead of being rd-continuous. As a matter of fact, we prove that all rd-continuous function satisfies all our hypotheses (see Lemma 7.2.8) and, since we are not considering condition (H2), our main result on the existence and uniqueness of a solution for dynamic equations on time scales is not a particular case of [13, Theorem 5.32].

Concerning a variation-of-constant formula for the nonhomogeneous dynamic equation (7.1), we notice that the results presented in [3, 10, 11, 37] also require that a is a regressive $n\times n$-matrix-valued function (see Definition 7.3.5) which is rd-continuous and $f:\mathbb{T}\to\mathbb{R}^n$ is rd-continuous. Here, we provide a variation-of-constant formula, in Section 7.2, where the functions a and f take values in an arbitrary Banach space and satisfy conditions (H1) and (H3) among others. We emphasize that our results on the existence and uniqueness of a solution for a linear nonhomogeneous dynamic equation and for a linear homogeneous dynamic equation, as well as the variation-of-constant formula, generalize the results from [3, 10, 11, 37]. Moreover, since we are considering Perron Δ-integrals instead of Riemann Δ-integrals, our integrands may be highly oscillating and have many discontinuities.

7.1 Existence and uniqueness of a solution

In order to establish results on existence and uniqueness of solutions for linear homogeneous and nonhomogeneous dynamic equations on time scales, we prove, in Subsection 7.1.1, the existence and uniqueness of a solution of a Volterra-Stieltljes integral equation. The results obtained here generalize those presented in [10].

7.1.1 Volterra-Stieltjes integral equations

The main goal of this subsection is to prove that the following linear VS integral equation

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v], \tag{7.4}$$

admits a unique solution, where X and Y are Banach spaces, $L(X,Y)$ is the Banach space of continuous linear mappings $T : X \to Y$, $J \subset \mathbb{R}$ is an interval containing $[t_0, v]$, $A : J \to L(X,Y)$, $h : J \to X$ and the integral in (7.4) is in the sense of Perron-Stieltjes.

We start by proving the following auxiliary result.

Theorem 7.1.1. Let $v \in J$ and $A : J \to L(X,Y)$ be locally of bounded variation. Then, the mapping $T : BV([t_0, v], X) \to BV([t_0, v], X)$ given by

$$Ty(t) = \int_{t_0}^{t} d[A(s)]y(s), \quad t \in [t_0, v]$$

is well-defined and it is a bounded compact linear operator on $BV([t_0, v], X)$.

Proof. At first, we notice that the existence of the Perron-Stieltjes $\int_{t_0}^{t} d[A(s)]y(s)$ is guaranteed by Proposition B.0.8 and, by linearity of the Perron-Stieltjes integral, T is linear. Let us prove that $Ty \in BV([t_0, v], X)$.

By Theorem A.0.4 and Proposition B.0.7, for every division $d = (t_i)$ of $[t_0, v]$ and every $y \in BV([t_0, v], X)$, we have

$$\begin{aligned}
\sum_{i=1}^{|d|} \|Ty(t_i) - Ty(t_{i-1})\| &= \sum_{i=1}^{|d|} \left\| \int_{t_0}^{t_i} d[A(s)]y(s) - \int_{t_0}^{t_{i-1}} d[A(s)]y(s) \right\| \\
&= \sum_{i=1}^{|d|} \left\| \int_{t_{i-1}}^{t_i} d[A(s)]y(s) \right\| \\
&\overset{\text{Prop.B.0.7}}{\leq} \sum_{i=1}^{|d|} \operatorname{var}_{t_{i-1}}^{t_i} A \|y\|_\infty \\
&\overset{\text{Th.A.0.4}}{=} \operatorname{var}_{t_0}^{v} A \|y\|_\infty,
\end{aligned}$$

where $\|y\|_\infty = \sup_{s \in [t_0, v]} \|y(s)\|$. Taking the supremum over all divisions $d = (t_i)$ of $[a, b]$, we obtain

$$\operatorname{var}_{t_0}^{v} Ty \leq \operatorname{var}_{t_0}^{v} A \|y\|_\infty. \tag{7.5}$$

Moreover,

$$\|y(s)\| \leq \|y(s) - y(t_0)\| + \|y(t_0)\| \leq \operatorname{var}_{t_0}^{s} y + \|y(t_0)\| \leq \operatorname{var}_{t_0}^{v} y + \|y(t_0)\|,$$

for all $s \in [t_0, v]$ and, hence,

$$\|y\|_\infty = \sup_{s\in[t_0,v]} \|y(s)\| \leq \text{var}_{t_0}^{v} y + \|y(t_0)\|. \tag{7.6}$$

Replacing (7.6) into (7.5), we obtain

$$\text{var}_{t_0}^{v} Ty \leq \text{var}_{t_0}^{v} A\|y\|_\infty \leq \text{var}_{t_0}^{v} A(\|y(t_0)\| + \text{var}_{t_0}^{v} y) = \text{var}_{t_0}^{v} A\|y\|_{BV} < \infty, \tag{7.7}$$

which shows that $Ty : [t_0, v] \to X$ is of bounded variation on $[t_0, v]$, for all $y \in BV([t_0, v], X)$ and, consequently, T is well-defined.

Moreover, T is bounded once, $\|Ty(t_0)\| = 0$ and, by (7.7), we have

$$\|Ty\|_{BV} = \|Ty(t_0)\| + \text{var}_{t_0}^{v} Ty < \text{var}_{t_0}^{v} A\|y\|_{BV}.$$

We target to prove that T is compact. Let $\{y_k\}_{k\in\mathbb{N}}$ be a uniformly bounded sequence in $BV([t_0, v], X)$, that is, there exists $C > 0$ such that $\|y_k\| \leq C$ for all $k \in \mathbb{N}$. By Helly's Choice Theorem (see Theorem A.0.6), the sequence $\{y_k\}_{k\in\mathbb{N}}$ contains a subsequence $\{y_{k_l}\}_{k_l\in\mathbb{N}}$ which converges pointwisely to a function $y_0 \in BV([t_0, v], X)$. To simplify the notation, we denote the subsequence $\{y_{k_l}\}_{k_l\in\mathbb{N}}$ by $\{y_k\}_{k\in\mathbb{N}}$. For all $k \in \mathbb{N}$, define $z_k : [t_0, v] \to X$ by

$$z_k(s) = y_k(s) - y_0(s), \quad s \in [t_0, v].$$

Then, $z_k \in BV([t_0, v], X)$ and

$$\lim_{k\to\infty} z_k(s) = 0, \quad s \in [t_0, v]. \tag{7.8}$$

Define $z : [t_0, v] \to X$ by

$$z(t) = \int_{t_0}^{t} d[A(s)]y_0(s), \quad s \in [t_0, v].$$

By Theorem A.0.4 and Proposition B.0.7, $z \in BV([t_0, v], X)$ and, for every division $d = (t_i)$ of $[t_0, v]$, we have

$$\begin{aligned}
\sum_{i=1}^{|d|} \|Ty_k(t_i) - z(t_i) - [Ty_k(t_{i-1}) - z(t_{i-1})]\| &= \sum_{i=1}^{|d|} \left\| \int_{t_{i-1}}^{t_i} d[A(s)]y_k(s) - \int_{t_{i-1}}^{t_i} d[A(s)]y_0(s) \right\| \\
&= \sum_{i=1}^{|d|} \left\| \int_{t_{i-1}}^{t_i} d[A(s)](y_k(s) - y_0(s)) \right\| \\
&= \sum_{i=1}^{|d|} \left\| \int_{t_{i-1}}^{t_i} d[A(s)]z_k(s) \right\| \\
&\leq \sum_{i=1}^{|d|} \text{var}_{t_{i-1}}^{t_i} A\|z_k\|_\infty \\
&= \text{var}_{t_0}^{v} A\|z_k\|_\infty
\end{aligned}$$

and, hence,

$$\text{var}_{t_0}^{v} (Ty_k - z) \leq \text{var}_{t_0}^{v} A\|z_k\|_\infty. \tag{7.9}$$

By equations (7.8) and (7.9), we can conclude

$$\begin{aligned}\lim_{k\to\infty}\|Ty_k - z\|_{BV} &= \lim_{k\to\infty}\|Ty_k(t_0) - z(t_0)\| + \lim_{k\to\infty}\operatorname{var}_{t_0}^{v}(Ty_k - z)\\ &= \lim_{k\to\infty}\operatorname{var}_{t_0}^{v}(Ty_k - z) = 0.\end{aligned}$$

Therefore, $\{Ty_k\}_{k\in\mathbb{N}}$ converges to z in $BV([t_0,v],X)$ and the operator T is compact. □

The next result is known as the Fredholm alternative for linear operators defined in Banach spaces and will be essential in the proof of the existence of solutions of the linear VS integral equation (7.4). For a proof of it, see [19, page 609].

Theorem 7.1.2 (Fredholm alternative)**.** Let X be a Banach space, $T \in L(X)$, $x \in X$ and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. If T is compact, then the equation $Tx - \lambda x = y$ has a solution for every $y \in X$ if and only if the equation $Tx = \lambda x$ has only the trivial solution.

In the sequel, we present a result on the existence and uniqueness of solution of the linear VS integral equation (7.4)

Theorem 7.1.3. Let $v \in J$ and $A : J \to L(X,Y)$ be locally of bounded variation on J and left-continuous. Then, the linear VS integral equation (7.4) admits a unique solution in $BV([t_0,v],X)$, for each given $h \in BV([t_0,v],X)$.

Proof. We start by proving that the homogeneous linear VS integral equation

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s), \quad t \in [t_0,v] \tag{7.10}$$

admits only the trivial solution in $BV([t_0,v],X)$.

Let $x : [t_0,v] \to X$ be a solution of (7.10) and define

$$C = \{t \in [t_0,v];\ \|x(t)\| = 0\}.$$

Then, C is a non-empty set, since $t_0 \in C$, and it is upper bounded.

Let $c = \sup C$. Then, for all $t \in [t_0,c)$, $\|x(t)\| = 0$ and, by Theorem B.0.6 and Proposition B.0.7, we have

$$\begin{aligned}\|x(c)\| &= \left\|\int_{t_0}^{c} d[A(s)]x(s)\right\|\\ &\leq \lim_{\tau\to c^-}\left\|\int_{t_0}^{\tau} d[A(s)]x(s)\right\| + \|A(c) - A(c^-)\|\|x(c)\|\\ &\leq \lim_{\tau\to c^-}\left(\operatorname{var}_{t_0}^{\tau} A \sup_{s\in[t_0,\tau]}\|x(s)\|\right) + \|A(c) - A(c^-)\|\|x(c)\| = 0,\end{aligned}$$

since A is left-continuous. Therefore, $\|x(c)\| = 0$.

Let us show that $c = v$. Assume that $c < v$ and consider two cases.

Case 1: A is right-continuous at c.

In this case, define $V:[t_0,v]\to X$ by

$$V(t)=\text{var}_{t_0}^{t}A,\quad \text{for all } t\in[t_0,v].$$

Thus, by hypothesis, V is well-defined. It is also nondecreasing and satisfies

$$V(t)-V(s)=\text{var}_{s}^{t}A,\quad \text{for all } s,t\in[t_0,v],\ s\leq t.$$

Since A is continuous at c, V is also continuous at c and, hence, there exists $\bar{t}\in(c,v]$ such that

$$V(\bar{t})-V(c)=\text{var}_{c}^{\bar{t}}A<\frac{1}{2}. \tag{7.11}$$

By Proposition B.0.7, for all $s_1,s_2\in[c,\bar{t}]$, $s_1<s_2$, we have

$$\|x(s_2)-x(s_1)\|=\left\|\int_{s_1}^{s_2}d[A(s)]x(s)\right\|\leq \text{var}_{s_1}^{s_2}A\sup_{s\in[s_1,s_2]}\|x(s)\|. \tag{7.12}$$

On the other hand,

$$\|x(s)\|\leq\|x(s)-x(c)\|+\|x(c)\|\leq \text{var}_{c}^{\bar{t}}x,\quad \text{for all } s\in[s_1,s_2]\quad \text{and}$$

$$\text{var}_{s_1}^{s_2}A=\text{var}_{c}^{\bar{t}}A-\text{var}_{s_1}^{c}A-\text{var}_{s_2}^{\bar{t}}A<\frac{1}{2}.$$

Then, (7.12) becomes

$$\|x(s_2)-x(s_1)\|\leq\frac{1}{2}\|x\|_{BV([c,\bar{t}],X)}$$

which implies

$$\|x\|_{BV([c,\bar{t}],X)}\leq\frac{1}{2}\|x\|_{BV([c,\bar{t}],X)}\quad \text{and}\quad \|x\|_{BV([c,\bar{t}],X)}=0.$$

Consequently, $\|x(t)\|=0$, for all $t\in[c,\bar{t}]$, which contradicts the fact that $c=\sup C$. Therefore, $c=v$ and $x(t)=0$, for all $t\in[t_0,v]$.

Case 2: A is not right-continuous at c.

Define $\widetilde{A}:[t_0,v]\to X$ by

$$\widetilde{A}(t)=\begin{cases} A(t), & t\in[t_0,c],\\ A(t)-A(c^+)+A(c), & t\in(c,v].\end{cases}$$

Then, $\widetilde{A}$ is continuous at c. By Theorem B.0.6 and by the fact that $\|x(c)\|=0$, for all $t\geq c$, we have

$$\begin{aligned}\left\|\int_c^t d[A(s)]x(s)\right\| &\leq \left\|\lim_{\tau\to c^+}\int_\tau^t d[A(s)]x(s))\right\|+\|A(c^+)-A(c)\|\|x(c)\|\\ &\leq \left\|\lim_{\tau\to c^+}\int_\tau^t d[A(s)-A(c^+)+A(c)]x(s))\right\|\\ &\quad+\left\|\lim_{\tau\to c^+}\int_\tau^t d[A(c^+)-A(c)]x(s)\right\|\\ &= \left\|\lim_{\tau\to c^+}\int_\tau^t d[\widetilde{A}(s)]x(s)\right\|=\left\|\int_c^t d[\widetilde{A}(s)]x(s)\right\|.\end{aligned} \tag{7.13}$$

Define $\widetilde{V} : [t_0, v] \to X$ by

$$\widetilde{V}(t) = \operatorname{var}_{t_0}^{t} \widetilde{A}, \quad \text{for all } t \in [t_0, v].$$

Thus,

$$\operatorname{var}_{c}^{t} \widetilde{A} \leq \operatorname{var}_{c}^{t} A, \quad \text{for all } t \geq c$$

and, hence, $\widetilde{V}$ is well-defined. Moreover, $\widetilde{V}$ is nondecreasing and satisfies

$$\widetilde{V}(t) - \widetilde{V}(s) = \operatorname{var}_{s}^{t} \widetilde{A}, \quad \text{for all } s, t \in [t_0, v] \; s \leq t.$$

Once $\widetilde{A}$ is continuous at c, $\widetilde{V}$ is also continuous at c. Then, there exists $\bar{t} \in (c, v]$ such that

$$\widetilde{V}(\bar{t}) - \widetilde{V}(c) = \operatorname{var}_{c}^{\bar{t}} \widetilde{A} < \frac{1}{2}.$$

By Proposition B.0.7 and equation (7.13), for all $s_1, s_2 \in [c, \bar{t}]$, $s_1 < s_2$, we have

$$\|x(s_2) - x(s_1)\| = \left\| \int_{s_1}^{s_2} d[A(s)]x(s) \right\| \leq \left\| \int_{s_1}^{s_2} d[\widetilde{A}(s)]x(s) \right\| \leq \operatorname{var}_{s_1}^{s_2} \widetilde{A} \sup_{s \in [s_1, s_2]} \|x(s)\|$$

and, analogously to Case 1, we conclude that $\|x(t)\| = 0$, for all $t \in [c, v]$. Therefore, $c = v$.

From Cases 1 and 2, the only solution of (7.10) is the trivial one.

Now, since equations (7.4) and (7.10) are equivalent to

$$y - Ty = h \quad \text{and} \quad y - Ty = 0 \tag{7.14}$$

respectively, and T is a compact operator (see Theorem 7.1.1), the existence of the solutions of the linear VS integral equation (7.4) is guaranteed by Theorem 7.1.2 with $\lambda = 1$. In order to prove the uniqueness, assume that there exist two solutions $y_1 : [t_0, v] \to X$ and $y_2 : [t_0, v] \to X$ of (7.4) and define $z(t) = y_1(t) - y_2(t)$, for all $t \in [t_0, v]$. Then,

$$z(t) = \int_{t_0}^{t} d[A(s)]y_1(s) - \int_{t_0}^{t} d[A(s)]y_2(s) = \int_{t_0}^{t} d[A(s)][y_1(s) - y_2(s)] = \int_{t_0}^{t} d[A(s)]z(s)$$

and, hence, z is solution of the linear VS integral equation (7.10) and $z(t) = 0$, for all $t \in [t_0, v]$. □

We end this subsection by presenting a version of Theorem 7.1.3 for finite dimensional Banach spaces, where we consider T be defined in the larger space of regulated functions instead of the space of bounded variation functions. Its proofs is analogous to the proof of Theorem 7.1.3 and, therefore, we omit it here and we refer the interested reader to [5, Theorem 4.5] for more details.

Theorem 7.1.4. Let $v \in J$ and $A : J \to L(\mathbb{R}^n, Y)$ be locally of bounded variation on J and left-continuous. Then, the linear VS integral equation (7.4) admits a unique solution in $G([t_0, v], \mathbb{R}^n)$, for each given $h \in G([t_0, v], \mathbb{R}^n)$.

7.1.2 Dynamic equations

In this subsection, we consider homogeneous and nonhomogeneous linear dynamic equations on time scales, whose functions are Perron Δ-integrable and we prove the existence and uniqueness of their solutions (see Theorem 7.1.12). Moreover, we give a relation between the solutions of our dynamic equations and the solutions of the VS integral equation (7.10) (see Theorem 7.1.13).

Throughout this subsection, X is a Banach space equipped with a norm $\|\cdot\|$, $L(X)$ is the Banach space of continuous linear mappings $T: X \to X$, $\mathbb{T}$ is a time scale, $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$ and $G_0(\mathbb{T}_0, X)$ is the vector space described in Definition D.0.3. Moreover, we consider the following norm

$$\|f\|_{\mathbb{T}_0} = \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)} \|f(s)\|,$$

for all $f \in G_0(\mathbb{T}_0, X)$.

Consider the nonhomogeneous linear dynamic equation

$$x^{\Delta} = a(t)x + f(t) \tag{7.15}$$

and its corresponding homogeneous linear dynamic equation

$$x^{\Delta} = a(t)x \tag{7.16}$$

on a time scale $\mathbb{T}$, where both functions $a: \mathbb{T} \to L(X)$ and $f: \mathbb{T} \to X$ satisfy the following conditions:

(T1) the Perron Δ-integrals

$$\int_{t_1}^{t_2} f(s)\Delta s \quad \text{and} \quad \int_{t_1}^{t_2} a(s)y(s)\Delta s$$

exist for all $t_1, t_2 \in \mathbb{T}_0$, whenever $y: \mathbb{T}_0 \to X$ is regulated;

(T2) there is a locally Perron Δ-integrable function $L: \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} a(s)[z(s) - y(s)]\Delta s \right\| \leq \|z - y\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s,$$

for all $z, y \in G_0(\mathbb{T}_0, X)$ and all $t_1, t_2 \in \mathbb{T}_0$;

(T3) there is a locally Perron Δ-integrable function $K: \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} f(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s,$$

for all $t_1, t_2 \in \mathbb{T}_0$.

In what follows, we present a definition of a solution for the dynamic equations (7.15) and (7.16).

Definition 7.1.5. Let $t_0 \in \mathbb{T}$ and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$. We say that a function $\bar{x} : \mathbb{T}_0 \to X$ is a *solution of the nonhomogeneous dynamic equation* (7.15) with initial condition $\bar{x}(t_0) = x_0 \in X$, if it is satisfies

$$\bar{x}(t) = x_0 + \int_{t_0}^{t} a(s)\bar{x}(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s, \quad \text{for all } t \in \mathbb{T}_0.$$

Moreover, a function $x : \mathbb{T}_0 \to X$ is said to be a *solution of the homogeneous dynamic equation* (7.16) with initial condition $x(t_0) = x_0 \in X$, if the equality

$$x(t) = x_0 + \int_{t_0}^{t} a(s)x(s)\Delta s$$

holds for all $t \in \mathbb{T}_0$.

The next result gives an interesting property of the solutions of the dynamic equations (7.15) and (7.16).

Theorem 7.1.6. The solutions of the dynamic equations (7.15) and (7.16) are rd-continuous.

Proof. Let $\bar{x} : \mathbb{T}_0 \to X$ be a solution of the nonhomogeneous dynamic equation (7.15). By Definition 7.1.5, for all $t \in \mathbb{T}_0$, we have

$$\bar{x}(t) = \bar{x}(t_0) + \int_{t_0}^{t} a(s)\bar{x}(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s.$$

By Theorem D.0.10, the Perron-Stieltjes integrals $\int_{t_0}^{t} a^*(s)\bar{x}^*(s)dg(s)$ and $\int_{t_0}^{t} f^*(s)dg(s)$ exist for all $t \in \mathbb{T}_0$, where $g(t) = t^*$ for all $t \in \mathbb{T}_0^*$. Moreover,

$$\int_{t_0}^{t} a^*(s)\bar{x}^*(s)dg(s) = \int_{t_0}^{t} a(s)\bar{x}(s)\Delta s \quad \text{and} \quad \int_{t_0}^{t} f^*(s)dg(s) = \int_{t_0}^{t} f(s)\Delta s,$$

for all $t \in \mathbb{T}_0$.

Since $g|_{\mathbb{T}_0}$ is the identity function, is it clear that $g|_{\mathbb{T}_0}$ is right-continuous and, by Lemma D.0.9, g is rd-continuous on $\mathbb{T}_0^*$. Then, by Theorem B.0.17, the functions

$$t \ni \mathbb{T}_0^* \mapsto \int_{t_0}^{t} a^*(s)\bar{x}^*(s)dg(s) \quad \text{and} \quad t \ni \mathbb{T}_0^* \mapsto \int_{t_0}^{t} f^*(s)dg(s)$$

are rd-continuous on $\mathbb{T}_0^*$ which, in turn, imply that $\bar{x}$ is rd-continuous.

The proof of the fact that the solution of homogeneous dynamic equation (7.16) is rd-continuous is analogous and, therefore, we omit it here. □

In the following remark, we show that the sets $G_0(\mathbb{T}_0^*, X)$ and $G_0(\mathbb{T}_0, X)$ can be related, where $G_0(\mathbb{T}_0^*, X)$ is given in Definition A.0.22-(iv)

Remark 7.1.7. Let $y \in G_0(\mathbb{T}_0^*, X)$. Since $\mathbb{T}_0 \subset \mathbb{T}_0^*$ we have

$$\|y\|_{\mathbb{T}_0} = \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)}\|y(s)\| \leq \sup_{s \in \mathbb{T}_0^*} e^{-(s-t_0)}\|y(s)\| = \|y\|_{\mathbb{T}_0^*} < \infty.$$

Therefore, $y|_{\mathbb{T}_0} \in G_0(\mathbb{T}_0, X)$.

We target to prove that the solutions of the dynamic equations (7.15) and (7.16) are related to solutions of a linear VS integral equations.

Define $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ by

$$A(t)y = \int_{t_0}^{t} a^*(s)y(s)dg(s), \tag{7.17}$$

for all $t \in \mathbb{T}_0^*$ and all $y \in G_0(\mathbb{T}_0^*, X)$, where $g(s) = s^*$ for every $s \in \mathbb{T}_0^*$, the integral in (7.17) is in the sense of Perron-Stieltjes and $L(G_0(\mathbb{T}_0^*, X), X)$ denotes the space of linear continuous mappings defined from $G_0(\mathbb{T}_0^*, X)$ into X.

Notice that, by Remark 7.1.7, $z = y|_{\mathbb{T}_0}$ belongs to $G_0(\mathbb{T}_0, X)$ for all $y \in G_0(\mathbb{T}_0^*, X)$ and, by condition (T1), the Perron Δ-integral $\int_{t_0}^{t} a(s)z(s)\Delta s$ exists for all $t \in \mathbb{T}_0$. By Theorem D.0.10, the Perron-Stieljes integral $\int_{t_0}^{t} a^*(s)z^*(s)dg(s) = \int_{t_0}^{t} a^*(s)y(s)dg(s)$ also exists for all $t \in \mathbb{T}_0^*$. Therefore, A is well-defined. Moreover, the next result shows that A is locally of bounded variation on $\mathbb{T}_0^*$.

Lemma 7.1.8. Assume that $a : \mathbb{T} \to L(X)$ satisfies condition (T2). Then, $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$, defined by (7.17), is locally of bounded variation on $\mathbb{T}_0^*$.

Proof. By the definition of $G_0(\mathbb{T}_0^*, X)$, $\|y\|_{\mathbb{T}_0^*} < \infty$ for all $y \in G_0(\mathbb{T}_0^*, X)$ (see Remark 7.1.7). Then, by condition (T2), for every $a, b \in \mathbb{T}_0^*$ and every division $d = (s_i)$ of $[a, b]$, we have

$$\begin{aligned}
\sum_{i=1}^{|d|} \|A(s_i)y - A(s_{i-1})y\| &= \sum_{i=1}^{|d|} \left\| \int_{t_0}^{s_i} a^*(s)y(s)dg(s) - \int_{t_0}^{s_{i-1}} a^*(s)y(s)dg(s) \right\| \\
&\overset{\text{Th. D.0.12}}{=} \sum_{i=1}^{|d|} \left\| \int_{t_0}^{s_i^*} a(s)y(s)\Delta s - \int_{t_0}^{s_{i-1}^*} a(s)y(s)\Delta s \right\| \\
&= \sum_{i=1}^{|d|} \left\| \int_{s_{i-1}^*}^{s_i^*} a(s)y(s)\Delta s \right\| \\
&\overset{(A3)}{\leq} \sum_{i=1}^{|d|} \left(\|y\|_{\mathbb{T}_0} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right) \\
&= \|y\|_{\mathbb{T}_0} \left(\sum_{i=1}^{|d|} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right) \\
&\overset{\text{Rem.7.1.7}}{\leq} \|y\|_{\mathbb{T}_0^*} \left(\sum_{i=1}^{|d|} \int_{s_{i-1}^*}^{s_i^*} L(s)\Delta s \right) \\
&= \|y\|_{\mathbb{T}_0^*} \int_{a^*}^{b^*} L(s)\Delta s < \infty.
\end{aligned}$$

Taking the supremum over all divisions $d = (s_i)$ of $[a, b]$, we obtain $\operatorname{var}_a^b A < \infty$ and the proof is complete, since the statement holds for all $a, b \in \mathbb{T}_0^*$. □

The next result ensures the existence of the Perron-Stieltjes integral of y with respect to A, whenever y is a regulated function defined on a closed interval.

Theorem 7.1.9. Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17). If $y \in G([t_0, v], X)$, then the Perron-Stieltjes integral

$$\int_{t_0}^{v} d[A(s)]y(s)$$

exists.

Proof. Consider $\widehat{X} = L(G_0(\mathbb{T}_0^*, X), X)$ and $\widehat{Y} = X = \widehat{Z}$. Then, $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ are Banach spaces once $G_0(\mathbb{T}_0^*, X)$ and X are Banach spaces (see Proposition A.0.23). Define $B : \widehat{X} \times \widehat{Y} \to \widehat{Z}$ by

$$B(F, x) = F z_x,$$

where $z_x : \mathbb{T}_0^* \to X$ is given by $z_x(t) = x$ for all $t \in \mathbb{T}_0^*$. It is clear that $z_x \in G_0(\mathbb{T}_0^*, X)$ and $\mathscr{B} = (\widehat{X}, \widehat{Y}, \widehat{Z})$ is a bilinear triple. Moreover, by Lemma 7.1.8, $\operatorname{var}_{t_0}^{v} A < \infty$ and, by Proposition B.0.8, the Perron-Stieltjes integral $\int_{t_0}^{v} d[A(s)]y(s)$ exists for all $y \in G([t_0, v], X)$. □

In the sequel, we present a relation between the Perron-Stieltjes integral $\int d[A(s)]y(s)$ and the Perron-Stieltjes Δ-integral $\int a(s)y(s)\Delta s$. Its proof follows the same ideas as in [21, Theorem 4.7].

Theorem 7.1.10. Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17). If $v \in \mathbb{T}_0^*$, then

$$\int_{t_0}^{t} d[A(s)]y(s) = \int_{t_0}^{t^*} a(s)y(s)\Delta s,$$

for all $t \in [t_0, v]$ and all $y \in G([t_0, v], X)$.

Proof. We start by proving that if $v \in \mathbb{T}_0^*$ and $\varphi : [t_0, t] \to X$ is a step function, then

$$\int_{t_0}^{t} d[A(s)]\varphi(s) = \int_{t_0}^{t^*} a(s)\varphi(s)\Delta s,$$

for all $t \in [t_0, v]$. By Remark A.0.8, φ is regulated on $[t_0, v]$ and, by Proposition B.0.8, the Perron-Stieltjes integral $\int_{t_0}^{t} d[A(s)]\varphi(s)$ exists for all $t \in [t_0, v]$. Let $t \in [t_0, v]$ be fixed. Since φ is a step function, there exists a division $d = (s_i)$ of $[t_0, t]$ and $c_1, \ldots, c_{|d|} \in X$ such that

$$\varphi(s) = c_i, \quad \text{for every } s \in (s_{i-1}, s_i).$$

Thus, by the definition of A, if $i \in \{1, \ldots, |d|\}$ and $s_{i-1} < t_1 < t_2 < s_i$, we have

$$\begin{aligned} \int_{t_1}^{t_2} d[A(s)]\varphi(s) &= \int_{t_1}^{t_2} d[A(s)]c_i = A(t_2)c_i - A(t_1)c_i \\ &= \int_{t_1}^{t_2} a^*(s)c_i dg(s) = \int_{t_1}^{t_2} a^*(s)\varphi(s)dg(s). \end{aligned} \tag{7.18}$$

By Theorem B.0.17, we get

$$\lim_{\xi \to s_{i-1}^+} \int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s) = a^*(s_{i-1})\varphi(s_{i-1})\Delta^+ g(s_{i-1}) \tag{7.19}$$

and

$$\lim_{\xi\to s_{i-1}^+}\int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s)=\int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s)-a^*(s_{i-1})\varphi(s_{i-1})\Delta^+g(s_{i-1}), \tag{7.20}$$

for all $\tau\in(s_{i-1},s_i)$.

Equation (7.19) together with equation (7.20), imply

$$\lim_{\xi\to s_{i-1}^+}\left(\int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s)+\int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s)\right)=\int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s), \tag{7.21}$$

or all $\tau\in(s_{i-1},s_i)$. Moreover, by Theorem B.0.6, if $\tau\in(s_{i-1},s_i)$, then

$$\begin{aligned}
\int_{s_{i-1}}^{\tau} d[A(s)]\varphi(s) &= \lim_{\xi\to s_{i-1}^+}\left(\int_{\xi}^{\tau} d[A(s)]\varphi(s)+A(\xi)\varphi(s_{i-1})-A(s_{i-1})\varphi(s_{i-1})\right)\\
&= \lim_{\xi\to s_{i-1}^+}\left(\int_{\xi}^{\tau} d[A(s)]\varphi(s)+\int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s)\right)\\
&\overset{(7.18)}{=} \lim_{\xi\to s_{i-1}^+}\left(\int_{\xi}^{\tau} a^*(s)\varphi(s)dg(s)+\int_{s_{i-1}}^{\xi} a^*(s)\varphi(s_{i-1})dg(s)\right)\\
&\overset{(7.21)}{=} \int_{s_{i-1}}^{\tau} a^*(s)\varphi(s)dg(s).
\end{aligned}$$

Therefore,

$$\int_{s_{i-1}}^{\tau} d[A(s)]\varphi(s)=\int_{s_{i-1}}^{\tau} a^*(s)\varphi(s_{i-1})dg(s), \tag{7.22}$$

for all $i\in\{1,\dots,|d|\}$ and all $\tau\in(s_{i-1},s_i)$.

Analogously,

$$\int_{\tau}^{s_i} d[A(s)]\varphi(s)=\int_{\tau}^{s_i} a^*(s)\varphi(s_{i-1})dg(s), \tag{7.23}$$

for all $i\in\{1,\dots,|d|\}$ and all $\tau\in(s_{i-1},s_i)$. By (7.22) and (7.23), we obtain

$$\int_{s_{i-1}}^{s_i} d[A(s)]\varphi(s)=\int_{s_{i-1}}^{s_i} a^*(s)\varphi(s)dg(s),\quad \text{for all } i\in\{1,\dots,|d|\}. \tag{7.24}$$

Therefore, by (7.24) and Theorem D.0.12, we conclude

$$\begin{aligned}
\int_{t_0}^{t} d[A(s)]\varphi(s) &= \sum_{i=1}^{|d|}\int_{s_{i-1}}^{s_i} d[A(s)]\varphi(s)=\sum_{i=1}^{|d|}\int_{s_{i-1}}^{s_i} a^*(s)\varphi(s)dg(s)\\
&= \int_{t_0}^{t} a^*(s)\varphi(s)dg(s)=\int_{t_0}^{t^*} a(s)\varphi(s)\Delta s.
\end{aligned} \tag{7.25}$$

In order to prove the assertion to regulated functions, we consider $y\in G([t_0,v],X)$ and a sequence of step functions $\varphi_k:[t_0,v]\to X$, $k\in\mathbb{N}$ such that

$$\lim_{k\to\infty}\|\varphi_k(s)-y(s)\|_{\infty}=0, \tag{7.26}$$

where the existence of such a sequence is guaranteed by Theorem A.0.9. By equation (7.25), we get

$$\int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s, \quad \text{for all } k \in \mathbb{N} \tag{7.27}$$

and, by Theorem B.0.5, we have

$$\lim_{k\to\infty} \int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t} d[A(s)]y(s). \tag{7.28}$$

Combining (7.27) with (7.28), we obtain

$$\lim_{k\to\infty} \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s = \lim_{k\to\infty} \int_{t_0}^{t} d[A(s)]\varphi_k(s) = \int_{t_0}^{t} d[A(s)]y(s).$$

We conclude the proof by showing

$$\lim_{k\to\infty} \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s = \int_{t_0}^{t^*} a(s)y(s)\Delta s. \tag{7.29}$$

At first, notice that the functions $\widehat{\varphi}_k, \widehat{y} : \mathbb{T}_0^* \to X$ defined by

$$\widehat{\varphi}_k(t) = \begin{cases} \varphi_k(t), & t \in [t_0, v] \\ \varphi_k(v) & t \in \mathbb{T}_0^* \setminus [t_0, v] \end{cases}$$

and

$$\widehat{y}(t) = \begin{cases} y(t), & t \in [t_0, v] \\ y(v) & t \in \mathbb{T}_0^* \setminus [t_0, v] \end{cases}$$

belong to $G_0(\mathbb{T}_0^*, X)$ and

$$\|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0} \le \|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0^*} \le \|\widehat{\varphi}_k - \widehat{y}\|_\infty = \|\varphi_k - y\|_\infty.$$

Moreover, condition (T2) implies

$$\begin{aligned} \left\| \int_{t_0}^{t^*} a(s)\varphi_k(s)\Delta s - \int_{t_0}^{t^*} a(s)y(s)\Delta s \right\| &= \left\| \int_{t_0}^{t^*} a(s)[\widehat{\varphi}_k(s) - \widehat{y}(s)]\Delta s \right\| \\ &\le \|\widehat{\varphi}_k - \widehat{y}\|_{\mathbb{T}_0} \int_{t_0}^{t^*} L(s)\Delta s \\ &\le \|\varphi_k - y\|_\infty \int_{t_0}^{t^*} L(s)\Delta s, \end{aligned} \tag{7.30}$$

and, hence, (7.29) follows by (7.26) and (7.30). □

In the next result, we prove the existence and uniqueness of a solution of a linear VS integral equation with which the dynamic equations (7.15) and (7.16) will be related to.

Theorem 7.1.11. Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17). Then, the linear VS integral equation

$$y(t) = \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v], \tag{7.31}$$

admits a unique solution in $BV([t_0, v], X)$, for any $h \in BV([t_0, v], X)$.

Proof. Once $g|_{\mathbb{T}_0}$ is the identity function, Lemma D.0.9 ensures that g is left-continuous on $\mathbb{T}_0^*$ and, by Theorem B.0.17, A is left-continuous. Therefore, the statement follows by Theorem 7.1.3 and Lemma 7.1.8. □

In what follows, we prove the existence and uniqueness of a solution for the dynamic equations (7.15) and (7.16). We point out that we do not require the rd-continuity of the functions involved and, for this reason, this result is more general than the ones found in the literature. See [3, 10, 11, 37], for example.

Theorem 7.1.12. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions (T1), (T2) and (T3). Then, the dynamic equations (7.15) and (7.16) admit unique solutions.

Proof. We start by proving the existence and uniqueness of a solution of the nonhomogeneous dynamic equation (7.15).

Define $\widetilde{h} : \mathbb{T}_0^* \to X$ by

$$\widetilde{h}(t) = \int_{t_0}^{t} f^*(s)dg(s), \quad \text{for all } t \in \mathbb{T}_0^*,$$

where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. By condition (T1) and Theorem D.0.10, the Perron-Stieltjes integral $\int_{t_0}^{t} f^*(s)dg(s)$ exists for all $t \in \mathbb{T}_0^*$ and, consequently, $\widetilde{h}$ is well-defined. Moreover, by Theorem D.0.12 and condition (T3), we have

$$\begin{aligned}
|\widetilde{h}(t_2) - \widetilde{h}(t_1)| &= \left\| \int_{t_0}^{t_2} f^*(s)dg(s) - \int_{t_0}^{t_1} f^*(s)dg(s) \right\| \\
&\overset{\text{Th.D.0.12}}{=} \left\| \int_{t_0}^{t_2^*} f(s)\Delta s - \int_{t_0}^{t_1^*} f(s)\Delta s \right\| \\
&= \left\| \int_{t_0}^{t_1^*} f(s)\Delta s + \int_{t_1^*}^{t_2^*} f(s)\Delta s - \int_{t_0}^{t_1^*} f(s)\Delta s \right\| \\
&= \left\| \int_{t_1^*}^{t_2^*} f(s)\Delta s \right\| \\
&\overset{\text{Cond.(A4)}}{\leq} \int_{t_1^*}^{t_2^*} K(s)\Delta s < \infty,
\end{aligned} \tag{7.32}$$

for all $t_1, t_2 \in \mathbb{T}_0^*$ and, hence, $\widetilde{h} \in BV([a,b], X)$ for all $[a,b] \subset \mathbb{T}_0^*$. Consequently, if $x_0 \in X$, then $h : \mathbb{T}_0^* \to X$, defined by

$$h(t) = \widetilde{h}(t) + x_0,$$

is locally of bounded variation on $\mathbb{T}_0^*$.

Let $v \in \mathbb{T}_0^*$ be arbitrary. By Theorem 7.1.11, there exists a unique solution $y : [t_0, v] \to X$ of the linear VS integral equation (7.31) in $BV([t_0, v], X)$, that is,

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s), \quad \text{for all } t \in [t_0, v], \tag{7.33}$$

where $A:\mathbb{T}_0^*\to L(G_0(\mathbb{T}_0^*,X),X)$ is given by (7.17). By Theorems D.0.12 and 7.1.10, we have

$$\int_{t_0}^{t} d[A(s)]y(s)=\int_{t_0}^{t^*} a(s)y(s)\Delta s,\quad \text{for all } t\in[t_0,v] \quad \text{and}$$

$$\int_{t_0}^{t} f^*(s)dg(s)=\int_{t_0}^{t^*} f(s)\Delta s\quad \text{for all } t\in[t_0,v].$$

Thus,

$$y(t)=x_0+\int_{t_0}^{t^*} a(s)y(s)\Delta s+\int_{t_0}^{t^*} f(s)\Delta s,\quad \text{for all } t\in[t_0,v]. \tag{7.34}$$

Consider the set

$$S:=\{y:[t_0,v_y]\to X;\ v_y\in\mathbb{T}_0^* \text{ and } y \text{ is a solution of (7.31) in } BV([t_0,v_y],X) \text{ with } y(t_0)=x_0\}.$$

Since (7.33) holds for every $v\in\mathbb{T}_0^*$, the set S is nonempty and $\mathbb{T}_0^*=\bigcup_{y\in S}[t_0,v_y]$. Moreover, if $y_1,y_2\in S$, then either $[t_0,v_{y_1}]\subset[t_0,v_{y_2}]$ or $[t_0,v_{y_2}]\subset[t_0,v_{y_1}]$. Assume, without loss of generality, that $[t_0,v_{y_1}]\subset[t_0,v_{y_2}]$. It is clear that $y_2|_{[t_0,v_{y_1}]}\in BV([t_0,v_{y_1}],X)$ and, by the uniqueness of a solution (see Theorem 7.1.3), $y_2|_{[t_0,v_{y_1}]}=y_1$. Therefore,

$$y_1(t)=y_2(t),\quad \text{for all } t\in[t_0,v_{y_1}]\cap[t_0,v_{y_2}] \text{ and all } y_1,y_2\in S. \tag{7.35}$$

Define $z:\mathbb{T}_0^*\to X$ by $z(t)=y(t)$, whenever $y\in S$ and $t\in[t_0,v_y]$. By (7.35), z is well-defined and, by (7.34), we have

$$z(t)=x_0+\int_{t_0}^{t^*} a(s)z(s)\Delta s+\int_{t_0}^{t^*} f(s)\Delta s,\quad \text{for all } t\in\mathbb{T}_0^*.$$

Moreover, owing to the fact that $t^*=t$ for all $t\in\mathbb{T}_0$, we have

$$z(t)=x_0+\int_{t_0}^{t} a(s)z(s)\Delta s+\int_{t_0}^{t} f(s)\Delta s,\quad \text{for all } t\in\mathbb{T}_0.$$

Consequently, $x:\mathbb{T}_0\to X$, defined by $x=z|_{\mathbb{T}_0}$, is a solution of the dynamic equation (7.15) with initial condition $x(t_0)=x_0$. The uniqueness of x follows directly from the uniqueness of a solution of (7.31).

The proof for the homogeneous dynamic equation (7.16) follows by taking $h(t)=x_0$, for all $t\in\mathbb{T}_0^*$ and, therefore, we omit it here. □

As a direct consequence of the proof of Theorem 7.1.12, we have the following result.

Theorem 7.1.13. Let $A:\mathbb{T}_0^*\to L(G_0(\mathbb{T}_0^*,X),X)$ be given by (7.17) and $x_0\in X$. Assume that $a:\mathbb{T}\to L(X)$ and $f:\mathbb{T}\to X$ satisfy conditions (T1), (T2) and (T3). Then, the following statements holds.

(i) If $y : \mathbb{T}_0^* \to X$ is a function such that the equality

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s) \tag{7.36}$$

holds for all $t \in \mathbb{T}_0$, then $x := y|_{\mathbb{T}_0}$ is the solution of the nonhomogeneous dynamic equation (7.15) with initial condition x_0. Conversely, if $x : \mathbb{T}_0 \to X$ is the solution of the nonhomogeneous dynamic equation (7.15) with initial condition x_0, then $x = y|_{\mathbb{T}_0}$, where y is given by (7.36).

(ii) If the function $y : \mathbb{T}_0^* \to X$ satisfies

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s), \tag{7.37}$$

for all $t \in \mathbb{T}$, then $x := y|_{\mathbb{T}_0}$ is the solution of the homogeneous dynamic equation (7.16) with initial condition x_0. Reciprocally, if $x : \mathbb{T}_0 \to X$ is the solution of the homogeneous dynamic equation (7.16) with initial condition x_0, then $x = y|_{\mathbb{T}_0}$, where y is given by (7.37).

In the sequel, we present a version of Theorem 7.1.12 when $X = \mathbb{R}^n$. We highlight that, in this case, condition (T3) is not required.

Theorem 7.1.14. Assume that $X = \mathbb{R}^n$ and let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17) and $x_0 \in X$. If $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions (T1) and (T2), then there exists a unique solution of the nonhomogeneous dynamic equation (7.15).

Proof. By condition (T1) and Theorems B.0.17 and D.0.12, for all $v \in \mathbb{T}_0^*$, the function $h : [t_0, v] \to X$, given by

$$h(t) = \int_{t_0}^{t} f^*(s)dg(s), \quad t \in [t_0, v],$$

where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$ is well-defined and $h \in G([t_0, v], X)$. Then, by Theorem 7.1.4, Lemma 7.1.8 and by the fact that $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$, defined by (7.17), is left-continuous, we conclude that for all $x_0 \in X$, there exists a unique function $y : [t_0, v] \to X$ for which

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + h(t), \quad t \in [t_0, v].$$

By Theorems D.0.12 and 7.1.10, we have

$$y(t) = x_0 + \int_{t_0}^{t^*} a(s)y(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s, \quad t \in [t_0, v].$$

Using the same ideas as in the proof of Theorem 7.1.12, we obtain a unique function $z : \mathbb{T}_0^* \to X$ such that

$$z(t) = x_0 + \int_{t_0}^{t^*} a(s)z(s)\Delta s + \int_{t_0}^{t^*} f(s)\Delta s, \quad t \in \mathbb{T}_0^*$$

and, consequently, $x = z|_{\mathbb{T}_0}$ is a solution of the nonhomogeneuous dynamic equation (7.15) with initial condition $x(t_0) = x_0$. □

7.2 Variation-of-constant formula

This section is devoted to establish a variation-of-constant formula for the nonhomogeneous dynamic equation (7.15). At first, we need some auxiliary results.

Theorem 7.2.1. Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17). Assume that $\Phi \in BV([t_0, v], L(X))$, then the Perron-Stieltjes integral

$$\int_{t_0}^{v} d[A(s)]\Phi(s)$$

exists.

Proof. Consider $\widehat{X} = L(G_0(\mathbb{T}_0^*, X), X)$ and $\widehat{Y} = L(X) = \widehat{Z}$. Since $G_0(\mathbb{T}_0^*, X)$ and X are Banach spaces (see Proposition A.0.23), $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ are also Banach spaces. Define $B : \widehat{X} \times \widehat{Y} \to \widehat{Z}$ by

$$B(F, G) = G \circ F.$$

It is clear that $\mathscr{B} = (\widehat{X}, \widehat{Y}, \widehat{Z})$ is a bilinear triple and, by Lemma 7.1.8, $\mathrm{var}_{t_0}^{v} A < \infty$. Therefore, by Proposition B.0.7, the Perron-Stieltjes integral $\int_{t_0}^{v} d[A(s)]\Phi(s)$ exists for all $\Phi \in G([t_0, v], L(X))$, in particular, $\int_{t_0}^{v} d[A(s)]\Phi(s)$ exists for all $\Phi \in BV([t_0, v], L(X))$. □

Henceforward, we denote by $I \in L(X)$ the identity operator. Using similar arguments as in Theorem 7.1.3, we obtain the following result.

Theorem 7.2.2. Let $v \in \mathbb{T}_0^*$ and $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17). Then, for a given $s \in [t_0, v]$, the equation

$$\Phi(t) = I + \int_{s}^{t} d[A(s)]\Phi(s), \quad t \in [t_0, v], \tag{7.38}$$

admits at least one nontrivial solution in $BV([t_0, v], L(X))$.

The next result ensures the existence of a fundamental operator associated to the dynamic equation (7.16).

Theorem 7.2.3. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions (T1), (T2) and (T3). Then, there exists a unique operator $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$, called *fundamental operator of the homogeneous linear dynamic equation* (7.16), such that $U(t, t) = I$, for all $t \in \mathbb{T}_0$, and if $x : \mathbb{T}_0 \to X$ is a solution of the homogeneous dynamic equation (7.16) with initial condition $x(t_0) = x_0 \in X$, then

$$x(t) = U(t, t_0)x_0,$$

for all $t \in \mathbb{T}_0$.

Proof. By Theorem 7.2.2, the set S, defined by

$$S := \{\Phi_i : [t_0, v_i] \to L(X);\ v_i \in \mathbb{T}_0^* \text{ and } \Phi_i \text{ is a solution of (7.38) in } BV([t_0, v_i], L(X))\},$$

is nonempty. Define $\Phi : \mathbb{T}_0^* \to L(X)$ by $\Phi(t) = \Phi_i(t)$, where $\Phi_i \in S$ and $t \in [t_0, v_i]$. Notice that Φ is well-defined since $\mathbb{T}_0^* = \bigcup_{\Phi_i \in S} [t_0, v_i]$ and, by the uniqueness of a solution of (7.38), if $\Phi_i, \Phi_j \in S$ with $i \neq j$, then $\Phi_i(t) = \Phi_j(t)$ for all $t \in [t_0, v_i] \cap [t_0, v_j]$. Thus, for all $t \in \mathbb{T}_0^*$, we have

$$\Phi(t) = I + \int_{t_0}^{t} d[A(s)]\Phi(s) \tag{7.39}$$

and Φ is locally of bounded variation on $\mathbb{T}_0^*$.

Define $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ by

$$V(t, \tau) = I + \int_{\tau}^{t} d[A(s)]\Phi(s), \quad \text{for all } t, \tau \in \mathbb{T}_0^*. \tag{7.40}$$

Then, $V(t, t_0) = \Phi(t)$ and $V(t, t) = I$ for all $t \in \mathbb{T}_0^*$.

Define $y : \mathbb{T}_0^* \to X$ by $y(t) = V(t, t_0)x_0$ for all $t \in \mathbb{T}_0^*$. Then,

$$\begin{aligned}
\int_{t_0}^{t} d[A(s)]y(s) &= \int_{t_0}^{t} d[A(s)]V(s, t_0)x_0 \\
&= \left(\int_{t_0}^{t} d[A(s)]\Phi(s) \right) x_0 \\
&= (\Phi(t) - I)x_0 = (V(t, t_0) - I)x_0 \\
&= y(t) - x_0
\end{aligned}$$

and, by Theorem 7.1.13-(i), $x := y|_{\mathbb{T}_0}$ is the solution of the homogeneous dynamic equation (7.16) with initial condition x_0. Therefore, the statement follows by defining $U := V|_{\mathbb{T}_0 \times \mathbb{T}_0}$. □

In what follows, we present some properties of the operator $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ defined by (7.40). Their proofs can be found in [16, Theorem 4.3].

Theorem 7.2.4. Let $s \in \mathbb{T}_0^*$ and $V(\cdot, s) : \mathbb{T}_0^* \to L(X)$ be defined by

$$V(t, s) = I + \int_{s}^{t} d[A(r)]\Phi(r), \quad \text{for all } t \in \mathbb{T}_0^*,$$

where Φ is given by (7.39). Then, V satisfies the following properties:

(i) for all $t, \tau \in \mathbb{T}_0^*$, we have

$$V(t, \tau) = I + \int_{\tau}^{t} d[A(r)]V(r, \tau);$$

(ii) $V(t, t) = I$ for all $t \in \mathbb{T}_0^*$;

(iii) $V(t, \tau) = V(t, r)V(r, \tau)$ for all $t, \tau, r \in \mathbb{T}_0^*$;

(iv) there exists $[V(t,\tau)]^{-1} \in L(X)$ and $[V(t,\tau)]^{-1} = V(\tau,t)$ for all $t,\tau \in \mathbb{T}_0^*$;

(v) $V(\cdot,\tau)$ and $V(\tau,\cdot)$ belongs to $BV([a,b],L(X))$ for all $[a,b] \subset \mathbb{T}_0^*$ and all $\tau \in \mathbb{T}_0^*$;

(vi) for every compact set $[a,b] \subset \mathbb{T}_0^*$, there exists a constant $M > 0$ such that

$$\|V(t,\tau)\| \leq M, \quad \text{for all } t,\tau \in [a,b].$$

Using similar ideas as in the proof of Theorem 7.1.10 and, by Theorem 7.2.4, we obtain the next result.

Theorem 7.2.5. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions (T1), (T2) and (T3). Then, the fundamental operator of the homogeneous linear dynamic equation (7.16) $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ given in Theorem 7.2.3, satisfies the following properties:

(i) for all $s,r \in \mathbb{T}_0$, we have

$$U(s,r) = I + \int_r^s a(\tau)U(\tau,r)\Delta\tau;$$

(ii) $U(t,t) = I$ for all $t \in \mathbb{T}_0$;

(iii) $U(t,\tau) = U(t,r)U(r,\tau)$ for all $t,\tau,r \in \mathbb{T}_0$;

(iv) there exists $[U(t,\tau)]^{-1} \in L(X)$ and $[U(t,\tau)]^{-1} = U(\tau,t)$ for all $t,\tau \in \mathbb{T}_0$;

(v) $U(\cdot,\tau)$ and $U(\tau,\cdot)$ are regulated on $[a,b]_{\mathbb{T}}$, for all $[a,b]_{\mathbb{T}} \subset \mathbb{T}_0$ and all $\tau \in \mathbb{T}_0$;

(vi) for every compact set $[a,b]_{\mathbb{T}} \subset \mathbb{T}_0$, there exists a constant $M > 0$ such that

$$\|U(t,\tau)\| \leq M, \quad \text{for all } t,\tau \in [a,b]_{\mathbb{T}}.$$

The next result is a variation-of-constant formula for a nonhomogeneous linear VS integral equation. Its proof is similar to the proof of [16, Theorem 4.10] and, therefore, we omit it here.

Theorem 7.2.6. Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*,X),X)$ be given by (7.17) and $h : \mathbb{T}_0^* \to X$ be a function such that $h \in BV([a,b],X)$ for all $[a,b] \subset \mathbb{T}_0^*$. If $y : \mathbb{T}_0^* \to X$ is a function such that the equality

$$y(t) = y(t_0) + \int_{t_0}^t d[A(s)]y(s) + h(t)$$

holds for all $t \in \mathbb{T}_0^*$, then

$$y(t) = V(t,t_0)y(t_0) + h(t) - h(t_0) - \int_{t_0}^t d_s[V(t,s)](h(s) - h(t_0)), \quad t \in \mathbb{T}_0^*,$$

where $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ is given by

$$V(t,\tau) = I + \int_\tau^t d[A(s)]V(s,\tau), \quad t,\tau \in \mathbb{T}_0^*.$$

Using Theorem 7.2.6 and the relation between the solution of the nonhomogeneous dynamic equation (7.15) and the solution of a linear VS integral equation, given in Theorem 7.1.13, we obtain a variation-of-constant formula for the dynamic equation (7.15).

Theorem 7.2.7. Let $A : \mathbb{T}_0^* \to L(G_0(\mathbb{T}_0^*, X), X)$ be given by (7.17) and $x_0 \in X$. Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions (T1), (T2) and (T3). Then, the solution of the dynamic equation (7.16), with initial condition x_0, is given by

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,s)f(s)\Delta s, \quad s \in \mathbb{T}_0,$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 7.2.3.

Proof. Let $x_0 \in X$. By Theorem 7.1.13-(i), if $x : \mathbb{T}_0 \to X$ is the unique solution of the dynamic equation (7.16) with initial condition x_0, then $x = y|_{\mathbb{T}_0}$, where $y : \mathbb{T}_0^* \to X$ is given by

$$y(t) = x_0 + \int_{t_0}^{t} d[A(s)]y(s) + \int_{t_0}^{t} f^*(s)dg(s), \quad t \in \mathbb{T}_0^*.$$

For all $t \in \mathbb{T}_0^*$, consider $h(t) = \int_{t_0}^{t} f^*(s)dg(s)$, where $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. Then, by Theorem D.0.12 and condition (T3), h is locally of bounded variation on $\mathbb{T}_0^*$ (see (7.32)) and, by Theorem 7.2.6, we have

$$y(t) = V(t,t_0)x_0 + h(t) - h(t_0) - \int_{t_0}^{t} d_s[V(t,s)](h(s) - h(t_0)), \quad t \in \mathbb{T}_0^*, \tag{7.41}$$

where $V : \mathbb{T}_0^* \times \mathbb{T}_0^* \to L(X)$ is given by

$$V(t,\tau) = I + \int_{\tau}^{t} d[A(s)]V(s,\tau), \quad t,\tau \in \mathbb{T}_0^*.$$

By Theorems B.0.9 and 7.2.4, for all $t \in \mathbb{T}_0^*$, we have

$$\begin{aligned} \int_{t_0}^{t} d_s[V(t,s)](h(s) - h(t_0)) &= \int_{t_0}^{t} d_s[V(t,s)]h(s) \\ &\overset{\text{Th.B.0.9}}{=} V(t,t)h(t) - V(t,t_0)h(t_0) \\ &\quad - \int_{t_0}^{t} V(t,s)f^*(s)dg(s) \\ &\overset{\text{Th.7.2.4}}{=} h(t) - \int_{t_0}^{t} V(t,s)f^*(s)dg(s). \end{aligned} \tag{7.42}$$

Thus, by (7.41), (7.42) and Theorem D.0.12, we obtain

$$y(t) = V(t,t_0)x_0 + \int_{t_0}^{t^*} V(t,s)f(s)\Delta(s), \quad \text{for all } t \in \mathbb{T}_0^*.$$

Therefore,

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,s)f(s)\Delta(s), \quad \text{for all } t \in \mathbb{T}_0,$$

since $x = y|_{\mathbb{T}_0}$, $U = V|_{\mathbb{T}_0 \times \mathbb{T}_0}$ and $t^* = t$ for all $t \in \mathbb{T}_0$. □

Our next goal is to show that our main results, namely Theorems 7.1.12 and 7.2.7, generalize [10, Theorems 5.8 and 5.24], [3, Theorem 3.1] and [11, Theorem 2.1]. To this end, we prove, in the next result, that if $A:\mathbb{T}\to\mathbb{R}^{n\times n}$ and $\overline{f}:\mathbb{T}\to\mathbb{R}^n$ are rd-continuous, then A and $\overline{f}$ satisfy conditions (T1), (T2) and (T3).

Lemma 7.2.8. Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0=[t_0,+\infty)\cap\mathbb{T}$, with $t_0\in\mathbb{T}$. If $\overline{f}:\mathbb{T}\to\mathbb{R}^n$ and $A:\mathbb{T}\to\mathbb{R}^{n\times n}$ are rd-continuous, then conditions (T1), (T2), (T3) are satisfied.

Proof. At first, we notice that since A is rd-continuous, for all regulated function $y:\mathbb{T}_0\to\mathbb{R}^n$, the product $Ay:\mathbb{T}_0\to\mathbb{R}^n$, given by $A(s)y(s)$ for all $s\in\mathbb{T}_0$, is also a regulated function. Moreover, every rd-continuous function is a regulated function, then $\overline{f}$ is also regulated and, by Proposition D.0.13, condition (T1) is satisfied.

Before proving condition (T2), we notice that the function $\mathbb{T}_0\ni t\mapsto\|A(t)\|e^{t-t_0}$ is regulated. Thus, by Proposition D.0.13, the Perron Δ-integral

$$\int_{t_1}^{t_2}\|A(s)\|e^{s-t_0}\Delta s$$

exists for all $t_1,t_2\in\mathbb{T}_0$.

Let $t_1,t_2\in\mathbb{T}_0$ be fixed and $y,z\in G_0(\mathbb{T}_0,X)$. Once the equality

$$\int_{t_1}^{t_2}A(s)[y(s)-z(s)]\Delta s=-\int_{t_2}^{t_1}A(s)[y(s)-z(s)]\Delta s$$

holds, we may assume that $t_1<t_2$.

By the definition of the Perron Δ-integral, for all $\varepsilon>0$, there exists a Δ-gauge δ on $[t_1,t_2]_{\mathbb{T}}$ such that

$$\left\|\sum_{i=1}^{|d|}A(\tau_i)[y(\tau_i)-z(\tau_i)](s_i-s_{i-1})-\int_{t_1}^{t_2}A(s)[y(s)-z(s)]\Delta s\right\|<\varepsilon\quad\text{and}$$

$$\sum_{i=1}^{|d|}\|A(\tau_i)\|e^{\tau_i-t_0}(s_i-s_{i-1})<\varepsilon+\int_{t_1}^{t_2}\|A(s)\|e^{s-t_0}\Delta s,$$

provided $d_{\mathbb{T}}=(\tau_i,[s_{i-1},s_i]_{\mathbb{T}})$ is a δ-fine tagged division of $[t_1,t_2]_{\mathbb{T}}$. Then,

$$\begin{aligned}\left\|\int_{t_1}^{t_2}A(s)[y(s)-z(s)]\Delta s\right\| &\leq \left\|\int_{t_1}^{t_2}A(s)[y(s)-z(s)]\Delta s-\sum_{i=1}^{|d|}A(\tau_i)[y(\tau_i)-z(\tau_i)](s_i-s_{i-1})\right\|\\ &\quad+\left\|\sum_{i=1}^{|d|}A(\tau_i)[y(\tau_i)-z(\tau_i)](s_i-s_{i-1})\right\|\\ &\overset{s_i>s_{i-1}}{\leq} \varepsilon+\sum_{i=1}^{|d|}\|A(\tau_i)\|\|y(\tau_i)-z(\tau_i)\|(s_i-s_{i-1})\\ &= \varepsilon+\sum_{i=1}^{|d|}\|A(\tau_i)\|\|y(\tau_i)-z(\tau_i)\|e^{-(\tau_i-t_0)}e^{(\tau_i-t_0)}(s_i-s_{i-1})\end{aligned}$$

$$\begin{aligned} &\leq \varepsilon + \|y-z\|_{\mathbb{T}_0} \sum_{i=1}^{|d|} \|A(\tau_i)\| e^{(\tau_i - t_0)} (s_i - s_{i-1}) \\ &\leq \varepsilon(1+\|y-z\|_{\mathbb{T}_0}) + \|y-z\|_{\mathbb{T}_0} \int_{t_1}^{t_2} \|A(s)\| e^{s-t_0} \Delta s. \end{aligned}$$

Therefore, condition (T2) holds by taking ε sufficiently small and $L(s) = \|A(s)\|e^{s-t_0}$ for all $s \in \mathbb{T}_0$.

Similarly, since $\overline{f}$ is regulated, $\|\overline{f}\|_{[t_1,t_2]_{\mathbb{T}}} = \sup_{t\in[t_1,t_2]_{\mathbb{T}}} \|\overline{f}(t)\| < \infty$, for all $t_1, t_2 \in \mathbb{T}_0$. Therefore, the function $K : \mathbb{T}_0 \to \mathbb{R}$ defined by

$$K(s) = \|\overline{f}(s)\|, \quad \text{for all } s \in \mathbb{T}_0,$$

is locally Perron Δ-integrable. Moreover, given $\varepsilon > 0$, there exists a Δ-gauge $\overline{\delta}$ on $[t_1,t_2]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{|\overline{d}|} \overline{f}(\tau_i)(s_i - s_{i-1}) - \int_{t_1}^{t_2} \overline{f}(s)\Delta s \right\| < \varepsilon \quad \text{and}$$

$$\sum_{i=1}^{|\overline{d}|} \|\overline{f}(\tau_i)\|(s_i - s_{i-1}) < \varepsilon + \int_{t_1}^{t_2} K(s)\Delta s,$$

provided $\overline{d}_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is a $\overline{\delta}$-fine tagged division of $[t_1,t_2]_{\mathbb{T}}$. Then,

$$\begin{aligned} \left\| \int_{t_1}^{t_2} \overline{f}(s)\Delta s \right\| &\leq \left\| \sum_{i=1}^{|\overline{d}|} \overline{f}(\tau_i)(s_i - s_{i-1}) - \int_{t_1}^{t_2} \overline{f}(s)\Delta s \right\| + \sum_{i=1}^{|\overline{d}|} \|\overline{f}(\tau_i)\|(s_i - s_{i-1}) \\ &\leq 2\varepsilon + \int_{t_1}^{t_2} K(s)\Delta s \end{aligned}$$

which proves condition (T3). □

In order to prove that Theorem 7.2.7 generalizes [10, Theorems 5.24] and [11, Theorem 2.1], we need the following auxiliary result.

Theorem 7.2.9. If $f : \mathbb{T}_0 \to \mathbb{R}^n$ and $a : \mathbb{T} \to \mathbb{R}^{n\times n}$ are rd-continuous, then

$$\int_{t_0}^{t} U(t,s)f(s)\Delta(s) = \int_{t_0}^{t} U(t,\sigma(s))f(s)\Delta s, \quad \text{for all } t \in \mathbb{T}_0, \tag{7.43}$$

where U is the fundamental operator of the homogeneous linear dynamic equation (7.16) given in Theorem 7.2.3.

Proof. At first, we notice that, by Lemma 7.2.8, a and f satisfies all the hypotheses of Theorems 7.1.12, 7.2.5 and 7.2.7. Then, by Theorem 7.2.5, we have

$$U(s,r) = I + \int_{r}^{s} a(\tau)U(\tau,r)\Delta\tau, \tag{7.44}$$

for all $r,s \in \mathbb{T}_0$. Moreover,

$$U(t,r)U(r,s) = U(t,s), \quad \text{for all } t,r,s \in \mathbb{T}_0. \tag{7.45}$$

By equations (7.44) and (7.45), we obtain

$$\begin{aligned} U(t,t_0) &= U(t,r)U(r,t_0) \\ &= \left(I+\int_r^t a(\tau)U(\tau,r)\Delta\tau\right)U(r,t_0) \\ &= U(r,t_0)+\int_r^t a(\tau)U(\tau,t_0)\Delta\tau, \end{aligned} \tag{7.46}$$

for all $t,r \in \mathbb{T}_0$.

Denote $U(t,t_0)$ by $\beta(t)$ for all $t \in \mathbb{T}_0$. By (7.46), we conclude

$$\beta(t)-\beta(r)=\int_r^t a(\tau)\beta(\tau)\Delta\tau, \quad \text{for all } t,r \in \mathbb{T}_0. \tag{7.47}$$

Therefore, β is a solution of the dynamic equation

$$\beta^{\Delta}(s)=a(s)\beta(s). \tag{7.48}$$

On the other hand, by Theorem 7.2.7, the function $x:\mathbb{T}_0 \to \mathbb{R}^n$, given by

$$x(t)=U(t,t_0)x_0+\int_{t_0}^t U(t,s)f(s)\Delta s, \quad t \in \mathbb{T}_0$$

is the unique solution of the dynamic equation

$$x^{\Delta}=a(t)x+f(t), \tag{7.49}$$

with initial condition $x(t_0)=x_0$. Let $y:\mathbb{T}_0 \to \mathbb{R}^n$ be defined by

$$y(t)=U(t,t_0)x_0+\int_{t_0}^t U(t,\sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}_0.$$

By equation (7.45), we can rewrite y by

$$y(t)=\beta(t)\alpha(t), \quad t \in \mathbb{T}_0, \tag{7.50}$$

where

$$\alpha(t)=x_0+\int_{t_0}^t U(t_0,\sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}_0.$$

By Theorem D.0.5, we obtain

$$\begin{aligned} y^{\Delta}(t) &= \beta^{\Delta}(t)\alpha(t)+\beta(\sigma(t))\alpha^{\Delta}(t) \\ &\overset{(7.48)}{=} a(t)\beta(t)\alpha(t)+\beta(\sigma(t))U(t_0,\sigma(t))f(t) \\ &\overset{(7.50)}{=} a(t)y(t)+U(\sigma(t),t_0)U(t_0,\sigma(t))f(t) \\ &\overset{(7.45)}{=} a(t)y(t)+U(\sigma(t),\sigma(t))f(t) \\ &\overset{(7.44)}{=} a(t)y(t)+If(t) \\ &= a(t)y(t)+f(t). \end{aligned}$$

Therefore, y is a solution of the dynamic equation (7.49) with initial condition $y(t_0)=x_0$ and, by the uniqueness of a solution (see Theorem 7.1.12), we have $y(t)=x(t)$ for all $t \in \mathbb{T}_0$. Consequently,

$$\int_{t_0}^t U(t,s)f(s)\Delta s=\int_{t_0}^t U(t,\sigma(s))f(s)\Delta s,$$

for all $t \in \mathbb{T}_0$. □

7.3 Controllability

The main goal of this section is to establish necessary and sufficient conditions for a nonhomogeneous linear dynamic equation to be controllable.

Let X and $\mathbb{U}$ be a Banach spaces and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, where $\mathbb{T}$ is a time scale and $t_0 \in \mathbb{T}$. Consider the control system on the time scale $\mathbb{T}$ described by

$$x^{\Delta} = a(t)x + B(t)u(t), \tag{7.51}$$

where $B : \mathbb{T} \to L(\mathbb{U}, X)$, $u : \mathbb{T} \to \mathbb{U}$ and $a : \mathbb{T} \to L(X)$. Furthermore, consider the following conditions:

(CT1) the Perron Δ-integrals

$$\int_{t_1}^{t_2} B(s)u(s)\Delta s \quad \text{and} \quad \int_{t_1}^{t_2} a(s)y(s)\Delta s$$

exist for all $t_1, t_2 \in \mathbb{T}$, whenever $y : \mathbb{T} \to X$ is regulated.

(CT2) there is a locally Perron Δ-integrable function $L : \mathbb{T} \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} a(s)[z(s) - y(s)]\Delta s \right\| \leq \|z - y\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s,$$

for all $z, y \in G_0(\mathbb{T}_0, X)$ and all $t_1, t_2 \in \mathbb{T}_0$.

(CT3) there is a locally Perron Δ-integrable function $K : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} B(s)u(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s,$$

for all $t_1, t_2 \in \mathbb{T}_0$.

We recall that the vector space $G_0(\mathbb{T}_0, X)$ and the norm $\|\cdot\|_{\mathbb{T}_0}$ are described in Definition D.0.3.

We denote by $\mathscr{U}$ the space of all *control functions*, $u : \mathbb{T} \to \mathbb{U}$, such that conditions (CT1) and (CT3) are satisfied.

By Theorems 7.1.12 and 7.2.7, the dynamic equation (7.51) admits a unique solution $x : \mathbb{T}_0 \to X$ with initial condition $\widetilde{x} \in X$ and control $u \in \mathscr{U}$. Moreover, x is given by

$$x(t) = U(t, t_0)\widetilde{x} + \int_{t_0}^{t} U(t, s)B(s)u(s)\Delta s, \quad s \in \mathbb{T}_0,$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 7.2.3 and we denote $x(\cdot)$ by $x(\cdot, \widetilde{x}, u)$.

In the sequel, we introduce a definition of controllability for the dynamic system (7.51).

Definition 7.3.1. Let $T \in \mathbb{T}_0$ be fixed and $S \subseteq X$ be such that $0 \in S$, where 0 denotes the neutral element of X. The state $d \in S$ is said to be

(i) *approximately controllable* at time T to a point $\widetilde{x} \in X$, if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{U}$ such that $x(T,d,u_n)$ goes to $\widetilde{x}$ as n goes to ∞;

(ii) *strictly controllable at time T* to a point $\widetilde{x} \in X$, if there exists $u \in \mathscr{U}$ such that $x(T,d,u) = \widetilde{x}$.

The dynamic system (7.51) is *approximately controllable* (*strictly controllable*) *at time T*, if all points of S are approximately controllable (strictly controllable) at time T to all points of X.

For all $t \in \mathbb{T}_0$ and all $d \in S$, define $G(t) : \mathscr{U} \to X$ by

$$G(t)u = \int_{t_0}^{t} U(t,s)B(s)u(s)\Delta s, \tag{7.52}$$

where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(X)$ is given in Theorem 7.2.3.

In what follows, we give necessary and sufficient conditions for the system (7.51) to be approximately controllable (strictly controllable).

Theorem 7.3.2. Let $T \in \mathbb{T}_0$ and $G(T) : \mathscr{U} \to X$ be given by (7.52). Then, the following statements hold.

(i) The dynamic system (7.51) is approximately controllable at time T if and only if the range of $G(T)$ is everywhere dense in X.

(ii) The dynamic system (7.51) is strictly controllable at time T if and only if the mapping $G(T)$ is onto.

Proof. We start by proving item (i). Let $\widetilde{x} \in X$ be arbitrary. Since the dynamic system (7.51) is approximately controllable at time T, Definition 7.3.1 yields all points in S are approximately controllable at time T to all points of X. In particular, $d = 0$ is approximately controllable at time T to $\widetilde{x}$. Therefore, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{U}$ such that $x(T,0,u_n)$ goes to $\widetilde{x}$ as n goes to ∞, that is, $G(T)u_n$ goes to $\widetilde{x}$, as n goes to ∞. Hence, the range of $G(T)$ is everywhere dense in X.

Reciprocally, for arbitrary $d \in S$ and $\widetilde{x} \in X$, $\widetilde{x} - U(T,t_0)d \in X$ and, if the range of $G(T)$ is everywhere dense in X, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{U}$ such that $G(T)u_n$ goes to $\widetilde{x} - U(T,t_0)d$ as n goes to ∞, that is, $x(T,d,u_n)$ goes to $\widetilde{x}$, as n goes to ∞, which in turn, implies that the dynamic system (7.51) is approximately controllable at time T.

As in item (i), in order to prove item (ii), we take $\widetilde{x}$ arbitrarily and $d = 0$. Once the dynamic system (7.51) is controllable at time T, there exists $u \in \mathscr{U}$ such that $x(T,0,u) = \widetilde{x}$, that is, $G(T)u = \widetilde{x}$. Therefore, $G(T)$ is onto.

Conversely, if $G(T)$ is onto, then for all $d \in S$ and all $\widetilde{x} \in X$, there exists $u \in \mathscr{U}$ such that $G(T)u = \widetilde{x} - U(T,t_0)d$, which implies that $x(T,d,u) = \widetilde{x}$. Therefore, the dynamic system (7.51) is strictly controllable at time T. □

In the next result, we consider $X=\mathbb{R}^n$, $\mathbb{U}=\mathbb{R}^p$, $p<n$, and use the notation M' to denote the transpose of a given matrix M.

Theorem 7.3.3. Let $\mathbb{T}$ be a time scale, $t_0\in\mathbb{T}$ and $\mathbb{T}_0=[t_0,+\infty)\cap\mathbb{T}_0$. Assume that $f:\mathbb{T}\to\mathbb{R}^n$ and $a:\mathbb{T}\to\mathbb{R}^{n\times n}$ are rd-continuous. Then, $G(T)$ is onto if and only if the rows of the matrix $U(t_0,\sigma(T))B(T)$ are linearly independent, where $T\in\mathbb{T}_0$, $G(T):\mathscr{U}\to\mathbb{R}^n$ is defined by (7.52) and $U:\mathbb{T}_0\times\mathbb{T}_0\to L(\mathbb{R}^n)$ is given in Theorem 7.2.3.

Proof. Let $T\in\mathbb{T}_0$ and suppose that the rows of the matrix $U(t_0,\sigma(T))B(T)$ are linearly independent. Then, the matrix

$$\mathscr{C}(t_0,T)=\int_{t_0}^{T}U(T,\sigma(s))B(T)B'(s)U'(t_0,s)\Delta s$$

is positive definite.

Let $x_0\in\mathbb{R}^n$ be arbitrary and define

$$u(s)=B'(s)U'(t_0,s)\mathscr{C}^{-1}(t_0,T)x_0,\quad \text{for all } s\in[t_0,T]_{\mathbb{T}}.$$

By (7.52) and Theorem 7.2.9, we get

$$G(T)u=\int_{t_0}^{T}U(T,\sigma(s))B(s)u(s)\Delta s=x_0,$$

which shows that $G(T)$ is onto.

Conversely, if the rows of the matrix $U(t_0,\sigma(T))B(T)$ are linearly dependent, then there exists $x_0\in\mathbb{R}^n$, $x_0\neq 0$, such that

$$x_0'U(t_0,\sigma(s))B(s)\equiv 0,\quad \text{for all } s\in[t_0,T]_{\mathbb{T}}. \tag{7.53}$$

Once $G(T)$ is onto, there exists u for which $G(T)u=x_0$. Therefore, by Theorem 7.2.9, we have

$$G(T)u=\int_{t_0}^{T}U(T,\sigma(s))B(s)u(s)\Delta s=x_0.$$

Multiplying the above equation by $x_0'U(t_0,\sigma(T))$, we obtain, by (7.53),

$$x_0'U(t_0,\sigma(T))x_0=\int_{t_0}^{T}x_0'U(t_0,\sigma(s))B(s)u(s)\Delta s=0,$$

which contradicts the fact that U is the fundamental operator of the dynamic equation (7.51). □

The next result is a straightforward consequence of Theorems 7.3.2 and 7.3.3. Unlike the proofs of the results presented in [12, Theorem 4.3], [17, Theorem 2.2] and [43, Theorem 1] which deals with the rank of some matrices, our result provides another characterization of strict controllability and its proof uses linear dependence of a much simpler matrix.

Corollary 7.3.4. Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}_0$, with $t_0 \in \mathbb{T}$. Assume that $f : \mathbb{T} \to \mathbb{R}^n$ and $a : \mathbb{T} \to \mathbb{R}^{n\times n}$ are rd-continuous. Then, the dynamic system (7.51) is strictly controllable at time $T \in \mathbb{T}_0$ if and only if the rows of the matrix $U(t_0, \sigma(T))B(T)$ are linearly independent, where $U : \mathbb{T}_0 \times \mathbb{T}_0 \to L(\mathbb{R}^n)$ is given in Theorem 7.2.3.

In the sequel, we present a definition of a regressive $n \times n$-matrix-valued function and some examples of fundamental operators borrowed from [10, Examples 5.9 and 5.19].

Definition 7.3.5. Let $\mathbb{T}$ be a time scale. An $n \times n$–matrix–valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called *regressive* (with respect to $\mathbb{T}$) if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$.

Example 7.3.6. Let I be the identity $n \times n$-matrix and $U(\cdot,\cdot)$ be the fundamental operator given in Theorem 7.2.3.

(i) If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ for all $t \in \mathbb{T}$. Moreover, any $n \times n$-matrix-valued function a on $\mathbb{T}$ is such that $I + \mu(t)a(t)$ is invertible for all $t \in \mathbb{T}^k$ and, hence, it is regressive. In this case, a matrix-valued function a is rd-continuous if and only if it is continuous. Then, the initial value problem
$$x^\Delta = a(t)x, \quad x(t_0) = x_0$$
has a unique solution provided a is continuous. Moreover, if a is a constant $n \times n$-matrix, then $U(t,t_0) = e^{a(t-t_0)}$ for all $t \in \mathbb{R}$.

(ii) If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and any $n \times n$-matrix-valued function a on $\mathbb{T}$ is rd-continuous. Moreover, in order for a matrix-valued function a on $\mathbb{T}$ to be regressive, the matrix $I + a(t)$ needs to be invertible for each $t \in \mathbb{Z}$. Furthermore, if a is a constant $n \times n$-matrix, then $U(t,t_0) = (I + a)^{(t-t_0)}$ for all $t \in \mathbb{Z}$.

We end this section by presenting two examples of strictly controllable dynamic systems on time scales.

Example 7.3.7. Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$ and $\mathscr{U}$ be the set of all regulated functions $u : \mathbb{T} \to \mathbb{R}$ such that $\|u\|_{\mathbb{T}_0} = \sup_{s\in\mathbb{T}_0} |u(s)| < c$ for some constant $c > 0$.

Consider the following control dynamic system
$$x^\Delta = ax + Bu(t) \tag{7.54}$$
on the time scale $\mathbb{T}$, where $u \in \mathscr{U}$,
$$a = \begin{pmatrix} -\dfrac{8}{45} & \dfrac{1}{30} \\ -\dfrac{1}{45} & -\dfrac{1}{30} \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

At first, we notice that, since a and B are real constant matrices, a and B are rd-continuous. Then, by Lemma 7.2.8, conditions (CT1) and (CT2) are fulfilled. Moreover, by Proposition D.0.13, we have

$$\left\| \int_{t_1}^{t_2} Bu(s)\Delta s \right\| \leq \|B\| \|u\|_{[t_1,t_2]_{\mathbb{T}}} [g(t_2) - g(t_1)] \leq \|B\| c[g(t_2) - g(t_1)], \tag{7.55}$$

for all $t_1, t_2 \in \mathbb{T}_0$ and all $u \in \mathscr{U}$, where $g(t) = t^*$ for all $t \in \mathbb{T}_0^*$. Since $g|_{\mathbb{T}_0}$ is the identity function, g is delta differentiable (see Definition D.0.4). Then, the function $K : \mathbb{T}_0 \to \mathbb{R}$ given by

$$K(t) = \|B\| c g^{\Delta}(t), \quad t \in \mathbb{T}_0$$

is well-defined and

$$\int_{t_1}^{t_2} K(s)\Delta s = \|B\| c[g(t_2) - g(t_1)], \tag{7.56}$$

for all $t_1, t_2 \in \mathbb{T}_0$. By equations (7.55) and (7.56), for all $t_1, t_2 \in \mathbb{T}_0$, we have

$$\left\| \int_{t_1}^{t_2} Bu(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s$$

which proves condition (CT3).

Let us consider two cases for the time scale $\mathbb{T}$.

(a) $\mathbb{T} = \mathbb{R}$.

Notice that a is continuous, since it is a real constant 2×2-matrix. Then, by Example 7.3.6-(i), there exists a solution of the dynamic equation (7.54) and $U(t,t_0) = e^{a(t-t_0)}$. On the other hand, it is clear that $U(t_0, \sigma(t))$ is invertible, for all $t \in \mathbb{T}_0$ and, consequently, the rows of $U(t_0, \sigma(t)) = U(t_0, \sigma(t))B(t)$ are linearly independent, for all $t \in \mathbb{T}_0$. Therefore, Corollary 7.3.4 guarantees that the dynamic system (7.54) is strictly controllable.

(b) $\mathbb{T} = \mathbb{Z}$.

Notice that

$$I + a = \begin{pmatrix} \frac{37}{45} & \frac{1}{30} \\ -\frac{1}{45} & \frac{29}{30} \end{pmatrix}$$

is invertible, since the determinant of $I + a$ is not zero. Therefore, by Example 7.3.6-(ii), a is regressive and, for all $t \in \mathbb{Z}$, $U(t,t_0) = (I+a)^{(t-t_0)}$ and $\sigma(t) = t+1$. On the other hand, once $(I+a)$ is invertible, $(I+a)^{(t-t_0)}$ is also invertible for all $t \in \mathbb{T}_0$ and, consequently, $U(\sigma(t), t_0) = U(t+1, t_0)$ is invertible, for all $t \in \mathbb{T}_0$, which implies that the rows of $U(\sigma(t), t_0) = U(\sigma(t), t_0)B(t)$ are linearly independent, for all $t \in \mathbb{T}_0$. Then, by Corollary 7.3.4, we conclude that the dynamic equation (7.54) is strictly controllable.

Example 7.3.8. Let X be a Banach space. Consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with $h > 0$. Let $t_0 = h$ and consider the homogeneous linear dynamic equation

$$x^{\Delta} = a(t)x \tag{7.57}$$

on the time scale $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, where $a : \mathbb{T} \to L(X)$ is given by

$$a(t)x = \begin{cases} 0, & (t,x) \in (\mathbb{T} \setminus \mathbb{T}_0) \times X \\ \frac{x}{t}, & (t,x) \in \mathbb{T}_0 \times X. \end{cases} \tag{7.58}$$

Since a is regulated, for all regulated functions $y : \mathbb{T}_0 \to X$, the product $ay : \mathbb{T}_0 \to X$, given by $a(s)y(s)$ for all $s \in \mathbb{T}_0$, is also a regulated function. Then, by Proposition D.0.13, the Perron Δ-integral $\int_{t_1}^{t_2} a(s)y(s)\Delta s$ exists for all $t_1, t_2 \in \mathbb{T}_0$ and all regulated functions $y : \mathbb{T}_0 \to X$. This leads to condition (CT1).

Let us prove condition (CT2). Consider $y, z \in G_0(\mathbb{T}_0, X)$ and define $L : \mathbb{T}_0 \to \mathbb{R}$ by

$$L(s) = \frac{e^{s-t_0}}{s}, \quad \text{for all } s \in \mathbb{T}_0.$$

Let $t_1, t_2 \in \mathbb{T}_0$ and assume that $t_1 < t_2$. Then, by Example D.0.8-(ii), we get

$$\begin{aligned}
\left\| \int_{t_1}^{t_2} a(s)[y(s) - z(s)]\Delta s \right\| &= \left\| \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} a(kh)[y(kh) - z(kh)]h \right\| \\
&= \left\| \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{y(kh) - z(kh)}{kh} h \right\| \\
&\leq \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{\|y(kh) - z(hk)\| e^{-(kh-t_0)} e^{kh-t_0}}{k} \\
&\leq \|y - z\|_{\mathbb{T}_0} \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{e^{kh-t_0}}{k} \\
&= \|y - z\|_{\mathbb{T}_0} \sum_{k=\frac{t_1}{h}}^{\frac{t_2}{h}-1} \frac{e^{kh-t_0}}{kh} h \\
&= \|y - z\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s,
\end{aligned}$$

and, hence, condition (CT2) is fulfilled.

On the other hand, if $x : \mathbb{T}_0 \to X$ is the solution of the dynamic equation (7.57), with initial condition $x(t_0) = x_0$, then, for all $t \in \mathbb{T}_0$, we have

$$x(t) = x_0 + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} \frac{x(kh)h}{kh} = x_0 + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} \frac{x(kh)}{k} = x_0 + \sum_{k=t_0}^{t-h} \frac{x(k)h}{k}.$$

Thus, a straightforward calculation shows that $U(t, t_0) = (I + h)^{\frac{t-t_0}{h}}$, where $I \in L(X)$ is the identity operator.

Now, let $\mathbb{U}$ be a Banach space, with $\mathbb{U} \subset X$, and $\mathscr{U}$ be the set of all regulated functions $u : \mathbb{T} \to \mathbb{U}$ such that for every interval $[a,b]_{\mathbb{T}} \subset \mathbb{T}_0$, there exists a constant $M > 0$ for which $\|u\|_{[a,b]_{\mathbb{T}}} < M$. Consider the nonhomogeneous dynamic system

$$x^{\Delta} = a(t)x + B(t)u(t), \tag{7.59}$$

where $B : \mathbb{T} \to L(\mathbb{U}, X)$ is defined by $B(t)y = y$ for all $y \in \mathbb{U}$ and all $t \in \mathbb{T}$, $a : \mathbb{T} \to L(X)$ is given by (7.58) and $u \in \mathscr{U}$. Then, by Proposition D.0.13, the Perron Δ-integral $\int_{t_1}^{t_2} u(s)\Delta s = \int_{t_1}^{t_2} B(s)u(s)\Delta s$ exists for all $t_1, t_2 \in \mathbb{T}_0$ and,

$$\left\| \int_{t_1}^{t_2} u(s)\Delta s \right\| \leq \|u\|_{[t_1,t_2]_{\mathbb{T}}}[g(t_2) - g(t_1)] \leq M[g(t_2) - g(t_1)] = \int_{t_1}^{t_2} K(s)\Delta s,$$

where $K(s) = Mg^{\Delta}(s)$ for all $s \in \mathbb{T}_0$ and $g(s) = s^*$ for all $s \in \mathbb{T}_0^*$. Therefore, condition (CT3) holds.

In what follows, we aim to prove that the dynamic system (7.59) is strictly controllable.

Let $t \in \mathbb{T}_0$ be fixed and $G(t) : \mathscr{U} \to X$ be given by

$$G(t)u = \int_{t_0}^{t} (I+h)^{\frac{t-s}{h}} u(s)\Delta s.$$

Notice that, if $x \in X$ and $u : \mathbb{T} \to \mathbb{U}$ is given by

$$u(s) = \begin{cases} \dfrac{U(t_0,t)x}{t-t_0} = \dfrac{(I+h)^{\frac{t_0-t}{h}} x}{t-t_0}, & \text{if } s \in \mathbb{T} \setminus \mathbb{T}_0, \\ \dfrac{U(s,t)x}{t-t_0} = \dfrac{(I+h)^{\frac{s-t}{h}} x}{t-t_0}, & \text{if } s \in \mathbb{T}_0, \end{cases}$$

then, by Theorem 7.2.5, u is well-defined and $u \in \mathscr{U}$. Moreover,

$$G(t)u = \int_{t_0}^{t} (I+h)^{\frac{t-s}{h}} \frac{(I+h)^{\frac{s-t}{h}}}{t-t_0} x\Delta s = \int_{t_0}^{t} \frac{x}{t-t_0}\Delta s = x$$

which implies that $G(t)$ is onto and, by Theorem 7.3.2, the dynamic system (7.59) is strictly controllable at time $t \in \mathbb{T}_0$. Since t is arbitrary, we conclude that the dynamic system (7.59) is strictly controllable.

APPENDIX

A

REGULATED FUNCTIONS

In this chapter, we provide the basic background material on regulated functions. In particular, we review definitions, introduce some notations and present known results.

Throughout this chapter, X is a Banach space equipped with a norm $\|\cdot\|$, $\mathscr{O} \subseteq X$ and, for every $a, b \in (-\infty, +\infty)$, $[a,b]$ denotes the corresponding compact interval of the real line.

At first, we present some auxiliary definitions.

Definition A.0.1. Let $-\infty < a < b < +\infty$. A *division* of the interval $[a,b]$ is a finite set of points of $[a,b]$, $d = \{s_0, s_1, \ldots, s_{|d|}\} \subset [a,b]$, where $a = s_0 < s_1 < \ldots < s_{|d|} = b$ and $|d| < +\infty$ represents the number of subintervals of $[a.b]$. The set of all divisions of $[a,b]$ is denoted by $\mathscr{D}[a,b]$.

Definition A.0.2. A function $f : [a,b] \to \mathscr{O}$ is called a *step function*, if there exist a division $d = \{s_1, s_2, \ldots, s_{|d|}\} \in \mathscr{D}[a,b]$ and $c_i \in X$, $i = 1, \ldots, |d|$, such that $f(t) = c_i$ for all $t \in (s_{i-1}, s_i)$, $i = 1, \ldots, |d|$. In this case, we write $f \in E([a,b], \mathscr{O})$.

Definition A.0.3. Let $f : [a,b] \to \mathscr{O}$ be a function. We define the *variation* of f on $[a,b]$ by

$$\mathrm{var}_a^b f = \sup_{d \in \mathscr{D}[a,b]} \sum_{i=1}^{|d|} \|f(s_i) - f(s_{i-1})\|.$$

Moreover, if $\mathrm{var}_a^b f < +\infty$, we say that f is a *function of bounded variation on* $[a,b]$. The set of all functions $f : [a,b] \to \mathscr{O}$ of bounded variation on $[a,b]$ is represented by $BV([a,b], \mathscr{O})$.

In what follows, we present two known results about the set $BV([a,b], \mathscr{O})$. The reader may consult [32, Theorems I.2.2 and I.2.3].

Theorem A.0.4. Let $f \in BV([a,b], \mathscr{O})$. Then, the following equality

$$\mathrm{var}_a^b f = \mathrm{var}_a^t f + \mathrm{var}_t^b f$$

holds for every $t \in [a,b]$.

Theorem A.0.5. The space $BV([a,b],X)$, equipped with the norm

$$\|f\|_{BV} = \|f(a)\| + \mathrm{var}_a^b f, \quad f \in BV([a,b],X),$$

is a Banach space.

In the sequel, we state a Helly's choice principle for Banach space-valued functions as presented in [33].

Theorem A.0.6 (Helly's choice)**.** If $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of functions defined from $[a,b]$ into X, then there exists a subsequence $\{f_{n_k}\}_{n_k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ converging pointwisely to a function $f \in BV([a,b],X)$, that is,

$$\lim_{n\to\infty} f_{n_k}(t) = f(t),$$

for all $t \in [a,b]$.

In the next lines, we present the definition of regulated functions.

Definition A.0.7. A function $f : [a,b] \to \mathscr{O}$ is called *regulated*, if at any point $t \in [a,b]$, it possesses one-sided limits, that is, the limit $\lim_{s\to t^-} f(s) = f(t^-) \in X$ exists for every $t \in (a,b]$ and the limit $\lim_{s\to t^+} f(s) = f(t^+) \in X$ exists for every $t \in [a,b)$. We denote by $G([a,b],\mathscr{O})$ the set of all regulated functions $f : [a,b] \to \mathscr{O}$.

The following statement can be found in [32, Theorem I.2.7](or [33, Corollary I.3.4]).

Remark A.0.8. It is well-known that

$$E([a,b],\mathscr{O}) \subset G([a,b],\mathscr{O})$$

and

$$BV([a,b],\mathscr{O}) \subset G([a,b],\mathscr{O}).$$

The next results give useful properties of regulated functions defined on compact intervals.

Theorem A.0.9 ([13, Theorem 1.4], [33, Theorem I.3.1] or [44, Theorem 4.5])**.** For every function $f \in G([a,b],X)$ (in particular, $f \in BV([a,b],X)$), there exists a sequence of step functions $\{\varphi_n\}_{n\in\mathbb{N}} \subset E([a,b],X)$ which converges uniformly to f, that is,

$$\sup_{s\in[a,b]} \|f(s) - \varphi_n(s)\| \overset{n\to\infty}{\to} 0.$$

Corollary A.0.10 ([44, Corollary 4.7])**.** If $f \in G([a,b],X)$, then for every $\varepsilon > 0$, the following sets

$$\{t \in [a,b) : \|f(t^+) - f(t)\| \geq \varepsilon\} \quad \text{and} \quad \{t \in (a,b] : \|f(t) - f(t^-)\| \geq \varepsilon\}$$

are finite.

Theorem A.0.11 ([44, Theorem 4.3])**.** If a sequence $\{f_n\}_{n\in\mathbb{N}} \subset G([a,b],X)$ converges uniformly to f, then $f \in G([a,b],X)$.

Theorem A.0.12 (Moore-Osgood Theorem, [9])**.** If a sequence $\{f_n\}_{n\in\mathbb{N}} \subset G([a,b],X)$ converges uniformly to f, then

$$lim_{s\to t^-} f(s) = \lim_{s\to t^-} \lim_{n\to+\infty} f_n(s) = \lim_{n\to+\infty} \lim_{s\to t^-} f_n(s), \quad t \in (a,b] \quad \text{and}$$

$$lim_{s\to t^+} f(s) = \lim_{s\to t^+} \lim_{n\to+\infty} f_n(s) = \lim_{n\to+\infty} \lim_{s\to t^+} f_n(s), \quad t \in [a,b).$$

Proposition A.0.13 ([28, Proposition 1.7] or [44, Corollary 4.6])**.** Every regulated function is bounded on compact intervals.

Proposition A.0.14. If $\Phi : X \to \mathbb{R}$ is a continuous function and $\bar{x} \in G([a,b],X)$, then $\Phi \circ \bar{x}$ belongs to G([a,b],X).

Proof. Let $\bar{x} \in G([a,b],X)$ and $\varepsilon > 0$ be given. Since Φ is continuous, there exists $\delta_0 > 0$ such that, if $y,z \in X$ satisfy

$$\|y-z\| < \delta_0, \tag{A.1}$$

then

$$|\Phi(y) - \Phi(z)| < \varepsilon. \tag{A.2}$$

Let us prove that the left-hand limit of Φ exists. Indeed, let $t \in (a,b]$. Since $\bar{x}$ is regulated, for this δ_0, there exists $\delta > 0$ such that, for all $s \in (t-\delta, t)$, we have

$$\|\bar{x}(s) - \bar{x}(t)\| < \delta_0. \tag{A.3}$$

By (A.1) and (A.3), we conclude

$$|\Phi(\bar{x}(s)) - \Phi(\bar{x}(t))| < \varepsilon, \quad \text{for all } s \in (t-\delta, t)$$

and, hence, $\lim_{s\to t^-} \Phi(\bar{x}(s))$ exists for all $t \in (a,b]$. Similarly, we can show that $\lim_{s\to t^+} \Phi(\bar{x}(s))$ exists for all $t \in [\alpha, \beta)$. □

Lemma A.0.15. If $f \in G([a,b],X)$, then

$$\sup_{s\in[a,b]} \|f(s)\| = c,$$

where $c = \|f(\sigma)\|$ for some $\sigma \in [a,b]$ or $c = \|f(\sigma^-)\|$ for some $\sigma \in (a,b]$ or $c = \|f(\sigma^+)\|$ for some $\sigma \in [a,b)$.

Proof. Let $c = \sup_{s\in[a,b]} \|f(s)\|$. By Proposition A.0.13, we can conclude that $c < +\infty$ and, by definition of the supremum, for all $n \in \mathbb{N}$, we can choose $t_n \in [a,b]$ such that

$$0 \leq c - \|f(t_n)\| < \frac{1}{n}$$

which implies

$$\lim_{n\to\infty} \|f(t_n)\| = c.$$

Once $\{t_n\}_{n\in\mathbb{N}} \subset [a,b]$, there exists a subsequence $\{t'_n\}_{n\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$ such that t'_n goes to $\sigma \in [a,b]$ as n goes to ∞. Since f is regulated, we have

$$c = \lim_{n\to\infty} \|f(t'_n)\| \in \{\|f(\sigma)\|, \|f(\sigma^-)\|, \|f(\sigma^+)\|\}$$

and the proof is complete. □

By Proposition A.0.13, the usual supremum norm, given by

$$\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|,$$

is well-defined for all $f \in G([a,b],X)$ and, in [33, Theorem I.3.6], the author proved the following result.

Theorem A.0.16. The space $G([a,b],X)$, equipped with the norm

$$\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|, \quad f \in G([a,b],X),$$

is a Banach space.

In what follows, we recall the definition of a left-continuous function.

Definition A.0.17. A function $f : [a,b] \to \mathscr{O}$ is said to be *left-continuous* on $(a,b]$, if

$$\lim_{s\to t^-} f(s) = f(t),$$

for every $t \in (a,b]$.

Remark A.0.18. The set of all functions $f : [a,b] \to \mathscr{O}$ such that $f \in G([a,b],\mathscr{O})$ and it is left-continuous on $(a,b]$ is denoted by $G^-([a,b],\mathscr{O})$.

In the sequel, we present important properties of the elements of $G^-([a,b],\mathbb{R})$.

Proposition A.0.19 ([13, Proposition 1.8] or [49, Proposition 10.11])**.** Let $f,g \in G^-([a,b],\mathbb{R})$ be given. If, for every $t \in [a,b)$, there exists $\delta(t) > 0$ such that for every $\eta \in (0,\delta(t))$, we have $f(t+\eta) - f(t) \leq g(t+\eta) - g(t)$, then

$$f(s) - f(a) \leq g(s) - g(a), \quad s \in [a,b].$$

We borrowed the next result from [47, Exercise 19.12-page 153].

Proposition A.0.20. Let $f:[a,b]\to\mathbb{R}$ be a function such that f is left-continuous on $(a,b]$ and $\sup_{s\in[a,t]} f(s)<\infty$, for all $t\in[a,b]$. If $g:[a,b]\to\mathbb{R}$ is defined by $g(t)=\sup_{s\in[a,t]} f(s)$, then g is nondecreasing and it is left-continuous on $(a,b]$. In particular, if $\widetilde{f}:[a,b]\to X$ is left-continuous on $(a,b]$ and $\sup_{s\in[a,t]}\|\widetilde{f}(s)\|<\infty$, for all $t\in[a,b]$, then the function $\widetilde{g}:[a,b]\to\mathbb{R}$, defined by $\widetilde{g}(t)=\sup_{s\in[a,t]}\|f(s)\|$, is nondreasing and left-continuous on $(a,b]$. Moreover, by Proposition A.0.13, we may consider $\widetilde{f}\in G^{-}([a,b],X)$.

The following result is a consequence of Proposition A.0.20. Therefore, we omit its proof here.

Proposition A.0.21. Let $f\in G^{-}([a,b],\mathbb{R})$ and $g:[a,b]\to\mathbb{R}$ be defined by $g(t)=\sup_{s\in[t,b]} f(s)$. Then, g is non-increasing and $g\in G^{-}([a,b],\mathbb{R})$. In particular, if $\widetilde{f}\in G^{-}([a,b],X)$, then the function $\widetilde{g}:[a,b]\to\mathbb{R}$, defined by $\widetilde{g}(t)=\sup_{s\in[a,t]}\|f(s)\|$, is non-increasing and left-continuous on $(a,b]$.

We end this chapter by presenting certain spaces of functions defined on unbounded intervals.

Definition A.0.22. Let $[t_0,+\infty)\subset\mathbb{R}$ be an arbitrary interval, $\mathscr{O}\subseteq X$ and $f:[t_0,+\infty)\to\mathscr{O}$ be a function. Then,

(i) f is *locally of bounded variation* on $[t_0,+\infty)$, if $f\in BV([a,b],\mathscr{O})$ for all $[a,b]\subset[t_0,+\infty)$;

(ii) f belongs to $G([t_0,+\infty),\mathscr{O})$, if $f\in G([a,b],\mathscr{O})$ for all $[a,b]\subset[t_0,+\infty)$;

(iii) f belongs to $BG([t_0,+\infty),\mathscr{O})$, if $G([t_0,+\infty),\mathscr{O})$ and f is bounded;

(iv) f belongs to $G_0([t_0,+\infty),\mathscr{O})$, if $f\in G([t_0,+\infty),\mathscr{O})$ and $\sup\limits_{s\in[t_0,+\infty)} e^{-(s-t_0)}\|f(t)\|<\infty$.

It is clear that $BG([t_0,+\infty),X)$ endowed with the usual supremum norm is a Banach space. Moreover, the next result shows that $G_0([t_0,+\infty),X)$ is a Banach space with respect to a special norm. Its proof can be found in [13, Proposition 1.9].

Proposition A.0.23. The space $G_0([t_0,+\infty),X)$, equipped with the norm

$$\|f\|_{[t_0,+\infty)}=\sup_{t\in[t_0,+\infty)} e^{-(t-t_0)}\|f(t)\|,\quad f\in G_0([t_0,+\infty),X),$$

is a Banach space.

APPENDIX

B

VECTOR INTEGRALS

The goal of this chapter is to present a brief overview of the theory of nonabsolute integration, due to Jaroslav Kurzweil and Ralph Henstock, for integrands taking values in Banach spaces.

Throughout this chapter, X, Y and Z are Banach spaces and $L(U,V)$ is the Banach space of continuous linear mappings $T: U \to V$, where $U,V \in \{X,Y,Z\}$. When $U = V$, we write simply $L(U)$ instead of $L(U,V)$.

At first, we recall the concept of a δ-fine tagged division of $[a,b] \subset \mathbb{R}$. See [49].

Definition B.0.1. A *tagged division* of $[a,b]$, with division points $a = s_0 \leq s_1 \leq \ldots \leq s_{|d|} = b$ and *tags* $\xi_i \in [s_{i-1}, s_i]$, $i = 1,2,\ldots,|d|$, is any finite collection of point-interval pairs $(\xi_i, [s_{i-1}, s_i])$. In this case, we write $d = (\xi_i, [s_{i-1}, s_i]) \in TD_{[a,b]}$, where $TD_{[a,b]}$ denotes the set of all tagged divisions of $[a,b]$. Any subset of a tagged division of $[a,b]$ is a *tagged partial division* of $[a,b]$ and the set of all tagged partial divisions of $[a,b]$ is denoted by $TPD_{[a,b]}$.

Given a positive function $\delta : [a,b] \to (0,+\infty)$, called a *gauge* on $[a,b]$, a tagged division $d = (\xi_i, [s_{i-1}, s_i])$ of $[a,b]$ is called δ*-fine*, whenever

$$[s_{i-1}, s_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i = 1,2,\ldots,|d|.$$

The following result deals with the existence of at least one δ-fine division. A proof of this fact can be found in [49, Lemma 1.4].

Lemma B.0.2 (Cousin's Lemma)**.** Given a gauge δ on $[a,b]$, there is a δ-fine tagged division of $[a,b]$.

In what follows, we recall the definition of a bilinear triple of Banach spaces.

Definition B.0.3. A triple of Banach spaces X,Y and Z is *bilinear*, if there exists a bilinear mapping $B : X \times Y \to Z$ such that

$$\|B(x,y)\| \leq \|x\|\|y\|, \quad \text{for all } (x,y) \in X \times Y.$$

We denote by $\mathscr{B}(X,Y,Z)$ the set of all bilinear triples.

It is not difficult to see that $\mathscr{B} = (L(X,Y),X,Y)$, with $B : L(X,Y) \times X \to Y$ defined by $B(F,x) = Fx \in Y$, is a bilinear triple. Another bilinear triple is $\mathscr{B} = (L(X,Y),L(Y,Z),L(X,Z))$ with $B : L(X,Y), \times L(Y,Z) \to L(X,Z)$ given by the composition $G \circ F \in L(X,Z)$ of the operators $F \in L(X,Y)$ and $G \in L(Y,Z)$.

In the sequel, we present the definition of the abstract Perron-Stieltjes integral. See, e.g., [31, 38, 40, 49–51].

Definition B.0.4. Let $\mathscr{B} = (X,Y,Z)$ be a bilinear triple. For given functions $f : [a,b] \to X$ and $g : [a,b] \to Y$ and a tagged division $d = (\xi_i, [s_{i-1}, s_i])$ of $[a,b]$, we define

$$S(f,dg,d) = \sum_{i=1}^{|d|} f(\xi_i)(g(s_i) - g(s_{i-1})) \quad \text{and}$$

$$S(df,g,d) = \sum_{i=1}^{|d|} [f(s_i) - f(s_{i-1})]g(\xi_i).$$

We say that $I \in Z$ is the *Perron-Stieltjes integral of f with respect to g*, if for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that

$$\|S(f,dg,d) - I\| < \varepsilon,$$

for all δ-fine tagged division $d = (\xi_i, [s_{i-1}, s_i])$ of $[a,b]$. Similarly $J \in Z$ is the *Perron-Stieltjes integral of g with respect to f*, if for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that

$$\|S(df,g,d) - J\| < \varepsilon,$$

for all δ-fine tagged division $d = (\xi_i, [s_{i-1}, s_i])$ of $[a,b]$. In these cases, we write

$$I = \int_a^b f(s)dg(s) \quad \text{and} \quad J = \int_a^b d[f(s)]g(s).$$

Moreover, we use the conventions

$$\int_a^b f(s)dg(s) = -\int_b^a f(s)dg(s) \quad \text{and} \quad \int_a^b d[f(s)]g(s) = -\int_b^a d[f(s)]g(s),$$

whenever $a \leq b$.

The integrals defined above are linear and additive with respect to intervals. For more details, see [50, Proposition 6].

In the next lines, we exhibit the uniform convergence theorem for Perron-Stieltjes integrals. Its proof can be found in [50, Theorem 11].

Theorem B.0.5. Let $\mathscr{B} = (X,Y,Z)$ be a bilinear triple and $f : [a,b] \to X$ be a function. If $x : [a,b] \to Y$ is the uniform limit of a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset E([a,b],X)$ such that the Perron-Stieltjes integral $\int_a^b d[f(s)]\varphi_n(s)$ exists for all $n \in \mathbb{N}$, then the integral $\int_a^b d[f(s)]x(s)$ exists and

$$\int_a^b d[f(s)]x(s) = \lim_{n\to\infty} \int_a^b d[f(s)]\varphi_n(s).$$

The following result shows that the indefinite Perron-Stieltjes integral is not continuous in general. The reader may consult [50, Theorem 17 and Remark 18] for a proof.

Theorem B.0.6. Assume that $\mathscr{B} = (X,Y,Z)$ is a bilinear triple and the functions $f : [a,b] \to X$ and $g : [a,b] \to Y$ are such that the integral $\int_a^b d[f(s)]g(s)$ exists. If $c \in [a,b]$, then

$$\lim_{s\to c^-} \left(\int_a^s d[f(s)]g(s) - f(s)g(c) + f(c)g(c) \right) = \int_a^c d[f(s)]g(s)$$

and

$$\lim_{s\to c^+} \left(\int_s^b d[f(s)]g(s) + f(s)g(c) - f(c)g(c) \right) = \int_c^b d[f(s)]g(s).$$

The next two results were borrowed from [50]. They relate the Perron-Stieltjes integral with the variation of its integrand.

Proposition B.0.7 ([50, Proposition 10])**.** Assume that $\mathscr{B} = (X,Y,Z)$ is a bilinear triple. If the functions $f : [a,b] \to X$ and $g : [a,b] \to Y$ are such that the Perron-Stieltjes integral $\int_a^b d[f(s)]g(s)$ exists and $\mathrm{var}_a^b f < \infty$, then

$$\left\| \int_a^b d[f(s)]g(s) \right\| \leq \mathrm{var}_a^b f \|g\|_\infty,$$

where $\|g\|_\infty = \sup_{s\in[a,b]} \|g(s)\|$.

Proposition B.0.8 ([50, Proposition 15])**.** Assume that $\mathscr{B} = (X,Y,Z)$ is a bilinear triple and $f : [a,b] \to X$ is such that $\mathrm{var}_a^b f < \infty$. If $g \in G([a,b],Y)$, then the Perron-Stieltjes integral $\int_a^b d[f(s)]g(s)$ exists.

In what follows, we present an integration-by-parts theorem for Perron-Stieltjes integrals. Its proof can be found in [13, Theorem 1.57] or [34, Theorem 1.15].

Theorem B.0.9. Let $f : [a,b] \to X$ be such that the integral $\int_a^b f(s)ds$ exists. If $\widetilde{f} : [a,b] \to X$ is given by $\widetilde{f}(t) = \int_a^t f(s)ds$ and $\alpha \in BV([a,b],L(X,Y))$, then the integrals $\int_a^b d[\alpha(s)]\widetilde{f}(s)$ and $\int_a^b \alpha(s)f(s)ds$ exist and the following equality

$$\int_a^b d[\alpha(s)]\widetilde{f}(s) = \alpha(b)\widetilde{f}(b) - \alpha(a)\widetilde{f}(a) - \int_a^b \alpha(s)f(s)ds$$

holds.

The result below represents a Gronwall-type inequality for the Perron-Stieltjes integral. A proof of it can be found in [49, Corollary 1.43].

Theorem B.0.10. Let $g : [a,b] \to [0,+\infty)$ be a nondecreasing left-continuous function, $k > 0$ and $l \geq 0$. Assume that $f : [a,b] \to [0,+\infty)$ is bounded and satisfies

$$f(t) \leq k + l \int_a^t f(s)dg(s), \quad \text{for all } t \in [a,b].$$

Then,

$$f(t) \leq ke^{l|g(t)-g(a)|}, \quad \text{for all } t \in [a,b].$$

In what follows, we present another definition of integral, introduced by Jaroslav Kurzweil, which will be crucial in the theory of generalized ordinary differential equations.

Definition B.0.11. A function $U : [a,b] \times [a,b] \to X$ is said to be *Kurzweil integrable* over $[a,b]$, if there is an element $I \in X$ such that, given $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for every δ-fine tagged division $d = (\xi_i, [s_{i-1}, s_i])$ of $[a,b]$, we have

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_{i=1}^{|d|} [U(\xi_i, s_i) - U(\xi_i, s_{i-1})]$. In this case,

$$I = \int_a^b DU(\tau,t).$$

Furthermore, we use the convention $\int_b^a DU(\tau,t) = - \int_a^b DU(\tau,t)$, whenever $a \leq b$.

The Kurzweil integral has the usual properties of uniqueness, linearity, additivity and integrability on subintervals. Moreover, we can extend the Kurzweil integral to unbounded intervals of $\mathbb{R}$ by defining δ-neighborhoods of $-\infty$ and $+\infty$. When X is an abstract Banach space, the interested reader may consult [13, Section 1.2] for more details or [49] for the particular case when $X = \mathbb{R}^n$.

In the sequel, we point out that the Kurzweil integral, given by Definition B.0.11, and the Perron-Stieltjes integrals, described in Definition B.0.4, can be related.

Remark B.0.12. Consider the bilinear triple $\mathscr{B} = (L(Y,X), Y, X)$ and let $f : [a,b] \to L(Y,X)$ and $g : [a,b] \to Y$ be given functions. If $U(\tau,t) = f(\tau)g(t)$, then the Kurzweil integral of U exists if and only if the Perron-Stieltjes integral of f with respect to g exists. In this case,

$$\int_a^b DU(\tau,t) = \int_a^b f(s)dg(s).$$

Similarly, if $U(\tau,t) = f(t)g(\tau)$, then

$$\int_a^b DU(\tau,t) = \int_a^b d[f(s)]g(s),$$

provided the integrals exist.

The following statement provides a useful tool in the theory of Kurzweil integral. For a proof, see [13, Lemma 1.7] when X is an infinite dimensional space or [49, Lemma 1.13] whenever X is a finite dimensional space.

Lemma B.0.13 (Saks-Henstock Lemma)**.** Let $U : [a,b] \times [a,b] \to X$ be Kurzweil integrable over $[a,b]$. Given $\varepsilon > 0$, let the gauge δ on $[a,b]$ be such that

$$\left\| \sum_{i=1}^{k} [U(\xi_i, s_i) - U(\xi_i, s_{i-1})] - \int_a^b DU(\tau,t) \right\| < \varepsilon,$$

for every δ-fine $d = (\xi_i, [s_{i-1}, s_i]) \in TD_{[a,b]}$ of $[a,b]$. If $d' = (\eta_i, [t_{i-1}, t_i]) \in TPD_{[a,b]}$ is δ-fine, that is,

$$\eta_i \in [t_{i-1}, t_i] \subset [\eta_i - \delta(\eta_i), \eta_i + \delta(\eta_i)], \ i = 1, 2, \dots, |d'|.$$

Then,

$$\left\| \sum_{i=1}^{k} [U(\eta_i, t_i) - U(\eta_i, t_{i-1})] - \int_{t_{i-1}}^{t_i} DU(\tau,t) \right\| < \varepsilon.$$

As a consequence of the Saks-Henstock Lemma, we have the next result. A proof of it can be found in [13, Corollary 2.8].

Corollary B.0.14. Let $U : [a,b] \times [a,b] \to X$ be Kurzweil integrable over $[a,b]$. Given $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that if $[\gamma, v]$ is a closed subinterval of $[a,b]$, then

(i) $(v - \gamma) < \delta(\gamma)$ implies $\left\| U(\gamma, v) - U(\gamma, \gamma) - \int_\gamma^v DU(\tau,t) \right\| < \varepsilon$;

(ii) $(v - \gamma) < \delta(v)$ implies $\left\| U(v, v) - U(v, \gamma) - \int_\gamma^v DU(\tau,t) \right\| < \varepsilon$.

The result below concerns the Cauchy extension for the Kurzweil integral and it is also known as Hake-type theorem for this integral. Its proof follows the same ideas of the proofs of [49, Theorems 1.14 and 1.16] with easy changes for Banach space-valued functions.

Theorem B.0.15. Let $U : [a,b] \times [a,b] \to X$ be Kurzweil integrable over $[a,b]$ and $c \in [a,b]$. Then,

$$\lim_{s \to c^-} \left[\int_a^s DU(\tau,t) - U(c,s) + U(c,c) \right] = \int_a^c DU(\tau,t),$$

$$\lim_{s \to c^+} \left[\int_c^b DU(\tau,t) + U(c,s) - U(c,c) \right] = \int_c^b DU(\tau,t)$$

and

$$\lim_{s \to c} \left[\int_a^s DU(\tau,t) - U(c,s) + U(c,c) \right] = \int_a^c DU(\tau,t).$$

Remark B.0.16. Theorem B.0.15 shows that the indefinite Kurzweil integral of U, defined by

$$[a,b] \ni s \mapsto \int_a^s DU(\tau,t),$$

may not be continuous and it is continuous at a point $c \in [a,b]$ if and only if the function $U(c,\cdot) : [a,b] \to X$ is continuous at c.

The next result is a particular case of Theorem B.0.15 and it is known as Hake-type theorem for the Perron-Stieltjes integral.

Theorem B.0.17. Consider functions $f : [a,b] \to X$ and $g \in G([a,b],\mathbb{R})$ such that the Perron-Stieltjes integral $\int_a^b f(s)dg(s)$ exists. Then, the functions

$$h(t) = \int_a^t f(s)dg(s), \quad t \in [a,b] \quad \text{and} \quad k(t) = \int_t^b f(s)dg(s), \quad t \in [a,b],$$

are regulated on $[a,b]$ and satisfy

$$h(t^+) = h(t) + f(t)\Delta^+ g(t), \quad k(t^+) = k(t) - f(t)\Delta^+ g(t), \quad t \in [a,b),$$

$$h(t^-) = h(t) - f(t)\Delta^- g(t), \quad k(t^-) = k(t) + f(t)\Delta^- g(t), \quad t \in (a,b],$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$.

APPENDIX

C

GENERALIZED ODES

This chapter is devoted to the fundamental properties and results of generalized ordinary differential equations for functions taking values in Banach spaces. We use the short form "generalized ODEs" to refer to these equations.

Let X be a Banach space, $\mathscr{O} \subseteq X$ be an open set, $I \subset \mathbb{R}$ be an interval, $\Omega = \mathscr{O} \times I$ and $F : \Omega \to X$ be a function.

Definition C.0.1. We say that $x : I \to X$ is a *solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t) \tag{C.1}$$

on the interval I, whenever $(x(t),t) \in \Omega$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),s), \quad \text{for all } s_1, s_2 \in I, \tag{C.2}$$

where the integral on the right-hand side of (C.2) has to be understood as the Kuzweil integral (see Definition B.0.11).

We sometimes refer the generalized generalized ODE (C.1) as *homogeneous nonlinear generalized ODE*. Moreover, we point out that any generalized ODE is a type of integral equation and the notation in equation (C.1) does not mean that x is differentiable with respect to τ. For example, if $r : [0,1] \to \mathbb{R}$ is a continuous function which is nowhere differentiable and $F(x,t) = r(t)$, then

$$\int_{s_1}^{s_2} DF(x(\tau),s) = r(s_2) - r(s_1)$$

and the function $x : [a,1] \to \mathbb{R}$, given by $x(s) = r(s)$ for all $s \in [0,1]$, is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) = \frac{dx}{d\tau} = Dr(t)$$

and it does not have a derivative at any point in $[0,1]$. See [49, Remark 3.2].

In the sequel, we describe a class of functions $F:\Omega\to X$ for which it is possible to get useful informations about the Kurzweil integral of F and the solutions of the generalized ODE (C.1).

Definition C.0.2. Let $h: I\to\mathbb{R}$ be a nondecreasing function and $\omega:[0,+\infty)\to[0,+\infty)$ be a continuous and increasing function such that $\omega(0)=0$. We say that a function $F:\Omega\to X$ belongs to the class $\mathscr{F}(\Omega,h,\omega)$ if, for all $(x,s_2),(x,s_1),(y,s_1),(y,s_2)\in\Omega$, we have

$$\|F(x,s_2)-F(x,s_1)\|\leq|h(s_2)-h(s_1)|, \quad \text{and} \tag{C.3}$$

$$\|F(x,s_2)-F(x,s_1)-F(y,s_2)+F(y,s_1)\|\leq\omega(\|x-y\|)|h(s_2)-h(s_1)|. \tag{C.4}$$

When $\omega:[0,+\infty)\to[0,+\infty)$ is the identity function, we write $\mathscr{F}(\Omega,h)$ instead of $\mathscr{F}(\Omega,h,\omega)$.

Next, we present sufficient conditions for the existence of the Kurzweil integral of F belonging to the class $\mathscr{F}(\Omega,h)$.

Theorem C.0.3 ([13, Theorem 4.7]). Let $F\in\mathscr{F}(\Omega,h)$ and $[a,b]\subset I$. If $y:[a,b]\to X$ is uniform limit of a sequence $\{y_n\}_{n\in\mathbb{N}}\subset E([a,b],X)$ such that $(y(t),t)\in\Omega$ and $(y_n(t),t)\in\Omega$ for all $t\in[a,b]$ and all $n\in\mathbb{N}$, then the Kurzweil integrals $\int_a^b DF(y(\tau),s)$ and $\int_a^b DF(y_n(\tau),s)$, $n\in\mathbb{N}$, exist and

$$\int_a^b DF(y(\tau),s)=\lim_{n\to\infty}\int_a^b DF(y_n(\tau),s).$$

Combining Theorems A.0.9 and C.0.3, the following result can be immediately obtained.

Corollary C.0.4. Let $[a,b]\subset I$ and $F\in\mathscr{F}(\Omega,h)$. If $y:[a,b]\to X$ is a regulated function (in particular, $y\in BV([a,b],X)$) and $(y(t),t)\in\Omega$ for all $t\in[a,b]$, then the Kurzweil integral $\int_a^b DF(y(\tau),s)$ exists.

The next two results give important estimates for the integral in (C.2).

Lemma C.0.5 ([13, Lemma 4.5]). Assume that $F:\Omega\to X$ satisfies condition (C.3) and let $[a,b]\subset I$. If $y:[a,b]\to X$ is such that $y(t)\in\mathscr{O}$ for all $t\in[a,b]$ and if the integral $\int_a^b DF(y(\tau),s)$ exists, then

$$\left\|\int_{s_1}^{s_2} DF(y(\tau),s)\right\|\leq|h(s_2)-h(s_1)|,$$

for all $s_1,s_2\in[a,b]$. Moreover, the function

$$t\mapsto\int_a^t DF(y(\tau),s)$$

is continuous at every point that the function h is continuous, is of bounded variation on $[a,b]$, and, hence, also regulated.

Lemma C.0.6 ([13, Lemma 4.6])**.** Let $[a,b] \subset I$ and $F : \Omega \to X$ be such that $F \in \mathscr{F}(\Omega, h, \omega)$ with $\Omega = \mathscr{O} \times I$. If $y, z \in G([a,b], \mathscr{O})$, then

$$\left\| \int_a^b D[F(y(\tau),s) - F(z(\tau),s)] \right\| \leq \int_a^b \omega(\|y(s) - z(s)\|) dh(s), \tag{C.5}$$

where the integral on the right-hand side of (C.5) has to be understood as the Perron-Stieltjes integral. See Definition B.0.4.

As a consequence of Corollary C.0.4 and Lemma C.0.5, we have the following result.

Lemma C.0.7. Let $[a,b] \subset I$, $y \in G^-([a,b],X)$ and $F : \Omega \to X$ be such that $F \in \mathscr{F}(\Omega, h)$, with $\Omega = \mathscr{O} \times I$. If $h : I \to \mathbb{R}$ is left-continuous on $(a,b]$, then the function $f : [a,b] \to X$ given by

$$f(t) = y(t) + \int_a^t DF(y(\tau),s), \quad t \in [a,b]$$

belongs to $G^-([a,b],X)$.

Proof. At first, notice that f is well-defined, since $G^-([a,b],X) \subset G([a,b],X)$ and the existence of the Kurzweil intregral $\int_a^t DF(y(\tau),s)$ is ensured by Corollary C.0.4. Once $y \in G([a,b],X)$, by Proposition A.0.13, there exists $M > 0$ such that

$$\sup_{t \in [a,b]} \|y(t)\| < M.$$

From this fact and Lemma C.0.5, for all $t_1, t_2 \in [a,b]$, we have

$$\begin{aligned} \|f(t_2) - f(t_1)\| &\leq \|y(t_2) - y(t_1)\| + |h(t_2) - h(t_1)| \leq \|y(t_2)\| + \|y(t_1)\| + |h(t_2) - h(t_1)| \\ &\leq 2M + |h(t_2) - h(t_1)|. \end{aligned}$$

Thus, $f \in BV([a,b],X) \subset G([a,b],X)$ (see Remark A.0.8).

On the other hand, let $c \in (a,b]$. Since y and h are left-continuous on $(a,b]$, there exists $\delta > 0$ such that, if $t \in (c - \delta, c)$ then,

$$\|y(t) - y(c)\| < \frac{\varepsilon}{2} \quad \text{and} \quad |h(t) - h(c)| < \frac{\varepsilon}{2}.$$

Consequently,

$$\|f(t) - f(c)\| \leq \|y(t) - y(c)\| + |h(t) - h(c)| < \varepsilon, \quad \text{for all } t \in (c - \delta, c)$$

which implies that f is left-continuous on $(a,b]$ and the proof is complete. □

The next result describes how the solutions of the generalized ODE (C.1) inherit the properties of the function $F : \Omega \to X$. In particular, if F is continuous with respect to the second variable, then any solution of the generalized ODE (C.1) is a continuous function. For a proof, see [13, Theorem 4.4] when X is an arbitrary Banach space or [49, Proposition 3.6] whenever $X = \mathbb{R}^n$.

Theorem C.0.8. Let $F \in \mathscr{F}(\Omega,h)$, $[a,b] \subset I$ and $x:[a,b] \to X$ be a solution of the generalized ODE (C.1). Then, for every $\sigma \in [a,b]$,

$$\lim_{s\to\sigma}[x(s)-F(x(\sigma),s)+F(x(\sigma),\sigma)] = x(\sigma).$$

The result below exhibits an important property of the solutions of the generalized ODE (C.1), when the function $F:\Omega \to X$ satisfies (C.3) (in particular, if $F \in \mathscr{F}(\Omega,h)$). Its proof can be found in [13, Lemma 4.9] when X is an arbitrary Banach space or [49, Lemma 3.10] whenever $X = \mathbb{R}^n$.

Lemma C.0.9. Let $F \in \mathscr{F}(\Omega,h)$, $[a,b] \subset I$ and $x:[a,b] \to X$ be a solution of the generalized ODE (C.1). Then,

$$\|x(s_2)-x(s_1)\| \leq |h(s_2)-h(s_1)|$$

for every $s_1,s_2 \in [a,b]$. Moreover, x is of bounded variation on $[a,b]$ and it is continuous at every point that h is continuous.

In what follows, we present another useful property of the solutions of the generalized ODE (C.1).

Remark C.0.10. If $x:[a,b] \to X$ is a solution of the generalized ODE (C.1), then,

$$\begin{aligned} x(s^+) - \lim_{\sigma\to s^+}\int_a^\sigma DF(x(\tau),t) &= \lim_{\sigma\to s^+}\left(x(\sigma)-\int_a^\sigma DF(x(\tau),t)\right) \\ &= \lim_{\sigma\to s^+}\left(x(a)+\int_a^\sigma DF(x(\tau),t)-\int_a^\sigma DF(x(\tau),t)\right) \\ &= x(a) = x(s)-\int_a^s DF(x(\tau),t). \end{aligned}$$

We target at addressing the existence and uniqueness of solutions of the generalized ODE (C.1). Toward this end, we consider

$$\Omega_F = \left\{(y,t)\in\Omega : y+F(y,t^+)-F(y,t)\in\mathscr{O}\right\}, \tag{C.6}$$

where $F(y,t^+) = \lim_{s\to t^+}F(y,s)$.

The following result gives sufficient conditions for the existence and uniqueness of a local solution for an initial value problem of the generalized ODE (C.1). For a proof of it, see [13, Theorem 5.1]. The reader may also consult [49, Theorem 4.2] for the finite dimensional case.

Theorem C.0.11 (Local existence and uniqueness)**.** Let $[a,b] \subset I$ and $F \in \mathscr{F}(\Omega,h)$, where $h: I \to \mathbb{R}$ is a nondecreasing and left-continuous function. If $(x_0,s_0) \in \Omega_F$, then there exists $\Delta > 0$ and a function $x:[s_0,s_0+\Delta] \to X$ which is the unique solution of the generalized ODE (C.1) with $x(s_0) = x_0$.

Our next goal is to present results which guarantee that the unique solution of the generalized ODE (C.1), whose existence is ensured by Theorem C.0.11, can be extended to intervals containing $[s_0, s_0+\Delta]$ up a maximal interval of existence. In order to do this, we consider $I = [t_0, +\infty) \subset \mathbb{R}$ and, for all $(x_0, s_0) \in \Omega$, we denote by S_{s_0,x_0} the set of all solutions, $x : I_x \to X$, of the generalized ODE (C.1) with $x(s_0) = x_0$, where $I_x \subset I$ is an interval with left endpoint s_0, that is, $I_x = [s_0, \omega)$, $I_x = [s_0, \omega]$ or $I_x = [s_0, +\infty)$ for some $\omega \in [s_0, +\infty)$. At first, we recall the definition, introduced in [21] , of a prolongation of solutions of the generalized ODE (C.1).

Definition C.0.12 (Prolongation of solutions)**.** Let $(x_0, s_0) \in \Omega$ and $x : I_x \to X$ be such that $x \in S_{s_0,x_0}$. A solution of the generalized ODE (C.1), $y : I_y \to X$, is called a *prolongation to the right of* x, if $y \in S_{s_0,x_0}$, $I_x \subset I_y$ and $x(t) = y(t)$ for all $t \in I_x$. Whenever $I_x \subsetneq I_y$, y is said to be a *proper prolongation of* x to the right.

Remark C.0.13. Given $x, y \in S_{s_0,x_0}$, we say that x is *smaller or equal* to y ($x \preccurlyeq y$) if and only if $I_x \subset I_y$ and $y(t) = x(t)$ for all $t \in I_x$. Moreover, $\preccurlyeq$ defines a partial order relation in S_{s_0,x_0}. See [21, Proposition 3.4].

Based on Definition C.0.12, the next result ensures the prolongation of a solution of the generalized ODE (C.1). For a proof, see [21, Theorem 3.1].

Theorem C.0.14. Let $F \in \mathscr{F}(\Omega, h)$, where $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function. Assume that $x : [a,b) \to X$ and $y : I_y \to X$ are solutions of the generalized ODE (C.1) on $[a,b)$ and I_y respectively, where $[a,b) \subset I$ and $I_y \in \{[b,\omega], [b,\omega), [b,+\infty); \omega \in (b,+\infty)\}$. If the limit $\lim_{t\to b^-} x(t)$ exists and $\lim_{t\to b^-} x(t) = y(b)$, then the function $z : [a,b) \cup I_y \to X$ defined by

$$z(t) = \begin{cases} x(t), & \text{if } t \in [a,b), \\ y(t), & \text{if } t \in I_y, \end{cases}$$

is a solution of the generalized ODE (C.1) on $[a,b) \cup I_y$.

The next definition was borrowed from [21].

Definition C.0.15 (Maximal solution)**.** Let $(x_0, s_0) \in \Omega$. We say that x is a *maximal forward solution* or simply *maximal solution* of the generalized ODE (C.1), with $x(s_0) = x_0$, if $x \in S_{s_0,x_0}$ and there is no proper prolongation of x to the right.

Remark C.0.16. When $x : I_x \to X$ is a maximal solution of the generalized ODE (C.1) and $\sup I_x = +\infty$, x is also known as a *global forward solution* on I_x.

The result below brings up sufficient conditions for the existence and uniqueness of a maximal solution of the generalized ODE (C.1). A version of such a result when $\Omega = B_c \times (a,b)$, with $B_c = \{x \in \mathbb{R}^n : \|x\| < c\}$ and $(a,b) \subset \mathbb{R}$, can be found in [49, Proposition 4.3]. For the infinite dimensional case, the reader may consult [13, Theorems 5.11 and 5.12] or [21, Theorems 3.9 and 3.10].

Theorem C.0.17. Let $F \in \mathscr{F}(\Omega,h)$, where $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function. For every $(x_0,s_0) \in \Omega_F$, there exists a unique maximal solution $x : [s_0,\omega(s_0,x_0)) \to X$ of the generalized ODE (C.1) with $x(s_0) = x_0$ and $\omega(s_0,x_0) \leq +\infty$.

Assuming the existence of a global forward solution of the generalized ODE (C.1), we obtain informations about the range of this solution. See [13, Theorem 5.13] or [21, Theorem 3.11].

Theorem C.0.18. Let $F \in \mathscr{F}(\Omega,h)$, where $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function. Suppose that $(x_0,s_0) \in \Omega_F$ and $x : [s_0,+\infty) \to X$ is the global forward solution of the generalized ODE (C.1) with $x(s_0) = x_0$. Then, for every compact set $K \subset \Omega$, there exists $t_K \in [s_0,\omega)$ for which $(x(t),t) \notin K$ for all $t \in (t_k,\omega)$, where $\omega \leq +\infty$.

In view of the previous theorem, the next result ensures that if the maximal solution of the generalized ODE (C.1) is taking values in a compact subset of $\mathscr{O}$, then it is defined on an unbounded interval, that is, the maximal solution is a global forward solution. Its proof follows from Theorems C.0.17 and C.0.18 and can be found in [13, Corollary 5.14] or [21, Corollary 3.12].

Corollary C.0.19. Let $F \in \mathscr{F}(\Omega,h)$, where $\Omega = \mathscr{O} \times I$ and $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function. Suppose that $(x_0,s_0) \in \Omega_F$ and $x : [s_0,\omega(s_0,x_0)) \to X$ is the maximal solution of the generalized ODE (C.1) with $x(s_0) = x_0$. If $x(t)$ belongs to a compact $N \subset \mathscr{O}$ for all $t \in [s_0,\omega(s_0,x_0))$, then $\omega(s_0,x_0) = +\infty$.

The next two results show that, when we consider a special Ω, we obtain the existence of a global forward solution of the generalized ODE (C.1).

Corollary C.0.20 ([13, Corollary 5.14] or [21, Corollary 3.14])**.** If $\Omega = X \times I$ and $F \in \mathscr{F}(\Omega,h)$, where $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function, then for every $(x_0,s_0) \in \Omega$, there exists a unique global forward solution $x : [s_0,+\infty) \to X$ of the generalized ODE (C.1) with $x(s_0) = x_0$.

Corollary C.0.21. Let $\Omega = N \times I$ and $F \in \mathscr{F}(\Omega,h)$, where $N \subset X$ is a compact set and $h : I \to \mathbb{R}$ is a nondecreasing and left-continuous function. Then, for every $(x_0,s_0) \in \Omega$, there exists a unique global forward solution $x : [s_0,+\infty) \to X$ of the generalized ODE (C.1) with $x(s_0) = x_0$.

Proof. For every $(x_0,x_0) \in \Omega_F$, Theorem C.0.17 guarantees the existence of a maximal solution $x : [t_0,\omega(s_0,x_0)) \to X$ of the generalized ODE (C.1) with $x(s_0) = x_0$, and, by the definition of a solution, $(x(t),t) \in \Omega$ for every $t \in I$. Then, by Corollary C.0.19, $\omega(s_0,x_0) = +\infty$ and, consequently, x is a global forward solution. □

The topic of the next results is the continuous dependence, with respect to initial conditions, of solutions of the generalized ODEs.

Theorem C.0.22 ([13, Theorem 7.2] or [49, Theorem 8.1])**.** Let $h:[s_0,+\infty)\to\mathbb{R}$ be a nondecreasing and left-continuous function and $\Omega=\mathscr{O}\times[s_0,+\infty)$. Assume that $F_k:\Omega\to X$ belongs to the class $\mathscr{F}(\Omega,h)$ for $k=0,1,\ldots$ and

$$\lim_{k\to+\infty} F_k(x,t)=F_0(x,t),$$

for all $(x,t)\in\Omega$. Moreover, for each $k=0,1,2,\ldots$, assume that the integral $\int_{t_0}^{t} DF_k(x_k(\tau),s)$ exists, where $x_k:[s_0,+\infty)\to X$ are functions such that $\lim_{k\to+\infty} x_k(s)=x_0(s)$. Then,

$$\lim_{k\to+\infty}\int_{s_0}^{t} DF_k(x_k(\tau),s)=\int_{s_0}^{t} DF_0(x_0(\tau),s).$$

Proposition C.0.23. Let $\Omega=\mathscr{O}\times[s_0,+\infty)$ and $h:[s_0,+\infty)\to\mathbb{R}$ be a nondecreasing and left-continuous function. Assume that $F_k:\Omega\to X$ belongs to the class $\mathscr{F}(\Omega,h)$ for $k=0,1,\ldots$ and

$$\lim_{k\to+\infty} F_k(x,t)=F_0(x,t), \tag{C.7}$$

for each $(x,t)\in\Omega$. Let $x_k:[s_0,+\infty)\to X$ be the unique global forward solution of the generalized ODE

$$\begin{cases} \dfrac{dx}{d\tau} &= DF_k(x,t), \quad k=1,2,\ldots, \\ x_k(s_0) &= v_k. \end{cases} \tag{C.8}$$

Assume that $\{v_k\}_{k\in\mathbb{N}}$ converges to a point $v_0\in X$. Then, there is a unique global forward solution $x_0:[s_0,+\infty)\to X$ of the generalized ODE

$$\begin{cases} \dfrac{dx}{d\tau} &= DF_0(x,t), \\ x_0(s_0) &= v_0 \end{cases} \tag{C.9}$$

such that $\lim_{k\to+\infty} x_k(s)=x_0(s)$ for all $s\in[s_0,+\infty)$.

Proof. We starting by the existence and uniqueness of a maximal solution of the generalized ODE (C.9). By Theorem C.0.17, it is enough to prove that $(s_0,v_0)\in\Omega_F$, that is,

$$v_0+F_0(v_0,s_0^+)-F_0(v_0,s_0)\in\mathscr{O}.$$

Since $F_k\in\mathscr{F}(\Omega,h)$ for all $k\in\mathbb{N}$, by (C.4), we have

$$\|F_k(v_k,s_0^+)-F_k(v_k,s_0)+F_k(v_0,s_0)-F_k(v_0,s_0^+)\|\leq\|v_k-v_0\||h(s_0^+)-h(s_0)|.$$

which implies

$$\lim_{k\to+\infty}\left(F_k(v_k,s_0^+)-F_k(v_k,s_0)+F_k(v_0,s_0)-F_k(v_0,s_0^+)\right)=0. \tag{C.10}$$

Combing (C.10) with (C.7) and the fact that $\{v_k\}_{k\in\mathbb{N}}$ converges to v_0 we obtain

$$\begin{aligned} v_0+F_0(v_0,s_0^+)-F_0(v_0,s_0) &= \lim_{k\to+\infty}\left(v_k+F_k(v_0,s_0^+)-F_k(v_0,s_0)\right) \\ &= \lim_{k\to+\infty}\left(v_k+F_k(v_k,s_0^+)-F_k(v_k,s_0)\right. \\ &\quad \left.-F_k(v_k,s_0^+)+F_k(v_k,s_0)+F_k(v_0,s_0^+)-F_k(v_0,s_0)\right) \quad (C.11) \\ &\overset{(C.10)}{=} \lim_{k\to+\infty}\left(v_k+F_k(v_k,s_0^+)-F_k(v_k,s_0)\right) \\ &= \lim_{k\to+\infty} x_k(s_0^+), \end{aligned}$$

where the last equality follows from Theorem C.0.8, since x_k is a solution of a generalized ODE for all $k\in\mathbb{N}$.

Let $z_0=\lim_{k\to+\infty}x_k(s_0^+)$, then given $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that

$$\|z_0-x_k(s_0^+)\|<\varepsilon, \quad \text{for all } k>k_0. \tag{C.12}$$

Take $\bar{k}>k_0$ be fixed. By the definition of the right-hand limit of $x_{\bar{k}}$, there exists $\delta_0>0$ such that if $t\in(s_0,s_0+\delta_0)$, then

$$\|x_{\bar{k}}(t)-x_{\bar{k}}(s_0^+)\|<\varepsilon. \tag{C.13}$$

Let $t_0\in(s_0,s_0+\delta_0)$ be fixed. By the definition of the solution of a generalized ODE, $x_{\bar{k}}(t_0)\in\mathscr{O}$ and, since $\mathscr{O}$ is an open set, there exists $\delta>0$ such that if $z\in X$ satisfies

$$\|z-x_{\bar{k}}(t_0)\|<\delta,$$

then $z\in\mathscr{O}$. By equations (C.12) and (C.13),

$$\|z_0-x_{\bar{k}}(t_0)\|\le\|z_0-x_{\bar{k}}(s_0^+)\|+\|x_{\bar{k}}(s_0^+)-x_k(t_0)\|<\varepsilon.$$

Since ε was taken arbitrarily, we conclude that $z_0\in\mathscr{O}$ and, by (C.11), $(v_0,s_0)\in\Omega_F$.

Then, by Theorem C.0.17, there exists a unique maximal solution, $x_0:[s_0,\omega(s_0,v_0))\to X$, of the generalized ODE (C.9) with $x_0(s_0)=v_0$. It remains to show that $x_0(t)=\lim_{k\to+\infty}x_k(t)$ for all $t\in[s_0,\omega(s_0,v_0))$ and $\omega(s_0,v_0)=+\infty$.

Let $y:[s_0,+\infty)\to X$ be defined by $y(t)=\lim_{k\to+\infty}x_k(t)$, for all $t\in[s_0,+\infty)$. Since $x_k\in G([a,b],\mathscr{O})$ for all $[a,b]\subset[s_0,+\infty)$ and all $k\in\mathbb{N}$ (see Lemma C.0.9), Theorem A.0.11 guarantees that $y\in G([a,b],\mathscr{O})$ for all $[a,b]\subset[s_0,+\infty)$. Moreover, by Corollary C.0.4, the Kurzweil integral $\int_{s_0}^t DF_0(y(\tau),s)$ exists for all $t\in[s_0,+\infty)$ and, by Theorem C.0.22 and by the definition of the solution of a generalized ODE, we have

$$\begin{aligned} y(t) &= \lim_{k\to+\infty}x_k(t)=\lim_{k\to+\infty}\left(v_k+\int_{s_0}^t DF_k(x_k(\tau),s)\right) \\ &= v_0+\int_{s_0}^t DF_0(y(\tau),s). \end{aligned}$$

Thus, y is a solution of the generalized ODE (C.9) on $[s_0,+\infty)$ with $y(s_0)=v_0$. By the uniqueness of solution, $x_0(t)=y(t)$ for all $t\in[t_0,\omega(s_0,v_0))$ and $\omega(s_0,v_0)=+\infty$. □

APPENDIX

D

THE TIME SCALES CALCULUS

In this Chapter, we present basic concepts and fundamental results of the theory of time scales. For more details, the reader may consult [10, 11].

A *time scale* is a closed nonempty subset $\mathbb{T}$ of the real line. For all $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}; s > t\}, \tag{D.1}$$

the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T}; s < t\}, \tag{D.2}$$

and the *graininess function* $\mu : \mathbb{T} \to [0, +\infty)$ by

$$\mu(t) = \sigma(t) - t. \tag{D.3}$$

In the sequel, we recall some standard definitions for the points in a given time scale.

Definition D.0.1. Let $\mathbb{T}$ be a time scale. Any point $t \in \mathbb{T}$ is called:

(i) *right-dense*, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$;

(ii) *right-scattered*, if $\sigma(t) > t$;

(iii) *left-dense*, if $t > \inf \mathbb{T}$ and $\rho(t) = t$;

(iv) *left-scattered*, if $\rho(t) < t$.

In the next definition, we present some concepts of a function defined on a time scale $\mathbb{T}$ and taking values in a Banach space X.

Definition D.0.2. Let $\mathbb{T}$ be a time scale. A function $f : \mathbb{T} \to X$ is said to be:

(i) *regulated*, provided its right-sided limit exists at all right-dense points in $\mathbb{T}$, and its left-sided limit exists at all left-dense points in $\mathbb{T}$;

(ii) *rd-continuous*, if it is regulated and continuous at right-dense points.

In what follows, we define an special set of regulated functions.

Definition D.0.3. Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, for some $t_0 \in \mathbb{T}$. A function $f : \mathbb{T}_0 \to X$ belongs to $G_0(\mathbb{T}, X)$ if f is regulated and

$$\|f\|_{\mathbb{T}_0} = \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)} \|f(s)\| < \infty.$$

In order to present the definition of derivatives in the framework of time scales, we define the set $\mathbb{T}^k$ by

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition D.0.4. Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to X$ be a function and $t \in \mathbb{T}^k$. We say that $f^{\Delta}(t)$ (provided it exists) is the *delta derivative of f at t*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\Big\| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \Big\| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in (t - \delta, t + \delta) \cap \mathbb{T}.$$

Moreover, we say that f is *delta differentiable on* $\mathbb{T}^k$, if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. In this case, the function $f^{\Delta} : \mathbb{T}^k \to X$ is called the *delta derivative of f on* $\mathbb{T}^k$.

The following result provides an useful property of delta derivatives. Its proof is a simple adaptation of [10, Theorem 1.20] to Banach space-valued functions. Therefore, we omit it here.

Theorem D.0.5. Let $\mathbb{T}$ be a time scale and assume that $f, g : \mathbb{T} \to X$ are delta differentiable at $t \in \mathbb{T}^k$. The following assertions hold.

(i) The sum $f + g : \mathbb{T} \to X$ is delta differentiable at t with $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.

(ii) For any constant $c \in \mathbb{T}$, $cf : \mathbb{T} \to X$ is delta differentiable at t with $(cf)^{\Delta}(t) = cf^{\Delta}(t)$.

(iii) The product $fg : \mathbb{T} \to X$ is delta differentiable at t with $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)$.

(iv) If f is a constant function, that is, $f(t) = c$ for all $t \in \mathbb{T}$ and for some constant $c \in X$, then $f^{\Delta}(t) = 0$.

In the next lines, we recall the definition of a special division of $[a, b]_{\mathbb{T}}$. See [46].

Definition D.0.6. Let $\mathbb{T}$ be a time scale. A *division* of $[a,b]_{\mathbb{T}}$ is a finite sequence of points $d_{\mathbb{T}} = \{s_0, s_1, \ldots, s_{|d|}\} \subset [a,b]_{\mathbb{T}}$, where $a = s_0 < s_1 < \ldots < s_{|d|} = b$. We say that $d_{\mathbb{T}}$ is a *tagged division* of $[a,b]_{\mathbb{T}}$, whenever

$$a = s_0 \leq \tau_1 \leq s_1 \leq \ldots \leq s_{|d|-1} \leq \tau_{|d|} \leq s_{|d|} = b,$$

with $s_i > s_{i-1}$, $s_i \in \mathbb{T}$ and $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$, for $1 \leq i \leq |d|$. We denote such tagged division by $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$, where τ_i is the associated tag point in $[s_{i-1}, s_i]_{\mathbb{T}}$.

We say that $\delta = (\delta_L, \delta_R)$ is a Δ-gauge of $[a,b]_{\mathbb{T}}$, provided $\delta_L(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \mu(t)$ for all $t \in [a,b)_{\mathbb{T}}$.

If δ is a Δ-gauge on $[a,b]_{\mathbb{T}}$, then a tagged division $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is called *δ-fine*, whenever

$$\tau_i - \delta_L(\tau_i) \leq s_{i-1} < s_i \leq \tau_i + \delta_R(\tau_i), \quad 1 \leq i \leq |d|.$$

Notice that, similarly as in real line case, that is, when $\mathbb{T} = \mathbb{R}$, we can ensure the existence of at least one δ-fine tagged divisions $d_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$. As a matter of fact, this result is a generalization of the Cousin Lemma (see Lemma B.0.2) for a Δ-gauge of a time scale interval and it can be found in [46, Lemma 1.9].

In what follows, we present a definition of Δ-integral of a function $f : [a,b]_{\mathbb{T}} \to X$ by means of δ-fine tagged divisions. We refer to this integral as Perron Δ-integral. See also [46].

Definition D.0.7. Let $\mathbb{T}$ be a time scale. We say that $f : [a,b]_{\mathbb{T}} \to X$ is *Perron Δ-integrable* on $[a,b]_{\mathbb{T}}$, if there is an element $I \in X$ such that given $\varepsilon > 0$, there exists a Δ-gauge δ on $[a,b]_{\mathbb{T}}$ for which

$$\left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon,$$

for all δ-fine tagged division $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ of $[a,b]_{\mathbb{T}}$. In this case, we write $I = \int_a^b f(t)\Delta t$ which is called the *Perron Δ-integral* of f. Moreover, a function $f : \mathbb{T} \to X$ is said to be *locally Perron Δ-integrable*, if the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists for all $[a,b]_{\mathbb{T}} \subset \mathbb{T}$.

The following example was borrowed from [10, Theorem 1.79], with obvious adaptation to Banach space-valued functions.

Example D.0.8. Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to X$ be such that the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right-hand side is the usual Perron integral.

(ii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a > b. \end{cases}$$

As in [25], in order to present a relation between Perron-Stieltjes integrals and Perron Δ-integrals, we define an *extension* of a given time scale $\mathbb{T}$ by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, +\infty), & \text{otherwise.} \end{cases}$$

Moreover, given a function $f : \mathbb{T} \to X$, an extension $f^* : \mathbb{T}^* \to X$ is defined by

$$f^*(t) = f(t^*), \quad \text{for all } t \in \mathbb{T}^*,$$

where t is a real number such that $t \leq \sup \mathbb{T}$ and $t^* = \inf\{s \in \mathbb{T}; s \geq t\}$. Since $\mathbb{T}$ is a closed set, it is clear that $t^* \in \mathbb{T}$ and, therefore, f^* is well-defined.

In what follows, we present a result which shows how f^* inherits some properties of f. Its proof, when $X = \mathbb{R}^n$, can be found in [52, Lemma 4] and, for arbitrary Banach space-valued functions, see [13, Lemma 3.20].

Lemma D.0.9. Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to X$ be a function and $f^* : \mathbb{T}^* \to X$ be the extension of f. Then, the following statements are true.

(i) If f is a regulated function on $\mathbb{T}$, then f^* is also regulated on $\mathbb{T}^*$.

(ii) If f is left-continuous on $\mathbb{T}$, then f^* is left-continuous on $\mathbb{T}^*$.

(iii) If f is right-continuous on $\mathbb{T}$, then f^* is right-continuous at right-dense points of $\mathbb{T}$.

The next result shows that the existence of Perron Δ-integrals and Perron-Stieltjes integrals are related. For a proof of this fact, see [13, Theorem 3.24] and, for the case when $X = \mathbb{R}^n$, see [25, Theorem 4.2].

Theorem D.0.10. Let $\mathbb{T}$ be a time scale and $f : [a,b]_{\mathbb{T}} \to X$ be a function. Define $g(t) = t^*$ for all $t \in [a,b]$. Then, the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists if and only if the Perron-Stieltjes integral $\int_a^b f^*(t)dg(t)$ exists. In this case, both integrals have the same value.

Remark D.0.11. By Theorem D.0.10 and the fact that the Perron-Stieltjes integral is linear and additive with respect to adjacent intervals, we conclude that the Perron Δ-integral is also linear and additive with respect to adjacent intervals. Moreover, if $f : \mathbb{T} \to X$ is a Perron Δ-integrable function on a time scale $\mathbb{T}$, then $\int_a^b f(s)\Delta s = -\int_b^a f(s)\Delta s$, whenever $a, b \in \mathbb{T}$ and $a \leq b$.

The next statement will be crucial to our proposes. Its proof can be found in [25, Theorem 4.1], for the case where $X = \mathbb{R}^n$, and in [13, Theorem 3.15] for Banach space-valued functions.

Theorem D.0.12. Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to X$ be a function such that the Perron Δ-integral $\int_a^b f(t)\Delta t$ exists for all $a, b \in \mathbb{T}$, $a < b$. Choose an arbitrary $a \in \mathbb{T}$ and define

$$F_1(t) = \int_a^t f(s)\Delta(s),\ t \in \mathbb{T},$$

$$F_2(t) = \int_a^t f^*(s)dg(s),\ t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then, $F_2 = F_1^*$.

We end this section by presenting a result which gives a sufficient condition for the existence of a Perron Δ-integral.

Proposition D.0.13. Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to X$ be a regulated function. Then, the Perron Δ-integral $\int_a^b f(s)\Delta s$ exists for all $a, b \in \mathbb{T}$, and

$$\left\| \int_a^b f(s)\Delta s \right\| \leq \|f\|_{[a,b]_{\mathbb{T}}}[g(b) - g(a)] \tag{D.4}$$

where $g(s) = s^*$ for all $s \in \mathbb{T}^*$ and $\|f\|_{[a,b]_{\mathbb{T}}} = \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\|$.

Proof. Let $f : \mathbb{T} \to X$ be a regulated function and define $g(s) = s^*$ for all $s \in \mathbb{T}^*$. Then, it is clear that g is nondecreasing and, by Lemma D.0.9, $f : \mathbb{T}^* \to X$ is also regulated. Moreover, by Proposition B.0.8, the Perron-Stieltjes integral $\int_a^b f^*(s)dg(s)$ exists for all $a, b \in \mathbb{T}$ and, by Theorem D.0.10, the Perron Δ-integral $\int_a^b f(s)\Delta s$ exists for all $a, b \in \mathbb{T}$.

Let us prove (D.4). Since $\int_a^b f(s)\Delta s = -\int_b^a f(s)\Delta s$, for all $a, b \in \mathbb{T}$ (see Remark D.0.11), we may assume, without loss of generality, that $a < b$. Let ε be given. Once the Perron Δ-integral $\int_a^b f(s)\Delta s$ exists, there is a Δ-gauge δ on $[a,b]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) - \int_a^b f(s)\Delta s \right\| < \varepsilon,$$

provided $d_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ is a δ-fine tagged division of $[a,b]_{\mathbb{T}}$. Then,

$$\begin{aligned} \left\| \int_a^b f(s)\Delta s \right\| &\leq \left\| \int_a^b f(s)\Delta s - \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\| + \left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\| \\ &\leq \varepsilon + \left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\|. \end{aligned} \tag{D.5}$$

Furthermore, since f^* is regulated, by Proposition A.0.13, we have

$$f(s) \leq \sup_{s\in[a,b]_{\mathbb{T}}} \|f(s)\| \leq \sup_{s\in[a,b]} \|f^*(s)\| < \infty, \quad \text{for all } s \in [a,b]_{\mathbb{T}}. \tag{D.6}$$

Equation (D.6) together with the fact that $g|_{\mathbb{T}}$ is the identity function yield

$$
\begin{aligned}
\left\| \sum_{i=1}^{|d|} f(\tau_i)(s_i - s_{i-1}) \right\| &\leq \sum_{i=1}^{|d|} \| f(\tau_i)(s_i - s_{i-1}) \| \\
&\leq \sup_{s\in[a,b]_{\mathbb{T}}} \| f(s) \| \sum_{i=1}^{|d|} |(s_i - s_{i-1})| \\
&\overset{s_i > s_{i-1}}{=} \sup_{s\in[a,b]_{\mathbb{T}}} \| f(s) \| \sum_{i=1}^{|d|} g(s_i) - g(s_{i-1}) \\
&= \| f \|_{[a,b]_{\mathbb{T}}} [g(b) - g(a)]
\end{aligned}
\tag{D.7}
$$

Finally, by (D.5) and (D.7), we obtain

$$
\left\| \int_a^b f(s)\Delta s \right\| \leq \varepsilon + \| f \|_{[a,b]_{\mathbb{T}}} [g(b) - g(a)]
$$

and the statement is proved once ε can be made arbitrarily small. □

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LIST OF SYMBOLS

Norms

$\|\cdot\|_\infty$	usual supremum norm, page 172
$\|\cdot\|_{BV}$	norm in $BV([a,b],X)$, page 170
$\|\cdot\|_{[t_0,+\infty)}$	norm in $G_0([t_0,+\infty),X)$, page 173
$\|\cdot\|_{\mathbb{T}_0}$	norm in $G_0(\mathbb{T}_0,X)$, page 190

Integrals

$\int_a^b DU(\tau,t)$	Kurzweil integral, page 178
$\int_a^b d[f(s)]g(s)$	Perron-Stieltjes integral of g with respect to f, page 176
$\int_a^b f(s)dg(s)$	Perron-Stieltjes integral of f with respect to g, page 176
$\int_a^b f(t)\Delta t$	Perron Δ-integral, page 191

Notations

M'	transpose of a matrix M, page 163
D^+U	upper right derivative of U, page 120
D^+V	upper right derivative of V, page 23
$d_{\mathbb{T}}$	tagged division of a time scale interval, page 190
$\mathscr{F}(\Omega,h)$	class of right-hand sides F of generalized ODEs, page 182
$\mathscr{F}(\Omega,h,\omega)$	class of right-hand sides F of generalized ODEs, page 182
f^*	extension of a function f defined on a time scale, page 192
$f^\Delta(t)$	delta derivative of f at t, page 190
μ	graininess function, page 189
ρ	Jump operator backward, page 189
σ	Jump operator forward, page 189
$\omega(s_0,x_0)$	supremum of the interval of a maximal solution, page 186
$\preccurlyeq$	partial order relation in S_{s_0,x_0}, page 185
$\mathbb{R}^+$	non-negative real line, page 22
$\mathbb{T}$	time scale, page 189
$\mathbb{T}^k$	especial set of a time scale, page 190
$\mathbb{T}^*$	extension of a time scale $\mathbb{T}$, page 192
$\mathrm{var}_a^b f$	variation of the function f in $[a,b]$, page 169
y_s	memory function, page 90

Sets

$\mathscr{B}(X,Y,Z)$ set of all bilinear triples, page 176

$\overline{B}_\rho$ closed ball of X, page 61

$\overline{B}_c$ closed ball of X, page 35

$BG([t_0,+\infty),\mathscr{O})$ set of regulated and bounded functions defined on unbounded intervals, page 173

$BV([a,b],\mathscr{O})$ set of functions of bounded variation in $[a,b]$, page 169

$\mathscr{D}[a,b]$ set of all divisions of the interval $[a,b]$, page 169

$\overline{E}_\rho$ closed ball of S, page 125

$E([a,b],\mathscr{O})$ set of step functions defined from $[a,b]$ into $\mathscr{O}$, page 169

$\overline{G}_\rho$ closed ball of $\mathbb{O}$, page 126

$G([a,b],\mathscr{O})$ set of regulated functions defined from $[a,b]$ into $\mathscr{O}$, page 170

$G([t_0,+\infty),\mathscr{O})$ set of regulated functions defined on unbounded intervals, page 173

$G^-([a,b],\mathscr{O})$ set of regulated and left-continuous functions defined from $[a,b]$ into $\mathscr{O}$, page 172

$G_0([t_0,+\infty),\mathscr{O})$ subset of $G([t_0,+\infty),\mathscr{O})$, page 173

$G_0(\mathbb{T}_0,X)$ subset of regulated function on a time scales equipped with the norm $\|\cdot\|_{\mathbb{T}_0}$, page 190

$L(U)$ Banach space of continuous linear mapping defined from U into U, page 175

$L(U,V)$ Banach space of continuous linear mapping defined from U into V, page 175

$\mathbb{O}$ special subset of $BG([t_0-r,+\infty),X)$, page 93

$\mathscr{O}$ open subset of X, page 181

O set with prolongation property, page 90

Ω_F set of initial conditions of generalized ODEs, page 184

S subset of $G([-r,0],X)$, page 90

S_{s_0,x_0} set of all solutions of a generalized ODE with initial condition $x(s_0)=x_0$, page 185

$TD_{[a,b]}$ set of all tagged divisions of $[a,b]$, page 175

$TPD_{[a,b]}$ set of all tagged partial divisions of $[a,b]$, page 175

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