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Line and plane congruences from a singularity theory viewpoint

Igor Chagas Santos

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Congruências de retas e planos do ponto de vista da teoria de singularidades

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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This work is dedicated to my parents and my sister,
who have supported me since I decided to come to São Carlos.

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"Las cosas, para hacerlas bien,
es preciso hacerlas dos veces,
porque la primera enseña a la segunda."
(Simón Bolívar)

RESUMO

SANTOS,I. C. **Congruências de retas e planos do ponto de vista da teoria de singularidades**. 2023. 122 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

Esta tese é dedicada ao estudo de congruências de retas e planos. Congruências de retas (resp. de planos) nada mais são que famílias parametrizadas de retas (resp. famílias parametrizadas de planos). No que diz respeito às congruências de retas, estudamos o caso a 3-parâmetros em $\mathbb{R}^4$ e classificamos as singularidades genéricas das congruências (caso geral), bem como as singularidades das congruências normais e normais Blaschke, neste último caso fornecendo uma resposta positiva para a conjectura apresentada por Izumiya, Saji e Takeuchi em 2003. Motivados pelo estudo das congruências normais Blaschke, também iniciamos o estudo de frontais sob o ponto de vista da geometria afim, generalizando a ideia de estrutura equiafim para frontais, definindo o campo Blaschke para frontais, fornecendo exemplos e um teorema fundamental para a teoria equiafim apresentada. Levando em conta o aspecto mais geométrico das congruências de retas apresentado na teoria introduzida por Ernst Kummer para o caso regular, estudamos congruências de retas nas quais a superfície diretora é um frontal, obtendo resultados que generalizam a teoria dada por Kummer. Além disso, considerando famílias parametrizadas de planos, apresentamos um teorema de classificação genérica das singularidades destas congruências, seguindo o método utilizado para o caso das famílias de retas.

Palavras-chave: Geometria diferencial, Geometria diferencial afim, Frontal, Congruência de retas, Congruência de planos.

ABSTRACT

SANTOS,I. C. **Line and plane congruences from a singularity theory viewpoint**. 2023. 122 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

This thesis is devoted to the study of line and plane congruences. Line congruences (resp. plane congruences) are nothing but parametric families of lines (resp. parametric families of planes). We study the case of 3-parameter line congruences in $\mathbb{R}^4$ in order to classify their generic singularities (general case) and the singularities of normal and Blaschke affine normal congruences, in this last case, providing a positive answer to the conjecture presented by Izumiya, Saji and Takeuchi in 2003. Motivated by the study of Blaschke line congruences, we study frontals from the differential affine geometry viewpoint, generalizing the idea of equiaffine structure, defining the Blaschke vector field of a frontal, providing examples and a fundamental theorem for the theory stated here. Taking into account Kummer's theory for line congruences in the regular case, we generalize some results to the case of line congruences for which the director surface is a frontal. Moreover, considering parametrized families of planes, we provide a classification of their generic singularities by using the same approach used for the case of lines.

Keywords: Differential geometry, Affine differential geometry, Frontal, Line congruence, Plane congruence.

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LIST OF SYMBOLS

$\mathcal{E}_n$ — Germs of smooth functions at $0 \in \mathbb{R}^n$

$\mathcal{R}$ — Group of right equivalences

$\mathcal{L}$ — Group of left equivalences

$\mathcal{A}$ — Left-right group

$\mathcal{K}$ — Contact group

$J\mathbf{f}(p)$ — Jacobian matrix of $\mathbf{f}$ at p

T_pM — Tangent space to M at p

∇ — Induced affine connection

$\mathbf{c}$ — Affine fundamental form

$F_{(\mathbf{x},\boldsymbol{\xi})}$ — Congruence generated by $(\mathbf{x},\boldsymbol{\xi})$

$C^\infty(U,\mathbb{R}^n)$ — Smooth maps from U to $\mathbb{R}^n$

$\mathcal{I}_p$ — Kummer first fundamental form at p

$\boldsymbol{\mathcal{I}}$ — Matrix of the Kummer first fundamental form

$\mathcal{II}_p$ — Kummer second fundamental form at p

$\boldsymbol{\mathcal{II}}$ — Matrix of the Kummer seconnd fundamental form

$\boldsymbol{\Omega}$ — Tangent moving basis

$\mathbf{I}$ — Matrix of the first fundamental form

$\mathbf{II}$ — Matrix of the seconnd fundamental form

K — Gaussian curvature

CONTENTS

CHAPTER

1 INTRODUCTION

Line congruences arose as a method of transforming one (hyper)-surface to another using lines. Then, a line congruence in $\mathbb{R}^n$ is a $n-1$-parameter family of (straight) lines, usually given by a pair $\{\mathbf{x}, \boldsymbol{\xi}\}$, where $\mathbf{x} : U \to \mathbb{R}^n$ and $\boldsymbol{\xi} : U \to \mathbb{R}^n \setminus \{0\}$ are smooth maps and U is an open subset of $\mathbb{R}^{n-1}$. A classical example of line congruence is that given by the normal lines to a regular surface, called *exact normal congruence*. The first record about line congruences appeared in "Mémoire sur la Théorie des Déblais et des Remblais" (1776,1784) where Gaspard Monge seeks to solve a minimizing cost problem of transporting an amount of land from one place to another, preserving the volume (see (GHYS, 2012) for historical notes). After Monge, Ernst Eduard Kummer in (KUMMER, 1859) was the first to deal with the general theory of line congruences. This theory is currently known as *K*ummer theory of line congruences and details can be found in (EISENHART, 1909) or (OGURA, 1916).

In recent years, the subject achieved an important development with contributions by (BARAJAS; CRAIZER; GARCIA, 2020), (CRAIZER; GARCIA, 2022a), (HONDA; IZUMIYA; TAKAHASHI, 2019), (IZUMIYA; SAJI; TAKEUCHI, 2003), (LOPES; RUAS; SANTOS, 2022) among others. From the singularity theory viewpoint, we look at a line congruence as a map $F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^{n-1}$ and I is an open interval thus, locally, a line congruence is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The case with $n = 3$ is studied in (IZUMIYA; SAJI; TAKEUCHI, 2003), where the authors classify the generic singularities of 2-parameter line congruences in $\mathbb{R}^3$, showing that these singularities are folds, cusps and swallowtails. They also show that the singularities which appear in generic normal line congruences are the Lagrangian stable ones (see table 1). Furthermore, considering the affine normal vector field (or Blaschke vector field) of a non-degenerate regular surface, the case of equiaffine normal congruences is studied and the authors present a conjecture which asserts that the generic singularities of Blaschke exact normal congruences are Lagrangian stable. More recently in (CRAIZER; GARCIA, 2022b), using the existence of an equiaffine pair defining a generic line congruence, the authors provide a geometric description of the singularities which appear in the classification given in (IZUMIYA; SAJI; TAKEUCHI, 2003).

From the affine differential geometry viewpoint, there is a particular interest in the Blaschke affine normal congruences, for instance, in (BARAJAS; CRAIZER; GARCIA, 2020) the authors study the affine principal lines on surfaces in 3-spaces near affine umbilic points. Taking into account equiaffine line congruences, i.e., line congruences for which the director surface is given by an equiaffine vector field transversal to the reference surface, in (CRAIZER; GARCIA, 2022a) the authors discuss the behavior of the curvature lines associated to this type of line congruence at isolated umbilic points.

Motivated by the results present in (IZUMIYA; SAJI; TAKEUCHI, 2003), in chapter 4, taking into account the case of 3-parameter line congruences in $\mathbb{R}^4$ we classify the generic singularities of 3-parameter line congruences and 3-parameter normal congruences in theorems 4.2.1 and 4.3.2, respectively. The comparison of these two theorems shows that the singularities of 3-parameter line congruences are different from the singularities of normal congruences. Furthermore, we show that singularities of corank 2 appear generically in both cases and the proof of theorem 4.2.1 relies on a refinement of $\mathcal{K}$-orbits by $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension 1. We also take a closer look at the case of Blaschke normal congruences, showing that their generic singularities are Lagrangian stable in corollary 4.4.1, providing a positive answer to the conjecture presented in (IZUMIYA; SAJI; TAKEUCHI, 2003). These results are part of a joint work with Débora Lopes and Maria Aparecida Soares Ruas and can also be found in (LOPES; RUAS; SANTOS, 2022).

The study of surfaces with singularities from the affine differential geometry viewpoint has not been much explored, mainly due to the difficulties which arise at singular points. Motivated by the classification of the singularities of Blaschke normal congruences, we explore this viewpoint in chapter 5, where we work with a special class of singular surfaces called frontals. If we take a surface S and we think of light as particles which propagate at unit speed in the direction of the normals of S, then at a given time t, this particles provide a new surface S'. We call S' the wave front of S. The notion of frontals arises as a generalization of wave fronts, when considering the case of hypersurfaces. In recent years, many papers are dedicated to the study of these singular surfaces, among them, see (FUKUNAGA; TAKAHASHI, 2019), (ISHIKAWA, 2018), (ISHIKAWA, 2020), (MARTINS *et al.*, 2016), (MEDINA-TEJEDA, 2020), (MEDINA-TEJEDA, 2022b), (MEDINA-TEJEDA, 2022a), (SAJI; TERAMOTO, 2021), (SAJI; UMEHARA; YAMADA, 2009). Other references can be found in the survey paper (ISHIKAWA, 2018).

In chapter 5, our goal is to extend the study of properties invariant under equiaffine transformations to the case of frontals, defining equiaffine structure on frontals, equiaffine transversal vector fields and the associated conormal vector field. With this, we seek to understand when it is possible to define a vector field along a frontal that, at regular points, plays the same role as the classical Blaschke vector field, then we define the Blaschke vector field of a frontal and we give, in theorem 5.2.1, necessary and sufficient conditions that a frontal needs to satisfy to have a Blaschke vector field. Furthermore, we obtain a version for frontals of the fundamental theorem of affine differential geometry for regular surfaces in a way that its proof relies on assuming the integrability conditions in the regular case (see theorem 5.3.1). In order to do this, we use the same approach applied in (MEDINA-TEJEDA, 2022a). These results are also in (SANTOS, 2022).

For a more geometric aspect of the line congruences we go back to Kummer's theory, which is briefly reviewed in chapter 3 section 3.3. The best-known results in Kummer's theory are formulated for congruences $\{\mathbf{x}, \boldsymbol{\xi}\}$ where $\mathbf{x}$ is a regular surface and $\boldsymbol{\xi}$ is an immersion. For instance, we discuss in Proposition 3.3.4 a nice way of defining lines of curvature using line congruences: lines of curvature on a smooth non-parabolic surface are those curves whose surfaces of congruence S_C are developable. Since the definition of line congruences admits pairs $\{\mathbf{x}, \boldsymbol{\xi}\}$, where $\mathbf{x}$ and $\boldsymbol{\xi}$ may have singularities, a natural question appears: what happens to the results of Kummer's theory when we have $\boldsymbol{\xi}$ being a frontal, for instance? In chapter 6 we extend this theory to the case of line congruences $\{\mathbf{x}, \boldsymbol{\xi}\}$ where $\mathbf{x}$ is a smooth map and $\boldsymbol{\xi}$ is a proper frontal, in order to answer this question. These results are part of a joint work with Débora Lopes, Maria Aparecida Soares Ruas and Tito Alexandro Medina Tejeda and can also be found in (LOPES *et al.*, 2022).

When working in $\mathbb{R}^4$ it seems natural to consider not only families of straight lines, but also families of planes over surfaces. The family of normal planes to a regular surface in $\mathbb{R}^4$ is a classical example of plane congruence and in this case it is not difficult to show that the generic singularities are the Lagrangian stable, since we can look at the plane congruence as a Lagrangian map associated to the family of distance squared functions, which is generically $\mathcal{P}$-$\mathcal{R}^+$-versal (see example 7.2.1). The classification of the generic singularities of plane congruences arises as a natural generalization of the results for the case of lines. We start chapter 7 with a more general case, taking into account r-surfaces in $\mathbb{R}^n$ (in the sense of (LIMA, 2004), chapter 7) and families of $n-r$-planes, where $n-1>r>1$, then we classify the generic singularities for the case $n=4$, $r=2$ in theorem 7.2.1. In this case we also have singularities of corank 1 and 2 and the proof relies on a refinement of $\mathcal{K}$-orbits by $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension less than or equal to 2.

CHAPTER

2

PRELIMINARIES FROM SINGULARITY THEORY

In this chapter, we present some basic results in singularity theory which help us in the next chapters. More details can be found in (GIBSON, 1979), (MOND; NUÑO-BALLESTEROS, 2020), (IZUMIYA *et al.*, 2016) and (WALL, 1981) .

2.1 Germs of smooth mappings

Let $U,V \subset \mathbb{R}^n$ be two open subsets of $\mathbb{R}^n$ containing a point $p \in \mathbb{R}^n$ and $f : U \to \mathbb{R}^p$ and $g : V \to \mathbb{R}^p$ be two smooth maps. We say that f is equivalent to g if there is an open set $W \subset U \cap V$ containing p such that $f_{|_W} = g_{|_W}$. This relation is an equivalence relation and an equivalent class is called a germ at p of a smooth map. A map-germ at p is denoted by

$$f : (\mathbb{R}^n, p) \to \mathbb{R}^p.$$

Let $\mathcal{E}_n$ denote the set of germs, at the origin $\mathbf{0}$ in $\mathbb{R}^n$, of smooth functions $(\mathbb{R}^n,\mathbf{0}) \to \mathbb{R}^n$,

$$\mathcal{E}_n = \{f : (\mathbb{R}^n,\mathbf{0}) \to \mathbb{R} : \text{f is the germ of a smooth function}\}.$$

With the addition and multiplication operations, $\mathcal{E}_n$ becomes a commutative ring with a unit. This ring is a local ring with maximal ideal, denoted by $\mathcal{M}_n$, given by

$$\mathcal{M}_n = \{f \in \mathcal{E}_n : f(0) = 0\}.$$

Sometimes it is important to look at the kth-power of $\mathcal{M}_n$, where k is a positive integer. This is the set of all $f \in \mathcal{M}_n$ with zero partial derivatives of order less than or equal to $k-1$ at the origin.

The set of all smooth map-germs $f : (\mathbb{R}^n, \mathbf{0}) \to \mathbb{R}^p$, denoted by $\mathcal{E}_{n,p}$, is a free $\mathcal{E}_n$-module given by

$$\mathcal{E}_{n,p} = \underbrace{\mathcal{E}_n \times \cdots \times \mathcal{E}_n}_{p} = (\mathcal{E}_n)^p.$$

Note that given a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ we obtain a mapping $f^* : \mathcal{E}_p \to \mathcal{E}_n$, called *induced algebra homomorphism*, given by $f^*(\lambda) = \lambda \circ f$. It is possible to show that if $p = n$ then f^* is an isomorphism if and only if f is invertible. For details, see chapter 4 in (GIBSON, 1979) or chapter 2 in (MOND; NUÑO-BALLESTEROS, 2020).

2.2 Mather's groups

Let $\mathcal{R}$ denote the group of germs of diffeomorphisms $(\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$. We refer to $\mathcal{R}$ as the "group of right equivalences" and this group acts smoothly on $\mathcal{E}(n, p)$ by

$$h \cdot f = f \circ h^{-1},$$

for all $h \in \mathcal{R}$ and $f \in \mathcal{E}(n, p)$.

The group $\mathcal{L}$ of germs of diffeomorphisms $(\mathbb{R}^p, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$ acts smoothly on $\mathcal{M}_n\mathcal{E}_{n,p}$ by

$$k \cdot f = k \circ f,$$

for all $k \in \mathcal{L}$ and $f \in \mathcal{M}_n\mathcal{E}_{n,p}$. We refer to $\mathcal{L}$ as the "group of left equivalences".

The "left-right" group $\mathcal{A}$ is given by the direct product of $\mathcal{R}$ and $\mathcal{L}$, i.e., $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. This group acts smoothly on $\mathcal{M}_n\mathcal{E}_{n,p}$ by

$$(h, k) \cdot f = k \circ f \circ h^{-1},$$

for all $(h, k) \in \mathcal{A}$ and $f \in \mathcal{M}_n\mathcal{E}_{n,p}$.

The group $\mathcal{K}$ is called the *contact group* and it is given by the germs of diffeomorphism $(\mathbb{R}^n \times \mathbb{R}^p, \mathbf{0}) \to (\mathbb{R}^n \times \mathbb{R}^p, \mathbf{0})$ which can be written in the form $H(x, y) = (h(x), H_1(x, y))$ such that $h \in \mathcal{R}$ and $H_1(x, 0) = 0$ for x near to $\mathbf{0}$. The group $\mathcal{K}$ acts on $\mathcal{M}_n\mathcal{E}_{n,p}$ as follows. Given $f, g \in \mathcal{M}_n\mathcal{E}_{n,p}$ and $(h, H) \in \mathcal{K}$, $g = (h, H) \cdot f$ if and only if

$$(x, g(x)) = H(h^{-1}(x), f \circ h^{-1}(x)).$$

Given map germs $f, g \in \mathcal{M}_n\mathcal{E}_{n,p}$, if there is $h \in \mathcal{R}$, such that $h^*(f^*(\mathcal{M}_p)) = g^*(\mathcal{M}_p)$, where $h^*(f^*(\mathcal{M}_p))$ is the ideal generated by the coordinate functions of $f \circ h$ and $g^*(\mathcal{M}_p)$ is the ideal generated by the coordinate functions of g, we have that f and g are $\mathcal{K}$-equivalent, denoted by,

$f \underset{\mathcal{K}}{\sim} g$ (see section 4.4 in (MOND; NUÑO-BALLESTEROS, 2020) for details). Let $J^k(n,p)$ be the k-jet space of map germs from $\mathbb{R}^n$ to $\mathbb{R}^p$. For any $j^k f(0)$, we set

$$\mathcal{K}^k(j^k f(0)) = \{j^k g(0) : f \underset{\mathcal{K}}{\sim} g\},$$

for the $\mathcal{K}$-orbit of f in the space of k-jets $J^k(n,p)$.

The Mather groups are not Lie groups and $\mathcal{E}(n,p)$ is not a finite dimensional manifold, but in order to define the tangent space to an orbit of one of the Mather's groups, we proceeded as follows (for details and more information, see section 3.5 in (IZUMIYA *et al.*, 2016) or chapters 3 and 4 in (MOND; NUÑO-BALLESTEROS, 2020)). Let $\pi : T\mathbb{R}^p \to \mathbb{R}^p$ be the tangent bundle over $\mathbb{R}^p$, thus a map-germ $\xi : (\mathbb{R}^n, \mathbf{0}) \to T\mathbb{R}^p$ is said to be a germ of *vector field along* $f \in \mathcal{E}_{n,p}$ if $\pi \circ \xi = f$. The tangent space θ_f to $\mathcal{E}_{n,p}$ at f is defined to be the $\mathcal{E}_n$-module of germs of vector fields along f.

Let $\theta_n = \theta_{id_{(\mathbb{R}^n,0)}}$ and $\theta_p = \theta_{id_{(\mathbb{R}^p,0)}}$, where $id_{(\mathbb{R}^n,0)}$ and $id_{(\mathbb{R}^p,0)}$ denote the germs of the identity maps on $(\mathbb{R}^n, \mathbf{0})$ and $(\mathbb{R}^p, \mathbf{0})$, respectively. Note that θ_n is nothing but the set of germs of the vector fields on $\mathbb{R}^n$ at the origin. Define the maps

$$\begin{aligned} tf : \theta_n &\to \theta_p \\ \phi &\mapsto df \circ \phi \\ \omega f : \theta_p &\to \theta_f \\ \psi &\mapsto \psi \circ f. \end{aligned}$$

Note that θ_n is a free module over $\mathcal{E}_n$ and θ_p is a free module over $\mathcal{E}_p$ with structure given by the homomorphism $f^* : \mathcal{E}_p \to \mathcal{E}_n$, defined by $f^*(\psi) = \psi \circ f$. Let $f^*(\mathcal{M}_p)$ denote the pullback of the maximal ideal in $\mathcal{E}_p$. The *tangent spaces* $L\mathcal{G} \cdot f$ to the $\mathcal{G}$-orbits of f at the germ f are defined as by:

$$L\mathcal{R} \cdot f = tf(\mathcal{M}_n \cdot \theta_n) \qquad L\mathcal{L} \cdot f = \omega f(\mathcal{M}_p \cdot \theta_p) \qquad L\mathcal{A} \cdot f = L\mathcal{R} \cdot f + L\mathcal{L} \cdot f$$
$$L\mathcal{K} \cdot f = L\mathcal{R} \cdot f + f^*(\mathcal{M}_p) \cdot \theta_f$$

If we choose a system of coordinates $(y_1, \cdots, y_p)$ in $\mathbb{R}^p$ then the germs of vector fields along f $\left(\dfrac{\partial}{\partial y_1}\right) \circ f, \cdots, \left(\dfrac{\partial}{\partial y_p}\right) \circ f$ form a free basis of θ_f. Then, θ_f can be identified canonically with $\mathcal{E}_{n,p}$, that is, θ_f is a free $\mathcal{E}_n$-module of rank p and we have

$$L\mathcal{R} \cdot f = \mathcal{M}_n \left\{ \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right\} \qquad L\mathcal{L} \cdot f = f^*(\mathcal{M}_p)\{e_1, \cdots, e_p\}$$
$$f^*(\mathcal{M}_p) \cdot \theta_f = f^*(\mathcal{M}_n) \cdot \mathcal{E}_{n,p}$$

where $e_1, \cdots, e_p$ are the standard basis vectors of $\mathbb{R}^p$ consider as elements of $\mathcal{E}_{n,p}$ and $(x_1, \cdots, x_n)$ is a coordinate system in $(\mathbb{R}^n, \mathbf{0})$.

When studying deformations, the singularity can move away from the origin and because of this, the *extended tangent spaces* are defined as follows, considering local coordinates,

$$L_e\mathcal{R}\cdot f = \mathcal{E}_n\left\{\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\right\} \qquad L_e\mathcal{L}\cdot f = f^*(\mathcal{E}_p)\{e_1,\cdots,e_p\}$$
$$L_e\mathcal{K}\cdot f = L_e\mathcal{R}\cdot f + f^*(\mathcal{M}_p)\cdot\mathcal{E}_{n,p}$$

For any of the groups $\mathcal{G}=\mathcal{R},\mathcal{L},\mathcal{A},\mathcal{K}$, the *codimension of the orbit* of f is defined by

$$\mathrm{cod}(f,\mathcal{G}) = \dim_{\mathbb{R}}\left(\frac{\mathcal{M}_n\mathcal{E}_{n,p}}{L\mathcal{G}\cdot f}\right)$$

and the *codimension of the extended orbit* of f is defined by

$$\mathrm{cod}_e(f,\mathcal{G}) = \dim_{\mathbb{R}}\left(\frac{\mathcal{E}_{n,p}}{L_e\mathcal{G}\cdot f}\right)$$

2.3 Unfoldings

Definition 2.3.1. Let $f\in\mathcal{E}(n,p)$. A r-parameter unfolding (r,F) of f is a map-germ

$$F:(\mathbb{R}^n\times\mathbb{R}^r,(\mathbf{0},\mathbf{0}))\to(\mathbb{R}^p\times\mathbb{R}^r,(\mathbf{0},\mathbf{0}))$$

in the form $F(x,y)=(\tilde{f}(x,y),y)$ with $\tilde{f}(x,0)=f(x)$. The family of map-germs $\tilde{f}$ is called a r-parameter deformation of f and we denote, for a fixed y_0, $\tilde{f}_{y_0}(x)=\tilde{f}(x,y_0)$.

Definition 2.3.2. Let $\mathcal{G}$ be a Mather group and I the identity in $\mathcal{G}$.

a) A *morphism* between two unfoldings (a,F) and (b,G) is a pair $(\alpha,\psi):(a,F)\to(b,G)$ with $\alpha:(\mathbb{R}^a,\mathbf{0})\to(\mathcal{G},I)$, $\psi:(\mathbb{R}^a,\mathbf{0})\to(\mathbb{R}^b,\mathbf{0})$, such that

$$\tilde{f}_y = \alpha(y)\cdot\tilde{g}_{\psi(y)}.$$

The unfolding (a,F) is then said to be induced from (b,G) by (α,ψ).

b) Two unfoldings (a,F) and (b,G) are $\mathcal{G}$-*equivalent* if there exists a morphism (α,ψ) : $(a,F)\to(b,G)$ where ψ is invertible (so, $a=b$).

c) An unfolding (a,F) of a map-germ f is said to be $\mathcal{G}$-*versal* if any unfolding (b,G) of f can be induced from (a,F).

d) An unfolding (a,F) of f is said to be $\mathcal{G}$-*trivial* if it is $\mathcal{G}$-equivalent to the constant unfolding (a,f).

Now, using definition 2.3.2 and the definition of codimension, we can define stability and infinitesimally stability.

Definition 2.3.3. A map-germ f is $\mathcal{G}$*-stable* (resp. $\mathcal{G}_e$*-stable*) if all of its unfoldings are $\mathcal{G}$-trivial (resp. $\mathcal{G}_e$-trivial).

Theorem 2.3.1. A map-germ f is $\mathcal{G}$*-stable* (resp. $\mathcal{G}_e$*-stable*) if and only if $\mathrm{cod}(f,\mathcal{G}) = 0$ (resp. $\mathrm{cod}_e(f,\mathcal{G}) = 0$).

Definition 2.3.4. Let $f : (N,x_0) \to (P,y_o)$ be a map germ between manifolds. An *unfolding* of f is a triple (F,i,j) of map germs, where $i : (N,x_0) \to (N',x'_0)$, $j : (P,y_0) \to (P',y'_0)$ are immersions and j is transverse to F, such that $F \circ i = j \circ f$ and $(i,f) : N \to \{(x',y) \in N' \times P : F(x') = j(y)\}$ is a diffeomorphism germ (see the associated diagram in figure 1). The dimension of the unfolding is $dim(N') - dim(N)$.

$$\begin{tikzcd}
(N',x'_0) \arrow[r, "F"] & (P',y'_0) \\
(N,x_0) \arrow[u, "i"] \arrow[r, "f"] & (P,y_0) \arrow[u, "j"']
\end{tikzcd}$$

Figure 1 – Associated diagram

Remark 2.3.1. The above definition of unfolding is locally equivalent to the usual parametrized one given in definition 2.3.1. For details, see chapter 3 in (GIBSON *et al.*, 2006).

Lemma 2.3.1. ((IZUMIYA; SAJI; TAKEUCHI, 2003), Lemma 3.1) Let $F : (\mathbb{R}^{n-1} \times \mathbb{R}, (\mathbf{0},0)) \to (\mathbb{R}^n, \mathbf{0})$ be a map germ with components $F_i(x,t)$, $i = 1,2,\cdots,n$, i.e.

$$F(x,t) = (F_1(x,t),\cdots,F_n(x,t)).$$

Suppose that $\dfrac{\partial F_n}{\partial t}(\mathbf{0},0) \neq 0$. We know by the Implicit Function Theorem that there is a germ of function $g : (\mathbb{R}^{n-1},\mathbf{0}) \to (\mathbb{R},0)$, such that

$$F_n^{-1}(0) = \{(x,g(x)) : x \in (\mathbb{R}^{n-1},\mathbf{0})\}.$$

Let us consider the immersion germs $i : (\mathbb{R}^{n-1},\mathbf{0}) \to (\mathbb{R}^n,(\mathbf{0},0))$, given by $i(x) = (x,g(x))$, $j : (\mathbb{R}^{n-1},\mathbf{0}) \to (\mathbb{R}^n,(\mathbf{0},0))$, given by $j(y) = (y,0)$ and a map germ $f : (\mathbb{R}^{n-1},\mathbf{0}) \to (\mathbb{R}^{n-1},\mathbf{0})$, given by $f(x) = (F_1(x,g(x)),\cdots,F_{n-1}(x,g(x)))$. Then the triple (F,i,j) is a one-dimensional unfolding of f.

For a map germ $f : (\mathbb{R}^n \times \mathbb{R}^r,\mathbf{0}) \to (\mathbb{R}^p,\mathbf{0})$, we define

$$\begin{aligned} j_1^k f : (\mathbb{R}^n \times \mathbb{R}^r,\mathbf{0}) &\to J^k(n,p) \\ (x,y) &\mapsto j_1^k f(x,y), \end{aligned}$$

where $j_1^k f(x,y)$ indicates the k-jet with respect to the first variable.

Lemma 2.3.2. ((IZUMIYA; TAKEUCHI, 2001), Lemma 3.3) Let $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}^p \times \mathbb{R}^r, \mathbf{0})$ be an unfolding of f_0 of the form $F(x,y) = (f(x,y),y)$. If $j_1^k f$ is transverse to $\mathcal{K}^k(j^k f_0(0))$ for a sufficiently large k, then F is infinitesimally $\mathcal{A}$-stable.

Let us denote by $C^\infty(M,\mathbb{R}^n)$ the space of smooth maps between the manifold M and $\mathbb{R}^n$ endowed with the so-called Whitney C^∞-topology (for more details on this topology see chapter 5 in (MOND; NUÑO-BALLESTEROS, 2020)).

Definition 2.3.5. Let $\mathcal{G}$ be one of Mather's subgroups of $\mathcal{K}$ and $\mathcal{B}$ a smooth manifold. A family of maps $F : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^k$, given by $F(x,y) = f_y(x)$, is said to be *locally $\mathcal{G}$-versal* if for every $(x,y) \in \mathbb{R}^n \times \mathcal{B}$, the germ of F at (x,y) is a $\mathcal{G}$-versal unfolding of f_y at x.

With notation as above, let $g : M \to \mathbb{R}^n$ be an immersion, where M is a smooth manifold, and denote by $\phi_g : M \times \mathcal{B} \to \mathbb{R}^k$ the map given by

$$\phi_g(z,y) = F(g(z),y).$$

Denote by $Imm(M,\mathbb{R}^n)$ the subset of $C^\infty(M,\mathbb{R}^n)$ whose elements are proper C^∞-immersions from M to $\mathbb{R}^n$.

Theorem 2.3.2. ((MONTALDI, 1991), Theorem 1) Suppose $F : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^k$ as above is locally $\mathcal{G}$-versal. Let $W \subset J^r(M,\mathbb{R}^k)$ be a $\mathcal{G}$-invariant submanifold, where M is a manifold and let

$$R_W = \{g \in Imm(M,\mathbb{R}^n) : j_1^r \phi_g \pitchfork W\}.$$

Then R_W is residual in $Imm(M,\mathbb{R}^n)$. Moreover, if $\mathcal{B}$ is compact and W is closed, then R_W is open and dense.

Lemma 2.3.3. ((GOLUBITSKY; GUILLEMIN, 2012), Lemma 4.6)(Basic Transversality Lemma) Let X, B and Y be smooth manifolds with W a submanifold of Y. Consider $j : B \to C^\infty(X,Y)$ a non-necessarily continuous map and define $\Phi : X \times B \to Y$ by $\Phi(x,b) = j(b)(x)$. Suppose Φ smooth and transversal to W, then the set

$$\{b \in B : j(b) \pitchfork W\}$$

is a dense subset of B.

When studying germs of functions it is important to consider also the direct product of the group $\mathcal{R}$ with translations, which we denote by $\mathcal{R}^+$.

Definition 2.3.6. Two families of germs of functions $F,G : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0})) \to (\mathbb{R},0)$ are $\mathcal{P}$-$\mathcal{R}^+$-equivalent if there exist a germ of diffeomorphism $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0})) \to (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0}))$ of the form $\Phi(x,y) = (\alpha(x,y), \psi(y)$ and a germ of function $c : (\mathbb{R}^r,\mathbf{0}) \to \mathbb{R}$ such that

$$G(x,y) = F \circ \Phi(x,y) + c(y).$$

Above, the letter $\mathcal{P}$ stands for parametrized, as we have a family of germs of diffeomorphisms $\alpha(,y)$ of $\mathbb{R}^n$ parametrized by y and the "$+$" stands for the addition of $c(y)$.

Definition 2.3.7. We say that a deformation $F : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0})) \to (\mathbb{R},0)$ of a germ of function $f \in \mathcal{M}_n$ is $\mathcal{R}^+$-versal if

$$L\mathcal{R}_e.f + \mathbb{R}.\{1, \dot{F}_1, \cdots, \dot{F}_r\} = \mathcal{E}_n,$$

where $\dot{F}_i = \dfrac{\partial F}{\partial y_i}(x,\mathbf{0})$, for $i = 1, \cdots, r$.

2.4 Lagrangian singularities

A skew-symmetric 2-form ω on a smooth manifold M is said to be a *symplectic form* if it is closed and non-degenerate and, that is, $d\omega = 0$ and for all $p \in M$, if $\omega_p(\mathbf{v},\mathbf{w}) = 0$, for all $\mathbf{w}$, then $\mathbf{v} = 0$. A manifold M equipped with a symplectic form ω is called a *symplectic manifold*. It follows from the definition of symplectic form that $d\omega = 0$ and ω^n is a volume form for M, so $\dim M = 2n$ for some positive integer n.

Example 2.4.1. Let N be a smooth manifold and T^*N its cotangent bundle. There is a canonical symplectic structure on T^*N. The canonical 1-form (or the Liouville form, or the Tautological form) λ on T^*N is defined, at each $(q,v) \in T^*N$, by $\lambda_{(q,v)} : T_{(q,v)}(T^*N) \to \mathbb{R}$, where

$$\lambda_{(q,v)}(w) = v\left(d\rho_{(q,v)}(w)\right)$$

and $\rho : T^*N \to N$ is the canonical projection defined by $\rho(q,v) = q$. The canonical symplectic structure on T^*N is given by the 2-form $\omega = -d\lambda$. Let $\mathbf{x} : U \to \mathbb{R}^n$ be a local system of coordinates of N, where $\mathbf{x} = (x_1, \cdots, x_n)$. The 1-forms $dx_i(q) : T_qN \to \mathbb{R}$, $i = 1, \cdots, n$ form a basis of T_q^*N, thus any $v \in_q^* N$ can be written in a unique way in the form

$$v = \sum_{i=1}^{n} p_i(q,v)dx_i(q).$$

Then, we obtain a local system of coordinates $\phi : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ of T_q^*N, where $\phi(q,v) = (x_1(q), \cdots, x_n(q), p_1(q,v), \cdots, p_n(q,v))$. In this system of coordinates, we have

$$\lambda = \sum_{i=1}^{n} p_i dx_i,$$

therefore,

$$\omega = -d\lambda = \sum_{i=1}^{n} dx_i \wedge dp_i.$$

Definition 2.4.1. Let (M_1,ω_1) and (M_2,ω_2) be two symplectic manifolds. A *symplectomorphism* between (M_1,ω_1) and (M_2,ω_2) is a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^*\omega_2 = \omega_1$. More precisely, $\omega_1(p)(\mathbf{v}_1,\mathbf{v}_2) = \omega_2(\phi(p))(d\phi_p(\mathbf{v}_1), d\phi_p(\mathbf{v}_2))$, for any $p \in M_1$ and $\mathbf{v}_1, \mathbf{v}_2 \in T_pM_1$.

Definition 2.4.2. Let M be a $2n$-dimensional smooth manifold and let ω be a symplectic form on M. We say that a smooth submanifold L of M is a Lagrangian submanifold if $\dim L = n$ and $\omega_{|_L} = 0$

Example 2.4.2. If we consider the cotangent bundle of a smooth manifold N with the canonical symplectic structure, given in example 2.4.1, then the fibers of $\rho : T^*N \to N$ are Lagrangian submanifolds.

Definition 2.4.3. Let $\pi : E \to N$ be a fiber bundle such that E is a symplectic manifold. We say that $\pi : E \to N$ is a *Lagrangian fibration* if its fibers are Lagrangian submanifolds of E.

Example 2.4.3. It follows from example 2.4.2 that $\rho : T^*N \to N$ is a Lagrangian fibration.

Definition 2.4.4. Let $\pi : E \to N$ and $\pi' : E' \to N'$ be a Lagrangian fibrations. A symplectomorphism $\Phi : E \to E'$ is said to be a *Lagrangian diffeomorphism* if there is a diffeomorphism $\phi : N \to N'$ such that $\pi' \circ \Phi = \phi \circ \pi$.

Definition 2.4.5.

a) Let $\pi : E \to N$ be a Lagrangian fibration and consider a Lagrangian immersion $i : L \to E$, that is, $i^*\omega = 0$. The restriction of π to $i(L)$, i.e., $\pi \circ i : L \to N$, is called a *Lagrangian map*.

b) The set of critical values of a Lagrangian map is said to be a *caustic*. We denote by $C(i(L))$ the caustic of the Lagrangian map $\pi \circ i : L \to N$.

c) We say that two Lagrangian maps $\pi \circ i : L \to N$ and $\pi' \circ i' : L' \to N'$ are *Lagrangian equivalent* if there is a Lagrangian diffeomorphism $\Phi : E \to E'$ such that $\Phi(i(L)) = i'(L')$.

Remark 2.4.1. If $\pi \circ i : L \to N$ and $\pi' \circ i' : L' \to N'$ are Lagrangian equivalent, then the caustics $C(i(L))$ and $C(i'(L'))$ are diffeomorphic. Also, it follows from the above proposition that if two Lagrangian maps are Lagrangian equivalent, then they are $\mathcal{A}$-equivalent.

It is known that all Lagrangian fibrations of a fixed dimension are locally Lagrangian diffeomorphic (see theorem 5.2 in (IZUMIYA *et al.*, 2016)), thus we can work on the cotangent bundle $\pi : T^*\mathbb{R}^r \to \mathbb{R}^r$ and all the results are valid on any Lagrangian fibration. Let $(x,p) = (x_1, \cdots, x_r, p_1, \cdots, p_r)$ denote the canonical coordinates on $T^*\mathbb{R}^r$, λ the canonical 1-form and ω the canonical symplectic form on $T^*\mathbb{R}^r$.

Definition 2.4.6. We say that an r-parameter family of germs of functions $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}, 0)$ is a *Morse family of functions* if the map germ $\Delta_F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$, given by

$$\Delta_F(x,y) = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n} \right)(x,y)$$

is not singular.

When F is a Morse family

$$(C_F,\mathbf{0}) = \{(x,y) \in (\mathbb{R}^n \times \mathbb{R}^r,\mathbf{0}) : \frac{\partial F}{\partial x_1}(x,y) = \cdots = \frac{\partial F}{\partial x_n(x,y)} = 0\}$$

is a germ of smooth submanifold of $(\mathbb{R}^n \times \mathbb{R}^r,\mathbf{0})$ of dimension r. Then, we immerse $(C_F,0)$ in the cotangent bundle $T^*\mathbb{R}^r$ using the map-germ $L(F) : (C_F,\mathbf{0}) \to T^*\mathbb{R}^r$ defined by

$$L(F)(x,y) = \left(y, \frac{\partial F}{\partial y_1}(x,y), \cdots, \frac{\partial F}{\partial y_r}(x,y)\right).$$

Note that $L(F)^*\lambda = \sum_{i=1}^r \frac{\partial F}{\partial y_i} dy_{i_{|C_F}} = dF_{|C_F}$, hence

$$L(F)^*\omega = -L(F)^*d\lambda = -dL(F)^*\lambda = -d(dF_{|C_F}) = -(ddF)_{|C_F} = 0.$$

As we know that $\dim C_F = r$, it follows that $L(F)(C_F)$ is a Lagrangian submanifold of $T^*\mathbb{R}^r$. We call F the *generating family* of the germ of Lagrangian submanifold $L(F)(C_F)$.

Example 2.4.4. Let $\mathbf{x} : U \to \mathbb{R}^4$ be a regular hypersurface in R^4, where $U \subset \mathbb{R}^3$ is open and $\mathbf{x}(U) = M$. Let $D : U \times \mathbb{R}^4 \to \mathbb{R}$, defined by

$$D(u,p) = \langle \mathbf{x}(u) - p, \mathbf{x}(u) - p\rangle$$

be the family of distance squared functions on M. The germ of D at each $(u_0, p_0) \in U \times \mathbb{R}^4$ is a Morse family of functions as follows. If we write $u = (u_1,u_2,u_3)$, $\mathbf{x}(u) = (x_1(u),x_2(u),x_3(u),x_4(u))$ and $p = (p_1, p_2, p_3, p_4)$, then

$$D(u,p) = \sum_{i=1}^{4} (x_i(u) - p_i)^2.$$

We need to prove that the map

$$\begin{aligned} \Delta_D : U \times \mathbb{R}^4 &\to \mathbb{R}^3 \\ (u,p) &\mapsto \left(\frac{\partial D}{\partial u_1}, \frac{\partial D}{\partial u_2}, \frac{\partial D}{\partial u_3}\right) \end{aligned}$$

is not singular. Its jacobian matrix is given by

$$J\Delta_D = \begin{pmatrix} a_{11} & a_{12} & a_{13} & -2(x_1)_{u_1} & -2(x_2)_{u_1} & -2(x_3)_{u_1} & -2(x_4)_{u_1} \\ a_{21} & a_{22} & a_{23} & -2(x_1)_{u_2} & -2(x_2)_{u_2} & -2(x_3)_{u_2} & -2(x_4)_{u_2} \\ a_{31} & a_{32} & a_{33} & -2(x_1)_{u_3} & -2(x_2)_{u_3} & -2(x_3)_{u_3} & -2(x_4)_{u_3} \end{pmatrix},$$

where $a_{ij} = 2\langle \mathbf{x}_{u_iu_j}(u), \mathbf{x}(u) - p\rangle + 2\langle \mathbf{x}_{u_i}(u), \mathbf{x}_{u_j}(u)\rangle$, $i = 1,2,3$. Since $\mathbf{x}$ is an embedding, the rank of the matrix

$$\begin{pmatrix} (x_1)_{u_1} & (x_2)_{u_1} & (x_3)_{u_1} & (x_4)_{u_1} \\ (x_1)_{u_2} & (x_2)_{u_2} & (x_3)_{u_2} & (x_4)_{u_2} \\ (x_1)_{u_3} & (x_2)_{u_3} & (x_3)_{u_3} & (x_4)_{u_2} \end{pmatrix}^T$$

is 3 at each point $u \in U$. Then Δ_D is not singular. Furthermore, the catastrophe set of D is given by

$$C_D = \{(u,p) : p = \mathbf{x}(u) + t\mathbf{n}(u), \text{ for some } t \in \mathbb{R}\},$$

where $\mathbf{n}$ denotes the unit normal vector field of M. Then, taking this into account, the Lagrangian immersion associated to D is given by

$$\begin{aligned} L(D) : (U \times I, (u_0,t_0)) &\to T^*\mathbb{R}^4 \\ (u,t) &\mapsto (\mathbf{x}(u) + t\mathbf{n}(u), 2t\mathbf{n}(u)), \end{aligned}$$

where $p_0 = \mathbf{x}(u_0) + t_0\mathbf{n}(u_0)$. Hence, the germ of Lagrangian map associated to D is $F_{(\mathbf{x},\mathbf{n})}(u,t) = \mathbf{x}(u) + t\mathbf{n}(u)$.

Theorem 2.4.1. ((IZUMIYA *et al.*, 2016), Theorem 4.8 (ii)) For an open and dense set of embeddings $\mathbf{x} : U \to \mathbb{R}^4$, the family D is locally $\mathcal{P}$-$\mathcal{R}^+$-versal.

Definition 2.4.7. A germ of Lagrangian immersion $i : (L,u) \to (T^*\mathbb{R}^r, p)$ (or a germ of Lagrangian map $\pi \circ i : (L,u) \to \mathbb{R}^r \pi(p))$ is said to be *Lagrangian stable* if for any representative $\bar{i} : V \to T^*\mathbb{R}^r$ of i, there is a neighborhood W of $\bar{i}$ (in the Whitney C^∞-topology on the subset of Lagrangian immersions considered as a subspace of $C^\infty(\mathbb{R}^r, T^*\mathbb{R}^r)$) and a neighborhood V of u such that for any Lagrangian immersion $\bar{j}$ in W, there exists $u' \in V$ with $\pi \circ i$ and $\pi \circ j$ Lagrangian equivalent, where $j : (L,u') \to (T^*\mathbb{R}^r, p')$ is the germ of $\bar{j}$ at u'.

Next we see that the notion of Lagrangian stability in terms of generating families.

Theorem 2.4.2. ((IZUMIYA *et al.*, 2016), Theorem 5.4 (1)) The Lagrangian map-germ $\pi \circ L(F)$ is Lagrangian stable if and only if F is an $\mathcal{R}^+$-versal unfolding of $f(x) = F(x,0)$.

Theorem 2.4.3. ((IZUMIYA *et al.*, 2016), Theorem 5.5) Let $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}, 0)$ be a Morse family of functions. Suppose that $L(F) : (C(F), \mathbf{0}) \to T^*R^r$ is Lagrangian stable and $r \leq 4$. Then $L(F)$ is Lagrangian equivalent to a germ of a Lagrangian submanifold whose generating family $G(x_1, \cdots, x_n, y_1, \cdots, y_r)$ is one of the following germs, where $Q(x_k, \cdots, x_n) = \pm x_k^2 \pm \cdots \pm x_n^2$,

a) $Q(x_2, \cdots, x_n) + x_1^3 + y_1x_1$

b) $Q(x_2, \cdots, x_n) + x_1^4 + y_1x_1 + y_2x_1^2$

c) $Q(x_2, \cdots, x_n) + x_1^5 + y_1x_1 + y_2x_1^2 + y_3x_1^3$

d) $Q(x_2, \cdots, x_n) + x_1^6 + y_1x_1 + y_2x_1^2 + y_3x_1^3 + u_4x_1^4$

e) $Q(x_3, \cdots, x_n) + x_1^3 + x_1x_2^2 + y_1x_1 + y_2x_2 + y_3(x_1^2 + x_2^2)$

f) $Q(x_3, \cdots, x_n) + x_1^3 + x_2^3 + y_1x_1 + y_2x_2 + y_3x_1x_2$

g) $Q(x_3,\cdots,x_n)+x_1^2x_2+x_2^4+y_1x_1+y_2x_2+y_3x_1^2+y_4x_2^2$.

The normal forms of the Lagrangian stable map-germs $\pi\circ L(G)$ for $r\leq 4$ are given in table 1.

G singularity type	**$\pi\circ L(G)$ singularity type**	**Normal form**
A_2	Fold	x_1^2
A_3	Cusp	$(x_1^3+y_2x_1,y_2)$
A_4	Swallowtail	$(x_1^4+y_2x_1+y_3x_1^2,y_2,y_3)$
A_5	Butterfly	$(x_1^5+y_2x_1+y_3x_1^2+y_4x_1^3,y_2,y_3,y_4)$
D_4^-	Elliptic Umbilic	$(x_1^2-x_2^2+y_3x_1,x_1x_2+y_3x_2,y_3)$
D_4^+	Hyperbolic Umbilic	$(x_1^2+y_3x_2,x_2^2+y_3x_1,y_3)$
D_5	Parabolic Umbilic	$(x_1x_2+y_3x_1,x_1^2+x_2^3+y_4x_2,y_3,y_4)$

Table 1 – Lagrangian stable singularities in R^r for $r\leq 4$

CHAPTER

3

PRELIMINARIES FROM DIFFERENTIAL GEOMETRY

In this chapter we review some well known results which play an important role for the understanding of this thesis. First, in section 3.1 we review some basic results on affine differential geometry, useful in chapter 4, where we classify generic singularities of Blaschke line congruences and chapter 5, where we generalize the idea of equiaffine structure for a special class of singular surfaces. In section 3.2 we summarize some results from (IZUMIYA; SAJI; TAKEUCHI, 2003) which are important in chapter 4 where we deal with the case of 3-parameter line congruences in $\mathbb{R}^4$. In section 3.3 Kummer's theory for line congruences is reviewed as preparation for chapter 6, where we generalize some of Kummer's results taking the director surface of the congruence as a singular surface. Finally, in section 3.4, definitions and properties of frontals are presented. These results are useful in chapter 5 as we want to study frontals from an affine viewpoint and chapter 6, since the singular surfaces we consider are proper frontals.

3.1 Affine differential geometry

Let us regard $\mathbb{R}^{n+1}$ as a $n+1$-dimensional affine space with volume element given by $\omega(e_1,\cdots,e_{n.1}) = \det(e_1,\cdots,e_{n+1})$, where $\{e_1,\cdots,e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$. Let D be the standard flat connection on $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^n$ and $\mathbf{x} : U \to \mathbb{R}^{n+1}$ be a regular hypersurface with $\mathbf{x}(U) = M$ and $\boldsymbol{\xi} : U \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ a vector field which is transversal to M. We can decompose the tangent space

$$T_p\mathbb{R}^{n+1} = T_pM \oplus \langle \boldsymbol{\xi}(u) \rangle_{\mathbb{R}},$$

where $\mathbf{x}(u) = p$. So, it follows that given X and Y vector fields on M, we have the decomposition

$$D_XY = \nabla_XY + \mathbf{c}(X,Y)\boldsymbol{\xi}, \tag{3.1}$$

where ∇ is the *induced affine connection* and $\mathbf{c}$ is the *affine fundamental form* induced by $\boldsymbol{\xi}$, which defines a symmetric bilinear form on each tangent space of M. We say that M is *non-degenerate* if $\mathbf{c}$ is non-degenerate which is equivalent to say that the Gaussian curvature of M never vanishes (see chapter 3 in (NOMIZU; KATSUMI; SASAKI, 1994)). Using the same idea, we decompose

$$D_X\boldsymbol{\xi} = -S(X) + \tau(X)\boldsymbol{\xi},$$

where S is the *shape operator* and τ is the *transversal connection form*. We say that $\boldsymbol{\xi}$ is an *equiaffine transversal vector field* if $\tau = 0$, i.e $D_X\boldsymbol{\xi}$ is tangent to M.

Using the volume element ω and the transversal vector field $\boldsymbol{\xi}$, we induce a volume element θ on M as follows

$$\theta(X_1, \cdots, X_n) = \omega(X_1, \cdots, X_n, \boldsymbol{\xi}),$$

where $X_1, \cdots, X_n$ are tangent to M.

Proposition 3.1.1. ((NOMIZU; KATSUMI; SASAKI, 1994), Proposition 1.4) We have

$$\nabla_X \theta = \tau(X)\theta, \text{ for all } X \in T_pM. \tag{3.2}$$

Consequently, the following two conditions are equivalent:

(a) $\nabla\theta = 0$.

(b) $\tau = 0$.

We say that M has a *parallel volume element* if there is a volume element θ on M such that $\nabla\theta = 0$, where

$$\nabla_X \theta(X_1, X_2) = X(\omega(X_1, X_2)) - \theta(X_1, \nabla_X X_2) - \theta(\nabla_X X_1, X_2)$$

for X, X_1, X_2 vector fields on M. Then, it follows from proposition 3.1.1 that a vector field $\boldsymbol{\xi}$, transversal to a non-parabolic surface, is equiaffine if and only if the induced volume element is parallel.

Given a non-degenerate hypersurface $\mathbf{x} : U \to \mathbb{R}^{n+1}$ and a vector field $\boldsymbol{\xi} : U \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ which is transversal to $M = \mathbf{x}(U)$, we take the line congruence generated by $(\mathbf{x}, \boldsymbol{\xi})$ and the map

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I &\to \mathbb{R}^{n+1} \\ (u,t) &\mapsto \mathbf{x}(u) + t\boldsymbol{\xi}(u), \end{aligned}$$

where I is an open interval.

Definition 3.1.1. A point $p = F(u,t)$ is called a focal point of multiplicity $m > 0$ if the differential dF has nullity m at (u,t), where nullity indicates the dimension of the kernel of dF.

The next proposition relates the shape operator S and the above definition.

Proposition 3.1.2. ((CECIL, 1994), Proposition 1) Let $\mathbf{x}: U \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface with transversal equiaffine vector field $\boldsymbol{\xi}$. Let S be the shape operator related to M and $\boldsymbol{\xi}$. A point $p = F(u,t)$ is a focal point of M of multiplicity $m > 0$ if and only if $1/t$ is an eigenvalue of S with eigenspace of dimension m at u.

For each $u \in U$ and $p \in \mathbb{R}^{n+1}$, we decompose $p - \mathbf{x}(u)$ into tangential and transversal components as follows

$$p - \mathbf{x}(u) = v(u) + \rho_p(u)\boldsymbol{\xi}(u), \tag{3.3}$$

where $v(u) \in T_{\mathbf{x}(u)}M$. The real function ρ_p is called an *affine support function* associated to M and $\boldsymbol{\xi}$.

Definition 3.1.2. Let $\mathbf{x}: U \to \mathbb{R}^{n+1}$, with $\mathbf{x}(U) = M$, be a non-degenerate hypersurface and take $\boldsymbol{\xi}: U \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ an equiaffine transversal vector field. Define $\boldsymbol{\nu}: U \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, such that for each $\mathbf{x}(u) = p \in M$ and $v \in T_p(M)$

$$\langle \boldsymbol{\nu}(u), \boldsymbol{\xi}(u) \rangle = 1 \text{ and } \langle \boldsymbol{\nu}(u), v \rangle = 0. \tag{3.4}$$

Each $\boldsymbol{\nu}(u)$ is called the *conormal vector* of $\mathbf{x}$ relative to $\boldsymbol{\xi}$ at p. The map $\boldsymbol{\nu}$ is called the *conormal map* of $\mathbf{x}$ relative to $\boldsymbol{\xi}$.

Remark 3.1.1. Using (3.3) and (3.4), we obtain

$$\rho_p(u) = \langle p - \mathbf{x}(u), \boldsymbol{\nu}(u) \rangle,$$

where ρ_p is the affine support function.

Proposition 3.1.3. ((CECIL, 1994), Proposition 2) Let $\mathbf{x}: U \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface and $\boldsymbol{\xi}$ an equiaffine transversal vector field. Then

a) The affine support function ρ_p has a critical point at u if and only if $p - \mathbf{x}(u)$ is a multiple of $\boldsymbol{\xi}(u)$.

b) If u is a critical point of ρ_p, then the Hessian of ρ_p at u has the form

$$H(X,Y) = \mathbf{c}(X, (I - \rho_p(u)S)Y), \ X, Y \in T_{\mathbf{x}(u)}M.$$

c) A critical point u of the function ρ_p is degenerate if and only if p is a focal point of M.

Given a non-degenerate hypersurface $\mathbf{x}(U) = M$, we know that the affine fundamental form $\mathbf{c}$ is non-degenerate, then it can be treated as a non-degenerate metric (not necessarily positive-definite) on M.

Definition 3.1.3. Let $\mathbf{x} : U \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface. A transversal vector field $\boldsymbol{\xi} : U \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ satisfying

a) $\boldsymbol{\xi}$ is equiaffine.

b) The induced volume element θ coincides with the volume element $\omega_{\mathbf{c}}$ of the non-degenerate metric $\mathbf{c}$.

is called *the Blaschke normal vector field* of M.

Remark 3.1.2. Given a non-degenerate hypersurface $\mathbf{x}(U) = M$, its Blaschke vector field is unique up to sign and is given by

$$\boldsymbol{\xi}(u) = |K(u)|^{1/n+2} N(u) + Z(u), \tag{3.5}$$

where K is the Gaussian curvature of M, N its unit normal and Z is a vector field on M, such that

$$II(Z,X) = -X(|K|^{1/n+2}), \forall X \in TM \tag{3.6}$$

where II denotes the second fundamental form of M (for details, see page 45 item (5) in (NOMIZU; KATSUMI; SASAKI, 1994)). Using (3.6) we can also write the vector field Z in terms of the coefficients of the second fundamental form and the partial derivatives of $|K|^{1/n+2}$. If we take the case $n = 2$, then $Z = a\mathbf{x}_{u_1} + b\mathbf{x}_{u_2}$, where

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}^{-1} \begin{pmatrix} -(|K|^{1/4})_{u_1} \\ -(|K|^{1/4})_{u_2} \end{pmatrix}. \tag{3.7}$$

Moreover, from (3.5) it follows that the conormal vector relative to the Blaschke vector field of a non-degenerate hypersurface in $\mathbb{R}^{n+1}$ is given by

$$\boldsymbol{\nu}(u) = |K(u)|^{-1/n+2} N(u) \tag{3.8}$$

3.2 2-parameter line congruences in $\mathbb{R}^3$

Here, we summarize some results from (IZUMIYA; SAJI; TAKEUCHI, 2003) which are generalized in chapter 4 to the case of 3-parameter line congruences in $\mathbb{R}^4$. We state in this section the conjecture 3.2.1 from (IZUMIYA; SAJI; TAKEUCHI, 2003), for which we give a positive answer in chapter 4. Along this section, U denotes an open subset of $\mathbb{R}^2$.

Definition 3.2.1. A *2-parameter line congruence in* $\mathbb{R}^3$ is a 2-parameter family of lines in $\mathbb{R}^3$. Locally, we write $\mathscr{C} = \{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$ and the line congruence is given by a smooth map

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I &\to \mathbb{R}^3 \\ (u,t) &\mapsto F(u,t) = \mathbf{x}(u) + t\boldsymbol{\xi}(u), \end{aligned}$$

where

- $\mathbf{x}: U \to \mathbb{R}^3$ is smooth and it is called a *reference surface of the congruence*;
- $\boldsymbol{\xi}: U \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is smooth and it is called the *director surface of the congruence*.

When there is no risk of confusion, we denote the line congruence just by F instead of $F_{(\mathbf{x},\boldsymbol{\xi})}$.

Lemma 3.2.1. The singular points of a line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ are the points (u,t) such that

$$t^2\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2}\rangle + t\langle \boldsymbol{\xi}, \mathbf{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} + \boldsymbol{\xi}_{u_1} \wedge \mathbf{x}_{u_2}\rangle + \langle \boldsymbol{\xi}, \mathbf{x}_{u_1} \wedge \mathbf{x}_{u_2}\rangle = 0.$$

Proof. The jacobian matrix of F is

$$JF = \begin{pmatrix} \mathbf{x}_{u_1} + t\boldsymbol{\xi}_{u_1} & \mathbf{x}_{u_2} + t\boldsymbol{\xi}_{u_2} & \boldsymbol{\xi} \end{pmatrix}.$$

As we know, (u,t) is a singular point of F if, and only if, $\det JF(u,t) = 0$, thus the result follows from

$$\det JF(u,t) = \langle \boldsymbol{\xi}, (\mathbf{x}_{u_1} + t\boldsymbol{\xi}_{u_1}) \wedge (\mathbf{x}_{u_2} + t\boldsymbol{\xi}_{u_2})\rangle = 0.$$

□

Definition 3.2.2. We say that $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$ is a *focal hypersurface* of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ if

$$\langle \boldsymbol{\xi}(u), \mathbf{y}_{u_1} \wedge \mathbf{y}_{u_2}\rangle = 0. \tag{3.9}$$

If $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$ is a focal hypersurface of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ then

$$t^2\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2}\rangle + t\langle \boldsymbol{\xi}, \mathbf{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} + \boldsymbol{\xi}_{u_1} \wedge \mathbf{x}_{u_2}\rangle + \langle \boldsymbol{\xi}, \mathbf{x}_{u_1} \wedge \mathbf{x}_{u_2}\rangle = 0.$$

3.2.1 2-parameter line congruences from the singularity theory viewpoint

In (IZUMIYA; SAJI; TAKEUCHI, 2003) the authors seek to classify the singularities of 2-parameter line congruences in $\mathbb{R}^3$. In order to do this, they consider some classes of congruences, like general line congruences, i.e., those for which there are no restrictions on $\mathbf{x}$ and $\boldsymbol{\xi}$, normal congruences and Blaschke affine normal congruences. First, the singularities of a general line congruence are classified in the following theorem.

Theorem 3.2.1. ((IZUMIYA; SAJI; TAKEUCHI, 2003), Theorem 1.2) There exists an open dense subset $\mathcal{O} \subset C^\infty(U, \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\mathbf{0}\})$ such that the germ of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0) \in U \times I$ is an immersive germ, or $\mathcal{A}$-equivalent to the fold, the cuspidal edge or the swallowtail for any $(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{O}$.

Here, the fold is the map germ defined by $(x,y,z) \mapsto (x,y,z^2)$, the cuspidal edge is the map germ defined by $(x,y,z) \mapsto (x,y,z^3+xz)$ and the swallowtail is defined by $(x,y,z) \mapsto (x,y,z^4+xz+yz^2)$.

An important and natural class of line congruence is the class of normal congruences, defined as follows.

Definition 3.2.3. A 3-parameter line congruence $\mathscr{C} = \{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$, for $u \in U \subset \mathbb{R}^2$, is said to be *normal* if for each point $u_0 \in U$ there is a neighborhood $\tilde{U}$ of u_0 and a regular hypersurface, given by $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$, whose normal vectors are parallel to $\boldsymbol{\xi}(u)$, for all $u \in \tilde{U}$. The congruence is an *exact normal* congruence if $\boldsymbol{\xi}(u)$ is a normal vector at $\mathbf{x}(u)$, for all $u \in U$.

The next proposition characterizes 2-parameter normal line congruences in $\mathbb{R}^3$.

Proposition 3.2.1. ((IZUMIYA; SAJI; TAKEUCHI, 2003), Proposition 5.1) A line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ is normal if and only if

$$\left\langle \mathbf{x}_{u_1}, \left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2} \right\rangle = \left\langle \mathbf{x}_{u_2}, \left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1} \right\rangle$$

.

Let us denote the space of the normal congruences by $N(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\}))$ with the Whitney C^∞-topology induced from $C^\infty\left(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\})\right)$. Then, we have the following theorem.

Theorem 3.2.2. ((IZUMIYA; SAJI; TAKEUCHI, 2003), Theorem 5.7) There exists an open dense subset $\mathcal{O} \subset N(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\}))$ such that the germ of the normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}$. Therefore, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is $\mathcal{A}$-equivalent to an immersion germ, the fold, the cuspidal edge, the swallowtail, the pyramid or the purse for any $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}$.

Another important class of line congruences considered in (IZUMIYA; SAJI; TAKEUCHI, 2003) is the Blaschke affine line congruence, i.e., the congruence given by a regular non-degenerate surface $\mathbf{x} : U \to \mathbb{R}^3$ and its Blaschke vector field $\boldsymbol{\xi} : U \to \mathbb{R}^3$ defined in definition 5.2.1. The authors observed that the affine evolute of a non-degenerate plane curve is the caustic of a certain Lagrangian submanifold in $T^*\mathbb{R}^2$. Using some results proved for normal congruences, the authors show some relations which suggest that something similar might occur for the case of non-parabolic surfaces (see section 6 in (IZUMIYA; SAJI; TAKEUCHI, 2003) for details). Taking this into account, they present the following conjecture, for which we give a positive answer in corollary 4.4.1.

Conjecture. ((IZUMIYA; SAJI; TAKEUCHI, 2003), Conjecture 6.5) Germs of generic Blaschke affine normal congruences at any points are Lagrangian stable.

3.3 Kummer's theory for 2-parameter line congruences in $\mathbb{R}^3$

Here, taking into account a Euclidean approach to line congruences $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$, where $\mathbf{x} : U \to \mathbb{R}^3$, $\boldsymbol{\xi} : U \to \mathbb{S}^2$ are smooth maps and $\boldsymbol{\xi}$ is an immersion, we present the theory developed by Ernst Eduard Kummer for line congruences. More content about line congruences can be found in (BIANCHI, 1894), (EISENHART, 1909) and (WEATHERBURN, 1955). We start by defining two quadratic forms associated to the line congruence $\mathcal{C}$.

Definition 3.3.1. Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence defined on U, an open subset of $\mathbb{R}^2$ and $S = \mathbf{x}(U)$ a regular surface. Let $\alpha : I \to S$, where I is an open interval, be a regular curve parametrized by arc length, such that $\alpha(s) = \mathbf{x}(u_1(s), u_2(s))$. If $\boldsymbol{\xi}(s) = \boldsymbol{\xi}(u_1(s), u_2(s))$, where $q = (u_1(0), u_2(0))$ and $\mathbf{v} = u_1'(0)\mathbf{x}_{u_1}(q) + u_2'(0)\mathbf{x}_{u_2}(q) \in T_pS$, where $\mathbf{x}(q) = p$, then we associate to $\mathcal{C}$ two quadratic forms, as follows:

(I) Kummer first fundamental form :

$$\begin{aligned} \mathcal{I}_p : T_pS &\to \mathbb{R} \\ \mathbf{v} &\mapsto \mathcal{I}_p(\mathbf{v}) = \mathscr{E}u_1'^2 + 2\mathscr{F}u_1'u_2' + \mathscr{G}u_2'^2, \end{aligned} \tag{3.10}$$

where $\mathscr{E} = \langle \boldsymbol{\xi}_{u_1}, \boldsymbol{\xi}_{u_1} \rangle$, $\mathscr{F} = \langle \boldsymbol{\xi}_{u_1}, \boldsymbol{\xi}_{u_2} \rangle$ and $\mathscr{G} = \langle \boldsymbol{\xi}_{u_2}, \boldsymbol{\xi}_{u_2} \rangle$. We denote by $\boldsymbol{\mathcal{I}}$ the associated matrix.

(II) Kummer second fundamental form:

$$\begin{aligned} \mathcal{II}_p : T_pS &\to \mathbb{R} \\ \mathbf{v} &\mapsto \mathcal{II}_p(\mathbf{v}) = \mathscr{L}u_1'^2 + (\mathscr{M}_1 + \mathscr{M}_2)u_1'u_2' + \mathscr{N}u_2'^2, \end{aligned} \tag{3.11}$$

where $\mathscr{L} = -\langle \mathbf{x}_{u_1}, \boldsymbol{\xi}_{u_1} \rangle$, $\mathscr{M}_2 = -\langle \mathbf{x}_{u_1}, \boldsymbol{\xi}_{u_2} \rangle$, $\mathscr{M}_1 = -\langle \mathbf{x}_{u_2}, \boldsymbol{\xi}_{u_1} \rangle$ and $\mathscr{N} = -\langle \mathbf{x}_{u_2}, \boldsymbol{\xi}_{u_2} \rangle$. We denote by $\boldsymbol{\mathcal{II}} = -D\boldsymbol{\xi}^T Dx$ the associated matrix.

Given a line $\{\mathbf{x}(q), \boldsymbol{\xi}(q)\}$ of the congruence $\mathcal{C}$, its *spherical representation* is given by the point $\boldsymbol{\xi}(q) \in \mathbb{S}^2$. If we have a curve C on the reference surface S its spherical representation is the curve on $\mathbb{S}^2$ given by the spherical representations of all the lines of the congruence through C.

Definition 3.3.2. The lines of the congruence passing through a curve C on the reference surface S form a ruled surface S_C called *surface of the congruence*.

3.3.1 Limit points and Kummer principal lines

If C is given by $\mathbf{x}(t) = \mathbf{x} \circ \alpha(t)$ where $\alpha(t) = (u_1(t), u_2(t))$ and $\boldsymbol{\xi}(t) = \boldsymbol{\xi} \circ \alpha(t)$, the surface of the congruence S_C can be written as

$$Y(t, w) = \mathbf{x}(t) + w\boldsymbol{\xi}(t), t \in I, w \in \mathbb{R}, \tag{3.12}$$

where the curve $\mathbf{x}(t)$ is called a *directrix* of S_C and for each fixed t the line L_t, which pass through $\alpha(t)$ and is parallel to $\boldsymbol{\xi}(t)$, is called a *generator* of the ruled surface S_C. If $\|\boldsymbol{\xi}(t)\| = 1$, we say that $\boldsymbol{\xi}(t)$ is the *spherical representation* of S_C. Since $\boldsymbol{\xi}$ is an immersion, $\|\boldsymbol{\xi}'(t)\| \neq 0$. Suppose $\|\boldsymbol{\xi}(t)\| = 1$, so the ruled surface considered is non-cylindrical. It is known (see section 3.5 in (CARMO, 2016)) that there exists a curve $\beta : I \to \mathbb{R}^3$, contained in the ruled surface S_C, parametrized by

$$\beta(t) = \mathbf{x}(t) + k(t)\boldsymbol{\xi}(t), \tag{3.13}$$

where $k(t) = -\dfrac{\langle \mathbf{x}'(t), \boldsymbol{\xi}'(t)\rangle}{\langle \boldsymbol{\xi}'(t), \boldsymbol{\xi}'(t)\rangle}$, whose tangent vector satisfies

$$\langle \beta'(t), \boldsymbol{\xi}'(t)\rangle = 0. \tag{3.14}$$

This special curve is called *striction line*. The intersection point of a generator with the striction line is called the *central point* of the generator. Given a generator L_t the coordinate of its central point is $k(t)$, given in (3.13)

Let $q = (u_1(0), u_2(0))$ and note that

$$\begin{aligned}
k(0) &= -\frac{\langle \mathbf{x}'(0), \boldsymbol{\xi}'(0)\rangle}{\langle \boldsymbol{\xi}'(0), \boldsymbol{\xi}'(0)\rangle} \\
&= -\frac{\langle u_1'(0)\mathbf{x}_{u_1}(q) + u_2'(0)\mathbf{x}_{u_2}(q), u_1'(0)\boldsymbol{\xi}_{u_1}(q) + u_2'(0)\boldsymbol{\xi}_{u_2}(q)\rangle}{\langle u_1'(0)\boldsymbol{\xi}_{u_1}(q) + u_2'(0)\boldsymbol{\xi}_{u_2}(q), u_1'(0)\boldsymbol{\xi}_{u_1}(q) + u_2'(0)\boldsymbol{\xi}_{u_2}(q)\rangle} \\
&= \frac{\mathscr{L} u_1'^2 + (\mathscr{M}_1 + \mathscr{M}_2) u_1' u_2' + \mathscr{N} u_2'^2}{\mathscr{E} u_1'^2 + 2\mathscr{F} u_1' u_2' + \mathscr{G} u_2'^2} \\
&= \frac{\mathcal{II}_p}{\mathcal{I}_p}, \text{ where } p = \mathbf{x}(q).
\end{aligned}$$

If we associate to $\mathbf{v} = \alpha'(0) = u_1'(0)\mathbf{x}_{u_1}(q) + u_2'(0)\mathbf{x}_{u_2}(q)$ its coordinates $(u_1'(0), u_2'(0))$, then it is possible to look at k as a function defined in T_pS, i.e

$$\mathscr{K}_p : T_pS \to \mathbb{R} \tag{3.15}$$

$$\mathbf{v} \mapsto \mathscr{K}_p(\mathbf{v}) = \frac{\mathcal{II}_p(\mathbf{v})}{\mathcal{I}_p(\mathbf{v})}, \tag{3.16}$$

which provides the coordinate of the central point of the generator L_0 associated to the surface of the congruence S_C. If we restrict $\mathscr{K}_p$ to a compact set, then this function have a maximum and a minimum values, which we denote by $\mathscr{K}_1$ and $\mathscr{K}_2$. Note that (3.15) depends only on p and the direction $\mathbf{v}$, so we can take the restriction $\mathscr{K}_p : \mathbb{S}^1 \to \mathbb{R}$.

Proposition 3.3.1. The extreme values of $\mathscr{K} : \mathbb{S}^1 \to \mathbb{R}$, denoted by $\mathscr{K}_1$ and $\mathscr{K}_2$, satisfy

$$\mathscr{K}_1 + \mathscr{K}_2 = \frac{-\mathscr{F}(\mathscr{M}_1 + \mathscr{M}_2) + \mathscr{G}\mathscr{L} + \mathscr{E}\mathscr{N}}{\mathscr{E}\mathscr{G} - \mathscr{F}^2} \tag{3.17a}$$

$$\mathscr{K}_1\mathscr{K}_2 = \frac{4\mathscr{L}\mathscr{N} - (\mathscr{M}_1 + \mathscr{M}_2)^2}{4(\mathscr{E}\mathscr{G} - \mathscr{F}^2)}. \tag{3.17b}$$

Proof. Suppose $\mathscr{K}_0 = \mathscr{K}_p(\lambda_0, \mu_0)$ an extreme value of

$$\mathscr{K}_p(\lambda,\mu) = \frac{\mathscr{L}\lambda^2 + (\mathscr{M}_1 + \mathscr{M}_2)\lambda\mu + \mathscr{N}\mu^2}{\mathscr{E}\lambda^2 + 2\mathscr{F}\lambda\mu + \mathscr{G}\mu^2},$$

where $(\lambda,\mu) \in \mathbb{S}^1$. Then,

$$0 = \frac{\partial \mathscr{K}_p}{\partial \lambda}(\lambda_0,\mu_0) = \lambda_0(-2\mathscr{L} + 2\mathscr{K}_0\mathscr{E}) + \mu_0(2\mathscr{K}_0\mathscr{F} - (\mathscr{M}_1 + \mathscr{M}_2)) \tag{3.18a}$$

$$0 = \frac{\partial \mathscr{K}_p}{\partial \mu}(\lambda_0,\mu_0) = \lambda_0(-(\mathscr{M}_1 + \mathscr{M}_2) + 2\mathscr{K}_0\mathscr{F}) + \mu_0(2\mathscr{K}_0\mathscr{G} - 2\mathscr{N}). \tag{3.18b}$$

From (3.18), we get

$$\begin{vmatrix} -2\mathscr{L} + 2\mathscr{K}_0\mathscr{E} & 2\mathscr{K}_0\mathscr{F} - (\mathscr{M}_1 + \mathscr{M}_2) \\ 2\mathscr{K}_0\mathscr{F} - (\mathscr{M}_1 + \mathscr{M}_2) & 2\mathscr{K}_0\mathscr{G} - 2\mathscr{N} \end{vmatrix} = 0,$$

thus,

$$\mathscr{K}_0^2(\mathscr{E}\mathscr{G} - \mathscr{F}^2) + \mathscr{K}_0(-\mathscr{E}\mathscr{N} - \mathscr{L}\mathscr{G} + (\mathscr{M}_1 + \mathscr{M}_2)\mathscr{F}) + \mathscr{L}\mathscr{N} - \frac{(\mathscr{M}_1 + \mathscr{M}_2)^2}{4} = 0.$$

From the above equation, we obtain (3.17). □

With notation as in proposition 3.3.1, the points on the line $\{\mathbf{x}(q), \boldsymbol{\xi}(q)\}$, where $\mathbf{x}(q) = p$, determined by $\mathscr{K}_1$ and $\mathscr{K}_2$ are called its limit points. They are boundaries of the segment of the line containing all other central points, associated to different directions taken in T_pS. If we take $\mathbf{x}(s) = \mathbf{x}(u_1(s), u_2(s))$ a regular curve on S, such that $(u_1'(s), u_2'(s))$ is associated to extreme values of $\mathscr{K}_{\mathbf{x}(t)}$, for all $t \in I$, an open interval, then from (3.18) isolating $\mathscr{K}_0$, we obtain the following binary differential equation

$$(2\mathscr{F}\mathscr{L} - \mathscr{E}(\mathscr{M}_1 + \mathscr{M}_2))u_1'^2 + 2(\mathscr{G}\mathscr{L} - \mathscr{E}\mathscr{N})u_1'u_2' + (\mathscr{G}(\mathscr{M}_1 + \mathscr{M}_2) - \mathscr{F}\mathscr{N})u_2'^2 = 0. \tag{3.19}$$

The curves which are solutions of (3.19) are called *Kummer principal lines*. A direction $\mathbf{v} \in T_pS \setminus \{\mathbf{0}\}$ associated to an extreme value of $\mathscr{K}_p$ is called a *Kummer principal direction* at p.

3.3.2 Focal points and developable surfaces of the congruence

Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence defined in U an open subset of $\mathbb{R}^2$, such that $\|\boldsymbol{\xi}\| = 1$. We can also look at the surfaces of the congruence which are developable. In this case, the central points are called *focal points* and the surface of the congruence is tangent to its striction line. If we parametrize the striction line by

$$\beta(s) = \mathbf{x}(u_1(s), u_2(s)) + \rho(u_1(s), u_2(s))\boldsymbol{\xi}(u_1(s), u_2(s)),$$

then β' is parallel to $\boldsymbol{\xi}$. Above, $\rho(u_1(s), u_2(s))$ is the coordinate of the focal point, for each $s \in I$, an open interval. Since $\boldsymbol{\xi}$ is unitary and β is parallel to $\boldsymbol{\xi}$, we get that $\langle \beta', \boldsymbol{\xi}_{u_1}\rangle = \langle \beta', \boldsymbol{\xi}_{u_2}\rangle = 0$, which is equivalent to

$$-(\mathscr{L}u_1' + \mathscr{M}_1u_2') + \rho(\mathscr{E}u_1' + \mathscr{F}u_2') = 0 \tag{3.20a}$$

$$-(\mathscr{M}_2u_1' + \mathscr{N}u_2') + \rho(\mathscr{F}u_1' + \mathscr{G}u_2') = 0. \tag{3.20b}$$

Proposition 3.3.2. The coordinates of the focal points of a given line of the congruence, denoted by ρ_1 andn ρ_2, satisfy:

$$\rho_1 + \rho_2 = \frac{-\mathscr{F}(\mathscr{M}_1 + \mathscr{M}_2) + \mathscr{G}\mathscr{L} + \mathscr{E}\mathscr{N}}{\mathscr{E}\mathscr{G} - \mathscr{F}^2} \tag{3.21a}$$

$$\rho_1\rho_2 = \frac{\mathscr{L}\mathscr{N} - \mathscr{M}_1\mathscr{M}_2}{\mathscr{E}\mathscr{G} - \mathscr{F}^2}. \tag{3.21b}$$

Proof. From (3.20), we get

$$-u_1'(-\mathscr{L} + \rho\mathscr{E}) = u_2'(\rho\mathscr{F} - \mathscr{M}_2)$$
$$-u_1'(-\mathscr{M}_1 + \rho\mathscr{F}) = u_2'(\rho\mathscr{G} - \mathscr{N}),$$

then

$$\begin{vmatrix} -\mathscr{L} + \rho\mathscr{E} & (-\mathscr{M}_1 + \rho\mathscr{F} \\ \rho\mathscr{F} - \mathscr{M}_2 & \rho\mathscr{G} - \mathscr{N} \end{vmatrix} = 0,$$

thus,

$$\rho^2(\mathscr{E}\mathscr{G} - \mathscr{F}^2) + \rho(-\mathscr{G}\mathscr{L} - \mathscr{E}\mathscr{N} + \mathscr{M}_1\mathscr{F} + \mathscr{M}_2\mathscr{F}) + \mathscr{L}\mathscr{N} - \mathscr{M}_1\mathscr{M}_2 = 0.$$

From the above equation, we get (3.21).

□

Proposition 3.3.3. Each line of the congruence admits at most two developable surfaces of the congruence through it.

Proof. From (3.20) we get

$$\begin{vmatrix} -\mathscr{L}u_1' - \mathscr{M}_2u_2' & -\mathscr{M}_1u_1' - \mathscr{N}u_2' \\ \mathscr{E}u_1' + \mathscr{F}u_2' & \mathscr{F}u_1' + \mathscr{G}u_2' \end{vmatrix} = 0$$

and form this determinant, we have the following binary differential equation

$$(-\mathscr{F}\mathscr{L} + \mathscr{E}\mathscr{M}_1)u_1'^2 + (-\mathscr{F}\mathscr{M}_2 - \mathscr{G}\mathscr{L} + \mathscr{E}\mathscr{N} + \mathscr{F}\mathscr{M}_1)u_1'u_2' + (-\mathscr{G}\mathscr{M}_2 + \mathscr{F}\mathscr{N})u_2'^2 = 0. \tag{3.22}$$

Thus, through each line of the congruence, we have possibly two curves which are directrices of developable surfaces of the congruence. □

We call equation (6.13) the *equation of the developable surfaces of the congruence*. It follows from proposition 3.2.1 that a 2-parameter line congruence $\{\mathbf{x}, \boldsymbol{\xi}\}$ is normal if and only if $\mathscr{M}_1 = \mathscr{M}_2$. Furthermore, if we compare (3.17) and (3.21), we have that, for normal congruences, the focal points and the limit points coincide. It follows also that he same happens to the directrices of the developable surfaces of the congruence and the Kummer principal lines, since the equations (6.13) and (3.19) are the same for $\mathscr{M}_1 = \mathscr{M}_2$.

Another important result from Kummer's theory is the following one, which relates the lines of curvature of a regular surface $\mathbf{x}: U \to \mathbb{R}^3$ to the Kummer principal lines, when considering the congruence given by $\mathbf{x}$ and its unit normal vector field. In this case, we say that the congruence is an *exact normal congruence*.

Proposition 3.3.4. Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be an exact normal congruence. A curve C on the reference surface parametrized by $\mathbf{x} \circ \alpha : I \to \mathbb{R}^3$, where $\alpha(s) = (u_1(s), u_2(s))$ is such that $(u_1', u_2') \neq (0,0)$, is a line of curvature if and only if the surface of the congruence $Y(s,w) = \mathbf{x}(s) + w\boldsymbol{\xi}(s)$ is developable.

Proof. Let S_C be the surface of the congruence parametrized by $Y(s,w) = \mathbf{x}(s) + w\boldsymbol{\xi}(s)$. It is known that the ruled surface S_C is developable if and only if $\left[\mathbf{x}', \boldsymbol{\xi}', \boldsymbol{\xi}\right] = 0$. We know that $\|\boldsymbol{\xi}\| = 1$ and $\langle \mathbf{x}', \boldsymbol{\xi} \rangle = 0$, thus $\left[\mathbf{x}', \boldsymbol{\xi}', \boldsymbol{\xi}\right] = 0$ if and only if $\boldsymbol{\xi}'(s) = k(s)\mathbf{x}'(s)$ and from Rodrigues' curvature formula (see section 3.2 in (CARMO, 2016)), $\mathbf{x}(s)$ is a line of curvature. □

3.4 Frontals

As we seek to study frontals from a differential affine geometry viewpoint in chapter 5 and generalize Kummer's theory for line congruences $\{\mathbf{x}, \boldsymbol{\xi}\}$, where $\mathbf{x}: U \to \mathbb{R}^3$ is a smooth map and $\boldsymbol{\xi}: U \to \mathbb{R}^3$ is a frontal in chapter 6, we give in this section some important notions related to this special class of singular surfaces. The main references in this section are (MEDINA-TEJEDA, 2022a) and (MEDINA-TEJEDA, 2022b). For properties of frontals and their geometrical invariants we also refer to (ISHIKAWA, 2018), (MARTINS *et al.*, 2016), (SAJI; TERAMOTO, 2021) and (SAJI; UMEHARA; YAMADA, 2009). A smooth map $\mathbf{x}: U \to \mathbb{R}^3$ is said to be a *frontal* if, for all $q \in U$ there is a vector field $\mathbf{n}: U_q \to \mathbb{R}^3$ where $q \in U_q$ is an open subset of U, such that $\|\mathbf{n}\| = 1$ and $\langle \mathbf{x}_{u_i}(u), \mathbf{n}(u) \rangle = 0$, for all $u \in U_q$, $i = 1,2$. This vector field is said to be a unit normal vector field along $\mathbf{x}$. We say that a frontal $\mathbf{x}$ is a *wave front* if the map $(\mathbf{x}, \mathbf{n}): U \to \mathbb{R}^3 \times \mathbb{S}^2$ is an immersion for all $q \in U$. Here, we consider mainly *proper frontals*, that is, frontals $\mathbf{x}$ for which the singular set $\Sigma(\mathbf{x}) = \{q \in U : \mathbf{x} \text{ is not immersive in } q\}$ has empty interior. This is equivalent to say that $U \setminus \Sigma(\mathbf{x})$ is an open dense set in U.

Definition 3.4.1. We call *moving basis* a smooth map $\boldsymbol{\Omega}: U \to \mathcal{M}_{3\times 2}(\mathbb{R})$ in which the columns $\mathbf{w}_1, \mathbf{w}_2 : U \to \mathbb{R}^3$ of the matrix $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ are linearly independent vector fields.

Definition 3.4.2. We call a *tangent moving basis* (tmb) of $\mathbf{x}$ a moving basis $\boldsymbol{\Omega} = (\mathbf{w}_1, \mathbf{w}_2)$, such that $\mathbf{x}_{u_1}, \mathbf{x}_{u_2} \in \langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbb{R}}$, where $\langle\ ,\rangle_{\mathbb{R}}$ denotes the linear span $\mathbb{R}$-vector space.

Next proposition provides a characterization of frontals in terms of tangent moving basis.

Proposition 3.4.1. ((MEDINA-TEJEDA, 2022a), Proposition 3.2) Let $\mathbf{x}: U \to \mathbb{R}^3$ be a smooth map with $U \subset \mathbb{R}^2$ an open set. Then, $\mathbf{x}$ is a frontal if and only if, for all $q \in U$, there are smooth maps $\boldsymbol{\Omega}: U_q \to \mathcal{M}_{3\times 2}(\mathbb{R})$ and $\boldsymbol{\Lambda}: U_q \to \mathcal{M}_{2\times 2}(\mathbb{R})$ with $\operatorname{rank}(\boldsymbol{\Omega}) = 2$ and $U_q \subset U$ a neighborhood of q, such that $D\mathbf{x}(\tilde{q}) = \boldsymbol{\Omega}\boldsymbol{\Lambda}_{\Omega}^T$, for all $\tilde{q} \in U_q$.

Since a tangent moving basis exist locally and we want to describe local properties, from now on we suppose that for a given frontal we have a global tangent moving basis. Then, if a frontal $\mathbf{x}$ satisfies $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, where $\mathbf{\Omega}$ is a tangent moving basis, we have that $\Sigma(\mathbf{x}) = \lambda_\Omega^{-1}(0)$, where $\lambda_\Omega := \det \mathbf{\Lambda}$.

Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ a tmb and denote by $\mathbf{n} = \dfrac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}$ the unit normal vector field induced by $\mathbf{\Omega}$. We set the matrices

$$\mathbf{I}_\Omega := \mathbf{\Omega}^T\mathbf{\Omega} = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} = \begin{pmatrix} \langle \mathbf{w}_1, \mathbf{w}_1 \rangle & \langle \mathbf{w}_1, \mathbf{w}_2 \rangle \\ \langle \mathbf{w}_2, \mathbf{w}_1 \rangle & \langle \mathbf{w}_2, \mathbf{w}_2 \rangle \end{pmatrix}, \tag{3.23}$$

$$\mathbf{II}_\Omega := -\mathbf{\Omega}^T D\mathbf{n} = \begin{pmatrix} e_\Omega & f_{1\Omega} \\ f_{2\Omega} & g_\Omega \end{pmatrix} = \begin{pmatrix} -\langle \mathbf{w}_1, \mathbf{n}_{u_1} \rangle & -\langle \mathbf{w}_1, \mathbf{n}_{u_2} \rangle \\ -\langle \mathbf{w}_2, \mathbf{n}_{u_1} \rangle & -\langle \mathbf{w}_2, \mathbf{n}_{u_2} \rangle \end{pmatrix}, \tag{3.24}$$

$$\boldsymbol{\mu}_\Omega := -\mathbf{II}_\Omega^T \mathbf{I}_\Omega^{-1}, \tag{3.25}$$

$$\boldsymbol{\alpha}_\Omega := \boldsymbol{\mu}_\Omega adj(\mathbf{\Lambda}_\Omega), \tag{3.26}$$

$$\mathcal{T}_1 := (\mathbf{\Omega}_{u_1}^T \mathbf{\Omega})\mathbf{I}_\Omega^{-1} = \begin{pmatrix} \mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^1 \end{pmatrix}, \tag{3.27}$$

$$\mathcal{T}_2 := (\mathbf{\Omega}_{u_2}^T \mathbf{\Omega})\mathbf{I}_\Omega^{-1} = \begin{pmatrix} \mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^1 \end{pmatrix}. \tag{3.28}$$

Given a frontal $\mathbf{x} : U \to \mathbb{R}^3$ and a tmb $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ it follows that $\langle \mathbf{w}_1, \mathbf{n} \rangle = \langle \mathbf{w}_2, \mathbf{n} \rangle = 0$. By taking this, we can rewrite

$$\mathbf{II}_\Omega = \begin{pmatrix} \langle (\mathbf{w}_1)_{u_1}, \mathbf{n} \rangle & \langle (\mathbf{w}_1)_{u_2}, \mathbf{n} \rangle \\ \langle (\mathbf{w}_2)_{u_1}, \mathbf{n} \rangle & \langle (\mathbf{w}_2)_{u_2}, \mathbf{n} \rangle \end{pmatrix}. \tag{3.29}$$

Remark 3.4.1. With notation as above, if $\mathbf{\Omega}$ is a tangent moving base of x, we have the decomposition $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}_\Omega^T$, then $\mathbf{\Lambda}_\Omega = D\mathbf{x}^T \mathbf{\Omega}\mathbf{I}_\Omega^{-1}$, namely $\mathbf{\Lambda}_\Omega$ is completely determined by x and $\mathbf{\Omega}$, therefore from now on, it will denote this matrix valued map. Also we write $T_\Omega = [w_1, w_2]_\mathbb{R}$ the plane generated by w_1 and w_2. Note that given two tangent moving basis $\mathbf{\Omega}$ and $\tilde{\Omega}$ of a proper frontal, we have $T_\Omega = T_{\tilde{\Omega}}$.

Definition 3.4.3. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal and Ω a tangent moving basis of $\mathbf{x}$, we define the $\mathbf{\Omega}$-*relative curvature* $K_\Omega := \det(\boldsymbol{\mu}_\Omega)$ and the $\mathbf{\Omega}$-*relative mean curvature* $H_\Omega := -\frac{1}{2}tr(\boldsymbol{\alpha}_\Omega)$, where $tr()$ is the trace and $adj()$ is the adjoint of a matrix. Also we call the functions $k_{1\Omega} := H_\Omega - \sqrt{H_\Omega^2 - \lambda_\Omega K_\Omega}$ and $k_{2\Omega} := H_\Omega + \sqrt{H_\Omega^2 - \lambda_\Omega K_\Omega}$ the $\mathbf{\Omega}$-*relative principal curvatures*.

According to (MEDINA-TEJEDA, 2020) it is possible to define smooth functions $k_1, k_2 : U \setminus \Sigma(\mathbf{x}) \to \mathbb{R}$, related to $k_{1\Omega}$ and $k_{2\Omega}$, which do not depend on the chosen tangent moving basis inducing the same orientation of the normal vector field. These functions have similar properties to the classical principal curvatures and in the case of non-degenerate singularities coincide with those functions defined in (TERAMOTO, 2016) (equation (2.6)), via a suitable change of coordinates.

Given a frontal $\mathbf{x}: U \to \mathbb{R}^3$ with a global unit normal vector field $\mathbf{n}: U \to \mathbb{R}^3$, we can also consider the matrices

$$\mathbf{I} := D\mathbf{x}^T D\mathbf{x} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{u_1}, \mathbf{x}_{u_1} \rangle & \langle \mathbf{x}_{u_1}, \mathbf{x}_{u_2} \rangle \\ \langle \mathbf{x}_{u_1}, \mathbf{x}_{u_1} \rangle & \langle \mathbf{x}_{u_2}, \mathbf{x}_{u_2} \rangle \end{pmatrix}, \tag{3.30}$$

$$\mathbf{II} := -D\mathbf{x}^T D\mathbf{n} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -\langle \mathbf{x}_{u_1}, \mathbf{n}_{u_1} \rangle & -\langle \mathbf{x}_{u_1}, \mathbf{n}_{u_2} \rangle \\ -\langle \mathbf{x}_{u_2}, \mathbf{n}_{u_1} \rangle & -\langle \mathbf{x}_{u_2}, \mathbf{n}_{u_2} \rangle \end{pmatrix}. \tag{3.31}$$

If we decompose $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, then $\mathbf{I} = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T$ and $\mathbf{II} = \mathbf{\Lambda}\mathbf{II}_\Omega$. Also, the classical normal curvature at a regular point $q \in U \setminus \Sigma(\mathbf{x})$ is given by

$$k_q(\vartheta) := \frac{\vartheta^T \mathbf{II} \vartheta}{\vartheta^T \mathbf{I} \vartheta}, \tag{3.32}$$

where $\vartheta \in \mathbb{R}^2 \setminus \{0\}$ are the coordinates of a vector in the tangent plane in the basis $\begin{pmatrix} \mathbf{x}_{u_1} & \mathbf{x}_{u_2} \end{pmatrix}$.

Definition 3.4.4. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$, we define the $\mathbf{\Omega}$-relative normal curvature by

$$k_q^\Omega(b) := \frac{b^T \mathbf{II}_\Omega adj(\mathbf{\Lambda}_\Omega^T) b}{b^T \mathbf{I}_\Omega b},$$

where $q \in U$ and $b \in \mathbb{R}^2 \setminus \{0\}$ represent the coordinates in the basis $\mathbf{\Omega}$ of vectors in $T_\Omega(q)$.

The directions defined by the vectors $\mathbf{v} \in T_\Omega$ represented by b in which $k_p^\Omega(b)$ achieves an extreme value are called *principal directions*.

Definition 3.4.5. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal and $\gamma: (-\varepsilon, \varepsilon) \to U$ a smooth curve. We say that γ is a *line of curvature* of $\mathbf{x}$ if $(\mathbf{x} \circ \gamma)'(t)$ defines a principal direction for every t such that $(\mathbf{x} \circ \gamma)'(t) \neq 0$.

The next proposition, given in (MEDINA-TEJEDA, 2022b), provides a differential equation associated to lines of curvature.

Proposition 3.4.2. ((MEDINA-TEJEDA, 2022b), Corollary 5.1) Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal and $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$. A smooth curve $\gamma: (-\varepsilon, \varepsilon) \to U$ is a line of curvature if and only if $\lambda_\Omega(\gamma)\gamma'^T \mathbf{P} \boldsymbol{\alpha}_\Omega^T(\gamma)\gamma' = 0$ on $(-\varepsilon, \varepsilon)$, where

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3.4.1 Frontals with extendable Gaussian curvature

Now, taking into account the results in (MEDINA-TEJEDA, 2022b), we investigate some classes of frontals for which the Gaussian curvature admits a smooth extension. These classes play an important role in chapter 5, where we define the Blaschke vector field of a frontal. Other important references for the study of Gaussian curvature are (MARTINS *et al.*, 2016) and (SAJI; UMEHARA; YAMADA, 2009).

3.4.1.1 Frontals with extendable normal curvature

Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal with extendable normal curvature. It follows from corollary 3.1 in (MEDINA-TEJEDA, 2022b) that if $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ is a frontal with extendable normal curvature and 0 a singularity of rank 1, then its Gaussian curvature has a smooth extension. Furthermore, this extension is non-vanishing if and only if $\mathbf{x} \sim \mathbf{n}$, where $\mathbf{n}$ is the unit normal vector field of $\mathbf{x}$ and $\sim$ indicates that there are $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ tmb of $\mathbf{x}$ and $\mathbf{n}$, respectively and a smooth matrix valued map $\mathbf{B} : U \to GL(2,\mathbb{R})$, such that $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1 \mathbf{B}$, where $D\mathbf{x} = \mathbf{\Omega}_1 \mathbf{\Lambda}_1^T$ and $D\mathbf{n} = \mathbf{\Omega}_2 \mathbf{\Lambda}_2^T$. The next theorem characterizes proper frontals of rank 1 with extendable normal curvature.

Theorem 3.4.1. ((MEDINA-TEJEDA, 2022b), Theorem 3.2) Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a proper frontal with extendable normal curvature and 0 a singularity of rank 1, then after a rigid motion and a change of coordinates on a neighborhood of 0, $\mathbf{x}$ can be represented by the formula:

$$\mathbf{y} = (u_1, b(u_1,u_2), \int_0^{u_2} \left(\int_0^{t_2} h(u_1,t_1) b_{u_2}(u_1,t_1) dt_1 \right) b_{u_2}(u_1,t_2) dt_2 + \int_0^{u_2} \left(\int_0^{u_1} l(t_1) dt_1 \right) b_{u_2}(u_1,t_2) dt_2 \tag{3.33}$$

$$+ \int_0^{u_1} \left(\int_0^{t_2} l(t_1) dt_1 \right) b_{u_1}(t_2,0) dt_2 + \int_0^{u_1} \left(\int_0^{t_2} r(t_1) dt_1 \right) dt_2),$$

where b,h,l,r are smooth function on neighborhoods of the origin in each case.

3.4.1.2 Wave fronts with extendable Gaussian curvature

If we look at a germ of wave front $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$, such that $0 \in \Sigma(\mathbf{x})$ and $\operatorname{rank} D\mathbf{x}(0) = 1$, then it follows from remark 4.1 in (MEDINA-TEJEDA, 2022b) that up to an isometry, $\mathbf{x}$ is $\mathcal{R}$-equivalent to

$$\mathbf{y} = (u_1, -h_{u_2}(u_1,u_2), \int_0^{u_1} (h_{u_1}(t,u_2) - u_2 h_{u_2 u_1}(t,u_2)) dt - \int_0^{u_2} t h_{u_2 u_2}(0,t) dt). \tag{3.34}$$

Note that $D\mathbf{y}$ has decomposition $D\mathbf{y} = \mathbf{\Omega}\mathbf{\Lambda}^T$, where

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_1 & u_2 \end{pmatrix} \text{ and } \mathbf{\Lambda}^T = \begin{pmatrix} 1 & 0 \\ b_{u_1} & b_{u_2} \end{pmatrix}, \tag{3.35}$$

for h, g_1 and b smooth functions such that $g_1 = h_{u_1}$ and $-b = h_{u_2}$. From this, we obtain that

$$K_\Omega = \frac{-h_{u_1 u_1}}{(1 + h_{u_1}^2 + u_2^2)^2} \text{ and } \lambda_\Omega = -h_{u_2 u_2}. \tag{3.36}$$

From corollary 4.3 in (MEDINA-TEJEDA, 2022b), it follows that if h in (3.34) satisfies the equation $h_{u_1 u_1} + c(u_1,u_2) h_{u_2 u_2} = 0$, where $c(u_1,u_2)$ is a smooth function, then $\mathbf{y}$ is a wave front of rank 1 with extendable Gaussian curvature. Furthermore, using this information in (3.36) and taking into account that in $U \setminus \Sigma(\mathbf{x})$ the Gaussian curvature is $K = \dfrac{K_\Omega}{\lambda_\Omega}$, we get, by the density of $U \setminus \Sigma(\mathbf{x})$, that its extension is given by

$$K = \frac{-c(u_1, u_2)}{(1 + h_{u_1}^2 + u_2^2)^2}. \tag{3.37}$$

Thus, if $c(u_1, u_2)$ is a non-vanishing function, we obtain a wave front of rank 1 for which the Gaussian curvature has a non-vanishing extension. It follows from proposition 3.4 in (MEDINA-TEJEDA, 2022b) that a wave front does not admit a smooth extension for its normal curvature, hence this class has no intersection with the class 3.4.1.1.

Remark 3.4.2. Note that the second order linear PDE

$$h_{u_1u_1} + c(u_1, u_2) h_{u_2u_2} = 0 \tag{3.38}$$

is an important step in order to obtain frontals with extendable non-vanishing Gaussian curvature. If $c(u_1, u_2) = 1$, then the equation (3.38) is the two dimensional *Laplace equation* (an elliptic equation), which was discovered by Euler in 1752. This PDE has been useful in many areas, like gravitational potential, propagation of heat, electricity and magnetism (for more information see chapter 7 in (GONZALEZ-VELASCO, 1996)). On the other hand, if we take $c(u_1, u_2) = -a^2$, where $a \neq 0$ is a constant, then (3.38) is the one-dimensional *wave equation* (a hyperbolic equation). The wave equation governs the dynamics of some physical systems, for instance, the guitar string, the longitudinal vibrations of an elastic bar, propagation of acoustic, fluid, and electromagnetic waves (see chapter 5 in (GONZALEZ-VELASCO, 1996)). Note in (3.36) that the sign of K is determined by the sign of $c(u_1, u_2)$, but this sign also identifies if the PDE (3.38) is hyperbolic or elliptic.

CHAPTER

4

SINGULARITIES OF 3-PARAMETER LINE CONGRUENCES IN $\mathbb{R}^4$

Here, we deal with three parameter families of lines in $\mathbb{R}^4$, i.e., 3-parameter line congruences in $\mathbb{R}^4$. More content about line congruences can be found in (BIANCHI, 1894), (EISENHART, 1909) and (WEATHERBURN, 1955). Our approach here is motivated by (IZUMIYA; SAJI; TAKEUCHI, 2003). Our goal is to classify generic singularities of 3-parameter line congruences, normal line congruences and Blaschke affine normal line congruences, in this last case, providing an answer to the conjecture presented in (IZUMIYA; SAJI; TAKEUCHI, 2003). Along this chapter, U denotes an open subset of $\mathbb{R}^3$

4.1 3-parameter line congruences in $\mathbb{R}^4$

Definition 4.1.1. A *3-parameter line congruence in* $\mathbb{R}^4$ is a 3-parameter family of lines in $\mathbb{R}^4$. Locally, we write $\mathscr{C} = \{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$ and the line congruence is given by a smooth map

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I &\to \mathbb{R}^4 \\ (u,t) &\mapsto F(u,t) = \mathbf{x}(u) + t\boldsymbol{\xi}(u), \end{aligned}$$

where

- $\mathbf{x} : U \to \mathbb{R}^4$ is smooth and it is called a *reference hypersurface of the congruence*;
- $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ is smooth and it is called the *director hypersurface of the congruence*.

When there is no risk of confusion, we denote the line congruence just by F instead of $F_{(\mathbf{x},\boldsymbol{\xi})}$.

Lemma 4.1.1. The singular points of a line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ are the points (u,t) such that

$$t^3\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\boldsymbol{\xi}_{u_3}\rangle + t^2\langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\mathbf{x}_{u_3}\rangle +$$
$$+t\langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \mathbf{x}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\mathbf{x}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\mathbf{x}_{u_3}\rangle + \langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\mathbf{x}_{u_3}\rangle = 0.$$

Proof. The jacobian matrix of F is

$$JF = \left[\mathbf{x}_{u_1}+t\boldsymbol{\xi}_{u_1} \quad \mathbf{x}_{u_2}+t\boldsymbol{\xi}_{u_2} \quad \mathbf{x}_{u_3}+t\boldsymbol{\xi}_{u_3} \quad \boldsymbol{\xi}\right].$$

As we know, (u,t) is a singular point of F if, and only if, $\det JF(u,t)=0$, thus the result follows from

$$\det JF(u,t) = \langle \boldsymbol{\xi}, (\mathbf{x}_{u_1}+t\boldsymbol{\xi}_{u_1})\wedge(\mathbf{x}_{u_2}+t\boldsymbol{\xi}_{u_2})\wedge(\mathbf{x}_{u_3}+t\boldsymbol{\xi}_{u_3})\rangle = 0.$$

□

Definition 4.1.2. We say that $\mathbf{y}(u)=\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u)$ is a *focal hypersurface* of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ if

$$\langle \boldsymbol{\xi}(u), \mathbf{y}_{u_1}\wedge\mathbf{y}_{u_2}\wedge\mathbf{y}_{u_3}\rangle = 0. \tag{4.1}$$

If $\mathbf{y}(u)=\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u)$ is a focal hypersurface of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ then

$$t^3\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\boldsymbol{\xi}_{u_3}\rangle + t^2\langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\mathbf{x}_{u_3}\rangle +$$
$$+t\langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\boldsymbol{\xi}_{u_3} + \mathbf{x}_{u_1}\wedge\boldsymbol{\xi}_{u_2}\wedge\mathbf{x}_{u_3} + \boldsymbol{\xi}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\mathbf{x}_{u_3}\rangle + \langle \boldsymbol{\xi}, \mathbf{x}_{u_1}\wedge\mathbf{x}_{u_2}\wedge\mathbf{x}_{u_3}\rangle = 0.$$

4.1.1 Ruled surfaces of the congruence

There is a geometric interpretation related to definition 4.1.2, when $\mathbf{x}$ is an embedding and $\boldsymbol{\xi}$ is an immersion, as follows. Let $\{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$ be a 3-parameter line congruence and C a regular curve on the reference hypersurface $\mathbf{x}$. If we restrict the director hypersurface $\boldsymbol{\xi}$ to this curve, we obtain a ruled surface associated to the 1-parameter family of lines $\{\mathbf{x}(s), \boldsymbol{\xi}(s)\}$, where s is the parameter of C, $\mathbf{x}(s)=\mathbf{x}(u(s))$ and $\boldsymbol{\xi}(s)=\boldsymbol{\xi}(u(s))$. The line obtained by fixing s is called a *generator of the ruled surface*. These kind of ruled surfaces are called *surfaces of the congruence* and since $\boldsymbol{\xi}'(s)\neq 0$, it is possible to define its striction curve (see section 3.5 in (CARMO, 2016) for details). In the special case where this ruled surface is developable, the points of contact of a generator with the striction curve are called *focal points*. Let us write $\alpha(s)=\mathbf{x}(u(s))+\rho(u(s))\boldsymbol{\xi}(u(s))$ as the striction curve, where $\rho(u(s))$ denotes the coordinate of the focal point relative to $\boldsymbol{\xi}(u(s))$. Suppose $\alpha'(s)\neq 0$ for all s, then it is possible to show that α' is parallel to $\boldsymbol{\xi}$ and assuming $\|\boldsymbol{\xi}\|=1$, α' is perpendicular to $\boldsymbol{\xi}_{u_i}$, $i=1,2,3$, thus

$$\begin{cases} u_1'(h_{11}+\rho g_{11})+u_2'(h_{21}+\rho g_{12})+u_3'(h_{31}+\rho g_{13})=0\\ u_1'(h_{12}+\rho g_{12})+u_2'(h_{22}+\rho g_{22})+u_3'(h_{32}+\rho g_{23})=0\\ u_1'(h_{13}+\rho g_{13})+u_2'(h_{23}+\rho g_{23})+u_3'(h_{33}+\rho g_{33})=0,\end{cases}$$

where $g_{ij} = \langle \boldsymbol{\xi}_{u_i}, \boldsymbol{\xi}_{u_j} \rangle$ and $h_{ij} = \langle \mathbf{x}_{u_i}, \boldsymbol{\xi}_{u_j} \rangle$. As we want to find a non-trivial solution for the above system, we obtain the cubic equation

$$\begin{vmatrix} h_{11}+\rho g_{11} & h_{21}+\rho g_{12} & h_{31}+\rho g_{13} \\ h_{12}+\rho g_{12} & h_{22}+\rho g_{22} & h_{32}+\rho g_{23} \\ h_{13}+\rho g_{13} & h_{23}+\rho g_{23} & h_{33}+\rho g_{33} \end{vmatrix} = 0,$$

from which we obtain the coordinates ρ_i of the focal points, $i = 1,2,3$. Hence, related to each line of the congruence we have (possibly) three focal points. We define a *focal set of the congruence* as

$$\mathbf{y}_i(u) = \mathbf{x}(u) + \rho_i(u)\boldsymbol{\xi}(u), \ i = 1,2,3.$$

Thus, for every u_0, $\mathbf{y}_i(u_0)$ is a focal point and there is a curve in this focal set (striction curve) $\alpha(s) = \mathbf{x}(u(s)) + \rho_i(u(s))\boldsymbol{\xi}(u(s))$, such that $\alpha(s_0) = \mathbf{y}_i(u_0)$ and $\alpha'(s_0)$ is parallel to $\boldsymbol{\xi}(u_0)$, then

$$\langle \boldsymbol{\xi}(u_0), \mathbf{y}_{iu_1} \wedge \mathbf{y}_{iu_2} \wedge \mathbf{y}_{iu_3} \rangle = 0. \tag{4.2}$$

Therefore, the focal points are located at the focal hypersurfaces defined 4.1.2.

4.2 Singularities of 3-parameter line congruences in $\mathbb{R}^4$

In this section we use methods of singularity theory to obtain the generic singularities of 3-parameter line congruences in $\mathbb{R}^4$. Our approach is the same as in (IZUMIYA; SAJI; TAKEUCHI, 2003), but here we are dealing with the case of 3 parameters in $\mathbb{R}^4$. Let $F_{(\mathbf{x},\boldsymbol{\xi})}$ be a line congruence and take x_i and ξ_i, $i = 1,2,3,4$, as the coordinate functions of $\mathbf{x}$ and $\boldsymbol{\xi}$, respectively, thus we have

$$F_{(\mathbf{x},\boldsymbol{\xi})}(u,t) = (x_1(u) + t\xi_1(u), x_2(u) + t\xi_2(u), x_3(u) + t\xi_3(u), x_4(u) + t\xi_4(u)).$$

If $(u_0,t_0) \in U \times I$ and $\xi_4(u_0) \neq 0$ then there exists $U_4 \subset U$ an open subset given by $\{u \in U : \xi_4(u) \neq 0\}$. Let us define

$$c_4(u) = -\frac{x_4(u) - a_0}{\xi_4(u)}, \tag{4.3}$$

where $u \in U_4$ and $a_0 = x_4(u_0) + t_0\xi_4(u_0)$. Therefore,

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})}(u,t) &= \mathbf{x}(u) + c_4(u)\boldsymbol{\xi}(u) + (t - c_4(u))\,\boldsymbol{\xi}(u) \\ &= \mathbf{x}(u) + c_4(u)\boldsymbol{\xi}(u) + \tilde{t}\boldsymbol{\xi}(u), \ where\ \tilde{t} = t - c_4(u). \end{aligned}$$

Then, if we look at $\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}(u,\tilde{t}) = \mathbf{x}(u) + c_4(u)\boldsymbol{\xi}(u) + \tilde{t}\boldsymbol{\xi}(u)$ we can see that its fourth coordinate, which is denoted by $\widetilde{F}_4$, is $x_4(u) + c_4(u)\xi_4(u) + \tilde{t}\xi_4(u) = a_0 + \tilde{t}\xi_4(u)$, by (4.3). Furthermore,

$\widetilde{F}_4^{-1}(a_0) = \{(u,0) : u \in U_4\}$ and via the Implicit Function Theorem and lemma 2.3.1, the germ of $\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}$ at $(u_0,0)$ is an one-dimensional unfolding of

$$\tilde{f}(u) = \tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}(u,0) = (x_1(u)+c_4(u)\xi_1(u), x_2(u)+c_4(u)\xi_2(u), x_3(u)+c_4(u)\xi_3(u)),$$

where $\tilde{\pi}_4(y_1,y_2,y_3,y_4) = (y_1,y_2,y_3)$.

Lemma 4.2.1. Let $F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I \to \mathbb{R}^4$ be a line congruence. With notation as above, the singularity of $\tilde{f}$ at u_0 is determined by $\tilde{\pi}_4 \circ \mathbf{x}$.

Proof. Let us suppose $\boldsymbol{\xi}_4(u_0) \neq 0$ (other cases are analogous), $(u_0,t_0) = (0,0) \in U \times I$ and $\boldsymbol{\xi}(0) = (0,0,0,1)$. Using the above notation, $c_4(0) = 0$, thus the jacobian matrix of $\tilde{f}$ at 0 is equal to the jacobian matrix of $\tilde{\pi}_4 \circ \mathbf{x}$ at 0. □

The above lemma is important because it shows that the singularity of $\tilde{f}$, and therefore the unfolding $\widetilde{F}$, is determined by $\tilde{\pi}_4 \circ \boldsymbol{\xi} : U \to \mathbb{R}^3$.

Lemma 4.2.2. Let $W \subset J^k(3,3)$ be a submanifold. For any fixed map germ $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{0\}$ and any fixed point $(u_0,t_0) \in U \times I$ with $\xi_4(u_0) \neq 0$, the set

$$T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0)} = \left\{ \mathbf{x} \in C^\infty(U,\mathbb{R}^4) : j^k_1\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right) \pitchfork W \;\; at \;\; (u_0,t_0) \right\}$$

is a residual subset of $C^\infty\left(U,\mathbb{R}^4\right)$.

Proof. We proceed as in (IZUMIYA; SAJI; TAKEUCHI, 2003), lemma 4.1. Let us identify $C^\infty(U,\mathbb{R}^4) \times C^\infty(U,\mathbb{R}^4 \setminus \{\mathbf{0}\}) = C^\infty(U,\mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\})$ and take the C^∞-Whitney Topology induced on $C^\infty(U,\mathbb{R}^4) \times \{\boldsymbol{\xi}\}$. Let us take $\{C_j\}_{j=1}^\infty$ a countable open cover for W, such that $\overline{C}_j$ is compact. Define

$$T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j} = \left\{ \mathbf{x} : j^k_1\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right) \pitchfork W, \text{ with } j^k_1\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)(u_0,t_0) \in \overline{C}_j \right\}.$$

The idea here is to show that $T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j}$ is an open subset of $C^\infty\left(U,\mathbb{R}^4\right)$. Note that the map

$$\hat{j}^k : C^\infty\left(U_4,\mathbb{R}^4\right) \to C^\infty\left(U_4 \times I, J^k(3,3)\right),$$

defined by $\hat{j}^k(\mathbf{x}) = j^k\left(\tilde{\pi}_4 \circ \tilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)$ is continuous, as follows. It is known that the k-jet map

$$j^k : C^\infty\left(U_4 \times I,\mathbb{R}^3\right) \to C^\infty\left(U_4 \times I, J^k(3,3)\right)$$

and the maps

$$\begin{aligned} s : C^\infty\left(U_4,\mathbb{R}^4\right) \times C^\infty\left(U_4,\mathbb{R}\right) \times C^\infty\left(I,\mathbb{R}\right) &\to C^\infty\left(U_4 \times I,\mathbb{R}^4\right) \\ (\mathbf{x}, c_4, \tilde{t}) &\mapsto \mathbf{x} + c_4\boldsymbol{\xi} + \tilde{t}\boldsymbol{\xi} \\ \pi_4 : C^\infty\left(U_4 \times I,\mathbb{R}^4\right) &\to C^\infty\left(U_4 \times I,\mathbb{R}^3\right) \\ (f_1,f_2,f_3,f_4) &\mapsto (f_1,f_2,f_3). \end{aligned}$$

are continuous, then as we have $\hat{j}^k = j^k \circ \pi_4 \circ s$, the continuity follows.

Define

$$\mathcal{O}_{W,C_j} = \left\{ g \in C^\infty \left(U_4 \times I, J^k(3,3) \right) : g \pitchfork W \ \ at \ \ (u_0,t_0), g(u_0,t_0) \in \overline{C}_j \right\},$$

which is open (see page 52 in (GOLUBITSKY; GUILLEMIN, 2012)). Considering that the restriction map

$$res|_{U_4} : C^\infty \left(U, \mathbb{R}^4 \right) \to C^\infty \left(U_4, \mathbb{R}^4 \right)$$

is continuous, it follows that

$$T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j} = \left(res|_{U_4} \right)^{-1} \circ \left(\hat{j}^k \right)^{-1} \left(\mathcal{O}_{W,C_j} \right)$$

is open. If we are able to show that $T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j}$ is a dense subset of $C^\infty \left(U, \mathbb{R}^4 \right)$, then we have

$$T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0)} = \bigcap_{j \in \mathbb{N}} T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j}$$

residual.

It is enough to show that

$$\begin{aligned} T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j,U_4} = \Big\{ \mathbf{x} \in C^\infty \left(U_4, \mathbb{R}^4 \right) : j_1^k \left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})} \right) \pitchfork W \ in \ (u_0,t_0), \\ j_1^k \left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})} \right) (u_0,t_0) \in \overline{C}_j \Big\} \end{aligned}$$

is a dense subset of $C^\infty \left(U_4, \mathbb{R}^4 \right)$.

Let us write

$$P(3,3,k) = \{(p_1,p_2,p_3) : p_i \text{ is a polynomial with } p_i(0) = 0 \text{ and } deg(p_i) \leq k\}.$$

Given $\mathbf{x} \in C^\infty \left(U_4, \mathbb{R}^4 \right)$ and $\boldsymbol{p} = (p_1,p_2,p_3) \in P(3,3,k)$, we define the map $f_{(\mathbf{x},\boldsymbol{p})} : U_4 \times I \to \mathbb{R}^3$ by

$$\begin{aligned} f_{(\mathbf{x},\boldsymbol{p})}(u,t) = (x_1(u) + p_1(u) + c_4(u)\xi_1(u) + t\xi_1(u), x_2(u) + p_2(u) + \\ + c_4(u)\xi_2(u) + t\xi_2(u), x_3(u) + p_3(u) + c_4(u)\xi_3(u) + t\xi_3(u)). \end{aligned}$$

We also define

$$\begin{aligned} \Phi : U_4 \times I \times P(3,3,k) &\to J^k(3,3) \\ (u,t,(p_1,p_2,p_3)) &\mapsto j_1^k f_{(\mathbf{x},\boldsymbol{p})}(u,t) = j^k f_{(\mathbf{x},\boldsymbol{p},t)}(u). \end{aligned}$$

Since we can look at $P(3,3,k)$ as $\mathbb{R}^N$, we identify $P(3,3,k)$ with $J^k(3,3)$, and their tangent spaces. Thus, we have that Φ is a submersion at any point and it follows that $\Phi \pitchfork W$. Using lemma 2.3.3, we have that

$$\{p = (p_1,p_2,p_3) \in P(3,3,k) : \Phi_p \pitchfork W \ \text{ at } \ (u_0,t_0), \text{such that } \Phi_p(u_0,t_0) \in \overline{C}_j\}$$

is dense in $P(3,3,k)$. Thus, there is a sequence $p_n = (p_1, p_2, p_3)_n$ in $P(3,3,k)$ such that $p_n \to (0,0,0)$ with $\Phi_{p_n} \pitchfork W$ in $\overline{C}_j$, for all $n \in \mathbb{N}$.

Note that $\tilde{x} = \mathbf{x} + ((p_1, p_2, p_3)_n, 0) \in T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j,U_4}$, $\forall\, n \in \mathbb{N}$, because

$$j_1^k\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\tilde{r},\boldsymbol{\xi})}\right) = j_1^k\left(f_{(\mathbf{x},p_n)}\right) = \Phi \pitchfork W.$$

Furthermore, $\lim_{n\to\infty} \mathbf{x} + ((p_1, p_2, p_3)_n, 0) = \mathbf{x}$ and $T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0),C_j,U_4}$ is dense in $C^\infty\left(U_4, \mathbb{R}^4\right)$.

□

If $\xi_j(u_0) \neq 0$, $j = 1,2,3$, we can define the set

$$T^{\boldsymbol{\xi}}_{j,W,(u_0,t_0)} = \left\{\mathbf{x} \in C^\infty(U, \mathbb{R}^4) : j_1^k\left(\tilde{\pi}_j \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right) \pitchfork W \;\; at \;\; (u_0,t_0)\right\},\ j = 1,2,3$$

where $\tilde{\pi}_j$ is the projection in the coordinates different than j. Thus, the above lemma holds for the sets $T^{\boldsymbol{\xi}}_{j,W,(u_0,t_0)}$, $j = 1,2,3,4$.

Remark 4.2.1. Define

$$\begin{aligned}\mathcal{O}_1 = \{&\boldsymbol{\xi} \in C^\infty\left(U, \mathbb{R}^4 \setminus \{\mathbf{0}\}\right) : \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi} \neq \mathbf{0},\ or\ \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_3} \wedge \boldsymbol{\xi} \neq \mathbf{0},\\ &or\ \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3} \wedge \boldsymbol{\xi} \neq \mathbf{0}, \forall\, u \in U\}\end{aligned}$$

Then, $\mathcal{O}_1$ is a residual subset of $C^\infty\left(U, \mathbb{R}^4 \setminus \{\mathbf{0}\}\right)$ as follows. If we denote by $\Sigma^i = \{\sigma \in J^1(4,4) :$ kernel rank$(\sigma) = i\}$, where kernel rank(σ) indicates the dimension of the kernel of σ, then $J^1(4,4) \setminus \mathcal{O}_1 = \Sigma^4 \cup \Sigma^3 \cup \Sigma^2$. It is known that Σ^i is a submanifold of $J^1(4,4)$ of codimension i^2 thus, since $U \subset \mathbb{R}^3$ is open, if we take $\boldsymbol{\xi}$ such that $j^1\boldsymbol{\xi} \pitchfork \Sigma^i$, then $j^1\boldsymbol{\xi}(U) \cap \Sigma^i = \emptyset$, $i = 2,3,4$ what happens if and only if $\boldsymbol{\xi} \in \mathcal{O}_1$. Therefore, by Thom's Transversality Theorem, $\mathcal{O}_1$ is residual. Note above that we are denoting $j^1\boldsymbol{\xi}(u) = [\boldsymbol{\xi}(u)\ \ \boldsymbol{\xi}_{u_1}(u)\ \ \boldsymbol{\xi}_{u_2}(u)\ \ \boldsymbol{\xi}_{u_3}(u)]$.

Finally, it follows from lemma 4.2.2 that

$$\widetilde{T}_{4,W,(u_0,t_0)} = \left\{(\mathbf{x}, \boldsymbol{\xi}) : j_1^k\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right) \pitchfork W\ at\ (u_0,t_0), \boldsymbol{\xi} \in \mathcal{O}_1\right\}$$

is residual.

Now, we are able to prove our first main theorem, which provides a classification of the generic singularities of 3-parameter line congruences in $\mathbb{R}^4$.

Theorem 4.2.1. There is an open dense set $\mathcal{O} \subset C^\infty\left(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\})\right)$, such that:

a) For all $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}$, the germ of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0) \in U \times I$ is stable;

b) For all $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}$, the germ of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0) \in U \times I$ is a 1-parameter versal unfolding of a germ $f : (\mathbb{R}^3, u_0) \to \mathbb{R}^3$ at $t = t_0$. Then, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is $\mathcal{A}$-equivalent to one of the normal forms below

- $(x,y,z,w) \mapsto (x,y,w,z^2)$ (Fold).
- $(x,y,z,w) \mapsto (x,y,w,z^3+xz)$ (Cusp).
- $(x,y,z,w) \mapsto (x,y,z^3+(x^2 \pm y^2)z+wz,w)$ (Lips/Beaks).
- $(x,y,z,w) \mapsto (x,y,w,z^4+xz+yz^2)$ (Swallowtail).
- $(x,y,z,w) \mapsto (x,y,w,z^4+xz \pm y^2z^2+wz^2)$.
- $(x,y,z,w) \mapsto (x,y,w,z^5+xz+yz^2+wz^3)$ (Butterfly).
- $(x,y,z,w) \mapsto (z,x^2+y^2+zx+wy,xy,w)$ (Hyperbolic Umbilic).
- $(x,y,z,w) \mapsto (z,x^2-y^2+zx+wy,xy,w)$ (Elliptic Umbilic).

Proof. We first prove item (a). Given $f \in \mathcal{E}_{3,3}$ and $z = j^k f(0)$, define

$$\mathcal{K}^k(z) = \{j^k g(0) : g \underset{\mathcal{K}}{\sim} f\}.$$

For a sufficiently large k, define

$$\Pi_k(3,3) = \{f \in J^k(3,3) : cod_e(\mathcal{K},f) \geq 5\}.$$

Consider

$$\Sigma^i = \{\sigma \in J^1(3,3) : \text{kernel rank}(\sigma) = i\} \subset J^1(3,3),$$

which is a submanifold of codimension i^2.

1. We look at the slice of $\Pi_k(3,3)$ in Σ^1, i.e., $f \in \Pi_k(3,3)$ such that *kernel rank*$(df(0)) = 1$. Then, we are dealing with $f \in \Pi_k(3,3)$ of *corank* 1. Therefore, we can write $f(x,y,z) = (x,y,g(x,y,z))$, where $g(0,0,z)$ has a singularity of A_r type, for some $5 \leq r \leq k-1$ and we call them $\mathcal{K}$-singularities of A_r-type. Note that if we regard the "good" set as the complement of $\Pi_k(3,3)$ in Σ^1, then its singularities are the $\mathcal{K}$-singularities of A_1, A_2, A_3 and A_4-type. Therefore, the slice $\Pi_k(3,3) \cap \Sigma^1$ is a semialgebraic set of codimension greater than or equal to 5, so it has a stratification $\{S_i^1\}_{i=1}^{m_1}$, with $codim(S_i^1) \geq 5$.
2. As we did in the first case, define $\Pi_k(3,3) \cap \Sigma^2$, i.e., the set of $f \in \Pi_k(3,3)$ of *corank* 2. We may assume that $f(x,y,z) = (z,g_1(x,y,z),g_2(x,y,z))$, where g_i has zero 1-jet and $(g_1(x,y,0),g_2(x,y,0))$ has 2-jet in $H^2(2,2)$, therefore, $(g_1(x,y,0),g_2(x,y,0))$ has 2-jet given by one of the normal forms below (See (GIBSON, 1979) or (MOND; NUÑO-BALLESTEROS, 2020)):

   $$(x^2+y^2,xy);\ (x^2-y^2,xy);\ (x^2,xy);\ (x^2,0);\ (x^2 \pm y^2,0);\ (0,0).$$

   Hence, by looking at the first two normal forms and its local algebras, f is $\mathcal{K}$-equivalent to one of the forms below:

- $W_1 : \left(z, x^2+y^2+xz, xy\right)$
- $W_2 : \left(z, x^2-y^2+xz, xy\right)$

and both of these forms have $cod_e(\mathcal{K}) = 4$. The other $\mathcal{K}$-orbits have $cod_e(\mathcal{K}) \geq 5$. Note that $\overline{\Sigma^2} \setminus (W_1 \cup W_2)$ is a semialgebraic set of codimension greater than or equal to 5. $\Pi_k(3,3) \cap \Sigma^2$ is a semialgebraic set contained in $\overline{\Sigma^2} \setminus (W_1 \cup W_2)$, then its codimension is greater than or equal to 5, thus, there is a stratification $\{S_i^2\}_{i=1}^{m_2}$ of it, with $codim(S_i^2) \geq 5$. Furthermore, the “good" set contains only W_1 and W_2.

3. In a similar way, we define $\Pi_k(3,3) \cap \Sigma^3$, i.e., the set of the k-jets $f \in \Pi_k(3,3)$ whose *corank* is 3. It is well-known that Σ^3 has codimension 9, so $\Pi_k(3,3) \cap \Sigma^3$ is a semialgebraic set of codimension greater than or equal to 9, hence, there is a stratification $\{S_i^3\}_{i=1}^{m_3}$, with $codim(S_i^3) > 5$.

   Then, it follows that the “good" set, i.e., the set of the $\mathcal{K}$-orbits of codimension less than or equal to 4, contains the following $\mathcal{K}$-orbits

   - type A_r, for $1 \leq r \leq 4$;
   - type W_1;
   - type W_2.

Applying lemma 4.2.2 and remark 4.2.1 to each strata of the above stratification, we obtain that

$$\mathcal{T}_j = \bigcap_{i=1}^{m_j} \widetilde{T}_{4,S_i^j,(u_0,t_0)},\ j = 1,2,3$$
$$\mathcal{T}_{3+r} = \widetilde{T}_{4,A_r,(u_0,t_0)},\ 1 \leq r \leq 4$$
$$\mathcal{T}_{7+i} = \widetilde{T}_{4,W_i,(u_0,t_0)},\ i = 1,2.$$

are residual subsets of $C^\infty(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$. Hence,

$$\mathcal{O}_{4,(u_0,t_0)} = \bigcap_{i=1}^{9} \mathcal{T}_i$$

is residual. The same is true for the sets $\mathcal{O}_{j,(u_0,t_0)}$, $j = 1,2,3$, defined in a similar way.

Since $\boldsymbol{\xi}(u) \neq 0$ for all $u \in U$, given a point $(u_0,t_0) \in U \times I$, $\xi_j(u_0) \neq 0$, for some j, there is a residual set $\mathcal{O}_{(u_0,t_0)} \subset C^\infty(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$, such that

$$(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}_{(u_0,t_0)} \Leftrightarrow j_1^k\left(\tilde{\pi}_j \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right) \pitchfork \mathcal{A}_r, W_1, W_2, S_i^j,\ j = 1,2,3,\ r = 1,\cdots,4.$$

It follows from what we already have done that the germ of $\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}$ at $(u_0, 0)$, which is equivalent to the germ of $F_{(\mathbf{x},\boldsymbol{\xi})}$ at (u_0,t_0), is a 1-dimensional unfolding of $\tilde{\pi}_j \circ \widetilde{F}(u,0)$ and it follows from lemma 2.3.2 that $F_{(\mathbf{x},\boldsymbol{\xi})}$ is $\mathcal{A}$-infinitesimally stable for all $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{O}_{(u_0,t_0)}$. Since a germ

$\mathcal{A}$-infinitesimally stable is $\mathcal{A}$-stable (see (MATHER, 1969)), there is a neighborhood $U_{u_0} \times I_{t_0}$ of (u_0,t_0) in $U \times I$, such that $F_{(\mathbf{x},\boldsymbol{\xi})}|_{U_{u_0}\times I_{t_0}}$ is $\mathcal{A}$-stable. This result holds independently of the fixed point (u_0,t_0), so we can consider a countable family of points $(u_i,t_i) \in U \times I$ and neighborhoods $U_{u_i} \times I_{t_i}$, $(i = 1,2,\cdots)$, such that $F_{(\mathbf{x},\boldsymbol{\xi})}|_{U_{u_i}\times I_{t_i}}$ is $\mathcal{A}$-stable and

$$U \times I = \bigcup_{i=1}^{\infty} U_{u_i} \times I_{t_i}.$$

Since $\mathcal{O}_{(u_i,t_i)}$ is a residual subset of $C^\infty(U,\mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$, it follows that

$$\mathcal{O}_2 = \bigcap_{i=1}^{\infty} \mathcal{O}_{(u_i,t_i)}$$

is residual. Furthermore, the germ of $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u,t) \in U \times I$ is $\mathcal{A}$-infinitesimally stable, for all $(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{O}_2$.

Since $\mathscr{F} : C^\infty(U,\mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\})) \to C^\infty(U \times I,\mathbb{R}^4)$, defined by $\mathscr{F}(\mathbf{x},\boldsymbol{\xi}) = F_{(\mathbf{x},\boldsymbol{\xi})}$, is continuous and

$$S = \{f \in C^\infty(U \times I,\mathbb{R}^4) : f \;\; \mathcal{A}\text{-infinitesimally stable}\}$$

is open (See (GOLUBITSKY; GUILLEMIN, 2012) p. 111), $\mathcal{O} = \mathscr{F}^{-1}(S)$ is open. By previous arguments $\mathcal{O}_2 \subset \mathcal{O}$ and $\mathcal{O}_2$ is dense, therefore $\mathcal{O}$ is an open dense subset.

To prove (b), we refine the $\mathcal{K}$-orbits of type A_2 and A_3 of the above stratification, by taking the $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension ≤ 1 inside these $\mathcal{K}$-orbits. Then, the relevant strata in this stratification are the $\mathcal{A}$-orbits of stable singularities A_k, $k = 1,2,3$, and the $\mathcal{A}$-orbits of singularities of $\mathcal{A}_e$-codimension 1 of type A_2, A_3, A_4 and D_4. The complement of their union is a semialgebraic set of codimension greater than or equal to 5.

1. $\mathcal{K}$-orbit of A_1 type

   $f(x,y,z) = (x,y,z^2)$ which is stable, hence, we have just this $\mathcal{A}$-orbit. Its suspension in $\mathbb{R}^4$ is the stable germ that we are looking for.

2. $\mathcal{K}$-orbits of A_2 type

   It follows from the classification made by Marar and Tari (MARAR; TARI, 1996), that the possible normal forms are

   $$f(x,y,z) = \left(x,y,z^3 + P(x,y)z\right),$$

   where $P(x,y)$ is one of the singularities A_k, D_k, E_6, E_7 or E_8 and $\mathrm{cod_e}\,(\mathcal{A},f) = \mu(P)$.

   As we are looking for f which have a versal unfolding of dimension 1 that is a stable germ, we must have $P(x,y) = x$ or $P(x,y) = x^2 \pm y^2$. Therefore, we have the $\mathcal{A}$-orbits

   $$f_1(x,y,z) = (x,y,z^3 + xz) \;\; \text{(Cusp)};$$
   $$f_2(x,y,z) = (x,y,z^3 + (x^2 \pm y^2)z) \;\; \text{(Lips(+) / Beaks(−))},$$

with $\operatorname{cod}_e(\mathcal{A}, f_1) = 0$ and $\operatorname{cod}_e(\mathcal{A}, f_2) = 1$. The stable germs $\mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ are, respectively

$$F_1(x,y,z,w) = (x,y,z^3+xz,w);$$
$$F_2(x,y,z,w) = (x,y,z^3+(x^2 \pm y^2)z+wz,w).$$

These germs are $\mathcal{A}$-equivalent, however they are considered separately, because they are versal unfoldings of f_1 and f_2, respectively, which are not $\mathcal{A}$-equivalent.

3. $\mathcal{K}$-orbits of A_3 type

   In a similar way, the possible normal forms are (see (MARAR; TARI, 1996), section 1)

$$(x,y,z^4+xz \pm y^k z^2), k \geq 1. \ \operatorname{cod}_e(\mathcal{A}) = k-1;$$
$$(x,y,z^4+(y^2 \pm x^k)z+xz^2), k \geq 2. \ \operatorname{cod}_e(\mathcal{A}) = k.$$

   Hence, the useful cases are those where $k = 1$ or $k = 2$ in the first type of orbit, i.e.,

$$f_1(x,y,z) = (x,y,z^4+xz+yz^2) \ \text{(Swallowtail)};$$
$$f_2(x,y,z) = (x,y,z^4+xz \pm y^2z^2),$$

   with $\operatorname{cod}_e(\mathcal{A}, f_1) = 0$ e $\operatorname{cod}_e(\mathcal{A}, f_2) = 1$. The stable germs $\mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ are, respectively

$$F_1(x,y,z,w) = (x,y,z^4+xz+yz^2,w)$$
$$F_2(x,y,z,w) = (x,y,z^4+xz \pm y^2z^2+wz^2,w).$$

4. $\mathcal{K}$-orbits of A_4 type

   Via (MARAR; TARI, 1996), the possible normal forms are

$$(x,y,z^5+xz+yz^2), \ \operatorname{cod}_e(\mathcal{A}) = 1;$$
$$(x,y,z^5+xz+y^2z^2+yz^3), \ \operatorname{cod}_e(\mathcal{A}) = 2;$$
$$(x,y,z^5+xz+yz^3), \ \operatorname{cod}_e(\mathcal{A}) = 3.$$

   Thus, the only case to be considered is

$$f(x,y,z) = (x,y,z^5+xz+yz^2),$$

   whose associated stable germ is

$$F(x,y,z,w) = (x,y,z^5+xz+yz^2+wz^3,w).$$

5. $\mathcal{K}$-orbits W_1 and W_2

The germs

$$F_1(x,y,z,w) = (z, x^2+y^2+zx+wy, xy, w);$$
$$F_2(x,y,z,w) = (z, x^2-y^2+zx+wy, xy, w).$$

are, respectively, 1-parameter versal unfoldings of (see (BRUCE, 1986), section 3)

$$f_1(x,y,z) = (z, x^2+y^2+zx, xy);$$
$$f_2(x,y,z) = \left(z, x^2-y^2+zx, xy\right),$$

where f_1 and f_2 are of the type W_1 e W_2, respectively and both have $\mathrm{cod_e}(\mathcal{A}) = 1$. Then, we conclude the proof.

□

4.3 Normal congruences

In this section, our approach is the same as in (IZUMIYA; SAJI; TAKEUCHI, 2003) and we seek to provide a classification of the generic singularities of 3-parameter normal congruences in $\mathbb{R}^4$. For this, it is necessary to characterize normal congruences and consider some aspects of Lagrangian singularities.

Definition 4.3.1. A 3-parameter line congruence $\mathscr{C} = \{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$, for $u \in U \subset \mathbb{R}^3$, is said to be *normal* if for each point $u_0 \in U$ there is a neighborhood $\tilde{U}$ of u_0 and a regular hypersurface, given by $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$, whose normal vectors are parallel to $\boldsymbol{\xi}(u)$, for all $u \in \tilde{U}$. The congruence is an *exact normal* congruence if $\boldsymbol{\xi}(u)$ is a normal vector at $\mathbf{x}(u)$, for all $u \in U$.

The next proposition characterizes 3-parameter normal line congruences in $\mathbb{R}^4$ and corresponds to the Proposition 5.1 in (IZUMIYA; SAJI; TAKEUCHI, 2003).

Proposition 4.3.1. Let $\mathscr{C} = \{\mathbf{x}(u), \boldsymbol{\xi}(u)\}$, $u \in U \subset \mathbb{R}^3$, be a 3-parameter line congruence in $\mathbb{R}^4$. $\mathscr{C}$ is normal if, and only if, $h_{ij}(u) = h_{ji}(u)$, $i,j \in \{1,2,3\}$, for all $u \in U$, where $h_{ij} = \left\langle \mathbf{x}_{u_i}, \left(\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_j} \right\rangle$.

Proof. Let $\mathscr{C}$ be a normal congruence and S' a hypersurface parameterized locally by $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$, whose normal vectors are parallel to $\boldsymbol{\xi}(u)$. Let us suppose that $\|\boldsymbol{\xi}(u)\| = 1$. Then, $y_{u_i}(u)$, $i = 1,2,3$ are orthogonal to $\boldsymbol{\xi}(u)$, therefore, $\langle \boldsymbol{\xi}, \mathbf{y}_{u_i}\rangle = 0$. From these expressions, we obtain

$$\begin{cases} t_{u_1} = -\langle \mathbf{x}_{u_1}, \boldsymbol{\xi}\rangle \\ t_{u_2} = -\langle \mathbf{x}_{u_2}, \boldsymbol{\xi}\rangle \\ t_{u_3} = -\langle \mathbf{x}_{u_3}, \boldsymbol{\xi}\rangle. \end{cases} \quad . \tag{4.4}$$

Since t is smooth, $t_{u_1u_2} = t_{u_2u_1}$, $t_{u_1u_3} = t_{u_3u_1}$ and $t_{u_2u_3} = t_{u_3u_2}$. From $t_{u_1u_2} = t_{u_2u_1}$, we obtain

$$-\langle \mathbf{x}_{u_1u_2}, \boldsymbol{\xi} \rangle - \langle \mathbf{x}_{u_1}, \boldsymbol{\xi}_{u_2} \rangle = -\langle \mathbf{x}_{u_1u_2}, \boldsymbol{\xi} \rangle - \langle \mathbf{x}_{u_2}, \boldsymbol{\xi}_{u_1} \rangle$$

Therefore, $h_{12} = \langle \mathbf{x}_{u_1}, \boldsymbol{\xi}_{u_2} \rangle = \langle \mathbf{x}_{u_2}, \boldsymbol{\xi}_{u_1} \rangle = h_{21}$. The other cases are analogous.

Reciprocally, suppose $h_{ij} = h_{ji}$, for $i, j = 1,2,3$. Taking into account the system (4.4), it follows from $h_{ij} = h_{ji}$ that $t_{u_1u_2} = t_{u_2u_1}$, $t_{u_1u_3} = t_{u_3u_1}$ and $t_{u_2u_3} = t_{u_3u_2}$. Therefore, this system is associated to an exact differential equation and it has a solution t. Write $\mathbf{y}(u) = \mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$. Note that

$$\begin{aligned} \langle \boldsymbol{\xi}, \mathbf{y}_{u_i} \rangle &= \langle \boldsymbol{\xi}, \mathbf{x}_{u_i} \rangle + t_{u_i} \\ &= \langle \boldsymbol{\xi}, \mathbf{x}_{u_i} \rangle - \langle \boldsymbol{\xi}, \mathbf{x}_{u_i} \rangle = 0. \end{aligned}$$

If $\mathbf{y}$ is not an immersion, there is a positive real number λ such that $\tilde{\mathbf{y}}(u) = \mathbf{x}(u) + (t(u) + \lambda)\boldsymbol{\xi}(u)$ is an immersion. For the last part, it is sufficient to look at the case when $\mathbf{y}(u)$ belongs to the focal set of the congruence. □

Denote by

$$Emb(U, \mathbb{R}^4) = \{\mathbf{x} : U \to \mathbb{R}^4 : \mathbf{x} \text{ is an embedding}\}$$

the space of the regular hypersurfaces in $\mathbb{R}^4$ with the Whitney C^∞-topology, and by

$$EN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) = \left\{(\mathbf{x}, \boldsymbol{\xi}) : \mathbf{x} \in Emb(U, \mathbb{R}^4),\ \boldsymbol{\xi}(u) \text{ is normal to } \mathbf{x} \text{ at } \mathbf{x}(u)\right\}$$

the space of the exact normal congruences. So, we have the following well known theorem.

Theorem 4.3.1. There is an open dense subset $O \subset Emb(U, \mathbb{R}^4)$, such that the germ of an exact normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $\mathbf{x} \in O$, i.e., $\forall \mathbf{x} \in O$, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table 2.

Singularity	**Normal form**
Fold	(x, y, w, z^2)
Cusp	$(x, y, w, z^3 + xz)$
Swallowtail	$(x, y, w, z^4 + xz + yz^2)$
Butterfly	$(x, y, w, z^5 + xz + yz^2 + wz^3)$
Elliptic Umbilic	$(z, w, x^2 - y^2 + zx, zy)$
Hyperbolic Umbilic	$(z, w, x^2 + zy, y^2 + zx, xy)$
Parabolic Umbilic	$(z, w, xy + xz, x^2 + y^3 + yw)$

Table 2 – Generic singularities of exact normal congruences

Proof. It follows from example 2.4.4 that the germ $F_{(\mathbf{x},\boldsymbol{\xi})}$ at (u_0,t_0), where $(\mathbf{x},\boldsymbol{\xi}) \in EN(U,\mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$, is the Lagrangian map associated to the germ of family of distance squared functions D on $M = \mathbf{x}(U)$, which is a Morse family of functions. Furthermore, from theorem 2.4.1 we know that for an open and dense subset of $Emb(U,\mathbb{R}^4)$ the family D is locally $\mathcal{P}$-$\mathcal{R}^+$-versal. Since, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is Lagrangian stable if and only if D is $\mathcal{P}$-$\mathcal{R}^+$-versal (see theorem 2.4.2), we have the result. □

Now, we define a natural map $P : EN\left(U,\mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) \to Emb(U,\mathbb{R}^4)$, given by $P(\mathbf{x},\boldsymbol{\xi}) = \mathbf{x}$. Taking into account the natural identification between $\mathbf{x} \in Emb(U,\mathbb{R}^4)$ and the pair $(\mathbf{x},\mathbf{n})$, where $\mathbf{n}$ is its unit normal vector field, it follows that this map is a retraction, so it is a continuous open map. Then, we have the following corollary, which provides a classification of the generic singularities of 3-parameter exact normal congruences.

Corollary 4.3.1. There is an open dense subset $O \subset EN\left(U,\mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right)$, such that the germ of an exact normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0) \in U \times I$ is a Lagrangian stable map germ, for all $(\mathbf{x},\boldsymbol{\xi}) \in O$.

Proof. It follows from the fact that $P : EN\left(U,\mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) \to Emb(U,\mathbb{R}^4)$ is an open continuous map and from theorem 4.3.1. □

Let us consider some aspects of Lagrangian singularities (see chapter 5 in (IZUMIYA *et al.*, 2016)). Take the cotangent bundle $\pi : T^*\mathbb{R}^4 \to \mathbb{R}^4$, whose symplectic structure is given locally by the 2-form $\omega = -d\lambda$, where λ is the Liouville 1-form, given locally by $\lambda = \sum_{i=1}^4 p_i dz_i$, where $(z_1,z_2,z_3,z_4,p_1,p_2,p_3,p_4)$ are the cotangent coordinates. For a given congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$, we define a smooth map $L_{(\mathbf{x},\boldsymbol{\xi})} : U \times I \to T^*\mathbb{R}^4 \simeq \mathbb{R}^4 \times (\mathbb{R}^4)^*$, given by

$$L_{(\mathbf{x},\boldsymbol{\xi})}(u,t) = \left(\mathbf{x}(u) + t\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u)\right).$$

Definition 4.3.2. We say that $F_{(\mathbf{x},\boldsymbol{\xi})}$ is a *Lagrangian Line Congruence* if $L_{(\mathbf{x},\boldsymbol{\xi})}$ is a Lagrangian immersion.

Proposition 4.3.2. Suppose that $L_{(\mathbf{x},\boldsymbol{\xi})}$ is an immersion. Then $F_{(\mathbf{x},\boldsymbol{\xi})}$ is a Lagrangian congruence if, and only if, is a normal congruence

Proof. Locally, the Liouville 1-form of $T^*\mathbb{R}^4$ is given by $\lambda = \sum_{i=1}^4 p_i dz_i$. So,

$$L^*_{(\mathbf{x},\boldsymbol{\xi})}(\lambda) = \sum_{i=1}^{4}\left(\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)dx_i(u) + t\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\right) + dt,$$

Therefore, being $\omega = -d\lambda$, we have

$$
\begin{aligned}
-L^*_{(\mathbf{x},\boldsymbol{\xi})}(\omega) = dL^*_{(\mathbf{x},\boldsymbol{\xi})}(\lambda) &= \sum_{i=1}^{4}\left(d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\wedge dx_i(u) + \frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)dt\wedge d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\right)\\
&= \left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_2}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_1}\right\rangle\right)du_1\wedge du_2+\\
&+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_3}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_1}\right\rangle\right)du_1\wedge du_3+\\
&+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_3}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_2}\right\rangle\right)du_2\wedge du_3+\\
&+\sum_{i=1}^{3}\left\langle\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|},\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_i}\right\rangle dt\wedge du_i,
\end{aligned}
$$

where $\mathbf{x}(u) = (x_1(u),x_2(u),x_3(u),x_4(u))$ and $\boldsymbol{\xi}(u) = (\xi_1(u),\xi_2(u),\xi_3(u),\xi_4(u))$. Thus

$$
\begin{aligned}
-L^*_{(\mathbf{x},\boldsymbol{\xi})}(\omega) &= \left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_2}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_1}\right\rangle\right)du_1\wedge du_2+\\
&+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_3}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_1}\right\rangle\right)du_1\wedge du_3+\\
&+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_3}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_2}\right\rangle\right)du_2\wedge du_3.
\end{aligned}
$$

Therefore, $L^*_{(x,e)}(\omega) = 0$ if, and only if,

$$
\begin{aligned}
h_{21} &= \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_2}\right\rangle = \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_1}\right\rangle = h_{12}\\
h_{31} &= \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\mathbf{x}_{u_3}\right\rangle = \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_1}\right\rangle = h_{13}\\
h_{32} &= \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\mathbf{x}_{u_3}\right\rangle = \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\mathbf{x}_{u_2}\right\rangle = h_{23}.
\end{aligned}
$$

□

By proposition 4.3.1, we can regard the space of the Lagrangian congruences as follows. A line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ is a Lagrangian congruence if, and only if, there is a smooth function $t : U \to \mathbb{R}$, such that $\mathbf{x}(u) + t(u)\boldsymbol{\xi}(u)$ is an immersion and the following conditions hold

$$
\begin{cases}
t_{u_1}(u) + \left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\mathbf{x}_{u_1}(u)\right\rangle = 0\\
t_{u_2}(u) + \left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\mathbf{x}_{u_2}(u)\right\rangle = 0\\
t_{u_3}(u) + \left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\mathbf{x}_{u_3}(u)\right\rangle = 0.
\end{cases}
\tag{4.5}
$$

So, we can define the space of the Lagrangian congruences

$$L(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))=\{(\mathbf{x},t,\boldsymbol{\xi}):\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u)\text{ is an immersion and (4.5) holds}\}$$

with the Whitney C^∞-topology. Our idea now is to show that the generic singularities of normal congruences are the same as the generic singularities of exact normal congruences, so, let us define the map

$$\begin{aligned}T_{rp}:C^\infty(U,\mathbb{R}^4\times\mathbb{R}\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))&\to C^\infty(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))\\(\mathbf{x}(u),t(u),\boldsymbol{\xi}(u))&\mapsto(\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u),\boldsymbol{\xi}(u)).\end{aligned}$$

Proposition 4.3.3. T_{rp} is an open continuous map under the Whitney C^∞-topology.

Proof. For any positive $k\in\mathbb{Z}$, the map

$$\begin{aligned}T_{rp}^k:J^k(U,\mathbb{R}^4\times\mathbb{R}\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))&\to J^k(U,\mathbb{R}^4\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))\\j^k(\mathbf{x},t,\boldsymbol{\xi})&\mapsto j^k(\mathbf{x}+t\boldsymbol{\xi},\boldsymbol{\xi})\end{aligned}$$

is a submersion, so it is an open map. From this, follows that T_{rp} is an open map. □

Now, take

$$N(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))=T_{rp}\left(L\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)\right)\subset C^\infty\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right),$$

with the Whitney C^∞-topology induced from $C^\infty\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$. Note that we can regard $N(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))$ as the space of the normal congruences. Then, we have the following theorem.

Theorem 4.3.2. There is an open dense set $O'\subset N(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))$, such that the germ of normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point (u_0,t_0) is a Lagrangian stable germ, for any $(\mathbf{x},\boldsymbol{\xi})\in O'$.

Proof. From Corollary (4.3.1), there is an open dense subset $O\subset EN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$, such that the germ of exact normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ is a Lagrangian stable germ for all $(\mathbf{x},\boldsymbol{\xi})\in O$ at any point $(u_0,t_0)\in U\times I$. As we know, T_{rp} is an open map, so we just need to take $O'=T_{rp}(O)$. □

4.4 Blaschke normal congruences

In this section we deal with one of the most important classes of equiaffine line congruences, which is the class of Blaschke normal congruences. Our goal is to provide a positive answer to the following conjecture from (IZUMIYA; SAJI; TAKEUCHI, 2003):

Conjecture. Germs of generic Blaschke affine normal congruences at any point are Lagrangian stable.

Taking this into account, let us regard $\mathbb{R}^4$ as a four-dimensional affine space with volume element given by $\omega(e_1,e_2,e_3,e_4) = \det(e_1,e_2,e_3,e_4)$, where $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{R}^4$. Let D be the standard flat connection on $\mathbb{R}^4$. Now, let us consider the affine support function ρ_p, given in (3.3) and fix an Euclidean inner product $\langle .,. \rangle$ in $\mathbb{R}^4$, then ρ_p is given by

$$\rho_p(u) = \langle p - \mathbf{x}(u), \boldsymbol{\nu}(u) \rangle, \tag{4.6}$$

where $\boldsymbol{\nu}$ is the conormal vector field relative to $\boldsymbol{\xi}$. Thus

$$\frac{\partial \rho}{\partial p_i}(u) = \boldsymbol{\nu}_i(u). \tag{4.7}$$

Remark 4.4.1. It follows from item 5.1.4 that the catastrophe set of ρ, which is also called the *Criminant set* of ρ, is

$$C_\rho = \{(u,p) : p = \mathbf{x}(u) + t\boldsymbol{\xi}(u), for some t \in \mathbb{R}\}.$$

Now we seek to prove that the family of affine support functions is a Morse family of functions. In order to do this, we prove first that the conormal vector field associated to an equiaffine vector field transversal to a non-degenerate hypersurface is an immersion.

Proposition 4.4.1. Let $\mathbf{x} : U \to \mathbb{R}^4$ be a non-degenerate hypersurface with transversal equiaffine vector field $\boldsymbol{\xi}$. The conormal vector field $\boldsymbol{\nu} : U \to \mathbb{R}^4$ relative to $\boldsymbol{\xi}$ is an immersion.

Proof. Let $\mathbf{x}(U) = M$. It follows from the fact that $\boldsymbol{\nu}$ is the conormal vector field of $\boldsymbol{\xi}$ that $\langle \boldsymbol{\nu}, \boldsymbol{\xi} \rangle = 1$ and $\langle \boldsymbol{\nu}, \mathbf{v} \rangle = 0$, for all tangent vector field $\mathbf{v}$. Thus, taking the derivative we get

$$\langle d\boldsymbol{\nu}_p(\mathbf{w}), \mathbf{v}(p) \rangle = -\langle \boldsymbol{\nu}(p), d\mathbf{v}_p(\mathbf{w}) \rangle = -\langle \boldsymbol{\nu}(p), (\nabla_{\mathbf{w}}\mathbf{v})_p + \mathbf{c}(\mathbf{w},\mathbf{v})_p\boldsymbol{\xi} \rangle = -\mathbf{c}(\mathbf{w},\mathbf{v})_p,$$

for all $p = \mathbf{x}(u)$ and $\mathbf{w} \in T_pM$. Then, $\mathbf{w}$ is a direction in the kernel of $d\boldsymbol{\nu}_p$ if and only if $\langle d\boldsymbol{\nu}_p(\mathbf{w}), \tilde{\mathbf{w}} \rangle = \langle \mathbf{0}, \tilde{\mathbf{w}} \rangle = -\mathbf{c}_p(\mathbf{w}, \tilde{\mathbf{w}}) = 0$, for all $\tilde{\mathbf{w}} \in T_pM$, but since $\mathbf{x}$ is non-degenerate it follows that $\mathbf{c}$ is non-degenerate, hence $\mathbf{w} = 0$ and $\boldsymbol{\nu}$ is an immersion. □

Proposition 4.4.2. Let $\mathbf{x} : U \to \mathbb{R}^4$ be a non-degenerate hypersurface with transversal equiaffine vector field $\boldsymbol{\xi}$. Then the family of germs of functions $\rho : \left(U \times \mathbb{R}^4, (u_0, p_0)\right) \to (\mathbb{R}, t_0)$, where $t_0 = \rho(u_0, p_0)$ and u_0 is a critical point of ρ_{p_0} is a Morse family of functions.

Proof. Let us denote $(u,p) = (u_1,u_2,u_3,p_1,p_2,p_3,p_4)$. From definition 2.4.6, in order to prove that ρ is a Morse family we need to prove that the map germ $\Delta : \left(U \times \mathbb{R}^4, (u_0,p_0)\right) \to \mathbb{R}^3$, given by

$$\Delta\rho(u,p) = \left(\frac{\partial \rho}{\partial u_1}, \frac{\partial \rho}{\partial u_2}, \frac{\partial \rho}{\partial u_3}\right)(u,p)$$

is not singular. Its jacobian matrix is given by

$$J(\Delta\rho)(u_0,p_0) = \begin{pmatrix} \dfrac{\partial^2\rho_{p_0}}{\partial u_1\partial u_1} & \dfrac{\partial^2\rho_{p_0}}{\partial u_1\partial u_2} & \dfrac{\partial^2\rho_{p_0}}{\partial u_1\partial u_3} & (\boldsymbol{\nu}_1)u_1 & (\boldsymbol{\nu}_2)u_1 & (\boldsymbol{\nu}_3)u_1 & (\boldsymbol{\nu}_4)u_1 \\ \dfrac{\partial^2\rho_{p_0}}{\partial u_1\partial u_2} & \dfrac{\partial^2\rho_{p_0}}{\partial u_2\partial u_2} & \dfrac{\partial^2\rho_{p_0}}{\partial u_2\partial u_3} & (\boldsymbol{\nu}_1)u_2 & (\boldsymbol{\nu}_2)u_2 & (\boldsymbol{\nu}_3)u_2 & (\boldsymbol{\nu}_4)u_2 \\ \dfrac{\partial^2\rho_{p_0}}{\partial u_1\partial u_3} & \dfrac{\partial^2\rho_{p_0}}{\partial u_2\partial u_3} & \dfrac{\partial^2\rho_{p_0}}{\partial u_3\partial u_3} & (\boldsymbol{\nu}_1)u_3 & (\boldsymbol{\nu}_2)u_3 & (\boldsymbol{\nu}_3)u_1 & (\boldsymbol{\nu}_4)u_3 \end{pmatrix}. \tag{4.8}$$

Since $\mathbf{x}: U \to \mathbb{R}^3$ is non-degenerate, it follows from proposition 4.4.1 that $\boldsymbol{\nu}: U \to \mathbb{R}^4$ is an immersion, therefore the jacobian matrix 4.8 has rank 3 and $\Delta\rho$ is not singular. □

The above proof is an alternative to that one presented in (LOPES; RUAS; SANTOS, 2022) for the same proposition.

Remark 4.4.2. It follows from the above proposition that the 4-parameter family of germs of functions $\rho : \big(U\times\mathbb{R}^4,(u_0,p_0)\big) \to (\mathbb{R},t_0)$, where u_0 is a critical point of ρ_{p_0}, is a Morse family. Furthermore, if $p_0 = \mathbf{x}(u_0)+t_0\boldsymbol{\xi}(u_0)$ (where $t_0 = \rho_{p_0}(u_0)$), the Lagrangian immersion associated to this Morse family is $L : (U\times\mathbb{R},(u_0,t_0)) \to T^*\mathbb{R}^4$, given by

$$L(u,t) = \left(\mathbf{x}(u)+t\boldsymbol{\xi}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u)\right),$$

whose Lagrangian map associated is $F_{(\mathbf{x},\boldsymbol{\xi})} = \pi\circ L(u,t) = \mathbf{x}(u)+t\boldsymbol{\xi}(u)$, where $\pi : T^*\mathbb{R}^4 \to \mathbb{R}^4$.

4.4.1 Blaschke Exact Normal Congruences

Here, we work with the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})} : U\times I \to \mathbb{R}^4$, where $\mathbf{x}: U\to\mathbb{R}^4$ is a non-degenerate regular surface and $\boldsymbol{\xi}: U\to\mathbb{R}^4$ is its Blaschke vector field (see definition 5.2.1). Let $Emb_{ng}(U,\mathbb{R}^4) = \{\mathbf{x}: U\to\mathbb{R}^4 : \mathbf{x}$ is a non-degenerate embedding$\}$ be the space of non-degenerate regular hypersurfaces with the Whitney C^∞- topology. Define the space of the Blaschke exact normal congruences as

$$BEN(U,\mathbb{R}^4\times(\mathbb{R}^4\setminus\{\mathbf{0}\})) = \big\{(\mathbf{x},\boldsymbol{\xi}) : \mathbf{x}\in Emb_{ng}(U,\mathbb{R}^4),\ \boldsymbol{\xi} \text{ is the Blaschke normal vector field of } \mathbf{x}\big\}.$$

Then, we identify (with the Whitney C^∞-topology) the spaces $Emb_{ng}(U,\mathbb{R}^4)$ and

$$S_{con}(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\}) = \big\{(\mathbf{x},\nu)\in C^\infty(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\}) : \mathbf{x}\in Emb_{ng}(U,\mathbb{R}^4) \text{ and } \boldsymbol{\nu} \text{ is the conormal of } \mathbf{x} \text{ relative to the Blaschke vector field}\big\}$$

Definition 4.4.1. Let $\mathbf{x}: U\to\mathbb{R}^4$, with $\mathbf{x}(U) = M$, be a non-degenerate hypersurface. We define the *conormal bundle* of M by

$$N^*_{\mathbf{x}} = \{(p,v) : p\in M,\ \langle v,w\rangle = 0,\ \forall\, w\in T_pM\} \subset T^*\mathbb{R}^4.$$

Remark 4.4.3. Note that we can look at $S_{con}(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\})$ as a section of the conormal bundle of M.

Let us define the following maps

$$\begin{aligned} H : \left(\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\times\mathbb{R}^4 &\to \mathbb{R} \\ (A,B,C) &\mapsto \langle B, C-A\rangle \end{aligned} \tag{4.9}$$

$$\begin{aligned} g : U &\to \mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\} \\ u &\mapsto (\mathbf{x}(u),\boldsymbol{\nu}(u)), \end{aligned} \tag{4.10}$$

where $g\in S_{con}(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\})$. If we fix a parameter C, $H_C:\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\}\to\mathbb{R}$ is a submersion, therefore, $H_C\circ g$ is a contact map. Finally, note that

$$\rho(u,p) = H\circ\left(g, I_d\big|_{\mathbb{R}^4}\right)(u,p).$$

Proposition 4.4.3. For a residual subset of $Emb_{ng}(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{\mathbf{0}\})$ the family ρ is locally $\mathcal{P}$-$\mathcal{R}^+$-versal.

Proof. Following the identification in remark 3.1.2 and the notation in remark 4.4.3 we can apply theorem 2.3.2 in order to show that there is a residual subset of $Emb_{ng}(U,\mathbb{R}^4\times\mathbb{R}^4\setminus\{0\})$ for which ρ is locally $\mathcal{P}$- $\mathcal{R}^+$-versal. □

Theorem 4.4.1. There is a residual subset $O\subset Emb_{ng}(U,\mathbb{R}^4)$ such that the germ of the Blaschke exact normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0)\in U\times I$ is a Lagrangian stable map germ for any $\mathbf{x}\in O$, i.e., $\forall\,\mathbf{x}\in O$, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table (2).

Proof. Let us take the map germ $F_{(\mathbf{x},\boldsymbol{\xi})}:(U\times\mathbb{R},(u_0,t_0))\to(\mathbb{R}^4,p_0)$. Thus u_0 is a critical point of ρ_{p_0}, by proposition 3.1.3. Then, $\rho:\left(U\times\mathbb{R}^3,(u_0,p_0)\right)\to(\mathbb{R},t_0)$ is a Morse family of functions. Furthermore, by Remark 4.4.1, the Lagrangian map related to this family is $F_{(\mathbf{x},\xi)}$. It is known that if ρ is $\mathcal{P}$-$\mathcal{R}^+$-versal, then $F_{(\mathbf{x},\boldsymbol{\xi})}$ is Lagrangian stable (see theorem 2.4.2), so the result follows from proposition 4.4.3. □

The map

$$\Pi : BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)\to Emb_{ng}(U,\mathbb{R}^4), \tag{4.11}$$

given by $\Pi(\mathbf{x},\boldsymbol{\xi})=\mathbf{x}$, is open and continuous. Thus, we obtain the following corollary, which proves the conjecture 3.2.1 given in (IZUMIYA; SAJI; TAKEUCHI, 2003).

Corollary 4.4.1. There is a residual subset $\mathcal{O}\subset BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$, such that the germ of the Blaschke exact normal congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $(u_0,t_0)\in U\times I$ is a Lagrangian stable map germ for any $(\mathbf{x},\boldsymbol{\xi})\in\mathcal{O}$, i.e., $\forall\,(\mathbf{x},\boldsymbol{\xi})\in\mathcal{O}$, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table (2).

4.4.2 Blaschke Normal Congruences

Let

$$BN(U,\mathbb{R}^4\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))=\{(\mathbf{x},\boldsymbol{\xi}):\exists t\in C^\infty(U,\mathbb{R}),\ s.t.\ \boldsymbol{y}(u)=\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u)\in Emb_{ng}(U,\mathbb{R}^4)\ \text{and}\ \boldsymbol{\xi}\ \text{is the Blaschke normal vector field of}\ \boldsymbol{y}\}$$

be the space of the Blaschke normal congruences. Alternatively we can look at this space as a subspace of $C^\infty(U,\mathbb{R}^4\times\mathbb{R}\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))$

$$BN(U,\mathbb{R}^4\times\mathbb{R}\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))=\{(\mathbf{x}(u),t(u),\boldsymbol{\xi}(u)):\boldsymbol{y}(u)=\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u)\in Emb_{ng}(U,\mathbb{R}^4)\ \text{and}\ \boldsymbol{\xi}\ \text{is the Blaschke normal vector field of}\ \boldsymbol{y}\}$$

In both cases, with the Whitney C^∞-topology.

The map

$$\begin{aligned}T_{rp}:C^\infty(U,\mathbb{R}^4\times\mathbb{R}\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))&\to C^\infty(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))\\(\mathbf{x}(u),t(u),\boldsymbol{\xi}(u))&\mapsto(\mathbf{x}(u)+t(u)\boldsymbol{\xi}(u),\boldsymbol{\xi}(u)),\end{aligned}$$

is open and continuous (see proposition 4.3.3) in the Whitney C^∞-topology. Notice that

$$BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)\subset C^\infty(U,\mathbb{R}^4\times\mathbb{R}\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))$$

with the following identification

$$BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)\ni(\mathbf{x},\boldsymbol{\xi})\sim(\mathbf{x},0,\boldsymbol{\xi}),$$

where $\mathbf{x}\in Emb_{ng}(U,\mathbb{R}^4)$ and $\boldsymbol{\xi}$ is its Blaschke normal vector field. Furthermore, we can look at the space of the Blaschke normal congruences as the space

$$\widetilde{BN}(U,\mathbb{R}^4\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))=T_{rp}\left(BN(U,\mathbb{R}^4\times\mathbb{R}\times(\mathbb{R}^4\setminus\{\mathbf{0}\}))\right). \tag{4.12}$$

Thus, $T_{rp}(BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right))=\widetilde{BN}\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$. Hence, we obtain the following theorem.

Theorem 4.4.2. There is a residual subset $\mathcal{O}'\subset\widetilde{BN}(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right))$, such that the germ of Blaschke normal congruence $F_{(\mathbf{x},\boldsymbol{e})}$ at any point $(u_0,t_0)\in U\times I$ is a Lagrangian stable map germ for any $(\mathbf{x},\boldsymbol{\xi})\in\mathcal{O}'$, i.e., $\forall(\mathbf{x},\boldsymbol{\xi})\in\mathcal{O}'$, $F_{(\mathbf{x},\boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table (2).

Proof. It is known that map T_{rp} is open and continuous and $T_{rp}(BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right))=\widetilde{BN}\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$. If $\mathcal{U}\subset BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$ is open and dense, then its image by T_{rp} is an open dense subset of $\widetilde{BN}\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$. Take $\mathcal{O}=\bigcap_{i\in\mathbb{N}}\mathcal{O}_i$ the residual subset of $BEN\left(U,\mathbb{R}^4\times\left(\mathbb{R}^4\setminus\{\mathbf{0}\}\right)\right)$ given in Corollary (4.4.1). We can show that $T_{rp}(\mathcal{O})=\mathcal{O}'=\bigcap_{i\in\mathbb{N}}\mathcal{O}'_i$, where $T_{rp}(\mathcal{O}_i)=\mathcal{O}'_i$, therefore $\mathcal{O}'$ is residual. □

Example 4.4.1. Taking into account (LEICHTWEISS, 1989)(section 2) and (LI *et al.*, 2015)(section 2.2.4) it is possible to parametrize a non-degenerate hypersurface M around an elliptic point, by considering not only $\mathcal{R}$-equivalence but also affine transformations of $\mathbb{R}^4$, as a graph of a function $h : U \to \mathbb{R}$, such that

$$\begin{aligned} h(u) = {} & 1/2(u_1^2+u_2^2+u_3^2)+a_{111}u_1u_2u_3+1/6\,(-a_{120}-a_{102})\,u_1^3+1/2a_{210}u_1^2u_2 \\ & +1/2a_{201}u_1^2u_3+1/6\,(-a_{210}-a_{012})\,u_2^3+1/2a_{120}u_1u_2^2+1/2a_{021}u_2^2u_3 \\ & +1/6\,(-a_{201}-a_{021})\,u_3^3+1/2a_{102}u_1u_3^2+1/2a_{012}u_2u_3^2+O(3). \end{aligned} \tag{4.13}$$

Here $O(3)$ means functions of order higher than 3. Since the group of affine transformations is different from the group of *Euclidean motions* (translations and rotations) it follows that this is not necessarily a local parametrization of M around an Euclidean umbilic point. Using this parametrization, the Blaschke normal vector of M at the origin is given by $(0,0,0,1)$. If we choose $a_{111} = a_{210} = a_{012} = a_{201} = 0$, $a_{120} = a_{102} = 1$ and $a_{021} = 2$, it follows that

$$h(u) = 1/2(u_1^2+u_2^2+u_3^2)-1/3u_1^3+1/2u_1u_2^2+1/2u_1u_3^2+u_2^2u_3-1/3u_3^3.$$

Using (3.5) we can compute the Blaschke normal vector field of M

$$\begin{aligned} \boldsymbol{\xi}(u) = {} & (6/5u_1+18/5u_1^2-17/5(u_2^2+u_3^2)+O(3), 2u_2-6u_1u_2-52/5u_2u_3+O(3), \\ & 2u_3-6u_1u_3-26/5(u_2^2-u_3^2)+O(3), 1+3/5u_1^2+u_2^2+u_3^2+O(3)). \end{aligned}$$

Furthermore, the congruence map $F_{(\mathbf{x},\boldsymbol{\xi})}(u,t) = \mathbf{x}(u_1,u_2,u_3)+t\boldsymbol{\xi}(u_1,u_2,u_3)$ has a singular point at $(0,0,0,-1/2)$ and its 2-jet at this point is given by

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})}(u,t) = {} & (2/5u_1-9/5u_1^2+17/10(u_2^2+u_3^2)+6/5\,(t+1/2)\,u_1, 3u_1u_2+26/5u_2u_3 \\ & +2\,(t+1/2)\,u_2, 3u_1u_3+13/5u_2^2-13/5u_3^2+2\,(t+1/2)\,u_3, t+1/5u_1^2). \end{aligned}$$

If we take $\lambda = s+\frac{1}{2} = t+\frac{1}{5}u_1^2$, then it is possible to verify that $F_{(\mathbf{x},\boldsymbol{\xi})}(u,\lambda)$ is an elliptic umbilic singularity.

Example 4.4.2. Let us take a non-degenerate hypersurface given by the graph of

$$\begin{aligned} h(u) = {} & -1/2u_1^2-1/2u_2^2+1/2u_3^2+1/6u_1^3-1/2u_1^2u_2 \\ & +1/2u_1u_3^2+1/3u_2^3+1/2u_2u_3^2. \end{aligned} \tag{4.14}$$

Then, in a similar way to the last example, it is possible to verify that the map $F_{(\mathbf{x},\boldsymbol{\xi})}$, where $\mathbf{x}(u_1,u_2,u_3) = (u_1,u_2,u_3,h(u_1,u_2,u_3))$ and $\boldsymbol{\xi}$ is the Blaschke normal vector field of $\mathbf{x}$, has a hyperbolic umbilic singularity at $(0,0,0,5/4)$.

Example 4.4.3. By taking a non-degenerate hypersurface given by the graph of

$$h(u) = 1/2(-u_1^2-u_2^2+u_3^2)+2u_1u_2u_3+1/2u_1u_2^2+1/2u_1u_3^2+1/4u_2^4 \tag{4.15}$$

it follows, in a similar way to the first example, that the map $F_{(\mathbf{x},\boldsymbol{\xi})}$, associated to the Blaschke exact normal congruence, has a parabolic umbilic singularity at $(0,0,0,-5/6)$.

CHAPTER

5

EQUIAFFINE STRUCTURE FOR FRONTALS

In this chapter we generalize the idea of equiaffine structure to the case of frontals and define the Blaschke vector field of a frontal. We also investigate some necessary and sufficient conditions that a frontal needs to satisfy to have a Blaschke vector field and provide some examples. Finally, taking the theory developed here into account a fundamental theorem, which is a version for frontals of the fundamental theorem of affine differential geometry, is shown. The results presented here can also be found in the paper (SANTOS, 2022), submitted for publication. In 5.4 we briefly discuss some problems we want to deal with in future research taking into consideration the theory developed here.

5.1 Equiaffine structure on frontals

In this section we define equiaffine transversal vector fields to a frontal similarly as defined in (NOMIZU; KATSUMI; SASAKI, 1994) when considering regular surfaces, however as we are dealing with frontals, we need to take into account tangent moving basis and the limiting tangent planes.

5.1.1 The case of the unit normal vector field

Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega} : U \to M_{3\times 2}(\mathbb{R})$ a tmb of $\mathbf{x}$, where $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ and $\mathbf{n} : U \to \mathbb{R}^3$ the unit normal vector field along $\mathbf{x}$. For each $q \in U$ we decompose

$$\mathbb{R}^3 = T_\Omega(q) \oplus \langle \mathbf{n}(q) \rangle_{\mathbb{R}}.$$

Using this decomposition we get

$$\mathbf{w}_{i_{u_j}} = \mathcal{T}^1_{ij}\mathbf{w}_1 + \mathcal{T}^2_{ij}\mathbf{w}_2 + p_{ij}\mathbf{n}, \tag{5.1}$$

where the symbols $\mathcal{T}^k_{ij}$, $i, j, k = 1, 2$ are those in (3.27) and (3.28). Note that $p_{ij} = \langle (\mathbf{w}_i)_{u_j}, \mathbf{n} \rangle$, thus the matrix $\begin{pmatrix} p_{ij} \end{pmatrix}$ coincide with the matrix (3.29).

Remark 5.1.1. If we define a bilinear form $p_\Omega(q) : T_\Omega \times T_\Omega \to \mathbb{R}$, given by $p_\Omega(q)(\mathbf{w}_i, \mathbf{w}_j) = p_{ij} = \langle \mathbf{w}_{i_{u_j}}(q), \mathbf{n}(q)\rangle$, then the matrix of p_Ω relative to the basis $\boldsymbol{\Omega}$ is $\mathbf{II}_\Omega$ and p_Ω is non-degenerate if and only if $\mathbf{II}_\Omega$ is non-singular, which is equivalent to say that the $\boldsymbol{\Omega}$-relative curvature K_Ω is non-zero (see definition 3.4.3).

Definition 5.1.1. A proper frontal is said to be a *non-parabolic frontal* if for some tmb $\boldsymbol{\Omega}$ the relative curvature K_Ω never vanishes.

Remark 5.1.2. It follows from corollary 3.23 in (MEDINA-TEJEDA, 2022a) that a frontal $\mathbf{x} : U \to \mathbb{R}^3$ is a wavefront if and only if, $(K_\Omega, H_\Omega) \neq \mathbf{0}$ on $\Sigma(\mathbf{x})$, for whatever tangent moving base $\boldsymbol{\Omega}$. Therefore, every non-parabolic frontal is a wavefront.

Proposition 5.1.1. The notion of non-parabolicity is independent of tmb.

Proof. Via proposition 3.18 in (MEDINA-TEJEDA, 2022a) the zeros of K_Ω are independent of tmb. □

The next proposition provides a representation formula for non-parabolic frontals.

Proposition 5.1.2. Let $\mathbf{x} : (U,0) \to (\mathbb{R}^3,0)$ be a germ of non-parabolic frontal, $\boldsymbol{\Omega}$ a tmb of $\mathbf{x}$ and $0 \in \Sigma(\mathbf{x})$. Then, up to an isometry $\mathbf{x}$ is $\mathcal{R}$-equivalent to

$$\mathbf{y}(u_1,u_2) = (a(u_1,u_2), b(u_1,u_2), \int_0^{u_1} (t_1 a_{u_1}(t_1,u_2) + u_1 b_{u_1}(t_1,u_2))dt_1 + \int_0^{u_2} t_2 b_{u_2}(0,t_2)dt_2),$$

where a, b are smooth functions such that $a_{u_2} = b_{u_1}$.

Proof. See proposition 4.1 in (MEDINA-TEJEDA, 2022b). □

5.1.2 The case of a transversal vector field

Let $\boldsymbol{\xi} : U \to \mathbb{R}^3$ be a vector field which is transversal to the frontal $\mathbf{x} : U \to \mathbb{R}^3$ i.e. $\boldsymbol{\xi}(q) \notin T_\Omega$ for all $q \in U$, where $\boldsymbol{\Omega} = \begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$ is a tmb. Thus, for each $q \in U$ we can decompose

$$\mathbb{R}^3 = T_\Omega(q) \oplus \langle \boldsymbol{\xi}(q)\rangle_{\mathbb{R}}.$$

In the same way that occurs with the unit normal vector field, we write

$$\mathbf{w}_{i_{u_j}} = \mathcal{D}^1_{ij}\mathbf{w}_1 + \mathcal{D}^2_{ij}\mathbf{w}_2 + h_{ij}\boldsymbol{\xi}. \tag{5.2}$$

and we obtain a bilinear form $h_\Omega(q) : T_\Omega \times T_\Omega \to \mathbb{R}$, such that $h_\Omega(q)(\mathbf{w}_i, \mathbf{w}_j) = h_{ij}(q)$. We call h_Ω the *relative affine fundamental form* of $\mathbf{x}$ induced by $\boldsymbol{\xi}$. In a similar way, we write

$$\boldsymbol{\xi}_{u_i} = -S^1_i\mathbf{w}_1 - S^2_i\mathbf{w}_2 + \tau_i\boldsymbol{\xi}. \tag{5.3}$$

Then for each $q \in U$ we have $S_\Omega(q) : T_\Omega(q) \to T_\Omega(q)$, such that $S_\Omega(q)(\mathbf{w}_i) = S^1_i\mathbf{w}_1 + S^2_i\mathbf{w}_2$ and $\tau_\Omega(q) : T_\Omega(q) \to \mathbb{R}$, such that $\tau_\Omega(q)(\mathbf{w}_i) = \tau_i$, $i = 1,2$. We call S_Ω the *relative shape operator* of $\boldsymbol{\xi}$ and τ_Ω the *relative transversal connection form*.

Definition 5.1.2. The vector $\boldsymbol{\xi}$ defines an *equiaffine structure* on $\mathbf{x}$ (or $\boldsymbol{\xi}$ is *equiaffine*) when the derivatives of $\boldsymbol{\xi}$ are in $T_\Omega(q)$, for all $q \in U$, i.e. when $\tau_\Omega \equiv 0$.

For a frontal $\mathbf{x} : U \to \mathbb{R}^3$ and an equiaffine transversal vector field $\boldsymbol{\xi} : U \to \mathbb{R}^3$ we say that the symbols $\mathcal{D}^k_{ij}$ and h_{ij}, given in (5.2), define an equiaffine structure on $\mathbf{x}$.

Definition 5.1.3. Given $\mathbf{v}_1, \mathbf{v}_2 \in T_\Omega$, we define

$$\theta(\mathbf{v}_1, \mathbf{v}_2) := \tilde{\omega}(\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\xi}),$$

where $\tilde{\omega}$ is the canonical volume element in $\mathbb{R}^3$, that is $\tilde{\omega}(e_1, e_2, e_3) = \det(e_1, e_2, e_3)$, for $\{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^3$. The volume element θ is called the *induced volume element*.

Proposition 5.1.3. If $\mathbf{x}$ is a non-parabolic frontal then the relative fundamental form induced by a transversal vector field is non-degenerate.

Proof. Let $\boldsymbol{\xi}$ be a transversal vector field, thus we can write

$$\boldsymbol{\xi} = \phi \mathbf{n} + Z,$$

where Z is tangent and $\phi : U \to \mathbb{R} \setminus 0$ is smooth. Hence,

$$\frac{\boldsymbol{\xi} - Z}{\phi} = \mathbf{n}$$

and

$$\begin{aligned}
\mathbf{w}_{iu_j} &= \mathcal{T}^1_{ij}\mathbf{w}_1 + \mathcal{T}^2_{ij}\mathbf{w}_2 + p_{ij}\mathbf{n} \\
&= \mathcal{T}^1_{ij}\mathbf{w}_1 + \mathcal{T}^2_{ij}\mathbf{w}_2 + p_{ij}\frac{\boldsymbol{\xi} - Z}{\phi} \\
&= \left(\mathcal{T}^1_{ij}\mathbf{w}_1 + \mathcal{T}^2_{ij}\mathbf{w}_2 - \frac{p_{ij}}{\phi}Z\right) + \frac{p_{ij}}{\phi}\boldsymbol{\xi} \\
&= \mathcal{D}^1_{ij}\mathbf{w}_1 + \mathcal{D}^2_{ij}\mathbf{w}_2 + h_{ij}\boldsymbol{\xi}.
\end{aligned}$$

Therefore, $h_{ij} = \dfrac{p_{ij}}{\phi}$, $i, j = 1, 2$. From this, $\left(p_{ij}\right)$ is non-singular if and only if $\left(h_{ij}\right)$ is non-singular. As $\mathbf{x}$ is non-parabolic, the result follows from remark 5.1.1. □

Proposition 5.1.4. Let $\boldsymbol{\xi} : U \to \mathbb{R}^3$ be a vector field which is transversal to a frontal $\mathbf{x} : U \to \mathbb{R}^3$. Let us suppose that

$$\boldsymbol{\xi} = \phi \mathbf{n} + Z,$$

where $Z(u) = a(u)\mathbf{w}_1(u) + b(u)\mathbf{w}_2(u) \in T_\Omega$, $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ and $\phi(u) \neq 0$, for all $u \in U$. Then h_Ω and τ_Ω satisfy

(a) $h_\Omega = \frac{1}{\phi} p_\Omega$.

(b) $\tau_\Omega(\mathbf{w}_i) = \frac{1}{\phi}\left(p_\Omega(Z, \mathbf{w}_i) + \phi_{u_i}\right)$.

Proof. Note that 5.1.4 was proved in proposition (5.1.3). In order to prove 6.8 let us write $B_\Omega = \begin{pmatrix} b_{ij} \end{pmatrix}$ the relative shape operator of $\mathbf{n}$. We know that

$$\boldsymbol{\xi}_{u_i} = -S_\Omega(\mathbf{w}_i) + \tau_\Omega(\mathbf{w}_i)\boldsymbol{\xi} = -S_\Omega(\mathbf{w}_i) + \tau_\Omega(\mathbf{w}_i)Z + \tau_\Omega(\mathbf{w}_i)\phi\mathbf{n}. \tag{5.4}$$

On the other hand,

$$\begin{aligned} \boldsymbol{\xi}_{u_i}(q) &= -\phi B_\Omega(q)(\mathbf{w}_i) + \phi_{u_i}\mathbf{n} + Z_{u_i} \\ &= \left(Z_{u_i}^{\top} - \phi B_\Omega(\mathbf{w}_i)\right) + \left(\phi_{u_i} + p_\Omega(Z, \mathbf{w}_i)\right)\mathbf{n}, \end{aligned} \tag{5.5}$$

where $Z_{u_i}^{\top}$ is the tangent component of Z_{u_i}. By comparing the normal components of (5.4) and (5.5), it follows that $\tau_\Omega(\mathbf{w}_i) = \frac{1}{\phi}\left(p_\Omega(Z, \mathbf{w}_i) + \phi_{u_i}\right)$. □

Remark 5.1.3. Let us take $\mathbf{x} : U \to \mathbb{R}$ a frontal and $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ a tmb. If $\boldsymbol{\xi} : U \to \mathbb{R}^3$, given by $\boldsymbol{\xi} = \phi\mathbf{n} + a\mathbf{w}_1 + b\mathbf{w}_2$, is an equiaffine transversal vector field, then $\tau_\Omega \equiv 0$. Via proposition 5.1.4, $\tau_\Omega(\mathbf{w}_i) = \frac{1}{\phi}\left(p_\Omega(a\mathbf{w}_1 + b\mathbf{w}_2, \mathbf{w}_i) + \phi_{u_i}\right)$. Since $\tau_\Omega \equiv 0$, we get in $U \setminus \Sigma(\mathbf{x})$

$$\begin{aligned} p_\Omega(a\mathbf{w}_1 + b\mathbf{w}_2, \mathbf{w}_1) &= a p_\Omega(\mathbf{w}_1, \mathbf{w}_1) + b p_\Omega(\mathbf{w}_2, \mathbf{w}_1) = a e_\Omega + b f_{2\Omega} = -\phi_{u_1} \\ p_\Omega(a\mathbf{w}_1 + b\mathbf{w}_2, \mathbf{w}_2) &= a p_\Omega(\mathbf{w}_1, \mathbf{w}_2) + b p_\Omega(\mathbf{w}_2, \mathbf{w}_2) = a f_{1\Omega} + b g_\Omega = -\phi_{u_2}, \end{aligned}$$

therefore,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix}^{-1} \begin{pmatrix} -\phi_{u_1} \\ -\phi_{u_2} \end{pmatrix} \text{ in } U \setminus \Sigma(\mathbf{x}). \tag{5.6}$$

Since $a, b \in C^\infty(U, \mathbb{R})$ and $U \setminus \Sigma(\mathbf{x})$ is dense, it follows that for a point $q \in \Sigma(\mathbf{x})$ we have

$$\begin{pmatrix} a(q) \\ b(q) \end{pmatrix} = \lim_{u \to q} \begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix}^{-1} \begin{pmatrix} -\phi_{u_1} \\ -\phi_{u_2} \end{pmatrix}.$$

Note in remark 3.1.2 that we could write the vector field W considering a tmb $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ instead of the usual tmb $D\mathbf{x} = \begin{pmatrix} \mathbf{x}_{u_1} & \mathbf{x}_{u_2} \end{pmatrix}$. In this case, we would obtain an expression like (5.6) instead of (3.7).

5.2 The Blaschke and the conormal vector fields of a frontal

The Blaschke vector field and the conormal vector field associated to an equiaffine transversal vector field play an important role when studying regular surfaces from the affine differential geometry viewpoint. With the Blaschke structure, for instance, we define proper and improper affine spheres (see chapter 2 in (NOMIZU; KATSUMI; SASAKI, 1994)) and the Blaschke line congruences (see 4.4.1 or section 6 in (LOPES; RUAS; SANTOS, 2022)). On the other hand, the conormal vector field makes calculations with the affine support function easier, for instance (see section 1 in (CECIL, 1994)). Taking into account the importance of these two objects, in this section we define the Blaschke vector field of a frontal and the conormal vector field associated to an equiaffine vector field transversal to a frontal.

5.2.1 The Blaschke vector field of a frontal

Definition 5.2.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal such that its Gaussian curvature never vanishes in $U \setminus \Sigma(\mathbf{x})$. We say that a transversal vector field $\boldsymbol{\xi}$ is the *Blaschke vector field* of $\mathbf{x}$ if it is a smooth extension of the usual Blaschke vector field defined in $U \setminus \Sigma(\mathbf{x})$.

It follows from the density of $U \setminus \Sigma(\mathbf{x})$ and from the fact that the Blaschke vector field is unique up to sign (see (NOMIZU; KATSUMI; SASAKI, 1994)) that the above extension is unique. Now, looking at the special affine group (or equiaffine group)

$$\mathbf{SA}(3,\mathbb{R}) = \{\Phi : x \mapsto \mathbf{A}x + \mathbf{b} : \mathbf{A} \in M_3(\mathbb{R}), \det \mathbf{A} = 1 \text{ and } \mathbf{b} \text{ is a constant vector}\}$$

we seek to show an invariance property for the Blaschke vector field defined here, in the following sense. Given a frontal $\mathbf{x}: U \to \mathbb{R}^3$ and $\Phi \in \mathbf{SA}(3,\mathbb{R})$, the Blaschke vector field of $\mathbf{y} = \Phi \circ \mathbf{x}$ is $\overline{\boldsymbol{\xi}}$, where

$$\overline{\boldsymbol{\xi}}(q) = \Phi_* \boldsymbol{\xi}(q), \text{ for all } q \in U.$$

Proposition 5.2.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal for which the Gaussian curvature has a non-vanishing smooth extension. If there exists, the Blaschke vector field of $\mathbf{x}$ is an equiaffine invariant.

Proof. Let $\Phi(x) = \mathbf{A}x + \mathbf{b}$ be an equiaffine transformation, then $\mathbf{y} = \Phi \circ \mathbf{x}$ has the same singular set of $\mathbf{x}$. Furthermore, it is known that in $U \setminus \Sigma(\mathbf{x})$ the usual Blaschke vector field of $\mathbf{y}$ is given by $\overline{\boldsymbol{\xi}} = \Phi_* \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the Blaschke vector field of $\mathbf{x}$. Let us keep the same notation for the extension of $\boldsymbol{\xi}$, then $\Phi_* \boldsymbol{\xi}$ is an extension of $\overline{\boldsymbol{\xi}}$ to U, so it is the Blaschke vector field of $\mathbf{y}$. □

Remark 5.2.1. Note that is not always possible to obtain a smooth extension of the Blaschke vector field. For instance, if $\mathbf{x}: U \to \mathbb{R}^3$ is a non-parabolic frontal, i.e., for all tmb $\boldsymbol{\Omega}$, the $\boldsymbol{\Omega}$-relative curvature $K_\Omega \neq 0$, then the Gaussian curvature is not extendable (see proposition 4.1 in (MEDINA-TEJEDA, 2020)). It is known that the usual Blaschke vector field is given in $U \setminus \Sigma(\mathbf{x})$ by $|K|^{1/4}\mathbf{n} + W$, where W is a tangent vector field (see remark 3.1.2), thus $\langle \boldsymbol{\xi}, \mathbf{n} \rangle = |K|^{1/4}$. Since it is not possible to extend K, it follows that is not possible to extend this vector field. From this, it follows that a necessary condition to obtain the Blaschke vector field of a frontal is that its Gaussian curvature is extendable.

Next, we characterize frontals for which it is possible to define the Blaschke vector field.

Theorem 5.2.1. A frontal $\mathbf{x}: U \to \mathbb{R}^3$ admits a Blaschke vector field if and only if its Gaussian curvature K has a non-vanishing extension to U and there are $a, b \in C^\infty(U, \mathbb{R})$ such that

$$\begin{pmatrix} a(q) \\ b(q) \end{pmatrix} = \lim_{u \to q} \begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix}^{-1} \begin{pmatrix} -\phi_{u_1} \\ -\phi_{u_2} \end{pmatrix}, \text{ for all } q \in U, \tag{5.7}$$

where $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ is a tmb and $\phi = |K|^{1/4}$.

Proof. Let $\boldsymbol{\xi} = \phi \mathbf{n} + a\mathbf{w}_1 + b\mathbf{w}_2$ be the Blaschke vector field of $\mathbf{x}$. Then, $\phi, a, b \in C^\infty(U, \mathbb{R})$ and $\phi = \langle \boldsymbol{\xi}, \mathbf{n} \rangle = |K|^{1/4}$ in $U \setminus \Sigma(\mathbf{x})$ (see remark 3.1.2). It follows from the density of $U \setminus \Sigma(\mathbf{x})$ and from the smoothness of ϕ that $\phi(q) = \lim_{u \to q} |K|^{1/4}$. From the fact that $\boldsymbol{\xi}$ is transversal to $\mathbf{x}$, it follows that $\phi \neq 0$ in U, consequently K admits a non-vanishing extension to U. As $\boldsymbol{\xi}$ is equiaffine in $U \setminus \Sigma(\mathbf{x})$, it follows from remark 5.1.3 that

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix}^{-1} \begin{pmatrix} -\phi_{u_1} \\ -\phi_{u_2} \end{pmatrix} \text{ in } U \setminus \Sigma(\mathbf{x}).$$

Since $a, b \in C^\infty(U, \mathbb{R})$ and $U \setminus \Sigma(\mathbf{x})$ is dense, we obtain for a point $q \in \Sigma(\mathbf{x})$ that

$$\begin{pmatrix} a(q) \\ b(q) \end{pmatrix} = \lim_{u \to q} \begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix}^{-1} \begin{pmatrix} -\phi_{u_1} \\ -\phi_{u_2} \end{pmatrix}.$$

Reciprocally, considering K the non-vanishing extension of the Gaussian curvature of $\mathbf{x}$, define $\phi = |K|^{1/4}$ and take $a, b \in C^\infty(U, \mathbb{R})$ satisfying (5.7), then it follows from definition 5.2.1 and from remarks 3.1.2 and 5.1.3 that $\boldsymbol{\xi}$ is the Blaschke vector field of $\mathbf{x}$. □

Remark 5.2.2.

(a) Let $\mathbf{x}: U \to \mathbb{R}^3$ be a frontal in the class 3.4.1.1, such that its Gaussian curvature K admits a non-vanishing extension, so in order to have a Blaschke vector field we just need to verify the condition (5.7) in theorem 5.2.1. One can verify that $K_{u_2} = \tilde{f} b_{u_2}$, for a smooth function $\tilde{f}$ is a sufficient condition for this to happen. For instance, with any of the choices below

- $b = u_2^2$, $r = 0$, $l = 1$ and $h = h(u_1, u_2)$ any smooth function (see example 5.2.1),
- $b = \frac{2}{5}u_2^5 + u_2^2$, $r = 0$, $l = 1$ and $h = h(u_1, u_2)$ any smooth function (see example 5.2.2),

we obtain $\mathbf{x}$ for which K admits a non-vanishing extension and condition (5.7) is verified, hence we have a Blaschke vector field.

(b) If $\mathbf{x}: U \to \mathbb{R}^3$ is a wave front given by the class 3.4.1.2, such that the Blaschke vector field exists in the regular part $U \setminus \Sigma(\mathbf{x})$, then it is given by $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$, where

$$\boldsymbol{\xi}_1 = -\frac{1}{4}\frac{c_{u_1}}{c^{3/4}h_{u_1u_1}}, \tag{5.8a}$$

$$\boldsymbol{\xi}_2 = -\frac{1}{4}\frac{c_{u_2}h_{u_1u_1} - c_{u_1}h_{u_1u_2}}{c^{3/4}h_{u_1u_1}}, \tag{5.8b}$$

$$\boldsymbol{\xi}_3 = \frac{1}{4}\frac{-u_2c_{u_2}h_{u_1u_1} + u_2c_{u_1}h_{u_1u_2} + 4ch_{u_1u_1} - c_{u_1}h_{u_1}}{c^{3/4}h_{u_1u_1}}. \tag{5.8c}$$

Note in (5.8) that is not always possible to extend $\boldsymbol{\xi}$, since $h_{u_1u_1}(q) = 0$ for all $q \in \Sigma(\mathbf{x})$. However, if we take for instance, c a smooth function such that $c_{u_1} = \tilde{g}h_{u_1u_1}$, for a smooth function $\tilde{g}$, then $\boldsymbol{\xi}$ admits an extension to the entire U. If $\tilde{g} = 0$, we get $c = c(u_2)$, which is satisfied for the case c constant, for instance (see example 5.2.3). It is worth observing that for c constant, we get $\boldsymbol{\xi} = (0,0,\rho)$, for some $\rho \in \mathbb{R}$ and if we think of frontal improper affine spheres as those frontals for which the Blaschke vector field is constant, then this choice of c provides a class of this type of frontals. This is an important class, specially if we seek to understand frontals from the affine viewpoint, and will be further discussed in future works. It is also important to remark that improper affine spheres with singularities is a topic of interest in differential geometry, see (CRAIZER; DOMITRZ; RIOS, 2020), (ISHIKAWA; MACHIDA, 2006) (MARTÍNEZ, 2005), (MILÁN, 2013) and (NAKAJO, 2009), for instance.

5.2.2 Examples

Now, we provide some examples taking into account the classes described in 3.4.1.

Example 5.2.1. Let $\mathbf{x}: U \to \mathbb{R}^3$ defined by $\mathbf{x} = (u_1, u_2^2, 4/15u_1u_2^5 + 1/2u_1^3u_2^4 + u_1u_2^2)$, where $U = (-1,1) \times (-4,4)$ (see figure 2). This frontal is a cuspidal cross-cap obtained from a $5/2$-cuspidal edge and satisfies $\mathbf{x} \sim \mathbf{n}$, where $\mathbf{n}$ is its unit normal. We decompose $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}^T$, where

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u_2^2(4/15u_2^3 + 3/2u_1^2u_2^2 + 1) & 1/3u_1\left(3u_1^2u_2^2 + 2u_2^3 + 3\right) \end{pmatrix} \text{ and } \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 2u_2 \end{pmatrix}.$$

We have that $\lambda_\Omega = 2u_2$ and

$$K_\Omega = \frac{18 \times 10^4 u_2\left(54{u_1}^4{u_2}^4 + 9{u_1}^2{u_2}^5 + 4{u_2}^6 + 54{u_1}^2{u_2}^2 + 12{u_2}^3 + 9\right)}{\mu^2},$$

where

$$\mu = 2025{u_1}^4{u_2}^8 + 720{u_1}^2{u_2}^9 + 900{u_1}^6{u_2}^4 + 64{u_2}^{10} + 1200{u_1}^4{u_2}^5 + 3100{u_1}^2{u_2}^6$$
$$+ 480{u_2}^7 + 1800{u_1}^4{u_2}^2 + 1200{u_1}^2{u_2}^3 + 900{u_2}^4 + 900{u_1}^2 + 900.$$

Therefore, considering that at a regular point the Gaussian curvature is given by $\dfrac{K_\Omega}{\lambda_\Omega}$, we obtain that the extension of the Gaussian curvature is

$$K = \frac{9\times 10^4\left(54u_1{}^4u_2{}^4+9u_1{}^2u_2{}^5+4u_2{}^6+54u_1{}^2u_2{}^2+12u_2{}^3+9\right)}{\mu^2}.$$

Then, writing $\phi = |K|^{1/4}$, the Blaschke vector field of $\mathbf{x}$ is given by $\boldsymbol{\xi} = \phi\mathbf{n} + a\mathbf{w}_1 + b\mathbf{w}_2$, where a and b are obtained using (5.6). Thus, $\boldsymbol{\xi} = \dfrac{1}{\rho^{7/4}}\left(\dfrac{-3\sqrt{3}}{8}\xi_1, \dfrac{9\sqrt{3}}{8}\xi_2, \dfrac{\sqrt{3}}{240}\xi_3\right)$, where

$$\begin{aligned}\xi_1 = {} & 216\,u_1{}^6u_2{}^4 - 189\,u_1{}^4u_2{}^5 + 66\,u_1{}^2u_2{}^6 + 16\,u_2{}^7 + 324\,u_1{}^4u_2{}^2 \\ & + 9\,u_1{}^2u_2{}^3 + 48\,u_2{}^4 + 108\,u_1{}^2 + 36\,u_2\end{aligned}$$

$$\begin{aligned}\xi_2 = {} & \left(216\,u_1{}^4u_2{}^4 + 87\,u_1{}^2u_2{}^5 - 16\,u_2{}^6 + 252\,u_1{}^2u_2{}^2 + 24\,u_2{}^3 + 72\right)u_2{}^2 \\ \xi_3 = {} & 145800\,u_1{}^8u_2{}^8 + 35721\,u_1{}^6u_2{}^9 + 25326\,u_1{}^4u_2{}^{10} + 4896\,u_1{}^2u_2{}^{11} + 277020\,u_1{}^6u_2{}^6 \\ & + 896\,u_2{}^{12} + 114129\,u_1{}^4u_2{}^7 + 39204\,u_1{}^2u_2{}^8 + 5088\,u_2{}^9 + 179820\,u_1{}^4u_2{}^4 + 88938\,u_1{}^2u_2{}^5 \\ & + 12096\,u_2{}^6 + 48600\,u_1{}^2u_2{}^2 + 14040\,u_2{}^3 + 6480 \\ \rho = {} & 54\,u_1{}^4u_2{}^4 + 9\,u_1{}^2u_2{}^5 + 4\,u_2{}^6 + 54\,u_1{}^2u_2{}^2 + 12\,u_2{}^3 + 9.\end{aligned}$$

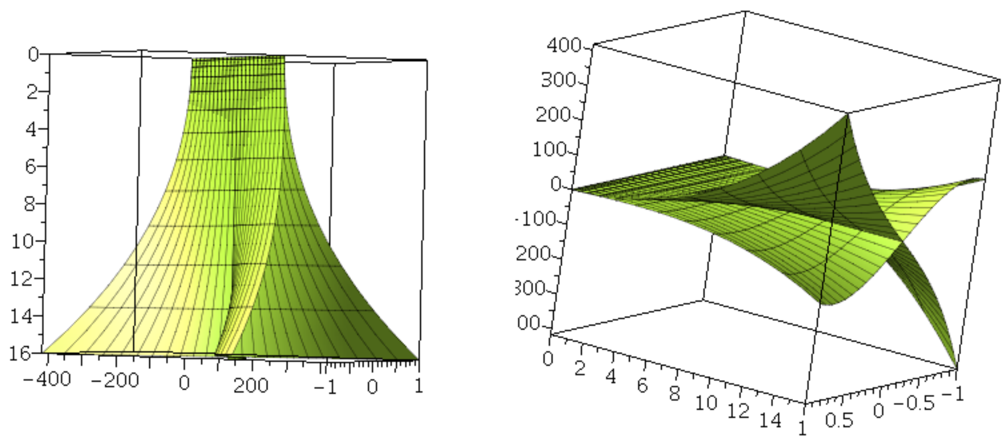

Figure 2 – Frontal with extendable non-vanishing Gaussian curvature.

Remark 5.2.3. It is worth observing that the Blaschke vector field $\boldsymbol{\xi}$ of a frontal is also a frontal, since $\boldsymbol{\xi}$ is equiaffine. In example 5.2.1, $\Sigma(\boldsymbol{\xi})$ is given in figure 3.

Then, the Blaschke vector field is a proper frontal and it is given in figure 4, restricting the domain to $(-1/10, 1/10)\times(-1/60, 1/60)$.

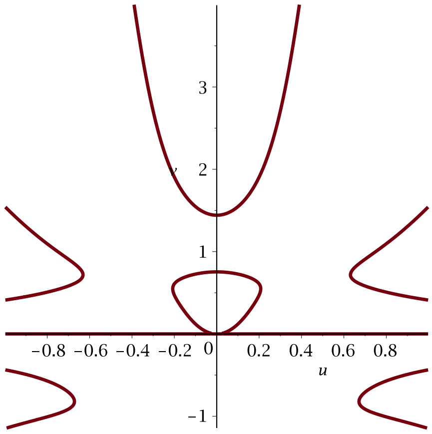

Figure 3 – Singular set $\Sigma(\boldsymbol{\xi})$ of the Blaschke vector field.

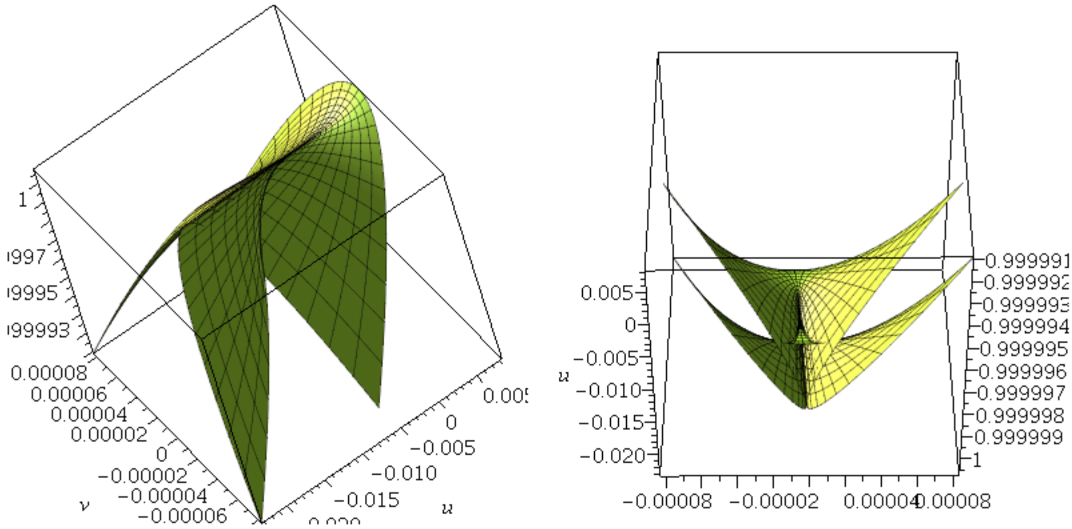

Figure 4 – The Blaschke vector field from example 5.2.1

Example 5.2.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ defined by $\mathbf{x} = \left(u_1, \frac{2}{5}u_2^5 + u_2^2, u_1 u_2^2\right)$, for $U = (-1,1) \times (-1,1)$ (see figure 8). This frontal satisfies $\mathbf{x} \sim \mathbf{n}$, where $\mathbf{n}$ is its unit normal. We decompose $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}^T$, where

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & u_2^3 + 1 \\ u_2^2 & u_1 \end{pmatrix} \text{ and } \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 2u_2 \end{pmatrix}.$$

We have that $\lambda_\Omega = 2u_2$ and

$$K_\Omega = \frac{2u_2(u_2+1)^2(u_2^2 - u_2 + 1)^2}{(u_2^{10} + 2u_2^7 + u_2^6 + u_2^4 + 2u_2^3 + u_1^2 + 1)^2}.$$

Then,

$$K = \frac{(u_2+1)^2(u_2^2 - u_2 + 1)^2}{(u_2^{10} + 2u_2^7 + u_2^6 + u_2^4 + 2u_2^3 + u_1^2 + 1)^2}$$

is the extension of the Gaussian curvature to U. In a similar way to example 5.2.1, we obtain that the Blaschke vector field of $\mathbf{x}$ is given by

$$\boldsymbol{\xi} = \frac{1}{4(u_2^2+u_2+1)^{3/2}(u_2+1)^{3/2}}(3u_2, 0, 7u_2^3+4).$$

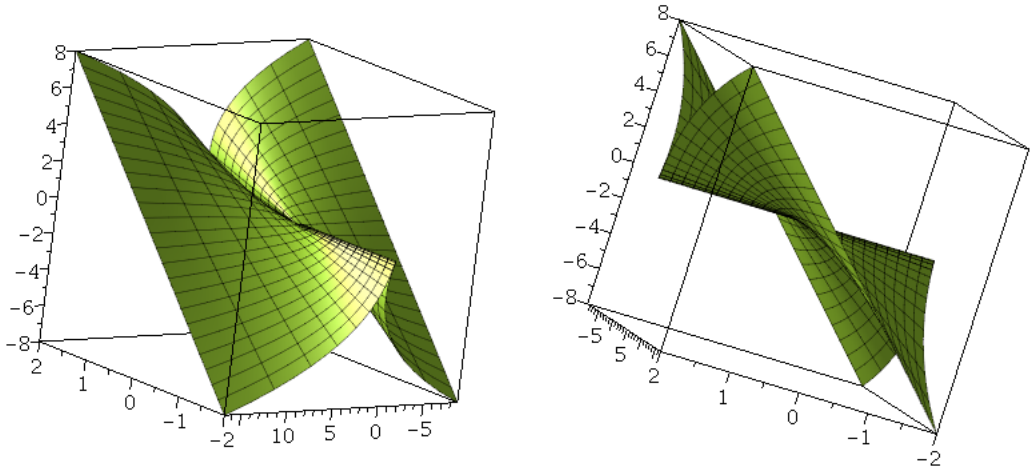

Figure 5 – Frontal with extendable non-vanishing Gaussian curvature.

Example 5.2.3. Let $\mathbf{x}:\mathbb{R}^2\to\mathbb{R}^3$ defined by $\mathbf{x}=(u_1, -12u_1{}^2u_2+4u_2{}^3, u_1{}^4+6u_1{}^2u_2{}^2-3u_2{}^4)$ (see figure 6). This is a wave front of rank 1 for which the Gaussian curvature admits a non-vanishing extension. We decompose $D\mathbf{x}=\boldsymbol{\Omega}\boldsymbol{\Lambda}^T$, where

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ -24u_1u_2 & 1 \\ 4u_1^3+12u_1u_2^2 & -u_2 \end{pmatrix} \text{ and } \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -12u_1^2+12u_2^2 \end{pmatrix}.$$

We have that $\lambda_\Omega = -12u_1^2+12u_2^2$ and

$$K_\Omega = \frac{12(u_1{}^2-u_2{}^2)}{(16u_1{}^6-96u_1{}^4u_2{}^2+144u_1{}^2u_2{}^4+u_2{}^2+1)^2}.$$

Then, the extension of the Gaussian curvature is

$$K = \frac{-1}{(16u_1{}^6-96u_1{}^4u_2{}^2+144u_1{}^2u_2{}^4+u_2{}^2+1)^2}.$$

In a similar way to example 5.2.1, we obtain that the Blaschke vector field of $\mathbf{x}$ is given by $\boldsymbol{\xi}=(0,0,1)$.

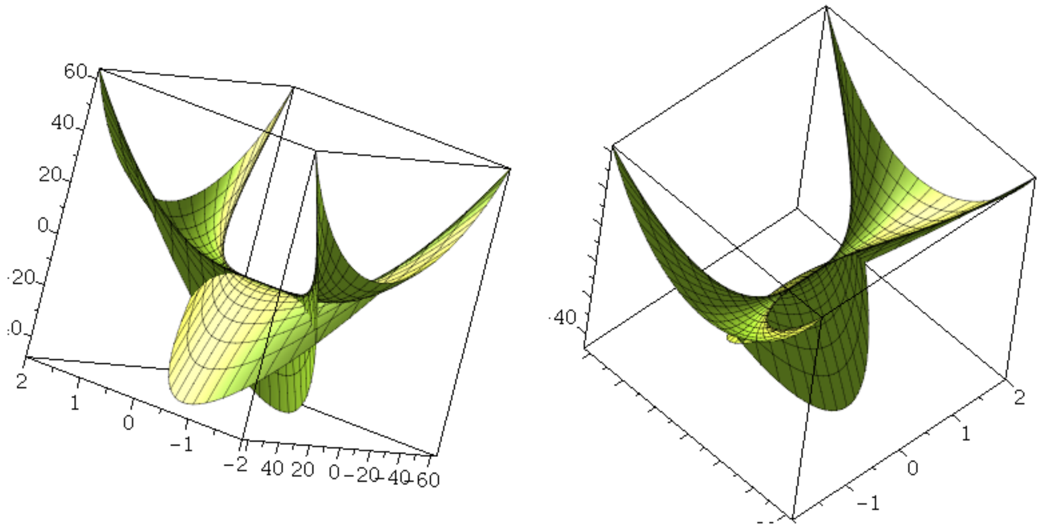

Figure 6 – Wave front of rank 1 with extendable non-vanishing Gaussian curvature.

5.2.3 Conormal vector field

Definition 5.2.2. Given a frontal $\mathbf{x}: U \to \mathbb{R}^3$, we define the affine conormal vector field of $\mathbf{x}$ relative to an equiaffine transversal vector field $\boldsymbol{\xi}$ as the vector field $\boldsymbol{\nu}: U \to \mathbb{R}^3 \setminus \mathbf{0}$, such that

$$\langle \boldsymbol{\nu}(u), \boldsymbol{\xi}(u) \rangle = 1$$
$$\langle \boldsymbol{\nu}(u), \mathbf{w} \rangle = 0, \text{ for all } \mathbf{w} \in T_\Omega(u),$$

for all $u \in U$.

Remark 5.2.4. It follows from the second condition above that the conormal vector field is always a multiple of the unit normal vector field $\mathbf{n}$ of $\mathbf{x}$.

The next proposition shows that an important property of the conormal vector field is still valid when we are working with the case of non-parabolic frontals.

Proposition 5.2.2. Given a frontal $\mathbf{x}: U \to \mathbb{R}^3$, a tmb $\boldsymbol{\Omega}$ and an equiaffine transversal vector field $\boldsymbol{\xi}$, we have that

$$\langle \boldsymbol{\nu}_{u_i}, \boldsymbol{\xi} \rangle = 0, i = 1, 2.$$
$$\langle \boldsymbol{\nu}_{u_i}, \mathbf{v} \rangle = -h_\Omega(\mathbf{v}, \mathbf{w}_i), \text{ where } \mathbf{v}(u) \in T_\Omega(u) \text{ for all } u \in U, i, j = 1, 2.$$

Furthermore, if $\mathbf{x}$ is non-parabolic then the conormal vector field is an immersion.

Proof. In order to simplify notation, we drop the subscript in the notation for the induced affine fundamental form and indicate only by h. We know that $\langle \boldsymbol{\nu}, \boldsymbol{\xi} \rangle = 1$ so, differentiating and using the fact that $\boldsymbol{\xi}$ is equiaffine, we get $\langle \boldsymbol{\nu}_{u_i}, \boldsymbol{\xi} \rangle = 0$.

From $\langle \boldsymbol{\nu}, \mathbf{v} \rangle = 0$, it follows that $\langle \boldsymbol{\nu}_{u_i}, \mathbf{v} \rangle = -\langle \boldsymbol{\nu}, \mathbf{v}_{u_i} \rangle$. We can write

$$-\mathbf{v}_{u_i} = -\mathbf{v}_{u_i}^{\top} - h(\mathbf{v}, \mathbf{w}_i)\boldsymbol{\xi},$$

where $\mathbf{v}_{u_i}^{\top}$ indicates the tangent component of $\mathbf{v}_{u_i}$, thus considering the properties of the conormal vector field, we obtain

$$\langle \boldsymbol{\nu}_{u_i}, \mathbf{v} \rangle = -\langle \boldsymbol{\nu}, \mathbf{v}_{u_i} \rangle = -h(\mathbf{v}, \mathbf{w}_i). \tag{5.9}$$

Now, let us suppose that $\mathbf{x}$ is non-parabolic and that $\boldsymbol{\nu}$ is not an immersion, hence there is $(a,b) \in \mathbb{R}^2 \setminus 0$ such that $a\boldsymbol{\nu}_{u_1} + b\boldsymbol{\nu}_{u_2} = 0$. Thus,

$$\langle a\boldsymbol{\nu}_{u_1} + b\boldsymbol{\nu}_{u_2}, \mathbf{v} \rangle = 0.$$

By using (5.9) and the above expression, we get

$$0 = -ah(\mathbf{v}, \mathbf{w}_1) - bh(\mathbf{v}, \mathbf{w}_2) = h(\mathbf{v}, -a\mathbf{w}_1 - b\mathbf{w}_2), \text{ for all } \mathbf{v},$$

but this contradicts the fact that h is non-degenerate. □

5.3 A fundamental theorem

In this section we provide, in theorem 5.3.1, a fundamental theorem for the theory developed in section 5.1. This theorem is a version for frontals of the fundamental theorem of affine differential geometry (see section 4.9 in (SIMON; SCHWENK-SCHELLSCHMIDT; VIESEL, 1991) for the classical result for regular surfaces). Thus, taking $U \subset \mathbb{R}^2$ an open subset and assuming the integrability conditions for the regular case are valid in an open dense subset of U, we obtain for each $q \in U$ a neighborhood $V \subset U$ of q, a frontal $\mathbf{x} : V \to \mathbb{R}^3$ and an equiaffine transversal vector field $\boldsymbol{\xi} : V \to \mathbb{R}^3$, in the sense of section 5.1. In order to do this, we use the same approach applied in (MEDINA-TEJEDA, 2022a).

5.3.1 The compatibility equations

Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal, $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ a tmb and $\mathbf{n} = \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|}$ the unit normal vector field induced by $\boldsymbol{\Omega}$. By considering the decomposition $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}^T$ and $\lambda_\Omega = \det \boldsymbol{\Lambda}$, one can get in $U \setminus \lambda_\Omega^{-1}(0)$ the following structural equations

$$\mathbf{x}_{u_1u_1} = \Gamma_{11}^1 \mathbf{x}_{u_1} + \Gamma_{11}^2 \mathbf{x}_{u_2} + e\mathbf{n} \tag{5.10a}$$
$$\mathbf{x}_{u_1u_2} = \Gamma_{21}^1 \mathbf{x}_{u_1} + \Gamma_{21}^2 \mathbf{x}_{u_2} + f\mathbf{n} \tag{5.10b}$$
$$\mathbf{x}_{u_2u_2} = \Gamma_{22}^1 \mathbf{x}_{u_1} + \Gamma_{22}^2 \mathbf{x}_{u_2} + g\mathbf{n}, \tag{5.10c}$$

where e, f and g are the coefficients of the second fundamental form of $\mathbf{x}$. In a similar way, if $\boldsymbol{\xi} : U \to \mathbb{R}^3$ is an equiaffine transversal vector field to $\mathbf{x}$, the following hold in $U \setminus \lambda_\Omega^{-1}(0)$

$$\mathbf{x}_{u_1u_1} = \widetilde{\Gamma}^1_{11}\mathbf{x}_{u_1} + \widetilde{\Gamma}^2_{11}\mathbf{x}_{u_2} + c_{11}\boldsymbol{\xi} \tag{5.11a}$$
$$\mathbf{x}_{u_1u_2} = \widetilde{\Gamma}^1_{21}\mathbf{x}_{u_1} + \widetilde{\Gamma}^2_{21}\mathbf{x}_{u_2} + c_{21}\boldsymbol{\xi} \tag{5.11b}$$
$$\mathbf{x}_{u_2u_2} = \widetilde{\Gamma}^1_{22}\mathbf{x}_{u_1} + \widetilde{\Gamma}^2_{22}\mathbf{x}_{u_2} + c_{22}\boldsymbol{\xi} \tag{5.11c}$$
$$\boldsymbol{\xi}_{u_1} = -b^1_1\mathbf{x}_{u_1} - b^2_1\mathbf{x}_{u_2} \tag{5.11d}$$
$$\boldsymbol{\xi}_{u_2} = -b^1_2\mathbf{x}_{u_1} - b^2_2\mathbf{x}_{u_2}. \tag{5.11e}$$

The symbols c_{ij} induce a symmetric bilinear form called the affine fundamental $\mathbf{c}$ relative to $\boldsymbol{\xi}$, while the symbols $\widetilde{\Gamma}^k_{ij}$ are associated to the induced affine connection ∇ (see (3.1)).

Proposition 5.3.1. If we write $\boldsymbol{\xi} = \phi\mathbf{n} + a\mathbf{x}_{u_1} + b\mathbf{x}_{u_2}$, where $\phi \neq 0$, then in $U \setminus \Sigma(\mathbf{x})$ we have:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} = \frac{1}{\phi}\begin{pmatrix} e & f \\ f & g \end{pmatrix}, \tag{5.12}$$

$$\widetilde{\Gamma}_1 = \begin{pmatrix} \widetilde{\Gamma}^1_{11} & \widetilde{\Gamma}^2_{11} \\ \widetilde{\Gamma}^1_{21} & \widetilde{\Gamma}^2_{21} \end{pmatrix} = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} \\ \Gamma^1_{21} & \Gamma^2_{21} \end{pmatrix} - \frac{1}{\phi}\begin{pmatrix} ae & be \\ af & bf \end{pmatrix} = \Gamma_1 - \frac{1}{\phi}\begin{pmatrix} ae & be \\ af & bf \end{pmatrix}, \tag{5.13}$$

$$\widetilde{\Gamma}_2 = \begin{pmatrix} \widetilde{\Gamma}^1_{21} & \widetilde{\Gamma}^2_{21} \\ \widetilde{\Gamma}^1_{22} & \widetilde{\Gamma}^2_{22} \end{pmatrix} = \begin{pmatrix} \Gamma^1_{21} & \Gamma^2_{21} \\ \Gamma^1_{22} & \Gamma^2_{22} \end{pmatrix} - \frac{1}{\phi}\begin{pmatrix} af & bf \\ ag & bg \end{pmatrix} = \Gamma_2 - \frac{1}{\phi}\begin{pmatrix} af & bf \\ ag & bg \end{pmatrix}. \tag{5.14}$$

Proof. If $\boldsymbol{\xi} = \phi\mathbf{n} + a\mathbf{x}_{u_1} + b\mathbf{x}_{u_2}$ in (5.11), then

$$\mathbf{x}_{u_1u_1} = (\widetilde{\Gamma}^1_{11} + c_{11}a)\mathbf{x}_{u_1} + (\widetilde{\Gamma}^2_{11} + c_{11}b)\mathbf{x}_{u_2} + \phi c_{11}\mathbf{n}$$
$$\mathbf{x}_{u_1u_2} = (\widetilde{\Gamma}^1_{21} + c_{21}a)\mathbf{x}_{u_1} + (\widetilde{\Gamma}^2_{21} + c_{21}b)\mathbf{x}_{u_2} + \phi c_{21}\mathbf{n}$$
$$\mathbf{x}_{u_2u_2} = (\widetilde{\Gamma}^1_{22} + c_{22}a)\mathbf{x}_{u_1} + (\widetilde{\Gamma}^2_{22} + c_{22}b)\mathbf{x}_{u_2} + \phi c_{22}\mathbf{n}.$$

Hence, comparing this to (5.10) we have the result. □

If we look at the basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{n}$ of $\mathbb{R}^3$, it follows from (5.1) that there are smooth functions $\mathcal{T}^k_{ij}$ defined in U, $i, j, k \in \{1, 2\}$, such that

$$\mathbf{w}_{1u_1} = \mathcal{T}^1_{11}\mathbf{w}_1 + \mathcal{T}^2_{11}\mathbf{w}_2 + e_\Omega\mathbf{n} \tag{5.15a}$$
$$\mathbf{w}_{2u_1} = \mathcal{T}^1_{21}\mathbf{w}_1 + \mathcal{T}^2_{21}\mathbf{w}_2 + f_{2\Omega}\mathbf{n} \tag{5.15b}$$
$$\mathbf{w}_{1u_2} = \mathcal{T}^1_{12}\mathbf{w}_1 + \mathcal{T}^2_{12}\mathbf{w}_2 + f_{1\Omega}\mathbf{n} \tag{5.15c}$$
$$\mathbf{w}_{2u_2} = \mathcal{T}^1_{22}\mathbf{w}_1 + \mathcal{T}^2_{22}\mathbf{w}_2 + g_\Omega\mathbf{n} \tag{5.15d}$$

We know that $\mathbf{w}_1, \mathbf{w}_2, \boldsymbol{\xi}$ is a basis of $\mathbb{R}^3$, then from (5.2), (5.3) and from the fact that $\boldsymbol{\xi}$ is equiaffine there are smooth functions $\mathcal{D}_{ij}^k$, h_{ij} and S_j^i defined in U, $i,j,k \in \{1,2\}$, such that

$$\mathbf{w}_{1u_1} = \mathcal{D}_{11}^1\mathbf{w}_1 + \mathcal{D}_{11}^2\mathbf{w}_2 + h_{11}\boldsymbol{\xi} \tag{5.16a}$$
$$\mathbf{w}_{2u_1} = \mathcal{D}_{21}^1\mathbf{w}_1 + \mathcal{D}_{21}^2\mathbf{w}_2 + h_{21}\boldsymbol{\xi} \tag{5.16b}$$
$$\mathbf{w}_{1u_2} = \mathcal{D}_{12}^1\mathbf{w}_1 + \mathcal{D}_{12}^2\mathbf{w}_2 + h_{12}\boldsymbol{\xi} \tag{5.16c}$$
$$\mathbf{w}_{2u_2} = \mathcal{D}_{22}^1\mathbf{w}_1 + \mathcal{D}_{22}^2\mathbf{w}_2 + h_{22}\boldsymbol{\xi} \tag{5.16d}$$
$$\boldsymbol{\xi}_{u_1} = -S_1^1\mathbf{w}_1 - S_1^2\mathbf{w}_2 \tag{5.16e}$$
$$\boldsymbol{\xi}_{u_2} = -S_2^1\mathbf{w}_1 - S_2^2\mathbf{w}_2. \tag{5.16f}$$

If we write $\boldsymbol{\xi} = \phi\mathbf{n} + a\mathbf{x}_{u_1} + b\mathbf{x}_{u_2}$ and $\boldsymbol{\xi} = \phi\mathbf{n} + \tilde{a}\mathbf{w}_1 + \tilde{b}\mathbf{w}_2$, where $\phi \neq 0$, then in $U \setminus \Sigma(\mathbf{x})$

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \mathbf{\Lambda}^T \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} \tilde{a}e_\Omega & \tilde{b}e_\Omega \\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega} \end{pmatrix} = \begin{pmatrix} ae_\Omega & be_\Omega \\ af_{2\Omega} & bf_{2\Omega} \end{pmatrix}\mathbf{\Lambda}. \tag{5.17}$$

Hence, taking (5.15), (5.16), (5.17) and using the method of the proof of proposition 5.3.1 one can get the following result:

Proposition 5.3.2. If we write $\boldsymbol{\xi} = \phi\mathbf{n} + \tilde{a}\mathbf{w}_1 + \tilde{b}\mathbf{w}_2$, for $\phi, \tilde{a}, \tilde{b} \in C^\infty(U,\mathbb{R})$ and $\phi \neq 0$, then

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \frac{1}{\phi}\begin{pmatrix} e_\Omega & f_{1\Omega} \\ f_{2\Omega} & g_\Omega \end{pmatrix} \text{ in } U, \tag{5.18}$$

$$\mathcal{D}_1 = \mathcal{T}_1 - \frac{1}{\phi}\begin{pmatrix} \tilde{a}e_\Omega & \tilde{b}e_\Omega \\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega} \end{pmatrix} = \mathcal{T}_1 - \frac{1}{\phi}\begin{pmatrix} ae_\Omega & be_\Omega \\ af_{2\Omega} & bf_{2\Omega} \end{pmatrix}\mathbf{\Lambda} \text{ in } U \setminus \Sigma(\mathbf{x}), \tag{5.19}$$

$$\mathcal{D}_2 = \mathcal{T}_2 - \frac{1}{\phi}\begin{pmatrix} \tilde{a}f_{1\Omega} & \tilde{b}f_{1\Omega} \\ \tilde{a}g_\Omega & \tilde{b}g_\Omega \end{pmatrix} = \mathcal{T}_2 - \frac{1}{\phi}\begin{pmatrix} af_{1\Omega} & bf_{1\Omega} \\ ae_\Omega & be_\Omega \end{pmatrix}\mathbf{\Lambda} \text{ in } U \setminus \Sigma(\mathbf{x}),$$

where $\mathcal{D}_1 = \begin{pmatrix} \mathcal{D}_{11}^1 & \mathcal{D}_{11}^2 \\ \mathcal{D}_{21}^1 & \mathcal{D}_{21}^2 \end{pmatrix}$, $\mathcal{D}_2 = \begin{pmatrix} \mathcal{D}_{12}^1 & \mathcal{D}_{12}^2 \\ \mathcal{D}_{22}^1 & \mathcal{D}_{22}^2 \end{pmatrix}$, $\mathcal{T}_1 = \begin{pmatrix} \mathcal{T}_{11}^1 & \mathcal{T}_{11}^2 \\ \mathcal{T}_{21}^1 & \mathcal{T}_{21}^2 \end{pmatrix}$ and $\mathcal{T}_2 = \begin{pmatrix} \mathcal{T}_{12}^1 & \mathcal{T}_{12}^2 \\ \mathcal{T}_{22}^1 & \mathcal{T}_{22}^2 \end{pmatrix}$.

Remark 5.3.1. With notation as in proposition (5.3.2), it follows from $\mathbf{II} = \mathbf{\Lambda}\mathbf{II}_\Omega$ (see 3.31), from 5.12 and from (5.18) that

$$\begin{pmatrix} ae & be \\ af & bf \end{pmatrix} = \begin{pmatrix} e & e \\ f & f \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathbf{\Lambda}\begin{pmatrix} e_\Omega & e_\Omega \\ f_{2\Omega} & f_{2\Omega} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{5.20}$$
$$\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} = \frac{1}{\phi}\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{\phi}\mathbf{\Lambda}\begin{pmatrix} e_\Omega & f_{2\Omega} \\ f_{1\Omega} & g_\Omega \end{pmatrix} = \mathbf{\Lambda}\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

in $U \setminus \Sigma(\mathbf{x})$. Furthermore, just taking the decomposition $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, we have that

$$\begin{pmatrix} S_1^1 & S_1^2 \\ S_2^1 & S_2^2 \end{pmatrix} = \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix}\mathbf{\Lambda}.$$

Proposition 5.3.3. Let $\mathbf{x}: U \to \mathbb{R}^3$ be a proper frontal and $\mathbf{\Omega}$ a tmb of $\mathbf{x}$. Then, in $U \setminus \Sigma(\mathbf{x})$, we can write

$$\mathcal{D}_1 = \mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1 \mathbf{\Lambda} - \mathbf{\Lambda}_{u_1}\right),$$
$$\mathcal{D}_2 = \mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_2 \mathbf{\Lambda} - \mathbf{\Lambda}_{u_2}\right).$$

Proof. It is known that the Christoffel symbols for the decomposition in the basis $\{\mathbf{x}_{u_1}, \mathbf{x}_{u_2}, \mathbf{n}\}$ are given by $\Gamma_1 = \left(D\mathbf{x}_{u_1}^T D\mathbf{x}\right)\mathbf{I}^{-1}$ (see section 4.3 in (CARMO, 2016)) thus, by taking (5.12), we get

$$\begin{aligned}
\widetilde{\Gamma}_1 = \Gamma_1 - \frac{1}{\phi}\begin{pmatrix} ae & be \\ af & bf \end{pmatrix} &= (D\mathbf{x}_{u_1}^T D\mathbf{x})\mathbf{I}^{-1} - \frac{1}{\phi}\begin{pmatrix} ae & be \\ af & bf \end{pmatrix}, \text{ from } D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T, \text{ we have} \\
&= \left((\mathbf{\Lambda}_{u_1}\mathbf{\Omega}^T + \mathbf{\Lambda}\mathbf{\Omega}_{u_1}^T)\mathbf{\Omega}\mathbf{\Lambda}^T\right)(\mathbf{\Lambda}^T)^{-1}\mathbf{I}_\Omega \mathbf{\Lambda}^{-1} - \frac{1}{\phi}\mathbf{\Lambda}\begin{pmatrix} ae_\Omega & be_\Omega \\ af_{2\Omega} & bf_{2\Omega} \end{pmatrix} \\
&= \left(\mathbf{\Lambda}_{u_1}\mathbf{\Omega}^T\mathbf{\Omega} + \mathbf{\Lambda}\mathbf{\Omega}_{u_1}^T\mathbf{\Omega}\right)(\mathbf{\Omega}^T\mathbf{\Omega})^{-1}\mathbf{\Lambda}^{-1} - \frac{1}{\phi}\mathbf{\Lambda}\begin{pmatrix} ae_\Omega & be_\Omega \\ af_{2\Omega} & bf_{2\Omega} \end{pmatrix}\mathbf{\Lambda}\mathbf{\Lambda}^{-1} \\
&= \left(\mathbf{\Lambda}_{u_1} + \mathbf{\Lambda}\left(\mathcal{T}_1 - \frac{1}{\phi}\begin{pmatrix} ae_\Omega & be_\Omega \\ af_{2\Omega} & bf_{2\Omega} \end{pmatrix}\mathbf{\Lambda}\right)\right)\mathbf{\Lambda}^{-1}, \text{ from (5.19), we have} \\
&= (\mathbf{\Lambda}_{u_1} + \mathbf{\Lambda}\mathcal{D}_1)\mathbf{\Lambda}^{-1},
\end{aligned}$$

since $\mathcal{T}_1 = \mathbf{\Omega}_{u_1}^T\mathbf{\Omega}(\mathbf{\Omega}^T\mathbf{\Omega})^{-1}$. From this, we obtain $\mathcal{D}_1 = \mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_{u_1}\right)$. Similarly, we prove the other case. □

From now on, if $\mathbf{A} \in M_{n\times n}(\mathbb{R})$, we denote by $\mathbf{A}_{(i)}$ the ith-row and by $\mathbf{A}^{(j)}$ the jth-column of $\mathbf{A}$.

Proposition 5.3.4. Let $\mathbf{I}, \mathbf{I}_\Omega, \mathbf{\Lambda}, \left(h_{ij}\right), \left(c_{ij}\right): U \to M_{2\times 2}(\mathbb{R})$ be arbitrary smooth maps, such that $\mathbf{I}_\Omega$ is symmetric and $i, j = 1, 2$. Consider also $\lambda_\Omega = \det\mathbf{\Lambda}$ and $\mathfrak{T}_\Omega$ the principal ideal generated by λ_Ω in the ring $C^\infty(U, \mathbb{R})$. Suppose that $U \setminus \lambda_\Omega^{-1}(0)$ is an open dense set and that

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^T \tag{5.21}$$

$$(c_{ij}) = \frac{1}{\phi}\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \mathbf{\Lambda}\left(h_{ij}\right),$$

where $\phi \in C^\infty(U, \mathbb{R}\setminus 0)$ and define in $U \setminus \lambda_\Omega^{-1}(0)$, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ by (5.13) and (5.14), respectively. Then,

(a) The map $\mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_{u_1}\right): U \setminus \lambda_\Omega^{-1}(0) \to M_{2\times 2}(\mathbb{R})$ has a unique C^∞ extension to U if and only if,

$$\mathbf{\Lambda}_{(1)u_1}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)}^T - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}_{(2)u_1}^T + E_{u_2} - F_{u_1} \in \mathfrak{T}_\Omega. \tag{5.22}$$

(b) The map $\mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_2\mathbf{\Lambda}-\mathbf{\Lambda}_{u_2}\right): U\setminus\lambda_\Omega^{-1}(0)\to M_{2\times 2}(\mathbb{R})$ has a unique C^∞ extension to U if and only if,

$$\mathbf{\Lambda}_{(1)u_2}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)}-\mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)u_2}+F_{u_2}-G_{u_1}\in\mathfrak{T}_\Omega.$$

Proof. (a) Let us suppose $\mathcal{D}_1$ the C^∞ extension of $\mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\mathbf{\Lambda}-\mathbf{\Lambda}_{u_1}\right)$ to U, thus

$$\mathbf{\Lambda}\mathcal{D}_1=\widetilde{\Gamma}_1\mathbf{\Lambda}-\mathbf{\Lambda}_{u_1}$$

in $U\setminus\lambda_\Omega^{-1}(0)$. From (5.13), it follows that

$$\mathbf{\Lambda}\mathcal{D}_1=\Gamma_1\mathbf{\Lambda}-\frac{1}{\phi}\begin{pmatrix}ae & be\\ af & bf\end{pmatrix}\mathbf{\Lambda}-\mathbf{\Lambda}_{u_1}.$$

It is known that $\Gamma_1=(\frac{1}{2}\mathbf{I}_{u_1}+\frac{1}{2}\mathbf{A}_1)\mathbf{I}^{-1}$, where $\mathbf{A}_1=\begin{pmatrix}0 & -(E_v-F_u)\\ E_v-F_u & 0\end{pmatrix}$. Hence

$$\mathbf{\Lambda}\mathcal{D}_1=(\frac{1}{2}\mathbf{I}_{u_1}+\frac{1}{2}\mathbf{A}_1)\mathbf{I}^{-1}\mathbf{\Lambda}-\frac{1}{\phi}\begin{pmatrix}ae & be\\ af & bf\end{pmatrix}\mathbf{\Lambda}-\mathbf{\Lambda}_{u_1}.$$

Via (5.21) we have an expression for $\mathbf{I}_{u_1}$, then multiplying the above expression by the right side with $2\mathbf{I}_\Omega\mathbf{\Lambda}^T$ and taking into account (5.19) and (5.20), we obtain that

$$\mathbf{\Lambda}\left(2\mathcal{D}_1\mathbf{I}_\Omega-\mathbf{I}_{\Omega u_1}+\frac{2}{\phi}\begin{pmatrix}\tilde{a}e_\Omega & \tilde{b}e_\Omega\\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega}\end{pmatrix}\mathbf{I}_\Omega\right)\mathbf{\Lambda}^T=\mathbf{\Lambda}\mathbf{I}_\Omega\mathbf{\Lambda}^t_{u_1}-\mathbf{\Lambda}_{u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^t+\mathbf{A}_1. \qquad (5.23)$$

Note that the right side of the above equation is a skew-symmetric matrix, which implies that $2\mathcal{D}_1\mathbf{I}_\Omega-\mathbf{I}_{\Omega u_1}+\frac{2}{\phi}\begin{pmatrix}\tilde{a}e_\Omega & \tilde{b}e_\Omega\\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega}\end{pmatrix}\mathbf{I}_\Omega$ is a skew-symmetric matrix in $U\setminus\lambda_\Omega^{-1}(0)$, but since $U\setminus\lambda_\Omega^{-1}(0)$ is dense, this is also true in U. So, there is $\omega_1\in C^\infty(U,\mathbb{R})$ such that

$$2\mathcal{D}_1\mathbf{I}_\Omega-\mathbf{I}_{\Omega u_1}+\frac{2}{\phi}\begin{pmatrix}\tilde{a}e_\Omega & \tilde{b}e_\Omega\\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega}\end{pmatrix}\mathbf{I}_\Omega=\begin{pmatrix}0 & -\omega_1\\ \omega_1 & 0\end{pmatrix}.$$

Multiplying (5.23) by the left side with $\begin{pmatrix}1 & 0\end{pmatrix}$ and by the right side with $\begin{pmatrix}0 & 1\end{pmatrix}^T$, we get

$$-\omega_1\lambda_\Omega=\mathbf{\Lambda}_{(1)}\begin{pmatrix}0 & -\omega_1\\ \omega_1 & 0\end{pmatrix}\mathbf{\Lambda}^T_{(2)}=\mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)u_1}-\mathbf{\Lambda}_{(1)u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)}-(E_{u_2}-F_{u_1})\in\mathfrak{T}_\Omega.$$

Reciprocally, suppose (5.22). Since $U\setminus\lambda_\Omega^{-1}(0)$ is a dense set, there is a unique $\omega_1\in C^\infty(U,\mathbb{R})$ such that

$$\mathbf{\Lambda}_{(1)u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)}-\mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)u_1}+E_{u_2}-F_{u_1}=\omega_1\lambda_\Omega\in\mathfrak{T}_\Omega.$$

From the above expression, it follows that $\mathbf{\Lambda I}_\Omega \mathbf{\Lambda}^t_{u_1} - \mathbf{\Lambda}_{u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^t + \mathbf{A}_1 = \mathbf{\Lambda}\begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix}\mathbf{\Lambda}^T$, since $\mathbf{\Lambda I}_\Omega \mathbf{\Lambda}^t_{u_1} - \mathbf{\Lambda}_{u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^t + \mathbf{A}_1$ is a skew-symmetric matrix. Define $\mathcal{D}_1 : U \to M_{2\times 2}(\mathbb{R})$, given by

$$\mathcal{D}_1 = \frac{1}{2}\left(\mathbf{I}_{\Omega u_1} - \frac{2}{\phi}\begin{pmatrix} \tilde{a}e_\Omega & \tilde{b}e_\Omega \\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega} \end{pmatrix}\mathbf{I}_\Omega + \begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix}\right)\mathbf{I}_\Omega^{-1},$$

thus

$$\mathbf{\Lambda}\left(2\mathcal{D}_1\mathbf{I}_\Omega - \mathbf{I}_{\Omega u_1} + \frac{2}{\phi}\begin{pmatrix} \tilde{a}e_\Omega & \tilde{b}e_\Omega \\ \tilde{a}f_{2\Omega} & \tilde{b}f_{2\Omega} \end{pmatrix}\mathbf{I}_\Omega\right)\mathbf{\Lambda}^T = \mathbf{\Lambda I}_\Omega \mathbf{\Lambda}^t_{u_1} - \mathbf{\Lambda}_{u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^t + \mathbf{A}_1.$$

Taking into account (5.21) and then the expression for $\mathbf{I}_{u_1}$ we show, using the above expression, that $\mathcal{D}_1 = \mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_{u_1}\right)$ in $U \setminus \lambda_\Omega^{-1}(0)$. Since $\mathcal{D}_1$ is smooth and $U \setminus \lambda_\Omega^{-1}(0)$ is dense, we have the result.

Item (b) follows analogously by considering the matrix $\mathbf{A}_2 := \begin{pmatrix} 0 & -(F_{u_2} - G_{u_1}) \\ F_{u_2} - G_{u_1} & 0 \end{pmatrix}$.

□

In preparation for the Fundamental theorem, let us set the matrices $\mathbf{W} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \boldsymbol{\xi} \end{pmatrix} \in GL(3)$,

$$\mathbf{D}_1 = \begin{pmatrix} \mathcal{D}^1_{11} & \mathcal{D}^2_{11} & h_{11} \\ \mathcal{D}^1_{21} & \mathcal{D}^2_{21} & h_{21} \\ -S^1_1 & -S^2_1 & 0 \end{pmatrix} \tag{5.24a}$$

$$\mathbf{D}_2 = \begin{pmatrix} \mathcal{D}^1_{12} & \mathcal{D}^2_{12} & h_{12} \\ \mathcal{D}^1_{22} & \mathcal{D}^2_{22} & h_{22} \\ -S^1_2 & -S^2_2 & 0 \end{pmatrix}. \tag{5.24b}$$

Then, the system (5.16) is represented by

$$\begin{cases} \mathbf{W}_{u_1} & = \mathbf{W}\mathbf{D}_1^T \\ \mathbf{W}_{u_2} & = \mathbf{W}\mathbf{D}_2^T. \end{cases} \tag{5.25}$$

It is known that the compatibility condition for the system (5.25) is $\mathbf{W}^T_{u_1u_2} = \mathbf{W}^T_{u_2u_1}$, from which we obtain

$$\mathbf{D}_1\mathbf{D}_2\mathbf{W}^T + \mathbf{D}_{1u_2}\mathbf{W}^T = \mathbf{D}_1\mathbf{W}^T_{u_2} + \mathbf{D}_{1u_2}\mathbf{W}^T = \mathbf{D}_2\mathbf{W}^T_{u_1} + \mathbf{D}_{2u_1}\mathbf{W}^T = \mathbf{D}_2\mathbf{D}_1\mathbf{W}^T + \mathbf{D}_{2u_1}\mathbf{W}^T,$$

that is equivalent to

$$(\mathbf{D}_1\mathbf{D}_2 + \mathbf{D}_{1u_2} - \mathbf{D}_2\mathbf{D}_1 - \mathbf{D}_{2u_1})\,\mathbf{W}^T = \mathbf{0}.$$

Since $\mathbf{W} \in GL(3)$, we get

$$\mathbf{D}_{1u_2} - \mathbf{D}_{2u_1} + [\mathbf{D}_1, \mathbf{D}_2] = \mathbf{0},$$

where $[\mathbf{D}_1, \mathbf{D}_2] = \mathbf{D}_1\mathbf{D}_2 - \mathbf{D}_2\mathbf{D}_1$.

We have next two auxiliary lemmas, which play an important role in the proof of theorem 5.3.1.

Lemma 5.3.1. The integrability conditions for the system

$$\begin{cases} \mathbf{x}_{u_1} = \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2 \\ \mathbf{x}_{u_2} = \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2 \\ \mathbf{x}(q) = p \end{cases} \tag{5.26}$$

are

- $\boldsymbol{\Lambda}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is symmetric;
- $\begin{pmatrix} 0 & 1 \end{pmatrix}(\boldsymbol{\Lambda}\mathcal{D}_1 + \boldsymbol{\Lambda}_{u_1}) = \begin{pmatrix} 1 & 0 \end{pmatrix}(\boldsymbol{\Lambda}\mathcal{D}_2 + \boldsymbol{\Lambda}_{u_2})$.

Proof. It is known that the integrability condition for the system (5.26) is $\mathbf{x}_{u_1u_2} = \mathbf{x}_{u_2u_1}$. If we set the matrices

$$\widetilde{\boldsymbol{\Lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{M} = \begin{pmatrix} \mathbf{x}_{u_1} & \mathbf{x}_{u_2} & \boldsymbol{\xi} \end{pmatrix},$$

then $\mathbf{M} = \mathbf{W}\widetilde{\boldsymbol{\Lambda}}^T$. Hence, the integrability condition is $\mathbf{M}_{u_1}\hat{\mathbf{j}} = \mathbf{M}_{u_2}\hat{\mathbf{i}}$, where $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ is the standard basis of $\mathbb{R}^3$. By using (5.25), $\mathbf{M} = \mathbf{W}\widetilde{\boldsymbol{\Lambda}}^T$ and $\mathbf{M}_{u_1}\hat{\mathbf{j}} = \mathbf{M}_{u_2}\hat{\mathbf{i}}$, we obtain

$$\mathbf{W}\mathbf{D}_1^T\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{j}} + \mathbf{W}\widetilde{\boldsymbol{\Lambda}}_{u_1}^T\hat{\mathbf{j}} = \mathbf{W}_{u_1}\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{j}} + \mathbf{W}\widetilde{\boldsymbol{\Lambda}}_{u_1}^T\hat{\mathbf{j}} = \mathbf{W}_{u_2}\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{i}} + \mathbf{W}\widetilde{\boldsymbol{\Lambda}}_{u_2}^T\hat{\mathbf{i}} = \mathbf{W}\mathbf{D}_2^T\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{i}} + \mathbf{W}\widetilde{\boldsymbol{\Lambda}}_{u_2}^T\hat{\mathbf{i}},$$

so, the integrability condition is equivalent to $\mathbf{D}_1^T\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{j}} + \widetilde{\boldsymbol{\Lambda}}_{u_1}^T\hat{\mathbf{j}} = \mathbf{D}_2^T\widetilde{\boldsymbol{\Lambda}}^T\hat{\mathbf{i}} + \widetilde{\boldsymbol{\Lambda}}_{u_2}^T\hat{\mathbf{i}}$. By taking each component of this expression we have

$$\lambda_{11u_2} - \lambda_{21u_1} = \mathcal{D}_{11}^1\lambda_{21} + \mathcal{D}_{21}^1\lambda_{22} - \mathcal{D}_{12}^1\lambda_{11} - \mathcal{D}_{22}^1\lambda_{12} \tag{5.27}$$

$$\lambda_{12u_2} - \lambda_{22u_1} = \mathcal{D}_{11}^2\lambda_{21} + \mathcal{D}_{21}^2\lambda_{22} - \mathcal{D}_{12}^2\lambda_{11} - \mathcal{D}_{22}^2\lambda_{12} \tag{5.28}$$

$$\lambda_{11}h_{12} + \lambda_{12}h_{22} = \lambda_{21}h_{11} + \lambda_{22}h_{21}. \tag{5.29}$$

Finally, note that (5.27) and (5.28) are equivalent to $\begin{pmatrix} 0 & 1 \end{pmatrix}(\boldsymbol{\Lambda}\mathcal{D}_1 + \boldsymbol{\Lambda}_{u_1}) = \begin{pmatrix} 1 & 0 \end{pmatrix}(\boldsymbol{\Lambda}\mathcal{D}_2 + \boldsymbol{\Lambda}_{u_2})$ and (5.29) is equivalent to say that $\boldsymbol{\Lambda}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is symmetric. □

Lemma 5.3.2. ((MEDINA-TEJEDA, 2022a), Lemma 5.2) If we have

$$\widetilde{\mathbf{\Lambda}}\mathbf{D}_1 = \mathbf{\Gamma}_1\widetilde{\mathbf{\Lambda}} - \widetilde{\mathbf{\Lambda}}_{u_1} \text{ and } \widetilde{\mathbf{\Lambda}}\mathbf{D}_2 = \mathbf{\Gamma}_2\widetilde{\mathbf{\Lambda}} - \widetilde{\mathbf{\Lambda}}_{u_2},$$

in which $\widetilde{\mathbf{\Lambda}}, \mathbf{D}_1, \mathbf{D}_2 : U \to M_{n\times n}(\mathbb{R})$ and $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2 : U \setminus \lambda_\Omega^{-1}(0) \to M_{n\times n}(\mathbb{R})$ are smooth maps with $int(\lambda_\Omega^{-1}(0)) = \emptyset$, where $\lambda_\Omega = \det \mathbf{\Lambda}$. Then,

$$\mathbf{\Gamma}_{1u_2} - \mathbf{\Gamma}_{2u_1} + [\mathbf{\Gamma}_1, \mathbf{\Gamma}_2] = \mathbf{0} \text{ is equivalent to } \mathbf{D}_{1u_2} - \mathbf{D}_{2u_1} + [\mathbf{D}_1, \mathbf{D}_2] = \mathbf{0} \text{ in } U.$$

Theorem 5.3.1. Let $\widetilde{\Gamma}^k_{ij}, b^i_j, e, f, g, \phi \in C^\infty(U, \mathbb{R})$, such that $\phi \neq 0$. Suppose that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}^T \tag{5.30}$$

$$\frac{1}{\phi}\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \tag{5.31}$$

and

$$\begin{pmatrix} S^1_1 & S^2_1 \\ S^1_2 & S^2_2 \end{pmatrix} = \begin{pmatrix} b^1_1 & b^2_1 \\ b^1_2 & b^2_2 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \tag{5.32}$$

where all the components above are C^∞ functions defined in U, $\lambda_\Omega = \det \mathbf{\Lambda}$ and $U \setminus \lambda_\Omega^{-1}(0)$ is a dense set, for $\mathbf{\Lambda} = \left(\lambda_{ij}\right)$. Suppose also that the compatibility equations for the system (5.11) are satisfied in $U \setminus \lambda_\Omega^{-1}(0)$ and that

$$\mathbf{\Lambda}_{(1)u_1}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)} - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)u_1} + E_{u_2} - F_{u_1} \in \mathfrak{T}_\Omega \tag{5.33a}$$

$$\mathbf{\Lambda}_{(1)u_2}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)} - \mathbf{\Lambda}_{(1)}\mathbf{I}_\Omega\mathbf{\Lambda}^T_{(2)u_2} + F_{u_2} - G_{u_1} \in \mathfrak{T}_\Omega, \tag{5.33b}$$

where $\mathfrak{T}_\Omega$ is the principal ideal generated by λ_Ω in the ring $C^\infty(U, \mathbb{R})$ and $\mathbf{I}_\Omega = \begin{pmatrix} E_\Omega & F_\Omega \\ F_\Omega & G_\Omega \end{pmatrix}$. Then,

(a) For each $q \in U$, there exists a neighborhood $V \subset U$ of q, a frontal $\mathbf{x} : V \to \mathbb{R}^3$ with tmb $\mathbf{\Omega}$, such that $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, and an equiaffine transversal vector field $\boldsymbol{\xi} : V \to \mathbb{R}^3$ with associated equiaffine structure given by $\left(h_{ij}\right), \mathcal{D}_1, \mathcal{D}_2$, where

$$\mathcal{D}_1 = \begin{pmatrix} \mathcal{D}^1_{11} & \mathcal{D}^2_{11} \\ \mathcal{D}^1_{21} & \mathcal{D}^2_{21} \end{pmatrix}$$

$$\mathcal{D}_2 = \begin{pmatrix} \mathcal{D}^1_{12} & \mathcal{D}^2_{12} \\ \mathcal{D}^1_{22} & \mathcal{D}^2_{22} \end{pmatrix},$$

are the unique C^∞ extensions of $\mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\mathbf{\Lambda} - \mathbf{\Lambda}_{u_1}\right)$ and $\mathbf{\Lambda}^{-1}\left(\widetilde{\Gamma}_2\mathbf{\Lambda} - \mathbf{\Lambda}_{u_2}\right)$ to U, respectively.

(b) If moreover, we suppose $eg - f^2 \neq 0$ in $V \setminus \lambda_\Omega^{-1}(0)$ and that the condition $\nabla \omega_{\mathbf{c}} = 0$ is satisfied in $V \setminus \lambda_\Omega^{-1}(0)$, where ∇ is the connection associated to the symbols $\frac{e}{\phi}, \frac{g}{\phi}, \frac{f}{\phi}, \widetilde{\Gamma}_{ij}^k$ and $\omega_{\mathbf{c}}$ is the volume element induced by the affine fundamental form $\mathbf{c}$, then there is a volume element ω in $\mathbb{R}^3$ such that $\boldsymbol{\xi}$ is the Blaschke vector field of the frontal $\mathbf{x}$.

(c) Let U be connected. Suppose that $\widetilde{\mathbf{x}} : U \to \mathbb{R}^3$ is another proper frontal, $\widetilde{\boldsymbol{\xi}}$ an equiaffine transversal vector field and $\widetilde{\boldsymbol{\Omega}}$ a tmb satisfying the same conditions that were obtained in (a). Then, $\mathbf{x}$ e $\widetilde{\mathbf{x}}$ are affinely equivalent.

Proof.

(a) It follows from (5.33) and from proposition (5.3.4) that the maps

$$\boldsymbol{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_{u_1}\right) \text{ and } \boldsymbol{\Lambda}^{-1}\left(\widetilde{\Gamma}_2\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_{u_2}\right),$$

defined in $U \setminus \lambda_\Omega^{-1}(0)$, admit unique C^∞ extensions to U, $\mathcal{D}_1$ e $\mathcal{D}_2$, respectively. Thus, in $U \setminus \lambda_\Omega^{-1}(0)$, we have

$$\mathcal{D}_1 = \boldsymbol{\Lambda}^{-1}\left(\widetilde{\Gamma}_1\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_{u_1}\right),$$
$$\mathcal{D}_2 = \boldsymbol{\Lambda}^{-1}\left(\widetilde{\Gamma}_2\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_{u_2}\right).$$

By using $\mathcal{D}_1$ and $\mathcal{D}_2$, we can build the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ as the matrices (5.24). Then, using (5.24), (5.30) and (5.32) we have that

$$\widetilde{\boldsymbol{\Lambda}}\mathbf{D}_1 = \widetilde{\boldsymbol{\Gamma}}_1\widetilde{\boldsymbol{\Lambda}} - \widetilde{\boldsymbol{\Lambda}}_{u_1} \text{ and } \widetilde{\boldsymbol{\Lambda}}\mathbf{D}_2 = \widetilde{\boldsymbol{\Gamma}}_2\widetilde{\boldsymbol{\Lambda}} - \widetilde{\boldsymbol{\Lambda}}_{u_2},$$

where

$$\widetilde{\boldsymbol{\Gamma}}_1 = \begin{pmatrix} \widetilde{\Gamma}_{11}^1 & \widetilde{\Gamma}_{11}^2 & \frac{e}{\phi} \\ \widetilde{\Gamma}_{21}^1 & \widetilde{\Gamma}_{21}^2 & \frac{f}{\phi} \\ -b_1^1 & -b_1^2 & 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{\Gamma}}_2 = \begin{pmatrix} \widetilde{\Gamma}_{12}^1 & \widetilde{\Gamma}_{12}^2 & \frac{f}{\phi} \\ \widetilde{\Gamma}_{22}^1 & \widetilde{\Gamma}_{22}^2 & \frac{g}{\phi} \\ -b_2^1 & -b_2^2 & 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{\Lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us consider the following system of PDE

$$\mathbf{W}_{u_1} = \mathbf{W}\mathbf{D}_1^T \tag{5.34a}$$

$$\mathbf{W}_{u_2} = \mathbf{W}\mathbf{D}_2^T \tag{5.34b}$$

$$\mathbf{W}(q) = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix}, \tag{5.34c}$$

where $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent vectors in $\mathbb{R}^3$ and $q \in U$ is a fixed point. The compatibility conditions for the system (5.11) are expressed by $\widetilde{\mathbf{\Gamma}}_{1u_2} - \widetilde{\mathbf{\Gamma}}_{2u_1} + \left[\widetilde{\mathbf{\Gamma}}_1, \widetilde{\mathbf{\Gamma}}_2\right] = \mathbf{0}$ and by hypothesis they are satisfied, so it follows from lemma 5.3.2 that $\mathbf{D}_{1u_2} - \mathbf{D}_{2u_1} + [\mathbf{D}_1, \mathbf{D}_2] = \mathbf{0}$, which is equivalent to the compatibility conditions for the system (5.34) (see 5.25). Thus, this system has a unique solution $\mathbf{W} : V \to GL(3)$, where $V \subset U$ is a neighborhood of q. If $\mathbf{W} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \boldsymbol{\xi} \end{pmatrix}$, it follows that the vector field $\boldsymbol{\xi}$ is transversal to $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbb{R}}$ and that $\boldsymbol{\xi}_{u_i} \in \langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbb{R}}$, $i = 1, 2$. Now, let us take the following system of PDE

$$\begin{cases} \mathbf{x}_{u_1} = \lambda_{11}\mathbf{w}_1 + \lambda_{12}\mathbf{w}_2 \\ \mathbf{x}_{u_2} = \lambda_{21}\mathbf{w}_1 + \lambda_{22}\mathbf{w}_2 \\ \mathbf{x}(q) = p, \end{cases} \tag{5.35}$$

for a fixed $p \in \mathbb{R}^3$. Note that

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (\mathbf{\Lambda}\mathcal{D}_1 + \mathbf{\Lambda}_{u_1}) = \begin{pmatrix} 0 & 1 \end{pmatrix} \widetilde{\Gamma}_1 \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \end{pmatrix} \widetilde{\Gamma}_2 \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{\Lambda}\mathcal{D}_2 + \mathbf{\Lambda}_{u_2})$$

in $\lambda_\Omega^{-1}(0)^c$, so, via density the equality holds in U. The above equality, together with the fact that $\mathbf{\Lambda} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is symmetric, means that the system (5.35) has a unique solution $\mathbf{x} : \widetilde{V} \to \mathbb{R}^3$, where $\widetilde{V} \subset V$ is a neighborhood of q (see lemma 5.3.1). As $D\mathbf{x} = \mathbf{\Omega}\mathbf{\Lambda}^T$, where $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ it follows that $\mathbf{x}$ is a frontal for which $\mathbf{\Omega}$ is a tmb, $\boldsymbol{\xi}$ is an equiaffine transversal vector field and the equiaffine structure is the desired one.

(b) Moreover, from $eg - f^2 \neq 0$ in $V \setminus \lambda_\Omega^{-1}(0)$ we obtain that $\mathbf{c}$ is non-degenerate in $V \setminus \lambda_\Omega^{-1}(0)$, thus let $\omega_{\mathbf{c}}$ be the volume element induced by the non-degenerate metric $\mathbf{c}$. Let ω_1 be the volume element in $\mathbb{R}^3$, so the volume element induced by the equiaffine vector field $\boldsymbol{\xi}$ is $\theta_1(\mathbf{v}_1, \mathbf{v}_2) = \omega_1(\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\xi})$, where $\mathbf{v}_1, \mathbf{v}_2$ are tangent vectors. Since $\boldsymbol{\xi}$ is equiaffine, θ_1 is a parallel volume element in the regular part of $\mathbf{x}$, i.e., $\nabla\theta_1 = 0$ (see proposition 3.1.1). The apolarity condition $\nabla\omega_{\mathbf{c}} = 0$ means that $\omega_{\mathbf{c}} = \mu\theta_1$, where μ is a positive constant, since a parallel volume element is unique up to a positive scalar multiple. If we take in $\mathbb{R}^3$ the volume element $\omega = \mu\omega_1$ and the new induced volume element θ, we get $\omega_{\mathbf{c}} = \theta$, therefore, $\boldsymbol{\xi}$ is the usual Blaschke vector field of $\mathbf{x}$ in the regular part. Since $\boldsymbol{\xi}$ is defined in V, it follows that $\boldsymbol{\xi}$ is the Blaschke vector field of $\mathbf{x}$ as defined in 5.2.1.

(c) If we write $\mathbf{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ and $\widetilde{\mathbf{\Omega}} = \begin{pmatrix} \widetilde{\mathbf{w}_1} & \widetilde{\mathbf{w}_2} \end{pmatrix}$, then for each $q \in U$ there is an isomorphism $L_q : \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$L_q(\mathbf{w}_i) = \widetilde{\mathbf{w}}_i,\ i = 1,2.$$
$$L_q(\boldsymbol{\xi}) = \widetilde{\boldsymbol{\xi}}.$$

We seek to show that L is constant. It follows from the fact that $\mathbf{x}$ e $\widetilde{\mathbf{x}}$ satisfy the same hypothesis given in item a) that both relative shape operators S_Ω and $\widetilde{S}_{\widetilde{\Omega}}$ are given by the matrix $\left(S_i^j\right)$, hence $L_q(S(\mathbf{w}_i)) = \widetilde{S}(\widetilde{\mathbf{w}}_i)$. Thus,

$$\begin{aligned} -\widetilde{S}(\widetilde{\mathbf{w}}_i) = \frac{\partial}{\partial u_i}\widetilde{\boldsymbol{\xi}} = \frac{\partial}{\partial u_i}L(\boldsymbol{\xi}) = \left(\frac{\partial}{\partial u_i}L\right)(\boldsymbol{\xi}) + L\left(\frac{\partial}{\partial u_i}\boldsymbol{\xi}\right) &= \left(\frac{\partial}{\partial u_i}L\right)(\boldsymbol{\xi}) - L(S(\mathbf{w}_i)) \\ &= \left(\frac{\partial}{\partial u_i}L\right)(\boldsymbol{\xi}) - \widetilde{S}(\widetilde{\mathbf{w}}_i), \end{aligned}$$

which means that $\left(\frac{\partial}{\partial u_i}L\right)(\boldsymbol{\xi}) = 0$, $i = 1,2$. Furthermore,

$$\begin{aligned} \widetilde{\mathbf{w}_1}_{u_1} = \frac{\partial}{\partial u_1}L(\mathbf{w}_1) &= \left(\frac{\partial}{\partial u_1}L\right)(\mathbf{w}_1) + L(\mathbf{w}_{1u_1}) \\ &= \left(\frac{\partial}{\partial u_1}L\right)(\mathbf{w}_1) + L(\mathcal{D}_{11}^1\mathbf{w}_1 + \mathcal{D}_{11}^2\mathbf{w}_2 + h_{11}\boldsymbol{\xi}) \\ &= \left(\frac{\partial}{\partial u_1}L\right)(\mathbf{w}_1) + \widetilde{\mathbf{w}_1}_{u_1}, \end{aligned}$$

so, $\left(\frac{\partial}{\partial u_1}L\right)(\mathbf{w}_1) = 0$. Analogously, we show that $\left(\frac{\partial}{\partial u_i}L\right)(\mathbf{w}_j) = 0$, $i, j = 1,2$. Therefore, L is constant and from $\widetilde{\mathbf{\Omega}} = L\mathbf{\Omega}$ we obtain

$$D\widetilde{\mathbf{x}} = \widetilde{\mathbf{\Omega}}\mathbf{\Lambda}^T = L\mathbf{\Omega}\mathbf{\Lambda}^T = LD\mathbf{x},$$

that is, $\widetilde{\mathbf{x}} = L\mathbf{x} + \mathbf{a}$, where $\mathbf{a}$ is a constant vector and $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear isomorphism.

$\square$

Remark 5.3.2. The approach used to prove item (a) in theorem 5.3.1 is the same as that applied to prove the existence part of the fundamental theorem in (MEDINA-TEJEDA, 2022a). Then, as we are considering here any equiaffine transversal vector field, it is possible to recover from item (a) the existence theorem from (MEDINA-TEJEDA, 2022a), taking the unit normal as the equiaffine vector field.

5.4 Future research

Taking into consideration the equiaffine structure on frontals stated here, we expect to work with other problems generally discussed only for regular surfaces. For instance, the study of the affine normal curvature defined in (DAVIS, 2009) and its extendibility when considering frontals. Furthermore, as improper affine spheres with singularities is a topic of interest in differential geometry, see for instance (CRAIZER; DOMITRZ; RIOS, 2020), (ISHIKAWA; MACHIDA, 2006), (MARTÍNEZ, 2005), (MILÁN, 2013) and (NAKAJO, 2009), we seek to understand better the class of frontal improper affine spheres described in remark 5.2.2 (b). We also look forward to understand how the conditions given in theorem 5.2.1 are related to invariants associated to frontals which were not explored in this thesis.

CHAPTER

6

A EUCLIDEAN APPROACH TO LINE CONGRUENCES WITH A FRONTAL DIRECTRIX SURFACE

Most of the results in Kummer's theory (see section 3.3) are proved for congruences $\{\mathbf{x},\boldsymbol{\xi}\}$, where $\boldsymbol{\xi}:U\to\mathbb{R}^3$ is an immersion and $U\subset\mathbb{R}^2$ is open. Our goal is to extend this theory to the case of line congruences where the director surface $\boldsymbol{\xi}:U\to\mathbb{R}^3$ is a proper frontal. For instance, we define principal surfaces and describe how their binary differential equation is related to the equation of developable surfaces. Furthermore, when considering the exact normal congruence given by a frontal $\mathbf{x}$ and its unit normal vector field $\mathbf{n}$ we study, in corollaries 6.2.3 and 6.2.4, how the Kummer principal lines (see definition 6.2.3) are related to the lines of curvature of $\mathbf{x}$ (see definition 3.4.5). We mentioned in section 2 that in the regular case it is possible to work using the tangent space of $\mathbf{x}$ or the tangent space of $\boldsymbol{\xi}$. Here, we replace these planes by the plane T_Ω, where $\boldsymbol{\Omega}$ is a tangent moving basis of $\boldsymbol{\xi}$. In this chapter, sometimes we deal with a pair of frontals $\mathbf{x}$ and $\boldsymbol{\xi}$, therefore in order to distinguish the matrix valued maps that have the same role of $\boldsymbol{\Lambda}_\Omega$, we denote $\boldsymbol{\Delta}_{\bar{\Omega}}$ the matrix such that $D\boldsymbol{\xi}=\bar{\boldsymbol{\Omega}}\boldsymbol{\Delta}_{\bar{\Omega}}^T$ and $\delta_{\bar{\Omega}}:=det(\boldsymbol{\Delta}_{\bar{\Omega}})$ where $\bar{\boldsymbol{\Omega}}$ is a tangent moving basis of $\boldsymbol{\xi}$. In particular, if we take $\boldsymbol{\xi}=\mathbf{n}$ the normal vector field induced by a tangent moving basis $\boldsymbol{\Omega}$ of $\mathbf{x}$, then is satisfied that $D\mathbf{n}=\boldsymbol{\Omega}\boldsymbol{\mu}_\Omega^T$ (see (MEDINA-TEJEDA, 2022a)), that is, $\boldsymbol{\Delta}_\Omega=\boldsymbol{\mu}_\Omega$. The results in this chapter are part of a joint work with Débora Lopes, Maria Aparecida Soares Ruas and Tito Alexandro Medina Tejeda (LOPES *et al.*, 2022).

6.1 Line congruences with a frontal director surface

Let us start with a special example of line congruence $\{\mathbf{x},\boldsymbol{\xi}\}$ for which $\mathbf{x}$ and $\boldsymbol{\xi}$ are frontals and $\boldsymbol{\xi}$ is an equiaffine transversal vector field (in the sense of chapter 5) different from the unit normal vector field of $\mathbf{x}$. This is an important class of line congruences which generalize the idea of equiaffine line congruence when considering proper frontals.

Example 6.1.1. Let $\mathbf{x} = (u_1, u_2^2, 4/15 u_1 u_2^5 + 1/2 u_1^3 u_2^4 + u_1 u_2^2)$ defined in $U \subset \mathbb{R}^2$, given by $U = (-1/10, 1/10) \times (-4, 4)$, (see figure 7) and $\boldsymbol{\xi} = \dfrac{1}{\rho^{7/4}} \left(\dfrac{-3\sqrt{3}}{8}\xi_1, \dfrac{9\sqrt{3}}{8}\xi_2, \dfrac{\sqrt{3}}{240}\xi_3 \right)$, where

$$\begin{aligned}
\xi_1 &= 216 u_1{}^6 u_2{}^4 - 189 u_1{}^4 u_2{}^5 + 66 u_1{}^2 u_2{}^6 + 16 u_2{}^7 + 324 u_1{}^4 u_2{}^2 \\
&\quad + 9 u_1{}^2 u_2{}^3 + 48 u_2{}^4 + 108 u_1{}^2 + 36 u_2 \\
\xi_2 &= \left(216 u_1{}^4 u_2{}^4 + 87 u_1{}^2 u_2{}^5 - 16 u_2{}^6 + 252 u_1{}^2 u_2{}^2 + 24 u_2{}^3 + 72\right) u_2{}^2 \\
\xi_3 &= 145800 u_1{}^8 u_2{}^8 + 35721 u_1{}^6 u_2{}^9 + 25326 u_1{}^4 u_2{}^{10} + 4896 u_1{}^2 u_2{}^{11} + 6480 \\
&\quad + 277020 u_1{}^6 u_2{}^6 + 896 u_2{}^{12} + 114129 u_1{}^4 u_2{}^7 + 39204 u_1{}^2 u_2{}^8 + 5088 u_2{}^9 \\
&\quad + 179820 u_1{}^4 u_2{}^4 + 88938 u_1{}^2 u_2{}^5 + 12096 u_2{}^6 + 48600 u_1{}^2 u_2{}^2 + 14040 u_2{}^3
\end{aligned}$$

$$\rho = 54 u_1{}^4 u_2{}^4 + 9 u_1{}^2 u_2{}^5 + 4 u_2{}^6 + 54 u_1{}^2 u_2{}^2 + 12 u_2{}^3 + 9.$$

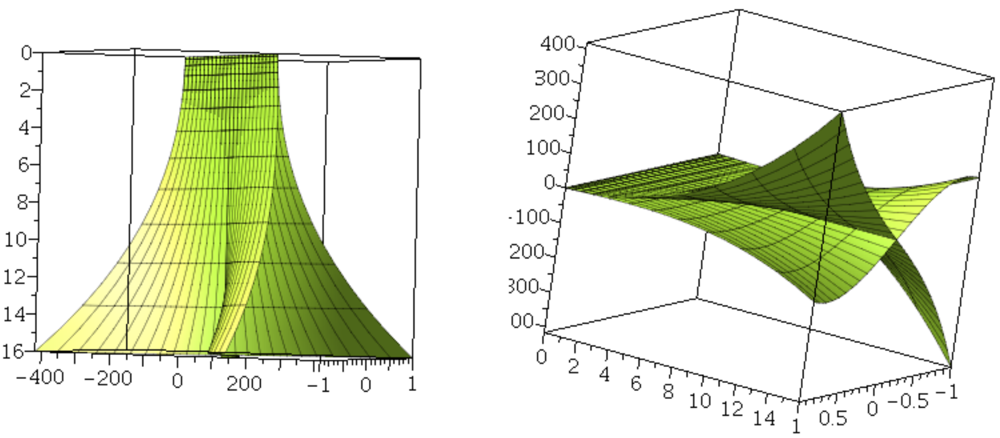

Figure 7 – Frontal which admits an equiaffine transversal vector field different from its unit normal vector field.

Remark 6.1.1. Along this chapter, we consider several times that $\boldsymbol{\xi}$ is a unitary frontal. In terms of family of lines, there is no difference considering $\boldsymbol{\xi}$ unitary or not, but we work with this restriction in order to extend some concepts of Kummer's theory to this context. Next, we define the relative quadratic forms associated to a line congruence.

6.1.1 Ω-Kummer fundamental forms

Definition 6.1.1. Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\mathbf{x} : U \to \mathbb{R}^3$ is a smooth map, $\boldsymbol{\xi} : U \to \mathbb{R}^3$ is a frontal and let $\boldsymbol{\Omega}$ be a tangent moving basis of $\boldsymbol{\xi}$. If $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$, we define the following quadratic forms:

$$\mathcal{I}_\Omega(\mathbf{v}) = \mathscr{E}_\Omega b_1^2 + 2\mathscr{F}_\Omega b_1 b_2 + \mathscr{G}_\Omega b_2^2, \tag{6.1}$$

where $\mathscr{E}_\Omega = \langle \mathbf{w}_1, \mathbf{w}_1 \rangle$, $\mathscr{F}_\Omega = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, $\mathscr{G}_\Omega = \langle \mathbf{w}_2, \mathbf{w}_2 \rangle$, $\mathbf{v} \in T_\Omega$ and (b_1, b_2) are the coordinates of $\mathbf{v}$ in the basis $\mathbf{w}_1, \mathbf{w}_2$. This form is called the $\boldsymbol{\Omega}$-*Kummer first fundamental form* of $\mathcal{C}$ and we denote by $\boldsymbol{\mathcal{I}}_\Omega := \boldsymbol{\Omega}^T \boldsymbol{\Omega}$ its associated matrix.

$$\mathcal{II}_\Omega(\mathbf{v}) = \mathscr{L}_\Omega b_1^2 + (\mathscr{M}_{1\Omega} + \mathscr{M}_{2\Omega}) b_1 b_2 + \mathscr{N}_\Omega b_2^2, \tag{6.2}$$

where $\mathscr{L}_\Omega = -\langle \mathbf{x}_{u_1}, \mathbf{w}_1 \rangle$, $\mathscr{M}_{2\Omega} = -\langle \mathbf{x}_{u_1}, \mathbf{w}_2 \rangle$, $\mathscr{M}_{1\Omega} = -\langle \mathbf{x}_{u_2}, \mathbf{w}_1 \rangle$ and $\mathscr{N}_\Omega = -\langle \mathbf{x}_{u_2}, \mathbf{w}_2 \rangle$. This form is called the $\boldsymbol{\Omega}$-*Kummer second fundamental form* of $\mathcal{C}$ and we denote by $\boldsymbol{\mathcal{II}}_\Omega := -\boldsymbol{\Omega}^T D\mathbf{x}$ the matrix of these last coefficients.

Remark 6.1.2. Note that $\mathcal{I}_\Omega$ is a positive-definite quadratic form.

Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to S^2$ is a frontal and let $\boldsymbol{\Omega}$ be a tangent moving basis of $\boldsymbol{\xi}$. Define the function $\mathscr{K}_q^\Omega : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\mathscr{K}_q^\Omega(b_1, b_2) = \frac{b^T \boldsymbol{\mathcal{II}}_\Omega adj(\boldsymbol{\Delta}_\Omega^T) b}{b^T \boldsymbol{\mathcal{I}}_\Omega b}, \tag{6.3}$$

where $q \in U$, $adj()$ denotes the adjoint of a matrix, $b^T = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$. Note that we can associate (b_1, b_2) to the coordinates of a vector $\mathbf{v} = b_1 \mathbf{w}_1(q) + b_2 \mathbf{w}_2(q)$, then we write $\mathscr{K}_q^\Omega(b_1, b_2) = \mathscr{K}_q^\Omega(\mathbf{v})$.

Proposition 6.1.1. Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi}$ is a frontal and let $\boldsymbol{\Omega}$ be a tangent moving basis of $\boldsymbol{\xi}$. Then

1. $\boldsymbol{\mathcal{I}} = \boldsymbol{\Delta}_\Omega \boldsymbol{\mathcal{I}}_\Omega \boldsymbol{\Delta}_\Omega^T$
2. $\boldsymbol{\mathcal{II}} = \boldsymbol{\Delta}_\Omega \boldsymbol{\mathcal{II}}_\Omega$.

Proof. Note that

$$\boldsymbol{\mathcal{I}} = D\boldsymbol{\xi}^T D\boldsymbol{\xi} = (\boldsymbol{\Omega}\boldsymbol{\Delta}_\Omega^T)^T (\boldsymbol{\Omega}\boldsymbol{\Delta}_\Omega^T) = \boldsymbol{\Delta}_\Omega \boldsymbol{\Omega}^T \boldsymbol{\Omega} \boldsymbol{\Delta}_\Omega^T = \boldsymbol{\Delta}_\Omega \boldsymbol{\mathcal{I}}_\Omega \boldsymbol{\Delta}_\Omega^T.$$

The case for $\boldsymbol{\mathcal{II}}$ follows analogously.

□

Given a line congruence $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$, where $\boldsymbol{\xi} : U \to S^2$ is a frontal and $\boldsymbol{\Omega}$ is a tangent moving basis of $\boldsymbol{\xi}$, the next proposition shows how the function $\mathscr{K}_q$, from Kummer's theory (given in 3.15) and the function $\mathscr{K}_q^\Omega$, defined in (6.3), are related when $q \notin \Sigma(\boldsymbol{\xi})$.

Proposition 6.1.2. Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to S^2$ is a frontal and let $\boldsymbol{\Omega}$ be a tangent moving basis of $\boldsymbol{\xi}$. Then, for each $q \notin \Sigma(\boldsymbol{\xi})$, $\delta_\Omega \mathscr{K}_q(a_1, a_2) = \mathscr{K}_q^\Omega(b_1, b_2)$, where we write $b^T = (b_1, b_2) \in \mathbb{R}^2$, $a^T = (a_1, a_2) = b^T \boldsymbol{\Delta}_\Omega^{-1}(q)$ and $\delta_\Omega = \det \boldsymbol{\Delta}_\Omega(q)$.

Proof. Since $q \notin \Sigma(\boldsymbol{\xi})$, let $\mathbf{w} = D\boldsymbol{\xi}_q a$, hence we can write $\mathbf{w} = \boldsymbol{\Omega}\boldsymbol{\Delta}_\Omega^T a$, where $a^T = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$. Thus $a = \boldsymbol{\Delta}_\Omega^{-T} b$, where $b^T = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$ are the coordinates of $\mathbf{w}$ relative to $\boldsymbol{\Omega}$. From proposition 6.1.1,

$$\begin{aligned}
\mathscr{K}_q(a_1, a_2) &= \frac{a^T \mathcal{II} a}{a^T \mathcal{I} a} = \frac{b^T \boldsymbol{\Delta}_\Omega^{-1} \mathcal{II} \boldsymbol{\Delta}_\Omega^{-T} b}{b^T \boldsymbol{\Delta}_\Omega^{-1} \mathcal{I} \boldsymbol{\Delta}_\Omega^{-T} b} = \frac{b^T \boldsymbol{\Delta}_\Omega^{-1} \left(\boldsymbol{\Delta}_\Omega \mathcal{II}_\Omega\right) \boldsymbol{\Delta}_\Omega^{-T} b}{b^T \boldsymbol{\Delta}_\Omega^{-1} \left(\boldsymbol{\Delta}_\Omega \mathcal{I}_\Omega \boldsymbol{\Delta}_\Omega^T\right) \boldsymbol{\Delta}_\Omega^{-T} b} \\
&= \frac{1}{\det(\boldsymbol{\Delta}_\Omega)} \frac{b^T \mathcal{II}_\Omega adj(\boldsymbol{\Delta}_\Omega^T) b}{b^T \boldsymbol{\Delta}_\Omega^{-1} \left(\boldsymbol{\Delta}_\Omega \mathcal{I}_\Omega \boldsymbol{\Delta}_\Omega^T\right) \boldsymbol{\Delta}_\Omega^{-T} b}
\end{aligned}$$

hence

$$\delta_\Omega \mathscr{K}_q(a_1, a_2) = \frac{b^T \mathcal{II}_\Omega adj(\boldsymbol{\Delta}_\Omega^T) b}{b^T \mathcal{I}_\Omega b} = \mathscr{K}_q^\Omega(b_1, b_2) \tag{6.4}$$

□

Note that $\mathscr{K}_q^\Omega(a_1, a_2) = \mathscr{K}_q^\Omega(\varphi(a_1, a_2))$, for all $\varphi \in \mathbb{R} \setminus 0$, thus we can consider $\mathscr{K}_q^\Omega : S^1 \to \mathbb{R}$.

Proposition 6.1.3. Let $\mathcal{C} = \{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to S^2$ is a frontal. The congruence $\mathcal{C}$ is normal if and only if the matrix $\mathcal{II}_\Omega adj(\boldsymbol{\Delta}_\Omega^T)$ is symmetric.

Proof. It follows from proposition 5.1 in (IZUMIYA; SAJI; TAKEUCHI, 2003) that $\mathcal{C}$ is normal if and only if $\mathcal{II} = \boldsymbol{\Delta}_\Omega \mathcal{II}_\Omega$ is symmetric. It can be shown by straightforward calculations that this is equivalent to say that $\mathcal{II}_\Omega adj(\boldsymbol{\Delta}_\Omega^T)$ is symmetric. □

6.2 Principal and developable surfaces of the congruence

6.2.1 Kummer principal directions

Definition 6.2.1. Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to \mathbb{R}^3$ is a frontal and let C be a curve on $\mathbf{x}$ parametrized by $\mathbf{x}(t) = \mathbf{x}(\alpha(t))$, where $\alpha : I \to U$ is smooth and $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(\alpha(t))$ is the restriction of $\boldsymbol{\xi}$ to C. The ruled surface S_C, parametrized by

$$Y(t, v) = \mathbf{x}(t) + v\boldsymbol{\xi}(t), \quad t \in I \subset \mathbb{R}, v \in \mathbb{R}, \tag{6.5}$$

is called a surface of the congruence.

Definition 6.2.2. Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to S^2$ is a proper frontal, $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{pmatrix}$ a tangent moving basis of $\boldsymbol{\xi}$. We say that a direction $\mathbf{w} \in T_\Omega(q)$ is a *Kummer principal direction* if $\mathscr{K}_q^\Omega(\mathbf{w})$ is an extreme value of $\mathscr{K}_q^\Omega$.

Remark 6.2.1. The Kummer principal directions do not depend on the chosen tangent moving basis. In fact, let $\{\mathbf{x},\boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi}: U \to \mathbb{R}^3$ is a proper frontal, $\boldsymbol{\Omega}$ and $\bar{\boldsymbol{\Omega}}$ are two tangent moving bases of $\boldsymbol{\xi}$. Thus there is $\mathbf{B}: U \to GL(2,\mathbb{R})$ such that $\boldsymbol{\Omega} = \bar{\boldsymbol{\Omega}}\mathbf{B}$. Note that $\boldsymbol{\mathcal{I}}_\Omega = \boldsymbol{\Omega}^T\boldsymbol{\Omega} = \mathbf{B}^T\bar{\boldsymbol{\Omega}}^T\bar{\boldsymbol{\Omega}}\mathbf{B} = \mathbf{B}^T\boldsymbol{\mathcal{I}}_{\bar{\Omega}}\mathbf{B}$ and $\boldsymbol{\mathcal{II}}_\Omega = \boldsymbol{\Omega}^T D\mathbf{x} = \mathbf{B}^T\bar{\boldsymbol{\Omega}}^T D\mathbf{x} = \mathbf{B}^T\boldsymbol{\mathcal{II}}_{\bar{\Omega}}$. Furthermore, if $D\boldsymbol{\xi} = \boldsymbol{\Omega}\boldsymbol{\Delta}_\Omega^T = \bar{\boldsymbol{\Omega}}\boldsymbol{\Delta}_{\bar{\Omega}}^T$, then $\boldsymbol{\Delta}_{\bar{\Omega}}^T = \mathbf{B}\boldsymbol{\Delta}_\Omega^T$. Using this and 6.3 we get $\mathscr{K}_q^{\bar{\Omega}}(\bar{b}_1,\bar{b}_2) = \det(\mathbf{B})\mathscr{K}_q^{\Omega}(b_1,b_2)$, where $(\bar{b}_1,\bar{b}_2)$ and (b_1,b_2) are the coordinates of a vector $\mathbf{v} \in T_\Omega(q) = T_{\bar{\Omega}}(q)$ in the bases $\boldsymbol{\Omega}$ and $\bar{\boldsymbol{\Omega}}$, respectively. Therefore, the extreme values of $\mathscr{K}_q^\Omega$ do not depend on the tangent moving basis.

Definition 6.2.3. Let $\mathcal{C} = \{\mathbf{x},\boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi}: U \to S^2$ is a proper frontal and $\boldsymbol{\Omega}$ a tangent moving basis of $\boldsymbol{\xi}$. Let S_C be a surface of the congruence, given by

$$Y(t,v) = \mathbf{x}(t) + v\boldsymbol{\xi}(t), \quad t \in I \subset \mathbb{R}, v \in \mathbb{R}, \tag{6.6}$$

where $\alpha: U \to I$, such that $\alpha(t) = (u_1(t),u_2(t))$ is smooth, $\mathbf{x}(t) = \mathbf{x}(\alpha(t))$ and $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(\alpha(t))$. We say that S_C is a *principal surface* if for all $t \in I$ such that $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{\Delta}_\Omega^T \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} \neq \mathbf{0}$, (b_1,b_2) determines a Kummer principal direction in T_Ω. We call $\alpha: I \to U$ a *Kummer principal line*.

Lemma 6.2.1. Let $\{\mathbf{x},\boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi}: U \to S^2$ is a proper frontal, $\boldsymbol{\Omega}$ is a tangent moving basis of $\boldsymbol{\xi}$ and we write $\boldsymbol{\Delta}_\Omega = \left(\delta_{ij}\right)$. Then, $\mathscr{K}_q^\Omega$ has an extreme value at (b_1,b_2) if and only if

$$\begin{aligned} & b_1(\delta_{22}\mathscr{L}_\Omega - \delta_{12}\mathscr{M}_1) + \frac{b_2}{2}(\delta_{11}\mathscr{M}_{1\Omega} - \delta_{21}\mathscr{L}_\Omega + \delta_{22}\mathscr{M}_{2\Omega} - \delta_{12}\mathscr{N}_\Omega) \\ & - k_0(b_1\mathscr{E}_\Omega + b_2\mathscr{F}_\Omega) = 0 \end{aligned} \tag{6.7}$$

$$\begin{aligned} & b_2(\delta_{11}\mathscr{N}_\Omega - \delta_{21}\mathscr{M}_{2\Omega}) + \frac{b_1}{2}(\delta_{11}\mathscr{M}_{1\Omega} - \delta_{21}\mathscr{L}_\Omega + \delta_{22}\mathscr{M}_{2\Omega} - \delta_{12}\mathscr{N}_\Omega) \\ & - k_0(b_2\mathscr{G}_\Omega + b_1\mathscr{F}_\Omega) = 0, \end{aligned} \tag{6.8}$$

where $k_0 = \mathscr{K}_q^\Omega(b_1,b_2)$.

Proof. Note that

$$\begin{aligned} \mathscr{K}_q^\Omega(b_1,b_2) = & \frac{b_1^2(\delta_{22}\mathscr{L}_\Omega - \delta_{12}\mathscr{M}_1) + b_2^2(\delta_{11}\mathscr{N}_\Omega - \delta_{21}\mathscr{M}_{2\Omega})}{b_1^2\mathscr{E}_\Omega + 2b_1b_2\mathscr{F}_\Omega + b_2^2\mathscr{G}_\Omega} \\ & + \frac{b_1b_2(\delta_{11}\mathscr{M}_{1\Omega} - \delta_{21}\mathscr{L}_\Omega + \delta_{22}\mathscr{M}_{2\Omega} - \delta_{12}\mathscr{N}_\Omega)}{b_1^2\mathscr{E}_\Omega + 2b_1b_2\mathscr{F}_\Omega + b_2^2\mathscr{G}_\Omega} \end{aligned}$$

Let us suppose that $\mathscr{K}_q^\Omega$ has an extreme value at (b_1,b_2). At an extreme value k_0 of $\mathscr{K}_q^\Omega$, we have $\frac{\partial \mathscr{K}_q^\Omega}{\partial b_i} = 0$, $i = 1,2$. From this, we get (6.7) and (6.8). Reciprocally, if (b_1,b_2) is such that (6.7) and (6.8) are valid, then we have directly that k_0 is an extreme value of $\mathscr{K}_q^\Omega$. Let us show that $\mathscr{K}_q^\Omega(b_1,b_2) = k_0$. Let us suppose $b_1 \neq 0$ and $b_2 \neq 0$ (other cases are analogous). If we sum

(6.7) multiplied by b_1 with (6.8) multiplied by b_2, we obtain

$$\begin{aligned} &b_1^2(\delta_{22}\mathscr{L}_\Omega-\delta_{12}\mathscr{M}_1)+b_1b_2(\delta_{11}\mathscr{M}_{1\Omega}-\delta_{21}\mathscr{L}_\Omega+\delta_{22}\mathscr{M}_{2\Omega}-\delta_{12}\mathscr{N}_\Omega)\\ &\quad+b_2^2(\delta_{11}\mathscr{N}_\Omega-\delta_{21}\mathscr{M}_{2\Omega})=k_0(b_1^2\mathscr{E}_\Omega+2b_1b_2\mathscr{F}_\Omega+b_2^2\mathscr{G}_\Omega). \end{aligned}$$

From this, we have $k_0=\mathscr{K}_q^{\Omega}(b_1,b_2)$. □

Proposition 6.2.1. Let $\{\mathbf{x},\boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi}:U\to S^2$ is a proper frontal, $\boldsymbol{\Omega}$ a tangent moving basis of $\boldsymbol{\xi}$ and we write $\boldsymbol{\Delta}_\Omega=\begin{pmatrix}\delta_{ij}\end{pmatrix}$. Then:

(1) A curve $(u_1(t),u_2(t))$ is a Kummer principal line if and only if this is a solution of

$$C_1b_1^2+C_2b_1b_2+C_3b_2^2=0,\ \text{for all } t, \tag{6.9}$$

where

$$\begin{aligned} C_1&=2\mathscr{F}_\Omega\left(\delta_{22}\mathscr{L}_\Omega-\delta_{12}\mathscr{M}_{1\Omega}\right)-\mathscr{E}_\Omega\left(\delta_{11}\mathscr{M}_{1\Omega}-\delta_{21}\mathscr{L}_\Omega+\delta_{22}\mathscr{M}_{2\Omega}-\delta_{12}\mathscr{N}_\Omega\right)\\ C_2&=2\mathscr{G}_\Omega\left(\delta_{22}\mathscr{L}_\Omega-\delta_{12}\mathscr{M}_{1\Omega}\right)-2\mathscr{E}_\Omega\left(\delta_{11}\mathscr{N}_\Omega-\delta_{21}\mathscr{M}_{2\Omega}\right)\\ C_3&=\mathscr{G}_\Omega\left(\delta_{11}\mathscr{M}_{1\Omega}-\delta_{21}\mathscr{L}_\Omega+\delta_{22}\mathscr{M}_{2\Omega}-\delta_{12}\mathscr{N}_\Omega\right)-2\mathscr{F}_\Omega\left(\delta_{11}\mathscr{N}_\Omega-\delta_{21}\mathscr{M}_{2\Omega}\right). \end{aligned}$$

We call (6.9) the *equation of principal surfaces* of the congruence.

(2) If the congruence is normal then (6.9) can be written as

$$\begin{pmatrix}u_1' & u_2'\end{pmatrix}\boldsymbol{\Delta}_\Omega\mathbf{P}adj(\boldsymbol{\mathcal{II}}_\Omega)^T\boldsymbol{\Delta}_\Omega\boldsymbol{\mathcal{I}}_\Omega\boldsymbol{\Delta}_\Omega^T\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=0, \tag{6.10}$$

where $\mathbf{P}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Proof. 1. From (6.7) and (6.8), we have that $(b_1(t),b_2(t))$ provides an extreme value of $\mathscr{K}_q^{\Omega}$ for all t if and only if

$$\begin{vmatrix} b_1(\delta_{22}\mathscr{L}_\Omega-\delta_{12}\mathscr{M}_1)+\dfrac{b_2}{2}\mathscr{M} & b_1\mathscr{E}_\Omega+b_2\mathscr{F}_\Omega\\ b_2(\delta_{11}\mathscr{N}_\Omega-\delta_{21}\mathscr{M}_{2\Omega})+\dfrac{b_1}{2}\mathscr{M} & b_2\mathscr{G}_\Omega+b_1\mathscr{F}_\Omega \end{vmatrix}=0,$$

where $\mathscr{M}=(\delta_{11}\mathscr{M}_{1\Omega}-\delta_{21}\mathscr{L}_\Omega+\delta_{22}\mathscr{M}_{2\Omega}-\delta_{12}\mathscr{N}_\Omega)$. The equation (6.9) is obtained directly from the above expression.

2. We know from proposition 6.1.3 that $\{\mathbf{x},\boldsymbol{\xi}\}$ is normal if and only if $\boldsymbol{\mathcal{II}}_\Omega adj(\boldsymbol{\Delta}_\Omega^T)$ is symmetric, which is equivalent to say that

$$\delta_{11}\mathscr{M}_{1\Omega}-\delta_{21}\mathscr{L}_\Omega=\delta_{22}\mathscr{M}_{2\Omega}-\delta_{12}\mathscr{N}_\Omega. \tag{6.11}$$

By using this condition in (6.9), we obtain (6.10).

□

Proposition 6.2.2. The discriminant $\mathscr{D} = C_2^2 - 4C_1C_3$ of the equation (6.9) is non-negative. This discriminant is zero if and only if the coefficients C_1, C_2 and C_3 are identically zero.

Proof. If we write

$$
\begin{aligned}
\mathscr{L} &= 2\left(\delta_{22}\mathscr{L}_\Omega - \delta_{12}\mathscr{M}_1\right),\\
\mathscr{N} &= 2\left(\delta_{11}\mathscr{N}_\Omega - \delta_{21}\mathscr{M}_{2\Omega}\right),\\
\mathscr{M} &= \left(\delta_{11}\mathscr{M}_{1\Omega} - \delta_{21}\mathscr{L}_\Omega + \delta_{22}\mathscr{M}_{2\Omega} - \delta_{12}\mathscr{N}_\Omega\right),
\end{aligned}
$$

then

$$
\begin{aligned}
C_1 &= \mathscr{F}_\Omega\mathscr{L} - \mathscr{E}_\Omega\mathscr{M},\\
C_2 &= \mathscr{G}_\Omega\mathscr{L} - \mathscr{E}_\Omega\mathscr{N},\\
C_3 &= \mathscr{G}_\Omega\mathscr{M} - \mathscr{F}_\Omega\mathscr{N}.
\end{aligned}
$$

Hence, $\mathscr{G}_\Omega C_1 - \mathscr{F}_\Omega C_2 + \mathscr{E}_\Omega C_3 = 0$, from which we get $C_3 = \dfrac{\mathscr{F}_\Omega C_2 - \mathscr{G}_\Omega C_1}{\mathscr{E}_\Omega}$. Thus

$$
\begin{aligned}
\mathscr{D} &= C_2^2 - 4\frac{C_1\mathscr{F}_\Omega C_2}{\mathscr{E}_\Omega} + 4\frac{\mathscr{G}_\Omega C_1^2}{\mathscr{E}_\Omega}\\
&= \left(C_2 - 2\frac{C_1\mathscr{F}_\Omega}{\mathscr{E}_\Omega}\right)^2 - 4\frac{C_1^2\mathscr{F}_\Omega^2}{\mathscr{E}^2} + 4\frac{\mathscr{G}_\Omega C_1^2}{\mathscr{E}_\Omega}\\
&= \left(C_2 - 2\frac{C_1\mathscr{F}_\Omega}{\mathscr{E}_\Omega}\right)^2 + 4\frac{C_1^2}{\mathscr{E}_\Omega^2}\left(\mathscr{E}_\Omega\mathscr{G}_\Omega - \mathscr{F}_\Omega\right) \geq 0.
\end{aligned}
$$

As $\mathscr{E}_\Omega\mathscr{G}_\Omega - \mathscr{F}_\Omega > 0$, we get $\mathscr{D} = 0$ if and only if $C_1 = C_2 = 0$, which implies that $C_3 = 0$. □

Proposition 6.2.2 asserts that at points where $\mathscr{D} > 0$ there are only two Kummer principal directions.

We can also look at the developable surfaces associated to a given line congruence $\{\mathbf{x}, \boldsymbol{\xi}\}$, where $\mathbf{x} : U \to \mathbb{R}^3$ is a smooth map and $\boldsymbol{\xi} : U \to \mathbb{R}^3$ is a unitary proper frontal. In order to do this, we have the next proposition.

Proposition 6.2.3. Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be a line congruence, where $\boldsymbol{\xi} : U \to S^2$ is a proper frontal, $\boldsymbol{\Omega}$ a tangent moving basis of $\boldsymbol{\xi}$. A surface of the congruence $Y(t,v) = x(u_1(t), u_2(t)) + v\boldsymbol{\xi}(u_1(t), u_2(t))$ is a developable surface if and only if $(u_1(t), u_2(t))$ is a solution of

$$
\begin{pmatrix} u_1' & u_2' \end{pmatrix} \mathbf{P}adj(\boldsymbol{\mathcal{I}\mathcal{I}}_\Omega)\boldsymbol{\mathcal{I}}_\Omega\boldsymbol{\Delta}_\Omega^T \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = 0. \tag{6.12}
$$

We call (6.12) the *equation of developable surfaces* of the congruence.

Proof. Let us suppose $\alpha : I \to U$ a smooth curve, given by $\alpha(t) = (u_1(t), u_2(t))$, such that the surface of the congruence $Y(t,v) = \mathbf{x}(t) + v\boldsymbol{\xi}(t)$ is developable, where $\mathbf{x}(t) = \mathbf{x}(\alpha(t))$ and

$\boldsymbol{\xi}(t)=\boldsymbol{\xi}(\alpha(t))$. Then it is known that $\left[\mathbf{x}',\boldsymbol{\xi}',\boldsymbol{\xi}\right]=0$ (See section 3.5 in (CARMO, 2016)). From this expression, we obtain the differential equation of developable surfaces

$$u_1'^2\left[\mathbf{x}_{u_1},\boldsymbol{\xi}_{u_1},\boldsymbol{\xi}\right]+u_1'u_2'\left(\left[\mathbf{x}_{u_1},\boldsymbol{\xi}_{u_2},\boldsymbol{\xi}\right]+\left[\mathbf{x}_{u_2},\boldsymbol{\xi}_{u_1},\boldsymbol{\xi}\right]\right)+u_2'^2\left[\mathbf{x}_{u_2},\boldsymbol{\xi}_{u_2},\boldsymbol{\xi}\right]=0. \tag{6.13}$$

By considering that $\boldsymbol{\xi}$ is unitary we have $\boldsymbol{\xi}=\dfrac{\mathbf{w}_1\times\mathbf{w}_2}{\|\mathbf{w}_1\times\mathbf{w}_2\|}$, where $\boldsymbol{\Omega}=\begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$. We also know that $D\boldsymbol{\xi}=\boldsymbol{\Omega}\boldsymbol{\Delta}_\Omega^T$, where $\boldsymbol{\Delta}_\Omega=\begin{pmatrix}\delta_{ij}\end{pmatrix}$, thus

$$\left[\mathbf{x}_{u_1},\boldsymbol{\xi}_{u_1},\boldsymbol{\xi}\right]=\frac{1}{\|\mathbf{w}_1\times\mathbf{w}_2\|}\langle\mathbf{x}_{u_1},(\delta_{11}\mathbf{w}_1+\delta_{12}\mathbf{w}_2)\times(\mathbf{w}_1\times\mathbf{w}_2)\rangle.$$

By using the coefficients of the first and second $\boldsymbol{\Omega}$-Kummer fundamental forms and the formula for the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\langle\mathbf{a},\mathbf{c}\rangle\mathbf{b}-\langle\mathbf{a},\mathbf{b}\rangle\mathbf{c}$, we get

$$\left[\mathbf{x}_{u_1},\boldsymbol{\xi}_{u_1},\boldsymbol{\xi}\right]=\delta_{11}(\mathscr{F}_\Omega\mathscr{L}_\Omega-\mathscr{E}_\Omega\mathscr{M}_{1\Omega})+\delta_{12}(\mathscr{L}_\Omega\mathscr{G}_\Omega-\mathscr{F}_\Omega\mathscr{M}_{1\Omega}). \tag{6.14}$$

In a similar way, we obtain

$$\begin{aligned}\left[\mathbf{x}_{u_1},\boldsymbol{\xi}_{u_2},\boldsymbol{\xi}\right]+\left[\mathbf{x}_{u_2},\boldsymbol{\xi}_{u_1},\boldsymbol{\xi}\right]=&\delta_{21}(\mathscr{F}_\Omega\mathscr{L}_\Omega-\mathscr{E}_\Omega\mathscr{M}_{1\Omega})+\delta_{22}(\mathscr{G}_\Omega\mathscr{L}_\Omega-\mathscr{F}_\Omega\mathscr{M}_{1\Omega}) \qquad (6.15)\\ &+\delta_{11}(\mathscr{F}_\Omega\mathscr{M}_{2\Omega}-\mathscr{E}_\Omega\mathscr{N}_\Omega)+\delta_{12}(\mathscr{G}_\Omega\mathscr{M}_{2\Omega}-\mathscr{F}_\Omega\mathscr{N}_\Omega)\\ \left[\mathbf{x}_{u_2},\boldsymbol{\xi}_{u_2},\boldsymbol{\xi}\right]=&\delta_{21}(\mathscr{F}_\Omega\mathscr{M}_{2\Omega}-\mathscr{E}_\Omega\mathscr{N}_\Omega)+\delta_{22}(\mathscr{G}_\Omega\mathscr{M}_{2\Omega}-\mathscr{F}_\Omega\mathscr{N}_\Omega). \qquad (6.16)\end{aligned}$$

Hence, from (6.14), (6.15) and (6.16) we can rewrite (6.13) as

$$\begin{pmatrix}u_1' & u_2'\end{pmatrix}\mathbf{P}adj(\mathcal{II}_\Omega)\mathcal{I}_\Omega\boldsymbol{\Delta}_\Omega^T\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=0.$$

□

Theorem 6.2.1. Let $\{\mathbf{x},\boldsymbol{\xi}\}$ be a normal line congruence, where $\boldsymbol{\xi}:U\to S^2$ is a proper frontal, $\boldsymbol{\Omega}$ a tangent moving basis of $\boldsymbol{\xi}$. Then the equation of principal surfaces is a multiple of the equation of developable surfaces by δ_Ω, where $\delta_\Omega=\det\boldsymbol{\Delta}_\Omega$. More precisely

$$\boldsymbol{\Delta}_\Omega\mathbf{P}adj(\mathcal{II}_\Omega)^T\boldsymbol{\Delta}_\Omega\mathcal{I}_\Omega\boldsymbol{\Delta}_\Omega^T=\delta_\Omega\mathbf{P}adj(\mathcal{II}_\Omega)\mathcal{I}_\Omega\boldsymbol{\Delta}_\Omega^T. \tag{6.17}$$

Proof. We know from propositions (6.2.1) and (6.2.3) that the binary differential equations which provide principal and developable surfaces of the congruence are, respectively, given by

$$\begin{pmatrix}u_1' & u_2'\end{pmatrix}\boldsymbol{\Delta}_\Omega\mathbf{P}adj(\mathcal{II}_\Omega)^T\boldsymbol{\Delta}_\Omega\mathcal{I}_\Omega\boldsymbol{\Delta}_\Omega^T\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=0, \tag{6.18}$$

$$\begin{pmatrix}u_1' & u_2'\end{pmatrix}\mathbf{P}adj(\mathcal{II}_\Omega)\mathcal{I}_\Omega\boldsymbol{\Delta}_\Omega\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=0. \tag{6.19}$$

It follows from proposition 6.1.3 that $\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T)$ is symmetric, since we have a normal congruence. It can be shown by straightforward calculations that this is equivalent to say that $adj(\mathcal{II}_\Omega)^T\mathbf{\Delta}_\Omega$ is symmetric. Hence, we have

$$\begin{aligned}
&\mathbf{\Delta}_\Omega \mathbf{P} adj(\mathcal{II}_\Omega)^T \mathbf{\Delta}_\Omega \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T \\
&= \mathbf{\Delta}_\Omega \mathbf{P} \mathbf{\Delta}_\Omega^T adj(\mathcal{II}_\Omega) \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T \\
&= -\mathbf{\Delta}_\Omega \mathbf{P} \mathbf{\Delta}_\Omega^T \mathbf{P}\mathbf{P} adj(\mathcal{II}_\Omega) \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T, \text{ since } -\mathbf{P} = \mathbf{P}^{-1} \\
&= \mathbf{\Delta}_\Omega adj(\mathbf{\Delta}_\Omega) \mathbf{P} adj(\mathcal{II}_\Omega) \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T, \text{ since } -\mathbf{P}\mathbf{\Delta}_\Omega^T\mathbf{P} = adj(\mathbf{\Delta}_\Omega) \\
&= \delta_\Omega \mathbf{P} adj(\mathcal{II}_\Omega) \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T.
\end{aligned}$$

□

Corollary 6.2.1. Let $\{\mathbf{x}, \boldsymbol{\xi}\}$ be a normal line congruence. If $\mathbf{x}$ and $\boldsymbol{\xi}$ are analytic, then an analytic solution of the equation of principal surfaces is either a branch of $\Sigma(\boldsymbol{\xi})$ or an analytic solution of the equation of developable surfaces.

Proof. Let $\gamma : I \to U$, given by $\gamma(t) = (u_1(t), u_2(t))$, be an analytic solution of the equation of principal surfaces, then $\delta_\Omega(\gamma)$ is an analytic mapping. If there is t_0 such that the derivatives $\delta_\Omega(\gamma)^{(j)}(t_0) = 0$, for all positive integer j, then $\delta_\Omega(\gamma)(t) = 0$, for all $t \in I$. Otherwise, the zeros of $\delta_\Omega(\gamma)$ are isolated and therefore γ is a solution of the equation of developable surfaces, once we have (6.17). □

From now on, we consider $\mathbf{x}$ a frontal, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $\boldsymbol{\xi}$ its normal vector field. Also note that we can take $\mathbf{\Omega}$ as a tangent moving basis of $\boldsymbol{\xi}$.

Proposition 6.2.4. Let $\mathbf{x} : U \to \mathbb{R}^3$ and $\boldsymbol{\xi} : U \to S^2$ be two proper frontals, such that $\boldsymbol{\xi}$ is the unit normal vector field of $\mathbf{x}$. Then, $\mathbf{w}$ is a principal direction if and only if is a Kummer principal direction associated to the congruence $\{\mathbf{x}, \boldsymbol{\xi}\}$.

Proof. It follows from (MEDINA-TEJEDA, 2022b) and from remark 6.2.1 that the principal directions (see definition 3.4.4) and the Kummer principal directions do not depend on the chosen tangent moving basis, so let us consider $\mathbf{\Omega}$ an orthonormal one, i.e, $\mathbf{\Omega}^T\mathbf{\Omega} = id_{\mathbb{R}^2}$. From lemma 3.1 and remark 3.1 in (MEDINA-TEJEDA, 2022b), it follows that $\mathbf{II}_\Omega adj(\Lambda_\Omega^T)$ is symmetric, then the principal directions are given by its eigenvectors. Since the congruence is normal, $\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T)$ is symmetric and we get analogously that the Kummer principal directions are given by its eigenvectors. As $D\boldsymbol{\xi} = \mathbf{\Omega}\mathbf{\Delta}_\Omega^T$ and $D\mathbf{x} = \mathbf{\Omega}\Lambda_\Omega^T$, then

$$\begin{aligned}
\mathbf{II}_\Omega adj(\mathbf{\Lambda}_\Omega^T) &= -\mathbf{\Omega}^T D\boldsymbol{\xi} adj(\mathbf{\Lambda}_\Omega^T) = -\mathbf{\Omega}^T\mathbf{\Omega}\mathbf{\Delta}_\Omega^T adj(\mathbf{\Lambda}_\Omega^T) = -\mathbf{\Delta}_\Omega^T adj(\mathbf{\Lambda}_\Omega^T) \\
\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T) &= -\mathbf{\Omega}^T D\mathbf{x} adj(\mathbf{\Delta}_\Omega^T) = -\mathbf{\Omega}^T\mathbf{\Omega}\mathbf{\Lambda}_\Omega^T adj(\mathbf{\Delta}_\Omega^T) = -\mathbf{\Lambda}_\Omega^T adj(\mathbf{\Delta}_\Omega^T).
\end{aligned}$$

Hence, $\mathbf{II}_\Omega adj(\Lambda_\Omega^T) = adj(\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T))$. Then, the eigenvectors of $\mathbf{II}_\Omega adj(\mathbf{\Lambda}_\Omega^T)$ are the eigenvectors of $\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T)$, that is, $\mathbf{w} \in T_\Omega$ is a principal direction if and only if is a Kummer principal direction. □

Remark 6.2.2. It follows from $\mathbf{II}_\Omega adj(\mathbf{\Lambda}_\Omega^T) = adj(\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T))$ that if $\mathbf{w}_1, \mathbf{w}_2$ are the eigenvectors of $\mathcal{II}_\Omega adj(\mathbf{\Delta}_\Omega^T)$ associated to the eigenvalues γ_1, γ_2, respectively, then $\mathbf{w}_1, \mathbf{w}_2$ are the eigenvectors of $\mathbf{II}_\Omega adj(\mathbf{\Lambda}_\Omega^T)$ associated to the eigenvalues γ_2, γ_1, respectively. Furthermore, if $\mathbf{w}_1$, for instance, is a Kummer principal direction associated to a maximum of $\mathscr{K}_q^\Omega$, then $\mathbf{w}_1$ is a principal direction associated to a minimum of the $\mathbf{\Omega}$-relative normal curvature, which is defined in (MEDINA-TEJEDA, 2022b). That is, if we denote by $\mathscr{K}_{1\Omega}$ and $\mathscr{K}_{2\Omega}$ the minimum and the maximum of $\mathscr{K}_q^\Omega$, respectively, then $k_{1\Omega} = \mathscr{K}_{2\Omega}$ and $k_{2\Omega} = \mathscr{K}_{1\Omega}$ where $k_{1\Omega}$ and $k_{2\Omega}$ are the $\mathbf{\Omega}$-relative principal curvatures (see definition 3.4.3).

Example 6.2.1. Despite proposition 6.2.4, it is not true that for a line congruence given by a frontal $\mathbf{x} : U \to \mathbb{R}^3$ and its unit normal vector field $\boldsymbol{\xi} : U \to S^2$, a curve is a Kummer principal line if and only if it is a line of curvature. Let us take, for instance, the congruence given by

$$\begin{aligned}
\mathbf{x}(u_1,u_2) &= \left(u_1, u_2, u_1^2 u_2 + u_2^2\right)\\
\boldsymbol{\xi}(u_1,u_2) &= \frac{1}{\sqrt{4u_1^2u_2^2 + u_1^4 + 4u_1^2u_2 + 4u_2^2 + 1}}\left(2u_1u_2, -u_1^2 - 2u_2, 1\right).
\end{aligned}$$

In this case, the Gaussian curvature is given by

$$K(u_1,u_2) = \frac{-4({u_1}^2 - u_2)}{\left({u_1}^4 + 4{u_1}^2{u_2}^2 + 4{u_1}^2u_2 + 4{u_2}^2 + 1\right)^2}$$

so $u_2 = u_1^2$ is a curve of parabolic points. The equation of the lines of curvature is given by

$$\begin{aligned}
&(2u_1^3u_2^2 - 4u_1u_2^3 + u_1)u_1'^2 + (-u_1^4u_2 - 4u_2^3 - u_2 + 1)u_2'u_1'\\
&+ (-u_1^5 - 2u_1^3u_2 - u_1)u_2'^2 = 0.
\end{aligned}$$

On the other hand, the equation of principal surfaces of the congruence is given by

$$\begin{aligned}
&\left(u_2 - u_1^2\right)\left[\left(2u_1^3u_2^2 - 4u_1u_2^3 + u_1\right)u_1'^2 + \left(-u_1^4u_2 - 4u_2^3 - u_2 + 1\right)u_2'u_1'\right.\\
&\left.+ \left(-u_1^5 - 2u_1^3u_2 - u_1\right)u_2'^2\right] = 0.
\end{aligned}$$

Then, the curve of parabolic points $u_2 = u_1^2$ is a Kummer principal line, but it is not a line of curvature.

Next, we have some results regarding the relation between the Kummer principal lines and the lines of curvature of a frontal.

Corollary 6.2.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a frontal, $\mathbf{\Omega}$ a tangent moving basis of $\mathbf{x}$ and $\boldsymbol{\xi} : U \to S^2$ a normal vector field induced by $\mathbf{\Omega}$. If $\boldsymbol{\xi}$ is a proper frontal, then the equation of principal surfaces is

$$\gamma'^T \mathbf{\Delta}_\Omega \mathbf{P} adj(\mathcal{II}_\Omega)^T \mathbf{\Delta}_\Omega \mathcal{I}_\Omega \mathbf{\Delta}_\Omega^T \gamma' = -K_\Omega \det(\mathbf{I}_\Omega)\gamma'^T \mathbf{P}\alpha_\Omega^T(\gamma)\gamma', \tag{6.20}$$

where K_Ω and α_Ω are related with $\mathbf{x}$ (see section 3.4).

Proof. Note that $\boldsymbol{\Delta}_\Omega = \mu_\Omega$, $K_\Omega = \det \boldsymbol{\Delta}_\Omega$, $\mathcal{I}_\Omega = \mathbf{I}_\Omega$ and $\mathcal{II}_\Omega = -\mathbf{I}_\Omega \boldsymbol{\Lambda}_\Omega^T$. Thus, from (6.17)

$$\begin{aligned}\boldsymbol{\Delta}_\Omega \mathbf{P} adj(\mathcal{II}_\Omega)^T \boldsymbol{\Delta}_\Omega \mathcal{I}_\Omega \boldsymbol{\Delta}_\Omega^T &= \delta_\Omega \mathbf{P} adj(\mathcal{II}_\Omega)\mathcal{I}_\Omega \boldsymbol{\Delta}_\Omega^T \\ &= K_\Omega \mathbf{P} adj(-\mathbf{I}_\Omega \boldsymbol{\Lambda}_\Omega^T)\mathbf{I}_\Omega \mu_\Omega^T \\ &= K_\Omega \mathbf{P} adj(\boldsymbol{\Lambda}_\Omega^T) adj(-\mathbf{I}_\Omega)\mathbf{I}_\Omega \mu_\Omega^T \\ &= -K_\Omega \det(\mathbf{I}_\Omega)\mathbf{P}\alpha_\Omega^T(\gamma),\end{aligned}$$

where $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}_\Omega^T$. □

Note that via corollary 6.2.2 we can express the equation of principal surfaces of the congruence (6.10) only considering quantities related to the frontal $\mathbf{x}$, when we have an exact normal congruence.

Remark 6.2.3. It is worth observing above that $-\det(\mathbf{I}_\Omega)\gamma'^T \mathbf{P}\alpha_\Omega^T(\gamma)\gamma' = 0$ is the equation of the developable surfaces of the congruence.

Corollary 6.2.3. Let $\mathbf{x} : U \to \mathbb{R}^3$ and $\boldsymbol{\xi} : U \to S^2$ be two proper frontals with the same singular sets, such that $\boldsymbol{\xi}$ is the unit normal vector field of $\mathbf{x}$. Then, a curve on $\mathbf{x}$ is a Kummer principal line if and only if it is a line of curvature of $\mathbf{x}$.

Proof. The results follows from the fact that $\lambda_\Omega \gamma'^T \mathbf{P}\alpha_\Omega^T(\gamma)\gamma' = 0$ is the equation of the lines of curvature (see proposition 3.4.2) and from $K_\Omega^{-1}(0) = \lambda_\Omega^{-1}(0)$. □

Corollary 6.2.4. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a proper frontal with extendable normal curvature, such that the extension of the Gaussian curvature K never vanishes. Then, the Kummer principal lines coincide with the lines of curvature of $\mathbf{x}$.

Proof. It follows from corollary 3.1 in (MEDINA-TEJEDA, 2022b) that $\mathbf{x}$ and its normal vector field have the same singular set, hence applying corollary 6.2.4 we have the result. □

Example 6.2.2. Let $\mathbf{x} : U \to \mathbb{R}^3$ defined by $x = \left(u_1, \frac{2}{5}u_2^5 + u_2^2, u_1 u_2^2\right)$, for $U = (-1,1) \times (-1,1)$ (Figure 8). Then $D\mathbf{x} = \boldsymbol{\Omega}\boldsymbol{\Lambda}_\Omega^T$, where

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & u_2^3 + 1 \\ u_2^2 & u_1 \end{pmatrix} \text{ and } \boldsymbol{\Lambda}_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 2u_2 \end{pmatrix}. \tag{6.21}$$

The unit normal vector field induced by $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\xi} = \frac{1}{\mu}(-{u_2}^2(u_2+1)({u_2}^2 - u_2 + 1), -u_1, (u_2+1)({u_2}^2 - u_2 + 1)),$$

where $\mu = \sqrt{{u_2}^{10} + 2{u_2}^7 + {u_2}^6 + {u_2}^4 + 2{u_2}^3 + {u_1}^2 + 1}$. The frontal $\mathbf{x}$ in this example is special, because it is a frontal with extendable normal curvature without false singularities (see comments after theorem 3.2 in (MEDINA-TEJEDA, 2022b)). Furthermore, $\lambda_\Omega = 2u_2$ and

$$K_\Omega = \frac{2u_2(u_2+1)^2(u_2^2-u_2+1)^2}{(u_2^{10}+2u_2^7+u_2^6+u_2^4+2u_2^3+u_1^2+1)^2},$$

therefore, considering that at a regular point the Gaussian curvature is given by $\dfrac{K_\Omega}{\lambda_\Omega}$, we obtain

$$K = \frac{(u_2+1)^2(u_2^2-u_2+1)^2}{(u_2^{10}+2u_2^7+u_2^6+u_2^4+2u_2^3+u_1^2+1)^2}$$

is the extension of the Gaussian curvature to U. Then, the Gaussian curvature also admits an extension to U and in this case, the extension is non-vanishing. By applying corollary 6.2.4, the Kummer principal lines coincide with the lines of curvature of $\mathbf{x}$, which are given by the implicit differential equation

$$\begin{aligned} &2u_2\left[\left({u_2}^7+{u_2}^4+{u_2}^3+1\right){u_1'}^2+\left(3u_1{u_2}^6+3u_1{u_2}^2\right)u_1'u_2'\right]\\ &+2u_2\left[\left(-4{u_2}^{11}-12{u_2}^8+2{u_1}^2{u_2}^5-12{u_2}^5-4{u_1}^2{u_2}^2-4{u_2}^2\right)u_2'^2\right]=0. \end{aligned}$$

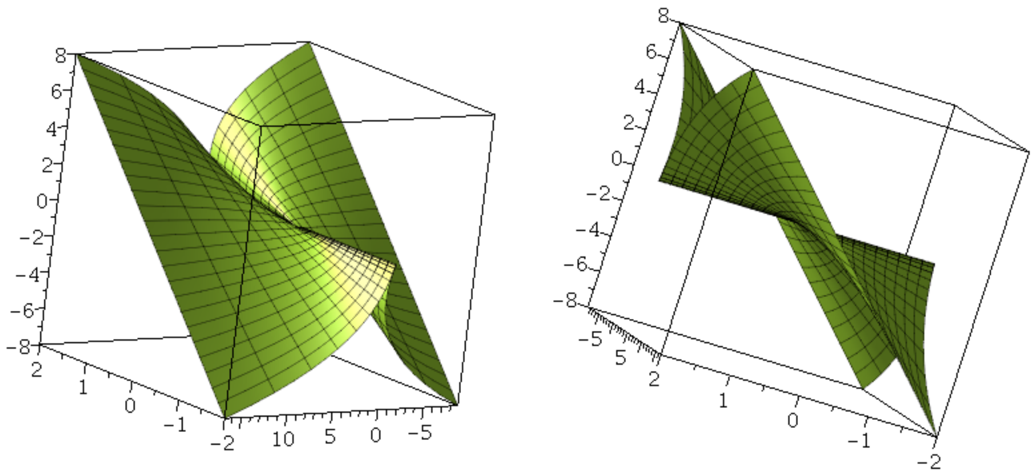

Figure 8 – Frontal for which the unit normal vector has the same singular set.

CHAPTER

7

SINGULARITIES OF 2-PARAMETER PLANE CONGRUENCES IN $\mathbb{R}^4$

In this chapter we briefly discuss parametric families of planes and their generic singularities. First, we start with a more general case, taking into account r-surfaces in $\mathbb{R}^n$ (in the sense of (LIMA, 2004), chapter 7) and families of $n-r$-planes, where $n-1>r>1$. The results in section 7.1 can be seen as generalizations of the results from section 4.2, when we assume codimension greater than or equal to 2 and parametrized families of planes. In section 7.2 we discuss 2-parameter family of planes and classify their generic singularities. In 7.3 we discuss some of the problems related to plane congruences we want to explore in the future.

7.1 A more general case

Let $\mathbf{x}: U \to \mathbb{R}^n$ and $\boldsymbol{\xi}^i : U \to \mathbb{R}^n \setminus \{\mathbf{0}\}$, where $U \subset \mathbb{R}^r$ is open and $i = 1, 2, \cdots, n-r$. Then, we associate to $(\mathbf{x}, \boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \cdots, \boldsymbol{\xi}^{n-r}) \in C^\infty(U, \mathbb{R}^n \times \underbrace{\mathbb{R}^n \setminus \{\mathbf{0}\} \times \cdots \times \mathbb{R}^n \setminus \{\mathbf{0}\}}_{n-r-\text{times}}) \equiv C^\infty(U, \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})^{n-r})$ the congruence

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi})} : U \times I_1 \times \cdots \times I_{n-r} &\to \mathbb{R}^n \\ (u, t^1, \cdots, t^{n-r}) &\mapsto \mathbf{x}(u) + t^1 \boldsymbol{\xi}^1 + \cdots t^{n-r} \boldsymbol{\xi}^{n-r}, \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{n-r})$, I_j is an open interval for all $j \in \{1, \cdots, n-r\}$ and the vector fields $\boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{n-r}$ are linearly independent. When there is no risk of confusion, we drop the subscript in $F_{(\mathbf{x},\boldsymbol{\xi})}$. Write

$$\mathscr{W}_{n-r} = \{\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{n-r}) \in C^\infty(U, (\mathbb{R}^n \setminus \{\mathbf{0}\})^{n-r}) : \boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{n-r} \text{ are linenarly independent}\}. \tag{7.1}$$

We can think of $\boldsymbol{\xi} \in \mathscr{W}_{n-r}$ as a parametrized family $n \times (n-r)$ of matrices, thus given $u \in U$ and $\boldsymbol{\xi} \in \mathscr{W}_{n-r}$ at least one of the $(n-r) \times (n-r)$ minors is not zero at u, for all $u \in U$. Let us suppose, without loss of generality, that the minor given by the first $n-r$ rows is not zero at u_0. Then, there is an open subset $U_0 \subset U$ such that this minor is not zero for all $u \in U_0$. Taking this into account there exists a germ of diffeomorphism $h : (\mathbb{R}^n, u_0, \mathbf{0}) \to (\mathbb{R}^n, u_0, t_0)$, where $t_0 = (t_0^1, t_0^2, \cdots, t_0^{n-r})$, such that the first $n-r$ coordinates of $\widetilde{F} = F \circ h$ are given by

$$\widetilde{F}_j = a_j + \sum_{i=1}^{n-r} t^1 \boldsymbol{\xi}_j^i,\ j = 1, \cdots, n-r, \tag{7.2}$$

where $\boldsymbol{\xi}_j^i$ indicates the j-th component of $\boldsymbol{\xi}^i$ and $a_k = \mathbf{x}_k(u_0) + \sum_{j=1}^{n-r} t_0^j \boldsymbol{\xi}_k^j(u_0)$, $k = 1, 2, \cdots, n-r$. Thus, if we write $a_0 = (a_1, \cdots, a_{n-r})$ and $G(u,t) = (\widetilde{F}_1, \cdots, \widetilde{F}_{n-r})$, we get $G^{-1}(a_0) = \{(u, \mathbf{0}) : u \in (U_0, u_0)\}$. Using this system of coordinates, we seek to prove that $\widetilde{F}$ is an $(n-r)$-dimensional unfolding of a germ from $\mathbb{R}^r$ to $\mathbb{R}^r$.

Let $\pi_{n-r} : \mathbb{R}^n \to \mathbb{R}^r$ be the projection in $\mathbb{R}^r$ given by $\pi(x_1, \cdots, x_{n-r}, x_{n-r+1}, \cdots, x_n) = (x_{n-r+1}, \cdots, x_n)$ and $\widetilde{f}(u) = \pi_{n-r} \circ \widetilde{F}(u, \mathbf{0})$.

Proposition 7.1.1. With notation as above, the map germ $\widetilde{F} : (\mathbb{R}^n, u_0, \mathbf{0}) \to \mathbb{R}^n$ is an $(n-r)$-dimensional unfolding of $\widetilde{f}(u) = \pi_{n-r} \circ \widetilde{F}(u, \mathbf{0})$.

Proof. Take

$$\begin{aligned} i : (\mathbb{R}^r, u_0) &\to \mathbb{R}^n \\ u &\mapsto (u, \mathbf{0}) \\ j : \mathbb{R}^r &\to \mathbb{R}^n \\ y &\mapsto (a_0, y) \end{aligned}$$

then, $\widetilde{F} \circ i = j \circ \widetilde{f}$. Note that $\widetilde{F}$ is transverse to j, since we are considering $\widetilde{F}_j$ written as in (7.2). Furthermore,

$$\{(u,t,y) : \widetilde{F}(u,t) = j(y)\} = \{(u, \mathbf{0}, \widetilde{f}(u)) : u \in (\mathbb{R}^r, u_0)\}.$$

Notice that $(i, f) : (\mathbb{R}^r, u_0) \to \{(u,t,y) : \widetilde{F}(u,t) = j(y)\}$ is given by $(i, \widetilde{f})(u) = (u, \mathbf{0}, \widetilde{f}(u))$, thus is a diffeomorphism. Therefore, $(\widetilde{F}, i, j)$ is a two dimensional unfolding of $\widetilde{f}$, from lemma 2.3.1. □

Lemma 7.1.1. Let $W \subset J^k(r,r)$ be a submanifold. For any fixed $\boldsymbol{\xi} \in \mathscr{W}_{n-r}$ and any fixed point $q_0 = (u_0, t_0) \in U \times I_1 \times \cdots \times I_{n-r}$, such that the minor given by the first $n-r$ rows of $\boldsymbol{\xi}$ is not zero at u_0, the set

$$T_{W,q_0}^{\boldsymbol{\xi}} = \left\{ \mathbf{x} \in C^\infty(U, \mathbb{R}^n) : j_1^k \left(\pi_{n-r} \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})} \right) \pitchfork W \text{ at } (u_0, t_0) \right\}$$

is a residual subset of $C^\infty(U, \mathbb{R}^n)$.

Proof. In what follows we identify $C^\infty(U,\mathbb{R}^n)\times\underbrace{C^\infty(U,\mathbb{R}^n\setminus\{\mathbf{0}\})\times\cdots\times C^\infty(U,\mathbb{R}^n\setminus\{\mathbf{0}\})}_{n-r-\text{times}}$ with $C^\infty(U,\mathbb{R}^n\times(\mathbb{R}^n\setminus\{\mathbf{0}\})^{n-r})$ and we take the C^∞-Whitney topology induced on $C^\infty(U,\mathbb{R}^n)\times\{\boldsymbol{\xi}\}$.

Let us take $\{C_j\}_{j=1}^\infty$ a countable open cover for W, such that $\overline{C_j}$ is compact for all $j\in\mathbb{N}$. Define

$$T^{\boldsymbol{\xi}}_{W,q_0,C_j}=\{\mathbf{x}\in C^\infty(U,\mathbb{R}^n): j_1^k\left(\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)\pitchfork W \text{ with } j_1^k\left(\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)(q_0)\in\overline{C_j}\}. \quad (7.3)$$

We claim that 7.3 is open. In fact, taking into account that the map $\hat{j}_1^k: C^\infty(U_0,\mathbb{R}^n)\to C^\infty(U_0\times I_1\times\cdots\times I_{n-r},J^k(r,r))$, given by $\hat{j}_1^k(\mathbf{x})=j_1^k\left(\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)$ is continuous, where $U_0\subset U$ is an open subset such that the minor given by the first $n-r$ rows of $\boldsymbol{\xi}$ is not zero for all $u\in U_0$, define

$$O_{W,C_j}=\{g\in C^\infty(U_0\times I_1\times\cdots\times I_{n-r},J^k(r,r)): g\pitchfork W \text{ at } q_0, g(q_0)\in\overline{C_j}\},$$

which is open. Thus, as the restriction map $res_{|U_0}: C^\infty(U,\mathbb{R}^n)\to C^\infty(U_0,\mathbb{R}^n)$ is also continuous, it follows that

$$T^{\boldsymbol{\xi}}_{W,q_0,C_j}=\left(res_{|U_0}\right)^{-1}\circ\left(\hat{j}^k\right)^{-1}(O_{W,C_j}) \text{ is open.}$$

If we show that $T^{\boldsymbol{\xi}}_{W,q_0,C_j}$ is dense, then $T^{\boldsymbol{\xi}}_{W,q_0}=\bigcap\limits_{j\in\mathbb{N}}T^{\boldsymbol{\xi}}_{W,q,C_j}$ is residual. Since the restriction map is surjective it is enough to show that

$$T_{W,q_0,C_j,U_0}=\{\mathbf{x}\in C^\infty(U_0,\mathbb{R}^n): j_1^k\left(\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)\pitchfork W \text{ with } j_1^k\left(\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}\right)(q_0)\in\overline{C_j}\}$$

is dense. Write $P(r,r,k)=\{(P_1,\cdots,P_r): P_i$ is a polynomial with $P_i(0)=0$ and $\deg(P_i)\le k=0, i=1,\cdots,r\}$. Given $\mathbf{x}\in C^\infty(U_0,\mathbb{R}^n)$ and $P=(P_1,\cdots,P_r)\in P(r,r,k)$, define $f_{(\mathbf{x},P)}: U_0\times I_1\cdots\times I_{n-r}\to\mathbb{R}^n$ by

$$f_{(\mathbf{x},P)}(u,t)=\pi_{n-r}\circ\widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})}(u,t)+P(u).$$

Define also

$$\begin{aligned}\Phi: U_0\times I_1\times\cdots\times I_{n-r}\times P(r,r,k)&\to J^k(r,r)\\ (u,t,P)&\mapsto j_1^k f_{\mathbf{x},P}(u,t)\end{aligned}$$

which is a submersion, thus $\Phi\pitchfork W$. Then, via lemma 2.3.3

$$\{P\in P(r,r,k):\Phi_P\pitchfork W \text{ at } q_0, \text{ such that } \Phi_P(q_0)\in\overline{C_j}\}$$

is dense in $P(r,r,k)$. Hence, there is $\{P_n\}$ a sequence in $P(r,r,k)$ such that $P_n\to 0$, with $\Phi_{P_n}\pitchfork W$, for all $n\in\mathbb{N}$. Note that $\mathbf{x}_n=\mathbf{x}+P_n\in T_{W,q_0,C_j,U_0}$ for all $n\in\mathbb{N}$ and $\mathbf{x}_n\to\mathbf{x}$, therefore, T_{W,q_0,C_j,U_0} is dense. $\square$

Remark 7.1.1. Let

$$\mathscr{O}_1 = \{\boldsymbol{\delta} \in \mathscr{W}_{n-r} : \begin{pmatrix} \boldsymbol{\delta} & \boldsymbol{\delta}_{u_1} & \cdots & \boldsymbol{\delta}_{u_r} \end{pmatrix} \text{ has rank} \geq 2\}.$$

Then, it follows analogously to remark 4.2.1 that $\mathscr{O}_1$ is a residual subset of $\mathscr{W}_{n-r}$.

Thus, it follows from lemma 7.1.1 that the set

$$T_{W,q_0} = \left\{ (\mathbf{x},\boldsymbol{\xi}) \in C^\infty(U, \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})^{n-r}) : j_1^k \left(\pi_{n-r} \circ \widetilde{F}_{(\mathbf{x},\boldsymbol{\xi})} \right) \pitchfork W \text{ at } q_0 \text{ and } \boldsymbol{\xi} \in \underbrace{\mathscr{O}_1 \times \cdots \times \mathscr{O}_1}_{n-r \text{ times}} \right\}$$

is residual. We proceed analogously if we start considering that any other minor of $\boldsymbol{\xi}$ is not zero in order to obtain a residual set as the above one. Then, we have the following theorem.

Theorem 7.1.1. There is an open dense set $\mathscr{O} \subset C^\infty(U, \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})^{n-r})$, such that for all $(\mathbf{x},\boldsymbol{\xi}) \in \mathscr{O}$ the germ of the line congruence $F_{(\mathbf{x},\boldsymbol{\xi})}$ at any point $q_0 \in U \times I_1 \times I_{n-r}$ is stable.

Proof. The proof follows the same steps of that of 4.2.1. □

7.2 2-parameter plane congruences in $\mathbb{R}^4$

Here, we deal with two parameter family of planes in $\mathbb{R}^4$, i.e., 2-parameter plane congruences in $\mathbb{R}^4$. Our approach for this case is motivated by section 3.2. Along this section $\widetilde{U}$ denotes an open subset of $\mathbb{R}^2$.

Definition 7.2.1. A 2-parameter plane congruence in $\mathbb{R}^4$ is a 2-parameter family of planes in $\mathbb{R}^4$. Locally, we write

$$\begin{aligned} F_{(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta})} : \widetilde{U} \times I \times J &\to \mathbb{R}^4 \\ (u,t,l) &\mapsto \mathbf{x}(u) + t\boldsymbol{\xi}(u) + l\boldsymbol{\delta}(u), \end{aligned}$$

where

- $\mathbf{x} : \widetilde{U} \to \mathbb{R}^4$ is smooth and it is called a *reference surface*.
- $\boldsymbol{\xi}, \boldsymbol{\delta} : \widetilde{U} \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ are smooth and linearly independent. We call $\boldsymbol{\xi}$ and $\boldsymbol{\delta}$ *director surfaces* of the congruence.

Let us write

$$\mathscr{W}_2 = \{(\boldsymbol{\xi},\boldsymbol{\delta}) \in C^\infty(\widetilde{U}, \mathbb{R}^4 \setminus \{\mathbf{0}\} \times \mathbb{R}^4 \setminus \{\mathbf{0}\}) : \boldsymbol{\xi} \text{ and } \boldsymbol{\delta} \text{ are linearly independent}\} \tag{7.4}$$

Example 7.2.1. A classical example of 2-parameter family of planes arises when we take a regular surface $\mathbf{x} : \widetilde{U} \to \mathbb{R}^4$, $\mathbf{x}(\widetilde{U}) = M$. Then, at each $q \in M$, $q = \mathbf{x}(u)$, there is a well defined normal plane N_qM. Analogously to example 2.4.4 it is possible to show that the family of distance squared functions on M, $D : \widetilde{U} \times \mathbb{R}^4 \to \mathbb{R}$, defined by

$$D(u,p) = \langle \mathbf{x}(u) - p, \mathbf{x}(u) - p\rangle$$

is a Morse family of functions and that the Lagrangian map associated is given by $F_{\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta}}(u,t,l) = \mathbf{x}(u) + t\boldsymbol{\xi}(u) + l\boldsymbol{\delta}(u)$, where $\{\boldsymbol{\xi}(u), \boldsymbol{\delta}(u)\}$ is a basis for the normal plane N_qM at $q = \mathbf{x}(u)$ (see section 7.9 in (IZUMIYA *et al.*, 2016)). Since the germ of family D is locally $\mathcal{P}$-$\mathcal{R}^+$-versal for an open and dense set of embeddings $\mathbf{x} \in C^\infty(\widetilde{U}, \mathbb{R}^4)$ (see theorem 4.8 in (IZUMIYA *et al.*, 2016)), it follows similarly to theorem 4.3.1 that the generic singularities of exact normal plane congruences are the Lagrangian stable (see table 2).

Lemma 7.2.1. The singular points of a line congruence $F_{(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta})}$ are the points (u,t,l) such that

$$\begin{aligned}
&t^2\langle \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle + l^2\langle \boldsymbol{\delta}_{u_1} \wedge \boldsymbol{\delta}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle + tl\left(\langle \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\delta}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle + \langle \boldsymbol{\delta}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle\right)\\
&+t\left(\langle \boldsymbol{\xi}_{u_1} \wedge \mathbf{x}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle + \langle \mathbf{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle\right) + l\left(\langle \boldsymbol{\delta}_{u_1} \wedge \mathbf{x}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle + \langle \mathbf{x}_{u_1} \wedge \boldsymbol{\delta}_{u_2} \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle\right)\\
&+\langle \mathbf{x}_{u_1} \wedge \mathbf{x}_{u_2} \boldsymbol{\xi}, \boldsymbol{\delta}\rangle = 0.
\end{aligned} \tag{7.5}$$

Proof. We know that $(u,t,l) \in \widetilde{U} \times I \times J$ is a singular point of $F_{(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta})}$ if and only if $\det JF(u,t,l) = 0$, where

$$JF(u,t,l) = \begin{bmatrix} \mathbf{x}_{u_1} + t\boldsymbol{\xi}_{u_1} + l\boldsymbol{\delta}_{u_1} & \mathbf{x}_{u_2} + t\boldsymbol{\xi}_{u_2} + l\boldsymbol{\delta}_{u_2} & \boldsymbol{\xi} & \boldsymbol{\delta} \end{bmatrix}_{4\times 4}$$

is the jacobian matrix of F at (u,t,l). From

$$\det JF(u,t,l) = \langle (\mathbf{x}_{u_1} + t\boldsymbol{\xi}_{u_1} + l\boldsymbol{\delta}_{u_1}) \wedge (\mathbf{x}_{u_2} + t\boldsymbol{\xi}_{u_2} + l\boldsymbol{\delta}_{u_2}) \wedge \boldsymbol{\xi}, \boldsymbol{\delta}\rangle = 0$$

we obtain (7.5). □

Theorem 7.2.1. There is an open dense set $\mathcal{O} \subset C^\infty(\widetilde{U}, \mathbb{R}^4) \times \mathscr{W}_2$, such that:

a) For all $(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\delta}) \in \mathcal{O}$ the germ of the plane congruence $F_{(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta})}$ at any point $(u_0, t_0, l_0) \in \widetilde{U} \times I \times J$ is stable;

b) For all $(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\delta}) \in \mathcal{O}$ the germ of the plane congruence $F_{(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\delta})}$ at any point $(u_0, t_0, l_0) \in \widetilde{U} \times I \times J$ is $\mathcal{A}$-equivalent to one of the normal forms below

- $(x,y,z,w) \mapsto (x,y,w,z^2)$ (Fold).
- $(x,y,z,w) \mapsto (x,y,w,z^3 + xz)$ (Cusp).
- $(x,y,z,w) \mapsto (x,y,w,z^3 \pm x^2z + yz)$ (Lips/Beaks).

- $(x,y,z,w) \mapsto (x,y,w,z^3+x^3z+yz+zxw)$ (Goose).
- $(x,y,z,w) \mapsto (x,y,w,z^4+xz+yz^2)$ (Swallowtail).
- $(x,y,z,w) \mapsto (x,y,w,xz^2+z^4+z^5+yz+wz^3)$ (Gulls).
- $(x,y,z,w) \mapsto (x,y,w,xz+z^5 \pm z^7+yz^2+wz^3)$ (Butterfly).
- $(x,y,z,w) \mapsto (z,w,x^2+y^3+zy,y^2+x^3+wx)$ (Hyperbolic Umbilic or Sharksfin).
- $(x,y,z,w) \mapsto (z,w,x^2-y^2+x^3+zx,xy+wx)$ (Elliptic Umbilic or Deltoid).

Proof. Item $a)$ follows directly from theorem 7.1.1, taking $n=4$ and $r=2$. To prove $b)$ we proceed as in theorem 4.2.1 item $b)$. The good set in item $a)$ contains the $\mathcal{K}$-orbits in $\{f \in J^k(2,2) : cod_e(\mathcal{K},f) \leq 4\}$, that is,

- $\mathcal{K}$-orbits of A_r type, for $1 \leq r \leq 4$;
- (x^2,y^2);
- (x^2-y^2,xy).

Then, we refine the $\mathcal{K}$-orbits on the above stratification, by taking the $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension ≤ 2 inside these $\mathcal{K}$-orbits. Then, the relevant strata in this stratification are the $\mathcal{A}$-orbits of stable singularities A_r, $r=1,2$ and the $\mathcal{A}$-orbits of singularities of $\mathcal{A}_e$-codimension 1 or 2 of type A_2, A_3, A_4 and D_4. The complement of their union is a semialgebraic set of codimension greater than or equal to 5. The normal forms and the versal unfoldings below were taken from (GIBSON; HAWES; HOBBS, 1994).

1. $\mathcal{K}$-orbit of A_1 type

   $f(x,y)=(x,y^2)$ which is stable, hence, we have just this $\mathcal{A}$-orbit. Its suspension in $\mathbb{R}^4$ is the stable germ that we are looking for.

2. $\mathcal{K}$-orbits of A_2 type

   The possible normal forms are

$$f_1(x,y)=(x,y^3+xy)\ (\text{Cusp}) - \mathrm{cod_e}\,(\mathcal{A},f_1)=0,$$
$$f_2(x,y)=(x,y^3 \pm x^2y)\ (\text{Lips(+)/Beaks(-)}) - \mathrm{cod_e}\,(\mathcal{A},f_1)=1,$$
$$f_3(x,y)=(x,y^3+x^3y)\ (\text{Goose} - \mathrm{cod_e}\,(\mathcal{A},f_1)=2.$$

   The versal unfoldings are given by (taking the suspension when it is necessary)

$$F_1(x,y,z,w)=(x,z,w,y^3+xy),$$
$$F_2(x,y,z,w)=(x,y,w,z^3 \pm x^2z+yz),$$
$$F_3(x,y,z,w)=(x,y,w,z^3+x^3z+yz+zxw),$$

respectively.

3. $\mathcal{K}$-orbits of A_3 type

   We get the following normal forms

$$f_1(x,y) = (x, y^4 + xy) \text{ (Swallowtail)} - \mathrm{cod_e}\,(\mathcal{A}, f_1) = 1,$$
$$f_2(x,y) = (x, y^4 + xy^2 + y^5) \text{ (Gulls)} - \mathrm{cod_e}\,(\mathcal{A}, f_1) = 2$$

   and the versal unfoldings are given by

$$F_1(x,y,z,w) = (x, y, w, xz + z^4 + yz^2),$$
$$F_2(x,y,z,w) = (x, y, w, xz^2 + z^4 + z^5 + yz + wz^3),$$

   respectively.

4. $\mathcal{K}$-orbits of A_4 type

   The only normal form is

$$f(x,y) = (x, y^5 + y^7 + xy) \text{ (Butterfly)} - \mathrm{cod_e}\,(\mathcal{A}, f_1) = 2,$$

   whose versal unfolding is

$$F(x,y,z,w) = (x, y, w, xz + z^5 \pm z^7 + yz^2 + wz^3).$$

5. $\mathcal{K}$-orbit (x^2, y^2)

   In (RIEGER; RUAS, 1991) it is shown that any germ contained in this $\mathcal{K}$ orbit is $\mathcal{A}$-equivalent to some member of the series of germs $I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1})$, $l \geq m \geq 1$, where the $\mathrm{cod_e}\left(\mathcal{A}, I_{2,2}^{l,m}\right) = l + m$. Hence, the only $\mathcal{A}$-orbit to be considered is $f_1(x,y) = (x^2 + y^3, y^2 + x^3)$, which has versal unfolding given by

$$F_1(x,y,z,w) = (z, w, x^2 + y^3 + zy, y^2 + x^3 + wx)$$

6. $\mathcal{K}$-orbit $(x^2 - y^2, xy)$

   Analogously to the last case, in (RIEGER; RUAS, 1991) it is shown that any germ contained in this $\mathcal{K}$ orbit is $\mathcal{A}$-equivalent to some member of the series $II_{2,2}^{l} = (x^2 - y^2 + x^{2l+1}, xy)$, $l \geq 1$, which has $\mathrm{cod_e}\left(\mathcal{A}, II_{2,2}^{l}\right) = 2l$. Thus, we only take into account the $\mathcal{A}$-orbit $f_2(x,y) = (x^2 - y^2 + x^3, xy)$ with versal unfolding given by

$$F_2(x,y,z,w) = (z, w, x^2 - y^2 + x^3 + zx, xy + wx)$$

□

7.3 Future research

In this thesis, when working with plane congruences, we only discuss the classification of their generic singularities, however, there are other problems associated to these congruences we expect to investigate in future research. For instance, taking into account an affine normal plane to a surface in $\mathbb{R}^4$, as those defined in (NOMIZU; VRANCKEN, 1993) and (NUNO-BALLESTEROS; SAIA; SANCHEZ, 2017) and the associated affine normal plane congruences, the classification of their generic singularities is still an open problem. Taking into account that line congruences are strongly related to ruled surfaces, another natural question is how plane congruences are related to non-degenerate 2-ruled hypersurfaces in 4-space (see (SAJI, 2002)). Furthermore, in (GUTIERREZ; RUAS, 2003) the authors conjectured that any locally strictly convex surface homeomorphic to the sphere has at least two inflections. Thus, as this conjecture is related to affine invariants and surfaces in $\mathbb{R}^4$ it seems natural to face this problem when studying affine plane congruences.

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