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The Banach-Mazur game and products of Baire spaces

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O jogo de Banach-Mazur e produto de espaços de Baire

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *VERSÃO REVISADA*

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ABSTRACT

MEDINA, G. A. **The Banach-Mazur game and products of Baire spaces**. 2021. 128 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

In this work we study Baire spaces and analyze the problem of product of Baire spaces. Then we present some conditions using the Banach-Mazur game to show that the Baire property is preserved in the product. Then we analyze the difference of the infinite product of Baire spaces, between the box product and Tychonoff product. We also present a multiboard version for this problem. Finally we present some open problems regarding the product of Baire spaces.

Keywords: Baire spaces, Banach-Mazur game, Product of Baire spaces, Multiboard topological games.

RESUMO

MEDINA, G. A. **O jogo de Banach-Mazur e produto de espaços de Baire**. 2021. 128 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Neste trabalho, estudamos os espaços de Baire e analisamos o problema do produto de espaços de Baire. Logo, apresentamos algumas condições usando o jogo Banach-Mazur para mostrar que o produto de espaços Baire é preservado. Analisamos a diferença do produto infinito dos espaços de Baire, entre o produto box e produto Tychonoff. Também apresentamos uma versão de um jogo topológico com vários tabuleiros para esse problema. Finalmente, apresentamos alguns problemas em aberto relacionados ao produto dos espaços de Baire.

Palavras-chave: Espaços de Baire, jogo de Banach-Mazur, Produto de espaços de Baire, Jogos topológicos com múltiplos tabuleiros.

CONTENTS

INTRODUCTION

A topological space is a Baire space provided that countable collections of dense open subsets have a dense intersection. Baire spaces constitute an important class in various branches of mathematics, this is the case in such well-known theorems as the Closed Graph Theorem, the Open Mapping Theorem and the Uniform Boundedness Theorem. In a sense, the Baire property is one of the weakest forms of topological completeness.

The problem of whether a product of a family of Baire spaces is Baire is an old one and is also well known that the answer to the problem is negative, even with fairly strong hypothesis. Indeed:

- In 1961, assuming the Continuum Hypothesis (CH), Oxtoby constructed the first example of a Baire space whose square is not Baire.
- In 1974, Krom, showed that if there exists a Baire space whose square is not Baire, then there exists a Baire metric space whose square is not Baire.
- Later, in 1976, using forcing techniques, Paul Cohen showed that only the usual axioms of Set Theory are needed to prove the existence of Baire spaces whose product is not Baire. That is, it is not necessary to add any set theoretic hypothesis to be able to construct two Baire spaces whose product is not Baire.
- Also, in 1986, Jan van Mill and Roman Pol showed that there are two normed Baire spaces whose product is not Baire.

However, there are several cases when products (finite, countable or arbitrary) of Baire spaces are again Baire. Some cases can be described in terms of games.

The Banach-Mazur game is the first infinite positional game of perfect information studied by mathematicians. The game was proposed in 1935 by the Polish mathematician Stanislaw Mazur and recorded in the Scottish Book (MAULDIN, 2015). The game, its solution and its importance went far beyond the Baire category classification. In fact, Baire spaces can be characterized via the Banach–Mazur game, then it is not surprising that topological games have been applied to attack the Baire product problem.

Therefore, one of our objectives in this work is, in addition to presenting results on the Baire product, to see when the product of Baire spaces is still Baire, giving conditions with the Banach-Mazur game (or some variation of it) over the spaces.

For this reason, we have structured the text as described below.

In the first chapter, we briefly review basic results of general topology, set theory and forcing. Along with that, we present the Baire spaces and basic results about them.

In the second chapter, we introduce the Banach-Mazur game and some of its applications, we also present some of its modifications.

In the third chapter, we present the problem of product of Baire spaces. In the first section, we present the examples of Cohen, Krom and Fleissner. These are some counterexamples of Baire spaces whose product is not Baire. In the second section, we present results of when the finite product of some Baire spaces is Baire. In the third section, we present the difference of phenomenon of being Baire in the infinite product (box product and Tychonoff product) of Baire spaces.

In the fourth chapter, we introduce multiboard games, these emerge as a possible solution to the problem of the infinite product of Baire spaces and we present some of its variations.

Finally, in the fifth chapter, we present some open problems related to the problem of the product of Baire spaces.

CHAPTER

1

PRELIMINARY RESULTS

In this chapter we will introduce the basic tools of topology, set theory and forcing to understand the Banach-Mazur game and some of its applications. We are going to start with some basic results in topology. For this part we follow the books of (WILLARD, 1970) and (WALDMANN, 2014) as main references.

1.1 Topology

1.1.1 Some definitions and basic facts

For this section we fix X a topological space.

Definition 1.1 (π-base)**.** A family $\mathscr{B}$ of non-empty open subsets of a topological space X is said to be a **π-base** (or pseudo-base) if for each non-empty open subset U of X there is an element $V \in \mathscr{B}$ such that $V \subseteq U$.

Note that every base of a topological space is a π-base.

Definition 1.2. A π-base $\mathscr{B}$ is called **locally countable** if each member of $\mathscr{B}$ contains only countably many members of $\mathscr{B}$.

Note that every second countable space has a locally countable π-base.

Definition 1.3. A subset A of X is a G_δ-set if it is a countable intersection of open sets and it is an F_σ if it is a countable union of closed sets.

Proposition 1.4.

(i) The complement of a G_δ is an F_σ and vice versa.

(ii) An F_σ can be written as the union of an **increasing** sequence $F_1 \subseteq F_2 \subseteq \cdots$ of closed sets. (Hence, a G_δ can be written as a decreasing intersection of open sets.)

(iii) A closed set in a metric space is a G_δ (hence, an open set is an F_σ.)

1.1.1.1 The Continuum Hypothesis for G_δ Sets

In this section we will show that G_δ sets in the real line satisfy the kind of continuum hypothesis in the sense that every G_δ set is either countable or has cardinality $\mathfrak{c}$. This sets will be of great importance later because we will see how the Banach-Mazur game works in the real line, specifically with this type of sets.

Definition 1.5. A set $A \subseteq \mathbb{R}$ is called

- **closed** if every limit point[1] of A is in A, i.e. if $A' \subseteq A$;
- **dense-in-itself** if every point of A is a limit point of A, i.e. if $A \subseteq A'$;
- **perfect** if it is both closed and dense-in.itself, i.e., if $A = A'$.

Definition 1.6. A family $J = \langle J_u : u \in \bigcup_{n\in\omega} 2^n \rangle$ is called a **Cantor system** if for each $u \in \bigcup_{n\in\omega} 2^n$:

1. J_u is a bounded proper closed interval, i.e., $J_u = [a,b]$ for some $a < b$;
2. $J_{u^\frown 0}, J_{u^\frown 1} \subseteq J_u$;
3. $J_{u^\frown 0} \cap J_{u^\frown 1} = \emptyset$;
4. For each $b \in 2^\omega$,
$$\lim_{n\to\infty} \ell(J_{b\restriction n}) = 0,$$
where $\ell(I)$ denotes the length of the interval I.

Definition 1.7. The set generated by the Cantor system $J = \langle J_u : u \in \bigcup_{n\in\omega} 2^n \rangle$ is the set P of real numbers defined by the condition:

$$x \in P \text{ if and only if there exists } b \in 2^\omega \text{ such that } x \in \bigcap_{n\in\omega} J_{b\restriction n}$$

Definition 1.8 (Generalized Cantor Sets)**.** A set is called a generalized Cantor set (or a Cantor-like set) if it is generated by some Cantor system.

Theorem 1.9. Every non-empty dense-in-itself G_δ set E contains a generalized Cantor set and so there is an injective $\varphi : 2^\omega \to E$ with $\varphi(2^\omega)$ being a perfect set. In particular, every non-empty dense-in-itself G_δ set has cardinality $\mathfrak{c}$.

[1] $x \in \mathbb{R}$ is a limit point of A if for each $\varepsilon > 0$, $(B_\varepsilon^{(x)} \setminus \{x\}) \cap A \neq \emptyset$, where $B_\varepsilon^{(x)} = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$.

Proof. The complete proof of this theorem can be found in (DASGUPTA, 2014), Theorem 1048. $\square$

Corollary 1.10. A non-empty perfect set in $\mathbb{R}$ has cardinality $\mathfrak{c}$.

Corollary 1.11. The set $\mathbb{Q}$ of rational numbers is not a G_δ set, and hence the set of irrational numbers is not an F_σ set.

We now have the result that the G_δ sets, and therefore the closed sets, satisfy the continuum hypothesis.

Corollary 1.12. Every uncountable G_δ set contains a generalized Cantor set and hence has cardinality $\mathfrak{c}$.

Proof. The complete proof of this corollary can be found in (DASGUPTA, 2014), Corollary 1051. $\square$

Corollary 1.13. Any uncountable closed subset of $\mathbb{R}$ contains a generalized Cantor set and hence has cardinality $\mathfrak{c}$.

Note that a set contains a generalized Cantor set if and only if it contains a non-empty perfect set. Hence we make the following definition.

Definition 1.14 (The Perfect Set Property)**.** A set is said to have the perfect set property if it is either countable or contains a perfect set (or equivalently, contains a generalized Cantor set). A collection of sets is said to have the perfect set property if every set in the family has the perfect set property.

For example closed sets and G_δ sets have the perfect set property.

1.1.1.2 Metric spaces and G_δ-sets

Definition 1.15. A sequence (x_n) in a metric space (M,d) is **Cauchy** if for each $\varepsilon > 0$, there is some positive integer N such that $d(x_n,x_m) < \varepsilon$ whenever $m,n \geq N$.

Definition 1.16. A metric space (M,d) is complete if every Cauchy sequence in M converges in M. We also say d is a complete metric for M. A topological space X is completely metrizable if there is a complete metric for X which generates its topology. Thus X is completely metrizable if it is homeomorphic to some complete metric space.

Note that while completeness is a property of metric spaces, complete metrizability is a property of topological spaces. For example, $]0,1[$ with the usual metric is not a complete metric space (consider the sequence $(\frac{1}{n})$), but is completely metrizable since it is homeomorphic to the complete space $\mathbb{R}$.

Definition 1.17. Metric spaces (M,d) and (N,d') are **isometric** if there is a one to one function f of M onto N such that $d'(f(x),f(y)) = d(x,y)$, for all $x,y \in M$. The mapping f is called an isometry.

A well known result of metric spaces mentions that every metric space can be completed in such a way that it is dense in the new space.

Theorem 1.18. Every metric space M can be isometrically embedded as a dense subset of a complete metric space. The resulting completion is unique up to isometry and is called the completion of M.

Proof. A complete proof of the theorem can be found in (WILLARD, 1970), Theorem 24.4. $\square$

We are now ready for the subspace theorem. Both are classical results from the 1920's. The first part is due to Alexandroff, the second to Mazurkiewicz. The full proof of these two theorems can be found in (WILLARD, 1970), Theorems 24.12 and 24.13.

Theorem 1.19. A G_δ-set in a complete metric space is completely metrizable. Conversely, if a subset A of a metric space M is completely metrizable, then it is a G_δ-set.

Theorem 1.20. For a metric space X the following are equivalent:

(i) X is completely metrizable,

(ii) X is a G_δ in its completion $\hat{X}$.

1.1.2 A little bit of Descriptive set theory

We begin this section by studying some new topological spaces and special sets, which can help us with examples for the Banach-Mazur game. For this part we follow the books of (KECHRIS, 1995) and (SRIVASTAVA, 1998).

1.1.2.1 Polish spaces

For the classic examples of Polish spaces we will need the following :

Proposition 1.21. A metrizable space is second countable if and only if it is separable.

Proof. A proof of this proposition can be found in (WILLARD, 1970), Theorem 16.11. □

Proposition 1.22. The product of any countable family of metrizable (resp. completely metrizable) spaces is a metrizable (resp. completely metrizable) space.

Proof. The complete proof of this proposition can be found in (WILLARD, 1970), Theorem 24.11. □

Definition 1.23 (Polish space)**.** A separable completely metrizable space is called **Polish.**

Proposition 1.24.

i) A closed subspace of a Polish space is Polish.

ii) The product of a countable sequence of Polish spaces is Polish.

Proof. For the first part, remember that a closed subspace contained in a complete metric space is complete. For the second part, let E be the product of a countable family $(E_n)_{n\in\omega}$ of Polish spaces. By Proposition 1.22, E is completely metrizable. Furthermore, if $\mathscr{B}_n$ is a countable basis for E_n, the topology of E is generated by the countable basis consisting of the finite intersections of open sets of the form $\prod_{n\in\omega} X_n$, where $X_n = E_n$ except for a finite number of indices, for which $X_n \in \mathscr{B}_n$. Therefore E is Polish. □

Example 1.

1) $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^\omega, \mathbb{C}^\omega$ are Polish.

2) The space A^ω, viewed as the product of infinitely many copies of A with the discrete topology, is completely metrizable and if A is countable it is Polish.

3) Of particular importance are the cases $A = 2 = \{0,1\}$ and $A = \omega$. We call $\mathscr{C} = 2^\omega$ the **Cantor space** and ω^ω the Baire space.

Theorem 1.25. Let X be a Polish space. Then there is a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f : F \to X$. In particular, if X is non-empty, there is a continuous surjection $g : \omega^\omega \to X$ extending f.

Proof. A proof of this theorem can be found in (KECHRIS, 1995), Theorem 7.9. □

Now we will give a characterization of the Baire space ω^ω.

Definition 1.26. A topological space X is **connected** if there is no partition $X = U \cup V, U \cap V = \emptyset$ where U, V are non-empty open sets. Or equivalently, if the only **clopen** (i.e., open and closed) sets are $\emptyset$ and X.

Definition 1.27. A topological space X is **zero-dimensional** if it is Hausdorff and has a basis consisting of clopen sets.

For example, the space A^ω is zero-dimensional since the standard basis $([s])_{s \in A^{<\omega}}$ consists of clopen sets.

Definition 1.28. A **Lusin scheme** on a set X is a family $\{A_s\}_{s \in \omega^{<\omega}}$ of subsets of X such that

i) $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$, if $s \in \omega^{<\omega}$, $i \neq j$ in ω;

ii) $A_{s^\frown i} \subseteq A_s$, if $s \in \omega^{<\omega}$, $i \in \omega$.

Definition 1.29. If (X, d) is a metric space and $\{A_s\}_{s \in \omega^{<\omega}}$ is a Lusin scheme on X, we say that $\{A_s\}_{s \in \omega^{<\omega}}$ has a **vanishing diameter** if $\lim_{n \to \infty} diam(A_{x \restriction n}) = 0$, for all $x \in \omega^\omega$. In this case if

$$D = \{x \in \omega^\omega : \bigcap_{n \in \omega} A_{x \restriction n} \neq \emptyset\},$$

define

$$\begin{array}{rccl} f & : & D & \longrightarrow & X \\ & & x & \longrightarrow & \bigcap_{n \in \omega} A_{x \restriction n} = \{f(x)\}. \end{array}$$

We call f the **associated map**.

Proposition 1.30. Let $\{A_s\}_{s \in \omega^{<\omega}}$ be a Lusin scheme on a metric space (X, d) that has vanishing diameter. If $f : D \to X$ is the associated map, then

i) f is injective and continuous.

ii) If (X, d) is complete and each A_s is closed, then D is closed.

iii) If each A_s is open then f is an embedding.

Proof. A complete proof of this proposition can be found in (KECHRIS, 1995), Proposition 7.6. □

Theorem 1.31 (Alexandrov-Urysohn)**.** The Baire space ω^ω is the unique, up to homeomorphism, non-empty Polish zero-dimensional space for which all compact subsets have empty interior.

Proof. A proof of this theorem can be found in (KECHRIS, 1995), Theorem 7.7. $\square$

1.1.2.2 Borel sets

Definition 1.32. An **algebra** on a set X is a collection $\mathscr{A}$ of subsets of X such that

(i) $X \in \mathscr{A}$;

(ii) whenever A belongs to $\mathscr{A}$ so does $X \setminus A$; i.e., $\mathscr{A}$ is closed under complements;

(iii) $\mathscr{A}$ is closed under finite unions.

Definition 1.33. An algebra closed under countable unions is called a **σ-algebra** on X.

Note that $\emptyset \in \mathscr{A}$ if $\mathscr{A}$ is an algebra and the intersection of a non-empty family of σ-algebras on a set X is a σ-algebra on X.

Definition 1.34. A **measurable space** is an ordered pair $(X, \mathscr{A})$ where X is a set and $\mathscr{A}$ a σ-algebra on X. Sets in $\mathscr{A}$ are called **measurable**.

Definition 1.35. Let $\mathscr{G}$ be any family of subsets of a set X. Let $\mathscr{S}$ be the family of all σ-algebras containing $\mathscr{G}$. Note that $\mathscr{S}$ contains the discrete σ-algebra $\mathscr{P}(X)$ and hence is not empty. Let $\sigma(\mathscr{G})$ be the intersection of all members of $\mathscr{S}$. Then $\sigma(\mathscr{G})$ is the smallest σ-algebra on X containing $\mathscr{G}$. $\sigma(\mathscr{G})$ is called the σ-algebra **generated** by $\mathscr{G}$ or $\mathscr{G}$ is a **generator** of $\sigma(\mathscr{G})$.

Let $\mathscr{D} \subseteq \mathscr{P}(X)$ and $Y \subseteq X$. We set

$$\mathscr{D}|_Y = \{B \cap Y : B \in \mathscr{D}\}.$$

Let $(X, \mathscr{B})$ be a measurable space and $Y \subseteq X$. Then $\mathscr{B}|_Y$ is a σ-algebra on Y, called the **trace** of $\mathscr{B}$ on Y.

If X is any metric space, or more generaly any topological space, the σ-algebra generated by the family of open sets in X is called the **Borel σ-algebra** on X and is denoted by $\mathscr{B}_X$. Its members are called **Borel sets**.

Definition 1.36. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A map $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is called **measurable** if $f^{-1}(B) \in \mathscr{A}$ for every $B \in \mathscr{B}$.

Definition 1.37. A measurable function $f : (X, \mathscr{B}_X) \to (Y, \mathscr{B}_Y)$ is called **Borel measurable**, or simply **Borel**.

Proposition 1.38. If X and Y are topological spaces, then every continuous function $f : X \to Y$ is Borel.

Proof. Remember that f is continuous iff $f^{-1}(U)$ is open in X for every open $U \subseteq Y$. $\square$

1.1.2.2.1 The Hierarchy of Borel sets

Let X be a set and $\mathscr{F}$ a family of subsets of X. We put

$$\mathscr{F}_\sigma = \left\{ \bigcup_{n\in\omega} A_n : A_n \in \mathscr{F} \right\}$$

and

$$\mathscr{F}_\delta = \left\{ \bigcap_{n\in\omega} A_n : A_n \in \mathscr{F} \right\}$$

So, $\mathscr{F}_\sigma(\mathscr{F}_\delta)$ is the family of countable unions (resp. countable intersections) of sets in $\mathscr{F}$. The family of finite unions (finite intersections) of sets in $\mathscr{F}$ will be denoted by $\mathscr{F}_s$ (resp. $\mathscr{F}_\delta$). Finally, $\neg\mathscr{F} = \{A \subseteq X : X \setminus A \in \mathscr{F}\}$. Note that $\mathscr{F}_s \subseteq \mathscr{F}_\sigma$, $\mathscr{F}_d \subseteq \mathscr{F}_\delta$, $\mathscr{F}_\sigma = \neg((\neg\mathscr{F})_\delta)$ and $\mathscr{F}_\delta = \neg((\neg\mathscr{F})_\sigma)$.

Let X be a metrizable space. For ordinals α, $\alpha < \omega_1$, we define the following classes by transfinite induction:

$$\Sigma_1^0(X) = \{U \subseteq X : U \text{ open}\}$$

$$\Pi_1^0(X) = \{F \subseteq X : F \text{ closed}\}$$

for $1 < \alpha < \omega_1$,

$$\Sigma_\alpha^0(X) = \left(\bigcup_{\beta<\alpha} \Pi_\beta^0(X) \right)_\sigma$$

$$\Pi_\alpha^0(X) = \left(\bigcup_{\beta<\alpha} \Sigma_\beta^0(X) \right)_\delta$$

Finally, for every $1 \leq \alpha < \omega_1$,

$$\Delta_\alpha^0(X) = \Sigma_\alpha^0(X) \cap \Pi_\alpha^0(X)$$

Note that

- $\Delta_1^0(X)$ is the family of all **clopen** subsets of X;
- $\Sigma_2^0(X)$ is the set of all F_σ subsets of X; and
- $\Pi_2^0(X)$ is the set of all G_δ sets in X.

The families $\Sigma^0_\alpha(X), \Pi^0_\alpha(X)$ and $\Delta^0_\alpha(X)$ are called **additive**, **multiplicative**, and **ambiguous classes** respectively. A set $A \in \Sigma^0_\alpha(X)$ is called an **additive class** α set. **Multiplicative class** α sets and **ambiguous class** α sets are similarly defined.

Some elementary facts.

(i) Additive classes are closed under countable unions, and multiplicative ones under countable intersection.

(ii) All the additive, multiplicative, and ambiguous classes are closed under finite unions and finite intersections.

(iii) For all $1 \leq \alpha < \omega_1$,
$$\Sigma^0_\alpha = \neg\Pi^0_\alpha \text{ (equivalently, } \Pi^0_\alpha = \neg\Sigma^0_\alpha\text{)}$$

(iv) For $\alpha \geq 1$, Δ^0_α is an algebra.

1.1.2.3 Analytic sets

Definition 1.39. Let X be a Polish space. A set $A \subseteq X$ is called **analytic** if there is a Polish space Y and a continuous function $f : Y \to X$ with $f(Y) = A$.

The empty set is analytic, by taking $Y = \emptyset$.

By Theorem 1.25, we can take in this definition $Y = \omega^\omega$ if $A \neq \emptyset$. The class of analytic sets in X is denoted by $\Sigma^1_1(X)$. The classical notation is $\mathbf{A}(X)$.

Proposition 1.40. Let X be a Polish space and $A \subseteq X$. The following statements are equivalent.

(i) A is analytic.

(ii) There is a continuous map $f : \omega^\omega \to X$ whose range is A.

(iii) There is a Polish space Y and a Borel set $B \subseteq X \times Y$ whose projection is A, that is, $A = proj_X(B)$.

(iv) There is a closed subset C of $X \times \omega^\omega$ whose projection is A, that is, $A = proj_X(C)$.

(v) For every uncountable Polish space Y there is a G_δ set B in $X \times Y$ whose projection is A, that is, $A = proj_X(B)$.

Proof. A proof of this proposition can be found in (SRIVASTAVA, 1998), Proposition 4.1.1. □

Theorem 1.41. Every uncountable analytic set contains a homeomorphic copy of the Cantor set and hence has cardinality $\mathfrak{c}$.

Proof. A complete proof of this theorem can be found in (SRIVASTAVA, 1998), Theorem 4.3.5. □

We can find a relationship between Borel and analytic sets. For this we need the following

Theorem 1.42 (Lusin-Souslin)**.** Let X be Polish and $A \subseteq X$ be Borel. There is a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f : F \to A$. In particular, if $A \neq \emptyset$, there is also a continuous surjection $g : \omega^\omega \to A$ extending f.

Proof. A proof of this theorem can be found in (KECHRIS, 1995), Theorem 13.7. □

Corollary 1.43. $\mathscr{B}_X \subseteq \Sigma^1_1(X)$.

1.1.3 Baire spaces

There are two approaches to study Baire spaces: one of them is to use first and second category sets and the other way is to use open and dense sets. In this first part we will discuss some results of the first approach of the Baire spaces, as they will help us later to characterize them using a modification of the Banach-Mazur game. Later we will use the second.

For this part we follow the books of (WALDMANN, 2014), (SINGH, 2013) and (HAWORTH; MCCOY, 1977).

Let X be a topological space, we start with some definitions and properties.

Definition 1.44. A set $A \subseteq X$ is **nowhere dense** in X if $Int(\overline{A}) = \emptyset$

Proposition 1.45. Let N be a subset of a space X. Then the following are equivalent:

(i) N is nowhere dense in X.

(ii) $X \setminus \overline{N}$ is dense in X.

(iii) For each non-empty open set U in X there exists a non-empty open set V such that $V \subseteq U$ and $V \cap N = \emptyset$.

Proof. $(i \Rightarrow ii)$ Let W be any open subset of X. Since $Int(\overline{N}) = \emptyset$, then $W \cap (X \setminus \overline{N}) \neq \emptyset$.

$(ii \Rightarrow iii)$ Consider $V = U \cap (X \setminus \overline{N})$.

$(iii \Rightarrow i)$ If $Int(\overline{N}) \neq \emptyset$, let $x \in Int(\overline{N})$, so there exists an non-empty open set A such that $x \in A \subseteq \overline{N}$, in particular $x \in \overline{N}$. Then $A \cap N \neq \emptyset$, contradiction.

□

Proposition 1.46. Let Y be a subspace of X, and let N be a subset of Y. If N is nowhere dense in Y, then N is nowhere dense in X. Conversely, if Y is open (or dense) in X and N is nowhere dense in X, then N is nowhere dense in Y.

Proof. Suppose that N is nowhere dense in Y. Let U be a non-empty open subset of X. If $U \cap Y = \emptyset$ we are through, so suppose that U intersects Y. Then there exists a non-empty open set V, open in Y, such that $V \subseteq U \cap Y$ and $V \cap N = \emptyset$. Now there is a set W, open in X, such that $V = W \cap Y$. Thus, $W \subseteq U$ and $W \cap N = \emptyset$, therefore N is nowhere dense in X.

Now suppose that Y is open in X and that N is nowhere dense in X. Let V be a non-empty open set in Y. Then V is open in X. Therefore, there exists a non-empty set U, open in X, such that $U \subseteq V$ and $U \cap N = \emptyset$. Thus, N is nowhere dense in Y since U is also open in Y.

□

Definition 1.47. A set $A \subseteq X$ is **meager** (or of **first category**) in X if $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense in X. A topological space X is called **meager in itself** if it can be written as a countable union of closed sets with empty interior.

The following proposition collects some basic properties of meager subsets:

Proposition 1.48. Let X be a topological space.

(i) A subset of a nowhere dense subset is again nowhere dense.

(ii) A finite union of nowhere dense subsets is again nowhere dense.

(iii) A subset of a meager subset is again meager.

(iv) A countable union of meager subsets is again meager.

Proof. A proof of this proposition can be found in (WALDMANN, 2014), Proposition 7.1.3. □

Corollary 1.49. Let X be a topological space and $A_1, A_2, \cdots, A_n \subseteq X$ be open and dense subsets. Then also $A_1 \cap A_2 \cap \cdots \cap A_n$ is open and dense.

Proposition 1.50. Let X be a topological space. Then the following statements are equivalent:

(i) Any countable union of closed subsets of X without interior points has no interior points.

(ii) Any countable intersection of open dense subsets of X is dense.

(iii) Every non-empty open subset of X is not meager.

(iv) The complement of every meager subset of X is dense.

Proof. A proof of this proposition can be found in (WALDMANN, 2014), Proposition 7.1.5. □

Proposition 1.51. In a topological space X, the union of any family of meager open sets is meager.

Proof. A complete proof of this proposition can be found in (HAWORTH; MCCOY, 1977), Theorem 1.6. □

Theorem 1.52. Let A be a subset of the space X, and suppose that for every non-empty open set U, there exists a non-empty open set V contained in U such that $V \cap A$ is of first category in X. Then A is of first category in X.

Proof. A complete proof of this theorem can be found in (HAWORTH; MCCOY, 1977), Theorem 1.7. □

Now we will focus on the second approach, which will be of more importance in order to introduce the Banach-Mazur game.

Definition 1.53 (Baire space). Let X be a topological space. Then X is called a Baire space if the intersection of each countable family of dense open sets in X is dense.

As we mentioned earlier, usually the categorical version of Baire spaces is part (iii) of Proposition 1.50, so we see that these are equivalent. Also note that a Baire space is not meager in itself.

We collect now some properties of Baire spaces:

Proposition 1.54. Let X be a non-empty Baire space.

(i) Let $\{A_n\}_{n\in\omega}$ be a countable closed cover of X. Then at least one A_n has non-empty interior, $Int(A_n) \neq \emptyset$.

(ii) Let $A \subseteq X$ be a non-empty open subset. Then A (with the subspace topology) is a Baire space again.

(iii) Let $B \subseteq X$ be a meager subset. Then $X \setminus B$ (with the subspace topology) is a Baire space again.

Proof. A complete proof of this proposition can be found in (WALDMANN, 2014), Proposition 7.1.8. □

In contrast, not every closed subspace of a Baire space is a Baire space, as can be seen by taking the space $\mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Note that $\{(x,0) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ is nowhere dense in $\mathbb{R}^2$, so by Proposition 1.54, (iii), $\mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ is a Baire space. Also the closed subspace $\{(x,0) : x \in \mathbb{Q}\}$ is meager in itself. Therefore $\{(x,0) : x \in \mathbb{Q}\}$ is not a Baire space.

This motivates the following definition.

Definition 1.55 (Hereditarily Baire space)**.** A Baire space X is **hereditarily Baire**[2] if every closed subspace of X is a Baire space.

For example, every complete metric space is hereditarily Baire. In a Baire space, the complement of any set of first category is called a **residual** (or **comeager**) set.

Proposition 1.56 (Oxtoby)**.** In a Baire space X, a set E is residual if and only if E contains a dense G_δ subset of X.

[2] Some authors use ‘completely Baire’ instead of ‘hereditarily Baire’.

Proof. Suppose $B = \bigcap_{n<\omega} G_n$, where each G_n is open, is a dense G_δ subset of E. Then each G_n is dense, and $X \setminus E \subseteq X \setminus G = \bigcup_{n<\omega}(X \setminus G_n)$ is of first category, so $X \setminus E$ is of first category.

Conversely, if $X \setminus E = \bigcup_{n<\omega} A_n \subseteq \bigcup_{n<\omega} \overline{A_n}$, where A_n is nowhere dense, let $B = \bigcap_{n<\omega}(X \setminus \overline{A_n})$. Then B is a G_δ set contained in E, also each $X \setminus \overline{A_n}$ is dense. As X is Baire, it follows that B is dense. □

Corollary 1.57. Let E be a subset of $\mathbb{R}$. Then E contains a dense G_δ subset of $\mathbb{R}$ if and only if E is residual.

We finalize this section defining productively Baire spaces. Later we will study the problem of the product of Baire spaces which is related to this last definition.

Definition 1.58. A Baire space X is **productively Baire** if $X \times Y$ is Baire for every Baire space Y.

1.2 Set theory

In this section we will introduce some basic concepts of set theory, which will help us later for some examples of product Baire spaces. For this part we follow the books of (JECH, 2003), (CIESIELSKI, 1997), (SCHIMMERLING, 2011) and (JUST; WEESE, 1997).

1.2.1 Some facts about ordinal and cardinal numbers

We begin with some results on cardinal arithmetic.

Proposition 1.59. If κ is an infinite cardinal and $|X_\alpha| \leq \kappa$ for all $\alpha < \kappa$ then

$$\left| \bigcup_{\alpha<\kappa} X_\alpha \right| \leq \kappa$$

Proof. A proof of this proposition can be found in (CIESIELSKI, 1997), Corollary 5.2.7. □

Let λ and κ be cardinals. We define

$$\lambda^{<\kappa} = \bigcup_{\alpha<\kappa} \lambda^\alpha.$$

For example, for a set A let $A^{<\omega} = \bigcup_{n<\omega} A^n$. Thus $A^{<\omega}$ is the set of all finite sequences with values in A.

Corollary 1.60. If κ is an infinite cardinal, then $|\kappa^{<\omega}| = \kappa$.

Theorem 1.61. If λ and κ are cardinal numbers such that $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.

In particular, $\lambda^\lambda = 2^\lambda$ for every infinite cardinal number λ.

Proof. A proof of this theorem can be found in (CIESIELSKI, 1997), Theorem 5.2.12. □

Proposition 1.62. For every infinite set X and nonzero cardinal $\kappa \leq |X|$

$$|[X]^\kappa| = |[X]^{\leq\kappa}| = |X|^\kappa.$$

Proof. A proof of this proposition can be found in (CIESIELSKI, 1997), Proposition 5.2.14. □

In particular $|[\mathbb{R}]^\omega| = (2^\omega)^\omega = 2^\omega = \mathfrak{c}$.

Definition 1.63. If γ is any limit ordinal, then the **cofinality** of γ is

$$cf(\gamma) = min\{type(X) : X \subseteq \gamma \wedge \sup(X) = \gamma\},$$

where $type(X)$ is the unique $\alpha \in ON$ such that $(X, \in) \cong (\alpha, \in)$.

Definition 1.64. Let γ be an ordinal number. γ is **regular** if $cf(\gamma) = \gamma$ and **singular** if $cf(\gamma) < \gamma$.

Definition 1.65. Let κ be a cardinal. The least cardinal $\lambda > \kappa$ is called the cardinal sucessor of κ, abbreviated by κ^+. A cardinal κ is called a sucessor cardinal if there is some cardinal $\mu < \kappa$ with $k = \mu^+$; otherwise κ is called a limit cardinal.

Proposition 1.66. For every infinite cardinal number κ, κ^+ is regular. In particular $\mathfrak{c}^+$ is regular.

Proof. A complete proof of this proposition can be found in (SCHIMMERLING, 2011), Lemma 4.32. □

Definition 1.67. Let χ be an ordinal number. A subset C of χ is called **club** if it is closed (in the order topology of χ) and unbounded. A subset A of χ is called **stationary** in χ if A has non-empty intersection with every C club in χ.

Example 2. (i) If $\alpha < \omega_1$, then $\{\beta < \omega_1 : \alpha < \beta\}$ is a club in ω_1.

(ii) $\{\alpha < \omega_1 : \alpha \text{ is a limit ordinal}\}$ is a club in ω_1.

Let κ be an uncountable regular cardinal. We have the following remarks:

1. A stationary set is unbounded in κ. Indeed, let $\gamma < \kappa$. Note that $[\gamma+1, \kappa[$ is a club in κ, then $S \cap [\gamma+1, \kappa[\neq \emptyset$, so there is $\xi \in S$ such that $\gamma < \gamma+1 \leq \xi$.
2. There are stationary sets that are not club in κ. In fact, consider the set $S = \kappa \setminus \{\omega\}$. We claim that S is stationary, otherwise, there is a club C in κ such that $S \cap C = (\kappa \setminus \{\omega\}) \cap C = \emptyset$ so $C \subseteq \{\omega\}$ which is bounded in κ, contradiction. Note that S is not closed, because $\omega \subseteq S$ and $\sup(\omega) = \omega \notin S$. Thus, S is a stationary set that is not a club.
3. Also note that if S is stationary in κ, and $S \subseteq T \subseteq \kappa$, then T is stationary in κ.

Proposition 1.68. Suppose that κ is a regular uncountable cardinal. If A and B are club in κ, then $A \cap B$ is club in κ.

Proof. A complete proof of this proposition can be found in (CUNNINGHAM, 2016), Theorem 9.3.7. □

In particular, if κ is an uncountable regular cardinal. Then, every club set is stationary in κ, because the intersection of two clubs in κ is a club in κ.

Proposition 1.69. Let κ be an uncountable regular cardinal. If $\theta < \kappa$ and $\langle C_\alpha : \alpha < \theta \rangle$ is a sequence of club subsets of κ, then the set

$$\bigcap \{C_\alpha : \alpha < \theta\}$$

is a club in κ.

Proof. A complete proof of this proposition can be found in (JECH, 2003), Theorem 8.3. □

Proposition 1.70. Let κ be an uncountable regular cardinal and $f : \kappa \to \kappa$ be a function. Then $\{\alpha < \kappa : f[\alpha] \subseteq \alpha\}$ is a club in κ.

Proof. Denote $C = \{\alpha < \kappa : f[\alpha] \subseteq \alpha\}$. We will first show that C is closed in κ. Indeed, let $\alpha < \kappa$ with $C \cap \alpha \neq \emptyset$. Note that $\sup(C \cap \alpha) < \kappa$, because κ is regular. Let $\beta < \sup(C \cap \alpha) = \bigcup(C \cap \alpha)$, so there exists $\gamma \in C \cap \alpha$ such that $\beta < \gamma$, in particular $f[\gamma] \subseteq \gamma < \alpha$ then $f(\beta) \in \gamma$, so $f[\sup(C \cap \alpha)] \subseteq \sup(C \cap \alpha)$.

Now we will show that C is unbounded in κ. Let $\sigma < \kappa$. First, consider $\sup(f[\sigma]) = \sup\{f(\beta) : \beta < \sigma\}$, since κ is regular, $\sup(f[\sigma]) + 1, \sigma + 1 \in \kappa$. Then

$$\beta_0 := \max\{\sigma + 1, \sup(f[\sigma]) + 1\} < \kappa$$

Assume that the monotone strictly increasing sequence $\langle \beta_j : j \leq n \rangle$ with $\beta_n < \kappa$ is already defined. Define

$$\beta_{n+1} := \max\{\beta_n + 1, \sup(f[\beta_n]) + 1\} < \kappa$$

Note that, for each $n \in \omega$, we have $f(\beta_n) \subseteq \beta_{n+1}$. As κ is uncountable regular, we have $\beta = \sup\{\beta_n : n \in \omega\} < \kappa$, and we obtain that

$$f[\beta] = f[\bigcup\{\beta_n : n \in \omega\}] = \bigcup\{f[\beta_n] : n \in \omega\} \subseteq \bigcup\{\beta_{n+1} : n < \omega\} = \beta.$$

Thus $\beta \in C$ and $\sigma < \beta$, therefore C is unbounded in κ.

□

Lemma 1.71. Let κ be a regular uncountable cardinal and let $\alpha \in \kappa$. If S is stationary in κ, then $S \setminus \alpha$ is stationary in κ.

Proof. Let C be a club in κ. Note that $[\alpha + 1, \kappa[$ is a club in κ then $C \cap [\alpha + 1, \kappa[$ is a club in κ so there exists $\gamma \in (C \cap [\alpha + 1, \kappa[) \cap S$. Therefore $\gamma \in (S \setminus \alpha) \cap C$. □

Finally, the following example of stationary set will be of vital importance later, as it will help us build examples of Baire spaces whose product is not Baire.

Definition 1.72. $C_\omega\chi$ is the subset of χ of ordinals of cofinality ω. That is,

$$C_\omega\chi = \{\beta < \chi : cf(\beta) = \omega\}.$$

Lemma 1.73. If χ is uncountable and regular, then $C_\omega\chi$ is stationary.

Proof. Let A be a club in χ. As χ is regular and A is unbounded, $|A| = \chi$. Let $(a_\alpha)_{\alpha<\chi}$ be an enumeration of A in strictly increasing order. As A is closed, $a_\omega = \sup\{a_n : n \in \omega\} \in A$; then $cf(a_\omega) = \omega$. Thus $a_\omega \in C_\omega\chi$ and $C_\omega\chi \cap A \neq \emptyset$. □

Theorem 1.74 (Solovay)**.** If $\chi > \omega$ is a regular cardinal, then any stationary subset of χ can be split into χ many disjoint stationary subsets of χ.

Proof. A complete proof of this theorem can be found in (JECH, 2003), Theorem 8.10. □

Lemma 1.75. If $\chi > \omega$ is regular, the union of less than χ many nonstationary sets is nonstationary.

Proof. Assume that $\{N_\alpha : \alpha < \gamma\}$ are nonstationary sets, where $\gamma < \chi$. By definition, there exist club sets $\{C_\alpha : \alpha < \gamma\}$ such that $N_\alpha \cap C_\alpha = \emptyset$ $(\alpha < \gamma)$. Set $N = \bigcup_{\alpha<\gamma} N_\alpha$, $C = \bigcap_{\alpha<\gamma} C_\alpha$. By Proposition 1.69, C is a club. Also note that $C \cap N = \emptyset$, so N is non-stationary, as claimed. □

Corollary 1.76. Suppose that κ is a regular uncountable cardinal and that $\gamma \in \kappa$. Let $\langle S_\alpha : \alpha \in \gamma \rangle$ be a γ-sequence of subsets of κ. Suppose that the set $\bigcup_{\alpha\in\gamma} S_\alpha$ is stationary in κ. Then S_α is stationary, for some $\alpha \in \gamma$.

Theorem 1.77 (Pressing Down Lemma or Fodor's Lemma)**.** Let κ be a regular uncountable cardinal, $S \subseteq \kappa$ be a stationary set and let $f : S \to \kappa$ be such that $f(\gamma) < \gamma$ for every $\gamma \in S$ (such a function is called a regressive function). Then there exists an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is stationary.

Proof. A complete proof of this theorem can be found in (JUST; WEESE, 1997), Theorem 21.12. □

When $cf(\chi) > \omega$, we can define a map $* : \chi^\omega \to \chi$, where $*(f) = f^*$ is the least α greater than $f(n)$ for all $n \in \omega$.

Proposition 1.78. Let $\chi > \omega$ be regular. If $K \subseteq \chi^\omega$ is closed, and $W = \{f^* : f \in K\}$ is stationary, then there is C club in χ such that $C \cap C_\omega\chi \subseteq W$

Proof. Let $\sigma \in \chi^{<\omega}$ and $W_\sigma = \{f^* : \sigma \subseteq f \in K\}$. Consider $\Sigma = \{\sigma : W_\sigma \text{ is stationary}\}$. By hypothesis $\Sigma \neq \emptyset$, because $\emptyset \in \Sigma$.

Claim 1.78.1. Using the Pressing Down Lemma one can build a function $\theta : \Sigma \times \chi \to \Sigma$ such that

(i) $\sigma \subseteq \theta(\sigma, \alpha)$;

(ii) $\theta(\sigma, \alpha) \notin \bigcup_{n\in\omega} \alpha^n$.

Proof. Indeed, let $\sigma \in \bigcup_{n\in\omega} \chi^n$ and $\alpha < \chi$. Consider $P = W_\sigma \setminus \alpha$. By Lemma 1.71, P is stationary in χ. Define

$$\begin{array}{cccl} g_\sigma & : & P & \longrightarrow & \chi \\ & & f^* & \longrightarrow & g_\sigma(f^*) = f(n), \end{array}$$

where $n = \min\{n \in \omega : f(n) \geq \alpha\}$. Note that $g_\sigma(f^*) < f^*$, for all $f^* \in P$, so by the Pressing Down Lemma (Theorem 1.77), there is $\gamma < \chi$ such that $g_\sigma^{-1}(\{\gamma\}) = \{f^* \in P : g_\sigma(f^*) = f(n) = \gamma\}$ is stationary. Note that $\gamma \geq \alpha$. Finally, define

$$\begin{array}{cccl} h & : & g_\sigma^{-1}(\{\gamma\}) & \longrightarrow & \omega \\ & & f^* & \longrightarrow & h(f^*) = n, \end{array}$$

where $n \in \omega$ is such that $g_\sigma(f^*) = f(n)$. Note that $g_\sigma^{-1}(\{\gamma\}) = \bigcup_{n\in\omega} h^{-1}(\{n\})$. Then, by Corollary 1.76, there is an $m \in \omega$ such that $h^{-1}(\{m\}) = \{f^* \in g_\sigma^{-1}(\{\gamma\}) : h(f^*) = m\} = \{f^* \in P : f(m) = \gamma\}$ is stationary.

If $m \in dom(\sigma)$, then $\theta(\sigma, \alpha) = \sigma$. In this case $\theta(\sigma, \alpha) \notin \bigcup_{n\in\omega} \alpha^n$, because $\sigma(m) = f(m) = \gamma \geq \alpha$.

If $m \notin dom(\sigma)$, so $m > |\sigma|$. We claim the following

Claim 1.78.2. There are a finite sequence of stationary sets $\langle S_0, \cdots, S_{m-|\sigma|-1}\rangle$ and a finite sequence of ordinals $\langle \beta_0, \cdots, \beta_{m-|\sigma|-1}\rangle$ such that $S_0 \subseteq S$ and, for $i < m - |\sigma| - 1$, then $S_{i+1} \subseteq S_i$ and if $f^* \in S_i$ then $f(i + |\sigma|) = \beta_i$.

Proof. In fact, for $i = 0$, consider

$$\begin{array}{cccl} g_0 & : & S & \longrightarrow & \chi \\ & & f^* & \longrightarrow & g_0(f^*) = f(|\sigma|) < f^*, \end{array}$$

where $S = \{f^* \in P : f(m) = \gamma\}$. By the Pressing Down Lemma, there exists $\beta_0 < \chi$ such that $g_0^{-1}(\{\beta_0\}) = S_0$.

For $0 < i < m - |\sigma| - 1$, consider

$$\begin{array}{cccl} g_i & : & S_{i-1} & \longrightarrow & \chi \\ & & f^* & \longrightarrow & g_i(f^*) = f(|\sigma| + i) < f^*, \end{array}$$

By the Pressing Down Lemma, there exists $\beta_i < \chi$ such that $g_i^{-1}(\{\beta_i\}) = S_i \subseteq S_{i-1}$. Note that, if $f^* \in S_i$, then $f(i + |\sigma|) = \beta_i$. □

Now we will build $\theta \in \chi^{m+1}$. Let $\theta|_{|\sigma|} = \sigma$ and $\theta(m) = \gamma$. Then, if $|\sigma| \leq i < m$, define $\theta(i) = \beta_{i-|\sigma|}$. Finally, note that $S_{m-|\sigma|-1} \subseteq W_\theta$. In fact, let $f^* \in S_{m-|\sigma|-1}$, in particular, $\sigma \subseteq f \in K$ and $f(m) = \gamma$. By Claim 1.78.2, $f(i + |\sigma|) = \beta_i$ for $i < m - |\sigma| - 1$, so $f \in W_\theta$. □

Consider $C = \{\gamma < \chi : \theta[(\Sigma \cap \gamma^{<\omega}) \times \gamma] \subseteq \gamma^{<\omega}\}$. We claim that C is a club in χ. Indeed,

- C is closed.

  Let $\gamma \in C'$, we will show that $\theta[(\Sigma \cap \gamma^{<\omega}) \times \gamma] \subseteq \gamma^{<\omega}$. Let $(\sigma, \alpha) \in (\Sigma \cap \gamma^{<\omega}) \times \gamma$, so there is $n_0 \in \omega$ such that $\sigma \in \gamma^{n_0}$. Consider $m = \max\{\sigma(n_0 - 1), \alpha\} < \gamma$; then there exists $\beta \in]m, \gamma+1[\cap (C \setminus \{\gamma\})$, so $\alpha < \beta < \gamma$ and $\sigma \in \beta^{<\omega}$, then $\theta(\sigma, \alpha) \in \theta[(\Sigma \cap \beta^{<\omega}) \times \beta] \subseteq \beta^{<\omega} \subseteq \gamma^{<\omega}$. Therefore $C' \subseteq C$, that is, C is closed.

- C is unbounded.

  For this, define

$$\begin{array}{rcccl} f & : & \chi & \longrightarrow & \chi \\ & & \gamma & \longrightarrow & f(\gamma) = \sup\{\theta^*(\sigma, \alpha) : \sigma \in \Sigma \cap \gamma^{<\omega}, \alpha < \gamma\}, \end{array}$$

  where $\theta^*(\sigma, \alpha) = \sup(ran(\theta(\sigma, \alpha)))$. Note that f is well defined, that is, $f(\gamma) = \sup\{\theta^*(\sigma, \alpha) : \sigma \in \Sigma \cap \gamma^{<\omega}, \alpha < \gamma\} < \chi$, because χ is an uncountable regular cardinal.

  By Proposition 1.70, $\{\gamma < \chi : f[\gamma] \subseteq \gamma\}$ is a club in χ. Then

$$\tilde{C} = \{\gamma < \chi : \gamma \text{ is a limit ordinal and } f[\gamma] \subseteq \gamma\}$$

  is a club in χ. Note that $\tilde{C} \subseteq C$. Indeed, let $\gamma \in \tilde{C}$ and let $(\sigma, \alpha) \in (\Sigma \cap \gamma^{<\omega}) \times \gamma$. As γ is a limit ordinal, there is $\alpha < \beta < \gamma$ such that $\sigma \in \beta^{<\omega}$; then $\theta^*(\sigma, \alpha) \leq f(\beta) < \gamma$, so $\theta(\sigma, \alpha) \in \gamma^{<\omega}$.

Finally, note that $C \cap C_\omega \chi \subseteq W$. Indeed, let $\gamma \in C \cap C_\omega \chi$. Then $cf(\gamma) = \omega$, so there exists a strictly increasing function $g : \omega \to \gamma$ whose range is cofinal in γ, that is, $\sup\{g(n) : n \in \omega\} = \gamma$ and $\theta[(\Sigma \cap \gamma^{<\omega}) \times \gamma] \subseteq \gamma^{<\omega}$. The main idea is to iterate θ, and g will help us keep going up to γ. So inductively build a sequence σ_n as follows:

(i) $\sigma_0 := \theta(\emptyset, g(0))$ and

(ii) $\sigma_{n+1} := \theta(\sigma_n, g(n))$.

Note that $\sigma_n \in \Sigma \cap \gamma^{<\omega}$ and for each $n \in \omega$, $\sigma_n \subseteq \sigma_{n+1} = \theta(\sigma_n, g(n))$. Consider

$$f = \bigcup_{n \in \omega} \sigma_n.$$

We claim that $f \in \chi^\omega$ and $f^* = \gamma$. In fact,

- $dom(f) = \omega$

  Note that for each $n \in \omega$, $dom(\sigma_n) \in \omega$ then $dom(f) \subseteq \omega$. Also $dom(f)$ is infinite, otherwise, $dom(f)$ is finite. Then consider $\beta = \max\{f(n) : n \in dom(f)\}$. As g is cofinal, there exists $m \in \omega$ such that $\beta < g(m)$. Also consider $\sigma_{m+1} = \theta(\sigma_m, g(m)) \notin g(m)^{<\omega}$ so there exists $m' \in dom(\sigma_{m+1}) \subseteq dom(f)$ such that $g(m) \leq \sigma_{m+1}(m') = f(m')$ so $\beta < g(m) \leq f(m')$, contradiction. Therefore $dom(f)$ is infinite, so $dom(f)$ is unbounded in ω. Then $\omega \subseteq dom(f)$. Indeed, let $m \in \omega$, then there exists $n \in dom(f)$ such that $m < n \in dom(f)$ so $m \in dom(f)$.

- $f^* = \sup\{f(m) : m \in \omega\} = \gamma$

  Let $\beta \in \gamma$, as $ran(g)$ is cofinal in γ, there is $m \in \omega$ such that $\beta < g(m)$. By construction, $\sigma_{m+1} = \theta(\sigma_m, g(m)) \subseteq f$ and $\sigma_{m+1} \notin g(m)^{<\omega} = \bigcup_{n\in\omega} g(m)^n$, then there exists $n \in dom(\sigma_{m+1}) \subseteq dom(f)$ such that $\beta < \sigma_{m+1}(n)$. Otherwise, $\sigma_{m+1} \in (\beta+1)^{<\omega} \subseteq (g(m))^{<\omega}$, contradiction. Therefore, $\beta < \sigma_{m+1}(n) = f(m)$. On the other hand, note that $\sup\{f(m) : m \in \omega\} \subseteq \gamma$, because $\sigma_m \in \gamma^{<\omega}$ for each $m \in \omega$.

Finally, note that $f \in K$. Indeed, for each $n \in \omega$ we have that $\sigma_n \in \Sigma$, that is, W_{σ_n} is stationary. In particular $W_{\sigma_n} \neq \emptyset$, so there exists $f_n \in K$ such that $\sigma_n \subseteq f_n$. We claim that $f_n \xrightarrow{n\to\infty} f$ in χ^ω. In fact, let $f \in N_s = \{h \in \chi^\omega : s \subseteq h\}$ where $s = (s(0), \cdots, s(n_s - 1)) \in \chi^{<\omega}$. As $s \subseteq f$, $n_s - 1 \in dom(f) = \bigcup_{m\in\omega} dom(\sigma_m)$ then there exists $m_0 \in \omega$ such that $s \subseteq \sigma_{m_0}$. Then, if $m > m_0$, $f_m \in N_\sigma$ therefore $f_n \xrightarrow{n\to\infty} f$.

$\square$

We have a generalization for finite products and the proof is similar to that of Lemma 1.78.

Corollary 1.79. Let $m < \omega$ and $\kappa > \omega$ be a regular cardinal. If $K \subseteq (k^\omega)^m$ is closed and

$$W = \{\alpha : \alpha = f_0^* = \cdots = f_{m-1}^* \text{ and } (f_0, \cdots, f_{m-1}) \in K\}$$

is stationary, then there is a club set C in k such that $C \cap C_\omega k \subseteq W$.

1.2.2 Combinatorial set theory

In this part we will see some consequences of Martin's axiom concerning the G_δ and meager subsets of the real line.

Definition 1.80. A family A is called a Δ-system if there is a set r such that $a \cap b = r$ whenever $a, b \in A$ and $a \neq b$.

Theorem 1.81. (Δ-System Lemma) Let κ and λ be infinite cardinals such that λ is regular and the inequality $\nu^{<\kappa} < \lambda$ holds for all $\nu < \lambda$. If B is a set of cardinality at least λ such that $|b| < \kappa$ for all $b \in B$, then there exists a Δ-system $A \subseteq B$ with $|A| = \lambda$.

Proof. A complete proof of this theorem can be found in (JUST; WEESE, 1997), Theorem (16.3). □

The following theorem is the most used version of the Δ-system lemma and it is a consequence of Theorem 1.81.

Theorem 1.82. Every uncountable family of finite sets contains an uncountable Δ-system.

Proof. A proof of this theorem can be found in (JUST; WEESE, 1997), Theorem (16.1). □

For this basic part we will use the first version, later for applications with the Banach-Mazur game and the infinite products of Baire spaces we will use the second version.

Definition 1.83. Let $\langle \mathbb{P}, \leq \rangle$ be a partially ordered set. A subset $D \subseteq \mathbb{P}$ is dense if

$$\forall p \in \mathbb{P}\, \exists q \in D\, (q \leq p).$$

Definition 1.84. A subset F of a partially ordered set $\langle \mathbb{P}, \leq \rangle$ is a filter in $\mathbb{P}$ if

(F1) for every $p, q \in F$ there is an $r \in F$ such that $r \leq p$ and $r \leq q$, and

(F2) if $q \in F$ and $p \in P$ are such that $q \leq p$ then $p \in F$.

Note that a simple induction argument shows that condition (F1) is equivalent to the following stronger condition.

(F1′) For every finite subset F_0 of F there exists an $r \in F$ such that $r \leq p$ for every $p \in F_0$.

Definition 1.85. If X is a non-empty set, then a filter on X is a subfamily $\mathscr{F}$ of $\mathscr{P}(X)$ such that

- $\mathscr{F}$ is closed under supersets, i.e.,

$$\forall Y \in \mathscr{F}\, \forall Z \subseteq X\, (Y \subseteq Z \to Z \in \mathscr{F});$$

- $\mathscr{F}$ is closed under finite intersections, i.e., $\bigcap H \in \mathscr{F}$ for all non-empty $H \in [\mathscr{F}]^{<\omega}$.

Note that if we consider $\langle \mathscr{P}(X), \subseteq \rangle$ and $\mathscr{F} \subseteq \mathscr{P}(X)$, this definition is a particular case of the previous definition in a partially ordered set.

Definition 1.86. Let $\langle \mathbb{P}, \leq \rangle$ be a partially ordered set, and let $\mathscr{D}$ be a family of dense subsets of $\mathbb{P}$. We say that a filter F in $\mathbb{P}$ is $\mathscr{D}$- generic if $F \cap D \neq \emptyset$ for all $D \in \mathscr{D}$.

Theorem 1.87 (Rasiowa–Sikorski lemma)**.** Let $\langle \mathbb{P}, \leq \rangle$ be a partially ordered set and $p \in \mathbb{P}$. If $\mathscr{D}$ is a countable family of dense subsets of $\mathbb{P}$ then there exists a $\mathscr{D}$-generic filter F in $\mathbb{P}$ such that $p \in F$.

Proof. A proof of this theorem can be found in (CIESIELSKI, 1997), Theorem 8.1.2. □

Definition 1.88. Let $\langle \mathbb{P}, \leq \rangle$ be a partially ordered set.

- $x, y \in \mathbb{P}$ are **comparable** if either $x \leq y$ or $y \leq x$. Thus a chain in $\mathbb{P}$ is a subset of $\mathbb{P}$ of pairwise-comparable elements.
- $x, y \in \mathbb{P}$ are **compatible** (in $\mathbb{P}$) if there exists a $z \in \mathbb{P}$ such that $z \leq x$ and $z \leq y$. In particular, condition (F1) from the definition of a filter says that any two elements of a filter F are compatible in F.
- $x, y \in \mathbb{P}$ are **incompatible** if they are not compatible. In this case we denote this fact by $x \perp y$.
- A subset A of $\mathbb{P}$ is an **antichain** (in $\mathbb{P}$) if every two distinct elements of A are incompatible. An antichain is **maximal** if it is not a proper subset of any other antichain. An elementary application of the Hausdorff maximal principle shows that every antichain in $\mathbb{P}$ is contained in some maximal antichain.
- A partially ordered set $\langle \mathbb{P}, \leq \rangle$ is **ccc** (or satisfies the countable chain condition) if every antichain of $\mathbb{P}$ is at most countable.

Consider the following axiom, known as Martin's axiom and usually abbreviated by **MA**.

Martin's axiom : Let $\langle \mathbb{P}, \leq \rangle$ be a ccc partially ordered set. If $\mathscr{D}$ is a family of dense subsets of $\mathbb{P}$ such that $|\mathscr{D}| < \mathfrak{c}$, then there exists a $\mathscr{D}$-generic filter in $\mathbb{P}$.

Note that the Continuum Hypothesis implies Martin's axiom. Now we will see some consequences of Martin's axiom in topology.

Theorem 1.89. Assume **MA**. If $X \in [\mathbb{R}]^{<\mathfrak{c}}$ then every subset Y of X is a G_δ subset of X, that is, there exists a G_δ set $G \subseteq \mathbb{R}$ such that $G \cap X = Y$.

Proof. Let $X \in [\mathbb{R}]^{<\mathfrak{c}}$ and fix $Y \subseteq X$. We will show that Y is a G_δ in X. Let $\mathscr{B} = \{B_n : n < \omega\}$ be a countable base for $\mathbb{R}$.

First notice that it is enough to find a set $\hat{A} \subseteq \omega$ such that for every $x \in X$

$$x \in Y \iff x \in B_n \text{ for infinitely many } n \text{ from } \hat{A} \qquad (*)$$

To see why, define for every $k < \omega$ an open set $G_k = \bigcup\{B_n : n \in \hat{A} \wedge n > k\}$ and put $G = \bigcap_{k<\omega} G_k$. Then G is a G_δ set and for every $x \in X$ we have

- if $x \in Y$, by $(*)$, $x \in B_n$ for infinitely many n from $\hat{A}$. Then for each $k < \omega$ there exists $m \in \hat{A}$ such that $x \in B_m$ and $m > k$, so $x \in G_k \forall k < \omega$.
- if $x \in G_k$ for all $k < \omega$ then for each $k < \omega$ there is $m_k \in \hat{A}$ such that $m_k > k$ and $x \in B_{m_k}$, that is, $x \in B_n$ for infinitely many n from $\hat{A}$.

In summary, for every $x \in X$ we have

$$x \in Y \iff x \in G_k \text{ for all } k < \omega$$

We define the partially ordered set $\langle \mathbb{P}, \leq \rangle$ by putting $\mathbb{P} = [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$ and for $\langle A_1, C_1 \rangle, \langle A_0, C_0 \rangle \in \mathbb{P}$ we define

$$\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$$

provided

(i) $A_1 \supset A_0$, $C_1 \supset C_0$ and

(ii) $c \notin B_m$ for all $m \in A_1 \setminus A_0$ and $c \in C_0$.

Now for $y \in Y$, $k < \omega$ and $z \in X \setminus Y$, define the following subsets of $\mathbb{P}$

$$D_y^k = \{\langle A, C \rangle \in \mathbb{P} : \exists m \in A \ (m \geq k \wedge y \in B_m)\}$$

and

$$E_z = \{\langle A, C \rangle \in \mathbb{P} : z \in C\}.$$

We will use Martin's axiom to find a $\mathscr{D}$-generic filter for

$$\mathscr{D} = \{D_y^k : y \in Y \wedge k < \omega\} \cup \{E_z : z \in X \setminus Y\}.$$

To use Martin's axiom, we have to check whether its assumptions are satisfied.

1. $\mathbb{P}$ **is ccc**.

   Indeed, suppose that there is $\{\langle A_\xi, C_\xi\rangle : \xi < \omega_1\}$ an uncountable antichain. Since $[\omega]^{<\omega}$ is countable, there are $A \in [\omega]^{<\omega}$ and $\zeta < \xi < \omega_1$ such that $A_\zeta = A = A_\xi$. Then $\langle A_\xi, C_\xi\rangle = \langle A, C_\xi\rangle$ and $\langle A_\zeta, C_\zeta\rangle = \langle A, C_\zeta\rangle$ are compatible, since $\langle A, C_\xi \cup C_\zeta\rangle$ extends them both, as condition (ii) is satisfied vacuously. Contradiction.

2. $\mathscr{D}$ **is a family of dense subsets of** $\mathbb{P}$.

   a) **For all** $y \in Y, k < \omega$**,** D^k_y **is dense in** $\mathbb{P}$.

      Indeed, take $\langle A, C\rangle \in \mathbb{P}$. Notice that there exist infinitely many basic open sets B_m such that
      $$y \in B_m \text{ and } C \cap B_m = \emptyset. \qquad (**)$$
      Take $m > k$ satisfying $(**)$, and notice that $\langle A \cup \{m\}, C\rangle \in D^k_y$ extends $\langle A, C\rangle$.

   b) **For all** $z \in X \setminus Y$**,** E_z **is dense in** $\mathbb{P}$.

      Indeed, take $\langle A, C\rangle \in \mathbb{P}$ and notice that $\langle A, C \cup \{z\}\rangle \in E_z$ extends $\langle A, C\rangle$.

3. $|\mathscr{D}| < \mathfrak{c}$.

   Note that $|\mathscr{D}| \leq |X| + \omega < \mathfrak{c}$.

Now apply Martin's axiom to find a $\mathscr{D}$-generic filter F in $\mathbb{P}$, and define
$$\hat{A} = \bigcup\{A : \langle A, C\rangle \in F\}.$$
We will show that $\hat{A}$ satisfies $(*)$. So let $x \in X$.

If $x \in Y$ then for every $k < \omega$ there exists $\langle A, C\rangle \in F \cap D^k_x$. In particular there exists $m_k \in A \subseteq \hat{A}$ with $m_k > k$ such that $x \in B_{m_k}$. So $x \in B_m$ for infinitely many m from $\hat{A}$.

If $x \in X \setminus Y$ then there exists $\langle A_0, C_0\rangle \in F \cap E_x$. In particular, $x \in C_0$. It is enough to prove that $x \notin B_m$ for every $m \in \hat{A} \setminus A_0$, because this implies that $\{m \in \hat{A} : x \in B_m\}$ is a finite set. So take $m \in \hat{A} \setminus A_0$. By the definition of $\hat{A}$ there exists $\langle A, C\rangle \in F$ such that $m \in A$. But, by the definition of a filter, there exists $\langle A_1, C_1\rangle \in F$ extending $\langle A, C\rangle$ and $\langle A_0, C_0\rangle$. Now $\langle A_1, C_1\rangle \leq \langle A_0, C_0\rangle$, $m \in A \subseteq A_1$, $m \notin A_0$ and $x \in C_0$. Hence, by (ii), $x \notin B_m$. □

Finally, if **MA** holds, we have control over the meager sets of the real line.

Theorem 1.90. If **MA** holds then a union of less than continuum many meager subsets of $\mathbb{R}$ is meager in $\mathbb{R}$.

Proof. A proof of this theorem can be found in (CIESIELSKI, 1997), Theorem (8.2.6). □

Corollary 1.91 (MA). Let A be a subset of the real line with $|A| < \mathfrak{c}$. Then A is meager.

Proof. Remember that for $x \in \mathbb{R}$, $\{x\}$ is nowhere dense, therefore meager. □

1.3 Forcing

In this section we will introduce some basic concepts of forcing. Since Cohen was the first to demonstrate, using forcing and without adding any more hypothesis (for example CH), that there are Baire spaces whose product is not Baire, we will study this example later. For the fundamental part of forcing and its properties we follow the books (KUNEN, 1980), (JECH, 2003) and (BELL, 2011).

Definition 1.92. We say that $(\mathscr{P}, \leq)$ or simply $\mathscr{P}$ is a partially ordered set (p.o) if:

- $p \leq p, \forall p \in \mathscr{P}$
- $(p \leq q$ and $q \leq p \to p = q), \forall p, q \in \mathscr{P}$
- $(p \leq q$ and $q \leq r \to p \leq r), \forall p, q, r \in \mathscr{P}$
- there is an element in $\mathscr{P}$, denoted by 1, such that $p \leq 1, \forall p \in \mathscr{P}$.

Definition 1.93. We say that $p \perp q$ (p **incompatible** with q) if there is no $r \leq p, q$. A partially ordered set $\mathscr{P}$ is a **forcing** if for each $p, q \in \mathscr{P}$ such that $q \not\leq p$, there exists $p' \leq q$ such that $p' \perp p$.

Suppose $\mathscr{M}$ is a countable standard transitive model of Zermelo-Fraenkel set theory (ZFC) and let $\mathscr{P}$ be a partially ordered set. We denote by $\mathscr{M}[G]$ the smallest model extending $\mathscr{M}$ and containing G as an element. We collect below some well-known facts.

The elements of the p.o. set $\mathscr{P}$ are often called conditions. We say that a condition p forces a sentence A (to be true in the model $\mathscr{M}[G]$) if A holds in $\mathscr{M}[G]$ whenever G contains p. In symbols this is written $p \Vdash A$.

Theorem 1.94 (Fundamental theorem of forcing)**.** A sentence A is satisfied in $\mathscr{M}[G]$ if and only if there is a condition $p \in G$ such that $p \Vdash A$.

Proof. A proof of this theorem can be found in (KUNEN, 1980), Lemma IV.2.24. □

From properties of generic subsets and the fundamental theorem of forcing it follows that, to prove that A holds in $\mathscr{M}[G]$, it suffices to prove that $\{p : p \Vdash A\}$ is a dense subset of $\mathbb{P}$.

Definition 1.95. Let $\mathscr{P}$ be a partially ordered set and let $p \in \mathscr{P}$. A set $D \subseteq \mathscr{P}$ is **dense below** p if for every $q \leq p$ there exists a $d \in D$ such that $d \leq q$.

Lemma 1.96. Let G be an $\mathscr{M}$-generic subset of $\mathscr{P}$, and φ be a sentence such that $p \Vdash \varphi$. If D is dense below p then $D \cap G \neq \emptyset$.

Proof. Consider $D' = D \cup \{q : q \text{ is incompatible with } p\}$. Note that D' is dense in $\mathbb{P}$, so there is $r \in D' \cap G$. If $r \in \{q : q \text{ is incompatible with } p\}$, we have a contradiction, because G is an $\mathscr{M}$-generic filter. Therefore $D \cap G \neq \emptyset$. $\square$

Proposition 1.97. The basic properties of the forcing relation are as follows.

(1) $p \Vdash \neg A$ if and only if no $q \leq p$ forces A;

We note that $p \Vdash \neg\neg A$ is equivalent to $p \Vdash A$, therefore,

(1′) $p \Vdash A$ if and only if no $q \leq p$ forces $\neg A$;

(2) $p \Vdash A \wedge B$ if and only if $p \Vdash A$ and $p \Vdash B$;

(3) $p \Vdash A \vee B$ if and only if $(\forall q \leq p)(\exists r \leq q)[r \Vdash A \text{ or } r \Vdash B]$;

(4) $p \Vdash \forall x A(x)$ if and only if $(\forall x \in \mathscr{M})[p \Vdash A(x)]$;

(5) $p \Vdash \exists x A(x)$ if and only if $(\forall q \leq p)(\exists r \leq q)(\exists x \in \mathscr{M})[r \Vdash A(x)]$.

An important property of the forcing relation is the following:

(6) for any sentence A and any $p \in \mathbb{P}$

$$(\exists q \leq p)[q \Vdash A \text{ or } q \Vdash \neg A].$$

Proof. A complete proof of this proposition can be found in (KUNEN, 1980), Lemma IV.2.30. $\square$

If a formula $A(x, \cdots, y)$ is satisfied in a model $\mathscr{M}$, we write

$$\mathscr{M} \Vdash A(x, \cdots, y).$$

In this notation the fundamental theorem of forcing can be written as follows: $\mathscr{M}[G] \Vdash A(x, \cdots, y)$ if and only if $(\exists p \in G)[p \Vdash A(x, \cdots, y)]$. If $\mathscr{M}[G] \Vdash A(x, \cdots, y)$ for any generic subset G of p.o. set $\mathbb{P}$, we write $\mathscr{M}^{\mathbb{P}} \Vdash A(x, \cdots, y)$.

An important class of forcings are those that, instead of any partial order, are given by a Boolean algebra in their usual order. In fact, it is possible to show that, given any partial order, it can be "immersed" in a "good way" in a complete Boolean algebra.

The present general version of the method of forcing, which uses the Boolean-valued models, is due to Solovay, Scott (SCOTT, 1967).

We present only one result on this new forcing point of view, as we will use it later.

Lemma 1.98. Let u be a nonzero element of B. For any partition $\{u_i : i \in I\}$ of u (i.e., $\sum_{i \in I} u_i = u$ and $u_i.u_j = 0$ for $i \neq j$) and any set $\{t_i : i \in I\}$ of elements of $\mathscr{M}^B$ there exists $t \in \mathscr{M}^B$ such that $u_i \leq$ "$t = t_i$" for all $i \in I$, *Proof.* A complete proof of this lemma can be found in (JECH, 1986), Lemma 49. □

1.3.1 Product forcing

If $\mathscr{P}$ and $\mathscr{Q}$ are partially ordered sets, then the cartesian product $P \times Q$ may be partially ordered pointwise to obtain a partially ordered set $\mathscr{P} \times \mathscr{Q}$

$$\langle p_0, q_0 \rangle \leq \langle p_1, q_1 \rangle \longleftrightarrow p_0 \leq p_1 \wedge q_0 \leq q_1$$

It easily seen that $\mathscr{P} \times \mathscr{Q}$, considered as a topological space, with the order topology, is **homeomorphic** to the product of topological spaces $\mathscr{P}$ and $\mathscr{Q}$.

Lemma 1.99. Let $\mathscr{P}$ and $\mathscr{Q}$ be two notions of forcing in $\mathscr{M}$ and let $G \subseteq \mathscr{P}$, $H \subseteq \mathscr{Q}$. The following statements are equivalent.

(a) $G \times H$ is $\mathscr{P} \times \mathscr{Q}$-generic over $\mathscr{M}$.

(b) G is $\mathscr{P}$-generic over $\mathscr{M}$ and H is $\mathscr{Q}$-generic over $\mathscr{M}[G]$.

Proof. A complete proof of this lemma can be found in (JECH, 2003), Lemma 15.9. □

Corollary 1.100. Under the conditions of the previous lemma, then the following are equivalent:

1. $G \times H$ is $\mathscr{P} \times \mathscr{Q}$-generic over $\mathscr{M}$.
2. G is $\mathscr{P}$-generic over $\mathscr{M}$ and H is $\mathscr{Q}$-generic over $\mathscr{M}[G]$.
3. H is $\mathscr{Q}$-generic over $\mathscr{M}$ and G is $\mathscr{P}$-generic over $\mathscr{M}[H]$.

Futhermore, if (1-3) hold, then $\mathscr{M}[G][H] = \mathscr{M}[H][G]$.

Lemma 1.101. If $\mathscr{P}$ and $\mathscr{Q}$ are forcings then the following are equivalent.

(a) $\mathscr{P} \times \mathscr{Q}$ is Baire in $\mathscr{M}$.

(b) $\mathscr{P}$ is Baire in $\mathscr{M}$, and whenever G is $\mathscr{P}$-generic over $\mathscr{M}$, then $\mathscr{Q}$ is Baire in $\mathscr{M}[G]$.

Proof. First, suppose that $\mathscr{P} \times \mathscr{Q}$ is Baire in $\mathscr{M}$ and G is a $\mathscr{P}$-generic filter over $\mathscr{M}$. Let H be a $\mathscr{Q}$-generic filter over $\mathscr{M}[G]$ and let a function $f : \omega \to \mathrm{Ord} \in (\mathscr{M}[G])[H]$. By Lemma 1.100, $f \in \mathscr{M}[G \times H] = (\mathscr{M}[G])[H]$, so $f \in \mathscr{M} \subseteq \mathscr{M}[G]$.

Now, let F be a $\mathscr{P} \times \mathscr{Q}$-generic over $\mathscr{M}$ and $f : \omega \to \mathrm{Ord} \in \mathscr{M}[F]$. By Lemma 1.100, $F = G \times H$, where G is $\mathscr{P}$-generic over $\mathscr{M}$ and H is $\mathscr{Q}$-generic over $\mathscr{M}[G]$. As $\mathscr{Q}$ is Baire in $\mathscr{M}[G]$ and $f \in \mathscr{M}[F] = (\mathscr{M}[G])[H]$, we have that $f \in \mathscr{M}[G]$ and as $\mathscr{P}$ is Baire in $\mathscr{M}$, then $f \in \mathscr{M}$.

□

CHAPTER

2

THE BANACH-MAZUR GAME

In this chapter we will study the topological game of Banach-Mazur and its applications. We will also analyze some of its variations. For the basic part of topological games we follow the article (AURICHI; DIAS, 2019) and the book (KECHRIS, 1995).

2.1 Definitions about topological games

In all of the games considered :

- there will be two players, Player I and Player II, playing against each other;
- there will be ω many innings — meaning that the innings will be numbered $0, 1, 2, 3, \cdots$, and that for each $n \in \omega$ there will be an n-th inning in the play;
- at the end of each complete play of the game, either Player I or Player II will be the winner — there are no draws.

Here we are assuming that the game at hand is a game of perfect information, meaning that, whenever a player must define their next move, it is assumed that they know all the previous moves made so far in the play.

Definition 2.1. Assume that G is an infinite positional game of perfect information, where Player I and Player II alternately choose some objects (e.g., points, sets, functions).

A **strategy** of a player is a function defined for those partial plays of G whose last move was made by the opponent. (Without loss of generality we may assume that the strategy is defined for the opponent's partial plays only, because the strategy determines uniquely the omitted moves of the player.)

Intuitively, a strategy is a way of playing the game. This means that a fixed strategy for one of the players must inform what decision should be taken for each possible situation that this player might encounter during a play of the game.

Definition 2.2 (**Winning strategies**). A winning strategy for a player is a strategy that wins the game, no matter how well the other player plays. In general, one player not having a winning strategy does not imply that the other player has one.

If Player (I or II) is a player of a game G, we denote by

$$\mathrm{I} \uparrow \mathsf{G} \text{ or } \mathrm{II} \uparrow \mathsf{G}$$

the fact that Player (I or II) has a winning strategy in G, and by Player $\not\uparrow \mathsf{G}$ the fact that Player (I or II) does not have a winning strategy in G.

Definition 2.3. A game G is

- **determined** if either $\mathrm{I} \uparrow \mathsf{G}$ or $\mathrm{II} \uparrow \mathsf{G}$;
- **undetermined** otherwise – i.e. if $\mathrm{I} \not\uparrow \mathsf{G}$ and $\mathrm{II} \not\uparrow \mathsf{G}$

Definition 2.4. Two games G and G' are **dual** if

- Player $\mathrm{I} \uparrow \mathsf{G} \iff$ Player $\mathrm{II} \uparrow \mathsf{G}'$

  and
- Player $\mathrm{II} \uparrow \mathsf{G} \iff$ Player $\mathrm{I} \uparrow \mathsf{G}'$

Definition 2.5. Two games G and G' are **equivalent** if

- Player $\mathrm{I} \uparrow \mathsf{G} \iff$ Player $\mathrm{I} \uparrow \mathsf{G}'$

  and
- Player $\mathrm{II} \uparrow \mathsf{G} \iff$ Player $\mathrm{II} \uparrow \mathsf{G}'$

2.2 The Banach-Mazur game

Definition 2.6. The Banach-Mazur game on a topological space X, denoted by $\mathsf{BM}(X)$, is played as follows: Players I and II play an inning per positive integer. In the n-th inning Player I chooses a nonempty open set A_n; Player II responds with a nonempty open set $B_n \subseteq A_n$. Player I must also obey the rule that for each n, $A_{n+1} \subseteq B_n$. A play $A_0, B_0, \cdots A_n, B_n, \cdots$ is won by Player II if $\bigcap_{n\in\omega} B_n \neq \emptyset$; otherwise, Player I wins.

An important observation, that will be of great importance later, is the following. Let $\mathscr{B}$ be a π-base for the topology of the space X. Then the Banach-Mazur game on X is equivalent to the $\mathscr{B}$-Banach-Mazur game on X, the latter being defined by the same rules as the former, with the extra restriction that both Player I and Player II must necessarily choose elements from $\mathscr{B}$ in their moves.

2.2.1 Applications of the Banach-Mazur game

We start with the game-theoretic characterization of the Baire spaces.

Theorem 2.7. A nonempty topological space X is a Baire space if and only if Player I has no winning strategy in the Banach-Mazur game $\mathsf{BM}(X)$.

Proof. First suppose that Player I has no winning strategy in $\mathsf{BM}(X)$. We will show that X is a Baire space. Note that this is equivalent to proving that if X is not Baire then Player I has a winning strategy in $\mathsf{BM}(X)$.

Therefore, suppose that X is not a Baire space. Then there is a sequence $(D_n : n \in \omega)$ of open dense sets in X such that $\bigcap_{n\in\omega} D_n$ is not dense, that is, there is a non-empty open set U such that $U \cap \bigcap_{n\in\omega} D_n = \emptyset$. Now, let us build a winning strategy σ for Player I in $\mathsf{BM}(X)$.

Indeed, in the first inning Player I plays $\sigma(\langle\ \rangle) = U$, so Player II responds B_0. In the second inning, Player I plays $\sigma(\langle B_0 \rangle) = D_0 \cap U$. Note that this is a valid move, because D_n is open and dense for each $n \in \omega$, so Player II responds B_1. Then, in the inning $n \in \omega$, Player I plays $\sigma(\langle B_0, \cdots, B_{n-1} \rangle) = (D_0 \cap \cdots \cap D_{n-1}) \cap B_{n-1}$. Note that this is a valid move, by Corollary 1.49, so Player II responds B_n, and so on. Then $\bigcap_{n\in\omega} B_n \subseteq \bigcap_{n\in\omega} D_n \cap U = \emptyset$, so σ is a winning strategy for Player I.

Now, suppose that X is a Baire space. We will show that Player I does not have a winning strategy in $\mathsf{BM}(X)$. For this let σ be a strategy for Player I. We will construct a nonempty pruned subtree $T \subseteq dom(\sigma)$ and in T we will find a play in which Player I does not win.

Claim 2.7.3. Let σ be a strategy for Player I in $\mathsf{BM}(X)$. If $t = (B_0, \cdots, B_n)$ is a sequence of open sets in the domain of σ, then there exists a maximal family $\mathscr{B}_t$ of open sets contained in $\sigma(t)$ such that $\{\sigma(t^\frown V) : V \in \mathscr{B}_t\}$ is a family of pairwise disjoint non-empty open sets.

Proof. Let $t \in dom(\sigma)$ and consider the family

$$\mathscr{F} = \{\mathscr{B} \subseteq \sigma(t) : \{\sigma(t^\frown V) : V \in \mathscr{B}\} \text{ is a family of pairwise disjoint non-empty open sets }\}.$$

Note that $(\mathscr{F}, \subseteq)$ is a partially ordered set, and $\mathscr{F} \neq \emptyset$, because $\{\sigma(t)\} \in \mathscr{F}$. Also, if $\mathscr{C} \subseteq \mathscr{F}$ is a chain, then $\bigcup \mathscr{C} \in \mathscr{F}$ is an upper bound for $\mathscr{C}$. Then, by the Kuratowski-Zorn Lemma, $\mathscr{F}$ has a maximal element. We will call this element by $\mathscr{B}_t$, for each $t \in dom(\sigma)$. □

To construct T we determine inductively which sequences from $dom(\sigma)$ of length n we put in T :

- $\langle\,\rangle \in T$
- if $t \in T$, then $t^\frown V \in T$ if and only if $V \in \mathscr{B}_t$.

Claim 2.7.4. $\bigcup_{V \in \mathscr{B}_t} \sigma(t^\frown V)$ is open and dense in $\sigma(t)$, for all $t \in T$.

Proof. Suppose otherwise, that is, there exists a non-empty open set $W \subseteq \sigma(t)$ such that $\bigcup_{V \in \mathscr{B}_t} \sigma(t^\frown V) \cap W = \emptyset$. Note that $W \notin \mathscr{B}_t$ and $\sigma(t^\frown W) \subseteq W$. Then $\mathscr{B}_t \cup \{W\}$ violates the maximality of $\mathscr{B}_t$. □

For each $n \in \omega$, define $\mathscr{A}_n = \{t \in T : |t| = n\}$ and $A_n = \bigcup_{t \in \mathscr{A}_n} \sigma(t)$.

Claim 2.7.5. For each $n \in \omega$, A_n is open and dense in $A_0 = \sigma(\langle\,\rangle)$.

Proof. Suppose by induction hypothesis that A_n is open and dense in $\sigma(\langle\,\rangle)$. We will show that A_{n+1} is open and dense in $\sigma(\langle\,\rangle)$. In fact, let $A \subseteq \sigma(\langle\,\rangle)$ a non-empty open set. So $\emptyset \neq A_n \cap A$, then there exists $t \in \mathscr{A}_n$ such that $\emptyset \neq \sigma(t) \cap A$. By Claim 2.7.4, there is $V \in \mathscr{B}_t$ such that $\emptyset \neq \sigma(t^\frown V) \cap A \subseteq A_{n+1} \cap A$, because $t^\frown V \in \mathscr{A}_{n+1}$. □

Note that, if $t \in \mathscr{A}_n$ and $s \in \mathscr{A}_{n+1}$, for some $n \in \omega$ and $\sigma(t) \cap \sigma(s) \neq \emptyset$. Then $s = t^\frown V$ for some $V \in \mathscr{B}_t$. Indeed, note that $\sigma(t) \cap \sigma(s)$ is a non-empty open set in $\sigma(t)$, so by Claim 2.7.4, there is a $V \in \mathscr{B}_t$ such that $\emptyset \neq \sigma(t^\frown V) \cap \sigma(s)$. As $\sigma[\mathscr{A}_n]$ is pairwise disjoint, $\sigma(t^\frown V) = \sigma(s)$. Also $t^\frown V = s$, because otherwise there is an $n_0 < n$ such that $s_{n_0} \neq (t^\frown V)_{n_0}$ and $\emptyset \neq \sigma(t^\frown V) = \sigma(s) \subseteq \sigma(s|_{n_0+1}) \cap \sigma((t^\frown V)|_{n_0+1}) = \emptyset$, contradiction.

Finally, as X is a Baire space, we have that the non-empty open subspace $\sigma(\langle\,\rangle) = U$ is Baire. As $\langle A_n : n \in \omega\rangle$ is a sequence of open dense sets in $A_0 = \sigma(\langle\,\rangle) = U$. Then $\bigcap_{n \in \omega} A_n$ is dense in A_0. In particular, there is $x \in \bigcap_{n \in \omega} A_n$, so $x \in \sigma(\langle\,\rangle) = U$. By the last observation, there is only one $V_0 \in \mathscr{B}_{\langle\,\rangle}$ such that $x \in \sigma(\langle V_0\rangle)$, also, $x \in A_1$. Again by the last observation, there is only one $V_1 \in \mathscr{B}_{\langle V_0\rangle}$ such that $x \in \sigma(\langle V_0, V_1\rangle)$, and so on. Then there exists a run $(\sigma(\langle\,\rangle), V_0, \sigma(\langle V_0\rangle), V_1, \cdots)$ such that $(V_n : n \in \omega) \in T$ and $x \in \bigcap_{n \in \omega} V_n$, that is, Player II wins this run. Then σ is not a winning strategy. Therefore, Player I has no winning strategy in $\mathsf{BM}(X)$. □

Theorem 2.8. In every complete metric space X, Player II has a winning strategy in $\mathsf{BM}(X)$.

Proof. Let X be a complete metric space, we are going to build a winning strategy δ for Player II in $\mathsf{BM}(X)$. Indeed, in the first inning Player I plays U_0 a non-empty open set. Let $x_0 \in U_0$ and $r_0 < 1$ such that $\overline{B_{r_0}^{(x_0)}} = \{y \in X : d(x_0,y) \leq r_0\} \subseteq U_0$. Then Player II plays $\delta(\langle U_0 \rangle) = B_{r_0}^{(x_0)}$. In the second inning Player I plays $U_1 \subseteq B_{r_0}^{(x_0)}$. Let $x_1 \in U_1$ and $r_1 < \frac{1}{2}$ such that $\overline{B_{r_1}^{(x_1)}} \subseteq U_1$. Then Player II plays $\delta(\langle U_0, U_1 \rangle) = B_{r_1}^{(x_1)}$. In the inning $n \in \omega$, if Player I plays U_n, let $x_n \in U_n$ and $r_n < \frac{1}{n+1}$ such that $\overline{B_{r_n}^{(x_n)}} \subseteq U_n$. Then Player II plays $\delta(\langle U_0, \cdots, U_n \rangle) = B_{r_n}^{(x_n)}$, and so on. Note that $(x_n)_{n\in\omega}$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$ and $n_0 \in \omega$ such that $\frac{1}{n_0+1} < \varepsilon$. Then, if $m, n > n_0$, $d(x_m, x_n) < \varepsilon$.

As X is a complete metric space, there exists $x \in X$ such that $(x_n)_{n\in\omega}$ converges to x. We claim that $x \in \overline{B_{r_{n+1}}^{(x_{n+1})}}$, for all $n \in \omega$. Indeed, suppose that $x \in \overline{B_{r_{n+1}}^{(x_{n+1})}}$. We will show that $x \in \overline{B_{n+2}} = \overline{B_{r_{n+2}}^{(x_{n+2})}}$. Note that $\overline{B_{n+2}}$ is closed. Consider the sub-sequence $(x_k)_{k\geq n+2} \subseteq \overline{B_{n+2}}$, and note that $(x_k)_{k\geq n+2} \subseteq \overline{B_{n+2}}$ also converges to $x \in \overline{B_{n+2}}$. Then $x \in \bigcap_{n\in\omega} B_{r_n}^{(x_n)}$ and therefore δ is a winning strategy for Player II in $\mathsf{BM}(X)$. □

Corollary 2.9. Every complete metric space is Baire.

Proof. Let X be a complete metric space. As Player II has a winning strategy in $\mathsf{BM}(X)$, we have that Player I has no winning strategy in $\mathsf{BM}(X)$. Therefore, by Theorem 2.7, X is a Baire space. □

Proposition 2.10. Let X be a topological space and let D be a G_δ and dense subset of X. Then Player II has a winning strategy in the game $\mathsf{BM}(X)$ if and only if Player II has a winning strategy in $\mathsf{BM}(D)$.

Proof. Let $D = \bigcap_{n<\omega} D_n$ be dense G_δ set, note that D_n is dense for each $n \in \omega$. Let δ be a winning strategy for Player II in $\mathsf{BM}(X)$. Now we are going to build a winning strategy δ' for Player II in $\mathsf{BM}(X)$.

Indeed, in the first inning in D, Player I plays $A^0 \cap D$, where A_0 is open in X. Now in X, in the first inning, Player I plays $A^0 \cap D_0$, then Player II responds $\delta(\langle A^0 \cap D_0 \rangle) = B^1$. Then in D, Player II responds $\delta'(\langle A^0 \cap D \rangle) = B^1 \cap D$.

In the second inning in D, Player I plays $A^2 \cap D$. Now in X, in the second inning, Player I plays $(A^2 \cap B^1) \cap D_1$, so Player II responds $\delta(\langle A^0 \cap D_0, (A^2 \cap B^1) \cap D_1 \rangle) = B^3$. Then in D, Player II responds $\delta'(\langle A^0 \cap D, A^2 \cap D \rangle) = B^3 \cap D$, and so on.

BM (D)

Player I	Player II
$A^0 \cap D$	
	$B^1 \cap D$
$A^2 \cap D$	
	$B^3 \cap D$
$\vdots$	$\vdots$

BM(X)

Player I	Player II
$A^0 \cap D_0$	
	$\delta(\langle A^0\rangle) = B^1$
$(A^2 \cap B^1) \cap D_1$	
	$\delta(\langle A^0 \cap D_0, (A^2 \cap B^1) \cap D_1\rangle) = B^3$
$\vdots$	$\vdots$

As δ is a winning strategy in X, then $\bigcap_{n<\omega} B^n \neq \emptyset$. Choose $x \in \bigcap_{n<\omega} B^n$. In particular $x \in D$, therefore $\bigcap_{n<\omega}(B^n \cap D) \neq \emptyset$. So δ' is a winning strategy for Player II in D.

Now suppose that Player II has a winning strategy δ' in $\mathsf{BM}(D)$. We will show that Player II has a winning strategy δ in $\mathsf{BM}(X)$.

Indeed, in the first inning in X, Player I plays A_0. Now in D, in the first inning, Player I plays $A_0 \cap D$, then Player II responds $\delta'(\langle A_0 \cap D\rangle) = B_1 \cap D$. Then in X, Player II responds $\delta(\langle A_0\rangle) = B_1 \cap A_0$.

In the second inning in X, Player I plays A_2. Now in D, in the second inning, Player I plays $A_2 \cap D$, next Player II responds $\delta(\langle A_0 \cap D, A_2 \cap D\rangle) = B_3 \cap D$. Then in X, Player II responds $\delta'(\langle A_0, A_2\rangle) = B_3 \cap A_2$, and so on.

BM (X)

Player I	Player II
A_0	
	$B_1 \cap A_0$
A_2	
	$B_3 \cap A_2$
$\vdots$	$\vdots$

BM(D)

Player I	Player II
$A_0 \cap D$	
	$B_1 \cap D$
$A_2 \cap D$	
	$B_3 \cap D$
$\vdots$	$\vdots$

As δ is a winning strategy for Player II in $\mathsf{BM}(D)$, then $\emptyset \neq \bigcap_{n\in\omega} B_{2n+1} \cap D \subseteq \bigcap_{n\in\omega} B_{2n+1} \cap A_{2n}$. Threfore δ is a winning strategy for Player II in $\mathsf{BM}(X)$, $\square$

Definition 2.11. A topological space X is defined to be **Choquet** if Player II has a winning strategy in the Banach-Mazur game $\mathsf{BM}(X)$.

Choquet spaces were introduced in 1975 by White who called them weakly α-favorable spaces. Note that every Choquet space is a Baire space this follows from Theorem 2.7.

Now we present the result of Oxtoby (OXTOBY, 1980), which gives us a characterization for metrizable Choquet spaces.

Theorem 2.12 (Oxtoby)**.** A metrizable space X is Choquet if, and only if, it contains a dense completely metrizable subspace.

Proof. First, suppose that X contains a dense completely metrizable subspace G. By Theorem 1.19, G is a G_δ-set and dense in X. Consider $\hat{X}$, the completion of X. In particular $\hat{X}$ is a Baire space, because it is a complete metric space. Note that G is also a G_δ-set and dense in $\hat{X}$. Put $G = \bigcap_{n\in\omega} G_n$. Note that G_n is open and dense in $\hat{X}$, for each $n \in \omega$.

As $\hat{X}$ is a complete metric space, it follows that Player II has a winning strategy in $\mathsf{BM}(\hat{X})$. Also, by Proposition 2.10, we have that Player II has a winning strategy δ' in $\mathsf{BM}(G)$, as G is a G_δ dense subset of X. Again by Proposition 2.10, Player II has a winning strategy in $\mathsf{BM}(X)$. Then X is Choquet.

Now, suppose that X is Choquet. We will show that X contains a dense completely metrizable subspace. Consider $\hat{X}$, the completion of X. Let δ be a winning strategy for Player II in $\mathsf{BM}(X)$. We start with some claims.

Claim 2.12.6. If δ is a winning strategy for Player II in $\mathsf{BM}(X)$, then there exists a winning strategy δ' for Player II in $\mathsf{BM}(X)$ such that for each $t = (U_0, \cdots, U_n) \in dom(\delta')$ we have that

(i) $\delta'(t) = \hat{V}_n \cap X$, where $\hat{V}_n$ is a non-empty open set in $\hat{X}$ with $diam(\hat{V}_n) \leq 2^{-n}$ and such that if $U_n = V_n \cap X$ then $\hat{V}_n \subseteq V_n$;

(ii) also, if $t^\frown U_{n+1} \in dom(\delta')$, that is, $\delta'(t^\frown U_{n+1}) = \hat{V}_{n+1} \cap X$. Then $\overline{\hat{V}_{n+1}} \subseteq \hat{V}_n$.

Proof. We will build δ' as follows:

In the first inning, if Player I plays $U_0 = V_0 \cap X$, where V_0 is a non-empty open set in $\hat{X}$, then Player II plays $\delta(\langle U_0 \rangle) = W_0 \cap X$. Let $x_0 \in V_0 \cap W_0 \cap X$ and $r_0 < \frac{1}{2}$ such that $\overline{B_{r_0}^{(x_0)}} \subseteq W_0 \cap V_0$ and set $\delta'(\langle U_0 \rangle) = B_{r_0}^{(x_0)} \cap X$. In the second inning, if Player I plays $U_1 = V_1 \cap X \subseteq B_{r_0}^{(x_0)} \cap X$ a non-empty open set in X, then Player II plays $\delta(\langle U_0, U_1 \rangle) = W_1 \cap X$. Consider the non-empty set $V_1 \cap W_1 \cap B_{r_0}^{(x_0)}$, let $x_1 \in V_1 \cap W_1 \cap B_{r_0}^{(x_0)}$ and $r_1 < \frac{1}{4}$ be such that $\overline{B_{r_1}^{(x_1)}} \subseteq V_1 \cap W_1 \cap B_{r_0}^{(x_0)}$ and set $\delta'(\langle U_0, U_1 \rangle) = B_{r_1}^{(x_1)} \cap X$. In the inning $n \in \omega$, if Player I plays $U_n = V_n \cap X \subseteq B_{r_{n-1}}^{(x_{n-1})} \cap X$, then Player II plays $\delta(\langle U_0, \cdots, U_n \rangle) = W_n \cap X$. Choose $x_n \in V_n \cap W_n \cap B_{r_{n-1}}^{(x_{n-1})}$ and $r_n < \frac{1}{2^{n+1}}$ be such that $\overline{B_{r_n}^{(x_n)}} \subseteq V_n \cap W_n \cap B_{r_{n-1}}^{(x_{n-1})}$ and set $\delta'(\langle U_0, \cdots, U_n \rangle) = B_{r_n}^{(x_n)} \cap X$, and so on.

As δ is a winning strategy for Player II, there exists $x \in \bigcap_{n\in\omega} W_n \cap X = \bigcap_{n\in\omega} U_n \subseteq \bigcap_{n\in\omega} B_{r_n}^{(x_n)} \cap X$. Then δ' is a winning strategy for Player II in $\mathsf{BM}(X)$. □

Claim 2.12.7. Let δ' and δ as above. If $s = (U_0, \cdots, U_{n-1}) \in dom(\delta)$, then there is a maximal family $\mathscr{B}_s$ contained in $\delta'(s)$ such that, if $\hat{V}_n \cap X = \delta'(s^\frown B)$, where $B \in \mathscr{B}_s$, then $\hat{\mathscr{V}}_s = \{\hat{V}_n : B \in \mathscr{B}_s\}$ is a family of pairwise disjoint open sets in $\hat{X}$.

Proof. Let $t \in dom(\delta)$ and consider the family

$$\mathscr{F} = \{\mathscr{B} \subseteq \delta(s) : \{\delta'(s^\frown B) : B \in \mathscr{B}\} \text{ is a family of pairwise disjoint non-empty open sets}\}$$

Note that $(\mathscr{F}, \subseteq)$ is a partially ordered set, and $\mathscr{F} \neq \emptyset$, because $\{U_{n-1}\} \in \mathscr{F}$. Also, if $\mathscr{C} \subseteq \mathscr{F}$ is a chain, then $\bigcup \mathscr{C} \in \mathscr{F}$ is an upper bound for $\mathscr{C}$. Then, by the Kuratowski-Zorn Lemma, $\mathscr{F}$ has a maximal element, we will call this element $\mathscr{B}_s$, for each $s \in dom(\delta)$. □

Also, by construction we have that $diam(\hat{V}_n) < 2^{-n}$ for all $\hat{V}_n \in \hat{\mathscr{V}}_s$.

Now we are going to build a subtree $S \subseteq dom(\delta')$ consisting of sequences of the form $(U_0, \hat{V}_0, U_1, \hat{V}_1, \cdots, U_n, \hat{V}_n)$, where U_i are non-empty open sets in X and $\hat{V}_i$ are non-empty open in $\hat{X}$. Also by Claim 2.12.6, we have that $\hat{V}_0 \supseteq \hat{V}_1 \supseteq \cdots$ and if $V_i = \hat{V}_i \cap X$, the run $(U_0, V_0, U_1, V_1, \cdots)$ is compatible with δ.

To construct S we determine inductively which sequences from $dom(\delta')$ of length n we put in S :

- $\{\langle U \rangle : U \in \mathscr{B}_0\} \in S$, where $\mathscr{B}_0$ is the maximal family of open sets of Claim 2.12.7.
- if $s \in S$, then $s^\frown B \in S$ if and only if $B \in \mathscr{B}_s$.

Claim 2.12.8. $\bigcup \hat{\mathscr{V}}_s = \bigcup\{\hat{V}_n : s \in S\}$ is dense in $\hat{V}_{n-1}$, for all $s = (U_0, \cdots, U_{n-1}) \in S$.

Proof. Suppose otherwise, that is, there exists a non-empty open set $W \subseteq \hat{V}_{n-1}$ such that $\bigcup \hat{\mathscr{V}}_s \cap W = \emptyset$. Note that $W \cap X \notin \mathscr{B}_s$, therefore $\hat{\mathscr{V}}_s \cup \{W \cap X\}$ violates the maximality of $\mathscr{B}_s$. □

For each $n \geq 1$, we define $\mathscr{W}_n = \{s \in S : |s| = n\}$ and $W_n = \bigcup\{\hat{V}_n : s \in S\}$.

Claim 2.12.9. For each $n \geq 1$, W_n is open and dense in $\hat{X}$.

Proof. Suppose by induction hypothesis that W_n is open and dense in $\hat{X}$. We will show that W_{n+1} is open and dense in $\hat{X}$. In fact, let A be a non-empty open set in $\hat{X}$. So $\emptyset \neq W_n \cap A$ and there exists $s \in S$ such that $\emptyset \neq \hat{V}_n \cap A$. By Claim 2.12.8, $\bigcup\{\hat{V}_{n+1} : s \in S\}$ is dense in $\hat{V}_n$, then there exists $s' \in S, |s'| = n+1$ such that $\emptyset \neq \hat{V}_{n+1} \cap (\hat{V}_n \cap A) \subseteq W_{n+1} \cap A$. □

Claim 2.12.10. $\bigcap_{n\geq 1} W_n \subseteq X$

Proof. Let $x \in \bigcap_{n\geq 1} W_n$, in particular $x \in W_1$. By Claim 2.12.7, there exists a unique $\hat{V}_1$ and there exists U_1 non-empty open in X such that $x \in \hat{V}_1$ and $\delta'(\langle U_1 \rangle) = \hat{V}_1 \cap X$. As $x \in W_1$, again by Claim 2.12.7, there exists a unique $\hat{V}_2$ and there is U_2 non-empty open in X such that $x \in \hat{V}_2$ and $\delta'(\langle U_1, U_2 \rangle) = \hat{V}_2 \cap X$, and so on. Then there exists a unique $(U_n)_{n\geq 1} \in S \subseteq dom(\delta')$ such that $x \in \bigcap_{n\geq 1} \hat{V}_n$. By construction, $diam(\hat{V}_n) < 2^{-n}$, therefore $\{x\} = \bigcap_{n\geq 1} \hat{V}_n$. Also, as δ' is a winning strategy for Player II, we have that $\emptyset \neq \bigcap_{n\geq 1} V_n = \bigcap_{n\geq 1} \hat{V}_n \cap X$, then $x \in X$. □

As $\hat{X}$ is Baire, we have that $W = \bigcap_{n\geq 1} W_n$ is dense in $\hat{X}$. So W is dense in X and is a G_δ-set, by Theorem 1.19, W is completely metrizable. Then X contains a dense completely metrizable subspace W.

□

Corollary 2.13. Let X be a dense subset of the real line. Then X is Choquet if, and only if, X is residual in $\mathbb{R}$.

Proof. By Theorem 2.12, X contains a dense completely metrizable subspace D. Note that D is dense completely metrizable in $\mathbb{R}$, by Theorem 1.19, D is a G_δ-set in $\mathbb{R}$. Therefore, by Proposition 1.56, X is residual in $\mathbb{R}$. □

Our motivation for this part is to characterize the spaces in which the Banach-Mazur game is undetermined.

Definition 2.14. Let X be a topological space. We say that X is an **undetermined space** if the Banach-Mazur game played on X is undetermined.

Lemma 2.15. Let $X \subseteq \mathbb{R}$ be a dense Baire space. Then $G \cap X$ is dense for each dense G_δ-set G in $\mathbb{R}$.

Proof. Let $G = \bigcap_{n\in\omega} G_n$ be a dense G_δ-set, so G_n is open and dense for each $n \in \omega$. Note that $G_n \cap X$ is open and dense in X for each $n \in \omega$. As X is a Baire space then $G \cap X = \bigcap_{n\in\omega}(G_n \cap X)$ is a dense set in X. □

Theorem 2.16. If $X \subseteq \mathbb{R}$ is a **dense** undetermined space then $G \cap X \neq \emptyset$ and $G \cap (\mathbb{R} \setminus X) \neq \emptyset$ for every dense G_δ-set $G \subseteq \mathbb{R}$.

Proof. Let G be a dense G_δ set. By Lemma 2.15, we have that $G \cap X$ is dense. In particular, $G \cap X \neq \emptyset$. Now, by Proposition 2.10, there exists δ_G be a winning strategy for Player II in the game BM(G). We will build a strategy $\tilde{\delta}$ for Player II in BM(X).

Indeed, in the first inning in X, Player I plays $A_0 \cap X$, where A_0 is open in $\mathbb{R}$. Now in G, in the first inning, Player I plays $A_0 \cap G$. Then Player II responds $\delta_G(\langle A_0 \cap G \rangle) = B_1 \cap G$, then in X, Player II responds $\tilde{\delta}(\langle A_0 \cap X \rangle) = B_1 \cap (G \cap X)$. In the second inning in X, Player I plays $A_2 \cap X$. Now in G, in the second inning, Player I plays $(A_2 \cap B_1) \cap G$, so Player II responds

$\delta_G(\langle A_0 \cap G, (A_2 \cap B_1) \cap G \rangle) = B_3 \cap G$. Then in X, Player II responds $\tilde{\delta}(\langle A_0 \cap X, A_2 \cap X \rangle) = B_1 \cap G$, and so on.

BM(X)		BM(G)	
Player I	Player II	Player I	Player II
$A_0 \cap X$		$A_0 \cap G$	
	$\tilde{\delta}(\langle A_0 \cap X \rangle) = B_1 \cap (G \cap X)$		$\delta_G(\langle A_0 \cap G \rangle) = B_1 \cap G$
$A_2 \cap X$		$(A_2 \cap B_1) \cap G$	
	$B_3 \cap (G \cap X)$		$\delta_G(\langle A_0 \cap G, (A_2 \cap B_1) \cap G \rangle) = B_3 \cap G$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

As δ_G is a winning strategy for Player II, then there exists $z \in \bigcap_{n<\omega} B_{2n+1} \cap G$. If $z \in X$ then $\tilde{\delta}$ is a winning strategy for Player II in $\mathsf{BM}(X)$, contradiction. Therefore $z \notin X$, then $z \in G \cap (\mathbb{R} \setminus X)$. $\square$

Corollary 2.17. If $X \subseteq \mathbb{R}$ is a **dense** undetermined space then $G \cap X$ is dense in X and $G \cap (\mathbb{R} \setminus X) \neq \emptyset$ for every dense G_δ-set $G \subseteq \mathbb{R}$.

Finally, joining the characterizations via games with Baire and Choquet spaces in the real line we have the following:

Corollary 2.18. Let X a dense subset of the real line. Then X is undetermined if, and only if, X is Baire and is not residual.

Later we will see an explicit example of an undetermined space on the real line this will be a Bernstein set. Furthermore, we will see that a Baire space that is not productively Baire is an undetermined space. In particular, the counterexamples mentioned in the introduction to this thesis are examples of undetermined spaces.

Theorem 2.19. If f is a continuous, open mapping of X onto Y, and X is Choquet, then Y is Choquet.

Proof. Let δ_X be a winning strategy for Player II in $\mathsf{BM}(X)$. We are going to build a winning strategy δ_Y for Player II in $\mathsf{BM}(Y)$.

Indeed, in the first inning in Y, Player I plays U_0 a non-empty open set in Y. Consider the non-empty open set $f^{-1}(U_0) \subseteq X$. In X, in the first inning, Player I plays $f^{-1}(U_0)$ and Player II responds $\delta_X(\langle f^{-1}(U_0)\rangle) = V_0$. Then, in Y, Player II plays $\delta_Y(\langle U_0\rangle) = f(\delta_Y(\langle U_0\rangle)) = f(V_0)$.

In the second inning, in Y, Player I plays $U_1 \subseteq f(V_0)$. Consider the non-empty open set $f^{-1}(U_1) \cap V_0$. In X, Player I plays $f^{-1}(U_1) \cap V_0$ and Player II responds $\delta_X(\langle f^{-1}(U_0), f^{-1}(U_1) \cap V_0\rangle) = V_1$ and, in Y, Player II responds $\delta_Y(\langle U_0, U_1\rangle) = f(V_1)$.

In the inning $n \in \omega$ in Y, if Player I plays $U_{n-1} \subseteq f(V_{n-2})$, consider the non-empty open set $f^{-1}(U_{n-1}) \cap V_{n-2}$. In X, Player I plays $f^{-1}(U_{n-1}) \cap V_{n-2}$ and Player II responds $\delta_X(\langle f^{-1}(U_0), \cdots, f^{-1}(U_{n-1}) \cap V_{n-2}\rangle) = V_{n-1}$ and, in Y, Player II responds $\delta_Y(\langle U_0, \cdots, U_{n-1}\rangle) = f(V_{n-1})$.

$\mathsf{BM}(Y)$

Player I	Player II
U_0	
	$\delta_Y(\langle U_0\rangle) = f(V_0)$
U_1	
	$\delta_Y(\langle U_0, U_1\rangle) = f(V_1)$
$\vdots$	$\vdots$

$\mathsf{BM}(X)$

Player I	Player II
$f^{-1}(U_0)$	
	$\delta_X(\langle A_0 \cap G\rangle) = V_0$
$f^{-1}(U_1) \cap V_0$	
	$\delta_X(\langle f^{-1}(U_0), f^{-1}(U_1) \cap V_0\rangle) = V_1$
$\vdots$	$\vdots$

As δ_X is a winning strategy then there exists $x \in \bigcap_{n\in\omega} V_n$, therefore $f(x) \in \bigcap_{n\in\omega} f(V_n)$, that is, δ_Y is a winning strategy for Player II in $\mathsf{BM}(Y)$.

$\square$

Corollary 2.20. Let $\{X_i : i \in I\}$ is a family of topological spaces such that $\prod_{i\in I} X_i$ is a Choquet space, then X_i is Choquet for each $i \in I$.

Theorem 2.21. Every Choquet metric space (X,d) without isolated points[1] contains a subspace homeomorphic to the Cantor set.

Proof. We are going to build a system $\mathscr{U} = \{U_s : s \in 2^{<\omega}\}$ on X such that

1. U_s is open non-empty;
2. $diam(U_s) \leq \frac{d}{2^{|s|}}$;
3. For $s \in 2^{<\omega}$ and $i \in \{0,1\}$, $U_{s\frown i} \subseteq U_s$ such that
   - $U_{s\frown 0} \cap U_{s\frown 1} = \emptyset$;
   - $diam(U_{s\frown i}) \leq \frac{d}{2^{|s|+1}}$.

We will use the Banach-Mazur game to build this family. Indeed, in the first inning Player I plays any open ball $U_\emptyset$ of diameter d. Then Player II plays $V_\emptyset$ such that $V_\emptyset$ is an open non-empty set and $V_\emptyset \subseteq U_\emptyset$. In the second inning, inside $V_\emptyset$ we build two open non-empty disjoint sets $U_{(0)}, U_{(1)}$ such that $U_{(0)} \cap U_{(1)} = \emptyset$ and $diam(U_{(0)}), diam(U_{(1)}) \leq \frac{d}{2}$. Player I can play any of the open sets $U_{(0)}, U_{(1)}$. Then Player II gives the respective responses $V_{(0)}, V_{(1)}$ for any move by Player I.

Player I	Player II	
$U_{(0)}$	$V_{(0)}$	where $V_{(0)} \subseteq U_{(0)}$
$U_{(1)}$	$V_{(1)}$	where $V_{(1)} \subseteq U_{(1)}$

Justification. Note that there is $x \in V_\emptyset$ and as $V_\emptyset$ is open there is an $\delta > 0$ such that $B_\delta^{(x)} \subseteq V_\emptyset$ and $\overline{B_\delta^{(x)}} \subseteq B_\delta^{[x]} \subsetneq V_\emptyset$. As X does not have isolated points, $B_\delta^{(x)} \cap X \setminus \{x\} \neq \emptyset$.

That is, there is $y \in B_\delta^{(x)} \subseteq V_\emptyset$ such that $x \neq y$ and as X is Hausdorff, there are two open disjoint sets A_0,A_1 such that $x \in A_0$ and $y \in A_1$, so $x \in A_0 \cap V_\emptyset$ and $y \in A_1 \cap V_\emptyset$ then there are two open balls such that $B_{\delta_0}^{(x)} \subseteq A_0 \cap B_\delta^{(x)} \subseteq A_0 \cap V_\emptyset$ and $B_{\delta_1}^{(y)} \subseteq A_1 \cap B_\delta^{(x)} \subseteq A_1 \cap V_\emptyset$, so $0 < diam(B_{\delta_0}^{(x)}), diam(B_{\delta_1}^{(y)}) \leq diam(U_\emptyset)$. Consider $r_0 = \frac{diam(B_{\delta_0}^{(x)})}{4}$ and $r_1 = \frac{diam(B_{\delta_1}^{(y)})}{4}$. Note that $0 < diam(B_{r_0}^{(x)}), diam(B_{r_1}^{(y)}) \leq \frac{d}{2}$.

Finally, we define $U_{(0)} = B_{r_0}^{(x)}$ and $U_{(1)} = B_{r_1}^{(y)}$. Note that $U_{(0)}, U_{(1)}$ are open and non-empty sets, $U_{(0)} \cap U_{(1)} = \emptyset$ and $diam(U_{(0)}), diam(U_{(1)}) \leq \frac{d}{2}$.

In the inning $|s|$, having defined U_s, we define $U_{s\frown 0}, U_{s\frown 1} \subseteq U_s$. In fact, suppose that Player I plays U_s then Player II responses V_s. Again (as in the initial case) inside V_s we build two open non-empty disjoint sets $U_{s\frown 0}, U_{s\frown 1}$ such that $diam(U_{s\frown 0}), diam(U_{s\frown 1}) \leq \frac{d}{2^{|s|+1}}$. Player I can play any of the open sets $U_{(s\frown 0)}, U_{(s\frown 1)}$ then Player II gives the respective responses

[1] A topological space without isolated points is the same as dense-in-itself space

$V_{s\frown 0}, V_{s\frown 1}$ for any move by Player I. Finally take $\mathscr{U} = \{U_s : s \in 2^{<\omega}\}$ and that is our family sought.

Let $r \in 2^\omega$, we define

$$U_r := \bigcap_{n\in\omega} U_{r\restriction n}.$$

Claim 2.21.11. U_r consists of exactly one point.

Proof. Note that by construction $U_r \neq \emptyset$, because Player II has a winning strategy in $\mathsf{BM}(X)$, that is, $\emptyset \neq \bigcap_{n\in\omega} V_n \subseteq U_r$. Note that $diam(U_r) \leq diam(U_{r\restriction n})$. Suppose that U_r contains more than one point, then $diam(U_r) > 0$. As

$$\lim_{n\to\infty} diam(U_{r\restriction n}) = 0,$$

because $diam(U_{r\restriction n}) \leq \frac{d}{2^n}, \forall n \in \omega$, then there exists $n_0 \in \omega$ such that $diam(U_r) \leq diam(U_{r\restriction n_0}) < diam(U_r)$, contradiction.

$\square$

Therefore, U_r consists of exactly one point. Let us call that point by x_r, that is, $\{x_r\} = U_r$. Define

$$f : 2^\omega \longrightarrow X$$

by

$$f(r) = x_r.$$

As 2^ω is compact and X is Hausdorff it is only necessary to show that f is injective and continuous.

Claim 2.21.12. f is injective.

Let $r, s \in 2^\omega$ be such that $r \neq s$ then $\{n \in \omega : r(n) \neq s(n)\} \neq \emptyset$. Consider $n_0 = \min\{n \in \omega : r(n) \neq s(n)\}$. In particular, $x_r \in U_r \subseteq U_{\langle r_0,..,r_{n_0-1},r_{n_o}\rangle}$ and $x_s \in U_s \subseteq U_{\langle r_0,..,r_{n_0-1},s_{n_o}\rangle}$, but for the construction

$$U_{\langle r_0,..,r_{n_0-1},r_{n_o}\rangle} \cap U_{\langle r_0,..,r_{n_0-1},s_{n_o}\rangle} = \emptyset.$$

Then $x_r \neq x_s$.

Claim 2.21.13. f is continuous.

Proof. Let $\varepsilon > 0$ and consider $B_\varepsilon^{(x_r)}$, as $\lim_{n\to\infty} diam(U_{r\restriction n}) = 0$ then there exists an $n_0 \in \omega$ such that $U_{r\restriction n_0} \subseteq B_\varepsilon^{(x_r)}$ and $x_r \in U_{r\restriction n_0}$. Consider $V_r = \{s \in 2^\omega : r \restriction n_0 \subseteq s\}$. Note that V is open in 2^ω and if $s \in V_r$ then $f(s) = x_s \in U_{r\restriction n_0} \subseteq B_\varepsilon^{(x_r)}$, so f is continuous. $\square$

Then f is an embedding of 2^ω into X, that is, $f(2^\omega) \subseteq X$ is homeomorphic to the Cantor set $\mathscr{C}$.

□

The Banach-Mazur game also has applications for productively Baire spaces. Later we will see that it also has applications for the infinite products of Baire spaces.

Proposition 2.22 (White)**.** Let X, Y be Choquet spaces. Then $X \times Y$ is Choquet.

Proof. Let δ_X and δ_Y be winning strategies for Player II in $\mathsf{BM}(X)$ and $\mathsf{BM}(Y)$ respectively. We will build a strategy δ for Player II in $\mathsf{BM}(X \times Y)$. Indeed, in the first inning in $X \times Y$, Player I plays U_0 a non-empty open set in $X \times Y$. Then there are non-empty open sets A_X^0 and A_Y^0 in X and Y such that $A_X^0 \times A_Y^0 \subseteq U_0$. In X, Player I plays A_X^0 and Player II responds $\delta_X(\langle A_X^0 \rangle)$ and, in Y, Player I plays A_Y^0 and Player II responds $\delta_Y(\langle A_Y^0 \rangle)$. Then Player II responds $\delta(\langle U_0 \rangle) = \delta_X(\langle A_X^0 \rangle) \times \delta_Y(\langle A_Y^0 \rangle)$.

In the second inning, Player I plays $U_1 \subseteq \delta(\langle U_0 \rangle) = \delta_X(\langle A_X^0 \rangle) \times \delta_Y(\langle A_Y^0 \rangle)$. As before, there are non-empty open sets A_X^1 and A_Y^1 in X and Y such that $A_X^1 \times A_Y^1 \subseteq U_1$. In X, Player I plays A_X^1 and Player II responds $\delta_X(\langle A_X^0, A_X^1 \rangle)$ and, in Y, Player I plays A_Y^1 and Player II responds $\delta_Y(\langle A_Y^0, A_Y^1 \rangle)$. Then Player II responds $\delta(\langle U_0, U_1 \rangle) = \delta_X(\langle A_X^0, A_X^1 \rangle) \times \delta_X(\langle A_X^0, A_X^1 \rangle)$, and so on.

As δ_X and δ_Y are winning strategies, we have that $\emptyset \neq \bigcap_{n\in\omega} \delta_X(\langle A_X^0, \cdots, A_X^n \rangle)$ and $\emptyset \neq \bigcap_{n\in\omega} \delta(\langle A_Y^0, \cdots, A_Y^n \rangle)$, then

$$\emptyset \neq \bigcap_{n\in\omega} \delta_X(\langle A_X^0, \cdots, A_X^n \rangle) \times \bigcap_{n\in\omega} \delta_Y(\langle A_Y^0, \cdots, A_Y^n \rangle) \subseteq \bigcap_{n\in\omega} \delta(\langle U_0, \cdots, U_n \rangle).$$

Therefore δ is a winning strategy for Player II in $\mathsf{BM}(X \times Y)$, that is, $X \times Y$ is a Choquet space. □

Proposition 2.23. Let X a Choquet topological space and let Y be a Baire space. Then $X \times Y$ is a Baire space. In other words Choquet spaces are productively Baire.

Proof. Suppose otherwise, that is, $X \times Y$ is not Baire. Then by Theorem 2.7, $\mathrm{I} \uparrow \mathsf{BM}(X \times Y)$, let us call this strategy for Player I in $\mathsf{BM}(X \times Y)$ by σ. We can assume that σ only gives basic open sets. Set ρ a winning strategy for Player II in $\mathsf{BM}(X)$. We will build a winning strategy φ for Player I in $\mathsf{BM}(Y)$.

Indeed, in the first inning in $X \times Y$, Player I plays $\sigma(\langle\rangle) = U_0 \times V_0$, where U_0 and V_0 are non-empty open sets in X and Y respectively. Now in X, in the first inning, Player I plays U_0 and Player II responds $\rho(\langle U_0 \rangle)$. In Y, Player I plays $\varphi(\langle\ \rangle) = V_0$ then Player II responds W_0. Then in $X \times Y$, Player II responds $\rho(\langle U_0 \rangle) \times W_0$.

In the second inning in $X \times Y$, Player I plays $\sigma(\langle \rho(\langle U_0 \rangle) \times W_0 \rangle) = U_1 \times V_1$. Now in X, in the second inning, Player I plays U_1, so Player II responds $\rho(\langle U_0, U_1 \rangle)$. Then in Y, Player I plays $\varphi(\langle W_0 \rangle) = V_1$ and Player II responds W_1. Then in $X \times Y$, Player II responds $\rho(\langle U_0, U_1 \rangle) \times W_1$. And so on.

		$BM(X \times Y)$	$BM(X)$	$BM(Y)$
0	I	$\sigma(\langle\,\rangle) = U_0 \times V_0$	U_0	$\varphi(\langle\,\rangle) = V_0$
	II	$\rho(\langle U_0 \rangle) \times W_0$	$\rho(\langle U_0 \rangle)$	W_0
1	I	$\sigma(\langle \rho(\langle U_0 \rangle) \times W_0 \rangle) = U_1 \times V_1$	U_1	$\varphi(\langle W_0 \rangle) = V_1$
	II	$\rho(\langle U_0, U_1 \rangle) \times W_1$	$\rho(\langle U_0, U_1 \rangle)$	W_1
2	I	$\sigma(\langle \rho(\langle U_0 \rangle) \times W_0, \rho(\langle U_0, U_1 \rangle) \times W_1 \rangle) = U_2 \times V_2$	U_2	$\varphi(\langle W_0, W_1 \rangle) = V_2$
	II	$\rho(\langle U_0, U_1, U_2 \rangle) \times W_2$	$\rho(\langle U_0, U_1, U_2 \rangle)$	W_2
		$\vdots$	$\vdots$	$\vdots$

As σ and ρ are winning strategies for Player I and Player II, respectively, we have that $\bigcap_{n\in\omega} \rho(\langle U_0, \ldots, U_n \rangle) \times W_n = \emptyset$ and $\bigcap_{n\in\omega} \rho(\langle U_0, \ldots, U_n \rangle) \neq \emptyset$. Then in $BM(Y)$ we have that

$$\bigcap_{n\in\omega} W_n = \emptyset.$$

Then φ is a winning strategy for Player I in $\mathsf{BM}(X \times Y)$ and this is a contradiction, because Y is a Baire space. Therefore $X \times Y$ is a Baire space.

$\square$

2.2.2 Modifications of the Banach-Mazur game

In this section we study some variations of the Banach-Mazur game. These will help us characterize new spaces and also continue to study the problem of the product of Baire spaces.

2.2.2.1 The MB(X) game

The game $\mathsf{MB}(X)$ is played like $\mathsf{BM}(X)$, except that now Player I wins if $\bigcap_{n\in\omega} B_n \neq \emptyset$, and Player II wins otherwise.

Proposition 2.24. For a topological space X the following are equivalent:

(1) Player II has a winning strategy in $\mathsf{MB}(X)$,

(2) For each non-empty open $U \subseteq X$, Player I has a winning strategy in $\mathsf{BM}(U)$,

(3) No open (non-empty) subspace of X is a Baire space.

Proof. $(1 \Rightarrow 2)$ Let δ a winning strategy for Player II in $\mathsf{MB}(X)$ and let $U \subseteq X$ be a non-empty open set. We are going to build a winning strategy δ_U for Player I in $\mathsf{BM}(U)$.

Indeed, in the first inning in X, if Player I plays U, next Player II responds $\delta(\langle U\rangle) = V_0 \subseteq U$. Now in U, in the first inning, Player I plays $\delta_U(\langle\rangle) = V_0$, then Player II responds $W_0 \subseteq V_0$. Then, in X, in the second inning, if Player I plays W_0, next Player II responds $\delta(\langle U, W_0\rangle) = V_1 \subseteq W_0$. In the second inning in U, Player I plays $\delta_U(\langle W_0\rangle) = V_1$, then Player II responds $W_1 \subseteq V_1$, and so on. As δ is a winning strategy for Player II, we have that $\bigcap_{n\in\omega} V_n = \emptyset$. Note that δ_U is a winning strategy for Player I in $\mathsf{BM}(U)$, because $\bigcap_{n\in\omega} \delta_U(\langle W_0, \cdots, W_{n-1}\rangle) = \bigcap_{n\in\omega} V_n = \emptyset$. By Theorem, we have that $(2 \Leftrightarrow 3)$. Now, suppose (3). We are going to build a winning strategy δ for Player II in $\mathsf{MB}(X)$.

Indeed, in $\mathsf{MB}(X)$, in the first inning, if Player I plays $A_0 \subseteq X$, as A_0 is not Baire, there exists a winning strategy δ_{A_0} for Player I in $\mathsf{BM}(A_0)$, so if $\delta_{A_0}(\langle\rangle) = V_0 \subseteq A_0$. So, in $\mathsf{MB}(X)$, Player II responds $\delta(\langle A_0\rangle) = V_0$. In the second inning, in $\mathsf{MB}(X)$, if Player I plays $A_1 \subseteq V_0$.

Next, in $\mathsf{BM}(A_0)$, in the first inning, if Player II plays A_1, and, in the second inning, in $\mathsf{BM}(A_0)$, Player I responds $\delta_{A_0}(\langle A_1\rangle) = V_1 \subseteq A_1$. Then, in the second inning, in $\mathsf{MB}(X)$, Player I plays $\delta(\langle A_0, A_1\rangle) = V_1$.

In $\mathsf{MB}(X)$, in the third inning, if Player I plays $A_2 \subseteq V_1$, So, in the second inning, in $\mathsf{BM}(A_0)$, Player II plays A_2. Next, in the third inning, in $\mathsf{BM}(A_0)$, Player I responds $\delta_{A_0}(\langle A_1, A_2\rangle) = V_2 \subseteq A_2$. Then, if Player I plays A_2, next Player II, and so on.

$\mathsf{MB}(X)$	
I	II
A_0	
	$\delta(\langle A_0\rangle) = V_0$
A_1	
	$\delta(\langle A_0, A_1\rangle) = V_1$
A_2	
	$\delta(\langle A_0, A_1\rangle) = V_2$
$\vdots$	$\vdots$

$\mathsf{BM}(A_0)$	
I	II
$\delta_{A_0}(\langle\rangle) = V_0 \subseteq A_0$	
	A_1
$\delta_{A_0}(\langle A_1\rangle) = V_1 \subseteq A_1$	
	A_2
$\delta_{A_0}(\langle A_1, A_2\rangle) = V_2 \subseteq A_2$	
$\vdots$	$\vdots$

As δ_{A_0} is a winning strategy for Player I, we have that $\bigcap_{n\in\omega} V_n = \emptyset$. Note that δ is a winning strategy for Player II in $\mathsf{MB}(X)$, because $\bigcap_{n\in\omega} \delta_{A_0}(\langle A_0, \cdots, A_{n-1}\rangle) = \bigcap_{n\in\omega} V_n = \emptyset$.

$\square$

Theorem 2.25. For a topological space X the following are equivalent:

(1) X is meager in itself.

(2) Player II has a winning strategy in $\mathsf{MB}(X)$.

Proof. First suppose that X is meager in itself, that is, there is a sequence $\langle N_n : n\in\omega\rangle$ of nowhere dense sets in X such that $X = \bigcup_{n\in\omega} N_n$, so $\emptyset = \bigcap_{n\in\omega} X\setminus N_n$. We can assume that N_n is closed with empty interior, for all $n\in\omega$, then $X\setminus N_n$ is open and dense in X. We are going to build a winning strategy δ for Player II in $\mathsf{MB}(X)$.

Indeed, in the first inning Player I plays A_0 and Player II responds $\delta(\langle A_0\rangle) = (X\setminus N_0)\cap A_0$. Note that this is a valid move because $X\setminus N_0$ is open dense. In the second inning, Player I plays $A_1 \subseteq (X\setminus N_0)\cap A_0$ and Player II plays $\delta(\langle A_0, A_1\rangle) = (X\setminus N_1)\cap A_1$, and so on.

In general, in the inning $n\in\omega$, Player I plays A_{n-1} and Player II responds $\delta(\langle A_0, \cdots, A_{n-1}\rangle) = (X\setminus N_{n-1})\cap A_{n-1}$.

Then $\bigcap_{n\in\omega} \delta(\langle A_0, \cdots, A_n\rangle) = \bigcap_{n\in\omega}(X\setminus N_n)\cap A_n \subseteq \bigcap_{n\in\omega}(X\setminus N_n) = \emptyset$. Therefore δ is a winning strategy for Player II in $\mathsf{MB}(X)$.

Now suppose that Player II has a winning strategy in $\mathsf{MB}(X)$. By Proposition 2.24, this is equivalent to no open (non-empty) subspace of X is a Baire space. We will show that X is meager in itself.

Indeed, for each non-empty open subset $A\subseteq X$ there are $\{B_1(A), B_2(A), \cdots\}$ a countable collection of open dense subsets in A and a non-empty open subset $B(A)\subseteq A$ such that

$$\bigcap_{n\in\mathbb{N}} B_n(A)\cap B(A) = \emptyset.$$

Let $\mathscr{A}$ be a maximal family of non-empty subsets such that $\{B(A) : A \in \mathscr{A}\}$ is a pairwise disjoint family.

Note that $A_0 = \bigcup_{A\in\mathscr{A}} B(A)$ is open dense in X. In fact, suppose otherwise, that is, there exists a non-empty subset $U \subseteq X$ such that $U \cap A_0 = \emptyset$, so $B(U) \cap B(A) = \emptyset$ for each $A \in \mathscr{A}$. Note that $U \notin \mathscr{A}$, then $\mathscr{A} \subsetneq \mathscr{A} \cup \{U\}$, contradicting that $\mathscr{A}$ is maximal.

Also, $A_1 = \bigcup_{A\in\mathscr{A}}(B_1(A) \cap B(A))$ is open and dense. In fact, let $V \subseteq X$ be a non-empty set, then there exists $A \in \mathscr{A}$ such that $V \cap B(A) \neq \emptyset$ (because A_0 is dense). As $B_1(A)$ is dense in A, we have that $\emptyset \neq (V \cap B(A)) \cap B_1(A) \subseteq V \cap A_1$. Then, for each $n \in \mathbb{N}$,

$$A_n = \bigcup_{A\in\mathscr{A}} (B_n(A) \cap B(A))$$

is open and dense for each $n \in \mathbb{N}$, by the same argument. Note that, $B(A) \cap A_n = B(A) \cap B_n(A)$.

Therefore,

$$\bigcap_{n\in\omega} A_n = \emptyset.$$

In fact, suppose otherwise, there exists

$$x \in B(A) \cap (\bigcap_{n>0} A_n) = \bigcap_{n>0} (B(A) \cap A_n) = \bigcap_{n>0} (B(A) \cap B_n(A)) = \emptyset$$

Then $X = \bigcup_{n\in\omega} X \setminus A_n$. Therefore X is meager in itself.

□

2.2.2.2 The Cantor game

The Cantor game on X, denoted by $2\mathsf{BM}(X)$, is played as follows: Player I and Player II play an inning per finite ordinal.

At the beginning, Player I plays $B_\emptyset$ a non-empty open set, and then Player II responds two non-empty open subsets $\mathscr{V}_0 = \{V_0, V_1\}$, with $V_0, V_1 \subseteq B_\emptyset$ and consider $W_0 = \bigcup \mathscr{V}_0$. Next, in the first inning, Player I plays $\{B_0, B_1\}$ two non-empty open sets, with $B_0 \subseteq V_0$ and $B_1 \subseteq V_1$ and Player II plays $\mathscr{V}_1 = \{V_{00}, V_{01}, V_{10}, V_{11}\}$ where V_{ij} are non-empty open sets, with $V_{00}, V_{01} \subseteq B_0$ and $V_{10}, V_{11} \subseteq B_1$, consider $W_1 = \bigcup \mathscr{V}_1$, and so on.

$2\mathsf{BM}(X)$

Player I	Player II
$\{B_\emptyset\}$	
	$\mathscr{V}_0 = \{V_0, V_1\}$ with $W_0 = \bigcup \mathscr{V}_0$
$\{B_0, B_1\}$	
	$\mathscr{V}_1 = \{V_{00}, V_{01}, V_{10}, V_{11}\}$ with $W_1 = \bigcup \mathscr{V}_1$
$\vdots$	$\vdots$

Player II wins the game $2\mathsf{BM}(X)$ if $\bigcap_{n\in\omega} W_n \neq \emptyset$, else Player I wins.

Note that in this variation of the game we have that if Player II has a winning strategy in the game $\mathsf{BM}(X)$ then Player II has a winning strategy in $2\mathsf{BM}(X)$.

Theorem 2.26. Let X be a topological space. Then $\mathrm{I} \uparrow \mathsf{BM}(X)$ if and only if $\mathrm{I} \uparrow 2\mathsf{BM}(X)$.

Proof. Suppose that $\mathrm{I} \uparrow \mathsf{BM}(X)$. We will show that Player I has a winning strategy σ in $2\mathsf{BM}(X)$. Indeed, by Theorem 2.7, X is not Baire, there exists a sequence $\langle D_n : n \in \omega\rangle$ of dense open set and exists a non-empty open U such that $\bigcap_{n\in\omega} D_n \cap U = \emptyset$.

Then in the first inning Player I plays $\sigma(\langle\ \rangle) = U$, next Player II responds $\mathscr{V}_0 = \{V_0, V_1\}$, with $V_i \subseteq U$ for $i \in \{0,1\}$. In the second inning, Player I plays $\sigma(\langle \mathscr{V}_0\rangle) = \{D_0 \cap V_0, D_0 \cap V_1\}$ and Player II plays $\mathscr{V}_1 = \{V_{00}, V_{01}, V_{10}, V_{11}\}$ with $V_{0i} \subseteq D_0 \cap V_0$ and $V_{1i} \subseteq D_1 \cap V_1$ for $i \in \{0,1\}$. In the third inning, Player I plays $\sigma(\langle \mathscr{V}_0, \mathscr{V}_1\rangle) = \{D_1 \cap V_{00}, D_1 \cap V_{01}, D_1 \cap V_{10}, D_1 \cap V_{11}\}$ and Player II responds $\mathscr{V}_2 = \{V_{000}, V_{001}, V_{010}, V_{011}, V_{100}, V_{101}, V_{110}, V_{111}\}$ with $V_{s^\frown i} \subseteq D_1 \cap V_s$, for all $s \in \{0,1\}^{\{0,1\}}$ and $i \in \{0,1\}$

In general, in the inning $n \in \omega$, Player I plays $\sigma(\langle \mathscr{V}_0, \cdots, \mathscr{V}_{n-2}\rangle) = \{D_n \cap V : V \in \mathscr{V}_{n-2}\}$.

2BM(X)

Player I	Player II
$\sigma(\langle\,\rangle) = U$	
	$\mathscr{V}_0 = \{V_0, V_1\}$
$\sigma(\langle \mathscr{V}_0 \rangle) = \{D_0 \cap V_0, D_0 \cap V_1\}$	
	$\mathscr{V}_1 = \{V_{00}, V_{01}, V_{10}, V_{11}\}$
$\vdots$	$\vdots$

Note that $\bigcup \mathscr{V}_0 \subseteq U$ and for $n \geq 1$, $\bigcup \mathscr{V}_n \subseteq D_n$, then $\bigcap_{n\in\omega} \bigcup V_n = \emptyset$. Therefore δ is a winning strategy for Player I in 2BM(X).

Now, suppose that Player I has a winning strategy σ in 2BM(X). We will show that X is not Baire. For this, we will use Theorem 2.7, that is, we will build a winning strategy σ' for Player I in BM(X).

Indeed, in the first inning, in 2BM(X), Player I plays $\sigma(\langle\,\rangle) = U_0$. Now in BM$(X)$, in the first inning, Player I plays $\sigma'(\langle\,\rangle) = U_0$, then Player II responds V_0, then in 2BM(X), Player II responds $\mathscr{V}_0 = \{V_0, V_0\} = \{V_0\}$. In the second inning, in 2BM(X), Player I plays $\sigma(\langle \mathscr{V}_0 \rangle) = \{\sigma_0(\langle V_0 \rangle), \sigma_1(\langle V_0 \rangle)\}$ with $\sigma_i(\langle V_0 \rangle) \subseteq V_i$ for $i \in \{0, 1\}$. Now in BM(X), in the second inning, Player I plays $\sigma'(\langle V_0 \rangle) = \sigma_0(\langle V_0 \rangle)$, so Player II responds V_1, then in 2BM(X), Player II responds $\mathscr{V}_1 = \{V_1, V_1, \sigma_1(\langle V_0 \rangle), \sigma_1(\langle V_0 \rangle)\} = \{V_1, \sigma_1(\langle V_0 \rangle)\}$ with $V_1 \subseteq \sigma_0(\langle V_0 \rangle)$.

In the third inning, in 2BM(X), Player I plays

$$\sigma(\langle \mathscr{V}_0, \mathscr{V}_1 \rangle) = \{\sigma_0(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, \sigma_1(\langle V_0 \rangle)\rangle), \sigma_1(\langle V_0, \sigma_1(\langle V_0 \rangle)\rangle)\}$$

with $\sigma_i(\langle V_0, V_1 \rangle) \subseteq V_1$ and $\sigma_i(\langle V_0, \sigma_1(\langle V_0 \rangle)\rangle) \subseteq \sigma_1(\langle V_0 \rangle)$ for $i \in \{0, 1\}$. Now in BM(X), in the third inning, Player I plays $\sigma'(\langle V_0, V_1 \rangle) = \sigma_0(\langle V_0, V_1 \rangle)$, so Player II responds V_2, then in 2BM(X), Player II responds

$$\mathscr{V}_2 = \{V_2, V_2, \sigma_1(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle)\}$$

with $V_2 \subseteq \sigma_0(\langle V_0, V_1 \rangle)$, and so on.

2BM(X)

Player I	Player II
U_0	
	$\mathscr{V}_0 = \{V_0, V_0\}$
$\{\sigma_0(\langle V_0 \rangle), \sigma_1(\langle V_0 \rangle)\}$	
	$\mathscr{V}_1 = \{V_1, V_1, \sigma_1(\langle V_0 \rangle), \sigma_1(\langle V_0 \rangle)\}$
$\{\sigma_0(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, \sigma_1(\langle V_0 \rangle)\rangle), \sigma_1(\langle V_0, \sigma_1(\langle V_0 \rangle)\rangle)\}$	
	$\mathscr{V}_2 = \{V_2, V_2, \sigma_1(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, V_1 \rangle), \sigma_0(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle), \sigma_1(\langle V_0, V_1 \rangle)\}$
$\vdots$	$\vdots$

As σ is a winning strategy for Player I, we have that $\bigcap_{n\in\omega}\bigcup\mathscr{V}_n = \emptyset$, note that $\bigcap_{n\in\omega} V_n \subseteq \bigcup\mathscr{V}_n = \emptyset$, then σ' is a winning strategy for Player I in $\mathsf{BM}(X)$, therefore X is not a Baire space.

$\mathsf{BM}(X)$

Player I	Player II
$\sigma'(\langle\,\rangle) = U_0$	
	V_0
$\sigma'(\langle V_0\rangle) = \sigma_0(\langle V_0\rangle)$	
	V_1
$\sigma'(\langle V_0, V_1\rangle) = \sigma_0(\langle V_0, V_1\rangle)$	
	V_2
$\vdots$	$\vdots$

□

Corollary 2.27. Let X be a topological space. Then X is a Baire space if and only if Player I does not have a winning strategy in $\mathsf{2BM}(X)$

The games $\mathsf{2BM}$ and BM are not equivalent, we will see later that Bernstein sets are an example of this.

2.2.2.3 The *-game

Let X be a non-empty perfect[2] Polish space with compatible complete metric d. Fix also a basis $\{V_n\}$ of non-empty open sets for X.

Given $A \subseteq X$, consider the following ***-game** $\mathsf{G}^*(\mathsf{A})$. In this game Player I starts by playing two basic open sets of diameter < 1 with disjoint closures and Player II next picks one of them. Then Player I plays two basic open sets of diameter $< \frac{1}{2}$, with disjoint closures, which are contained in the set that II picked before, and then II picks one of them, and so on. The sets that Player II picked define a unique x. Then I wins iff $x \in A$.

$\mathsf{G}^*(\mathsf{A})$

Player I	Player II
$(U_0^{(0)}, U_1^{(0)})$	
	i_0
$(U_0^{(1)}, U_1^{(1)})$	
	i_1
$\vdots$	$\vdots$

More specifically, $U_i^{(n)}$ are basic open sets with $diam(U_i^{(n)}) < 2^{-n}$, $\overline{U_0^{(n)}} \cap \overline{U_1^{(n)}} = \emptyset$, $i_n \in \{0,1\}$, and $\overline{U_0^{(n+1)} \cup U_1^{(n+1)}} \subseteq U_{i_n}^{(n)}$. Let $x \in X$ be defined by $\{x\} = \bigcap_n U_{i_n}^{(n)}$. Then Player I wins iff $x \in A$.

Theorem 2.28. Let X be a non-empty perfect Polish space and $A \subseteq X$. Then Player I has a winning strategy in $\mathsf{G}^*(\mathsf{A})$ iff A contains a Cantor set.

Proof. (1). Let σ be a winning strategy for Player I, σ induces a Cantor scheme, as follows:

- **Inning** 0

  Player I plays $\sigma(\langle\,\rangle) = (U^{(0)}, U^{(1)})$, then Player II can respond with $U^{(0)}$ or $U^{(1)}$.

- **Inning** 1

  In any case, Player I plays $\sigma(\langle U^{(0)}\rangle) = (U^{(00)}, U^{(01)})$ or $\sigma(\langle U^{(1)}\rangle) = (U^{(10)}, U^{(11)})$, then Player II can respond with $U^{(00)}$, $U^{(01)}$, $U^{(10)}$ or $U^{(11)}$, and so on.

[2] A topological space is **perfect** if all its points are limit points.

By the rules of the game, we have that for each $s \in 2^{<\omega} \setminus \{\emptyset\}$, U^s is open, $\overline{U^{s^\frown 0} \cup U^{s^\frown 1}} \subseteq U^s$, $diam(U^s) < 2^{-|s|}$ and $\overline{U^{s^\frown 0}} \cap \overline{U^{s^\frown 1}} = \emptyset$. Then $\{U^s : s \in 2^{<\omega}\}$ is a Cantor scheme [3]. Also for each $x \in 2^\omega$, if $\{p_x\} = \bigcap_{n\in\omega} U^{x\restriction n}$, as σ is a winning strategy $p_x \in A$. Then the function

$$f : 2^\omega \to A$$

defined as $f(x) = p_x$ is injective and continuous, so A contains a Cantor set.

Now suppose that A contains a Cantor set $\mathscr{C}$. We can find σ be a winning strategy for Player I as follows :

- **Inning 0**

  Let $x \in \mathscr{C}$ and consider $B^{(x)}_{\frac{1}{2}}$. As $\mathscr{C}$ is perfect, there is $y \in \mathscr{C} \cap B^{(x)}_{\frac{1}{2}} \setminus \{x\}$. As X is Hausdorff, it follows that there are two basic open sets $U^{(0)}_0$ and $U^{(1)}_0$ of diameter < 1 with disjoint closures, such that $x \in U^{(0)}_0$ and $y \in U^{(1)}_0$. Note that $U^{(0)}_0 \cap \mathscr{C} \neq \emptyset$ and $U^{(1)}_0 \cap \mathscr{C} \neq \emptyset$. Finally Player I plays $\sigma(\langle\,\rangle) = (U^{(0)}_0, U^{(1)}_0)$, next Player II chooses one of them, say $U^{(0)}_{i_0}$, with $i_0 \in \{0,1\}$. Put $x_0 \in U^{(0)}_{i_0} \cap \mathscr{C}$.

- **Inning 1**

  By construction $\mathscr{C} \cap U^{(0)}_{i_0} \neq \emptyset$. Let $z \in \mathscr{C} \cap U^{(0)}_{i_0}$. As $\mathscr{C}$ is perfect, then $\mathscr{C} \cap B^{(z)}_{\frac{1}{4}} \cap U^{(0)}_{i_0} \setminus \{z\} \neq \emptyset$, let $w \in \mathscr{C} \cap B^{(z)}_{\frac{1}{4}} \cap U^{(0)}_{i_0} \setminus \{z\}$. Again as X is Hausdorff it follows that there are two basic open sets $U^{(1)}_0$ and $U^{(1)}_1$ of diameter $< \frac{1}{2}$ with disjoint closures, which are contained in the set that Player II picked before. Note that $\mathscr{C} \cap U^{(1)}_0 \neq \emptyset$ and $\mathscr{C} \cap U^{(1)}_1 \neq \emptyset$. Then Player I plays $\sigma(\langle U^{(0)}_{i_0} \rangle) = (U^{(1)}_0, U^{(1)}_1)$, next Player II chooses one of them, say $U^{(1)}_{i_1}$. Put $x_1 \in U^{(1)}_{i_1} \cap \mathscr{C}$, and so on.

We claim that σ is a winning strategy. Indeed, let $x \in X$ be defined by $\{x\} = \bigcap_n U^{(n)}_{i_n}$. Note that x_n converges to x, so $x \in \mathscr{C} \subseteq A$, then σ is a winning strategy for Player I.

□

[3] We can define $U^\emptyset = X$.

2.2.3 An undetermined space

Remember that a topological space X is an **undetermined space** if the Banach-Mazur gamed played on X is undetermined.

2.2.3.1 Bernstein sets

In (BERNSTEIN, 1907) Felix Bernstein utilizes the method of transfinite recursion and defines a subset B of $\mathbb{R}$ such that both sets B and $\mathbb{R}\setminus B$ meet every nonempty perfect set in $\mathbb{R}$; so both B and $\mathbb{R}\setminus B$ turn out to be non-measurable with respect to the Lebesgue measure. The above mentioned construction is based on appropriate uncountable forms of the Axiom of Choice, which were radically rejected by Lebesgue in that time. Namely, Bernstein utilizes the fact that there exists a well ordering of the family of all uncountable closed subsets of $\mathbb{R}$.

Our goal here is to show that Bernstein sets are Baire spaces but not Choquet, and hence are spaces in which the Banach-Mazur game is undetermined.

Definition 2.29. Let $B\subseteq\mathbb{R}$ we say that B is a **Bernstein set** if for all uncountable closed set $F\subseteq\mathbb{R}$, we have that $F\cap B\neq\emptyset$ and $F\cap\mathbb{R}\setminus B\neq\emptyset$.

Note that if B is a Bernstein set then $\mathbb{R}\setminus B$ is a Bernstein set.

Proposition 2.30. A set B is a Bernstein set if neither B nor its complement $\mathbb{R}\setminus B$ contains any nonempty perfect set. In other words a set B is a Bernstein subset of $\mathbb{R}$ if for every non-empty perfect set $P\subseteq\mathbb{R}$ both sets $P\cap B, P\cap(\mathbb{R}\setminus B)$ are non-empty.

Proposition 2.31. Let $B\subseteq\mathbb{R}$ a Bernstein set. Then:

(i) B has no isolated points.

(ii) B is a dense subset of $\mathbb{R}$.

Proof. (i) Suppose otherwise, that is, there are $x\in B\setminus B'$ and $\varepsilon>0$ such that $B_{\varepsilon}^{(x)}\cap B=\{x\}$. Then $\emptyset\neq\overline{B_{\frac{\varepsilon}{2}}^{(x)}}\cap(\mathbb{R}\setminus B)=\emptyset$, contradiction.

(ii) Let $x\in\mathbb{R}$ and $\varepsilon>0$, then $\emptyset\neq B\cap\overline{B_{\frac{\varepsilon}{2}}^{(x)}}\subseteq B\cap B_{\varepsilon}^{(x)}$. $\square$

As we mentioned earlier the objective is to demonstrate that the Bernstein set is an undeterminated space for the Banach-Mazur game. For this, we will use the Cantor game. Before starting, let us remember the following fact of the topology of the real line.

Lemma 2.32. Let $\langle K_n : n\in\omega\rangle$ be a sequence of non-empty compact sets in $\mathbb{R}$ such that $K_0\supseteq K_1\supseteq K_2\supseteq\cdots$. Then $\bigcap_{n\in\omega}K_n\neq\emptyset$.

Theorem 2.33. Let $B\subseteq\mathbb{R}$ be a Bernstein set. Then Player II has a winning strategy in $\mathsf{2BM}(X)$.

Proof. Let B be a Bernstein set, we are going to build a winning strategy δ for Player II in $2\mathsf{BM}(X)$. Indeed,

In the first inning, Player I plays $U_\emptyset = A_0 \cap B$ where $A_0 \subseteq \mathbb{R}$ is a non-empty open set, let $a_0 \in A_0 \cap B$, in particular there is $r > 0$ such that $\overline{B_r^{(a_0)}} \subseteq A_0$. Note that $\emptyset \neq \overline{B_r^{(a_0)}} \cap (\mathbb{R} \setminus B)$, let $b_0 \in \overline{B_r^{(a_0)}} \cap (\mathbb{R} \setminus B)$. Then choose two non-empty open subsets V_0, V_1 such that

- $\overline{V_0}, \overline{V_1} \subseteq \overline{B_r^{(a_0)}}$,
- $\overline{V_0} \cap \overline{V_1} = \emptyset$
- $diam(\overline{V_0}), diam(\overline{V_1}) \leq \frac{r}{2}$ and

Then Player II responds $\delta(\langle\{U_\emptyset\}\rangle) = \{V_0 \cap B, V_1 \cap B\}$ and consider $\mathscr{V}_0 = \bigcup \delta(\langle U_\emptyset \rangle)$ and $W_0 = \overline{V_0} \cup \overline{V_1}$.

In the second inning, Player I plays $\{U_0, U_1\}$ with $U_0 \subseteq V_0 \cap B$ and $U_1 \subseteq V_1 \cap B$. For each $i \in \{0,1\}$, as in the previous case, let $a_{i1} \in U_i$ and choose $r_{i1} < \frac{r}{2^2}$ with $a_{i1} \in \overline{B_{r_{i1}}^{(a_{i1})}} \subseteq V_i$ and let $b_{i1} \in \overline{B_{r_{i1}}^{(a_{i1})}} \cap (\mathbb{R} \setminus B)$. Then choose four non-empty open subsets $V_{00}, V_{01}, V_{10}, V_{11}$ such that

- $\overline{V_{00}}, \overline{V_{01}} \subseteq V_0$ and $\overline{V_{10}}, \overline{V_{11}} \subseteq V_1$
- $\{\overline{V_{00}}, \overline{V_{01}}, \overline{V_{10}}, \overline{V_{11}}\}$ is a disjoint pairwise family and
- $diam(\overline{V_{00}}), diam(\overline{V_{01}}), diam(\overline{V_{10}}), diam(\overline{V_{11}}) \leq \frac{r}{2^2}$.

Then Player II responds $\delta(\langle\{U_\emptyset\}, \{U_0, U_1\}\rangle) = \{V_{00} \cap B, V_{01} \cap B, V_{10} \cap B, V_{11} \cap B\}$ and consider $\mathscr{V}_1 = \bigcup \delta(\langle\{U_\emptyset\}, \{U_0, U_1\}\rangle)$ and $W_1 = \overline{V_{00}} \cup \overline{V_{01}} \cup \overline{V_{10}} \cup \overline{V_{11}}$.

In the inning $n \in \omega$, if Player I plays $\{U_s : s \in 2^{<\omega}, |s| = n-1\}$. Suppose defined r_s, a_s and b_s for $s \in 2^{<\omega}$ with $|s| = n-1$. Then, as before, let $r_{s^\frown n} < \frac{r}{2^n}$, $a_{s^\frown 0}, a_{s^\frown 1}$ and $b_{s^\frown 0}, b_{s^\frown 1}$. Then choose an open family $\{V_{s^\frown 0}, V_{s^\frown 1} : |s| = n-1\}$ such that

- $\overline{V_{s^\frown i}} \subseteq V_s$ for $i \in \{0,1\}$;
- $\{\overline{V_{s^\frown 0}}, \overline{V_{s^\frown 1}} : |s| = n-1\}$ is a pairwise disjoint family;
- $diam(\overline{V_{s^\frown 0}}), diam(\overline{V_{s^\frown 1}}) \leq \frac{r}{2^n}$.

Then Player II responds $\delta(\langle\{U_\emptyset\}, \{U_0, U_1\}, \cdots, \{U_s : s \in 2^{<\omega}, |s| = n-1\}\rangle) = \{V_{s^\frown 0} \cap B, V_{s^\frown 1} \cap B : |s| = n-1\}$ and consider $\mathscr{V}_{n-1} = \bigcup \delta(\langle\{U_\emptyset\}, \{U_0, U_1\}, \cdots, \{U_s : s \in 2^{<\omega}, |s| = n-1\}\rangle)$ and $W_{n-1} = \bigcup\{\overline{V_{s^\frown 0}}, \overline{V_{s^\frown 1}} : |s| = n-1\}$.

Note that, for each $n \in \omega$, we have that $W_n \subseteq \mathbb{R}$ is a compact and $W_{n+1} \subseteq W_n$. Also, by construction, $W_n \subseteq \mathscr{V}_n$.

Claim 2.33.14. $\bigcap_{n\in\omega} W_n$ is closed and uncountable.

Proof. Let $f \in 2^\omega$. Define $D_f = \bigcap_{n\in\omega} \overline{V_{f|_n}} \subseteq \bigcap_{n\in\omega} W_n$. By Lemma 2.32, $\emptyset \neq D_f$. As $diam(\overline{V_{f|_n}}) \leq \frac{r}{2^n}$, so $D_f = \{x_f\}$. Finally, define $g : 2^\omega \to \bigcap_{n\in\omega} W_n$ by $g(f) = x_f$ and note that g is injective. □

Then there exist $x \in \bigcap_{n\in\omega} W_n \cap B$, in particular $x \in W_n \cap B \subseteq \mathscr{V}_n$. Therefore δ is a winning strategy for Player II in $2\mathsf{BM}(X)$.

□

Corollary 2.34. Let $B \subseteq \mathbb{R}$ be a Bernstein set. Then Player I has no winning strategy in $2\mathsf{BM}(X)$. In particular, B is a Baire space.

Proof. By Theorem 2.33, Player II has a winning strategy in $2\mathsf{BM}(B)$. Therefore Player I has no winning strategy in $2\mathsf{BM}(B)$. So by Corollary 2.27, Player I has no winning strategy in $\mathsf{BM}(B)$. Then by Theorem 2.7, B is a Baire space. □

Proposition 2.35. Let $B \subseteq \mathbb{R}$ be a Bernstein set. Then Player II has no winning strategy in $BM(X)$.

Proof. Suppose otherwise, that is, Player II has a winning strategy in $\mathsf{BM}(B)$. As B has no isolated points, by Theorem 2.21, there is a set $C \subseteq B$ homeomorphic to the Cantor set, in particular C is closed. Note that $C \cap (\mathbb{R} \setminus B) = \emptyset$, but as B is a Bernstein set, $C \cap (\mathbb{R} \setminus B) \neq \emptyset$, contradiction. □

Corollary 2.36. The Banach-Mazur game is undetermined when is played in a Bernstein set in the real line.

CHAPTER

3

PRODUCTS OF BAIRE SPACES

In this section we will study the problem of when the product of two spaces is Baire. We will start with examples of Baire spaces whose product is not Baire. Then we will give conditions on the spaces to make his product a Baire space.

3.1 Counterexamples

3.1.1 Two Baire spaces whose product is not Baire. An example in ZFC with forcing.

In this first section we present the article (COHEN, 1976), in which it is shown, using forcing, that in ZFC, there are two Baire spaces whose product is not a Baire space.

Let $\mathscr{P} = \langle P, \leq \rangle$ be a p.o. set. Two elements p and q of it are called compatible if there is an $r \in \mathscr{P}$ such that $r \leq p$ and $r \leq q$; otherwise they are called incompatible. A subset D of $\mathscr{P}$ is said to be **dense** in $\mathscr{P}$ if for each $p \in P$ there is a $d \in D$ such that $d \leq p$.

Remember that a partially ordered set $\mathscr{P} = \langle P, \leq \rangle$ is a **forcing** if for each $p, q \in \mathscr{P}$ such that $q \not\leq p$, there exists $p' \leq q$ such that p', p are incompatible.

We define on P a topology $\tau_{\leq}$ by declaring each set $\{q : q \leq p\}$ to be open. Note that if the space is derived from a p.o. set as above, then any such countable intersection of open sets is necessarily open.

Furthermore if $A \subseteq P$ then in this topology

(a) $x \in \operatorname{int}(A)$ iff $\downarrow x = \{y \in P : y \leq x\} \subseteq A$

(b) $x \in \overline{A}$ iff $\downarrow x \cap A \neq \emptyset$

(c) A is dense in P iff $(\forall x \in P)[\downarrow x \cap A \neq \emptyset]$

Now let $\mathscr{M}$ be any model and $\mathscr{P}$ any p.o. set in $\mathscr{M}$, let G be an $\mathscr{M}$-generic subset of $\mathscr{P}$, and $\mathscr{M}[G]$ the corresponding generic extension of $\mathscr{M}$.

The most important connection between forcing and topology is as follows:

Lemma 3.1. $(P,\tau_{\leq})$ is a Baire space if and only if for every $\mathscr{M}$-generic subset G of $\mathbb{P}$ no new ω-sequences of ordinals occur in $\mathscr{M}[G]$.

Proof. First, suppose that $(P,\tau_{\leq})$ is a Baire space, and let $f \in \mathscr{M}[G]$ with $domf = \omega$, whose values are ordinals, as the formula $f : \omega \to \text{Ord}$ is a function in $\mathscr{M}[G]$ is satisfied, then by Theorem 1.94, there exists $p' \in G$ such that p' forces it. For every $n \in \omega$ consider the set $D_n = \{p \in P : (\exists \alpha \in \text{Ord})(p \Vdash \text{“}f(\check{n}) = \check{\alpha}\text{''})\}$.

Claim 3.1.15. For each $n \in \omega$, D_n is open and dense below p'.

Proof. Let $q \leq p'$, σ and $q' \leq q$ such that $q' \Vdash f(n) = \sigma$, as $q' \leq p'$, $q' \Vdash \sigma$ is an ordinal , so there is a $q'' \leq q$ such that $q'' \Vdash \sigma = \alpha$. □

As $\{q : q \leq p'\}$ is open, then it is a Baire space, so $\bigcap_{n\in\omega} D_n$ is dense below p'. By Lemma 1.96, it follows that $\bigcap_{n\in\omega} D_n$ is dense in P. Then $G \cap \bigcap_{n\in\omega} D_n \neq \emptyset$, so let $p \in G \cap \bigcap_{n\in\omega} D_n$, then for every $n \in \omega$, there is an $\alpha_n \in \text{Ord}$ such that $p \Vdash \text{“}f(\check{n}) = \check{\alpha}_n\text{''}$. Finally define $\varphi(n) = \alpha_n$, note that $\varphi \in \mathscr{M}$, and $p \Vdash \text{“}f = \check{\varphi}\text{''}$, so $f \in \mathscr{M}$.

Now, suposse that $(P,\tau_{\leq})$ is not a Baire space. Then there exists a sequence of open dense subsets $\{D_n : n \in \omega\}$ and $q_0 \in P$ such that $\bigcap_{n\in\omega} D_n \cap \{q : q \leq q_0\} = \emptyset$. For each $n \in \omega$, there exists $I_n = \{r^n_\alpha : \alpha < k_n\}$ be a maximal family of pairwise incompatible contained in D_n, consider $D'_n = \{p : (\exists \alpha < k_n)(p \leq r^n_\alpha)\}$, as D_n is open, we have that $D'_n \subseteq D_n$. Also note that D'_n is open and dense. Indeed, let $p \in P$ then there is a $d_n \in D_n$ such that $d_n \leq p$, note that there is a r^n_α compatible with d_n, otherwise we would have a contradiction with the maximality of I_n, so there is a $r \in P$ such that $r \leq r^n_\alpha$ and $r \leq d_n$, so $r \in D'_n$ and $r \leq p$, therefore D'_n is open and dense. Then $\bigcap_{n\in\omega} D'_n \cap \{q : q \leq q_0\} \subseteq \bigcap_{n\in\omega} D_n \cap \{q : q \leq q_0\} = \emptyset$. For each $n \in \omega$, consider I_n and $\{\alpha : \alpha < k_n\}$, by Lemma 1.98, there is a $t \in \mathscr{M}^B$ such that $r^n_\alpha \Vdash \text{“}t = \alpha\text{''}$, for all $\alpha \in k_n$, that is, $r^n_\alpha \Vdash t(\check{n}) = \check{\alpha}$. By hypothesis, $\{q : q \text{ decides } t(n) \text{ for all } n \in \omega\}$ is dense, so there is $q_1 \leq q_0$ such that q_1 decides $t(n)$ for all $n \in \omega$, that is, q_1 forces that $t(n)$ is an ordinal. Then there is $n_0 \in \omega$ such that $q_1 \notin D'_{n_0}$, also there is $\alpha < k_{n_0}$ such that $q_1 \Vdash t(\check{n}_0) = \alpha$, so $q_1 \not\leq r^{n_0}_\alpha$. Therefore there is a $r \leq q_1$ such that r is incompatible with $r^{n_0}_\alpha$, so there is a β such that $r^{n_0}_\beta$ is compatible with r, because otherwise this would be a contradiction with the maximality of I_n. Therefore there is a $s \leq r, r^{n_0}_\beta$ such that $s \Vdash t(n_0) = \alpha$ and $s \Vdash t(n_0) = \beta$, contradiction. □

3.1.1.1 The construction

We begin this part with the following facts of stationary sets of ω_1 and product forcing.

Proposition 3.2. The intersection of countably many club sets is a club set.

Lemma 3.3 (Banach)**.** There are two disjoint stationary subsets of ω_1.

Proof. Let $\{x_\alpha : \alpha < \omega_1\}$ be a set of irrationals, where $x_\alpha \neq x_\beta$ for distinct $\alpha, \beta < \omega_1$. For each rational q, let $S_q(0) = \{\alpha \in \omega_1 : x_\alpha < q\}$ and $S_q(1) = \{\alpha \in \omega_1 : x_\alpha > q\}$.

Suppose that for all $q \in \mathbb{Q}$, there is $i_q \in \{0, 1\}$ such that $S_q(i_q)$ contains a club in ω_1.

Then, by Proposition 3.2,

$$C = \bigcap_{q \in \mathbb{Q}} S_q(i_q)$$

contains a club, so C is uncountable, while $|C| \leq 1$, contradiction.

Therefore, there is a $q \in \mathbb{Q}$ such that $S_q(0)$ and $S_q(1)$ do not contain any club. In particular, $S_q(0)$ and $S_q(1)$ are disjoint stationary sets.

$\square$

Now from a stationary set S of ω_1 we construct a p.o. set $\mathscr{P}_S$ of conditions:

- a condition $p \in \mathscr{P}_S$ is a countable subset of S that is closed in the order topology of ω_1. In particular each member p of $\mathscr{P}_S$ has a maximum.

$$\mathscr{P}_S = \{p \subseteq S : |p| \leq \aleph_0 \text{ and } p \text{ is closed in } \omega_1\}$$

- If $p, q \in \mathscr{P}_S$, then

$$p \leq q \quad \text{iff} \quad q \subseteq p \text{ and } (p \setminus q) \cap \bigcup q = \emptyset,$$

which is equivalent to the fact that $\alpha > \beta$ for all $\alpha \in p \setminus q$ and $\beta \in q$.

Claim 3.3.16. $\mathscr{P}_S$ is a forcing.

Proof. Let $p, q \in \mathscr{P}_S$ and suppose that $q \not\leq p$. Then $p \not\subseteq q$ or $(q \setminus p) \cap \bigcup p \neq \emptyset$. Choose $\beta \in S \setminus \bigcup(p \cup q)$ this is possible because S is stationary.

In the first case, let $\alpha \in p \setminus q$. Consider $r = q \cup \{\beta\}$. Note that $r \leq q$. Suppose that r and p are compatible. Then there is a $s \in \mathscr{P}_S$ such that $s \leq r, p$. Note that $\alpha < \beta \leq \bigcup r$ therefore, $\alpha \in (s \setminus r) \cap \bigcup r = \emptyset$, contradiction. Then r are p are not compatible.

In the second case, let $\alpha \in (q \setminus p) \cap \bigcup p$. We claim that q and p are incompatible. Indeed, suposse otherwise, there is a $s \leq p, q$. Note that $\alpha \in (s \setminus p) \cap \bigcup p$, contradiction.

$\square$

Claim 3.3.17. In any generic extension $\mathscr{M}[G]$ by means of an $\mathscr{M}$-generic subset G of $\mathscr{P}_S$ no new sequences of ordinals appear.

Proof. Let t be a function in $\mathscr{M}[G]$ from ω to Ord then there is a $p \in G$ such that $p \Vdash t : \omega \to$ Ord is a function. In order to finish the proof we need only show that there is a $q \leq p$ such that $q \Vdash t(n)$ for all $n \in \omega$.

We define by induction R_α and η_α as follows :

1. $\{\eta_\alpha : \alpha < \omega_1\}$ is a continuous[1] increasing sequence of countable ordinals.
2. $\{R_\alpha : \alpha < \omega_1\}$ is a continuous increasing sequence of countable subsets of $\mathscr{P}_S$.
3. $R_\alpha \subseteq \{r \in \mathscr{P}_S : r \subseteq \eta_\alpha \wedge r \leq p\}$
4. $(\forall r \in R_\alpha)(\forall n \in \omega)(\exists s \in R_{\alpha+1})[s < r \wedge s \subseteq \eta_\alpha \wedge s \text{ decides } t(n)]$

Now, consider $C = \{\eta_\alpha : \alpha \text{ is a limit ordinal}\}$. Note that C is a club in ω_1. Indeed, as $|C| = \omega_1$, we have that C is unbounded. Now, let $\eta_{\alpha_0} < \eta_{\alpha_1} < \cdots < \eta_{\alpha_\xi} < \cdots (\xi < \gamma)$ be a sequence of elements of C, of length $\gamma < \omega_1$, then $\sup\{\eta_{\alpha_\xi} : \xi < \gamma\} = \bigcup_{\xi<\gamma} \eta_{\alpha_\xi} = \eta_{\sup\{\alpha_\xi : \xi<\gamma\}}$. Then there is a limit ordinal $\alpha < \omega_1$ such that $\eta_\alpha \in C \cap S$. As $\alpha < \omega_1$ is a limit ordinal, there is an strictly increasing sequence $\langle \alpha_n : n \in \omega \rangle$ which converges to α.

Now, let $r_0 \in R_{\alpha_0}$. By Condition 4, there is $s_0 \in R_{\alpha_0+1}$ such that $s_0 < r_0$, $s_0 \subseteq \eta_{\alpha_0}$ and $s_0 \Vdash t(0)$. As $\alpha_0 + 1 \leq \alpha_1$, we have that $s_0 \in R_{\alpha_1}$, so there is a $s_1 \in R_{\alpha_1+1}$ such that $s_1 < s_0$ and $s_1 \Vdash t(1)$, and so on. Then we have a decreasing sequence $\langle s_n : n \in \omega \rangle$ such that $s_n \in R_{\alpha_n+1}$, $s_n \subseteq \eta_{\alpha_n}$ and $s_n \Vdash t(n)$.

Consider $q = \{\eta_\alpha\} \cup \bigcup\{s_n : n \in \omega\}$. Note that $q \leq s_n$, for all $n \in \omega$. Indeed, let $n \in \omega$ and suppose that there is a $x \in (q \setminus s_n) \cap \bigcup s_n$. In particular, there is an $a \in s_n$ such that $x \in a$. Then $x = \eta_\alpha$, or there is a $N \in \omega$ such that $x \in s_N$. In the first case, by construction it follows that $a \in \eta_{\alpha_n}$, so $\eta_\alpha = x < a < \eta_{\alpha_n} \leq \eta_\alpha$, contradiction. In the other case, note that $n \leq N$, in particular $s_N \setminus s_n \cap \bigcup s_n = \emptyset$, so $x \notin \bigcup s_n$, contradiction. Then $q \leq s_n$, for all $n \in \omega$. Therefore q decides $t(n)$ for every $n \in \omega$, then $t \in \mathscr{M}$.

□

Therefore $\mathscr{P}_S$ is a Baire space. Also we have that:

Claim 3.3.18. In $\mathscr{M}[G]$ the set S contains an uncountable closed subset.

Proof. Indeed, our candidate is $\bigcup G \in \mathscr{M}[G]$. We have that:

[1] A sequence of ordinals $\langle \gamma_\alpha : \alpha \in \text{Ord} \rangle$ is continuous, if for every limit α, $\gamma_\alpha = \sup\{\gamma_\xi : \xi < \alpha\}$

- $\bigcup G \subseteq S$. For this, let $x \in \bigcup G$, there is a $g \in G$ such that $x \in g$, as $G \subseteq \mathscr{P}_S$, $g \in \mathscr{P}_S$, so $x \in S$.

- $\bigcup G$ **is unbounded**. For this, note that for each $\alpha < \omega_1$, the set $\{p \in \mathscr{P}_S : \max p > \alpha\}$ is dense. Indeed, let $s \in \mathscr{P}_S$, if $\max s < \alpha + 1$, consider $p = s \cup \{\alpha + 1\}$, so $p \leq s$. Finally let $\alpha < \omega_1$, as $\{p : \max p > \alpha\}$ is dense, there is a $x \in G \cap \{p : \max p > \alpha\}$, so $\alpha < \max x \in x$. Then $\bigcup G$ is unbounded, so it is uncountable.

- $\bigcup G$ **is closed**. Indeed, let $\beta \in \overline{\bigcup G}$. Then there is a $x \in \downarrow \beta \cap \bigcup G$, so there exists $p \in G \subseteq \mathscr{P}_S$ such that $x \in \downarrow \beta \cap p$, so $\beta \in \overline{p}$. As p is closed, $\beta \in p$. Then $\beta \in \bigcup G$.

$\square$

Thus, in the generic extension by means of $\mathscr{P}_S$ no new ω-sequences of ordinals appear, but a new uncountable subset of ω_1 contained in S occurs.

By Lemma 3.3, take two disjoint stationary subsets S_1 and S_2 in ω_1 and we have two p.o sets $\mathscr{P}_{S_1}$ and $\mathscr{P}_{S_2}$ defined like $\mathscr{P}_S$ above. Then

Claim 3.3.19. $\mathscr{P}_{S_1} \times \mathscr{P}_{S_2}$ is not Baire.

Proof. Suppose that $\mathscr{P}_{S_1} \times \mathscr{P}_{S_2}$ is Baire in $\mathscr{M}$. By Lemma 1.100, let $G = G_1 \times G_2$ be a $\mathscr{P}_{S_1} \times \mathscr{P}_{S_2}$-generic over $\mathscr{M}$, where $G_1 \subseteq \mathscr{P}_{S_1}$ is $\mathscr{P}_{S_1}$-generic over $\mathscr{M}$ and $G_2 \subseteq \mathscr{P}_{S_2}$ is $\mathscr{P}_{S_2}$-generic over $\mathscr{M}[G_1]$, also G_1 is $\mathscr{P}_{S_1}$-generic over $\mathscr{M}[G_2]$ and $\mathscr{M}[G] = (\mathscr{M}[G_2])[G_1] = (\mathscr{M}[G_1])[G_2]$. We know that $\mathscr{P}_{S_1}$ and $\mathscr{P}_{S_2}$ are Baire spaces in $\mathscr{M}$. As G_1 is $\mathscr{P}_{S_1}$-generic over $\mathscr{M}$, by Lemma 1.101, $\mathscr{P}_{S_2}$ is Baire in $\mathscr{M}[G_1]$. As G_2 is $\mathscr{P}_{S_2}$-generic over $\mathscr{M}[G_1]$, we have that no new sequences of ordinals appear in $(\mathscr{M}[G_1])[G_2]$. Also in $\mathscr{M}[G_1]$ we have that there is a closed uncountable set A_1 contained in S_1, therefore A_1 is closed uncountable in $(\mathscr{M}[G_1])[G_2]$. Indeed,

- A_1 **is closed in** $(\mathscr{M}[G_1])[G_2]$. Otherwise, there is a $x \in \overline{A_1} \setminus A_1$. As ω_1 is first countable, there is a $\langle \gamma_n : n \in \omega \rangle \subseteq A_1$ such that $\gamma_n \to x$. Note that $\langle \gamma_n : n \in \omega \rangle, x \in \mathscr{M}[G_1]$ therefore $x \in A_1$, because A_1 is closed in $\mathscr{M}[G_1]$. Contradiction.

- A_1 **is uncountable in** $(\mathscr{M}[G_1])[G_2]$. Otherwise, there is an injection $f : \omega \to A_1$, so $f \in \mathscr{M}[G_1]$ and $A_1 \in \mathscr{M}[G_1]$. Then A_1 is countable in $\mathscr{M}[G_1]$, contradiction.

Also $\mathscr{P}_{S_2} \times \mathscr{P}_{S_1}$ is Baire, similarly as before there exists a closed uncountable $A_2 \subseteq S_2$ in $(\mathscr{M}[G_2])[G_1]$. Then in $(\mathscr{M}[G_1])[G_2]$ we have that there are closed uncountable sets $A_1 \subseteq S_1$ and $A_2 \subseteq S_2$. By Proposition 3.2, $A_1 \cap A_2$ is a club, in particular $\emptyset \neq A_1 \cap A_2 \subseteq S_1 \cap S_2 = \emptyset$, contradiction. $\square$

Finally collecting all of the above we have the following

Theorem 3.4 (Cohen)**.** There are two Baire spaces whose product is not a Baire space.

3.1.2 Two metric Baire spaces whose product is not Baire.

Assuming that there are two Baire topological spaces whose product is not Baire, Krom showed that there are two Baire metric spaces whose product is also not Baire. For this Krom associated a ultrametric space with a topological space. Unlike the previous example we will use the Banach-Mazur game to demonstrate the basic properties of this new metric space. In this part we study the article (KROM, 1974).

3.1.2.1 The Krom space

Definition 3.5. For any sets S, T and for $n \in \omega \setminus \{0\}$ let ${}^{S}T$ be the set of all functions from S into T and let ${}^{n}T$ be the set of all functions from $\{0, \ldots, n-1\}$ into T. For a set S of sets and $n \in (\omega \setminus \{0\}) \cup \{\omega\}$ let

$$\downarrow {}^{n}S = \{\sigma \in {}^{n}S \mid \sigma(h) \subseteq \sigma(h-1) \text{ for all } 0 < h < n\}$$

Definition 3.6 (Krom space)**.** For any topological space X and base $\mathscr{B}$ for X such that $\emptyset \notin \mathscr{B}$, the associated **countable sequence space** $\mathscr{K}(X)$ is defined by

$$\mathscr{K}_{\mathscr{B}}(X) = \left\{ \sigma \in \downarrow^{\omega}\mathscr{B} : \bigcap_{n \in \omega} \sigma(n) \neq \emptyset \right\},$$

and the topology is that given by the Baire metric, for $\sigma \neq \rho$ the distance $d(\sigma, \rho) = \frac{1}{n+1}$ where n is the least integer in $\{h \in \omega : \sigma(h) \neq \rho(h)\}$.

Let $X, \mathscr{B}$ and $\mathscr{K}(X)$ be as indicated. For any $\sigma \in \mathscr{K}(X) \subseteq \downarrow^{\omega}\mathscr{B}$ and $n \in \omega \setminus \{0\}$ let $B^n(\sigma) = \{\rho \in \mathscr{K}(X) : \sigma \restriction_n = \rho \restriction_n\}$, consider $\mathscr{B}^* = \{B^n(\sigma) : \sigma \in \mathscr{K}(X), n \in \omega \setminus \{0\}\}$. Note that $\mathscr{B}^*$ is a base for $\mathscr{K}(X)$.

Put differently, a base for $\mathscr{K}(X)$ is the family of all sets $[f]$, $f \in \bigcup_{n \in \mathbb{N}} \downarrow^{n}\mathscr{B}$ where, if $n < \omega$ and $f \in \downarrow^{n}\mathscr{B}$, then

$$[f] = \{g \in \mathscr{K}(X) : g \restriction_n = f\}.$$

Proposition 3.7. $\tilde{\mathscr{B}} = \{[f] : f \in \bigcup_{n \in \mathbb{N}} \downarrow^{n}\mathscr{B}\}$ is a base for $\mathscr{K}(X)$.

Proof. Note that each member of $\tilde{\mathscr{B}}$ is an open set in $\mathscr{K}(X)$. Now, if U is an open subset of $\mathscr{K}(X)$ and $\rho \in U$, then there exists $r > 0$ such that $B_r^{(\rho)} \subseteq U$. By the Archimedean property, there is a $n_0 \in \omega$ such that $\frac{1}{n_0+1} < r$; then $[\rho \restriction_{n_0}] \subseteq B_r^{(\rho)} \subseteq U$. □

Corollary 3.8. Let X be a topological space with a countable base $\mathscr{B}$. Then $\mathscr{K}(X)$ is a second countable metric space.

Proof. It follows from $\bigcup_{n \in \mathbb{N}} \downarrow^{n}\mathscr{B} \subseteq \bigcup_{n \in \mathbb{N}} {}^{n}\mathscr{B} = \mathscr{B}^{<\omega}$ and $|\mathscr{B}^{<\omega}| = |\mathscr{B}| = \omega$. □

Now we will see an application of the Banach-Mazur game and the Krom space.

Theorem 3.9 (Krom)**.** For any topological spaces X,Y and any base $\mathscr{B}$ for X, $X\times Y$ is a Baire space if and only if $\mathscr{K}(X)\times Y$ is a Baire space where $\mathscr{K}(X)$ is the countable sequence space associated with X and $\mathscr{B}$.

Proof. Assume that $X\times Y$ is not Baire, then $\mathrm{I}\uparrow \mathsf{BM}(X\times Y)$; call σ this strategy. We will build a winning strategy σ' for Player I in $\mathsf{BM}(\mathscr{K}(X)\times Y)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(X\times Y)$, Player I plays $\sigma(\langle\,\rangle)=A_0\times B_0$. Then, in $\mathsf{BM}(\mathscr{K}(X)\times Y)$, Player I plays $\sigma'(\langle\,\rangle)=[\langle A_0\rangle]\times B_0$, where $\sigma_0=\langle A_0\rangle\in\downarrow^{1}\mathscr{B}$. Next Player II responds $[\delta_0]\times V_0$, with $\delta_0\in\downarrow^{n_0}\mathscr{B}$ and $\sigma_0\subseteq\delta_0$, so Player II plays $\delta_0(n_0-1)\times V_0$.

- **Inning 1**

  Player I plays $\sigma(\langle\delta_0(n_0-1)\times V_0\rangle)=A_1\times B_1$. Then in $\mathsf{BM}(\mathscr{K}(X)\times Y)$, Player I plays $\sigma'(\langle[\delta_0]\times V_0\rangle)=[\sigma_1]\times B_1$, where $\sigma_1=\delta_0{}^\frown A_1\in\downarrow^{n_0+1}\mathscr{B}$. So Player II responds $[\delta_1]\times V_1$, with $\delta_1\in\downarrow^{n_1}\mathscr{B}$ and $\sigma_1\subseteq\delta_1$, also we can suppose that $n_1-1\geq n_0$. Then in $\mathsf{BM}(X\times Y)$, Player II plays $\delta_1(n_1-1)\times V_1$.

- **Inning 2**

  Player I plays $\sigma(\langle\delta_0(n_0-1)\times V_0,\delta_0(n_1-1)\times V_1\rangle)=A_2\times B_2$. Then in $\mathsf{BM}(\mathscr{K}(X)\times Y)$, Player I plays $\sigma'(\langle[\delta_0]\times V_0,[\delta_1]\times V_1\rangle)=[\sigma_2]\times B_2$, where $\sigma_2=\delta_1{}^\frown A_2\in\downarrow^{n_1+1}\mathscr{B}$. So Player II responds $[\delta_2]\times V_2$, with $\delta_2\in\downarrow^{n_2}\mathscr{B}$ and $\sigma_2\subseteq\delta_2$. Again we can suppose that $n_2-1\geq n_1$. Then in $\mathsf{BM}(X\times Y)$, Player II plays $\delta_2(n_2-1)\times V_2$, and so on.

$\mathsf{BM}(X\times Y)$

Player I	Player II
$\sigma(\langle\,\rangle)=A_0\times B_0$	
	$\delta_0(n_0-1)\times V_0$
$A_1\times B_1$	
	$\delta_1(n_1-1)\times V_1$
$A_2\times B_2$	
	$\delta_2(n_2-1)\times V_2$
$\vdots$	$\vdots$

$\mathsf{BM}(\mathscr{K}(X)\times Y)$

Player I	Player II
$\sigma'(\langle\,\rangle)=[\sigma_0]\times B_0$	
	$[\delta_0]\times V_0$
$[\sigma_1]\times B_1$	
	$[\delta_1]\times V_1$
$[\sigma_2]\times B_2$	
	$[\delta_2]\times V_2$
$\vdots$	$\vdots$

Claim 3.9.20. $\bigcap_{k\in\omega}[\delta_k]\times V_k=\emptyset$

Proof. Suppose otherwise, that is, there exists $(h,y)\in[\delta_k]\times V_k, \forall k\in\omega$. In particular $h\in\bigcap_{k\in\omega}[\delta_k]$. Now let us see what happens in $X\times Y$. As σ is a winning strategy, we have that $\bigcap_{k\in\omega}\delta_k(n_k-1)\times V_k=\emptyset$. As $h\in\mathcal{K}(X)$, we have that $\bigcap_{k\in\omega}h(k)\neq\emptyset$. Put $x\in h(k), \forall k\in\omega$. In particular, for each $k\in\omega$, $\delta_k\subseteq h$ and therefore $x\in h(n_k-1)=\delta_k(n_k-1)$, then $(x,y)\in\bigcap_{k\in\omega}\delta_k(n_k-1)\times V_k$, contradiction. □

Then σ' is a winning strategy for Player I in the game $\mathsf{BM}(\mathcal{K}(X)\times Y)$, therefore $\mathcal{K}(X)\times Y$ is not a Baire space.

Now assume that $\mathcal{K}(X)\times Y$ is not a Baire space, so $\mathrm{I}\uparrow\mathsf{BM}(\mathcal{K}(X)\times Y)$. Let be σ' be a winning strategy for Player I in $\mathsf{BM}(\mathcal{K}(X)\times Y)$. We build a winning strategy σ for Player I in $\mathsf{BM}(X\times Y)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\mathcal{K}(X)\times Y)$, Player I plays $\sigma'(\langle\,\rangle)=[\sigma_0]\times B_0$, with $\sigma_0\in\downarrow^{n_0}\mathcal{B}$. Then, in $\mathsf{BM}(X\times Y)$, Player I plays $\sigma(\langle\,\rangle)=\sigma_0(n_0-1)\times B_0$. Next Player II responds $W_0\times V_0$, so Player II plays $[\delta_0]\times V_0$, where $\delta_0=\sigma_0{}^\frown W_0\in\downarrow^{n_0+1}\mathcal{B}$.

- **Inning 1**

  Player I plays $\sigma'(\langle[\delta_0]\times V_0\rangle)=[\sigma_1]\times B_1$, with $\sigma_1\in\downarrow^{n_1}\mathcal{B}$. Note that we can suppose that $n_1-1\geq n_0$. Then in $\mathsf{BM}(X\times Y)$, Player I plays $\sigma(\langle W_0\times V_0\rangle)=\sigma_1(n_1-1)\times B_1$. Next Player II plays $W_1\times B_1$. Then Player II plays $[\delta_1]\times V_1$, where $\delta_1=\sigma_1{}^\frown W_1\downarrow^{n_1+1}\mathcal{B}$.

- **Inning 2**

  Player I plays $\sigma'(\langle[\delta_0]\times V_0,[\delta_1]\times V_1\rangle)=[\sigma_2]\times B_2$, with $\sigma_2\in\downarrow^{n_2}\mathcal{B}$. Again we can suppose that $n_2-1\geq n_1$. Then in $\mathsf{BM}(X\times Y)$, Player I plays $\sigma(\langle W_0\times V_0,W_1\times V_1\rangle)=\sigma_2(n_2-1)\times B_2$. Next, Player II plays $W_2\times B_2$. Then Player II plays $[\delta_2]\times V_2$, where $\delta_2=\sigma_2{}^\frown W_2\downarrow^{n_2+1}\mathcal{B}$, and so on.

$\mathsf{BM}(\mathcal{K}(X)\times Y)$

Player I	Player II
$\sigma'(\langle\,\rangle)=[\sigma_0]\times B_0$	
	$[\delta_0]\times V_0$
$[\sigma_1]\times B_1$	
	$[\delta_1]\times V_1$
$[\sigma_2]\times B_2$	
	$[\delta_2]\times V_2$
$\vdots$	$\vdots$

$\mathsf{BM}(X\times Y)$

Player I	Player II
$\sigma(\langle\,\rangle)=\sigma_0(n_0-1)\times B_0$	
	$W_0\times V_0$
$\sigma_1(n_1-1)\times B_1$	
	$W_1\times V_1$
$\sigma_2(n_2-1)\times B_2$	
	$W_2\times V_2$
$\vdots$	$\vdots$

Claim 3.9.21. $\bigcap_{k\in\omega} W_k \times V_k = \emptyset$

Proof. Suppose otherwise, that is, there exists $(x,y) \in W_k \times V_k, \forall k \in \omega$. Define $\rho = \bigcup_{k\in\omega} \sigma_k$. Note that $\rho \in \mathscr{K}(X) \subseteq\downarrow^{\omega}\mathscr{B}$, because $x \in \bigcap_{k\in\omega} W_k$. Then $(\rho, y) \in \bigcap_{k\in\omega}[\delta_k] \times V_k$, and this is a contradiction. □

Therefore σ' is a winning strategy for Player I in $\mathsf{BM}(X \times Y)$, therefore $X \times Y$ is a Baire space.

□

Corollary 3.10. Any topological space is a Baire space if and only if all of its associated countable sequence spaces are Baire.

Proof. Consider the trivial one element space $Y = \{y\}$ in the Theorem 3.9. □

Corollary 3.11. Let X be a topological space and let $\mathscr{K}(X)$ be an associated countable sequence space. Then $\mathrm{I}\uparrow\mathsf{BM}(X)$ if and only if $\mathrm{I}\uparrow\mathsf{BM}(\mathscr{K}(X))$.

Proposition 3.12. Let X be a topological space with base $\mathscr{B}$ and $\mathscr{K}(X)$ its associated Krom space. Then $\mathrm{II}\uparrow\mathsf{BM}(X)$ if and only if $\mathrm{II}\uparrow\mathsf{BM}(\mathscr{K}(X))$.

Proof. Let δ be a winning strategy for Player II in $\mathsf{BM}(X)$. We are going to build a winning strategy δ' for Player II in $\mathsf{BM}(\mathscr{K}(X))$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\mathscr{K}(X))$, Player I plays $[\sigma_0]$, with $\sigma_0 \in\downarrow^{n_0}\mathscr{B}$. Then, in $\mathsf{BM}(X)$, Player I plays $\sigma_0(n_0-1)$. Next Player II responds $\delta(\langle\sigma_0(n_0-1)\rangle) = V_0$, so Player II plays $\delta'(\langle[\sigma_0]\rangle) = [\sigma_0{}^\frown V_0]$.

- **Inning 1**

  Player I plays $[\sigma_1]$, with $\sigma_1 \in\downarrow^{n_1}\mathscr{B}$, with $\sigma_0{}^\frown V_0 \subseteq \sigma_1$. Also we can suppose that $n_1 - 1 \geq n_0$. Then in $\mathsf{BM}(X)$, Player I plays $\sigma_1(n_1-1)$. Next Player II responds $\delta(\langle\sigma_0(n_0-1), \sigma_1(n_1-1)\rangle) = V_1$. Then in $\mathsf{BM}(\mathscr{K}(X))$, Player II plays $\delta'(\langle[\sigma_0],[\sigma_1]\rangle) = [\sigma_1{}^\frown V_1]$.

- **Inning 2**

  Player I plays $[\sigma_2]$, with $\sigma_2 \in\downarrow^{n_2}\mathscr{B}$, with $\sigma_1{}^\frown V_1 \subseteq \sigma_2$ and again we can suppose that $n_2 - 1 \geq n_1$. Then in $\mathsf{BM}(X)$, Player I plays $\sigma_2(n_2-1)$. Next Player II responds $\delta(\langle\sigma_0(n_0-1), \sigma_1(n_1-1), \sigma_2(n_2-1)\rangle) = V_2$. Then in $\mathsf{BM}(\mathscr{K}(X))$, Player II plays $\delta'(\langle[\sigma_0],[\sigma_1],[\sigma_2]\rangle) = [\sigma_2{}^\frown V_2]$, and so on.

$\mathsf{BM}(X)$

Player I	Player II
$\sigma_0(n_0-1)$	
	$\delta(\langle\sigma_0(n_0-1)\rangle)=V_0$
$\sigma_1(n_1-1)$	
	V_1
$\sigma_2(n_2-1)$	
	V_2
$\vdots$	$\vdots$

$\mathsf{BM}(\mathscr{K}(X))$

Player I	Player II
$[\sigma_0]$	
	$\delta'(\langle[\sigma_0]\rangle)=[\sigma_0{}^\frown V_0]$
$[\sigma_1]$	
	$[\sigma_1{}^\frown V_1]$
$[\sigma_2]$	
	$[\sigma_2{}^\frown V_2]$
$\vdots$	$\vdots$

As δ is a winning strategy for Player II then $\bigcap_{k\in\omega}V_k\neq\emptyset$. Choose $x\in\bigcap_{k\in\omega}V_k$. Consider $\rho=\bigcup_{k\in\omega}\sigma_k{}^\frown V_k$. Note that $\rho\in\mathscr{K}(X)$, because $x\in\bigcap_{k\in\omega}V_k$. Then $\bigcap_{k\in\omega}[\sigma_k{}^\frown V_k]\neq\emptyset$. Then δ' is a winning strategy for Player II in the game $\mathsf{BM}(\mathscr{K}(X))$.

Now assume that Player II has a winning strategy δ' in the game $\mathsf{BM}(\mathscr{K}(X))$. We build a winning strategy δ for Player II in $\mathsf{BM}(X)$. Indeed,

- **Inning 0**, in $\mathsf{BM}(X)$, Player I plays $A_0\in\mathscr{B}$; then, in $\mathsf{BM}(\mathscr{K}(X))$, Player I plays $[\langle A_0\rangle]$. Next Player II responds $\delta'(\langle[\langle A_0\rangle]\rangle)=[\delta_0]$, with $\delta_0\in\downarrow^{n_0}\mathscr{B}$ and $\langle A_0\rangle\subseteq\delta_0$. Then Player II plays $\delta(\langle A_0\rangle)=\delta_0(n_0-1)$.
- **Inning 1**, Player I plays $A_1\in\mathscr{B}$. Then, in $\mathsf{BM}(\mathscr{K}(X))$, Player I plays $[\delta_0{}^\frown A_1]$. Next Player II responds $\delta'(\langle[\langle A_0\rangle],[\delta_0{}^\frown A_1]\rangle)=[\delta_1]$, with $\delta_1\in\downarrow^{n_1}\mathscr{B}$ and $\delta_0{}^\frown A_1\subseteq\delta_1$. Note that we can suppose $n_1-1\geq n_0$. Then Player II plays $\delta(\langle A_0,A_1\rangle)=\delta_1(n_1-1)$.
- **Inning 2**, then Player I plays $A_2\in\mathscr{B}$ so, in $\mathsf{BM}(\mathscr{K}(X))$, Player I plays $[\delta_1{}^\frown A_2]$. Next Player II responds $\delta'(\langle[\langle A_0\rangle],[\delta_0{}^\frown A_1],[\delta_1{}^\frown A_2]\rangle)=[\delta_2]$, with $\delta_2\in\downarrow^{n_2}\mathscr{B}$ and $\delta_1{}^\frown A_2\subseteq\delta_2$, note that we can suppose $n_2-1\geq n_1$. Then Player II plays $\delta(\langle A_0,A_1,A_2\rangle)=\delta_2(n_2-1)$, and so on.

$\mathsf{BM}(\mathscr{K}(X))$

Player I	Player II
$[\langle A_0\rangle]$	
	$\delta'(\langle[\langle A_0\rangle]\rangle)=[\delta_0]$
$[\delta_0{}^\frown A_1]$	
	$[\delta_1]$
$[\delta_1{}^\frown A_2]$	
	$[\delta_2]$
$\vdots$	$\vdots$

$\mathsf{BM}(X)$

Player I	Player II
A_0	
	$\delta(\langle A_0\rangle)=\delta_0(n_0-1)$
A_1	
	$\delta_1(n_1-1)$
A_2	
	$\delta_2(n_2-1)$
$\vdots$	$\vdots$

Again as σ' is a winning strategy for Player II in $\mathsf{BM}(\mathscr{K}(X))$. Then $\bigcap_{k\in\omega}[\delta_k] \neq \emptyset$. Choose $f \in [\delta_k], \forall k \in \omega$. In particular, there exists $x \in \bigcap_{k\in\omega} f(k)$. Also note that for each $k \in \omega$, $x \in f(n_k - 1) = \delta_k(n_k - 1)$. Then $x \in \bigcap_k \delta_k(n_k - 1)$, therefore σ is a winning strategy for Player II in the game $\mathsf{BM}(\mathscr{K}(X))$. □

In other words,

Corollary 3.13. Let X be a topological space with base $\mathscr{B}$ and $\mathscr{K}(X)$ its associated Krom space. Then the games $\mathsf{BM}(\mathscr{K}(X))$ and $\mathsf{BM}(X)$ are equivalents.

Corollary 3.14. Let X be a topological space. The following are equivalent:

(a) X is productively Baire;

(b) for each base $\mathscr{B}$ for X, its associated Krom space $\mathscr{K}_{\mathscr{B}}(X)$ is Baire;

(c) there exists a base $\mathscr{B}$ for X such that its associated Krom space $\mathscr{K}_{\mathscr{B}}(X)$ is Baire.

Finally we present the result of Krom, commented at the beginning of this section.

Theorem 3.15. There are two ultrametric Baire spaces such that their Cartesian product is not a Baire space.

Proof. By Cohen's Theorem (Theorem 3.4) there are two Baire spaces X, Y such that $X \times Y$ is not Baire. For each of these spaces we associate respective Krom spaces $\mathscr{K}(X)$ and $\mathscr{K}(Y)$. Then, by Theorem 3.9, we have that $\mathscr{K}(X) \times Y$ is not a Baire. Again by Theorem 3.9, we have that $\mathscr{K}(X) \times \mathscr{K}(Y)$ is not a Baire space. □

3.1.2.2 A counterexample with $C_\omega\mathfrak{c}^+$

Finally we present an example of a Baire space whose square is not a Baire space. This example appears in the article (FLEISSNER; KUNEN, 1978). Also for this part we follow the notation and results of the Section 1.2.1.

Remember that $C_\omega\mathfrak{c}^+$ is the subset of $\mathfrak{c}^+$ of ordinals of cofinality ω. Also, as $\mathfrak{c}^+$ is a regular uncountable cardinal, then $C_\omega\mathfrak{c}^+$ is stationary. So by Solovay's theorem (Theorem 1.74), $C_\omega\mathfrak{c}^+$ can be split into many $\mathfrak{c}^+$ many mutually disjoint stationary subsets of $\mathfrak{c}^+$.

So let $\{A_\chi : \chi \in 2^\omega\}$ be mutually disjoint stationary **subsets** of $C_\omega\mathfrak{c}^+$. Let $M = 2^\omega \times (\mathfrak{c}^+)^\omega$. Our space is

$$Y = \{\langle \chi, f\rangle \in M : f^* \in A_\chi\}$$

Proposition 3.16. Y is a Baire space.

Proof. Let $\mathscr{D} = \{D_i : i \in \omega\}$ be a family of dense open sets of M and let V be a non-empty open set of M. Let

$$W = \{f^* : \langle \chi, f\rangle \in V \cap \bigcap \mathscr{D}\}.$$

Claim 3.16.22. W is a stationary set in $\mathfrak{c}^+$.

Proof. Let C be a club in $\mathfrak{c}^+$. As V is a non-empty open set of M, there is a basic[2]open set $B_0 := N_{s_0} \times N_{t_0} \subseteq V$, where $s_0 \in 2^{<\omega}$ and $t_0 \in (\mathfrak{c}^+)^{<\omega}$. Define $s_1 = s_0{}^\frown(s_0)^* \in 2^{<\omega}$ and $t_1 = t_0{}^\frown a_0 \in (\mathfrak{c}^+)^{<\omega}$, where $a_0 = min\{x \in C : x > t_0{}^*\}$. Then define $B_1 := N_{s_1} \times N_{t_1} \subseteq B_0$. Also $D_0 \cap B_1$ is a non-empty open set of M, then choose $B_2 := N_{s_2} \times N_{t_2} \subseteq D_0 \cap B_1$ with $s_1 \subseteq s_2$ and $t_1 \subseteq t_2$. Define $s_3 = s_2{}^\frown(s_2)^* \in 2^{<\omega}$ and $t_3 = t_2{}^\frown a_2 \in (\mathfrak{c}^+)^{<\omega}$, where $a_2 = min\{x \in C : x > t_2{}^*\}$, then define $B_3 := N_{s_3} \times N_{t_3} \subseteq B_2$, and so on. Note that $(B_n)_{n\in\omega}$ is a decreasing sequence of non-empty open sets, such that $B_0 \subseteq V$ and $B_{2n+2} \subseteq D_n$, for all $n \in \omega$. Note that $x := \bigcup_{n\in\omega} s_n \in 2^\omega$ and $f := \bigcup_{n\in\omega} t_n \in J_{\mathfrak{c}^+}$. Also $f^* = \sup\{a_{2n} : n \in \omega\} \in C$, because C is closed in $\mathfrak{c}^+$. Then $\langle x, f\rangle \in \bigcap_{n\in\omega} B_n \subseteq V \cap \bigcap \mathscr{D}$, so $f^* \in C \cap W$. □

Now for $\langle \chi, f\rangle \in M$, $h : \omega \to \omega$, and $i \in \omega$, let $B(\chi, f, h, i)$ be the ball of radius $2^{-h(i)}$ around $\langle \chi, f\rangle$. Explicity,

$$B(\chi, f, h, i) = \{\langle \chi', f'\rangle \in M : \chi \restriction h(i) = \chi' \restriction h(i),\ f \restriction h(i) = f' \restriction h(i)\}.$$

Let

$$W_{\chi h} = \{f^* : f \in K_{\chi h}\},$$

where

$$K_{\chi h} = \{f \in J_{\mathfrak{c}^+} : B(\chi, f, h, i) \subseteq D_i \cap V \text{ for all } i \in \omega\}.$$

[2] Remember that the standard basis for the topology of A^ω consists of the sets $N_s = \{x \in A^{<\omega} : s \subseteq x\}$, where $s \in A^{<\omega}$ and A is endowed with the discrete topology.

Claim 3.16.23. We have the following properties:

(a.) $W = \bigcup\{W_{\chi h} : \chi \in J_2, h \in \omega^\omega\}$;

(b.) $K_{\chi h}$ is closed in $J_{\mathfrak{c}^+}$;

(c.) There are $\chi \in J_2$ and $h \in \omega^\omega$ such that $W_{\chi h}$ is a stationary set.

Proof. $(a.)$ Note that $\bigcup\{W_{\chi h} : \chi \in J_2, h \in \omega^\omega\} \subseteq W$.

On the other hand, let $f^* \in W$, so $\langle \chi, f \rangle \in V \cap \bigcap \mathscr{D}$. Let $i \in \omega$. By definition $\langle \chi, f \rangle \in V \cap D_i$. As $V \cap D_i$ is non-empty open set, then there are $s_i \in 2^{<\omega}$ and $t_i \in {\mathfrak{c}^+}^{<\omega}$ such that $\langle \chi, f \rangle \in N_{s_i} \times N_{t_i} \subseteq V \cap D_i$.

Define $h : \omega \to \omega$ as $h(i) = max\{dom(s_i), dom(t_i)\}$. Note that $B(\chi, f, h, i) \subseteq N_{s_i} \times N_{t_i} \subseteq D_i \cap V$, for all $i \in \omega$. Then $f \in K_{\chi h}$ and so $f^* \in W_{\chi h}$.

$(b.)$ We will show that $J_{\mathfrak{c}^+} \setminus K_{\chi h}$ is open. Let $f \in J_{\mathfrak{c}^+} \setminus K_{\chi h}$. Then there exists $i_0 \in \omega$ such that $B(\chi, f, h, i_0) \not\subseteq D_{i_0} \cap V$, so there is a $\langle \chi', f' \rangle \in B(\chi, f, h, i_0)$ such that $\langle \chi', f' \rangle \notin D_{i_0} \cap V$. Note that $N_{f' \restriction h(i_0)} \subseteq J_{\mathfrak{c}^+} \setminus K_{\chi h}$, otherwise there is a $g \in N_{f' \restriction h(i_0)}$ such that $B(\chi, g, h, i) \subseteq V \cap D_i$ for all $i \in \omega$. In particular $B(\chi, g, h, i_0) \subseteq V \cap D_{i_0}$. But $\langle \chi', f' \rangle \in B(\chi, g, h, i_0)$, contradiction.

$(c.)$ Otherwise $W_{\chi h}$ is non-stationary for all $\chi \in J_2$ and $h \in \omega^\omega$. By part $(a.)$, W is the union of $\mathfrak{c}$ non-stationary sets. But by Claim 3.16.22, W is a stationary subset of $\mathfrak{c}^+$. This is a contradiction with Lemma 1.75.

□

Finally, by part $(c.)$ of Claim 3.16.23 and by Proposition 1.78, there is a club C such that $C \cap C_\omega \mathfrak{c}^+ \subseteq W_{\chi h}$. Note that $\emptyset \neq C \cap A_\chi \subseteq C \cap C_\omega \mathfrak{c}^+ \subseteq W_{\chi h}$. Then, $A_\chi \cap W_{\chi h} \neq \emptyset$. So there is a $\langle \chi, f \rangle \in Y \cap V \cap \bigcap \mathscr{D}$, and Y is Baire.

□

Theorem 3.17. *Y^2 is meager in itself*

Proof. Consider

$$D_i = \left\{ \langle \langle \chi, f \rangle, \langle \chi', f' \rangle \rangle \in Y^2 : \begin{array}{l} \chi \neq \chi' \text{ and} \\ min(f^*, (f')^*) > max(f(i), f'(i)) \end{array} \right\}$$

Claim 3.17.24. For all $i \in \omega$, D_i is open and dense in Y^2.

Proof. Fix $i \in \omega$, we have the following facts.

- D_i **is open.**

  Let $\langle\langle\chi, f\rangle, \langle\chi', f'\rangle\rangle \in D_i$. Then there is $j \in \omega$ such that $\chi \restriction j = \chi' \restriction j$ and $\chi_j \neq \chi'_j$. Call $s_1 = \chi \restriction_{j+1}$ and $s_2 = \chi' \restriction_{j+1}$. Also $max(f(i), f'(i)) < min(f^*, (f')^*) \leq f^*, (f')^*$. Then there are $n_1, n_2 \in \omega$ such that $\max\{f(i), g(i)\} < f(n_1), g(n_2)$. Consider $k_i = \max\{i + 1, n_1 + 1, n_2 + 1\}$, $\rho_i = f \restriction_{k_i}$ and $\sigma_i = g \restriction_{k_i}$.

  Finally, note that

  $$\langle\langle\chi, f\rangle, \langle\chi', f'\rangle\rangle \in [(N_{s_1} \times N_{\rho_i}) \times (N_{s_1} \times N_{\rho_i})] \cap Y^2 \subseteq D_i.$$

- D_i **is dense.**

  Let $\langle\langle x_1, f_1\rangle, \langle x_2, f_2\rangle\rangle \in Y^2$. Consider the non-empty basic open set

  $$[(N_{x_1 \restriction_{i_1}} \times N_{f_1 \restriction_{j_1}}) \times (N_{x_2 \restriction_{i_2}} \times N_{f_2 \restriction_{j_2}})] \cap Y^2.$$

  Consider $k = max(i_1, i_2)$ and define

  $$x^1 = (x_1 \restriction_k)^\frown 0$$

  and

  $$x^2 = (x_2 \restriction_k)^\frown 1.$$

  Note that $x^1, x^2 \in 2^\omega$ and $x^1 \neq x^2$.

  Consider $m_1 = max(j_1, i+1)$ and $m_2 = max(j_2, i+1)$, then define

  $$f^1 = (f_1 \restriction_{m_1})^\frown max((f_1 \restriction_{m_1})^*, (f_2 \restriction_{m_2})^*)^\frown min\{x \in A_{x^1} : x > max((f_1 \restriction_{m_1})^*, (f_2 \restriction_{m_2})^*)\}$$

  and

  $$f^2 = (f_2 \restriction_{m_2})^\frown max((f_1 \restriction_{m_1})^*, (f_2 \restriction_{m_2})^*)^\frown min\{x \in A_{x^2} : x > max((f_1 \restriction_{m_1})^*, (f_2 \restriction_{m_2})^*)\}.$$

  Note that $f^1, f^2 \in J_{\mathfrak{c}^+}$, so $(f^1)^* \in A_{x^1}$ and $(f^2)^* \in A_{x^2}$, also $max(f^1(i), f^2(i)) = max(f_1(i), f_2(i)) < (f^1)^*, (f^2)^*$ then $max(f^1(i), f^2(i)) < min((f^1)^*, (f^2)^*)$, therefore

  $$\langle\langle x^1, f^1\rangle, \langle x^2, f^2\rangle\rangle \in [(N_{x_1 \restriction_{i_1}} \times N_{f_1 \restriction_{j_1}}) \times (N_{x_2 \restriction_{i_2}} \times N_{f_2 \restriction_{j_2}})] \cap Y^2 \cap D_i.$$

$\square$

Claim 3.17.25. $\bigcap_{i\in\omega} D_i = \emptyset$.

Proof. Otherwise, there exists $\langle\langle\chi, f\rangle, \langle\chi', f'\rangle\rangle \in D_i$, for all $i \in \omega$. Then $\chi \neq \chi'$ and $min(f^*, (f')^*) > max(f(i), f'(i))$, for all $i \in \omega$. By definition, as $\chi \neq \chi'$ then $f^* \neq (f')^*$. Note that for all $i \in \omega$, we have that $f(i), f'(i) \leq max(f(i), f'(i)) < min(f^*, (f')^*)$. Then $f, f^* \leq min(f^*, (f')^*) \leq f, f^*$, so $f = f^*$, contradiction. □

Note that by the previous claims, Y^2 is meager in itself, in particular Y^2 is not Baire.

□

3.2 Conditions for the product to be a Baire space.

As we have seen before, there are examples of Baire spaces whose product is not Baire and whose product is even meager in itself. Now we present conditions on one of the spaces, which makes your product a Baire space.

Lemma 3.18. Let X,Y be Baire spaces with Y having a countable π-base. Then for every sequence $G_1,G_2,\cdots$ of open dense subsets of $X\times Y$ we have that $\bigcap_{m\in\omega} G_m \neq \emptyset$.

Proof. Let $\mathscr{U}=\{U_n : n\in\omega\}$ be a countable π-base of Y. Consider the projection $\pi_X : X\times Y \to X$. Note that π_X is open and continuous. Let $m,n\in\omega$, and define $U(m,n)=\pi_X[G_m\cap(X\times U_n)]$.

Claim 3.18.26. $U(m.n)$ is open and dense in X, for each $m,n\in\omega$.

Proof. As $G_m\cap(X\times U_n)$ is open, then $U(m,n)=\pi_X[G_m\cap(X\times U_n)]$ is open. Now, let O be a non-empty open set in X, so $O\times U_n$ is a non-empty open set in $X\times Y$. Then there is a $(x,y)\in G_m\cap(O\times U_n)$, so $x\in U(m,n)\cap O$. □

As X is Baire, we have that $\bigcap_{m,n\in\omega} U(m,n)$ is dense in X. In particular, there is a $x_0\in\bigcap_{m,n\in\omega} U(m,n)$. For each $m\in\omega$, we define $H_m=\{y\in Y : (x_0,y)\in G_m\}$.

Claim 3.18.27. For each $m\in\omega$, H_m is open and dense in Y.

Proof. H_m is open. Indeed, let $y\in H_m$, so $(x_0,y)\in G_m$, then there is a basic non-empty open set $U\times V$ in $X\times Y$ such that $(x_0,y)\in U\times V\subseteq G_m$, then $y\in V\subseteq H_m$. Now, we will show that H_m is dense in Y. Indeed, let $U_n\in\mathscr{U}$ be a non-empty open set of Y. As $x_0\in U(m,n)=\pi_X[G_m\cap(X\times U_n)]$, therefore there is a $y\in Y$ such that $(x_0,y)\in G_m\cap(X\times U_n)$, then $y\in H_m\cap U_n$. □

As Y is Baire, we have that $\bigcap_{m\in\omega} H_m$ is dense in Y. In particular, there is a $y_0\in\bigcap_m H_m$. Finally, note that $(x_0,y_0)\in\bigcap_{m\in\omega} G_m$.

□

Theorem 3.19. The Cartesian product $X\times Y$ of a Baire space X and a Baire space Y having a countable π-base is a Baire space.

Proof. Let $\langle G_n : n\in\omega\rangle$ be a sequence of open dense sets in $X\times Y$. We will show that $\bigcap_{n\in\omega} G_n$ is dense in $X\times Y$. For this, let $U\times V$ be a basic non-empty open set in $X\times Y$ we must show that $\bigcap_{n\in\omega} G_n\cap(U\times V)\neq\emptyset$. Indeed, note that U is Baire, because is open in X, also $V\subseteq Y$ is second countable, so consider the sequence $\langle G_n\cap(U\times V) : n\in\omega\rangle$ of open dense sets in $U\times V$. By Lemma 3.18, we have that $\emptyset\neq\bigcap_{n\in\omega} G_n\cap(U\times V)$.

□

Corollary 3.20. If X is a second countable Baire space and Y is a Baire space, then $X \times Y$ is Baire.

In particular, if $B \subseteq \mathbb{R}$ is a Bernstein set, remember that B is a second countable Baire space, then B is productively Baire.

Also, remember the following proposition that was proved in the applications part of the section on the Banach-Mazur game.

Proposition 3.21. Let X be a Choquet topological space and let Y a Baire space. Then $X \times Y$ is a Baire space.

Remark 1. The converse is not true, because Bernstein sets are productively Baire but Player II has no winning strategy in the game BM.

Now we present a result due to Moors (MOORS, 2006) that, together with the hereditary spaces, Baire provides us with information about the product of two Baire spaces. We will also use the Banach-Mazur game to prove it.

Lemma 3.22. Let X be a topological space, let (Y,d) be a metric space and let O be a dense open subset of $X \times Y$. Then given any finite subset Z of Y, $\varepsilon > 0$ and non-empty open subset of U of X, there exist a finite subset Y' of Y and a non-empty open subset V of U such that

(i) for each $z \in Z$ there exists a $y \in Y'$ with $d(y,z) < \varepsilon$ and

(ii) $V \times Y' \subseteq O$.

Proof. We will demonstrate this result through a direct induction argument about the number of elements of Z.

In fact, firstly suppose that $Z = \{z\} \subseteq Y$, let $\varepsilon > 0$ and $U \subseteq X$ as above. Consider the non-empty open set $U \times B_\varepsilon^{(z)}$ in $X \times Y$. So there is a $(u,y') \in (U \times B_\varepsilon^{(z)}) \cap O$. Then there are non-empty open sets V and W in X and Y respectively such that $(u,y') \in V \times W \subseteq (U \times B_\varepsilon^{(z)}) \cap O$. Finally, consider $Y' = \{y'\}$ and $V \subseteq U$. Notice that these sets satisfy (i) and (ii).

Now suppose that $Z = \{z_1, z_2\} \subseteq Y$, let $\varepsilon > 0$ and $U \subseteq X$ as before. Applying the previous case to $\{z_1\}$ we have that there is a finite subset Y_1 of Y and a non-empty open subset V_1 of U satisfying (i) and (ii). In particular, there is $y_1 \in Y_1$ with $d(z_1, y_1) < \varepsilon$. Consider the non-empty open set $V_1 \times B_\varepsilon^{(z_2)}$, so there is $(a_2, b_2) \in (V_1 \times B_\varepsilon^{(z_2)}) \cap O$ and a non-empty basic open set $V_1' \times W_2$ such that $(a_2, b_2) \in V_1' \times W_2 \subseteq (V_1 \times B_\varepsilon^{(z_2)}) \cap O$. Finally, consider $Y' = \{y_1, b_2\}$ and $V = V_1'$. Note that these new sets also satisfy (i) and (ii).

Suppose the result is valid for finite sets of cardinality $n \in \omega$, and let $Z = \{z_1, \cdots, z_{n+1}\}$, let $\varepsilon > 0$ and $U \subseteq X$ as before. Consider $Z' = \{z_1, \cdots, z_n\}$, so $Z = Z' \cup \{z_{n+1}\}$. By the inductive hypothesis for Z' there exists a finite Y'' and a non-empty open subset V' of U such that

(i) for all $j \in \{1, \cdots, n\}$, there exists a $y_j \in Y''$ with $d(z_j, y_j) < \varepsilon$ and

(ii) $V' \times Y'' \subseteq O$.

Also, by construction, we can suppose that there are non-empty open sets $W_1, \cdots, W_n$ such that $V' \times W_j \subseteq O$, for all $j \in \{1, \cdots, n\}$. Consider the non-empty open set $V' \times B_\varepsilon^{(z_{n+1})}$. So there is $(a_{n+1}, b_{n+1}) \in (V' \times B_\varepsilon^{(z_{n+1})}) \cap O$ and a non-empty basic open set $V'' \times W_{n+1}$ such that $(a_{n+1}, b_{n+1}) \in V'' \times W_{n+1} \subseteq (V' \times B_\varepsilon^{(z_{n+1})}) \cap O$. Finally, consider $Y' = \{y_1, \cdots, y_n, b_{n+1}\}$ and $V = V''$. Note that these new sets also satisfy (i) and (ii).

□

Remember that every complete metric space is productively Baire (Theorem 2.8 and Proposition 2.23), also every complete metric space is hereditarily Baire. The following theorem generalizes these facts.

Theorem 3.23 (Moors)**.** Let X be a Baire space and let (Y, d) be a hereditarily Baire metric space. Then $X \times Y$ is a Baire space.

Proof. Let $\langle O_n : n \in \omega \rangle$ be a sequence of open dense sets in $X \times Y$. Note that we can assume that the sequence is decreasing. We will show that $\bigcap_{n\in\omega} O_n$ is dense in $X \times Y$. Indeed, let U and V be non-empty open sets in X and Y respectively, we will show that $\emptyset \neq \bigcap_{n\in\omega} O_n \cap (U \times V)$.

Let us define a strategy σ for Player I in the Banach-Mazur game played on X to build a strategy σ for Player I.

In the first inning, let $(x, y) \in O_1 \cap (U \times V)$. Then there are O_1^1 and O_2^1 non-empty open sets in X and Y respectively such that $(x, y) \in O_1^1 \times O_2^1 \subseteq O_1 \cap (U \times V)$, so $(U \cap O_1^1) \times \{y\} \subseteq O_1^1 \times O_2^1 \subseteq O_1$. Define $U_\emptyset = U \cap O_1^1$, $Y_\emptyset = \{y\}$ and $Z_\emptyset = \{y\} = Y_\emptyset$. Finally Player I plays $\sigma(\langle\,\rangle) = U_\emptyset$. Let $A_1 \subseteq U_\emptyset$ be the answer of Player II.

In the second inning, consider the space $A_1 \times V$. Note that $O_2 \cap (A_1 \times V)$ is open dense in $X \times V$. Also consider the finite set $Z_\emptyset = \{y\} \subseteq V$, $\varepsilon = \frac{1}{2}$ and the same open A_1. Then by Lemma 3.22, there exists a finite set $Y_{(A_1)} \subseteq V$, and a non-empty open $U_{(A_1)} \subseteq A_1$ such that

(i) for each $z \in Z_\emptyset$ there is a $y \in Y_{(A_1)}$ with $d(y, z) < \frac{1}{2}$ and

(ii) $U_{(A_1)} \times Y_{(A_1)} \subseteq O_2 \cap (X \times V) \subseteq O_2$.

Then Player I plays $\sigma(\langle A_1 \rangle) = U_{(A_1)} \subseteq A_1$. Let $A_2 \subseteq U_{(A_1)}$ be the answer of Player II.

In the third inning, consider the space $A_2 \times V$. Note that $O_3 \cap (A_2 \times V)$ is open dense in $A_2 \times V$, also consider the finite set $Z_{(A_1)} = Y_{(A_1)} \cup Z_\emptyset$, $\varepsilon = \frac{1}{3}$ and the open A_2. Then by Lemma 3.22, there exists a finite set $Y_{(A_1,A_2)} \subseteq V$, and a non-empty open $U_{(A_1,A_2)} \subseteq A_2$ such that

(i) for each $z \in Z_{(A_1)}$ there is $y \in Y_{(A_1,A_2)}$ with $d(y, z) < \frac{1}{3}$ and

(ii) $U_{(A_1,A_2)} \times Y_{(A_1,A_2)} \subseteq O_3 \cap (X \times V) \subseteq O_3$.

Then Player I plays $\sigma(\langle A_1, A_2\rangle) = U_{(A_1,A_2)} \subseteq A_2$ and let $A_3 \subseteq U_{(A_1,A_2)}$ be the answer of Player II.

Finally, in the n-th inning, consider the space $X \times V$, and suppose that the finite subsets $Y_{(A_1,\cdots,A_j)}$ and $Z_{(A_1,\cdots,A_j)}$ of V, the non-empty open subset $U_{(A_1,\cdots,A_j)}$ of A_n and σ have been defined for each $(A_1,\cdots,A_j)$ of length j with $1 \leq j \leq (n-1)$ so that:

(i) for each $z \in Z_{(A_1,\cdots,A_{n-1})}$ there is $y \in Y_{(A_1,\cdots,A_n)}$ with $d(y,z) < \frac{1}{n+1}$ and

(ii) $U_{(A_1,\cdots,A_n)} \times Y_{(A_1,\cdots,A_n)} \subseteq O_{n+1} \cap (X \times V) \subseteq O_{n+1}$.

Then Player I plays $\sigma(\langle A_1,\cdots A_n\rangle) = U_{(A_1,\cdots,A_n)}$ and Player II responds A_n. Define $Z_{(A_1,\cdots,A_n)} = Y_{(A_1,\cdots,A_n)} \cup Z_{(A_1,\cdots,A_{n-1})}$.

This completes the definition of the strategy σ for Player I in $\mathsf{BM}(X)$. Note that $Z_\emptyset \subseteq Z_{(A_1)} \subseteq Z_{(A_1,A_2)} \subseteq Z_{(A_1,A_2,A_3)} \subseteq \cdots \subseteq V$.

As X is a Baire space, then by Theorem 2.7, σ is not a winning strategy for Player I. That is, there is a sequence $\langle A_n : n \in \omega\rangle$ of open sets in X such that $\bigcap_{n\in\omega} A_n \neq \emptyset$. Let $x \in \bigcap_{n\in\omega} A_n$. Note that $x \in A_n \subseteq U_{(A_1,\cdots,A_{n-1})} \subseteq U_\emptyset \subseteq U$.

Let $n \in \omega$, so $x \in A_n \subseteq U_{(A_1,\cdots,A_{n-1})}$. By construction, $U_{(A_1,\cdots,A_{n-1})} \times Y_{(A_1,\cdots,A_{n-1})} \subseteq O_n$. Define $W_n = \pi_Y[(\{x\} \times Y) \cap O_n]$. We claim that W_n is open in Y. Indeed, let $w \in W_n$, so $(x,w) \in O_n$, as O_n is open, there are non-empty open sets B^1 and B^2 in X and Y respectively with $(x,w) \in B^1 \times B^2 \subseteq O_n$. Finally note that $w \in B^2 \subseteq W_n$.

Then, for each $n \in \omega$, we have defined the open set $W_n \subseteq Y$ such that $\{x\} \times W_n = (\{x\} \times Y) \cap O_n$. Note that for each $n \in \omega$, we have that $W_{n+1} \subseteq W_n$.

Consider $Z = \bigcup\{Z_{(A_1,\cdots,A_{n-1})} : n \in \omega\} \subseteq V \subseteq Y$. As Y is hereditarily Baire, $\overline{Z}$ is Baire.

Claim 3.23.28. For each $n \in \omega$, $W_n \cap Z$ is dense in Z.

Proof. Let $n \in \omega$. We will show that $Z \subseteq \overline{W_n \cap Z}$. Indeed, let $z \in Z$. Then there is a $k \in \omega$ such that $z \in Z_{(A_1,\cdots,A_{k-1})}$. By construction, there is a $y \in Y_{(A_1,\cdots,A_k)} \subseteq Z_{(A_1,\cdots,A_k)} \subseteq Z$ with $d(y,z) < \frac{1}{k+1}$. Also $(x,y) \in U_{(A_1,\cdots,A_k)} \times Y_{(A_1,\cdots,A_k)} \subseteq O_{k+1}$. Then $(x,y) \in O_{k+1} \cap (\{x\} \times Y) = \{x\} \times W_{k+1}$, so $y \in W_{k+1}$.

Now, let $\varepsilon > 0$. We must show that $\emptyset \neq B_\varepsilon^{(z)} \cap (W_n \cap Z)$. Let $j \in \omega$ such that $\frac{1}{j} < \varepsilon$. We have the following cases:

(a) if $n \leq k$, so $y \in W_{k+1} \subseteq W_n$, also we have the sub-cases,

- $j \leq k$. So $d(y,z) < \frac{1}{k+1} < \varepsilon$, therefore $y \in B_\varepsilon^{(z)} \cap (W_n \cap Z)$.
- $k < j$. Note that $z \in Z_{(A_1,\cdots,A_{k-1})} \subseteq Z_{(A_1,\cdots,A_{j-1})}$. By construction, there exists $y' \in Y_{(A_1,\cdots,A_j)}$ with $d(y',z) < \frac{1}{j+1} < \varepsilon$. Note that $y' \in W_{j+1} \subseteq W_n$, then $y' \in B_\varepsilon^{(z)} \cap (W_n \cap Z)$.

(b) if $k < n$, we have that $z \in Z_{(A_1,\cdots,A_{k-1})} \subseteq Z_{(A_1,\cdots,A_{n-1})}$ and there exists $y' \in Y_{(A_1,\cdots,A_n)} \subseteq Z_{(A_1,\cdots,A_n)} \subseteq Z$ with $d(y',z) < \frac{1}{n+1}$. Note that $y' \in W_{n+1} \subseteq W_n$.

- $j \leq n$. so $d(y',z) < \frac{1}{n+1} < \varepsilon$, therefore $y' \in B_{\varepsilon}^{(z)} \cap (W_n \cap Z)$.
- $n < j$. Note that $z \in Z_{(A_1,\cdots,A_{n-1})} \subseteq Z_{(A_1,\cdots,A_{j-1})}$. By construction, there exists $y'' \in Y_{(A_1,\cdots,A_j)}$ with $d(y'',z) < \frac{1}{j+1} < \varepsilon$. Note that $y'' \in W_{j+1} \subseteq W_n$, then $y'' \in B_{\varepsilon}^{(z)} \cap (W_n \cap Z)$.

Finally, in any case we have shown that $\emptyset \neq B_{\varepsilon}^{(z)} \cap (W_n \cap Z)$, therefore $Z \subseteq \overline{W_n \cap Z}$. Then $\overline{W_n \cap Z}^Z = \overline{W_n \cap Z} \cap Z = Z$, that is, $W_n \cap Z$ is dense in Z. □

Also for each $n \in \omega$, we have that $\overline{Z} \subseteq \overline{W_n \cap Z} \subseteq \overline{W_n \cap \overline{Z}}$. Then $\langle W_n \cap \overline{Z} : n \in \omega \rangle$ is a sequence of open dense sets in $\overline{Z}$. Since $\overline{Z}$ is Baire, we have that $\bigcap_{n\in\omega} W_n \cap \overline{Z}$ is dense in $\overline{Z}$. In particular, since $V \cap \overline{Z}$ is a non-empty open set in $\overline{Z}$, there is a $y \in \bigcap_{n\in\omega}(W_n \cap \overline{Z}) \cap V$. Therefore for each $n \in \omega$, $(x,y) \in (\{x\} \times Y) \cap O_n$, then $(x,y) \in \bigcap_{n\in\omega} O_n \cap (U \times V)$.

□

3.3 Infinite products of Baire spaces

In this part we will see how the property of being Baire can change when we consider the infinite products in the usual topology and in the box topology.

Remember some basic definitions and properties of infinite products. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces and put $X = \prod_{\lambda\in\Lambda} X_\lambda$.

The **Tychonoff product topology** on X is the topology having the collection of sets of the form

$$\prod_{\lambda\in M} U_\lambda \times \prod_{\mu\in\Lambda\setminus M} X_\mu,$$

where U_λ is an open set in X_λ for each $\lambda \in M$ and M is a finite subset of Λ, as a **base**. We will denote this topological space by $\prod_{\lambda\in\Lambda} X_\lambda$. In the case that $|\Lambda| = \kappa$ and $X_\lambda = X$ for all $\lambda \in \Lambda$ we denote $\prod_{\lambda\in\Lambda} X_\lambda$ by X^κ.

The **box topology** on X is the topology having the collection of sets of the form

$$\prod_{\lambda\in\Lambda} U_\lambda$$

where U_λ is an open set in X_λ for each $\lambda \in \Lambda$, as a base. If U_λ is an open subset of X_λ for each λ, then $\prod_{\lambda\in\Lambda} U_\lambda$ is called a box in $\prod_{\lambda\in\Lambda} X_\lambda$. We will denote this topological space by $\square_{\lambda\in\Lambda} X_\lambda$. In the case that $|\Lambda| = \kappa$ and $X_\lambda = X$ for all $\lambda \in \Lambda$ we denote $\square_{\lambda\in\Lambda} X_\lambda$ by $\square^\kappa X$.

Lemma 3.24. Let $\mathscr{B}$ be a π-base of X and let κ a cardinal, then the set of products of κ elements chosen from $\mathscr{B}$ is a π-base for $\square^\kappa X$.

Also let us fix a notation for a base we have a base for the countable Tychonoff power. Let $\tau^*(X)$ be the family of all nonempty open sets of a space X, and let $[\tau^*(X)]^{<\omega}$ be the family of all finite subsets of $\tau^*(X)$. For each $\mathscr{U} = \{U_0, \cdots, U_{n-1}\} \in [\tau^*(X)]^{<\omega}$, let

$$[\mathscr{U}] = [U_0, \cdots, U_{n-1}] := \prod_{j=0}^{n-1} U_j \times X^{\omega\setminus n}$$

be the basic open set in X^ω defined by $\mathscr{U}$ in this particular order. If $\mathscr{V} = \{V_0, \cdots, V_{m-1}\} \in [\tau^*(X)]^{<\omega}$ is disjoint from $\{U_0, \ldots, U_{n-1}\}$, then $[U_0, \cdots, U_{n-1}, \mathscr{V}]$ is defined by

$$[U_0, \cdots, U_{n-1}, \mathscr{V}] := \prod_{j=0}^{n-1} U_j \times \prod_{k=0}^{m-1} V_k \times X^{\omega\setminus(n\cup m)}.$$

Furthermore, we put

$$\mathscr{B}(X^\omega) := \{[\mathscr{U}] : \mathscr{U} \in [\tau^*(X)]^{<\omega}\}.$$

Also remember that, for each $\mu \in \Lambda$, the map $\pi_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu$ defined by the relation $\pi_\mu((x_\lambda)_{\lambda\in\Lambda}) = x_\mu$ is called the projection on X_μ. Each π_μ is continuous and an open map in both Tychonoff and box topologies.

3.3.1 Counterexamples with infinite products of Baire spaces.

In this first part we present the example of a Baire X space whose Tychonoff power X^ω is meager in itself and its finite power X^n is Baire, for all $n \in \omega$. This example appears in the article (FLEISSNER; KUNEN, 1978).

Let $\{A_y : y \in \omega^\omega\}$ be disjoint stationary subsets of $C_{\omega \mathfrak{c}^+}$. Let

$$C_y = \bigcup \{A_{y'} : y' \in \omega^\omega \text{ and } y'(0) \neq y(0)\}.$$

Let

$$X = \{\langle y, f\rangle \in \omega^\omega \times J_{\mathfrak{c}^+} : f^* \in C_y\}.$$

Theorem 3.25. X^ω is meager in itself .

Proof. To se this, for any $i, j, k < \omega$ let us define $D_{ijk} \subseteq X^\omega$ by

$$D_{ijk} = \{\langle\langle y_0, f_0\rangle, \cdots\rangle \in X^\omega : \min(f_i^*, f_j^*) > \max(f_i(k), f_j(k))\}.$$

In addition, for each $l < \omega$, we define $E_l \subseteq X^\omega$ by

$$E_l = \{\langle\langle y_0, f_0\rangle, \cdots\rangle \in X^\omega : l \subseteq \{y_0(0), \cdots, y_m(0)\} \text{ for some } m < \omega\}.$$

Claim 3.25.29. $D_{ijk}, E_l \subseteq X^\omega$ are open and dense sets, for all $i, j, l, k < \omega$.

Proof. Fix $i, j, l, k < \omega$.

- **D_{ijk} is open.**

  Let $\langle\langle y_0, f_0\rangle, \cdots\rangle \in D_{ijk}$, so $max(f(i), f'(i)) < min(f^*, (f')^*) \leq f^*, (f')^*$. Then there are $n_1, n_2 \in \omega$ such that $\max\{f(i), g(i)\} < f(n_1), g(n_2)$. Consider $k_i = \max\{i+1, n_1+1, n_2+1\}$, $\rho_i = f\restriction_{k_i}$ and $\sigma_i = g\restriction_{k_i}$.

  Finally, note that

  $$\langle\langle y_0, f_0\rangle, \cdots\rangle \in [(J_\omega \times J_{\mathfrak{c}^+}) \times \cdots \times (J_\omega \times N_{\rho_i}) \times \cdots \times (J_\omega \times N_{\sigma_i}) \times (J_\omega \times J_{\mathfrak{c}^+}) \times \cdots] \cap X^\omega \subseteq D_i.$$

- **D_{ijk} is dense.**

  Let $\langle\langle y_0, f_0\rangle, \cdots, \langle y_{m-1}, f_{m-1}\rangle\rangle \in X^m$. By definition $f_i^* \in A_{y_i'}$ and $f_j^* \in A_{y_j'}$ for some $A_{y_i'}$, $A_{y_j'}$ with $y_i'(0) \neq y_i(0)$ and $y_i'(0) \neq y_i(0)$. Consider the non-empty basic open set

  $$\left[\prod_{p=0}^{m-1} (N_{y_p\restriction_{s_p}} \times N_{f_p\restriction_{t_p}}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega\setminus m}\right] \cap X^\omega.$$

  Define $k_i = max(k+1, t_i)$, $k_j = max(k+1, t_j)$ [3] and consider

  $$f^i = (f_i\restriction_{k_i})^\frown(max((f_i\restriction_{k_i})^*, (f_j\restriction_{k_j})^*))^\frown min\{x \in A_{y_i'} : x > max((f_i\restriction_{k_i})^*, (f_j\restriction_{k_j})^*)\}$$

[3] Note that we could have the case where $i, j \notin m$. In this case $k_i = k_j = k+1$.

and

$$f^j = (f_j \restriction_{k_j})^\frown (max((f_i \restriction_{k_i})^*, (f_j \restriction_{k_j})^*))^\frown min\{x \in A_{y'_j} : x > max((f_i \restriction_{k_i})^*, (f_j \restriction_{k_j})^*)\}.$$

Note that $f^i \in N_{f_i \restriction_{k_i}} \cap X$ and $f^j \in N_{f_j \restriction_{k_j}} \cap X$. Finally, note that

$$\langle \langle y_0, f_0 \rangle, \cdots, \langle y_i, f^i \rangle, \cdots, \langle y_j, f^j \rangle, \cdots \rangle \in D_{ijk} \cap \left[\prod_{p=0}^{m-1} (N_{y_p \restriction_{s_p}} \times N_{f_p \restriction_{t_p}}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega \setminus m} \right] \cap X^\omega.$$

- E_l **is open**. Let $\langle \langle y_0, f_0 \rangle, \cdots \rangle \in E_l$. Then there exists $m < \omega$ such that $l \subseteq \{y_0(0), \cdots, y_m(0)\}$. Define $s_j = y_j \restriction_1$ for all $j < m+1$. Note that

$$\left[\prod_{j=0}^{m+1} (N_{s_j} \times J_{\mathfrak{c}^+}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega \setminus m+1} \right] \cap X^\omega \subseteq E_l.$$

- E_l **is dense**. Consider the non-empty basic open set

$$\left[\prod_{j=0}^{m-1} (N_{s_j} \times N_{t_j}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega \setminus m} \right] \cap X^\omega.$$

  Let $\langle \langle y_0, f_0 \rangle, \cdots \rangle \in \left[\prod_{j=0}^{m-1} (N_{s_j} \times N_{t_j}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega \setminus m} \right] \cap X^\omega$. For each $p < l$, consider $\langle c_p, g_p \rangle \in X$, where $c_p(n) = p$ and $g_p(n) = x_p \in A_{c_{p+1}}$, for all $n \in \omega$.

  Finally note that $\langle \langle y_0, f_0 \rangle, \cdots, \langle y_{m-1}, f_{m-1} \rangle, \langle c_0, g_0 \rangle, \cdots, \langle c_{l-1}, g_{l-1} \rangle, \langle y_{m+l}, f_{m+l} \rangle, \cdots \rangle \in \left[\prod_{j=0}^{m-1} (N_{s_j} \times N_{t_j}) \times (J_\omega \times J_{\mathfrak{c}^+})^{\omega \setminus m} \right] \cap X^\omega \cap E_l$.

□

Claim 3.25.30. $\bigcap_{i,j,l,k<\omega} D_{ijk} \cap E_l = \emptyset$

Proof. Otherwise, let $\langle \langle y_0, f_0 \rangle, \cdots \rangle \in \bigcap_{i,j,l,k<\omega} D_{ijk} \cap E_l$. Note that $f_0^* = f_1^* = \cdots = \gamma \in C_\omega \mathfrak{c}^+$. Indeed, let $i, j < \omega$. Then $f_i(k), f_j(k) \leq max(f_i(k), f_j(k)) < min(f_i^*, f_j^*)$ for every $k \in \omega$, therefore $f_i^*, f_j^* = min(f_i^*, f_j^*) = f_i^*, f_j^*$.

Also, by definition $\gamma \in C_{y_n}$ for all $n < \omega$, in particular there exists $z \in J_\omega$ such that $\gamma \in A_z$.

We claim that $z(0) \neq y_n(0)$ for all $n < \omega$, otherwise, suppose that $\exists p < \omega$ such that $z(0) = y_p(0)$. Note that $\gamma \in C_{y_p} \cap A_z$; then, there exists $y' \in J_\omega$ such that $z(0) = y'(0) \neq y_p(0)$, contradiction. In particular, $z(0) \notin \{y_n(0) : n < \omega\}$.

On the other hand, by the definition of the E_l's, we have that $\omega = \{y_n(0) : n < \omega\}$. This a contradiction, therefore we have the result. □

Note that by the previous claims, X^ω is meager in itself , in particular X^ω is not Baire.

□

Now let us show the second part of the initial statement.

Theorem 3.26. Let $n < \omega$, then X^n is a Baire space.

Proof. Let $\mathscr{D} = \{D_i : i \in \omega\}$ be a family of dense open sets in $(\omega^\omega \times J_{\mathfrak{c}^+})^n$ and V a non-empty open subset in $(\omega^\omega \times J_{\mathfrak{c}^+})^n$. Put

$$W = \{\alpha < \mathfrak{c}^+ : \alpha = f_0^* = \cdots = f_{n-1}^* \text{ and } \langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in V \cap \bigcap_{i<\omega} D_i\}$$

Claim 3.26.31. W is a stationary subset of $\mathfrak{c}^+$.

Proof. Let C be a club in $\mathfrak{c}^+$. As V is a non-empty open set of $(J_\omega \times J_{\mathfrak{c}^+})^n$, there is a basic open set $B_0 := \prod_{j=0}^{n-1} N_{s_j^0} \times N_{t_j^0} \subseteq V$, where $s_j^0 \in \omega^{<\omega}$ and $t_j^0 \in (\mathfrak{c}^+)^{<\omega}$, for all $j < n$.

Define $s_j^1 = s_j^0{}^\frown(s_j^0)^* \in \omega^{<\omega}$ and $t_j^1 = t_j^0{}^\frown a_0 \in (\mathfrak{c}^+)^{<\omega}$, where $a_0 = min\{x \in C : x > \max\{(t_j^0)^* : j < n\}\}$, for all $j < n$. Then define $B_1 := \prod_{j=0}^{n-1} N_{s_j^1} \times N_{t_j^1} \subseteq B_0$. Also $D_0 \cap B_1$ is a non-empty open set of $(J_\omega \times J_{\mathfrak{c}^+})^n$. Then choose $B_2 := \prod_{j=0}^{n-1} N_{s_j^2} \times N_{t_j^2} \subseteq V \subseteq D_0 \cap B_1$ with $s_j^1 \subseteq s_j^2$ and $t_j^1 \subseteq t_j^2$, for all $j < n$.

Define $s_j^3 = s_j^2{}^\frown(s_j^2)^* \in \omega^{<\omega}$ and $t_j^3 = t_j^2{}^\frown a_2 \in (\mathfrak{c}^+)^{<\omega}$, where $a_2 = min\{x \in C : x > \max\{(t_j^2)^* : j < n\}\}$, for all $j < n$. Then define $B_3 := N_{s_3} \times N_{t_3} \subseteq B_2$, and so on. We have that $(B_n)_{n\in\omega}$ is a decreasing sequence of non-empty open sets, such that $B_0 \subseteq V$ and $B_{2n+2} \subseteq D_n$, for all $n \in \omega$. Note that, for each $j < n$, we have that $x^j := \bigcup_{m\in\omega} s_j^m \in \omega^\omega$ and $f^j := \bigcup_{m\in\omega} t_j^m \in J_{\mathfrak{c}^+}$, also $\alpha = (f^j)^* = \sup\{a_{2n} : n \in \omega\} \in C$, because C is closed in $\mathfrak{c}^+$. Then $\langle\langle x^0, f^0\rangle, \cdots, \langle x^{n-1}, f^{n-1}\rangle\rangle \in \bigcap_{n\in\omega} B_n \subseteq V \cap \bigcap \mathscr{D}$, so $\alpha \in C \cap W$.

□

For a point $p = \langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in (J_\omega \times J_{\mathfrak{c}^+})^n$ and $h \in \omega^\omega$ and an $i \in \omega$, $B(p, 2^{-h(i)})$ denotes the ball centered at p with the radius $2^{-h(i)}$, i.e.,

$$\langle\langle \overline{y}_0, \overline{f}_0\rangle, \cdots, \langle \overline{y}_{n-1}, \overline{f}_{n-1}\rangle\rangle \in B(p, 2^{-h(i)})$$

if and only if $\overline{y}_j \restriction_{h(i)} = y_j \restriction_{h(i)}, \overline{f}_j \restriction_{h(i)} = f_j \restriction_{h(i)}$ for all $j < n$.

For each $y = (y_0, \cdots, y_{n-1}) \in (J_\omega)^n$ and $h \in \omega^\omega$ define

$$W_{yh} = \{\alpha < \mathfrak{c}^+ : \alpha = f_0^* = \cdots = f_{n-1}^* \text{ and } f = (f_0, \cdots, f_{n-1}) \in K_{yh}\},$$

where

$$K_{yh} = \{f \in (J_{\mathfrak{c}^+})^n : B(p_f^y, 2^{-h(i)}) \subseteq D_i \cap V \text{ for all } i \in \omega\},$$

where $p_f^y = \langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in (J_\omega \times J_{\mathfrak{c}^+})^n$.

Claim 3.26.32. We have the following properties:

(a.) $W = \bigcup\{W_{yh} : y \in J_\omega, h \in \omega^\omega\}$;

(b.) K_{yh} is closed in $(J_{\mathfrak{c}^+})^n$;

(c.) There are $y \in J_\omega$ and $h \in \omega^\omega$ such that W_{yh} is a stationary set.

Proof. $(a.)$ Note that $\bigcup\{W_{yh} : y \in J_m, h \in \omega^\omega\} \subseteq W$.

On the other hand, let $\alpha \in W$, so there exists $p_f^y = \langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in V \cap \bigcap_{i<\omega} D_i$ with $\alpha = f_0^* = \cdots = f_{n-1}^*$. We have that for any $i \in \omega$, $V \cap D_i$ is a non-empty open subset of $(\omega^\omega \times J_{\mathfrak{c}^+})^n$. Then, there are $s_0, \cdots, s_{n-1} \in \omega^{<\omega}$ and $t_0, \cdots, t_{n-1} \in (\mathfrak{c}^+)^{<\omega}$ such that $p_f^y \in [(N_{s_0} \times N_{t_0}) \times \cdots \times (N_{s_{n-1}} \times N_{t_{n-1}})] \subseteq V \cap D_i$. Consider $s = max\{dom(s_k) : k < n\}$ and $t = max\{dom(t_k) : k < n\}$, then define $h : \omega \to \omega$ as $h(i) = max\{s, t\}$. Note that $B(p_f^y, 2^{-h(i)}) \subseteq D_i \cap V$, for all $i \in \omega$. Then $f \in K_{yh}$ and so $f^* \in W_{yh}$.

$(b.)$ We will show that $(J_{\mathfrak{c}^+})^n \setminus K_{yh}$ is open. Let $f \in (J_{\mathfrak{c}^+})^n \setminus K_{yh}$. Then there exists $i_0 \in \omega$ such that $B(p_f^y, 2^{-h(i_0)}) \nsubseteq D_{i_0} \cap V$, so there is $\langle\langle \overline{y}_0, \overline{f}_0\rangle, \cdots, \langle \overline{y}_{n-1}, \overline{f}_{n-1}\rangle\rangle \in B(p_f^y, 2^{-h(i)})$ such that $\langle\langle \overline{y}_0, \overline{f}_0\rangle, \cdots, \langle \overline{y}_{n-1}, \overline{f}_{n-1}\rangle\rangle \notin D_{i_0} \cap V$. Note that $N_{\overline{f}_0 \restriction h(i_0)} \times \cdots \times N_{\overline{f}_{n-1} \restriction h(i_0)} \subseteq (J_{\mathfrak{c}^+})^n \setminus K_{yh}$, otherwise there is $g = (g_1, \cdots, g_{n-1}) \in N_{\overline{f}_0 \restriction h(i_0)} \times \cdots \times N_{\overline{f}_{n-1} \restriction h(i_0)}$ such that $B(p_g^y, 2^{-h(i)}) \subseteq V \cap D_i$ for all $i \in \omega$. In particular $B(p_g^y, 2^{-h(i_0)}) \subseteq V \cap D_{i_0}$, but $\langle\langle \overline{y}_0, \overline{f}_0\rangle, \cdots, \langle \overline{y}_{n-1}, \overline{f}_{n-1}\rangle\rangle \in B(p_g^y, 2^{-h(i_0)})$. Contradiction.

$(c.)$ Otherwise W_{yh} is non-stationary for all $y \in J_\omega$ and $h \in \omega^\omega$. Now, by Claim 3.26.31, W is a stationary subset of $\mathfrak{c}^+$. By part $(a.)$, W is the union of $\mathfrak{c}$ non-stationary sets. This is a contradiction with Lemma 1.75.

□

Now by part $(c.)$ of Claim 3.26.32 and by Lemma 1.79, there is a club C such that $C \cap C_\omega \mathfrak{c}^+ \subseteq W_{yh}$.

Choose $\hat{y} \in (n+1)^\omega \subseteq J_\omega = \omega^\omega$ such that $\hat{y}(0) \notin \{y_0(0), \cdots, y_{n-1}(0)\}$. Then by definition of C_{y_i}'s, we have $A_{\hat{y}} \subseteq C_{y_i}$ for all $i < n$.

Note that $\emptyset \neq C \cap A_{\hat{y}} \subseteq C \cap C_\omega \mathfrak{c}^+ \subseteq W_{yh}$, let $\beta \in C \cap A_{\hat{y}} = C \cap A_{\hat{y}} \cap C_\omega \mathfrak{c}^+ \subseteq A_{\hat{y}} \cap W_{yh}$. Then there exists a point $(f_0, \cdots, f_{n-1}) \in (J_{\mathfrak{c}^+})^n$ such that $\langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in V \cap \bigcap_{i\in\omega} D_i$ and $f_0^* = \cdots = f_{n-1}^* = \beta$. Since $f_i^* \in A_{\hat{y}} \subseteq C_{y_i}$ for all $i < n$, we have $(y_i, f_i) \in X$ for all $i < n$. It follows that

$$\langle\langle y_0, f_0\rangle, \cdots, \langle y_{n-1}, f_{n-1}\rangle\rangle \in V \cap \bigcap_{i\in\omega} D_i \cap X^n,$$

which implies that $\bigcap_{i\in\omega} D_i \cap X^n$ is dense in X^n, and thus X^n is a Baire space. □

3.3.2 Conditions for infinite product of Baire spaces to be Baire.

In this part we will give conditions on a topological space X so that their infinite powers, in the box and Tychonoff product topology, are Baire. Again the Banach-Mazur game will be of great importance to demonstrate some of these results. It is important to mention that the phenomenon of being Baire in product can change depending on which topology we choose (box or Tychonoff).

3.3.2.1 Tychonoff products

Theorem 3.27 (Choquet)**.** Tychonoff products of Choquet spaces are Choquet and therefore they are Baire.

Proof. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Choquet spaces and let δ_α a winning strategy for Player II in $\mathsf{BM}(X_\alpha)$. We will build a winning strategy δ for Player II in $\mathsf{BM}(\prod_{\alpha\in\Lambda} X_\alpha)$. Indeed,

- **Inning 0**

  Player I plays $U_0 = \prod_{\alpha\in A_0} U_\alpha^0 \times \prod_{\alpha\notin A_0} X_\alpha$ where A_0 is a finite subset of Λ and U_α^0 is a non-empty open subset of X_α. Next, Player II responds $\delta(\langle U_0\rangle) = \prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0\rangle) \times \prod_{\alpha\notin A_0} X_\alpha$.

- **Inning 1**

  Player I plays $U_1 = \prod_{\alpha\in A_1} U_\alpha^1 \times \prod_{\alpha\notin A_1} X_\alpha \subseteq \delta(\langle U_0\rangle)$ where $A_1 \supseteq A_0$ is a finite subset of Λ and U_α^1 is a non-empty open subset of X_α. Next, Player II responds $\delta(\langle U_0, U_1\rangle) = \prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0, U_\alpha^1\rangle) \times \prod_{\alpha\in A_1\setminus A_0} \delta_\alpha(\langle U_\alpha^1\rangle) \times \prod_{\alpha\notin A_1} X_\alpha$.

- **Inning 2**

  Player I plays $U_1 = \prod_{\alpha\in A_1} U_\alpha^1 \times \prod_{\alpha\notin A_1} X_\alpha \subseteq \delta(\langle U_0\rangle)$ where $A_2 \supseteq A_1$ is a finite subset of Λ and U_α^2 is a non-empty open subset of X_α. Next, Player II responds $\delta(\langle U_0, U_1, U_2\rangle) = \prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0, U_\alpha^1, U_\alpha^2\rangle) \times \prod_{\alpha\in A_1\setminus A_0} \delta_\alpha(\langle U_\alpha^1, U_\alpha^2\rangle) \times \prod_{\alpha\in A_2\setminus A_1} \delta_\alpha(\langle U_\alpha^2\rangle) \times \prod_{\alpha\notin A_2} X_\alpha$, and so on.

$\mathsf{BM}(\prod_{\alpha\in\Lambda} X_\alpha)$

Player I	Player II
$\prod_{\alpha\in A_0} U_\alpha^0 \times \prod_{\alpha\notin A_0} X_\alpha$	
	$\prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0\rangle) \times \prod_{\alpha\notin A_0} X_\alpha$
$\prod_{\alpha\in A_1} U_\alpha^1 \times \prod_{\alpha\notin A_1} X_\alpha$	
	$\prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0, U_\alpha^1\rangle) \times \prod_{\alpha\in A_1\setminus A_0} \delta_\alpha(\langle U_\alpha^1\rangle) \times \prod_{\alpha\notin A_1} X_\alpha$
$\prod_{\alpha\in A_2} U_\alpha^2 \times \prod_{\alpha\notin A_2} X_\alpha$	
	$\prod_{\alpha\in A_0} \delta_\alpha(\langle U_\alpha^0, U_\alpha^1, U_\alpha^2\rangle) \times \prod_{\alpha\in A_1\setminus A_0} \delta_\alpha(\langle U_\alpha^1, U_\alpha^2\rangle) \times \prod_{\alpha\in A_2\setminus A_1} \delta_\alpha(\langle U_\alpha^2\rangle) \times \prod_{\alpha\notin A_2} X_\alpha$
$\vdots$	$\vdots$

Note that for each $\alpha \in \bigcup_{m\in\omega} A_m$, as δ_α is a winning strategy, there exists $x_\alpha \in \delta_\alpha(\langle U_\alpha^0, \cdots, U_\alpha^n\rangle)$ for all $n \in \omega$. Choose any point $x_* \in X$ and define $x_\alpha = x_*$ for $\alpha \in \kappa \setminus \bigcup_{m\in\omega} A_m$.

Then $x = (x_\alpha)_{\alpha\in\Lambda} \in \bigcap_{n\in\omega} \delta(\langle U_0, \cdots, U_n\rangle)$ and therefore δ is a winning strategy for Player II in $\mathsf{BM}(\prod_{\alpha\in\Lambda} X_\alpha)$, so $\prod_{\alpha\in\Lambda} X_\alpha$ is a Choquet space. $\square$

In this part we present a result of the article (LI; ZSILINSZKY, 2017) by Rui Li and László Zsilinszky which generalizes Theorem 3.9 for infinite Tychonoff products.

Theorem 3.28. Let I be an index set. Then $\prod_{i\in I} X_i$ is a Baire space if and only if $\prod_{i\in I} \mathscr{K}_{\mathscr{B}_i}(X_i)$ is a Baire space, for some (equivalently, for every) choice of bases $\mathscr{B}_i$ for X_i.

Proof. First, assume that Player I has a winning strategy σ in $\mathsf{BM}(\prod_{i\in I} X_i)$. We are going to build a winning strategy σ' for Player I in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$ as follows:

- **First inning**

  In $\mathsf{BM}(\prod_{i\in I} X_i)$, if Player I plays $\sigma(\langle\,\rangle) = \prod_{i\in I_0} V_{0,i} \times \prod_{i\notin I_0} X_i$ for some finite $I_0 \subseteq I$ and $V_{0,i} \in \mathscr{B}_i$, then, in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, Player I plays $\sigma'(\langle\,\rangle) = \prod_{i\in I_0} V_{0,i}^* \times \prod_{i\notin I_0} \mathscr{K}(X_i)$ where $V_{0,i}^* = [\langle V_{0,i}\rangle]$. Next, Player II plays $U_0^* = \prod_{i\in J_0} U_{0,i}^* \times \prod_{i\notin J_0} \mathscr{K}(X_i)$ for some finite $J_0 \supseteq I_0$, and for all $i \in J_0$, $U_{0,i}^* = [\langle U_{0,i}(0), \cdots, U_{0,i}(m_{0,i})\rangle]$ with $\langle U_{0,i}(0), \cdots, U_{0,i}(m_{0,i})\rangle \in \downarrow^{m_{0,i}+1} \mathscr{B}_i$, $m_{0,i} \geq 0$ and $U_{0,i}(0) = V_{0,i}$ for all $i \in I_0$. Then, in $\mathsf{BM}(\prod_{i\in I} X_i)$, Player II plays $U_0 = \prod_{i\in J_0} U_{0,i}(m_{0,i}) \times \prod_{i\notin J_0} X_i$.

- **Second inning**

  In $\mathsf{BM}(\prod_{i\in I} X_i)$, Player I plays $\sigma(\langle U_0\rangle) = \prod_{i\in I_1} V_{1,i} \times \prod_{i\notin I_1} X_i$ where $I_1 \supseteq J_0$ is finite, $V_{i,1} \in \mathscr{B}_i$ for each $i \in I_1$ and $V_{i,1} \subseteq U_{0,i}(m_{0,i})$ whenever $i \in J_0$. Then, in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, Player I plays $\sigma'(\langle U_0^*\rangle) = \prod_{i\in I_1} V_{1,i}^* \times \prod_{i\notin I_1} \mathscr{K}(X_i)$ where

$$V_{i,1}^* = \begin{cases} [\langle U_{0,i}(0), \cdots, U_{0,i}(m_{0,i}), V_{i,1}\rangle] & \text{if } i \in J_0 \\ [\langle V_{1,i}\rangle] & \text{if } i \in I_1 \setminus J_0. \end{cases}$$

Next Player II plays $U_1^* = \prod_{i\in J_1} U_{1,i}^* \times \prod_{i\notin J_1} \mathscr{K}(X_i)$ for some finite $J_1 \supseteq I_1$, and for all $i \in J_1$, $U_{1,i}^* = [\langle U_{1,i}(0), \cdots, U_{1,i}(m_{1,i})\rangle]$ with $\langle U_{1,i}(0), \cdots, U_{1,i}(m_{1,i})\rangle \in \downarrow^{m_{1,i}+1} \mathscr{B}_i$. Note that

(i) $\langle U_{1,i}(0), \cdots, U_{1,i}(m_{1,i})\rangle \supseteq \langle U_{0,1}(0), \cdots, U_{0,i}(m_{0,i}), V_{i,1}\rangle$ for $i \in J_0$ and

(ii) $U_{1,i}(0) = V_{1,i}$ for $i \in I_1 \setminus J_0$.

Then, in BM($\prod_{i\in I} X_i$), Player II plays $U_1 = \prod_{i\in J_1} U_{1,i}(m_{1,i}) \times \prod_{i\notin J_1} X_i$, and so on.

BM($\prod_{i\in I} X_i$)

Player I	Player II
$\sigma(\langle\ \rangle) = \prod_{i\in I_0} V_{0,i} \times \prod_{i\notin I_0} X_i$	
	$U_0 = \prod_{i\in J_0} U_{0,i}(m_{0,i}) \times \prod_{i\notin J_0} X_i$
$\prod_{i\in I_1} V_{1,i} \times \prod_{i\notin I_1} X_i$	
	$U_1 = \prod_{i\in J_1} U_{1,i}(m_{1,i}) \times \prod_{i\notin J_1} X_i$
$\vdots$	$\vdots$

BM($\prod_{i\in I} \mathscr{K}(X_i)$)

Player I	Player II
$\sigma'(\langle\ \rangle) = \prod_{i\in I_0} V_{0,i}^* \times \prod_{i\notin I_0} \mathscr{K}(X_i)$	
	$U_0^* = \prod_{i\in J_0} U_{0,i}^* \times \prod_{i\notin J_0} \mathscr{K}(X_i)$
$\prod_{i\in I_1} V_{1,i}^* \times \prod_{i\notin I_1} \mathscr{K}(X_i)$	
	$U_1^* = \prod_{i\in J_1} U_{1,i}^* \times \prod_{i\notin J_1} \mathscr{K}(X_i)$
$\vdots$	$\vdots$

Proceeding inductively, we can define σ' so that whenever $k < \omega$, and $U_k^* = \prod_{i\in J_k} U_{k,i}^* \times \prod_{i\notin J_k} \mathscr{K}(X_i)$ is given for some finite J_k, and for all $i \in J_k$, $U_{k,i}^* = [\langle U_{k,i}(0), \cdots, U_{k,i}(m_{k,i})\rangle]$ for $\langle U_{k,i}(0), \cdots, U_{k,i}(m_{k,i})\rangle \in \downarrow^{m_{k,i}+1} \mathscr{B}_i$ and $m_{k,i} \geq 0$, then $\sigma'(\langle U_0^*, \cdots, U_k^*\rangle) = \prod_{i\in I_{k+1}} V_{k+1,i}^* \times \prod_{i\notin I_{k+1}} \mathscr{K}(X_i)$ have been chosen, where $I_{k+1} \supseteq J_k$ is finite, and

$$V_{k+1,i}^* = \begin{cases} [\langle U_{k,i}(0), \cdots, U_{k,i}(m_{k,i}), V_{k+1,i}\rangle] & \text{if } i \in J_k \\ [\langle V_{k+1,i}\rangle] & \text{if } i \in I_{k+1} \setminus J_k \end{cases}$$

is such that $\sigma(\langle U_0,\cdots,U_k\rangle) = \prod_{i\in I_{k+1}} V_{k+1,i} \times \prod_{i\notin I_{k+1}} X_i$ where $U_j = \prod_{i\in J_j} U_{j,i}(m_{j,i}) \times \prod_{i\notin J_j} X_i$ for all $j \leq k$.

As σ is a winning strategy for Player I in $\mathsf{BM}(\prod_{i\in I} X_i)$, we have that

$$\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \sigma(\langle U_0,\cdots,U_n\rangle) = \emptyset$$

for each play $\sigma(\langle\ \rangle), U_0, \sigma(\langle U_0\rangle), U_1, \cdots, U_n, \sigma(\langle U_0,\cdots,U_n\rangle), \cdots$ of $\mathsf{BM}(\prod_{i\in I} X_i)$.

Claim 3.28.33. σ' is a winning strategy for Player I in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$.

Proof. Let $\sigma'(\langle\ \rangle), U_0^*, \sigma'(\langle U_0^*\rangle), U_1^*, \cdots, U_n^*, \sigma'(\langle U_0^*,\cdots,U_n^*\rangle), \cdots$ be a play of $\mathsf{BM}(\mathscr{K}(X))$ and assume there exists $f \in \bigcap_{n\in\omega} \sigma'(\langle U_0^*,\cdots,U_n^*\rangle) = \bigcap_{n\in\omega} U_n^*$.

Then for each $i \in I$, $f(i) \in \mathscr{K}(X_i)$, so we can pick some $x_i \in \bigcap_{n\in\omega} f(i)(n)$. Moreover, if $i \in I_k$ for a given $k < \omega$, then $x_i \in V_{k,i}$, so $(x_i)_{i\in I} \in \prod_{i\in I_k} V_{k,i} \times \prod_{i\notin I_k} X_i$. Thus, $(x_i)_{i\in I} \in \bigcap_{n\in\omega} \sigma(\langle U_0,\cdots,U_n\rangle)$, contradiction. □

Therefore $\prod_{i\in I} \mathscr{K}(X_i)$ is not a Baire space.

Now assume that Player I has a winning strategy σ' in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$. We are going to build a winning strategy σ for Player I in $\mathsf{BM}(\prod_{i\in I} X_i)$ as follows:

- **First inning**

  In $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, if Player I plays $\sigma'(\langle\ \rangle) = \prod_{i\in I_0} V_{0,i}^* \times \prod_{i\notin I_0} \mathscr{K}(X_i)$ for some finite $I_0 \subseteq I$ where for all $i \in I_0$, $V_{0,i}^* = [\langle V_{0,i}(0),\cdots,V_{0,i}(m_{0,i})\rangle]$. Then, in $\mathsf{BM}(\prod_{i\in I} X_i)$, Player I plays $\sigma(\langle\ \rangle) = \prod_{i\in I_0} V_{0,i}(m_{0,i}) \times \prod_{i\notin I_0} X_i$. Next, Player II plays $U_0 = \prod_{i\in J_0} U_{0,i} \times \prod_{i\notin J_0} X_i$ for some finite $J_0 \supseteq I_0$ and $U_{0,i} \subseteq V_{0,i}(m_{0,i})$ for all $i \in I_0$. Define:

$$U_{0,i}^* = \begin{cases} [\langle V_{0,i}(0),\cdots,V_{0,i}(m_{0,i}), U_{0,i}\rangle] & \text{for all } \ i \in I_0 \\ [\langle U_{0,i}\rangle] & \text{for all } \ i \in J_0 \setminus I_0 \end{cases}$$

  Then, in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, Player II plays $U_0^* = \prod_{i\in J_0} U_{0,i}^* \times \prod_{i\notin J_0} \mathscr{K}(X_i)$.

- **Second inning**

  In $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, Player I plays $\sigma'(\langle U_0^*\rangle) = \prod_{i\in I_1} V_{1,i}^* \times \prod_{i\notin I_1} \mathscr{K}(X_i)$ where $I_1 \supseteq J_0$ is finite and $V_{1,i}^* = [\langle V_{1,i}(0),\cdots,V_{1,i}(m_{1,i})\rangle]$ whenever $i \in I_1$. Also note that

  (i) $\langle V_{1,i}(0),\cdots,V_{1,i}(m_{1,i})\rangle \supseteq \langle V_{0,i}(0),\cdots,V_{0,i}(m_{0,i}), U_{0,i}\rangle$ for $i \in I_0$ and

  (ii) $V_{1,i}(0) = U_{0,i}$ for $i \in J_0 \setminus I_0$.

  Then, in $\mathsf{BM}(\prod_{i\in I} X_i)$, Player I plays $\sigma(\langle U_0\rangle) = \prod_{i\in I_1} V_{1,i}(m_{1,i}) \times \prod_{i\notin I_1} X_i$. Next Player II plays $U_1 = \prod_{i\in J_1} U_{1,i} \times \prod_{i\notin J_1} X_i$.

Define:

$$U^*_{1,i} = \begin{cases} [\langle V_{1,i}(0), \cdots, V_{1,i}(m_{1,i}), U_{1,i}\rangle] & \text{for all } \ i \in I_1 \\ [\langle U_{1,i}\rangle] & \text{for all } \ i \in J_1 \setminus I_1 \end{cases}$$

Then, in $\mathsf{BM}(\prod_{i\in I} X_i)$, Player II plays $U_1 = \prod_{i\in J_1} U_{1,i}(m_{1,i}) \times \prod_{i\notin J_1} X_i$, and so on.

$\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$

Player I	Player II
$\sigma'(\langle\,\rangle) = \prod_{i\in I_0} V^*_{0,i} \times \prod_{i\notin I_0} \mathscr{K}(X_i)$	
	$U^*_0 = \prod_{i\in J_0} U^*_{0,i} \times \prod_{i\notin J_0} \mathscr{K}(X_i)$
$\prod_{i\in I_1} V^*_{1,i} \times \prod_{i\notin I_1} \mathscr{K}(X_i)$	
	$U^*_1 = \prod_{i\in J_1} U^*_{1,i} \times \prod_{i\notin J_1} \mathscr{K}(X_i)$
$\vdots$	$\vdots$

$\mathsf{BM}(\prod_{i\in I} X_i)$

Player I	Player II
$\sigma(\langle\,\rangle) = \prod_{i\in I_0} V_{0,i}(m_{0,i}) \times \prod_{i\notin I_0} X_i$	
	$U_0 = \prod_{i\in J_0} U_{0,i} \times \prod_{i\notin J_0} X_i$
$\prod_{i\in I_1} V_{1,i}(m_{1,i}) \times \prod_{i\notin I_1} X_i$	
	$U_1 = \prod_{i\in J_1} U_{1,i} \times \prod_{i\notin J_1} X_i$
$\vdots$	$\vdots$

As σ' is a winning strategy for Player I in $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$, we have that

$$\bigcap_{n\in\omega} U^*_n = \bigcap_{n\in\omega} \sigma'(\langle U^*_0, \cdots, U^*_n\rangle) = \emptyset$$

for each play $\sigma'(\langle\,\rangle), U^*_0, \sigma'(\langle U^*_0\rangle), U^*_1, \cdots, U_n, \sigma'(\langle U^*_0, \cdots, U^*_n\rangle), \cdots$ of $\mathsf{BM}(\prod_{i\in I} \mathscr{K}(X_i))$.

Claim 3.28.34. σ is a winning strategy for Player I in $\mathsf{BM}(\prod_{i\in I} X_i)$.

Proof. Let $\sigma(\langle\,\rangle), U_0, \sigma(\langle U_0\rangle), U_1, \cdots, U_n, \sigma(\langle U_0, \cdots, U_n\rangle), \cdots$ be a play of $\mathsf{BM}(\prod_{i\in I} X_i)$ and assume there exists $(x_i)_{i\in I} \in \bigcap_{n\in\omega} \sigma(\langle U_0, \cdots, U_n\rangle) = \bigcap_{n\in\omega} U_n$.

Let $k \in \omega$ and $i \in I_k$ then define $f(i) = \bigcup_{k\in\omega} [\langle V_{k,i}(0), \cdots, V_{k,i}(m_{k,i})\rangle]$. Note that for each $i \in I_k$ and $k \in \omega$, $f(i) \in \mathscr{K}(X_i)$, because $x_i \in \bigcap_{n\in\omega} f(i)(n)$. Now, if $i \notin I \setminus \bigcup_{k\in\omega} I_k$ put $f(i) = \langle X_i\rangle_{n\in\omega}$. Then $f = (f(i))_{i\in I} \in \bigcap_{n\in\omega} \sigma'(\langle U^*_0, \cdots, U^*_n\rangle)$, contradiction. □

Therefore $\prod_{i\in I} X_i$ is not a Baire space.

□

Now we present the result of (OXTOBY, 1961) that shows that arbitrary product of Baire spaces with countable π-bases are Baire.

Lemma 3.29. The Tychonoff product of any **countable** family of spaces, each of which has a countable π-base, has a countable π-base. Futhermore, if each space is a Baire space, then the product is a Baire space.

Proof. The complete proof of this lemma can be found in (HAWORTH; MCCOY, 1977), Lemma 5.6. □

Remember that a topological space X satisfies the **countable chain condition** iff every family of disjoint open subsets of X is countable. For example every separable space has the countable chain condition.

Lemma 3.30. The product of every family of spaces, each of which has a countable π-base, has the countable chain condition.

Proof. The complete proof of this lemma can be found in (HAWORTH; MCCOY, 1977), Lemma 5.8. □

Lemma 3.31. Let $\{X_\alpha : \alpha \in A\}$ be a family of Baire spaces such that the product of any countable subcollection is a Baire space and such that $\prod_{\alpha\in A} X_\alpha$ has the countable chain condition. Then $\prod_{\alpha\in A} X_\alpha$ is a Baire space.

Proof. Let $\{G_n\}_{n\in\omega}$ be a sequence of dense open subsets of $X = \prod_{\alpha\in A} X_\alpha$. By Zorn's lemma, each G_n contains a maximal pairwise disjoint family of **basic** open sets, $\{U^n_m : m \in \omega\}$, which is countable since X has the countable chain condition. Therefore,

$$H_n = \bigcup_{m\in\omega} U^n_m \subseteq G_n$$

is a dense open subset of X.

Note that each U^n_m is of the form $\prod_{\alpha\in A^n_m} U_\alpha \times \prod_{\alpha\in A\setminus A^n_m} X_\alpha$, where $\mathsf{supp}(U^n_m) = A^n_m$ is a finite subset of A. Let $B = \bigcup_{n,m\in\omega} A^n_m$. Note that B is a countable subset of A. Now each H_n is of the form $K_n \times \prod_{\alpha\in A\setminus B} X_\alpha$ where K_n is an open subset of $\prod_{\alpha\in B} X_\alpha$.

Since each H_n is dense in X, each K_n is dense in $\prod_{\alpha\in B} X_\alpha$. Indeed, let V be a basic non-empty open subset of $\prod_{\alpha\in B} X_\alpha$, so $V = \prod_{\alpha\in\mathsf{supp}(V)} V_\alpha \times \prod_{\alpha\in B\setminus\mathsf{supp}(V)} X_\alpha$. This induces the non-empty basic open set V' in X, that is , $V' = \prod_{\alpha\in\mathsf{supp}(V)} V_\alpha \times \prod_{\alpha\in A\setminus\mathsf{supp}(V)} X_\alpha$, so $H_n \cap V' \neq \emptyset$. Then $\emptyset \neq p_B(H_n \cap V') \subseteq p_B(H_n) \cap p_B(V') = K_n \cap V$, where $p_B : X \to \prod_{\alpha\in B} X_\alpha$ is the projection map.

As $\prod_{\alpha\in B} X_\alpha$ is a Baire space then $\bigcap_{n\in\omega} K_n$ is dense in $\prod_{\alpha\in B} X_\alpha$. Hence $\bigcap_{n\in\omega} H_n$ is dense in X, and, therefore $\bigcap_{n\in\omega} G_n$ is dense in X. Indeed, let W be a non-empty basic open subset of X, so $W = \prod_{\alpha\in\mathsf{supp}(W)} W_\alpha \times \prod_{\alpha\in A\setminus\mathsf{supp}(W)} X_\alpha$. As $p_B(W)$ is a non-empty open set in $\prod_{\alpha\in B} X_\alpha$, then $p_B(W)\cap\bigcap_{n\in\omega} K_n \neq \emptyset$. Let $x = (x_\alpha)_{\alpha\in B} \in p_B(W)\cap\bigcap_{n\in\omega} K_n$, extending x to all A, that is, if $\alpha\in\mathsf{supp}(W)\setminus B$, put $x_\alpha = y_\alpha \in W_\alpha$, and, if $\alpha\in A\setminus(B\cup\mathsf{supp}(W)), x_\alpha = z_\alpha \in X_\alpha$, so $(x_\alpha)_{\alpha\in A} \in W\cap\bigcap_{n\in\omega} H_n$. □

Finally we have that

Theorem 3.32 (Oxtoby)**.** The product of every family of Baire spaces, each of which has a countable π-base, is a Baire space.

Proof. Let $\{X_\alpha : \alpha\in\Lambda\}$ be a family of Baire spaces, each of which has a countable π-base. Note that by Lemma, $\prod_{\alpha\in\Lambda} X_\alpha$ has the countable chain condition and each product of any countable subcollection of Λ also has the countable chain condition. Then, by Lemma, $\prod_{\alpha\in\Lambda} X_\alpha$ is a Baire space. □

Corollary 3.33. Any Tychonoff product of second countable Baire spaces is Baire.

Corollary 3.34. Let $B\subseteq\mathbb{R}$ be a Bernstein set. Then, for each $\kappa\geq 2$, the Tychonoff power B^κ is a Baire space. Therefore $\mathrm{I}\not\uparrow\mathsf{BM}(B^\kappa)$.

Finally we present two results that appear in the article (FLEISSNER; KUNEN, 1978) of William Fleissner and Kenneth Kunen. The first is a new application of the Banach-Mazur game and the second again relates the cellularity and the meager in itself spaces.

Theorem 3.35 (Kunen-Fleissner)**.** Let $\kappa\geq\omega$. If X^ω is Baire, then X^κ is Baire, where the powers are considered in the Tychonoff product.

Proof. Let $\kappa>\omega$, we will show that, if X^κ is not Baire then X^ω is not Baire. For this, let σ be a winning strategy for Player I in $\mathsf{BM}(X^\kappa)$. We are going to build a winning strategy $\tilde{\sigma}$ for Player I in $\mathsf{BM}(X^\omega)$.

- **Inning 0**

  In $\mathsf{BM}(X^\kappa)$, Player I plays $\sigma(\langle\,\rangle) = \prod_{i\in\mathsf{N}_0} U_i^0 \times X^{\kappa\setminus\mathsf{N}_0}$, where $\mathsf{N}_0 = \{k_0^0,\cdots,k_{n_0-1}^0\}$ is the support of $\sigma(\langle\,\rangle)$. Now in $\mathsf{BM}(X^\omega)$, Player I plays $\tilde{\sigma}(\langle\,\rangle) = \prod_{j=0}^{n_0-1}\tilde{U}_j^0 \times X^{\omega\setminus n_0}$, where for each $j\in\{0,\cdots,n_0-1\}$ we rename $\tilde{U}_j^0 := U_{k_j^0}^0$. Next Player II responds $\prod_{j=0}^{m_0-1}\tilde{V}_j^0 \times X^{\omega\setminus m_0}$ with $m_0\geq n_0$. Now, we will rename the plays of Player II, that is, we define

$$\begin{cases} V_{k_j^0}^0 := \tilde{V}_j^0 & \text{if } j\in\{0,\cdots,n_0-1\} \\ V_{k_{n_0-1}^0+1+j}^0 := \tilde{V}_{n_0+j}^0 & \text{if } j\in\{0,\cdots,m_0-1-n_0\}. \end{cases}$$

  Also, put $\mathsf{M}_0 = \mathsf{N}_0\cup\{k_{n_0-1}^0+1+j : j\in\{0,\cdots,m_0-1-n_0\}\}$. Returning to $\mathsf{BM}(X^\kappa)$, Player II responds $\prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}$.

- **Inning 1**

In $\mathsf{BM}(X^\kappa)$, Player I plays $\sigma(\langle\prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}\rangle) = \prod_{i\in\mathsf{N}_1} U_i^1 \times X^{\kappa\setminus\mathsf{N}_1}$, where $\mathsf{N}_1 = \mathsf{M}_0 \cup \{k_0^1, \cdots, k_{n_1-1}^1\}$ is the support of $\sigma(\langle\prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}\rangle)$. We will rename the plays of Player I, that is, we define

$$\begin{cases} \tilde{U}_j^1 := U_{k_j^0}^1 & \text{if } j \in \{0, \cdots, n_0 - 1\} \\ \tilde{U}_{n_0+j}^1 := U_{k_{n_0-1}^0+1+j}^1 & \text{if } j \in \{0, \cdots, m_0 - 1 - n_0\} \\ \tilde{U}_{m_0+j}^1 := U_{k_j^1}^1 & \text{if } j \in \{0, \cdots, n_1 - 1\}. \end{cases}$$

The first two lines tell us that the $\tilde{U}$'s and U's are the same in m_0 and the last line tells us that after m_0 we complete with the U's from $\{k_0^1, \cdots, k_{n_1-1}^1\}$ to $m_0 + n_1$ in X^ω.

Now in $\mathsf{BM}(X^\omega)$, Player I plays $\tilde{\sigma}(\langle\prod_{j=0}^{m_0-1} \tilde{V}_j^0 \times X^{\omega\setminus m_0}\rangle) = \prod_{j=0}^{m_0+n_1-1} \tilde{U}_j^1 \times X^{\omega\setminus(m_0+n_1)}$; next Player II responds $\prod_{j=0}^{m_1-1} \tilde{V}_j^1 \times X^{\omega\setminus m_1}$, with $m_1 \geq m_0 + n_1$. Again, we will rename the plays of Player II, that is, we define

$$\begin{cases} V_{k_j^0}^1 := \tilde{V}_j^1 & \text{if } j \in \{0, \cdots, n_0 - 1\} \\ V_{k_{n_0-1}^0+1+j}^1 := \tilde{V}_{n_0+j}^1 & \text{if } j \in \{0, \cdots, m_0 - 1 - n_0\} \\ V_{k_j^1}^1 := \tilde{V}_{m_0+j}^1 & \text{if } j \in \{0, \cdots, n_1 - 1\} \\ V_{k^*+1+j}^1 := \tilde{V}_{m_0+n_1+j}^1 & \text{if} j \in \{0, \cdots, m_1 - m_0 - 1 - n_1\} \\ & \text{where } k^* = \max\{k_{n_0-1}^0 + 1 + m_0 - 1 - n_0, k_{n_1-1}^1\}. \end{cases}$$

Put $\mathsf{M}_1 = \mathsf{N}_1 \cup \{k^* + 1 + j : j \in \{0, \cdots, m_1 - m_0 - 1 - n_1\}\}$. Returning to $\mathsf{BM}(X^\kappa)$, Player II responds $\prod_{i\in\mathsf{M}_1} V_i^1 \times X^{\kappa\setminus\mathsf{M}_1}$, and so on.

$\mathsf{BM}(X^\kappa)$

Player I	Player II
$\sigma(\langle\,\rangle) = \prod_{i\in\mathsf{N}_0} U_i^0 \times X^{\kappa\setminus\mathsf{N}_0}$	
	$\prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}$
$\prod_{i\in\mathsf{N}_1} U_i^1 \times X^{\kappa\setminus\mathsf{N}_1}$	
	$\prod_{i\in\mathsf{M}_1} V_i^1 \times X^{\kappa\setminus\mathsf{M}_1}$
$\vdots$	$\vdots$

$\mathsf{BM}(X^\omega)$

Player I	Player II
$\tilde{\sigma}(\langle\,\rangle) = \prod_{j=0}^{n_0-1} \tilde{U}_j^0 \times X^{\omega\setminus n_0}$	
	$\prod_{j=0}^{m_0-1} \tilde{V}_j^0 \times X^{\omega\setminus m_0}$
$\prod_{j=0}^{m_0+n_1-1} \tilde{U}_j^1 \times X^{\omega\setminus(m_0+n_1)}$	
	$\prod_{j=0}^{m_1-1} \tilde{V}_j^1 \times X^{\omega\setminus m_1}$
$\vdots$	$\vdots$

As σ is a winning strategy for Player I in $\mathsf{BM}(X^\kappa)$, we have that

$$\bigcap_{n\in\omega}\prod_{i\in\mathsf{M}_n} V_i^n \times X^{\kappa\setminus\mathsf{M}_n} = \emptyset$$

for each play $\sigma(\langle\,\rangle), \prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}, \sigma(\langle\prod_{i\in\mathsf{M}_0} V_i^0 \times X^{\kappa\setminus\mathsf{M}_0}\rangle), \prod_{i\in\mathsf{M}_1} V_i^1 \times X^{\kappa\setminus\mathsf{M}_1}, \cdots$ of $\mathsf{BM}(X^\kappa)$.

Claim 3.35.35. $\tilde{\sigma}$ is a winning strategy for Player I in $\mathsf{BM}(X^\omega)$.

Proof. Let $\tilde{\sigma}(\langle\,\rangle), \prod_{j=0}^{m_0-1} \tilde{V}_j^0 \times X^{\omega\setminus m_0}, \tilde{\sigma}(\langle\prod_{j=0}^{m_0-1} \tilde{V}_j^0 \times X^{\omega\setminus m_0}\rangle), \prod_{j=0}^{m_1-1} \tilde{V}_j^1 \times X^{\omega\setminus m_1}, \cdots$ be a play of $\mathsf{BM}(X^\omega)$ and assume there exists $x = (x_j)_{j\in\omega} \in \bigcap_{n\in\omega}\prod_{j=0}^{m_n-1} \tilde{V}_j^n \times X^{\omega\setminus m_1}$. Now define

$$\begin{cases} x_{k_j^0} := x_j & \text{if } j \in \{0,\cdots,n_0-1\} \\ x_{k_{n_0-1}^0+1+j} := x_{n_0+j} & \text{if } j \in \{0,\cdots,m_0-1-n_0\} \\ x_{k_j^1} := x_{m_0+j} & \text{if } j \in \{0,\cdots,n_1-1\} \\ x_{k^*+1+j} := x_{m_0+n_1+j} & \text{if } j \in \{0,\cdots,m_1-m_0-1-n_1\}, \text{ and so on.} \end{cases}$$

Choose any point $x_* \in X$ and define $x_\alpha = x_*$ for $\alpha \in \kappa\setminus\omega$. Then completing $x \in X^\omega$ to $\tilde{x} = (x_\alpha)_{\alpha\in\kappa} \in X^\kappa$, we have that $\tilde{x} \in \bigcap_{n\in\omega}\prod_{i\in\mathsf{M}_n} V_i^n \times X^{\kappa\setminus\mathsf{M}_n}$, contradicting the fact that Player I has a winning strategy in $\mathsf{BM}(X^\kappa)$. Therefore $\tilde{\sigma}$ is a winning strategy for Player I in the game $\mathsf{BM}(X^\omega)$. □

Therefore X^ω is not a Baire space. □

Definition 3.36. Let $\varkappa$ be a cardinal. A topological space X has **cellularity** $\varkappa$ if every family of disjoint open sets of X has cardinality $\leq \varkappa$.

Theorem 3.37. Suppose for all $\beta \in I$, X_β has a π-base of cardinality $\leq \varkappa$. Then if $X = \prod\{X_\beta : \beta \in I\}$ is meager in itself, there is $I' \subseteq I$, $|I'| \leq \varkappa$, such that $\prod\{X_\beta : \beta \in I'\}$ is meager in itself.

Proof. Direct from Lemmas 3.38 and 3.40. □

Lemma 3.38. If each X_β has a π-base of cardinality $\leq \varkappa$ and I is **finite**, then $X = \prod_{\beta\in I} X_\beta$ has cellularity $\varkappa$.

Proof. Let $\mathscr{B}_i$ be a π-base of X_i, for all $i \in I$. By hyphotesis, for each $i \in I$, $|\mathscr{B}_i| \leq \varkappa$. Define the π-base for X, as $\mathscr{B} = \{\prod_{i\in I} B_i : B_i \in \mathscr{B}_i, \forall i \in I\}$, note that $|\mathscr{B}| \leq \varkappa$, because, as I is finite then $|\mathscr{B}|^{|I|} = |\mathscr{B}| \leq \varkappa$.

Now, suppose otherwise that there is a family of disjoint non-empty open sets $\mathscr{F}$ with $|\mathscr{F}| > \varkappa$. Then for each $F \in \mathscr{F}$, there is a $B_F \in \mathscr{B}$ such that $B_F \subseteq F$, so $\{B_F : F \in \mathscr{F}\} \subseteq \mathscr{B}$. But $|\{B_F : F \in \mathscr{F}\}| > \varkappa$, contradiction.

□

Corollary 3.39. If each X_β has a π-base of cardinality $\leq \varkappa$ and I is **infinite**, then $X = \prod_{\beta \in I} X_\beta$ has cellularity $\varkappa$.

Proof. Assume towards a contradiction that $\{U_\beta : \beta < \varkappa^+\}$ is a family of pairwise disjoint, non-empty open subsets of $\prod_{i \in I} X_i$. By shrinking the U_β's if necessary, we may assume that each U_β is a basic open set. Then U_β depends on a finite set of coordinates, $b_\beta \subseteq I$.

Applying the Δ-system lemma (Theorem 1.81) for $\kappa = \omega$ and $\lambda = \varkappa^+$ we have that there a Δ-system $B \subseteq \{b_\beta : \beta < \varkappa^+\}$ with root Δ such that $|B| = \varkappa^+$. Note that $\Delta \subseteq I$ cannot be empty, since $b_\alpha \cap b_\beta = \emptyset$ implies that $U_\alpha \cap U_\beta \neq \emptyset$. Note that by Lemma 3.38, $\prod_{\beta \in \Delta} X_\beta$ has cellularity $\varkappa$. Let $\pi[U_\beta]$ be the projection of U_β onto $\prod_{\beta \in \Delta} X_\beta$. Then $\{\pi[U_\beta] : \beta \in B\}$ forms a disjont family of non-empty sets in $X = \prod_{\beta \in \Delta} X_\beta$, contradiction. □

Lemma 3.40. Suppose $X = \prod\{X_\beta : \beta \in I\}$ has cellularity $\varkappa$ and is meager in itself. Then there is a $I' \subseteq I$, $|I'| \leq \varkappa$, such that $\prod\{X_\beta : \beta \in I'\}$ is meager in itself.

Proof. Let $\mathscr{D} = \{D_n : n \in \omega\}$ be a family of dense open sers of X, $\bigcap \mathscr{D} = \emptyset$. Let $\{G^n_\beta : \beta \in K_n\}$ be a **maximal** collection of disjoint basic open subsets of D_n.

Claim 3.40.36. $\bigcup\{G^n_\beta : \beta \in K_n\}$ is dense open.

Proof. Suppose otherwise; then there is a non-empty open set V in X such that $V \cap \bigcup\{G^n_\beta : \beta \in K_n\} = \emptyset$. Choose W a basic non-empty open set in X such that $W \subseteq D_n \cap V$. Then $\{W\} \cup \{G^n_\beta : \beta \in K_n\}$ is a collection of disjoint basic open subsets of D_n, contradiction. □

Define $\tilde{D}_n := \bigcup\{G^n_\beta : \beta \in K_n\}$ and let $I' = \bigcup\{\operatorname{supp} G^n_\beta : \beta \in K_n, n \in \omega\}$. Consider $\pi : X \to \prod\{X_\beta : \beta \in I'\}$ the projection onto $\prod\{X_\beta : \beta \in I'\}$. Then $\{\pi[\tilde{D}_n] : n \in \omega\}$ is a family of dense open sets in $\prod\{X_\beta : \beta \in I'\}$.

□

3.3.2.2 Box products

Theorem 3.41. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Choquet spaces. Then $\square_{\alpha\in\Lambda}X_\alpha$ is a Choquet space.

Proof. Let $\alpha \in \Lambda$ and δ_α be a winning strategy for Player II in $\mathsf{BM}(X_\alpha)$. We are going to build a winning strategy δ' for Player II in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$.

Indeed, in the first inning in $\square_{\alpha\in\Lambda}X_\alpha$, Player I plays $\square_{\alpha\in\Lambda}U_\alpha^0$, where U_α^0 is a non-empty open set in X_α for all $\alpha \in \Lambda$. Then Player II responds $\delta'(\langle\square_{\alpha\in\Lambda}U_\alpha^0\rangle) = \square_{\alpha\in\Lambda}\delta_\alpha(\langle U_\alpha^0\rangle)$. In the second inning, Player I plays $\square_{\alpha\in\Lambda}U_\alpha^1 \subseteq \square_{\alpha\in\Lambda}\delta_\alpha(\langle U_\alpha^0\rangle)$; next, Player II plays $\delta'(\langle\square_{\alpha\in\Lambda}U_\alpha^0, \square_{\alpha\in\Lambda}U_\alpha^1\rangle) = \square_{\alpha\in\Lambda}\delta_\alpha(\langle U_\alpha^0, U_\alpha^1\rangle)$. In the inning $n \in \omega$, if Player I plays $\square_{\alpha\in\Lambda}U_\alpha^{n-1}$ then Player II responds $\delta'(\langle\square_{\alpha\in\Lambda}U_\alpha^0, \cdots, \square_{\alpha\in\Lambda}U_\alpha^{n-1}\rangle) = \square_{\alpha\in\Lambda}\delta_\alpha(\langle U_\alpha^0, \cdots, U_\alpha^{n-1}\rangle)$, and so on.

For each $\alpha \in \Lambda$, as δ_α is a winning strategy for Player II, then there exists

$$x_\alpha \in \bigcap_{n\in\omega} \delta_\alpha(\langle U_\alpha^0, U_\alpha^1, \cdots, U_\alpha^n\rangle);$$

then

$$x = (x_\alpha)_{\alpha\in\Lambda} \in \bigcap_{n\in\omega} \square_{\alpha\in\Lambda}\delta_\alpha(\langle U_\alpha^0, U_\alpha^1, \cdots, U_\alpha^n\rangle) = \bigcap_{n\in\omega} \delta'(\langle\square_{\alpha\in\Lambda}U_\alpha^0, \cdots, \square_{\alpha\in\Lambda}U_\alpha^{n-1}\rangle).$$

Therefore δ' is a winning strategy for Player II in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, so $\square_{\alpha\in\Lambda}X_\alpha$ is Choquet. $\square$

Corollary 3.42. If a space is Choquet, then all powers of that space, considered in the box product topology, are Choquet spaces.

Corollary 3.43 (White)**.** If Player II has a winning strategy in the Banach-Mazur game on a space, then all powers of that space, considered in the box product topology, are Baire spaces.

In the article (ZSILINSZKY, 2004) by László Zsilinszky, it is commented that by making a slight modification in the proof of the main result the following result is obtained

Theorem 3.44 (Zsilinszky)**.** If X_i is a Baire space having a locally countable π-base for each $i \in \omega$, then $\square_{i\in\omega}X_i$ is a Baire space.

Corollary 3.45. The countable box power of a second countable Baire space is Baire.

Corollary 3.46. Let $B \subseteq \mathbb{R}$ be a Bernstein set. Then, for each $n \leq \omega$, $\square^n B$ is a Baire space, therefore I $\not\uparrow \mathsf{BM}(\square^n B)$.

Also we can generalize Theorem 3.9 for infinite box products.

Theorem 3.47. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces with $\mathscr{B}_\alpha$ a base for X_α and let $\{\mathscr{K}(X_\alpha) : \alpha \in \Lambda\}$ be their associated Krom spaces. Then $\square_{\alpha\in\Lambda}X_\alpha$ is Baire if and only if $\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha)$ is Baire.

Proof. First we will show that if $\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha)$ is not Baire then $\square_{\alpha\in\Lambda}X_\alpha$ is not Baire. Let σ be a winning strategy for Player I in $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$. We will build a winning strategy σ' for Player I in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player I plays $\sigma(\langle\,\rangle) = \square_{\alpha\in\Lambda}[\delta_0^\alpha]$ where for each $\alpha\in\Lambda$, $\delta_0^\alpha \in\downarrow^{n_0^\alpha}\mathscr{B}_\alpha$, $n_0^\alpha\in\omega$. Then, in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}\delta_0^\alpha(n_0^\alpha - 1)$, next Player II responds $\square_{\alpha\in\Lambda}U_0^\alpha$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player II responds $\square_{\alpha\in\Lambda}[\delta_0^\alpha{}^\frown U_0^\alpha]$.

- **Inning 1**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player I plays $\sigma(\langle\square_{\alpha\in\Lambda}[\delta_0^\alpha{}^\frown U_0^\alpha]\rangle) = \square_{\alpha\in\Lambda}[\delta_1^\alpha]$ where for each $\alpha\in\Lambda$, $\delta_1^\alpha \in\downarrow^{n_1^\alpha}\mathscr{B}_\alpha$, $n_1^\alpha\in\omega$, also we can assume that $\delta_1^\alpha \supseteq \delta_0^\alpha{}^\frown U_0^\alpha$, in particular $n_1^\alpha - 1 \geq n_0^\alpha$; then, in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}\delta_1^\alpha(n_1^\alpha - 1)$, next Player II responds $\square_{\alpha\in\Lambda}U_1^\alpha$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player II responds $\square_{\alpha\in\Lambda}[\delta_1^\alpha{}^\frown U_1^\alpha]$.

- **Inning 2**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player I plays $\sigma(\langle\square_{\alpha\in\Lambda}[\delta_0^\alpha{}^\frown U_0^\alpha], \square_{\alpha\in\Lambda}[\delta_1^\alpha{}^\frown U_1^\alpha]\rangle) = \square_{\alpha\in\Lambda}[\delta_2^\alpha]$ where for each $\alpha\in\Lambda$, $\delta_2^\alpha \in\downarrow^{n_2^\alpha}\mathscr{B}_\alpha$, $\delta_2^\alpha \supseteq \delta_1^\alpha{}^\frown U_1^\alpha$, in particular $n_2^\alpha - 1 \geq n_1^\alpha$; then, in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}\delta_2^\alpha(n_2^\alpha - 1)$, next Player II responds $\square_{\alpha\in\Lambda}U_2^\alpha$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$, Player II responds $\square_{\alpha\in\Lambda}[\delta_2^\alpha{}^\frown U_2^\alpha]$, and so on.

$\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$

Player I	Player II
$\square_{\alpha\in\Lambda}[\delta_0^\alpha]$	
	$\square_{\alpha\in\Lambda}[\delta_0^\alpha{}^\frown U_0^\alpha]$
$\square_{\alpha\in\Lambda}[\delta_1^\alpha]$	
	$\square_{\alpha\in\Lambda}[\delta_1^\alpha{}^\frown U_1^\alpha]$
$\square_{\alpha\in\Lambda}[\delta_2^\alpha]$	
	$\square_{\alpha\in\Lambda}[\delta_2^\alpha{}^\frown U_2^\alpha]$
$\vdots$	$\vdots$

$\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$

Player I	Player II
$\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}\delta_0^\alpha(n_0^\alpha - 1)$	
	$\square_{\alpha\in\Lambda}U_0^\alpha$
$\square_{\alpha\in\Lambda}\delta_1^\alpha(n_1^\alpha - 1)$	
	$\square_{\alpha\in\Lambda}U_1^\alpha$
$\square_{\alpha\in\Lambda}\delta_2^\alpha(n_2^\alpha - 1)$	
	$\square_{\alpha\in\Lambda}U_2^\alpha$
$\vdots$	$\vdots$

As, in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, σ is a winning strategy for Player I we have that

$$\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}[\delta_n^\alpha{}^\frown U_n^\alpha] = \emptyset.$$

Claim 3.47.37. $\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}U_n^\alpha = \emptyset$.

Proof. Suppose otherwise for a contradiction. There exists $(x_\alpha)_{\alpha\in\Lambda} \in \bigcap_{n\in\omega} \square_{\alpha\in\Lambda} U_n^\alpha = \emptyset$, so $x_\alpha \in U_n^\alpha$ for each $\alpha \in \Lambda$ and $n \in \omega$. Let $\alpha \in \Lambda$, and consider $\rho_\alpha = \bigcup_{n\in\omega}[\delta_n^\alpha {}^\frown U_n^\alpha]$. Note that $\rho_\alpha \in \mathscr{K}(X)$, because $x_\alpha \in \bigcap_{n\in\omega} \rho_\alpha(n)$. Then $(\rho_\alpha)_{\alpha\in\Lambda} \in \bigcap_{n\in\omega} \square_{\alpha\in\Lambda}[\delta_n^\alpha {}^\frown U_n^\alpha]$, contradiction. □

Therefore σ' is a winning strategy for Player I in $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, so $\square_{\alpha\in\Lambda} X_\alpha$ is not a Baire space.

Now we will show that if $\square^\kappa X$ is not Baire then $\square^\kappa \mathscr{K}(X)$ is not Baire. Let σ be a winning strategy for Player I in $\mathsf{BM}(\square^\kappa X)$, we will build a winning strategy σ' for Player I in $\mathsf{BM}(\square_{\alpha\in\Lambda} \mathscr{K}(X_\alpha))$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player I plays $\sigma(\langle\,\rangle) = \square_{\alpha\in\Lambda} U_0^\alpha$. Then, in $\mathsf{BM}(\square_{\alpha\in\Lambda} \mathscr{K}(X_\alpha))$, Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}[\langle U_0^\alpha\rangle]$, next Player II responds $\square_{\alpha\in\Lambda}[\delta_0^\alpha]$ where for each $\alpha \in \Lambda$, $\delta_0^\alpha \in \downarrow^{n_0^\alpha} \mathscr{B}_\alpha$, $n_0^\alpha \in \omega$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player II responds $\square_{\alpha\in\Lambda} \delta_0^\alpha(n_0^\alpha - 1)$.

- **Inning 1**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player I plays $\sigma(\langle \square_{\alpha\in\Lambda} \delta_0^\alpha(n_0^\alpha - 1)\rangle) = \square_{\alpha\in\Lambda} U_1^\alpha$. Then, in $\mathsf{BM}(\square_{\alpha\in\Lambda} \mathscr{K}(X_\alpha))$, Player I plays $\sigma'(\langle \square_{\alpha\in\Lambda}[\delta_0^\alpha]\rangle) = \square_{\alpha\in\Lambda}[\delta_0^\alpha {}^\frown U_1^\alpha]$, next Player II responds $\square_{\alpha\in\Lambda}[\delta_1^\alpha]$ where for each $\alpha \in \kappa$, $\delta_1^\alpha \in \downarrow^{n_1^\alpha} \mathscr{B}_\alpha$, $n_1^\alpha \in \omega$. Also we can assume that $\delta_1^\alpha \supseteq \delta_0^\alpha {}^\frown U_1^\alpha$, in particular $n_1^\alpha - 1 \geq n_0^\alpha$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player II responds $\square_{\alpha\in\Lambda} \delta_1^\alpha(n_1^\alpha - 1)$.

- **Inning 2**

  In $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player I plays $\sigma(\langle \square_{\alpha\in\Lambda} \delta_0^\alpha(n_0^\alpha - 1), \square_{\alpha\in\Lambda} \delta_1^\alpha(n_1^\alpha - 1)\rangle) = \square_{\alpha\in\Lambda} U_2^\alpha$. Then, in $\mathsf{BM}(\square_{\alpha\in\Lambda} \mathscr{K}(X_\alpha))$, Player I plays $\sigma'(\langle \square_{\alpha\in\Lambda}[\delta_0^\alpha], \square_{\alpha\in\Lambda}[\delta_1^\alpha]\rangle) = \square_{\alpha\in\Lambda}[\delta_1^\alpha {}^\frown U_2^\alpha]$, next Player II responds $\square_{\alpha\in\Lambda}[\delta_2^\alpha]$ where for each $\alpha \in \kappa$, $\delta_2^\alpha \in \downarrow^{n_2^\alpha} \mathscr{B}_\alpha$, $n_2^\alpha \in \omega$. Also we can assume that $\delta_2^\alpha \supseteq \delta_1^\alpha {}^\frown U_2^\alpha$, in particular $n_2^\alpha - 1 \geq n_1^\alpha$. Now, in $\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$, Player II responds $\square_{\alpha\in\Lambda} \delta_2^\alpha(n_2^\alpha - 1)$, and so on.

$\mathsf{BM}(\square_{\alpha\in\Lambda} X_\alpha)$

Player I	Player II
$\square_{\alpha\in\Lambda} U_0^\alpha$	
	$\square_{\alpha\in\Lambda} \delta_0^\alpha(n_0^\alpha - 1)$
$\square_{\alpha\in\Lambda} U_1^\alpha$	
	$\square_{\alpha\in\Lambda} \delta_1^\alpha(n_1^\alpha - 1)$
$\square_{\alpha\in\Lambda} U_2^\alpha$	
	$\square_{\alpha\in\Lambda} \delta_2^\alpha(n_2^\alpha - 1)$
$\vdots$	$\vdots$

$\mathsf{BM}(\square_{\alpha\in\Lambda} \mathscr{K}(X_\alpha))$

Player I	Player II
$\sigma'(\langle\,\rangle) = \square_{\alpha\in\Lambda}[\langle U_0^\alpha\rangle]$	
	$\square_{\alpha\in\Lambda}[\delta_0^\alpha]$
$\square_{\alpha\in\Lambda}[\delta_0^\alpha {}^\frown U_1^\alpha]$	
	$\square_{\alpha\in\Lambda}[\delta_1^\alpha]$
$\square_{\alpha\in\Lambda}[\delta_1^\alpha {}^\frown U_2^\alpha]$	
	$\square_{\alpha\in\Lambda}[\delta_2^\alpha]$
$\vdots$	$\vdots$

As, in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, σ is a winning strategy for Player I we have that

$$\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}[\delta_n^{\alpha\frown}U_n^\alpha]=\emptyset.$$

Claim 3.47.38. $\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}U_n^\alpha=\emptyset$

Proof. Suppose otherwise for a contradiction. There exists $(x_\alpha)_{\alpha\in\Lambda}\in\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}U_n^\alpha=\emptyset$, so $x_\alpha\in U_n^\alpha$ for each $\alpha\in\Lambda$ and $n\in\omega$. Let $\alpha\in\Lambda$, and consider $\rho_\alpha=\bigcup_{n\in\omega}[\delta_n^{\alpha\frown}U_n^\alpha]$, note that $\rho_\alpha\in\mathscr{K}(X)$, because $x_\alpha\in\bigcap_{n\in\omega}\rho_\alpha(n)$, then $(\rho_\alpha)_{\alpha\in\Lambda}\in\bigcap_{n\in\omega}\square_{\alpha\in\Lambda}[\delta_n^{\alpha\frown}U_n^\alpha]$, contradiction. □

Therefore σ' is a winning strategy for Player I in $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$, then $\square_{\alpha\in\Lambda}X_\alpha$ is not a Baire space.

□

Corollary 3.48. Let $\{X_\alpha:\alpha\in\Lambda\}$ be a family of topological spaces with $\mathscr{B}_\alpha$ a base for X_α and let $\{\mathscr{K}(X_\alpha):\alpha\in\Lambda\}$ be their associated Krom spaces. Then I $\uparrow\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$ if and only if I $\uparrow\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$.

Corollary 3.49. Let κ be a infinite cardinal and let X be a topological space with $\mathscr{B}$ a base for X such that $\emptyset\notin\mathscr{B}$ and let $\mathscr{K}(X)$ be its associated Krom space. Then $\square^\kappa X$ is Baire if and only if $\square^\kappa\mathscr{K}(X)$ is Baire.

Proposition 3.50. Let $\{X_\alpha:\alpha\in\Lambda\}$ be a family of topological spaces with $\mathscr{B}_\alpha$ a base for X_α and let $\{\mathscr{K}(X_\alpha):\alpha\in\Lambda\}$ its associated Krom spaces. Then $\square_{\alpha\in\Lambda}X_\alpha$ is Choquet if and only if $\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha)$ is Choquet.

Proof. First suppose that $\square_{\alpha\in\Lambda}X_\alpha$ is Choquet, then, by Theorem 2.19, X_α is Choquet. Now by Proposition 3.12, $\mathscr{K}(X_\alpha)$ is Choquet, so, by Corollary 3.42, $\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha)$ is Choquet and reciprocally. □

Corollary 3.51. Let κ be a infinite cardinal and let X be a topological space with $\mathscr{B}$ a base for X such that $\emptyset\notin\mathscr{B}$ and let $\mathscr{K}(X)$ be its associated Krom space. Then $\square^\kappa X$ is Choquet if and only if $\square^\kappa\mathscr{K}(X)$ is Choquet.

Corollary 3.52. Let κ be a infinite cardinal and let X be a topological space with $\mathscr{B}$ a base for X such that $\emptyset\notin\mathscr{B}$ and let $\mathscr{K}(X)$ be its associated Krom space. Then the games $\mathsf{BM}(\square_{\alpha\in\Lambda}X_\alpha)$ and $\mathsf{BM}(\square_{\alpha\in\Lambda}\mathscr{K}(X_\alpha))$ are equivalent.

In the article (GALVIN; SCHEEPERS, 2016) of Fred Galvin and Marion Scheepers, using other games and measurable cardinals the following is proved:

Theorem 3.53 (Galvin and Scheepers)**.** If it is consistent there is a proper class of measurable cardinals, then it is consistent that if all box powers of a space are Baire, then the space is Choquet.

This motivates to define the following

Definition 3.54. The theory of Galvin-Scheepers it's simply **ZFC + "if all box powers of a space are Baire, then the space is Choquet"**.

In this new theory, we have the following results:

Corollary 3.55. In the theory of Galvin and Scheepers, if the all box powers of a Baire space X are Baire spaces then its Tychonoff powers are Baire spaces.

Proof. Let X be a Baire space whose all box powers are Baire spaces. Then (in the theory of Galvin and Scheepers), X is Choquet. Since Tychonoff products of Choquet spaces are Baire, we have that all Tychonoff powers of X are Baire spaces. $\square$

Corollary 3.56. In the theory of Galvin and Scheepers, there are a second countable Baire space X and a cardinal κ such that the box power $\square^\kappa X$ is not Baire.

Proof. Let $B \subseteq \mathbb{R}$ be a Bernstein set. Remember that B is a second countable Baire space and is not Choquet. We claim that there is a cardinal κ such that the box power $\square^\kappa B$ is not Baire. Otherwise, all box powers of B are Baire then (in this theory), B is Choquet, contradiction. $\square$

CHAPTER

4

MULTIBOARD TOPOLOGICAL GAMES

In this chapter we introduce the multiboard topological games. This idea emerged in the article (GALVIN; SCHEEPERS, 2016) of Fred Galvin and Marion Scheepers and it will be used to study infinite products of Baire spaces.

4.1 Some versions of multiboard topological games

We will see versions of multiboard topological games. Intuitively we are playing the Banach-Mazur game simultaneously in multiple boards. Let X be a non-empty topological space and let $\kappa \geq 1$ be a cardinal.

Definition 4.1 (Version 1: κ-multiboard Banach-Mazur game). The Version 1 of the κ-multiboard Banach-Mazur game is defined as follows:

Player I and Player II play an inning per finite ordinal.

- At the beginning, Player I first selects a sequence $(B^0_\alpha)_{\alpha<\kappa}$ of nonempty open sets, and then Player II responds with a sequence $(B^1_\alpha)_{\alpha<\kappa}$ of nonempty open sets such that $B^1_\alpha \subseteq B^0_\alpha, \forall \alpha < \kappa$.
- Later, in each inning $n \in \omega$, Player I chooses a sequence $(B^{2n}_\alpha)_{\alpha<\kappa}$ of nonempty open sets such that $B^{2n}_\alpha \subseteq B^{2n-1}_\alpha, \forall \alpha < \kappa$, then Player II responds with a sequence $(B^{2n+1}_\alpha)_{\alpha<\kappa}$ of nonempty open sets such that $B^{2n+1}_\alpha \subseteq B^{2n}_\alpha, \forall \alpha < \kappa$.

  Player I wins this play if there exists $\alpha < \kappa$ such that $\bigcap_{n<\omega} B^{2n+1}_\alpha = \emptyset$. Else Player II wins. We denote this game by $\mathsf{BM}^\kappa_1(X)$.

We have the following simple observations:

1. Let $2 \leq \lambda < \kappa$ be cardinal numbers, if Player I has a winning strategy in the game $\mathsf{BM}_1^\lambda(X)$ then Player I has a winning strategy in the game $\mathsf{BM}_1^\kappa(X)$.

2. Let $2 \leq \lambda < \kappa$ be cardinal numbers, if Player II has a winning strategy in the game $\mathsf{BM}_1^\kappa(X)$ then Player II has a winning strategy in the game $\mathsf{BM}_1^\lambda(X)$.

3. If Player I has a winning strategy in $\mathsf{BM}(\mathsf{X})$ then Player I has a winning strategy in $\mathsf{BM}_1^\kappa(X)$ for every cardinal κ.

4. Player II has a winning strategy in $\mathsf{BM}(\mathsf{X})$ if and only if Player II has a winning strategy in $\mathsf{BM}_1^\kappa(X)$ for every (equivalently, some) cardinal κ.

Proposition 4.2. Let X be a topological space and κ be a cardinal. Then the games $\mathsf{BM}_1^\kappa(X)$ and $\mathsf{BM}(\square^\kappa X)$ are equivalent.

Proof. First suppose that Player I has a winning strategy σ in $\mathsf{BM}_1^\kappa(X)$. We will build a winning strategy σ' for Player I in $\mathsf{BM}(\square^\kappa X)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}_1^\kappa(X)$, Player I plays $\sigma(\langle\,\rangle) = (U_\alpha^0)_{\alpha<\kappa}$ where U_α^0 is a non-empty open subset of X for each $\alpha < \kappa$. Then, in $\mathsf{BM}(\square^\kappa X)$, Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha<\kappa} U_\alpha^0$. Next Player II plays $\square_{\alpha<\kappa} V_\alpha^0$. Returning to $\mathsf{BM}_1^\kappa(X)$, Player II plays $(V_\alpha^0)_{\alpha<\kappa}$.

- **Inning 1**

  In $\mathsf{BM}_1^\kappa(X)$, Player I plays $\sigma(\langle (V_\alpha^0)_{\alpha<\kappa}\rangle) = (U_\alpha^1)_{\alpha<\kappa}$ where U_α^1 is a non-empty open subset of V_α^0 for each $\alpha < \kappa$. Then, in $\mathsf{BM}(\square^\kappa X)$,

α-board	
Player I	Player II
U_α^0	
	V_α^0
U_α^1	
	V_α^1
$\vdots$	$\vdots$

$\mathsf{BM}(\square_\kappa X)$	
Player I	Player II
$\sigma'(\langle\,\rangle) = \square_{\alpha<\kappa} U_\alpha^0$	
	$\square_{\alpha<\kappa} V_\alpha^0$
$\square_{\alpha<\kappa} U_\alpha^1$	
	$\square_{\alpha<\kappa} V_\alpha^1$
$\vdots$	$\vdots$

We claim that σ' is a winning strategy. Indeed, let

$$\sigma'(\langle\,\rangle), \square_{\alpha<\kappa} V_\alpha^0, \sigma'(\square_{\alpha<\kappa} V_\alpha^0), \square_{\alpha<\kappa} V_\alpha^1, \cdots$$

be a play in $\mathsf{BM}(\square^\kappa X)$, and suppose that $\bigcap_{n\in\omega} \square_{\alpha<\kappa} V_\alpha^n \neq \emptyset$, that is, there exists $(x_\alpha)_{\alpha<\kappa} \in \square_{\alpha<\kappa} V_\alpha^n$ for all $n \in \omega$. Then for each $\alpha < \kappa$ we have that $x_\alpha \in \bigcap_{n\in\omega} V_\alpha^n$, contradiction.

Now suppose that Player I has a winning strategy σ in $\mathsf{BM}(\square^\kappa X)$. We will build a winning strategy σ' for Player I in $\mathsf{BM}_1^\kappa(X)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\square^\kappa X)$, Player I plays $\sigma(\langle\,\rangle) = \square_{\alpha<\kappa} U_\alpha^0$ where U_α^0 is a non-empty open subset of X for each $\alpha < \kappa$. Then, in $\mathsf{BM}_1^\kappa(X)$, Player I plays $\sigma'(\langle\,\rangle) = (U_\alpha^0)_{\alpha<\kappa}$. Next Player II plays $(V_\alpha^0)_{\alpha<\kappa}$. Returning to $\mathsf{BM}(\square^\kappa X)$, Player II plays $\square_{\alpha<\kappa} V_\alpha^0$.

- **Inning 1**

  In $\mathsf{BM}(\square^\kappa X)$, Player I plays $\sigma(\langle\,\rangle) = \square_{\alpha<\kappa} U_\alpha^1$ where U_α^1 is a non-empty open subset of V_α^0 for each $\alpha < \kappa$. Then, in $\mathsf{BM}_1^\kappa(X)$, Player I plays $\sigma'(\langle(V_\alpha^0)_{\alpha<\kappa}\rangle) = (U_\alpha^1)_{\alpha<\kappa}$. Next Player II plays $(V_\alpha^1)_{\alpha<\kappa}$. Returning to $\mathsf{BM}(\square^\kappa X)$, Player II plays $\square_{\alpha<\kappa} V_\alpha^1$, and so on.

$\mathsf{BM}(\square X^\kappa)$	
Player I	Player II
$\square_{\alpha<\kappa} U_\alpha^0$	
	$\square_{\alpha<\kappa} V_\alpha^0$
$\square_{\alpha<\kappa} U_\alpha^1$	
	$\square_{\alpha<\kappa} V_\alpha^1$
$\vdots$	$\vdots$

α-board	
Player I	Player II
$\sigma'(\langle\,\rangle) = U_\alpha^0$	
	V_α^0
U_α^1	
	V_α^1
$\vdots$	$\vdots$

We claim that σ' is a winning strategy. Indeed, let

$$\sigma'(\langle\,\rangle), (V_\alpha^0)_{\alpha<\kappa}, \sigma'((V_\alpha^0)_{\alpha<\kappa}), (V_\alpha^1)_{\alpha<\kappa}, \cdots$$

be a play in $\mathsf{BM}(\square^\kappa X)$, and suppose that Player II wins, that is, for each $\alpha < \kappa$, $\bigcap_{n\in\omega} V_\alpha^n \neq \emptyset$, i.e., there exists $x_\alpha \in V_\alpha^n$ for all $n \in \omega$. Then $x = (x_\alpha)_{\alpha<\kappa} \in \bigcap_{n\in\omega} \square_{\alpha<\kappa} V_\alpha^n$, contradicting the fact that σ is a winning strategy for Player I in $\mathsf{BM}(\square X^\kappa)$.

Now we will prove the second part of equivalence. Suppose that Player II has a winning strategy δ in $\mathsf{BM}_1^\kappa(X)$. We will build a winning strategy δ' for Player II in $\mathsf{BM}(\square^\kappa X)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}(\square^\kappa X)$, Player I plays $\square_{\alpha<\kappa} U_\alpha^0$ where U_α^0 is a non-empty open subset of X for each $\alpha < \kappa$. Then, in $\mathsf{BM}_1^\kappa(X)$, Player I plays $(U_\alpha^0)_{\alpha<\kappa}$. Next Player II plays $\delta(\langle (U_\alpha^0)_{\alpha<\kappa}\rangle) = (V_\alpha^0)_{\alpha<\kappa}$. Returning to $\mathsf{BM}(\square^\kappa X)$, Player II plays $\delta'(\langle \square_{\alpha<\kappa} U_\alpha^0\rangle) = \square_{\alpha<\kappa} V_\alpha^0$.

- **Inning 1**

  In $\mathsf{BM}(\square^\kappa X)$, Player I plays $\square_{\alpha<\kappa} U_\alpha^1$ where U_α^1 is a non-empty open subset of V_α^0 for each $\alpha < \kappa$. Then, in $\mathsf{BM}_1^\kappa(X)$, Player I plays $(U_\alpha^1)_{\alpha<\kappa}$. Next Player II plays $\delta(\langle (U_\alpha^0)_{\alpha<\kappa}, (U_\alpha^1)_{\alpha<\kappa}\rangle) = (V_\alpha^1)_{\alpha<\kappa}$. Returning to $\mathsf{BM}(\square^\kappa X)$, Player II plays $\delta'(\langle \square_{\alpha<\kappa} U_\alpha^0, \square_{\alpha<\kappa} U_\alpha^1\rangle) = \square_{\alpha<\kappa} V_\alpha^1$, and so on.

α-board	
Player I	Player II
U_α^0	
	V_α^0
U_α^1	
	V_α^1
$\vdots$	$\vdots$

$\mathsf{BM}(\square X^\kappa)$	
Player I	Player II
$\square_{\alpha<\kappa} U_\alpha^0$	
	$\square_{\alpha<\kappa} V_\alpha^0$
$\square_{\alpha<\kappa} U_\alpha^1$	
	$\square_{\alpha<\kappa} V_\alpha^1$
$\vdots$	$\vdots$

We claim that δ' is a winning strategy. Indeed, let

$$\square_{\alpha<\kappa} U_\alpha^0, \delta'(\langle \square_{\alpha<\kappa} U_\alpha^0\rangle), \square_{\alpha<\kappa} U_\alpha^1, \delta'(\langle \square_{\alpha<\kappa} U_\alpha^0, \square_{\alpha<\kappa} U_\alpha^1\rangle), \cdots$$

be a play in $\mathsf{BM}(\square^\kappa X)$. As δ is a winning strategy, we have that for each $\alpha < \kappa$, $\bigcap_{n\in\omega} V_\alpha^n = \emptyset$; therefore, $\bigcap_{n\in\omega} \square_{\alpha<\kappa} V_\alpha^n = \bigcap_{n\in\omega} \square_{\alpha<\kappa} \delta'(\langle \square_{\alpha<\kappa} U_\alpha^0, \cdots, \square_{\alpha<\kappa} U_\alpha^n\rangle) = \emptyset$.

Now suppose that Player II has a winning strategy δ in $\mathsf{BM}(\square^\kappa X)$, We will build a winning strategy δ' for Player II in $\mathsf{BM}_1^\kappa(X)$. Indeed,

- **Inning 0**

  In $\mathsf{BM}_1^\kappa(X)$, Player I plays $(U_\alpha^0)_{\alpha<\kappa}$ where U_α^0 is a non-empty open subset of X for each $\alpha < \kappa$. Then, in $\mathsf{BM}(\square^\kappa X)$, Player I plays $\square_{\alpha<\kappa} U_\alpha^0$. Next Player II plays $\delta(\langle \square_{\alpha<\kappa} U_\alpha^0\rangle) = \square_{\alpha<\kappa} V_\alpha^0$. Returning to $\mathsf{BM}_1^\kappa(X)$, Player II plays $\delta'(\langle (U_\alpha^0)_{\alpha<\kappa}\rangle) = (V_\alpha^0)_{\alpha<\kappa}$.

- **Inning 1**

  In $\mathsf{BM}_1^\kappa(X)$, Player I plays $(U_\alpha^1)_{\alpha<\kappa}$ where U_α^1 is a non-empty open subset of V_α^0 for each $\alpha<\kappa$. Then, in $\mathsf{BM}(\square^\kappa X)$, Player I plays $\square_{\alpha<\kappa}U_\alpha^1$. Next Player II plays $\delta(\langle\square_{\alpha<\kappa}U_\alpha^0,\square_{\alpha<\kappa}U_\alpha^1\rangle)=\square_{\alpha<\kappa}V_\alpha^1$. Returning to $\mathsf{BM}_1^\kappa(X)$, Player II plays $\delta'(\langle(U_\alpha^0)_{\alpha<\kappa},(U_\alpha^1)_{\alpha<\kappa}\rangle)=(V_\alpha^1)_{\alpha<\kappa}$, and so on.

$\mathsf{BM}(\square X^\kappa)$	
Player I	Player II
$\square_{\alpha<\kappa}U_\alpha^0$	
	$\square_{\alpha<\kappa}V_\alpha^0$
$\square_{\alpha<\kappa}U_\alpha^1$	
	$\square_{\alpha<\kappa}V_\alpha^1$
$\vdots$	$\vdots$

α-board	
Player I	Player II
U_α^0	
	V_α^0
U_α^1	
	V_α^1
$\vdots$	$\vdots$

We claim that δ' is a winning strategy. Indeed, let

$$(U_\alpha^0)_{\alpha<\kappa},\delta'(\langle(U_\alpha^0)_{\alpha<\kappa}\rangle),(U_\alpha^1)_{\alpha<\kappa},\delta'(\langle(U_\alpha^0)_{\alpha<\kappa},(U_\alpha^1)_{\alpha<\kappa}\rangle),\cdots$$

be a play in $\mathsf{BM}_1^\kappa(X)$. As δ is a winning strategy, we have that there exists $(x_\alpha)_{\alpha<\kappa}\in\bigcap_{n\in\omega}\square_{\alpha<\kappa}V_\alpha^n$ then $x_\alpha\in\bigcap_{n\in\omega}V_\alpha^n$ for each $\alpha<\kappa$. $\square$

Finally we present a summary of the results obtained in this section

- $\mathrm{I}\uparrow\mathsf{BM}_1^\lambda(X)\overset{\lambda\leq\kappa}{\Longrightarrow}\mathrm{I}\uparrow\mathsf{BM}_1^\kappa(X)$
- $\mathrm{II}\uparrow\mathsf{BM}_1^\kappa(X)\overset{\lambda\leq\kappa}{\Longrightarrow}\mathrm{II}\uparrow\mathsf{BM}_1^\lambda(X)$
- $\mathrm{I}\uparrow\mathsf{BM}(X)\Longrightarrow\mathrm{I}\uparrow\mathsf{BM}_1^\kappa(X)$
- $\mathrm{II}\uparrow\mathsf{BM}(X)\Longleftrightarrow\mathrm{II}\uparrow\mathsf{BM}_1^\kappa(X)$
- $\mathrm{I}\uparrow\mathsf{BM}_1^\kappa(X)\Longleftrightarrow\mathrm{I}\uparrow\mathsf{BM}(\square^\kappa X)$.
- $\mathrm{II}\uparrow\mathsf{BM}_1^\kappa(X)\Longleftrightarrow\mathrm{II}\uparrow\mathsf{BM}(\square^\kappa X)$.

Definition 4.3 (**Version 2 : κ-multiboard Banach-Mazur game $\mathsf{BM}_2^\kappa(X)$**). The same rules of the previous game, only that the criterion of victory changes, that is , Player II wins this game if there exists $\alpha < \kappa$ such that $\bigcap_{n<\omega} B_\alpha^{2n+1} \neq \emptyset$. Else Player I wins. We denote this game by $\mathsf{BM}_2^\kappa(X)$.

We have the following simple observations:

1. Let $2 \leq \lambda < \kappa$ be cardinal numbers. If Player I has a winning strategy in the game $\mathsf{BM}_2^\kappa(X)$ then Player I has a winning strategy in the $\mathsf{BM}_2^\lambda(X)$ game.

2. Let $2 \leq \lambda < \kappa$ be cardinal numbers. If Player II has a winning strategy in the game $\mathsf{BM}_2^\lambda(X)$ then Player II has a winning strategy in the $\mathsf{BM}_2^\kappa(X)$ game.

3. Player I has a winning strategy in the game $\mathsf{BM}_2^\kappa(X)$ for every (equivalently, some) cardinal κ if and only if Player I has a winning strategy in $\mathsf{BM}(X)$.

4. If Player II has a winning strategy in $\mathsf{BM}(\mathrm{X})$ then Player II has a winning strategy in $\mathsf{BM}_2^\kappa(X)$ for every cardinal κ.

Theorem 4.4. Let X be a topological space, and κ be a cardinal.

(1) If Player I has a winning strategy in $\mathsf{BM}_2^\kappa(X)$ then Player I has a winning strategy in $\mathsf{BM}(\square^\kappa X)$.

(2) If Player II has a winning strategy in $\mathsf{BM}(\square^\kappa X)$, then Player II has a winning strategy in $\mathsf{BM}_2^\kappa(X)$.

Proof. First, let σ be a winning strategy for Player I in $\mathsf{BM}_2^\kappa(X)$, we are going to build a winning strategy σ' for Player I in $\mathsf{BM}(X^\kappa)$, as follows:

- **Inning 0**

  Player I plays $\sigma(\langle\,\rangle) = (B_\alpha^0)_{\alpha<\kappa}$, so Player I plays $\sigma'(\langle\,\rangle) = \square_{\alpha<\kappa} B_\alpha^0$. Next Player II responds $\square_{\alpha<\kappa} B_\alpha^1$. This induces that Player II plays $(B_\alpha^1)_{\alpha<\kappa}$ in $\mathsf{BM}_2^\kappa(X)$.

- **Inning 1**

  In $\mathsf{BM}_2^\kappa(X)$, Player I plays $\sigma(\langle (B_\alpha^1)_{\alpha<\kappa}\rangle) = (B_\alpha^2)_{\alpha<\kappa}$. Then Player I plays $\sigma'(\langle \square_{\alpha<\kappa} B_\alpha^1\rangle) = \square_{\alpha<\kappa} B_\alpha^2$. Next, Player II responds $\square_{\alpha<\kappa} B_\alpha^3$. This induces that Player II plays $(B_\alpha^3)_{\alpha<\kappa}$ in $\mathsf{BM}_2^\kappa(X)$, and so on.

$\mathsf{BM}_2^\kappa(X)$

Player I	Player II
$\sigma(\langle\,\rangle) = (B^0_\alpha)_{\alpha<\kappa}$	
	$(B^1_\alpha)_{\alpha<\kappa}$
$\sigma(\langle (B^1_\alpha)_{\alpha<\kappa}\rangle) = (B^2_\alpha)_{\alpha<\kappa}$	
	$(B^3_\alpha)_{\alpha<\kappa}$
$\vdots$	$\vdots$

$\mathsf{BM}(X^\kappa)$

Player I	Player II
$\sigma'(\langle\,\rangle) = \square_{\alpha<\kappa} B^0_\alpha$	
	$\square_{\alpha<\kappa} B^1_\alpha$
$\sigma'(\langle \square_{\alpha<\kappa} B^1_\alpha\rangle) = \square_{\alpha<\kappa} B^2_\alpha$	
	$\square_{\alpha<\kappa} B^3_\alpha$
$\vdots$	$\vdots$

As σ is a winning strategy for Player I, we have that for each $\alpha < \kappa$,

$$\bigcap_{n<\omega} B^{2n+1}_\alpha = \emptyset,$$

so

$$\bigcap_{n\in\omega} \square_{\alpha<\kappa} B^{2n+1}_\alpha = \emptyset,$$

then σ' is a winning strategy for Player I in $\mathsf{BM}(X^\kappa)$.

Now let δ be a winning strategy for Player II in $\mathsf{BM}(\square^\kappa X)$, we are going to build a winning strategy δ' for Player II in $\mathsf{BM}_2^\kappa(X)$, as follows:

- **Inning 0**

  Player I plays $(B^0_\alpha)_{\alpha<\kappa}$. This induces that Player I plays $\square_{\alpha<\kappa} B^0_\alpha$. Next, Player II responds $\delta(\langle\,\rangle) = \square_{\alpha<\kappa} B^1_\alpha$. This induces that Player II plays $\delta'(\langle (B^0_\alpha)_{\alpha<\kappa}\rangle) = (B^1_\alpha)_{\alpha<\kappa}$ in $\mathsf{BM}_2^\kappa(X)$.

- **Inning 1**

  In $\mathsf{BM}_2^\kappa(X)$, Player I plays $(B^2_\alpha)_{\alpha<\kappa}$. Then Player I plays $\square_{\alpha<\kappa} B^2_\alpha$. Next Player II responds $\delta(\langle \square_{\alpha<\kappa} B^0_\alpha, \square_{\alpha<\kappa} B^2_\alpha\rangle) = \square_{\alpha<\kappa} B^3_\alpha$. This induces that Player II plays $\delta'(\langle (B^0_\alpha)_{\alpha<\kappa}, (B^2_\alpha)_{\alpha<\kappa}\rangle) = (B^3_\alpha)_{\alpha<\kappa}$ in $\mathsf{BM}_2^\kappa(X)$, and so on.

$\mathsf{BM}_2^\kappa(X)$

Player I	Player II
$(B^0_\alpha)_{\alpha<\kappa}$	
	$\delta'(\langle (B^0_\alpha)_{\alpha<\kappa}\rangle) = (B^1_\alpha)_{\alpha<\kappa}$
$(B^2_\alpha)_{\alpha<\kappa}$	
	$\delta'(\langle (B^0_\alpha)_{\alpha<\kappa}, (B^2_\alpha)_{\alpha<\kappa}\rangle) = (B^3_\alpha)_{\alpha<\kappa}$
$\vdots$	$\vdots$

$\mathsf{BM}(X^\kappa)$

Player I	Player II
$\square_{\alpha<\kappa} B^0_\alpha$	
	$\delta(\langle\,\rangle) = \square_{\alpha<\kappa} B^1_\alpha$
$\Pi_{\alpha<\kappa} B^2_\alpha$	
	$\delta(\langle \square_{\alpha<\kappa} B^0_\alpha, \square_{\alpha<\kappa} B^2_\alpha\rangle) = \square_{\alpha<\kappa} B^3_\alpha$
$\vdots$	$\vdots$

As δ is a winning strategy for Player II, we have that

$$\bigcap_{n<\omega} \square_{\alpha<\kappa} B_\alpha^{2n+1} \neq \emptyset,$$

so in this case, for all $\alpha < \kappa$,

$$\bigcap_{n\in\omega} B_\alpha^{2n+1} = \emptyset,$$

then δ' is a winning strategy for Player II in $\mathsf{BM}_2^\kappa(X)$. $\square$

Finally we present a summary of the results obtained in this section

- $\mathrm{I} \uparrow \mathsf{BM}_2^\kappa(X) \overset{\lambda\leq\kappa}{\Longrightarrow} \mathrm{I} \uparrow \mathsf{BM}_2^\lambda(X)$
- $\mathrm{II} \uparrow \mathsf{BM}_2^\lambda(X) \overset{\lambda\leq\kappa}{\Longrightarrow} \mathrm{II} \uparrow \mathsf{BM}_2^\kappa(X)$
- $\mathrm{I} \uparrow \mathsf{BM}(X) \Longleftrightarrow \mathrm{I} \uparrow \mathsf{BM}_2^\kappa(X)$
- $\mathrm{II} \uparrow \mathsf{BM}(X) \Longrightarrow \mathrm{II} \uparrow \mathsf{BM}_2^\kappa(X)$
- $\mathrm{I} \uparrow \mathsf{BM}_2^\kappa(X) \Longrightarrow \mathrm{I} \uparrow \mathsf{BM}(\square^\kappa X)$
- $\mathrm{II} \uparrow \mathsf{BM}(\square^\kappa X) \Longrightarrow \mathrm{II} \uparrow \mathsf{BM}_2^\kappa(X)$

Motivated by solving the problem of the infinite product of Baire spaces, Professor Leandro Aurichi presented for me the next new version of the multiboard game, which is very different from the previous ones. The motivation of this new version is by the proof that a Bernstein set on the real line is undeterminated, because in a part of the proof we use that a Bernstein set cannot contain a Cantor set, which has cardinality $\mathfrak{c}$.

Definition 4.5 (Version 3 : $\mathfrak{c}$-modified multiboard Banach-Mazur game). The Version 3 of the $\mathfrak{c}$-multiboard Banach-Mazur game is defined as follows:

Player I and Player II play an inning per finite ordinal.

- At the beginning, Player I first selects $(B_\alpha^0)_{\alpha<\mathfrak{c}}$ a sequence of nonempty open sets, and then Player II responds with $(B_\alpha^1)_{\alpha<\mathfrak{c}}$ a sequence of nonempty open sets such that $B_\alpha^1 \subseteq B_\alpha^0, \forall\alpha < \mathfrak{c}$.
- Later, in each inning $n \in \omega$, Player I choose $(B_\alpha^{2n})_{\alpha<\mathfrak{c}}$ a sequence of nonempty open sets such that $B_\alpha^{2n} \subseteq B_\alpha^{2n-1}, \forall\alpha < \mathfrak{c}$ then Player II responds with $(B_\alpha^{2n+1})_{\alpha<\mathfrak{c}}$ a sequence of nonempty open sets such that $B_\alpha^{2n+1} \subseteq B_\alpha^{2n}, \forall\alpha < \mathfrak{c}$.

- For each $\alpha < \mathfrak{c}$, in the α-board define $B^\alpha = \bigcap_{n<\omega} B_\alpha^{2n+1}$. Consider
$$P = \bigcup_{\alpha<\mathfrak{c}} B^\alpha.$$
Player II wins this play if $|P| \geq \mathfrak{c}$. Else Player I wins. We denote this game by $\mathtt{mod}\ \mathsf{BM}^{\mathfrak{c}}(X)$.

Note that if Player II has a winning strategy in $\mathtt{mod}\ \mathsf{BM}^{\mathfrak{c}}(X)$ then Player II has a winning strategy in $\mathsf{BM}_2^{\mathfrak{c}}(X)$.

Theorem 4.6. If the Continuum Hypothesis holds then Player II has winning strategy in $\mathtt{mod}\ \mathsf{BM}^{\mathfrak{c}}(\mathbb{R})$.

Proof. Write $\mathbb{R} = \{x_\alpha : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$ consider the set $Y_\alpha = \{x_\beta : \beta \geq \alpha\}$. Note that for each $\alpha \in \omega_1$, Y_α is a G_δ set and dense in $\mathbb{R}$. As Player II has a winning strategy δ in $\mathsf{BM}(\mathbb{R})$ then Player II has a winning strategy in $\mathsf{BM}(Y_\alpha)$ for each $\alpha \in \omega_1$, call δ_α this strategy for Player II. We will build a winning strategy for Player II in $\mathtt{mod}\ \mathsf{BM}^{\mathfrak{c}}(\mathbb{R})$. Indeed,

- **Inning 0**

  Player I plays $(B_\alpha^0)_{\alpha<\omega_1}$. Now at the same instant we play in $\mathsf{BM}(Y_\alpha)$ for each $\alpha \in \omega_1$. Player I plays $B_\alpha^0 \cap Y_\alpha$ in each $\mathsf{BM}(Y_\alpha)$, then Player II responds with $\delta_\alpha(\langle B_\alpha^0 \cap Y_\alpha\rangle)$ open non-empty in Y_α, that is, there is W_α^1 open in $\mathbb{R}$ such that $\delta_\alpha(\langle B_\alpha^0 \cap Y_\alpha\rangle) = W_\alpha^1 \cap Y_\alpha$. Then, in $\mathtt{mod}\ \mathsf{BM}_2^{\mathfrak{c}}(\mathbb{R})$, Player II responds $\delta(\langle (B_\alpha^0)_{\alpha<\omega_1}\rangle) = (W_\alpha^1 \cap B_\alpha^0)_{\alpha<\omega_1}$.

- **Inning 1**

  Player I plays $(B_\alpha^2)_{\alpha<\omega_1}$, with $B_\alpha^2 \subseteq W_\alpha^1 \cap B_\alpha^0$, for each $\alpha \in \omega_1$. Now at the same instant we play in each $\mathsf{BM}(Y_\alpha)$ for each $\alpha \in \omega_1$. Player I plays $B_\alpha^2 \cap Y_\alpha$, then Player II responds $\delta_\alpha(\langle B_\alpha^0 \cap Y_\alpha, B_\alpha^2 \cap Y_\alpha\rangle)$ open non-empty in Y_α, that is, there is W_α^3 open in $\mathbb{R}$ such that $\delta_\alpha(\langle B_\alpha^0 \cap Y_\alpha, B_\alpha^2 \cap Y_\alpha\rangle) = W_\alpha^3 \cap Y_\alpha$. Then, in $\mathtt{mod}\ \mathsf{BM}_2^{\mathfrak{c}}(\mathbb{R})$, Player II responds $\delta(\langle (B_\alpha^0)_{\alpha<\omega_1}, (B_\alpha^2)_{\alpha<\omega_1}\rangle) = (W_\alpha^3 \cap B_\alpha^2)_{\alpha<\omega_1}$, and so on.

$\alpha-$board

Player I	Player II
B_α^0	
	$W_\alpha^1 \cap B_\alpha^0$
B_α^2	
	$W_\alpha^3 \cap B_\alpha^2$
$\vdots$	$\vdots$

$\mathsf{BM}(Y_\alpha)$

Player I	Player II
$B_\alpha^0 \cap Y_\alpha$	
	$\delta_\alpha(\langle B_\alpha^0 \cap X_\alpha\rangle) = W_\alpha^1 \cap Y_\alpha$
$B_\alpha^2 \cap Y_\alpha$	
	$\delta_\alpha(\langle B_\alpha^0 \cap X_\alpha, B_\alpha^2 \cap X_\alpha\rangle) = W_\alpha^3 \cap Y_\alpha$
$\vdots$	$\vdots$

As Player II has a winning strategy in Y_α, then

$$\bigcap_{n<\omega}(W_\alpha^{2n+1}\cap Y_\alpha)\neq\emptyset, \forall\alpha\in\omega_1.$$

Note that for each $\alpha<\omega_1$,

$$B^\alpha=\bigcap_{n<\omega}(W_\alpha^{2n+1}\cap B_\alpha^{2n})\supseteq\bigcap_{n<\omega}(W_\alpha^{2n+1}\cap Y_\alpha)\neq\emptyset.$$

Also for each $\alpha<\omega_1$, choose $y_\alpha\in\bigcap_{n<\omega}(W_\alpha^{2n+1}\cap Y_\alpha)\subseteq B^\alpha$; then $\{y_\alpha:\alpha<\omega_1\}\subseteq P$. In particular $y_\alpha\in Y_\alpha$, then there exists $\alpha'<\omega_1$ such that $y_\alpha=x_{\alpha'}\in Y_\alpha$, so $\alpha'\geq\alpha$.

Claim 4.6.39. $Y=\{y_\alpha:\alpha<\omega_1\}$ is an uncountable set.

Proof. Otherwise, Y is countable then there are $k\leq\omega$ and a bijection $g:Y\longrightarrow k$, also we have a surjective function $f:\omega_1\longrightarrow Y$, so there exists a surjective function $h=g\circ f:\omega_1\longrightarrow k$, then

$$\omega_1=\bigcup_{n<k}h^{-1}(n).$$

Note that there is $n_0<k$ such that $|h^{-1}(n_0)|=|f^{-1}(g^{-1}(n_0))|=\omega_1$. Then y_α is the same for each $\alpha\in f^{-1}(g^{-1}(n_0))$. Also if $\alpha,\beta\in f^{-1}(g^{-1}(n_0))$ we have that there are $\alpha',\beta'\in\omega_1$ such that $\alpha'\geq\alpha$, $\beta'\geq\beta$ and $x_{\alpha'}=y_\alpha=y_\beta=x_{\beta'}$. Therefore $\alpha'=\beta'$. Then there is $\gamma<\omega_1$ such that $\alpha<\gamma,\forall\alpha\in f^{-1}(g^{-1}(n_0))$, contradiction. □

Therefore $\{y_\alpha:\alpha\in\omega_1\}$ is uncountable; then $2^\omega=\omega_1\leq|P|$, so δ is a winning strategy for Player II in $\mathtt{mod}\ \mathsf{BM}^{\mathfrak{c}}(\mathbb{R})$. □

CHAPTER

5

OPEN PROBLEMS

In this final part, we present some open problems about the Banach-Mazur game and product of Baire spaces.

At the beginning of this section we present some counterexamples of Baire spaces whose product is not Baire, in the article (HERNáNDEZ; MEDINA; TKACHENKO, 2015), the following question arises:

Question 1 : Do there exist separable (regular, Tychonoff) Baire spaces X and Y such that the product $X \times Y$ fails to be Baire?

As we mentioned earlier Galvin and Scheepers note that White showed that all box powers of Choquet spaces are Baire, and then prove Theorem 3.53. That is why the following question arises in the article (TALL, 2016).

Question 2 : Are large cardinals necessary for Theorem 3.53?

They then ask whether there are any consistent counterexamples.

Also remember that Oxtoby proved that any Tychonoff product of Baire spaces, each with a countable π-base, in particular, each second countable, is Baire, but that a Bernstein set of reals is Baire but not Choquet, so in the Theorem 3.53, Tychonoff powers are not enough.

Fleissner raises the question of whether, if the box product of a collection of Baire spaces is Baire, its Tychonoff product is Baire. Note that, by Corollary 3.55, for box powers, in the **theory of Galvin and Scheepers**, this is true. That is why in the article (FLEISSNER; KUNEN, 1978) the following question arises.

Question 3 : Can one prove in ZFC that if a box product of a collection of Baire spaces is Baire, then its Tychonoff product is Baire?

In the same article, Fleissner also asks whether the box product of Baire spaces with a countable base is Baire. That is,

Question 4 : Is the box product of second countable Baire spaces Baire?

Note that, by Corollary 3.56, in the theory of Galvin and Scheepers, this is not true.

As we have previously noted with the Banach-Mazur game, we could not characterize the productively Baire spaces, since the Bernstein set is productively Baire but it is an undeterminated space. That is why we ask the following question.

Question 5 : Is there a game-theoretical characterization for the property of being productively Baire?

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