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On k-folding map-germs and hidden symmetries of curves in the euclidean plane

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Germes de aplicações k-dobras e simetrias ocultas de curvas no plano euclidiano

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestra em Ciências – Matemática. *VERSÃO REVISADA*

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To my mother Apolonia, my sister Alana and my brother Josué.

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"Return to your rest, my soul, for the Lord has been good to you."
(Psalm 116;7)

ABSTRACT

FALQUETO, A. D. **On k-folding map-germs and hidden symmetries of curves in the euclidean plane**. 2023. 70 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

The aim of this work is to study the local singularities of germs of k-folds for $k \geq 3$ and derive from them hidden symmetries of curves in the Euclidean plane. We used the Complete Transversal Method in order to classify the $\mathscr{A}$-simple singularities of map-germs $\mathbb{C},0 \to \mathbb{C}^2,0$. We then prove that all the simple singularities of such germs can be realised by k-folding maps and that any k-folding map-germ can have an $\mathscr{A}$-simple singularity. This does not occur in the case of surfaces, as proved in (PEÑAFORT SANCHIS; TARI, 2023). Finally, we proved that the singularities of k-folding map-germs reveal information about the local symmetry of the curve.

Keywords: Germs, K-folding maps, Singularities, Symmetries.

RESUMO

FALQUETO, A. D. **Germes de aplicações k-dobras e simetrias ocultas de curvas no plano euclidiano**. 2023. 70 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

O objetivo deste trabalho é estudar as singularidades locais dos germes de aplicações k-dobras para $k \geq 3$ e derivar delas simetrias ocultas de curvas no plano euclidiano.

Primeiramente, utilizamos o Método da Transversal Completa para classificar as singularidades $\mathscr{A}$-simples dos germes $\mathbb{C},0 \to \mathbb{C}^2,0$. Em seguida, provamos que todas estas singularidades podem ser realizadas pelas aplicações k-dobras e que qualquer aplicação k-dobra pode ter uma singularidade $\mathscr{A}$-simples, o que não ocorre no caso de superfícies, conforme provado em (PEÑAFORT SANCHIS; TARI, 2023). Por fim, provamos que as singularidades dos germes de k-dobras revelam informações a respeito da simetria da curva.

Palavras-chave: Germes, K-dobras, Singularidades, Simetrias.

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CHAPTER 1

INTRODUCTION

The 2-folding map-germs, also called folding map-germs, have been studied by Bruce and Wilkinson in order to describe some geometric features of surfaces in $\mathbb{R}^3$. In (PEÑAFORT SANCHIS; TARI, 2023), the authors generalised that work by considering k-folding maps on surfaces in $\mathbb{R}^3$ with $k \geq 3$. They recovered many known geometrical features of surfaces and also discovered new ones.

In (TARI, 1990), it was shown how a family of 2-folding map-germs, also called folding map-germs, on a plane curve can be used to study the local singularities of the duals of symmetry sets of plane curves and families of such curves, that is, the infinitesimal axes of symmetry. Given a plane curve (or a family of plane curves) and a family of folding-maps on it, since the dual of the symmetry set can be identified as a part of the bifurcation set of the family of folding maps, it was shown in (TARI, 1990) that:

- the dual of an ordinary inflexion on the symmetry set is a cusp;
- for a generic 1-parameter family of curves, the dual of a higher inflexion on the symmetry set undergoes swallowtail transitions;
- for a generic 1-parameter families of curves the dual of the symmetry set when the bitangent circle is biosculating undergoes lips or beaks transitions;
- at an ordinary vertex on the curve, the dual of the symmetry set is a curve with an ending point;
- at a higher vertex on the curve, the dual of the symmetry set is some sections of the product of (t^2, t^5) by a line.

In this work, we consider k-folding map-germs on plane curves, for $k \geq 3$. We show that some singularities of k-folding maps reveal information about the hidden symmetries of plane curves. Since those symmetries cannot be viewed in the real case, which is why we call them "hidden", we consider the curve in the complex plane or, in the case where the curve is in $\mathbb{R}^2$, we complexify the curve if it is analytic or a certain jet of its parametrization.

In Chapter 2, we give a brief introduction to the Singularity Theory, showing concepts used in this work.

In Chapter 3, we give the proof and correct some misprints of the classification of $\mathscr{A}$-simple singularities of germs $\mathbb{C},0 \to \mathbb{C}^2,0$ given by C.G. Gibson and C. A. Hobbs in (GIBSON; HOBBS, 1983), using the Complete Transversal Method.

Chapter 4 is dedicated to the study of the k-folding map-germs on plane curves and their local singularities. We proved that all $\mathscr{A}$-simple singularities can be realised by k-folding map-germs and that all k-folding map-germs can have $\mathscr{A}$-simple singularities. We use the result to obtain information about the hidden local symmetry of a plane curve. We observe that the results in Chapter 4 are original.

Finally, we give in the Appendix some results in Singularity Theory applied to the geometry of curves.

CHAPTER

2

BASIC RESULTS IN SINGULARITY THEORY

We give in this chapter some results in Singularity Theory which we used in Chapter 3 for classifying the $\mathscr{A}$-simple singularities of germs $\mathbb{C},0 \to \mathbb{C}^2,0$.

In what follows, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$. All the maps are considered smooth (if $\mathbb{K}=\mathbb{R}$) or holomorphic (if $\mathbb{K}=\mathbb{C}$). Our main reference is the most recent book to date on Singularity Theory (MOND; NUÑO-BALLESTEROS, 2020).

2.1 Action of Lie groups

Definition 2.1.1. (GIBSON, 1979) A Lie group G is a group which is a smooth manifold and the group operations of multiplication $G\times G \to G$, given by $(g_1,g_2)\mapsto g_1g_2$ and inversion $G \to G^{-1}$, given by $g \mapsto g^{-1}$ are smooth maps.

Definition 2.1.2. (GIBSON, 1979) Let G be a group and $M \neq \emptyset$ be a set. An action of G on M is a map $\phi : G\times M \to M$, given by $\phi(g,x) = g\cdot x$, such that

$$\text{(a) } 1\cdot x = x,\ \text{where 1 is the identity of } G;$$
$$\text{(b) } (gh)\cdot x = g\cdot(h\cdot x), \forall x \in M, \forall g,h \in G.$$

Example 2.1.3. The general linear group $Gl(n)$ is a Lie group (see for example (GIBSON, 1979)). Indeed, $Gl(n)$ is a group with multiplication of matrices. Moreover, since it can be identified as the group of non-singular real $n\times n$ matrices, it is an open subset of the vector space $M(n)$ of all real $n\times n$ matrices. Thus, $Gl(n)$ is a smooth manifold. The group operation of multiplication is smooth since the multiplication of matrices in $M(n)$ is a polynomial mapping. Matrix inversion in $Gl(n)$ is a rational mapping with nonzero denominator, hence, smooth.

Given an action of a group G on M, we can define an equivalence relation in M. For $x,y \in M$, we say that x is equivalent to y if there exists $g \in G$ such that $y = g\cdot x$. The equivalence classes of this relation are called orbits. Given $x \in M$, the orbit of x is the set

$$G \cdot x = \{g \cdot x, g \in G\}.$$

Definition 2.1.4. (GIBSON, 1979) Let G be a Lie group acting on a smooth manifold M. We say that $\phi : G \times M \to M$ is a smooth action if ϕ is smooth.

Theorem 2.1.5. (GIBSON, 1979) Let G be a Lie group acting smoothly on a manifold M. Then the orbits are immersed submanifolds in M.

In the study of actions of Lie groups on smooth manifolds it is useful to know the tangent space to an orbit at a point. The next proposition shows how this tangent space can be described.

Proposition 2.1.6. (GIBSON, 1979) Let $\phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. Suppose that all the orbits are smooth submanifolds of M. Then for any point $x \in M$ the natural mapping $\phi_x : G \to G \cdot x$ given by $g \to g \cdot x$ is a submersion.

Proof. First, we show that rank$(D\phi_x(h))$ = rank$(D\phi_x(1))$, for all $h \in G$. Let $h \in G$ and consider the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\phi_x} & G \cdot x \\ \downarrow{\scriptstyle \theta_h} & & \downarrow{\scriptstyle \bar{\theta}_h} \\ G & \xrightarrow[\phi_x]{} & G \cdot x \end{array}$$

where $\theta_h(g) = h \cdot g$ and $\bar{\theta}_h(y) = h \cdot y$. Observe that θ_h and $\bar{\theta}_h$ are diffeomorphisms since $(\theta_h)^{-1} = \theta_{h^{-1}}$ and $(\bar{\theta}_h)^{-1} = \bar{\theta}_{h^{-1}}$. Also, the diagram above commutes. Indeed,

$$(\bar{\theta}_h \circ \phi_x)(g) = h \cdot (g \cdot x) = (hg) \cdot x = (\phi_x \circ \theta_h)(g).$$

This comuting diagram gives rise to the comuting diagram of differentials

$$\begin{array}{ccc} T_1 G & \xrightarrow{D\phi_x(1)} & T_x G \cdot x \\ \downarrow{\scriptstyle D\theta_h(1)} & & \downarrow{\scriptstyle D\bar{\theta}_h(x)} \\ T_h G & \xrightarrow[D\phi_x(h)]{} & T_{h \cdot x} G \cdot x \end{array}$$

The vertical arrows in the diagram of differentials are linear isomorphisms since θ_h and $\bar{\theta}_h$ are diffeomorphisms. Therefore,

$$rank(D\phi_x(1)) = rank(D\phi_x(h)), \text{ for all } h \in G.$$

According to Sard's Theorem (GOLUBITSKY; GUILLEMIN, 1973), the set of regular values of the map ϕ_x is dense. So, there exists x_0 such that $\phi_x(x_0)$ is a submersion. Since rank$(D\phi_x)$ is constant and equal to rank$(D\phi_{x_0})$, the map $D\phi_x$ is surjective. Hence, ϕ_x is a submersion for all $x \in M$. □

Corollary 2.1.7. If G is a Lie group acting on a smooth manifold M and suppose that the orbits are smooth submanifolds. Then $\dim G\cdot x \leq \dim G$ for all $x \in M$.

Lemma 2.1.8. (Mather's Lemma, (MOND; NUÑO-BALLESTEROS, 2020)) Suppose the Lie group G acts smoothly on the manifold M, and that $N \subset M$ is a smooth connected submanifold. Then a necessary and sufficient condition for N to be contained in a single orbit is that

1. for all $p \in N, T_pN \subset T_pG\cdot p$;
2. the dimension of $T_pG\cdot p$ is the same, for all $p \in N$.

Proof. See (MOND; NUÑO-BALLESTEROS, 2020). □

2.2 Map-germs $\mathbb{K}^n,0 \to \mathbb{K}^p,0$

Definition 2.2.1. (MOND; NUÑO-BALLESTEROS, 2020) Consider $f : U \to \mathbb{K}^p$ and $g : V \to \mathbb{K}^p$ maps defined on neighbourhoods U and V of a point $q \in \mathbb{K}^n$. We say that f and g are equivalent if and only if there exists a neighbourhood W of q, with $W \subset U \cap V$, such that $f \restriction_W = g \restriction_W$. An equivalent class of such maps is called a germ of mapping or map-germ at q.

Notation: $f : \mathbb{K}^n,x \to \mathbb{K}^p,y$, where $y = f(x)$.

Remark 2.2.2. (MOND; NUÑO-BALLESTEROS, 2020) The set of all germs $\mathbb{K}^n,0 \to \mathbb{K}^p$ is denoted by $\mathscr{O}(n,p)$. When $p = 1$, this set is denoted by $\mathscr{O}_n$ and it is a local ring with maximal ideal $\mathscr{M}_n$, where $\mathscr{M}_n = \{f \in \mathscr{O}_n; f(0) = 0\}$.

There are some special groups that act on $\mathscr{M}_n\mathscr{O}(n,p)$. They are called the Mather's groups.

2.3 Mather's Groups and Tangent Spaces

Definition 2.3.1. (MOND; NUÑO-BALLESTEROS, 2020) The group $\mathscr{R}$ is the group of the germs of diffeomorphisms $\mathbb{K}^n,0 \to \mathbb{K}^n,0$. The group $\mathscr{L}$ is the group of the germs of diffeomorphisms $\mathbb{K}^p,0 \to \mathbb{K}^p,0$ and $\mathscr{A}$ is the direct product $\mathscr{R}\times\mathscr{L}$. The actions of the above groups on $\mathscr{M}_n\mathscr{O}(n,p)$ are given by

$$h\cdot f = f\circ h^{-1}, h \in \mathscr{R}$$
$$k\cdot f = k\circ f, k \in \mathscr{L}$$
$$(h,k)\cdot f = k\circ f\circ h^{-1}, (h,k) \in \mathscr{A},$$

where $f \in \mathscr{M}_n\mathscr{O}(n,p)$. The $\mathscr{R}$ (respectively, $\mathscr{L}$) is also called the groups of germs of changes of coordinates in the source (respectively, target).

Definition 2.3.2. (MOND; NUÑO-BALLESTEROS, 2020) The group $\mathscr{C}$ is the group of germs of diffeomorphisms $H : \mathbb{K}^n \times \mathbb{K}^p, 0 \to \mathbb{K}^n \times \mathbb{K}^p, 0$ such that $H(x,y) = (x, H'(x,y))$ with $H'(x,0) = 0$ for $x \in \mathbb{K}^n$ close to the origin. The action of $\mathscr{C}$ on $\mathscr{M}_n\mathscr{O}(n,p)$ is given by

$$H \cdot f(x) = H(x, f(x)), H \in \mathscr{C}, f \in \mathscr{M}_n\mathscr{O}(n,p).$$

The group $\mathscr{K}$, also called the contact group, is the group of the germs of diffeomorphisms $H : \mathbb{K}^n \times \mathbb{K}^p, 0 \to \mathbb{K}^n \times \mathbb{K}^p, 0$ given by $H(x,y) = (h(x), H'(x,y))$, where $h \in \mathscr{R}$, $H'(x,0) = 0$ for $x \in \mathbb{K}^n$ close to the origin. The action of $\mathscr{K}$ on $\mathscr{M}_n\mathscr{O}(n,p)$ is given by

$$H \cdot f(x) = H(h^{-1}(x), f(h^{-1}(x))), H \in \mathscr{K}, f \in \mathscr{M}_n\mathscr{O}(n,p).$$

Definition 2.3.3. The groups $\mathscr{R}, \mathscr{L}, \mathscr{A}, \mathscr{C}$ and $\mathscr{K}$ are called Mather's groups.

If $\mathscr{G}$ is a Lie group acting smoothly on a manifold M, the orbits are immersed submanifolds and the tangent spaces are the image of the differential of the orbit map. However, the Mather's groups are not finite dimensional and the orbits are not manifolds. In that case, the tangent space to an orbit is defined by considering derivatives of paths (MOND; NUÑO-BALLESTEROS, 2020).

Definition 2.3.4. (MOND; NUÑO-BALLESTEROS, 2020) Let $f \in \mathscr{O}(n,p)$, $T\mathbb{K}^n$ and $T\mathbb{K}^p$ the tangent bundle of $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively. Consider $\pi_1 : T\mathbb{K}^n \to \mathbb{K}^n$ and $\pi_2 : T\mathbb{K}^p \to \mathbb{K}^p$ the germs of natural projections. A vector field ζ along f is a germ such that $\pi_2 \circ \zeta = f$, where $\xi : (\mathbb{K}^n, 0) \to T\mathbb{K}^p$.

The set of all vector fields along f is denoted by θf.

Remark 2.3.5. The set of the germs of vector fields in $\mathbb{K}^n$ at the identity is denoted by $\theta_n = \theta_{1\mathbb{K}^n}$. Analogously, $\theta_p = \theta_{1\mathbb{K}^p}$ defines the set of vector fields in $\mathbb{K}^p$ at the identity.

Consider the maps

$$tf : \theta_n \to \theta_f$$
$$\phi \mapsto df \circ \phi$$

and

$$wf : \theta_p \to \theta_f$$
$$\psi \mapsto \psi \circ f$$

The map wf is induced by

$$f^* : \mathscr{O}_p \to \mathscr{O}_n$$
$$\alpha \mapsto \alpha \circ f.$$

One can identify θ_f as $\mathscr{O}(n,p)$. The following definition gives the tangent spaces to an orbits of an action of a Mather group on $\mathscr{M}_n\mathscr{O}(n,p)$ in that case.

Definition 2.3.6. (IZUMIYA *et al.*, 2015) Let $f \in \mathscr{M}_n\mathscr{O}(n,p)$ and $\mathscr{G}$ be one of the Mather's Group. The tangent space to the orbit $\mathscr{G} \cdot f$ is defined as:

$$\begin{gathered} T\mathscr{L} \cdot f = f^*(\mathscr{M}_p) \cdot \{e_1, ..., e_p\}, \\ T\mathscr{R} \cdot f = \mathscr{M}_n\{\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\}, \\ T\mathscr{A} \cdot f = T\mathscr{R} \cdot f + T\mathscr{L} \cdot f, \\ T\mathscr{C} \cdot f = f^*(\mathscr{M}_p) \cdot \mathscr{O}_n \cdot \{e_1, ..., e_p\}, \\ T\mathscr{K} \cdot f = T\mathscr{R} \cdot f + T\mathscr{C} \cdot f, \end{gathered}$$

where $e_1, ..., e_p$ are the elements of the standard basis of $\mathbb{K}^p$(considered as elements of $\mathscr{O}(n,p)$.

By allowing the vector fields involved in the definition of the tangent space not to fix the origin, the extended tangent spaces are given as follows

$$\begin{gathered} T\mathscr{L}_e \cdot f = f^*(\mathscr{O}_p) \cdot \{e_1, ..., e_p\}, \\ T\mathscr{R}_e \cdot f = \mathscr{O}_n\{\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\}, \\ T\mathscr{A}_e \cdot f = T\mathscr{R}_e \cdot f + T\mathscr{L}_e \cdot f, \\ T\mathscr{C}_e \cdot f = f^*(\mathscr{M}_p) \cdot \mathscr{O}_n \cdot \{e_1, ..., e_p\}, \\ T\mathscr{K}_e \cdot f = T\mathscr{R}_e \cdot f + T\mathscr{C}_e \cdot f. \end{gathered}$$

Definition 2.3.7. (IZUMIYA *et al.*, 2015) The $\mathscr{G}$-codimension of f is defined by

$$\mathscr{G} - cod(f) = \dim_{\mathbb{K}} \frac{\mathscr{M}_n \cdot \mathscr{O}(n,p)}{T\mathscr{G} \cdot f}.$$

and its extended codimension is defined by

$$\mathscr{G}_e - cod(f) = \dim_{\mathbb{K}} \frac{\mathscr{O}(n,p)}{T_e\mathscr{G} \cdot f}.$$

2.4 Jet space

Definition 2.4.1. (MOND; NUÑO-BALLESTEROS, 2020) Denote by $J^k(n,p)$ the space of p-tuples of polynomials of degree less than or equal to k in n variables with no constant term. A map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ determines a germ of a map $j^k f : (\mathbb{K}^n, 0) \to J^k(n,p)$, the k-jet extension of f, defined by

$$j^k f(x) = \text{ degree } k \text{ Taylor polynomial of } f \text{ at } x, \text{ without its constant term.}$$

Definition 2.4.2. (MOND; NUÑO-BALLESTEROS, 2020) The subspace $H^{k+1}(n,p)$ of $J^{k+1}(n,p)$ is the subspace of the p-tuples of homogeneous polynomials of degree $k+1$ in n variables.

Definition 2.4.3. (IZUMIYA *et al.*, 2015) Let $\mathscr{G}$ be a Mather group. We denote by $\mathscr{G}_k$ the subgroup of $\mathscr{G}$ of the elements of $\mathscr{G}$ with the identity as the k-jet. The $\mathscr{G}_k$ is a normal subgroup of $\mathscr{G}$.

Remark 2.4.4. An important subgroup of $\mathscr{A}$ is the group $\mathscr{A}_1$, which is the group of the elements of $\mathscr{A}$ with the identity as the 1-jet.

Definition 2.4.5. (IZUMIYA *et al.*, 2015) The set of the k-jets of elements of $\mathscr{G}$ is denoted by $\mathscr{G}^{(k)} = \mathscr{G}/\mathscr{G}_k$, which is a Lie group.

The action of a Mather's group $\mathscr{G}$ on $\mathscr{O}(n,p)$ induces an action of $\mathscr{G}^{(k)}$ in $J^k(n,p)$ defined as follows.

Proposition 2.4.6. (IZUMIYA *et al.*, 2015) For $j^k f \in J^k(n,p)$ and $j^k h \in \mathscr{G}^{(k)}$, the action of $\mathscr{G}^{(k)}$ on $J^k(n,p)$ is given by

$$j^k h \cdot j^k f = j^k(h \cdot f)$$

Therefore, in order to understand the action of $\mathscr{G}$ on $\mathscr{M}_n\mathscr{O}(n,p)$ one can study the action of $\mathscr{G}^{(k)}$ on $J^k(n,p)$.

In Singularity Theory, it is useful to classify the germs up to equivalence under the action of a Mather's group. By the finite determinacy, the problem of classification is reduced to a space of finite dimension, that is, the jet space.

2.5 Finite Determinacy

Let $\mathscr{G}$ be a Mather group.

Definition 2.5.1. (MOND; NUÑO-BALLESTEROS, 2020) We say that $f \in \mathscr{M}_n\mathscr{O}(n,p)$ is $k-\mathscr{G}$-determined if any other germ $g \in \mathscr{M}_n\mathscr{O}(n,p)$ such that $j^k f = j^k g$ is $\mathscr{G}$-equivalent to f. Also, f is finitely $\mathscr{G}$-determined if it is k-determined for some $k < \infty$. The $\mathscr{G}$- determinacy degree of f is the lowest k such that f is k-determined.

Theorem 2.5.2. (MOND; NUÑO-BALLESTEROS, 2020) The following statements are equivalent:

(a) f is finitely $\mathscr{G}$-determined;

(b) for some k, $\mathscr{M}_n^k\mathscr{O}(n,p) \subset T\mathscr{G} \cdot f$;

(c) for some k, $\mathscr{M}_n^k\mathscr{O}(n,p) \subset T\mathscr{G}_e \cdot f$;

(d) $\mathscr{G} - cod(f) < \infty$;

(e) $\mathscr{G}_e - cod(f) < \infty$;

In the context of finite determinacy, it is also interesting to find the degree of determinacy of a given map-germ. This problem is solved in (BRUCE; PLESSIS; WALL, 1987). In our case, we are going to use the group $\mathscr{G} = \mathscr{A}$. The following corollary is useful for estimating the degree of $\mathscr{A}$-determinacy of a map-germ.

Corollary 2.5.3. (BRUCE; PLESSIS; WALL, 1987) If $f \in \mathscr{M}_n\mathscr{O}(n,p)$ satisfies

$$\mathscr{M}_n^l\mathscr{O}(n,p) \subset T\mathscr{K} \cdot f;$$
$$\mathscr{M}_n^{r+1}\mathscr{O}(n,p) \subset T\mathscr{A}_1 \cdot f + \mathscr{M}_n^{l+r+1}\mathscr{O}(n,p),$$

then f is $r-\mathscr{A}_1$-determined.

2.6 Unfoldings

Given a $\mathscr{G}$-finitely determined f, we can consider its deformations and study the singularities that appear in such deformations. This is the idea of unfoldings.

Definition 2.6.1. (MOND; NUÑO-BALLESTEROS, 2020) Consider a germ $f_0 \in \mathscr{M}_n\mathscr{O}(n,p)$. An s-parameter unfolding of f_0 is a germ $F : \mathbb{R}^n \times \mathbb{R}^s, 0 \to \mathbb{R}^p \times \mathbb{R}^s, 0$ of the form

$$F(x,u) = (f(x,u),u),$$

where $f(.,0) = f_0$. The family f is called a deformation of f_0.

Definition 2.6.2. (MOND; NUÑO-BALLESTEROS, 2020) Let $F,G : \mathbb{R}^n \times \mathbb{R}^s, 0 \to \mathbb{R}^p \times \mathbb{R}^s, 0$ two s-parameter unfoldings of f_0. We say that F and G are isomorphic if there exist unfoldings of the germs of the identity

$$\phi : \mathbb{R}^n \times \mathbb{R}^s, 0 \to \mathbb{R}^n \times \mathbb{R}^s, 0$$
$$\psi : \mathbb{R}^p \times \mathbb{R}^s, 0 \to \mathbb{R}^p \times \mathbb{R}^s, 0$$

such that F is $\mathscr{A}$-equivalent to G via ϕ and ψ.

Remark 2.6.3. The definition of isomorphic unfoldings can be applied to any Mather group.

Given an unfolding, we can obtain another one by a change of parameter. That is the case of induced unfoldings.

Definition 2.6.4. (MOND; NUÑO-BALLESTEROS, 2020) Consider a germ $h : \mathbb{R}^t, 0 \to \mathbb{R}^s, 0$. The pull-back of F by h, denoted by h^*F, is the t-parameter unfolding given by

$$h^*F(x,v) = (f(x,h(v)),v).$$

Definition 2.6.5. (GIBSON, 1979) Let $F : \mathbb{R}^n \times \mathbb{R}^s, 0 \to \mathbb{R}^p \times \mathbb{R}^s$ and $G : \mathbb{R}^n \times \mathbb{R}^t, 0 \to \mathbb{R}^p \times \mathbb{R}^t$ be two unfoldings of f_0. We say that G is induced by F if there exists a germ $h : \mathbb{R}^t, 0 \to \mathbb{R}^s, 0$ such that G is isomorphic to h^*F. If $h : \mathbb{R}^s, 0 \to \mathbb{R}^s, 0$ is a germ of a diffeomorphism and F is isomorphic to h^*G, then F and G are called equivalent unfoldings.

Definition 2.6.6. (MOND; NUÑO-BALLESTEROS, 2020) Let F be an unfolding of the germ f_0. We say that F is a versal unfolding if any other unfolding of f_0 is induced by F.

2.7 The tangent space to the orbit of the action of $\mathscr{G}^{(k)}$ on $J^k(n,p)$

With the theory of unfoldings, one can describe the tangent spaces to the orbit of the action of $\mathscr{G}^{(k)}$ on $J^k(n,p)$, for a Mather Group $\mathscr{G}$. First, we need the following definition.

Definition 2.7.1. (MOND; NUÑO-BALLESTEROS, 2020) Let

$$F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$$
$$F(x,t) = (f_t(x), t)$$

be any origin-preserving unfolding of a germ f and $\eta = j^k f$. Define a curve $\gamma : (\mathbb{K}, 0) \to J^k(n,p)$ where $\gamma(t) = j^k f_t$. Notice that

$$\gamma(0) = \eta;$$
$$\gamma'(0) = \tfrac{d}{dt}\{j^k f_t\}|_{t=0} = j^k(\tfrac{d}{dt} f_t|_{t=0}).$$

Therefore, k-jets of 1-parameter unfoldings of f that preserve the origin are tangent vectors to $J^k(n,p)$. Conversely, any tangent vector to that space can be obtained in this way. Indeed, for $l \geq k$, consider

$$\pi_l^k : J^l(n,p) \to J^k(n,p)$$
$$\pi_l^k(j^l f) = j^k f$$

Since π_l^k is an epimorphism and a linear map, it is a submersion when we consider the manifold structures.

Lemma 2.7.2. (MOND; NUÑO-BALLESTEROS, 2020) For each $\eta = j^k f \in J^k(n,p)$, the orbit $\mathscr{G}^{(k)} \cdot \eta$ is a submanifold of $J^k(n,p)$ whose tangent space at η is

$$T_\eta(\mathscr{G}^{(k)} \cdot \eta) = j^k(T\mathscr{G} \cdot f)$$

Proof. Since this is an action of a Lie group on a manifold, the tangent space of the orbit is the image of the differential of the orbit map, that is,

$$T_\eta(\mathscr{G}^{(k)} \cdot \eta) = d_e \alpha_\eta(T_e \mathscr{G}^{(k)}),$$

where α_η is the orbit map, given by

$$\alpha_\eta : \mathscr{G}^{(k)} \to J^k(n,p)$$
$$\alpha_\eta(\psi) = \psi\eta.$$

Therefore, according to Definition 2.7.1, the vectors in $T_\eta(\mathscr{G}^{(k)} \cdot \eta)$ can be described by

$$d_e \alpha_\eta(j^k(\tfrac{d}{dt}\psi_t\}|_{t=0}),$$

where ψ_t is any origin-preserving unfolding of identity in $\mathscr{G}$.

Using the linearity of the differential map and the definition of the orbit map, we obtain

$$d_e\alpha_\eta(j^k(\tfrac{d}{dt}\psi_t\}|_{t=0}) = d_e\alpha_\eta(\tfrac{d}{dt}\{j^k\psi_t\}|_{t=0}) = \tfrac{d}{dt}\{\alpha_\eta(j^k\psi_t)\}|_{t=0} = \tfrac{d}{dt}\{j^k(\psi_t f))\}|_{t=0} = j^k(\tfrac{d}{dt}(\psi_t f)|_{t=0}.$$

The claim is that the tangent space $T\mathscr{G}\cdot f$ is the set of all vector fields obtained as $\frac{d}{dt}(\psi_t f)|_{t=0}$, where ψ_t is an origin-preserving unfolding of the identity in $\mathscr{G}$. We give the proof for $\mathscr{G}=\mathscr{A}$; the proof of the other groups is analogous.

Let ϕ_t and ψ_t be origin-preserving of the identity in $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively. According to the definitions of the maps tf and wf, we have

$$\tfrac{d}{dt}\{\psi_t\circ f\circ\phi_t^{-1}\}|_{t=0} = tf(\tfrac{d}{dt}\phi_t^{-1}|_{t=0}) + wf(\tfrac{d}{dt}\psi_t|_{t=0}) \in T\mathscr{A}\cdot f.$$

Conversely, let $v\in T\mathscr{A}\cdot f$. Since $T\mathscr{A}\cdot f = T\mathscr{R}\cdot f + T\mathscr{L}\cdot f$, there exist $\xi\in\mathscr{M}_n\theta_n$ and $\eta\in\mathscr{M}_p\theta_p$ such that $v = d_f\circ\xi + w\circ\eta$. Let ϕ_t and ψ_t be origin-preserving unfolding of identity in $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively, such that $\xi = \frac{d\phi_t^1}{dt}|_{t=0}$ and $\eta = \frac{d\psi_t}{dt}|_{t=0}$. Then,

$$v = tf(\tfrac{d\phi_t^{-1}}{dt}|_{t=0}) + wf(\tfrac{d\psi_t}{dt}|_{t=0}) = \tfrac{d}{dt}\{\psi_t\circ f\circ\phi_t^{-1}\}|_{t=0}.$$

□

2.8 Simple germs

Definition 2.8.1. (ARNOLD; GUSEIN-ZADE; VARCHENKO, 1982) The modality m of a point $x\in X$ under the action of a Lie group G on a manifold X is the least number such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be simple, if its modality is 0, that is, if its neighbourhood intersects only a finite number of orbits.

Remark 2.8.2. The modality of a finitely determined map-germ is the modality of a sufficient jet in the jet-space under the action of the jet-group.

Example 2.8.3. It is shown in (ARNOLD; GUSEIN-ZADE; VARCHENKO, 1982) that the simple singularities of germs of functions are A_k (for $k\geq 1$), D_k (for $k\geq 3$), E_6, E_7, E_8.

2.9 Complete Transversals

The method of complete transversals is a powerful tool for classification of singularities of map-germs. The classification is carried out inductively on the jet level. First, we fix a k-jet j^kf and describe all $\mathscr{G}$-orbits of $k+1$-jets which have the same k- jet as j^kf using the Complete Transversal Theorem. If the orbits are finitely determined, the process stops. Otherwise, we continue the method describing all $\mathscr{G}$-orbits of $(k+2)$-jets with its $(k+1)$-jet equal to $j^{k+1}f$ and so on.

We state the Complete Transversal's theorem for the group $\mathscr{A}$.

Theorem 2.9.1. (BRUCE; KIRK; PLESSIS, 1997) Let $f \in J^k(n,p)$ and $T \subset H^{k+1}(n,p)$ such that

$$\mathscr{M}_n^{k+1}\mathscr{O}(n,p) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_n^{k+2}\mathscr{O}(n,p).$$

Then, every $g \in J^{k+1}(n,p)$ with $j^k g(0) = f$ is $\mathscr{A}_1^{k+1}$-equivalent to $f+\beta$, with $\beta \in T$. The subspace T is called k-complete transversal.

CHAPTER

3

CLASSIFICATION OF $\mathscr{A}$-SIMPLE SINGULARITIES OF GERMS $\mathbb{C},0 \to \mathbb{C}^2,0$

A germ of a curve parametrized by $f:\mathbb{C},0 \to \mathbb{C}^2,0$ can be described as a fiber $g^{-1}(0)$ of a germ of function $g:\mathbb{C}^2,0 \to \mathbb{C},0$. Bruce and Gaffney related the $\mathscr{K}$-singularities of g to the $\mathscr{A}$-singularities of f and obtained the following result.

Theorem 3.0.1. (BRUCE; GAFFNEY, 1982) The following are representatives of the $\mathscr{A}$-simple singularities of map-germs $\mathbb{C},0 \to \mathbb{C}^2,0$:

Table 1 – $\mathscr{A}$-simple singularities of germs $\mathbb{C}$, 0 $\to \mathbb{C}^2$,0

Name	Singularity
A_{2k}	(t^2,t^{2k+1}), $k \geq 1$
E_{6k}	(t^3,t^{3k+1}), $k \geq 1$
E_{6k+2}	(t^3,t^{3k+2}), $k \geq 1$
E_{6k}	$(t^3,t^{3k+1}+t^{3p+2})$, $k \leq p < (2k-1)$
E_{6k+2}	$(t^3,t^{3k+1}+t^{3p+2})$, $p < k \leq (2p-1)$
W_{12}	(t^4,t^5)
W_{12}	(t^4,t^5+t^7)
$W^{\#}_{1.2k-5}$	(t^4,t^6+t^{2k+1}), $k \geq 3$
W_{18}	(t^4,t^7)
W_{18}	(t^4,t^7+t^9)
W_{18}	(t^4,t^7+t^{13})

In this chapter, we give the proof of the classification of $\mathscr{A}$-simple singularities of germs $(\mathbb{C},0) \to (\mathbb{C}^2,0)$ given in Theorem 3.0.1 and also in (GIBSON; HOBBS, 1983). The classification is carried out inductively on the jet-level using the Complete Transversal Method.

3.1 The classification of $\mathscr{A}$-simple germs $\mathbb{C},0 \to \mathbb{C}^2,0$

Consider $f \in J^1(1,2)$. Therefore, $f = (at, bt)$, for $a, b \in \mathbb{C}$.

If $a \neq 0$ or $b \neq 0$, f is $\mathscr{A}$-equivalent to $(t,0)$. Otherwise, f is $\mathscr{A}$-equivalent to $(0,0)$.

The 1-jet $(t,0)$

Proposition 3.1.1. The germ $g = (t,0)$ is 1-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1\mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(1,0)] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Also,

$$\mathscr{M}_1^2\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(1,0)] + g^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^3\mathscr{O}(1,2).$$

Therefore, according to Corollary 2.5.3, the germ $(t,0)$ is 1-$\mathscr{A}_1$-determined and, hence, 1-$\mathscr{A}$-determined. □

The 1-jet $(0,0)$

Any 2-jet with 1-jet $(0,0)$ is given by (at^2, bt^2), $a, b \in \mathbb{C}$.

If $a \neq 0$ or $b \neq 0$, (at^2, bt^2) is $\mathscr{A}^{(2)}$-equivalent to $(t^2, 0)$. Otherwise, it is $\mathscr{A}^{(2)}$-equivalent to $(0,0)$.

The 2-jet $(t^2,0)$

Any $2k$-jet with $(2k-1)$-jet $f = (t^2,0)$ is $\mathscr{A}^{(2k)}$-equivalent to f. Any $(2k+1)$-jet with $2k$-jet $(t^2,0)$ is $\mathscr{A}^{(2k+1)}$-equivalent to $(t^2, at^{2k+1}), a \in \mathbb{C}$. Indeed, the tangent space $T\mathscr{A}_1 \cdot f$ is given by

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(2t,0)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Thus, we have

$$\mathscr{M}_1^{2k+1}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{2k+2}\mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^{2k+1})\}$. It follows by the Complete Transversal Theorem that every $g \in J^{2k+1}(1,2)$ with $j^{2k}g(0) = (t^2,0)$ is $\mathscr{A}^{(2k+1)}$-equivalent to (t^2, at^{2k+1}), for $a \in \mathbb{C}$. If $a \neq 0$, by a change of coordinates in the target we can prove that the germ (t^2, at^{2k+1}) is $\mathscr{A}^{(2k+1)}$-equivalent to (t^2, t^{2k+1}). Otherwise, it is $\mathscr{A}^{(2k+1)}$-equivalent to $(t^2, 0)$.

Proposition 3.1.2. The germ $\phi = (t^2, t^{2k+1})$ is $(2k+1)$-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^2\mathscr{O}(1,2) \subset T\mathscr{K} \cdot \phi,$$

where $T\mathscr{K} \cdot \phi = \mathscr{M}_1[(2t, (2k+1)t^{2k})] + \phi^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Furthermore,

$$\mathscr{M}_1^{2k+2}\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(2t, (2k+1)t^{2k})] + \phi^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^{2k+4}\mathscr{O}(1,2),$$

Thus, by Corollary 2.5.3 the germ (t^2, t^{2k+1}) is $(2k+1)$-.$\mathscr{A}_1$-determined, hence $(2k+1)$- $\mathscr{A}$-determined. This germ is not p-$\mathscr{A}$-determined, for $p < (2k+1)$, since the p-jet of (t^2, t^{2k+1}) is the same as the p-jet of $(t^2, 0)$, and those germs are not $\mathscr{A}$-equivalent. □

The 2-jet $(0,0)$

Any 3-jet with 2-jet $(0,0)$ is given by (at^3, bt^3), for $a, b \in \mathbb{C}$.

If $a \neq 0$ or $b \neq 0$, (at^3, bt^3) is $\mathscr{A}^{(3)}$-equivalent to $(t^3, 0)$. Otherwise, (at^3, bt^3) is $\mathscr{A}^{(3)}$-equivalent to $(0,0)$.

The 3-jet $(t^3, 0)$

Consider the p-jet with $(p-1)$-jet $f = (t^3, 0)$.

If $p = 0 \bmod 3$, any p-jet with $(p-1)$-jet $(t^3, 0)$ is $\mathscr{A}^{(p)}$-equivalent to $(t^3, 0)$.

If $p = 1 \bmod 3$, $p = 3k+1$, the orbits in the p-jet are (t^3, t^{3k+1}), $(t^3, 0)$.

If $p = 2 \bmod 3$, $p = 3k+2$, the orbits in the p-jet are (t^3, t^{3k+2}), $(t^3, 0)$.

Indeed, the tangent space to the $\mathscr{A}_1$-orbit of f is given by

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(3t^2, 0)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Then, we have

$$\mathscr{M}_1^{3k+1}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{3k+2}\mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^{3k+1})\}$. It follows by Theorem 2.9.1 that every germ $g \in J^{3k+1}(1,2)$ with $j^{3k}g(0) = (t^3, 0)$ is $\mathscr{A}^{(3k+1)}$-equivalent to (t^3, at^{3k+1}), with a $\in \mathbb{C}$. If $a \neq 0$, then (t^3, at^{3k+1}) is $\mathscr{A}^{(3k+1)}$-equivalent to (t^3, t^{3k+1}). Otherwise, it is $\mathscr{A}^{(3k+1)}$-equivalent to $(t^3, 0)$.

Also, we have

$$\mathscr{M}_1^{3k+2}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{3k+3}\mathscr{O}(1,2)$$

where $T = \mathbb{C}\{(0, t^{3k+2})\}$. Therefore, every germ in $J^{3k+2}(1,2)$ with $(3k)$-jet $(t^3, 0)$ is $\mathscr{A}^{(3k+2)}$-equivalent to (t^3, at^{3k+2}). If $a \neq 0$, it is $\mathscr{A}^{(3k+2)}$-equivalent to (t^3, t^{3k+2}). Otherwise, it is $\mathscr{A}^{(3k+2)}$-equivalent to $(t^3, 0)$.

Proposition 3.1.3. The germ $g = (t^3, t^{3k+1})$ is $(3k+1)$-$\mathscr{A}$-determined, for $k = 1$.

Proof. Observe that

$$\mathscr{M}_1^3\mathscr{O}(1,2) \subset T\mathscr{K} \cdot f.$$

Also,

$$\mathscr{M}_1^6\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(3t^2, 4t^3)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^9\mathscr{O}(1,2).$$

Therefore, the germ (t^3, t^4) is 5-$\mathscr{A}_1$-determined and, thus, 5-$\mathscr{A}$-determined.

Let $N = \{(t^3, t^4 + at^5), a \in \mathbb{C}\}$ and $G = \mathscr{A}^{(5)}$. Then, N is a connected manifold, $T_pN = \{(0,t^5)\}$ and

$$T_pN \subset T_pG \cdot p.$$

Indeed, since $T_pG \cdot p = J^5(T\mathscr{A} \cdot f)$ and

$$\begin{array}{rcl} v_1 = \left(\frac{3t^4}{4}, t^5 + \frac{5at^6}{4}\right) & = & \frac{t^2}{4}(3t^2, 4t^3 + 5at^4) \in T_pG \cdot p, \\ v_2 = \left(\frac{3t^4}{4} + \frac{3at^5}{4}, 0\right) & = & \frac{3}{4}(t^4 + at^5, 0) \in T_pG \cdot p, \\ v_3 = \left(\frac{3at^5}{4}, \frac{4at^6 + 5a^2t^7}{4}\right) & = & \frac{at^3}{4}(3t^2, 4t^5 + 5at^4) \in T_pG \cdot p, \end{array}$$

we have

$$(0, t^5 + \frac{at^6}{4} + \frac{5a^2t^7}{4}) = v_1 - v_2 + v_3 \in T_pG \cdot p.$$

Therefore, the first condition of Mather's Lemma is satisfied. Also, since

$$T_pG \cdot p = \{(t^3,0), (0,t^3), (t^4,0), (0,t^4), (t^5,0), (0,t^5)\}$$

and its dimension does not depend on a, the second condition of Mather's Lemma is also satisfied. Then, N is contained in a unique orbit. Therefore, the germ (t^3, t^4) is 4-$\mathscr{A}$-determined. □

Proposition 3.1.4. The germ (t^3, t^{3k+2}) is $(3k+2)$-$\mathscr{A}$-determined, for $k = 1$.

Proof. We have

$$\mathscr{M}_1^3 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot f.$$

Also,

$$\mathscr{M}_1^8 \mathscr{O}(1,2) \subset \mathscr{M}_1^2[(3t^2, 5t^4)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^{11} \mathscr{O}(1,2).$$

Therefore, the germ (t^3, t^5) is 7-$\mathscr{A}_1$-determined and, thus, 7-$\mathscr{A}$-determined. Actually, this germ is 5-$\mathscr{A}$-determined. Indeed, consider $f = (t^3, t^5 + a_1t^6 + a_2t^7)$, with $a_1, a_2 \in \mathbb{C}$. Then, f is $\mathscr{L}$-equivalent to $(t^3, t^5 + a_2t^7)$.

We claim that $(t^3, t^5 + a_2t^7)$ is in the same orbit as (t^3, t^5). Let $N = \{(t^3, t^5 + a_2t^7), a_2 \in \mathbb{C}\} \subset J^7(1,2)$ and $G = \mathscr{A}^{(7)}$. Then, N is a connected manifold and $T_pN = \mathbb{C}\{(0,t^7)\}$. Observe that

$$\begin{array}{rcl} v_1 = \left(\frac{3t^5}{5}, t^7 + \frac{7a_2t^9}{5}\right) & = & \frac{t^3}{5}\left(3t^2, 5t^4 + 7a_2t^6\right) \in T_pG \cdot p; \\ v_2 = \left(\frac{3t^5 + 21a_2t^7}{5}, 0\right) & = & \frac{3}{5}\left(t^5 + 7a_2t^7, 0\right) \in T_pG \cdot p; \\ v_3 = \left(\frac{3a_2t^5}{5}, a_2t^9 + \frac{7a_2^2t^{11}}{5}\right) & = & \left(\frac{3a_2t^7}{5}, \frac{5a_2t^9 + 7a_2^2t^{11}}{5}\right) \in T_pG \cdot p; \end{array}$$

and

$$\left(0, t^7 + \frac{12a_2t^9 + 7a_2^2t^{11}}{5}\right) = v_1 - v_2 + v_3 \in T_pG \cdot p;$$

Therefore, the first condition of Mather's Lemma is satisfied. Moreover, since

$$T_pG \cdot p = \mathbb{C}\{(t^3,0), (0,t^3), (t^4,0), (t^5,0), (0,t^5), (t^6,0), (0,t^6), (t^7,0), (0,t^7)\},$$

the dimension of the tangent space $T_pG \cdot p$ does not depend on a_2. Therefore, the second condition of Mather's Lemma is also satisfied. Then, N is contained in a unique orbit. □

In what follows, we consider $k > 1$.

The $(3k+1)$**-jet** (t^3, t^{3k+1})

Consider the q-jet with $(q-1)$-jet $f = (t^3, t^{3k+1})$.

If $q = 2 \bmod 3$, $q = 3p+2$, with $k \leq p < 2k-1$, any q-jet with $(q-1)$-jet (t^3, t^{3k+1}) is $\mathscr{A}^{(q)}$-equivalent to $(t^3, t^{3k+1} + at^{3p+2})$. Otherwise, the complete transversal is empty. Indeed, the tangent space $T\mathscr{A}_1 \cdot f$ is described by

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(3t^2, (3k+1)t^{3k})] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Since we have

$$\mathscr{M}_1^{3p+2}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{3p+3}\mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^{3p+2})\}$, every germ in $J^{3p+2}(1,2)$ with $(3p+1)$-jet (t^3, t^{3k+1}) is $\mathscr{A}^{(3p+2)}$-equivalent to $(t^3, t^{3k+1} + at^{3p+2})$, $a \in \mathbb{C}$.

If $a \neq 0$ and $\phi = (t^3, t^{3k+1} + at^{3p+2})$, consider the change of coordinate in the source given by $h(t) = (\frac{1}{a})^{\frac{1}{(3(p-k)+1)}}$t. Then, we have

$$(\phi \circ h)(t) = \left(\frac{t^3}{a^{\frac{3}{3(p-k)+1}}}, \frac{t^{3k+1}}{a^{\frac{3k+1}{3(p-k)+1}}} + \frac{t^{3p+2}}{a^{\frac{3k+1}{3(p-k)+1}}} \right)$$

By the change of coordinates in the target $k(x,y) = (a^{\frac{3}{3(p-k)+1}}x, a^{\frac{3k+1}{3(p-k)+1}}y)$, we obtain

$$(k \circ \phi \circ h)(t) = (t^3, t^{3k+1} + t^{3p+2})$$

Therefore, ϕ is $\mathscr{A}^{(3p+2)}$-equivalent to $(t^3, t^{3k+1} + t^{3p+2})$.

Proposition 3.1.5. The germ $g = (t^3,\ t^{3k+1} + t^{3p+2})$, with $k \leq p < 2k-1$ is $(3p+2)$-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^3\mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1})] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Moreover, we have

$$\mathscr{M}_1^{3p+3}\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1})] + g^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^{3p+6}\mathscr{O}(1,2),$$

since

$v_1 = (t^{3p+3}, 0) = (t^{3(p+1)}, 0)$,
$v_2 = (0, t^{3p+3}) = (0, t^{3(p+1)})$,
$v_3 = \frac{t^{3p+2}}{3}\big(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1}\big) - \big(0, t^{3(2p+1)}\big)$,
$v_4 = \frac{t^{3p-3k+4}}{(3k+1)}\big(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1}\big) - \big(\frac{3t^{3p-3k+6}}{(3k+1)}, 0\big) - \frac{(3p+2)}{(3k+1)}\big(0, t^{3p+4} + t^{6p-3k+5}\big)$,
$v_5 = \frac{t^{3p+3}}{3}\big(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1}\big) - \big(0, \frac{(3k+1)t^{3k+3p+3}}{3}\big)$,
$v_6 = \frac{t^4}{(3p+2)}\big(3t^2, (3k+1)t^{3k} + (3p+2)t^{3p+1}\big) - \big(\frac{3t^6}{(3p+2)}, 0\big) - \big(0, \frac{(3k+1)(t^{3k+4} + t^{3p+5})}{(3p+2)}\big)$

are in $T\mathscr{A}_1 \cdot g + \mathscr{M}_1^{3p+6}\mathscr{O}(1,2)$. Hence, the germ $(t^3, t^{3k+1} + t^{3p+2})$ is $(3p+2)$-$\mathscr{A}_1$-determined, and thus, $(3p+2)$-$\mathscr{A}$-determined. It cannot be s-$\mathscr{A}$-determined, for $s < (3p+2)$ since the s-jet of $(t^3, t^{3k+1} + t^{3p+2})$ is the same as the s-jet of (t^3, t^{3k+1}) and those germs are not $\mathscr{A}$-equivalent. □

Proposition 3.1.6. The germ $\phi = (t^3, t^{3k+1})$ is $(6k-4)$-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^3\mathscr{O}(1,2) \subset T\mathscr{K} \cdot \phi,$$

where $T\mathscr{K} \cdot \phi = \mathscr{M}_1[(3t^2, (3k+1)t^{3k})] + \phi^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Also, we have

$$\mathscr{M}_1^{6k}\mathscr{O}(1,2) \subset \mathscr{M}_1[(3t^2, (3k+1)t^{3k})] + \phi^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^{6k+3}\mathscr{O}(1,2),$$

since

$$\begin{array}{rcl} (t^{6k}, 0) &=& (t^{3(2k)}, 0), \\ (0, t^{6k}) &=& (0, t^{3(2k)}), \\ (t^{6k+1}, 0) &=& (t^{3k+(3k+1)}, 0), \\ (0, t^{6k+1}) &=& (0, t^{3k+(3k+1)}), \\ (t^{6k+2}, 0) &=& (t^{2(3k+1)}, 0), \\ (0, t^{6k+2}) &=& (0, t^{2(3k+1)}) \end{array}$$

are in $T\mathscr{A}_1 \cdot \phi + \mathscr{M}_1^{6k+3}\mathscr{O}(1,2)$. It follows by Corollary 2.5.3 that the germ (t^3, t^{3k+1}) is $(6k-1)$-$\mathscr{A}_1$-determined. Actually, this germ is $(6k-4)$-$\mathscr{A}$-determined. Indeed, consider $f = (t^3, t^{3k+1} + a_1 t^{6k-3} + a_2 t^{6k-2} + a_3 t^{6k-1})$, with $a_1, a_2, a_3 \in \mathbb{C}$. Then, f is $\mathscr{L}$-equivalent to $(t^3, t^{3k+1} + a_3 t^{6k-1})$.

We claim that $(t^3, t^{3k+1} + a_3 t^{6k-1})$ is in the same orbit as (t^3, t^{3k+1}). Let $N = \{p = (t^3, t^{3k+1} + a_3 t^{6k-1}),\ a_3 \in \mathbb{C}\} \subset J^{6k-1}(1,2)$ and $G = \mathscr{A}^{(6k-1)}$. Then, N is a connected manifold and $T_pN = \mathbb{C}\{(0, t^{6k-1})\}$. Observe that

$$\begin{aligned} v_1 &= \left(\tfrac{3t^{3k+1}}{(3k+1)}, t^{6k-1} + \tfrac{(6k-1)a_3 t^{9k-3}}{(3k+1)}\right) = \tfrac{t^{3k-1}}{(3k+1)}(3t^2, (3k+1)t^{3k} + (6k-1)a_3 t^{6k-2}) \in T_pG \cdot p; \\ v_2 &= \left(\tfrac{3t^{3k+1}}{(3k+1)} + \tfrac{3a_3 t^{6k-1}}{(3k+1)}, 0\right) = \tfrac{3}{(3k+1)}(t^{3k+1} + a_3 t^{6k-1}, 0) \in T_pG \cdot p; \\ v_3 &= \left(\tfrac{3a_3 t^{6k-1}}{(3k+1)}, a_3 t^{9k-3} + \tfrac{a_3^2(6k-1)t^{12k-5}}{(3k+1)}\right) = \tfrac{a_3 t^{6k-3}}{(3k+1)}(3t^2, (3k+1)t^{3k} + (6k-1)a_3 t^{6k-2}) \in T_pG \cdot p; \end{aligned}$$

and

$$\left(0, t^{6k-1} + \left(\tfrac{(6k-1)a_3}{(3k+1)} + a_3\right)t^{9k-3} + \tfrac{a_3^2(6k-1)t^{12k-5}}{(3k+1)}\right) = v_1 - v_2 + v_3 \in T_pG \cdot p.$$

Hence, the first condition of Mather's Lemma is satisfied. Moreover, $T_pG \cdot p$ is generated by $(t^3,0)$,$(0,t^3)$, $(t^6,0)$, $(0,t^6)$, ...,$(t^{6k-3},0)$, $(0,t^{6k-3})$, $(t^5,0)$, $(t^8,0)$, $(0,t^{3k+1})$,$(0,t^{3k+1})$, $(t^{3k+2},0)$, $(t^{3k+4},0)$,$(0,t^{3k+4})$, $(t^{3k+5},0)$, $(t^{6k-2},0)$, $(0,t^{6k-2})$, $(t^{6k-1},0)$ and $(0,t^{6k-1})$ and its dimension does not depend on a_3. Therefore, the second condition of Mather's Lemma is also satisfied. Then, N is contained in a unique orbit. □

Remark 3.1.7. In (GIBSON; HOBBS, 1983), the $\mathscr{A}$-singularities $(t^3, t^{3k+1} + t^{3p+2})$, with $k \leq p < 2k$ were classified as $\mathscr{A}$-simple singularities. However, since the germ (t^3, t^{3k+1}) is $(6k-4)$-$\mathscr{A}$-determined, only the singularities $(t^3, t^{3k+1} + t^{3p+2})$, with $k \leq p < 2k-1$ should be considered rather than $(t^3, t^{3k+1} + t^{3p+2})$, with $k \le p < 2k$. Our classification agrees with that of (BRUCE; GAFFNEY, 1982).

The $(3k+2)$-jet (t^3, t^{3k+2})

Consider the q-jet with $(q-1)$-jet $f = (t^3, t^{3k+2})$.

If $q = 1 \bmod 3$, $q = 3p+1$, with $k < p \le 2k-1$, any q-jet with $(q-1)$-jet (t^3, t^{3k+2}) is $\mathscr{A}^{(3p+1)}$-equivalent to $(t^3, t^{3k+2} + at^{3p+1})$. Otherwise, the complete transversal is empty. Indeed, the tangent space $T\mathscr{A}_1 \cdot f$ is given by

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(3t^3, (3k+2)t^{3k+1})] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Hence, we have

$$\mathscr{M}_1^{3p+1}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{3p+2}\mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^{3p+1})\}$. Therefore, every germ in the $(3p+1)$-jet with $3p$-jet (t^3, t^{3k+2}) is $\mathscr{A}^{(3p+1)}$-equivalent to $(t^3, t^{3k+2} + at^{3p+1})$. If $a \neq 0$, $(t^3, t^{3k+2} + at^{3p+1})$ is $\mathscr{A}^{(3p+1)}$-equivalent to $(t^3, t^{3k+2} + t^{3p+1})$. Otherwise, it is $\mathscr{A}^{(3p+1)}$-equivalent to (t^3, t^{3k+2}).

Proposition 3.1.8. The germ $g = (t^3, t^{3k+2} + t^{3p+1})$, with $k < p \le 2k-1$ is $(3p+1)$-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^3\mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(3t^2, (3k+2)t^{3k+1} + (3p+1)t^{3p})] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Moreover,

$$\mathscr{M}_1^{3p+2}\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(3t^2, (3k+2)t^{3k+1} + (3p+1)t^{3p})] + g^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\} + \mathscr{M}_1^{3p+5}\mathscr{O}(1,2),$$

since

$$\begin{aligned}
v_1 &= t^{3p}(3t^2, (3k+1)t^{3k+1} + (3p+1)t^{3p}) - (0, (3p+1)t^{6p}),\\
v_2 &= t^{3(p-k)+1}(3t^2, (3k+2)t^{3k+1} + (3p+1)t^{3p}) - (0, (3p+1)t^{3p+2} + (3p+1)t^{6p-3k+1}),\\
v_3 &= (t^{3p+3}, 0) = (t^{3(p+1)}, 0),\\
v_4 &= (0, t^{3p+3}) = (0, t^{3(p+1)}),\\
v_5 &= t^{3p+2}(3t^2, (3k+2)t^{3k+1} + (3p+1)t^{3p}) - (0, (3k+2)t^{3p+3k+3}),\\
v_4 &= (0, t^{3p+4}) = t^4(3t^2, (3k+2)t^{3k+1} + (3p+1)t^{3p}) - (0, (3k+2)t^{3k+5} + (3k+2)t^{3p+4})
\end{aligned}$$

are in $T\mathscr{A}_1 \cdot g + \mathscr{M}_1^{3p+5}\mathscr{O}(1,2)$. Therefore, the germ $(t^3, t^{3k+2} + t^{3p+1})$, with $k < p \le 2k-1$ is $(3p+1)$-$\mathscr{A}_1$-determined, then, $(3p+1)$-$\mathscr{A}$-determined. It cannot be s-$\mathscr{A}$-determined, for $s < (3p+1)$, since the s-jet of $(t^3, t^{3k+2} + t^{3p+1})$ is the same as the s-jet of (t^3, t^{3k+2}) and those germs are not $\mathscr{A}$-equivalent. □

Proposition 3.1.9. The germ $g = (t^3, t^{3k+2})$ is $(6k-2)$-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^3\mathscr{O}(1,2) \subset T\mathscr{K}\cdot g,$$

where $T\mathscr{K}\cdot g = \mathscr{M}_1[(3t^2,(3k+2)t^{3k+1})] + g^*(\mathscr{M}_2)\cdot\{e_1,e_2\}\mathscr{O}_1$. Also, we have

$$\mathscr{M}_1^{6k+2}\mathscr{O}(1,2) \subset \mathscr{M}_1^2[(3t^2,(3k+2)t^{3k+1})]+g^*(\mathscr{M}_2^2)\cdot\{e_1,e_2\}+\mathscr{M}_1^{6k+5}\mathscr{O}(1,2),$$

since

$$\begin{aligned}
v_1 &= (t^{6k+2},0) = (t^{3k+(3k+2)},0),\\
v_2 &= (0,t^{6k+2}) = (0,t^{(3k+2)+3k}),\\
v_3 &= (t^{6k+3},0) = (t^{3(2k+1)},0),\\
v_4 &= (0,t^{6k+3}) = (0,t^{3(2k+1)}),\\
v_5 &= (t^{6k+4},0) = \tfrac{t^{6k+2}}{3}\left(3t^2,(3k+2)t^{3k+1}\right) - \left(0,\tfrac{(3k+2)t^{21k+5}}{3(3k+2)}\right),\\
v_6 &= (0,t^{6k+4}) = \tfrac{t^{3k+3}}{(3k+2)}\left(3t^2,(3k+2)t^{3k+1}\right) - \left(\tfrac{3t^{3k+5}}{(3k+2)},0\right)
\end{aligned}$$

are in $T\mathscr{A}_1\cdot g+\mathscr{M}_1^{6k+5}\mathscr{O}(1,2)$. Therefore, the germ (t^3,t^{3k+2}) is $(6k+1)$-$\mathscr{A}_1$-determined, and hence, $(6k+1)$-$\mathscr{A}$-determined. Actually, this germ is $(6k-2)$-$\mathscr{A}$-determined. Indeed, consider $f = (t^3,t^{3k+2}+a_1t^{6k-1}+a_2t^{6k}+a_3t^{6k+1})$, with $a_1,a_2,a_3\in\mathbb{C}$. Then, f is $\mathscr{L}$-equivalent to $(t^3,t^{3k+2}+a_3t^{6k+1})$.

We claim that $(t^3,t^{3k+2}+a_3t^{6k+1})$, for $a_3\in\mathbb{C}$ is in the same orbit as (t^3,t^{3k+2}). Let $N=\{p=(t^3,t^{3k+2}+a_3t^{6k+1}),a_3\in\mathbb{C}\}\subset J^{6k+1}(1,2)$ and $G=\mathscr{A}^{(6k+1)}$. Then, N is a connected manifold and $T_pN=\mathbb{C}\{(0,t^{6k+1})\}$. Observe that

$$\begin{aligned}
u_1 &= \left(\tfrac{3t^{3k+2}}{(3k+2)},t^{6k+1}+\tfrac{(6k+1)a_3t^{9k}}{(3k+2)}\right) &&= \tfrac{t^{3k}}{(3k+2)}(3t^2,(3k+2)t^{3k+1}+a_3(6k+1)t^{6k})\in T_pG\cdot p;\\
u_2 &= \left(\tfrac{3t^{3k+2}}{(3k+2)}+\tfrac{3a_3t^{6k+1}}{(3k+2)},0\right) &&= \tfrac{3}{(3k+2)}\left(t^{3k+2}+a_3t^{6k+1},0\right)\in T_pG\cdot p;\\
u_3 &= \left(\tfrac{3a_3t^{6k+1}}{(3k+2)},a_3t^{9k}+\tfrac{(6k+1)a_3^2t^{12k-1}}{(3k+2)}\right) &&= \tfrac{a_3t^{6k-1}}{(3k+2)}\left(3t^2,(3k+2)t^{3k+1}+a_3(6k+1)t^{6k}\right)\in T_pG\cdot p;
\end{aligned}$$

and

$$\left(0,t^{6k+1}+\tfrac{(9k+3)a_3t^{9k}}{(3k+2)}+\tfrac{(6k+1)a_3^2t^{12k-1}}{(3k+2)}\right)=u_1-u_2+u_3\in T_pG\cdot p.$$

Therefore, the first condition of Mather's Lemma is satisfied. Moreover, the tangent space $T_pG\cdot p$ is generated by $(t^3,0)$, $(0,t^3)$,...,$(t^{6k},0)$, $(0,t^{6k})$, $(t^4,0)$, ...,$(t^{6k+1},0)$, $(t^5,0)$, $(0,t^5)$,...,$(t^{6k-1},0)$ and $(0,t^{6k-1})$. Since the dimension of $T_pG\cdot p$ is a constant, the second condition of Mather's Lemma is also satisfied. Then, N is contained in a unique orbit, that is $(t^3,t^{3k+2}+a_3t^{6k+1})$ is in the same orbit as (t^3,t^{3k+2}). □

Remark 3.1.10. In (GIBSON; HOBBS, 1983), the $\mathscr{A}$-singularities $(t^3,t^{3k+2}+t^{3p+1})$, with $k<p\le 2k$ were classified as $\mathscr{A}$-simple singularities. However, since the germ (t^3,t^{3k+2}) is $(6k-2)$-$\mathscr{A}$-determined, only the singularities $(t^3,t^{3k+2}+t^{3p+1})$, with $k<p\le 2k-1$ should be considered rather than $(t^3,t^{3k+2}+t^{3p+2})$, $k<p\le 2k$. Our classification agrees with that of (BRUCE; GAFFNEY, 1982).

The 3-jet $(0,0)$

Any 4-jet with 3-jet $(0,0)$ is given by $(at^4, bt^4), a, b \in \mathbb{C}$.

If $a \neq 0$ or $b \neq 0$, (at^4, bt^4) is $\mathscr{A}^{(4)}$-equivalent to $(t^4, 0)$. Otherwise, it is $\mathscr{A}^{(4)}$-equivalent to $(0,0)$.

The 4-jet $(t^4, 0)$

- Any 5-jet with 4-jet $f = (t^4, 0)$ is $\mathscr{A}^{(5)}$-equivalent to $(t^4, at^5), a \in \mathbb{C}$. We have

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(4t^3, 0)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Then, we have

$$\mathscr{M}_1^5 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^6 \mathscr{O}(1,2),$$

with $T = \mathbb{C}\{(0, t^5)\}$. According to Theorem 2.9.1, any 5-jet with 4-jet $(t^4, 0)$ is $\mathscr{A}^{(5)}$-equivalent to (t^4, at^5), $a \in \mathbb{C}$. If $a \neq 0$, (t^4, at^5) is $\mathscr{A}^{(5)}$-equivalent to (t^4, t^5). Otherwise, it is $\mathscr{A}^{(5)}$-equivalent to $(t^4, 0)$.

- Any 6-jet with 5-jet $f = (t^4, 0)$ is $\mathscr{A}^{(6)}$-equivalent to (t^4, at^6), for $a \in \mathbb{C}$. We have

$$\mathscr{M}_1^6 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^7 \mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^6)\}$. Therefore, every $g \in J^6(1,2)$, with $j^5 g(0) = (t^4, 0)$ is $\mathscr{A}^{(6)}$-equivalent to (t^4, at^6), $a \in \mathbb{C}$. If $a \neq 0$, (t^4, at^6) is $\mathscr{A}^{(6)}$-equivalent to (t^4, t^6). Otherwise, it is $\mathscr{A}^{(6)}$-equivalent to $(t^4, 0)$.

- Any 7-jet with 6-jet $f = (t^4, 0)$ is $\mathscr{A}^{(7)}$-equivalent to (t^4, at^7). We have

$$\mathscr{M}_1^7 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^8 \mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^7)\}$. Therefore, every germ in $J^7(1,2)$ with 6-jet $(t^4, 0)$ is $\mathscr{A}^{(7)}$-equivalent to (t^4, at^7), for $a \in \mathbb{C}$. If $a \neq 0$, (t^4, at^7) is $\mathscr{A}^{(7)}$-equivalent to (t^4, t^7). Otherwise, it is $\mathscr{A}^{(7)}$-equivalent to $(t^4, 0)$.

The 5-jet (t^4, t^5)

- Any 6-jet with 5-jet $f = (t^4, t^5)$ is $\mathscr{A}^{(6)}$-equivalent to $(t^4, t^5 + at^6)$, for $a \in \mathbb{C}$. Indeed, the tangent space $T\mathscr{A}_1 \cdot f$ is given by

$$T\mathscr{A}_1 \cdot f = \mathscr{M}_1^2[(4t^3, 5t^4)] + f^*(\mathscr{M}_2^2) \cdot \{e_1, e_2\}.$$

Then, we have

$$\mathscr{M}_1^6 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^7 \mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0, t^6\}$. It follows by Theorem 2.9.1 that every $g \in J^6(1,2)$, with $j^5 g(0) = (t^4, t^5)$ is $\mathscr{A}^{(6)}$-equivalent to $(t^4, t^5 + at^6)$, for $a \in \mathbb{C}$.

We will show that $(t^4, t^5 + at^6)$ is in the same orbit as (t^4, t^5), using Mather's Lemma. Let $N = \{p = (t^4, t^5 + at^6), a \in \mathbb{C}\} \subset J^6(1,2)$. Then, N is a connected manifold and $T_p N = \mathbb{C}\{(0, t^6)\}$. Consider $G = \mathscr{A}^{(6)}$. Thus,

$$T_p N \subset T_p G \cdot p$$

Indeed, it follows by Lemma 2.7.2 that the tangent space $T_pG \cdot p$ is $J^6(T\mathscr{A} \cdot p)$. Notice that $(0,t^6)$ is in $T_pG \cdot p$. We have

$$v_1 = \frac{t^2}{5}[(4t^3, 5t^4 + 6at^5)] \in T_pG \cdot p;$$
$$v_2 = \frac{1}{5}(4t^5, 0) = \frac{1}{5}(4t^5 + 4at^6, 0) - \frac{at^3}{5}(4t^3, 5t^4 + 6at^5) \in T_pG \cdot p;$$

and

$$(0, t^6 + \frac{6at^7}{5}) = v_1 - v_2 \in T_pG \cdot p.$$

So, the first condition of Mather's Lemma is satisfied. Since

$$T_pG \cdot p = \mathbb{C}\{(t^4,0),(0,t^4),(t^5,0),(0,t^5),(t^6,0),(0,t^6)\}, \forall p \in N,$$

the tangent space $T_pG \cdot p$ is a 6-dimensional subspace. Thus, the second condition of Mather's Lemma is also satisfied. Therefore, $(t^4, t^5 + at^6)$ and (t^4, t^5) are in the same orbit, for all $a \in \mathbb{C}$.

- Any 7-jet with 6-jet $f = (t^4, t^5)$ is $\mathscr{A}^{(7)}$-equivalent to $(t^4, t^5 + at^7)$, for $a \in \mathbb{C}$. We have

$$\mathscr{M}_1^7 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^8 \mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0,t^7)\}$. According to Complete Transversal's Theorem, every germ in $J^7(1,2)$ with 6-jet (t^4, t^5) is $\mathscr{A}^{(7)}$-equivalent to $(t^4, t^5 + at^7)$, $a \in \mathbb{C}$.

If $a \neq 0$, we obtain that $g = (t^4, t^5 + at^7)$ is $\mathscr{A}^{(7)}$-equivalent to $(t^4, t^5 + t^7)$. Indeed, consider the maps $h(t) = \frac{t}{\sqrt{a}}$ and $k(x,y) = (a^2x, (\sqrt{a})^5 y)$. Then, we have

$$(k \circ g \circ h)(t) = (t^4, t^5 + t^7).$$

If $a = 0$, $(t^4, t^5 + at^7)$ is $\mathscr{A}^{(7)}$-equivalent to (t^4, t^5).

Proposition 3.1.11. The germ $\phi = (t^4, t^5 + t^7)$ is 7-$\mathscr{A}$-determined.

Proof. Observe that

$$\mathscr{M}_1^4 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot \phi,$$

where $T\mathscr{K} \cdot \phi = \mathscr{M}_1[(4t^3, 5t^4 + 7t^6)] + \phi^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Also, we have

$$\mathscr{M}_1^8 \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot \phi + \mathscr{M}_1^{12} \mathscr{O}(1,2).$$

It follows by Corollary 2.5.3 that the germ $(t^4, t^5 + t^7)$ is 7-$\mathscr{A}_1$-determined. Hence, it is 7-$\mathscr{A}$-determined. □

Proposition 3.1.12. The germ $\phi = (t^4, t^5)$ is 7-$\mathscr{A}$- determined.

Proof. We have

$$\mathscr{M}_1^4 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot \phi,$$

where $T\mathscr{K}\cdot\phi = \mathscr{M}_1[(4t^3,5t^4)] + \phi^*(\mathscr{M}_2)\cdot\{e_1,e_2\}\mathscr{O}_1$. Moreover, we have

$$\mathscr{M}_1^8\mathscr{O}(1,2) \subset T\mathscr{A}_1\cdot\phi + \mathscr{M}_1^{12}\mathscr{O}(1,2).$$

According to Corollary 2.5.3, the germ (t^4,t^5) is 7-$\mathscr{A}_1$-determined. Therefore, it is 7-$\mathscr{A}$-determined. □

The 6-jet (t^4,t^6)

• If $p = 0 \mod 2$, $p = 2k$, with $k \geq 3$, any p-jet with $(p-1)$-jet $f = (t^4,t^6)$ is $\mathscr{A}^{(p)}$-equivalent to (t^4,t^6).

• If $p = 1 \mod 2$, $p = 2k+1$, with $k \geq 3$, any p-jet with $(p-1)$-jet $f = (t^4,t^6)$ is $\mathscr{A}^{(p)}$-equivalent to (t^4,t^6+at^{2k+1}), $a \in \mathbb{C}$. Indeed, the tangent space $T\mathscr{A}_1\cdot f$ is given by

$$T\mathscr{A}_1\cdot f = \mathscr{M}_1^2[(4t^3,6t^5)] + f^*(\mathscr{M}_2^2)\cdot\{e_1,e_2\}.$$

Moreover, we have

$$\mathscr{M}_1^{2k+1}\mathscr{O}(1,2) \subset T\mathscr{A}_1\cdot f + T + \mathscr{M}_1^{2k+3}\mathscr{O}(1,2),$$

with $T = \mathbb{C}\{(0,t^{2k+1})\}$. Therefore, every p-jet with $(p-1)$-jet (t^4,t^6) is $\mathscr{A}^{(p)}$-equivalent to (t^4,t^6+at^{2k+1}). If $a \neq 0$, (t^4,t^6+at^{2k+1}) is $\mathscr{A}$-equivalent to (t^4,t^6+t^{2k+1}). Otherwise, it is $\mathscr{A}$-equivalent to (t^4,t^6).

Proposition 3.1.13. The germ $g = (t^4,t^6+t^{2k+1})$ is $(2k+1)$- $\mathscr{A}$-determined.

Proof. Since

$$\mathscr{M}_1^4\mathscr{O}(1,2) \subset T\mathscr{K}\cdot g,$$

where $T\mathscr{K}\cdot g = \mathscr{M}_1[(4t^3,6t^5+(2k+1)t^{2k})] + g^*(\mathscr{M}_2)\cdot\{e_1,e_2\}\mathscr{O}_1$ and

$$\mathscr{M}_1^{2k+4}\mathscr{O}(1,2) \subset T\mathscr{A}_1\cdot g + \mathscr{M}_1^{2k+8}\mathscr{O}(1,2),$$

the germ (t^4,t^6+t^{2k+1}) is $(2k+3)$-$\mathscr{A}_1$-determined. Actually, this germ is $(2k+1)$-$\mathscr{A}$-determined. Indeed, by the action of group $\mathscr{L}_1$, we can prove that any $(2k+2)$-jet with $(2k+1)$-jet (t^4,t^6+t^{2k+1}) is $\mathscr{A}^{(2k+2)}$-equivalent to (t^4,t^6+t^{2k+1}). Also, any $(2k+3)$-jet with $(2k+2)$-jet (t^4,t^6+t^{2k+1}) is $\mathscr{A}^{(2k+3)}$-equivalent to $(t^4,t^6+t^{2k+1}+at^{2k+3})$, for $a \in \mathbb{C}$, which by Mather's Lemma is in the same orbit as (t^4,t^6+t^{2k+1}). □

The 7-jet (t^4,t^7)

• If $p = 8$ or $p = 12$ any p-jet with $(p-1)$-jet $f = (t^4,t^7)$ is $\mathscr{A}^{(p)}$-equivalent to f.

• Any 9-jet with 8-jet $f = (t^4,t^7)$ is $\mathscr{A}^{(9)}$-equivalent to (t^4,t^7+at^9), for $a \in \mathbb{C}$. We have

$$T\mathscr{A}_1\cdot f = \mathscr{M}_1^2[(4t^3,7t^6)] + f^*(\mathscr{M}_2^2)\cdot\{e_1,e_2\}.$$

Thus, we have

$$\mathscr{M}_1^9\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{10}\mathscr{O}(1,2),$$

and the complete transversal $T = \mathbb{C}\{(0,t^9)\}$. It follows by Theorem 2.9.1 that every germ in $J^9(1,2)$ with 8-jet (t^4,t^7) is $\mathscr{A}^{(9)}$-equivalent to (t^4,t^7+at^9), $a \in \mathbb{C}$.

If $a \neq 0$, we can prove that the germ (t^4,t^7+at^9) and (t^4,t^7+t^9) are $\mathscr{A}^{(9)}$-equivalent. Otherwise, it is $\mathscr{A}^{(9)}$-equivalent to (t^4,t^7).

- Any 10-jet with 9-jet $f = (t^4,t^7)$ is $\mathscr{A}^{(10)}$-equivalent to (t^4,t^7). Since we have

$$\mathscr{M}_1^{10}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{11}\mathscr{O}(1,2),$$

the complete transversal is the subspace $\mathbb{C}\{(0,t^{10})\}$. Therefore, every germ $g \in J^{10}(1,2)$ with $j^9g(0) = (t^4,t^7)$ is $\mathscr{A}^{(10)}$-equivalent to (t^4,t^7+at^{10}), with $a \in \mathbb{C}$.

We claim that (t^4,t^7+at^{10}) is in the same orbit as (t^4,t^7).

Let $N = \{p = (t^4,t^7+at^{10}), a \in \mathbb{C}\} \subset J^{10}(1,2)$. Then, N is a connected manifold and $T_pN = \mathbb{C}\{(0,t^{10})\}$. Consider $G = \mathscr{A}^{(10)}$. It follows by Lemma 2.7.2 that the tangent space $T_pG \cdot p$ is $J^{10}(T\mathscr{A} \cdot p)$. Observe that

$$\begin{aligned} v_1 &= t^4(4t^3, 7t^6+10at^9) \in T_pG\cdot p;\\ v_2 &= t^7(4t^3, 7t^6+10at^9) \in T_pG\cdot p;\\ v_3 &= (t^7+at^{10},0) \in T_pG\cdot p; \end{aligned}$$

and

$$\left(0, t^{10} + \tfrac{3at^{13}}{7} + \tfrac{10a^2t^{16}}{7}\right) = \tfrac{1}{7}v_1 - \tfrac{a}{7}v_2 - \tfrac{4}{7}v_3 \in T_pG\cdot p.$$

Therefore, $(0,t^{10}) \in T_pG \cdot p$ and $T_pN \subset T_pG \cdot p$, that is, the first condition of Mather's Lemma is satisfied. Also, we have

$$T_pG\cdot p = \mathbb{C}\{(t^4,0),(0,t^4),(t^5,0),(t^7,0),(0,t^7),(t^8,0),(0,t^8),(t^9,0),(0,t^{10}),(t^{10},0)\}$$

Then, the dimension of the tangent space $T_pG \cdot p$ is a constant. Since it does not depend on a, the second condition of Mather's Lemma is satisfied. Thus, N is contained in a unique orbit, that is, (t^4,t^7+at^{10}) is in the same orbit as (t^4,t^7).

- Any 11-jet with 10-jet $f = (t^4,t^7)$ is $\mathscr{A}^{(11)}$-equivalent to f, since the 11-complete transversal is empty.

- Any 13-jet with 12-jet $f = (t^4,t^7)$ is $\mathscr{A}^{(13)}$-equivalent to (t^4,t^7+at^{13}), for $a \in \mathbb{C}$. We have

$$\mathscr{M}_1^{13}\mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot f + T + \mathscr{M}_1^{14}\mathscr{O}(1,2),$$

where $T = \mathbb{C}\{(0,t^{13})\}$. It follows by Theorem 2.9.1 that every germ in $J^{13}(1,2)$ with 12-jet (t^4,t^7) is $\mathscr{A}^{(13)}$-equivalent to (t^4,t^7+at^{13}), $a \in \mathbb{C}$. If $a \neq 0$, the germ (t^4,t^7+at^{13}) is $\mathscr{A}^{(13)}$-equivalent to (t^4,t^7+t^{13}), by changes of coordinates in the source and in the target. Otherwise, it is $\mathscr{A}^{(13)}$-equivalent to (t^4,t^7).

Proposition 3.1.14. The germ $g = (t^4, t^7 + t^9)$ is 9-$\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^4 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(4t^3, 7t^6 + 9t^8)] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$ and

$$\mathscr{M}_1^{11} \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot g + \mathscr{M}_1^{15} \mathscr{O}(1,2),$$

according to Corollary 2.5.3, the germ $(t^4, t^7 + t^9)$ is 10-$\mathscr{A}_1$-determined. Actually, this germ is 9-$\mathscr{A}$-determined, since any 10-jet with 9-jet $(t^4, t^7 + t^9)$ is $\mathscr{A}^{(10)}$-equivalent to $(t^4, t^7 + t^9 + at^{10})$, for $a \in \mathbb{C}$, which is in the same orbit as $(t^4, t^7 + t^9)$ by Mather's Lemma. □

Proposition 3.1.15. The germ $g = (t^4, t^7 + t^{13})$ is 13- $\mathscr{A}$-determined.

Proof. We have

$$\mathscr{M}_1^4 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(4t^3, 7t^6 + 13t^{12})] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Also, we have

$$\mathscr{M}_1^{14} \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot g + \mathscr{M}_1^{18} \mathscr{O}(1,2).$$

Therefore, by Corollary 2.5.3, the germ $(t^4, t^7 + t^{13})$ is 13-$\mathscr{A}_1$-determined. Hence, 13-$\mathscr{A}$-determined. □

Proposition 3.1.16. The germ $g = (t^4, t^7)$ is 13-$\mathscr{A}$-determined.

Proof. Notice that

$$\mathscr{M}_1^4 \mathscr{O}(1,2) \subset T\mathscr{K} \cdot g,$$

where $T\mathscr{K} \cdot g = \mathscr{M}_1[(4t^3, 7t^6)] + g^*(\mathscr{M}_2) \cdot \{e_1, e_2\}\mathscr{O}_1$. Also, we have

$$\mathscr{M}_1^{14} \mathscr{O}(1,2) \subset T\mathscr{A}_1 \cdot g + \mathscr{M}_1^{8} \mathscr{O}(1,2).$$

It follows by the Corollary 2.5.3 that the germ (t^4, t^7) is 13-$\mathscr{A}$-determined, hence, 13-$\mathscr{A}$-determined. □

Remark 3.1.17. The 8-jet $(t^4, 0)$ leads to none-simple germs. Also, the 4-jets $(0,0)$ leads to none-simple germs.(See Lemma 3.1 in (BRUCE; GAFFNEY, 1982)).

CHAPTER

4

k-FOLDING MAPS

In this chapter, we consider k-folding map-germs on plane curves, for $k \geq 3$. We first study the $\mathscr{A}$-simple singularities of k-folding map-germs. We then relate the singularities of k-folding map-germs to the local geometry of the curve and show that they reveal information about the hidden symmetries of the curve.

4.1 Introduction

Definition 4.1.1. The k-folding map-germ is the map-germ $\mathbb{C},0 \to \mathbb{C}^2,0$ of the form $\omega_k(x,y) = (x,y^k)$. It "folds" the plane $\mathbb{C}^2$ along the line $y=0$, gluing the points $(x,y),(x,\xi y),\ldots,(x,\xi^{k-1}y)$, where $\xi = e^{2\pi i/k}$ is a primitive k-th root of unity. Observe that $Fix(\omega_k) = \{y=0\}$.

We focus on the study of the local geometry of curves, so given a point on a curve and a line in $\mathbb{C}^2$, we take the curve locally as the graph of a function f parametrized by $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$ with $\gamma(t) = (f(t),t)$. The k-folding map ω_k in Definition 4.1.1 restricted to γ is the map-germ $F_k : \mathbb{C},0 \to \mathbb{C}^2,0$ given by

$$F_k(t) = (f(t),t^k).$$

We shall call F_k the k-folding map (meaning the restriction of ω_k to γ).

Remark 4.1.2. Since the k-folding map-germ F_k is constructed by the map ω_k, which is related to the k-th roots of unity, we need to consider $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$, i.e., as a curve in the complex plane. In the case of $\gamma : \mathbb{R},0 \to \mathbb{R}^2,0$, we consider γ an analytic curve and complexify it. However, the results also hold for smooth curves $\gamma : \mathbb{R},0 \to \mathbb{R}^2,0$, by complexifying a certain jet of its parametrization.

Proposition 4.1.3. Consider a curve $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$. The k-folding map-germ F_k is singular if and only if $\gamma'(0)$ is in the kernel of $d\omega_k$, i.e., $\gamma'(0)$ is parallel to the vector $(0,1)$. Equivalently, the orthogonal direction to $\mathrm{Fix}(\omega_k)$ is a normal direction to γ at $t=0$.

Proof. Since γ is an analytic curve, we can write $\gamma(t) = (f(t), t)$, with $f(t) = \sum_{j=0}^{\infty} a_j t^j$ a convergent power series. The Jacobian matrix of the k-folding map $\omega_k(x,y) = (x, y^k)$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & ky^{k-1} \end{pmatrix}$$

Thus, $\ker J_{\omega_k}(0,0) = \mathbb{C} \cdot (0,1)$. Since the jacobian matrix of k-folding map-germ F_k is given by $\begin{pmatrix} f'(t) \\ kt^{k-1} \end{pmatrix}$, F_k is singular at the origin if and only if $f'(0) = 0$. Then, $\gamma'(0) = (0,1)$, which is in the kernel of $J\omega_k$. □

Example 4.1.4. The folding map-germ on a curve glues the points (x,y) and $(x,-y)$. After restricting the folding map-germ to it, we obtain a singular curve, as shown in the following figure.

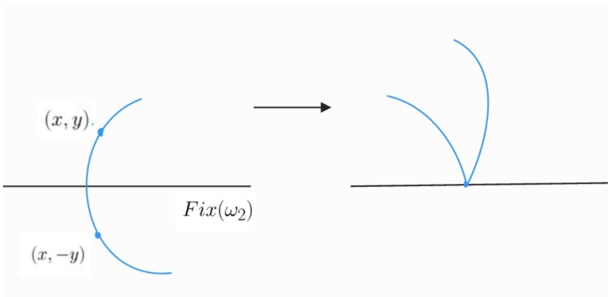

Figure 1 – The folding map-germ restricted to a curve

We consider next the $\mathscr{A}$-simple singularities of the k-folding map-germ F_k. For that we need the following auxiliary result.

Lemma 4.1.5. (GRADSHTEYN; RYZHIK, 2007) Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with $a_0 \neq 0$. Then, for every $k \in \mathbb{N}$, $h(z) = (\sum_{j=0}^{\infty} a_j z^j)^k = \sum_{j=0}^{\infty} c_j z^j$, with

$$c_0 = (a_0)^k,$$
$$c_m = \frac{1}{ma_0} \sum_{j=0}^{m-1} [k(m-j) - j] a_{m-j} c_j, m \geq 1.$$

Proof. It follows by differentiation that $h'(z) = k(g(z))^{k-1} g'(z)$. Therefore,

$$g(z)h'(z) = kg'(z)(g(z))^k = kg'(z)h(z). \tag{4.1}$$

Write $g(z)h'(z) = \sum_{j=0}^{\infty} d_j z^j$ and $g'(z)h(z) = \sum_{j=0}^{\infty} e_j z^j$. Then by the formula for the product of the power series (NETO, 2008) we have

$$d_m = \sum_{j=0}^{m} (j+1) a_{m-j} c_{(j+1)},$$
$$e_m = \sum_{j=0}^{m} (m-j+1) c_j a_{m-j+1}.$$

Using (4.1), we get by considering the coefficients of z^{m-1},

$$\sum_{j=1}^{m} ja_{m-j}c_j = k\sum_{j=0}^{m-1}(m-j)c_ja_{m-j}, \forall m \geq 1.$$

Therefore,

$$mc_ma_0 = \sum_{j=0}^{m-1}(k(m-j)-j)a_{m-j}c_j.$$

The coefficient c_0 can be determined using the fact that $h(0) = (a_0)^k = c_0$. □

4.2 $\mathscr{A}$-simple singularities of a k-folding map-germ

In (PEÑAFORT SANCHIS; TARI, 2023), the authors proved that k-folding map-germs on surfaces do not have $\mathscr{A}$-simple singularities for $k \geq 5$. In the case of k-folding map-germs on plane curves, we obtain a different result.

We take $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$, $\gamma(t) = (f(t),t)$ with $f(t) = a_1t + a_2t^2 + a_3t^3 + O(4)$. From Proposition 4.1.3 if $a_1 \neq 0$, F_k is an immersion at the origin, so from now on, we set $a_1 = 0$.

Suppose that $a_2 \neq 0$. Then, $F_k(t) = (a_2t^2(1 + \frac{a_3}{a_2}t + \frac{a_4}{a_2}t^2 + O(3)), t^k)$. Let $T = tg(t)$, with

$$g(t) = \left(\frac{1}{a_2t^2}f(t)\right)^{\frac{1}{2}}$$

and consider the function $H(t,T) = tg(t) - T$. We have $\frac{\partial H}{\partial t}(0,0) = g(0) \neq 0$, so by the Implicit Function Theorem we can write $t = h(T)$ for some germ of an analytic function h. We write $h(T) = T\sum_{j=0}^{\infty} b_jT^j$, where $b_j = \frac{h^{(j+1)}(0)}{(j+1)!}$. Observe that $b_0 = h'(0) = -\frac{\frac{\partial H(0,0)}{\partial T}}{\frac{\partial H(0,0)}{\partial t}} = -1 \neq 0$. By Lemma 4.1.5, $(h(T))^k = T^k(\sum_{j=0}^{\infty} b_jT^j)^k = T^k\sum_{j=0}^{\infty} c_jT^j$, where

$$c_0 = (b_0)^k,$$
$$c_m = \frac{1}{mb_0}\sum_{j=0}^{m-1} k(m-j)b_{m-j}c_j, m \geq 1.$$

We have $F_k(h(T)) = (T^2, (h(T))^k) = (T^2, T^k\sum_{j=0}^{\infty} c_jT^j)$.

In order to determine the singularity type of F_k when $a_1 = 0$ and $a_2 \neq 0$, we need the following results.

Lemma 4.2.1. Let n be an even number. Suppose that the even-order derivatives of h up to order $n-2$ are equal to zero at the origin and assume $h^{(l)}(0)g(h(0)) = -lg^{(l-1)}(h(0))(h'(0))^l$, for l even, with $2 \leq l \leq n-2$. Then, for p odd, with $1 \leq p \leq n-3$, $(g \circ h)^{(p)}(0) = 0$ and $(g \circ h)^{(n-1)}(0) = g^{(n-1)}(0)(h'(0))^{n-1}$.

Proof. By Faà di Bruno's formula (ROMAN, 1980) which generalizes the chain rule

$$\frac{\partial^p(g \circ h)(T)}{\partial T^p} = \sum \frac{p!}{j_1!j_2!\dots j_p!}g^{(j)}(h(T))(h'(T))^{j_1}(\frac{h"(T)}{2!})^{j_2}\dots(\frac{h^{(p)}(T)}{p!})^{j_p},$$

where the sum is over none-negative integers $j_1, j_2, ..., j_p$ satisfying $j_1 + j_2 + ... + j_p = j$ and $j_1 + 2j_2 + ... + pj_p = p$.

Consider p odd, $p \leq n-1$. Since the derivatives of even order of h up to $n-2$ are zero at the origin and all $j_i's$ are such that $i \leq p \leq n-1$, we can take $j_i = 0$, for all i even. Then, if j is even, we have

$$\begin{aligned} j_1 + j_3 + j_5 + ... + j_p &= j, \\ j_1 + 3j_3 + 5j_5 + ... + pj_p &= p. \end{aligned}$$

Subtracting the two equations, gives

$$2j_3 + 4j_5 + ... + (p-1)j_p = p - j.$$

The term $2j_3 + 4j_5 + ... + (p-1)j_p$ is a sum of even numbers, so it is even. Since p is odd and j is even, $p - j$ is odd. Therefore, for j even, the terms of the form $(g^{(j)}(h(0)))$ are multiplied by some derivative of even order of h at the origin, which is zero.

If j is odd, it follows by the hypothesis and by the fact that even order derivatives of h up to order $n-2$ are zero at the origin that $g^{(j)}(h(0)) = 0$ for all j odd, with $1 \leq j \leq n-3$.

Thus, by Faà di Bruno's formula, $(g \circ h)^{(p)}(0) = 0$, for p odd , $1 \leq p \leq n-3$. Moreover, for $j = p = n-1$, that is, for $j_1 = p$ and $j_2 = ... = j_p = 0$, we obtain $(g \circ h)^{(n-1)}(0) = g^{(n-1)}(h(0))(h'(0))^{n-1}$. □

We need the following formula, generalizing the product rule.

Lemma 4.2.2. Consider $h(T)g(h(T)) = T$. It follows by the generalization of the product rule, also called Leibniz Rule (ROMAN, 1980) that, for $p \geq 2$

$$\begin{aligned} &\big(h(T)g(h(T))\big)^{(p)} = \tbinom{p}{0} h^{(p)}(T)\big(g(h(T))\big) + \tbinom{p}{1} h^{(p-1)}(T)\big(g(h(T))\big)' + \\ &\tbinom{p}{2} h^{(p-2)}(T)\big(g(h(T))\big)''(T) + ... + \tbinom{p}{p}\big(g(h(T))\big)^{(p)} h(T) = 0. \end{aligned}$$

Theorem 4.2.3. Let n be an even number and suppose that $h^{(2p)}(0) = 0$, for $0 \leq p < \frac{n-1}{2}$. Then, $h^{(n)}(0)g(0) = -ng^{(n-1)}(0)(h'(0))^n$.

Proof. It follows by Lemma 4.2.2 that

$$h'(T)g(h(T)) + g'(h(T))h'(T)h(T) = 1.$$

and

$$h''(T)g(h(T)) + 2g'(h(T))(h'(T))^2 + g''(h(T))(h'(T))^2 h(T) + h''(T)h(T)g'(h(T)) = 0.$$

Since $h(0) = 0$, $h''(0)g(0) = -2g'(0)(h'(0))^2$. Therefore, the statement holds for $n = 2$.

Let $n > 2$ be an even number. Suppose that $h^{(l)}(0)g(0) = -lg^{(l-1)}(0)(h'(0))^l$, for $l = 2, 4, ..., n-2$ and that the derivatives of h of even order up to $n-2$ are zero at the origin.

According to Lemma 4.2.2,

$$\binom{n}{0}h^{(n)}(0)\big(g(h(0))\big)+\binom{n}{1}h^{(n-1)}(0)\big(g\circ h\big)'(0)+\binom{n}{3}h^{(n-3)}(0)\big(g\circ h\big)^{(3)}(0)+\ldots+ \binom{n}{n-1}h'(0)\big(g\circ h\big)^{(n-1)}(0)+\binom{n}{n}\big(g\circ h\big)^{(n)}(0)h(0)=0.$$

It follows by Lemma 4.2.1 and from the fact that $h(0)=0$ that

$$\begin{aligned}\binom{n}{0}h^{(n)}(0)\big(g(h(0))\big) &= -\binom{n}{n-1}h'(0)\big(g\circ h\big)^{(n-1)}(0)\\ &= -nh'(0)(g^{(n-1)}(0))(h'(0))^{n-1}\\ &= -ng^{(n-1)}(0)(h'(0))^{n}.\end{aligned}$$

Hence, $\binom{n}{0}h^{(n)}(0)(g(0))=-ng^{(n-1)}(0)(h'(0))^{n}$. □

Theorem 4.2.4. Let n be an odd integer. Suppose that $n=1$ or $g^{(i)}(0)=0$, for $1\leq i\leq n-2$, with i odd. Then

$$g^{(n)}(0)=\frac{n!a_{n+2}}{2a_2g(0)}.$$

Proof. We have $(g(t))^2=\frac{1}{a_2}\sum_{n=2}^{\infty}a_nt^{n-2}$. Then, $2g(t)g'(t)=\frac{1}{a_2}\sum_{n=3}^{\infty}a_n(n-2)t^{n-3}$ and consequently $2(g'(0))g(0)=\dfrac{a_3}{a_2}$.

Let $n\geq 3$ be an odd integer and suppose $g^{(i)}(0)=0$, for $1\leq i\leq n-2$. Let $y(t)=g^2(t)=\frac{1}{a_2}\sum_{n=0}^{\infty}a_nt^{n-2}$, $t\in\mathbb{C},0$. Since g is analytic, the function y is analytic and $y^{(n)}(0)=n!\frac{a_{n+2}}{a_2}$.

By the generalization of the product rule, $y^{(n)}(0)=(g\cdot g)^{(n)}=\sum_{j=0}^{n}\binom{n}{j}g^{(j)}(0)g^{(n-j)}(0)$ and using the hypothesis we get $y^{(n)}(0)=2g^{(n)}(0)g(0)$. It follows that $g^{(n)}(0)=\dfrac{n!a_{n+2}}{2a_2g(0)}$. □

We can now prove the following result.

Theorem 4.2.5. If $a_1=0$ and $a_2\neq 0$, the k-folding map has an A_{k+2p}-singularity if and only if $a_1=a_3=\ldots=a_{2p+1}=0$ and $a_{2p+3}\neq 0$, when k is even. When k is odd, the k-folding map has an A_{k-1}-singularity.

Proof. If k is odd, it is enough to consider the coefficients c_j with even indices, since those with odd indices can be eliminated by changes of coordinates in the target, i.e, using the action of the left group $\mathscr{L}$.

Since $c_0=(b_0)^k\neq 0$, it follows that $F_k(T)$ is $\mathscr{A}$-equivalent to (T^2,T^k) and has an A_{k-1}-singularity.

If k is even, we consider the coefficients with odd indices, since for j even, $k+j$ is even and the coefficients c_{k+j} can be eliminated by a change of coordinates in the target.

- Case 1: $b_1\neq 0$

It follows by Lemma 4.1.5 that $c_1=\frac{kb_1c_0}{a_0}\neq 0$, so the map-germ F_k is $\mathscr{A}$-equivalent to (T^2,T^{k+1}), that is, F_k has an A_k-singularity.

• Case 2: $b_1 = 0$

We claim that if m is an odd number and $b_{2i+1} = 0$, for $i = 0, \ldots, \frac{m-3}{2}$, then $c_{2i+1} = 0, i = 0, \ldots, \frac{m-3}{2}$ and $c_m = \frac{kb_m c_0}{b_0}$.

For $m = 3$, it follows by Lemma 4.1.5 that

$$c_3 = \tfrac{1}{3b_0}\big(3kb_3c_0 + (2k-1)b_2c_1 + (k-2)b_1c_2\big).$$

Since $c_1 = 0$, $c_3 = \frac{kb_3c_0}{b_0}$.

Suppose by induction that the claim holds for all c_j, with j odd and $j \le m$ and suppose that $b_{2i+1} = 0$, $i = 0, \ldots, \frac{m-1}{2}$.

According to Lemma 4.1.5, the coefficient c_{m+2} is given by

$$c_{m+2} = \frac{1}{(m+2)b_0}\big(k(m+2)b_{m+2}c_0 + \sum_{j=1}^{m+1}(k(m+2-j)-j)b_{m+2-j}c_j\big). \tag{4.2}$$

If $j \ge 1$ is odd, by the induction hypothesis, we obtain $c_j = \dfrac{kb_jc_0}{b_0}$ for $j \le m$. Therefore, for $j < m+2$, $(k(m+2-j)-j)b_{m+2-j}c_j = 0$, since the b_j's are zero. If j is even, with $j \ge 2$, the coefficient b_{m+2-j} is zero since $(m+2-j)$ is odd and $(m+2-j) \le m$. Thus, $(k(m+2-j)-j)b_{m+2-j}c_j) = 0$. It follows by (4.2), that $c_{m+2} = \frac{kb_{m+2}c_0}{b_0}$.

Now if $b_1 = \ldots = b_m = 0$ and $b_{m+2} \neq 0$, for m odd, then $c_1 = \ldots = c_m = 0$ and $c_{m+2} \neq 0$. Consequently, F_k is $\mathscr{A}$-equivalent to $(T^2, T^{k+(m+2)})$, that is, it has an A_{k+m+1}-singularity.

By Lemma 4.2.1, Lemma 4.2.2, Theorem 4.2.3 and Theorem 4.2.4 it follows that the conditions on b_j's are equivalent to $a_3 = \ldots = a_{m+2} = 0$ and $a_{m+4} \neq 0$. Setting $m = 2p-1$ for some p, we have F_k is $\mathscr{A}$-equivalent to (T^2, T^{k+2p+1}) if and only if $a_3 = \ldots = a_{2p+1} = 0$ and $a_{2p+3} \neq 0$. □

Now suppose that $a_1 = a_2 = 0$ and $a_3 \neq 0$. Then, $F_k(t) = (a_3t^3(1 + \frac{a_4}{a_3}t + \frac{a_5}{a_3}t^2 + O(3)), t^k)$. We change coordinates so that $F_k(T) = (T^3, (h(T))^k)$. For that we set $T = tg(t)$ with

$$g(t) = \Big(\tfrac{1}{a_3t^3}f(t)\Big)^{\frac{1}{3}}.$$

In order to determine the singularity type of F_k in this case we need the following results.

Proposition 4.2.6. If l is an integer such that $3 \nmid l$ and $c_i = 0$, with $1 \le i \le l-1$, $3 \nmid i$, then $c_l = \frac{kb_lc_0}{b_0}$. Also, $c_1 = \frac{kb_1c_0}{b_0}$.

Proof. It follows by Lemma 4.1.5 that $c_1 = \frac{kb_1c_0}{b_0}$.

Now consider l an integer such that $3 \nmid l$. Then, $l = 3p+1$ or $l = 3p+2$, for some p.

If $l = 1$ mod 3, for $l = 4$ and $c_1 = c_2 = 0$, it follows by Lemma 4.1.5 that $c_4 = \frac{kb_4c_0}{b_0}$.

Suppose that $c_i = \frac{kb_ic_0}{b_0}$, for $1 \leq i \leq l$ such that $3 \nmid i$ and consider $c_i = 0$, with $1 \leq i \leq l+1$, $3 \nmid i$. According to Lemma 4.1.5,

$$\begin{aligned} c_{l+3}(l+3)b_0 &= k(l+3)b_{l+3}c_0 + (k(l+2)-1)b_{l+2}c_1 + \ldots + (k-(l+2))b_1c_{l+2} \\ &= k(l+3)b_{l+3}c_0 + (kl-3)b_lc_3 + \ldots + (k-(l+2))b_1c_{l+2}. \end{aligned}$$

Since $m+n = l+3$, with m and n the indices of b and c, respectively, and $m = 1 \text{mod } 3$, $1 \leq m \leq l$, by the induction hypothesis we have $c_m = \frac{kb_mc_0}{b_0} = 0$. Therefore, such $b_m's$ are zero and, hence, $c_{l+3} = \frac{kb_{l+3}c_0}{b_0}$.

If $l = 2$ mod 3, following Lemma 4.1.5, the statement holds for $l = 2$.

Suppose that $c_i = \frac{kb_ic_0}{b_0}$, for $1 \leq i \leq l$, and $3 \nmid i$. Consider $c_i = 0$, with $1 \leq i \leq l+2$, $3 \nmid i$. Then,

$$\begin{aligned} c_{l+3}(l+3)b_0 &= k(l+3)b_{l+3}c_0 + (k(l+2)-1)b_{l+2}c_1 + \ldots + (k-(l+2))b_1c_{l+2} \\ &= k(l+3)b_{l+3}c_0 + (kl-3)b_lc_3 + \ldots + (k-(l+2))b_1c_{l+2}. \end{aligned}$$

Since $m+n = l+3$, with m and n indices of b and c, respectively, and $m = 2$ mod 3, $1 \leq m \leq l$, by the induction hypothesis we have $c_m = \frac{kb_mc_0}{b_0} = 0$. Then, such $b_m's$ are zero and, hence, $c_{l+3} = \frac{kb_{l+3}c_0}{b_0}$. □

Proposition 4.2.7. Let p be a positive integer. Suppose that $h^{(i)}(0) = 0$, with $i \neq 1 \text{mod } 3$ and $0 \leq i < 3p+l$, for $l = 0$ or 2. Then, $h^{(3p+l)}(0)g(0) = -(3p+l)(h'(0))^{3p+l}g^{(3p+l-1)}(0)$.

Proof. Consider $l = 0$. If $p = 1$, suppose $h^{(i)}(0) = 0$, for $i \neq 1$ mod 3, with $0 < i < 3$. According to Leibniz Rule,

$$\binom{3}{0}h^{(3)}(0)g(0) + \binom{3}{1}h''(0)\big(g \circ h\big)'(0) + \binom{3}{2}h'(0)\big(g \circ h\big)''(0) = 0.$$

Since $h''(0) = 0$ and $\big(g \circ h\big)''(0) = g''(0)(h'(0))^2 + h''(0)g'(0)$, we obtain

$$\begin{aligned} \binom{3}{0}h^{(3)}(0)g(0) &= -\binom{3}{2}(h'(0))^3g''(0). \\ h^{(3)}(0)g(0) &= -3(h'(0))^3g''(0). \end{aligned}$$

Suppose that the statement holds up to the order $3p$, that is, $h^{(3i)}(0)g(0) = -3i(h'(0))^{3i}g^{(3i-1)}(0)$, for $i \leq p$. We also suppose $h^{(i)}(0) = 0$, for $i \neq 1$ mod 3 and $0 < i < 3p+3$. It follows by Leibniz Rule that

$$\begin{aligned} \binom{3p+3}{0}h^{(3p+3)}(0)g(0) + \binom{3p+2}{1}h^{(3p+2)}(0)\big(g \circ h\big)'(0) + \ldots + \binom{3p+3}{3p+2}h'(0)\big(g \circ h\big)^{(3p+2)}(0) \\ + \binom{3p+3}{3p+3}h(0)\big(g \circ h\big)^{(3p+3)}(0) = 0. \end{aligned}$$

Since $h^{(i)}(0) = 0$, for $i \neq 1$ mod 3, $0 \leq i < 3p+3$, the expression above can be rewritten as follows

$$\begin{aligned} \binom{3p+3}{0}h^{(3p+3)}(0)g(0) + \binom{3p+1}{2}h^{(3p+1)}(0)\big(g \circ h\big)''(0) + \ldots + \binom{3p-2}{5}h^{(3p-2)}(0)\big(g \circ h\big)^{(5)}(0) \\ + \ldots + \binom{3p+3}{3p+2}h'(0)\big(g \circ h\big)^{(3p+2)}(0) = 0. \end{aligned}$$

We need to consider the derivatives $\left(g\circ h\right)^{(3j+2)}(0)$, for $0\le j\le p$. By Faà di Bruno's Formula,

$$\frac{\partial^{3j+2}(g\circ h)(T)}{\partial T}=\sum\tfrac{(3j+2)!}{m_1!m_2!...m_{3j+2}!}g^{(m)}(h(T))(h'(T))^{m_1}(\tfrac{h"(T)}{2!})^{m_2}...(\tfrac{h^{(3j+2)}(T)}{(3j+2)!})^{m_{3j+2}},$$

where the sum is over none-negative integers $m_1,m_2,...,m_{3j+2}$ satisfying $m_1+m_2+...+m_{3j+2}=m$ and $m_1+2m_2+...+(3j+2)m_{3j+2}=3j+2$. Since $h^{(i)}(0)=0$, for $i\neq 1\text{mod }3$ and $1<i<3p+3$, we can take $m_i=0$ zero for all i satisfying those conditions. Then,

$$\left(g\circ h\right)^{(3j+2)}(0)=\sum\tfrac{(3j+2)!}{m_1!m_2!...m_{3j+2}!}g^{(m)}(0)(h'(0))^{m_1}(h^{(4)}(0))^{m_4}...(h^{(3j+1)}(0))^{m_{3j+1}},$$

where

$$\begin{aligned}m_1+m_4+...+m_{3j+1}&=m\\ m_1+4m_4+...+(3j+1)m_{3j+1}&=3j+2.\end{aligned}$$

Subtracting the two equations, we obtain

$$3m_4+...+3m_{3j+1}=3j+2-m.$$

Notice that $3m_4+...+3m_{3j+1}$ is a multiple of three. Then, $3\mid(3j+2-m)$, that is, $m=2$ mod 3. According to the induction hypothesis, $h^{(3i)}(0)g(0)=-3i(h'(0))^{3i}g^{(3i-1)}(0)$, with $1\le i\le p$. Since $h^{(i)}(0)=0$, for $0\le i<3p+3$ and $i\neq 1$ mod 3, $g^{(3i-1)}(0)=0$, where $1\le i\le p$. Then, $g^{(3(i-1)+2)}(0)=0$, with $1\le i\le p$. It follows by Faà di Bruno's formula that $\left(g\circ h\right)^{(3j+2)}(0)=0$, for $0\le j\le p-1$.

For $j=p$, since $m=2$ mod 3, with $m\le 3p+2$ and $g^{(3j+2)}(0)=0$, for $0\le j\le p-1$, according to Faà di Bruno's formula, we are going to consider $m=3p+2$. Then, it follows by

$$\begin{aligned}m_1+m_4+...+m_{3p+1}&=3p+2\\ m_1+4m_4+...+(3p+1)m_{3p+1}&=3p+2\end{aligned}$$

that

$$3m_4+...+3pm_{3p+1}=0.$$

Since $m_i\ge 0$, we have $m_i=0$, for $4\le i\le 3p+1$, $i=1$ mod 3. Then, $m_1=3p+2$ and,

$$\left(g\circ h\right)^{(3p+2)}(0)=g^{(3p+2)}(0)(h'(0))^{3p+2}.$$

Thus,

$$\begin{aligned}\tbinom{3p+3}{0}h^{(3p+3)}(0)g(0)&=-\tbinom{3p+3}{3p+2}(h'(0))^{3p+3}g^{(3p+2)}(0)\\ h^{(3p+3)}(0)g(0)&=-(3p+3)(h'(0))^{3p+3}g^{(3p+2)}(0).\end{aligned}$$

Now consider $l=2$. If $p=0$, it follows by $h(T)g(h(T))=T$ that

$$h'(T)g(h(T))+g'(h(T))(h'(T))h(T)=1$$

and

$$\begin{aligned}&h''(T)g(h(T))+g'(h(T))(h'(T))^2+g''(h(T))(h'(T))^2h(T)+\\ &g'(h(T))h''(T)h(T)+g'(h(T))(h'(T))^2=0.\end{aligned}$$

Since $h(0)=0$, we get $h''(0)g(0)=-2g'(0)(h'(0))^2$.

Suppose that the statement holds up to p, that is,

$$h^{(3j+2)}(0)g(0)=-(3j+2)(h'(0))^{3j+2}g^{(3j+1)}(0),$$

with $j\leq p$. Consider $h^{(i)}(0)=0$, for $i<3p+5$ and $i\neq 1$ mod 3. Notice that

$$\binom{3p+5}{0}h^{(3p+5)}(0)g(0)+\binom{3p+5}{1}h^{(3p+4)}(0)\big(g\circ h\big)'(0)+...+\binom{3p+5}{3p+3}h''(0)\big(g\circ h\big)^{(3p+3)}(0)$$
$$+\binom{3p+5}{3p+4}h'(0)\big(g\circ h\big)^{(3p+4)}(0)+\binom{3p+5}{3p+5}h(0)\big(g\circ h\big)^{(3p+5)}(0)=0.$$

Since $h^{(i)}(0)=0$ for $0\leq i<3p+5$, with $i\neq 1$ mod 3,

$$\binom{3p+5}{0}h^{(3p+5)}(0)g(0)+\binom{3p+5}{1}h^{(3p+4)}(0)\big(g\circ h\big)'(0)+...+\binom{3p+5}{3p+4}h'(0)\big(g\circ h\big)^{(3p+4)}(0)=0.$$

By Faà di Bruno's formula,

$$\big(g\circ h\big)^{(3j+1)}(0)=\sum\frac{(3j+1)!}{m_1!...m_{3j+1}!}g^{(m)}(0)(h'(0))^{m_1}(h^{(4)}(0))^{m_4}...(h^{(3j+1)}(0))^{m_{3j+1}},$$

where

$$\begin{aligned} m_1+m_4+...+m_{3j+1} &= m\\ m_1+4m_4+...+3j+1m_{3j+1} &= 3j+1.\end{aligned}$$

Subtracting the two equations, we obtain

$$3m_4+...+3jm_{3j+1}=3j+1-m$$

and, thus, $m=1$mod 3.

It follows by the induction hypothesis and $h^{(i)}(0)=0$, for $i\neq 1$ mod 3 with $0\leq i<3p+5$ that $g^{(3j+1)}(0)=0$ for $0\leq j\leq p$. Then, since $m=1$ mod 3, we can consider $m=3p+4$.

By Faà di Bruno's formula, $\big(g\circ h\big)^{(3j+1)}(0)=0$, with $0\leq j\leq p$. For $j=p+1$, we obtain

$$\begin{aligned} m_1+m_4+...+m_{3j+1} &= 3p+4\\ m_1+4m_4+...+3j+1m_{3j+1} &= 3p+4.\end{aligned}$$

Then, $m_4=...=m_{3m+1}=0$ and $m_1=3p+4$.

According to Faà di Bruno's formula,

$$\big(g\circ h\big)^{(3p+4)}(0)=g^{(3p+4)}(0)(h'(0))^{3p+4}.$$

Then,

$$\begin{aligned} \binom{3p+5}{0}h^{(3p+5)}(0)g(0) &= -\binom{3p+5}{3p+4}g^{(3p+4)}(0)(h'(0))^{3p+5}\\ h^{(3p+5)}(0)g(0) &= -(3p+5)g^{(3p+4)}(0)(h'(0))^{3p+5}.\end{aligned}$$

□

Theorem 4.2.8. Consider $n=1$ mod 3. If $n\neq 1$, suppose that $g^{(m)}(0)=0$, for $1\leq m<n$, with $3\nmid m$. Then, $g^{(n)}(0)=\dfrac{n!a_{n+3}}{3a_3(g(0))^2}$. Also, $g'(0)=\dfrac{a_4}{3a_3(g(0))^2}$.

Proof. Let $y(t) = g^3(t) = \frac{1}{a_3}\sum_{j=0}^{\infty} a_{j+3}t^j, t \in \mathbb{C}, 0$. It follows by the analyticity of y that $y^{(n)}(0) = \frac{n!a_{n+3}}{a_3}$. In particular, $y'(0) = 3(g(0))^2 g'(0) = \frac{a_4}{a_3}$.

Also, according to Leibniz Rule, $y^{(n)}(0) = (g^2 \cdot g)^{(n)}(0) = \sum_{j=0}^{n} \binom{n}{j} (g^2)^{(j)}(0) g^{(n-j)}(0)$, and since $n = 1 \bmod 3$, we get $g^{(n-j)}(0) = 0$, for $3 \mid j$, with $j > 0$ or $j = 2 \bmod 3$. Then, it is sufficient to consider the cases when $j = 1 \bmod 3$ and when $j = 0$. Firstly, consider $j < n$.

For $j = 1 \bmod 3$, it follows by Leibniz Rule that $(g^2)^{(j)}(0) = \sum_{i=0}^{j} \binom{j}{i} g^{(i)}(0) g^{(j-i)}(0)$.

a) For $3 \mid i$, we have $(j-i) = 1 \bmod 3$. Since $j - i \leq j < n$, we obtain $g^{(j-i)}(0) = 0$.

b) Let $i = 1 \bmod 3$. Since $i \leq j < n$, it follows by the hypothesis that $g^{(i)}(0) = 0$.

c) If $i = 2 \bmod 3$, it follows by $i \leq j < n$ that $g^{(i)}(0) = 0$.

Therefore, for $j < n$, $(g^2)^{(j)}(0) = 0$.

For $j = n$:

a) If $3 \nmid i$ and $i < n$, it follows by the hypothesis that $g^{(i)}(0) = 0$.

b) If $3 \mid i$ and $i > 0$, since $(j-i) < n$ and $(j-i) = 1 \bmod 3$ we have $g^{(j-i)}(0) = 0$.

c) If $i = 0$ or $i = n$, according to Leibniz Rule, we have $\binom{n}{0} g(0) g^{(n)}(0)$ and $\binom{n}{n} g^{(n)}(0) g(0)$. Thus, for $j = n$, we have $(g^2)^{(n)}(0) = \binom{n}{0} g(0) g^{(n)}(0) + \binom{n}{n} g^{(n)}(0) g(0)$.

For $j = 0$, we obtain $\binom{n}{0} g^2(0) g^{(n)}(0)$. Therefore,

$$\begin{aligned} y^{(n)}(0) &= \tbinom{n}{0} g^2(0) g^{(n)}(0) + \tbinom{n}{n} (g^2)^{(n)} g(0) \\ &= \tbinom{n}{0} g^2(0) g^{(n)}(0) + \tbinom{n}{n} g(0) [\tbinom{n}{0} g(0) g^{(n)}(0) + \tbinom{n}{n} g(0) g^{(n)}(0)] \\ &= 3g^2(0) g^{(n)}(0). \end{aligned}$$

It follows that $3(g(0))^2 g^{(n)}(0) = \frac{n!a_{n+3}}{a_3}$, so $g^{(n)}(0) = \frac{n!a_{n+3}}{3a_3(g(0))^2}$. □

Theorem 4.2.9. Consider n such that $n = 2 \bmod 3$. Suppose that $g^{(m)}(0) = 0$, $1 \leq m < n$, with $3 \nmid m$. Then, $g^{(n)}(0) = \frac{n!a_{n+3}}{3a_3(g(0))^2}$.

Proof. Let $y(t) = g^3(t) = \frac{1}{a_3}\sum_{j=0}^{\infty} a_{j+3}t^j, t \in \mathbb{C}, 0$. It follows by the analiticity of y that $y^{(n)}(0) = \frac{n!a_{n+3}}{a_3}$.

Also, according to Leibniz Rule, $y^{(n)}(0) = (g^2 \cdot g)^{(n)}(0) = \sum_{j=0}^{n} \binom{n}{j} (g^2)^{(j)}(0) g^{(n-j)}(0)$. Since $n = 2 \bmod 3$, it follows by the hypothesis that $g^{(n-j)}(0) = 0$, for $j = 1 \bmod 3$ or $j = 0 \bmod 3$, with $j > 0$. We need to consider the case $j = 2 \bmod 3$ and when $j = 0$. According to Leibniz Rule $(g^2)^{(j)}(0) = \sum_{i=0}^{j} \binom{j}{i} g^{(i)}(0) g^{(j-i)}(0)$.

For $j < n$ and $j = 2 \bmod 3$:

a) If $3 \mid i$, we have $(j-i) = 2 \bmod 3$. Since $j - i \leq j < n$, we obtain $g^{(j-i)}(0) = 0$;

b) If $i = 1 \bmod 3$, it follows by $(j-i) = 1 \bmod 3$ that $g^{(j-i)}(0) = 0$;

c) For $i = 2 \bmod 3$, we obtain $g^{(i)}(0) = 0$, since $i \leq j < n$.

Therefore, for $j < n$, $(g^2)^{(j)}(0) = 0$.

For $j = n$:

a) If $3 \nmid i$ and $i < n$, it follows by the hypothesis that $g^{(i)}(0) = 0$;

b) If $3 \mid i$, we have $j - i < n$ and $(j-i) = 2 \bmod 3$. Then, $g^{(j-i)}(0) = 0$;

c) For $i = 0$ and $i = n$, we obtain $\binom{n}{0}\ g(0)g^{(n)}(0)$ and $\binom{n}{n}\ g^{(n)}(0)g(0)$, by the Leibniz Rule.

Thus, for $j = n, (g^2)^{(n)}(0) = \binom{n}{0}\ g(0)g^{(n)}(0) + \binom{n}{n}\ g^{(n)}(0)g(0)$.

For $j = 0$, we have $\binom{n}{0}\ (g(0))^2 g^{(n)}(0)$. Therefore,

$$\begin{aligned} y^{(n)}(0) &= \tbinom{n}{0}(g(0))^2 g^{(n)}(0) + \tbinom{n}{0}(g(0))^2 g^{(n)}(0) + \tbinom{n}{n} g^{(n)}(0)(g(0))^2 \\ &= 3(g(0))^2 g^{(n)}(0). \end{aligned}$$

It follows that $3(g(0))^2 g^{(n)}(0) = \dfrac{n!a_{n+3}}{a_3}$, so $g^{(n)}(0) = \dfrac{n!a_{n+3}}{3a_3(g(0))^2}$. □

We can now prove the following result.

Theorem 4.2.10. Suppose that $a_1 = a_2 = 0$ and $a_3 \neq 0$. If $3 \mid k$, say $k = 3m$, then the singularity of F_k is of the type

$$\begin{aligned} E_{6(m+j)} &\Leftrightarrow a_4 = a_5 = a_7 = \ldots = a_{3j+2} = 0, \text{ and } a_{3j+4} \neq 0. \\ E_{6(m+j)+2} &\Leftrightarrow a_4 = a_5 = a_7 = \ldots = a_{3j+4} = 0 \text{ and } a_{3j+5} \neq 0. \end{aligned}$$

If $k = 3m+1$ (resp. $k = 3m+2$), then F_k has an E_{6m} (resp. E_{6m+2})-singularity.

Proof. We have two cases:

• The case $3 \nmid k$:

Since k is not divisible by three, we have two possibilities: $k = 1 \bmod 3$ or $k = 2 \bmod 3$. Also, $F_k(T) = (T^3, c_0T^k + c_1T^{k+1} + O(k+2))$, where $c_0 = (b_0)^k = (h'(0))^k \neq 0$. It follows from the classification results in Chapter 3 that F_k is finitely $\mathscr{A}$-determined.

If $k = 3m+1$ for some integer m, F_k is $\mathscr{A}$-equivalent to $(T^3, T^{3m+1} + T^{3l+2})$ if $c_{3(l-m)+1} \neq 0$ or (T^3, T^{3m+1}), if all the coefficients $c_{3(l-m)+1}$ are zero, for $m \leq l < (2m-1)$. If $k = 3m+2$, then F_k is $\mathscr{A}$-equivalent to $(T^3, T^{3m+2} + T^{3l+1})$ if $c_{3(l-m)-1} \neq 0$ or (T^3, T^{3m+2}), if all the coefficients $c_{3(l-m)-1}$ are zero, with $m < l \leq (2m-1)$.

• The case $3|k$:

In this case, we can eliminate the terms of degrees divisible by three by changes of coordinates in the target.

Now if $k = 3m$ for some m, F_k is $\mathscr{A}$-equivalent to $(T^3, T^{3(m+j)+1} + T^{3l+2})$, with $(m+j) \leq l < 2(m+j) - 1$ or $(T^3, T^{3(m+j)+1})$ if all the coefficients $c_1 = c_2 = c_4 = ... = c_{3j-1} = 0$ and $c_{3j+1} \neq 0$. Also, F_k is $\mathscr{A}$-equivalent to $(T^3, T^{3(m+j)+2} + T^{3l+1})$, with $(m+j) < l \leq 2(m+j) - 1$ or $(T^3, T^{3(m+j)+2})$ if all the coefficients $c_1 = c_2 = c_4 = ... = c_{3j+1} = 0$ and $c_{3j+2} \neq 0$.

According to Proposition 4.2.6, the conditions on the coefficients c_j's above determine some conditions on $b'_j s$. It follows by Proposition 4.2.7, Theorem 4.2.8 and Theorem 4.2.9 that the conditions on b_j's are equivalent to $a_4 = a_5 = a_7 = ... = a_{3j+2} = 0$ and $a_{3j+4} \neq 0$ for $E_{6(m+j)}$-singularity and $a_4 = a_5 = a_7 = ... = a_{3j+4} = 0$ and $a_{3j+5} \neq 0$ for $E_{6(m+j)+2}$-singularity. □

Theorem 4.2.11. Suppose that $a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$. Then, the 4-folding map-germ can have a singularity of type

$$(t^4, t^5), (t^4, t^5 + t^7), (t^4, t^6 + t^{2l+1}), (t^4, t^7 + t^9), (t^4, t^7), (t^4, t^7 + t^{13}).$$

Proof. The statement follows by the classification results in Chapter 3 as the coefficients a_i can take any values in $\mathbb{C}$. □

Remark 4.2.12. If $a_5 = a_6 = a_7 = 0$, the 4-folding map-germ does not have an $\mathscr{A}$-simple singularity.

Remark 4.2.13. Those singularities in Theorem 4.2.11 are of type $W_{12}, W^{\#}_{1.2k-5}$ and W_{18}, as shown in Theorem 3.0.1 of Chapter 3.

Theorem 4.2.14. Suppose that $a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$. Then, the 5-folding map-germ can have a singularity of the form

$$(t^4, t^5 + t^7) \text{ and } (t^4, t^5).$$

Proof. The statement follows by the classification results in Chapter 3 as the coefficients a_i can take any values in $\mathbb{C}$. □

Remark 4.2.15. Those singularities in Theorem 4.2.14 are of type W_{12}, as shown in Theorem 3.0.1 of Chapter 3.

Theorem 4.2.16. Any $\mathscr{A}$-simple singularity of map-germs $\mathbb{C}, 0 \to \mathbb{C}^2, 0$ can be realised as a singularity of a k-folding map-germ for some $k \geq 3$. Also, any k-folding map-germs for $k \geq 3$ can have an $\mathscr{A}$-simple singularity.

Proof. The proof follows by Theorem 4.2.5, Theorem 4.2.10, Theorem 4.2.11 and Theorem 4.2.14. □

Remark 4.2.17. If $a_1 = a_2 = a_3 = a_4 = 0$, then the singularities of the k-folding map-germs are not $\mathscr{A}$-simple, when $k \geq 5$.

4.3 k-folding map-germs on generic plane curves

The local geometry of a generic plane curve reveal aspect of the local singularities of k-folding map-germs on it. For more details about genericity of curves, see Appendix A.

Theorem 4.3.1. Let $\gamma : \mathbb{R},0 \to \mathbb{R}^2,0$ be a generic plane curve (that is, in an open and dense set of plane curves), so if the origin is an inflexion or a vertex, then it is an ordinary one.

a) If the tangent line to γ at the origin is not orthogonal to $Fix(\omega_k) = \{y = 0\}$, the k-folding map is an immersion at the origin.

Suppose now that $\gamma'(0)$ is orthogonal to $Fix(\omega_k)$ at $t = t_0$.

b) If the origin is neither a vertex nor an inflexion, then the k-folding map has an A_k singularity at the origin if k is even and an A_{k-1} singularity if k is odd.

c) If γ has an ordinary vertex at the origin, then the k-folding map has an A_{k-1} singularity at the origin when k is odd and an A_{k+2} singularity when k is even.

d) If γ has an ordinary inflexion at the origin and $3 \nmid k$, then the k-folding map has an E_{6p} (resp.E_{6p+2}) singularity at the origin, if $k = 3p+1$ (resp. $k = 3p+2$) for some p. If $3 \mid k$, $k = 3p$, for some p, the k-folding map has an E_{6p} singularity at the origin.

Proof. We choose an appropriate system of coordinates so that $\gamma : \mathbb{R},0 \to \mathbb{R}^2,0$ is the graph of a function $\bar{f} : \mathbb{R},0 \to \mathbb{R}$, that is, $\gamma(t) = (\bar{f}(t),t)$. Take the l-jet of $\bar{f}$ at the origin, for l high enough. Since $j^l\bar{f}(0) = f$ is a polynomial function, it can be complexified and considered as a germ of a holomorphic function $f : \mathbb{C},0 \to \mathbb{C},0$. We can consider the curve $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$, given by $\gamma(t) = (f(t),t)$, with $f(t) = \sum_{j=1}^{l} a_j t^j$, as the singularities of F_k depend only on some jet of $\bar{f}$.

a) The statement follows by Proposition 4.1.3.

For the remaining part of the proof we need the following. We suppose now that $a_1 = 0$ so that $\gamma'(0) = (0,1)$ is orthogonal to $Fix(\omega_k)$. Then, the curvature of γ is given by

$$\kappa(t) = -\frac{2a_2+6a_3t+12a_4t^2+O(3)}{((2a_2t+3a_3t^2+4a_4t^3+O(4))^2+1)^{\frac{3}{2}}},$$

and its derivatives are given by

$$\kappa'(t) = \frac{p_1(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{3}{2}}} - \frac{p_2(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{5}{2}})},$$

with

$$p_1(t) = -24a_4t - 6a_3 + O(2)$$
$$p_2(t) = -3(12a_4t^2 + 6a_3t + 2a_2 + O(3))^2(4a_4t^3 + 3a_3t^2 + 2a_2t + O(4)).$$

We have

$$\begin{aligned}\kappa''(t) \;=\; & -\frac{p_3(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{3}{2}}} - \frac{p_4(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{5}{2}}}\\ & -\frac{p_5(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{5}{2}}} - \frac{p_6(t))}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{5}{2}}}\\ & +\frac{p_7(t)}{((4a_4t^3+3a_3t^2+2a_2t+O(4))^2+1)^{\frac{7}{2}}},\end{aligned}$$

with

$$
\begin{aligned}
p_3(t) &= 24a_4 + O(1),\\
p_4(t) &= 3(-12a_4t^2 - 6a_3t - 2a_2 + O(3))(12a_4t^2 + 6a_3t + 2a_2 + O(3))^2,\\
p_5(t) &= 6(-24a_4t - 6a_3 + O(2))(12a_4t^2 + 6a_3t + 2a_2 + O(3))(4a_4t^3 + 3a_3t^2 + 2a_2t + O(4)),\\
p_6(t) &= 3(24a_4t + 6a_3 + O(2))(-12a_4t^2 - 6a_3t - 2a_2 + O(3))(4a_4t^3 + 3a_3t^2 + 2a_2t + O(4),\\
p_7(t) &= -15(12a_4t^2 + 6a_3t + 2a_2 + O(3))^3(4a_4t^3 + 3a_3t^2 + 2a_2t + O(4))^2.
\end{aligned}
$$

b) It follows by Definition A.1.1 that the origin is an inflexion point if and only if $\kappa(0) = 0$, that is, $a_2 = 0$. Moreover, also by Definition A.1.1 the origin is a vertex if and only if $\kappa(0) \neq 0$ and $\kappa'(0) = 0$, that is, $a_2 \neq 0$ and $a_3 = 0$. Therefore, the origin is neither a vertex nor an inflexion if and only if $a_2 \neq 0$ and $a_3 \neq 0$. According to Theorem 4.2.5, the k-folding map-germ has an A_{k-1}-singularity for k odd and an A_k-singularity when k is even.

c) According to Definition A.1.1, if the origin is an ordinary vertex of γ, then $\kappa(0) \neq 0$, $\kappa'(0) = 0$ and $\kappa''(0) \neq 0$. In particular, $a_2 \neq 0$ and $a_3 = 0$. Then, if k is odd, the k-folding map-germ has an A_{k-1} singularity, as proved in Theorem 4.2.5.

Now consider k even. Since for a generic plane curve only one condition is allowed, as $a_3 = 0$, we have $a_5 \neq 0$. Therefore, it follows by Theorem 4.2.5 that the k-folding map has an A_{k+2} singularity.

d) It follows by Definition A.1.1 that the ordinary inflexion at the origin occurs when $\kappa(0) = 0$ and $\kappa'(0) \neq 0$, that is, if $a_2 = 0$ and $a_3 \neq 0$. In this case, for $3 \nmid k$, according to Theorem 4.2.10, there exist the following possibilities:

- The k-folding map-germ has an E_{6p}-singularity, if $k = 3p+1$;

- The k-folding map-germ has an E_{6p+2}-singularity, if $k = 3p+2$;

Now consider $k = 3p$, for some p. Since the curve is generic and $a_2 = 0$, then we have $a_4 \neq 0$. Therefore, it follows by Theorem 4.2.10 that the k-folding map has an E_{6p}-singularity. □

Example 4.3.2. Consider the curve $\gamma : \mathbb{C},0 \to \mathbb{C}^2,0$ given by $\gamma(t) = (t^2 + t^3, t)$. Since $a_2 \neq 0$ and $a_3 \neq 0$, the origin is neither a vertex nor an inflexion. According to the Theorem 4.3.1, we have $F_4(T) \sim_{\mathscr{A}} (T^2, T^5)$. The following figure represents γ and the real part of F_4.

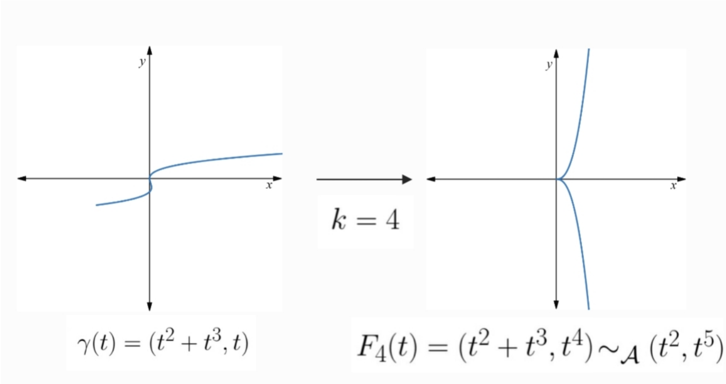

Figure 2 – 4-Folding map-germ on $\gamma(t) = (t^2 + t^3, t)$

4.4 k-folding map-germs and the symmetries of the curve

In this section, we use the singularities of the k-folding map-germ to study the local symmetry of a plane curve, that is, the contact between the curve γ and its reflected curves $R_{\xi^j} \circ \gamma$, $1 \leq j \leq k-1$, where $R_{\xi^j} : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ is given by $R_{\xi^j}(x, y) = (x, \xi^j y)$ and $\xi = e^{2\pi i/k}$. The following figure shows some reflections of the curve γ.

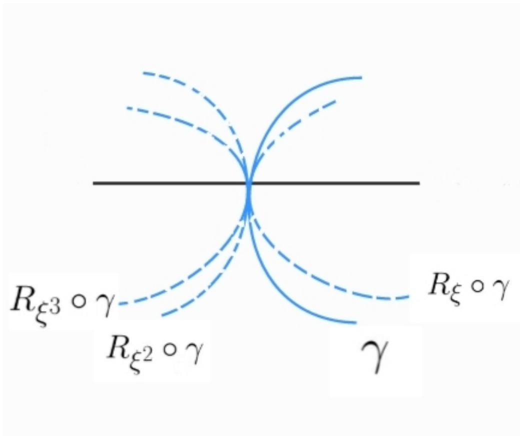

Figure 3 – The curve γ and its reflected curves

In what follows, we take $\gamma : \mathbb{C}, 0 \to \mathbb{C}^2, 0$ with $\gamma'(0)$ orthogonal to $Fix(\omega_k)$. For a definition of contact between two curves see Definition A.2.1.

Theorem 4.4.1. Let $\gamma : \mathbb{C}, 0 \to \mathbb{C}^2, 0$ be a plane curve without an inflexion at the origin. If k is even, then F_k has an A_{k+2p}-singularity if and only if γ and the curve $R_{\xi^{\frac{k}{2}}} \circ \gamma$ have $(2p+3)$-point contact at the origin.

Proof. According to the proof of Theorem 4.2.16, for k even and γ without an inflexion, the k-folding map has an A_{k+2p}-singularity if and only if $a_3 = \ldots = a_{2p+1} = 0$ and $a_{2p+3} \neq 0$. Suppose that this is the case. Then, γ is given by

$$\gamma(t) = (a_2 t^2 + a_4 t^4 + \ldots + a_{2p+2} t^{2p+2} + a_{2p+3} t^{2p+3} + a_{2p+4} t^{2p+4} + a_{2p+5} t^{2p+5} + O(2p+7), t).$$

For $j=\frac{k}{2}$, we have the curve $R_{\xi^j}\circ\gamma=(f(t),\xi^j t)$. We consider the contact between γ and $R_{\xi^j}\circ\gamma$. Let $g:\mathbb{C}^2,0\to\mathbb{C}$, $g(x,y)=x-f(y)$. Then, $\gamma=g^{-1}(0)$ and the contact between the two curves is given by the singularity of

$$\begin{aligned} g(R_{\xi^j}\circ\gamma)(t) &= f(t)-f(\xi^j t) \\ &= a_2(1-\xi^{2j})t^2+\ldots+a_{2p+3}(1-\xi^{(2p+3)j})t^{2p+3}+O(2p+4). \end{aligned}$$

Since $\xi^{2j}=1$, we have $\big(g(R_{\xi^j}\circ\gamma)\big)^{(i)}(0)=0$, for $0\le i\le 2p+2$ and $\big(g(R_{\xi^j}\circ\gamma)\big)^{(2p+3)}(0)\neq 0$. Therefore, γ and $R_{\xi^j}\circ\gamma$ have $(2p+3)$-point contact at the origin.

Now suppose that for $j=\frac{k}{2}$ γ and $R_{\xi^j}\circ\gamma$ have $(2p+3)$-point contact at the origin. Then, $\big(g(R_{\xi^j}\circ\gamma)\big)^{(i)}(0)=0$ for $0\le i\le 2p+2$ and $\big(g(R_{\xi^j}\circ\gamma)\big)^{(2p+3)}(0)\neq 0$. Since $(1-\xi^{3j})\neq 0$, ..., $(1-\xi^{(2p+1)j})\neq 0$, we obtain $a_3=\ldots=a_{2p+1}=0$ and $a_{2p+3}\neq 0$. Therefore, for k even, F_k has an A_{k+2p}-singularity. □

Remark 4.4.2. When γ does not have an inflexion at the origin, γ and $R_{\xi^j}\circ\gamma$ have 2-point contact at the origin, for $j\neq\frac{k}{2}$. Indeed,

$$g(R_{\xi^j}\circ\gamma)(t) \;=\; f(t)-f(\xi^j t)=a_2(1-\xi^{2j})t^2+O(3).$$

Since $(1-\xi^{2j})\neq 0$ and $a_2\neq 0$, the two curves have clearly 2-point contact at the origin.

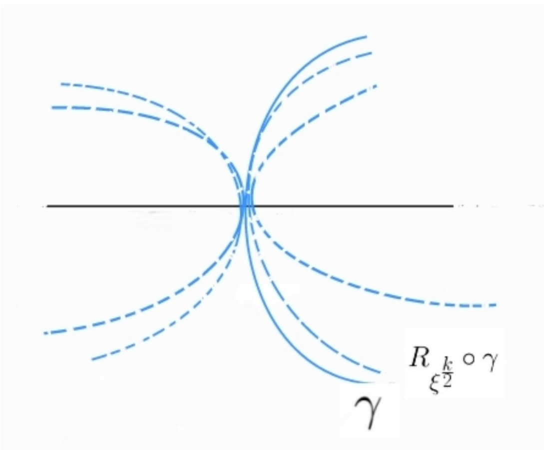

Figure 4 – The curve γ and its reflected curves, when $F_k(T)\sim_A(T^2,T^{k+2p+1})$, with k even

Theorem 4.4.3. Consider $\gamma:\mathbb{C},0\to\mathbb{C}^2,0$ a plane curve with an ordinary inflexion at the origin. If $3\nmid k$, then γ and all the reflected curves $R_{\xi^j}\circ\gamma$ have 3-point contact at the origin, with $1\le j\le k-1$.

Proof. Since γ has an ordinary inflexion at the origin, we have $\gamma(t)=(a_3t^3+a_4t^4+a_5t^5+a_6t^6+O(7),t)$ with $a_3\neq 0$. Consider $g:\mathbb{C}^2\to\mathbb{C}$ given by $g(x,y)=x-f(y)$. Then,

$$\begin{aligned} \big(g(R_{\xi^j}\circ\gamma)\big)(t) &= f(t)-f(\xi^j t) \\ &= a_3(1-\xi^{3j})t^3+a_4(1-\xi^{4j})t^4+a_5(1-\xi^{5j})t^5+O(6). \end{aligned}$$

Observe that $\xi^{3j} = 1$ if and only if $k \mid 3j$. If $3 \nmid k$, in order to have $\xi^{3j} = 1$ we need that $k \mid j$. But that does not hold since $1 \leq j \leq k-1$. Therefore, γ and the reflected curves $R_{\xi^j} \circ \gamma$, with $1 \leq j \leq k-1$, have 3-point contact at the origin. $\square$

Theorem 4.4.4. Suppose that $3 \mid k$, so that $k = 3m$ for some m and suppose that γ has an ordinary inflexion at the origin. The k-folding map-germ is $\mathscr{A}$-equivalent to $(t^3, t^{3(m+j)+1} + t^{3p+2})$, with $m+j \leq p < 2(m+j)-1$ or $(t^3, t^{3(m+j)+1})$, for some j, if and only if γ and $R_{\xi^m} \circ \gamma$ have $(3j+4)$-point contact at the origin.

Proof. According to the proof of Theorem 4.2.10, if $3 \mid k$, the k-folding map-germ is $\mathscr{A}$-equivalent to $(t^3, t^{3(m+j)+1} + t^{3p+2})$, with $m+j \leq p < 2(m+j)-1$ or $(t^3, t^{3(m+j)+1})$, for some j, if and only if the coefficients $a_4 = a_5 = a_7 = \ldots = a_{3j+2} = 0$ and $a_{3j+4} \neq 0$. Suppose that this is the case. Then, the curve γ is given by

$$\gamma(t) = (a_3t^3 + a_6t^6 + \ldots + a_{3j+3}t^{3j+3} + a_{3j+4}t^{3j+4} + O(3j+5), t).$$

We have

$$\begin{aligned} \big(g(R_{\xi^m} \circ \gamma)\big)(t) &= f(t) - f(\xi^m t) \\ &= a_3(1-\xi^{3m})t^3 + a_6t^6(1-\xi^{6m})t^6 + \ldots + a_{3j+4}(1-\xi^{(3j+4)m})t^{3j+4} + O(3j+5) \end{aligned}$$

Since $k = 3m$, we obtain $\big(g(R_{\xi^m} \circ \gamma)\big)^{(n)}(0) = 0$ for $0 \leq n \leq 3j+3$ and $\big(g(R_{\xi^m} \circ \gamma)\big)^{(3j+4)}(0) \neq 0$. Hence, γ and the curve $R_{\xi^m} \circ \gamma$ have $(3j+4)$-point contact at the origin.

Now suppose that the curves γ and $R_{\xi^m} \circ \gamma$ have $(3j+4)$-point contact at the origin. Then, $\big(g(R_{\xi^m} \circ \gamma)\big)^{(n)}(0) = 0$ for $0 \leq n \leq 3j+3$ and $\big(g(R_{\xi^m} \circ \gamma)\big)^{(3j+4)}(0) \neq 0$. Since

$$\begin{aligned} \big(g(R_{\xi^m} \circ \gamma)\big)(t) &= f(t) - f(\xi^m t) \\ &= a_3(1-\xi^{3m})t^3 + a_4(1-\xi^{4m})t^4 + \ldots + a_{3j+4}(1-\xi^{(3j+4)m})t^{3j+4} + O(3j+5), \end{aligned}$$

we obtain $a_4 = a_5 = a_7 = \ldots = a_{3j+2} = 0$ and $a_{3j+4} \neq 0$. $\square$

Theorem 4.4.5. Suppose that $3 \mid k$, so that $k = 3m$ for some m and suppose that γ has an ordinary inflexion at the origin. The k-folding map-germ is $\mathscr{A}$-equivalent to $(t^3, t^{3n+1} + t^{3(m+j)+2})$, with $m+j < n \leq 2(m+j)-1$ or to $(t^3, t^{3(m+j)+2})$, for some j, if and only if γ and the curve $R_{\xi^m} \circ \gamma$ have $(3j+5)$-point contact at the origin.

Proof. According to the proof of Theorem 4.2.10, for $3 \mid k$, the k-folding map is $\mathscr{A}$-equivalent to $(t^3, t^{3n+1} + t^{3(m+j)+2})$, with $m+j < n \leq 2(m+j)-1$ or $(t^3, t^{3(m+j)+2})$, for some j, if and only if the coefficients of f $a_4 = a_5 = a_7 = \ldots = a_{3j+4} = 0$ and $a_{3j+5} \neq 0$. Suppose that this is the case. Then, the curve γ is given by

$$\gamma(t) = (a_3t^3 + a_6t^6 + \ldots + a_{3j+5}t^{3j+5} + O(3j+6), t).$$

Consider $g:\mathbb{C},0\to\mathbb{C}^2,0$, $g(x,y)=x-f(y)$. Then, $\gamma=g^{-1}(0)$ and

$$\begin{aligned}(g(R_{\xi^m}\circ\gamma))(t) &= f(t)-f(\xi^m t)\\ &= a_3(1-\xi^{3m})t^3+a_6(1-\xi^{6m})t^6+\ldots+a_{3j+5}(1-\xi^{(3j+5)m})t^{3j+5}+O(3j+6).\end{aligned}$$

Since $k=3m$, $\big(g(R_{\xi^m}\circ\gamma)\big)^{(n)}(0)=0$, for $0\leq n\leq 3j+4$ and $\big(g(R_{\xi^m}\circ\gamma)\big)^{(3j+5)}(0)\neq 0$. Therefore, γ and the curve $R_{\xi^m}\circ\gamma$ have $(3j+5)$-point contact at the origin.

Conversely, suppose that γ and $R_{\xi^m}\circ\gamma$ have $(3j+5)$-point contact at the origin, for some j. Then, $\big(g(R_{\xi^m}\circ\gamma)\big)^{(n)}(0)=0$, for $0\leq n\leq 3j+4$ and $\big(g(R_{\xi^m}\circ\gamma)\big)^{(3j+5)}(0)\neq 0$. Since

$$\begin{aligned}\big(g(R_{\xi^m}\circ\gamma)\big)(t) &= f(t)-f(\xi^m t)\\ &= a_3(1-\xi^{3m})t^3+\ldots+a_{3j+5}(1-\xi^{(3j+5)m})t^{3j+5}+O(3j+6),\end{aligned}$$

we obtain $a_4=a_5=a_7=\ldots=a_{3j+4}=0$ and $a_{3j+5}\neq 0$. □

Theorem 4.4.6. Let $\gamma:\mathbb{C},0\to\mathbb{C}^2,0$ be a plane curve with a second order inflexion at the origin, that is, $a_2=a_3=0$ and $a_4\neq 0$. Then, the relationship between the local singularities of the 4-folding map-germ and the order of contact of γ and the reflected curves $R_{\xi^j}\circ\gamma$, with $1\leq j\leq 3$ is given by:

a) If the 4-folding map-germ is $\mathscr{A}$-equivalent to (t^4,t^5+t^7) or (t^4,t^5), then the curve γ and all reflected curves $R_{\xi^j}\circ\gamma$, $1\leq j\leq 3$ have 5-point contact at the origin.

b) If the 4-folding map-germ is $\mathscr{A}$-equivalent to (t^4,t^7), (t^4,t^7+t^9) or (t^4,t^7+t^{13}), then γ and all reflected curves $R_{\xi^j}\circ\gamma$, with $1\leq j\leq 3$ have 7-point contact at the origin.

Proof. a) For $g:\mathbb{C}^2\to\mathbb{C}; g(x,y)=x-f(y)$, we obtain $\gamma=g^{-1}(0)$ and the contact between this curve and $R_{\xi^j}\circ\gamma$, $1\leq j\leq 3$ is given by the derivatives of

$$\big(g(R_{\xi^j}\circ\gamma)\big)(t)=f(t)-f(\xi^j t)=a_4(1-\xi^{4j})t^4+a_5(1-\xi^{5j})t^5+a_6(1-\xi^{6j})t^{6j}+O(7).$$

If the 4-folding map-germ is $\mathscr{A}$-equivalent to (t^4,t^5) or (t^4,t^5+t^7), then $a_5\neq 0$.

Since $\xi^{4j}=1$, for all j, we obtain $(1-\xi^{4j})=0$ for all j. Hence,

$$\big(g(R_{\xi^j}\circ\gamma)\big)^{(i)}(0)=0, 0\leq i\leq 4 \text{ and } \big(g(R_{\xi^j}\circ\gamma)\big)^{(5)}(0)\neq 0, \text{ with } 1\leq j\leq 3$$

and the result follows.

b) Consider $g:\mathbb{C}\to\mathbb{C}^2; g(x,y)=x-f(y)$. Then, $\gamma=g^{-1}(0)$. If the 4-folding map-germ is $\mathscr{A}$-equivalent to (t^4,t^7+t^9), (t^4,t^7+t^{13}) or (t^4,t^7), then $a_5=a_6=0$ and $a_7\neq 0$. Moreover, the contact between γ and the reflected curves $R_{\xi^j}\circ\gamma$, with $1\leq j\leq 3$, is given by the derivatives of

$$\begin{aligned}\big(g(R_{\xi^j}\circ\gamma)\big) &= f(t)-f(\xi^j t)\\ &= a_4(1-\xi^{4j})t^4+a_5(1-\xi^{5j})t^5+a_6(1-\xi^{6j})t^{6j}+a_7(1-\xi^{7j})t^7+O(8).\end{aligned}$$

and for $a_5 = a_6 = 0$, $a_7 \neq 0$, we obtain

$$\big(g(R_{\xi^j} \circ \gamma)\big)^{(i)}(0) = 0, 0 \leq i \leq 6 \text{ and } \big(g(R_{\xi^j} \circ \gamma)\big)^{(7)}(0) \neq 0, \text{ with } 1 \leq j \leq 3$$

and the result follows. □

Theorem 4.4.7. Let $\gamma : \mathbb{C}, 0 \to \mathbb{C}^2, 0$ be a plane curve with a second order inflexion at the origin. If the 5-folding map-germ is $\mathscr{A}$-equivalent to $(t^4, t^5 + t^7)$ or (t^4, t^5), then γ and all the curves $R_{\xi^j} \circ \gamma$ have 4-point contact, for $1 \leq j \leq 4$.

Proof. Denoting $\gamma = g^{-1}(0)$, with $g : \mathbb{C} \to \mathbb{C}^2, g(x,y) = x - f(y)$. If the 5-folding map is $\mathscr{A}$-equivalent to $(t^4, t^5 + t^7)$ or (t^4, t^5), then $a_4 \neq 0$.

The contact between γ and $R_{\xi^j} \circ \gamma$ is given by the derivatives of

$$\big(g(R_{\xi^j} \circ \gamma)\big)(t) = a_4(1 - \xi^{4j})t^4 + a_5(1 - \xi^{5j})t^5 + a_6(1 - \xi^{6j})t^6 + a_7(1 - \xi^{7j})t^7 + O(8),$$

and the result follows. □

CHAPTER

5

CONCLUSION

With this work, we studied k-folding map-germs on plane curves, generalising the case $k = 2$. Also, we showed that all k-folding map-germs for $k \geq 3$ can have all the possible $\mathscr{A}$-simple singularities and those singularities reveal information about the hidden symmetries of a plane curve. Furthermore, we showed that inflexions and vertices of a generic plane curve can be used to describe those singularities of a k-folding map-germ.

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APPENDIX

A

In this appendix, we enunciate some results in Singularity Theory applied to the geometry of curves. The main reference is (BRUCE; GIBLIN, 1992).

A.1 Special points on plane curves

Definition A.1.1. Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $\kappa(t)$ the curvature of γ at t. Given $t_0 \in I$, we say that

(a) $\gamma(t_0)$ is a vertex if $\kappa(t_0) \neq 0$ and $\kappa'(t_0) = 0$. The point $\gamma(t_0)$ is an ordinary vertex if furthermore $\kappa''(t_0) \neq 0$ and a higher vertex if $\kappa''(t_0) = 0$.

(b) $\gamma(t_0)$ is an inflexion if $\kappa(t_0) = 0$. The point $\gamma(t_0)$ is an ordinary inflexion if furthermore $\kappa'(t_0) \neq 0$ and a higher inflexion if $\kappa'(t_0) = 0$.

A.2 Contact between curves

Definition A.2.1. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve, with $\gamma(t_0) = 0$ and let $F : \mathbb{R}^n \to \mathbb{R}^p$ be submersion at the origin. We say that γ and $F^{-1}(0)$ have k-point contact at $t = t_0$ if the function $g = F \circ \gamma$ satisfies

$$g(t_0) = g'(t_0) = ... = g^{(k-1)}(t_0) = 0 \text{ and } g^{(k)}(t_0) \neq 0.$$

The contact between γ and lines or circles can be studied analyzing the derivatives of two special functions, respectively: the height functions and the distance-squared functions.

Definition A.2.2. Let $u \in \mathbb{R}^n$. The distance-squared function on γ from u is the function $f_d : I \to \mathbb{R}$ defined by

$$f_d(t) = ||\gamma(t) - u||^2 = (\gamma(t) - u) \cdot (\gamma(t) - u)$$

Definition A.2.3. Let $u \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. The height function on γ in the direction u is the function $f_h : I \to \mathbb{R}$ defined by

$$f_h(t) = \gamma(t) \cdot u.$$

Note that $\gamma(t)\cdot u$ is the distance from $\gamma(t)$ to the hyperplane through 0 perpendicular to u.

Proposition A.2.4. Let $n=2$ in Definition A.2.3 and Definition A.2.2. We say that γ has k-point contact at $t=t_0$ with the circle centered u and passing through $\gamma(t_0)$ if and only if the distance-squared function f_d on γ from u satisfies $f_d^{(i)}(t_0)=0, i=1,\ldots,k-1$ and $f_d^{(k)}(t_0)\neq 0$. In particular, γ has an ordinary (resp. higher) vertex at $t=t_0$ if and only if $k=4$ (resp. some $k\geq 5$), where $u=\gamma(t_0)+\frac{1}{\kappa(t_0)}N(t_0)$.

The curve γ has k-point contact at $t=t_0$ with its tangent line at t_0 if and only if the height function f_h on γ in the direction u perpendicular to the tangent vector $T(t_0)$ satisfies $f_h^{(i)}(t_0)=0, i=1,\ldots,k-1$ and $f_h^{(k)}(t_0)\neq 0$. In particular, γ has an ordinary (resp. higher) inflexion at $t=t_0$ if and only $k=3$ (resp. some $k\geq 4$).

A.3 A word on genericity

Definition A.3.1. We say that a property P is dense or holds for a dense set of (regular) plane curves $\gamma: I\to\mathbb{R}^2$ if there exists a neighbourhood U of 0 in some Euclidean space $\mathbb{R}^N$ and a family of regular plane curves $\bar{\gamma}: I\times U\to\mathbb{R}^2$ such that:

- (a) $\bar{\gamma}(t,0)=\gamma(t)$;
- (b) if $\{u_n\}$ is a sequence in U with $lim_{n\to\infty}u_n=0$, then the property P holds for the sequence of curves γ_n, defined by $\gamma_n(t)=\bar{\gamma}(t,u_n)$.

Definition A.3.2. The property P is open or holds for an open set of (regular) plane curves if given a curve $\gamma: I\to\mathbb{R}^2$ with property P and a family $\bar{\gamma}: I\times U\to\mathbb{R}^2$ of (regular) curves $\bar{\gamma}_u$, the property P holds for all curves $\bar{\gamma}_u$ with u in some neighbourhood U of 0.

Definition A.3.3. A property P is said to be generic or to hold for a generic set of curves if it is both dense and open.

Proposition A.3.4. In an open and dense set of regular curves $\gamma: S^1\to\mathbb{R}^2$ there exist only finitely many ordinary inflexions and ordinary vertices and no higher inflexions or higher vertices, that is, these properties are generic.

Proof. See [(BRUCE; GIBLIN, 1992)]. □

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