Symmetries in binary differential equations

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Simetrias em equações diferenciais binárias

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To my Mom, my Dad and my brothers
Azizi and Mariana.

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“Se as coisas são inatingíveis...ora!
não é motivo para não querê-las...
Que triste os caminhos se não fora
A mágica presença das estrelas! ”
(Mario Quintana - Espelho Mágico)

Resumo

P. TEMPESTA. **Symmetries in binary differential equations** . 2017. 99 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

O objetivo desta tese é introduzir o estudo sistemático de simetrias em equações diferenciais binárias (EDBs). Neste trabalho formalizamos o conceito de EDB simétrica sobre a ação de um grupo de Lie compacto. Um dos principais resultados é uma fórmula que relaciona o efeito geométrico e álgebrico das simetrias presentes no problema. Utilizando ferramentas da teoria invariante e de representação para grupos compactos deduzimos as formas gerais para EDBs equivariantes. Um estudo sobre o comportamento das retas invariantes na configuração de EDBs com coeficientes homogêneos de grau n é feito com ênfase nos casos de grau 0 e 1, ainda no caso de grau 1 são apresentadas suas formas normais. Simetrias de 1-formas lineares são também estudadas e relacionadas com as simetrias dos seus campos tangente e ortogonal.

Palavras-chave: equação diferencial binária, simetria, 1-forma quadrática equivariante, grupo de Lie compacto, teoria de representação.

Abstract

P. TEMPESTA. **Symmetries in binary differential equations** . 2017. 99 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

The purpose of this thesis is to introduce the systematic study of symmetries in binary differential equations (BDEs). We formalize the concept of a symmetric BDE, under the linear action of a compact Lie group. One of the main results establishes a formula that relates the algebraic and geometric effects of the occurrence of the symmetry in the problem. Using tools from invariant theory and representation theory for compact Lie groups we deduce the general forms of equivariant binary differential equations under compact subgroups of $\mathbf{O}(2)$. A study about the behavior of the invariant straight lines on the configuration of homogeneous BDEs of degree n is done with emphasis on cases in which $n = 0$ and $n = 1$. Also for the linear case $(n = 1)$ the equivariant normal forms are presented. Symmetries of linear 1-forms are also studied and related with symmetries of tangent and orthogonal vectors fields associated with it.

Key-words: binary differential equation, symmetry, equivariant quadratic 1-form, compact Lie group, representation theory.

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BDE — binary differential equation;

$T\mathbb{R}^2$ — tangent bundle of $\mathbb{R}^2$;

$\mathscr{Q}(\mathbb{R}^2)$ — set of smooth quadratic 1-forms $T\mathbb{R}^2 \to \mathbb{R}$;

(a,b,c) — coefficients of the BDE;

Γ, Λ — compact Lie groups;

δ — discriminant function of the BDE;

Δ — discriminant set of the BDE;

$\mathscr{Q}[\Gamma,\eta]$ — set of the Γ-equivariant quadratic 1-forms;

$\mathscr{F}_i$ — foliations associated with a BDE;

$\mathscr{L}(\mathbb{R}^2)$ — set of linear differential 1-forms;

$\mathscr{L}[\Lambda,\sigma]$ — set of Λ-equivariant linear 1-forms;

(ρ, V) — vector spaces V under the representation ρ of Γ;

CONTENTS

INTRODUCTION

A binary differential equation on the plane, or a BDE , is an implicit quadratic differential equation of the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0, \tag{1}$$

where the coefficients a,b,c are smooth real functions on $\mathbb{R}^2$. Let $\mathscr{Q}(\mathbb{R}^2)$ denote the set of C^∞ quadratic 1-forms $\omega : T\mathbb{R}^2 \to \mathbb{R}$ of the form $\omega(x,y,dx,dy) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, where $T\mathbb{R}^2$ denote the tangent bundle. The function $\delta : \mathbb{R}^2 \to \mathbb{R}$, $\delta(x,y) = (b^2 - ac)(x,y)$, is the *discriminant function* and its zero set

$$\Delta = \{(x,y) \in \mathbb{R}^2 : (b^2 - ac)(x,y) = 0\}$$

is the *discriminant set* of the BDE. At points where $\delta > 0$, (1) defines a pair of transversal directions, and by the *configuration* associated with the BDE we mean the distribution of all solution curves tangent to these directions.

The geometry of a BDE configuration is a subject of great interest, with important applications in differential geometry, as the equations of lines of curvature, characteristic curves and asymptotic curves of smooth surfaces as we can see in the survey [32] and references therein. Conditions for local stability of positive binary differential equations ($\delta > 0$) and their classification are given in [22] and [23], with a description of the topological patterns that bifurcate in one-parameter families of these equations. Singular points of a class of positive binary equations associated with a smooth surface are also studied in [22] and [29], the coefficients of the BDE being given in terms of the coefficients of the first and second fundamental forms of the surface. In [19] the authors study isolated singularities of binary differential equations of degree n, we also give a classification of phase portraits of the n-web around a generic singular point which are totally real. Challapa in [14] introduces the definition of index for a class of equations which coincides with the classical Hopf definition for positive BDEs. Determining models of configurations associated with BDEs has been also addressed in many works; for example, in [10, 11, 12, 13, 15, 18, 24, 30, 31] the classification of BDEs is performed up to topological, formal, analytic or smooth equivalences.

This thesis is motivated by the recognition of symmetries in most normal forms that appear in the works mentioned above. Symmetries are transformations that take solutions of the equation into solutions and are, therefore, directly associated with an invariant property of the configuration associated with the BDE. We assume that the set Γ of all these elements, with the composition operation, is a compact Lie group whose structure is described by group representation theory. We remark that, in the linear case, for example, namely when the coefficients of (1) are linear functions, the symmetry group is necessary nontrivial; in fact, we have that the minus identity is always an element of this group, that tells us which groups are not symmetry groups of a linear BDE.

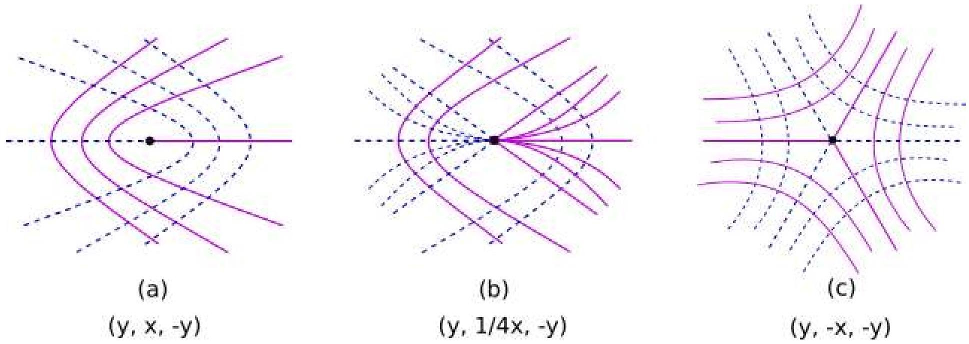


Figure 1 – Configurations of symmetric BDEs. In (a) and (b) the symmetry group is $\mathbf{Z}_2 \times \mathbf{Z}_2$ and in (c) the symmetry group is $\mathbf{D}_6$.

To remark on evidences of occurrence of symmetries in configurations associated with BDEs, let us consider the configurations that appear in [10, 29], which are generic topological structures of principal direction fields at umbilic points of surfaces on Euclidean spaces. The normal forms are given as triples (a,b,c) for $c = \pm a$ in (1) and their configurations are reproduced in Fig. 1, the so-called (a) lemon, (b) monstar and (c) star, also called Darbouxians. The solution curves determine two foliations on the plane distinguished by blue dashed lines and magenta solid lines, and the black point is the discriminant set. The pictures clearly suggest an invariance of the three configurations under reflection with respect to the x-axis. There is another invariance with respect to the y-axis, which is given by this operation followed by a change of colour. As a consequence, the composition of these two elements (minus identity) must be a symmetry which interchanges colour. In fact, we should recognize *a priori* minus identity in the set of symmetries of all these cases by their linearity, as mentioned above. The third picture has also six rotational symmetries, three of which are colour-preserving and the other three are colour-interchanging. In fact, the full symmetry group of pictures (a) and (b) is $\mathbf{Z}_2 \times \mathbf{Z}_2$, generated by the reflections across the axes, and the full symmetry group of picture (c) is the dihedral group $\mathbf{D}_6$, generated by a reflection and a rotation of order six. As these examples illustrate, the group action must be defined taking colour changes into consideration at the region on the plane where (1) defines a bivalued direction field.

This thesis introduces the systematic study of symmetries in binary differential equations and it is organized as follows:

In Chapter 1 we introduce the notion of symmetries in a BDE, namely when the equation is invariant under the linear action of a subgroup Γ of the orthogonal group $\mathbf{O}(2)$. We formalize the concept using group representation theory on the tangent bundle on which the associated quadratic 1-form is defined. The main result is Theorem 1.1.5, which establishes a formula that reveals the effect of a symmetry in the configuration geometry in simple algebraic terms. For BDEs whose coefficients do not vanish simultaneously at any point we define a surface associated with it and, via an induced action (Proposition 1.2.3) it is possible to observe the interchange of foliations of the configuration by a permutation of disjoint components of the surface. We also make an investigation of symmetries of linear 1-forms. This chapter finishes raising an attempt to relate symmetries of quadratic 1-form with symmetries of its associated pair of linear 1-forms, which is an open question by now.

Chapter 2 is dedicated to find the general forms of symmetric quadratic 1-forms. The investigation of occurrence of symmetries is converted in purely algebraic terms: the set of equivariant quadratic 1-forms is identified with the spaces of equivariant matrix-valued mappings, which is a finitely generated module over the ring of invariant functions. Then, the problem to find the general forms is translated to the problem to find generators for a module. In this chapter we generalize the symbolic algorithm developed by Antoneli *et.al.* in [5] and use this generalization to deduce the general forms of equivariant quadratic 1-forms under any compact subgroup of the orthogonal group $\mathbf{O}(2)$. For each case we illustrate with an example and at the end we present a summarizing table.

In Chapter 3 we study BDEs whose coefficients are homogeneous polynomial functions of degree n. This particular class of BDEs has the property that the symmetry group is always nontrivial. We see that the invariant straight lines that can occur have different behavior depending on parity of the degree of the coefficients. Indeed, if n is even the invariant straight line belongs to one foliation, whereas, if n is odd it splits into pieces on the two foliations and on the discriminant set. A special attention is given to constant and linear cases, namely when the coefficient functions have degree 0 and 1, respectively, to which, in addition of the invariant straight lines, we characterize the groups that can be realizable as a symmetry group in both cases.

The aim of Chapter 4 is to obtain the normal forms of BDEs whose coefficients are linear functions. We define the Γ-equivalence between such equations, and deduce the normal forms for each group that can be realizable as a symmetry group of this type of binary differential equations. The results are summarized in a table at the end of the chapter. Some normal forms present a modal parameter, which does not occur when the symmetries are not taken into account.

In the end of the thesis we present some open questions that we intend to address as future works.

CHAPTER

1

SYMMETRIC QUADRATIC 1-FORMS

In this chapter we formalize the concept of a symmetric binary differential equation, under the linear action of a compact Lie subgroup Γ of the orthogonal group $\mathbf{O}(2)$ via group representation theory on the tangent bundle. The main result in this chapter is Theorem 1.1.5 which establishes a formula that reveals the effect of a symmetry on the geometry configuration in simple algebraic terms. We also relate the symmetries of the discriminant set to the symmetries of the quadratic 1-form. The combination of this information gives us a way to detect the symmetry group of a BDE through its associated configuration (Remark 1.1.4).

In Section 1.2, for a quadratic 1-form whose the coefficients do not vanish at any point, we induce an action on surface M associated with the BDE (Definition 1.2.1). The interchange of foliations observed in it configuration is translated into a permutation of the disjoints components of the surface. Finally, in Section 1.3 we discuss the symmetries of a linear 1-form. We prove that the tangent and orthogonal vector fields associated with a symmetric linear 1- form inherit its symmetries but in different way, as we can see in Propositions 1.3.2 and 1.3.3. Although the connection between the symmetries of the linear 1-form and the associated vectors field is well-established, the relation of symmetries of a quadratic 1-form and the pair of tangent vector fields associated with it is not obvious, as is shown in the Subsection 1.3.1.

1.1 The symmetry group of a binary differential equation

Consider a BDE of the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0, \tag{1.1}$$

where the coefficients a, b, c are smooth real functions on $\mathbb{R}^2$. We mention that all that is done in this work holds equality for BDEs defined on an open set of $\mathbb{R}^2$.

The function $\delta : \mathbb{R}^2 \to \mathbb{R}$, $\delta(x,y) = (b^2 - ac)(x,y)$, is the *discriminant function* and its zero set $\Delta = \{(x,y) \in \mathbb{R}^2 : (b^2-ac)(x,y) = 0\}$ is the *discriminant set* of the BDE. Let $\mathscr{Q}(\mathbb{R}^2)$ denote the set of real C^∞ quadratic differential 1-forms on $\mathbb{R}^2$, $\omega : T\mathbb{R}^2 \to \mathbb{R}$,

$$\omega(x,y,dx,dy) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2, \tag{1.2}$$

with a,b,c C^∞ functions on $\mathbb{R}^2$. The BDE (1.1) is so given by the equation $\omega = 0$.

Let Γ be a compact Lie group acting linearly on $\mathbb{R}^2$. This induces an action of Γ on the tangent bundle $T\mathbb{R}^2 = \{((x,y),(X,Y)) : (x,y),(X,Y) \in \mathbb{R}^2\}$,

$$\begin{array}{rcl} \Gamma \times T\mathbb{R}^2 & \to & T\mathbb{R}^2 \\ (\gamma,(x,y),(X,Y)) & \mapsto & \gamma \cdot ((x,y),(X,Y)) \;=\; \big(\gamma(x,y),(d\gamma)_{(x,y)}(X,Y)\big), \end{array} \tag{1.3}$$

where, $(d\gamma)_{(x,y)}(X,Y)$ is in fact simply $\gamma(X,Y)$ by the linearity of the action.

The symmetry group of a binary differential equation $\omega = 0$ is a subgroup of $\mathbf{O}(2)$ that leaves invariant the configuration of its integral curves. Now, there is an action of Γ on $\mathscr{Q}(\mathbb{R}^2)$ given by $(\gamma\omega)(X) = \omega(\gamma^{-1}X)$, $X \in T\mathbb{R}^2$. For each $\gamma \in \Gamma$, we have that $\gamma\omega = \pm\omega$, since the only nontrivial one-dimensional representation of a compact Lie group is the sign representation. This representation on $\mathbb{R}$ is then defined as $\eta : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}$ by requiring that $\eta(\gamma)\omega = \gamma\omega$. In a geometrical point of view, the tangent space at (x,y) is divided into cones $C_\pm$ such that $\omega(X) > 0$ (respectively < 0) on the interior of C_+ (respectively C_-). We change sign when tangent cones are mapped to opposite sign cones by γ. Of course, the foliations (determined by cone bounderies) do not depend on sign, which is why we can use a bigger group of symmetries. We then define:

Definition 1.1.1. *Let Γ be a compact Lie group acting linearly on $\mathbb{R}^2$ and $\eta : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}$ a one-dimensional representation of Γ. An element $\omega \in \mathscr{Q}(\mathbb{R}^2)$ is Γ-equivariant if, for all $\gamma \in \Gamma$,*

$$\omega(\gamma \cdot (x,y,dx,dy)) \;=\; \eta(\gamma)\omega(x,y,dx,dy). \tag{1.4}$$

If ω is Γ-equivariant, then the equation $\omega = 0$ is Γ-invariant or, as we shall also say, Γ is the symmetry group of the BDE. We denote by $\mathscr{Q}[\Gamma,\eta]$ the set of Γ-equivariant quadratic 1-forms. The group of symmetries of a BDE generally admits, by its nature, an index-2 normal subgroup, which is precisely the case when the group homomorphism η in Definition 1.1.1 is nontrivial.

Proposition 1.1.2. *Let $\omega \in \mathscr{Q}[\Gamma,\eta]$. The discriminant function $\delta : \mathbb{R}^2 \to \mathbb{R}$ of ω is invariant under the action of Γ, that is, $\delta(\gamma(x,y)) = \delta(x,y)$, $\forall \gamma \in \Gamma$.*

Proof. : Observe that $\eta^2 \equiv 1$, so by (1.4) the result follows. $\square$

Definition 1.1.3. *The symmetry group of a set $W \subset \mathbb{R}^2$ is the subgroup $\Sigma(W)$ of $\mathbf{O}(2)$ that leaves W setwise invariant, that is*

$$\gamma w \in W, \ \forall w \in W, \ \forall \gamma \in \Sigma(W).$$

Remark 1.1.4. *Let $\Sigma(\Delta) \leq \mathbf{O}(2)$ denote the group of symmetries of the discriminant set Δ of $\omega \in \mathscr{Q}[\Gamma, \eta]$. Then*

$$\Gamma \leq \Sigma(\Delta). \tag{1.5}$$

In other words, symmetries of a BDE are at most the symmetries of the discriminant set. This can be of practical use when detecting the symmetry group of the equation if we know the shape of Δ. Clearly the equality in (1.5) is not always true: in the example where the configuration is given by Figure 1(a), the discriminant set is the origin, so $\Sigma(\Delta) = \mathbf{O}(2)$ whereas $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$.

It is not obvious whether $\eta(\gamma) = 1$ or -1 for each γ in the symmetry group of a BDE. As we shall see in Theorem 1.1.5, this depends not only whether γ preserves or interchanges foliations, but also whether it preserves or inverts orientation on the plane. This theorem is the main result of this chapter.

Solutions of (1.1) are nonoriented curves, associated with direction fields. At the region on the plane where the discriminant function is positive, these form a pair of foliations $\mathscr{F}_1$ and $\mathscr{F}_2$. This pair is, in turn, associated with the vector fields

$$F_i(x,y) \ = \ (a(x,y), -b(x,y) + (-1)^i \sqrt{\delta(x,y)}), \ \ i = 1,2, \tag{1.6}$$

To state the result, we introduce another homomorphism: consider the open region on the plane

$$\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ \delta(x,y) > 0 \text{ and } a(x,y) \neq 0\}, \tag{1.7}$$

and consider the restriction of the action of Γ on Ω. This is well-defined since the discriminant set Δ is Γ-invariant and by the equality (1.4) if $a(x,y) \neq 0$ then $a(\gamma(x,y))$ so is. For BDEs (1.1) for which Ω is not empty, we introduce the homomorphism $\lambda : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}$,

$$\lambda(\gamma) = \begin{cases} 1, & \gamma(\mathscr{F}_i) = \mathscr{F}_i \\ -1, & \gamma(\mathscr{F}_i) = \mathscr{F}_j, \ j \neq i, \end{cases} \tag{1.8}$$

$i, j \in \{1,2\}$. It follows directly from this definition that the subgroup of symmetries of each foliation $\mathscr{F}_i$, $i = 1,2$, is $\Sigma(\mathscr{F}_i) = \ker \lambda$.

Theorem 1.1.5. *Let η, $\lambda : \Gamma \to \mathbf{Z}_2$ be the two group homomorphisms of Definition 1.1.1 and of (1.8). Then, for all $\gamma \in \Gamma$,*

$$\lambda(\gamma) = \det(\gamma)\, \eta(\gamma).$$

Proof. At $(x,y) \in \Omega$ consider the pair of tangent vectors given in (1.6), that are also transversal since δ is positive and $a \neq 0$. From the definition of the action Γ on $T\mathbb{R}^2$, the pair of tangent vectors to the two solution curves at $\gamma(x,y)$ is given by

$$\gamma F_i(x,y),\ i = 1,2.$$

On the other hand, from the equivariance of ω under Γ, the vectors

$$v_i = (\eta(\gamma)a(x,y), -\eta(\gamma)b(x,y) + (-1)^i\sqrt{\delta(x,y)}),\ i = 1,2,$$

are also tangent vectors to the two solution curves at $\gamma(x,y)$. By symmetry it follows that these two pairs must be parallel, i.e., there exists a nonzero α such that, for $i \neq j \in \{1,2\}$,

$$\gamma F_i(x,y) = \begin{cases} \alpha v_i, & \text{if } \lambda(\gamma) = 1 \\ \alpha v_j, & \text{if } \lambda(\gamma) = -1. \end{cases} \tag{1.9}$$

Also, by the orthogonality of the action, we have $\alpha = \pm 1$. Now, consider the two matrices M_1 and M_2 whose columns are the vectors $\gamma F_1, \gamma F_2$ and v_1, v_2 both calculated at (x,y), respectively, that is,

$$M_1 = \begin{pmatrix} \gamma F_1 & \gamma F_2 \end{pmatrix},\ M_2 = \begin{pmatrix} \eta(\gamma)a & \eta(\gamma)a \\ -\eta(\gamma)b - \sqrt{\delta} & -\eta(\gamma)b + \sqrt{\delta} \end{pmatrix}.$$

From (1.9) it follows that

$$\det(M_1) = \alpha^2\lambda(\gamma)\det(M_2) = \lambda(\gamma)\det(M_2).$$

Finally, $\det(M_1) = \det(\gamma)\ 2a\sqrt{\delta}$, and $\det(M_2) = \eta(\gamma)2a\sqrt{\delta}$. Hence,

$$\det(\gamma) = \eta(\gamma)\lambda(\gamma),$$

which implies the result since these are all group homomorphisms $\Gamma \to \mathbf{Z}_2 = \{\pm 1\}$ and the $\det(M_1), \det(M_2)$ not vanish at any $(x,y) \in \Omega$. □

Corollary 1.1.6. *If* $\ker\lambda \cap \ker\eta$ *is finite, then it is a cyclic subgroup of* Γ.

Proof. Let $\gamma \in \ker\lambda \cap \ker\eta$, so $\det(\gamma) = 1$. Hence $\ker\lambda \cap \ker\eta \subset \mathbf{SO}(2)$, thus, by the finitude of the intersection the result holds. □

This corollary gives a simple way to determine the existence, or nonexistence, of a BDE with a given pair of a group and a homomorphism η. For instance, there is no BDE with symmetry group $\mathbf{D}_n$, n even, for which the subgroup $\mathbf{D}_{n/2}$ preserves foliations, *i.e.* $\ker\lambda = \mathbf{D}_{n/2}$, if $\ker\eta = \mathbf{D}_n$.

A direct consequence of the theorem above is the following:

Remark 1.1.7. *Theorem 1.1.5 adds information to the inclusion (1.5) when detecting the symmetry group* Γ *of a BDE. In fact, it provides the construction of the homomorphism* η *by the geometrical investigation of whether each element* $\gamma \in \Gamma$ *preserves the foliations (*$\lambda(\gamma) = 1$*) or interchanges the foliations (*$\lambda(\gamma) = -1$*). To illustrate, consider the pictures in Figure 1 on the Introduction. In (a) and (b), foliations are interchanged by* κ_y*, the reflexion with respect to y-axis, whereas they are preserved by* κ_x*, the reflection with respect to x-axis, and now we use* $\det(\kappa_x) = \det(\kappa_y) = -1$ *to conclude by Theorem 1.1.5 that* $\eta(\kappa_x) = -\eta(\kappa_y) = -1$*. These are the generators of the symmetry group* $\mathbf{Z}_2 \times \mathbf{Z}_2$*, and so the homomorphism* η *is well-established for these examples. In (c) foliations are interchanged by* κ_y *and rotation of* $\pi/3$*; since these are orientation reserving and orientation preserving respectively, it follows that* η *assumes* 1 *and* -1*, respectively. These are the generators of the symmetry group* $\mathbf{D}_6$*, and so the homomorphism* η *is well-established.*

1.2 A surface associated with the BDE

Let $\omega \in \mathscr{Q}(\mathbb{R}^2)$ be a quadratic differential 1-form with coefficient functions a, b, c and suppose $a \neq 0$. Set $U = \Omega \cup \Delta$, Ω in (1.7) and Δ the discriminant set. Define the functions $p_i : U \to \mathbb{R}$, given by

$$p_i(x,y) = \frac{-b(x,y) + i\sqrt{\delta(x,y)}}{a(x,y)}, \quad i = -1, 1. \tag{1.10}$$

Since the function a does not vanish on U, we can write the tangent vector fields to the foliations associated to the BDE as

$$F_i(x,y) = \big(1, p_i(x,y)\big), i = -1, 1, \tag{1.11}$$

In other words, the functions p_i give the slopes of the tangent lines at each leaf, passing through the point (x,y), with respect to the x-axis. Now we define $\phi_i : U \to \mathbb{R}^3$ by

$$\phi_i(x,y) = (x, y, p_i(x,y)), \; i = -1, 1. \tag{1.12}$$

Definition 1.2.1. *Let* $\omega \in \mathscr{Q}(\mathbb{R}^2)$ *be a quadratic 1-form with coefficients* a, b, c *and* $a \neq 0$*. Let* $\phi_i, i = -1, 1$*, as in (1.12). The surface*

$$M = \phi_1(U) \cup \phi_{-1}(U)$$

is called the surface of the BDE $\omega = 0$*.*

A decomposition of M into disjoint components is given in the following proposition. This good decomposition allows us to see the effect of the symmetries of the BDE on M.

Proposition 1.2.2. *Let* $\omega \in \mathscr{Q}(\mathbb{R}^2)$ *with coefficients* a, b, c *and* $a \neq 0$*. Let* $\phi_i, i = -1, 1$*, as in (1.12). The surface* M *of* $\omega = 0$ *can be decomposed into the form*

$$M = \phi_1(\Omega) \,\dot\cup\, \phi_{-1}(\Omega) \,\dot\cup\, \phi_1(\Delta). \tag{1.13}$$

Proof. This is direct from the fact that the equality of the functions $\phi_i, i = -1, 1$, occurs on Δ, so *i. e.*, $\phi_1(U) \cap \phi_{-1}(U) = \phi_1(\Delta)$. $\square$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the standard projection $(x,y,z) \mapsto (x,y)$. By construction of the surface M, π restricted to M is a double covering map of Ω. Moreover, by the continuity of the functions $\phi_i, i = -1, 1$, $\phi_1(\Delta)$ is a curve on M and its projection is the discriminant curve Δ.

It is expected that symmetries of a quadratic 1-form ω reflect on the surface M. We investigate here the action of the symmetry group on the components $\phi_i(\Omega), i = -1, 1$, and on $\phi_1(\Delta)$. The configuration is Γ-invariant, that is, the set of integral curves remains the same under the action of Γ, so it is expected that M is also Γ-invariant. Moreover, the action on M must take into account the interchange of foliations that may occur in the configuration associated with the BDE.

The foliations of a BDE are in a one-to-one correspondence with the vector fields $F_i, i = -1, 1$ in (1.11), which, in turn, are in one-to-one correspondence with the functions $p_i, i = -1, 1$. By Theorem 1.1.5 we know that the homomorphism $\lambda : \Gamma \to \mathbf{Z}_2$ governs the interchange of foliations at the configuration associated with the BDE. Since U is invariant under the action of Γ, for each $(x,y) \in U$ and $\gamma \in \Gamma$ we define

$$\gamma \cdot p_i(x,y) = p_{i\lambda(\gamma)}(\gamma(x,y)) = \frac{-b(\gamma(x,y)) + i\lambda(\gamma)\sqrt{\delta(\gamma(x,y))}}{a(\gamma(x,y))}, \quad i = -1, 1.$$

Therefore, the action of Γ on M, induced from the action on the quadratic 1-form, is given by:

Proposition 1.2.3. *Let $\omega \in \mathscr{Q}[\Gamma, \eta]$ with coefficients a, b, c and $a \neq 0$. Let M be the surface of the equation $\omega = 0$ and $\phi_i, i = -1, 1$ as in (1.12). The mapping*

$$\begin{array}{ccl} \Gamma \times M & \to & M \\ \big(\gamma, (x,y,p_i(x,y))\big) & \mapsto & \gamma \star \big(x,y,p_i(x,y)\big) = \big(\gamma(x,y), \gamma \cdot p_i(x,y)\big) \end{array}$$

defines an action of Γ on M.

Proof. The first step is prove that $\big(\gamma(x,y), \gamma \cdot p_i(x,y)\big) \in M$. In fact, U is Γ-invariant, so $\gamma(x,y) \in U$, and observe that

$$\big(\gamma(x,y), \gamma \cdot p_i(x,y)\big) = \phi_{i\lambda(\gamma)}(\gamma(x,y)), \quad i = -1, 1, \tag{1.14}$$

where $i\lambda(\gamma) = -1, 1$. Thus, $\big(\gamma(x,y), \gamma \cdot p_i(x,y)\big) \in M$.

The second step is to show that, for $\gamma_1, \gamma_2 \in \Gamma$,

$$(\gamma_1\gamma_2) \star \big(x,y,p_i(x,y)\big) = \gamma_1 \star \big(\gamma_2 \star \big(x,y,p_i(x,y)\big)\big), \quad \forall (x,y) \in U,$$

which holds since λ is a group homomorphism, that is, $\lambda(\gamma_1\gamma_2) = \lambda(\gamma_1)\lambda(\gamma_2)$. $\square$

The proof of the proposition above shows us how the group acts on the components of the surface M. This is what we present in the next corollary:

Corollary 1.2.4. *From (1.14), if $\lambda(\gamma) = 1$, γ leaves the components $\phi_i(\Omega), i = -1, 1$ invariant, whereas if $\lambda(\gamma) = -1$, γ takes $\phi_i(\Omega)$ into $\phi_{i\lambda(\gamma)}(\Omega), i = -1, 1$. In other words, the homomorphism λ permutes the components $\phi_i(\Omega)$ of M. Moreover, from the invariance of the discriminant function the component $\phi_1(\Delta)$ does not "feel" the action of the group.*

We can see this behavior in the next two examples:

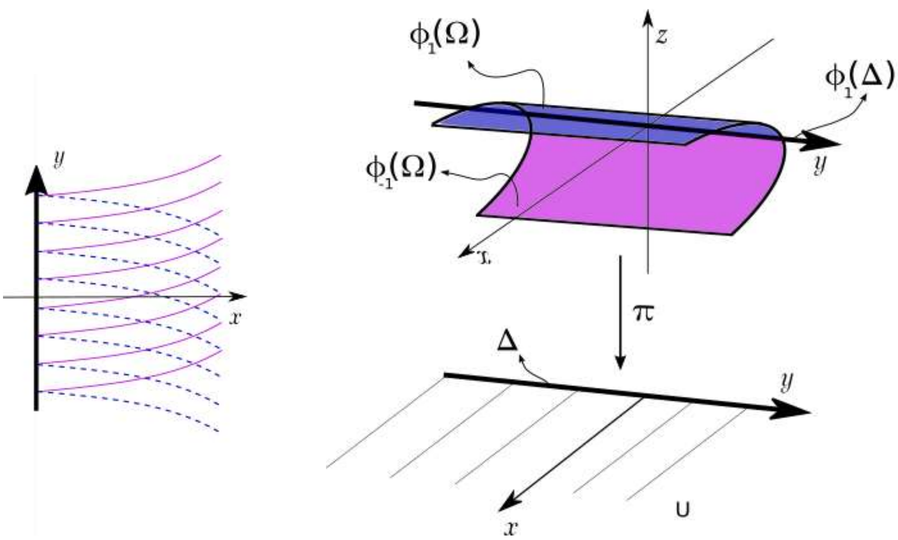
Example 1.2.5. *Consider the equation*

$$\omega(x, y, dx, dy) = dy^2 - x dx^2 = 0, \tag{1.15}$$

which has symmetry group $\Gamma = \mathbf{Z}_2(\kappa_x)$, that is, the group generated by the reflection with respect to the x-axis. The homomorphism η is trivial and $\ker \lambda = \{I\}$, by Theorem 1.1.5, that is, the reflection κ_x interchanges foliations. The sets U, Ω and Δ are: $U = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}; \Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\Delta = \{(0, y) : y \in \mathbb{R}\}$. The functions $p_i : U \to \mathbb{R}$ and $\phi_i : U \to \mathbb{R}^3$ are given by $p_i(x, y) = i\sqrt{x}$ and

$$\phi_i(x, y) = (x, y, i\sqrt{x}), \quad i = -1, 1.$$

The vector fields F_i tangent to the foliations $\mathscr{F}_i$ are $F_i(x, y) = (1, p_i(x, y)), i = -1, 1$, respectively. The disjoint components of M are $\phi_{-1}(\Omega) = \{z^2 - x = 0 : x > 0, z < 0\}$, $\phi_1(\Omega) = \{z^2 - x = 0 : x, z > 0\}$ and $\phi_1(\Delta) = \{(0, y, 0), y \in \mathbb{R}\}$.



(a) (b)

Figure 2 – Configuration and surface associated with the BDE $(1, 0, -x)$.

In Figure 2(a) we have the configuration of (1.15), in blue dashed lines we have the corresponding solutions to the foliation $\mathscr{F}_1$, in magenta solid lines the corresponding solutions *to the foliation $\mathscr{F}_{-1}$. The thick black line is the discriminant set. In Figure 2(b) we have the surface associated with (1.15), in blue we have the component $\phi_1(\Omega)$, in magenta the component $\phi_{-1}(\Omega)$ and the thick black line represents $\phi_{-1}(\Delta)$.*

Example 1.2.6. *Consider the equation*

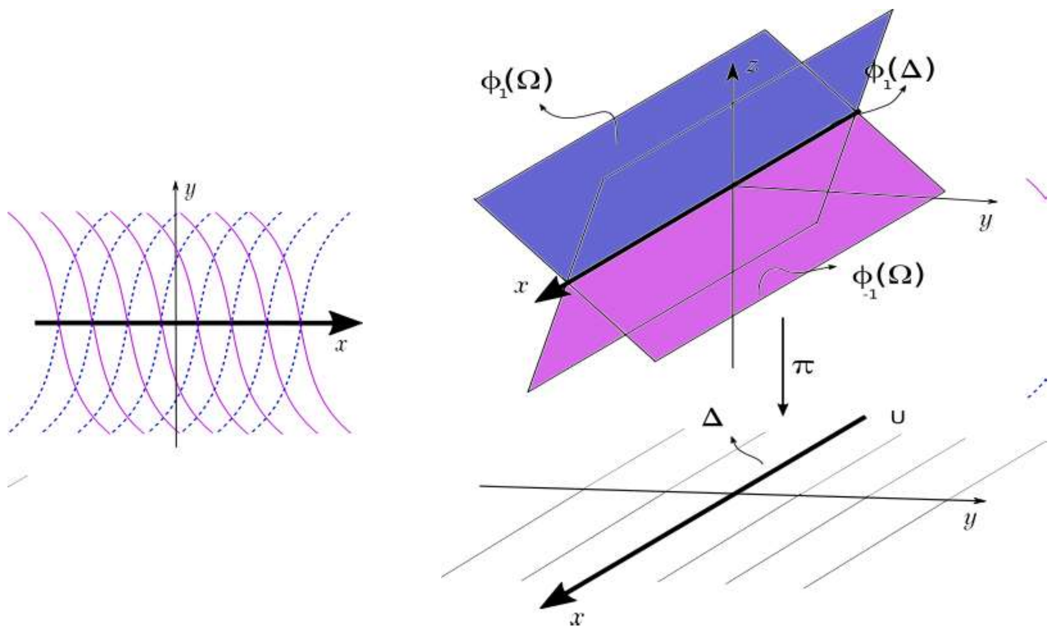
$$\omega(x,y,dx,dy) = dy^2 - y^2dx^2 = 0, \tag{1.16}$$

which has symmetry group $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$, that is, the group generated by the reflections with respect to the x-axis and to the y-axis. The homomorphism η is trivial and $\ker\lambda = \mathbf{Z}_2(-I)$, which means that the reflections across the axis interchange foliations. Here $U = \mathbb{R}^2$, $\Delta = \{(x,0) : x \in \mathbb{R}\}$ and $\Omega = \mathbb{R}^2 \setminus \Delta$. The functions $p_i : U \to \mathbb{R}$, and $\phi_i : U \to \mathbb{R}^3$ are given by $p_i(x,y) = i\sqrt{y^2}$ and

$$\phi_i(x,y) = (x,y,i\sqrt{y^2}), \;\; i = -1,1.$$

The vector fields F_i tangent to the foliations $\mathscr{F}_i$ are $F_i(x,y) = (1,p_i(x,y)), i = -1,1,$ respectively. The disjoint components of M are $\phi_{-1}(\Omega) = \{z^2 - y^2 = 0 : z < 0\}$, $\phi_1(\Omega) = \{z^2 - y^2 = 0 : z > 0\}$ and $\phi_1(\Delta) = \{(x,0,0), x \in \mathbb{R}\}$.

In Figure 3(a) we have the configuration associated with the BDE (1.16) and in Figure 3(b) we have the surface associated with this BDE.



(a) (b)

Figure 3 – Configuration and surface associated with the BDE $(1,0,-y^2)$.

The surface M is also given by the zeros of the implicit differential equation

$$F(x,y,p) = a(x,y)p^2 + 2b(x,y)p + c(x,y), \qquad p = \frac{dy}{dx}. \tag{1.17}$$

Looking at $M = F^{-1}(0)$ the next result concerns about the regularity of M.

Lemma 1.2.7. *If the coefficient function a does not vanish, then* 0 *is a regular value of F if and only if* 0 *is a regular value of the discriminant function δ.*

Proof. Since a does not vanish we can suppose $a \equiv 1$ and rewrite (1.17) as

$$F(x,y,p) = p^2 + 2b(x,y)p + c(x,y), \quad p = \frac{dy}{dx},$$

the discriminant function being given by $\delta(x,y) = b^2(x,y) - c(x,y)$. We denote by the subscripts x,y,p the partial derivative with respect to the variables x,y and p, respectively. By definition, 0 is a regular value of F if and only if $F_x(x,y,p), F_y(x,y,p), F_p(x,y,p)$ do not vanish simultaneously for any $(x,y,p) \in F^{-1}(0)$, where $F_x(x,y,p) = 2b_x(x,y)p + c_x(x,y)$, $F_y(x,y,p) = 2b_y(x,y)p + c_y(x,y)$ and $F_p(x,y,p) = 2p + 2b(x,y)$. Suppose that $F_p = 0$, so $p = -b$ and the expressions of F_x and F_y become

$$F_x(x,y,p) = -2b(x,y)b_x(x,y) + c_x(x,y) = -\delta_x(x,y),$$

$$F_y(x,y,p) = -2b(x,y)b_y(x,y) + c_y(x,y) = -\delta_y(x,y).$$

Therefore, $F_x(x,y,p) = F_y(x,y,p) = F_p(x,y,p) = 0$ if and only if $\delta_x(x,y,p) = \delta_y(x,y,p) = 0$. So the result holds. □

We remark here that Lemma 1.2.7 above is a sufficient criterion for smoothness. In fact, in Example 1.2.6 we have $\delta(x,y) = y^2$, so all points of the discriminant curve are critical points of δ, although, $\Delta = \{(x,0), x \in \mathbb{R}\}$ is smooth.

Remark 1.2.8. *The surface M as presented in (1.17) appears in the qualitative study of implicit differential equations as it can be seen in [6] for example. Generally speaking, the technique used to obtaining local topological models of BDEs consists of lifting the bivalued direction field given by the BDE $\omega = 0$ to a single valued vector field on M. See [16] for the case of BDEs whose coefficients not vanish simultaneously at any point. For the case of BDEs whose coefficients vanish at the origin see [10] (for $c = -a$) and [11] (for general case).*

1.3 Symmetric linear 1-forms

Let Λ be a compact subgroup of $\mathbf{O}(2)$ acting linearly on $\mathbb{R}^2$ and consider its induced action on $T\mathbb{R}^2$ given in (1.3). Let $\mathscr{L}(\mathbb{R}^2)$ denote the set of real C^∞ linear 1-forms on $\mathbb{R}^2$ $\alpha : T\mathbb{R}^2 \to \mathbb{R}$,

$$\alpha(x,y,dx,dy) = A(x,y)dy + B(x,y)dx, \tag{1.18}$$

with A,B smooth functions. We say that α is Λ-equivariant if

$$\alpha(\gamma.(x,y,dx,dy)) = \sigma(\gamma)\alpha(x,y,dx,dy), \forall \gamma \in \Lambda,$$

where $\sigma : \Lambda \to \mathbf{Z}_2 = \{\pm 1\}$ is a group homomorphism. We denote by $\mathscr{L}[\Lambda, \sigma]$ the set of Λ-equivariant linear 1-forms.

We investigate, in this section, the relationship between symmetries of a linear 1-form and symmetries of two special vector fields associated with it, namely the tangent and orthogonal vector fields defined below.

Definition 1.3.1. *Let $\alpha \in \mathscr{L}(\mathbb{R}^2)$. The vector fields $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$F(x,y) = (A(x,y), -B(x,y)) \ \ \text{and} \ \ G(x,y) = (B(x,y), A(x,y)),$$

are called tangent vector field associated with α and orthogonal vector field associated with α, respectively.

We can write (1.18) as

$$\alpha(x,y,dx,dy) = \big\langle G(x,y), (dx,dy) \big\rangle. \tag{1.19}$$

To state the results we introduce the following definition. A mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ is $[\Lambda, \sigma]$-*equivariant* if for all $\gamma \in \Lambda$ and $(x,y) \in \mathbb{R}^2$,

$$G(\gamma(x,y)) = \sigma(\gamma)\gamma G(x,y). \tag{1.20}$$

The Λ-equivariance of α reflects into equivariance conditions in both vector fields, F and G, as we can see in the following two propositions.

Proposition 1.3.2. *Let $\alpha(x,y,dx,dy) = A(x,y)dy + B(x,y)dx$ be a linear 1-form and $G : \mathbb{R}^2 \to \mathbb{R}^2$ the orthogonal vector field associated with it. Then $\alpha \in \mathscr{L}[\Lambda, \sigma]$ if and only if G is $[\Lambda, \sigma]$-equivariant.*

Proof. Suppose G $[\Lambda, \sigma]$-equivariant. By (1.19) we have

$$\begin{aligned}
\alpha(\gamma(x,y,dx,dy)) &= \big\langle G(\gamma(x,y)), \gamma(dx,dy) \big\rangle = \big\langle \sigma(\gamma)\gamma G(x,y), \gamma(dx,dy) \big\rangle \\
&= \sigma(\gamma)\big\langle \gamma G(x,y), \gamma(dx,dy) \big\rangle = \sigma(\gamma)\big\langle G(x,y), (dx,dy) \big\rangle \\
&= \sigma(\gamma)\alpha(x,y,dx,dy),
\end{aligned}$$

for all $\gamma \in \Lambda$. Now suppose that $\alpha \in \mathscr{L}[\Lambda, \sigma]$, then

$$\big\langle G(\gamma(x,y)), \gamma(dx,dy) \big\rangle - \sigma(\gamma)\big\langle G(x,y), (dx,dy) \big\rangle = 0. \tag{1.21}$$

Since $\langle , \rangle$ is Λ-invariant ($\Lambda \subseteq \mathbf{O}(2)$) we can rewrite (1.21) as

$$\big\langle \gamma^t G(\gamma(x,y)) - \sigma(\gamma)G(x,y), (dx,dy) \big\rangle = 0, \ \forall (x,y) \in \mathbb{R}^2, \ \forall \, \gamma \in \Lambda. \tag{1.22}$$

Thus, $\gamma^t G(\gamma(x,y)) - \sigma(\gamma)G(x,y) = 0$, which is equivalent to

$$G(\gamma(x,y)) = \sigma(\gamma)\gamma G(x,y),$$

for all $\gamma \in \Lambda, (x,y) \in \mathbb{R}^2$. $\square$

Proposition 1.3.3. *Let $\alpha(x,y,dx,dy) = A(x,y)dy + B(x,y)dx$ be a linear 1-form and $F,G : \mathbb{R}^2 \to \mathbb{R}^2$ the tangent and orthogonal maps associated with α, respectively. Then G is $[\Lambda,\sigma]$-equivariant if and only if F is $[\Lambda,\xi]$-equivariant, where $\xi(\gamma) = \sigma(\gamma)\det(\gamma), \forall \gamma \in \Lambda$.*

Proof. Suppose G $[\Lambda,\sigma]$-equivariant and write $\gamma \in \Lambda$ as $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so $G(\gamma(x,y)) = \sigma(\gamma)\gamma G(x,y)$ can be written as

$$\begin{pmatrix} B(\gamma(x,y)) \\ A(\gamma(x,y)) \end{pmatrix} = \sigma(\gamma) \begin{pmatrix} aB(x,y) + bA(x,y) \\ cB(x,y) + dA(x,y) \end{pmatrix}$$

Thus, $F(\gamma(x,y)) = (A(\gamma(x,u)), -B(\gamma(x,y)))$, that is,

$$\begin{pmatrix} A(\gamma(x,y)) \\ -B(\gamma(x,y)) \end{pmatrix} = \sigma(\gamma) \begin{pmatrix} cB(x,y) + dA(x,y) \\ -aB(x,y) - bA(x,y) \end{pmatrix} = \sigma(\gamma) \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} A(x,y) \\ -B(x,y) \end{pmatrix}.$$

Set $M = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. Direct calculations show that $M = \det(\gamma)(\gamma^{-1})^t$. However, $\gamma \in \mathbf{O}(2)$ so $\gamma^{-1} = \gamma^t$. Therefore,

$$F(\gamma(x,y)) = \sigma(\gamma)\det(\gamma)\gamma F(x,y) = \xi(\gamma)F(x,y). \tag{1.23}$$

Suppose now that F is $[\Lambda,\xi]$-equivariant and write $\gamma \in \Lambda$ as before. The relation (1.23) allows us to write $G(\gamma(x,y)) = (B(\gamma(x,y)), A(\gamma(x,y)))$ as

$$\begin{pmatrix} B(\gamma(x,y)) \\ A(\gamma(x,y)) \end{pmatrix} = \xi(\gamma) \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B(x,y) \\ A(x,y) \end{pmatrix} = \xi(\gamma) M \begin{pmatrix} B(x,y) \\ A(x,y) \end{pmatrix}.$$

Now, $M = \det(\gamma)(\gamma^{-1})^t$, $\xi(\gamma) = \sigma(\gamma)\det(\gamma)$ and $\det^2(\gamma) = 1$ since $\gamma \in \mathbf{O}(2)$, so we conclude that

$$G(\gamma(x,y)) = \sigma(\gamma)\gamma G(x,y),$$

for all $\gamma \in \Lambda, (x,y) \in \mathbb{R}^2$. □

Remark 1.3.4. *A vector field that satisfies the relation (1.20) is known in the literature as a reversible-equivariant vector field, see [5] for example. A dynamical system governed by a reversible-equivariant vector field, that is, in the presence of symmetries (equivariances) and reversing symmetries (reversibilities) is called reversible-equivariant system. In terms of the dynamics, recall that both symmetries and reversing symmetries take trajectories into trajectories, the first ones preserving direction, whereas the others revert direction. Proposition 1.3.3 gives an interesting information about the symmetries of a vector field and its orthogonal vector field. For instance, a vector field and its orthogonal vector field are $[\Lambda,\sigma]$-equivariant if and only if each element in Λ is orientation preserving. Yet, if a vector field has only symmetries ($\xi \equiv 1$), in order to its orthogonal vector field to have reversibilities, it is enough (and by Proposition 1.3.3 also necessery) that one element of the group to act as a time reversing ($\sigma(\gamma) = \det(\gamma)$).*

1.3.1 Quadratic 1-forms and linear 1-forms

Let $\omega \in \mathscr{Q}[\Gamma,\eta]$ be a quadratic 1-form with coefficients a,b and c. The solutions of $\omega = 0$ form a pair of transverse foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on Ω. The vector fields $F_i, i = 1,2$, given in (1.6), tangent to these foliations can be associated with the linear 1-forms

$$\alpha_i(x,y,dx,dy) = a(x,y)dy - \big(-b(x,y) + (-1)^i\sqrt{\delta(x,y)}\big)dx, \ \ i = 1,2.$$

A natural question is what the symmetries of a pair of linear 1-forms, and consequently, of a pair of vector fields relate with the symmetries of the quadratic 1-form generated by this pair and vise-versa. Consider the example,

$$\omega(x,y,dx,dy) = ydy^2 + 2xdxdy - ydx^2 = 0,$$

which has symmetry group $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ and the homomorphisms $\eta, \lambda : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}$ are characterized by $\ker\eta = \mathbf{Z}_2(\kappa_y)$ and $\ker\lambda = \mathbf{Z}_2(\kappa_x)$. The associated vector fields are

$$F_i(x,y) \ = \ (y, -x + (-1)^i\sqrt{x^2+y^2}), \ \ i = 1,2.$$

The configuration of this equation is given in Fig. 1(a). Both F_1 and F_2 are $[\Lambda,\sigma]$-equivariant vector fields where $\Lambda = \mathbf{Z}_2(\kappa_x)$ and $\sigma(\kappa_x) = -1$, that is,

$$F_i(\kappa_x(x,y)) \ = \ -\kappa_x F_i(x,y), i = 1,2.$$

As the picture suggests, this reflection is in fact a symmetry of the BDE. Now, the combination of the two foliations adds symmetries to the whole picture, leading to a configuration which is also symmetric with respect to the reflection on the y-axis. By the nature of this additional symmetry, this element should invert foliations. In fact, we prove that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is the symmetry group of the BDE.

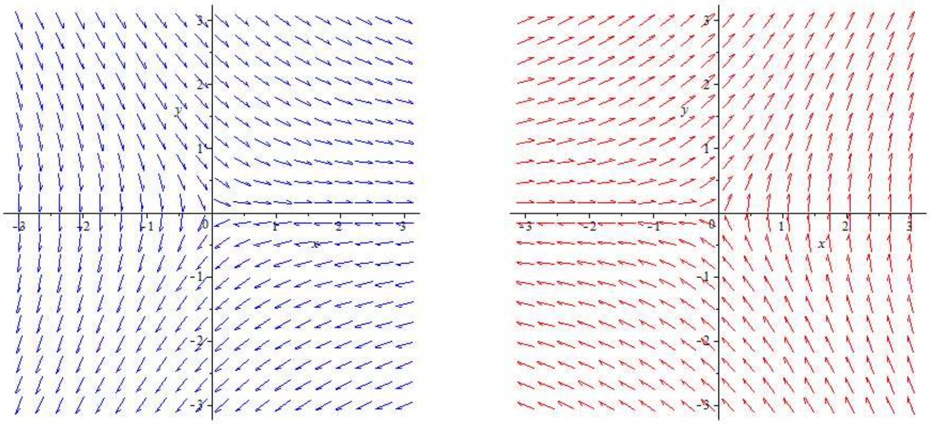
Consider now the example,

$$\omega(x,y,dx,dy) = ydy^2 - 2xdxdy - ydx^2 = 0,$$

which has symmetry group $\Gamma = \mathbf{D}_6$, $\ker\eta = \mathbf{D}_3(\kappa_y)$, the group generated by the reflection κ_y and by the rotation of angle $2\pi/3$, and $\ker\lambda = \mathbf{D}_3(\kappa_x)$ generated by the reflection κ_x with respect to the x-axis and the rotation of angle $2\pi/3$. The associated vector fields are

$$F_i(x,y) \ = \ (y, x + (-1)^i\sqrt{x^2+y^2}), \ \ i = 1,2. \tag{1.24}$$

The configuration of this equation is given in Fig. 1(c). However, both F_1 and F_2 are not $[\mathbf{D}_3(\kappa_x),\sigma]$-equivariant vector fields for any homomorphism $\sigma : \mathbf{D}_3(\kappa_x) \to \mathbf{Z}_2 = \{\pm 1\}$, as we can see in Figure 4(a) and 4(b). Indeed, the only normal index-2 subgroup of $\mathbf{D}_3(\kappa_x)$ is the cyclic group $\mathbf{Z}_3$. Thus $\sigma(R_{2\pi/3})$ must be equal to 1, but this does not occur in both vector fields,



(a) (b)

Figure 4 – Vector fields F_1 and F_2 given in (1.24), respectively.

F_1, F_2. As consequence of Proposition 1.3.2, the linear 1-forms associated with the BDE are not $[\mathbf{D}_3(\kappa_x), \sigma]$-equivariant.

The examples presented in this section show how delicate is the study of symmetries of a quadratic differential 1-form via the symmetries of a pair of associated linear 1-forms. They also show the difficulty to realize how each element of a symmetry group should act on a quadratic 1-form ω generated by this pair, since the symmetry group of the BDE may be larger than the symmetry group of the linear 1-forms. This is an issue that we intend to investigate in near future.

CHAPTER

2

GENERAL FORMS OF SYMMETRIC BDES

One important step in the study of equations under symmetry is to find their general forms. The aim of this chapter is to present the algebraic forms of BDEs symmetric under compact subgroups Γ of $\mathbf{O}(2)$ with its standard action on the plane. The results presented here are also in [25].

We start with a brief discussion about theory of representation and the invariant theory of polynomials. They have proved to be a powerful tool in the algebraic approach of problems involving symmetries. For details see [8] and [21]. In Subsection 2.1.1 we generalize the results of [5] given to Γ-equivariant mappings for possibly distinct representations in the source and target. These results allow us to deduce the general forms of invariant binary differential equations when the homomorphism η is not trivial in Definition 1.1.1. If η is trivial we find the generators of the model of equivariant matrix-value mappings $\mathbb{R}^2 \to M_2(\mathbb{R}^2)$, and project onto the space of mappings $\mathbb{R}^2 \to Sym_2$, where Sym_2 denotes the space of order-2 symmetric matrices, via the projection (2.7).

The deduction of the equivariant general forms under compact subgroups of $\mathbf{O}(2)$ is given from Section 2.3 to Section 2.8. At each section an example is presented to illustrate the configuration associated with each group for distinct choices of the homomorphism η. All figures in this thesis have been made using a computer program written by A. Montesinos [4]. We finish the chapter presenting a table, Table 1, which summarizes the results of the previous sections.

2.1 Algebraic statements

Let Γ be a compact Lie group acting linearly on a real vector space V of finite dimension n. Denote by $GL(n)$ the group of nonsingular $n\times n$ matrices over $\mathbb{R}$. There is a Γ-invariant inner product on V under which the associated representation $\rho:\Gamma\to GL(n)$, $\rho(\gamma)x=\gamma x$, is orthogonal, *i.e.* for $\gamma\in\Gamma$, $\rho(\gamma)\in\mathbf{O}(n)$, the group of orthogonal matrices of order n ([21, XII, Proposition 1.3]). Hence, Lie groups in this section are the closed subgroups of $\mathbf{O}(n)$. We denote by (ρ,V) the representation of Γ on V.

Definition 2.1.1. *Let (ρ,V) be a representation of Γ on V. A real function $f:V\to\mathbb{R}$ is Γ-invariant if*

$$f(\rho(\gamma)x)=f(x),\ \ \forall\,\gamma\in\Gamma,\ \forall\,x\in V.$$

The set $\mathscr{P}(\Gamma)$ of Γ-invariant polynomials is a ring over $\mathbb{R}$. A finite set $\{u_1,...u_s\}$ of Γ-invariants generating this ring is called a *Hilbert basis* for $\mathscr{P}(\Gamma)$. The existence of a Hilbert basis was proved by Weyl in 1946 (see [21, XII, Proposition 4.2]).

Definition 2.1.2. *Let (ρ,V) and (ν,W) be representations of Γ on V and on W, respectively. A mapping $g:V\to W$ is Γ-equivariant if*

$$g(\rho(\gamma)x)=\nu(\gamma)g(x),\ \ \forall\,\gamma\in\Gamma,\ \forall\,x\in V.$$

The set $\vec{\mathscr{P}}(\Gamma)$ of Γ-equivariant mappings $(\rho,V)\to(\nu,W)$ with polynomial entries is a module over $\mathscr{P}(\Gamma)$. Poénaru in 1976 [21, XII, Proposition 6.8] proved that $\vec{\mathscr{P}}(\Gamma)$ is finitely generated over the ring $\mathscr{P}(\Gamma)$.

2.1.1 Reynolds operators and algorithm

Consider a one-dimensional representation of Γ,

$$\eta:\Gamma\to\mathbf{Z}_2=\{\pm1\}, \tag{2.1}$$

which is a group homomorphism. We note that $\Gamma_+=\ker\eta$ is a normal subgroup of Γ of index 2 if η is nontrivial. The η*-dual representation* of (ν,W), denoted by ν_η, is defined by the product

$$\gamma\mapsto\nu_\eta(\gamma)=\eta(\gamma)\nu(\gamma).$$

Definition 2.1.3. *Let $\eta:\Gamma\to\mathbf{Z}_2$ be a group homomorphism as in (2.1) and (ρ,V) and (ν,W) representations of Γ. We define the $\mathscr{P}(\Gamma)$-modules of polynomial mappings*

$$\mathscr{P}^\eta(\Gamma):=\{f:V\to\mathbb{R}:f(\rho(\gamma)x)=\eta(\gamma)f(x),\ \ \forall\,\gamma\in\Gamma,\ \forall\,x\in V\}$$

and

$$\vec{\mathscr{P}}^\eta(\Gamma):=\{g:V\to W:g(\rho(\gamma)x)=\eta(\gamma)\nu(\gamma)g(x),\ \ \forall\,\gamma\in\Gamma,\ \forall\,x\in V\}.$$

Any function $f \in \mathscr{P}^{\eta}(\Gamma)$ is Γ-equivariant from (ρ, V) to $(\eta, \mathbb{R})$, and any polynomial mapping $g \in \overrightarrow{\mathscr{P}}^{\eta}(\Gamma)$ is Γ-equivariant from (ρ, V) to (ν_η, W), so the finitude of generators for each, as $\mathscr{P}(\Gamma)$-modules, follows by Poénaru's theorem mentioned above.

A connection is established in [5] between the invariant theory for Γ and $\Gamma_+ = \ker \eta$. This is done through an algebraic algorithm to compute generators of $\overrightarrow{\mathscr{P}}^{\eta}(\Gamma)$ from the knowledge of generators of $\overrightarrow{\mathscr{P}}(\Gamma_+)$, when source and target spaces are the same. In Proposition 2.1.4 and Algorithm 2.1.5 we generalize this, with a similar algorithm to compute generators of Γ-equivariants with possibly distinct source and target.

We follow the notation used in [5] to introduce the Reynolds operators $R : \mathscr{P}(\Gamma_+) \to \mathscr{P}(\Gamma_+)$ and $\overrightarrow{R} : \overrightarrow{\mathscr{P}}(\Gamma_+) \to \overrightarrow{\mathscr{P}}(\Gamma_+)$,

$$R(f)(x) = 1/2(f(x) + f(\rho(\delta)x)) \quad \text{and} \quad \overrightarrow{R}(g)(x) = 1/2(g(x) + \nu(\delta)^{-1} g(\rho(\delta)x)),$$

and, the η-Reynolds operators on $\mathscr{P}(\Gamma_+)$ and on $\overrightarrow{\mathscr{P}}(\Gamma_+)$, $S : \mathscr{P}(\Gamma_+) \to \mathscr{P}(\Gamma_+)$ and $\overrightarrow{S} : \overrightarrow{\mathscr{P}}(\Gamma_+) \to \overrightarrow{\mathscr{P}}(\Gamma_+)$,

$$S(f)(x) = 1/2(f(x) - f(\rho(\delta)x)) \quad \text{and} \quad \overrightarrow{S}(g)(x) = 1/2(g(x) - \nu(\delta)^{-1} g(\rho(\delta)x)),$$

for an arbitrary fixed $\delta \in \Gamma \setminus \Gamma_+$.

Let us denote by $I_{\mathscr{P}(\Gamma_+)}$ and $I_{\overrightarrow{\mathscr{P}}(\Gamma_+)}$ the identity maps on $\mathscr{P}(\Gamma_+)$ and on $\overrightarrow{\mathscr{P}}(\Gamma_+)$, respectively.

Proposition 2.1.4. *The operators above satisfy the following:*

(a) *They are homomorphisms of $\mathscr{P}(\Gamma)$-modules and*

$$R + S = I_{\mathscr{P}(\Gamma_+)} \quad \textit{and} \quad \overrightarrow{R} + \overrightarrow{S} = I_{\overrightarrow{\mathscr{P}}(\Gamma_+)}.$$

(b) *They are idempotent projections and the following direct sum decompositions of $\mathscr{P}(\Gamma)$-modules hold:*

$$\mathscr{P}(\Gamma_+) = \mathscr{P}(\Gamma) \oplus \mathscr{P}^{\eta}(\Gamma) \quad \textit{and} \quad \overrightarrow{\mathscr{P}}(\Gamma_+) = \overrightarrow{\mathscr{P}}(\Gamma) \oplus \overrightarrow{\mathscr{P}}^{\eta}(\Gamma). \tag{2.2}$$

Proof. Analogous to Propositions 2.3 and 2.4 in [5]. □

The Algorithm 2.1.5 is based on the decompositions (2.2) and on the projection operators S and $\overrightarrow{S}$ applied to a given Hilbert basis of $\mathscr{P}(\Gamma_+)$ and a set of generators of $\overrightarrow{\mathscr{P}}(\Gamma_+)$. The procedure is:

Algorithm 2.1.5. *Let Γ be a closed subgroup of $\mathbf{O}(n)$ and $\eta : \Gamma \to \mathbf{Z}_2$ a homomorphism with $\ker \eta = \Gamma_+$, and let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathscr{P}(\Gamma_+)$ and $\{H_0, \ldots, H_r\}$ a generator set of $\overrightarrow{\mathscr{P}}(\Gamma_+)$ as a $\mathscr{P}(\Gamma_+)$-module;*

1 *Fix* $\delta \in \Gamma \setminus \Gamma_+$ *arbitrary;*

2 *For* $i \in \{1,...,s\}$, *do* $\tilde{u}_i = S(u_i), \tilde{u_0} := 1$;

3 *For* $i \in \{0,...,s\}$ *and* $j \in \{0,...,r\}$, *do* $H_{ij} = \tilde{u}_i H_j$;

4 *For* $i \in \{0,...,s\}$ *and* $j \in \{0,...,r\}$, *do* $\tilde{H}_{ij} = \overrightarrow{S}(H_{ij})$.

Result: $\{\tilde{H}_{ij} : \ 0 \le i \le s, 0 \le j \le r\}$ *is a generator set of* $\overrightarrow{\mathscr{P}}^{\eta}(\Gamma)$ *as a* $\mathscr{P}(\Gamma)$*-module.*

As proved in [5], step 2 above provides a generator set of the $\mathscr{P}(\Gamma)$-module $\mathscr{P}^{\eta}(\Gamma)$ (these are the anti-invariants in that paper). What we also remark at this point is that replacing $\overrightarrow{S}$ by the projection operator $\overrightarrow{R}$ in step 4 we obtain, as expected, a direct way to compute a set of generators for the equivariants under the whole group Γ from the knowledge of equivariants under the subgroup Γ_+. This is formalized below:

Proposition 2.1.6. *Let* Γ *be a compact Lie group acting on* V *and on* W *and* $\{H_{ij} = \tilde{u}_i H_j, 0 \le i \le s, 0 \le j \le r\}$ *a generator set of* $\overrightarrow{\mathscr{P}}(\Gamma_+)$ *as a* $\mathscr{P}(\Gamma)$*-module given by step* 3 *in Algorithm* 2.1.5. *Then*

$$\{\overrightarrow{R}(H_{ij}), 0 \le i \le s, 0 \le j \le r\}$$

generates $\overrightarrow{\mathscr{P}}(\Gamma)$ *as a* $\mathscr{P}(\Gamma)$*-module.*

Proof. Let $g \in \overrightarrow{\mathscr{P}}(\Gamma) \subset \overrightarrow{\mathscr{P}}(\Gamma_+)$. Then $g = \sum_{i,j}^{s,r} p_{ij} H_{ij}, \quad p_{ij} \in \mathscr{P}(\Gamma),\ 0 \le i \le s$ e $0 \le j \le r$. Since $\overrightarrow{R}$ is a $\mathscr{P}(\Gamma)$-homomorphism and $\overrightarrow{R}(g) = g$, then

$$g = \overrightarrow{R}(g) = \overrightarrow{R}\left(\sum_{i,j}^{s,r} p_{ij} H_{ij}\right) = \sum_{i,j}^{s,r} p_{ij} \overrightarrow{R}(H_{ij}).$$

□

2.2 General forms of symmetric matrix-valued mappings

The quadratic differential 1-forms

$$\omega(x,y,dx,dy) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2, \tag{2.3}$$

with a,b,c polynomial functions on $\mathbb{R}^2$, are in one-to-one correspondence with the matrix-valued mapping $B : \mathbb{R}^2 \to Sym_2$,

$$B(x,y) = \begin{pmatrix} c(x,y) & b(x,y) \\ b(x,y) & a(x,y) \end{pmatrix}, \tag{2.4}$$

where Sym_2 denotes the space of symmetric matrices of order 2. The quadratic 1-form (2.3) can be written as

$$\omega \ = \ \begin{pmatrix} dx \\ dy \end{pmatrix}^t B(x,y) \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where superscript t denotes transposition. Let $\omega \in \mathscr{Q}[\Gamma,\eta]$ be a Γ-equivariant quadratic 1-form, and denote the representation of each $\gamma \in \Gamma$ on $\mathbb{R}^2$ by γ itself. Replacing the equivariance condition (1.4) into (2.4), the equivariance condition of B is obtained, with the action on the target given by the homomorphism η and conjugacy:

$$B\big(\gamma(x,y)\big) = \eta(\gamma)\gamma B(x,y)\gamma^t, \ \forall\, \gamma \in \Gamma. \tag{2.5}$$

B satisfying (2.5) is Γ-equivariant, being the action of Γ on Sym_2 given by $\gamma \cdot M = \gamma M \gamma^t, \forall M \in Sym_2$. From now on we shall use this matricial notation to investigate symmetries in BDEs. From this, the problem to find the general form for the Ↄ-equibariant quadratic 1-forms is converted to the problem of finding a set of generators for the module of Γ-equivariant mappings $B : \mathbb{R}^2 \to Sym_2$. These modules are $\overrightarrow{\mathscr{P}}(\Gamma)$ or $\overrightarrow{\mathscr{P}}^\eta(\Gamma)$, depending on whether the group homomorphism $\eta : \Gamma \to \mathbf{Z}_2$ is trivial or nontrivial.

For the computations in the following sections we shall use the action of (subgroups of) $\mathbf{O}(2)$ on $\mathbb{R}^2 \simeq \mathbb{C}$ as the usual semi-direct product of $\mathbf{SO}(2)$ and $\mathbf{Z}_2(\kappa_x)$, using complex coordinates,

$$\theta \cdot z = e^{i\theta} z, \ \text{ and } \ \kappa_x z = \bar{z}, \ \ \theta \in [0,2\pi], \ z \in \mathbb{C}.$$

In Sections 2.3-2.8 we derive the general forms of symmetric BDEs under all closed subgroups Γ of $\mathbf{O}(2)$, up to isomorphic representations, for all possible homomorphisms η. In Subsection 2.9 we summarize the general forms in Table 1.

For a given choice of a group Γ and a homomorphism η we use from now the notation $\Gamma[\ker\eta]$ when η is nontrivial. This notation is motivated by the fact that the definition of η is determined by the subgroup $\ker\eta$, which also motivates the use of the notation $\mathscr{P}[\Gamma,\ker\eta]$ for $\mathscr{P}^\eta(\Gamma)$, $\overrightarrow{\mathscr{P}}[\Gamma,\ker\eta]$ for $\overrightarrow{\mathscr{P}}^\eta(\Gamma)$ and $\mathscr{Q}[\Gamma,\ker\eta]$ for $\mathscr{Q}[\Gamma,\eta]$.

2.3 SO(2)-equivariant quadratic forms

Here we consider the group $\mathbf{SO}(2)$. The only possibility for η is to be trivial, since $\mathbf{SO}(2)$ does not have one index-2 subgroup. We compute generators of $\overrightarrow{\mathscr{P}}(\mathbf{SO}(2))$ by computing generators of $\mathscr{M}(\mathbf{SO}(2))$, the module of $\mathbf{SO}(2)$-equivariant matrix-valued mappings $\mathbb{R}^2 \to M_2(\mathbb{R}^2)$, and projecting onto the space of mappings $\mathbb{R}^2 \to Sym_2$. In complex coordinates we write any element of $\mathscr{M}(\mathbf{SO}(2))$ as

$$w \mapsto \alpha(z)w + \beta(z)\bar{w}, \ \forall w \in \mathbb{C}^2, \tag{2.6}$$

for functions $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, with α_j, β_j, $j = 1,2$, real functions. Associating it with the real matrix

$$M = \begin{pmatrix} \alpha_1+\beta_1 & \beta_2-\alpha_2 \\ \alpha_2+\beta_2 & \alpha_1-\beta_1 \end{pmatrix},$$

the desired quadratic forms are obtained by the projection

$$M \mapsto B = (M+M^t)/2, \tag{2.7}$$

after imposing the $\mathbf{SO}(2)$-symmetry condition. Write (2.6) as

$$M(z)w = \sum \alpha_{jk} z^j \bar{z}^k w + \sum \beta_{jk} z^j \bar{z}^k \bar{w}, \quad \alpha_{jk}, \beta_{jk} \in \mathbb{C}.$$

The equivariance with respect to $\theta \in \mathbf{SO}(2)$ gives

$$M(z)w = \sum \alpha_{jk} e^{i\theta(j-k)} z^j \bar{z}^k w + \sum \beta_{jk} e^{i\theta(j-k-2)} z^j \bar{z}^k \bar{w}. \tag{2.8}$$

So $\alpha_{jk} = \alpha_{jk} e^{i\theta(j-k)}$ and $\beta_{jk} = \beta_{jk} e^{i\theta(j-k-2)}$ for all $\theta \in (0, 2\pi]$, and so

$$\alpha_{jk} = 0 \text{ ou } j = k \quad \text{and} \quad \beta_{jk} = 0 \text{ ou } \ j = k + 2 (\text{mod n}). \tag{2.9}$$

A Hilbert basis for $\mathscr{P}(\mathbf{SO}(2))$ is given in [21],

$$\{u_1 z = z\bar{z}\}.$$

Factor out $z\bar{z}$ in (2.8) and use (2.9) to get

$$M(z)w \;\; = \;\; \sum_k \alpha_k (z\bar{z})^k w + \sum_k \beta_{k+2} (z\bar{z})^k z^2 \bar{w},$$

where $\alpha_k, \beta_k \in \mathbb{C}$, to conclude that a set of generators of $\mathscr{M}_2(\mathbf{SO}(2))$ over $\mathscr{P}(\mathbf{SO}(2))$ is given by the elements

$$M_1(z)w = w, \;\; M_2(z)w = iw, \;\; M_3(z)w = z^2\bar{w}, \;\; M_4(z)w = iz^2\bar{w}.$$

We now apply the projection (2.7) to the elements above to find generators of $\overrightarrow{\mathscr{P}}(\mathbf{SO}(2))$,

$$B_1(z)w = w, \;\; B_3(z)w = z^2\bar{w}, \quad B_4(z)w = iz^2\bar{w}.$$

Rewriting in (x,y)-coordinates we have the generators of the $\mathbf{SO}(2)$-invariant BDEs:

Theorem 2.3.1. *If* $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{SO}(2), \mathbf{SO}(2)]$, *then*

$$a = p_1 + (y^2 - x^2)p_2 + 2xyp_3, \;\; b = 2xyp_2 + (x^2 - y^2)p_3, \;\; c = p_1 + (x^2 - y^2)p_2 - 2xyp_3, \tag{2.10}$$

where $p_i \in \mathscr{P}(\mathbf{SO}(2)), i = 1,2,3.$

We finish this section with an example of an $\mathbf{SO}(2)$-invariant configuration. We choose $p_1 \equiv p_2 \equiv p_3 \equiv 1$ in (2.10), so that the differential form is

$$(1 + y^2 - x^2 + 2xy, \; x^2 - y^2 + 2xy, \; 1 + x^2 - y^2 - 2xy).$$

The homomorphism λ is trivial and the discriminant function is the $\mathbf{O}(2)$-invariant given by

$$\delta(x,y) = 2(x^2 + y^2)^2 - 1.$$

The configuration is illustrated in Fig. 5.

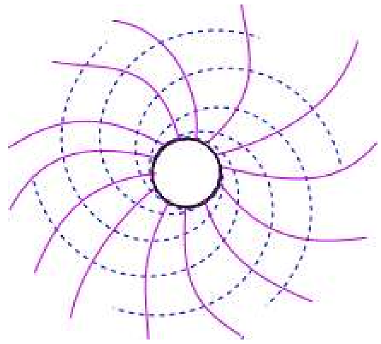


Figure 5 – Configuration with symmetry **SO**(2).

2.4 O(2)-equivariant quadratic forms

Here we consider the group $\mathbf{O}(2)$, with η trivial. The case $\ker\eta = \mathbf{SO}(2)$ is dealt with in Subsection 2.4.1. As in previous section, we compute generators of $\overrightarrow{\mathscr{P}}(\mathbf{O}(2))$ by computing generators of $\mathscr{M}(\mathbf{O}(2))$, the module of $\mathbf{O}(2)$-equivariant matrix-valued mappings $\mathbb{R}^2 \to M_2(\mathbb{R}^2)$, and project onto the space of mappings $\mathbb{R}^2 \to Sym_2$. In complex coordinates we write any element of $\mathscr{M}(\mathbf{O}(2))$ as in (2.6) and we write it as

$$M(z)w = \sum \alpha_{jk} z^j \bar{z}^k w + \sum \beta_{jk} z^j \bar{z}^k \bar{w}, \quad \alpha_{jk}, \beta_{jk} \in \mathbb{C}.$$

The equivariance with respect to reflection κ_x gives

$$M(z)w = \sum \bar{\alpha}_{jk} z^j \bar{z}^k w + \sum \bar{\beta}_{jk} z^j \bar{z}^k \bar{w}. \tag{2.11}$$

So $\alpha_{jk}, \beta_{jk} \in \mathbb{R}$. Now, the equivariance with respect to θ gives

$$M(z)w = \sum \alpha_{jk} e^{i\theta(j-k)} z^j \bar{z}^k w + \sum \beta_{jk} e^{i\theta(j-k-2)} z^j \bar{z}^k \bar{w}. \tag{2.12}$$

So $\alpha_{jk} = \alpha_{jk} e^{i\theta(j-k)}$ and $\beta_{jk} = \beta_{jk} e^{i\theta(j-k-2)}$ for all $\theta \in (0, 2\pi]$, and so

$$\alpha_{jk} = 0 \text{ ou } j = k \quad \text{and} \quad \beta_{jk} = 0 \text{ ou } \ j = k+2. \tag{2.13}$$

A Hilbert basis for $\mathscr{P}(\mathbf{O}(2))$ is given in [21],

$$\{u_1(z) = z\bar{z}\}.$$

Factor out $z\bar{z}$ in (2.12) and use (2.13) to get

$$M(z)w \quad = \quad \sum_j \alpha_j (z\bar{z})^j w + \sum_k \beta_k (z\bar{z})^k z^2 \bar{w},$$

where $\alpha_j, \beta_k \in \mathbb{R}$. It follows that a set of generators of $\mathscr{M}(\mathbf{O}(2))$ over $\mathscr{P}(\mathbf{O}(2))$ is given by the elements

$$M_1(z)w = w, \ M_2(z)w = z^2\bar{w}.$$

We now apply the projection (2.7) to the elements above to find generators of $\overrightarrow{\mathscr{P}}(\mathbf{O}(2))$,

$$B_1(z)w = w, \text{ and } B_2(z)w = z^2\bar{w}.$$

Rewriting in (x, y)-coordinates we have the generators of the $\mathbf{O}(2)$-invariant BDEs:

Theorem 2.4.1. *If* $\omega = (a,b,c) \in \mathcal{Q}[\mathbf{O}(2),\mathbf{O}(2)]$, *then*

$$a = p_1 + (y^2 - x^2)p_2,\ b = 2xyp_2 \text{ and } c = p_1 + (x^2 - y^2)p_2, \tag{2.14}$$

where $p_i \in \mathscr{P}(\mathbf{O}(2)), i = 1,2.$

An example of an $\mathbf{O}(2)$-invariant configuration is given in Figure 6(a), for which we have $p_1 \equiv 0, p_2 \equiv 1$.

2.4.1 $\mathbf{O}(2)[\mathbf{SO}(2)]$*-equivariant quadratic forms*

In this case, $\ker\eta = \mathbf{SO}(2)$. From Section 2.3 we extract

$$H_0(z)w = w,\ H_1(z)w = z^2\bar{w} \text{ and } H_2(z)w = iz^2\bar{w}$$

as generators of $\overrightarrow{\mathscr{P}}(\mathbf{SO}(2))$ over the ring $\mathscr{P}(\mathbf{SO}(2))$ whose Hilbert basis is

$$\{u_1(z) = z\bar{z}\}.$$

We now apply Algorithm 2.1.5:

1. Fix $\kappa_x \in \mathbf{O}(2) \setminus \mathbf{SO}(2)$;

2. Generators of $\mathscr{P}[\mathbf{O}(2),\mathbf{SO}(2)]$ over $\mathscr{P}(\mathbf{O}(2))$:

$$\tilde{u}_1(z) = S(u_1)(z) = \frac{1}{2}(z\bar{z} - \bar{z}z) = 0.$$

3. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{O}(2),\mathbf{SO}(2)]$ over $\mathscr{P}(\mathbf{SO}(2))$: set $\tilde{u}_0(z) = 1$,

$$H_{0j}(z)w = \tilde{u}_0(z)H_j(z)w = H_j(z)w,\ j = 0,1,2;$$

$$H_{1j}(z)w = \tilde{u}_1(z)H_j(z)w = 0,\ j = 0,1,2.$$

So the generators are

$$H_{00}(z)w = w, H_{01}(z)w = z^2\bar{w} \text{ and } H_{02}(z)w = iz^2\bar{w}.$$

4. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{O}(2),\mathbf{SO}(2)]$ over $\mathscr{P}(\mathbf{O}(2))$:

$$\tilde{H}_{00}(z)w = \overrightarrow{S}(H_{00})w = 0;$$

$$\tilde{H}_{01}(z)w = \overrightarrow{S}(H_{01})w = 0;$$

$$\tilde{H}_{02}(z)w = \overrightarrow{S}(H_{02})w = iz^2\bar{w}.$$

Hence, $\overrightarrow{\mathscr{P}}[\mathbf{O}(2),\mathbf{SO}(2)]$ is the $\mathscr{P}(\mathbf{O}(2))$-module generated by

$$\tilde{B}(z)w = iz^2\bar{w}.$$

Rewriting in (x,y)-coordinates we have the generators of the $\mathbf{O}(2)[\mathbf{SO}(2)]$-invariant BDEs:

Theorem 2.4.2. *If* $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{O}(2),\mathbf{SO}(2)]$, *then*

$$a = 2xyp, \;\; b = (x^2-y^2)p, \;\; c = -2xyp, \tag{2.15}$$

where $p \in \mathscr{P}(\mathbf{O}(2))$.

We finish with examples of $\mathbf{O}(2)$-invariant and $\mathbf{O}(2)[\mathbf{SO}(2)]$-invariant configurations, respectively. For $\mathbf{O}(2)$, we considered $p_1 \equiv 0, p_2 \equiv 1$ in (2.14), so the differential form is

$$(y^2-x^2, 2xy, x^2-y^2).$$

The homomorphism λ is such that $\ker\lambda = \mathbf{SO}(2)$ and the discriminant function is the $\mathbf{O}(2)$-invariant given by

$$\delta(x,y) = (x^2+y^2)^2.$$

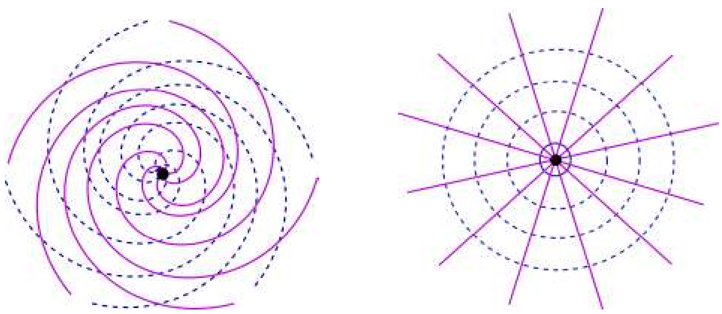
This is illustrated in Fig. 6(a).

The configuration in Fig. 6(b) is $\mathbf{O}(2)[\mathbf{SO}(2)]$ symmetric, whose quadratic 1-form has been chosen by taking $p \equiv 1$ in (2.15), that is,

$$(2xy, x^2-y^2, -2xy).$$

The homomorphism λ is trivial and the discriminant function is the $\mathbf{O}(2)$-invariant given by

$$\delta(x,y) = (x^2+y^2)^2.$$



(a) (b)

Figure 6 – Configurations with symmetry (a) $\mathbf{O}(2)$ and (b) $\mathbf{O}(2)[\mathbf{SO}(2)]$.

2.5 $\mathbf{Z}_n$-equivariant quadratic forms

Here we consider the cyclic group $\mathbf{Z}_n$, $n \geq 3$, with η trivial. We compute generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_n)$ by computing generators of $\mathscr{M}(\mathbf{Z}_n)$, the module of $\mathbf{Z}_n$-equivariant matrix-valued mappings $\mathbb{R}^2 \to M_2(\mathbb{R}^2)$, and projecting onto the space of mappings $\mathbb{R}^2 \to Sym_2$, as is done in Section 2.3. Write (2.6) as

$$M(z)w = \sum \alpha_{jk} z^j \bar{z}^k w + \sum \beta_{jk} z^j \bar{z}^k \bar{w}, \quad \alpha_{jk}, \beta_{jk} \in \mathbb{C}.$$

The equivariance with respect to $\theta \in \mathbf{Z}_n$ gives

$$M(z)w = \sum \alpha_{jk} e^{i\theta(j-k)} z^j \bar{z}^k w + \sum \beta_{jk} e^{i\theta(j-k-2)} z^j \bar{z}^k \bar{w}. \tag{2.16}$$

So $\alpha_{jk} = \alpha_{jk} e^{i\theta(j-k)}$ and $\beta_{jk} = \beta_{jk} e^{i\theta(j-k-2)}$ for $\theta = 2k\pi/n,\ k = 1,\ldots,n$, and so

$$\alpha_{jk} = 0 \text{ ou } j \equiv k (\text{mod n}) \quad \text{and} \quad \beta_{jk} = 0 \text{ ou } \ j \equiv k+2 (\text{mod n}). \tag{2.17}$$

A Hilbert basis for $\mathscr{P}(\mathbf{Z}_n)$ is given in [17]

$$\{u_1(z) = z\bar{z}, u_2(z) = z^n + \bar{z}^n, u_3(z) = i(z^n - \bar{z}^n)\}.$$

Factor out $z\bar{z}$ in (2.16) and use (2.17) to get

$$\begin{aligned} M(z)w &= \sum_{j \geq k} \alpha_{jk} (z\bar{z})^k z^{j-k} w + \sum_{j<k} \alpha_{jk} (z\bar{z})^j \bar{z}^{k-j} w + \sum_{j \geq k} \beta_{jk} (z\bar{z})^k z^{j-k} \bar{w} + \\ &+ \sum_{j<k} \beta_{jk} (z\bar{z})^j \bar{z}^{k-j} \bar{w} \ = \sum c^1_{kl_1} (z\bar{z})^k z^{l_1 n} w + \sum c^2_{jl_2} (z\bar{z})^j \bar{z}^{l_2 n} w + \\ &+ \sum c^3_{kl_3} (z\bar{z})^k z^{l_3 n+2} \bar{w} + \sum c^4_{jl_4} (z\bar{z})^j \bar{z}^{l_4 n-2} \bar{w}, \end{aligned}$$

where, $c^t \in \mathbb{C}$, $l_t \in \mathbb{N}$, $t = 1,\ldots,4$, $l_1, l_2, l_3 \geq 0$ and $l_4 \geq 1$. We now use the identities

$$\begin{aligned} z^{ln} &= (z^n + \bar{z}^n) z^{(l-1)n} - (z\bar{z})^n z^{(l-2)n} \\ \bar{z}^{ln} &= (z^n + \bar{z}^n) \bar{z}^{(l-1)n} - (z\bar{z})^n \bar{z}^{(l-2)n} \\ z^{ln+2} &= (z^n + \bar{z}^n) z^{(l-1)n+2} - (z\bar{z})^n z^{(l-2)n+2} \\ \bar{z}^{ln-2} &= (z^n + \bar{z}^n) \bar{z}^{(l-1)n-2} - (z\bar{z})^n \bar{z}^{(l-2)n-2} \\ \bar{z}^n &= (z^n + \bar{z}^n) - z^n \\ z^{n+2} &= (z^n + \bar{z}^n) z^2 - (z\bar{z})^2 \bar{z}^{n-2} \\ \bar{z}^{(l+1)n-2} &= (z^{ln} + \bar{z}^{ln}) \bar{z}^{n-2} - (z\bar{z})^{n-2} z^{(l-1)n+2} \end{aligned}$$

to conclude that a set of generators of $\mathscr{M}_2(\mathbf{Z}_n)$ over $\mathscr{P}(\mathbf{Z}_n)$ is given by the elements

$$M_1(z)w = w,\ M_2(z)w = iw,\ M_3(z)w = z^2\bar{w},\ M_4(z)w = iz^2\bar{w},\ M_5(z)w = \bar{z}^{n-2}\bar{w},$$

$$M_6(z)w = i\bar{z}^{n-2}\bar{w},\ M_7(z)w = z^n w,\ M_8(z)w = iz^n w.$$

We now apply the projection (2.7) to the elements above to find generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_n)$,

$$B_1(z)w = w,\ B_3(z)w = z^2\bar{w},\ B_4(z)w = iz^2\bar{w},\ B_5(z)w = \bar{z}^{n-2}\bar{w},\ B_6(z)w = i\bar{z}^{n-2}\bar{w}.$$

Rewriting in (x,y)-coordinates we have the generators of the $\mathbf{Z}_n$-invariant BDEs:

Theorem 2.5.1. *If $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{Z}_n, \mathbf{Z}_n]$, then*

$$\begin{aligned} a &= p_1 + (y^2 - x^2)p_2 + 2xyp_3 - A_1 p_4 - A_2 p_5, \\ b &= 2xyp_2 + (x^2 - y^2)p_3 + A_1 p_5 - A_2 p_4, \\ c &= p_1 + (x^2 - y^2)p_2 - 2xyp_3 + A_1 p_4 + A_2 p_5, \end{aligned} \tag{2.18}$$

where $p_i \in \mathscr{P}(\mathbf{Z}_n), i = 1, \ldots, 5,\ A_1 = Re(z^{n-2})$ and $A_2 = Im(z^{n-2})$.

An example of an $\mathbf{O}(2)$-invariant configuration is given in Figure 7(a), for which $n = 5$, $p_1 \equiv p_2 \equiv p_5 \equiv 1$ and $p_3 \equiv p_4 \equiv p_5 \equiv 0$.

2.5.1 $\mathbf{Z}_n[\mathbf{Z}_{n/2}]$-equivariant quadratic forms, for $n \geq 4$ even

In this case, $\ker \eta = \mathbf{Z}_{n/2}$. From the preceding subsection we extract

$$H_0(z)w = w, H_1(z)w = z^2\bar{w}, H_2(z)w = iz^2\bar{w}, H_3(z)w = \bar{z}^{n/2-2}\bar{w}, H_4(z)w = i\bar{z}^{n/2-2}\bar{w}$$

as generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_{n/2})$ over the ring $\mathscr{P}(\mathbf{Z}_{n/2})$ whose Hilbert basis is

$$\{u_1(z) = z\bar{z}, u_2(z) = z^{n/2} + \bar{z}^{n/2}, u_3(z) = i(z^{n/2} - \bar{z}^{n/2})\},$$

We now apply Algorithm 2.1.5:

1. Fix $\delta = e^{2\pi i/n} \in \mathbf{Z}_n \setminus \mathbf{Z}_{n/2}$;
2. Generators of $\mathscr{P}[\mathbf{Z}_n, \mathbf{Z}_{n/2}]$ over $\mathscr{P}(\mathbf{Z}_n)$:

$$\tilde{u}_1(z) = S(u_1)(z) = \frac{1}{2}(z\bar{z} - (e^{2\pi i/n}z)(e^{-2\pi i/n}\bar{z})) = 0.$$

$$\tilde{u}_2(z) = S(u_2)(z) = \frac{1}{2}(z^{n/2} + \bar{z}^{n/2} - (e^{\pi i}z^{n/2} + e^{-\pi i}\bar{z}^{n/2}) = z^{n/2} + \bar{z}^{n/2}.$$

$$\tilde{u}_3(z) = S(u_3)(z) = \frac{1}{2}(i(z^{n/2} - \bar{z}^{n/2}) - i((e^{\pi i}z^{n/2} - e^{-2\pi i}\bar{z}^{n/2}) = i(z^{n/2} - \bar{z}^{n/2}).$$

3. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{Z}_n, \mathbf{Z}_{n/2}]$ over $\mathscr{P}(\mathbf{Z}_n/2)$: set $\tilde{u}_0(z) = 1$,

$$H_{0j}(z)w = \tilde{u}_0(z)H_j(z)w = H_j(z)w,\ j = 0, \ldots, 4;$$

$$H_{1j}(z)w = \tilde{u}_1(z)H_j(z)w = 0,\ j = 0, \ldots, 4;$$

$$H_{20}(z)w = \tilde{u}_2(z)H_0(z)w = (z^{n/2} + \bar{z}^{n/2})w;$$

$$H_{21}(z)w = \tilde{u}_2(z)H_1(z)w = (z^{n/2+2} + (z\bar{z})^2\bar{z}^{n/2-2})\bar{w};$$

$$H_{22}(z)w = \tilde{u}_2(z)H_2(z)w = i(z^{n/2+2} + (z\bar{z})^2\bar{z}^{n/2-2})\bar{w};$$

$$H_{23}(z)w = \tilde{u}_2(z)H_3(z)w = (\bar{z}^{n/2-2} + (z\bar{z})^{n/2-2}z^2)\bar{w};$$

$$H_{24}(z)w = \tilde{u}_2(z)H_4(z)w = i(\bar{z}^{n/2-2} + (z\bar{z})^{n/2-2}z^2)\bar{w};$$
$$H_{30}(z)w = \tilde{u}_3(z)H_0(z)w = i(z^{n/2} - \bar{z}^{n/2})w;$$
$$H_{31}(z)w = \tilde{u}_3(z)H_1(z)w = i(z^{n/2+2} - (z\bar{z})^2\bar{z}^{n/2-2})\bar{w};$$
$$H_{32}(z)w = \tilde{u}_3(z)H_2(z)w = (-z^{n/2+2} + (z\bar{z})^2\bar{z}^{n/2-2})\bar{w};$$
$$H_{33}(z)w = \tilde{u}_3(z)H_3(z)w = i(-\bar{z}^{n-2} + (z\bar{z})^{n/2-2}z^2)\bar{w};$$
$$H_{34}(z)w = \tilde{u}_3(z)H_4(z)w = (\bar{z}^{n-2} - (z\bar{z})^{n/2-2}z^2)\bar{w},$$

which, as an intermediate step, we simplify to the reduced list

$$H_{00}(z)w = w, H_{01}(z)w = z^2\bar{w}, H_{02}(z)w = iz^2\bar{w}, H_{03}(z)w = \bar{z}^{n/2-2}\bar{w},$$
$$H_{04}(z)w = i\bar{z}^{n/2-2}\bar{w}, H_{20}(z)w = (z^{n/2} - \bar{z}^{n/2})w, H_{21}(z)w = z^{n/2+2}\bar{w}, H_{22}(z)w = iz^{n/2+2}\bar{w},$$
$$H_{23}(z)w = \bar{z}^{n-2}\bar{w}, H_{24}(z)w = i\bar{z}^{n-2}\bar{w}, H_{30}(z)w = i(z^{n/2} - \bar{z}^{n/2})w.$$

4. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{Z}_n, \mathbf{Z}_{n/2}]$ over $\mathscr{P}(\mathbf{Z}_n)$:

$$\tilde{H}_{00}(z)w = \tilde{H}_{01}(z)w = \tilde{H}_{02}(z)w = 0;$$
$$\tilde{H}_{03}(z)w = \bar{z}^{n/2-2}\bar{w};$$
$$\tilde{H}_{04}(z)w = i\bar{z}^{n/2-2}\bar{w};$$
$$\tilde{H}_{20}(z)w = (z^{n/2} + \bar{z}^{n/2})w;$$
$$\tilde{H}_{21}(z)w = z^{n/2+2}\bar{w};$$
$$\tilde{H}_{22}(z)w = iz^{n/2+2}\bar{w};$$
$$\tilde{H}_{23}(z)w = \tilde{H}_{24}(z)w = 0;$$
$$\tilde{H}_{30}(z)w = i(z^{n/2} - \bar{z}^{n/2})w,$$

where $\tilde{H}_{ij} = \overrightarrow{S}(H_{ij})$ for $i = 0,1,2$ and $\tilde{H}_{30} = \overrightarrow{S}(H_{30})$.

Therefore, $\overrightarrow{\mathscr{P}}[\mathbf{Z}_n, \mathbf{Z}_{n/2}]$ is the $\mathscr{P}(\mathbf{Z}_n)$-module generated by

$$\tilde{B}_1(z)w = \bar{z}^{n/2-2}\bar{w},\ \tilde{B}_2(z)w = i\bar{z}^{n/2-2}\bar{w},\ \tilde{B}_3(z)w = (z^{n/2} + \bar{z}^{n/2})w,$$
$$\tilde{B}_4(z)w = z^{n/2+2}\bar{w},\ \tilde{B}_5(z)w = iz^{n/2+2}\bar{w},\ \tilde{B}_6(z)w = i(z^{n/2} - \bar{z}^{n/2})w.$$

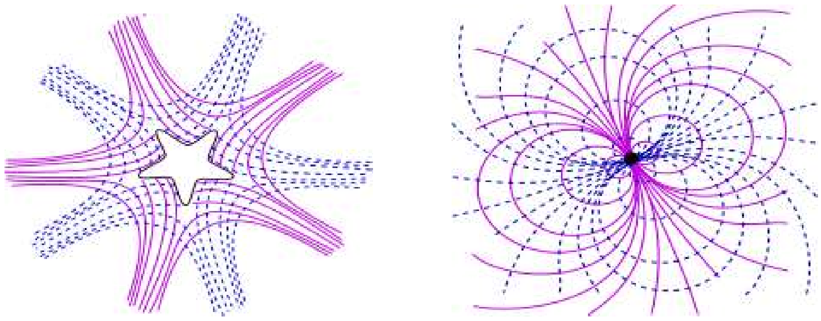
Rewriting in (x,y)-coordinates we have the generators of the $\mathbf{Z}_n[\mathbf{Z}_{n/2}]$-invariant BDEs:

Theorem 2.5.2. *If $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{Z}_n, \mathbf{Z}_{n/2}]$, then*

$$\begin{aligned} a &= -A_3p_1 - A_4p_2 + A_5p_3 - A_7p_4 + A_8p_5 + A_6p_6,\\ b &= -A_4p_1 + A_3p_2 + A_8p_4 + A_7p_5,\\ c &= A_3p_1 + A_4p_2 + A_5p_3 + A_7p_4 - A_8p_5 + A_6p_6, \end{aligned} \tag{2.19}$$

where $p_i \in \mathscr{P}(\mathbf{Z}_n), i = 1,\dots,6$, $A_3 = Re(z^{n/2-2}), A_4 = Im(z^{n/2-2}), A_5 = Re(z^{n/2}), A_6 = Im(z^{n/2})$, $A_7 = Re(z^{n/2+2})$ and $A_8 = Im(z^{n/2+2})$.

We finish with examples of $\mathbf{Z}_n$-invariant and $\mathbf{Z}_n[\mathbf{Z}_{z/2}]$-invariant configurations, respectively.



(a) (b)

Figure 7 – Configurations with symmetry group given by (a) $\mathbf{Z}_5$ and (b) $\mathbf{Z}_4[\mathbf{Z}_2]$.

We consider $n = 5$, taking $p_1 \equiv p_2 \equiv p_5 \equiv 1$ and $p_3 \equiv p_4 \equiv p_5 \equiv 0$ in (2.19). The differential form is

$$(1+y^2-x^2-3x^2y+y^3, 2xy+x^3-3xy^2, 1+x^2-y^2+3x^2y-y^3).$$

The homomorphism λ is necessarily trivial. The discriminant function is the $\mathbf{Z}_5$-invariant given by

$$\delta(x,y) = (x^2+y^2)^3 + 10x^4y - 20x^2y^3 + 2y^5 + (x^2+y^2)^2 - 1.$$

The picture for this case is shown in Fig. 7(a). The star shape of the discriminant set is in fact $\mathbf{Z}_5$-symmetric without reflectional symmetries, as it is easily checked by direct calculation.

Fig. 7(b) is a $\mathbf{Z}_4[\mathbf{Z}_2]$ case, considering $p_1 \equiv p_2 \equiv p_4 \equiv p_5 \equiv 1$ and $p_3 \equiv p_6 \equiv 0$ in (2.19), so that the differential form is

$$(-x^4+6x^2y^2-y^4+4x^3y-4xy^3, x^4-6x^2y^2+y^4+4x^3y-4xy^3, x^4-6x^2y^2+y^4-4x^3y+4xy^3).$$

The homomorphism λ must be such that $\ker\lambda = \mathbf{Z}_2$. The discriminant set is the origin, given by the zero set of (the $\mathbf{O}(2)$-invariant)

$$\delta(x,y) = 2(x^2+y^2)^4.$$

2.6 $\mathbf{D}_n$-equivariant quadratic forms, for $n \geq 3$

Consider the dihedral group $\mathbf{D}_n, n \geq 3$, generated by the rotation of angle $2\pi/n$ and the reflection κ_x with respect to the x-axis. Here the homomorphism η is trivial and the other possibilities for $\ker\eta$ are dealt with in following subsections. The technique is the same used for computing the generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_n)$. We compute the generators of $\mathscr{M}(\mathbf{D}_n)$, the module of $\mathbf{D}_n$-equivariant matrix-valued mappings $\mathbb{R}^2 \to M_2(\mathbb{R}^2)$, and project onto the space of mappings $\mathbb{R}^2 \to Sym_2$.

Golubitsky *et. al.* in ([21, XIV,§3]) give a set of generators of $\mathscr{M}(\mathbf{D}_n)$ over $\mathscr{P}(\mathbf{D}_n)$,

$$M_1(z)w = w,\ M_2(z)w = z^2\bar{w}, M_3(z)w = \bar{z}^{n-2}\bar{w},\ M_4(z)w = z^n w.$$

We now apply the projection (2.7) to the elements above to find generators of $\overrightarrow{\mathscr{P}}(\mathbf{D}_n)$,

$$B_1(z)w = w,\ B_2(z)w = z^2\bar{w},\ \text{and}\ B_3(z)w = \bar{z}^{n-2}\bar{w}.$$

Rewriting in (x,y)-coordinates the generators of the $\mathbf{D}_n$-invariant BDEs are:

Theorem 2.6.1. *If $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{D}_n,\mathbf{D}_n]$, then*

$$a = p_1 + (y^2 - x^2)p_2 - A_1 p_3,\ \ b = 2xyp_2 - A_2 p_3,\ c = p_1 + (x^2 - y^2)p_2 + A_1 p_3, \tag{2.20}$$

where $p_i \in \mathscr{P}(\mathbf{D}_n), i = 1,2,3, A_1 = Re(z^{n-2})$ and $A_2 = Im(z^{n-2})$.

An example of an $\mathbf{D}_5$-invariant configuration is given in Figure 8(a), for which $p_1 \equiv p_2 \equiv p_3 \equiv 1$.

2.6.1 $\mathbf{D}_n[\mathbf{Z}_n]$*-equivariant quadratic forms, for* $n \geq 3$

In this case, $\ker\eta = \mathbf{Z}_n, n \geq 3$. From Section 2.5 we extract

$$H_0(z)w = w, H_1(z)w = z^2\bar{w}, H_2(z)w = iz^2\bar{w}, H_3(z)w = \bar{z}^{n-2}\bar{w}, H_4(z)w = i\bar{z}^{n-2}\bar{w}$$

as generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_n)$ over the ring $\mathscr{P}(\mathbf{Z}_n)$ whose Hilbert basis is

$$\{u_1(z) = z\bar{z}, u_2(z) = z^n + \bar{z}^n, u_3(z) = i(z^n - \bar{z}^n)\},$$

We now apply Algorithm 2.1.5:

1. Fix $\kappa_x \in \mathbf{D}_n \setminus \mathbf{Z}_n$;
2. Generators of $\mathscr{P}[\mathbf{D}_n,\mathbf{Z}_n]$ over $\mathscr{P}(\mathbf{D}_n)$:

$$\tilde{u}_1(z) = S(u_1)(z) = \frac{1}{2}(z\bar{z} - \bar{z}z) = 0.$$

$$\tilde{u}_2(z) = S(u_2)(z) = \frac{1}{2}(z^n + \bar{z}^n - (\bar{z}^n + z^n)) = 0.$$

$$\tilde{u}_3(z) = S(u_3)(z) = \frac{1}{2}(i(z^n - \bar{z}^n) - i(\bar{z}^n - z^n)) = i(z^n - \bar{z}^n).$$

3. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{D}_n,\mathbf{Z}_n]$ over $\mathscr{P}(\mathbf{Z}_n)$: set $\tilde{u}_0(z) = 1$,

$$H_{0j}(z)w = \tilde{u}_0(z)H_j(z)w = H_j(z)w,\ j = 0,\ldots,4;$$

$$H_{1j}(z)w = H_{2j}(z)w = 0,\ j = 0,\ldots,4;$$

$$H_{30}(z)w = \tilde{u}_3(z)H_0(z)w = i(z^n - \bar{z}^n)w;$$

$$H_{31}(z)w = \tilde{u}_3(z)H_1(z)w = i(z^{n+2} - (z\bar{z})^2\bar{z}^{n-2})\bar{w};$$

$$H_{32}(z)w = \tilde{u}_3(z)H_2(z)w = (-z^{n+2} + (z\bar{z})^2\bar{z}^{n-2})\bar{w};$$

$$H_{33}(z)w = \tilde{u}_3(z)H_3(z)w = i(-\bar{z}^{2n-2} + (z\bar{z})^{n-2}z^2)\bar{w};$$

$$H_{34}(z)w = \tilde{u}_3(z)H_4(z)w = (\bar{z}^{2n-2} - (z\bar{z})^{n-2}z^2)\bar{w}.$$

We use now the identities

$$\begin{array}{rcl} z^{n+2} & = & (z^n + \bar{z}^n)z^2 - (z\bar{z})^2\bar{z}^{n-2} \\ iz^{n+2} & = & i(z^n + \bar{z}^n)z^2 - i(z\bar{z})^2\bar{z}^{n-2} \\ \bar{z}^{2n-2} & = & (z^n + \bar{z}^n)\bar{z}^{n-2} - (z\bar{z})^{n-2}z^2 \\ i\bar{z}^{2n-2} & = & i(z^n + \bar{z}^n)\bar{z}^{n-2} - (z\bar{z})^{n-2}z^2 \end{array}$$

to conclude that the generators of $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{Z}_n]$ over $\mathscr{P}(\mathbf{Z}_n)$ are

$$H_{00}(z)w = w, H_{01}(z)w = z^2\bar{w}, H_{02}(z)w = iz^2\bar{w}, H_{03}(z)w = \bar{z}^{n-2}\bar{w},$$

$$H_{04}(z)w = i\bar{z}^{n-2}\bar{w} \text{ and } H_{30}(z)w = i(z^n - \bar{z}^n)w.$$

4. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{Z}_n]$ over $\mathscr{P}(\mathbf{D}_n)$:

$$\tilde{H}_{00}(z)w = \tilde{H}_{01}(z)w = \tilde{H}_{03}(z)w = 0;$$

$$\tilde{H}_{02}(z)w = iz^2\bar{w};$$

$$\tilde{H}_{04}(z)w = i\bar{z}^{n-2}\bar{w};$$

$$\tilde{H}_{30}(z)w = i(z^n - \bar{z}^n)w.$$

Therefore, $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{Z}_n]$ is the $\mathscr{P}(\mathbf{D}_n)$-module generated by

$$\tilde{B}_1(z)w = iz^2\bar{w},\ \tilde{B}_2(z)w = i\bar{z}^{n-2}\bar{w} \text{ and } \tilde{B}_3(z)w = i(z^n - \bar{z}^n)w.$$

Rewriting in (x, y)-coordinates we have the generators of the $\mathbf{D}_n[\mathbf{Z}_n]$-invariant BDEs:

Theorem 2.6.2. *If $\omega = (a, b, c) \in \mathscr{Q}[\mathbf{D}_n, \mathbf{Z}_n]$, then*

$$a = 2xyp_1 - A_2p_2 + A_9p_3,\ b = (x^2 - y^2)p_1 + A_1p_2,\ c = -2xyp_1 + A_2p_2 + A_9p_3, \tag{2.21}$$

where $p_i \in \mathscr{P}(\mathbf{D}_n), i = 1, 2, 3,\ A_1 = Re(z^{n-2}),\ A_2 = Im(z^{n-2})$ and $A_9 = Im(z^n)$.

An example of a $\mathbf{D}_6[\mathbf{Z}_6]$-invariant configuration is given in Figure 8(b), for which we choose $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv 0$.

2.6.2 $\mathbf{D}_n[\mathbf{D}_{n/2}(\kappa_x)]$-equivariant quadratic forms, for $n \geq 4$ even

Here $\ker \eta = \mathbf{D}_{n/2}(\kappa_x), n \geq 4$, the dihedral group generated by the rotation of angle $4\pi/n$ and the reflection κ_x. From Section 2.6 we extract

$$H_0(z)w = w,\ H_1(z)w = z^2\bar{w} \text{ and } H_2(z)w = \bar{z}^{n/2-2}\bar{w}$$

as generators of $\overrightarrow{\mathscr{P}}(\mathbf{D}_{n/2}(\kappa_x))$ over the ring $\mathscr{P}(\mathbf{D}_{n/2}(\kappa_x))$ whose Hilbert basis is

$$\{u_1(z) = z\bar{z},\ u_2(z) = z^{n/2} + \bar{z}^{n/2}\}$$

We now apply Algorithm 2.1.5:

1. Fix $\delta = e^{2\pi i/n} \in \mathbf{D}_n \setminus \mathbf{D}_{n/2}(\kappa_x)$;
2. Generators of $\mathscr{P}[\mathbf{D}_n, \mathbf{D}_{n/2}(\kappa_x)]$ over $\mathscr{P}(\mathbf{D}_n)$:
$$\tilde{u}_1(z) = S(u_1)(z) = \frac{1}{2}(z\bar{z} - (e^{2\pi i/n}z)(e^{-2\pi i/n}\bar{z})) = 0.$$
$$\tilde{u}_2(z) = S(u_2)(z) = \frac{1}{2}(z^{n/2} + \bar{z}^{n/2} - (e^{\pi i}z^{n/2} + e^{-\pi i}\bar{z}^{n/2}) = z^{n/2} + \bar{z}^{n/2}.$$
3. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{D}_{n/2}(\kappa_x)]$ over $\mathscr{P}(\mathbf{D}_n/2(\kappa_x))$: set $\tilde{u}_0(z) = 1$,
$$H_{0j}(z)w = \tilde{u}_0(z)H_j(z)w = H_j(z)w,\ j = 0,1,2;$$
$$H_{1j}(z)w = \tilde{u}_1(z)H_j(z)w = 0,\ j = 0,1,2;$$
$$H_{20}(z)w = \tilde{u}_2(z)H_0(z)w = (z^{n/2} + \bar{z}^{n/2})w;$$
$$H_{21}(z)w = \tilde{u}_2(z)H_1(z)w = (z^{n/2+2} + (z\bar{z})^2\bar{z}^{n/2-2})\bar{w};$$
$$H_{22}(z)w = \tilde{u}_2(z)H_2(z)w = (\bar{z}^{n/2-2} + (z\bar{z})^{n/2-2}z^2)\bar{w},$$
which, as an intermediate step, we simplify to the reduced list
$$H_{00}(z)w = w, H_{01}(z)w = z^2\bar{w}, H_{02}(z)w = \bar{z}^{n/2-2}\bar{w},$$
$$H_{20}(z)w = (z^{n/2} + \bar{z}^{n/2})w, H_{21}(z)w = z^{n/2+2}\bar{w}, H_{22}(z)w = z^{n/2-2}\bar{w}.$$
4. Generators of $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{D}_{n/2}(\kappa_x)]$ over $\mathscr{P}(\mathbf{D}_n)$:
$$\tilde{H}_{00}(z)w = \tilde{H}_{01}(z)w = \tilde{H}_{22}(z)w = 0;$$
$$\tilde{H}_{02}(z)w = \bar{z}^{n/2-2}\bar{w};$$
$$\tilde{H}_{20}(z)w = (z^{n/2} + \bar{z}^{n/2})w;$$
$$\tilde{H}_{21}(z)w = z^{n/2+2}\bar{w}.$$

Therefore, $\overrightarrow{\mathscr{P}}[\mathbf{D}_n, \mathbf{D}_{n/2}(\kappa_x)]$ is the $\mathscr{P}(\mathbf{D}_n)$-module generated by

$$\tilde{B}_1(z)w = \bar{z}^{n/2-2}\bar{w},\ \tilde{B}_2(z)w = (z^{n/2} + \bar{z}^{n/2})w,\ \text{and } \tilde{B}_3(z)w = z^{n/2+2}\bar{w}.$$

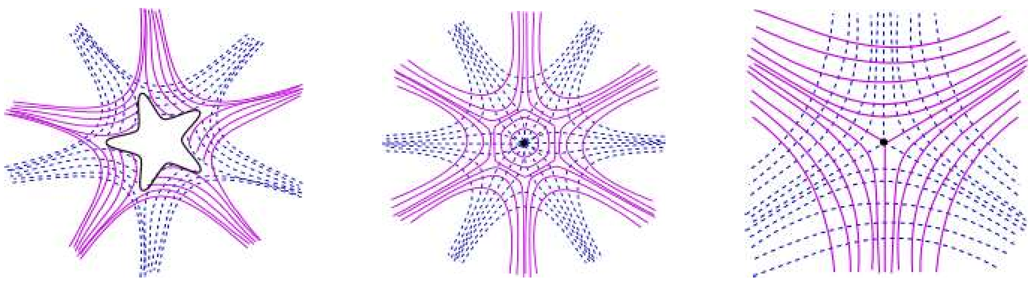
Rewriting in (x,y)-coordinates we have the generators of the $\mathbf{D}_n[\mathbf{D}_{n/2}(\kappa_x)]$-invariant BDEs:

Theorem 2.6.3. *If $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{D}_n, \mathbf{D}_{n/2}]$, then*

$$a = -A_3p_1 + A_5p_2 - A_7p_3,\quad b = -A_4p_1 + A_8p_3 \quad c = A_3p_1 + A_5p_2 + A_7p_3, \tag{2.22}$$

where $p_i \in \mathscr{P}(\mathbf{D}_n), i = 1,2,3$, $A_3 = Re(z^{n/2-2}), A_4 = Im(z^{n/2-2}), A_5 = Re(z^{n/2}), A_7 = Re(z^{n/2+2})$ and $A_8 = Im(z^{n/2+2})$.

We finish with examples of $\mathbf{D}_5$-invariant, $\mathbf{D}_6[\mathbf{Z}_6]$ and $\mathbf{D}_6[\mathbf{D}_3(\kappa_x)]$-invariant configurations, respectively.



(a) (b) (c)

Figure 8 – Configurations with symmetry groups $\mathbf{D}_5$, $\mathbf{D}_6[\mathbf{Z}_6]$ and $\mathbf{D}_6[\mathbf{D}_3(\kappa_x)]$.

We consider $\mathbf{D}_5$, with $p_1 \equiv p_2 \equiv p_3 \equiv 1$ in (2.20) so that the differential form is

$$(1 + y^2 - x^2 - x^3 + 3xy^2, 2xy - 3x^2y + y^3, 1 - y^2 + x^2 + x^3 - 3xy^2).$$

In this case, $\ker\lambda = \mathbf{Z}_5$ and the discriminant function is the $\mathbf{D}_5$-invariant given by

$$\delta(x,y) = (x^2+y^2)^3 + 2x^5 - 20x^3y^2 + 10xy^4 + (x^2+y^2)^2 - 1.$$

The configuration is illustrated in Fig. 8(a).

We now consider $\mathbf{D}_6[\mathbf{Z}_6]$ choosing $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv 0$ in (2.21), so that the form is

$$(2xy - 4x^3y + 4xy^3, x^2 - y^2 + x^4 - 6x^2y^2 + y^4, -2xy + 4x^3y - 4xy^3).$$

In this case λ is trivial and the discriminant function is $\mathbf{D}_6$-invariant and given by

$$\delta(x,y) = (x^2+y^2)^4 + 2x^6 - 30x^4y^2 + 30x^2y^4 + 2y^6 + (x^2+y^2)^2.$$

The picture is given in Fig. 8(b).

We now turn to $\mathbf{D}_6[\mathbf{D}_3(\kappa_x)]$ taking $p_1 \equiv 1$ and $p_2 \equiv p_3 \equiv 0$ in (2.22), so that the form is

$$(-x,-y,x).$$

In this case, $\ker\lambda = \mathbf{D}_3(\kappa_y)$ and the discriminant set is the origin, given by the zero set of

$$\delta(x,y) = x^2 + y^2.$$

The picture is given in Fig. 8(c).

2.7 $\mathbf{Z}_2$-equivariant quadratic forms

Let $\mathbf{Z}_2$ be the group generated by the reflection κ_x on the x-axis. First we consider $\eta : \mathbf{Z}_2(\kappa_x) \to \mathbf{Z}_2$ trivial. Imposing the $\mathbf{Z}_2$-equivariance to the matrix-valued mapping $B : \mathbb{R}^2 \to Sym_2$ in (2.4) we have

$$\begin{pmatrix} c(x,-y) & b(x,-y) \\ b(x,-y) & a(x,-y) \end{pmatrix} = \begin{pmatrix} c(x,y) & -b(x,y) \\ -b(x,y) & a(x,y) \end{pmatrix}.$$

Thus, a and c are $\mathbf{Z}_2$ -invariant and b is $\mathbf{Z}_2$-equivariant. Therefore, by [21, XIV §3] the generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_2)$ under $\mathscr{P}(\mathbf{Z}_2)$ are

$$(x,y) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ (x,y) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ (x,y) \mapsto \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}.$$

It follows that the generators of $\mathbf{Z}_2$-invariant BDEs are:

Theorem 2.7.1. *If $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{Z}_2, \mathbf{Z}_2]$, then*

$$a = p_1, \quad b = yp_2 \quad c = p_3 \tag{2.23}$$

where $p_i \in \mathscr{P}(\mathbf{Z}_2), i = 1,\dots,3$.

An example of a $\mathbf{Z}_2$-invariant configuration is given in Figure 9(a), for which $p_1 \equiv p_2 \equiv p_3 \equiv 1$.

2.7.1 $\mathbf{Z}_2[1]$-equivariant quadratic forms

Assume now η nontrivial, so $\ker\eta = \mathbf{1}$. Imposing the $\mathbf{Z}_2[\mathbf{1}]$-equivariance to (2.4) gives

$$\begin{pmatrix} c(x,-y) & b(x,-y) \\ b(x,-y) & a(x,-y) \end{pmatrix} = \begin{pmatrix} -c(x,y) & b(x,y) \\ b(x,y) & -a(x,y) \end{pmatrix}.$$

Hence b is $\mathbf{Z}_2$-invariant and the functions a and c are $\mathbf{Z}_2$-equivariant. Therefore, the generators for $\overrightarrow{\mathscr{P}}(\mathbf{Z}_2, \mathbf{1})$ under $\mathscr{P}(\mathbf{Z}_2)$ are

$$(x,y) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ (x,y) \mapsto \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},\ (x,y) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

It follows that the generators of the $\mathbf{Z}_2[\mathbf{1}]$-invariant BDEs are:

Theorem 2.7.2. *If* $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{Z}_2, \mathbf{1}]$, *then*

$$a = yp_1, \quad b = p_2 \quad c = yp_3, \tag{2.24}$$

where $p_i \in \mathscr{P}(\mathbf{Z}_2), i = 1, \ldots, 3.$

We finish with examples of $\mathbf{Z}_2$-invariant and $\mathbf{Z}_2[\mathbf{1}]$-invariant configurations, respectively. We consider $\mathbf{Z}_2$, and take $p_1 \equiv p_2 \equiv p_3 \equiv 1$ in (2.23), so that the form is

$$(1, y, 1).$$

We have $\ker \lambda = \{\mathbf{1}\}$ and the discriminant function is $\mathbf{Z}_2$-invariant and given by

$$\delta(x,y) = y^2 - 1.$$

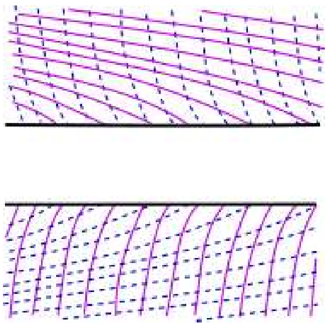
See the configuration of this case in Fig. 9(a).

Fig. 9(b) is a $\mathbf{Z}_2[\mathbf{1}]$ case, for which we have chosen $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$ in (2.24), so that the form is

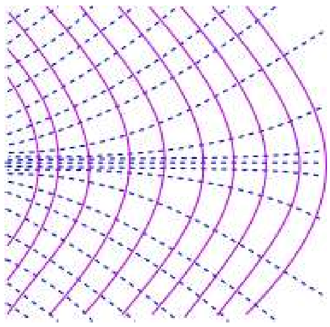
$$(y, 1, -y).$$

The homomorphism λ is trivial and the discriminant function is $\mathbf{Z}_2$-invariant and given by

$$\delta(x,y) = y^2 + 1.$$



(a)



(b)

Figure 9 – Configurations with symmetry groups $\mathbf{Z}_2$ and $\mathbf{Z}_2[\mathbf{1}]$.

2.8 $\mathbf{Z}_2 \times \mathbf{Z}_2$-equivariant quadratic forms

Let $\mathbf{Z}_2 \times \mathbf{Z}_2$ be the group generated by the reflections κ_x and κ_y on the x-axis and y-axis, respectively. First we consider $\eta : \mathbf{Z}_2 \times \mathbf{Z}_2 \to \mathbf{Z}_2$ trivial. In ([21, XIV,§3]) we have a set of generators of the matrix-valued mapping $M : \mathbb{R} \to \mathscr{M}(\mathbb{R}^2)$.

$$(x,y) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ (x,y) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ (x,y) \mapsto \begin{pmatrix} 0 & xy \\ 0 & 0 \end{pmatrix}, \ (x,y) \mapsto \begin{pmatrix} 0 & 0 \\ xy & 0 \end{pmatrix}.$$

Therefore, by the projection (2.7) the generators of $\overrightarrow{\mathscr{P}}(\mathbf{Z}_2\times\mathbf{Z}_2)$ under $\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)])$ are

$$(x,y)\mapsto\begin{pmatrix}1&0\\0&0\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&0\\0&1\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&xy\\xy&0\end{pmatrix}.$$

It follows that the generators of the $\mathbf{Z}_2\times\mathbf{Z}_2$-invariant BDEs are:

Theorem 2.8.1. *If* $\omega=(a,b,c)\in\mathscr{Q}[\mathbf{Z}_2\times\mathbf{Z}_2,\mathbf{Z}_2\times\mathbf{Z}_2]$, *then*

$$a=p_1,\quad b=xyp_2\quad c=p_3, \tag{2.25}$$

where $p_i\in\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2), i=1,\ldots,3$.

An example of an $\mathbf{Z}_2\times\mathbf{Z}_2$-invariant configuration is given in Figure 10(a), for which $p_1\equiv p_2\equiv 1$ and $p_3\equiv -1$.

2.8.1 $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$*-equivariant quadratic forms*

Assume here η nontrivial with $\ker\eta=\mathbf{Z}_2(-I)$, the cyclic group generated by minus identity. Imposing the $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$-equivariance to (2.4) gives the following condition on functions a,b and c: b is $\mathbf{Z}_2\times\mathbf{Z}_2$-invariant; a and c are $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$-equivariant. Therefore, the generators for $\overrightarrow{\mathscr{P}}(\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)])$ under $\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2)$ are

$$(x,y)\mapsto\begin{pmatrix}xy&0\\0&0\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&0\\0&xy\end{pmatrix}.$$

It follows that the generators of the $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$-invariant BDEs are:

Theorem 2.8.2. *If* $\omega=(a,b,c)\in\mathscr{Q}[\mathbf{Z}_2\times\mathbf{Z}_2,\mathbf{Z}_2(-I)]$, *then*

$$a=xyp_1,\quad b=p_2,\quad c=xyp_3, \tag{2.26}$$

where $p_i\in\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2), i=1,\ldots,3$.

An example of an $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$-invariant configuration is given in Figure 10(b), for which $p_1\equiv p_2\equiv 1$ and $p_3\equiv -1$.

2.8.2 $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$*-equivariant quadratic forms*

Finally, we assume η nontrivial with $\ker\eta=\mathbf{Z}_2(\kappa_x)$, the cyclic group generated by the reflection on x-axis. Imposing the $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariance to (2.4) gives the following condition on functions a,b and c: b is $[\mathbf{Z}_2,\mathbf{Z}_2(\kappa_x)]$-equivariant, whereas a and c are $[\mathbf{Z}_2,\mathbf{Z}_2(\kappa_y)]$-equivariant. Therefore, the generators for $\overrightarrow{\mathscr{P}}(\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ under $\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2)$ are

$$(x,y)\mapsto\begin{pmatrix}x&0\\0&0\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&y\\y&0\end{pmatrix},\ (x,y)\mapsto\begin{pmatrix}0&0\\0&x\end{pmatrix}.$$

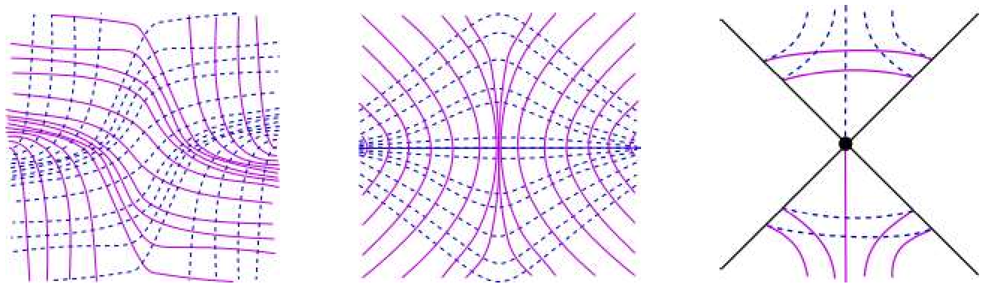
It follows that the generator of the $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-invariant BDEs are:

Theorem 2.8.3. *If* $\omega = (a,b,c) \in \mathscr{Q}[\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2(-I)]$, *then*

$$a = xp_1, \quad b = yp_2, \quad c = xp_3, \tag{2.27}$$

where $p_i \in \mathscr{P}(\mathbf{Z}_2 \times \mathbf{Z}_2), i = 1, \ldots, 3.$

We finish with examples of $\mathbf{Z}_2 \times \mathbf{Z}_2$-invariant, $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(-I)]$-invariant and $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)$-invariant configurations, respectively.



(a) (b) (c)

Figure 10 – Configurations with symmetry groups $\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(-I)]$ and $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$.

For $\mathbf{Z}_2 \times \mathbf{Z}_2$ in (2.25), we take $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$, so that the differential form is

$$(1, xy, -1).$$

In this case $\ker \lambda = \mathbf{Z}_2$ and the discriminant function is the $\mathbf{Z}_2 \times \mathbf{Z}_2$-invariant given by

$$\delta(x,y) = x^2y^2 + 1.$$

This is illustrated in Fig. 10(a).

We now consider $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(-I)]$ choosing $p_1 \equiv p_2 \equiv 1$ and $p_3 \equiv -1$ in (2.26), so that the form is

$$(xy, 1, -xy).$$

In this case λ is trivial and the discriminant function is $\mathbf{Z}_2 \times \mathbf{Z}_2$-invariant and given by

$$\delta(x,y) = x^2y^2 + 1.$$

The picture is given in Fig 10(b).

Finally, consider $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ taking $p_1 \equiv p_2 \equiv p_3 \equiv 1$ in (**??**), so that differential form is

$$(x, y, x).$$

In this case $\ker \lambda = \mathbf{Z}_2(\kappa_y)$ and the discriminant function is given by

$$\delta(x,y) = y^2 - x^2.$$

See the illustration of this case in Fig. 10(c).

2.9 Summarizing table and conclusions

In this section we list the general forms of symmetric quadratic differential 1-forms deduced in the previous sections. Table 1 shows each group Γ with all possible values of $\ker\eta$, denoted by $\Gamma[\ker\eta]$. Following the previous notation, when η is trivial the group is denoted simply by Γ. Also, $\mathbf{D}_n(\kappa_x)$ and $\mathbf{D}_n(\kappa_y)$ shall denote the dihedral groups generated by the rotation of angle $2\pi/n$ and by the reflections with respect to the x-axis or y-axis, respectively.

$\Gamma[\ker\eta]$	$\ker\lambda$	General form
$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	$a=p_1+(y^2-x^2)p_2+2xyp_3$; $b=2xyp_2+(x^2-y^2)p_3$; $c=p_1+(x^2-y^2)p_2-2xyp_3$, $p_i\in\mathscr{P}(\mathbf{SO}(2)), i=1,2,3$.
$\mathbf{O}(2)$	$\mathbf{SO}(2)$	$a=p_1+(y^2-x^2)p_2$; $b=2xyp_2$; $c=p_1+(x^2-y^2)p_2,\ p_i\in\mathscr{P}(\mathbf{O}(2)), i=1,2$.
$\mathbf{O}(2)[\mathbf{SO}(2)]$	$\mathbf{O}(2)$	$a=2xyp$; $b=(x^2-y^2)p$; $c=-2xyp, p\in\mathscr{P}(\mathbf{O}(2))$.
$\mathbf{Z}_n$, $n\geq 3$	$\mathbf{Z}_n$	$a=p_1+(y^2-x^2)p_2+2xyp_3-A_1p_4-A_2p_5$; $b=2xyp_2+(x^2-y^2)p_3+A_1p_5-A_2p_4$; $c=p_1+(x^2-y^2)p_2-2xyp_3+A_1p_4+A_2p_5$, $p_i\in\mathscr{P}(\mathbf{Z}_n), i=1,\dots,5$.
$\mathbf{Z}_n[\mathbf{Z}_{n/2}]$, $n\geq 4$ even	$\mathbf{Z}_{n/2}$	$a=-A_3p_1-A_4p_2+A_5p_3-A_7p_4+A_8p_5+A_6p_6$; $b=-A_4p_1+A_3p_2+A_8p_4+A_7p_5$; $c=A_3p_1+A_4p_2+A_5p_3+A_7p_4-A_8p_5+A_6p_6$, $p_i\in\mathscr{P}(\mathbf{Z}_n), i=1,\dots,6$.
$\mathbf{D}_n$, $n\geq 3$	$\mathbf{Z}_n$	$a=p_1+(y^2-x^2)p_2-A_1p_3$; $b=2xyp_2-A_2p_3$; $c=p_1+(x^2-y^2)p_2+A_1p_3,\ p_i\in\mathscr{P}(\mathbf{D}_n), i=1,2,3$.
$\mathbf{D}_n[\mathbf{Z}_n]$, $n\geq 3$	$\mathbf{D}_n$	$a=2xyp_1-A_2p_2+A_9p_3$; $b=(x^2-y^2)p_1+A_1p_2$; $c=-2xyp_1+A_2p_2+A_9p_3,\ p_i\in\mathscr{P}(\mathbf{D}_n), i=1,2,3$.
$\mathbf{D}_n[\mathbf{D}_{n/2}(\kappa_x)]$, $n\geq 4$ even	$\mathbf{D}_{n/2}(\kappa_y)]$	$a=-A_3p_1+A_5p_2-A_7p_3$; $b=-A_4p_1+A_8p_3$; $c=A_3p_1+A_5p_2+A_7p_3,\ p_i\in\mathscr{P}(\mathbf{D}_n), i=1,2,3$.
$\mathbf{Z}_2$	$\mathbf{1}$	$a=p_1$; $b=yp_2$; $c=p_3,\ p_i\in\mathscr{P}(\mathbf{Z}_2), i=1,2,3$.
$\mathbf{Z}_2[\mathbf{1}]$	$\mathbf{Z}_2$	$a=yp_1$; $b=p_2$; $c=yp_3,\ p_i\in\mathscr{P}(\mathbf{Z}_2), i=1,2,3$.
$\mathbf{Z}_2\times\mathbf{Z}_2$	$\mathbf{Z}_2(-I)$	$a=p_1$; $b=xyp_2$; $c=p_3,\ p_i\in\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2), i=1,2,3$.
$\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$a=xyp_1$; $b=p_2$; $c=xyp_3, p_i\in\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2), i=1,2,3$.
$\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$	$\mathbf{Z}_2(\kappa_y)$	$a=xp_1$; $b=yp_2$; $c=xp_3, p_i\in\mathscr{P}(\mathbf{Z}_2\times\mathbf{Z}_2), i=1,2,3$.
$A_1=\mathrm{Re}(z^{n-2})$, $A_2=\mathrm{Im}(z^{n-2})$, $A_3=\mathrm{Re}(z^{n/2-2})$, $A_4=\mathrm{Im}(z^{n/2-2})$, $A_5=\mathrm{Re}(z^{n/2})$,		
$A_6=\mathrm{Im}(z^{n/2})$, $A_7=\mathrm{Re}(z^{n/2+2})$, $A_8=\mathrm{Im}(z^{n/2+2})$, $A_9=\mathrm{Im}(z^n)$.		

Table 1 – General forms of equivariant quadratic differential 1-forms on the plane under closed subgroups of $\mathbf{O}(2)$.

Remark 2.9.1. *The symmetry group of the configuration shown in Fig.1(c) is $\mathbf{D}_6[\mathbf{D}_3(\kappa_y)]$. Whose* *quadratic form $(y, x, -y)$ appears in Table 1 by interchanging the variables x and y and taking $p_1 \equiv 1$ and $p_2 \equiv p_3 \equiv 0$ in the general form for the group $\mathbf{D}_6[\mathbf{D}_3(\kappa_x)]$. Similarly, the symmetry group of the configurations in Fig. 1(a) and (b) is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_y)]$, whose quadratic forms appear from the data for $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ in Table 1 by interchanging x and y and taking $p_1 \equiv p_2 \equiv 1, p_3 \equiv -1$, and $p_1 \equiv 1, p_2 \equiv \frac{1}{4}, p_3 \equiv -1$, respectively.*

In [11] the authors consider BDEs whose discriminant function is of Morse type. In this case, the discriminant set is a pair of transversal straight lines by the origin or the origin itself. They prove that these BDEs are topologically equivalent to their linear part. We remark that all the normal forms they obtain must be equivariant under a finite symmetry group. In fact, it follows from Table 1 that there are no BDEs with linear coefficients with infinite group of symmetries. As it appears in [11], the Morse condition is given in terms of the coefficients of the linear part of the smooth functions a, b and c. More precisely, if we write $a = a_1x + a_2y + \mathbf{o}(2)$, $b = b_1x + b_2y + \mathbf{o}(2)$ and $c = c_1x + c_2y + \mathbf{o}(2)$, then the condition is

$$(c_2a_1 - c_1a_2)^2 - 4(b_2a_1 - b_1a_2)(c_2b_1 - c_1b_2) \neq 0. \tag{2.28}$$

From Table 1, the possible symmetry groups of BDEs whose linear parts satisfy (2.28) are

$$\mathbf{Z}_3,\ \mathbf{Z}_6[\mathbf{Z}_3],\ \mathbf{D}_3, \mathbf{D}_3[\mathbf{Z}_3],\ \mathbf{D}_6[\mathbf{D}_3] \tag{2.29}$$

or

$$\mathbf{Z}_2,\ \mathbf{Z}_2[\mathbf{1}],\ \mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]. \tag{2.30}$$

Recall from Remark 1.1.4 that the set of all symmetries of a BDE is at most the symmetry group $\Sigma(\Delta)$ of the discriminant set. Hence, for the Morse cases it follows that if Δ is the origin, then the possible nontrivial symmetry groups are all the ones listed in (2.29) and (2.30), whereas when the discriminant set is a pair of transversal straight lines, the possible groups are only the groups listed in (2.30). We also point out that the finiteness of the symmetry group also holds for equations with constant coefficients. A classification of these two types of BDEs is done in Chapter 3, including an analysis of the corresponding group of symmetries of the equation with possible number of invariant lines in the associated configuration.

CHAPTER

3

BDES WITH HOMOGENEOUS COEFFICIENTS

This chapter is dedicated to study of binary differential equations whose coefficients are homogeneous polynomial functions of any degree. This special class of BDEs has the property that the symmetry group is always nontrivial, and furthermore, if the degree of the coefficient functions is odd, then there exists an element in the symmetry group that interchanges foliations. This is the statement of Theorem 3.1.2.

The problem to find the number of invariant straight lines on polynomial differential systems is a known issue on the literature, as we can see in [7]. An analysis of the number of invariant lines in the configuration associated with a homogeneous BDE, just as the study of its behavior, is done in Section 3.2. In Section 3.3 we present the closed subgroups of $\mathbf{O}(2)$ realized as symmetry groups of BDEs with coefficients of degree 0. In Section 3.4 we characterize BDEs with coefficients of degree 1, via the number of invariant straight lines.

3.1 Algebraic structure

Let $\omega : T\mathbb{R}^2 \to \mathbb{R}$ be a quadratic 1-form,

$$\omega(x,y,dx,dy) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2, \tag{3.1}$$

with a, b and c C^∞-functions.

Definition 3.1.1. *A homogeneous quadratic 1-form ω of degree n is a quadratic 1-form whose coefficients a, b and c are homogeneous polynomial functions of degree n. The BDE $\omega = 0$ is called a homogeneous BDE of degree n.*

Theorem 3.1.2. *If $\omega = 0$ is a homogeneous BDE of degree n, then the symmetry group Γ is nontrivial.*

Proof. Let $B : \mathbb{R}^2 \to Sym_2$ be a matrix-valued mapping associated with ω. If I denotes the identity matrix of order 2, then

$$B(-I(x,y)) = B(-x,-y) = (-1)^n B(x,y) = (-1)^n (-I) B(x,y) (-I)^t. \tag{3.2}$$

Hence, $-I \in \Gamma$ and $\eta(-I) = -1$ if n is odd and $\eta(-I) = 1$ if n is even. □

As direct consequences we have:

Corollary 3.1.3. *If n is odd, then the homomorphisms η and λ are not trivial.*

Corollary 3.1.4. *If n is odd, the group Γ can be decomposed as*

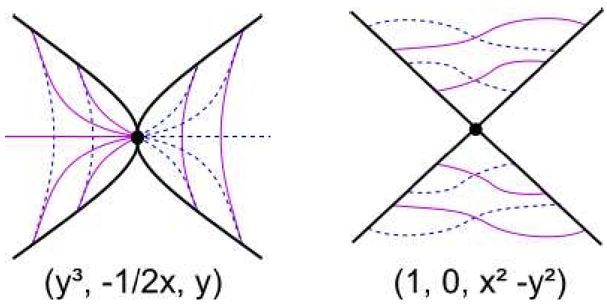
$$\Gamma = \ker\eta \ \dot\cup -I\ker\eta = \ker\lambda \ \dot\cup -I\ker\lambda.$$

The above results say that the configuration associated with a homogeneous BDE always is symmetric with respect to the origin. Moreover, if n is odd then minus identity interchanges foliations while if n is even, it preserves foliations. Hence, the symmetry group of a homogeneous BDE of degree odd always admits an index-2 normal subgroup. As a consequence of this fact, there is no homogeneous BDE $\mathbf{Z}_n$-equivariant with n odd. This fact also be deduced from Table 1.

Remark 3.1.5. *Theorem 3.1.2 and its consequences hold for a larger class of BDEs than the homogeneous ones. In fact, from (3.2), it is enough that the matrix-valued mapping $B : \mathbb{R}^2 \to Sym_2$ to satisfy*

$$B(-x,-y) = (-1)^n B(x,y) \tag{3.3}$$

for some natural number n. For example, in Figure 11(a) we have the configuration associated with $(y^3, -x/2, y)$ that satisfies the relation (3.3) with $n = 1$, $\ker\eta = \mathbf{Z}_2(\kappa_y)$ and $\ker\lambda = \mathbf{Z}_2(\kappa_x)$. In Figure 11(b) we have the configuration associated with $(1, 0, x^2 - y^2)$ that satisfies the relation (3.3) with $n = 2$, the homomorphism η is trivial so $\ker\lambda = \mathbf{Z}_2(-I)$.



(a) (b)

Figure 11 – Configurations with symmetry groups (a) $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_y)]$ and (b) $\mathbf{Z}_2 \times \mathbf{Z}_2$, respectively.

The discriminant function $\delta(x,y) = (b^2 - ac)(x,y)$ of a homogeneous BDE of degree $n, n \geq 1$, is a homogeneous function of degree $2n$, so it is an invariant function under the action of $-I$. Consequently, the set Δ is not empty, ($\delta(0,0)=0$) and, by Remark 1.1.4, it is symmetric with respect to the origin.

3.2 Invariant straight lines

Consider a regular parametrization $r:\mathbb{R}\to\mathbb{R}^2$ of a straight line through the origin

$$r(t)=(\mu \mathrm{x}(t),\beta \mathrm{x}(t)), \tag{3.4}$$

where $\mu,\beta\in\mathbb{R}$ and , $\mathrm{x}:\mathbb{R}\to\mathbb{R}$ is a smooth function with $\mathrm{x}(t)=0$ if and only if $t=0$ and $\mathrm{x}'(t)\neq 0,\ \forall t\in\mathbb{R}$.

Definition 3.2.1. *Let $\omega\in\mathscr{Q}(\mathbb{R}^2)$ be a quadratic 1-form. We say that r in (3.4) is an invariant straight line of the BDE $\omega=0$ if*

$$\omega(r(t),r'(t))=0,\ \ \forall t\in\ R. \tag{3.5}$$

To state the next results, given a homogeneous BDE $\omega=0$ we write the coefficients a,b and c in the form

$$a(x,y)=\sum_{i=0}^{n}a_ix^{n-i}y^i,\ \ b(x,y)=\sum_{i=0}^{n}b_ix^{n-i}y^i\ \text{ and }\ c(x,y)=\sum_{i=0}^{n}c_ix^{n-i}y^i, \tag{3.6}$$

where $a_i,b_i,c_i\in\mathbb{R}, i=1,\dots,n$.

Proposition 3.2.2. *Let $\omega=0$ be a homogeneous BDE of degree n with coefficients a,b and c as in (3.6). If $a_n\neq 0$ or $c_0\neq 0$, then the number of invariant straight lines through the origin is at most $n+2$.*

Proof. From (3.5) we have

$$\left(\sum_{i=0}^{n}a_i(\mu\mathrm{x}(t))^{n-i}(\beta\mathrm{x}(t))^i\right)\beta^2(\mathrm{x}'(t))^2+2\left(\sum_{i=0}^{n}b_i(\mu\mathrm{x}(t))^{n-i}(\beta\mathrm{x}(t))^i\right)\mu\beta(\mathrm{x}'(t))^2+$$

$$\left(\sum_{i=0}^{n}c_i(\mu\mathrm{x}(t))^{n-i}(\beta\mathrm{x}(t))^i\right)\mu^2(\mathrm{x}'(t))^2=0\quad\Longleftrightarrow$$

$$(\mathrm{x}(t))^n(\mathrm{x}'(t))^2\left(\sum_{i=0}^{n}a_i\beta^{i+2}\mu^{n-i}+2\sum_{i=0}^{n}\mu^{n-i+1}b_i\beta^{i+1}+\sum_{i=0}^{n}c_i\mu^{n-i+2}\beta^i\right)=0\quad\Longleftrightarrow$$

$$a_n\beta^{n+2}+(2b_n+a_{n-1})\mu\beta^{n+1}+\sum_{i=2}^{n}(a_{i-2}+2b_{i-1}+c_i)\mu^{n-i+2}\beta^i+(2b_0+c_1)\mu^{n+1}\beta+c_0\mu^{n+2}=0. \tag{3.7}$$

The last equality holds since r is regular. Therefore, the number of zeros of (3.7) is precisely the number of invariant straight lines through the origin. Hence, if $a_n\neq 0$ or $c_0\neq 0$ the result follows. $\square$

The equation (3.7) is called *equation of the invariant straight lines*. As a direct consequence of the previous proposition we have:

Corollary 3.2.3. *The configuration associated with a homogeneous BDE has* $0,1,2,...,$ $n+2$ *or infinite invariant straight lines through the origin, depending on the values of the coefficients* $a_i, b_i, c_i, i=1,...,n$.

Remark 3.2.4. *It is important to note that the deduction of the equation of the invariant straight lines (3.7) does not take into account whether the invariant straight lines are solutions of the discriminant function. Because of this, the invariant straight lines that can be seen in the configuration are those that are contained in* $\Omega = \{(x,y) \in \mathbb{R} : \delta(x,y) > 0\}$.

Consider $\omega \in \mathscr{Q}[\Gamma, \eta]$ a Γ-equivariant homogeneous quadratic 1-form of degree n. Depending on the parity of the degree we have different behaviors of the straight lines. In fact, if n is odd, then $-I$ interchanges foliations, so an invariant straight line splits into peaces in the two foliations and on the discriminant set, whereas if n is even, then $-I$ preserves foliation, so the invariant straight line splits in peaces into the same foliation.

Now, if the group Γ contains a rotation R distinct from the identity or the minus identity and if there exists one invariant straight line, then by the linearity of the action, the rotation forces the existence of other invariant straight lines in the configuration. More precisely, since $-I$ belongs to Γ and the rotation $R \notin \{I, -I\}$, then we must have $\mathbf{Z}_\ell \subset \Gamma$, for $\ell \geq 4$ even. Moreover, if there is one invariant straight line r, $-I$ leaves r setwise invariant, that is, $-Ir = r$. So, by symmetry there exist at least $\ell/2$ invariant straight lines. From this argumentation, the next two propositions follow:

Proposition 3.2.5. *Let* $\omega \in \mathscr{Q}[\mathbf{Z}_\ell, \eta]$ *and* $\ell \geq 4$ *even. If there exists an invariant straight line in the configuration associated with* $\omega = 0$*, then there exist at least* $\ell/2$ *invariant straight lines.*

Proposition 3.2.6. *Let* $\omega \in \mathscr{Q}[\mathbf{D}_\ell(\kappa_x), \eta]$ *and* $\ell \geq 4$ *even. If there exists an invariant straight line in the configuration associated with* $\omega = 0$*, then there exist at least* $\ell/2$ *invariant straight lines.*

We finish this section with an example that illustrates the results exposed here.

Example 3.2.7. *Consider the homogeneous BDE of degree* 4 *with coefficients,*

$$(-2x^3y + 6xy^3, 2x^4 - 6x^2y^2, 2x^3y - 6xy^3)$$

whose symmetry group is $\mathbf{D}_6(\kappa_x)[\mathbf{Z}_6]$ *and* λ *is trivial, i.e., there is no interchange of foliations. The discriminant function is given by*

$$\delta(x,y) = 4x^2(x^2+y^2)(x^2-3y^2)^2.$$

The discriminant set is the union of three straight lines, $x = 0$ $x = \pm\sqrt{3}y$*. The equation that gives the invariant straight lines is*

$$2\mu\beta\left(3\beta^4 - 10\mu^2\beta^2 + 3\mu^4\right) = 0,$$

so there are 6 *invariant straight lines,* $x = 0,\ y = 0,\ x = \pm\sqrt{3}y$ *and* $x = \pm\sqrt{3}y/3$*. Three of them belongs to the discriminant set, as we can see in Figure 12.*

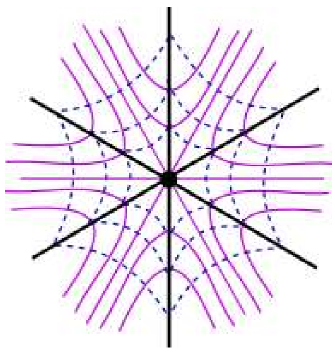


Figure 12 – Configuration of $(-2x^3y + 6xy^3, 2x^4 - 6x^2y^2, 2x^3y - 6xy^3)$.

3.3 BDEs with coefficients of degree 0

Here we consider a binary differential equation of the form $\omega = (a, b, c)$ where $a, b, c \in \mathbb{R}$, which we shall call the constant case. The discriminant function $\delta = b^2 - ac$ is constant and we have only two possibilities for the discriminant set: either the empty set (if $\delta > 0$ or $\delta < 0$) or the whole plane (if $\delta = 0$). So, in this section, we shall consider $b^2 - ac > 0$.

Proposition 3.3.1. *For the constant case with* $b^2 - ac > 0$*, the number of invariant straight lines through the origin in its configuration is always equal to* 2*.*

Proof. From equation (3.7) the number of invariant straight lines through the origin is given by the number of zeros of

$$a\beta^2 + 2b\mu\beta + c\mu^2 = 0, \tag{3.8}$$

where $\mu, \beta \in \mathbb{R}$. If the coordinate axes are not invariant lines of the BDE, that is, $a \neq 0$ or $c \neq 0$, then we can write (3.8) as

$$ap^2 + 2bp + c = 0 \ \text{ or } \ a + 2bq + cq^2 = 0, \ \ p = \beta/\mu, \ \ q = \mu/\beta, \tag{3.9}$$

which has two distinct roots since $b^2 - ac > 0$. If $a = 0$ or $c = 0$, one of the invariant straight lines is a coordinate axis while the other one is $(x, -cx/2b)$ or $(x, -2bx/a)$, respectively. When $b = 0$ we have $ac < 0$ and the straight lines are $(x, \pm\sqrt{c/a}y)$. □

We use the standard representation of $\mathbf{O}(2)$ on $\mathbb{R}^2$ for $R_\theta \in \mathbf{SO}(2)$ of angle $\theta \in [0, 2\pi]$ and the reflexion κ_x with respect to the x-axis. We have:

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{and} \quad \kappa_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.10}$$

From the condition of equivariance given in (2.5), if $\gamma \in \mathbf{O}(2)$ belongs to the symmetry group Γ, the matrix-valued mapping $B : \mathbb{R}^2 \to Sym_2$ associated with the BDE, that in this case is a constant matrix, must satisfy

$$B = \eta(\gamma)\gamma B \gamma^t, \ \forall \gamma \in \Gamma, \tag{3.11}$$

where $\eta(\gamma) = \pm 1$. The next result uses (3.10) and the relation (3.11) to explicitly calculate the admissible symmetry groups of a BDE with constant coefficients, namely groups of some binary differential equation.

Theorem 3.3.2. *The admissible symmetry groups of a BDE with constant coefficients are*

$$\mathbf{Z}_2(-I), \ \ \mathbf{Z}_2 \times \mathbf{Z}_2, \ \ \mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(-I)], \ \ \mathbf{Z}_4[\mathbf{Z}_2(-I)], \ \ \mathbf{D}_4(\kappa_x)[\mathbf{D}_2(\kappa_x)].$$

Proof. Let B be the matrix-valued mapping associated with $\omega = 0$ and consider $R_\theta \in \mathbf{SO}(2)$. If $\eta(R_\theta) = 1$, the relation

$$B = R_\theta B R_\theta^t$$

is satisfied for $\theta = 0, \pi$, for all a, b, c with $b^2 - ac > 0$. If $\eta(R_\theta) = -1$, the relation

$$B = -R_\theta B R_\theta^t$$

is satisfied for $\theta = \pi/2, 3\pi/2$ and $c = -a$. Now, for the reflection κ_x we have that the equivariant conditions are satisfied for $\eta(\kappa_x) = 1$ if and only if $b = 0$ and, for $\eta(\kappa_x) = -1$ if and only if $a = c = 0$.

So, we conclude that the admissible groups are: $\mathbf{Z}_2(-I)$, with η trivial; $\mathbf{Z}_2 \times \mathbf{Z}_2$, with $\ker \eta = \mathbf{Z}_2(-I)$ and $a = c = 0$; $\mathbf{Z}_2 \times \mathbf{Z}_2$ with $\eta =$ trivial and $b = 0$; $\mathbf{Z}_4$ with $\ker \eta = \mathbf{Z}_2(-I)$ and $c = -a$ and $\mathbf{D}_4(\kappa_x)$ with $\ker \eta = \mathbf{D}_2(\kappa_x)$, $b = 0$ and $c = -a$. □

We organize the results of Theorem 3.3.2 in Table 2. In Figure 13 we present an example of each case of Table 2.

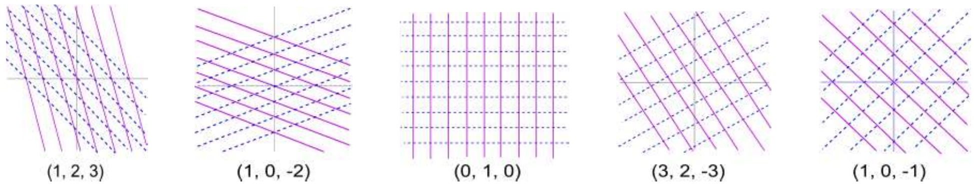
3.4 BDEs with coefficients of degree 1

In this section we consider quadratic 1-forms $\omega = (a, b, c)$ where the coefficients $a, b, c : \mathbb{R}^2 \to \mathbb{R}$ are linear functions

$$a(x,y) = a_0 x + a_1 y, \ \ b(x,y) = b_0 x + b_1 y \ \ c(x,y) = c_0 x + c_1 y,$$

$\Gamma[\ker\eta]$	$\ker\lambda$	General form
$\mathbf{Z}_2(-I)$	$\mathbf{Z}_2(-I)$	$(a,b,c),\ \ b^2-ac>0.$
$\mathbf{Z}_2\times\mathbf{Z}_2$	$\mathbf{Z}_2(-I)$	$(a,0,c),\ \ ac<0.$
$\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$(0,b,0),\ \ b\neq 0.$
$\mathbf{Z}_4[\mathbf{Z}_2(-I)]$	$\mathbf{Z}_2(-I)$	$(a,b,-a),\ \ a\neq 0.$
$\mathbf{D}_4(\kappa_x)[\mathbf{D}_2(\kappa_x)]$	$\mathbf{Z}_2(-I)$	$(a,0,-a),\ \ a\neq 0.$

Table 2 – General form of BDEs with constant coefficients.



(a) (b) (c) (d) (e)

Figure 13 – Configurations associated with BDEs with constant coefficients and their symmetry group (a) $\mathbf{Z}_2(-I)$, (b) $\mathbf{Z}_2\times\mathbf{Z}_2$, (c) $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(-I)]$, (d) $\mathbf{Z}_4[\mathbf{Z}_2(-I)]$ and (e) $\mathbf{D}_4(\kappa_x)[\mathbf{D}_2(\kappa_x)]$.

$a_i, b_i, c_i \in \mathbb{R}, i = 0, 1$. The discriminant function

$$\delta(x,y) = x^2(b_0^2 - a_0c_0) + xy(2b_0b_1 - a_1c_0 - a_0c_1) + y^2(b_1^2 - a_1c_1)$$

is a homogeneous function of degree 2, so it is an invariant function under the action of $-I$. Consequently, the discriminant set $\Delta = \delta^{-1}(0)$ is not empty ($\delta(0,0) = 0$) and symmetric with respect to the origin. Thus we have 3 possibilities: Δ is either a point (the origin), a straight line through the origin or a pair of transversal straight lines through the origin.

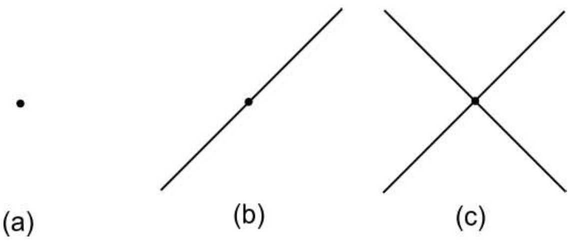


Figure 14 – Configurations of discriminant set of BDE with linear coefficients

From Remark 1.1.4, in Figure 14(a) the symmetry group must be $\mathbf{O}(2)$, in 14(b) either $\mathbf{Z}_2$ or $\mathbf{Z}_2\times\mathbf{Z}_2$, and in 14(c) either $\mathbf{Z}_2\times\mathbf{Z}_2$ or $\mathbf{D}_4$, depending on the position of the lines in the plane. By using the same technique used in the proof of Theorem 3.3.2, we have:

Theorem 3.4.1. *The admissible symmetry groups of a BDE with linear coefficients are*

$$\mathbf{Z}_2(-I)[\boldsymbol{I}],\ \mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)],\ \mathbf{Z}_6[\mathbf{Z}_3],\ \mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)].$$

Proof. The result follows from applying R_θ and κ_x in the equivariance condition (2.5). Imposing $B(R_\theta(x,y)) = \eta(R_\theta)R_\theta B(x,y)(R_\theta)^t$ we have

- if $\eta(R_\theta) = 1$
  $\theta = 0,\ \forall\, a_i, b_i, c_i \in\ \mathbb{R}, i = 0,1.$
  $\theta = 2\pi/3$ and $\theta = 4\pi/3, b_0 = -a_1,\ b_1 = a_0,\ c_0 = -a_0,\ c_1 = -a_1,\ \forall\, a_i \in\ \mathbb{R}, i = 0,1.$

- if $\eta(R_\theta) = -1$
  $\theta = \pi,\ \forall\, a_i, b_i, c_i \in\ \mathbb{R}, i = 0,1.$
  $\theta = \pi/3$ and $\theta = 5\pi/3, b_0 = -a_1,\ b_1 = a_0,\ c_0 = -a_0,\ c_1 = -a_1,\ \forall\, a_i \in\ \mathbb{R}, i = 0,1.$

Imposing $B(\kappa_x(x,y)) = \eta(\kappa_x)\kappa_x B(x,y)(\kappa_x)^t$,

- if $\eta(R_\theta \kappa_x) = 1$
  $\theta = 0,\ a_0 = b_1 = c_0 = 0$ and $a_1,\ b_0,\ c_1 \in \mathbb{R}.$
  $\theta = \pi,\ a_1 = b_0 = c_1 = 0$ and $a_0,\ b_1,\ c_0 \in \mathbb{R}.$
  $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}, b_0 = -a_1,\ b_1 = a_0,\ c_0 = -a_0,\ c_1 = -a_1,\ \forall\, a_i \in \mathbb{R}, i = 0,1.$
  $\theta = \frac{\pi}{2},\ c_0 = -a_1,\ b_0 = -b_1,\ c_1 = -a_0, a_i,\ b_i,\ c_i \in \mathbb{R}, i = 0,1.$
  $\theta = \frac{3\pi}{2},\ c_0 = a_1,\ b_0 = b_1,\ c_1 = a_0,\ a_i,\ b_i,\ c_i \in \mathbb{R}, i = 0,1.$

- if $\eta(R_\theta \kappa_x) = -1$
  $\theta = 0,\ a_1 = b_0 = c_1 = 0$ and $a_0,\ b_1,\ c_0 \in \mathbb{R}.$
  $\theta = \pi,\ a_0 = b_1 = c_0 = 0$ and $a_1,\ b_0,\ c_1 \in \mathbb{R}.$
  $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}, b_0 = -a_1,\ b_1 = a_0,\ c_0 = -a_0,\ c_1 = -a_1,\ \forall\, a_i \in\ \mathbb{R}, i = 0,1.$
  $\theta = \frac{\pi}{2},\ c_0 = a_1,\ b_0 = b_1,\ c_1 = a_0, a_i,\ b_i,\ c_i \in \mathbb{R}, i = 0,1.$
  $\theta = \frac{3\pi}{2},\ c_0 = -a_1,\ b_0 = -b_1,\ c_1 = -a_0, a_i,\ b_i,\ c_i \in \mathbb{R}, i = 0,1.$

Then, we have the result. □

The general forms obtained in Theorem 3.4.1 are organized in Table 3.

$\Gamma[\ker\eta]$	$\ker\lambda$	General form
$\mathbf{Z}_2(-I)[1]$	1	$(a_0x + a_1y, b_0x + b_1y, c_0x + c_1y),\ \ a_i, b_i, c_i \in \mathbb{R}, i = 0,1.$
$\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$	$\mathbf{Z}_2(\kappa_y)$	$(a_0x, b_1y, c_0x),\ a_0, b_1, c_0 \in \mathbb{R}.$
$\mathbf{Z}_6[\mathbf{Z}_3]$	$\mathbf{Z}_3$	$(-a_0x - a_1y, a_1x - a_0y, a_0x + a_1y),\ \ a_0, a_1 \in \mathbb{R}, a_0^2 + a_1^2 \neq 0.$
$\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$	$\mathbf{D}_3(\kappa_y)$	$(-a_0x, -a_0y, a_0x),\ \ a_0 \neq 0.$

Table 3 – General form of BDEs with linear coefficients.

From equation (3.7) the number of invariant straight lines through the origin on a configuration associated with a BDE with linear coefficients is given by the zeros of the cubic

$$\beta^3 a_1 + \mu\beta^2(2b_1 + a_0) + \mu^2\beta(2b_0 + c_1) + c_0\mu^3. \tag{3.12}$$

So, the number of invariant straight lines that can occur is $0,1,2,3$ or infinite, depending on the values of the coefficients $a_i, b_i, c_i, i = 1,2$. Moreover, let $r(t)$ be an regular parametrization of an invariant straight line. Since $-I$ interchanges foliations, if $r(t) \in \mathscr{F}_i,\ t > 0$ then $r(t) \in \mathscr{F}_j,\ t < 0$ for $i \neq j$.

We present now a characterization of such BDEs according to the number of invariant straight lines trough the origin that appear on its configuration.

1. **Infinite straight lines:**
   This is only possible if $a_1 = a_0 + 2b_1 = 2b_0 + c_1 = c_0 = 0$, which implies that the discriminant set is a straight line. The general form for this case is

$$(-2b_1x, b_0x + b_1y, -2b_0y),$$

   where $b_i, \in \mathbb{R}^*, i = 1,2$. The symmetry group is $\mathbf{Z}_2(-I)[\mathbf{1}]$. We can see an example in Figure 15(a), for $b_0 = b_1 = 1$.

2. **No straight lines:**
   Here we must have $a_1 = 0, 2b_1 + a_0 \neq 0$ and $(2b_0 + c_1)^2 - 4c_0(2b_1 + a_0) < 0$. The discriminant set Δ can be any of the forms in Figure 14, depending on the choice of the coefficients. The symmetry group is $\mathbf{Z}_2(-I)[1]$ or $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$. The general form of the BDEs under each group is, respectively,

$$(a_0x + a_1y, b_0x + b_1y, c_0x + c_1y) \text{ and } (a_0x, b_1y, c_0x),$$

   where $a_i, b_i, c_i \in \mathbb{R}$. For the group $\mathbf{Z}_2(-I)[1]$ we can see an example in Figure 15(b).

3. **1 or 2 straight lines:**
   In either case, Δ can be any of the forms as in Figure 14, depending on the choice of the coefficients. The existence of only one or two invariant straight lines and Theorem 3.4.1 implies that the symmetry group can be $\mathbf{Z}_2(-I)[1]$ or $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$, respectively. The general form of the BDEs for each group is, respectively,

$$(a_0x + a_1y, b_0x + b_1y, c_0x + c_1y) \text{ and } (a_0x, 2b_1y, c_0x),$$

   where $a_i, b_i, c_i \in \mathbb{R}$. For 1 straight line we can see examples in Figure 15(c), and in Figure 15(d) for 2 straight lines. In the last case one of them coincide with the discriminant set.

4. **3 straight lines:**
   Here, the discriminant set Δ also can be any of the forms as in Fig.14. When Δ is the origin, and only in this case, the groups $\mathbf{Z}_6[\mathbf{Z}_3]$ and $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$ can appear. The general forms for these last groups are, respectively,

$$(-a_0x - a_1y, a_1x - a_0y, a_0x + a_1y), \text{ and } (-a_0x, -a_0y, a_0x),$$

   where $a_i \in \mathbb{R}^*, i = 1,2$. We illustrate the cases in Figure 15(e) and 15(f). In the first one the symmetry group is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ and in the second one is $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$.

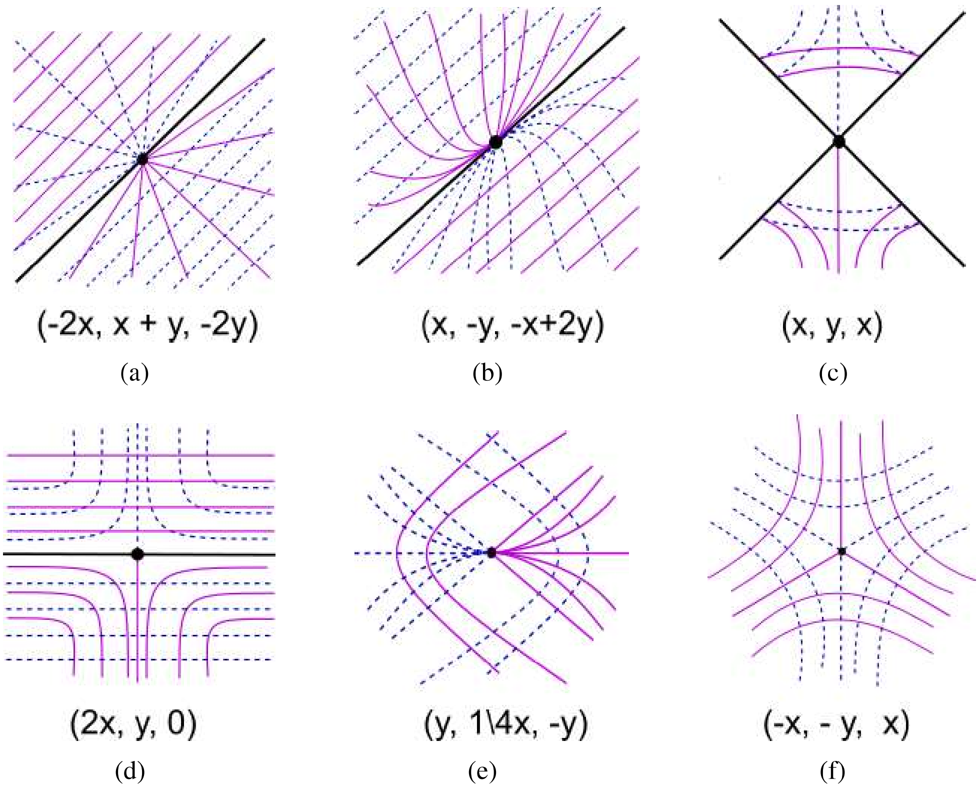


Figure 15 – Configurations with (a) infinity, (b) zero, (c) one, (d) two and (e)-(f) three invariant straight lines through the origin.

CHAPTER

4

EQUIVARIANT LINEAR NORMAL FORMS

A normal form of a (germ of) a mapping, generally speaking, is a simplified expression obtained by defining a notion of equivalence between them (often a change of coordinates), that is considered to preserve its essential features. Our class of objects is the set of Γ-equivariant matrix-valued mappings associated with a BDE and the equivalence relation must preserve the symmetry group. In section 4.1 we define the Γ-equivalence between Γ-equivariant quadratic 1-forms with linear coefficients (Definition 4.1.1).

The purpose of this chapter is to obtain the normal forms, namely representatives of the classes of the equivalence relation, for BDEs with linear coefficients, that is when a, b and c are linear functions. The deduction, for each closed subgroup of $\mathbf{O}(2)$ given in Theorem 3.4.1 is presented. We highlight the Subsections 4.1.3 and 4.1.4, where some of the equivariant normal forms obtained present a modal parameter, which does not happen when the symmetries are not taken into account. The modal parameter is associated with the slop of the invariant straight lines in the configuration associated with the BDE. Section 4.2 finishes the chapter summarizing the results in Table 4.

4.1 Γ-equivalence

Let Γ be a compact subgroup of $\mathbf{O}(2)$ and consider the action induced on the tangent bundle $T\mathbb{R}^2$ as in (1.3). The set of Γ-equivariant linear diffeomorphism is denoted by $\overrightarrow{\mathscr{R}}(\Gamma)$,

$$\overrightarrow{\mathscr{R}}(\Gamma) := \{\phi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ linear difeomorphism} : \phi(\gamma(x,y)) = \gamma\phi(x,y), \ \forall\gamma \in \Gamma, \forall(x,y) \in \mathbb{R}^2\}.$$

Definition 4.1.1. *Let $\omega_1, \omega_2 \in \mathscr{Q}[\Gamma, \eta]$ be Γ-equivariant quadratic 1-forms with linear coefficients. They are Γ-equivalent if there exist a linear diffeomorphism $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2 \in$*

$\overrightarrow{\mathscr{R}}(\Gamma)$ such that

$$\omega_2 = \phi^* \omega_1, \tag{4.1}$$

that is, $\omega_2(x,y,dx,dy) = \omega_1\big(\phi_1(x,y), \phi_2(x,y), d\phi_1(x,y), d\phi_2(x,y)\big)$, *where* $\begin{pmatrix} d\phi_1 \\ d\phi_2 \end{pmatrix} = d\phi \begin{pmatrix} dx \\ dy \end{pmatrix}$.

Let $\overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma)$ denote the submodule $\overrightarrow{\mathscr{P}}^{\eta}(\Gamma)$ of the Γ-equivariant matrix-valued mappings whose entries are linear functions. Definition 4.1.1 induces an action of $\overrightarrow{\mathscr{R}}(\Gamma)$ on $\overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma)$ given by

$$\begin{array}{ccccc} \overrightarrow{\mathscr{R}}(\Gamma) \times \overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma) & \to & & & \overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma) \\ (\phi, B) & \mapsto & (\phi \cdot B) & = & (d\phi^{-1})^t B(\phi^{-1})(d\phi^{-1}). \end{array} \tag{4.2}$$

The action is well defined, since $\phi \in \overrightarrow{\mathscr{R}}(\Gamma)$ implies that $\phi^{-1} \in \overrightarrow{\mathscr{R}}(\Gamma)$ and $(d\phi)_{\gamma(x,y)}\gamma = \gamma(d\phi)_{(x,y)}$. In fact, for all $\gamma \in \Gamma$ and $(x,y) \in \mathbb{R}^2$ we have

$$\begin{aligned} (\phi \cdot B)(\gamma(x,y)) &= (d\phi^{-1})^t_{\gamma(x,y)} B(\phi^{-1}(\gamma(x,y)))(d\phi^{-1})_{\gamma(x,y)} \\ &= \gamma(d\phi^{-1})^{-1}_{(x,y)} \gamma^t B(\gamma\phi^{-1}(x,y))\gamma(d\phi^{-1})_{(x,y)}\gamma^t \\ &= \gamma(d\phi^{-1})^{-1}_{(x,y)} \gamma^t \eta(\gamma)\gamma B(\phi^{-1}(x,y))\gamma^t \gamma(d\phi^{-1})_{(x,y)}\gamma^t \\ &= \eta(\gamma)\gamma(d\phi^{-1})^t_{(x,y)} B(\phi^{-1}(x,y))(d\phi^{-1})_{(x,y)}\gamma^t \\ &= \eta(\gamma)\gamma(\phi \cdot B)(x,y)\gamma^t. \end{aligned}$$

The *orbit* of the action of $\overrightarrow{\mathscr{R}}(\Gamma)$ on $B \in \overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma)$ is the set $\overrightarrow{\mathscr{R}}(\Gamma)B = \{\phi \cdot B : \phi \in \overrightarrow{\mathscr{R}}(\Gamma)\}$. Mappings on the same orbit of $\overrightarrow{\mathscr{R}}(\Gamma)$ have the same discriminant set. Indeed, if we denote by δ_B and $\delta_{\phi \cdot B}$ the discriminant function associated with B and $\phi \cdot B$, respectively, then $\delta^{-1}_{\phi \cdot B}(0) = (\delta_B \circ \phi^{-1})^{-1}(0) = \phi(\delta_B^{-1}(0))$.

From here on $\mathscr{Q}_{\ell}(\Gamma, \eta)$ denotes the set of Γ-equivariant quadratic 1-forms ω with linear coefficients; the equation $\omega = 0$ is denoted by the triple (a,b,c) where $a(x,y) = a_0 x + a_1 y$, $b(x,y) = b_0 x + b_1 y$ and $c(x,y) = c_0 x + c_1 y$. We identify $\overrightarrow{\mathscr{R}}(\Gamma)$ with the space of Γ-equivariant 2×2 matrices. So we can rewrite the equivalence relation (4.1) as

$$\phi \cdot B = \phi^t (B \circ \phi)\phi. \tag{4.3}$$

By using the relation (4.3) we compute, in next four subsections the normal form for each symmetry group Γ of the Table 3, namely $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$, $\mathbf{Z}_6[\mathbf{Z}_3]$, $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ and $\mathbf{Z}_2(-I)[1]$. As in the previous chapters we denote $\overrightarrow{\mathscr{P}}^{\eta}_{\ell}(\Gamma)$ by $\overrightarrow{\mathscr{P}}_{\ell}[\Gamma, \ker\eta]$. We finish with a table containing all the linear equivariant normal forms obtained.

4.1.1 $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$*-equivariant normal forms*

Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{D}_6(\kappa_x))$. So $\phi = sI$ $s \neq 0 \in \mathbb{R}$, where I denotes the identity matrix.

Let $B \in \overrightarrow{\mathscr{P}}_\ell[\mathbf{D}_6(\kappa_x), \mathbf{D}_3(\kappa_x)]$. According to the Table 3 there exists $a_0 \neq 0 \in \mathbb{R}$ such that $B(x,y) = a_0 B_1(x,y)$, where $B_1(x,y) = \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}$. From (4.3) we have

$$\phi^t B(\phi(x,y))\phi = s^3 a_0 B_1(x,y).$$

The discriminant function of $\phi \cdot B$ is given by $\delta(x,y) = s^6 a_0^2(x^2+y^2)$ so the discriminant set is always the origin. The equation of invariant straight lines (see (3.7)) is given by

$$s^3 a_0 \mu(-3\beta^2 + \mu^2) = 0.$$

There are exactly 3 invariant straight lines through the origin for any $a_0 \neq 0 \in \mathbb{R}$, namely $x = 0$ and $(\pm\sqrt{3}x, x)$. Now, taking $s = 1/\sqrt[3]{a_0}$, that is always possible because $a_0 \neq 0$, we have that B is $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$-equivalent to B_1 for all a_0. Therefore, there is only one no trivial orbit of the action of $\overrightarrow{\mathscr{R}}(\mathbf{D}_6)$ on $\overrightarrow{\mathscr{P}}_\ell(\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)])$ and we conclude:

Proposition 4.1.2. *Let $\omega \in \mathscr{Q}_\ell(\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)])$ be a quadratic 1-form of the form $(-a_0x, -a_0y, a_1y)$, $a_0 \neq 0$. Then ω is $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$-equivalent to*

$$(-x, -y, x).$$

In Figure 16 we illustrate the configuration associated with a $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$-invariant BDE.

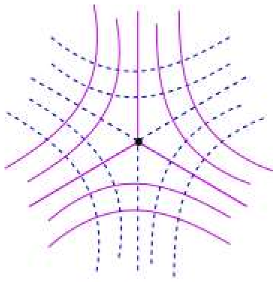


Figure 16 – $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$-equivariant linear normal form.

4.1.2 $\mathbf{Z}_6[\mathbf{Z}_3]$-equivariant normal forms

Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_6)$ be a linear diffeomorphism, so

$$\phi = \begin{pmatrix} A & -D \\ D & A \end{pmatrix}, \ A^2 + D^2 \neq 0, \ \ A, D \in \mathbb{R}. \tag{4.4}$$

Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_6, [\mathbf{Z}_3])$. According to Table 3 there exist $a_0, a_1 \in \mathbb{R}$ such that $a_0^2 + a_1^2 \neq 0$ and $B(x,y) = \begin{pmatrix} a_0x + a_1y & a_1x - a_0y \\ a_1x - a_0y & -a_0x - a_1y \end{pmatrix}$. From (4.3) we have

$$\phi^t B(\phi(x,y))\phi = \begin{pmatrix} A_0x + A_1y & A_1x - A_0y \\ A_1x - A_0y & -A_0x - A_1y \end{pmatrix}, \tag{4.5}$$

where

$$A_0 = a_0A^3 + 3a_1A^2D - 3a_0AD^2 - a_1D^3 \text{ and } A_1 = a_1A^3 - 3a_0A^2D - 3a_1AD^2 + a_0D^3. \quad (4.6)$$

The discriminant function of $\phi \cdot B$ is $\delta(x,y) = (A_0^2 + A_1^2)(x^2+y^2)$ so the discriminant set is always the origin since we suppose $A_0^2 + A_1^2 \neq 0$. The equation of invariant straight lines through the origin is given by

$$A_1\beta^3 - 3A_0\mu\beta^2 + 3A_1\mu^2\beta + A_0\mu^3 = 0,$$

and has 3 real roots, since $A_0^2 + A_1^2 \neq 0$. To deduce the equivariant normal forms we study the cases $a_0a_1 \neq 0$ and $a_0a_1 = 0$ separately.

4.1.2.1 *Case* $a_0a_1 = 0$

Consider $a_0 = 0$. Then $B(x,y) = \begin{pmatrix} a_1y & a_1x \\ a_1x & -a_1y \end{pmatrix}, a_1 \neq 0 \in \mathbb{R}$, and the coefficients A_0, A_1 in (4.6) are: $A_0 = 3a_1A^2D - a_1D^3$ and $A_1 = a_1A^3 - 3a_1AD^2$.

Setting $A = 0$ and $D = -\sqrt[3]{1/a_1}$ we have $A_0 = 1$ and $A_1 = 0$. Then, from (4.5), B is $\mathbf{Z}_6[\mathbf{Z}_3]$-equivalent to

$$B_1(x,y) = \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}.$$

Now, if we take $\psi = \begin{pmatrix} -1/\sqrt[3]{2} & 1/\sqrt[3]{2} \\ -1/\sqrt[3]{2} & -1/\sqrt[3]{2} \end{pmatrix}$, then $\big(\psi^t(B_1 \circ \psi)\psi\big)(x,y) = B_2(x,y)$ where

$$B_2(x,y) = \begin{pmatrix} x+y & x-y \\ x-y & -x-y \end{pmatrix}.$$

If we suppose $a_1 = 0$, is enough to put $A = \sqrt[3]{1/a_0}$, $D = 0$ and the result holds.

4.1.2.2 *Case* $a_0a_1 \neq 0$

Take ϕ in (4.4) with $D \neq 0$. So we can see A_0 and A_1 as polynomials in the variable $p = A/D$, that is,

$$A_0(p) = a_0p^3 + 3a_1p^2 - 3a_0p - a_1 \text{ and } A_1(p) = a_1p^3 - 3a_0p^2 - 3a_1p + a_0.$$

The cubic A_1 has at least one real root. Let p_1 be one of them, so $A_1(p_1) = 0$. Now, the resultant of A_0, A_1 is $R(A_0,A_1) = 64(a_0^2 + a_1^2)^3$ that is always different from zero, since $a_0^2 + a_1^2 \neq 0$. This means that there is no common roots between $A_0(p)$ and $A_1(p)$ (for a reference on the resultant of two polynomials see [20]). So, setting $D = 1$ and $A = p_1$ from (4.5) B is $\mathbf{Z}_6[\mathbf{Z}_3]$-equivalent to B_1 where

$$B_1(x,y) = \begin{pmatrix} A_0(p_1)x & -A_0(p_1)y \\ -A_0(p_1)y & -A_0(p_1)x \end{pmatrix}.$$

Now, taking $\psi = \begin{pmatrix} -1/\sqrt[3]{2A_0(p_1)} & 1/\sqrt[3]{2A_0(p_1)} \\ -1/\sqrt[3]{2A_0(p_1)} & -1/\sqrt[3]{2A_0(p_1)} \end{pmatrix}$, B_1 is $\mathbf{Z}_6[\mathbf{Z}_3]$-equivalent to B_2, via ψ, where

$$B_2(x,y) = \begin{pmatrix} x+y & x-y \\ x-y & -x-y \end{pmatrix}.$$

From the discussion above we conclude that there is only one nontrivial orbit of the action $\overrightarrow{\mathscr{R}}(\mathbf{Z}_6)$ on $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_6, [\mathbf{Z}_3])$:

Proposition 4.1.3. *Let ω be a $\mathbf{Z}_6[\mathbf{Z}_3]$-equivariant quadratic 1-form of the form $(-a_0x - a_1y, a_1x - a_0y, a_0x + a_1y)$, $a_0^2 + a_1^2 \neq 0$. Then ω is $\mathbf{Z}_6[\mathbf{Z}_3]$-equivalent to*

$$(-x-y, x-y, x+y).$$

In Figure 17 we illustrate the configuration associated with a $\mathbf{Z}_6[\mathbf{Z}_3]$-invariant BDE.

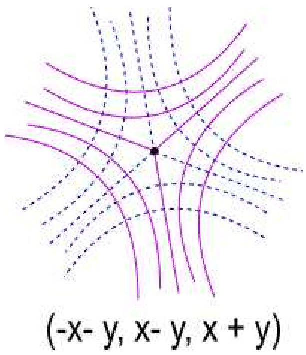


Figure 17 – $\mathbf{Z}_6[\mathbf{Z}_3]$-equivariant linear normal form.

Remark 4.1.4. *Consider the quadratic 1-forms*

$$\omega_1 = (-a_0x, -a_0y, a_0y) \ \text{ and } \ \omega_1 = (a_1y, -a_1x, -a_1y),$$

where $a_0, a_1 \in \mathbb{R}^*$. By Proposition 4.1.3 they are $\mathbf{Z}_6[\mathbf{Z}_3]$-equivalents. It is interesting to note that they are both symmetric under the larger order-6 dihedral group, the first one being $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_x)]$-equivariant and the second $\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_y)]$-equivariant.

4.1.3 $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant normal forms

Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ be a linear diffeomorphism, so ϕ is of the form $\phi = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $AD \neq 0$, $A, D \in \mathbb{R}$. Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$, then

$$B(x,y) = \begin{pmatrix} c_0x & b_1y \\ b_1y & a_0x \end{pmatrix},$$

where $a_0, b_1, c_0 \in \mathbb{R}$. From (4.3) we have

$$\phi^t B(\phi(x,y))\phi = \begin{pmatrix} c_0 A^3 x & b_1 A D^2 y \\ b_1 A D^2 y & a_0 A D^2 x \end{pmatrix}. \tag{4.7}$$

The discriminant function of B is $\delta(x,y) = b_1^2 y^2 - c_0 a_0 x^2$, then the discriminant set can be the origin, if $a_0 b_1 c_0 \neq 0$ and $c_0 a_0 < 0$; two transversal straight lines through the origin if $a_0 b_1 c_0 \neq 0$ and $c_0 a_0 > 0$; the y-axis if $a_0 c_0 \neq 0$ and $b_1 = 0$ or the x-axis if $b_1 \neq 0$ and $c_0 a_0 = 0$.

The equation of invariant straight lines is given by

$$c_0 \mu^3 + \beta^2 \mu (2b_1 + a_0) = 0, \tag{4.8}$$

then the y-axis is always an invariant straight line and the existence of other invariant straight lines depends on the choice of the coefficients $a_0, b_1, c_0 \in \mathbb{R}$.

We divide the study in five cases according to the coefficients of the BDE: $b_1 = 0$ and $a_0 c_0 \neq 0$; $a_0 = 0$ and $b_1 c_0 \neq 0$; $a_0 = c_0 = 0$ and $b_1 \neq 0$; $c_0 = 0$ and $a_0 b_1 \neq 0$ and finally $a_0 b_1 c_0 \neq 0$.

4.1.3.1 Case $b_1 = 0$ and $a_0 c_0 \neq 0$

Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ of the form

$$B(x,y) = \begin{pmatrix} c_0 x & 0 \\ 0 & a_0 x \end{pmatrix}.$$

The discriminant function is $\delta(x,y) = -a_0 c_0 x^2$, so the discriminant set is $\Delta = \{(0,y), y \in \mathbb{R}\}$. From (4.8) the equation of invariant straight lines is given by

$$\mu(c_0 \mu^2 + a_0 \beta^2) = 0.$$

So, when $a_0 c_0 > 0$ we have one invariant straight line which coincides with the discriminant set and, when $a_0 c_0 < 0$, we have 3 invariant straight lines, where again, one of them coincide with the discriminant set .

Let $\varepsilon = sign(a_0 c_0) = \pm 1$. For $\varepsilon = 1$ setting $A = 1/\sqrt[3]{c_0}$ and $D = \sqrt[2]{\sqrt[3]{c_0}/a_0}$ in (4.7) then B is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to B_1 where

$$B_1(x,y) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

For $\varepsilon = -1$ setting $A = 1/\sqrt[3]{c_0}$ and $D = \sqrt[2]{-\sqrt[3]{c_0}/a_0}$ in (4.7) then B is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to B_2 where

$$B_2(x,y) = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}.$$

Therefore, we conclude that:

Proposition 4.1.5. *Let ω be a $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant quadratic 1-form of the form $(a_0x,0,c_0x)$. Then ω is $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to*

$$(\varepsilon x,\ 0,\ x),\ \ \varepsilon=sign(a_0c_0)=\pm 1.$$

In Figure 18 we illustrate the configurations associated with the equivariant normal form of the Proposition 4.1.5 with $\varepsilon=1$ and $\varepsilon=-1$, respectively.

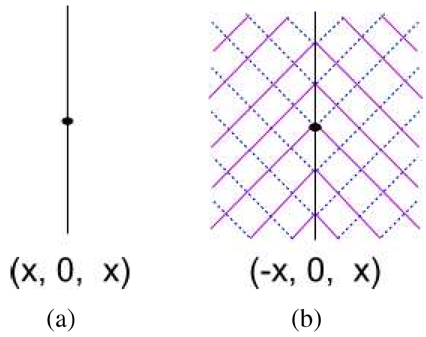


Figure 18 – $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant normal forms.

4.1.3.2 Case $a_0=0$ and $b_1c_0\neq 0$

Let $B\in\overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ of the form

$$B(x,y)=\begin{pmatrix} c_0x & b_1y\\ b_1y & 0\end{pmatrix}.$$

The discriminant function is $\delta(x,y)=b_1^2y^2$ so the discriminant set is $\Delta=\{(x,0),x\in\mathbb{R}\}$. From (4.8) the equation of the invariant straight lines is given by

$$\mu(c_0\mu^2+2b_1\beta^2)=0.$$

When $b_1c_0>0$ we have one invariant straight line through the origin and when $b_1c_0<0$ we have 3 invariant straight lines, and none of them coincide with the discriminant set .

Let $\varepsilon=sign(b_1c_0)$. Setting $A=1/\sqrt[3]{c_0}$ and $D=\sqrt{\varepsilon\sqrt[3]{c_0}/b_1}$ in (4.7) we have that B is $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to B_1 where

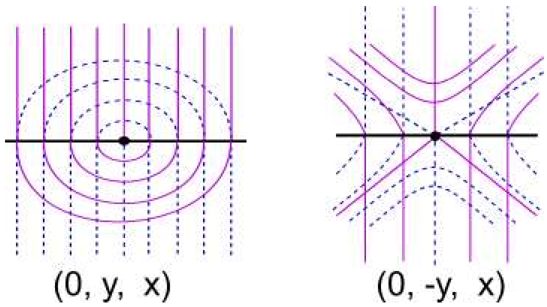
$$B_1(x,y)=\begin{pmatrix} x & \varepsilon y\\ \varepsilon y & 0\end{pmatrix}.$$

We conclude that:

Proposition 4.1.6. *Let ω be a $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant quadratic 1-form of the form $(0,b_1y,c_0x)$. Then ω is $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to*

$$(0,\ \varepsilon y,\ x),\ \ \varepsilon=sign(b_1c_0)=\pm 1.$$

In Figure 19 we illustrate the configurations associated with $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-invariant BDEs of the Proposition 4.1.6 with $\varepsilon=1$ and $\varepsilon=-1$, respectively.



(a) (b)

Figure 19 – $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant normal forms.

4.1.3.3 *Case* $a_0 = c_0 = 0$ *and* $b_1 \neq 0$

Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ with $a_0 = c_0 = 0$ and $b_1 \neq 0$, so

$$B(x,y) = \begin{pmatrix} 0 & b_1 y \\ b_1 y & 0 \end{pmatrix}.$$

The discriminant function is $\delta(x,y) = b_1^2 y^2$ and the discriminant set is $\Delta = \{(x,0), x \in \mathbb{R}\}$. From (4.8) the equation of invariant straight lines is

$$2b_1 \mu \beta^2 = 0.$$

So the coordinate axes always are the invariant straight lines through the origin, and one of them coincide with the discriminant set .

Setting $A = 1/b_1$ and $D = 1$ in (4.7), B is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to B_1 where

$$B_1(x,y) = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}.$$

Therefore we conclude that:

Proposition 4.1.7. *Let ω be a $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant quadratic 1-form of the form $(0, b_1 y, 0)$. Then ω is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to*

$$(0,\ y,\ 0).$$

In Figure 20 we illustrate the configuration associated with the equivariant normal form in Proposition 4.1.7.

4.1.3.4 *Case* $c_0 = 0$ *and* $a_0 b_1 \neq 0$

Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ with $c_0 = 0$ and $a_0 b_1 \neq 0$, so

$$B(x,y) = \begin{pmatrix} 0 & b_1 y \\ b_1 y & a_0 x \end{pmatrix}.$$

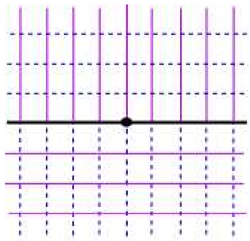


Figure 20 – $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant linear normal form.

The discriminant function is $\delta(x,y) = b_1^2 y^2$ and the discriminant set is $\Delta = \{(x,0), x \in \mathbb{R}\}$. From (4.8) the equation of invariant straight lines is

$$\mu\beta^2(2b_1 + a_0) = 0. \tag{4.9}$$

So, when $2b_1 + a_0 \neq 0$ the invariant straight lines are the coordinates axis and, when $2b_1 + a_0 = 0$, we have infinity invariant straight lines through the origin.

Setting $A = 1/b_1$ and $D = 1$ in (4.7), B is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to B_1 where

$$B_1(x,y) = \begin{pmatrix} 0 & y \\ y & a_0/b_1 x \end{pmatrix}.$$

Now, denote B by $B(a_0,b_1)$ to explicit the coefficients a_0, b_1. From (4.7), the mappings $B(a_0,b_1)$ and $\tilde{B}(\tilde{a}_0,\tilde{b}_1)$ are equivalents if and only if there exist $A, D \in \mathbb{R}$, $AD \neq 0$ such that

$$\begin{pmatrix} 0 & y \\ y & a_0/b_1 x \end{pmatrix} = \begin{pmatrix} 0 & AD^2 y \\ AD^2 y & AD^2 \tilde{a}_0/\tilde{b}_1 x \end{pmatrix},$$

that is, if and only if

$$a_0/b_1 = \tilde{a}_0/\tilde{b}_1.$$

So we conclude that:

Proposition 4.1.8. *Let ω be a $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant quadratic 1-form of the form $(a_0 x, b_1 y, 0)$. Then ω is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to*

$$(mx, y, 0), \quad 0 \neq m \in \mathbb{R}.$$

According to the expression of the normal form in Proposition 4.1.8 the equation of invariant straight lines (4.9) can be rewritten as $\mu\beta^2(2+m) = 0$. So, when $m = -2$ we have infinite invariant straight lines through the origin, otherwise we have two straight lines, one being the discriminant set. The foliations are given by the solutions of

$$mxdy^2 + 2ydxdy = 0.$$

One foliation is given by the curves $y = c$ and $\ln|yx^{2m}| = c$, for $c > 0$ and, the curves of the other foliation have the same equations for $c < 0$. The parameter m is called *modal parameter* or *moduli*. In Figure 21 we can see its geometrical effect on the configuration associated with a $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-invariant BDE of the form $(mx, y, 0)$.

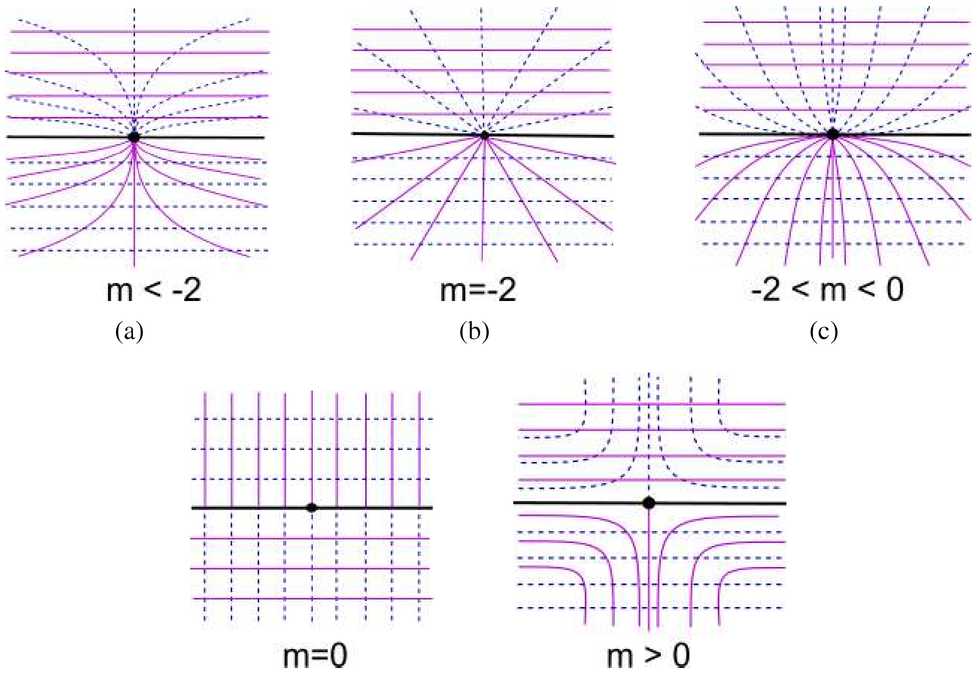


Figure 21 – Configurations associated with $(mx, y, 0)$, $m \in \mathbb{R}$.

4.1.3.5 Case $a_0 b_1 c_0 \neq 0$

Let $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)])$ of the form

$$B(x,y) = \begin{pmatrix} c_0 x & b_1 y \\ b_1 y & a_0 x \end{pmatrix}.$$

We denote B by $B(a_0, b_1, c_0)$. The discriminant function is $\delta(x,y) = b_1^2 y^2 - a_0 c_0 x^2$ and the discriminant set is the origin if $a_0 c_0 < 0$ or two transversal straight lines through the origin if $a_0 c_0 > 0$. Let $\varepsilon = sign(a_0 c_0)$. From (4.7), if we take $A = 1/\sqrt[3]{c_0}$ and $D = \sqrt[6]{\varepsilon c_0 / a_0^3}$ we have that $B(a_0, b_1, c_0)$ is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ equivalent to $B(\varepsilon, \varepsilon b_1/a_0, 1)$.

Lemma 4.1.9. *Let $B(\varepsilon, \varepsilon b_1/a_0, 1)$, $\varepsilon = sign(a_0 c_0)$, and $\tilde{B}(\tilde{\varepsilon}, \tilde{\varepsilon}\tilde{b}_1/\tilde{a}_0, 1)$, $\tilde{\varepsilon} = sign(\tilde{a}_0 \tilde{c}_0)$. The mappings B and $\tilde{B}$ are $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$ equivalent if and only if*

$$\varepsilon = \tilde{\varepsilon} \ \ and \ \ b_1/a_0 = \tilde{b}_1/\tilde{a}_0.$$

Proof. From (4.7), the mappings are equivalent if and only if there exist $A, D \in \mathbb{R}$, $AD \neq 0$ such that

$$\begin{pmatrix} A^3 x & \varepsilon b_1/a_0 A D^2 y \\ \varepsilon b_1/a_0 A D^2 y & \varepsilon A D^2 x \end{pmatrix} = \begin{pmatrix} x & \tilde{\varepsilon}\tilde{b}_1/\tilde{a}_0 y \\ \tilde{\varepsilon}\tilde{b}_1/\tilde{a}_0 y & \tilde{\varepsilon} x \end{pmatrix}.$$

Then $A = 1$, $D^2 = \tilde{\varepsilon}/\varepsilon > 0$ and so the result holds. □

As the consequence of Lemma 4.1.9 we have that:

Proposition 4.1.10. *Let ω be a $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant quadratic 1-form of the form (a_0x, b_1y, c_0x). Then ω is $\mathbf{Z}_2 \times \mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivalent to*

$$(\varepsilon x, \varepsilon m y, x), \;\; 0 \neq m \in \mathbb{R} \;\; and \;\; \varepsilon = sign(a_0 c_0).$$

According to Proposition 4.1.10 the discriminant function is

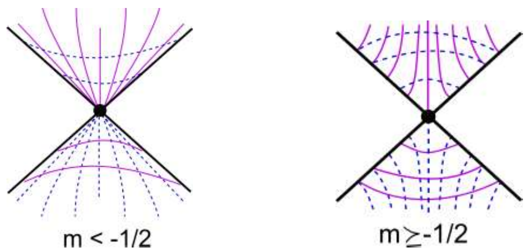
$$\delta(x,y) = m^2y^2 - \varepsilon x^2,$$

and the equation of invariant straight lines is given by

$$\mu(\mu^2 + \beta^2(2\varepsilon m + \varepsilon)) = 0.$$

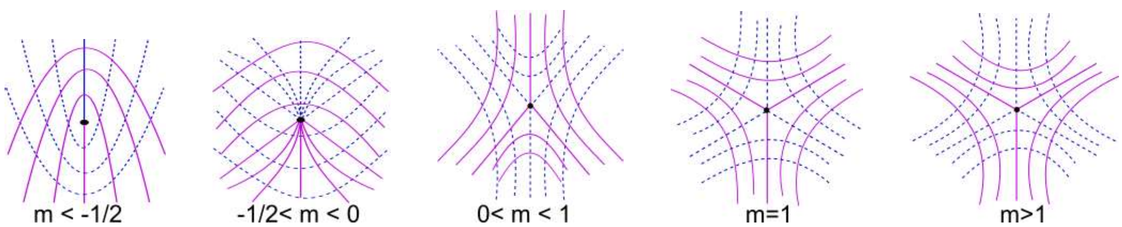
Consider $\varepsilon = 1$. The discriminant set is two straight lines through the origin, $x = \pm my$. We have 1 invariant straight line trough the origin, the y-axis, if $2m+1 \geq 0$. If $2m+1 < 0$ we have 3 invariant straight lines, but from Remark 3.2.4 they appear in the configuration associated with the BDE if and only if they are contained in $\Omega : \{(x,y) \in \mathbb{R} : \delta(x,y) > 0\}$. So, direct calculations show us that we see three invariant straight lines except if $m = -1$, in whose case two of them coincide with the discriminant set. Now, consider $\varepsilon = -1$. Then the discriminant set is the origin and we have 1 straight line through the origin if $2m+1 \leq 0$ and 3 if $2m+1 > 0$.

In Figure 22 and Figure 23 we illustrate the variation of the modal parameter m in each case, $\varepsilon = \pm 1$, respectively. In both cases the modal parameter m control the position of invariant straight lines in the configuration associated with the BDE.



(a) (b)

Figure 22 – Configurations associated with (x, my, x) $m \neq 0 \in \mathbb{R}$.



(a) (b) (c) (d) (e)

Figure 23 – Configurations associated with $(-x, -my, x)$, $m \neq 0 \in \mathbb{R}$.

4.1.4 $\mathbf{Z}_2(-I)[1]$-equivariant normal forms

The Γ-equivalence relation given in (4.3) when $\Gamma = \mathbf{Z}_2(-I)$ is the linear change of coordinates used by Bruce and Tari in [11] to do a formal reduction of a linear part of BDE with discriminant function of Morse type. This means that the discriminant set is the origin or a pair of transversal straight lines through the origin.

We use de results of [11] to direct our search to the equivariant normal forms in this section. From [11, Proposition 3.2] we have the prenormal form

$$(y, b_0x + b_1y, \varepsilon y), \quad \varepsilon = \pm 1. \tag{4.10}$$

The discriminant function is given by

$$\delta(x,y) = b_0^2x^2 + 2b_0b_1xy + (b_1^2 - \varepsilon)y^2.$$

Let $D := (2b_0b_1)^2 - 4b_0^2((b_1^2 - \varepsilon)) = 4(\varepsilon b_0^2)$. Since δ is a conic we have that the discriminant set is: the origin when $D < 0$, that is, $\varepsilon = -1$ and $b_0 \neq 0$; one straight line through the origin when $D = 0$, that is, $b_0 = 0$; two straight lines through the origin when $D > 0$ or $(b_1^2 - \varepsilon) = 0$, that is, $\varepsilon = 1$ and $b_0 \neq 0$.

The equation of invariant straight lines is given by

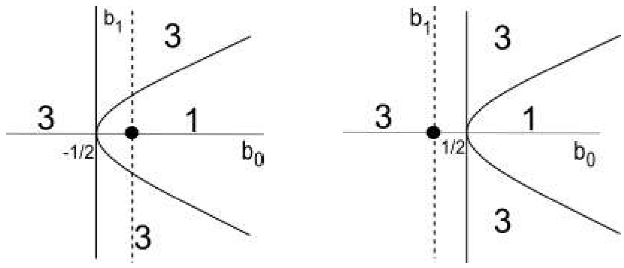
$$\beta\big(\beta^2 + 2b_1\beta\mu + \mu^2(2b_0 + \varepsilon)\big). \tag{4.11}$$

Thus, the x-axis is always an invariant line and the others are given by the zeros of $\beta^2 + 2b_1\beta\mu + \mu^2(2b_0 + \varepsilon)$. Then, we obtain 2 distinct invariant straight lines when $b_1^2 - 2b_0 - \varepsilon = 0$ or $2b_0 + \varepsilon = 0$; one invariant straight lines when $b_1^2 - 2b_0 - \varepsilon < 0$ and 3 distinct invariant straight lines when $b_1^2 - 2b_0 - \varepsilon > 0$.

The conditions discussed above give us special curves partitioning the (b_0, b_1)-plane on regions, which direct our search to equivariant normal form. Namely:
The curves for $\varepsilon = 1$ are
(i) $b_0 = 0$;
(ii) $2b_0 + 1 = 0$ or $b_1^2 - 2b_0 - 1 = 0$;

The curves for $\varepsilon = -1$ are
(i) $b_0 = 0$.
(ii) $2b_0 - 1 = 0$ or $b_1^2 - 2b_0 + 1 = 0$.

In Figure 24 we have the partition of the (b_0, b_1)-plane for $\varepsilon = 1$ and $\varepsilon = -1$, respectively. The dashed lines are the points where the discriminant set is a straight line through the origin, on the other points the discriminant set is a point in Figure 24(a) and two straight lines in Figure 24(b). The number of invariant straight lines through the origin is explicit on the figures and on the solid lines this number is equal to two.



(a) (b)

Figure 24 – Partition of the (b_0, b_1)-plane for $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

We denote a matrix-valued mapping $B \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$ associated with (4.10) by $B(b_0, b_1)$. We divide the deduction of the $\mathbf{Z}_2(-1)[1]$-equivaraint normal forms in three principal cases: $\varepsilon = 1$, $\varepsilon = -1$ and when the pair (b_0, b_1) does not satisfy any of the conditions listed above. From here, if $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ is a linear diffeomorphism, then

$$\phi = \begin{pmatrix} \alpha & \sigma \\ \vartheta & \xi \end{pmatrix}, \tag{4.12}$$

for $\alpha\xi - \sigma\vartheta \neq 0,\ \alpha, \vartheta, \xi, \sigma \in \mathbb{R}$.

4.1.4.1 Equivariant normal forms of $(y, b_0x + b_1y, y)$

We deduce here the $\mathbf{Z}_2(-I)[1]$-equivariant normal forms for the pair (b_0, b_1) in each one of the conditions: $b_0 = 0$, $2b_0 + 1 = 0$ and $b_1^2 - 2b_0 - 1 = 0$.

1. **Case** $b_0 = 0$

   Let $B(0, b_1) \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$. The discriminant function in given by

   $$\delta(x, y) = (b_1^2 - 1)y^2,$$

   so the discriminant set is the x-axis if $b_1 \neq \pm 1$ and all the plane otherwise. According to (4.11) the equation of invariant straight lines is given by

   $$\beta\left(\beta^2 + 2\mu\beta b_1 + \mu^2\right) = 0.$$

   So, we have 1 invariant straight line if $-1 < b_1 < 1$; 2 if $b_1 = \pm 1$ and 3 if $|b_1| > 1$.

   Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From equivalence relation (4.3) we have

   $$\phi^t(B \circ \phi(x, y))\phi = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

   where

   $$C_0 = \vartheta(2b_1\alpha\vartheta + \alpha^2 + \vartheta^2) \text{ and } C_1 = \xi(2b_1\alpha\vartheta + \alpha^2 + \vartheta^2),$$

$$B_0 = \vartheta(b_1\xi\alpha + b_1\sigma\vartheta + \xi\vartheta + \alpha\sigma) \text{ and } B_1 = \xi(b_1\xi\alpha + b_1\sigma\vartheta + \xi\vartheta + \alpha\sigma),$$

$$A_0 = \vartheta(2b_1\xi\sigma + \xi^2 + \sigma^2) \text{ and } A_1 = \xi(2b_1\xi\sigma + \xi^2 + \sigma^2).$$

The condition $A_0 = C_0 = B_0 = 0$ is satisfies if and only if $\vartheta = 0$. Now the condition $C_1 = A_1 = 1$ implies that $\xi = 1/\alpha^2$ and $\sigma = (-b_1 + \sqrt{\alpha^6 + b_1^2 - 1})/\alpha^2$. The expression of B_1 becomes

$$B_1 = \sqrt{\alpha^6 + b_1^2 - 1}/\alpha^3,$$

Therefore, $B(0, b_1)$ is $\mathbf{Z}_2(-I)[1]$ equivalent to $B(0, \tilde{b_1})$ if and only if $B_1 = \tilde{b_1}$. Thinking of B_1 as a function of α we find three equivalence class: one if $-1 < b_1 < 1$, other if $|b_1| > 1$ and, if $b_1 = 1$ is enough to set $\vartheta = 0$, $\xi = 1/\alpha^2$ and $\sigma = (-1 \pm \alpha^3)/\alpha^2$ and $B(0,1)$ is equivalent to $B(0, \pm 1)$.

Then we have 3 normal forms:

$$(y, 2y, y), \quad (y, y/2, y) \text{ and } (y, y, y).$$

2. **Case** $2b_0 + 1 = 0$

Let $B(-1/2, b_1) \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$. The discriminant function in given by

$$\delta(x,y) = x^2/4 - b_1xy + (b_1^2 - 1)y^2$$

so the discriminant set is the reunion of two transversal lines across the origin for all $0 \neq b_1 \in \mathbb{R}$. According to (4.11) the equation of invariant straight lines is given by

$$\beta^2(\beta + 2b_1\mu) = 0.$$

So, we have 2 invariant straight lines for all $b_1 \in \mathbb{R}$. Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From the equivalence relation (4.3) we have

$$\phi^t(B \circ \phi(x,y))\phi = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

where

$$C_0 = \vartheta^2(2b_1\alpha + \vartheta) \text{ and } C_1 = 2b_1\xi\alpha\vartheta + \xi\alpha^2 + \xi\vartheta^2 - \alpha\sigma\vartheta,$$

$$B_0 = 1/2\alpha\sigma\vartheta + \sigma b_1\vartheta^2 - 1/2\xi\alpha^2 + \alpha b_1\xi\vartheta + \xi\vartheta^2 \text{ and}$$

$$B_1 = 1/2\sigma\xi\alpha - 1/2\vartheta\sigma^2 + \sigma\vartheta b_1\xi + \xi^2\alpha b_1 + \xi^2\vartheta,$$

$$A_0 = 2b_1\xi\sigma\vartheta + \xi^2\vartheta - \xi\alpha\sigma + \sigma^2\vartheta \text{ and } A_1 = \xi^2(2b_1\sigma + \xi).$$

The condition $C_0 = A_0 = 0$ implies that $\vartheta = \sigma = 0$. The expressions of C_1 and A_1 become $C_1 = \xi\alpha^2$ and $A_1 = \xi^3$. So, $C_1 = A_1 = 1$ implies that $\xi = 1$ and $\alpha^2 = 1$ and thus

$$B_0 = -1/2 \text{ and } B_1 = \alpha b_1.$$

Therefore, the normal form in this case is

$$(y, -x/2 + my, y), \; m > 0,$$

where the modal parameter m gives the position of the invariant straight lines in the configuration associated with the BDE.

3. **Case** $b_1^2 - 2b_0 - 1 = 0$

Let $B(b_0, b_1) \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$ with $b_1^2 - 2b_0 - 1 = 0$. The discriminant function is given by

$$\delta(x,y) = (b_1^2/2 - 1/2)^2 x^2 + 2b_1(b_1^2/2 - 1/2)xy + (b_1^2 - 1)y^2$$

the discriminant set is the union of two transversal lines across the origin unless $b_1 = \pm 1$. According to (4.11) the equation of invariant straight lines is given by

$$\beta\left(\beta^2 + 2\mu\beta b_1 + \mu^2 b_1^2\right) = 0.$$

So, we have 2 invariant straight lines for all $0 \neq b_1 \in \mathbb{R}$. Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From equivalence relation (4.3) we have

$$\phi^t(B \circ \phi(x,y))\phi = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

where

$$C_0 = \vartheta(b_1\alpha + \vartheta)^2 \;\text{ and }\; C_1 = b_1^2\alpha\sigma\vartheta + 2b_1\xi\alpha\vartheta + \xi\alpha^2 + \xi\vartheta^2 - \alpha\sigma\vartheta,$$

$$B_0 = 1/2\alpha\sigma\vartheta + 1/2b_1^2\alpha\sigma\vartheta + \sigma b_1\vartheta^2 + 1/2\xi\alpha^2 b_1^2 - 1/2\xi\alpha^2 + b_1\xi\alpha\vartheta + \xi\vartheta^2 \;\text{ and}$$

$$B_1 = 1/2\sigma\xi\alpha + 1/2\vartheta\sigma^2 b_1^2 - 1/2\vartheta\sigma^2 + \sigma\vartheta b_1\xi + 1/2b_1^2\xi\alpha\sigma + \xi^2 b_1\alpha + \xi^2\vartheta,$$

$$A_0 = b_1^2\xi\alpha\sigma + 2b_1\xi\sigma\vartheta + \xi^2\vartheta - \xi\alpha\sigma + \sigma^2\vartheta \;\text{ and }\; A_1 = \xi(b_1\sigma + \xi)^2.$$

The condition $C_0 = A_0 = 0$ implies that $\vartheta = \sigma = 0$. The expressions of C_1 and A_1 become $C_1 = \xi\alpha^2$ and $A_1 = \xi^3$. So, $A_1 = C_1 = 1$ implies that

$$B_0 = 1/2(b_1^2 - 1) \;\text{ and }\; B_1 = \alpha b_1.$$

So the normal form is given by

$$(y, (m^2 - 1)x/2 + my, y), m > 0 \text{ and } m \neq 1.$$

When $b_1 = 1$ we return for the case $b_0 = 0$, and the normal form is (y, y, y).

4.1.4.2 Equivariant normal forms of $(y, b_0x + b_1y, -y)$

We deduce here the $\mathbf{Z}_2(-I)[1]$-equivariant normal forms for the pair (b_0, b_1) in each one of the conditions: $b_0 = 0$, $2b_0 - 1 = 0$ and $b_1^2 - 2b_0 + 1 = 0$.

1. **Case** $b_0 = 0$

   Let $B(0, b_1) \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$. The discriminant function in given by

   $$\delta(x,y) = (b_1^2 + 1)y^2,$$

   so the discriminant set is the x-axis for all $b_1 \in \mathbb{R}$. According to (4.11) the equation of invariant straight lines is given by

   $$\beta\left(\beta^2 + 2\mu\beta b_1 - \mu^2\right) = 0.$$

   So, we have 3 invariant straight lines through the origin.
   Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From the equivalence relation (4.3) we have

   $$\phi^t(B \circ \phi(x,y))\phi = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

   where

   $$C_0 = \vartheta(2b_1\alpha\vartheta - \alpha^2 + \vartheta^2) \text{ and } C_1 = \xi(2b_1\alpha\vartheta - \alpha^2 + \vartheta^2),$$
   $$B_0 = \vartheta(b_1\xi\alpha + b_1\sigma\vartheta + \xi\vartheta - \alpha\sigma) \text{ and } B_1 = \xi(b_1\xi\alpha + b_1\sigma\vartheta + \xi\vartheta - \alpha\sigma),$$
   $$A_0 = \vartheta(2b_1\xi\sigma + \xi^2 - \sigma^2) \text{ and } A_1 = \xi(2b_1\xi\sigma + \xi^2 - \sigma^2).$$

   The condition $A_0 = C_0 = B_0 = 0$ is satisfied if and only if $\vartheta = 0$. Now the conditions $C_1 = -1$ and $A_1 = 1$ imply that $\xi = 1/\alpha^2$ and $\sigma = (b_1 + \sqrt{-\alpha^6 + b_1^2 + 1})/\alpha^2$. The expression of B_1 becomes

   $$B_1 = -\sqrt{-\alpha^6 + b_1^2 + 1}/\alpha^3,$$

   Therefore, $B(0, b_1)$ is $\mathbf{Z}_2(-I)[1]$ equivalent to $B(0,1)$ if and only if $B_1 = 1$ which is always possible. In fact, take $\alpha = -(1/2)\sqrt{6}(32b_1^2 + 32)$ so $B_1 = 1$ and $\alpha^6 = 1/2(b_1^2 + 1) < b_1^2 + 1$, then σ is well define. We conclude that there exists only one nontrivial equivariant normal form:

   $$(y, y, -y).$$

2. **Case** $2b_0 - 1 = 0$

   Let $B(1/2, b_1) \in \overrightarrow{\mathscr{P}}(\mathbf{Z}_2(-I)[1])$. The discriminant function in given by

   $$\delta(x,y) = x^2/4 + b_1xy + (b_1^2 + 1)y^2,$$

so the discriminant set is the origin for all $0 \neq b_1 \in \mathbb{R}$. According to (4.11) the equation of invariant straight lines is given by

$$\beta^2(\beta + 2b_1\mu) = 0.$$

So, we have 2 invariant straight lines for all $b_1 \in \mathbb{R}$.

Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From the equivalence relation we have

$$\phi^t(B \circ \phi(x,y)) = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

where

$$C_0 = \vartheta^2(2b_1\alpha + \vartheta) \text{ and } C_1 = 2b_1\xi\alpha\vartheta - \xi\alpha^2 + \xi\vartheta^2 + \alpha\sigma\vartheta,$$

$$B_0 = -1/2\alpha\sigma\vartheta + \sigma b_1\vartheta^2 + 1/2\xi\alpha^2 + \alpha b_1\xi\vartheta + \xi\vartheta^2 \text{ and}$$

$$B_1 = -1/2\sigma\xi\alpha + 1/2\vartheta\sigma^2 + \sigma\vartheta b_1\xi + \xi^2\alpha b_1 + \xi^2\vartheta,$$

$$A_0 = 2b_1\xi\sigma\vartheta + \xi^2\vartheta + \xi\alpha\sigma - \sigma^2\vartheta \text{ and } A_1 = \xi^2(2b_1\sigma + \xi).$$

The condition $C_0 = A_0$ implies that $\vartheta = \sigma = 0$. The expressions of C_1 and A_1 become $C_1 = -\xi\alpha^2$ and $A_1 = \xi^3$. So, $C_1 = -1$ and $A_1 = 1$ implies that $\xi = 1$ and $\alpha^2 = 1$ and thus

$$B_0 = 1/2 \text{ and } B_1 = \alpha b_1.$$

Therefore, the normal form in this case is

$$(y, x/2 + my, -y), \; m > 0,$$

where the modal parameter m gives the position of the invariants straight lines in the configuration associated with the BDE.

3. **Case** $b_1^2 - 2b_0 + 1 = 0$

Let $B(b_0, b_1) \in \overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$ with $b_1^2 - 2b_0 + 1 = 0$. The discriminant function in given by

$$\delta(x,y) = (b_1^2/2 + 1/2)x^2 + b_1(b_1^2 + 1)xy + (b_1^2 + 1)y^2$$

and the discriminant set is the origin for all $b_1 \in \mathbb{R}$. According to (4.11) the equation of invariant straight lines is given by

$$\beta(\beta^2 + 2\mu\beta b_1 + \mu^2 b_1^2) = 0.$$

So, we have 2 invariant straight lines for all $0 \neq b_1 \in \mathbb{R}$. Let $\phi \in \overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From the equivalence relation (4.3) we have

$$\phi^t(B \circ \phi(x,y))\phi = \begin{pmatrix} C_0x + C_1y & B_0x + B_1y \\ B_0x + B_1y & A_0x + A_1y \end{pmatrix},$$

where

$$C_0=\vartheta(b_1\alpha+\vartheta)^2 \text{ and } C_1=b_1^2\alpha\sigma\vartheta+2b_1\xi\alpha\vartheta-\xi\alpha^2+\xi\vartheta^2+\alpha\sigma\vartheta,$$

$$B_0=-1/2\alpha\sigma\vartheta+1/2b_1^2\alpha\sigma\vartheta+\sigma b_1\vartheta^2+1/2\xi\alpha^2b_1^2+1/2\xi\alpha^2+b_1\xi\alpha\vartheta+\xi\vartheta^2 \text{ and}$$

$$B_1=-1/2\sigma\xi\alpha+1/2\vartheta\sigma^2b_1^2+1/2\vartheta\sigma^2+\sigma\vartheta b_1\xi+1/2b_1^2\xi\alpha\sigma+\xi^2b_1\alpha+\xi^2\vartheta,$$

$$A_0=b_1^2\xi\alpha\sigma+2b_1\xi\sigma\vartheta+\xi^2\vartheta+\xi\alpha\sigma-\sigma^2\vartheta \text{ and } A_1=\xi(b_1\sigma+\xi)^2.$$

The conditions $C_0=A_0=0$ imply that $\vartheta=\sigma=0$. The expressions of C_1 and A_1 become $C_1=-\xi\alpha^2$ and $A_1=\xi^3$. So, $A_1=1$ and $C_1=1$ imply that

$$B_0=(b_1^2+1)/2 \text{ and } B_1=\alpha b_1,\ \alpha^2=1.$$

So the normal form is given by

$$(y,(m^2+1)x/2+my,-y),\ m>0.$$

4.1.4.3 Equivariant nornal form of $(y,b_0x+b_1y,\varepsilon y)$

Let $B(b_0,b_1)\in\overrightarrow{\mathscr{P}}_\ell(\mathbf{Z}_2(-I)[1])$ with the pair (b_0,b_1) does not satisfy any of the condition listed in Subsection 4.1.4. Let $\phi\in\overrightarrow{\mathscr{R}}(\mathbf{Z}_2(-I))$ as in (4.12). From equivalence relation (4.3) we have

$$\phi^t(B\circ\phi(x,y))\phi=\begin{pmatrix} C_0x+C_1y & B_0x+B_1y \\ B_0x+B_1y & A_0x+A_1y\end{pmatrix},$$

where

$$C_0=\vartheta(2b_0\alpha^2+2b_1\alpha\vartheta\alpha^2\varepsilon+\vartheta^2) \text{ and } C_1=2b_0\alpha\sigma\vartheta+2b_1\xi\alpha\vartheta+\xi\alpha^2\varepsilon+\xi\vartheta^2,$$

$$B_0=b_0\xi\alpha^2+b_0\alpha\sigma\vartheta+b_1\xi\alpha\vartheta+b_1\sigma\vartheta^2+\alpha\varepsilon\sigma\vartheta+\xi\vartheta^2 \text{ and}$$

$$B_1=b_0\xi\alpha\sigma+b_0\sigma^2\vartheta+b_1\xi^2\alpha+b_1\xi\sigma\vartheta+\xi\alpha\varepsilon\sigma+\xi^2\vartheta,$$

$$A_0=2b_0\xi\alpha\sigma+2b_1\xi\sigma\vartheta+\varepsilon\sigma^2\vartheta+\xi^2\vartheta \text{ and } A_1=\xi(2b_0\sigma^2+2b_1\xi\sigma+\varepsilon\sigma^2+\xi^2)$$

The conditions $C_0=A_0=0$ imply that $\vartheta=\sigma=0$. The expressions of C_1 and A_1 becomes $C_1=\xi\alpha^2\varepsilon$ and $A_1=\xi^3$. So, $A_1=1$ and $C_1=\varepsilon$ imply that

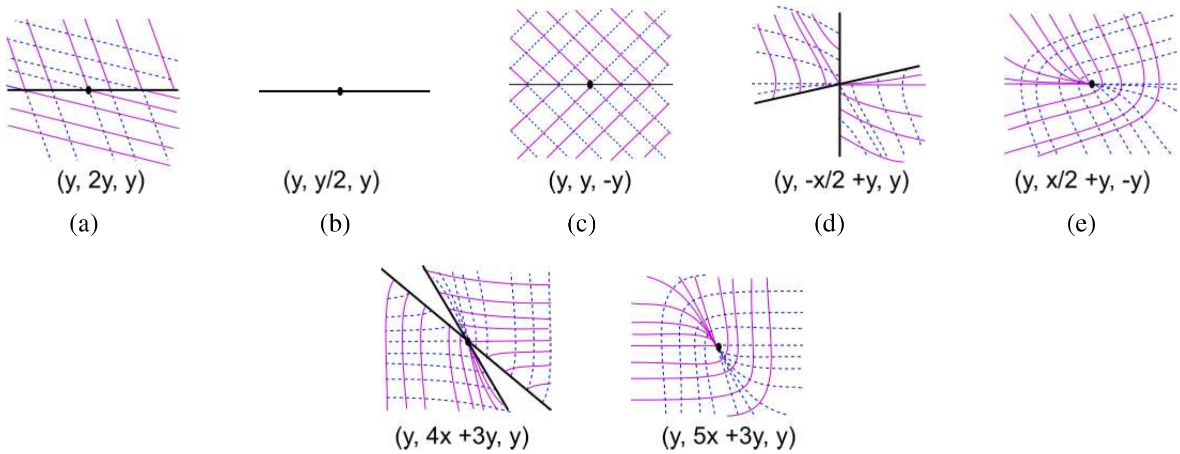
$$B_0=b_0 \text{ and } B_1=\alpha b_1,\ \alpha^2=1.$$

So the normal form is given by,

$$(y,nx+my,\varepsilon y),\ (m,n)\in\Upsilon,$$

where $\Upsilon=\mathbb{R}^2\setminus\{(n,m)\in\mathbb{R}^2 : m>0, n=0,\ 2n+\varepsilon=0,\ m^2-2n-\varepsilon=0\}$.

We finish with some examples that illustrate the configurations of the linear general form symmetric under $\mathbf{Z}_2(-I)$.



(f) (g)

Figure 25 – Configurations associated with $\mathbf{Z}_2(-I)$-equivariant linear BDEs.

4.2 Summarizing table

In this section we present the equivariant normal forms of symmetric quadratic differential 1-forms $\omega = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dy^2$ where a, b and c are linear functions. Table 4 shows each group $\Gamma[\ker\eta]$ and the kernel of the homomorphism λ.

$\Gamma[\ker\eta]$	$\ker\lambda$	General form
$\mathbf{D}_6(\kappa_x)[\mathbf{D}_3(\kappa_y)]$	$\mathbf{D}_3(\kappa_y)$	$(-x,-y,x)$.
$\mathbf{Z}_6[\mathbf{Z}_3]$	$\mathbf{Z}_3$	$(-x-y, x-y, x+y)$.
$\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$	$\mathbf{Z}_2(\kappa_y)$	$(\varepsilon x, 0, x),\ \varepsilon=\pm 1$. $(0,\varepsilon y, x),\ \varepsilon=\pm 1$. $(0,y,0)$. $(mx, y, 0),\ m\in\mathbb{R}^*$. $(\varepsilon x, \varepsilon m y, x),\ \varepsilon=\pm 1,\ m\in\mathbb{R}^*$.
$\mathbf{Z}_2(-I)[1]$	$\mathbf{Z}_2(-I)$	$(y,2y,y)$. $(y,y/2,y)$. $(y,y,\varepsilon y),\ \varepsilon=\pm 1$. $(y,-\varepsilon x/2+my,\varepsilon y),\ m>0, \varepsilon=\pm 1$. $(y,(m^2-\varepsilon)x/2+my,\varepsilon y),\ m>0,\varepsilon=\pm 1$. $(y, nx+my, \varepsilon y),\ \varepsilon=\pm 1, (m,n)\in\Upsilon$.
$\Upsilon = \mathbb{R}^2\setminus\{(n,m)\in\mathbb{R}^2 : m>0, n=0,\ 2n+\varepsilon=0,\ m^2-2n-\varepsilon=0\}$.		

Table 4 – Linear equivariant normal forms.

Remark 4.2.1. *If we consider the quadratic 1-forms*

$$\omega_1 = (a_1 y, b_0 x, c_1 y) \ \ \textit{and} \ \ \omega_1 = (c_1 x, b_0 y, a_1 x),$$

where $a_1, b_0, c_1 \in \mathbb{R}$, we note that they are $\mathbf{Z}_2(-I)[1]$-equivalent. In fact, take ϕ in (4.12) with $\alpha=\xi=0$ and $\vartheta=\sigma=1$. It is interesting to note that the first one is $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_y)]$-equivariant and the second is $\mathbf{Z}_2\times\mathbf{Z}_2[\mathbf{Z}_2(\kappa_x)]$-equivariant.

Remarks 4.1.4 and 4.2.1 give examples of the same ideia of the existence of a relationship between the linear quadratic 1-forms symmetric under the same group Γ but with distinct homomorphisms η, the relationship being an equivalence which is equivariant under a common subgroup.

CONCLUSION

The development of this thesis proved to be fruitful and pave ways for new research projects. With the definition of equivariance in the space of the quadratic 1-forms and having established the algebraic relationship that controls the interchanges of foliations, some questions arise.

Given a pair of foliations on the plane associated with a binary differential equation, how can we relate their symmetries with the symmetries of the BDE? This pair of foliations is associated with two linear 1-forms, whose product is a quadratic 1-form and whose symmetries we have studied in Chapter 1 of this thesis. The classification of pairs of linear 1-forms have been treated in [26] and [27] under the singularity theory point of view. We intend to relate our study of symmetries of pairs of 1-forms done in Section 1.3 to this symmetries of the quadratic 1-form. As discussed at the end of this chapter this is a sensitive issue and we intend to answer this question.

A natural continuation of the study proposed in Chapter 4 is to classify the nonlinear Γ-equivariant mappings $B : \mathbb{R}^2 \to Sym_2$. For the case without symmetries we cite the paper [18], where the author gives a classification of simple space curve singularities which are not complete intersections. Bruce in [9] obtain a list of all simple germs of mappings of this type, under a natural notion of equivalence, and investigate their geometry.

Another line of investigation is to relate symmetries in differential forms to pairs of foliations of a special classes of surfaces. On this subject some questions were made to us by Ronaldo Garcia and Kentaro Saji: for a given equivariant BDE, we ask whether this can be realized as an equation of lines of curvatures or of asymptotic lines of a surface immersed on $\mathbb{R}^n$ for some n. It was also asked to us by R. Garcia, how the prior knowledge of the symmetries of the problem can contribute.

Another line is to consider the occurrence of symmetries in n-webs. For $n = 3$, for example, solutions of an cubic implicit differential equation defines three foliations on the plane. We think that, as in the case of a pair of foliations, we can establish a relation to give the interchange of foliations by an element of the symmetry group Γ. This will correspond to the homomorphism $\lambda : \Gamma \to \mathbf{Z}_2$ of this thesis. For references on webs we have been reading [28] and also [1, 2, 3].

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