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Cohomology of quasi-coherent sheaves over projective schemes

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Daniel Alberto Aguilar Alvarez

**Cohomologia de feixes quasi-coherentes sobre esquemas
projetivos**

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RESUMO

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O objetivo deste trabalho é apresentar ao leitor o estudo de algumas ferramentas matemáticas utilizadas nos problemas atuais da geometria algébrica, pressupondo apenas alguns conhecimentos em álgebra e topologia. Expõe conceitos e resultados básicos na teoria de feixes e esquemas, que logo são usados para entender a correspondência que existe entre a cohomologia local e a cohomologia de feixes, no caso de feixes quasi-coherentes sobre esquemas projetivos. Finalmente enunciamos alguns problemas em aberto relacionados com o polinômio de Hilbert e a regularidade de Castelnuovo-Mumford de um feixe coerente.

Palavras-chave: Cohomologia local, Cohomologia de feixes, Feixe quasi-coerente, Esquema projetivo.

ABSTRACT

AGUILAR ALVAREZ, D. A. **Cohomology of quasi-coherent sheaves over projective schemes**. 2021. 70 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

The objective of this work is to present the reader with the study of some mathematical tools used in current problems of algebraic geometry, assuming only some knowledge in algebra and topology. We treat basic concepts and results in the theory of sheaves and schemes that we later use to understand the correspondence between local cohomology and sheaf cohomology of quasi-coherent sheaves over projective schemes. Then, with this background we are able to state some open problems that are related to the Hilbert polynomial and to the Castelnuovo-Mumford regularity of a coherent sheaf.

Keywords: Local cohomology, Sheaf cohomology, Quasi-coherent sheaf, Projective scheme.

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INTRODUCTION

Modern algebraic geometry is a broad area of study in mathematics closely related to other areas such as commutative algebra, topology, sheaf theory, complex analysis and number theory. The approach to the subject developed in the twentieth century by mathematicians like Alexander Grothendieck and Jean P. Serre, redefine basic geometric objects and generalize them to the language of sheaves and schemes, which has shown to be very useful in proving classical results like the Bezout's theorem and new results such as those concerning invariants like the Hilbert polynomial or the Castelnuovo-Mumford regularity. The purpose of this thesis is to understand some mathematical tools that are used in the current study of algebraic geometry, such as local cohomology and sheaf cohomology.

Chapter 2 is an introduction to the concept of sheaves. We define sheaves on an arbitrary topological space, but our examples are mainly oriented to the study of the Zariski topology on the set of prime ideals of a ring, which is our main object of study. We highlight the difference between presheaves and sheaves using the concept of stalk, the so called "sheafification" of a presheaf allows us to study the topological space "locally", that is, at the level of stalks, where many properties of sheaves can be more easily proven.

In chapter 3 we focus our attention to schemes. A scheme is an example of a locally ringed space, that is, a topological space together with a sheaf of rings such that the stalks are local rings. In addition, a scheme looks locally like the spectrum of some ring A , by spectrum we mean the set of prime ideals of A , which we denote by $\text{Spec } A$. The simplest example of scheme is what we call an affine scheme. Its underlying topological space corresponds to the spectrum of some ring endowed with the Zariski topology (ATIYAH; MACDONALD, 1969, I), and the sheaf of rings corresponds to the set of regular functions defined in every open set. This mimics the construction of the set of regular functions in classic algebraic geometry (HARTSHORNE, 1977, I.3). As a consequence of proposition 3.14 we have an equivalence between the category of affine schemes and the category of rings, similar to the correspondence between affine algebraic varieties and finitely generated k -algebras which is a domain and k is an algebraically closed

field.

Another important example of scheme is that of a projective scheme, which we construct with a graded ring R and underlying topological space $\text{Proj } R \subseteq \text{Spec } R$. Here, the construction of the sheaf structure mimics the definition of the set of regular functions for quasi-projective varieties (HARTSHORNE, 1977, I). Our first examples of schemes are those who come from open and closed sets of the original topological space. We introduce the concept of open and closed subscheme and give a characterization of the notion of noetherian scheme.

In chapter 4 we extend our examples of sheaves defining sheaves of modules over an arbitrary ringed space. In the case of an affine scheme $\text{Spec } A$, given an arbitrary A -module M we define the sheaf associated to M , denoted by \tilde{M} , this is an example of a quasi-coherent sheaf. Corollary 4.7 establishes an equivalence between the category of quasi-coherent sheaves on $\text{Spec } A$ and the category of A modules. Similarly to the affine case, given a graded ring R and a graded R -module M we define a quasi-coherent sheaf of modules \tilde{M} and we use it to define the twisted sheaf of any sheaf of modules. This sheaf will be crucial in the study of cohomology of quasi-coherent sheaves. In the case where R is a finitely generated algebra we will have a correspondence between quasi-coherent sheaves and graded modules (see proposition 4.18). Finally we show that any quasi-coherent sheaf of ideals on a scheme X define a uniquely determined closed subscheme of X , and that every closed subscheme defines a quasi-coherent sheaf of ideals on X .

In chapter 5 we define the concept of injective sheaf over an arbitrary ringed space X and construct an injective resolution for any sheaf of modules over X . We use this injective resolution to construct the sheaf cohomology groups for any sheaf of modules. In the case where $X = \text{Spec } A$ and A is a noetherian ring and the sheaf is quasi-coherent, this cohomology groups will vanish (theorem 5.12).

Given an arbitrary ring A and an A -module M , for any ideal $\mathfrak{a} \subseteq A$ we define the \mathfrak{a} -torsion functor $\Gamma_{\mathfrak{a}}(-)$ from the category of A -modules to itself, and we apply it to an injective resolution of M to construct the local cohomology groups of M .

Given a projective scheme $X = \text{Proj } R$, where R is a finitely generated positively graded ring, and a quasi-coherent sheaf associated to a graded module M , we use Čech cohomology to prove the Serre-Grothendieck correspondence between the sheaf cohomology groups of \tilde{M} and the local cohomology groups of M (see theorem 5.22). As a consequence of this correspondence we obtain the Serre finiteness theorem, which is a powerful result giving conditions to when the cohomology groups of a coherent sheaf vanish and when they are finitely generated.

The last part of the chapter is devoted to some open problems on the subject. For this we introduce the concepts of Hilbert polynomial of a quasi-coherent sheaf on an projective scheme $\text{Proj } R$ and the Castelnuovo-Mumford regularity of a sheaf. We ask what kind of sheaves have regularity bounded by the Hilbert coefficients and what closed subschemes satisfy the Eisenbud-

Goto conjecture. In our purpose of studying modern techniques used in current research, we chose to focus on understanding this kind of open problems, which seem to be a very active field.

We assume the reader is familiar with basic results of commutative algebra concerning rings, ideals and modules. Any result from commutative or homological algebra needed in the text will be stated with a reference to guide the reader. We also use some examples of classic algebraic geometry which foundations may be found in ([HARTSHORNE, 1977](#), I). We try to avoid the language of category theory but sometimes we include an explanation of what a result says in categorical terms for the reader who is familiar. Throughout the text any ring will be commutative with identity 1.

SHEAVES

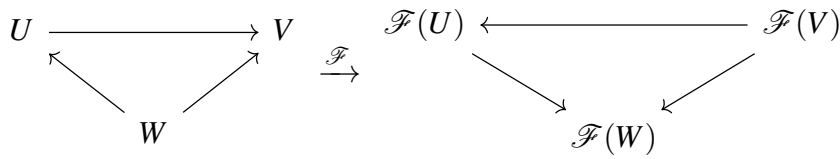
In this chapter we introduce the concept of presheaves and sheaves on an arbitrary topological space. We highlight the role of the stalk of a sheaf and prove that any presheaf can be extended to a uniquely determined sheaf, this allows us to see every sheaf as a sheaf of functions locally determined by its stalk, see proposition 2.18.

2.1 Presheaves and sheaves

As a motivational example let \mathbb{R}^n be the euclidean n -space with the usual topology. For every open set $U \subseteq \mathbb{R}^n$ consider the abelian group $\mathcal{F}(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$, note that if $U \subseteq V$ then there is a natural group morphism $\rho_{VU} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ sending an element f to its restriction $f|_U$. Now, if we have two such functions $f, g \in \mathcal{F}(U)$ and an open cover $\{U_i\}_{i \in I}$ of U , that is $U_i \subseteq \mathbb{R}^n$ are open and $U = \bigcup_{i \in I} U_i$, such that $f|_{U_i} \equiv g|_{U_i}$ for every i then we must have $f \equiv g$ to start with, that is we can identify a continuous function by looking at it in smaller open sets that form a covering. Similarly assume that for every U_i in this open cover there is a function $f_i \in \mathcal{F}(U_i)$ such that for any i, j we have $f_i|_{U_i \cap U_j} \equiv f_j|_{U_i \cap U_j}$. Then we can extend these functions to the whole open set U and obtain a continuous function $f \in \mathcal{F}(U)$ such that $f|_{U_i} \equiv f_i$ for all i . We generalize this example with the notion of presheaf and sheaf.

Definition 2.1. Let X be a topological space. A **pre-sheaf** \mathcal{F} of abelian groups on X consists of the following data: an abelian group $\mathcal{F}(U)$ for every open set $U \subseteq X$ and, for every pair of open sets $U \subseteq V$, a homomorphism of abelian groups $\rho_{VU} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ such that

- (i) If $W \subseteq U \subseteq V$ are three open subsets of X then $\rho_{VW} = \rho_{UW} \circ \rho_{VU}$.
- (ii) $\rho_{UU} : \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ is the identity map.
- (iii) $\mathcal{F}(\emptyset) = 0$.



In the language of categories the presheaf \mathcal{F} is a contravariant functor from the category of open sets of X to the category of abelian groups. The category of abelian groups can be replaced by any other category. Important cases for us are the category of rings and modules over a ring. We follow Hartshorne (1977) treating the case of abelian groups in this chapter since there are no complications in extending to any of this categories. If \mathcal{F} is a presheaf on X we refer to $\mathcal{F}(U)$ as the **sections** of \mathcal{F} over the open set U and is sometimes denoted by $\Gamma(U, \mathcal{F})$, when $X = U$ we call $\Gamma(X, \mathcal{F})$ the **global sections** of \mathcal{F} . We call ρ_{VU} the **restriction maps** of the sheaf \mathcal{F} , and we use the notation $s|_U = \rho_{VU}(s)$ for $s \in \mathcal{F}(V)$ and $U \subseteq V$, it is read "s restricted to U ".

Definition 2.2. A presheaf \mathcal{F} is called a **sheaf** if for every open set $U \subseteq X$ and any open covering $\{U_i\}_{i \in I}$ of U , the following conditions hold:

- (i) (*Identity axiom*) If $s \in \mathcal{F}(U)$ is such that $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.
- (ii) (*Gluability axiom*) If $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

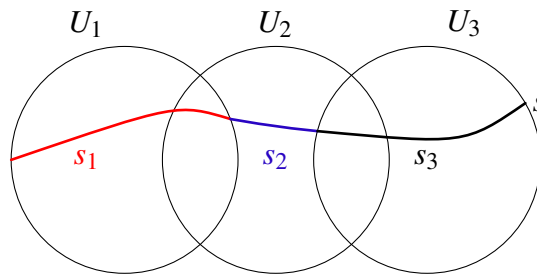


Figure 1 – Note that the identity axiom implies uniqueness in the gluability axiom.

Observation 2.3. Let \mathcal{F} a presheaf on X . For any open set U and any open cover $\{U_i\}_{i \in I}$ of U consider the sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\varepsilon} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{d^0} \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

where ε and d^0 are define using the restriction maps $\varepsilon(f) = (f|_{U_i})_{i \in I}$ and $d^0(f)_{(i,j) \in I \times I} = (f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j})$. Note that this sequence is exact, that is $\ker \varepsilon = 0$ and $\text{im } \varepsilon = \ker d^0$, for every open set $U \subset X$ if and only if \mathcal{F} is a sheaf.

Example 2.4. *Restriction of a sheaf.* For any open subset $U \subset X$ and any presheaf \mathcal{F} on X we define the **restriction of \mathcal{F} to U** by $\mathcal{F}|_U(W) = \mathcal{F}(W)$ for every open set $W \subset U$. It is immediate to check that $\mathcal{F}|_U$ is a presheaf on U and if \mathcal{F} is a sheaf then $\mathcal{F}|_U$ is also a sheaf.

Example 2.5. Let $X = \mathbb{C}^n$ be the complex n -space with the usual euclidean geometry. For every open set $U \subseteq X$ let $\mathcal{F}(U) = \{f : U \rightarrow \mathbb{C} \mid f \text{ is holomorphic and bounded}\}$. Clearly \mathcal{F} is a presheaf of abelian groups but it fails to be a sheaf since, by Liouville's theorem, the only holomorphic bounded functions defined in X are constant, thus we fail to glue different holomorphic functions along X . We will see a way to construct a sheaf from any given presheaf (see proposition 2.18).

Example 2.6. *Constant sheaf.* Let X be a topological space, consider any abelian group A as a topological space with the discrete topology. The **constant sheaf \mathcal{C}** defined by A in any open subset U of X , is $\mathcal{C}(U)$ the group of all continuous maps from U to A . Note that if U is connected then $\mathcal{C}(U) \cong A$, since for $f \in \mathcal{C}(U)$ if $a \in f(U)$ then $U = f^{-1}(a) \cup f^{-1}(A \setminus \{a\})$ is the union of two disjoint open sets, this implies that $f^{-1}(A \setminus \{a\}) = \emptyset$, thus f is constant and can be identify with a . If $X = \mathbb{C}$ with the usual topology, every connected component of a open set is open¹, then for every open set $U = \bigcup_{i \in I} U_i$, where the U_i are its connected components, one has $\mathcal{C}(U) \cong \prod_{i \in I} A$.

Let k be an algebraically closed field. The **affine n -space** over k , denoted by \mathbb{A}_k^n , is defined to be the set of all n -tuples of elements of k . Let $A = k[X_1, \dots, X_n]$ be the ring of polynomials in n variables over k , elements in A may be seen as functions from \mathbb{A}_k^n to k . Let $T \subseteq A$ be a set of polynomials, sets of the form $V(T) = \{P \in \mathbb{A}_k^n \mid f(P) = 0 \text{ for all } f \in T\} \subseteq \mathbb{A}_k^n$ are called algebraic sets, and their complements are the open sets in the so called Zariski topology on \mathbb{A}_k^n (see Hartshorne (1977, I)). An irreducible algebraic subset of \mathbb{A}_k^n with the induced topology is called an **affine variety** and any open subset of an affine variety is called a **quasi-affine variety**.

Example 2.7. *Affine variety.* Let X be a quasi-affine variety. Let $U \subseteq X$ be an open set, a function on $f : U \rightarrow k$ is called regular if for every point $P \in U$ exists a neighborhood $W \subseteq U$ of P and polynomials $g, h \in A$ such that h is nowhere zero on W and $f|_W \equiv g/h$. The set of regular functions on U denoted by $\mathcal{O}_X(U)$ is a ring. If $U \subseteq V$ then the natural restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ is a homomorphism of rings. This turns \mathcal{O}_X into a sheaf of rings on X .

Consider \mathbb{A}_k^{n+1} , the affine $n+1$ -space over an algebraically closed field k . The **projective n -space**, denoted by \mathbb{P}_k^n is the set $\mathbb{A}_k^{n+1} \setminus \{0\} / \sim$ where the equivalence relation \sim is defined on $\mathbb{A}_k^{n+1} \setminus \{0\}$ by $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if and only if $x_i = \lambda y_i$ for some $\lambda \in k \setminus \{0\}$. Let $R = k[X_0, \dots, X_n]$ be the ring of polynomials in $n+1$ variables over k . Polynomials over R do not define functions as in the affine case since a polynomial may attain different values at different

¹ This is a consequence of the fact that \mathbb{C} is locally connected, that is, there exists a basis for the topology consisting of open connected sets.

elements representing the same class. To correct this consider homogeneous polynomials² that are well defined in a sense: we say that a homogeneous polynomial $f \in R$ is zero at $P \in \mathbb{P}_k^n$ if $f(x_0, \dots, x_n) = 0$ for any and then for all representatives (x_0, \dots, x_n) of P . Let $T \subseteq R$ be a set of homogeneous polynomials, sets of the form $V(T) = \{P \in \mathbb{P}_k^n \mid f(P) = 0 \text{ for every } f \in T\}$ are called algebraic sets and their complements form the collection of open sets for the Zariski topology on \mathbb{P}_k^n (see [Hartshorne \(1977, I\)](#)). An irreducible algebraic set with the induced topology is called a **projective variety** and an open subset of a projective variety is called a **quasi-projective variety**.

Example 2.8. *Projective variety.* Let $X \subseteq \mathbb{P}_k^n$ be a quasi projective variety. Let $U \subseteq X$ be an open set. A function $f : U \rightarrow k$ is a regular function on U if for every $P \in U$ exists a neighborhood $W \subseteq U$ of P and homogeneous polynomials $g, h \in S$ of the same degree such that h is nowhere zero on W and $f|_W \equiv g/h$ ³. The set of regular functions on U , denoted by $\mathcal{O}_X(U)$, is a ring. If $U \subseteq V$ the natural restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ is a homomorphism of rings. This turns \mathcal{O}_X into a sheaf of rings on X .

Example 2.9. (*Sheaf of holomorphic functions*) Let \mathbb{C}^n the topological space with the usual metric and $W \subseteq \mathbb{C}^n$ an open subset. For any open subset $U \subseteq W$ define $\mathcal{O}_W^{\text{hol}}(U) = \{f : U \rightarrow \mathbb{C} \text{ holomorphic}\}$. For open subsets $V \subseteq U$, define $\rho_{UV} : \mathcal{O}_W^{\text{hol}}(U) \rightarrow \mathcal{O}_W^{\text{hol}}(V)$ as the restriction $f \rightarrow f|_V$ of a map. Then $\mathcal{O}_W^{\text{hol}}(U)$ is a sheaf of rings on W . Moreover, it is a sheaf of \mathbb{C} -algebras.

Example 2.10. *Sheaf structure of an integral domain.* Let A be an integral domain. Consider the set $X = \text{Spec } A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}$ with the *Zariski topology* as defined in [Atiyah and Macdonald \(1969, I, Exercise 15\)](#). The sets $D(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \notin \mathfrak{p}\}$ for $f \in A$ form a basis for this topology. We will define a sheaf of rings on X . Let K be the fraction field of A and for every $U \subseteq X$ define

$$\mathcal{O}_A(U) := \bigcap_{\mathfrak{p} \in U} A_{\mathfrak{p}} \subseteq K$$

note that $\mathcal{O}_A(U)$ is a ring, if $U \subseteq V$ note that $\mathcal{O}_A(V) \subseteq \mathcal{O}_A(U)$, so take the inclusion map $\mathcal{O}_A(V) \hookrightarrow \mathcal{O}_A(U)$ as the restriction map. Since the restriction maps are all injective, it is easy to check the sheaf axioms, thus \mathcal{O}_A is a sheaf on X .

Definition 2.11. A **morphism** $\theta : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on a topological space X is defined to be a collection of homomorphisms $\{\theta_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U) \mid U \subseteq X \text{ open}\}$ such that $\theta_V(s)|_U = \theta_U(s|_U)$ for all $s \in \mathcal{F}(V)$, $U \subset V$, i.e. the following diagram

² $f \in R$ is homogeneous of degree d if $f(tX_0, \dots, tX_n) = t^d f(X_0, \dots, X_n)$ for every $t \in k \setminus \{0\}$, see [Hartshorne \(1977, I\)](#).

³ Note that g and h are not well defined functions on W but since they are of the same degree, their quotient is a well defined function.

$$\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\theta_V} & \mathcal{G}(V) \\
\downarrow \rho_{VU} & & \downarrow \rho'_{VU} \\
\mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{G}(U)
\end{array}$$

commutes, where ρ and ρ' are the restriction maps of \mathcal{F} and \mathcal{G} respectively. If there is a morphism of sheaves $\eta : \mathcal{G} \rightarrow \mathcal{F}$ such that $\eta_U \circ \theta_U = id_{\mathcal{F}(U)}$ and $\theta_U \circ \eta_U = id_{\mathcal{G}(U)}$ for all open set $U \subseteq X$ we say θ is an **isomorphism**. If \mathcal{F} and \mathcal{G} are just presheaves on X , we use the same definition for morphism of presheaves.

Remark 2.12. It is straightforward to check that $\theta : \mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism of sheaves if and only if $\theta_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an isomorphism for every open set $U \subseteq X$. The inverse of θ is the collection of homomorphisms $\{\theta_U^{-1}\}_{U \subseteq X}$, which are compatible with the restriction maps.

Notation 2.13. Let \mathcal{F} be a sheaf on a topological space X , we use the notation $\Gamma(X, \mathcal{F})$ to refer to the group $\mathcal{F}(X)$, it is called the group of **global sections** of \mathcal{F} . If $\theta : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on X we denote the morphism between global sections by $\Gamma(X, \theta) : \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G})$.

2.2 Germs and stalk

Definition 2.14. Let \mathcal{F} be a presheaf of rings on a topological space X . The **stalk** of \mathcal{F} at $P \in X$ is

$$\mathcal{F}_P := \varinjlim_{P \in U} \mathcal{F}(U)$$

the direct limit of the groups $\mathcal{F}(U)$ such that $P \in U$.

In other words, let X be a topological space and \mathcal{F} a sheaf of abelian groups on X . For $P \in X$ consider the set $\Omega_P := \{(U, s) \mid U \subseteq X \text{ is open and } s \in \mathcal{F}(U)\}$ with the equivalence relation \sim given by $(U, s) \sim (V, t)$ if and only if there exists $W \subseteq U \cap V$ such that $s|_W = t|_W$. A **germ** is an equivalence class $[U, s]$ and is also denoted by s_P . We call the set of germs Ω_P / \sim the **stalk** of \mathcal{F} at P and denote it by \mathcal{F}_P . Note that \mathcal{F}_P is an abelian group with the operation $s_P + t_P = [U, s] + [V, t] = [U \cap V, s|_{U \cap V} + t|_{U \cap V}] = (s + t)_P$. For any open neighborhood $U \subseteq X$ of P and any $s \in \mathcal{F}(U)$ consider the natural projection $\mathcal{F}(U) \rightarrow \mathcal{F}_P$ given by $s \mapsto s_P$. This is a group homomorphism and is such that the diagram

$$\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\rho_{VU}} & \mathcal{F}(U) \\
& \searrow & \downarrow \\
& & \mathcal{F}_P
\end{array}$$

commutes for $U \subseteq V$.

The notion of stalks has a familiar geometric content, it is an abstraction of the notion of rings of germs. For example,

Example 2.15. Assume X is an affine variety see example 2.7, the ideal associated to X is $I(X) = \{f \in A \mid f(P) = 0 \text{ for all } P \in X\}$ and its *coordinate ring* is the quotient $A(X) := A/I(X)$. By Hartshorne (1977, I, Theorem 3.2) we have $\mathcal{O}_X(X) \cong A(X)$ and $\mathcal{O}_{X,P} \cong A(X)_{m_P}$, for all $P \in X$, where $A(X)_{m_P}$ is the localization at the maximal ideal m_P defined by P , that is the ideal generated by elements $f \in A(X)$ such that $f(P) = 0$. Similarly, let X be a projective variety see example 2.8. We define its associated homogeneous ideal by $I(X) := \{f \in S \mid f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in X\}$ and its associated *coordinate ring* by $S(X) := S/I(X)$. By Hartshorne (1977, I, Theorem 3.4) we have $\mathcal{O}_X(X) \cong k$ and $\mathcal{O}_{X,P} \cong S(X)_{(m_P)}$ for all $P \in X$ where $S(X)_{(m_P)}$ is the subring of $S(X)_{m_P}$ form by elements g/h such that g and h are homogeneous polynomials of the same degree.

Example 2.16. Let X be an open subset of \mathbb{C}^n and $\mathcal{O}_X^{\text{hol}}$ as in example 2.9, the stalk of \mathcal{O}_X at $w \in X$ is the ring of germs of holomorphic functions at w , that is, the ring of convergent power series in n variables. Indeed, two holomorphic functions f, g defined in open neighborhoods U and V of w that agree in some domain⁴ $D \subseteq U \cap V$ have the same Taylor expansion around w , so the stalk at w is given by

$$\mathcal{O}_{X,w}^{\text{hol}} = \left\{ \sum_{v \in \mathbb{N}^n} a_v (z-w)^v \mid \text{has a positive radius of convergence} \right\}$$

where $a_v \in \mathbb{C}$, and $(z-w)^v = (z_1 - w_1)^{v_1} \cdots (z_n - w_n)^{v_n}$ if the open set U is connected, then by the identity theorem⁵ the morphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,w}$ is injective.

Note that a morphism of (pre)sheaves $\theta : \mathcal{F} \rightarrow \mathcal{G}$ induces a morphism on the stalks $\theta_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$ given by $\theta_P([(U, t)]) = [(U, \theta_U(t))]$, one can check this is a homomorphism of groups and that for $P \in U$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow{\theta_P} & \mathcal{G}_P \end{array}$$

where the vertical arrows are the natural projection. The following proposition is an important property of this induced map, see Hartshorne (1977, II, Proposition 1.1).

Proposition 2.17. *Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on a topological sapce X . Then θ is an isomorphism if and only if the induce map on the stalk $\theta_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$ is an isomorphism for every $P \in X$.*

⁴ By domain we refer to an open and connected set.

⁵ The identity theorem states that if two holomorphic functions f, g on a domain D coincide in a set $S \subseteq D$, where S has an accumulation point, then $f = g$ on D .

Proof. First suppose θ is an isomorphism, we show θ_P is bijective for every $P \in X$. For surjectivity, let $t_P \in \mathcal{G}_P$, take U an open neighborhood of P . Since θ_U is surjective, there is an element $s \in \mathcal{F}(U)$ such that $\theta_U(s) = t$, now, $s_P \in \mathcal{F}_P$ satisfies $\theta_P(s_P) = t_P$. For injectivity, suppose $\theta_P(s_P) = 0$, by shrinking if necessary, we can find a neighborhood U of P such that $\theta_U(s) = 0$ in $\mathcal{G}(U)$, since θ_U is injective, $s = 0$ in $\mathcal{F}(U)$ and then $s_P = 0$ as required.

For the converse assume θ_P is an isomorphism for every P . We will show $\theta_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an isomorphism, that is, injective and surjective, for every open set U . By doing this we'll be able to define inverse morphisms $\mathcal{G}(U) \rightarrow \mathcal{F}(U)$ for every open set U , and therefore an inverse morphism $\mathcal{G} \rightarrow \mathcal{F}$ for θ . First we show θ_U is injective. Suppose $\theta_U(s) = 0$. The commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow{\theta_P} & \mathcal{G}_P \end{array}$$

together with the injectivity of θ_P , implies that $[(U, s)] = 0$ in \mathcal{F}_P . Then there exists an open neighborhood $W_P \subset U$ of P such that $s|_{W_P} = 0$ in $\mathcal{F}(W_P)$. Since the sets W_P cover U , by the first property of sheaves, $s = 0$ in U .

Next we prove θ_U is surjective. Let $t \in \mathcal{G}(U)$. Since θ_P is surjective for all P , $[U, t] = [W_P, \theta_{W_P}(s(P))]$ in \mathcal{F}_P for some $s(P) \in \mathcal{F}(W_P)$, by shrinking W_P if necessarily we have $t|_{W_P} = s(P)$. For two points $P, Q \in U$ note that

$$\theta_{W_P \cap W_Q}(s(P)|_{W_P \cap W_Q}) = t|_{W_P \cap W_Q} = \theta_{W_P \cap W_Q}(s(Q)|_{W_P \cap W_Q})$$

By the previous part, $\theta_{W_P \cap W_Q}$ is injective, then $s(P)|_{W_P \cap W_Q} = s(Q)|_{W_P \cap W_Q}$, again the sets W_P form an open cover of U so there exists $s \in \mathcal{F}(U)$ such that $s|_{W_P} = s(P)$ for all $p \in U$, furthermore

$$\theta_U(s)|_{W_P} = \theta_{W_P}(s|_{W_P}) = \theta_{W_P}(s(P)) = t|_{W_P}$$

and once more by the properties of sheaves $\theta_U(s) = t$ and we have proven surjectivity. \square

2.3 Sheafification and properties of sheaves

Let \mathcal{F} be a presheaf on a topological space X . We want to associate to \mathcal{F} a uniquely determined sheaf. Let $U \subseteq X$ be an open set, we call a function $\varphi : U \rightarrow \bigsqcup_{P \in U} \mathcal{F}_P$ a **regular function** if for all $P \in U$ we have $\varphi(P) \in \mathcal{F}_P$ and there exists an open neighborhood $W \subseteq U$ of P and a section $t \in \mathcal{F}(W)$ such that $\varphi(Q) = t_Q$ for all $Q \in W$. Consider the following set of functions

$$\begin{aligned} \mathcal{F}^+(U) &:= \left\{ \varphi : U \rightarrow \bigsqcup_{P \in U} \mathcal{F}_P \mid \varphi \text{ is a regular function} \right\} \\ \mathcal{F}^+(\emptyset) &:= \{0\} \end{aligned}$$

Note that by definition \mathcal{F}^+ is a sheaf of abelian groups together with the natural restriction maps and it is uniquely determined by the following universal property (see Hartshorne (1977, II, Proposition-Definition 1.2.)).

Proposition 2.18. (Sheafification) *Let \mathcal{F} be a presheaf. There exists a sheaf \mathcal{F}^+ and a morphism $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$ with the property that for any sheaf \mathcal{G} , and any morphism $\eta : \mathcal{F} \rightarrow \mathcal{G}$, there is a unique morphism $\eta^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $\eta = \eta^+ \circ \theta$. The pair (\mathcal{F}^+, θ) is unique up to isomorphism. The sheaf \mathcal{F}^+ is called the **sheaf associated** to the presheaf \mathcal{F} or the **sheafification** of \mathcal{F} .*

Proof. Let \mathcal{F}^+ be the sheaf defined above. Define the map θ on an arbitrary open set U by $\theta_U : \mathcal{F}(U) \rightarrow \mathcal{F}^+(U)$, by $s \mapsto \varphi_s$ where $\varphi_s(P) = s_P$ for every $P \in U$. It is straightforward to verify that θ_U is a well defined homomorphism and that θ is a morphism of sheaves. Now suppose $\eta : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism and \mathcal{G} is a sheaf, we will construct a morphism $\eta^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ that satisfies the property described. Let U be any open set and $\varphi \in \mathcal{F}^+(U)$. For every $P \in U$ there is an open set W_P containing P and an element $s(P) \in \mathcal{F}(W_P)$, such that $\varphi|_{W_P} = s(P)$. Let $t(P) \in \mathcal{G}(W_P)$ such that $\eta_{W_P}(s(P)) = t(P)$. Note that $s(P)|_{W_P \cap W_Q} = s(Q)|_{W_P \cap W_Q}$ since $s(P)_{Q'} = s(Q)_{Q'}$ for all $Q' \in W_P \cap W_Q$, therefore

$$\eta_{W_P}(s(P))|_{W_P \cap W_Q} = \eta_{W_P}(s(P)|_{W_P \cap W_Q}) = \eta_{W_P}(s(Q)|_{W_P \cap W_Q}) = \eta_{W_Q}(s(Q))|_{W_P \cap W_Q}$$

This implies $t(P)|_{W_P \cap W_Q} = t(Q)|_{W_P \cap W_Q}$. Since \mathcal{G} is a sheaf there exists $t \in \mathcal{G}(U)$ such that $t|_{W_P} = t(P)$. Define $\eta^+(\varphi) := t$. Since \mathcal{G} is a sheaf, it follows that η^+ is a well defined morphism of sheaves that satisfies $\eta = \eta^+ \circ \theta$. Uniqueness of \mathcal{F}^+ is a consequence of the universal property. \square

Example 2.19. Let A be an abelian group and X a topological space, the constant sheaf on X defined by A , example 2.6, is the sheafification of

$$U \longmapsto \{f : U \rightarrow A \mid f \text{ is a constant function}\}$$

Remark 2.20. Note that $\mathcal{F}_P \cong \mathcal{F}_P^+$, this implies that if \mathcal{F} is already a sheaf, then it is isomorphic to \mathcal{F}^+ .

Definition 2.21. Let $f : X \rightarrow Y$ be a continuous map between topological spaces. Let \mathcal{F} be a sheaf on X . The **direct image** $f_*\mathcal{F}$ of \mathcal{F} is the sheaf on Y given by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}V)$$

for every open set $V \in Y$ with restriction maps those from \mathcal{F} . It is straightforward to check $f_*\mathcal{F}$ is a sheaf on Y .

In the following example we see a morphism between sheaves of affine varieties induced by an homeomorphism of the underlying topological spaces, by computations on the stalks we show that it is not an isomorphism of sheaves.

Example 2.22. Let k be an algebraically closed field. Let $X = \mathbb{A}_k^1$ and $Y = V(y^2 - x^3) \subseteq \mathbb{A}_k^2$ be affine algebraic varieties as in example 2.7. Consider the map

$$\begin{aligned} f : X &\longrightarrow Y \\ t &\longmapsto (t^2, t^3) \end{aligned}$$

see figure 2 below for a picture in the real case, this is a well defined homeomorphism in the Zariski topology. Consider the direct image $f_*\mathcal{O}_X$ as a sheaf on Y and for every open set $U \subseteq Y$ define the map

$$\begin{aligned} f_U^\# : \mathcal{O}_Y(U) &\longrightarrow f_*\mathcal{O}_X(U) = \mathcal{O}_Y(f^{-1}U) \\ g &\longmapsto g \circ f \end{aligned}$$

this is a well defined homomorphism of rings since for any $P \in U$, if $W \subseteq U$ is an open neighborhood of P where $g = a/b$, $a, b \in k[x, y]$ and b nowhere zero on W , we have that $g \circ f = a(t^2, t^3)/b(t^2, t^3)$ on $f^{-1}W$, also $b(t^2, t^3)$ is nowhere zero on $f^{-1}W$ since $(t^2, t^3) \in Y$ implies $b(t^2, t^3) \neq 0$, this implies that $g \circ f$ is a regular function on $f^{-1}U$. The induced morphism on the stalks $f_0^\# : \mathcal{O}_{Y, f(0)} \longrightarrow \mathcal{O}_{X, 0}$ in the case $t = 0$ is not an isomorphism since $\mathcal{O}_{Y, f(0)} \cong \mathcal{O}_{Y, (0,0)} \cong (k[x, y]/(y^2 - x^3))_{\mathfrak{m}_{(0,0)}}$ and $\mathcal{O}_{X, 0} \cong k[T]_{\mathfrak{m}_0}$, it can be proven by explicit calculation that $\dim_k(\mathfrak{m}_{(0,0)}/\mathfrak{m}_{(0,0)}^2) = 2$ and $\dim_k(\mathfrak{m}_0/\mathfrak{m}_0^2) = 1$ which implies that the two local rings are not isomorphic (with some abuse of notation we are assuming that $\mathfrak{m}_{(0,0)}$ and \mathfrak{m}_0 are the respective maximal ideals of $\mathcal{O}_{Y, (0,0)}$ and $\mathcal{O}_{X, 0}$).

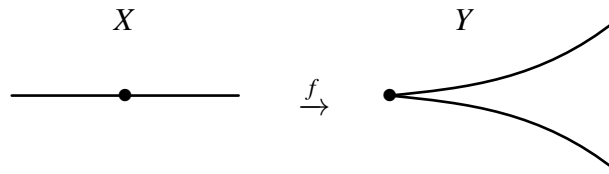


Figure 2

Definition 2.23. A **subsheaf** \mathcal{F}' of a sheaf \mathcal{F} is a sheaf such that $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ for all open set U . The restriction maps are induced by those of \mathcal{F} . The **quotient sheaf** of \mathcal{F} by the subsheaf \mathcal{F}' is the sheafification of the presheaf

$$U \mapsto \frac{\mathcal{F}(U)}{\mathcal{F}'(U)}.$$

For the next example, remember the following definitions. A space $X \subseteq \mathbb{C}^n$ is called **locally analytic**, if for any point $p \in X$, there exists an open subset U of p in \mathbb{C}^n , and finitely many holomorphic functions f_1, \dots, f_t defined on U such that $X \cap U = \{x \in U : f_1(x) = \dots = f_t(x) = 0\}$. A set $X \subseteq U$ is called an **analytic set** of U , if X is locally analytic, and closed in U .

The next example, shows a large class of sheaves defined on analytical sets, this class has an immense theoretical content. For more details see examples in (ISHII, 2018).

Example 2.24. Let $W \subseteq \mathbb{C}^n$ be an open subset and define $X = \{x \in W : f_1(x) = \cdots = f_t(x) = 0\}$ be an analytic set. Thus, let U be an open subset of W , we define

$$\mathcal{I}(U) := \{f \mid f \text{ is an holomorphic function on } U \text{ such that } f|_{X \cap U} = 0\}.$$

Define ρ_{UV} as the restriction of functions as before. Then \mathcal{I} is a sheaf of abelian groups and it is a subsheaf of $\mathcal{O}_W^{\text{hol}}$. Here, as $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_W^{\text{hol}}(U)$, the presheaf defined by $\mathcal{O}_W^{\text{hol}}(U)/\mathcal{I}(U)$ is a presheaf of rings. Therefore the sheafification $\mathcal{O}_W^{\text{hol}}/\mathcal{I}$ is a sheaf of rings. This is a sheaf on W , but it is also considered as a sheaf on X . Indeed a subset V of X is represented as $V = U \cap X$ by using an open subset U of W . Define $\mathcal{O}_W^{\text{hol}}(V) := \mathcal{O}_W^{\text{hol}}(V)/\mathcal{I}(U)$. Then the right-hand side is independent of a choice of an open subset U , therefore $\mathcal{O}_X^{\text{hol}}$ is a sheaf of rings on X , we denote by \mathcal{O}_X .

Remark 2.25. The morphism $i : \mathcal{F}' \rightarrow \mathcal{F}$ defined by the inclusions $i_U : \mathcal{F}'(U) \rightarrow \mathcal{F}(U)$ induces a homomorphism $i_P : \mathcal{F}'_P \rightarrow \mathcal{F}_P$ which is injective for every $P \in X$, thus, we can identify \mathcal{F}'_P as a subgroup of \mathcal{F}_P . It follows from this identification that for any point $P \in X$ there is an isomorphism of stalks $(\mathcal{F}/\mathcal{F}')_P \cong \mathcal{F}_P/\mathcal{F}'_P$.

Proposition 2.26. Let \mathcal{F}' be a subsheaf of a sheaf \mathcal{F} on a topological space X .

- (a) Let $U \subset X$ be open and $s \in \mathcal{F}(U)$. Then $s \in \mathcal{F}'(U)$ if and only if $s_P \in \mathcal{F}'_P$ for all $P \in U$.
- (b) $\mathcal{F}' = \mathcal{F}$ if and only if $\mathcal{F}'_P = \mathcal{F}_P$ for all $P \in X$.
- (c) $\mathcal{F} = 0$ if and only if $\mathcal{F}_P = 0$ for all $P \in X$.

Proof. (a) If $s \in \mathcal{F}'(U)$ it is straightforward that $s_P \in \mathcal{F}'_P$ for all $P \in U$. Conversely, let $s_P \in \mathcal{F}'_P$ for all $P \in U$. Then, there exists a neighborhood $W_P \subseteq U$ of P and $t(P) \in \mathcal{F}'(W_P)$ such that $s|_{W_P} = t(P)$. Note that the sets W_P cover U and $t(P)|_{W_P \cap W_Q} = s|_{W_P \cap W_Q} = t(Q)|_{W_P \cap W_Q}$. Since \mathcal{F}' is a sheaf, there exists $t \in \mathcal{F}'(U)$ such that $t|_{W_P} = t(P)|_{W_P} = s|_{W_P}$ for every $P \in U$, thus $s = t \in \mathcal{F}'(U)$.

(b) By part (a), $\mathcal{F}'_P = \mathcal{F}_P$ for all $P \in X$ if and only if $\mathcal{F}'(U) = \mathcal{F}(U)$ for all $U \subseteq X$.

(c) Take $\mathcal{F}' = 0$ in part (b). □

Definition 2.27. Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. The **kernel sheaf** of θ is the subsheaf

$$\ker \theta(U) := \ker(\mathcal{F}(U) \xrightarrow{\theta_U} \mathcal{G}(U))$$

of \mathcal{F} . Note that this is in fact a sheaf. The morphism θ is **injective** if $\ker \theta = 0$, thus θ is injective if and only if θ_U is injective for all $U \subseteq X$.

Definition 2.28. Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. The **image sheaf** of θ is the sheafification of the presheaf

$$U \mapsto \text{im}(\mathcal{F}(U) \xrightarrow{\theta_U} \mathcal{G}(U))$$

denoted by $\text{im } \theta$. We say θ is **surjective** if $\text{im } \theta \cong \mathcal{G}$.

Remark 2.29. Note that by the universal property of the sheafification, there is an injective morphism $\text{im } \theta \rightarrow \mathcal{G}$, thus we can identify $\text{im } \theta$ with a subsheaf of \mathcal{G} . If θ is surjective it is not necessarily true that the maps θ_U are surjective, see example 5.8.

Proposition 2.30. Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves.

(a) $(\ker \theta)_P \cong \ker(\theta_P)$ and $(\text{im } \theta)_P \cong \text{im}(\theta_P)$.

(b) θ is injective (resp. surjective) if and only if θ_P is injective (resp. surjective) for all $P \in X$.

(c) The sequence of sheafs and morphisms

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\theta^{i-1}} \mathcal{F}^i \xrightarrow{\theta^i} \mathcal{F}^{i+1} \rightarrow \dots$$

is exact (i.e. $\ker \theta^i = \text{im } \theta^{i-1}$) if and only if the sequence

$$\dots \rightarrow \mathcal{F}_P^{i-1} \xrightarrow{\theta_P^{i-1}} \mathcal{F}_P^i \xrightarrow{\theta_P^i} \mathcal{F}_P^{i+1} \rightarrow \dots$$

is exact for all $P \in X$.

Proof. (a) There is a well defined natural inclusion $i_P : (\ker \theta)_P \hookrightarrow \mathcal{F}_P$ sending an element in $(\ker \theta)_P$ to its class in \mathcal{F}_P . So what we want to show is that $i_P((\ker \theta)_P) = \ker \theta_P \subseteq \mathcal{F}_P$.

This is a consequence of the commutative diagram

$$\begin{array}{ccccc} \ker \theta(U) & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow & & \downarrow \\ (\ker \theta)_P & \xrightarrow{i_P} & \mathcal{F}_P & \xrightarrow{\theta_P} & \mathcal{G}_P \end{array}$$

For the other part, we also have a well defined inclusion $i_P : (\text{im } \theta)_P \hookrightarrow \mathcal{G}_P$, so what we want to show is that $i_P((\text{im } \theta)_P) = \text{im } \theta_P \subseteq \mathcal{G}_P$. This equality is a consequence of the commutative diagrams

$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{G}(U) & \longleftarrow & \text{im } \theta(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow{\theta_P} & \mathcal{G}_P & \longleftarrow_{i_P} & (\text{im } \theta)_P \end{array}$$

(b) By proposition 2.26, $\ker \theta = 0$ if and only if $(\ker \theta)_P = \ker \theta_P = 0$ for every $P \in X$ and $\text{im } \theta = \mathcal{G}$ if and only if $(\text{im } \theta)_P = \text{im } \theta_P = \mathcal{G}_P$ for every $P \in X$.

(c) Also by proposition 2.26, $\ker \theta_i = \text{im } \theta_{i-1}$ if and only if $\ker (\theta_i)_P = \text{im } (\theta_{i-1})_P$ for all $P \in X$. \square

Example 2.31. *Sheaf of ideals.* Let X be a projective or affine variety over an algebraically closed field k . Let $Y \subseteq X$ be a closed subset. For every open set $U \subseteq X$ define

$$\mathcal{I}_Y(U) := \{\varphi \in \mathcal{O}_X(U) \text{ such that } \varphi|_{U \cap Y} \equiv 0\}$$

this is in fact a sheaf on X and $\mathcal{I}_Y(U)$ is an ideal of the ring $\mathcal{O}_X(U)$, we call \mathcal{I}_Y the **sheaf of ideals** of Y on X , it is a subsheaf of \mathcal{O}_X . Assume Y is a subvariety, that is an irreducible closed set and consider the sheaf of regular functions \mathcal{O}_Y on Y . The inclusion map $i : Y \rightarrow X$ induces a morphism of sheaves $i^\sharp : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Y$, at the level of stalks it is

$$\begin{aligned} i_P^\sharp : \mathcal{O}_{X,P} &\longrightarrow \mathcal{O}_{Y,P} \\ \varphi_P &\longmapsto (\varphi|_{Y \cap U})_P \end{aligned}$$

for any open neighborhood U of P . Note that $\varphi_P \in \ker i_P^\sharp$ if and only if $\varphi|_{Y \cap U} \equiv 0$ for some open neighborhood U of P , so $\ker i_P^\sharp \cong \mathcal{I}_{Y,P}$, this implies $\mathcal{O}_X / \ker i^\sharp \cong \mathcal{O}_X / \mathcal{I}_Y$. Therefore, for every open set $U \subseteq X$ we have a well defined morphism

$$\frac{\mathcal{O}_X(U)}{\mathcal{I}_Y(U)} \longrightarrow i_*\mathcal{O}_Y(U)$$

this gives a morphism of sheaves $\mathcal{O}_X / \mathcal{I}_Y \longrightarrow i_*\mathcal{O}_Y$. At the level of stalks we have an isomorphism $\mathcal{O}_{X,P} / \mathcal{I}_{Y,P} \longrightarrow \mathcal{O}_{Y,P}$, so we arrive to the isomorphism $\mathcal{O}_X / \mathcal{I}_Y \cong i_*\mathcal{O}_Y$.

SCHEMES

This chapter is devoted to the concept of scheme. We present first the larger category of locally ringed spaces. Then we construct affine schemes and projective schemes and prove they are locally ringed spaces. Then we treat our first examples of schemes, open and closed subschemes which will play an important role in the discussion of open problems in chapter 5. Finally the notion of noetherian scheme and dimension are given together with a characterization of noetherian schemes that says that being noetherian is a local property.

3.1 Locally ringed spaces

Let X be a topological space and consider the set of real numbers \mathbb{R} with the usual euclidean topology. As a motivational example define the set of functions $\mathcal{C}_X(U) := \{f : U \rightarrow \mathbb{R}, \text{ continuous}\}$ for any open set $U \subseteq X$. With the natural restriction maps \mathcal{C}_X is a sheaf of \mathbb{R} -algebras on X . Let $x_0 \in X$ and consider the stalk \mathcal{C}_{X,x_0} at x_0 , if $f_{x_0} \in \mathcal{C}_{X,x_0}$ is not zero then there exists a neighborhood of x_0 where f do not vanish, that is f_{x_0} is invertible, so the ideal $\mathfrak{m}_{x_0} = \{g_{x_0} \in \mathcal{C}_{X,x_0} \mid g(x_0) = 0\}$ is the unique maximal ideal of \mathcal{C}_{X,x_0} , moreover the ring homomorphism $\mathcal{C}_{X,x_0} \rightarrow \mathbb{R}$ sending f_{x_0} to $f(x_0)$ is a surjective ring homomorphism and identifies $\mathcal{C}_{X,x_0}/\mathfrak{m}_{x_0}$ with \mathbb{R} .

Let $\varphi : X \rightarrow Y$ be a continuous function between topological spaces, for any open set $V \subseteq Y$ consider the homomorphism of \mathbb{R} -algebras

$$\begin{aligned} \varphi_V^\sharp : \mathcal{C}_Y(V) &\longrightarrow \varphi_* \mathcal{C}_X(V) = \mathcal{C}_X(\varphi^{-1}V) \\ f &\longmapsto f \circ \varphi \end{aligned}$$

this defines a morphism of sheaves $\varphi^\sharp : \mathcal{C}_Y \rightarrow \varphi_* \mathcal{C}_X$ that induces a homomorphism on the stalks $\varphi_{x_0}^\sharp : \mathcal{C}_{Y,f(x_0)} \rightarrow \mathcal{C}_{X,x_0}$, note that in this case we have $\varphi_{x_0}^\sharp(\mathfrak{m}_{f(x_0)}) \subseteq \mathfrak{m}_{x_0}$ this is what we call a local homomorphism.

Definition 3.1. A homomorphism $\varphi : A \rightarrow B$ of local rings is called **local** if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ where \mathfrak{m}_A and \mathfrak{m}_B are the maximal ideals of A and B respectively.

Definition 3.2. A **ringed space** is a pair (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf of rings on X . A **morphism** of ringed spaces is a pair $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ where $f : X \rightarrow Y$ is a continuous map and $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves on Y .

Definition 3.3. A ringed space (X, \mathcal{O}_X) is a **locally ringed space** if for every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. A **morphism** of locally ringed spaces is a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ such that the induced map of local rings

$$\begin{aligned} f_x^\# : \mathcal{O}_{Y,f(x)} &\longrightarrow (f_*\mathcal{O}_X)_{f(x)} \cong \mathcal{O}_{X,x} \\ [(V, h)] &\longmapsto [(f^{-1}V, f_V^\#(h))] \end{aligned}$$

is a local homomorphism for every $x \in X$. It is called an **isomorphism** if f is a homeomorphism between topological spaces and $f^\#$ is an isomorphism of sheaves.

Example 3.4. Let X be an algebraic variety and \mathcal{O}_X its sheaf of regular functions. If X is an affine variety with coordinate ring $A(X)$ then (X, \mathcal{O}_X) is a locally ringed space since $\mathcal{O}_{X,P} \cong A(X)_{\mathfrak{m}_P}$. Similarly, if X is a projective variety with homogeneous coordinate ring $S(X)$, then (X, \mathcal{O}_X) is a locally ringed space since $\mathcal{O}_{X,P} \cong S(X)_{(\mathfrak{m}_P)}$, see example 2.15.

Example 3.5. Let U be an open set in \mathbb{C}^n and let f_1, \dots, f_t be holomorphic functions defined on U . Let X be an analytic set defined by the holomorphic functions f_1, \dots, f_t on U . Then (X, \mathcal{O}_X) is a ringed space together with the euclidean topology, and the sheaf is defined in the example 2.24. Thus, with this, is easily checked that (X, \mathcal{O}_X) is a locally ringed space since $\mathcal{O}_{X,P} \cong \left[\frac{\mathcal{O}_U^{\text{hol}}}{(f_1, \dots, f_t)|_U} \right]_{\mathfrak{m}_P}$.

In particular, if X is open, the space $(X, \mathcal{O}_X^{\text{hol}})$ of holomorphic functions is a locally ringed space. The stalk $\mathcal{O}_{X,w}^{\text{hol}}$ of convergent power series at w is a local ring with maximal ideal the set of holomorphic functions that vanish at w , see example 2.16.

3.2 Affine schemes

Let A be a ring. In this section we associate to A a topological space $X = \text{Spec } A$ whose underlying set will be the set of all prime ideals of A . This association is such that for a given homomorphism of rings $\varphi : A \rightarrow B$ the map $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$ defined by $\varphi^*(\mathfrak{q}) := \varphi^{-1}(\mathfrak{q})$, where $\mathfrak{q} \in \text{Spec } B$, is a continuous map. Then we construct the *structure sheaf* of X denoted by \mathcal{O}_X , which will turn (X, \mathcal{O}_X) into a locally ringed space. Any (X, \mathcal{O}_X) isomorphic to $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ as locally ringed spaces for some ring A will be called an *affine scheme*. We will see that any ring homomorphism $\varphi : A \rightarrow B$ induces a morphism of locally ringed spaces $(\varphi^*, \varphi^\#)$, and any morphism of affine schemes $(f, f^\#)$ is induced by a unique ring

homomorphism that coincides with f . In the category language we have an anti-equivalence between the category of rings and the category of affine schemes (that is an arrow reversing equivalence).

Definition 3.6. Let $\text{Spec } A := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}$.

Now we endow $X = \text{Spec } A$ with a topological space structure. For each subset $S \subseteq A$ let $V(S) := \{\mathfrak{p} \in X : S \subseteq \mathfrak{p}\}$. Note that if \mathfrak{a} is the ideal generated by S then $V(S) = V(\mathfrak{a})$. For an element $f \in A$ we write $V(f)$ instead of $V(\{f\})$.

Lemma 3.7. (a) $V(1) = \emptyset$ and $V(0) = X$.

(b) If $\{\mathfrak{a}_i\}_{i \in I}$ is a family of ideals of A , then $V(\bigcup_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$.

(c) For ideals \mathfrak{a} and \mathfrak{b} of A we have $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(d) If $\mathfrak{a} \subseteq A$ is any ideal then $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

Proof. Assertions (a) and (b) are immediate, for (c) just note that a prime ideal contains \mathfrak{a} or \mathfrak{b} if and only if it contains $\mathfrak{a} \cap \mathfrak{b}$ or equivalently if and only if it contains $\mathfrak{a}\mathfrak{b}$. For (d) note that $\mathfrak{a} \subseteq \mathfrak{p} \Leftrightarrow \sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$. \square

These lemma shows that the sets $V(\mathfrak{a})$ satisfy the axioms for closed sets in a topological space. We call this topology the **Zariski Topology** on $\text{Spec } A$. For any element $f \in A$, let $D(f) := X \setminus V(f) = \{\mathfrak{p} \in X : f \notin \mathfrak{p}\}$.

Proposition 3.8. The sets $D(f)$ form a basis¹ for the Zariski topology and are quasi-compact² (therefore X is quasi-compact).

Proof. For a proof of this fact we refer the reader to [Atiyah and Macdonald \(1969, I, Exercise 17\)](#). \square

Given an open set $U \subset X = \text{Spec } A$ in the Zariski topology, we define $\mathcal{O}_X(U)$ to be the set of functions

$$\varphi : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that³

- i. $\varphi(\mathfrak{p}) \in A_{\mathfrak{p}}$.

¹ Every open set in $\text{Spec } A$ is the union of elements of the form $D(f)$.

² A topological space is quasi-compact if every open cover has a finite subcover (see ([HARTSHORNE, 1977, II, Exercise 2.13](#))).

³ The symbol $\bigsqcup A_{\mathfrak{p}}$ refers to a disjoint union of sets $A_{\mathfrak{p}}$.

- ii. $\forall p \in U$ there exists $U' \subseteq U$ containing p and $f, g \in A$ such that $\varphi|_{U'} \equiv f/g \in A_p$, that is $\varphi(q) = f/g \in A_q$ for all $q \in U'$.

The following propositions will imply that the functions $\varphi \in \mathcal{O}_X(U)$ are regular functions in the sense of section 2.3.

Proposition 3.9. *Let A be a ring, $X = \text{Spec } A$, then \mathcal{O}_X is a sheaf of rings on $X = \text{Spec } A$ with the usual restriction maps*

$$\begin{aligned} \mathcal{O}_X(V) &\longrightarrow \mathcal{O}_X(U) \\ \varphi &\longmapsto \varphi|_U \end{aligned}$$

where $U \subseteq V$

Proof. Let $U \subseteq X$ be an open set. Sums and products of regular functions in U are again regular functions. The regular function defined to be $1 \in A_p$ for each $p \in U$ is an identity for the set of regular functions and if it is defined to be $0 \in A_p$ for each $p \in U$ is a zero for the set of regular functions, hence every $\mathcal{O}_X(U)$ is a ring. The usual restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ for $U \subseteq V$ is a ring homomorphism, thus \mathcal{O}_X is a presheaf.

To see it is a sheaf let $U = \bigcup_{i \in I} U_i$ be an open cover of U . If $\varphi \in \mathcal{O}_X(U)$ satisfies $\varphi|_{U_i} \equiv 0$ for every i , then $\varphi \equiv 0$ since it is the zero function. If there are $\varphi_i \in \mathcal{O}_X(U_i)$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ then we can define $\varphi \in \mathcal{O}_X(U)$ by $\varphi(p) := \varphi_i(p)$ if $p \in U_i$ to obtain a well defined regular function. \square

Definition 3.10. Let A be a ring and $X = \text{Spec } A$, we call \mathcal{O}_X the **structure sheaf** of X .

Let $p \in X$ and consider the stalk $\mathcal{O}_{X,p}$ at p (see definition 2.14). An element of $\mathcal{O}_{X,p}$ is an equivalent class of regular functions defined at open neighborhoods of p , if $\varphi \sim \psi$ are two such regular functions, where $\varphi \in \mathcal{O}_X(U)$ and $\psi \in \mathcal{O}_X(V)$ then we have an open set $W \subset U \cap V$ containing p such that $\varphi|_W \equiv \psi|_W$. We refer to each class by $[U, \varphi]$ or by φ_p . There is a natural morphism from $\mathcal{O}_{X,p}$ to A_p defined by

$$[U, \varphi] \longmapsto \varphi(p).$$

This is a well defined ring homomorphism since $(\varphi, U) \sim (\psi, V)$ implies $\varphi|_W \equiv \psi|_W$ for $W \subset U \cap V$ an open set containing p , in particular $\varphi(p) = \psi(p)$. This map is also injective since for $\varphi(p) = 0 \in A_p$ there is an open set U containing p such that $\varphi(q) = a/f$ for all $q \in U$, this implies that there exists $h \notin p$ such that $ha = 0$ in A , consider the open set $W = U \cap D(h)$ and let $q \in W$, note that $\varphi(q) = a/f$ and $h \notin q$, furthermore $ha = 0$ in A so $a/f = 0$ in A_q , and this is true for all $q \in W$, therefore $(\varphi, U) \sim (0, W)$. This map is also surjective since given $a/b \in A_p$ consider the class of $(a/b, D(b))$ in $\mathcal{O}_{X,p}$ to be its pre-image. Therefore this map is a ring isomorphism. Now let $f \in A$ be a non nilpotent element. Consider the homomorphism $H : A_f \rightarrow \mathcal{O}_X(D(f))$

sending each element $a/f^n \in A_f$ to $H(a/f^n) \in \mathcal{O}_X(D(f))$ which assigns to each \mathfrak{p} the image of a/f^n in $A_{\mathfrak{p}}$ under the natural homomorphism $A_f \rightarrow A_{\mathfrak{p}}$. The next proposition says that this assignment is also a ring isomorphism, for a complete proof of this see [Hartshorne \(1977, II, Proposition 2.2.\)](#).

Proposition 3.11. *Let A be a ring, $X = \text{Spec } A$, \mathcal{O}_X its structure sheaf and $f \in A$ a non nilpotent element, then*

$$(a) \ \mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

$$(b) \ \mathcal{O}_X(D(f)) \cong A_f.$$

Proposition 3.11 implies that (X, \mathcal{O}_X) is a locally ringed space for $X = \text{Spec } A$. For the case $f = 1$, proposition 3.11 says that $\Gamma(X, \mathcal{O}_X) = A$, is the ring of global sections of \mathcal{O}_X .

Definition 3.12. An **affine scheme** is a locally ringed space (X, \mathcal{O}_X) isomorphic (as locally ringed space) to $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ for some ring A . A **scheme** is a locally ringed space (X, \mathcal{O}_X) in which every point $\mathfrak{p} \in X$ has an open neighborhood U such that the topological space U together with the sheaf structure $\mathcal{O}_X|_U$ is an affine scheme. A **morphism** of schemes is a morphism of locally ringed spaces.

Remark 3.13. For simplicity we refer to a scheme (X, \mathcal{O}_X) as X , and to a morphism of schemes $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ as $f : X \rightarrow Y$.

Let $\varphi : A \rightarrow B$ be a ring homomorphism and denote $X = \text{Spec } A$ and $Y = \text{Spec } B$, we will define a morphism $(\varphi^*, \varphi^\#) : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ of locally ringed spaces. Let $\varphi^* : Y \rightarrow X$ be the induced continuous map defined by $\varphi^*(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$. Now we define a morphism of sheaves

$$\varphi^\# : \mathcal{O}_X \rightarrow (\varphi^*)_* \mathcal{O}_Y.$$

It is enough to define this morphism for open sets of the form $D(f) \subseteq X$ since they form a basis for the topology by proposition 3.8. Let $f \in A$, by [Atiyah and Macdonald \(1969\)](#) we have $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$ so we define $\varphi^\#_{D(f)}$ as the ring homomorphism induced by φ :

$$\begin{aligned} \mathcal{O}_X(D(f)) = A_f &\xrightarrow{\varphi^\#_{D(f)}} B_{\varphi(f)} = ((\varphi^*)_* \mathcal{O}_Y)(D(f)) \\ a/f^n &\longmapsto \varphi(a)/\varphi(f)^n \end{aligned}$$

Note that $\varphi^\#$ is compatible with the restrictions to elements of the basis $D(g) \subseteq D(f)$, thus it is a morphism of sheaves of rings. Note that we can recover the morphism φ from $\varphi^\#$ taking the open set $X = D(1)$, that is $\varphi \equiv \varphi^\#_X$. For every element $\mathfrak{q} \in Y$ the induced map

$$\begin{aligned} \varphi^\#_{\mathfrak{q}} : \mathcal{O}_{X, \varphi^*(\mathfrak{q})} = A_{\varphi^*(\mathfrak{p})} &\longrightarrow B_{\mathfrak{q}} = \mathcal{O}_{Y, \varphi(\mathfrak{q})} \\ a/f &\longmapsto \varphi(a)/\varphi(f) \end{aligned}$$

coincides with the local homomorphism $\varphi_{\varphi^*(\mathfrak{q})} : A_{\varphi^*(\mathfrak{q})} \rightarrow B_{\mathfrak{q}}$ induced by φ , thus $\varphi_{\mathfrak{q}}^{\sharp}$ is a local homomorphism of rings, this proves that $(\varphi^*, \varphi^{\sharp})$ is a homomorphism of locally ringed spaces.

Now, let $(f, f^{\sharp}) : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ where $Y = \text{Spec } B$ and $X = \text{Spec } A$ be a morphism of affine schemes. We obtain a ring homomorphism $\Gamma(f)$ given by

$$\begin{aligned} \Gamma(X, \mathcal{O}_X) = A &\xrightarrow{\Gamma(f)} B = \Gamma(Y, \mathcal{O}_Y) \\ a &\longmapsto f_X^{\sharp}(a) \end{aligned}$$

So we have a correspondence between rings and affine schemes which is actually a bijection as we summarize in the following proposition, see [Hartshorne \(1977, II, Proposition 2.3\)](#).

Proposition 3.14. (a) *Any morphism of rings $\varphi : A \rightarrow B$ induces a natural morphism of locally ringed spaces*

$$(\varphi^*, \varphi^{\sharp}) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A}).$$

(b) *Any morphism of locally ringed spaces $(f, f^{\sharp}) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is induced by a homomorphism of rings $\varphi : A \rightarrow B$ as in (a).*

Let X be a scheme and $U \subseteq X$ an open set. Consider the restriction of \mathcal{O}_X to U given by

$$\mathcal{O}_X|_U(V) = \mathcal{O}_X(U \cap V)$$

for $V \subseteq X$. It defines a scheme structure on $(U, \mathcal{O}_X|_U)$. In fact, note that it is easily seen to be a locally ringed space. To see U can be covered by affine schemes, let $X = \bigcup \text{Spec } A_i$ be an open cover of X by affine schemes, then $U = \bigcup (U \cap \text{Spec } A_i)$ is an open cover of U , now by proposition 3.8 each $U \cap \text{Spec } A_i$ can be covered by sets of the form $\text{Spec } (A_i)_f$ for $f \in A_i$ since the sets $D(f)$ form a basis for $\text{Spec } A_i$.

Remark 3.15. Thus, we have shown

- (i) (U, \mathcal{O}_X) is a scheme.
- (ii) The open affine sets⁴ form a basis for the topology of X .

Definition 3.16. A morphism of schemes $i : Y \rightarrow X$ is called a **open immersion** if it induces a homeomorphism between Y and an open set $U \subseteq X$ and such that the morphism of sheaves $i^{\sharp} : \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y$ induces an isomorphism $\mathcal{O}_X|_U \cong i_* \mathcal{O}_Y$ of sheaves on U . Any scheme Y such that $i : Y \rightarrow X$ is an open immersion is called an **open subscheme** of X .

Example 3.17. (*Principal open subschemes of an affine scheme.*) Let $X = \text{Spec } A$ be an affine scheme and f a non nilpotent element of A . Consider the canonical homomorphism $\varphi : A \rightarrow A_f$. This induces a morphism of locally ringed spaces $(\varphi^*, \varphi^{\sharp}) : (\text{Spec } A_f, \mathcal{O}_{\text{Spec } A_f}) \rightarrow (X, \mathcal{O}_X)$.

⁴ $U \subseteq X$ is an open affine set if it is open and $U = \text{Spec } A$ for some ring A .

The morphism φ^* induces an homeomorphism between $\text{Spec } A_f$ and $D(f) \subseteq X$, see [Atiyah and Macdonald \(1969, 5, Exercise 21\)](#). Now let $\mathfrak{q} \in \text{Spec } A_f$, the induced map on the stalks $\varphi_{\mathfrak{q}}^{\sharp} : \mathcal{O}_{X, \varphi^{-1}(\mathfrak{q})} \cong A_{\varphi^{-1}(\mathfrak{q})} \longrightarrow (A_f)_{\mathfrak{q}} \cong \mathcal{O}_{\text{Spec } A_f, \mathfrak{q}}$ is an isomorphism since $f \notin \varphi^{-1}(\mathfrak{q})$, so the induced morphism $(\varphi^*, \varphi^{\sharp}|_{D(f)}) : (\text{Spec } A_f, \mathcal{O}_{\text{Spec } A_f}) \longrightarrow (D(f), \mathcal{O}_X|_{D(f)})$ is an isomorphism of locally ringed spaces.

Example 3.18. (*Closed subschemes of affine schemes.*) Let $X = \text{Spec } A$ be an affine scheme. For any ideal $I \subseteq A$ consider the canonical homomorphism $\varphi : A \rightarrow A/I$. The induced continuous map $\varphi^* : \text{Spec } A/I \rightarrow X$ induces a homeomorphism between $i : \text{Spec } A/I \rightarrow V(I)$ ([ATIYAH; MACDONALD, 1969, 1, Exercise 21](#)). We use the direct image of $\mathcal{O}_{\text{Spec } A/I}$ under i to define a structure of a locally ringed space on $V(I)$. That is, for any open set $V \subseteq V(I)$ we have $i_* \mathcal{O}_{\text{Spec } A/I}(V) = \mathcal{O}_{\text{Spec } A/I}(i^{-1}(V))$, note that $(V(a), i_* \mathcal{O}_{\text{Spec } A/I})$ is a locally ringed space since the induced map on the stalks induces an isomorphism $(i_* \mathcal{O}_{\text{Spec } A/I})_{\mathfrak{p}} \cong \mathcal{O}_{\text{Spec } A/I, i^{-1}(\mathfrak{p})} \cong (A/I)_{i^{-1}(\mathfrak{p})}$ for every $\mathfrak{p} \subseteq A$ such that $I \subseteq \mathfrak{p}$. The scheme $(\text{Spec } A/I, \mathcal{O}_{\text{Spec } A/I})$ is called a *closed subscheme* of X , see definition [3.28](#).

Example 3.19. Let $A = k[x, y]/(y - x^2, y)$. Following [example 3.18](#) the canonical morphism of rings $k[x, y] \rightarrow A$ induces a closed subscheme structure on $\text{Spec } A$. Since $(y - x^2, y) = (x^2, y)$ we have that $A = k[x, y]/(x^2, y) \cong k[x]/(x^2)$, thus $\text{Spec } A = \{\mathfrak{p}\}$ where $\mathfrak{p} = (x)$ is the ideal generated by x , furthermore since $\{(x)\}$ is the only nonempty open set of $\text{Spec } A$ we have that the stalk

$$\mathcal{O}_{\text{Spec } R, \mathfrak{p}} \cong \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } R}) \cong \frac{k[x]}{(x^2)}$$

is a local ring with nilpotent elements. Note that $\dim_k \mathcal{O}_{\text{Spec } R, \mathfrak{p}} = 2$, we say that \mathfrak{p} is a double point of $\text{Spec } A$.

We finish this subsection with the notion of topological dimension of a scheme.

Definition 3.20. The **dimension** of a scheme X , $\dim X$, is its dimension as a topological space. If Z is an irreducible closed subset of X , the **codimension** of Z in X , denoted $\text{codim}(Z, X)$ is the maximum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \dots < Z_n$$

of distinct closed irreducible subsets of X , beginning with Z . If Y is any closed subset of X , we also define

$$\text{codim}(Y, X) = \inf_{Z \subset Y} \text{codim}(Z, X)$$

where the sets Z range over all closed irreducible sets of Y .

Observation 3.21. Note that if $X = \text{Spec } A$ is an affine scheme, the dimension of X is the same as the Krull dimension of A . This can be seen since given a maximal chain of prime ideals in A , $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n$, we get a chain of closed sets

$$V(\mathfrak{p}_0) \supsetneq V(\mathfrak{p}_1) \supsetneq \dots \supsetneq V(\mathfrak{p}_n)$$

By the correspondence between prime ideals of A and irreducible sets of $\text{Spec } A$, the sets $V(\mathfrak{p}_i)$ are irreducible. Thus, $\text{krull dim } A \leq \dim X$. Conversely, given a chain of closed irreducible sets

$$V(\mathfrak{a}_0) \subsetneq V(\mathfrak{a}_1) \subsetneq \cdots \subsetneq V(\mathfrak{a}_n)$$

we get the chain $\sqrt{\mathfrak{a}_0} \supsetneq \sqrt{\mathfrak{a}_1} \supsetneq \cdots \supsetneq \sqrt{\mathfrak{a}_n}$ of prime ideals, thus $\text{krull dim } A \geq \dim X$.

3.3 Projective schemes

Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring, in this section we define a special kind of scheme using the set of homogeneous prime ideals as the base set of the topological space and constructing a sheaf of rings in a similar way to that of the affine case. Let $R_+ = \bigoplus_{n > 0} R_n$ the ideal of elements with positive degree, also called the **irrelevant ideal** of R .

Definition 3.22. Let $\text{Proj } R := \{\mathfrak{p} \subseteq R : \mathfrak{p} \text{ is homogeneous prime ideal and } R_+ \not\subseteq \mathfrak{p}\}$.

Now we endow $X = \text{Proj } R$ with a topological space structure. For a set $T \subseteq R$ of homogeneous elements we define $V_+(T) = \{\mathfrak{p} \in \text{Proj } R \mid T \subseteq \mathfrak{p}\}$. If \mathfrak{a} is the ideal generated by T then \mathfrak{a} is homogeneous and $V_+(T) = V_+(\mathfrak{a})$. For an homogeneous element $f \in R$ we write $V_+(f)$ instead of $V_+(\{f\})$.

Lemma 3.23. (a) $V_+(R_+) = \emptyset$ and $V_+(0) = X$.

(b) If $\{\mathfrak{a}_i\}_{i \in I}$ is a family of ideals of R , then $V_+(\bigcup_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V_+(\mathfrak{a}_i)$.

(c) For ideals \mathfrak{a} and \mathfrak{b} of R we have $V_+(\mathfrak{a} \cap \mathfrak{b}) = V_+(\mathfrak{a}\mathfrak{b}) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$.

(d) If $\mathfrak{a} \subseteq R$ is any ideal then $V_+(\mathfrak{a}) = V_+(\sqrt{\mathfrak{a}})$.

Proof. Assertions (a) and (b) are immediate, for (c) just note that a prime ideal contains \mathfrak{a} or \mathfrak{b} if and only if it contains $\mathfrak{a} \cap \mathfrak{b}$ or equivalently if and only if it contains $\mathfrak{a}\mathfrak{b}$. For (d) note that $\mathfrak{a} \subseteq \mathfrak{p}$ if and only if $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$. \square

Observation 3.24. Since $V_+(I) = \emptyset$ if and only if $R_+ \subseteq \sqrt{I}$ we do not include ideals that contain the irrelevant ideal in the definition of $\text{Proj } R$.

Therefore, sets of the form $V_+(\mathfrak{a})$ where \mathfrak{a} is a homogeneous ideal satisfy the axioms for closed sets in a topological space, this endows $X := \text{Proj } R$ with a topological space structure. Let $f \in R_+$ be a homogeneous element, define the open sets $D_+(f) := \text{Proj } R \setminus V_+(f) = \{\mathfrak{p} \in \text{Proj } R : f \notin \mathfrak{p}\}$. The topology just defined in $\text{Proj } R$ coincides with the induced topology that comes from the inclusion $\text{Proj } R \subseteq \text{Spec } R$, since for every homogeneous ideal \mathfrak{a} we have $V_+(\mathfrak{a}) = V(\mathfrak{a}) \cap \text{Proj } R$, that is, every closed set of $\text{Proj } R$ comes from the intersection of a closed set of $\text{Spec } R$ and $\text{Proj } R$. Also, for any ideal $\mathfrak{a} \subseteq R$ we have $V(\mathfrak{a}) \cap \text{Proj } R = V_+(\mathfrak{a}^h)$

where \mathfrak{a}^h is the homogeneous ideal generated by \mathfrak{a} , that is, every closed set in $\text{Spec } R$ gives a closed set in $\text{Proj } R$. The sets $D_+(f)$ with $f \in R_+$ homogeneous form a basis for the topology of $\text{Proj } R$ (see [Görtz and Wedhorn \(2010, Proposition 13.4\)](#)).

Next we want to define a sheaf of rings on $X = \text{Proj } R$. For each $\mathfrak{p} \in X$ let T be the set of homogeneous elements of $R - \mathfrak{p}$. With this assumption, the localization $T^{-1}R$ has a natural structure of graded ring where an element $a/f \in T^{-1}R$ has degree = degree a - degree f , we denote the subring of elements of degree zero of $T^{-1}R$ by $R_{(\mathfrak{p})}$, that is, the set of elements a/f where $a \in R$ and $f \in R - \mathfrak{p}$ are homogeneous elements of the same degree. Now for every open set $U \subseteq \text{Proj } R$ we define $\mathcal{O}_X(U)$ to be the set of functions $\varphi : U \mapsto \bigsqcup_{\mathfrak{p} \in U} R_{(\mathfrak{p})}$ such that for every $\mathfrak{p} \in U$, $\varphi(\mathfrak{p})$ belongs to $R_{(\mathfrak{p})}$, and there exists an open neighborhood $V \subseteq U$ of \mathfrak{p} and elements $f, g \in R$ such that $\varphi|_V \equiv f/g \in R_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$. As in the affine case it is an easy task to verify that \mathcal{O}_X is a sheaf of rings for $X = \text{Proj } R$. [Proposition 3.26](#) will imply that the functions $\varphi \in \mathcal{O}_X(U)$ are regular functions in the sense of [section 2.3](#).

Definition 3.25. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and $X = \text{Proj } R$, we call \mathcal{O}_X the **structure sheaf** of X .

Let $\mathfrak{p} \in X$ and consider the stalk $\mathcal{O}_{X,\mathfrak{p}}$ at \mathfrak{p} . The map $\mathcal{O}_{X,\mathfrak{p}} \rightarrow R_{(\mathfrak{p})}$ sending an element $[U, \varphi] \in \mathcal{O}_{X,\mathfrak{p}}$ to $\varphi(\mathfrak{p}) \in R_{(\mathfrak{p})}$ is a well define ring homomorphism, in fact it is an isomorphism of local rings, this can be seen similarly as in the affine case in [Proposition 3.11](#).

If $f \in R_+$ is an homogeneous non nilpotent element, then the localization R_f has a natural structure of \mathbb{Z} -graded ring given by degree $a/f^n = \text{degree } a - (\text{degree } f + n)$, define the set $R_{(f)}$ to be the subring of elements of degree 0 in R_f . The map

$$\begin{aligned} H : D_+(f) &\longrightarrow \text{Spec } R_{(f)} \\ \mathfrak{p} &\longmapsto (\mathfrak{p}R_f) \cap R_{(f)} \end{aligned}$$

is an homeomorphism of topological spaces, where $D_+(f)$ has the induced topology from $\text{Proj } R$ and $\text{Spec } R_{(f)}$ has the Zariski topology (see [Hartshorne \(1977, Proposition 2.5\)](#)). From the map H we construct an isomorphism of sheaves

$$H^\sharp : \mathcal{O}_{\text{Spec } R_{(f)}} \rightarrow H_*(\mathcal{O}_X|_{D_+(f)}).$$

We first look at the stalks of the sheaves above. We have $\mathcal{O}_{\text{Spec } R_{(f)},\mathfrak{q}} \cong (R_{(f)})_{\mathfrak{q}}$ and $(\mathcal{O}_X|_{D_+(f)})_{\mathfrak{p}} \cong \mathcal{O}_{X,\mathfrak{p}} \cong S_{\mathfrak{p}}$. Let $\mathfrak{p} = H(\mathfrak{q})$, there is an isomorphism between these two rings given by

$$\begin{array}{ccc} h_{\mathfrak{p}} : (R_{(f)})_{\mathfrak{q}} &\longrightarrow & R_{(\mathfrak{p})} & & h_{\mathfrak{p}}^{-1} : R_{(\mathfrak{p})} &\longrightarrow & (R_{(f)})_{\mathfrak{q}} \\ \frac{a/f^r}{b/f^s} &\longmapsto & \frac{af^s}{bf^r} & & \frac{a}{b} &\longmapsto & \frac{ab^{d-1}/f^e}{b^d/f^e} \end{array}$$

where $d = \text{degree } f$ and $e = \text{degree } a = \text{degree } b$. Let $U \subseteq X$ be an open set, define the map $H_U^\sharp : \mathcal{O}_{\text{Spec } R_{(f)}}(U) \rightarrow \mathcal{O}_X(H^{-1}(U))$ by

$$H_U^\sharp(s)(\mathfrak{p}) := h_{\mathfrak{p}}(s(H(\mathfrak{p})))$$

for $s \in \mathcal{O}_{\text{Spec } R_{(f)}}(U)$ and $\mathfrak{p} \in H^{-1}(U)$. The proof of the next proposition implies that $(H, H^\sharp) : (D_+(f), \mathcal{O}|_{D_+(f)}) \rightarrow (\text{Spec } R_{(f)}, \mathcal{O}_{\text{Spec } R_{(f)}})$ is an isomorphism of affine schemes.

Proposition 3.26. *Let $X = \text{Proj } R$ where $R = \bigoplus_{n \geq 0} R_n$ is a graded ring, let $f \in R_+$ be a homogeneous non nilpotent element, then*

- (a) $\mathcal{O}_{X, \mathfrak{p}} \cong S_{(\mathfrak{p})}$ for every $\mathfrak{p} \in \text{Proj } R$.
- (b) $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong (\text{Spec } R_{(f)}, \mathcal{O}_{\text{Spec } R_{(f)}})$ is an isomorphism of locally ringed spaces. For simplicity we write $D_+(f) \cong \text{Spec } R_{(f)}$.
- (c) (X, \mathcal{O}_X) is a scheme.

Proof. See Hartshorne (1977, II, Proposition 2.5) □

Example 3.27. Let A be a ring. Define the **affine r -space** over A to be the scheme $\mathbb{A}_A^r := \text{Spec } A[y_1, \dots, y_r]$ together with the structure sheaf, here $A[y_1, \dots, y_r]$ is the ring of polynomials on r variables over A . The **projective r -space** over A is defined to be the scheme $\mathbb{P}_A^r := \text{Proj } A[x_0, \dots, x_r]$ together with its structure sheaf. There are canonical isomorphisms

$$A[x_0, \dots, x_r]_{(x_i)} \cong A \left[\frac{x_0}{x_i}, \dots, \frac{x_r}{x_i} \right] \cong A[y_1, \dots, y_r]$$

which induce an isomorphism of schemes $D_+(x_i) \cong \text{Spec } A[y_1, \dots, y_r]$, that is, \mathbb{P}_A^r has a open cover of $r + 1$ copies of \mathbb{A}_A^r .

3.4 Subschemes and noetherian schemes

Let $\varphi : A \rightarrow B$ be a surjective homomorphism of rings, by Atiyah and Macdonald (1969, I, Exercise 21) the induced continuous map $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$ induces a homeomorphism between $\text{Spec } B$ and the closed set $V(\ker \varphi) \subseteq \text{Spec } A$, we say that $\text{Spec } B$ is a closed subscheme of $\text{Spec } A$ and that φ^* is a closed immersion, we generalize this idea as follows.

Definition 3.28. A **closed immersion** is a morphism of schemes $f : Z \rightarrow X$ such that it induces a homeomorphism between Z and a closed subset of X and such that the morphism of sheaves $f^\sharp : \mathcal{O}_X \rightarrow f_* \mathcal{O}_Z$ is surjective. A **closed subscheme** of a scheme X is given by a closed subset $Z \subseteq X$ and a scheme structure (Z, \mathcal{O}_Z) such that the inclusion map $i : Z \hookrightarrow X$ is a closed immersion.

Example 3.29. Let R be a graded ring and $I \subseteq R$ a homogeneous ideal, the natural map $\varphi : R \rightarrow R/I$ is a surjective homomorphism of graded rings preserving degrees, moreover $\varphi(R_+) \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \text{Proj}(R/I)$ and $\varphi^* : \text{Proj}(R/I) \rightarrow \text{Proj}(R)$ sending \mathfrak{p} to $\varphi^{-1}(\mathfrak{p})$ is a well defined continuous map, in fact, together with the morphism of sheaves $\mathcal{O}_{\text{Proj } R} \rightarrow (\varphi^*)_* \mathcal{O}_{\text{Proj } R/I}$, it is a closed immersion. The map φ^* defines a homeomorphism between $\text{Proj } R/I$ and $V_+(I)$ and we identify $(\text{Proj } R/I, \mathcal{O}_{\text{Proj } R/I})$ with the closed subscheme $(V_+(I), (\varphi^*)_* \mathcal{O}_{\text{Proj } R/I})$.

Example 3.30. Consider the homogeneous ideal $I = (y^2z - x^2(x+z), y) \subseteq k[x, y, z]$ and set $R = k[x, y, z]/I$. We identify the closed scheme $Z = \text{Proj } R$ with $V_+(I)$. Note that $V_+(I) \cap V_+(z) = \emptyset$ since every prime in $V_+(I) \cap V_+(z)$ contains R_+ , thus $V_+(z) \subseteq D_+(z) \cong \text{Spec } R_{(z)} \cong k[x, y, z]_{(z)} \cong k[x, y]$. Therefore $V_+(I)$ is identified with the closed set $V(y^2 - x^2(x+1), y) = V(x^2(x+1), y) \subseteq \text{Spec } k[x, y]$ and we have

$$\Gamma(Z, \mathcal{O}_Z) \cong \frac{k[x, y]}{(x^2(x+1), y)} \cong \frac{k[x]}{x^2(x+1)} \cong \frac{k[x]}{x^2} \times \frac{k[x]}{(x+1)}$$

Definition 3.31. A scheme X is **locally noetherian** if it can be covered by open affine subsets $\text{Spec } A_i$, where each A_i is a noetherian ring. X is called **noetherian** if it is locally noetherian and quasi-compact⁵.

Observation 3.32. X is noetherian if it can be covered by a finite number of open affine subsets $\text{Spec } A_i$, where each A_i is a noetherian ring.

If A is a noetherian ring then $\text{Spec } A$ is a noetherian topological space. Note that for every descending chain of closed sets $Z_1 \supseteq Z_2 \supseteq \dots$ we get an stationary ascending chain of ideals $I(Z_1) \subseteq I(Z_2) \subseteq \dots$ in A . Also if R is a graded noetherian ring then $X = \text{Proj } R$ is a locally noetherian scheme since it can be covered by open affine schemes $\text{Spec } R_{(f)}$ where $R_{(f)}$ is noetherian.

If X is a noetherian scheme, then its underlying topological space is a noetherian topological space. To see this suppose we have a chain of closed sets $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \dots$ in X . Write $X = \bigcup_{i=1}^n \text{Spec } A_i$ where A_i are noetherian rings. Then $\text{Spec } A_i$ is a noetherian topological space and we have a chain of closed sets in $\text{Spec } A_i$, $Z_1 \cap \text{Spec } A_i \supseteq Z_2 \cap \text{Spec } A_i \supseteq \dots$ and for some r , $\text{Spec } A_i \cap Z_r = \text{Spec } A_i \cap Z_{r+1} = \dots$, and therefore we can write $Z_r = \bigcup_{i=1}^n \text{Spec } A_i \cap Z_r = \bigcup_{i=1}^n \text{Spec } A_i \cap Z_{r+1} = Z_{r+1}$ and so the chain is stationary.

In the definition of locally noetherian scheme we do not require that every open affine subset is the spectrum of a noetherian ring. Nevertheless there is a base for the topology consisting of the spec of noetherian rings. In fact, we can cover X by open affine sets $U = \text{Spec } B$ with B noetherian, then the open subsets $D(f) = \text{Spec } B_f \subseteq \text{Spec } B$ form a basis for the topology of $\text{Spec } B$ and the B_f are also noetherian. The following proposition states that every affine open set is the spec of a noetherian ring, that is, locally noetherian is a "local property", see [Hartshorne \(1977, II, Proposition 3.2\)](#).

Proposition 3.33. *A scheme X is locally noetherian if and only if for every open affine subset $U = \text{Spec } A$, A is a noetherian ring. Particularly, an affine scheme $X = \text{Spec } A$ is noetherian if and only if A is a noetherian ring.*

Example 3.34. By proposition 3.33 the scheme $\text{Spec } \mathbb{Z}$ is noetherian, its elements are (0) and (p) for p a prime number. Note that (p) is a closed point in $\text{Spec } \mathbb{Z}$ ($\{p\} = V(p)$) while (0)

⁵ A topological space is quasi-compact if every open cover has a finite subcover, see [Hartshorne \(1977, II, Exercise 2.13\)](#).

is a generic point whose closure is $\text{Spec } \mathbb{Z}$. Furthermore, the scheme $\text{Spec } \mathbb{Z}[x_1, \dots, x_r]$ is a noetherian scheme. In particular the prime ideals of $\mathbb{Z}[x]$ are (0) , (p) with $p \in \mathbb{Z}$ a prime number, (f) with $f \in \mathbb{Z}[x]$ irreducible over \mathbb{Q} such that its coefficients have no common factors on \mathbb{Z} and (p, f) where p is prime and f is monic whose reduction modulo p is irreducible. For example the ideal $(x^2 - 3) \subseteq \text{Spec } \mathbb{Z}[x]$ defines a closed subscheme $\text{Spec } \mathbb{Z}[x]/(x^2 - 3) \cong \text{Spec } \mathbb{Z}[\sqrt{3}]$. The ring $A = \mathbb{Z}[\sqrt{3}]$ is called the ring of integers of the number field $K = \mathbb{Q}[\sqrt{3}]$. As for \mathbb{Z} , the prime ideals of A correspond to closed points $\mathfrak{p} = V(\mathfrak{p})$ for nonzero $\mathfrak{p} \in \text{Spec } A$ and to a generic point (0) , this is described in [Eisenbud and Harris \(2000, II.4.2\)](#) using the morphism $\text{Spec } A \rightarrow \text{Spec } \mathbb{Z}$ induced by the canonical homomorphism $\mathbb{Z} \rightarrow A$.

SHEAVES OF MODULES

In this chapter we consider a sheaf \mathcal{F} on a scheme (X, \mathcal{O}_X) such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module for every open $U \subseteq X$. Given an affine or projective scheme $\text{Spec } A$ or $\text{Proj } R$ and any A or R -module M , we construct a sheaf of modules \tilde{M} associated to M , every sheaf that looks locally like this will be called a quasi-coherent sheaf. When X is affine there will be an equivalence between the category of \mathcal{O}_X -modules and the category of A -modules. This is not the case for $X = \text{Proj } R$, here we will construct a graded R -module $\Gamma_*(\mathcal{F})$ such that the sheaf associated to this module will be isomorphic to \mathcal{F} . Finally we prove that any closed subscheme induces a quasi-coherent sheaf with its sheaf of ideals, and that any quasi-coherent sheaf of ideals on a scheme induces a uniquely determined closed subscheme.

Definition 4.1. Let (X, \mathcal{O}_X) be a ringed space. A **sheaf of \mathcal{O}_X -modules** over X (or a **\mathcal{O}_X -module**) is a sheaf \mathcal{F} such that for every open set $U \subset X$, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$ -module and for every inclusion $U \subset V$ the restriction map $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is compatible with the respective ring structures through the restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$. More precisely if $s \in \mathcal{O}_X(V)$ and $m \in \mathcal{F}(V)$ then $(s \cdot m)|_U = s|_U \cdot m|_U$. A **morphism** of sheaves of \mathcal{O}_X -modules, $\theta : \mathcal{F} \rightarrow \mathcal{G}$, is a morphism of sheaves such that $\theta_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ are morphisms of $\mathcal{O}_X(U)$ -modules for all open set $U \subseteq X$.

Note that for any $x \in X$ the abelian group \mathcal{F}_x inherits a $\mathcal{O}_{X,x}$ -module structure given by $s_x \cdot m_x = (s \cdot m)_x$.

Let (X, \mathcal{O}_X) be a scheme, then we say that an \mathcal{O}_X -module \mathcal{F} is **locally free** if there is an open affine cover U_i of X such that $\mathcal{F}|_{U_i}$ is isomorphic to a direct sum of copies of \mathcal{O}_{U_i} . If the number of copies r is finite and constant, then \mathcal{F} is called locally free of rank r (vector bundle). If \mathcal{F} is locally free of rank one then we way say that \mathcal{F} is invertible (line bundle).

Example 4.2. (Trivial examples). A ringed space \mathcal{O} is itself a \mathcal{O}_X -module. Also, there is the

free \mathcal{O}_X -module of rank n :

$$\mathcal{O}_X^n \cong \underbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}_{n\text{-times}}$$

4.1 Quasi-coherent sheaves

Let A be a ring and M an A -module. Let $X = \text{Spec } A$ and $U \subseteq X$ any open set in the Zariski topology, consider functions $s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ from U to the disjoint union of the modules $M_{\mathfrak{p}}$ ¹ such that for every $\mathfrak{p} \in U$

- (i) $s(\mathfrak{p}) \in M_{\mathfrak{p}}$.
- (ii) Exists an open neighborhood $W \subseteq U$ of \mathfrak{p} and elements $m \in M$, $f \in A$ satisfying $s(\mathfrak{q}) = m/f \in M_{\mathfrak{q}}$ for all $\mathfrak{q} \in W$.

We denote the set of this functions by $\tilde{M}(U)$. It is easy to check that the association $U \mapsto \tilde{M}(U)$ together with the maps $\tilde{M}(V) \rightarrow \tilde{M}(U)$ sending a function $s \in \tilde{M}(V)$ to its restriction $s|_U \in \tilde{M}(U)$ when $U \subseteq V$, defines a sheaf of modules on X . We call \tilde{M} the **sheaf associated** to M .

Proposition 4.4 implies that the functions defined above correspond to regular functions in the sense of section 2.3.

Definition 4.3. Let (X, \mathcal{O}_X) be a scheme. An \mathcal{O}_X -module \mathcal{F} is **quasi-coherent** if X can be covered by open sets $U_i = \text{Spec } A_i$, A_i a collection of rings, such that for every i there is an A_i -module M_i with $\mathcal{F}|_{U_i} = \tilde{M}_i$. We say \mathcal{F} is **coherent** if the M_i are finitely generated as A_i -modules.

Proposition 4.4. Let A be a ring, M an A -module and $X = \text{Spec } A$.

- (a) The stalk $(\tilde{M})_{\mathfrak{p}}$ is isomorphic to the localized A -module $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$.
- (b) $\tilde{M}(D(f))$ and M_f are isomorphic A_f -modules. In particular, $\Gamma(X, \tilde{M}) \cong M$ as A -modules.
- (c) \tilde{M} is a quasi-coherent sheaf on X .

Proof. For part (a) consider the A -module homomorphism $H : (\tilde{M})_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$ sending $[U, s] \mapsto s(\mathfrak{p})$. The proof that this is an isomorphism is very similar to that of proposition 3.11 item (a), just replacing A by M when needed.

For part (b), note that $\tilde{M}(D(f))$ is a $\mathcal{O}_X(D(f)) \cong A_f$ -module, define the morphism $M_f \rightarrow \tilde{M}(D(f))$ by sending m/f to the function that assigns to every $\mathfrak{p} \in D(f)$ the element $m/f \in M_{\mathfrak{p}}$. The prove that this is an isomorphism is identical to the proof of proposition 3.11 item (b). If We obtain $\Gamma(X, \tilde{M}) \cong M$ when $f = 1$.

(c) comes from the fact that the sets $D(f)$ cover X . □

¹ Here we localize M at the multiplicative subset $A \setminus \{\mathfrak{p}\}$.

Let A be a ring. For any morphism $\varphi : M \rightarrow N$ of A -modules we will construct a morphism of sheaves $\tilde{\varphi} : \tilde{M} \rightarrow \tilde{N}$. For an element $s \in \tilde{M}(U)$, $\mathfrak{p} \in U$ define $\tilde{\varphi}(U)(s)(\mathfrak{p}) = \varphi_{\mathfrak{p}}(s(\mathfrak{p})) \in N_{\mathfrak{p}}$ where the morphism of $A_{\mathfrak{p}}$ -modules $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is $\varphi_{\mathfrak{p}}(m/f) = \varphi(m)/f$. Note that if we have an exact sequence

$$M \xrightarrow{\varphi} N \xrightarrow{\phi} P$$

the sequence

$$\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N} \xrightarrow{\tilde{\phi}} \tilde{P}$$

is also exact since $M_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} P_{\mathfrak{p}}$ is exact for all $\mathfrak{p} \in X$.

The correspondence $\varphi \mapsto \tilde{\varphi}$ is a bijection. It is clearly one to one since $\tilde{\varphi} = \tilde{\phi}$ implies $\varphi_{\mathfrak{p}} = \phi_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$. To see it is onto let $\theta : \tilde{M} \rightarrow \tilde{N}$ a morphism of \mathcal{O}_X -modules. Taking global sections we get a morphism of A -modules

$$\varphi : M \cong \Gamma(X, \tilde{M}) \xrightarrow{\theta_X} \Gamma(X, \tilde{N}) \cong N$$

We summarize this in proposition 4.5, see Hartshorne (1977, II, Proposition 5.2).

Proposition 4.5. *Let $\varphi : M \rightarrow N$ a morphism of A -modules, there exists an associated morphism $\tilde{\varphi} : \tilde{M} \rightarrow \tilde{N}$ such that the correspondence $\varphi \mapsto \tilde{\varphi}$ is bijective and for every exact sequence $M \rightarrow N \rightarrow P$ the sequence $\tilde{M} \rightarrow \tilde{N} \rightarrow \tilde{P}$ is also exact.*

In the language of category theory proposition 4.5 says that the correspondence $M \mapsto \tilde{M}$ is an exact fully faithful covariant functor between the category of A -modules and the category of \mathcal{O}_X -modules for $X = \text{Spec } A$.

A quasi-coherent sheaf on a scheme X is actually of the form \tilde{M} on any affine open subset $U = \text{Spec } A \subseteq X$ where M is an A -module, as we state in proposition 4.6, see Hartshorne (1977, II, Proposition 5.4).

Proposition 4.6. *Let X be a scheme. An \mathcal{O}_X -module \mathcal{F} is quasi-coherent if and only if for any open affine subset $U = \text{Spec } A$, there exists an A -module M such that $\mathcal{F}|_U = \tilde{M}$. If X is noetherian then \mathcal{F} is coherent if and only if the same is true and each module M may be assume to be finitely generated A -module.*

If \mathcal{F} is a quasi-coherent sheaf over a scheme X and U an open affine subset of X , then $\mathcal{F}|_U$ is quasi-coherent, so in proving proposition 4.6 we may assume $X = \text{Spec } A$. Now, if $M = \Gamma(X, \mathcal{F})$ then $\mathcal{F} \cong \tilde{M}$ and for every $f \in A$ this isomorphism on the basis sets $\theta_{D(f)} : M_f \rightarrow \mathcal{F}(D(f))$ is $\theta_{D(f)}(m/f^k) = (1/f^k)m|_{D(f)}$.

Corollary 4.7. *Let $X = \text{Spec } A$. Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ be morphism between quasi-coherent sheaves, consider the associated morphism of A -modules $\Gamma(X, \theta) : \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G})$. The correspondence $\theta \mapsto \Gamma(X, \theta)$ is bijective and for every quasi-coherent \mathcal{O}_X -module we have $\mathcal{F} \cong \Gamma(X, \mathcal{F})$.*

Proof. The map $\theta \mapsto \theta_X$ is bijective since it is inverse to the map given in proposition 4.5, and \mathcal{O}_X -module we have $\mathcal{F} \cong \Gamma(X, \mathcal{F})^\sim$ by the proof of the previous proposition. \square

In the language of category theory corollary 4.7 says that the category of quasi-coherent \mathcal{O}_X -modules is equivalent² to the category of A -modules. If A is a noetherian ring then we have an equivalence between coherent \mathcal{O}_X -modules and finitely generated A -modules.

4.2 Quasi-coherent sheaves on Proj R

In this section given $R = \bigoplus_{n \geq 0} R_n$ a positively graded ring we associate to every graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a quasi-coherent sheaf \tilde{M} . This construction is similar to that of the structure sheaf \mathcal{O}_X given in section 3.3. Then given a \mathcal{O}_X -module \mathcal{F} we construct a graded module $\Gamma_*(\mathcal{F})$ and we obtain an isomorphism $\Gamma_*(\mathcal{F})^\sim \cong \mathcal{F}$ in the case that $R = R_0 + R_1 + \dots$ is positively graded and R is finitely generated by R_1 over R_0 .

Let M be a graded R -module and $\mathfrak{p} \in \text{Proj } R$ an ideal, define $M_{(\mathfrak{p})}$ to be the set of elements m/f where $m \in M$ and $f \in R$ are homogeneous and have the same degree. That is, if we consider T as the set of homogeneous elements not in \mathfrak{p} , then $T^{-1}M$ has a natural grading degree $m/f = \text{degree } m - \text{degree } f$, and $M_{(\mathfrak{p})}$ is the submodule of degree 0 elements of $T^{-1}M$. Note that $M_{(\mathfrak{p})}$ is a $R_{(\mathfrak{p})}$ -module. Similarly, for a fixed homogeneous element $f \in R_+$, M_f has a graded R_f -module structure, its degree zero submodule, denoted by $M_{(f)}$, is the set of elements m/f^n where m is homogeneous and degree $m = \text{degree } f^n$. In particular, $M_{(f)}$ is a $R_{(f)}$ -module.

Definition 4.8. Let R be a positively graded ring, M a graded R -module and $X = \text{Proj } R$. We define the \mathcal{O}_X -module \tilde{M} as follows: for every open set $U \subseteq X$ let $\tilde{M}(U)$ be the set of functions $s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ such that for every $\mathfrak{p} \in U$

- (i) $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$.
- (ii) Exists an open set $W \subset U$ containing \mathfrak{p} and homogeneous elements $m \in M$, $f \in R$ of the same degree satisfying $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$ for all $\mathfrak{q} \in W$.

Here the maps $\tilde{M}(V) \rightarrow \tilde{M}(U)$ are given by the restriction $s \mapsto s|_U$ whenever $U \subset V$ and $s \in \tilde{M}(V)$.

Proposition 4.9 implies that $\tilde{M}(U)$ are regular functions in the sense of section 2.3.

Proposition 4.9. Let R be a graded ring, M a graded R -module and $X = \text{Proj } R$.

- (a) The stalk $(\tilde{M})_{\mathfrak{p}}$ is isomorphic to $M_{(\mathfrak{p})}$ for every $\mathfrak{p} \in X$.

² See Lane (2013, pp. 93) for a definition of equivalent categories

- (b) For every homogeneous element $f \in R_+$, $\widetilde{M}|_{D_+(f)} \cong (M_{(f)})^\sim$ via the isomorphism of $D_+(f) \cong \text{Spec } R_{(f)}$.
- (c) \widetilde{M} is quasi-coherent. Furthermore, if R is noetherian and M is finitely generated, \widetilde{M} is coherent.

Proof. (a) It suffices to prove that the morphism $(\widetilde{M})_{\mathfrak{p}} \rightarrow M_{(\mathfrak{p})}$ given by $[U, s] \mapsto s(\mathfrak{p})$ is an isomorphism, see proposition 3.26 item (a) for an analogous proof of this fact.

(b) We want to prove $\widetilde{M}_{(f)} \cong H_*\widetilde{M}|_{D_+(f)}$ as $\mathcal{O}_{\text{Spec } R_{(f)}}$ -modules. Since there is a natural isomorphism $(M_{(f)})_{H(\mathfrak{p})} \rightarrow M_{(\mathfrak{p})}$ for every $\mathfrak{p} \in D_+(f)$, we can construct an isomorphism of sheaves

$$\widetilde{M}_{(f)} \rightarrow H_*\widetilde{M}|_{D_+(f)}$$

analogously as the construction of the isomorphism $H^\sharp: \mathcal{O}_{D_+(f)} \rightarrow \mathcal{O}_{\text{Spec } R_{(f)}}$ given in the proof of proposition 3.26 item (b).

(c) This is a consequence of (b) since $M_{(f)}$ is a $R_{(f)}$ -module and $D_+(f) \cong \text{Spec } R_{(f)}$ is affine. Now if R is noetherian and M is finitely generated, $R_{(f)}$ is also noetherian and $M_{(f)}$ is finitely generated, and therefore \widetilde{M} is coherent. \square

Definition 4.10. The **tensor product** $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two sheaves of \mathcal{O}_X -modules is the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Lemma 4.11. If R is generated by R_1 as R_0 -algebra then $(M \otimes_R N)^\sim \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ for any pair of graded R -modules M and N .

Proof. This follows from the fact that $(M \otimes_R N)_{(f)} \cong M_{(f)} \otimes_{R_{(f)}} N_{(f)}$. \square

Definition 4.12. Let R be a graded ring and $X = \text{Proj } R$. For any graded R -module M and any $n \in \mathbb{Z}$ we define the **twisted module** $M(n)$ of M by $M(n)_d := M_{n+d}$. For any sheaf of \mathcal{O}_X -modules \mathcal{F} , we denote by $\mathcal{F}(n)$ the **twisted sheaf** $\mathcal{F} \otimes_{\mathcal{O}_X} R(n)^\sim$, i. e.,

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} R(n)^\sim.$$

Note that $\mathcal{O}_X(n) \cong R(n)^\sim$. We call $\mathcal{O}_X(1)$ the **twisting sheaf** of Serre.

Note that the morphism $M \otimes_R R(n) \rightarrow M(n)$ defined by $m \otimes t \mapsto tm$ is an isomorphism of graded R -modules, hence $(M \otimes_R R(n))^\sim \cong M(n)^\sim$. Then, by the previous lemma

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M} \otimes_{\mathcal{O}_X} R(n)^\sim \cong (M \otimes_R R(n))^\sim \cong M(n)^\sim.$$

In particular

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = R(n)^\sim \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = R(n)^\sim(m) \cong (S(n)(m))^\sim \cong R(n+m)^\sim = \mathcal{O}_X(n+m).$$

Definition 4.13. Let R be a graded ring, $X = \text{Proj } R$ and \mathcal{F} an \mathcal{O}_X -module. We define the abelian group $\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$.

Remark 4.14. $\Gamma_*(\mathcal{O}_X)$ is a graded ring, to see this consider the group homomorphism

$$\Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathcal{O}_X(m)) \rightarrow \Gamma(X, \mathcal{O}_X(n) \otimes \mathcal{O}_X(m))$$

since $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$, we get a homomorphism

$$\alpha_{n,m} : \Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathcal{O}_X(m)) \rightarrow \Gamma(X, \mathcal{O}_X(m+n)).$$

Now, given $f = (f_n)_{n \in \mathbb{Z}}, g = (g_n)_{n \in \mathbb{Z}} \in \Gamma_*(\mathcal{O}_X)$, define the product $f \cdot g := (\alpha_{n,m}(f_n \otimes g_m))_{n,m \in \mathbb{Z}}$. This product respects the grading on $\Gamma_*(\mathcal{O}_X)$.

Remark 4.15. $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$ -module. To see this consider the homomorphism of abelian groups

$$\Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathcal{F}(m)) \rightarrow \Gamma(X, \mathcal{O}_X(n) \otimes \mathcal{F}(m))$$

and since $\mathcal{O}_X(n) \otimes \mathcal{F}(m) \cong \mathcal{F}(n+m)$ we get the homomorphism

$$\beta_{n,m} : \Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathcal{F}(m)) \rightarrow \Gamma(X, \mathcal{F}(n+m)).$$

Now, given an element $f = (f_n)_{n \in \mathbb{Z}} \in \Gamma_*(\mathcal{O}_X)$ and an element $s = (s_n)_{n \in \mathbb{Z}} \in \Gamma_*(\mathcal{F})$ we define the operation $f \cdot s := (\beta_{n,m}(f_n \otimes s_m))_{n,m \in \mathbb{Z}}$. This converts $\Gamma_*(\mathcal{F})$ into a graded $\Gamma_*(\mathcal{O}_X)$ -module.

Remark 4.16. $\Gamma_*(\mathcal{F})$ is a graded R -module. Note that there is a natural graded homomorphism of rings $\alpha : R \rightarrow \Gamma_*(\mathcal{O}_X)$ such that for any element $s = (s_n)_{n \geq 0} \in R$ it associates to each s_n the constant global section in $\Gamma(X, \mathcal{O}_X(n))$ given by $s_n \mapsto s_n/1 \in S(n)_{(\mathfrak{p})}$, this gives $\Gamma_*(\mathcal{F})$ a structure of graded R -module.

Let $R = R_0[x_0, \dots, x_r]$ be the polynomial ring of $r+1$ variables over R_0 and $X = \text{Proj } R$. Let $\alpha : R \rightarrow \Gamma_*(\mathcal{O}_X)$ be the graded morphism of remark 4.16, it is in fact an isomorphism, It is sufficient to show each induced group homomorphism $\alpha_n : R_n \rightarrow \Gamma(X, \mathcal{O}_X(n))$ is bijective, see [Hartshorne \(1977, II, Proposition 5.13\)](#). Then

$$R \cong \Gamma_*(\mathcal{O}_X) \tag{4.1}$$

therefore we obtain

$$\Gamma(\mathbb{P}_{R_0}^r, \mathcal{O}_X(n)) \cong \begin{cases} 0 & \text{if } n < 0 \\ R_n & \text{if } n \geq 0 \end{cases}$$

Definition 4.17. A **standard positively graded ring** is a positively graded ring $R = \bigoplus_{n \geq 0} R_n$ finitely generated by R_1 as a R_0 -algebra, that is $R = R_0[a_0, \dots, a_r]$ for some $a_i \in R_1$.

Let R be a standard positively graded ring. Given a graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ we have a natural homomorphism of R_0 -modules $M_n \rightarrow \Gamma(X, \tilde{M}(n))$ since every element $s \in M_n$ defines a global section $s/1 \in \Gamma(X, \tilde{M}(n))$. This gives a map between graded rings

$$M \xrightarrow{\zeta(M)} \Gamma_*(\tilde{M})$$

which is naturally a graded homomorphism of R -modules. Conversely given a quasi-coherent sheaf \mathcal{F} over X there is a morphism of sheaves $\xi : \Gamma_*(\mathcal{F})^\sim \rightarrow \mathcal{F}$. Let $f \in R_n$, then $\xi_{D_+(f)}$ looks like

$$\begin{aligned} \Gamma_*(\mathcal{F})^\sim(D_+(f)) &\cong \Gamma_*(\mathcal{F})_{(f)} \xrightarrow{\xi_{D_+(f)}} \mathcal{F}(D_+(f)) \\ \frac{x}{f^d} &\longmapsto \frac{x|_{D_+(f)}}{\alpha_{dn}(f^d)|_{D_+(f)}} \end{aligned}$$

where $x \in \Gamma(X, \mathcal{F}(dn))$ and $\alpha_{dn} : R_{dn} \rightarrow \Gamma(X, \mathcal{O}_X(dn))$ is the part of degree dn of the homomorphism $\alpha : R \rightarrow \Gamma_*(\mathcal{F}_X)$ in remark 4.16, note that this is an invertible element in $\mathcal{O}_X(dn)$. In fact this morphism is an isomorphism as we state in the following proposition.

Proposition 4.18. (*HARTSHORNE, 1977, II, Proposition 5.15*). *Let R be a standard positively graded ring. Let \mathcal{F} be a quasi-coherent sheaf on $X = \text{Proj } R$, then there is a natural isomorphism $\Gamma_*(\mathcal{F})^\sim \rightarrow \mathcal{F}$.*

4.3 Quasi-coherent sheaves and closed subschemes

Definition 4.19. A **sheaf of ideals** on a scheme X is a sheaf \mathcal{I} such that $\mathcal{I}(U)$ is an ideal of the ring $\mathcal{O}_X(U)$, in this case note that \mathcal{I} is a subsheaf of \mathcal{O}_X .

Definition 4.20. Let Z be a closed subscheme of X and $i : Z \rightarrow X$ the inclusion map, we define the **ideal sheaf** of Z to be

$$\mathcal{I}_Z := \ker i^\sharp : \mathcal{O}_X \rightarrow i_* \mathcal{O}_Z$$

Let Z be a closed subscheme of X with $i : Z \hookrightarrow X$ the inclusion map, by Hartshorne (1977, II, Proposition 5.8) the sheaf $i_* \mathcal{O}_Z$ is quasi-coherent as well as the sheaf of ideals \mathcal{I}_Z . In the case where X is a noetherian scheme, for every open affine set $U = \text{Spec } A$ we have $\mathcal{I}_Z(U) \subseteq \mathcal{O}_X(U) \cong A$, since A is noetherian we have that $\mathcal{I}_Z(U)$ is finitely generated and therefore \mathcal{I}_Z is coherent. Conversely, suppose \mathcal{I} is a quasi-coherent sheaf of ideals on a scheme X . Let $Z = \text{Supp}(\mathcal{O}_X/\mathcal{I}) := \{\mathfrak{p} \in X \mid \mathcal{O}_{X,\mathfrak{p}}/\mathcal{I}_{\mathfrak{p}} \neq 0\}$ the support of $\mathcal{O}_X/\mathcal{I}$. We claim that Z is a closed subset of X , to see this let $U = \text{Spec } A \subseteq X$ and suppose $\mathcal{I}|_U \cong \tilde{I}$ where $I \subseteq A$ is an ideal, then $(\mathcal{O}_X/\mathcal{I})|_U \cong (A/I)^\sim$ and $\text{Supp}(\mathcal{O}_X/\mathcal{I}) \cap U \cong \text{Supp}(A/I)^\sim = V(\text{Ann}_A(A/I))$ ³, since

³ Here $\text{Ann}_A(M) := \{a \in A \mid am = 0 \text{ for all } m \in M\}$ for every A -module M .

being closed is a local property we conclude that $\text{Supp}(\mathcal{O}_X/\mathcal{I})$ is closed. Now we claim that $(Y, \mathcal{O}_X/\mathcal{I})$ is a unique closed subscheme of X with ideal sheaf \mathcal{I} . We need to check that

$$i^\sharp : \mathcal{O}_X \rightarrow i_*(\mathcal{O}_X/\mathcal{I})$$

is surjective and that $\ker i^\sharp \cong \mathcal{I}$, both things can be checked at the level of stalks. Let $\mathfrak{p} \in X$ and choose an affine neighborhood $U = \text{Spec } A$ of \mathfrak{p} , here the induced map on the stalks is $i_{\mathfrak{p}}^\sharp : A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/I_{\mathfrak{p}}$ for some ideal $I \subseteq A$ such that $\mathcal{I}|_U \cong \tilde{I}$, which is surjective and $\ker i_{\mathfrak{p}}^\sharp \cong \mathcal{I}_{\mathfrak{p}}$, we have proven the following proposition, see [Hartshorne \(1977, II, Proposition 5.9\)](#).

Proposition 4.21. *Let X be a scheme. For every closed subscheme Z of X the corresponding ideal sheaf \mathcal{I}_Z is a quasi-coherent sheaf of ideals on X . If X is noetherian it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X .*

If $X = \text{Spec } A$ then the previous proposition establishes a one to one correspondence between the ideals of A and closed subschemes of X . Given an ideal $I \subseteq A$ the correspondent closed subscheme is $(\text{Spec}(A/I), \mathcal{O}_{\text{Spec}(A/I)})$, note that $\mathcal{O}_X/\tilde{I} \cong \mathcal{O}_{\text{Spec}(A/I)}$, conversely, given a closed subscheme (Y, \mathcal{O}_Y) of X , since \mathcal{I}_Y is quasi-coherent, there is an ideal $I \subseteq A$ such that $\tilde{I} \cong \mathcal{I}_Y$, note that $\mathcal{O}_X/\tilde{I} \cong i_*\mathcal{O}_Y$.

Let A be a ring and Z a closed subscheme of $X = \mathbb{P}_A^r$, the projective r -space see example [3.27](#). We claim that there exists a homogeneous ideal $I \subseteq R = A[X_0, \dots, X_r]$ such that $\mathcal{I}_Z \cong \tilde{I}$. We know that $\Gamma_*(\mathcal{I}_Z)$ is a graded submodule of $\Gamma_*(\mathcal{O}_X)$ and $\Gamma_*(\mathcal{O}_X) \cong R$ by equation [4.1](#). This implies that $\Gamma_*(\mathcal{I}_Z)$ is a homogeneous ideal of R , we know then that it defines a closed subscheme of X see example [3.29](#), by proposition [4.18](#) we have $\Gamma_*(\mathcal{I}_Z) \cong \mathcal{I}_Z$ and by proposition [4.21](#) this two sheaves define the same closed subscheme, thus $I := \Gamma_*(\mathcal{I}_Z)$ is the desired ideal.

COHOMOLOGY

In this chapter we introduce sheaf cohomology and local cohomology and prove the classical form of the Serre-Grothendieck correspondence, which establishes an isomorphism between the local cohomology groups of a module and the sheaf cohomology groups of the sheaf associated to this module. This allows us to prove another basic result in the theory of quasi-coherent sheaf over projective schemes, the Serre finiteness theorem, it says that the cohomology groups of the twisted sheaf of a quasi-coherent sheaf are finitely generated and eventually vanish. With this, we discuss some open problems that arise from the study of the regularity and the Hilbert function of a sheaf.

5.1 Cohomology of sheaves

Let (X, \mathcal{O}_X) be a ringed space. For any morphism $\theta : \mathcal{F} \rightarrow \mathcal{G}$ of \mathcal{O}_X -modules we have a morphism between the global section modules

$$\Gamma(X, \theta) : \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}).$$

$\Gamma(X, -)$ is a covariant functor from the category of \mathcal{O}_X -modules to the category of $\Gamma(X, \mathcal{O}_X)$ -modules, in fact it is a left exact functor as we show in the following proposition.

Proposition 5.1. *Let X be a topological space and $0 \rightarrow \mathcal{F}' \xrightarrow{\theta} \mathcal{F} \xrightarrow{\eta} \mathcal{F}''$ an exact sequence of sheaves on X . For any open set $U \subseteq X$ the sequence*

$$0 \rightarrow \mathcal{F}'(U) \xrightarrow{\theta_U} \mathcal{F}(U) \xrightarrow{\eta_U} \mathcal{F}''(U)$$

is exact.

Proof. Assume $0 \rightarrow \mathcal{F}' \xrightarrow{\theta} \mathcal{F} \xrightarrow{\eta} \mathcal{F}''$ is exact and fix an open set $U \subseteq X$. The morphism θ_U is injective since θ is injective. Now we prove $\text{im } \theta_U = \ker \eta_U$. We know that $0 \rightarrow \mathcal{F}'_P \xrightarrow{\theta_P} \mathcal{F}_P \xrightarrow{\eta_P}$

\mathcal{F}_P'' is exact for every $P \in X$. Let $s \in \mathcal{F}'(U)$, note that $(\eta_U(\theta_U(s)))|_P = \eta_P(\theta_P(s_P)) = 0$ for every $P \in U$, this implies that $\eta_U(\theta_U(s)) = 0$, so $\text{im } \theta_U \subseteq \ker \eta_U$. For the remainder inclusion let $r \in \ker \eta_U$, thus $\eta_U(r) = 0$ and $\eta_P(r_P) = (\eta_U(r))|_P = 0$ for every $P \in U$, since the sequence on the stalks is exact there exists $s_P \in \mathcal{F}'_P$ such that $\theta_P(s_P) = r_P$ for every $P \in U$, thus, there is an open neighborhood W_P of P such that $\theta_{W_P}(s(P)) = r|_{W_P}$ for some $s(P) \in \mathcal{F}'(W_P)$. Write W_{PQ} for $W_P \cap W_Q$ and note that $\theta_{W_{PQ}}(s(P)|_{W_{PQ}}) = r|_{W_{PQ}} = \theta_{W_{PQ}}(s(Q)|_{W_{PQ}})$, by injectivity of θ we have $s(P)|_{W_{PQ}} = s(Q)|_{W_{PQ}}$ and since the sets W_P cover U there exists $s \in \mathcal{F}'(U)$ such that $s|_{W_P} = s(P)$. Note also that $\theta_U(s) = r$ since $(\theta_U(s))|_{W_P} = \theta_{W_P}(s(P)) = r|_{W_P}$ for every $P \in U$, therefore $r \in \text{im } (\theta_U)$ and $\ker \eta_U \subseteq \text{im } \theta_U$. \square

Definition 5.2. Let (X, \mathcal{O}_X) be a ringed space and \mathcal{G} be a sheaf of \mathcal{O}_X -modules. A sheaf \mathcal{I} of \mathcal{O}_X -modules is called **injective** if for any exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

the sequence

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \longrightarrow 0$$

is exact. In other words, the functor $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact.

Equivalently, \mathcal{I} is injective if and only if for any pair of sheaves \mathcal{F}, \mathcal{G} , any morphism $h : \mathcal{F} \rightarrow \mathcal{I}$ and monomorphism $f : \mathcal{F} \rightarrow \mathcal{G}$, there exists a morphism $g : \mathcal{G} \rightarrow \mathcal{I}$ such that $h = g \circ f$, that is, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ & & & \searrow h & \downarrow g \\ & & & & \mathcal{I} \end{array}$$

commutes.

Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. We will construct an injective sheaf \mathcal{I} and an injective morphism of sheaves $\mathcal{F} \rightarrow \mathcal{I}$. For each $x \in X$, \mathcal{F}_x is an $\mathcal{O}_{X,x}$ -module, so there is an injective $\mathcal{O}_{X,x}$ -module \mathcal{I}_x and a monomorphism $\mathcal{F}_x \rightarrow \mathcal{I}_x$ (ROTMAN, 2009, Theorem 3.38). For each $x \in X$ consider the topological space $\{x\}$ consisting of one element and the discrete topology. Let $j : \{x\} \hookrightarrow X$ be the inclusion map and consider the constant sheaf defined on $\{x\}$ by \mathcal{I}_x (see Example 2.6). The direct image $j_*\mathcal{I}_x$ ¹ has a natural \mathcal{O}_X -module structure, in fact, for any open subset $U \subseteq X$ containing x and any $r \in \mathcal{O}_X(U)$, $s \in j_*\mathcal{I}_x(U) \cong \mathcal{I}_x$, we have $r \cdot s = r_x \cdot s$. Define the \mathcal{O}_X -module $\mathcal{I} = \prod_{x \in X} j_*(\mathcal{I}_x)$ by

$$\mathcal{I}(U) := \prod_{x \in X} j_*(\mathcal{I}_x)(U) = \prod_{x \in U} \mathcal{I}_x$$

¹ $(j_*\mathcal{I}_x)(U)$ is \mathcal{I}_x if $x \in U$ and 0 otherwise.

with the natural projection as the restriction map. It is straightforward that \mathcal{I} is a sheaf and it has a natural structure of \mathcal{O}_X -module defined by $r \cdot s = r \cdot (s_x)_{x \in U} := (r_x \cdot s_x)_{x \in U}$, also we have that the stalks satisfy $(\mathcal{I})_x = \mathcal{I}_x$.

It follows easily that for any \mathcal{O}_X -module \mathcal{G} there is a canonical isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \cong \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_*(\mathcal{I}_x))$$

moreover, there is an isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_*(\mathcal{I}_x)) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{I}_x)$$

for every $x \in X$, therefore the injective homomorphisms of modules $\mathcal{F}_x \rightarrow \mathcal{I}_x$ give rise to a morphism of \mathcal{O}_X -modules $\mathcal{F} \rightarrow \mathcal{I}$, which by proposition 2.30 is also injective.

Note that \mathcal{I} is an injective \mathcal{O}_X -module. In fact, let $\mathcal{G}' \rightarrow \mathcal{G}$ be an injective morphism of \mathcal{O}_X -modules, we then have an injective morphism of $\mathcal{O}_{X,x}$ -modules $\mathcal{G}'_x \rightarrow \mathcal{G}_x$ for every $x \in X$, since \mathcal{I}_x is injective we have a surjective morphism

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{I}_x) \rightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}'_x, \mathcal{I}_x).$$

Taking direct product over all points of X we get a surjection

$$\prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{I}_x) \rightarrow \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}'_x, \mathcal{I}_x)$$

and by the isomorphism discussed above we obtain a surjective map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \rightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}', \mathcal{I})$$

and therefore \mathcal{I} is an injective \mathcal{O}_X -module. As a consequence we obtain the following proposition.

Proposition 5.3. (*HARTSHORNE, 1977, III, Proposition 2.2*). *If (X, \mathcal{O}_X) is a ringed space and \mathcal{F} a sheaf of \mathcal{O}_X -modules, then there exists an injective \mathcal{O}_X -module \mathcal{I} and an injective morphism of sheaves $\mathcal{F} \rightarrow \mathcal{I}$. The sheaf \mathcal{I} is said to be an **injective extension** of \mathcal{F} .*

Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules and $\varepsilon_0 : \mathcal{F} \rightarrow \mathcal{I}_0$ an injective extension. Identifying \mathcal{F} with $\mathrm{im} \varepsilon_0$ define the sheaf $\mathcal{F}_1 := \mathcal{I}_0 / \mathcal{F}$. The sheaf \mathcal{F}_1 has a injective extension $\varepsilon_1 : \mathcal{F}_1 \rightarrow \mathcal{I}_1$. Similarly we may form the quotient $\mathcal{F}_2 := \mathcal{I}_1 / \mathcal{F}_1$ together with its injective extension $\varepsilon_2 : \mathcal{F}_2 \rightarrow \mathcal{I}_2$. Repeating this process we obtain an exact sequence

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{\varepsilon_n} \mathcal{I}_n \xrightarrow{\pi_n} \mathcal{F}_{n+1} \longrightarrow 0$$

for every $n \geq 0$ where π_n is the natural projection.

If we define the map $\alpha_n := \varepsilon_{n+1} \circ \pi_n : \mathcal{I}_n \rightarrow \mathcal{I}_{n+1}$ for all $n \geq 0$, we obtain the sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F} & \xrightarrow{\varepsilon_0} & \mathcal{I}_0 & \xrightarrow{\alpha_0} & \mathcal{I}_1 & \xrightarrow{\alpha_1} & \mathcal{I}_2 & \xrightarrow{\alpha_2} & \dots \\
& & & & \searrow \pi_0 & & \nearrow \varepsilon_1 & & \searrow \pi_1 & & \nearrow \varepsilon_2 \\
& & & & & & \mathcal{F}_1 & & & & \mathcal{F}_2
\end{array}$$

which is exact since $\ker \alpha_{n+1} = \ker \pi_{n+1} = \text{im } \varepsilon_{n+1} = \text{im } \alpha_n$. The exact sequence

$$\mathbf{I}^\bullet : 0 \rightarrow \mathcal{F} \xrightarrow{\varepsilon_0} \mathcal{I}_0 \xrightarrow{\alpha_0} \mathcal{I}_1 \xrightarrow{\alpha_1} \mathcal{I}_2 \xrightarrow{\alpha_2} \dots \quad (5.1)$$

is said to be an **injective resolution** of \mathcal{F} . We have shown that every \mathcal{O}_X -module has an injective resolution.

Now, let \mathcal{F} be a sheaf of \mathcal{O}_X -modules on a ringed space (X, \mathcal{O}_X) and \mathbf{I}^\bullet an injective resolution of \mathcal{F} . Apply the functor $\Gamma(X, \cdot)$ to $\mathbf{I}^\bullet_{\mathcal{F}} := 0 \rightarrow \mathcal{I}_0 \xrightarrow{\alpha_0} \mathcal{I}_1 \xrightarrow{\alpha_1} \mathcal{I}_2 \xrightarrow{\alpha_2} \dots$ to obtain

$$\Gamma(X, \mathbf{I}^\bullet_{\mathcal{F}}) : 0 \longrightarrow \Gamma(X, \mathcal{I}_0) \xrightarrow{\Gamma(X, \alpha_0)} \Gamma(X, \mathcal{I}_1) \xrightarrow{\Gamma(X, \alpha_1)} \dots$$

This sequence is no longer exact but it satisfies $\text{im } \Gamma(X, \alpha_{n-1}) \subseteq \ker \Gamma(X, \alpha_n)$, this is what we call a **complex**². Define the **sheaf cohomology group** of \mathcal{F} to be the $\Gamma(X, \mathcal{O}_X)$ -modules

$$H^n(X, \mathcal{F}) := \frac{\ker \Gamma(X, \alpha_n)}{\text{im } \Gamma(X, \alpha_{n-1})}$$

and for $n = -1$ we assume $\Gamma(X, \alpha_{-1}) := 0 \rightarrow \Gamma(X, \mathcal{I}_0)$ is the trivial map. The following theorem is a basic result in cohomology theory over an abelian category, it will be used in later proofs, for a slightly more general statement where the set X is replaced by any open set $U \subseteq X$, we refer the reader to [Itaka \(1982, Theorem 4.1\)](#).

Theorem 5.4. (a) *The groups $H^n(X, \mathcal{F})$ do not depend on the choice of the injective resolution of \mathcal{F} .*

(b) *Let $\theta : \mathcal{F} \rightarrow \mathcal{G}$ and be a morphism of sheaves, then, for all $n \geq 0$, there is a well define natural homomorphism $H^n(X, \theta) : H^n(X, \mathcal{F}) \rightarrow H^n(X, \mathcal{G})$ such that $H^n(X, \text{id}) = \text{id}$ and for any morphism $\eta : \mathcal{G} \rightarrow \mathcal{H}$ of sheaves, $H^n(X, \eta \circ \theta) = H^n(X, \eta) \circ H^n(X, \theta)$.*

(c) *For any short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\theta} \mathcal{G} \xrightarrow{\eta} \mathcal{H} \longrightarrow 0$$

*there are associated **connecting morphisms** $\delta^n : H^n(X, \mathcal{H}) \rightarrow H^{n+1}(X, \mathcal{F})$ for every $n \geq 0$ such that*

² A complex in an abelian category is a collection of objects A^i and morphisms $d^i : A^i \rightarrow A^{i+1}$, $i \in \mathbb{Z}$, such that $d^{i+1} \circ d^i = 0$.

$$\begin{aligned}
0 &\longrightarrow H^0(X, \mathcal{F}) \xrightarrow{H^0(X, \theta)} H^0(X, \mathcal{G}) \xrightarrow{H^0(X, \eta)} H^0(X, \mathcal{H}) \xrightarrow{\delta^0} H^1(X, \mathcal{F}) \longrightarrow \dots \\
\dots &\xrightarrow{\delta^{n-1}} H^n(X, \mathcal{F}) \xrightarrow{H^n(X, \theta)} H^n(X, \mathcal{G}) \xrightarrow{H^n(X, \eta)} H^n(X, \mathcal{H}) \xrightarrow{\delta^n} \dots
\end{aligned}$$

is a long exact sequence.

Corollary 5.5. For every sheaf \mathcal{F} on (X, \mathcal{O}_X) we have $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$. If \mathcal{F} is an injective sheaf then $H^n(X, \mathcal{F}) = 0$ for every $n > 0$.

Proof. Let I^\bullet be an injective resolution of \mathcal{F} as in equation 5.1. We have $H^0(X, \mathcal{F}) \cong \ker \Gamma(X, \alpha_0) \cong \ker \Gamma(X, \pi_0) \cong \Gamma(X, \mathcal{F})$. Suppose \mathcal{F} is injective, since the constant sheaf $\{0\}$ is injective the resolution

$$0 \longrightarrow \mathcal{F} \xrightarrow{id} \mathcal{F} \longrightarrow 0 \longrightarrow \dots$$

is also injective, hence $H^n(X, \mathcal{F}) = 0$ for every $n > 0$. \square

Definition 5.6. Let X be a topological space. A sheaf \mathcal{F} on X is said to be **flasque** if for every pair of open subsets $U, V \subseteq X$ such that $U \subseteq V$ the restriction map $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is surjective.

Example 5.7. The constant sheaf of example 2.6 is flasque.

Example 5.8. Let $X = \mathbb{C}$ with the usual euclidean topology and let $\underline{\mathbb{Z}}$ be the constant sheaf of example 2.6 defined by \mathbb{Z} on X . Consider the sequence of sheaves

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \cdot} \mathcal{O}_X^{\text{hol}} \xrightarrow{\exp(\cdot)} \mathcal{O}_X^{\text{hol}*} \rightarrow 1$$

where 0 is the constant sheaf on X defined by the additive group $\{0\}$ and 1 is the constant sheaf on X defined by the multiplicative group $\{1\}$. The first map is just multiplication by $2\pi i$ and the second map is composition with the complex exponential map. It is easy to see that this is an exact sequence of sheaves³. Thus the morphism $\mathcal{O}_X^{\text{hol}} \xrightarrow{\exp(\cdot)} \mathcal{O}_X^{\text{hol}*}$ is surjective but it fails to be flasque since, for $g(z) = z$ defined in $U = \mathbb{C} \setminus \{0\}$, we have $g \in \mathcal{O}_X^{\text{hol}*}(U)$ but there is no holomorphic function satisfying $\exp(f(z)) = z$ on U , therefore \exp_U is not surjective.

Remark 5.9. If $0 \rightarrow \mathcal{F}' \xrightarrow{\theta} \mathcal{F} \xrightarrow{\eta} \mathcal{F}'' \rightarrow 0$ is exact and \mathcal{F}' is flasque then $0 \rightarrow \mathcal{F}'(U) \xrightarrow{\theta_U} \mathcal{F}(U) \xrightarrow{\eta_U} \mathcal{F}''(U) \rightarrow 0$ is exact for every open set $U \subseteq X$.

By proposition 5.1 is sufficient to prove that $\mathcal{F}(U) \xrightarrow{\eta_U} \mathcal{F}''(U)$ is surjective. Let $U \subseteq X$ be an open set and $t \in \mathcal{F}''(U)$. Since η is surjective, for every $x \in U$ there is an open neighborhood

³ The first map is injective since $2\pi i n = 0$ implies $n = 0$. Since $\exp(f) = 1$ if and only if $f = 2\pi i \mathbb{Z}$, we conclude that $\ker \exp(\cdot) = \text{Im } 2\pi i \cdot$. Now, to see $\exp(\cdot)$ is surjective, let $P \in X$ be any point and consider $g_P \in \mathcal{O}_{X,P}^{\text{hol}*}$. There exists an open neighborhood U of P such that $g|_U \neq 0$. In this case the complex logarithm of g is well defined in U and the function $f = \frac{1}{2\pi i} \log g$ is a well defined holomorphic function such that $\exp_P(f_P) = g_P$, since this is the case for any P we conclude that $\exp(\cdot)$ is surjective.

$W \subseteq U$ of x such that $t|_W = \eta|_W(s)$ for some $s \in \mathcal{F}(W)$. Therefore we can cover U by open sets $\{U_i\}_{i \in I}$ with $U_i \subseteq U$, and elements $s_i \in \mathcal{F}(U_i)$ such that $t|_{U_i} = \eta_{U_i}(s_i)$. Let $\Sigma = \{(\bigcup_{i \in J} U_i, s) : J \subseteq I, s \in \mathcal{F}(\bigcup_{i \in J} U_i), \eta_{\bigcup_{i \in J} U_i}(s) = t|_{\bigcup_{i \in J} U_i}\}$. Consider the partial order on Σ given by $(\bigcup_{i \in J_1} U_i, s_1) \geq (\bigcup_{i \in J_2} U_i, s_2)$ if and only if $\bigcup_{i \in J_1} U_i \subseteq \bigcup_{i \in J_2} U_i$ and $s_2|_{\bigcup_{i \in J_1} U_i} = s_1$. It is easy to verify Σ is closed under increasing chains. By Zorn's Lemma, there is a maximal element $(\bigcup_{i \in J} U_i, s) \in \Sigma$. Denote $W = \bigcup_{i \in J} U_i$, we claim $W = U$. Suppose on the contrary that there exists $k \in I$ such that $U_k \not\subseteq W$. Note that $\eta_W(s) = t|_W$ and $\eta_{U_k}(s_k) = t|_{U_k}$ implies that $\eta|_{W \cap U_k}(s|_{W \cap U_k} - s_k|_{W \cap U_k}) = 0$ and by exactness of $\theta_{W \cap U_k}, s|_{W \cap U_k} - s_k|_{W \cap U_k} \in \text{Im } \theta_{W \cap U_k}$. That is, there exists $r \in \mathcal{F}'(W \cap U_k)$ such that $\theta_{W \cap U_k}(r) = s|_{W \cap U_k} - s_k|_{W \cap U_k}$. Since \mathcal{F}' is flasque, there exists $r' \in \mathcal{F}'(W \cup U_k)$ such that $r'|_{W \cap U_k} = r$. Now consider the cover W and U_k of $W \cup U_k$, note that $(s - \theta_W(r'|_W))|_{W \cap U_k} = s_k|_{W \cap U_k}$, so there exists $z \in \mathcal{F}(W \cup U_k)$ such that $z|_W = s - \theta_W(r'|_W)$ and $z|_{U_k} = s_k$. Note that $\eta_W(z|_W) = \eta_W(s) = t|_W$ and $\eta_{U_k}(z|_{U_k}) = t|_{U_k}$, then $\eta_{W \cup U_k}(z) = t|_{W \cup U_k}$. Then, we have $\eta_{W \cup U_k}(z + \theta_{W \cup U_k}(r')) = \eta_{W \cup U_k}(z) = t|_{W \cup U_k}$ and $(z + \theta_{W \cup U_k}(r'))|_W = s$. This implies $(W \cup U_k, z) \in \Sigma$, which contradicts the maximality assumption. Hence $W = U$ and $\eta_U(s) = t$ as required.

We will use flasque sheaves instead of injective sheaves to calculate the cohomology groups. Actually, every injective sheaf \mathcal{I} is flasque (HARTSHORNE, 1977, III, Lemma 2.4). Flasque sheaves are also **acyclic** with respect to the sheaf cohomology groups, that is, every flasque sheaf \mathcal{F} satisfy $H^n(X, \mathcal{F}) = 0$ for every $n \geq 1$ as we show in the following proposition.

Proposition 5.10. *If \mathcal{F} is flasque then $H^n(X, \mathcal{F}) = 0$ for all $n \geq 1$.*

Proof. Consider the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0$$

where \mathcal{I} is injective and $\mathcal{G} = \mathcal{I}/\mathcal{F}$. Since \mathcal{I} is also flasque we have \mathcal{G} is flasque (see Hartshorne (1977, II, Exercise 1.16 (c))). By remark 5.9 the sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}) \rightarrow \Gamma(X, \mathcal{G}) \rightarrow 0$$

or, equivalently, the sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{I}) \rightarrow H^0(X, \mathcal{G}) \rightarrow 0$$

is exact. By corollary 5.5 $H^n(X, \mathcal{I}) = 0$ for $n > 0$, and by theorem 5.4 item (c) we have the exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{I}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{F}) \rightarrow 0$$

where the third homomorphism is surjective, this implies that the fourth homomorphism $H^0(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{F})$ is the zero map, but it is also surjective, hence $H^1(X, \mathcal{F}) = 0$. For $n > 1$ we have the exact sequence

$$0 \rightarrow H^{n-1}(X, \mathcal{G}) \rightarrow H^n(X, \mathcal{F}) \rightarrow 0$$

thus, $H^{n-1}(X, \mathcal{G}) \cong H^n(X, \mathcal{F})$. If we argue by induction, we have shown that $H^1(X, \mathcal{F}) = 0$. Suppose the result is valid for $n - 1$ and every flasque sheaf, since \mathcal{G} is flasque $0 = H^{n-1}(X, \mathcal{G})$ so $H^n(X, \mathcal{F}) \cong 0$ as required. \square

Furthermore, for any exact sequence of \mathcal{O}_X -modules

$$\mathbf{J}^\bullet : 0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{G}_0 \xrightarrow{\beta_0} \mathcal{G}_1 \xrightarrow{\beta_1} \mathcal{G}_2 \xrightarrow{\beta_2} \dots$$

which is acyclic with respect to the sheaf cohomology⁴, there is an isomorphism

$$H^n(X, \mathcal{F}) \cong \frac{\ker \Gamma(X, \beta_n)}{\operatorname{im} \Gamma(X, \beta_{n-1})}$$

where $\beta_{-1} : 0 \rightarrow \mathcal{G}_0$ is the trivial map. To see this first note that $0 \rightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{G}_0 \xrightarrow{\beta_0} \mathcal{G}_1$ is exact, by proposition 5.1 the sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\Gamma(X, \varepsilon)} \Gamma(X, \mathcal{G}_0) \xrightarrow{\Gamma(X, \beta_0)} \Gamma(X, \mathcal{G}_1)$$

is exact, thus $\ker \Gamma(X, \beta_0) = \operatorname{im} \Gamma(X, \varepsilon) = \Gamma(X, \mathcal{F}) \cong H^0(X, \mathcal{F})$. For $n > 1$, consider the following exact sequences

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{G}_0 \xrightarrow{\beta_0} \operatorname{im} \beta_0 \longrightarrow 0 \quad (5.2)$$

$$0 \longrightarrow \operatorname{im} \beta_0 \xrightarrow{i} \mathcal{G}_1 \xrightarrow{\beta_1} \mathcal{G}_2 \xrightarrow{\beta_2} \dots \quad (5.3)$$

Since $H^n(X, \mathcal{G}_0) = 0$ for every $n > 1$, we apply theorem 5.4 item (c) to the sequence in equation 5.2, we obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\Gamma(X, \varepsilon)} \Gamma(X, \mathcal{G}_0) \xrightarrow{\Gamma(X, \beta_0)} \Gamma(X, \operatorname{im} \beta_0) \xrightarrow{\delta^0} H^1(X, \mathcal{F}) \longrightarrow 0 \longrightarrow \dots \\ \dots \longrightarrow 0 \longrightarrow H^n(X, \operatorname{im} \beta_0) \longrightarrow H^{n+1}(X, \mathcal{F}) \longrightarrow 0 \longrightarrow \dots \end{aligned}$$

From this, and using the exactness of equation 5.3, we deduce

$$H^1(X, \mathcal{F}) \cong \frac{\Gamma(X, \operatorname{im} \beta_0)}{\ker \delta^0} \cong \frac{\Gamma(X, \operatorname{im} \beta_0)}{\operatorname{im} \Gamma(X, \beta_0)} \cong \frac{\ker \Gamma(X, \beta_1)}{\operatorname{im} \Gamma(X, \beta_0)}$$

which solves for $n = 1$. So far we have proven the result to be true in the case $n = 1$ for any sheaf of \mathcal{O}_X -modules. Lets assume by induction that the result is valid for some fixed $n > 1$ and any sheaf of \mathcal{O}_X -modules and lets prove the case $n + 1$. For $n > 1$ we have

$$H^n(X, \operatorname{im} \beta^0) \cong H^{n+1}(X, \mathcal{F}).$$

⁴ \mathbf{J} is called **acyclic** with respect to the sheaf cohomology if and only if $H^n(X, \mathcal{G}_i) = 0$ for all $n \geq 1$ and every $i \geq 0$.

If we apply the induction hypotheses to $\text{im } \beta_0$ then from equation 5.3 we obtain

$$\frac{\ker \Gamma(X, \beta_{n+1})}{\text{im } \Gamma(X, \beta_n)} \cong H^n(X, \text{im } \beta_0),$$

thus

$$H^{n+1}(X, \mathcal{F}) \cong \frac{\ker \Gamma(X, \beta_{n+1})}{\text{im } \Gamma(X, \beta_n)}.$$

So, we have proven the following proposition.

Proposition 5.11. (*IITAKA, 1982, Proposition 4.3*). *Let (X, \mathcal{O}_X) be a ringed space and*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{G}^0 \xrightarrow{\beta^0} \mathcal{G}^1 \xrightarrow{\beta^1} \mathcal{G}^2 \xrightarrow{\beta^2} \dots$$

an exact sequence of \mathcal{O}_X -modules such that $H^n(X, \mathcal{G}^i) = 0$ for all $n > 0$ and every $i \geq 0$ (for example flasque sheaves), then

$$H^n(X, \mathcal{F}) \cong \frac{\ker \beta_X^n}{\text{Im } \beta_X^{n-1}}$$

where $\beta^{-1} : 0 \rightarrow \mathcal{G}^0$.

Now we calculate the sheaf cohomology groups for a quasi-coherent sheaf in the case where $X = \text{Spec } A$ is an affine scheme and A is a noetherian ring.

Theorem 5.12. *Let $X = \text{Spec } A$ with A noetherian. Then for every quasi-coherent sheaf \mathcal{F} and for every $i > 0$, $H^i(X, \mathcal{F}) = 0$.*

Proof. Let $M = \Gamma(X, \mathcal{F})$ then $\mathcal{F} \cong \tilde{M}$ (see corollary 4.7). Consider $0 \rightarrow M \rightarrow I^\bullet$ an injective resolution of M as A -module. The association $M \rightarrow \tilde{M}$ is exact by proposition 4.5, so the sequence $0 \rightarrow \tilde{M} \rightarrow I^\bullet$ is exact. Since A is a noetherian ring, by Hartshorne (1977, III, Proposition 3.4), each \tilde{I}^i is flasque and therefore we can use them to calculate the cohomology of \mathcal{F} (see proposition 5.11). Applying $\Gamma(X, -)$ we obtain the original injective resolution $0 \rightarrow M \rightarrow I^\bullet$, then calculating its cohomology we get $H^0(X, \mathcal{F}) = M$ and $H^i(X, \mathcal{F}) = 0$ for $i > 0$. \square

5.2 Local cohomology

In this section we introduce the concept of local cohomology of a given module with respect to some ideal. To construct this cohomology groups we will use an injective resolution of the module M and then we will apply a left exact functor. This is done in order to establish a connection between cohomology of sheaves and local cohomology of modules so that we can write properties of sheaves in algebraic terms. This cohomology groups will be closely related to the sheaf cohomology groups of the sheaf \tilde{M} defined by M . Lets start with the following definition. For more details on the theory of local cohomology of modules see for example (BRODMANN; SHARP, 1998) and (IYENGAR *et al.*, 2007).

Definition 5.13. Let A be a ring and M an A -module, for any ideal $\mathfrak{a} \subseteq A$ we define the \mathfrak{a} -torsion module $\Gamma_{\mathfrak{a}}(M)$ by

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M : \mathfrak{a}^n m = 0 \text{ for some } n \geq 0\}.$$

The module $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M . The following notation will be useful in stating a few properties of this module, let N be a submodule of M , define the set

$$(N :_M \mathfrak{a}) := \{m \in M \mid \mathfrak{a}m \subseteq N\},$$

which is also a submodule of M . It is easy to verify that

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n).$$

Remark 5.14. Let $\mathfrak{a} \subseteq A$ be an ideal, we say that an A -module M is \mathfrak{a} -torsion if $\Gamma_{\mathfrak{a}}(M) = M$. If $\Gamma_{\mathfrak{a}}(M) = 0$ we say M is \mathfrak{a} -torsion free. Since $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$ we have that $\Gamma_{\mathfrak{a}}(M)$ is \mathfrak{a} -torsion.

In the language of category theory the next result tells us that $\Gamma_{\mathfrak{a}}(-)$ is a left exact additive covariant functor from the category of A -modules to itself, called \mathfrak{a} -torsion functor, or, **Gamma functor**.

Proposition 5.15. For every morphism of A -modules $f : M \rightarrow N$ there exists a morphism $\Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) \rightarrow \Gamma_{\mathfrak{a}}(N)$ of A -modules such that for any exact sequence of A -modules $0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} P$ the sequence $0 \rightarrow \Gamma_{\mathfrak{a}}(N) \xrightarrow{\Gamma_{\mathfrak{a}}(f)} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(g)} \Gamma_{\mathfrak{a}}(P)$ is exact.

Proof. Let $f : M \rightarrow N$ an homomorphism of A -modules. Now, since $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$, so there is an A -homomorphism

$$\begin{aligned} \Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) &\longrightarrow \Gamma_{\mathfrak{a}}(N) \\ m &\longmapsto f(m). \end{aligned}$$

Let $0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} P$ be a short exact sequence of A -modules. We claim that

$$0 \rightarrow \Gamma_{\mathfrak{a}}(N) \xrightarrow{\Gamma_{\mathfrak{a}}(f)} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\Gamma_{\mathfrak{a}}(g)} \Gamma_{\mathfrak{a}}(P)$$

is exact. For $x \in \Gamma_{\mathfrak{a}}(N)$ we have $\Gamma_{\mathfrak{a}}(f)(x) = f(x)$, thus injectivity follows.

We have $\text{im}(\Gamma_{\mathfrak{a}}(f)) \subseteq \ker(\Gamma_{\mathfrak{a}}(g))$ since $g \circ f = 0$ implies

$$\Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f) = \Gamma_{\mathfrak{a}}(g \circ f) = \Gamma_{\mathfrak{a}}(0) = 0.$$

For the remaining inclusion let $y \in \ker(\Gamma_{\mathfrak{a}}(g))$, this implies that $g(y) = 0$ and that there exists $x \in N$ such that $f(x) = y$, we only need to show that $x \in \Gamma_{\mathfrak{a}}(M)$. There is $n \in \mathbb{N}$ such that $\mathfrak{a}^n y = 0$, this implies $f(\mathfrak{a}^n x) = \mathfrak{a}^n y = 0$ and by injectivity of f we have $\mathfrak{a}^n x = 0$ as required. \square

By [Rotman \(2009, Proposition 6.4\)](#) for every A -module M and every ideal $\mathfrak{a} \subseteq A$ there exists an injective resolution⁵

$$\mathbf{I}^\bullet : 0 \longrightarrow M \xrightarrow{\alpha} I_0 \xrightarrow{\alpha_0} I_1 \xrightarrow{\alpha_1} \cdots \longrightarrow I_i \xrightarrow{\alpha_i} I_{i+1} \longrightarrow \cdots \quad (5.4)$$

of M . Now we apply $\Gamma_{\mathfrak{a}}(-)$ on

$$\mathbf{I}^{\bullet, M} : 0 \longrightarrow I_0 \xrightarrow{\alpha_0} I_1 \xrightarrow{\alpha_1} \cdots \longrightarrow I_i \xrightarrow{\alpha_i} I_{i+1} \longrightarrow \cdots$$

to obtain

$$\Gamma_{\mathfrak{a}}(\mathbf{I}^{\bullet, M}) : 0 \longrightarrow \Gamma_{\mathfrak{a}}(I_0) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha_0)} \Gamma_{\mathfrak{a}}(I_1) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha_1)} \cdots \longrightarrow \Gamma_{\mathfrak{a}}(I_i) \xrightarrow{\Gamma_{\mathfrak{a}}(\alpha_i)} \Gamma_{\mathfrak{a}}(I_{i+1}) \longrightarrow \cdots .$$

Define the i -th local cohomology module $H_{\mathfrak{a}}^i(M)$ of M with respect to \mathfrak{a} as

$$H_{\mathfrak{a}}^i(M) := \ker(\Gamma_{\mathfrak{a}}(\alpha_i)) / \text{im}(\Gamma_{\mathfrak{a}}(\alpha_{i-1})),$$

for all integer $i \geq 0$ where $\Gamma_{\mathfrak{a}}(\alpha_{-1})$ is the trivial map $0 \rightarrow \Gamma_{\mathfrak{a}}(I_0)$.

We are concern in the case where $R = \bigoplus_{n \geq 0} R_n$ is a positively graded noetherian ring, $\mathfrak{a} = R_+ = \bigoplus_{n > 0} R_n$ is the irrelevant ideal and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded R -module. We can assume that the groups $H_{R_+}^i(M)$ are graded R -modules, this is because the category of graded R -modules has enough injectives and projective objects, this allows us to apply techniques of homological algebra to this category, in particular equation 5.4 can be replace by an exact sequence of graded R -modules \mathbf{I}^\bullet , applying the functor $\Gamma_{R_+}(-)$ to $\mathbf{I}^{\bullet, M}$ and its right derived functor $H_{R_+}^i(-)$ in this category gives us a grading $H_{R_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(M)_n$ for every $i \geq 0$, for details in this matter see [Brodmann and Sharp \(1998, Chapter 12\)](#). We state the following proposition that will be the essence of the Serre finiteness theorem 5.24 , for a proof of this see ([BRODMANN; SHARP, 1998, Proposition 15.1.5](#)).

Proposition 5.16. *Let R be a positively graded noetherian ring and M a finitely generated graded R -module, then*

(a) *For all $i \geq 0$ the n -th graded component $H_{R_+}^i(M)_n$ of $H_{R_+}^i(M)$ is a finitely generated R_0 -module for all $n \in \mathbb{Z}$.*

(b) *$H_{R_+}^i(M)_n$ vanishes for all $n \gg 0$ and all $i \geq 0$.*

5.3 Čech complexes

In this section we define the Čech cohomology for a quasi-coherent sheaf over a projective scheme $\text{Proj } R$ where R is positively graded ring and we explore its relation with the

⁵ An **injective resolution** of M is an exact sequence $0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{\alpha^0} I^1 \xrightarrow{\alpha^1} \cdots \longrightarrow I^i \xrightarrow{\alpha^i} I^{i+1} \longrightarrow \cdots$ of A -modules such that every I^i is injective.

sheaf cohomology of section 5.1. Theorem 5.18 establishes an isomorphism between the two cohomology groups. Similarly, given an arbitrary ring A we define a Čech cohomology for a module with respect to some finitely generated ideal. Theorem 5.20 establishes an isomorphisms between these Čech cohomology groups and the local cohomology groups from section 5.2.

Let $R = \bigoplus_{n \geq 0} R_n$ be a standard positively graded ring, write $R = R_0[x_0, \dots, x_r]$ where $x_i \in R_1$. Let $X = \text{Proj } R$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded R -module. We know that the open sets $U_i = D_+(x_i)$ are affine and cover X , denote by $\mathfrak{A} = \{D_+(x_i)\}_{i=0}^r$ this open cover and for every set of sub-indexes $\{i_0, i_1, \dots, i_t\} \subseteq \{0, \dots, r\}$ such that $i_0 < \dots < i_t$ set $U_{i_0 \dots i_t} = U_{i_0} \cap \dots \cap U_{i_t}$. The t -th **Čech module** of the sheaf \tilde{M} with respect to \mathfrak{A} is define to be

$$\check{C}^t(\mathfrak{A}, \tilde{M}) := \prod_{i_0 < \dots < i_t} \tilde{M}(U_{i_0 \dots i_t}) \cong \prod_{i_0 < \dots < i_t} M_{(x_{i_0} \dots x_{i_t})}$$

where the product is taken over all $(t+1)$ -tuples $\{i_0, \dots, i_t\}$ such that $i_0 < \dots < i_t$. Define the **co-boundary map** $d^t : \check{C}^t(\mathfrak{A}, \tilde{M}) \rightarrow \check{C}^{t+1}(\mathfrak{A}, \tilde{M})$ for $\alpha = (\alpha_{i_0, \dots, i_t})_{\{i_0 < \dots < i_t\}} \in \check{C}^t(\mathfrak{A}, \tilde{M})$ by

$$\alpha \mapsto \left(\sum_{k=0}^{t+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{t+1}} \right)_{i_0, \dots, i_{t+1}} \quad (5.5)$$

where the symbol \hat{i}_k means we are omitting the index i_k . Note that every $\alpha_{i_0, \dots, \hat{i}_k, \dots, i_t} \in M_{(x_{i_0} \dots x_{i_k} \dots x_{i_t})}$ can be seen canonically as an element of $M_{(x_{i_0} \dots x_{i_{t+1}})}$. For example if $r = 2$ we have the maps

$$M_{(x_0)} \times M_{(x_1)} \times M_{(x_2)} \xrightarrow{d^0} M_{(x_0 x_1)} \times M_{(x_0 x_2)} \times M_{(x_1 x_2)} \xrightarrow{d^1} M_{(x_0 x_1 x_2)}$$

$$\left(\frac{f}{x_0^t}, \frac{g}{x_1^t}, \frac{h}{x_2^t} \right) \longmapsto \left(\frac{g}{x_1^t} - \frac{f}{x_0^t}, \frac{h}{x_2^t} - \frac{f}{x_0^t}, \frac{h}{x_2^t} - \frac{g}{x_1^t} \right)$$

$$\left(\frac{F}{x_0^t x_1^t}, \frac{G}{x_0^t x_2^t}, \frac{H}{x_1^t x_2^t} \right) \longmapsto \frac{H}{x_1^t x_2^t} - \frac{G}{x_0^t x_2^t} + \frac{F}{x_0^t x_1^t}$$

note that in this case we have $d^1 \circ d^0 = 0$, it can be checked that $d^{t+1} \circ d^t = 0$ for every t , thus

$$0 \rightarrow \check{C}^0(\mathfrak{A}, \tilde{M}) \xrightarrow{d^0} \dots \xrightarrow{d^{t-1}} \check{C}^t(\mathfrak{A}, \tilde{M}) \xrightarrow{d^t} \dots \xrightarrow{d^{r-1}} \check{C}^r(\mathfrak{A}, \tilde{M}) \rightarrow 0$$

is a complex. Consequently we define the t -th **Čech cohomology group** of \tilde{M} with respect to the open covering \mathfrak{A} by

$$\check{H}^t(\mathfrak{A}, \tilde{M}) := \frac{\ker d^t}{\text{im } d^{t-1}}$$

Remark 5.17. $\check{H}^0(\mathfrak{A}, \tilde{M}) \cong \Gamma(X, \tilde{M}) = M$. In fact, $\check{H}^0(\mathfrak{A}, \mathcal{F}) = \ker d^0$. Let $\alpha = (\alpha_i)_{i=0}^r \in \ker d^0$, then $\alpha_i|_{D_+(x_i) \cap D_+(x_j)} = \alpha_j|_{D_+(x_i) \cap D_+(x_j)}$ and by the gluability axiom of \tilde{M} , we can find a unique $\bar{\alpha} \in \Gamma(X, \tilde{M})$ such that $\bar{\alpha}|_{D_+(x_i)} = \alpha_i$. Hence we have a bijective group homomorphism $\check{H}^0(\mathfrak{A}, \tilde{M}) \rightarrow \Gamma(X, \tilde{M})$ defined by $\alpha \mapsto \bar{\alpha}$.

With X and M as above we construct a sheaf version of the Čech module. For any open set $U \subseteq X$ define

$$\mathcal{C}^t(\mathfrak{A}, \tilde{M})(U) := \prod_{i_0 < \dots < i_t} \tilde{M}|_{U_{i_0 \dots i_t}}(U) = \prod_{i_0 < \dots < i_t} \tilde{M}(U_{i_0 \dots i_t} \cap U)$$

this turns $\mathcal{C}^t(\mathfrak{A}, \tilde{M})$ into a sheaf of \mathcal{O}_X -modules for all t . There is a morphism of sheaves $d^t : \mathcal{C}^t(\mathfrak{A}, \tilde{M}) \rightarrow \mathcal{C}^{t+1}(\mathfrak{A}, \tilde{M})$ defined in a similar way as (5.5)

$$d^t : \mathcal{C}^t(\mathfrak{A}, \tilde{M})(U) \longrightarrow \mathcal{C}^{t+1}(\mathfrak{A}, \tilde{M})(U)$$

$$\alpha \longmapsto \left(\sum_{k=0}^{t+1} (-1)^k \alpha|_{U_{i_0 \dots \hat{i}_k \dots i_{t+1}}} \right)_{i_0, \dots, i_{t+1}}.$$

This morphism satisfies $d^{t+1} \circ d^t = 0$, and $\Gamma(X, \mathcal{C}^t(\mathfrak{A}, \tilde{M})) = \mathcal{C}^t(\mathfrak{A}, \tilde{M})(X) = \check{C}^t(\mathfrak{A}, \tilde{M})$. There is a natural morphism of sheaves $\varepsilon : \tilde{M} \rightarrow \mathcal{C}^0(\mathfrak{A}, \tilde{M})$ defined by $\varepsilon_U(\alpha) := (\alpha|_{U \cap D_+(x_i)})_{i=0}^r$ for any $U \subset X$ and $\alpha \in \tilde{M}(U)$. ε is injective since $D_+(x_i)$ cover X and \tilde{M} is a sheaf. It is easy to verify that $\varepsilon(\mathcal{F}) = \ker d^0$. In fact

$$0 \rightarrow \tilde{M} \xrightarrow{\varepsilon} \mathcal{C}^0(\mathfrak{A}, \tilde{M}) \xrightarrow{d^0} \mathcal{C}^1(\mathfrak{A}, \tilde{M}) \xrightarrow{d^1} \dots$$

is an exact sequence of sheaves, see Hartshorne (1977, III, Lemma 4.2). Now, let $\mathbf{I}^\bullet : 0 \rightarrow \tilde{M} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$ be an injective resolution of \tilde{M} , by Rotman (2009, Theorem 6.16) there exist morphisms of sheaves $\mathcal{C}^t(\mathfrak{A}, \tilde{M}) \rightarrow \mathcal{I}^t$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{M} & \xrightarrow{\varepsilon} & \mathcal{C}^0(\mathfrak{A}, \tilde{M}) & \xrightarrow{d^0} & \mathcal{C}^1(\mathfrak{A}, \tilde{M}) & \xrightarrow{d^1} & \dots \\ & & \downarrow id & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{M} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \dots \end{array}$$

commutes. Applying $\Gamma(X, -)$ to the above diagram we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\Gamma(X, \varepsilon)} & \check{C}^0(\mathfrak{A}, \tilde{M}) & \xrightarrow{d^0} & \check{C}^1(\mathfrak{A}, \tilde{M}) & \xrightarrow{d^1} & \dots \\ & & \downarrow id & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \Gamma(X, \mathcal{I}^0) & \longrightarrow & \Gamma(X, \mathcal{I}^1) & \longrightarrow & \dots \end{array}$$

which induces a morphism between its cohomology groups, see Rotman (2009, Proposition 6.8). That is, we have morphisms

$$\check{H}^t(\mathfrak{A}, \tilde{M}) \longrightarrow H^t(X, \tilde{M})$$

In the case where R is a noetherian ring, the following theorem implies that these morphisms are in fact isomorphisms for every t , this is a consequence of Hartshorne (1977, III, Theorem 4.5) where the result is proven for a noetherian separated scheme X , an affine cover \mathfrak{A} and a quasi-coherent sheaf \mathcal{F} , a slightly more general case than our own. Here the hypothesis of the scheme being separated is needed to guaranteed that the intersection of two open affine sets is affine, which is our case since $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$.

Theorem 5.18. . Let $R = \bigoplus_{n \geq 0} R_n$ be standard positively graded (see definition 4.17) noetherian ring. Let $X = \text{Proj } R$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded R -module. Let $\mathfrak{A} = \{D_+(x_i)\}_{i=0}^r$, then the above homomorphism

$$\check{H}^t(\mathfrak{A}, \tilde{M}) \longrightarrow H^t(X, \tilde{M})$$

is an isomorphism for every $t = 0, \dots, r$.

In particular the Čech cohomology groups do not depend on the choice of generators for R_+ and $H^t(X, \tilde{M}) = 0$ for every $t < 0$ and $t > r$.

Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal finitely generated by the elements (a_1, \dots, a_r) . Let M be an arbitrary A -module. We define a Čech complex $\check{C}^\bullet(M)$ of M with respect to the sequence of elements a_1, \dots, a_r as follows. First we define the modules

$$\begin{aligned} \check{C}^0(M) &:= M \\ \check{C}^t(M) &:= \prod_{i_1 < \dots < i_t} M_{a_{i_1} \dots a_{i_t}} \text{ for } t = 1, \dots, r \end{aligned}$$

together with the morphisms

$$\begin{aligned} d^0 : \check{C}^0(M) &\longrightarrow \check{C}^1(M) \\ m &\longmapsto \left(\frac{m}{1}, \dots, \frac{m}{1} \right) \end{aligned}$$

and

$$\begin{aligned} d^t : \check{C}^t(M) &\longrightarrow \check{C}^{t+1}(M) \\ \alpha &\longmapsto \left(\sum_{k=1}^{t+1} (-1)^k \alpha_{i_1, \dots, \hat{i}_k, \dots, i_{t+1}} \right)_{i_1, \dots, i_{t+1}} \end{aligned}$$

where each element $\alpha_{i_1, \dots, \hat{i}_k, \dots, i_{t+1}} \in \check{C}^t(M)$ may be seen as an element of $\check{C}^{t+1}(M)$ written as $x_{i_k}^s \alpha_{i_1, \dots, \hat{i}_k, \dots, i_{t+1}} / x_{i_k}^s$ for some adequate $s \in \mathbb{N}$. We have $d^{t+1} \circ d^t = 0$ for every $t = 1, \dots, r-2$, see [Brodmann and Sharp \(1998, 5.1.5 Proposition and Definition\)](#). Thus we define the **Čech complex** of M with respect to a_1, \dots, a_r by

$$\check{C}^\bullet(M) : 0 \rightarrow \check{C}^0(M) \xrightarrow{d^0} \check{C}^1(M) \xrightarrow{d^1} \dots \xrightarrow{d^{r-1}} \check{C}^r(M) \rightarrow 0.$$

We define

$$\check{H}^t(M) := \ker d^t / \text{Im } d^{t-1}$$

as the t -th **Čech cohomology module** of the Čech complex of M with respect to the ideal \mathfrak{a} with generators a_1, \dots, a_r .

Remark 5.19. Note that

$$\begin{aligned} \check{H}^0(M) &= \ker d^0 \\ &= \{m \in M \mid a_i^n m = 0 \text{ for all } i = 1, \dots, r \text{ and some } n \in \mathbb{N}\} \\ &= \Gamma_{\mathfrak{a}}(M) \end{aligned}$$

Sometimes we write $\check{C}^\bullet(a_1, \dots, a_n; M)$, $\check{C}^t(a_1, \dots, a_r; M)$ and $\check{H}^t(a_1, \dots, a_r; M)$ to specify what sequence of generators of \mathfrak{a} we are using.

Now we state the following important theorem.

Theorem 5.20. (*BRODMANN; SHARP, 1998, 5.1.19 Theorem*) *Let A be a noetherian ring and a_1, \dots, a_r be generators of an ideal \mathfrak{a} of A . For any A -module M there are isomorphisms*

$$\check{H}^t(a_1, \dots, a_r; M) = \check{H}^t(M) \xrightarrow{\sim} H_{\mathfrak{a}}^t(M)$$

for every $t \in \mathbb{N}$.

In particular $H_{\mathfrak{a}}^t(M) = 0$ for every $t < 0$ and $t > r$.

Assume $R = \bigoplus_{n \geq 0} R_n$ is a standard positively graded noetherian ring and M a graded R -module. Let $x_0, \dots, x_r \subseteq R_1$ be generators of the irrelevant ideal $R_+ = \bigoplus_{n > 0} R_n$. The modules $\check{C}^t(x_0, \dots, x_r; M)$ gain naturally a structure of graded module together with homogeneous homomorphisms d^t , therefore the modules $\check{H}^t(x_0, \dots, x_r; M)$ gain also a natural grading. Theorem 5.20 endows $H_{R_+}^t(M)$ with a graded R -module structure, this structure coincides with the grading discussed in section 5.2, to see a proof of this we refer the reader to Brodmann and Sharp (1998, Chapter 12), in particular the local cohomology modules are independent of the choice of generators for R_+ .

Remark 5.21. There is a natural way to relate the Čech cohomology groups $\check{H}^t(X, \tilde{M})$ of the quasi-coherent sheaf \tilde{M} with the Čech cohomology groups $\check{H}^t(x_0, \dots, x_r; M)$ where the irrelevant ideal R_+ is generated by $x_0, \dots, x_r \in R_1$. In theorem 5.22 we use this relation to establish a correspondence between the local cohomology of M with respect to R_+ and the sheaf cohomology of \tilde{M} .

5.4 Serre - Grothendieck correspondence

In this section we prove the classical form of the Serre-Grothendieck correspondence between Local cohomology and sheaf cohomology using the tools developed in section 5.3. Then we prove using this correspondence that any coherent sheaf \mathcal{F} on $X = \text{Proj } R$ is of the form \tilde{M} for some finitely generated graded R -module M . Finally we prove the Serre's finiteness theorem that states that the cohomology groups $H^i(X, \mathcal{F}(n))$ are finitely generated R_0 -modules and they all vanish for all $i > 0$ and sufficiently large n .

Let $R = \bigoplus_{n \geq 0} R_n$ be a standard positively graded noetherian ring, see definition 4.17. By Atiyah and Macdonald (1969, Theorem 10.7) this is the same as saying R_0 is a noetherian ring and R is finitely generated as a R_0 -module. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules on $X = \text{Proj } R$, recall definition 4.13 for $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$, $\Gamma_*(\mathcal{F})$ is a graded R -module.

The following theorem establishes an isomorphism between the groups that arise from the sheaf cohomology and the local cohomology.

Theorem 5.22. (Classical form of the Serre - Grothendieck Correspondence (IYENGAR et al., 2007, Theorem 13.21)). Let R be a standard positively graded noetherian ring and $X = \text{Proj } R$, for every graded R -module M and every $n \in \mathbb{Z}$ we obtain an exact sequence of graded R -modules

$$0 \longrightarrow H_{R_+}^0(M) \longrightarrow M \xrightarrow{\zeta(M)} \Gamma_*(\tilde{M}) \longrightarrow H_{R_+}^1(M) \longrightarrow 0$$

and a graded R -module isomorphism

$$H_{R_+}^{i+1}(M) \cong \bigoplus_{n \in \mathbb{Z}} H^i(X, \tilde{M}(n)), \quad \forall i \geq 1$$

in particular $H_{R_+}^{i+1}(M)_n \cong H^i(X, \tilde{M}(n))$ for every $n \in \mathbb{Z}$.

Proof. Suppose $\{x_0, \dots, x_r\} \subset R_1$ is a set of generators of R as a R_0 -algebra. Let $U_i = \text{Spec } S_{(x_i)}$ and $\mathfrak{U} = \{U_0, \dots, U_r\}$ be an open cover of X . Consider the complex

$$\begin{aligned} & C(x_0, \dots, x_r; M)[-1] : \\ & 0 \longrightarrow 0 \longrightarrow \bigoplus_{i=0}^r M_{x_i} \longrightarrow \cdots \longrightarrow \bigoplus_{i_0 < \dots < i_s} M_{x_{i_0} \dots x_{i_s}} \longrightarrow \cdots \longrightarrow M_{x_0 \dots x_r} \longrightarrow 0 \end{aligned}$$

which comes from the Čech complex associated to M using the maximal ideal $R_+ = (x_0, \dots, x_r)$ deleting $\check{C}^0(x_0, \dots, x_r; M) = M$ and shifting by -1 . Consider also the Čech complex

$$\begin{aligned} & \check{C}(x_0, \dots, x_r; M) : \\ & 0 \longrightarrow M \longrightarrow \bigoplus_{i=0}^r M_{x_i} \longrightarrow \cdots \longrightarrow \bigoplus_{i_0 < \dots < i_s} M_{x_{i_0} \dots x_{i_s}} \longrightarrow \cdots \longrightarrow M_{x_0 \dots x_r} \longrightarrow 0 \end{aligned}$$

and the complex

$$M : 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

We get a short exact sequence of complexes

$$0 \longrightarrow C(x_0, \dots, x_r; M)[-1] \longrightarrow \check{C}(x_0, \dots, x_r; M) \longrightarrow M \longrightarrow 0.$$

To finish the proof we pass to the associated long exact sequence of the respective cohomologies. In degree 0 we have $H^0(C(x_0, \dots, x_r; M)[-1]) = 0$, $H^0(\check{C}(x_0, \dots, x_r; M)) = H_{R_+}^0(M)$ and

$H^0(M) = M$. In degree 1 we have

$$\begin{aligned}
H^1(C(x_0, \dots, x_r; M)[-1]) &= \ker\left(\bigoplus_{i=0}^r M_{x_i} \rightarrow \bigoplus_{0 \leq i < j \leq r} M_{x_i x_j}\right) \\
&= \ker\left(\bigoplus_{i=0}^r \bigoplus_{n \in \mathbb{Z}} M(n)_{(x_i)} \rightarrow \bigoplus_{0 \leq i < j \leq r} \bigoplus_{n \in \mathbb{Z}} M(n)_{(x_i x_j)}\right) \\
&= \bigoplus_{n \in \mathbb{Z}} \ker\left(\bigoplus_{i=0}^r M(n)_{(x_i)} \rightarrow \bigoplus_{0 \leq i < j \leq r} M(n)_{(x_i x_j)}\right) \\
&= \bigoplus_{n \in \mathbb{Z}} H^0(X, \tilde{M}(n)) \\
&= \Gamma_*(\tilde{M})
\end{aligned}$$

and for the remaining complexes we have $H^1(\check{C}(x_0, \dots, x_r; M)) = H_{R_+}^1(M)$ and $H^1(M) = 0$, this gives the exact sequence

$$0 \longrightarrow H_{R_+}^0(M) \longrightarrow M \longrightarrow \Gamma_*(\tilde{M}) \longrightarrow H_{R_+}^1(M) \longrightarrow 0$$

as required. Now, note that the cohomology of the complex M is 0 for $i \geq 2$, this gives short exact sequences

$$0 \longrightarrow H^i(C(x_0, \dots, x_r; M)[-1]) \longrightarrow H^i(\check{C}(x_0, \dots, x_r; M)) \longrightarrow 0.$$

Similar to the degree 0 case we also have

$$H^i(C(x_0, \dots, x_r; M)[-1]) \cong \bigoplus_{n \in \mathbb{Z}} H^{i-1}(X, \tilde{M}(n))$$

and

$$H^i(\check{C}(x_0, \dots, x_r; M)) = H_{R_+}^i(M)$$

so, the above short exact sequence yields the isomorphism

$$H_{R_+}^i(M) \cong \bigoplus_{n \in \mathbb{Z}} H^{i-1}(X, \tilde{M}(n))$$

for every $i \geq 2$ as required. □

Proposition 5.23. *Let R be a standard positively graded noetherian ring, $X = \text{Proj } R$ and \mathcal{F} be a coherent sheaf on X . Then there exists a finitely generated graded R -module M such that $\mathcal{F} \cong \tilde{M}$.*

Proof. By proposition 4.18 there is a graded R -module N such that $\mathcal{F} \cong \tilde{N}$. Write $R = R_0[f_0, \dots, f_r]$ where $f_i \in R_1 \setminus \{0\}$. Note that $X = \cup_{i=0}^r D_+(f_i)$. By proposition 4.6 we have $\tilde{N}|_{D_+(f_i)} \cong \tilde{M}_i$ for some finitely generated $R_{(f_i)}$ -module M_i , thus, taking global sections we obtain by proposition 4.9 that $N_{(f_i)} \cong M_i$ and therefore $N_{(f_i)}$ is finitely generated $R_{(f_i)}$ -module for each $i =$

$0, \dots, r$. Let $\{\frac{m_{i_1}}{f_i^{n_{i_1}}}, \dots, \frac{m_{i_l}}{f_i^{n_{i_l}}}\}$ generate $N_{(f_i)}$ as a $R_{(f_i)}$ -module, note that the m_{i_j} 's are homogeneous, thus it makes sense to consider the graded R -module M to be the one generated by $\{m_{0_1}, \dots, m_{0_l}, \dots, m_{r_1}, \dots, m_{r_l}\}$. We show that $\Gamma_{R_+}(N/M) = N/M$, that is, N/M is R_+ -torsion. There is only left to show the inclusion " \supseteq ". For this, let $m \in N$, it is enough to prove that there is $n' \geq 0$ such that $R_+^{n'} m \in M$. We may assume m is homogeneous. For every $i = 0, \dots, r$ pick n_i such that $m/f_i^{n_i} \in M_{(f_i)}$, thus there exist $u_1, \dots, u_l \in R_{(f_i)}$ such that

$$\frac{m}{f_i^{n_i}} = u_1 \frac{m_{i_1}}{f_i^{n_{i_1}}} + \dots + u_l \frac{m_{i_l}}{f_i^{n_{i_l}}}.$$

Take $n'_i \geq 0$ such that

$$f_i^{n'_i} m = g_1 m_{i_1} + \dots + g_l m_{i_l}$$

where $g_j \in R$. Now take n' great enough such that $f_i^{n'} m \in M$ for all $i = 0, \dots, r$, since R_+ is the ideal generated by $\{f_0, \dots, f_r\}$ we have $R_+^{n'} m \in M$ and the inclusion is proved. So we have that N/M is R_+ -torsion, we show now that $(N/M)^\sim = 0$. Note that for $\mathfrak{p} \in X$ we have $(N/M)^\sim_{\mathfrak{p}} = 0$. In fact for every $\mathfrak{p} \in X$ and every $s_{\mathfrak{p}} \in (N/M)_{\mathfrak{p}}$ we can choose i such that $\mathfrak{p} \in D_+(f_i)$, thus $s|_{D_+(f_i)} \in (N/M)_{(f_i)}$ and we have $f_i^{n'} s|_{D_+(f_i)} = 0$ which implies that $s_{\mathfrak{p}} = 0$. As a consequence of proposition 2.26 we deduce that $\tilde{M} \cong \tilde{N}$ since they coincide at every stalk. \square

Theorem 5.24. (*BRODMANN; SHARP, 1998, 20.4.6 Theorem*)(*Serre's Finiteness Theorem*)
Let R be a standard positively graded noetherian ring, $X = \text{Proj } R$ and \mathcal{F} a coherent sheaf on X , then

- (a) $H^i(X, \mathcal{F}(n))$ is a finitely generated R_0 -module, for all $i \geq 0$ and all $n \in \mathbb{Z}$;
- (b) There exists $n_0 \in \mathbb{Z}$ such that $n \geq n_0$ implies $H^i(X, \mathcal{F}(n)) = 0$ for all $i \geq 1$.

Proof. By proposition 5.23 there exists a finitely generated graded R -module M such that $\mathcal{F} \cong \tilde{M}$. By theorem 5.22 we have an exact sequence

$$0 \longrightarrow H_{R_+}^0(M)_n \longrightarrow M_n \longrightarrow H^0(X, \mathcal{F}(n)) \longrightarrow H_{R_+}^1(M)_n \longrightarrow 0$$

since M_n is finitely generated and by proposition 5.16 item (a) so is $H_{R_+}^1(M)_n$ we have that $H^0(X, \mathcal{F}(n))$ is finitely generated as R_0 -module. For $i \geq 1$ theorem 5.22 implies that $H^i(X, \mathcal{F}(n)) \cong H_{R_+}^{i+1}(M)_n$, thus, by proposition 5.16, we conclude that $H^i(X, \mathcal{F}(n))$ are finitely generated R_0 -modules and they vanish for $n \gg 0$ and all $i > 0$. \square

5.5 Regularity and Eisenbud-Goto conjecture

5.5.1 Hilbert polynomials

One of the main tools used in geometry are Hilbert-Serre polynomials, multiplicities and Hilbert-Samuel coefficients. Before defining them, we enunciate a classic theorem in graded

modules that relates to polynomial, degree and dimension.

Let R be a standard positively graded ring and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded R -module. Suppose that $\ell_{R_0}(M_n) < \infty$ for all $n \in \mathbb{Z}$. Define the **Hilbert function** $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$ of M by

$$H_M(n) := \ell_{R_0}(M_n) \text{ for all } n \in \mathbb{Z}.$$

Theorem 5.25. (*BRUNS; HERZOG, 1998, 1, 4.1.3*). *Let R be a standard positively graded ring and suppose R_0 is an artinian local ring. Let M be a non-zero finitely generated graded R -module. Then, there exists a polynomial $P_M(t) \in \mathbb{Q}[t]$ so that for all large n , $H_M(n) = P_M(n)$ and $d := \deg(P_M(t)) = \dim(M) - 1$.*

Here $\dim(M) := \dim(R/\text{ann}(M))$ as in [Bruns and Herzog \(1998, Appendix\)](#). Due to the above theorem, we define the following polynomial whose coefficients are very important in algebraic geometry and commutative algebra.

Definition 5.26. Let R be a standard positively graded ring and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded R -module. Suppose that $\ell_{R_0}(M_n) < \infty$ for all $n \in \mathbb{Z}$. The polynomial $P_M(t) \in \mathbb{Q}[t]$ such that, for all $n \gg 0$, $H_M(n) = P_M(n)$ is called the **Hilbert polynomial** of M .

Now, let $R = \bigoplus_{n \geq 0} R_n$ be a standard positively graded ring such that (R_0, \mathfrak{m}_0) is an artinian local ring. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R -module. We know by ([BRUNS; HERZOG, 1998, 1, 4.1.6](#)) that for $n \gg 0$ the numerical function

$$SF_M(n) = \sum_{i=0}^{n-1} H_M(i)$$

is a polynomial function $HP_M(x) \in \mathbb{Z}[x]$ of degree $d = \dim(M)$ that is also called the Hilbert polynomial of M . It can be written in the form

$$HP_M(n) = \sum_{i=0}^d (-1)^i e_i(M) \binom{n-d-i}{d-i} = e_0(M) \frac{n^d}{d!} + \text{lower order terms}$$

with integer coefficients $e_i(M)$ called the **Hilbert coefficients** of M and $e_0(M)$ is called the **Hilbert-Samuel multiplicity** of M .

Assume that \mathcal{F} is the coherent sheaf induced by M on $X = \text{Proj } R$. We know that the cohomology groups $H^i(X, \mathcal{F}(n))$ are finitely generated by [theorem 5.24](#) item (a). Thus, the modules $H^i(X, \mathcal{F}(n))$ are both artinian and noetherian and hence of finite length as R_0 -modules. For $i \geq 0$ and $n \in \mathbb{Z}$ define its length

$$h_{X, \mathcal{F}}^i(n) = h_{\mathcal{F}}^i(n) := \ell_{R_0}(H^i(X, \mathcal{F}(n))).$$

By [theorem 5.24](#) item (b), there is $n_0 \in \mathbb{Z}$ such that $h_{\mathcal{F}}^i(n) = 0$ for all $n \geq n_0$. Therefore it makes sense to define the **characteristic function** $\chi_{\mathcal{F}} : \mathbb{Z} \rightarrow \mathbb{Z}$ of \mathcal{F} by setting

$$\chi_{\mathcal{F}}(n) = \sum_{i=0}^{\infty} (-1)^i h_{\mathcal{F}}^i(n) \quad \text{for all } n \in \mathbb{Z}.$$

Note that, for all $n \in \mathbb{Z}$, we get

$$h_{\mathcal{F}}^0(n) = \ell_{R_0}(M_n) + h_{\mathcal{F}}^1(n) - h_{\mathcal{F}}^0(n)$$

by proposition 5.24 and theorem 5.22, so that for any integer $d = r \geq \dim(M)$

$$\begin{aligned} \chi_{\mathcal{F}}(n) &= \sum_{i=0}^d (-1)^i h_{\mathcal{F}}^i(n) \\ &= h_{\mathcal{F}}^0(n) - \sum_{i=2}^d (-1)^i h_{\mathcal{F}}^i(n) \\ &= \ell_{R_0}(M_n) - \sum_{i=0}^d (-1)^i h_{\mathcal{F}}^i(n). \end{aligned}$$

Thus, by theorem 5.24, $\chi_{\mathcal{F}}(n) = \ell_{R_0}(M_n)$ for all $n \gg 0$. Therefore, observe that for $n \gg 0$, by theorem 5.25 and proposition 5.23, $\chi_{\mathcal{F}}(n)$ is a polynomial of degree $\dim(\mathcal{F}) - 1$. So, by (BRUNS; HERZOG, 1998, 1, 4.1.6), for $n \gg 0$, the numerical function

$$H_{\mathcal{F}}(n) = \sum_{j=0}^n \ell_{R_0}(M_j)$$

is a polynomial function $HP_{\mathcal{F}}(x) \in \mathbb{Z}[x]$ of degree $d = \dim(\mathcal{F})$. It can be written in the form

$$HP_{\mathcal{F}}(n) = \sum_{i=0}^d (-1)^i e_i(\mathcal{F}) \binom{n-d-i}{d-i}$$

with integer coefficients $e_i(\mathcal{F})$, called the **Hilbert coefficients** of \mathcal{F} . There, $e_0(\mathcal{F})$ is called the **Hilbert-Samuel multiplicity** of \mathcal{F} and $e_0(\mathcal{F}) \geq 0$.

Next, we introduce the invariants

$$a_i(\mathcal{F}) := (-1)^{\dim(\mathcal{F})-i} e_{\dim(\mathcal{F})-i}(\mathcal{F}) \quad \text{for all } i = 0, \dots, \dim(\mathcal{F}),$$

so that,

$$HP_{\mathcal{F}}(n) = \sum_{i=0}^d \binom{n+i}{i} a_i(\mathcal{F}).$$

5.5.2 Regularity

Assume R is a standard positively graded noetherian ring and set $X = \text{Proj } R$. Let \mathcal{F} be a coherent sheaf on X . Many results in algebraic geometry can be written in the form of vanishing and non vanishing statements of the groups $H^i(X, \mathcal{F}(n))$. We say that \mathcal{F} is **m -regular** if

$$H^i(X, \mathcal{F}(m-i)) = 0$$

for all $i > 0$. This definition was originally given by Mumford and Bergman (1966, Lecture 14). By Hartshorne (1977, III, Theorem 2.7) we have that $H^i(X, \mathcal{F}(n)) = 0$ for every $i > \dim(X)$, by

theorem 5.24 item (b) there exists $n_0 \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}(n)) = 0$ for $n \geq n_0$ and every $i > 0$. Thus if we take $m = n_0 + \dim(X)$ we have that $H^i(X, \mathcal{F}(m-i)) = 0$ for every $i > 0$. This implies that every coherent sheaf is m -regular for some m . The **Castelnuovo-Mumford regularity** for the sheaf \mathcal{F} is defined to be

$$\text{reg}(\mathcal{F}) := \min\{m \mid \mathcal{F} \text{ is } m\text{-regular}\}$$

or $-\infty$ if \mathcal{F} is m -regular for every m .

5.5.3 Eisenbud-Goto Conjecture

When studying multiplicities and regularity, in the algebraic context there is an infinity of developed theory and an infinity of open problems. Thanks to Serre-Grothendieck's Correspondence theorem the same questions can be brought into the geometric algebraic context. Below we list some of these problems and for the sake of time we will not go into much detail. Let $X = \text{Proj } R$ where R is a standard graded noetherian ring. Let \mathcal{F} be a coherent sheaf on X , we ask the following question.

Problem 5.27. What is the class of sheaves \mathcal{F} in which $\text{reg}(\mathcal{F})$ is polynomially bounded in terms of the Hilbert coefficients $e_i(\mathcal{F})$ of \mathcal{F} ?

In order to study this question let us remember the definition of associated primes of a module. If M is an R -module we say that a prime ideal $\mathfrak{p} \subseteq R$ is associated to M if $\mathfrak{p} = \text{Ann}(m) = \{a \in R \mid rm = 0\}$ for some $m \in M$. In our context we say that $\mathfrak{p} \in X$ is **associated to \mathcal{F}** if the maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ of the local ring $\mathcal{O}_{X,\mathfrak{p}}$ is associated to the $\mathcal{O}_{X,\mathfrak{p}}$ -module $\mathcal{F}_{\mathfrak{p}}$. We denote by $\text{Ass}_X(\mathcal{F})$ the set of associated points to \mathcal{F} .

Definition 5.28. A sequence of global sections $f_1, \dots, f_r \in H^0(X, \mathcal{O}_X(1))$ is said to be **\mathcal{F} -regular**, if

$$H_i \cap \text{Ass}_X(\mathcal{F}|_{H_1 \cap \dots \cap H_{i-1}}) = \emptyset \text{ for } i = 1, \dots, r,$$

where $H_i \subseteq X$ is the subscheme defined by f_i . It is equivalent to say, that the natural homomorphisms of sheaves

$$f_i : \mathcal{F}|_{H_1 \cap \dots \cap H_{i-1}}(n) \rightarrow \mathcal{F}|_{H_1 \cap \dots \cap H_{i-1}}(n+1)$$

are injective for all $i = 0, \dots, r$ and for all $n \in \mathbb{Z}$.

Define $\dim(\mathcal{F}) := \sup\{i \geq 0 \mid H^i(X, \mathcal{F}(n)) \neq 0 \text{ for some } n \in \mathbb{Z}\}$. Assume R_0 is artinian, let $r \geq 0$, $\underline{b} = (b_0, \dots, b_r) \in \mathbb{Z}^{r+1}$ and $b_i \geq 0$. Then, \mathcal{F} is said to be a **\underline{b} -sheaf**, if $\dim(\mathcal{F}) \leq r$ and

$$h_{\mathcal{F}|_{H_1 \cap \dots \cap H_{i-1}}}^0(-1) \leq b_i \text{ for } i = 0, \dots, r$$

where $H_i \subseteq X$ is the subscheme defined by f_i .

With these notations and definitions Kleiman answers the question on b -sheaves that says.

Theorem 5.29. (*BERTHELOT et al., 1971, EX XIII, Theorem 1.11*) Let $r \in \mathbb{N}_0$. Then, there is a polynomial $P_r \in \mathbb{Q}[t_1, \dots, t_r]$ such that for each algebraically closed field k , each projective scheme X over k and each coherent sheaf of \mathcal{O}_X -modules which is a $b = (b_0, \dots, b_r)$ -sheaf of dimension r we have

$$\text{reg}(\mathcal{F}) \leq P_r(b_0 - a_0(\mathcal{F}), \dots, b_r - a_r(\mathcal{F})).$$

A scheme (X, \mathcal{O}_X) is **irreducible** if the corresponding topological space is irreducible. It is **reduced** if the rings $\mathcal{O}_X(U)$ have no nilpotent elements for every open set $U \subseteq X$, or equivalently if the stalks $\mathcal{O}_{X,P}$ have no nilpotent elements.

Let $R = k[x_0, x_1, \dots, x_r]$ be a polynomial ring and let $X = \mathbb{P}_k^r$ where $r > 1$ and k is an algebraically closed field (see example 3.27). Let Y be an irreducible reduced closed subscheme of X and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ the ideal sheaf⁶ associated to Y , suppose Y is a non degenerate variety, that is, it is not contained in any hyperplane⁷. In section 4.3 we saw that $I = \Gamma_*(\mathcal{I}_Y) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{I}_Y(n))$ is the defining homogeneous ideal of Y , that is $\tilde{I} \cong \mathcal{I}_Y$.

Now, define $k[Y] := R/I = \bigoplus_{n \geq 0} k[Y]_n$ to be the homogeneous **coordinate ring** of Y , where $k[Y]_n = R_n/(I_X \cap R_n)$. We can see $k[Y]$ as a finitely generated graded R -module and consider. In this case, the Hilbert function of $k[Y]$ is $H_{k[Y]}(n) = \dim_k(k[Y]_n)$. Then, for all $n \gg 0$, its Hilbert polynomial of $k[Y]$ is denoted by $P_{k[Y]}(t) \in \mathbb{Q}(t)$. It can be written in the form

$$P_{k[Y]}(n) = e_0(k[Y]) \frac{n^{\dim(Y)-1}}{(\dim(Y)-1)!} + \text{lower order terms}$$

where the integer $e_0(k[Y])$ is called the **degree** of Y . This can also be denoted or calculated as

$$\text{deg}(Y) := \lim_{n \rightarrow \infty} \frac{(\dim(Y)-1)! P_{k[Y]}(n)}{n^{\dim(Y)-1}} = e_0(k[Y]),$$

where $(\dim(Y)-1)! = 1 \cdot 2 \cdot \dots \cdot \dim(Y) - 1$ and $0! = 1$. Set the Castelnuovo-Mumford regularity of Y as

$$\text{reg}(Y) := \text{reg}(\mathcal{I}_Y).$$

The conjecture of Eisenbud-Goto says:

Problem 5.30.

$$\text{reg}(Y) \leq \text{deg}(Y) - \text{codim}(Y) + 1.$$

Castelnuovo (1893) proved the conjecture to be true for smooth curves in the projective space, that is, when $Y \subseteq \mathbb{P}_k^3$ is a projective smooth variety of dimension 1 and degree d we have

⁶ See definition 4.20 of ideal sheaf.

⁷ A hyperplane is a closed set $V_+(F)$ where F is a homogeneous polynomial of degree 1.

$\text{reg}(Y) \leq d - 1$. A century later it was proven by [Gruson, Lazarsfeld and Peskine \(1983\)](#) that any curve (not necessarily smooth) $Y \subseteq \mathbb{P}_k^r$ satisfies the inequality, and they gave necessary and sufficient conditions under which the equality holds. [Pinkham \(1986\)](#) and [Lazarsfeld et al. \(1987\)](#) proved the equality also holds for smooth surfaces when the characteristic of k is 0. [Ran et al. \(1990\)](#) proved the conjecture for smooth varieties of dimension 3 (smooth threefolds) contained in \mathbb{P}_k^r where $r \geq 9$ and characteristic of k is 0. [Kwak \(1999\)](#) proved the conjecture holds for smooth threefolds $X \subseteq \mathbb{P}_k^5$ when k has characteristic 0. [McCullough and Peeva \(2018\)](#) gives counterexamples to the Eisenbud-Goto conjecture, they construct two non smooth threefolds in \mathbb{P}_k^5 , the first one has degree 375 and regularity greater than 418 and the second one has degree 31 and regularity 38, both when characteristic of k is 0. In arbitrary characteristic this inequality has been established for a large class of surfaces $Y \subseteq \mathbb{P}_k^r$ of degree $r + 1$ see ([BRODMANN; EISENBUD, 1999](#)) but not for all of them, an open problem is whether

Problem 5.31. *all projective surfaces $Y \subseteq \mathbb{P}_k^r$ satisfy Eisenbud-Goto inequality?*

Two counterexamples to the Eisenbud-Goto conjecture that hold in arbitrary characteristic can be found in [McCullough and Peeva \(2018, 1.8\)](#).

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