UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Topological complexity and the Lusternik-Schnirelmann category

Matias de Jong van Lier

Dissertação de Mestrado do Programa de Pós-Graduação em Matemática (PPG-Mat)

ICMC USP
SÃO CARLOS

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: ______________________

Matias de Jong van Lier

Topological complexity and the Lusternik-Schnirelmann category

Dissertation submitted to the Instituto de Ciências Matemáticas e de Computação – ICMC-USP – in accordance with the requirements of the Mathematics Graduate Program, for the degree of Master in Science. *EXAMINATION BOARD PRESENTATION COPY*

Concentration Area: Mathematics

Advisor: Profa. Dra. Denise de Mattos

USP – São Carlos
May 2021

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

d716t de Jong van Lier, Matias
Topological Complexity and the Lusternik-Schnirelmann Category / Matias de Jong van Lier; orientadora Denise de Mattos. -- São Carlos, 2021.
163 p.

Dissertação (Mestrado - Programa de Pós-Graduação em Matemática) -- Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2021.

1. Topologia Algébrica. 2. Complexidade Topológica. 3. Categoria de Lusternik-Schnirelmann. 4. Planejamento de movimento. 5. Topologia Fibrewise. I. de Mattos, Denise, orient. II. Título.

Bibliotecários responsáveis pela estrutura de catalogação da publicação de acordo com a AACR2:
Gláucia Maria Saia Cristianini - CRB - 8/4938
Juliana de Souza Moraes - CRB - 8/6176

Matias de Jong van Lier

Complexidade topológica e categoria de Lusternik-Schnirelmann

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

Área de Concentração: Matemática

Orientadora: Profa. Dra. Denise de Mattos

USP – São Carlos
Maio de 2021

ACKNOWLEDGEMENTS

I would first like to express my profound gratitude to my academic advisor, Professor Denise de Mattos, whose expertise and experience helped me immensely. I am thankful for all the insightful discussions we had and all the valuable suggestions she gave me. I am also thankful to Professor Edivaldo Lopes dos Santos and Cesar Augusto Ipanaque Zapata for participating in many of the seminars I presented and providing me with valuable advice.

I am extremely grateful to my internship supervisor, Professor Norio Iwase, for his guidance during my studies at Kyushu University, and all his help in my adaptation in Japan. I am also thankful for all the friends I made during my time in Japan and all the help I received from Kyushu University's staff members.

I am deeply grateful to my girlfriend, Carolina, for always staying by my side and supporting me in the best way possible, to my father, mother, and sister, for supporting all my decisions and always being there for me, and to all my friends, for all the good experiences we had.

Special thanks to Instituto de Ciências Matemáticas e de Computação – Universidade de São Paulo[1] and to Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP)[2] for funding my master's project[3] and my research internship in Japan[4].

[1] https://www.icmc.usp.br/
[2] https://fapesp.br/
[3] project number: 2018/23410-3
[4] project number: 2019/20061-0

ABSTRACT

LIER, M. J. **Topological complexity and the Lusternik-Schnirelmann category**. 2021. 163 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

In recent years, a new field integrating robotics and topology was born, referred to by many as Topological Robotics, in which the main strategy is to use algebraic topological tools to get some insight into robotics problems. One of those problems is called the robot motion planning problem and is the main motivation for this work.

We present an in-depth study of Topological Complexity, discussing how it relates to the Motion Planning Problem, and the main methods for computing it for CW complexes and Smooth Manifolds, spaces of great interest in robotics. The concept of Lusternik-Schnirelmann category is introduced due to its connection with Topological complexity, both being particular cases of the more general concept of the Schwarz Genus of a fibration.

Keywords: Topological Complexity, Lusternik-Schnirelmann category, Schwarz Genus, Algebraic Topology, Motion Planning Problem, Fibrewise Topology.

RESUMO

LIER, M. J. **Complexidade topológica e categoria de Lusternik-Schnirelmann**. 2021. 163 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Nos últimos anos, nasceu um novo campo de pesquisa integrando robótica e topologia, referido por muitos como Robótica Topológica, no qual a principal estratégia é utilizar ferramentas da topologia algébrica para obter uma maior intuição sobre certos problemas da robótica. Um desses problemas chama-se problema do planejamento do movimento de um robô e é a principal motivação para este trabalho.

Apresentamos um estudo aprofundado da Complexidade Topológica, discutindo como ela se relaciona com o Problema do Planejamento de Movimento, e os principais métodos para a sua computação para complexos CW e variedades suaves, espaços de grande interesse da robótica. O conceito da categoria de Lusternik-Schnirelmann é introduzido devido à sua ligação com a complexidade topológica, sendo ambos casos particulares do conceito mais geral do Gênero de Schwarz de uma fibração.

Palavras-chave: Complexidade Topológica, categoria de Lusternik-Schnirelmann, Schwarz Genus, Topologia Algébrica, Problema do Planejamento de Movimento, Topologia Fibrewise.

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

CHP	Covering homotopy property
ENR	Euclidean neighborhood retract
HLP	Homotopy lifting property
LS category	Lusternik-Schnirelmann category
PID	Principal Ideal Domain
TC	Topological Complexity

LIST OF SYMBOLS

X — Usually denotes a topological space

$\mathrm{Supp}(f)$ — Support of a real valued function $f : X \to \mathbb{R}$

$\mathbb{R}$ — The real numbers with the usual topology

M^n — M is an n-manifold

$\overline{D}^n$ — The real closed n-disk $\overline{D}^n = \{x \in \mathbb{R}^n \ : \ \|x\| \leq 1\}$

Δ^n — The standard n-simplex

∂ — The boundary homomorphism

$H_n(X;R)$ — The nth singular homology R-module of X

$H^n(X;M)$ — The nth singular cohomology R-module of X with coefficients in M

$\tilde{H}_n(X;R)$ — Reduced nth singular homology R-module

$\tilde{H}^n(X;M)$ — Reduced nth singular cohomology R-module with coefficients in M

$H^n(X,A;M)$ — nth singular cohomology R-module of X relative to A with coefficients in M

$\smile$ — Cup product

$\times$ — Cross product

X^I — Path space of X

M_f — Mapping cylinder of a map $f : X \to Y$

SX — Suspension of X

ΣX — Reduced suspension of X

$\mathbb{RP}^n$ — Real projective space

$\mathbb{CP}^n$ — Complex projective space

$\mathbb{HP}^n$ — Quaternionic projective space

$\mathrm{cat}(X)$ — Lusternik-Schnirelmann Category of X

$\mathrm{cup}_R(X)$ — R-cuplength of X

$T^k(X)$ — Fat wedge of X of order k

$\tilde{p}_n : \tilde{G}_n(X) \to X$ — nth Ganea fibration

$p_n : G_n(X) \to X$ — Ganea fibration (fibre-cofibre construction)

$\pi : X^I \to X \times X$ — Path fibration

$TC(X)$ — Topological complexity of X

$g(\mathfrak{B})$ — Schwarz genus of a fibre space $\mathfrak{B}$

$g(p)$ — Schwarz genus of a fibration $p : E \to B$

$l_R(\mathfrak{B})$ — Length of the fibration $\mathfrak{B}$ with coefficients in R

$l_R(p)$ — Length of the fibration $p : E \to B$ with coefficients in R

$\mathrm{zcl}_K(X)$ — Zero divisors cup length of $H^*(X; K)$

$SO(n)$ — Special orthogonal group in dimension n

$SE(n)$ — Special Euclidean group in dimension n

$\mathrm{OI}(X)$ — Order of instability of X

$\mathrm{TC}^{\mathrm{ENR}}(X)$ — Topological complexity of X considering coverings by ENR subsets

$\mathrm{TC}^M(X)$ — Monoidal topological complexity

$\mathrm{TC}^S(X)$ — Symmetric topological complexity

$\mathrm{cat}_B^B(X)$ — Fibrewise pointed LS category

$\mathrm{cat}_B^*(X)$ — Fibrewise unpointed LS category

$d(X)$ — Fibre space $d(X) = (X \times X, p_X, X, s_X)$

$F(S)$ — Free module generated by S

$M \otimes_R N$ — Tensor product of two R-modules M and N

$\mathsf{Tor}_n(M,N)$ — The nth Tor R-module from the pair of R-modules M and N

$\mathsf{Ext}_n(M,N)$ — The nth Ext R-module from the pair of R-modules M and N

CONTENTS

INTRODUCTION

Topology was born in Euler's paper on the Seven Bridges of Königsberg in 1736 (EULER, 1736), and its modern name was coined by the German mathematician Johann Benedict Listing, in 1848 (LISTING, 1848). It is best described as the study of sets in which one can establish a notion of "proximity" between points. These sets are called Topological Spaces, and their importance lies in the fact that we can define an idea of continuity in them.

In 1895, Henri Poincaré revolutionized mathematics by introducing the concepts of homotopy and homology groups, in a paper entitled "Analysis Situs" (POINCARÉ, 1895), thereby inaugurating a major field of study, nowadays called Algebraic Topology, which can be described as the use of algebraic tools in solving topological problems. In the last century, efforts from numerous renowned mathematicians have made Algebraic Topology what it is today, a solid strong theory with vast applications in many areas of mathematics and science (HATCHER, 2002; WHITEHEAD, 1978).

Parallel to this story, in the world of robotics, the interest in manufacturing autonomous robots, i.e., robots that can complete tasks without the need for human control, became greater than ever. To properly operate, an autonomous robot must have a sophisticated algorithm for deciding on how to proceed from one configuration to the next. This fundamental question of how to set up such an algorithm is currently known as the Robot Motion Planning Problem and is a major field of study in robot engineering, thoroughly discussed in (LATOMBE, 1991).

At the beginning of the 21st century, Michael Farber started applying topological tools to solve problems in robotics and engineering, creating the field of Topological Robotics (FARBER, 2008). In this process, Farber came across the famous Robot Motion Planning Problem. At first, this might seem to be a simple problem, and in fact, it is for spaces such as a point-like robot moving on a plane with no obstacles. Nevertheless, in real applications, one has spaces with many obstacles and humanoid robots with many joints and complex structures, with the ability to move and rotate in a three-dimensional setting. For such complex spaces, there is no simple answer to the motion planning problem.

Motivated by that, Farber introduced a numerical invariant called the Topological Complexity (FARBER, 2003), which gives a lower bound for how "complex" any motion planning algorithm will be for a given topological space. Usually, the spaces of interest are the ones that describe the possible configurations of a robot, but the theory can be developed for any topological space.

The difficulty in building motion planning algorithms comes mainly from the fact that we are always interested in the most "stable" algorithm, meaning that the motion planner functions should be continuous. In (FARBER, 2003), it was shown that the only spaces where one can produce a motion planning algorithm with a single continuous motion planning function for the entire space are the contractible spaces (TC=1). For any non-contractible topological space, depending on the value of TC, one needs an algorithm with at least that number of motion planning functions, which always results in an algorithm with some problematic points, where the motion planner may be discontinuous, meaning that small variations in initial and final configurations can produce completely unrelated paths. To sum it up, the topological complexity provides us with a lower bound for how "problematic" any motion planning algorithm will be in a given space.

Two geometrically different spaces can have the same topology, which is what we call homeomorphic spaces, for example, the letters M and N are homeomorphic, but they are not homeomorphic to the letter X. There is a less restrictive concept called homotopy type, which only depends on the topology of a space, but spaces with different topology may have the same homotopy type, in the previous example, M, N, and X have the same homotopy type (HATCHER, 2002).

We can simplify many problems in robotics by solving them up to homotopy type, meaning that we solve them for a homotopy equivalent space. A simple example would be a robot that can move on a plane with one obstacle in the middle of it, this space is homotopy equivalent to S^1, which is much easier to work with. One would also expect that the complexity of a motion planner only depends on the homotopy type of the space, and in fact, this is true (theorem 3.2.3).

When introducing topological complexity, in (FARBER, 2003), Farber showed that there are important upper bounds depending on the dimension and the Lusternik-Schnirelmann category of the space (theorem 3.2.6). Furthermore, by using results due to Albert Schwarz (SCHWARZ, 1966), Farber obtained a cohomological lower bound for TC. These upper and lower bounds prove to be useful in determining the complexity of n-spheres, projective spaces, robot arms, n-dimensional rigid bodies, and others (see chapter 3).

Chapter 1 of this dissertation is dedicated to the introduction of some crucial background one needs before properly understanding the theory of Topological Complexity. The most important part of this preliminary chapter is the discussion about (Co)Homology Theory, in which we introduce the cup product and the cohomology ring, which are essential later on when producing a lower bound for TC via cuplength (chapter 3). Due to the importance of this preliminary topic, we also present some auxiliary results on Homological Algebra, in appendix A.

In Chapter 2, we present the concept of Lusternik-Schnirelmann Category (or LS category). This theory has a strong relation with TC, due to the fact that both invariants are particular cases of the more general concept of Schwarz Genus (SCHWARZ, 1966) (chapter 3). Hence, by better understanding the LS category we can gain some insight into the topic of Topological Complexity. The most important result relating the two invariants is theorem 3.2.6, in which is also shown that there is a close relationship between the topological complexity of a space and its covering dimension.

The LS category was introduced by Lazar Lusternik and Lev Schnirelmann in (LUSTERNIK; SCHNIRELMANN, 1934), initially called simply "category", with its first definition contemplating only smooth manifolds. Its importance lays in the fact that the category of a manifold is an upper bound for the number of critical points of any smooth function on it. Later on, Ralph H. Fox gave a new formulation to what we now know as the Lusternik-Schnirelmann Category (FOX, 1941).

Chapter 3 is dedicated to the theory of Topological Complexity. We start by properly formulating the Motion Planning Problem mathematically, which motivates the definition of topological complexity. In section 3.2, we present some key results regarding TC, for example, the fact that it is a homotopy invariant (theorem 3.2.3), and its relation with LS category and covering dimension (theorem 3.2.6).

In section 3.3, we introduce what is called the Schwarz genus of a fibration, and show that both TC and LS category are particular cases of this broader concept. We give a detailed proof for a cohomological lower bound for the Schwarz genus of any fibration (theorem 3.3.7), and by applying this to the path fibration we show that if K is a field, then the topological complexity of any space is bounded from below by what we call the zero divisors cup length of $H^*(X;K)$ (theorem 3.3.11).

After obtaining this important lower bound for TC, we give two practical examples of computations, the first one being for the case of a robot arm with n joints, and the second one is the case of a rigid body moving freely, either in a 2D space or a 3D space.

We close chapter 3 by presenting the concept of "order of instability" (both locally and globally) of a motion planner, which measures how unstable a motion planning algorithm is at a given point (or as a whole, in the global case). In theorem 3.4.6, we prove that for nice enough spaces (connected C^∞-smooth manifolds) the minimum order of instability for a given space coincides with its topological complexity.

We finish by presenting the fibrewise method for topological complexity, in chapter 4. The idea, presented in (IWASE; SAKAI, 2010; IWASE; SAKAI, 2012), is that topological complexity is the same as a newly formulated LS category in the context of fibrewise topological spaces.

It is important to notice that the numerical invariant topological complexity has no relation to “optimal distance paths”, i.e., paths that minimize distance, it is simply a measure of how “problematic” any motion planning algorithm in a given space will be. Nevertheless, there are some variations in the definitions of topological complexity that incorporate some restrictions to how the paths must be. Two quite simple and important versions are the monoidal topological complexity and the symmetric topological complexity. Both these concepts are introduced in chapter 4, and we finish this last chapter by using the fibrewise category to obtain a relation between topological complexity and the monoidal version of topological complexity (IWASE; SAKAI, 2010).

CHAPTER

1

PRELIMINARIES

In this first chapter, we present many important concepts for the development of both LS category and topological complexity. We develop the theory of (Co)homology in more detail since it will be important when fabricating a lower bound for TC in chapter 3.

1.1 Paracompact and Normal Spaces

Definition 1.1.1 ((MUNKRES, 2000))**.** If X is a topological space, then a collection $\{A_\alpha\}_\alpha$ of subspaces of X is said to be **locally finite** if for every point $x \in X$ there exists an open neighborhood V_x of x, which intersects a finite number of elements from $\{A_\alpha\}_\alpha$.

Remember that if $\{U_\alpha\}$ is an open covering of X, then an **open refinement** of that covering is another open covering $\{V_\alpha\}$ such that for each α we have $V_\alpha \subset U_\alpha$.

Definition 1.1.2 ((MUNKRES, 2000))**.** A topological space X is **paracompact** if every open covering of X has a locally finite open refinement that covers X.

Definition 1.1.3 ((MUNKRES, 2000))**.** Let $\{U_\alpha\}_\alpha$ be an open covering of a topological space X. A **partition of unity subordinate to** $\{U_\alpha\}_\alpha$ is a collection of functions $\{\phi_\alpha\}_\alpha$, $\phi_\alpha : X \to [0,1]$, such that

(i) $\mathrm{Supp}(\phi_\alpha) \subset U_\alpha$, for all α. Remember that $\mathrm{Supp}(\phi_\alpha) = \overline{\{x \in X \mid \phi_\alpha(x) \neq 0\}}$

(ii) The family $\{\mathrm{Supp}(\phi_\alpha)\}_\alpha$ is locally finite.

(iii) $\sum_\alpha \phi_\alpha(x) = 1$, for all $x \in X$.

If X is a smooth manifold (see section 1.2), then the partition above will be called a **smooth partition of unity** if all ϕ_α are smooth maps.

Theorem 1.1.4 ((MUNKRES, 2000))**.** Consider X a paracompact Hausdorff space and $\{U_\alpha\}_\alpha$ an open covering of X. Then there exists a partition of unity subordinate to $\{U_\alpha\}_\alpha$.

Definition 1.1.5 ((DUGUNDJI, 1966))**.** A topological space X is said to be **normal** if it is a Hausdorff space in which for every pair of disjoint closed sets $E, F \subset X$, there are disjoint open sets U, V with $E \subset U$ and $F \subset V$.

Theorem 1.1.6 ((DUGUNDJI, 1966))**.** Let X be a Hausdorff space, then the following statements are equivalent:

1. X is normal
2. For any locally finite open covering $\{U_\alpha\}_\alpha$ of X there exists an open refinement $\{V_\alpha\}_\alpha$ that covers X, such that $\overline{V}_\alpha \subset U_\alpha$ for all α, and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$.

For the proof of this statement refer to Theorem 6.1 in (DUGUNDJI, 1966).

Definition 1.1.7 ((CORNEA *et al.*, 2003))**.** A topological space X is called **completely normal** whenever the following equivalent conditions are satisfied:

1. Any pair of subsets $A, B \subset X$, such that $\overline{A} \cap B = \emptyset$ and $B \cap \overline{A} = \emptyset$, can be separated by disjoint open sets.
2. Every subspace of X is normal.

1.2 Topological and Smooth Manifolds

Definition 1.2.1 ((TU, 2011))**.** A topological space M is said to be **locally Euclidean of dimension** n, if each point $x \in M$ has an open neighborhood U and a homeomorphism $\phi : U \to V$ onto an open subset $V \subset \mathbb{R}^n$. The pair $(U, \phi : U \to V)$ is called a **chart** of M, U is a **coordinate neighborhood** and ϕ a **coordinate map**.

Lemma 1.2.2 ((LEE, 2011))**.** For a topological space X the following statements are equivalent

1. X is locally Euclidean of dimension n.
2. Every point of X has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$.
3. Every point of X has a neighborhood homeomorphic to $\mathbb{R}^n$.

Definition 1.2.3 ((TU, 2011))**.** A **topological manifold of dimension** n, or a **topological n-manifold**, is a locally euclidean topological space of dimension n, which is also second countable and Hausdorff.

In many cases we shall denote a topological manifold by M^n, by which we mean that M is a n-manifold, not to be confused with the n-fold Cartesian product $M \times \cdots \times M$.

Recall that we define the closed n-disk in $\mathbb{R}^n$ as the subset $\overline{D}^n = \{x \in \mathbb{R}^n \,:\, \|x\| \leq 1\}$. Given a Topological n-manifold, to introduce the concept of a closed n-disk in M we proceed as follows.

Definition 1.2.4. A **closed n-disk** in a topological n-manifold M is a subset $D \subset M$ homeomorphic to $\overline{D}^n$ and such that there exists a coordinate neighborhood $U \subset M$, with $D \subset U$, with D being a deformation retract of U.

The last part in the previous definition is motivated by the fact that the closed n-disk in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n$, with the deformation retraction $r : \mathbb{R}^n \to \overline{D}^n$ defined by

$$r(x) = \begin{cases} x, & \text{if } x \in \overline{D}^n \\ \frac{x}{\|x\|}, & \text{if } x \in \mathbb{R}^n \setminus \overline{D}^n \end{cases} .$$

Definition 1.2.5 ((LEE, 2012))**.** If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets, then a function $F : U \to V$ is said to be **smooth** (or C^∞) if all its components have continuous partial derivatives of all orders. Furthermore, if F is bijective with smooth inverse, we call it a **diffeomorphism**.

Definition 1.2.6 ((LEE, 2012))**.** Let M be a topological n-manifold, and suppose (U, φ) and (V, ψ) are charts. Then we say that these two charts are **smoothly compatible** if either $U \cap V = \emptyset$ or $\psi\varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Definition 1.2.7 ((LEE, 2012))**.** An **atlas** for a topological manifold M is a collection of charts whose coordinate neighborhoods cover M. If all the charts are smoothly compatible, then we have a **smooth atlas**. Furthermore, we say that a smooth atlas is **maximal** if it is not properly contained in any larger smooth atlas.

Definition 1.2.8 ((LEE, 2012))**.** A **smooth manifold** is a topological manifold together with a maximal smooth atlas, sometimes called the **smooth structure** on the manifold.

Next, we present a sequence of well-known theorems regarding Topological and Smooth Manifolds, for the proof of these results refer to the literature mentioned in each theorem.

Theorem 1.2.9 ((LEE, 2011) Theorem 4.77)**.** If X is a second countable, locally compact Hausdorff space, then it is paracompact. In particular, if X is a topological manifold, it is paracompact.

Theorem 1.2.10 (**Smirnov metrization theorem** - (MUNKRES, 2000) Theorem 42.1)**.** Let X be a topological space. Then X is metrizable if and only if it is paracompact, Hausdorff, and locally metrizable. In particular, every topological manifold is metrizable.

Theorem 1.2.11 ((LEE, 2012) Examples 1.8 and 1.34)**.** If $M_1,\ldots,M_k$ are topological manifolds (respectively smooth manifolds) of dimension $n_1,\ldots,n_k$, then $M_1\times\cdots\times M_k$ is a topological manifold (respectively smooth manifold) of dimension $n_1+\cdots+n_k$.

Theorem 1.2.12 ((LEE, 2012) Theorem 6.15)**.** Any smooth n-manifold is homeomorphic to a subset of $\mathbb{R}^{2n+1}$.

Theorem 1.2.13 (**Sard's Theorem**. (LEE, 2012) Theorem 6.10)**.** If $f: M\to N$ is a smooth map between smooth manifolds, then the set of critical values of f has measure zero in N.

Theorem 1.2.14 ((LEE, 2012) Theorem 2.23)**.** If M is a smooth manifold with an open covering $\mathscr{U}=\{U_\alpha\}_\alpha$, then there is a smooth partition of unity subordinate to $\mathscr{U}$.

Theorem 1.2.15 ((LEE, 2012) Theorem 2.29)**.** For any closed subset F of a smooth manifold M there is a smooth non negative function $f: M\to\mathbb{R}$ such that $f^{-1}(0)=F$.

1.3 (Co)Homology Theory

This section is dedicated to the introduction of two central theories in Algebraic Topology. The first one being Singular Homology, and the second one its dual notion of Singular Cohomology. These two theories have a strong link which will be made evident by the Universal Coefficient Theorem. Many classical theorems will be presented without proof, for deeper understanding refer to (HATCHER, 2002).

We shall develop both theories using the concept of free R-modules, which reduces to the case of free abelian groups if we choose the ring R to be $\mathbb{Z}$. For basic definitions and results regarding R-modules refer to Appendix A.1.

Singular Homology and Cohomology

Definition 1.3.1 ((HATCHER, 2002))**.** The **standard n-simplex** is the topological space given by

$$\Delta^n \doteq \left\{(t_0,\ldots,t_n)\in\mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i\geq 0, \text{ for all } i=0,\ldots,n\right\}.$$

Definition 1.3.2 ((HATCHER, 2002))**.** A **singular n-simplex** in X is any continuous function $\sigma:\Delta^n\to X$, and we denote by $S_n(X)$ the set of all singular n-simplexes in X.

Define the continuous functions (embeddings) $\varepsilon_{i,n-1}:\Delta^{n-1}\to\Delta^n$ given by

$$\varepsilon_{i,n-1}(t_0,\ldots,t_{n-1}) = (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}).$$

We define the i^{th} **face operator** to be the function $\partial_n^i : S_n(X) \to S_{n-1}(X)$ given by $\partial_n^i(\sigma) = \sigma\varepsilon_{i,n-1}$ (sometimes denoted simply by ∂^i).

Let R be a commutative ring with unity, we define $C_n(X;R)$ to be the free R-module generated by $S_n(X)$ (see definition A.1.9).

Define for each $n = 0, 1, \ldots$ the function $\partial_n : S_n(X;R) \to C_{n-1}(X;R)$ given by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_n^i(\sigma),$$

which extends uniquely to an R-homomorphism $\partial_n : C_n(X;R) \to C_{n-1}(X;R)$ (see definition A.1.7) called the **boundary homomorphism**, additionally, it is easy to prove that $\partial_n \partial_{n+1} = 0$ for all n, hence we have a chain complex (see definition A.2.1)

$$\cdots \longrightarrow C_{n+1}(X;R) \xrightarrow{\partial_{n+1}} C_n(X;R) \xrightarrow{\partial_n} C_{n-1}(X;R) \longrightarrow \cdots .$$

In many occasions we may drop the index and write simply $\partial : C_n(X;R) \to C_{n-1}(X;R)$. We usually use the notation $C(X;R)$, without index, to refer to the whole chain complex above.

The homology R-modules (see definition A.2.2) of the chain complex above are called the **singular homology R-modules of X**, denoted by

$$H_n(X;R) = \frac{Z_n(X;R)}{B_n(X;R)},$$

in which $Z_n(X;R) = \ker(\partial)$ and $B_n(X;R) = \mathrm{Im}(\partial)$. Usually, elements of $Z_n(X;R)$ and $B_n(X;R)$ are called **n-cycles** and **n-boundaries** of X, respectively. In particular, $H_n(X;R)$ is called the **n^{th} singular homology R-module of X**.

Given M an R-module, one can produce a dual chain complex by applying the functor $\mathrm{Hom}_R(\cdot, M)$ (see definition A.2.13), by doing so, we obtain the chain complex (sometimes called a cochain complex)

$$\cdots \longrightarrow C^{n-1}(X;M) \xrightarrow{\delta_n} C^n(X;M) \xrightarrow{\delta_{n+1}} C^{n+1}(X;M) \longrightarrow \cdots ,$$

in which $C^n(X;M) \doteq (C_n(X;R))^* = \mathrm{Hom}_R(C_n(X;R);M)$ and $\delta_n(\varphi) \doteq (\partial_n)^*(\varphi) = \varphi\partial_n$. In many occasions we may drop the indexes and write simply δ.

We define the n-cocycles and n-coboundaries submodules of $C^n(X;M)$ to be $Z^n(X;M) \doteq \ker(\delta)$ and $B^n(X;M) \doteq \mathrm{Im}(\delta)$, respectively. The quotient R-module

$$H^n(X;M) \doteq \frac{Z^n(X;M)}{B^n(X;M)}$$

is called the **n^{th} singular cohomology R-module of X with coefficients in M**.

Proposition 1.3.3 ((HATCHER, 2002))**.** A continuous map $f : X \to Y$ between topological spaces X and Y, induces a chain map $f_\# : C_n(X) \to C_n(Y)$ given by $f_\#(\sigma) = f\sigma$, for σ a singular n-simplex. Hence it also induces a cochain map $f^\# : C_n^*(Y) \to C_n^*(X)$ given by $f^\#(\varphi) = \varphi f_\#$. This implies that f induces a map on singular homology and cohomology, namely $f_* : H_n(X;R) \to H_n(Y;R)$ and $f_*^* : H^n(Y;R) \to H^n(X;R)$ given by $f_*([z]) = [f_\#(z)]$ and $f_*^*([\varphi]) = [f^\#(\varphi)]$.

We may use the notation f^* instead of f_*^* for the map induced in cohomology.

Proposition 1.3.4. Let $f : X \to Y$ be a continuous map between topological spaces, $A \subset X$ any subset of X and $f_o = f|_A : A \to Y$. If $u \in \ker(f_*^*)$, then $u \in \ker(f_{o*}^*)$.

Proof. Notice that for any $\varphi \in C_n^*(Y;R)$ and any singular n-simplex $\sigma \in C_n(A;R) \subset C_n(X;R)$, we have $f_o^\#(\varphi)(\sigma) = \varphi f_{o\#}(\sigma) = \varphi f_o \sigma = \varphi f \sigma = \varphi f_\#(\sigma) = f^\#(\varphi)(\sigma)$. Clearly, this means that $\varphi f_{o\#} = (\varphi f_\#)|_{C_n(A;R)}$

If we had $f_*^*([\varphi]) = 0$, then $f^\#\varphi = \varphi f_\# = \delta\psi = \psi\partial$, for some $\psi \in C_{n-1}(X;R)$. Then, we have $\varphi f_{o\#} = (\varphi f_\#)|_{C_n(A;R)} = (\psi\partial)|_{C_n(A;R)} = \psi|_{C_{n-1}(A;R)}\partial|_{C_n(A;R)}$. So, if we define $\psi_o = \psi|_{C_{n-1}(A;R)}$, we have $f_o^\#(\varphi) = \delta\psi_o$, hence $f_{o*}^*([\varphi]) = [f_o^\#(\varphi)] = [\delta\psi_o] = 0$. □

Next, we will present the most important results regarding singular homology, many of which have a dual version for singular cohomology.

Homotopy invariance

Theorem 1.3.5 ((HATCHER, 2002))**.** If $f, g : X \to Y$ are homotopic maps, then the induced maps in singular homology and cohomology are the same, $f_* = g_*$ and $f^* = g^*$.

Proof. Notice that, in view of theorem A.2.22, to prove this theorem we only have to show that $f_\#, g_\# : C_n(X;R) \to C_n(Y;R)$ are chain homotopic.

Let $F : X \times I \to Y$ be a homotopy from f to g. Define the maps $\zeta_i^n : \Delta^n \to \Delta^{n-1} \times I$ given by $\zeta_i^n(t_0, \dots, t_n) = \big((t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n), t_{i+1} + \cdots + t_n\big)$, and define an R-homomorphism $P : C_{n-1}(X) \to C_n(Y)$ by

$$P(\sigma) = \sum_{i=0}^{n} F(\sigma \times id)\zeta_i^n,$$

for σ a singular $(n-1)$-simplex. One can check that $\partial P + P\partial = f_\# - g_\#$, which proves the theorem. □

Theorem 1.3.6 ((HATCHER, 2002))**.** If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X;R) \to H_n(Y;R)$ and $f_*^* : H^n(Y;M) \to H^n(X;M)$ are isomorphisms for any n, any commutative ring R and R-module M. In other words, the homology and cohomology of a space X only depends on its homotopy type.

Proof. The statement is a direct consequence of A.2.22. □

Reduced Homology and Cohomology

It is easy to explicitly compute the homology of a single point space, the result being $H_0(\{p\};R) = R$ and $H_n(\{p\};R) = 0$ for $n \geq 1$, so, by theorem 1.3.6, we know that this is the homology of any contractible space. In many situations, it is preferable to have a functor as the homology functor that takes a contractible space to the trivial module in every level. Singular homology fails in this aspect only for H_0, for this reason, we define another version of homology.

Definition 1.3.7 ((HATCHER, 2002))**.** For any topological space X let $\varepsilon : C_0(X;R) \to R$ be the homomorphism $\varepsilon\left(\sum_\sigma r_\sigma \sigma\right) = \sum_\sigma r_\sigma$, we clearly have $\varepsilon\partial = 0$, hence we have a chain complex

$$\cdots \longrightarrow C_{n+1}(X;R) \xrightarrow{\partial} C_n(X;R) \longrightarrow \cdots \longrightarrow C_1(X;R) \xrightarrow{\partial} C_0(X;R) \xrightarrow{\varepsilon} R \longrightarrow 0 \ ,$$

the homology modules of which are called the **reduced singular homology modules** of X with coefficients in R, denoted by $\tilde{H}_n(X;R)$.

It is easy to see, from the previous definition, that we basically have $\tilde{H}_n(X;R) = H_n(X;R)$ for $n \geq 1$, and $H_0(X;R) \approx \tilde{H}_0(X;R) \oplus R$. To see that this last part is true, consider the short exact sequence $0 \longrightarrow \ker(\varepsilon) \hookrightarrow C_0(X;R) \xrightarrow{\varepsilon} R \longrightarrow 0$, which splits, since R is free, hence $C_0(X;R) \approx \ker(\varepsilon) \oplus R$, so we get

$$\frac{C_0(X;R)}{\mathrm{Im}(\partial)} = \frac{\ker(\varepsilon) \oplus R}{\mathrm{Im}(\partial)},$$

but since $\mathrm{Im}(\partial) \subset \ker(\varepsilon)$ we have

$$\frac{\ker(\varepsilon) \oplus R}{\mathrm{Im}(\partial)} = \frac{\ker(\varepsilon)}{\mathrm{Im}(\partial)} \oplus R,$$

thus we concluded that $H_0(X;R) \approx \tilde{H}_0(X;R) \oplus R$.

In a similar fashion, we define the **reduced singular cohomology modules** of X with coefficients in an R-module M, denoted by $\tilde{H}^n(X;M)$, to be the cohomology modules of the cochain complex

$$0 \longrightarrow R^* \xrightarrow{\varepsilon^*} C_0^*(X;R) \xrightarrow{\delta} C_1^*(X;R) \longrightarrow \cdots \ .$$

It is easy to see that $\tilde{H}^n(X;M) = H^n(X;M)$, for $n \geq 1$, and $\tilde{H}^0(X;M) = H^0(X;M)/\mathrm{Im}(\varepsilon^*)$.

Relative homology and Cohomology

If A is a subspace of X, let $C_n(A;R)$ be the submodule of $C_n(X;R)$ generated by all singular n-simplexes σ with $\mathrm{Im}(\sigma) \subset A$. We can define the quotient R module

$$C_n(X,A;R) \doteq \frac{C_n(X;R)}{C_n(A;R)},$$

and since we clearly have $\partial(C_n(A;R)) \subset C_{n-1}(A;R)$, the boundary map naturally induces another boundary map $\partial : C_n(X,A;R) \to C_{n-1}(X,A;R)$ given by $\partial([c]) = [\partial(c)]$, hence we have a chain complex

$$\cdots \longrightarrow C_n(X,A;R) \xrightarrow{\partial} C_{n-1}(X,A;R) \longrightarrow \cdots \longrightarrow C_1(X,A;R) \xrightarrow{\partial} C_0(X,A;R) \longrightarrow 0 \ ,$$

the homology of which, denoted by $H_n(X,A;R)$, is called the **homology module of X relative to A**. By dualizing the previous chain complex, we can define the cohomology modules of the new chain complex, denoted by $H^n(X,A;M)$, and called the **nth singular cohomology module of X relative to A with coefficients in M**.

Proposition 1.3.8. The quotient R-modules $C_n(X,A;R)$ are free for all n.

The previous proposition is a direct consequence of proposition A.1.8. From this we conclude, by lemma A.2.9, that the sequence

$$0 \longrightarrow C_n(A;R) \xrightarrow{i} C_n(X;R) \xrightarrow{p} C_n(X,A;R) \longrightarrow 0$$

is a short exact split sequence, in which i is the canonical inclusion and p is the projection homomorphism. Lemma A.2.19 implies that by dualizing with $\mathrm{Hom}_R(\cdot,N)$, generates a short exact split sequence

$$0 \longrightarrow C_n^*(X,A;R) \xrightarrow{p^*} C_n^*(X;R) \xrightarrow{i^*} C_n^*(A;R) \longrightarrow 0 \ .$$

Long sequence of a pair

Notice that

$$0 \longrightarrow C(A;R) \xrightarrow{i} C(X;R) \xrightarrow{p} C(X,A;R) \longrightarrow 0$$

is a short exact split sequence of chain complexes and chain maps, and by consequence so is

$$0 \longrightarrow C^*(X,A;R) \xrightarrow{p^*} C^*(X;R) \xrightarrow{i^*} C^*(A;R) \longrightarrow 0 \ ,$$

hence, by theorem A.2.16 and corollary A.2.20 there are long exact sequences

$$\cdots \longrightarrow H_n(A;R) \xrightarrow{i_*} H_n(X;R) \xrightarrow{p_*} H_n(X,A;R) \xrightarrow{\alpha} H_{n-1}(A;R) \longrightarrow \cdots \ ,$$

and

$$\cdots \longrightarrow H^n(X,A;N) \xrightarrow{p^*_*} H^n(X;N) \xrightarrow{i^*_*} H^n(A;N) \xrightarrow{\beta} H^{n+1}(X,A;N) \longrightarrow \cdots \ .$$

In this case it is rather easy to understand the connecting homomorphism α. Notice that, if we take $z + C_n(A;R)$ in $Z_n(X,A;R)$ any cycle, then $\partial(z) + C_{n-1}(A;R) = [0]$ implies that $\partial(z) \in Z_{n-1}(A;R)$, hence we can define a homomorphism $H_n(X,A;R) \to H_{n-1}(A;R)$ by sending $[z]$ into $[\partial(z)]$, and it is not difficult to see that this is precisely the connecting homomorphism as we defined in the proof of theorem A.2.16, so we have

$$\alpha\big((z+C_n(A;R))+B_n(X,A;R)\big) = \partial(z)+B_n(A;R),$$

for this reason many authors write ∂ for the connecting homomorphism, but we shall keep using α to avoid confusion.

The two previous long sequences are called the **long exact homology sequence of the pair** (X,A) and the **long exact cohomology sequence of the pair** (X,A), respectively.

If A is a point, then the long exact homology sequence of the pair gives us $\tilde{H}_n(X;R)\approx H_n(X,\{pt\};R)$.

Long exact sequence of a triple

Notice that given a triple (X,A,B) of topological spaces, with $B\subset A\subset X$ we can construct the following short exact split sequence

$$0 \longrightarrow C_n(A,B;R) \xrightarrow{\ i\ } C_n(X,B;R) \xrightarrow{\ p\ } C_n(X,A;R) \longrightarrow 0 \ ,$$

in which $i(c+C_n(B)) = c+C_n(B)$ and $p(c+C_n(B)) = c+C_n(A)$, with both functions well defined, since $B\subset A\subset X$. The sequence is clearly exact, and the fact that it splits follows easily, since $C_n(X,A;R)$ is free. One can see that i and p define chain maps in the natural way, and from theorem A.2.16 we get the following long exact sequence

$$\cdots \longrightarrow H_n(A,B;R) \xrightarrow{\ i_*\ } H_n(X,B;R) \xrightarrow{\ p_*\ } H_n(X,A;R) \xrightarrow{\ \alpha\ } H_{n-1}(A,B;R) \longrightarrow \cdots \ ,$$

which is called the **long exact homology sequence of the triple** (X,A,B). Just like in the previous case of the long exact sequence of a pair, here we can understand the homomorphism α in a really simple way. Notice that if $z\in C_n(X,A;R)$ is a cycle, it means that $\partial(z)\in C_{n-1}(A;R)$, hence the class $\partial(z)+C_{n-1}(B;R)$ is a cycle in $C_{n-1}(A,B;R)$. By using the definition of the connecting homomorphism, one gets

$$\alpha((z+C_n(A;R))+B_n(X,A;R)) = (\partial(z)+C_{n-1}(B;R))+B_{n-1}(A,B;R).$$

If N is any R-module, from corollary A.2.20 we get the long exact sequence

$$\cdots \longrightarrow H^n(X,A;N) \xrightarrow{\ p_*^*\ } H^n(X,B;N) \xrightarrow{\ i_*^*\ } H^n(A,B;N) \xrightarrow{\ \beta\ } H^{n+1}(X,A;N) \longrightarrow \cdots \ ,$$

called the **long exact cohomology sequence of the triple** (X,A,B).

Excision

Theorem 1.3.9 ((HATCHER, 2002), **The Excision Theorem 1).** Let (X,A,B) be a triple of spaces with $B \subset A \subset X$, such that the closure of B lies in the interior of A ($\overline{B} \subset \mathrm{int}(A)$). Then the inclusion map $i : (X \setminus B, A \setminus B) \hookrightarrow (X,A)$ induces isomorphisms

$$i_* : H_n(X \setminus B, A \setminus B) \to H_n(X,A),$$

for all n.

Theorem 1.3.10 ((HATCHER, 2002), **The Excision Theorem 2).** If we have subspaces $Z,Y \subset X$ such that their interiors cover X, $\mathrm{int}(Z) \cup \mathrm{int}(Y) = X$, then the inclusion $j : (Z, Y \cap Z) \hookrightarrow (X,Y)$, induces isomorphisms

$$j_* : H_n(Z, Y \cap Z) \to H_n(X,Y),$$

for all n.

To see that theorems 1.3.9 and 1.3.10 are equivalent, simply note that in the notation of both theorems we have to take $A = Y$ and $X \setminus B = Z$ to go back and forth from one to the other.

If $\mathscr{U}$ is a collections of subsets of X, such that their interiors cover X, we denote by $C_n^{\mathscr{U}}(X;R)$ the collection of elements in $C_n(X;R)$ which are sums of singular chains in each subset of $\mathscr{U}$. The boundary map naturally restricts to $\partial : C_n^{\mathscr{U}}(X;R) \to C_{n-1}^{\mathscr{U}}(X;R)$, thus we have a new chain complex, which gives us homology R-modules $H_n^{\mathscr{U}}(X;R)$.

We will not give the proof for the excision theorem here, since it is too technical and long. One important part in proving the theorem is showing the following result.

Proposition 1.3.11 ((HATCHER, 2002)). The inclusion map $i : C_n^{\mathscr{U}}(X;R) \to C_n(X;R)$ induces an isomorphism $i_* : H_n^{\mathscr{U}}(X;R) \to H_n(X;R)$ for all integers n.

The basic idea behind the proof of the previous proposition is that we can take a class $[z] \in H_n(X;R)$ and write it as a sum of classes $[z_1] + \cdots + [z_k]$ such that for each i we have $z_i \in C_n(U_i;R)$ for some $U_i \in \mathscr{U}$. In the case $n = 1$, we can see this process as taking a path in X and writing it as the product of finitely many paths, each path with its image contained in an element of $\mathscr{U}$ (it is always possible to take finitely many paths since the initial path is a compact subspace of X).

The excision theorem is useful when computing the homology of wedge sums, in what follows we shall present some results in this direction.

Definition 1.3.12 ((HATCHER, 2002)). A pair of spaces (X,A) is said to be a **good pair** if A is a nonempty closed subset of X, such that there exists a neighborhood U of A, to which A is a deformation retract, that is, there is a homotopy $F: U \times I \to A$, such that $F|_{U\times\{0\}} = id_U$, $F(U,1) \subset A$ and $F(a,t) = a$, for all $a \in A$.

Proposition 1.3.13 ((HATCHER, 2002))**.** If (X,A) is a good pair and R a commutative ring, then the quotient map $q:(X,A) \to (X/A, A/A)$ induces an isomorphism $q_*: H_n(X,A;R) \to H_n(X/A,A/A;R) \approx \tilde{H}_n(X/A;R)$.

Proof. Since (X,A) is a good pair we know there is an open subset $V \subset X$ such that $A \subset V$ is a deformation retract of V. Consider the commutative diagram

$$\begin{tikzcd}
(X,A) \arrow[r, hook, "i"] \arrow[d, "q"] & (X,V) \arrow[d, "q"] & (X\setminus A, V\setminus A) \arrow[l, hook', "k"'] \arrow[d, "\tilde{q}=q|_{X\setminus A}"] \\
(X/A,A/A) \arrow[r, hook, "j"] & (X/A,V/A) & ((X/A)\setminus(A/A),(V/A)\setminus(A/A)) \arrow[l, hook', "l"']
\end{tikzcd}$$

by applying the homology functor we get the commutative diagram

$$\begin{tikzcd}
H_n(X,A;R) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(X,V;R) \arrow[d, "q_*"] & H_n(X\setminus A, V\setminus A;R) \arrow[l, "k_*"'] \arrow[d, "\tilde{q}_*"] \\
H_n(X/A,A/A;R) \arrow[r, "j_*"] & H_n(X/A,V/A;R) & H_n((X/A)\setminus(A/A),(V/A)\setminus(A/A);R) \arrow[l, "l_*"']
\end{tikzcd}$$

Our first claim is that i_*, j_*, k_* and l_* are isomorphisms.

That k_* and l_* are isomorphisms follows directly from the Excision Theorem 1.3.9.

For i_*, consider the homology long exact sequence of the triple (X,V,A)

$$\cdots \longrightarrow H_n(V,A;R) \longrightarrow H_n(X,A;R) \xrightarrow{i_*} H_n(X,V;R) \longrightarrow H_{n-1}(V,A;R) \longrightarrow \cdots,$$

since A is a deformation retract of V we have that the pairs (V,A) and (A,A) are homotopic, hence $H_k(V,A;R) \approx H_k(A,A;R) = 0$, for all n. Hence, by the exact sequence above, we conclude that i_* is an isomorphism.

Similarly, for j_* we consider the long exact sequence of the triple $(X/A,V/A,A/A)$, and since $(V/A,A/A)$ is homotopic to $(A/A,A/A)$ we conclude that j_* must be an isomorphism.

Now we claim that $\tilde{q}_*$ is also an isomorphism. To see why this is true, simply notice that $\tilde{q}: X\setminus A \to (X/A)\setminus(A/A)$ is an homeomorphism, given by $\tilde{q}(x) = [x]$, for all $x \in X\setminus A$, which is clearly bijective onto $(X/A)\setminus(A/A)$, and the continuous map $r:(X/A)\setminus(A/A) \to X\setminus A$, given by $r([x]) = x$ is the inverse homeomorphism, hence $\tilde{q}_*$ is in fact an isomorphism.

Finally, the commutativity of the diagram above implies that both q_*s are isomorphisms, hence we get $H_n(X,A;R) \approx H_n(X/A,A/A;R) \approx \tilde{H}_n(X/A;R)$. □

Lemma 1.3.14. Let R be a commutative ring. If $\{(X_\alpha,A_\alpha)\}_\alpha$ is a family of pairs of spaces, then

$$H_n\left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha A_\alpha; R\right) \approx \bigoplus_\alpha H_n(X_\alpha,A_\alpha;R).$$

Proof. Notice that each singular simplex $\sigma : \Delta^n \to \bigsqcup_\alpha X_\alpha$ has a path connected image, hence every element of $C_n(\sqcup_\alpha X_\alpha; R)$ is a finite formal sum $\sum_i \sigma_i$ with each $\sigma_i \in C_n(X_{\alpha_i}; R)$ for some α_i, which is exactly the description of the module $\bigoplus_\alpha C_n(X_\alpha; R)$, hence we get

$$C_n(\sqcup_\alpha X_\alpha; R) \approx \bigoplus_\alpha C_n(X_\alpha; R)$$
$$Z_n(\sqcup_\alpha X_\alpha; R) \approx \bigoplus_\alpha Z_n(X_\alpha; R)$$
$$B_n(\sqcup_\alpha X_\alpha; R) \approx \bigoplus_\alpha B_n(X_\alpha; R)$$

and the same is valid for A_α.

Recall that, by the first isomorphism theorem for R-modules, for any family of R-modules M_α, with submodules $N_\alpha \subset M_\alpha$, we have $\frac{\bigoplus_\alpha M_\alpha}{\bigoplus_\alpha N_\alpha} \approx \bigoplus_\alpha \frac{M_\alpha}{N_\alpha}$. To prove this fact, simply notice that the kernel of the projection homomorphism $\bigoplus_\alpha M_\alpha \to \bigoplus_\alpha \frac{M_\alpha}{N_\alpha}$ is exactly $\bigoplus_\alpha N_\alpha$.

So we have

$$C_n(\sqcup_\alpha X_\alpha, \sqcup_\alpha A_\alpha; R) \approx \frac{\bigoplus_\alpha C_n(X_\alpha; R)}{\bigoplus_\alpha C_n(A_\alpha; R)} \approx \bigoplus_\alpha \frac{C_n(X_\alpha; R)}{C_n(A_\alpha; R)} = \bigoplus_\alpha C_n(X_\alpha, A_\alpha; R),$$

similarly $Z_n(\sqcup_\alpha X_\alpha, \sqcup_\alpha A_\alpha; R) \approx \bigoplus_\alpha Z_n(X_\alpha, A_\alpha; R)$ and $B_n(\sqcup_\alpha X_\alpha, \sqcup_\alpha A_\alpha; R) \approx \bigoplus_\alpha B_n(X_\alpha, A_\alpha; R)$, hence

$$H_n\left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha A_\alpha; R\right) \approx \bigoplus_\alpha H_n(X_\alpha, A_\alpha; R).$$

□

Corollary 1.3.15 ((HATCHER, 2002))**.** Suppose $\{(X_\alpha, x_\alpha)\}_\alpha$ is a collection of good pairs (in particular, if X_α are all CW complexes), then the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$\oplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{H}_n(X_\alpha; R) \to \tilde{H}_n\left(\bigvee_\alpha X_\alpha; R\right).$$

Proof. To show this corollary, we simply apply proposition 1.3.13. Mimicking the notation in the proposition, consider

$$(X, A) = \left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}\right),$$

then clearly $X/A = \bigvee_\alpha X_\alpha$, and the proposition states that the quotient map

$$q : \left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}\right) \to \left(\bigvee_\alpha X_\alpha, *\right)$$

induces an isomorphism

$$q_* : H_n\left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}; R\right) \to H_n\left(\bigvee_\alpha X_\alpha, *; R\right) \approx \tilde{H}_n\left(\bigvee_\alpha X_\alpha; R\right),$$

and by applying lemma 1.3.14 we conclude

$$\bigoplus_{\alpha}\tilde{H}_n(X_\alpha;R)\approx\bigoplus_{\alpha}H_n(X_\alpha,\{x_\alpha\};R)\approx\tilde{H}_n\left(\bigvee_{\alpha}X_\alpha;R\right).$$

□

Example 1.3.16. Since the n-sphere S^n is a CW complex, for any point $x\in S^n$ the pair $(S^n,\{x\})$ is a good pair, hence corollary 1.3.15 gives us

$$\tilde{H}_j\left(\bigvee_{i=1}^{k}S^n;R\right)=\begin{cases}R^k, & \text{if } j=n\\ 0, & \text{otherwise}\end{cases}.$$

Mayer-Vietoris Sequence

When one has a collection of two subsets $\mathscr{U}=\{A,B\}$ of a space X, such that $\operatorname{int}(A)\cup\operatorname{int}(B)=X$, it is common to write $C_n(A+B;R)$ instead of $C_n^{\mathscr{U}}(X;R)$.

Consider the exact sequence

$$0\longrightarrow C_n(A\cap B;R)\xrightarrow{\ f\ }C_n(A;R)\oplus C_n(B;R)\xrightarrow{\ g\ }C_n(A+B;R)\longrightarrow 0\ ,$$

in which $f(x)=(x,-x)$ and $g(x,y)=x+y$. Both f and g are chain maps, and by theorem A.2.16 we have the long exact sequence

$$\cdots\longrightarrow H_n(A\cap B;R)\xrightarrow{f_*}H_n(A;R)\oplus H_n(B;R)\xrightarrow{g_*}H_n(A+B;R)\xrightarrow{\alpha}H_{n-1}(A\cap B;R)\longrightarrow\cdots,$$

and by proposition 1.3.11 we know that $H_n(A+B;R)\approx H_n(X;R)$, so the exact sequence becomes

$$\cdots\longrightarrow H_n(A\cap B;R)\xrightarrow{f_*}H_n(A;R)\oplus H_n(B;R)\xrightarrow{g_*}H_n(X;R)\xrightarrow{\alpha}H_{n-1}(A\cap B;R)\longrightarrow\cdots,$$

which is the **Mayer-Vietoris sequence**.

The maps f_* and g_* are easily induced from f and g, namely $f_*([z])=([z],[-z])$ and $g([z],[y])=[z+y]$. The connecting homomorphism is not that complicated, simply note that any n-cycle $z\in C_n(X;R)$ is homologous to a sum $x+y\in C_n(A+B)$, and since $\partial(x+y)=0$, we have $\partial y=-\partial x\in A\cap B$, and by the definition of the connecting homomorphism we have $\alpha([z])=[\partial x]=[\partial y]$.

The Universal Coefficient Theorem for Cohomology

The idea of this section is to establish a relation between Cohomology and Homology modules, this is done via the Universal Coefficient Theorem, which asserts that Cohomology is the dual of Homology (via the Hom functor) plus an extra factor given by the Ext functor, which is introduced in the homological algebra appendix A.2.

In the entirety of this section, we shall be assuming that R is a Principal Ideal Domain (PID), it will be made clear very soon why this requirement is needed for the proof of the Universal Coefficient Theorem.

The initial intuition that one might have for constructing this theorem is that since we generated the cohomology R-modules by applying the Hom functor to the chain complex modules, which gave rise to the singular cochain complex modules, one natural question is whether the cohomology modules are simply the dual of the homology modules, in other words, we want to know if there is an isomorphism $H^n(X;M) \approx \mathrm{Hom}_R(H_n(X;R);M)$. The answer to this question is given by the Universal Coefficient Theorem.

Theorem 1.3.17 ((HATCHER, 2002))**.** If X is a topological space, R is a Principal Ideal Domain and M is an R-module, then the following is a split exact sequence:

$$0 \longrightarrow \mathrm{Ext}_R(H_{n-1}(X;R),M) \longrightarrow H^n(X;M) \longrightarrow \mathrm{Hom}_R(H_n(X;R),M) \longrightarrow 0 \ .$$

So we do not have an isomorphism per se, but we conclude that the cohomology R-modules are given by

$$H^n(X;M) \approx \mathrm{Hom}_R(H_n(X),M) \oplus \mathrm{Ext}_R(H_{n-1}(X),M),$$

and by using the commonly known properties of Hom and Ext, as presented in appendix A.2, one can easily obtain the cohomology modules once the homology of a space is known.

Instead of proving the universal coefficient theorem for topological spaces, we shall prove it for chain complex of free R-modules, which will automatically imply the topological version.

Theorem 1.3.18 ((HATCHER, 2002))**.** If R is a Principal Ideal Domain, C is a chain complex of free R-modules, and M is an R-module, then the following is a split exact sequence:

$$0 \longrightarrow \mathrm{Ext}_R(H_{n-1}(C;R),M) \longrightarrow H^n(C;M) \longrightarrow \mathrm{Hom}_R(H_n(C;R),M) \longrightarrow 0 \ .$$

We start by defining a natural homomorphism

$$h : H^n(C;M) \to \mathrm{Hom}_R(H_n(C;R),M),$$

to do so, denote by Z^n and B^n the submodules of cocycles and coboundaries and by Z_n and B_n the cycles and boundaries of C_n. If $[\varphi] \in H^n(C;M)$, then $\varphi \in Z^n$, which means $\varphi : C_n \to M$ is a homomorphism with $\delta\varphi = \varphi\partial = 0$, which basically means that $\varphi(B_n) = 0$.

With this in mind, we can define the restriction $\varphi_o = \varphi|_{Z_n}$, and since $\varphi_o(B_n) = 0$, lemma A.1.6 implies that φ_o induces a homomorphism

$$\overline{\varphi}_o : \frac{Z_n}{B_n} \to M,$$

which is clearly an element of $\text{Hom}_R(H_n(C;R),M)$, hence we have our candidate for homomorphism

$$\begin{aligned} h : H^n(C;M) &\to \text{Hom}_R(H_n(C;R),M) \\ \varphi &\mapsto \overline{\varphi}_o, \end{aligned}$$

in other words, $h([\varphi])(z+B_n) = \overline{\varphi}_o(z+B_n) = \varphi(z)$.

Notice that h is well defined, if $[\varphi] = [\psi]$ are two elements in $H^n(C;M)$, then $\varphi - \psi \in B^n$, i.e, $\varphi - \psi = \delta\gamma$ for some $\gamma \in C^*_{n-1}$, which implies that for any $z \in Z_n$ we have $(\varphi - \psi)(z) = \delta\gamma(z) = \gamma(\partial z) = \gamma(0) = 0$, hence $\varphi|_{Z_n} = \psi|_{Z_n}$, and consequently $\overline{\varphi}_o = \overline{\psi}_o$.

The map h is clearly an R-homomorphism, since

$$\begin{aligned} h([r\varphi + \psi])(z+B_n) &= (r\varphi + \psi)(z) \\ &= r\varphi(z) + \psi(z) \\ &= rh([\varphi])(z+B_n) + h([\psi])(z+B_n) \end{aligned}$$

for any $\varphi,\ \psi \in Z_n$, $r \in R$, and $z \in Z_n$.

Next, we will show that h is an epimorphism, by showing that it has a right inverse. This is the part where the assumption of R being a PID plays a big role. In appendix A.1 we have shown that if R is a PID, then all the submodules of a free R-module are also free (Theorem A.1.11), thus B_n and Z_n are free R-modules, once C_n is a free R-module.

The sequence

$$0 \longrightarrow Z_n \xrightarrow{\ i\ } C_n \xrightarrow{\ \partial\ } B_{n-1} \longrightarrow 0\ ,$$

is short exact, and the fact that B_{n-1} is free implies that the sequence above splits (lemma A.2.9). Hence there is a homomorphism $p : C_n \to Z_n$ which restricted to Z_n is the identity $p|_{Z_n} = id_{Z_n}$.

Define a homomorphism

$$g : \text{Hom}_R(H_n(C;R),M) \to H^n(C;M),$$

in the following way. If $\psi : H_n(C;R) \to M$ is an R-homomorphism, then it defines a homomorphism $\tilde{\psi} : Z_n \to M$, given by $\tilde{\psi}(z) = \psi(z+B_n)$, hence $\tilde{\psi}(B_n) = 0$. Now we extend this homomorphism by defining $\psi_1 = \tilde{\psi}p : C_n \to M$, and since $p|_{Z_n} = id_{Z_n}$ and $B_n \subset Z_n$ we still have $\psi_1(B_n) = 0$, which is the same as $\delta\psi_1 = 0$, i.e., $\psi_1 \in \ker(\delta)$, whence $[\psi_1] \in H^n(C;M)$. Thus we define $g(\psi) = [\psi_1]$. Notice that

$$\begin{aligned} hg(\psi)(z+B_n) &= h([\psi_1])(z+B_n) \\ &= \psi_1(z) = \tilde{\psi}(z) = \psi(z+B_n), \end{aligned}$$

hence $hg = id_{\mathrm{Hom}_R(H_n(C;R),M)}$, thus showing that h is surjective.

From the previous discussion we conclude that the sequence

$$0 \longrightarrow \ker(h) \hookrightarrow H^n(C;M) \xrightarrow{h} \mathrm{Hom}(H_n(C);M) \longrightarrow 0 \ ,$$

is short split exact, which already shows that $H^n(C;M)$ is equal to the direct sum of $\mathrm{Hom}(H_n(C);M)$ and an extra factor $\ker(h)$, which we will show is the Ext functor factor as stated in the theorem.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, hook] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

if we dualize it by applying the $\mathrm{Hom}(\_,M)$ functor we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B_n^* \arrow[r, "\delta"] & C_{n+1}^* \arrow[r] & Z_{n+1}^* \arrow[r] & 0 \\
0 \arrow[r] & B_{n-1}^* \arrow[r, "\delta"] \arrow[u, "0"] & C_n^* \arrow[r] \arrow[u, "\delta"] & Z_n^* \arrow[r] \arrow[u, "0"] & 0
\end{tikzcd}$$

Notice that the rows in the original diagram are split exact (since each B_n is free), and by lemma A.2.19 we have that the dual of a split exact sequence is itself split exact, hence the rows of the second diagram above are split exact, therefore, by corollary A.2.17, we get a long exact sequence of homology from the dualized diagram above

$$\cdots \longrightarrow B_{n-1}^* \longrightarrow H^n(C;M) \longrightarrow Z_n^* \longrightarrow B_n^* \longrightarrow \cdots \ .$$

Notice that the connecting homomorphism $\alpha : Z_n^* \to B_n^*$ is simply a restriction, i.e., if $\varphi \in Z_n^*$, then $\alpha(\varphi) = \varphi|_{B_n}$. This can be easily shown by applying the definition of the connecting homomorphism, by surjectivity of $C_n^* \to Z_n^*$, if $\varphi \in Z_n^*$, there is a $\Phi \in C_n^*$ such that $\Phi|_{Z_n} = \varphi$, and then we clearly have $\delta(\Phi) = \delta(\varphi|_{B_n})$.

So if $i_n : B_n \hookrightarrow Z_n$ is the canonical inclusion map, what we concluded above is that $\alpha = i_n^* : Z_n^* \to B_n^*$.

From a long exact sequence, as the previous one, we can always extract short exact sequences

$$0 \longrightarrow \mathrm{Coker}(i_{n-1}^*) \longrightarrow H^n(C;M) \longrightarrow \ker(i_n^*) \longrightarrow 0 \ .$$

Notice that we can identify $\ker(i_n^*)$ with $\mathrm{Hom}_R(H_n(C;R),M)$, with the following isomorphism

$$\begin{aligned}
\ker(i_n^*) &\longrightarrow \mathrm{Hom}_R(H_n(C;R),M) \\
(\varphi : Z_n \to M) &\longmapsto (\overline{\varphi} : Z_n/B_n \to M),
\end{aligned}$$

since $\varphi \in \ker(i_n^*)$ means exactly that $\varphi|_{B_n} = 0$, we define $\overline{\varphi}(z+B_n) = \varphi(z)$, and the previous map is well defined. The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Hom}(H_n(C;R),M) &\longrightarrow \ker(i_n^*) \\ (\psi : Z_n/B_n \to M) &\longmapsto (\tilde{\psi} : Z_n \to M) \end{aligned}$$

in which $\tilde{\psi}(z) = \psi(z+B_n)$.

If $k : Z_n \hookrightarrow C_n$ is the inclusion map, then $k^* : C_n^* \to Z_n^*$ is the restriction map, i.e., $k^*(\varphi) = \varphi|_{Z_n}$, and the map induced in homology $k_*^* : H^n(C;M) \to Z_n^*$, is given by $k_*^*([\varphi]) = \varphi|_{Z_n}$, this homomorphism can be restricted to $k_*^* : H^n(C;M) \to \ker(i_n^*)$, as in the previous exact sequence, and we get that

$$\ker(k_*^*) = \{[\varphi] \mid \varphi|_{Z_n} = 0\},$$

which is exactly the kernel of the homomorphism h we defined earlier, hence we can substitute the previous short exact sequence by

$$0 \longrightarrow \mathrm{Coker}(i_{n-1}^*) \longrightarrow H^n(C;M) \xrightarrow{\ h\ } \mathrm{Hom}_R(H_n(C;R),M) \longrightarrow 0 \ ,$$

and since h has a right inverse this is a split short exact sequence.

Now we proceed to show that $\mathrm{Coker}(i_{n-1}^*)$ is the Ext factor presented in the theorem.

From the discussion about the Ext functor in appendix A.2, we can see that

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C;R)$$

is a free resolution of $H^{n-1}(C;M)$, by dualizing it with $\mathrm{Hom}(\_,M)$ we get

$$0 \longrightarrow (H_{n-1}(C;R))^* \longrightarrow Z_{n-1}^* \xrightarrow{i_{n-1}^*} B_{n-1}^* \longrightarrow 0 \ ,$$

whence, by definition

$$\mathrm{Ext}_R(H_{n-1}(C;R),M) = \frac{B_{n-1}^*}{\mathrm{Im}(i_{n-1}^*)} = \mathrm{Coker}(i_{n-1}^*).$$

This concludes the proof of theorem 1.3.18, which immediately implies theorem 1.3.17.

Notice that, since the Ext functor is trivial for free R-modules (proposition A.2.30 property E2), we get the following corollaries.

Corollary 1.3.19. Let R be a PID, M an R-module, and C is a chain complex of free R-modules such that all its homology groups are free, then

$$H^n(C;M) \approx \mathrm{Hom}_R(H_n(C;R),M).$$

Corollary 1.3.20. Let K be a field, M a K-module, and C a chain complex of K-modules, then

$$H^n(C;M) \approx \mathrm{Hom}_K(H_n(C;K),M).$$

Example 1.3.21. One good example for using the universal coefficient theorem is to compute the cohomology modules of the n-sphere S^n. It is common to compute the homology modules of S^n by using the Mayer-Vietoris Sequence, if R is a PID with unit, we get

$$H_j(S^n;R) = \begin{cases} R, & \text{if } j=0 \text{ or } n; \\ 0, & \text{otherwise.} \end{cases}$$

Since R is a free R-module, we can apply corollary 1.3.19, and by using the fact that $\mathrm{Hom}_R(R,M) \approx M$ (from proposition A.2.23) we get

$$H^j(S^n;M) = \begin{cases} M, & \text{if } j=0 \text{ or } n; \\ 0, & \text{otherwise.} \end{cases}$$

in other words, the homology and cohomology modules are the same for S^n.

Corollary 1.3.22. Suppose R is a DIP, M is an R-module and $\{(X_\alpha,x_\alpha)\}_\alpha$ is a collection of good pairs, then

$$\bigoplus_\alpha \tilde{H}^n(X_\alpha;R) \approx \tilde{H}^n\left(\bigvee_\alpha X_\alpha;R\right).$$

Proof. We already know that this is valid in the case of homology, from corollary 1.3.15, and by combining that result with the universal coefficient theorem we get

$$\begin{aligned}\tilde{H}^n\left(\bigvee_\alpha X_\alpha;M\right) &\approx \mathrm{Hom}_R\left(\bigoplus_\alpha \tilde{H}_n(X_\alpha;R),M\right) \oplus \mathrm{Ext}_R\left(\bigoplus_\alpha \tilde{H}_{n-1}(X_\alpha;R),M\right) \\ &\approx \bigoplus_\alpha \left[\mathrm{Hom}_R(\tilde{H}_n(X_\alpha;R),M) \oplus \mathrm{Ext}_R(\tilde{H}_{n-1}(X_\alpha;R),M)\right] \\ &\approx \bigoplus_\alpha \tilde{H}^n(X_\alpha;M).\end{aligned}$$

□

Example 1.3.23. From the previous corollary we can compute the cohomology of the wedge of spheres

$$\tilde{H}^j\left(\bigvee_{i=1}^k S^n;M\right) = \begin{cases} M^k, & \text{if } j=n; \\ 0, & \text{otherwise.} \end{cases}$$

Cup Product

In this section our goal is to define a product between cohomology classes, making it possible to introduce the cohomology ring later on.

We will first define the cup product in singular cohomology and prove it is a well-defined product later on.

Definition 1.3.24 ((HATCHER, 2002))**.** If X is a topological space with singular cohomology R-modules $H^k(X;R)$ (with coefficients in a commutative ring R), we define the **cup product** by

$$\smile : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$$
$$([\varphi],[\psi]) \mapsto [\varphi \smile \psi]$$

in which $\varphi : C_k(X;R) \to R$ and $\psi : C_l(X;R) \to R$ are cocycles, and $\varphi \smile \psi : C_{k+l}(X;R) \to R$ is the cocycle given by

$$\varphi \smile \psi(\sigma) = (\varphi(\sigma\varepsilon_k^{k+l}))(\psi(\sigma\xi_l^{k+l})),$$

in which σ is a singular $(k+l)$-simplex $\varepsilon_k^{k+l} : \Delta^k \to \Delta^{k+l}$ and $\xi_l^{k+l} : \Delta^l \to \Delta^{k+l}$ are the canonical embeddings

$$\varepsilon_k^{k+l}(t_0,\dots,t_k) = (t_0,\dots,t_k,0,\dots,0),$$
$$\xi_l^{k+l}(t_0,\dots,t_l) = (0,\dots,0,t_0,\dots,t_l),$$

and in the general case we have

$$\varphi \smile \psi\left(\sum_\sigma r_\sigma \sigma\right) = \sum_\sigma r_\sigma^2(\varphi(\sigma\varepsilon_k^{k+l}))(\psi(\sigma\xi_l^{k+l})).$$

Now, we need to show that the cup product is a well defined function, by showing that $\varphi \smile \psi$ is in fact a cocycle, if both φ and ψ are cocycles, and that if $([\varphi_1],[\psi_1]) = ([\varphi_2],[\psi_2])$, then $[\varphi_1 \smile \psi_1] = [\varphi_2 \smile \psi_2]$.

First, notice that if the cup product is well defined, then it will be an R-bilinear function.

Proposition 1.3.25. The cup product is an R-bilinear map, in other words, the distributive laws apply: $(r\varphi_1 + \varphi_2) \smile \psi = r(\varphi_1 \smile \psi) + (\varphi_2 \smile \psi)$ and $\varphi \smile (r\psi_1 + \psi_2) = r(\varphi \smile \psi_1) + (\varphi \smile \psi_2)$, for $\varphi, \varphi_1, \varphi_2 \in C_k^*(X;R)$, $\psi, \psi_1, \psi_2 \in C_l^*(X;R)$ and $r \in R$.

Proof. Just for this proof let us denote $\varepsilon \doteq \varepsilon_k^{k+l}$ and $\xi \doteq \xi_l^{k+l}$. From the definition, if σ is a singular $(k+l)$-simplex we have

$$\begin{aligned}(r\varphi_1 + \varphi_2) \smile \psi(\sigma) &= (r\varphi_1(\sigma\varepsilon) + \varphi_2(\sigma\varepsilon))(\psi(\sigma\xi)) \\ &= r(\varphi_1(\sigma\varepsilon))(\psi(\sigma\xi)) + (\varphi_2(\sigma\varepsilon))(\psi(\sigma\xi)) \\ &= (r(\varphi_1 \smile \psi) + (\varphi_2 \smile \psi))(\sigma),\end{aligned}$$

and analogously $\varphi \smile (r\psi_1+\psi_2) = r(\varphi \smile \psi_1)+(\varphi \smile \psi_2)$. $\square$

The next lemma will be useful in proving that the cup product is well defined.

Lemma 1.3.26 ((HATCHER, 2002))**.** If $\varphi \in C_k^*(X;R)$ and $\psi \in C_l^*(X;R)$, then

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi).$$

Proof. Let σ be a singular $(k+l+1)$-simplex, then

$$\begin{aligned}
(\delta\varphi) \smile \psi(\sigma) &= (\delta\varphi(\sigma\varepsilon_{k+1}^{k+l+1}))(\psi(\sigma\xi_l^{k+l+1})) = (\varphi(\partial(\sigma\varepsilon_{k+1}^{k+l+1})))(\psi(\sigma\xi_l^{k+l+1}))\\
&= \sum_{j=0}^{k+1}(-1)^j(\varphi(\partial^j(\sigma\varepsilon_{k+1}^{k+l+1})))(\psi(\sigma\xi_l^{k+l+1}))\\
&= \left[\sum_{j=0}^{k}(-1)^j(\varphi(\partial^j(\sigma\varepsilon_{k+1}^{k+l+1})))(\psi(\sigma\xi_l^{k+l+1}))\right]\\
&+(-1)^{k+1}(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\sigma\xi_l^{k+l+1})),
\end{aligned}$$

in the last step, we used the fact that $\partial^{k+1}(\sigma\varepsilon_{k+1}^{k+l+1}) = \sigma\varepsilon_k^{k+l+1}$. On the other hand we have

$$\begin{aligned}
(-1)^k\varphi \smile (\delta\psi)(\sigma) &= (-1)^k(\varphi(\sigma\varepsilon_k^{k+l+1}))(\delta\psi(\sigma\xi_{l+1}^{k+l+1}))\\
&= (-1)^k(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\partial(\sigma\xi_{l+1})^{k+l+1}))\\
&= (-1)^k\sum_{j=0}^{l+1}(-1)^j(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\partial^j(\sigma\xi_{l+1}^{k+l+1})))\\
&= \sum_{j=k}^{k+l+1}(-1)^j(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\partial^{j-k}(\sigma\xi_{l+1}^{k+l+1})))\\
&= (-1)^k(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\sigma\xi_l^{k+l+1}))\\
&+\left[\sum_{j=k+1}^{k+l+1}(-1)^j(\varphi(\sigma\varepsilon_k^{k+l+1}))(\psi(\partial^{j-k}(\sigma\xi_{l+1}^{k+l+1})))\right],
\end{aligned}$$

this time, in the last step, we used the fact that $\partial^0(\sigma\xi_{l+1}^{k+l+1}) = \sigma\xi_l^{k+l+1}$. It is not difficult to see that we have the following equalities

- $\partial^j(\sigma\varepsilon_{k+1}^{k+l+1}) = (\partial^j\sigma)\varepsilon_k^{k+l}$, for $j=0,\dots,k$
- $\sigma\varepsilon_k^{k+l+1} = (\partial^j\sigma)\varepsilon_k^{k+l}$, for $j=k+1,\dots,k+l+1$
- $\sigma\xi_l^{k+l+1} = (\partial^j\sigma)\xi_l^{k+l}$, for $j=0,\dots,k$
- $\partial^{j-k}(\sigma\xi_{l+1}^{k+l+1}) = (\partial^j\sigma)\xi_l^{k+l}$, for $j=k+1,\dots,k+l+1$

So we get

$$\begin{aligned}[(\delta\varphi)\smile\psi+(-1)^k\varphi\smile(\delta\psi)](\sigma) &= \sum_{j=0}^{k+l+1}(-1)^j(\varphi((\partial^j\sigma)\varepsilon_k^{k+l}))(\psi((\partial^j\sigma)\xi_l^{k+l})) \\ &= \sum_{j=0}^{k+l+1}(-1)^j\varphi\smile\psi(\partial^i\sigma) \\ &= \varphi\smile\psi(\partial\sigma)=\delta(\varphi\smile\psi)(\sigma).\end{aligned}$$

□

This lemma implies that if both φ and ψ are cocycles than $\varphi\smile\psi$ is a cocycle, and if φ is a coboundary ($\varphi=\delta\phi$, $\phi: C_{k-1}(X;R)\to R$) and ψ is a cocycle, then $\varphi\smile\psi=\delta\phi\smile\psi=\delta(\phi\smile\psi)$ (the analogous happens if φ is a cocycle and ψ is a coboundary). These properties show that the cup product is a well defined function.

It is also possible to define a relative version of the cup product. If A and B are open subsets of X we define the map

$$H^k(X,A;R)\times H^l(X,B;R)\to H^{k+l}(X,A\cup B;R)$$

by first restricting the usual cup product to the map

$$H^k(X,A;R)\times H^l(X,B;R)\to H^{k+l}(X,A+B;R),$$

in which $H^n(X,A+B;R)$ has as elements equivalence classes of cocycles that vanish in sums of chains in A and chains in B. Notice that the inclusion $C_n^*(X,A\cup B;R)\hookrightarrow C_n^*(X,A+B;R)$ induces an isomorphism in cohomology, $H^n(X,A\cup B;R)\approx H^n(X,A+B;R)$, thus giving the relative cup product as desired.

Notice that the cup product $H^k(X,A;R)\times H^l(X,B;R)\to H^{k+l}(X,A+B;R)$ is well defined because if $\phi\in C_k^*(X;R)$ vanishes in $C_k(A;R)$ and $\psi\in C_l(X;R)$ vanishes in $C_l(B;R)$, then $\psi\smile\phi$ obviously vanishes in $C_{k+l}(A;R)$ and $C_{k+l}(B;R)$, hence it also vanishes on sums of elements $C_{k+l}(A+B;R)$.

Theorem 1.3.27 ((HATCHER, 2002))**.** If R is a commutative ring, then for any $u,v\in H^*(X,A;R)$ we have $u\smile v=(-1)^{|u||v|}v\smile u$, in which $|u|$ is the cohomology level of u, i.e., if $u\in H^k(X,A;R)$, then $|u|=k$.

Proposition 1.3.28 ((HATCHER, 2002))**.** Given R a commutative ring and $f:(X,A)\to(Y,B)$ a continuous map between pairs of spaces, the cohomology induced map, $f_*^*: H^n(Y,B;R)\to H^n(X,A;R)$, satisfies $f_*^*(u\smile v)=f_*^*(u)\smile f_*^*(v)$.

The Cohomology ring

With the cup product in our hands, we can easily turn $H^*(X;R)$ into a ring (or even an R-algebra), called the **Cohomology ring** of X with coefficients in R.

We simply define $H^*(X;R)$ to be the direct sum $\bigoplus_i H^i(X;R)$, with the multiplication of two elements $\sum_i u_i$ and $\sum_j v_j$ defined by $(\sum_i u_i)(\sum_j v_j) = \sum_{i,j} u_i \smile v_j$. It is clear that $H^*(X;R)$ is an R-module, and since the cup product is an R-bilinear associative binary operation, we have that it is, in fact, an R-algebra.

Remark 1.3.29. Although $H^*(X;R)$ has an R-algebra structure, in many situations we will only use the fact that it has a ring structure.

Cross Product

Another important notion in cohomology is the cross product. For its definition, we use the cup product, as introduced earlier. Many authors follow the inverse order, defining first the cross product and then using it to produce the cup product. For practicality and simplicity, we chose to define the cup product first, using the singular simplexes definition.

Definition 1.3.30 ((HATCHER, 2002))**.** Given two topological spaces X and Y, and a commutative ring R, the **cross product** is the homomorphism

$$\mu : H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R)$$
$$u \otimes v \mapsto u \times v,$$

in which $u \times v = p_1{}_*^*(u) \smile p_2{}_*^*(v)$, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the canonical projections.

If we remember the definition of tensor product in section A.1, we can confirm that the cross product, as defined above, is, in fact, a homomorphism, since the R-bilinearity of the cup product implies that $(u,v) \mapsto u \times v$ is R-bilinear, hence defines a unique homomorphism μ.

It is possible to define a product which turns $H^*(X;R) \otimes H^*(Y;R)$ into a ring, such that the cross product becomes a ring homomorphism. This multiplication is given by $(u_1 \otimes v_1)(u_2 \otimes v_2) = (-1)^{|v_1||u_2|}(u_1 \smile u_2) \otimes (v_1 \smile v_2)$, and it is not difficult to verify, by using the definition of the cross product and theorem 1.3.27, that in this condition the cross product becomes a ring homomorphism.

We can establish a direct relation between the cross product and the cup product by means of the diagonal map

$$\Delta : X \to X \times X$$
$$x \mapsto (x,x),$$

if we compose its induced map in cohomology with the cross product we get

$$\begin{aligned}\Delta_*^*(u \times v) &= \Delta_*^*(p_{1*}^*(u) \smile p_{2*}^*(v)) \\ &= \Delta_*^*(p_{1*}^*(u)) \smile \Delta_*^*(p_{2*}^*(v)) \\ &= (p_1\Delta)_*^*(u) \smile (p_2\Delta)_*^*(v) \\ &= u \smile v,\end{aligned}$$

for any $u, v \in H^*(X;R)$. In the previous equation we used the fact that $f_*^*(u \smile v) = f_*^*(u) \smile f_*^*(v)$ (proposition 1.3.28), and that $p_1\Delta = p_2\Delta = id_X$.

The Künneth Theorem

We can extend the cross product to the following homomorphism

$$\begin{aligned}\mu : \bigoplus_{i=0}^{n} H^i(X;R) \otimes H^{n-i}(Y;R) &\to H^n(X \times Y;R) \\ \sum_j u_i \otimes v_j &\mapsto \sum_j u_i \times v_j\end{aligned}$$

which we shall simply call the cross product homomorphism.

As shown in (HATCHER, 2002), this homomorphism is an isomorphism in the case where R is a field (usually called the **Künneth isomorphism**), this means that the cohomologies of X and Y completely determine the cohomology of the Cartesian product $X \times Y$ (the same is valid for homology). This also results in an isomorphism between the cohomology rings

$$\mu : H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R).$$

In the more general case, when we only ask for R to be a PID, we get the following statement, called the **Künneth Theorem**.

Theorem 1.3.31 ((DAVIS; KIRK, 2001))**.** If R is a PID, the following is a split exact sequence

$$0 \to \bigoplus_{i=0}^{n} H^i(X;R) \otimes H^{n-i}(Y;R) \to H^n(X \times Y;R) \to \bigoplus_{i=0}^{n-1} \mathrm{Tor}(H^i(X;R), H^{n-1-i}(Y;R)) \to 0 .$$

1.4 Homotopy Theory

In many moments throughout this section we shall write diagrams like

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

and ask if such a diagram has a solution, meaning that all the unbroken arrows are known morphisms which commute in the diagram, and the question is whether there is a dashed arrow morphism, such that the whole diagram commutes.

Fibrations

Definition 1.4.1 ((ARKOWITZ, 2011))**.** A continuous map $p: E \to B$ has the **homotopy lifting property (HLP)**, or **covering homotopy property (CHP)**, with respect to a space X, if the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution for any maps F and f. Furthermore, p is called a **fibration**, or **fibre map**, if it satisfies the HLP for any space X. If p satisfies the HLP for any CW-complex X (or, equivalently, for any cube I^n), then it is called a **Serre fibration**, or **weak fibration**.

It is important to note that many authors use the nomenclature **Hurewicz fibration** to what we defined as a fibration, to further distinguish it from the Serre fibration.

For a fibration $p: E \to B$ we usually call E the **total space** and B the **base** of the fibration. If $b \in B$, we denote $F_b = p^{-1}(b)$, which is called the **fiber** over b of the fibration p. In the case of based spaces, with base point $* \in B$, we usually say that $F = p^{-1}(*)$ is the fiber of p, and we say that (E, B, F, p) is a **fiber space**. In the particular case where the base B is path connected we have that all fibers F_b are of the same homotopy type, thus for the means of homotopy theory the fiber is the same for all $b \in B$. Sometimes we simply write one Gothic letter for a fiber space $\mathfrak{B} \doteq (E, B, F, p)$.

Proposition 1.4.2. If $p: E \to B$ is a fibration and $A \subset B$, then $p_A : p^{-1}(A) \to A$ is also a fibration.

Proof. Let X be a topological space, and consider the following commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & p^{-1}(A) \arrow[r, hook, "i"] \arrow[d, "p_A"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] & A \arrow[r, hook, "j"] & B \end{tikzcd}$$

in which f and F are arbitrary continuous maps, such that $F|_{X\times\{0\}} = p_A f$.

We clearly have $p(if) = jF|_{X\times\{0\}}$, and since p is a fibration we can apply the homotopy lifting property, which gives us a map

$$G : X \times I \to E,$$

such that $G|_{X\times\{0\}} = if$ and $pG = jF$. From this last equality we get $p(G(X \times I)) \subset j(A) = A$, hence $G(X \times I) \subset p^{-1}(A)$. So we can define the map

$$\begin{aligned} H : X \times I &\to p^{-1}(A) \\ (x,t) &\mapsto G(x,t) \end{aligned}$$

which is well defined since $G(x,t) \in p^{-1}(A)$, for all $(x,t) \in X \times I$. Notice that $H|_{X\times\{0\}} = f$ and $p_A H = F$. Hereby we have shown that p_A has the homotopy lifting property for any space X, hence it is a fibration. □

Example 1.4.3. Let X be a topological space and consider its path space $X^I = \{\gamma : I \to X \mid \gamma \text{ continuous}\}$ (sometimes denoted by PX) with the usual compact-open topology. The map $\pi : X^I \to X \times X$ given by $\pi(\gamma) = (\gamma(0), \gamma(1))$ is a fibration, usually called the **path fibration** of X. To see that it is in fact a fibration let us show that it satisfies the homotopy lifting property for an arbitrary space W. Consider the commutative diagram

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook] & X^I \arrow[d, "\pi"] \\ W \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X \times X \end{tikzcd}$$

for which we want to find a map $\tilde{F}$ so that it still commutes.

Notice that $F = (F_1, F_2)$ with $F_1, F_2 : W \times I \to X$, and we can define the map

$$\begin{aligned} G : W \times I &\to X \\ (w,t) &\mapsto f(w)(t), \end{aligned}$$

The commutativity of the previous diagram gives us $F|_{W\times 0} = \pi f$, which means

$$(F_1(w,0), F_2(w,0)) = F(w,0) = \pi f(w) = (f(w)(0), f(w)(1)),$$

hence $F_1(w,0) = f(w)(0)$ and $F_2(w,0) = f(w,1)$. Thus we can define the function $\tilde{F} : W \times I \to X^I$ by

$$\tilde{F}(w,t)(s) = \begin{cases} F_1(w, -3s+t), \text{ for } 0 \leq s \leq t/3; \\ f(w, (3s-t)/(3-2t)), \text{ for } t/3 \leq s \leq 1 - t/3; \\ F_2(w, 3(s-1)+t), \text{ for } 1 - t/3 \leq s \leq 1; \end{cases}$$

which is continuous, with $\tilde{F}|_{W\times\{0\}} = f$ and $\pi\tilde{F}(w,t) = (\tilde{F}(w,t)(0), \tilde{F}(w,t)(1)) = (F_1(w,t), F_2(w,t)) = F(w,t)$, whence π has the homotopy lifting property for any space. □

If we apply proposition 1.4.2 to the previous example we get the following corollary.

Corollary 1.4.4. If X is a topological space with subsets $A, B \subset X$, consider the following subspace of the path space

$$E(X;A,B) = \{\gamma \in X^I \mid \gamma(0) \in A \text{ and } \gamma(1) \in B\},$$

and let $\overline{\pi} : E(X;A,B) \to A \times B$ be defined by $\overline{\pi}(\gamma) = (\gamma(0), \gamma(1))$, then $\overline{\pi}$ is a fibration.

Proof. Simply notice that $E(X;A,B) = \pi^{-1}(A \times B)$ (with π defined in example 1.4.3), and that $\overline{\pi}$ is simply a restriction of π, thus we can apply proposition 1.4.2 to conclude that $\overline{\pi}$ is a fibration. □

Proposition 1.4.5. A fibration $p : E \to B$ has a section $s : B \to E$ if and only if there is a map $r : B \to E$ such that $pr \simeq id_B$.

Proof. The "only if" part of the proposition is trivial. To show the "if" part, suppose there is a map $r : B \to E$ such that $pr \simeq id_B$, and let $F : B \times I \to B$ be a homotopy from pr to id_B, then the diagram

$$\begin{tikzcd} B \times \{0\} \arrow[r, "r"] \arrow[d, hook] & E \arrow[d, "p"] \\ B \times I \arrow[r, "F"] & X \end{tikzcd}$$

commutes, and by the homotopy lifting property there is a homotopy $G : B \times I \to E$ such that $pG = F$, hence $pG(x,1) = x$ and $s : B \to E$ given $s(x) = G(x,1)$ is a section of p. □

Cofibrations

We say that (X,A) is a **pair of spaces** if X is a topological space and $A \subset X$ is a subspace with the induced topology from X. Particularly, when A is a closed subset of X, we will call (X,A) a **closed pair**.

Definition 1.4.6 ((ARKOWITZ, 2011), definition 1.5.12)**.** We say a pair of spaces (X,A) has the **homotopy extension property** if the diagram

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "G"] \arrow[d, hook, "i"] & Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

has a solution for any space Y and any map G.

In other words, (X,A) has the homotopy extension property if any homotopy from A, which has an extension at the starting point, can be extended throughout the whole interval I.

If the pair (X,A) has the homotopy extension property, then it is called a **cofibred pair** ((JAMES, 2012)). The inclusion map $A \hookrightarrow X$ is a particular case of what we call a cofibration, generalized in the following definition.

Definition 1.4.7 ((HATCHER, 2002))**.** A map $u : A \to X$ is called a **cofibration** if for any space E, any map $f : X \to E$ and any homotopy $g_t : A \to E$ such that $g_0 = f \circ u$ there exists a homotopy $f_t : X \to E$ such that $g_t = f_t \circ u$, for all $t \in I$.

Proposition 1.4.8 ((HATCHER, 2002)). Given $u : A \to X$ a cofibration, then u is a homeomorphism onto its image $u(A)$. Hence all cofibrations can be thought of as inclusion maps.

Fibrations and Cofibrations main results

Before presenting some important results regarding fibrations and cofibrations, let us introduce some useful concepts.

Definition 1.4.9 ((ARKOWITZ, 2011)). Given a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ in a category $\mathscr{C}$, its **pullback** is given by an object P and morphisms u, v such that the following diagram (called a **pullback square**) commutes

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

and for any object Q and morphisms u' v' such the unbroken arrow diagram

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists!\ \omega"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\ & P \arrow[r, "v"] \arrow[d, "u"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

commutes, there exists a unique morphism $\omega : Q \to P$ such that the whole diagram commutes. We may refer to P or (P, u, v) as the pullback of the diagram.

The dual definition, starting with a diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$, is called a **pushout**.

Proposition 1.4.10 ((ARKOWITZ, 2011) proposition 3.2.9). In the category of topological spaces and continuous maps the pullback of any diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ exists and is unique (analogously for the pushout).

For topological spaces one can write the pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$ explicitly as

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

and the maps $u : P \to X$ and $v : P \to Y$ are the canonical projections (for a proof see proposition 3.3.11 in (ARKOWITZ, 2011)). Similarly, the pushout of a diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ can be written as

$$P = X \vee Y / \sim ,$$

in which $f(z) \sim g(z)$, for all $z \in Z$, and $u : X \to P$ and $v : Y \to P$ are the canonical quotient maps (for a proof see proposition 3.2.9 in (ARKOWITZ, 2011)).

Proposition 1.4.11 ((ARKOWITZ, 2011) proposition 3.3.12)**.** If

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a pullback square and f is a fibration, then v is a fibration.

Next, we present some results which guarantee that any map between topological spaces can be replaced by a fibration or a cofibration, what we mean by "replacing" a map will soon become clear.

Definition 1.4.12 ((ARKOWITZ, 2011))**.** Given a map $f : X \to Y$, we define the **mapping path of** f, denoted E_f, as the pullback of $X \xrightarrow{f} Y \xleftarrow{p_0} Y^I$, in which $p_0(\gamma) = \gamma(0)$, for all $\gamma \in Y^I$, i.e.,

$$E_f = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}.$$

Proposition 1.4.13 ((ARKOWITZ, 2011) proposition 3.5.8)**.** If we define $f' : X \to E_f$ and $f'' : E_f \to Y$ by $f'(x) = (x, \gamma_{f(x)})$ and $f''(x, \gamma) = \gamma(1)$, in which $\gamma_{f(x)}$ is the constant path at $f(x)$, then the diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[rr, bend left, "f"] & E_f \arrow[r, "f''"] & Y \end{tikzcd}$$

commutes and

1. f' is a homotopy equivalence.

2. f'' is a fibration.

The previous proposition clarifies what we mean by **replacing a map** f **with a fibration** f''. From the point of view of homotopy theory, X and E_f represent the same thing, once they are homotopy equivalent. Furthermore, since f' is a homotopy equivalence and $f''f' = f$, we have that f and f'' are the same map up to homotopy equivalence, hence for homotopy theory it is all the same map, i.e., any homotopic property of f is shared by f'' and vise versa, for example, if f is null homotopic, so is f'', if f induces an isomorphism in some level of homotopy groups or homology modules, so does f'', and so on.

Next, we present the dual process of replacing a map with a cofibration.

Definition 1.4.14 ((ARKOWITZ, 2011))**.** Given a map $f: X \to Y$, we define the **mapping cylinder of** f to be the space

$$M_f = (X \times I \sqcup Y)/\sim,$$

in which $(x,0) \sim f(x)$.

Proposition 1.4.15 ((ARKOWITZ, 2011) proposition 3.5.2)**.** Given a map $f: X \to Y$, we can define the maps $f': X \to M_f$ and $f'': M_f \to Y$, by $f'(x) = [(x,1)]$, $f''([(x,t)]) = f(x)$ and $f''([y]) = y$, then $f = f''f'$ and

1. f' is a cofibration.
2. f'' is a homotopy equivalence.

Definition 1.4.16 ((ARKOWITZ, 2011))**.** The **homotopy pullback** of a diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is given by

$$P = \{(x,\gamma,y) \in X \times Z^I \times Y \mid \gamma(0) = f(x) \text{ and } \gamma(1) = g(y)\},$$

and maps $u: P \to X$ and $v: P \to Y$, the canonical projections.

Equivalently, the homotopy pullback of the given diagram is the pullback of $E_f \xrightarrow{f''} Z \xleftarrow{g} Y$, in which f'' is a fibration equivalent to f.

Notice that proposition 1.4.11 implies that both u and v in the definition of homotopy pullback (1.4.16) are fibrations.

Proposition 1.4.17 ((ARKOWITZ, 2011))**.** If (P,u,v) is the homotopy pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$, then $fu \simeq gv$. Furthermore, if (P',u',v') is another triple with these properties, then there is a map $\omega: P' \to P$ such that the following diagram homotopy commutes

$$\begin{tikzcd}
P' \arrow[dr, dashed, "\omega"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\
& P \arrow[r, "v"] \arrow[d, "u"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

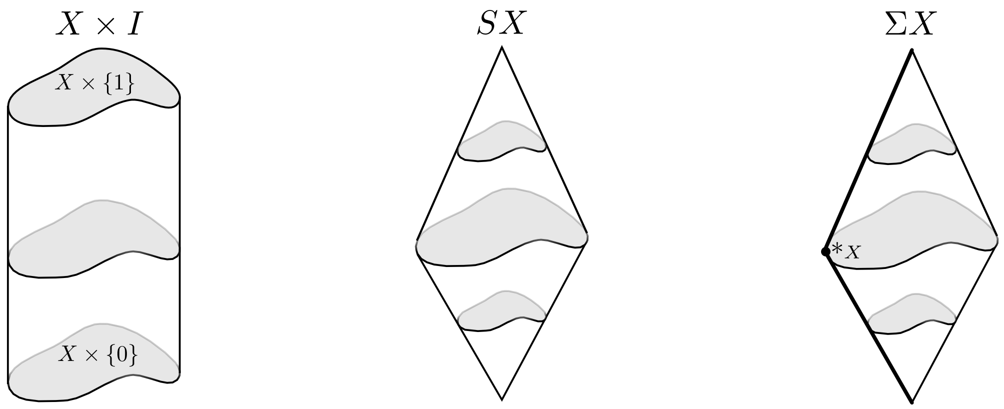
Suspensions and Loop Spaces

Definition 1.4.18 ((HATCHER, 2002))**.** The **suspension** of a topological space X, denoted by SX, is the quotient space $X \times I/\sim$, in which $\sim$ is the equivalence relation that identifies all points of $X \times \{0\}$ and $X \times \{1\}$. So we are basically collapsing the ends of the "cylinder" $X \times I$ to points (see figure 1).

Definition 1.4.19 ((HATCHER, 2002))**.** The **reduced suspension** ΣX of a based space $(X, *_X)$, is defined as the quotient of the suspension SX by an equivalence relation that collapses $*_X \times I$ to a single point.

Example 1.4.20. The suspension of the n-sphere S^n is homeomorphic to the $(n+1)$-sphere, $SS^n = S^{n+1}$.

Figure 1 – Representation of the suspension SX and the reduced suspension ΣX of a space X.



Source: Elaborated by the author.

Definition 1.4.21 ((HATCHER, 2002))**.** The loop space of a based topological space $(X, *_X)$ is the subspace $\Omega X \subset X^I$, given by

$$\Omega X = \{\gamma \in X^I \mid \gamma(0) = \gamma(1) = *_X\}.$$

NDR pairs

Definition 1.4.22 ((WHITEHEAD, 1978))**.** Let (X,A) be a pair of spaces, we say that (X,A) is an **NDR pair** if there exists a pair of continuous functions (u,h), with $u : X \to I$ and $h : X \times I \to X$, such that:

1. $A = u^{-1}(0)$.
2. h is a homotopy relative to A, starting at the identity map, i.e. $h(x,0) = x$, for $x \in X$.
3. If $x \in X$ and $u(x) < 1$ then $h(x,1) \in A$.

If condition (3) is replaced by $h(x,1) \in A$, for all $x \in X$, then (X,A) is called a **DR pair**.

Definition 1.4.23. A subset $A \subset X$ is called a **neighborhood retract** of X if there exists an open subset $U \subset X$, with $A \subset U$, such that A is a retract of U.

Notice that in the case of (X,A) being an NDR pair, we have, as a consequence of conditions 2 and 3, that A is a neighborhood retract of X, with the neighborhood being $U = u^{-1}([0,1))$, and the retraction is given by $h(x,1)$ with $x \in U$.

Theorem 1.4.24 ((WHITEHEAD, 1978))**.** Given a closed pair (X,A) the following statements are equivalent

1. (X,A) is an NDR pair.
2. $(I \times X, 0 \times X \cup I \times A)$ is a DR pair.
3. $0 \times X \cup I \times A$ is a retract of $I \times X$.
4. (X,A) is a cofibred pair.

Let us prove the above-stated theorem by proving some simpler lemmas.

Lemma 1.4.25. The pair $(I,\{0\})$ is a DR pair.

Proof. Simply define the function $u : I \to I$ as $u(x) = x/2$ and the homotopy $h(t,x) = (1-t)x$, it is not difficult to see that this pair of functions satisfy the conditions of definition 1.4.22 for DR pairs. □

Lemma 1.4.26. If both (X,A) and (Y,B) are NDR pairs, then so is $(X \times Y, X \times B \cup A \times Y)$. And if either (X,A) or (Y,B) is a DR pair, then the resulting pair is also DR.

Proof. Suppose (u,h) and (v,g) are pairs of functions as in definition 1.4.22 for the pairs (X,A) and (Y,B) respectively. We wish to show that $(X \times Y, X \times B \cup A \times Y)$ is an NDR pair. For this reason we define the function $w : X \times Y \to I$ by

$$w(x,y) = \min\{u(x), v(y)\},$$

and since $u^{-1}(0) = A$ and $v^{-1}(0) = B$ we clearly have that

1. $w^{-1}(0) = X \times B \cup A \times Y$.
2. If we define the map $f : X \times Y \times I \to X \times Y$ by

$$f(x,y,t) = \begin{cases} (x,y), \text{ if } v(x) = u(y) = 0 \text{ (i.e } (x,y) \in A \times B); \\ (h(x,t), g(y, tu(x)/v(y))), \text{ if } v(y) > 0 \text{ and } v(y) \geq u(x); \\ (h(x, tv(y)/u(x)), g(y,t)), \text{ if } u(x) > 0 \text{ and } u(x) \geq v(y); \end{cases}$$

we have that f is a homotopy (since it is continuous), and $f(x,y,0) = (x,y)$ for all $(x,y) \in X \times Y$. What is left to prove is that condition 2 from definition 1.4.22 is valid, i.e., that this homotopy is relative to $X \times B \cup A \times Y$. Take $(x,y) \in X \times B \cup A \times Y$. If $(x,y) \in A \times B$ we clearly have $f(x,y) = (x,y)$. If $(x,y) \in (X \setminus A) \cup B$ we fall in the third case of the definition of f, thus $f(x,y) = (h(x,0), g(y,t)) = (x,y)$. The last case is $(x,y) \in A \times (Y \setminus B)$ which is completely analogous to the previous case.

3. Notice that

$$f(x,y,1) = \begin{cases} (x,y), \text{ if } v(x) = u(y) = 0 \text{ (i.e } (x,y) \in A \times B); \\ (h(x,1), g(y, u(x)/v(y))), \text{ if } v(y) > 0 \text{ and } v(y) \geq u(x); \\ (h(x, v(y)/u(x)), g(y,1)), \text{ if } u(x) > 0 \text{ and } u(x) \geq v(y); \end{cases}$$

so if $w(x,y) < 1$, we have that $u(x) < 1$ or $v(y) < 1$, which implies that $h(x,1) \in A$ or $g(y,1) \in B$, hence from the definition of f we get that $f(x,y,1) \in X \times B \cup A \times Y$, whence concluding that $(X \times Y, X \times B \cup A \times Y)$ is an NDR pair.

Suppose that (X,A) is a DR pair, then all the prove above is still valid, and we have the additional information that $h(x,1) \in A$ for all $x \in X$, this clearly implies that $f(x,y,1) \in X \times B \cup A \times Y$, for all $(x,y) \in X \times Y$, hence $(X \times Y, X \times B \cup A \times Y)$ is a DR pair. □

Lemma 1.4.27. If (X,A) is a closed pair, then $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ if and only if (X,A) is a cofibred pair.

Proof. Suppose there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$. Let us show that (X,A) is a cofibred pair. First of all, consider arbitrary maps $f_0 : X \to Y$ and $g_t : A \to Y$, with $t \in I$, such that g_t is a homotopy and $g_0 = f_0|_A$. Remember that we can write this homotopy g_t as a function $G : A \times I \to Y$ defined as $G(a,t) = g_t(a)$. Then we can define a function $H : X \times \{0\} \cup A \times I \to Y$, which is basically the combination of f_0 and G, that is, $H(x,0) = f_0(x)$, for all $x \in X$, and $H(a,t) = G(a,t)$, for all $(a,t) \in A \times I$. The fact that A is closed in X implies that H is continuous, as shown below.

Notice first that $X \times \{0\}$ and $A \times I$ are both closed in $X \times I$, whence $X \times \{0\} \cup A \times I$ is also closed. Now, if $F \subset U$ is a closed subset, then $H^{-1}(F) = f_0^{-1}(F) \times \{0\} \cup G^{-1}(F)$, and from the continuity of f_0 we have that $f_0^{-1}(F) \times \{0\}$ is closed in $X \times \{0\}$, hence it is closed in $X \times I$. Similarly $G^{-1}(F)$ is closed in $A \times I$, therefore it must be closed in $X \times I$. Finally we conclude that $H^{-1}(F) = f_0^{-1}(F) \times \{0\} \cup G^{-1}(F)$ is closed in $X \times I$, in particular, it is closed in $X \times \{0\} \cup A \times I$, thus proving that H is continuous.

To finish this part of the proof, consider the homotopy $H \circ r : X \times I \to Y$, which shows that (X,A) is a cofibred pair.

Conversely, suppose that (X,A) is a cofibred pair, then define $Y \doteq X \times \{0\} \cup A \times I$, $f_0 : X \to Y$ given by $f_0(x) = (x,0)$, and $g_t : A \to Y$ given by $g_t(a) = (a,t)$. By hypothesis there exists a function $F : X \times I \to Y$, such that $F(x,0) = (x,0)$ and $F(a,t) = (a,t)$, for all $x \in X$, $a \in A$ and $t \in I$. Hence F is a retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$.

□

Proof of theorem 1.4.24. (1) $\implies$ (2): Since (X,A) is NDR and $(I,\{0\})$ is DR (Lemma 1.4.25), we have by lemma 1.4.26 that $(X\times I, X\times\{0\}\cup A\times I)$ is a DR pair.

(2) $\implies$ (3): This is always true, if any pair (X,A) is a DR pair then A is a retract of X, since there is a continuous function $h: X\times I\to X$ with $h(X\times\{1\})\subset A$ and $h(x,1)=x$, for all $x\in X$.

(3) $\iff$ (4): Lemma 1.4.27

(3) $\implies$ (1): Suppose we have the retraction

$$r: X\times I\to X\times\{0\}\cup A\times I.$$

Let $\pi_1: X\times I\to X$ and $\pi_2: X\times I\to I$ be the projections onto the first and second coordinates respectively. Now, we can define the pair of functions (u,h), which shall satisfy the NDR conditions. First, define $u: X\to I$ as

$$u(x)=\sup\{t-\pi_2\circ r(x,t)|t\in I\},$$

and $h: X\times I\to X$ as $h(x,t)=\pi_1\circ r(x,t)$. We clearly have that $u(x)\geq 0$, since for $t=0$ we have $t-\pi_2\circ r(x,t)=0$. Now we proceed to check the three NDR conditions.

(i) $A=u^{-1}(0)$: If $a\in A$ we have $t-\pi_2\circ r(a,t)=t-\pi_2(a,t)=t-t=0$ for all $t\in I$, therefore $u(a)=0$. Conversely if $x\in X$ is such that $u(x)=0$, then $t-\pi_2\circ r(x,t)=0$ for all $t\in I$, in particular, for $t>0$, hence $\pi_2\circ r(x,t)=t>0$, if $t>0$, and since the codomain of r is $X\times\{0\}\cup A\times I$, we conclude that $r(x,t)\in A\times I$ for $t>0$, and the fact that A is closed implies that $r(x,0)=(x,0)\in A\times I$, hence $x\in A$.

(ii) h is clearly continuous, since it is the combination of continuous functions, so we have

$$h(x,0)=\pi_1\circ r(x,0)=\pi_1(x,0)=x$$

and

$$h(a,t)=\pi_1\circ r(a,t)=\pi_1(a,t)=a,$$

for all $x\in X$, $a\in A$, and $t\in I$.

(iii) $u(x)<1 \implies \pi_2\circ r(x,1)>0 \implies r(x,1)\in A\times I \implies h(x,1)=\pi_1\circ r(x,1)\in A$.

Hence (X,A) is an NDR pair. ∎

Proposition 1.4.28. The pair $(I,\{0,1\})$ is an NDR pair.

Proof. We define $u: I\to I$ by

$$u(t)=\begin{cases}4t & ,\text{if } 0\leq t\leq 1/4;\\ 1 & ,\text{if } 1/4\leq t\leq 3/4;\\ 4(1-t) & ,\text{if } 3/4\leq t\leq 1;\end{cases}$$

and $h: I \times I \to I$ by

$$h(x,t) = \begin{cases} t(x-1/2)+x & \text{,if } t/(2+2t) \le x \le (2+t)/(2+2t); \\ 0 & \text{,if } x \le t/(2+2t); \\ 1 & \text{,if } x \ge (2+t)/(2+2t); \end{cases}$$

It is not difficult to see that both u and h are continuous, and that they satisfy the conditions of definition 1.4.22. □

As a consequence of the previous proposition and lemma 1.4.26, we get the following corollary.

Corollary 1.4.29. If (X,A) is an NDR pair, then $(X \times I, X \times \{0,1\} \cup A \times I)$ is also an NDR pair.

With the previous corollary, we can prove the following theorem.

Theorem 1.4.30. If the inclusion $i: A \hookrightarrow X$ is both a homotopy equivalence and a cofibration, then X strongly deformation retract onto A.

Proof. Let $r': X \to A$ be the homotopy inverse of the inclusion $i: A \hookrightarrow X$, i.e., $ir' \simeq id_X$ and $r'i \simeq id_A$. Let $F': A \times I \to A$ be a homotopy such that $F'|_{A\times\{0\}} = r'i$ and $F'|_{A\times\{1\}} = id_A$. Since F' is exactly r' when restricted to $A \times \{0\}$, and $A \hookrightarrow X$ is a cofibration, we have, by the homotopy extension property, that there is a homotopy $G': X \times I \to X$, such that $G'|_{X\times\{0\}} = r'$ and $G'|_{A\times I} = F'$. Thus if we define $r: X \to A$ by $r(x) = G'(x,1)$, we get $ri(a) = G'(a,1) = F'(a,1) = id_A(a) = a$, for all $a \in A$, and G' is a homotopy between $r \simeq r'$, hence $ir \simeq ir' \simeq id_X$.

Now let $F: X \times I \to X$ be a homotopy between id_X and ir, that is, $F(x,0) = x$ and $F(x,1) = ir(x)$, for all $x \in X$. Define $G: ((X \times \{0,1\}) \cup (A \times I)) \times I \to X$ by

$$\begin{aligned} G(x,0,t) &= x, \\ G(x,1,t) &= F(r(x), 1-t), \\ G(a,s,t) &= F(a,(1-t)s), \end{aligned}$$

for all $x \in X$, $a \in A$ and $s,t \in I$. It is not hard to see that G is well defined and continuous.

By corollary 1.4.29, we know that $A \hookrightarrow X$ being a cofibration implies that $X \times \{0,1\} \cup A \times I \hookrightarrow X \times I$ is a cofibration. Since the map G restricted to $(X \times \{0,1\} \cup A \times I) \times \{0\}$ is equal to F, we can apply the homotopy extension property and obtain a map

$$\tilde{G}: X \times I \times I \to X,$$

with $\tilde{G}(x,s,0) = F(x,s)$ and $\tilde{G}|_{((X\times\{0,1\})\cup(A\times I)\times I)} = G$.

Now we can finally define the strong deformation retraction $H : X \times I \to X$ by

$$H(x,s) = \tilde{G}(x,t,1),$$

then we have the following equations

$$\begin{aligned} H(x,0) &= \tilde{G}(x,0,1) = G(x,0,1) = x, \\ H(x,1) &= \tilde{G}(x,1,1) = G(x,1,1) = F(r(x),0) = r(x) \in A, \\ H(a,t) &= \tilde{G}(a,t,1) = G(a,t,1) = F(a,0) = a, \end{aligned}$$

for all $x \in X$, $a \in A$ and $t \in I$, whence H is a strong deformation retraction of X into A. □

The Strøm Structure

The Strøm Structure is very similar to the NDR pairs previously introduced, a close relationship will be shown in a theorem later on.

Definition 1.4.31. By a **Strøm Structure** on a pair (X,A), we mean a pair of maps (α,h), with $\alpha : X \to I$ and $h : X \times I \to X$, such that:

(i) $\alpha(A) = 0$.

(ii) h is a homotopy relative to A, starting at the identity map, and with $h(x,t) \in A$, when $t > \alpha(x)$.

If A is closed, then condition (ii) implies that $A = \alpha^{-1}(0)$.

Theorem 1.4.32. If (X,A) is a closed pair, then (X,A) is cofibred if and only if there exists a Strøm Structure on (X,A).

We will postpone the proof of this theorem to chapter 4, in which we shall prove it in the fibrewise case.

Notice that theorems 1.4.32 and 1.4.24 imply that in the case of closed pairs (X,A), the concepts of NDR and Strøm Structure are equivalent.

1.5 Dimension Theory

Definition 1.5.1 ((MUNKRES, 2000))**.** Let $\mathscr{A}$ be a family of subsets of X, the **order** of $\mathscr{A}$ is the integer n, such that there is some $x \in X$ contained in n elements of $\mathscr{A}$, but there is no $y \in X$ contained in more than n of those elements. If no such integer exists we say the order is ∞.

Definition 1.5.2 ((MUNKRES, 2000))**.** The **dimension** of a topological space X is the smallest positive integer m such that for any open covering $\mathscr{A}$ of X there exists an open refinement $\mathscr{B}$ of order $m+1$, in this case we denote $\dim(X) = m$. The dimension of the empty set is defined to be -1, and if $\dim(X) > n$, for all $n \in \mathbb{N}$, then we write $\dim(X) = \infty$.

In the literature, this dimension may be referred to as **covering dimension**, or **Lebesgue covering dimension**.

Example 1.5.3. Any discrete topological space X has dimension zero, since $\{\{x\}\}_{x\in X}$ is an open refinement of any open covering of X, and it has order 1.

Theorem 1.5.4 ((PEARS, 1975), Theorem 2.7)**.** For the n-dimensional euclidean space we have $\dim(\mathbb{R}^n) = n$.

As one would expect, the dimension of a topological n-manifold is precisely n, in other words, it coincides with the usual notion of dimension in a manifold. It is not a triviality to prove this fact, one possible direction is to first define the inductive dimensions as follows.

Definition 1.5.5 ((PEARS, 1975))**.** The **small inductive dimension** of a space X, denoted by $\mathrm{ind}(X) \in \mathbb{N} \cup \{-1, \infty\}$, is defined inductively in the following manner.

1. $\mathrm{ind}(X) = -1 \iff X = \emptyset$.
2. for $n \geq 0$, $\mathrm{ind}(X) \leq n$ if for every $x \in X$ and an open neighborhood V_x of x, there is an open neighborhood U_x of x, with $U_x \subset V_x$ such that $\mathrm{ind}(\partial U_x) \leq n-1$.
3. if $\mathrm{ind}(X) \leq n$ and $\mathrm{ind}(X) \nleq n-1$, then $\mathrm{ind}(X) = n$.
4. if $X \neq \emptyset$ and $\mathrm{ind}(X) \nleq n$ for all $n \in \mathbb{N}$, then $\mathrm{ind}(X) = \infty$.

Definition 1.5.6 ((PEARS, 1975))**.** The **large inductive dimension** of a space X, denoted by $\mathrm{Ind}(X) \in \mathbb{N} \cup \{-1, \infty\}$, is defined inductively in the following manner.

1. $\mathrm{Ind}(X) = -1 \iff X = \emptyset$.
2. for $n \geq 0$, $\mathrm{Ind}(X) \leq n$ if for every pair of closed subset $A \subset X$ and open subset $B \subset X$, such that $A \subset B$, there exists an open set U, such that $A \subset U \subset B$ and $\mathrm{Ind}(\partial U) \leq n-1$.
3. if $\mathrm{Ind}(X) \leq n$ and $\mathrm{Ind}(X) \nleq n-1$, then $\mathrm{Ind}(X) = n$.
4. if $X \neq \emptyset$ and $\mathrm{Ind}(X) \nleq n$, for all $n \in \mathbb{N}$, then $\mathrm{Ind}(X) = \infty$.

Notice that the only difference between definitions 1.5.5 and 1.5.6 is the second item, while in the first we consider points of X and its neighborhoods, in the second definition we change the points with closed sets. Therefore if X is a Hausdorff space, or more generally a T_1 space, we have $\mathrm{ind}(X) \leq \mathrm{Ind}(X)$.

Proposition 1.5.7. Let X be a topological space, if $A \subset X$, then $\mathrm{ind}(A) \leq \mathrm{ind}(X)$.

Theorem 1.5.8 ((NAGATA, 1981) Theorem II.7)**.** *If X is a metrizable space, then $\mathrm{Ind}(X) = \dim(X)$.*

Theorem 1.5.9 ((NAGATA, 1981) Theorem IV.1)**.** *If X is a metrizable, separable space, then $\mathrm{Ind}(X) = \mathrm{ind}(X) = \dim(X)$. In particular, $\mathrm{Ind}(\mathbb{R}^n) = \mathrm{ind}(\mathbb{R}^n) = \dim(\mathbb{R}^n) = n$.*

This last theorem, together with theorem 1.2.10, implies that that for M^n a topological n-manifold $\dim(M^n) = \mathrm{ind}(M^n)$. Now we can show that $\mathrm{ind}(M^n) = n$, and finally, we obtain the result $\dim(M^n) = n$.

Theorem 1.5.10. *If M^n is a topological n-manifold, then $\mathrm{ind}(M^n) = n$.*

Proof. We prove this by induction on n.

n=1: If M^1 is a 1-manifold, take $x \in M^1$ and U a neighborhood of x, then there is another neighborhood V, such that $V \subset U$, and $\phi : V \to A$ is a homeomorphism, with $A \subset \mathbb{R}$ open. Now, let B be an open ball (i.e., an interval) in $\mathbb{R}$ such that $\phi(x) \in B \subset \overline{B} \subset A$. Notice that $x \in \phi^{-1}(B) \subset U$ and $\partial\phi^{-1}(B) \cong \partial B$, which consists of two isolated points, therefore $\mathrm{ind}(\partial\phi^{-1}(B)) = 0$. Also, for any open neighborhood W of x with $W \subset \phi^{-1}(B)$, we have $\mathrm{ind}(\partial W) = 0$, hence $\mathrm{ind}(M^1) \leq 1$ and $\mathrm{ind}(M^1) \nleq 0$, therefore $\mathrm{ind}(M^1) = 1$.

Induction hypothesis: Suppose $\mathrm{ind}(M^k) = k$ for $k = 1, \ldots, n-1$.

Let M^n be a topological n-manifold, take $x \in M^n$ and U a neighborhood of x, then there is another neighborhood V, such that $V \subset U$, and $\phi : V \to A$ is a homeomorphism, with $A \subset \mathbb{R}^n$ open. Let B be an open ball in $\mathbb{R}^n$ such that $\phi(x) \in B \subset \overline{B} \subset A$. We have $x \in \phi^{-1}(B) \subset U$, and $\partial\phi^{-1}(B) \cong \partial B \cong S^{n-1}$, which is a topological $(n-1)$-manifold, from the induction hypothesis we get $\mathrm{ind}(\partial\phi^{-1}(B)) = n-1$. So we know that $\mathrm{ind}(M^n) \leq n$. Notice that $\phi^{-1}(B) \cong B \cong \mathbb{R}^n$, therefore $\mathrm{ind}(\phi^{-1}(B)) = \mathrm{ind}(\mathbb{R}^n) = n$, and from proposition 1.5.7 we have $n = \mathrm{ind}(\phi^{-1}(B)) \leq \mathrm{ind}(M^n) \leq n$, whence we have $\mathrm{ind}(M^n) = n$. ☐

Corollary 1.5.11. *If M^n is a topological n-manifold, then $\dim(M^n) = n = \mathrm{Ind}(M^n)$.*

1.6 CW-complexes

Introduction

In this section, we will define the important class of topological spaces called CW-complexes, which plays an important role in homotopy theory. Before defining these spaces, which have an inductive definition, let us introduce some useful concepts.

Definition 1.6.1 ((MUNKRES, 1984), Chapter 1, Section 2, pg. 10)**.** Given a topological space X and a family of subspaces of $\{A_\alpha\}_\alpha$ of X, the topology of X is said to be **coherent** with the family $\{A_\alpha\}_\alpha$ if U is open in X if and only if $U \cap A_\alpha$ is open in each

A_α (equivalently we can change open with closed). When this happens, we say that X has the **coherent topology** in relation to $\{A_\alpha\}_\alpha$.

Theorem 1.6.2. Given a topological space X and a family of subspaces $\{A_\alpha\}_\alpha$, the following statements are equivalent.

(i) X has the coherent topology in relations to $\{A_\alpha\}_\alpha$.

(ii) X has the finest topology such that all the inclusions $i_\alpha : A_\alpha \to X$ are continuous.

(iii) A function $f : X \to Y$ is continuous if and only if $fi_\alpha : A_\alpha \to Y$ is continuous, for all α.

Proof. It is easy to see that (i) and (ii) are equivalent by simply noticing that if $U \subset X$, then $i_\alpha^{-1}(U) = U \cap A_\alpha$. Let us show that (i) implies (iii), suppose (i) is true, clearly f continuous implies that the restrictions fi_α are all continuous, conversely if all fi_α are continuous, let $V \subset Y$ be open, then $(fi_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha$ are all open, and by te definition of coherent topology $f^{-1}(V)$ is open in X, hence f is continuous. Finally, to show that (iii) implies (ii), suppose (iii) is true, then the inclusions $i_\alpha : A_\alpha \to X$ are all continuous, since $id : X \to X$ is obviously continuous, it remains to show that X has the finest topology with this property. Let X' be equal to X as a set, equipped with the coherent topology in relations to $\{X_\alpha\}_\alpha$, we want to show that $X = X'$. Notice that, since (i) and (ii) are equivalent, the inclusions $X_\alpha \hookrightarrow X'$ are continuous, hence by (iii) the identity map (as sets) $X \to X'$ is continuous, but since the topology of X' is finner than the topology of X, this can only mean that $X = X'$. $\square$

Definition 1.6.3 ((ARKOWITZ, 2011))**.** Given two topological spaces X and Y, define the **disjoint union** $X \sqcup Y$ to be the topological space which as a set is the usual disjoint union, and is equipped with the coherent topology in relation to $\{X, Y\}$.

Notice that by defining the coherent topology on the disjoint union, as above, we have that both inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ are continuous, it is the largest topology with this property.

Analogously, for an arbitrary family of spaces $\{X_\alpha\}_\alpha$ we define the disjoint union $\bigsqcup_\alpha X_\alpha$, in which $A \subset \bigsqcup_\alpha X_\alpha$ is an open subset if and only if each $A \cap X_\alpha$ is open in X_α. This will also be the largest topology such that each inclusion $X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ is continuous.

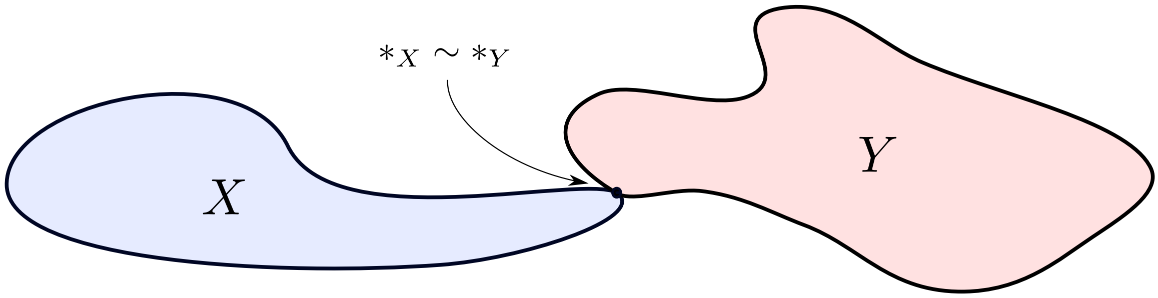
Remember that if X is a topological space and $p : X \to S$ is a surjection onto a set S, then we can define the **quotient topology** on S as follows: $U \in S$ is open $\iff$ $p^{-1}(U)$ is open in X. This is particularly useful when we have a space X with a certain equivalence relation $\sim$. Then we can equip the quotient $X/\sim$ with the quotient topology for the canonical projection $\pi : X \to X/\sim$. In all the remainder of this work, we shall assume that this is the topology of quotient spaces, unless otherwise stated.

If in a space X we fix a certain point $*_X$, then we say that $(X,*_X)$ is a **based space** with **basepoint** $*_X$.

Definition 1.6.4. Given X and Y two based spaces we define their **wedge product** (or wedge sum), $X \vee Y$, to be $X \sqcup Y/\sim$, in which every point is equivalent only to itself, except for the basepoints $*_X$ and $*_Y$ which are also equivalent to each other.

There is an intuitive way to understand the construction above, we are taking two spaces X and Y and gluing one point of X to one point of Y as depicted in figure 2.

Figure 2 – The wedge of two based spaces.



Source: Elaborated by the author.

Notice that once we take the wedge of X and Y we still have a based space $X \vee Y$ with basepoint $[*_X] = [*_Y]$.

One can easily extend the definition of wedge to an arbitrary collection $\{X_\alpha\}_\alpha$ of based spaces. The wedge of such a collection, denoted by $\bigvee_\alpha X_\alpha$, is the disjoint union quotiented by the equivalence relation which associates all the basepoints. This new space has basepoint $[*_{X_\alpha}]$ for any α in the family of indices.

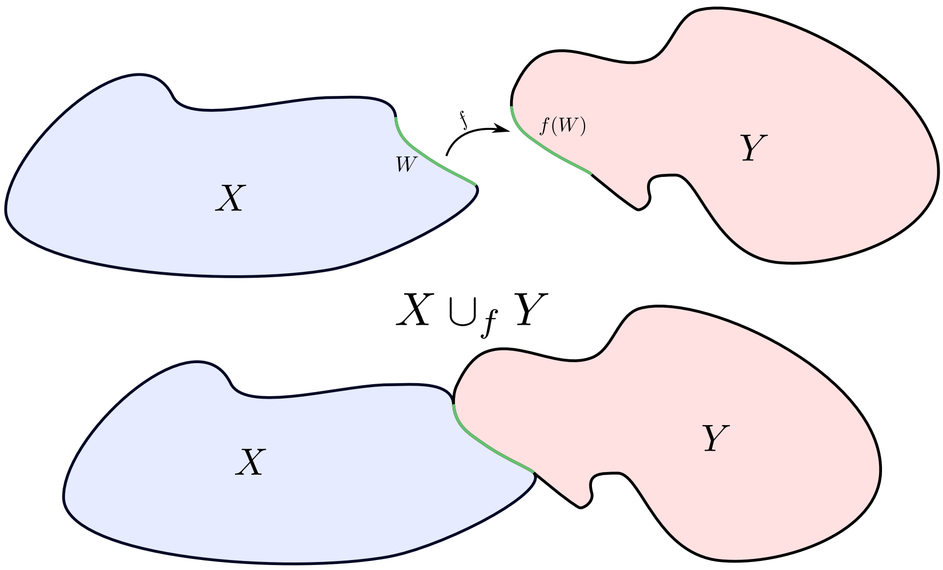
Definition 1.6.5. Given two topological spaces X and Y, a subset $W \subset X$, and a continuous function $f : W \to Y$, we define the **adjunction space** of X and Y with respect to f, $X \cup_f Y$, to be the quotient space $X \sqcup Y/\sim$ in which the equivalence relation is given by $w \sim f(w)$.

As seen in figure 3, in an adjunction space we are basically gluing the subspace W to its image $f(W)$, thereby connecting X and Y.

In what follows we shall call an **open n-cell** any topological space homeomorphic to the open n-disc, $D^n = \{x \in \mathbb{R}^n \ : \ \|x\| < 1\}$, and a **closed n-cell** any space homeomorphic to the closed n-disc $\overline{D}^n$. If E^n is a closed n-cell then there is a homeomorphism $\varphi : \overline{D}^n \to E^n$, and we define its boundary ∂E^n to be the image $\varphi(S^{n-1})$.

Definition 1.6.6. Suppose X is a topological spaces, $\{E^n_\alpha\}_\alpha$ is an indexed family of n-cells and $\phi_\alpha : \partial E^n_\alpha \to X$ are continuous functions from the boundary of E_α to X for each

Figure 3 – Pictorial representation of an adjunction space.



Source: Elaborated by the author.

index α. Then we can define a function $\phi : \bigsqcup_\alpha \partial E^n_\alpha \to X$, in which $\bigsqcup_\alpha \partial E^n_\alpha \subset \bigsqcup_\alpha E^n_\alpha$ and $\phi|_{\partial E^n_\alpha} = \phi_\alpha$. And we define the adjunction space $X \cup_\phi \bigsqcup_\alpha E^n_\alpha$ which is called: "the space obtained from X by **attaching n-cells** through the functions ϕ_α (see figure 4).

CW complexes

Definition 1.6.7 ((ARKOWITZ, 2011))**.** Consider a topological space X.

1. if $X = \emptyset$, then it is a **CW complex**.
2. if $X \neq \emptyset$, then this will be a **CW complex** if it is a Hausdorff space together with a sequence of subspaces called **skeleta**

$$X^0 \subset X^1 \subset X^2 \subset \cdots,$$

whose union is X, such that the following conditions are satisfied.

a) X^0 is a nonempty set of points. And we proceed inductively, so that X^1 is obtained from X^0 by attaching 1-cells, X^2 is obtained from X^1 by attaching 2-cells, and so on.

Before stating the second condition, let us clarify some notation. From (a) we know that X^n is obtained from X^{n-1} by attaching n-cells. To write more explicitly, there exists a set $\{E^n_\alpha\}_\alpha$ of closed n-cells and maps ϕ^n_α which induce a map $\phi : \bigsqcup_\alpha \partial E^n_\alpha \to X^{n-1}$ such that

$$X^n = X^{n-1} \cup_\phi \bigsqcup_\alpha E^n_\alpha.$$

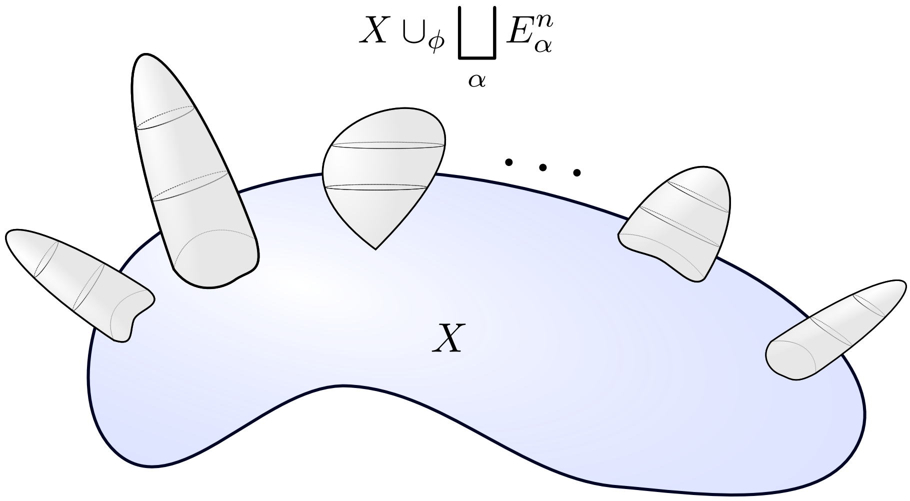
Now, we can define a map from E^n_α to X^n by composing the following inclusion and projection

$$E^n_\alpha \xhookrightarrow{i} X^{n-1} \sqcup (\textstyle\bigsqcup_\alpha E^n_\alpha) \xrightarrow{p} X^n = X^{n-1} \cup_\phi \textstyle\bigsqcup_\alpha E^n_\alpha \ ,$$

this composition we shall denote by Φ^n_α, and the subset $\Phi^n_\alpha(E^n_\alpha) \subset X$ is called an **closed n-cell** of X and is denoted $\bar{e}^n_a$.

b) X has the coherent topology with respect to the set of all closed cells $\{\bar{e}^n_\alpha\}^n_\alpha$. This means $A \subset X$ is open $\iff$ $A \cap \bar{e}^n_\alpha$ is open for all cells $\bar{e}^n_\alpha$.

Figure 4 – A rough representations of the space obtained from X by attaching n-cells



Source: Elaborated by the author.

Notations and definitions regarding CW complexes.

- The maps $\phi^n_\alpha : \partial E^n_\alpha \to X^{n-1}$ are called **attaching maps**.
- The maps $\Phi^n_\alpha : E^n_\alpha \to X^n$ are called **characteristic maps**, notice that $\Phi^n_\alpha|_{\partial E^n} = \phi^n_\alpha$.
- The image $\Phi(E^n_\alpha \setminus \partial E^n_\alpha)$ is called an **open n-cell** and is denoted e^n_α.
- X is said to be a **finite CW complex** if it has finitely many closed (or open) cells.
- X is said to be a **finite dimensional CW complex** if for some N we have that X is equal to its N-skeleton $X = X^N$. The smallest N satisfying this condition is called the dimension of X.
- X is said to be a **locally finite CW complex** if the set of open (or closed) n-cells is locally finite.

Remark: An open n-cell e^n_α need not be open in X. The reason for it being called "open", is the fact that it is homeomorphic to the open n-disk D^n, since we have the homeomorphism $\Phi^n_\alpha|_{E^n_\alpha \setminus \partial E^n_\alpha} : E^n_\alpha \setminus \partial E^n_\alpha \to e^n_\alpha$.

Proposition 1.6.8. Given X a CW complex and an open and respective closed n-cells e^n_α and $\overline{e}^n_\alpha$, we have that the closure of e^n_α in X is $\overline{e}^n_\alpha$ (justifying the notation).

Proof. Denote the closure of e^n_α by $Cl(e^n_\alpha)$. Note that $\overline{e}^n_\alpha$ is compact, since $\overline{e}^n_\alpha = \Phi(E^n_\alpha)$. Given that X is Hausdorff, if $y \in X \setminus \overline{e}^n_\alpha$, there must be disjoint open sets separating y and $\overline{e}^n_\alpha$, hence $y \notin Cl(e^n_\alpha)$, and we conclude that $Cl(e^n_\alpha) \subset \overline{e}^n_\alpha$. Now, notice that if an open subset $U \subset X$ intersects $\overline{e}^n_\alpha$, then it also intersects e^n_α, since $(\Phi^n_\alpha)^{-1}(U)$ is open in E^n_α. Hence $\overline{e}^n_\alpha = Cl(e^n_\alpha)$. □

Lemma 1.6.9 ((ARKOWITZ, 2011) Lemma 1.5.6)**.** Given X a CW complex we have.

1. X has the coherent topology with respect to the set of skeleta $\{X_n\}_n$.
2. If $K \subset X$ is a compact subset, then K intersects only finitely many open (or closed) cells of X.

Since a closed n-cell $\overline{e}^n_\alpha$ is the image of the compact space E^n_α through the continuous map $\Phi^n_\alpha : E^n_\alpha \to X^n$, it is itself a compact subspace of X. Whence by lemma 1.6.9 we have that $\overline{e}^n_\alpha$ intersects finitely many open cells of X.

The name CW complex comes from the following properties.

(C) Closure-finiteness: The closure of each cell intersects finitely many cells.

(W) Weak topology: The topology on a CW complex is the coherent topology with respect to its cells (many authors use the term weak topology instead of coherent topology, but we prefer the latter one to avoid confusion with the functional analysis concept of weak topology).

Proposition 1.6.10 ((LEE, 2011) proposition 5.4)**.** For X a locally finite CW complex we do not need to check for condition (b) in definition 1.6.7 (the coherent topology condition), it is a consequence of the rest of the definition.

Lemma 1.6.11 ((ARKOWITZ, 2011) Lemma 1.5.7)**.** Given X a CW complex, Y a topological space and $f : X \to Y$ a function, the following statements are equivalent.

1. f is continuous.
2. $f|_{\overline{e}^n_\alpha} : \overline{e}^n_\alpha \to Y$ is continuous for all closed cells of X.
3. $f \circ \Phi^n_\alpha : E^n_\alpha \to Y$ is continuous for all characteristic maps Φ^n_α.

4. $f|_{X^n} : X^n \to Y$ is continuous for all the skeleta of X.

Proposition 1.6.12. Any CW complex is:

1. a normal space [(HATCHER, 2002), Proposition A3];
2. locally contractible [(HATCHER, 2002), Proposition A4];
3. paracompact [(LEE, 2011), Theorem 5.22];
4. completely normal [(LUNDELL; WEINGRAM, 1969), Propostition 4.3] - Lundell and Weingram actually prove that any CW complex is perfectly normal, which is an even stronger condition than completely normal.

Given X a topological space we can always consider a family of subsets $\mathscr{F}$ of X, and generate a topological space given by X with the coherent topology in relation to the family $\mathscr{F}$. One particularly interesting case is when $\mathscr{F}$ is the family of all compact subsets of X, in which we denote the new space by X_C. Notice that X and X_C have the same compact subsets. In general, a space that has the coherent topology with respect to its compact subspaces is called a **Compactly Generated Space**.

Remark 1.6.13. Some authors, for example, (WHITEHEAD, 1978), use the nomenclature compactly generated space only for Hausdorff spaces with the above-mentioned property. This is particularly interesting when developing homotopy theory on a fixed category.

Proposition 1.6.14. The following are compactly generated spaces.

1. CW complexes.
2. Metric spaces.
3. Locally compact spaces.

Proof. 1. Suppose X is a CW complex, if $F \subset X$ is a subset whose intersection with each compact subspace of X is closed, then its intersection with all closed cells of X are closed (since a closed cell is compact), hence F is closed in X. Whence X is compactly generated.

2. Suppose X is a metric space. If $A \subset X$ is not open in X, then there exists a non-interior point $x \in A$. Thus there is a sequence $(x_n)_n$ in $X \setminus A$ converging to x. The set $S \doteq \{x_n\}_n \cup \{x\}$ is a compact subset of X whose intersection with A is not open in S. Therefore X is compactly generated.

3. Suppose X is locally compact. If $A \subset X$ is not open in X, then there exists a non-interior point $x \in A$. Let K be a compact neighborhood of x ($x \in int(K)$), then $A \cap K$

cannot be open in K, otherwise, we would have $A \cap K = U \cap K$, for U some open subset of X, and so $U \cap int(K)$ would be an open subset of X, containing x and contained in A, contradicting the hypothesis that x is not an interior point of A. Hence X is compactly generated. □

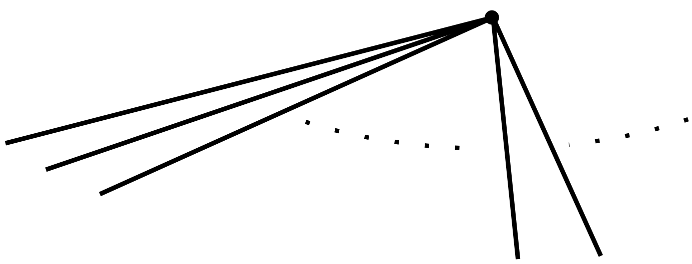
Given X and Y two CW complexes, one might be interested in taking the product $X \times Y$. The immediate question that arises is whether this product is itself a CW complex. Unfortunately, the answer is not always positive, nonetheless, we can define a topology pretty naturally on $X \times Y$ which makes it a CW complex, which will be exactly the compactly generated space $(X \times Y)_C$, more explicitly we have the following theorem.

Theorem 1.6.15 ((HATCHER, 2002), Theorem A.6)**.** For CW complexes X and Y we can define a CW complex structure on $X \times Y$ by considering its closed cells to be $\overline{e}^n_\alpha \times \overline{f}^m_\beta$, in which $\overline{e}^n_\alpha$ and $\overline{f}^m_\beta$ are the closed cells of X and Y, respectively. The characteristic maps are defined as $\Phi^n_\alpha \times \Psi^m_\beta$, in which Φ^n_α and Ψ^m_β are the characteristic maps of $\overline{e}^n_\alpha$ and $\overline{f}^m_\beta$, respectively. Then this CW complex is exactly $(X \times Y)_C$. If either X or Y is locally compact, then $(X \times Y)_C = X \times Y$. If both X and Y have countably many cells, then $(X \times Y)_C = X \times Y$.

Example 1.6.16. Here we discuss a case in which $(X \times Y)_C \neq X \times Y$. This example was constructed in (DOWKER, 1952).

Consider X and Y CW complexes of dimension 1, constructed in the following manner. X has uncountably many 1 cells, all with one common point (0 cell), more explicitly $X = \bigvee_s I_s$, in which s runs through all the sequences of positive integers $s = (s_1, s_2, \dots)$ and I_s is the unit interval $[0,1]$ with base point 0. Y is defined similarly, $Y = \bigvee_j I_j$, with the indexes j running through the positive integers.

Figure 5 – Representation of the spaces X and Y from example 1.6.16.



Source: Elaborated by the author.

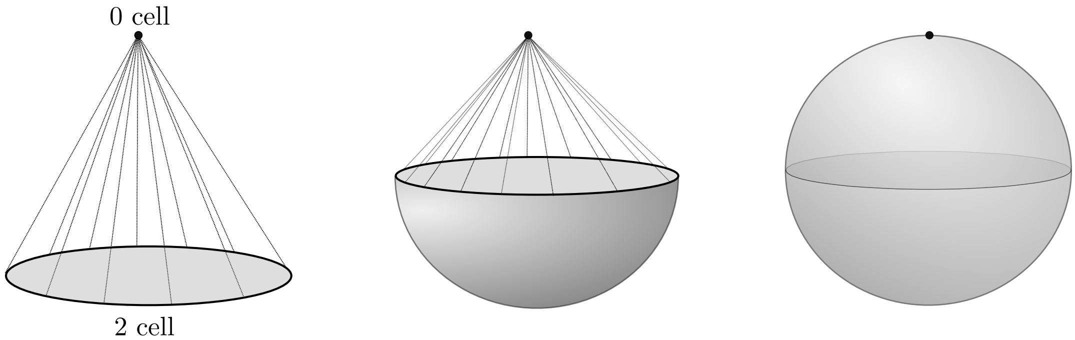
Given an index $s = (s_1, s_2, \dots)$ and $j \in \mathbb{N}$ consider the point $p_{sj} \doteq (1/s_j, 1/s_j)$ in $I_s \times I_j \subset X \times Y$, and define $P \subset X \times Y$ to be the subset containing all the points p_{sj}. Notice that the intersection of P with the 0-cells and 1-cells of $(X \times Y)_C$ is empty, and with any 2-cell $I_s \times I_j$ it is exactly one point p_{sj}, hence they are all closed subsets of the cells, thus, by theorem 1.6.15, P is a closed subset of $(X \times Y)_C$.

Let us show that P is not closed in $X \times Y$ with the usual product topology. To prove this, we will show that $(0,0) \in P$, in which 0 is the point attaching all the unit intervals in each CW complex. An open neighborhood of 0 in X is of the form $U \doteq \bigvee_s [0, a_s]$, and similarly a neighborhood of 0 in Y is of the form $V \doteq \bigvee_j [0, b_j]$, and we consider the open neighborhood of $(0,0)$ given by $U \times V$. Now we wish to show that the intersection of P and $U \times V$ is not empty. Choose an index $s = (s_1, s_2, \dots)$ with $s_j > \max\{j, 1/b_j\}$ and choose an integer $k > 1/a_s$. Then $s_k > k > 1/a_s$, which implies $1/s_k < a_s$. Since $1/s_k < b_k$, we have $p_{sk} = (1/s_k, 1/s_k) \in [0, a_s] \times [0, b_j] \subset U \times V$. Therefore P is not closed, and we conclude $(X \times Y)_C \neq X \times Y$.

Example 1.6.17. There is a particular interest in the product of a CW complex X with the unit interval $I = [0,1]$, which is also a CW complex with two 0 cells $\{0\}$ and $\{1\}$, and a single closed 1-cell $[0,1]$. Since I is compact, by theorem 1.6.15, we have that $(X \times I)_C = X \times I$.

Example 1.6.18. Every n-sphere S^n has the structure of a CW complex, which has a single 0 cell and an n cell, with attaching map $\phi : \partial E^n \to *$ the constant map.

Figure 6 – Representation of the CW structure of a 2-sphere. The boundary of a 2-disk is identified with a single point outside of it, and the space we get is homeomorphic to the 2-sphere.



Source: Elaborated by the author.

Some other classical examples of CW complexes are $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, compact connected surfaces and simplicial complexes.

Definition 1.6.19 ((HATCHER, 2002))**.** If X and Y are CW complexes, then $f : X \to Y$ is said to be a **cellular map**, if for any integer, k, the k-skeleton of X is mapped into the k-skeleton of Y, i.e., $f(X^k) \subset Y^k$

Theorem 1.6.20 (**Cellular approximation theorem**. (HATCHER, 2002) theorem 4.8)**.** Any map $f : X \to Y$ between CW complexes is homotopic to a cellular map.

Δ-*complexes and simplicial complexes*

Definition 1.6.21 ((VICK, 2012))**.** The **convex hull** of a set $A \subset \mathbb{R}^n$ is the smallest convex subset of $\mathbb{R}^n$ containing A.

Definition 1.6.22 ((MUNKRES, 1984))**.** A finite set of points $\{x_0, x_1, \ldots, x_k\} \subset \mathbb{R}^n$ is said to be **geometrically independent**, if the set of vectors $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent.

Definition 1.6.23 ((HATCHER, 2002), Chapter 2, Section 2.1, page 103)**.** An **n-simplex** is the convex hull of a geometrically independent set $\{x_0, \ldots, x_n\}$ in some $\mathbb{R}^m$, with $m \geq n$. The points $x_0, \ldots, x_n$ are called the **vertices** of the n-simplex.

We usually consider the vertices of an n-simplex to have an ordering and we write $[x_0, x_1, \ldots, x_n]$, by which we mean that this is an n-simplex with vertices $x_0, \ldots, x_n$, and those vertices follow that specified order, if we switched, for example, x_0 and x_1, we would get $[x_1, x_0, x_2, \ldots, x_n]$, which is essentially the same n-simplex (geometrically), but if we take the ordering into consideration, this is a different n-simplex then the one we started with.

A **subsimplex** of an n-simplex, $[x_0, x_1, \ldots, x_n]$, is a simplex spanned by a subset of those vertices, following the order inherited from the n-simplex. A particular example are the **faces** of the n-simplex, which are subsimplexes with n vertices, more specifically, we call $[x_0, \ldots, \hat{x}_i, \ldots, x_n]$ the ith face of the n-simplex.

Definition 1.6.24 ((HATCHER, 2002))**.** The **standard n-simplex**, usually written Δ^n, is the the n-simplex with its vertices being the canonical base of $\mathbb{R}^{n+1}$, that is $\Delta^n = [e_0, e_1, \ldots, e_n]$.

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

We usually denote the ith face of Δ^n by $(\Delta^n)^i$. Notice that $(\Delta^n)^i = [e_0, \ldots, \hat{e}_i, \ldots, e_n]$ is homeomorphic to Δ^{n-1} by the canonical homeomorphism $h : \Delta^{n-1} \to (\Delta^n)^i$ given by

$$h(t_0, \ldots, t_{n-1}) = (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \tag{1.1}$$

The **boundary** of Δ^n is defined as the union of all its faces $\partial\Delta^n = \cup_i [e_0, \ldots, \hat{e}_i, \ldots, e_n]$.

$$\partial\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0 \text{ for all } i, \text{ and at least one } t_i \text{ is zero} \right\}.$$

The **open n-simplex** is defined as $\mathring{\Delta}^n \doteq \Delta^n \setminus \partial\Delta^n$.

Definition 1.6.25 ((HATCHER, 2002))**.** A topological space X is called a **Δ-complex** (or as some authors may write, it has a Δ-complex structure) if there are maps $\psi_\alpha^n : \Delta^n \to X$, in such a manner that:

1. The restrictions $\psi_\alpha^n|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \to X$ are injective, and each point of x lies in the image of exactly one such map.

2. For all maps $\psi_\alpha^n : \Delta^n \to X$ and each face $(\Delta^n)^i$ we have $\psi_\alpha^n|_{(\Delta^n)^i} \circ h = \psi_\beta^{n-1} : \Delta^{n-1} \to X$, for some map ψ_β^{n-1}. In which h is the canonical homeomorphism as in equation 1.1.

3. A subset $A \subset X$ is open if and only if each $(\psi_\alpha^n)^{-1}(A)$ is open in Δ^n.

As a consequence of condition 3 above, a Δ-complex can be built inductively as a quotient space, similarly to the manner we built CW-complexes previously. We start with a set X^0 of 0-simplexes, each with a map $\psi_\alpha^0 : \Delta_\alpha^0 \to X^0$ (here Δ_α^0 is simply Δ^0). Then we attach a family of 1-simplexes Δ_α^1, we do this by considering the disjoint union $X_0 \sqcup (\bigsqcup_\alpha \Delta_\alpha^1)$, and taking the quotient by identifying each vertex of Δ_α^1 with an element of X_0, so that we get a space X^1 with maps $\psi_\alpha^1 : \Delta_\alpha^1 \to X^1$ which restricted to a vertex of Δ_α^1 becomes one of the maps ψ_α^0 (considering the canonical homeomorphism 1.1). If X^{n-1} has been constructed, we proceed to get X^n by considering a collection of n-simplexes Δ_α^n, and attaching them similarly to the case described above.

With the construction above one can show that every Δ-complex is a CW complex, with its open cells being $e_\alpha^n = \psi_\alpha^n(\mathring{\Delta}_\alpha^n)$, and characteristic maps ψ_α^n.

Definition 1.6.26. A **simplicial complex** is a Δ-complex with the extra condition that each simplex is uniquely determined by its vertices. More explicitly, if X is a simplicial complex and $\psi_\alpha^n : \Delta_\alpha^n \to X$ is a map as in definition 1.6.25, then the image by ψ_α^n of the vertices of Δ_α^n has n elements, and no other such map has exactly these elements in the image of its vertices.

1.7 Euclidean Neighborhood Retract

Definition 1.7.1 ((DOLD, 2012))**.** A topological space X is said to be an Euclidean neighborhood retract (ENR), if it is homeomorphic to $X' \subset \mathbb{R}^n$, for some n, such that there is an open subset of $U \subset \mathbb{R}^n$ containing X', to which X' is a retract.

Clearly, the most simple examples of ENRs are the open subsets of any Euclidean space $\mathbb{R}^n$.

Lemma 1.7.2. The disjoint union of a finite number of ENRs is still an ENR.

Proof. Let A and B be two ENRs and consider $X = A \sqcup B$ to be the disjoint union of the two. Since both are *ENRs*, there exists euclidean subsets $A' \subset \mathbb{R}^n$ and $B' \subset \mathbb{R}^m$, with open sets containing them, U and V, respectively, such that A and B are homoeomorphic to A' and B', which are retracts of U and V, respectively. The subset A is also homeomorphic to $\tilde{A} \doteq A' \times \{0\}^m \times \{0\} \subset \mathbb{R}^{n+m+1}$ and B is homoeomorphic to $\tilde{B} \doteq \{0\}^n \times B \times \{1\} \subset \mathbb{R}^{n+m+1}$. Clearly $\tilde{A}$ and $\tilde{B}$ are disjoint in R^{n+m+1}, hence $X = A \sqcup B$ is homeomorphic to $\tilde{A} \cup \tilde{B}$, which clearly is a retract of the open subset $U \times (-1/3, 1/3)^m \times (-1/3, 1/3) \cup (-1/3, 1/3)^n \times V \times (2/3, 4/3)$. Whence X is an ENR. □

Example 1.7.3. The union of ENRs is not necessarily an ENR. Take for example the two subsets of $\mathbb{R}$:

$$A = \{0\}, \qquad B = \{1/n \in \mathbb{R} \mid n \in \mathbb{N}_{>0}\},$$

both are clearly ENRs, since A is a retract of $\mathbb{R}$ itself and B is a retract of the open set $\cup_{n=1}^{\infty}(1/n - \varepsilon_n, 1/n + \varepsilon_n)$, in which $\varepsilon_n = \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})$. But it is easy to see that $X = A \cup B$ is not an ENR, by contradiction suppose $r : U \to X$ is a retraction of an open subspace U onto X, then there is a path-connected open subset $V \subset U$ with $0 \in V$. Consider $s \doteq r|_V : V \to X$, since s is continuous, $s(V)$ must be path-connected, but since $0 \in s(V)$ and $1/n \in s(V)$, for all n sufficiently large, we have a contradiction.

Lemma 1.7.4. The cartesian product $X \times Y$ of two ENRs is still an ENR.

Proof. If A and B are ENRs, with retractions from open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, $r : U \to A'$ and $s : V \to B'$, onto $A' \subset \mathbb{R}^n$ and $B' \subset \mathbb{R}^m$, homeomorphic to A and B, respectively. Then clearly $A \times B$ is homeomorphic to $A' \times B' \subset \mathbb{R}^{n+m}$, and $r \times s : U \times V \to A' \times B'$ is a retraction. Hence $A \times B$ is an ENR. □

Lemma 1.7.5 ((DOLD, 2012))**.** Let X be an ENR and Y a topological space with a subspace $B \subset Y$. If $f_0, f_1 : Y \to X$ are maps such that $f_0|_B = f_1|_B$, then there is an open neighborhood W of B and a homotopy $F : f_0|_W \simeq f_{1W}$, such that $F(b,t) = f_0(b)$, for all $b \in B$ and $t \in I$.

Lemma 1.7.6 ((DOLD, 2012))**.** Let X be an ENR with a subset B which is itself an ENR. Then there is an open subset U, with $B \subset U \subset X$, and a retraction $r : U \to B$, such that, if we denote by $i : B \hookrightarrow X$ the canonical inclusion, then $ir : U \to X$ is homotopic to the inclusion $j : U \hookrightarrow X$.

Theorem 1.7.7 ((DOLD, 2012) proposition 8.12)**.** Any locally contractible and locally compact subspace of $\mathbb{R}^n$ is an ENR.

CHAPTER

2

THE LUSTERNIK-SCHNIRELMANN CATEGORY

In this chapter, we will introduce the concept of Lusternik-Schnirelmann Category. The main references here are (CORNEA *et al.*, 2003; JAMES, 1978).

2.1 Lusternik-Schnirelmann category

Definition 2.1.1 ((JAMES, 1978))**.** Given X a topological space and $A \subset X$, we say that A is a **categorical subset** (or just **categorical**) if A is contractible in X, in other words, the inclusion $A \hookrightarrow X$ is null-homotopic. A covering of X is said to be **categorical** if all its elements are categorical subsets of X.

Definition 2.1.2 ((JAMES, 1978))**.** The **Lusternik-Schnirelmann Category**(or **LS Category**) of a space X, denoted by $\mathrm{cat}(X)$, is the smallest positive integer k such that there exists an open categorical covering of X of cardinality k. If there is no such k we write $\mathrm{cat}(X) = \infty$.

Some authors prefer to define the LS category to be one less than what we defined above, this would not change much in the theory that follows. To avoid confusion we shall always prefer the definition given above throughout the text unless otherwise stated.

One thing one might immediately conclude from the definition of LS category is that X is a contractible space if and only if $\mathrm{cat}(X) = 1$.

There is also the concept of category for subspaces of X. The **subspace category** of $A \subset X$ is the least integer k such that there are open subsets $U_1, \ldots, U_k$ of X, which cover A and are contractible in X. This invariant is denoted by $\mathrm{cat}_X(A)$, and as in the previous case, if no such integer exists we write $\mathrm{cat}_X(A) = \infty$.

A covering of X with all open subsets contractible in X (as in definition 2.1.2) is usually called a **categorical covering**. So we can reformulate the LS category as being the cardinality of the smallest categorical covering of X.

The LS category is an important topological invariant, and the following proposition shows that it is stronger than that, as it is a homotopy invariant.

Proposition 2.1.3 ((JAMES, 1978))**.** The Lusternik-Schnirelmann Category depends only on the homotopy type of the space.

Proof. Suppose X and Y have the same homotopy type, and that $f : X \to Y$ is a homotopy equivalence with $g : Y \to X$ its homotopy inverse. Suppose $\mathrm{cat}(X) = k$, and let $U_1, \dots, U_k$ be a categorical open covering of X. Consider the open covering of Y given by $V_j = g^{-1}(U_j)$, $j = 1, \dots, k$. Let us show that V_j is categorical. We know that U_j is categorical, so the inclusion $i_{U_j} : U_j \hookrightarrow X$ is null-homotopic, which in turn implies that $f|_{U_j} = f \circ i_{U_j}$ is null-homotopic, whence $f|_{U_j} \circ g|_{V_j} = f \circ g|_{V_j}$ is null-homotopic, and since $f \circ g \simeq id_Y \implies f \circ g|_{V_j} \simeq i_{V_j} : V_j \hookrightarrow Y$, we conclude that V_j is categorical. Hence $\mathrm{cat}(Y) \leq \mathrm{cat}(X)$. The inverse inequality is completely analogous. □

Next, we provide a comparison between LS category and the dimension of a given space X. Dimension here always means covering dimension, as discussed in section 1.5, in the case of Manifolds and CW complexes, this coincides with the usual concept of dimension.

Theorem 2.1.4 ((JAMES, 1978))**.** Let X be a path-connected, paracompact, and locally contractible Hausdorff space, then

$$\mathrm{cat}(X) \leq \dim(X) + 1. \tag{2.1}$$

Proof. If $\dim(X) = \infty$ there is nothing to be proven. Assume that $\dim(X) = n < \infty$. Since we are supposing X locally contractible, there is a categorical covering $\{V_\beta\}_{\beta \in B}$ of X, and since the dimension of X is n there is a locally finite refinement $\{U_\alpha\}_{\alpha \in A}$, of order $n+1$. Since each U_α is contained in a V_β, which is contractible in X, the path connectedness of X implies U_α is also categorical, hence $\{U_\alpha\}_\alpha$ is a categorical covering.

Since X is a Hausdorff paracompact space, by theorem 1.1.4 we know that there is a partition of unity subordinate to the covering $\{U_\alpha\}_\alpha$, let $\{\pi_\alpha\}_\alpha$ be such a partition.

Now, for each $x \in X$ define

$$S(x) = \{\alpha \in A \mid \pi_\alpha(x) \neq 0\},$$

which is clearly a finite set. For each finite subset $S \subset A$ define

$$W(S) = \{x \in X \mid \pi_\alpha(x) > 0 \text{ and } \pi_\alpha(x) > \pi_\beta(x), \text{ for all } \beta \notin S \text{ and } \alpha \in S\}.$$

Before proceeding, let us show that $W(S)$ is open for any finite set $S \subset A$. Suppose $x \in W(S)$, since $\{U_\alpha\}_\alpha$ is locally finite there is an open neighborhood N of x which intersects finitely many elements of the open covering, let us denote those elements by $U_{\alpha_1}, \dots, U_{\alpha_k}$, and denote by $T = \{\alpha_1, \dots, \alpha_k\} \subset A$. Consequently, the only nontrivial functions among the collection $\{\pi_\alpha\}_\alpha$ in N are π_α for $\alpha \in T$. If $T \setminus S = \emptyset$, then we have that

$$N \cap \left(\bigcap_{\alpha \in S} \pi_\alpha^{-1}((0,\infty)) \right)$$

is an open subset of $W(S)$ containing x. And when $T \setminus S \neq \emptyset$, the subspace

$$N \cap \left(\bigcap_{\alpha \in S, \beta \in T \setminus S} (\pi_\alpha - \pi_\beta)^{-1}((0,\infty)) \right)$$

is an open subset of $W(S)$ containing x, thus showing that $W(S)$ is in fact open. In both cases above we are using the fact that S and T are finite, so we are taking finite intersections of open substes, which are itself open.

For any finite subset $S \subset A$ let $|S|$ denote the cardinality of S. If for two different subsets of A we have $|S| = |S'|$, then $W(S)$ and $W(S')$ are disjoint. This is easily seen since if $x \in W(S)$, then there are $\alpha \in S \setminus S'$ and $\beta \in S' \setminus S$, and we have $\pi_\alpha(x) > \pi_\beta(x)$, which implies $x \notin W(S')$, and by a completely analogous argument, if $y \in W(S')$, then $y \notin W(S)$.

If $\alpha \in S$ we clearly have $W(S) \subset \{x \in X \mid \pi_\alpha(x) > 0\} \subset U_\alpha$, and since U_α is categorical and X is path connected, we have that each $W(S)$ is an open categorical subset of X.

Now, notice that $x \in W(S(x))$, for all $x \in X$, hence $\{W(S(x))\}_{x \in X}$ covers X. Also notice that $|S(x)| \leq n+1$ for any $x \in X$, since $\{U_\alpha\}_\alpha$ has order $n+1$. For $k = 1, \dots, n+1$ we define the following subspaces of X

$$W_k = \bigcup_{|S(x)|=k} W(S(x)).$$

Remember that for $x \neq y$ in X such that $|S(x)| = |S(y)|$ we either have $W(S(x)) = W(S(y))$ or $W(S(x))$ and $W(S(y))$ are disjoint, so W_k is a disjoint union of open categorical subspaces, hence it is itself an open categorical subspace of X. Thus we have found an open categorical covering $W_1, \dots, W_{n+1}$ of X, which implies $\mathrm{cat}(X) \leq n+1 = \dim(X)+1$.

□

Remark 2.1.5. In James' article (JAMES, 1978) Theorem 2.1.4 was stated without the assumption of X being locally contractible, however Farber showed that this hypothesis cannot be ignored (FARBER, 2003). For a counterexample consider X to be the subset of $\mathbb{R}^2$ given by the union of $C_n = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y - 1/n)^2 = 1/n^2\}$, for $n \in \mathbb{N}$. There exists no contractible neighborhood of $(0,0)$ in X, since any neighborhood of this point will certainly contain a circle centered in $(0, 1/n)$ of radius $1/n$ for a big enough n, whence $\mathrm{cat}(X) = \infty$, although $\dim(X) = 1$.

Definition 2.1.6 ((CORNEA *et al.*, 2003))**.** Given X a topological space, the Geometric Category of X is defined as the least number k such that there is an open covering, of cardinality k, of X by contractible subsets. This invariant is usually written $\mathrm{gcat}(X)$. If no such integer exists we write $\mathrm{gcat}(X) = \infty$.

Notice that in our initial definition of LS category we considered open coverings by subsets contractible in X, and in this new definition of Geometric Category, we are asking for each subset to be contractible in itself. Since this last condition creates a stronger restriction on the possible open coverings, it is straightforward that for any space X we have

$$\mathrm{cat}(X) \leq \mathrm{gcat}(X).$$

Definition 2.1.7. Given X a compact topological n-manifold we define the **ball-category** of X, as the lowest cardinality of a covering of X by closed n-disks (see definition 1.2.4). This is usually written as $\mathrm{ballcat}(X)$. If no such integer exists we write $\mathrm{ballcat}(X) = \infty$.

Since closed n-disks are always contractible, we clearly have

$$\mathrm{cat}(X) \leq \mathrm{gcat}(X) \leq \mathrm{balcat}(X).$$

Next, we use the structure of the cohomology ring of a space X, to obtain a lower bound for its LS category.

Definition 2.1.8 ((CORNEA *et al.*, 2003))**.** Let R be a commutative ring and X a topological space, the R-cuplength of X, $\mathrm{cup}_R(X)$, is defined as the least integer k such that for any collection $u_1, \ldots, u_k \in \tilde{H}^*(X;R)$, we have $u_1 \smile \cdots \smile u_k = 0$.

Theorem 2.1.9 ((CORNEA *et al.*, 2003))**.** Given R a commutative ring and X a topological space we have

$$\mathrm{cup}_R(X) \leq \mathrm{cat}(X).$$

Proof. We begin by supposing that $\mathrm{cat}(X) = n$, and our goal is to show that this will imply that the R-cuplength of X is lower than or equal to n. Let $\{U_1, \ldots, U_n\}$ be a categorical open covering of X. Let $u_1, \ldots, u_n$ be cohomology classes in $H^*(X;R)$.

For each (X, U_i) we may consider the cohomology exact sequence of the pair, as described in section 1.3, given by

$$\cdots \longrightarrow H^m(X, U_i; R) \xrightarrow{q_{i*}^*} H^m(X;R) \xrightarrow{j_{i*}^*} H^m(U_i;R) \longrightarrow \cdots ,$$

in which $q_i : X \to (X, U_i)$ and $j_i : U_i \to X$ are the canonical inclusions.

By definition, U_i being categorical means that the inclusion map $j_i : U_i \to X$ is nullhomotopic, and by the homotopy invariance of cohomology this implies that $j_{i*}^* = 0$, which by exactness of the previous sequence implies that q_{i*}^* is surjective. Hence for $u_i \in H^*(X;R)$ there exists $\overline{u}_i \in H^*(X;U_i;R)$ such that $q_{i*}^*(\overline{u_i}) = u_i$. We may find such a $\overline{u_i}$ for each u_i, $i = 1,\dots,n$.

Remember the relation between the cup product and the cross product via the diagonal map, described in section 1.3, and notice that it easily extends to the general relative case, more explicitly, for subspaces $A, B \subset X$, the cup product becomes

$$\begin{aligned}\smile : H^*(X,A;R) \otimes H^*(X,B;R) &\to H^*(X,A\cup B;R)\\ u\otimes v &\mapsto u \smile v,\end{aligned}$$

while the cross product and the diagonal map are given by

$$\begin{aligned}\mu : H^*(X,A;R) \otimes H^*(X,B;R) &\to H^*(X\times X, A\times X \cup X\times B;R)\\ u\otimes v &\mapsto u \times v,\end{aligned}$$

and

$$\begin{aligned}\Delta : (X, A\cup B) &\to (X\times X, A\times X\cup X\times B)\\ x &\mapsto (x,x),\end{aligned}$$

respectively. And we still have $u \smile v = \Delta_*^*(u\times v)$. On top of that, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "\Delta"] & (X, A\cup B) \arrow[d, "\Delta"] \\ X\times X \arrow[r, "r_1\times r_2"] & (X\times X, A\times X\cup X\times B) \end{tikzcd}$$

in which r, r_1 and r_2 are the canonical inclusions. This diagram together with the cross product yields the following commutative diagram

$$\begin{tikzcd} H^*(X;R) & H^*(X,A\cup B;R) \arrow[l, "r_*^*"'] \\ H^*(X\times X;R) \arrow[u, "\Delta_*^*"] & H^*(X\times X, A\times X\cup X\times B;R) \arrow[l, "(r_1\times r_2)^*"'] \arrow[u, "\Delta_*^*"'] \\ H^*(X;R)\otimes H^*(X;R) \arrow[u, "\mu"] & H^*(X,A;R)\otimes H^*(X,B;R) \arrow[l, "r_{1*}^*\otimes r_{2*}^*"'] \arrow[u, "\mu"'] \end{tikzcd}$$

and using the fact that the cup product is exactly $\Delta_*^*\mu$ we get $r_*^*(u\smile v) = r_{1*}^*(u) \smile r_{2*}^*(v)$, for any cohomology classes $u \in H^*(X,A;R)$ and $v \in H^*(X,B;R)$.

Now we just apply the equality above to our case of interest, namely we have n inclusions $q_i : X \to (X, U_i)$ and $q : X \to (X, \bigcup_{i=1}^n U_i)$, so we get

$$q_*^*(\overline{u}_1 \smile \cdots \smile \overline{u}_n) = q_{0*}^*(\overline{u}_1) \smile \cdots \smile q_{n*}^*(\overline{u}_n) = u_1 \smile \cdots \smile u_n.$$

But since $\bigcup_{i=1}^n U_i = X$, we have $H^*(X, \bigcup_{i=1}^n U_i;R) = 0$, hence $q_*^* = 0$, thus $u_1 \smile \cdots \smile u_n = 0$, and we conclude that $\mathrm{cup}_R(X) \le n = \mathrm{cat}(X)$. $\square$

Often in homotopy theory and when discussing topological complexity, we are interested in spaces of the homotopy type of a CW-complex. Such a space satisfies the conditions of theorem 2.1.4, so for CW-complexes we have

$$\mathrm{cup}_R(X) \leq \mathrm{cat}(X) \leq \dim(X) + 1,$$

in which R is a commutative ring and $\dim(X)$ is simply the CW-complex dimension.

Example 2.1.10. The cohomology ring $\tilde{H}^*(S^n;R)$ only has one generator at the level n, hence $\mathrm{cup}_R(S^n) = 2 \leq \mathrm{cat}(S^n)$. Furthermore, it is easy to construct a categorical covering of S^n with two open sets, namely if $N, S \in S^n$ are the north and south pole, consider the covering given by $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$, it is well known that these spaces are homeomorphic to $\mathbb{R}^n$, hence they are contractible, and $\mathrm{cat}(S^n) \leq 2$. With the two inequalities we conclude that $\mathrm{cat}(S^n) = 2$ for any integer $n \geq 1$.

Example 2.1.11. Let M_g be a closed orientable surface of genus $g \geq 1$. Hatcher shows in Example 3.7 of (HATCHER, 2002) that the cup product $H^1(M_g;\mathbb{Z}) \otimes H^1(M_g;\mathbb{Z}) \to H^2(M_g;\mathbb{Z})$ is not trivial, by which we conclude that $3 \leq \mathrm{cup}_{\mathbb{Z}}(M_g) \leq \mathrm{cat}(M_g) \leq \dim(M_g) + 1 = 3$, hence $\mathrm{cat}(M_g) = 3$.

Example 2.1.12. Let $T^n = \prod_{i=1}^n S^1$ be the n-torus, then clearly $\dim(T^n) = n.\dim(S^1) = n$, hence $\mathrm{cat}(T^n) \leq n+1$. Now we wish to show that for a field K we have $\mathrm{cup}_K(T^n) = n+1$. To show this we will use the fact that the cross product is an isomorphism (as a consequence of theorem 1.3.31, since K is a field). First notice that the Künneth theorem implies that $H^n(T^n;K) \approx H^1(S^1;K) \otimes_K H^{n-1}(T^{n-1};K)$, in which we are using the fact that $T^n = S^1 \times T^{n-1}$, thus, by induction, knowing that $H^1(S^1;K) = K$, we get $H^n(T^n;K) = K$. Furthermore, we have the cross product isomorphism

$$\times : H^1(S^1;K) \otimes H^1(S^1;K) \to H^2(T^2;K),$$

which implies that all elements of $H^2(T^2;K)$ are of the form $a \smile b$, in which $a = p_1{}^*_*(a')$ and $b = p_2{}^*_*(b')$, for some $a', b' \in H^1(S^1;K)$ and p_1 and p_2 the cononical first and second coordinate projections of T^2 onto S^1.

As an induction hypothesis suppose that every element of $H^l(T^l;K)$ is of the form $a_1 \smile \cdots \smile a_l$ for $l = 1, \ldots, n-1$, then the cross product isomorphism,

$$\times : H^1(S^1;K) \otimes H^{n-1}(T^{n-1};K) \to H^n(T^n;K),$$

implies that every element of $H^n(T^n;K)$ is of the form

$$p_1{}^*_*(a) \smile p_2{}^*_*(a_1 \smile \cdots \smile a_{n-1}) = p_1{}^*_*(a) \smile p_2{}^*_*(a_1) \smile \cdots \smile p_2{}^*_*(a_{n-1}),$$

hence $cup_K(T^n) = n+1$ and we conclude that $\mathrm{cat}(T^n) = n+1$.

Notice that when $g = 1$ in example 2.1.11, we have the same space as when $n = 2$ in example 2.1.12, i.e., $M_1 = T^2$, and the results obtained in both examples are consistent $\mathrm{cat}(M_1) = \mathrm{cat}(T^2) = 3$

Example 2.1.12 shows us that both the dimension upper bound, $\mathrm{cat}(X) \leq \dim(X) + 1$, and the cup length lower bound, $\mathrm{cup}_R(X) \leq \mathrm{cat}(X)$, may be reached by some spaces, and even stronger we showed that there is a space for which $\mathrm{cup}_R(X) = \dim(X) + 1$.

Definition 2.1.13 ((CORNEA *et al.*, 2003))**.** A based space X is said to have a **non-degenerate basepoint** x_o if (X, x_o) is a cofibre pair, in other words, the inclusion $x_o \hookrightarrow X$ has to be a cofibration.

If the subset $\{x_o\}$ is closed in X (for instance, if X is Hausdorff) and x_o is a non-degenerate basepoint, we have, by theorem 1.4.24, that (X, x_o) is an NDR pair, and, as showed in section 1.4, there is an open neighborhood of x_o that strong deformation retracts to x_o. This is always the case for CW-complexes, their basepoints are always non-degenerate, which is a consequence of the fact that any CW-pair (X, A) is an NDR pair, since A is closed in X and $A \hookrightarrow X$ is a cofibration (see, for example, proposition 1.5.17 in (ARKOWITZ, 2011)).

Lemma 2.1.14 ((CORNEA *et al.*, 2003))**.** If (X, x_o) is a path connected normal cofibre pair, with $\mathrm{cat}(X) \leq n$, then we can find a categorical open covering $\{V_1, \ldots, V_n\}$ of X with $x_o \in V_i$, for all i, and each V_i is contractible to x_o relative to x_o, meaning that there is a homotopy $F : V_i \times I \to X$ from the inclusion $V_i \hookrightarrow X$ to the constant map equal to x_o, for which $F(x_o, t) = x_o$, for all $t \in I$.

Proof. First, let us quickly discuss the case when $n = 1$, which would immediately imply $\mathrm{cat}(X) = 1$. In this case, we know that X is contractible, in other words, the inclusion $x_o \hookrightarrow X$ is a homotopy equivalence, and, by hypothesis, it is also a cofibration, so theorem 1.4.30 applies and we conclude that X strongly deformation retracts into x_o, which solves the case $n = 1$ for the current lemma.

Now suppose $n \geq 2$ and consider a categorical open covering $\{U_i\}_{i=1}^n$, from which we shall construct a new covering that satisfies the lemma's conditions.

Notice that we can choose the elements U_i of the covering above, in a way that at least one of them does not contain the basepoint x_o. In case all of them were to contain x_o we can simply redefine U_n to be $U_n \setminus x_o$, which is still an open categorical set. In addition, we can enumerate the covering $\{U_i\}_i$ in a way that $x \in U_i$, for $i = 1, \ldots, k$ and $x \notin U_i$, for $i = k+1, \ldots, n$, for some integer $1 \leq k < n$.

Since X is a normal space and $\{U_i\}_i$ is finite, theorem 1.1.6 guarantees that there is another open covering $\{W_i\}_i$ with $\overline{W}_i \subset U_i$ and $W_i \neq \emptyset$, whenever $U_i \neq \emptyset$. And the fact that (X,x_o) is a cofibre pair implies, via theorem 1.4.24, that there is a neighborhood of x_o, $N \subset X$, that strong deformation retracts to x_o. Consider the subspace

$$\mathscr{N} = N \cap U_1 \cap \cdots \cap U_k \cap (X \setminus \overline{W}_{k+1}) \cap \cdots \cap (X \setminus \overline{W}_n),$$

and notice that, since $\mathscr{N} \subset N$, this subspace also strongly deformation retracts to x_o, and it is open, once it is a finite intersection of open subspaces.

Since X is normal, there is an open set M such that

$$x_o \in M \subset \overline{M} \subset \mathscr{N} \subset U_j,$$

for $j = 1, \ldots, n$.

We define a new categorical open covering $\{V_j\}_{j=1}^n$ by

$$V_j = (U_j \cap (X \setminus \overline{M})) \cup M, \text{ for } \ j = 0, \ldots, k,$$

and

$$V_j = W_j \cup \mathscr{N}, \text{ for } \ j = k+1, \ldots, n.$$

Now we just have to check that this is an open covering and that all subsets are categorical. That all V_j are open subsets is immediate. To see that $\{V_j\}_j$ covers X, first remember that $\{W_j\}_j$ covers X, and $W_j \subset V_j$, for $j = k+1, \ldots, n$, so the only problem may be in V_j, for $j = 1, \ldots, k$. Notice that for those cases the only part of U_j that is not in V_j is $\overline{M} \setminus M$, but we have $\overline{M} \subset \mathscr{N} \subset V_n$, whence $\{V_j\}_j$ does in fact cover X.

Finally, let us show that each V_j is categorical, and even more strongly, let us show that each V_j contracts to x_o in X via a contracting homotopy relative to x_o. To construct those homotopies, firs consider the contracting homotopies

$$G_j : U_j \times I \to X,$$

such that $G(u,0) = u$ and $G(u,1) = c_j$, for all $u \in U$, in which c_j is some point in X.

For each c_j, $j = 1, \ldots, n$, define a path

$$\gamma_j : I \to X,$$

such that $\gamma(0) = c_j$ and $\gamma(1) = x_o$. This is possible, since X is path connected.

Let

$$H_j : N \times I \to X$$

be a homotopy such that $H_j(x,0) = x$, $H_j(x,1) = x_o$ and $H_j(x_o,t) = x_o$, for all $x \in N$ and $t \in I$. Such an H exists since x_o is a strong deformation retract of N.

For $j = 1, \ldots, k$ define $F_j : V_j \times I \to X$ by

$$F_j(x,t) = \begin{cases} H_j(x,t), & \text{if } x \in M; \\ G_j(x,2t), & \text{if } x \in (U_j \cap (X \setminus \overline{M})) \text{ and } 0 \leq t \leq 1/2; \\ \gamma_j(2t-1), & \text{if } x \in (U_j \cap (X \setminus \overline{M})) \text{ and } 1/2 \leq t \leq 1; \end{cases}$$

and for $j = k+1, \ldots, n$ define $F_j : V_j \times I \to X$ by

$$F_j(x,t) = \begin{cases} H_j(x,t), & \text{if } x \in \mathscr{N}; \\ G_j(x,2t), & \text{if } x \in W_j \text{ and } 0 \leq t \leq 1/2; \\ \gamma_j(2t-1), & \text{if } x \in W_j \text{ and } 1/2 \leq t \leq 1. \end{cases}$$

It is not difficult to see that F_j is a contracting homotopy relative to x_o, for all j. □

Remark 2.1.15. A categorical covering as in lemma 2.1.14, for which each categorical subset contracts to the basepoint via a contracting homotopy relative to the basepoint, is called a **based categorical covering**.

Proposition 2.1.16 ((CORNEA *et al.*, 2003))**.** If U and V form an open covering of X, then $\text{cat}(X) \leq \text{cat}(U) + \text{cat}(V)$.

Proof. Suppose $\text{cat}(U) = m$ and $\text{cat}(V) = n$. Let $\{U_1, \ldots, U_m\}$ and $\{V_1, \ldots, V_n\}$ be categorical open coverings for U and V, respectively. Notice that, since U_i is open in U, which is open in X, we have that U_i is itself open in X, and the same is true for V_j. Since any U_i or V_j is contractible in U or V, they are clearly contractible in X. We conclude that $\{U_1, \ldots, U_m, V_1, \ldots, V_n\}$ is a categorical open covering of X, since it has $m+n$ elements we have $\text{cat}(X) \leq m+n = \text{cat}(U) + \text{cat}(V)$. □

Proposition 2.1.17 ((CORNEA *et al.*, 2003))**.** For X and Y normal path-connected spaces with non-degenerate basepoints, we have $\text{cat}(X \vee Y) = \max(\text{cat}(X), \text{cat}(Y))$.

Proof. Let us denote $\text{cat}(X) = n$ and $\text{cat}(Y) = m$. Let $\{U_j\}_{j=1}^n$ and $\{V_j\}_{j=1}^m$ be the based categorical open coverings of X and Y, respectively.

For any $1 \leq i \leq n$ and $1 \leq j \leq m$, it is easy to see that $U_i \cup V_j$ is an open subset of $X \vee Y$. Furthermore, U_i has a contracting homotopy relative to the basepoint $x_o \in X$, likewise V_j has a contracting homotopy relative to the basepoint $y_o \in Y$. Since the basepoints are fixed in these homotopies, they easily define a homotopy in $U_i \cup V_j \subset X \vee Y$ which contracts $U_i \cup V_j$ to the basepoint $[x_o] = [y_o]$, hence $U_i \cup V_j$ is an open categorical subset of $X \vee Y$.

If $n \geq m$ (the other case is analogous) take the based categorical open covering

$$\{U_1 \cup V_1, \ldots, U_m \cup V_m, U_{m+1} \cup V_m, \ldots, U_n \cup V_m\},$$

since it has n elements we get $\text{cat}(X \vee Y) \leq \max(\text{cat}(X), \text{cat}(Y))$.

For the reverse inequality, consider the canonical maps $k : X \vee Y \to X$ and $j : X \to X \vee Y$. Suppose $\text{cat}(X \vee Y) = l$, with a categorical open covering $\{U_j\}_{j=1}^{l}$. Define $V_j = j^{-1}(U_j) = X \cap U_j$. Notice that each V_j is categorical, indeed if $H : U_i \times I \to X \vee Y$ is a homotopy contracting to the basepoint $[x_o]$, which is always possible by path conectedness, then define $G : V_j \times I \to X$ by $G(v,t) = k(H(j(v),t))$, and notice that $G(v,0) = k(H(j(v),0)) = k(j(v)) = v$, since $kj = id_X$, and $G(v,1) = k(H(j(v),1)) = k([x_o]) = x_o$. Whence $\text{cat}(X) \leq l = \text{cat}(X \vee Y)$, and a completely analogous argument shows that $\text{cat}(Y) \leq \text{cat}(X \vee Y)$. □

Example 2.1.18. For any positive integer n we have $\text{cat}(S^n \vee \cdots \vee S^n) = \text{cat}(S^n) = 2$.

In an even more general case, we may consider positive integers $n_1, \ldots, n_m$ and we have $\text{cat}(S^{n_1} \vee \cdots \vee S^{n_m}) = 2$.

2.2 LS category of Products

In topology, one really important construction is that of a cartesian product of two (or more) topological spaces. Next, we shall present some interesting results for the LS category of such spaces.

For the proof of the first theorem in this section, we shall develop a slightly different way to define LS category, by means of what we call a categorical sequence, as presented next.

Definition 2.2.1 ((CORNEA *et al.*, 2003))**.** Given X a topological space, we say that a nested sequence $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_k = X$ of open subsets is a **categorical sequence**, if for every $i = 1, \ldots, k$ we have $O_i \setminus O_{i-1} \subset U_i$, for some categorical open set U_i. Furthermore, the integer k, as above, is called the length of the categorical sequence.

Lemma 2.2.2 ((CORNEA *et al.*, 2003))**.** For any topological space, X, we have $cat(X) \leq k$ if and only if X has a categorical sequence of length k.

Proof. If $cat(X) \leq k$, let $\{U_i\}_{i=1}^{k}$ be a categorical open covering of X, define $O_0 = \emptyset$ and $O_j = \bigcup_{i=1}^{j} U_i$, for $j = 1, \ldots, k$, then $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_k = X$ and $O_j \setminus O_{j-1} \subset U_j$, for $j = 1, \ldots, k$. Thus we have a categorical sequence of length k.

Conversely, suppose there is a categorical sequence of length k, $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_k = X$, since this is a nested sequence with $O_k = X$, it must be the case that $\{O_j \setminus O_{j-1}\}_{j=1}^{k}$ covers X, hence $\{U_j\}_{j=1}^{k}$ covers X, in which U_j is the open categorical set containing $O_j \setminus O_{j-1}$, therefore $cat(X) \leq k$. □

Remark 2.2.3. We can conclude from lemma 2.2.2 that the LS category of a space X is the smallest integer k such that there exists a categorical sequence of length k in X.

Now we are ready to proceed to one of the important theorems in this section. We will actually present two versions of the next theorem, in each we will impose some conditions in the spaces X, Y and the product $X \times Y$ so that $\mathrm{cat}(X \times Y) < \mathrm{cat}(X) + \mathrm{cat}(Y)$. In both versions presented it is not difficult to see that X and Y being CW-complexes satisfies the requested conditions.

Remember that if all subspaces of a space X are normal, then X is called a completely normal space (definition 1.1.7), which is equivalent to the condition that all pairs of subsets $A, B \subset X$, such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, should be separable by disjoint open subsets. In particular, every metric space is normal, and all its subspaces are metric, hence normal, thus all metric spaces are completely normal. And, as stated in proposition 1.6.12, we have that all CW complexes are completely normal.

Theorem 2.2.4 ((CORNEA *et al.*, 2003))**.** Suppose X and Y are path-connected spaces with $X \times Y$ completely normal (in particular, if X and Y are metric spaces or CW complexes). Then

$$\mathrm{cat}(X \times Y) < \mathrm{cat}(X) + \mathrm{cat}(Y).$$

Proof. Suppose $\mathrm{cat}(X) = n$ and $\mathrm{cat}(Y) = m$, by lemma 2.2.2 there are categorical sequences $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_n = X$ and $\emptyset = P_0 \subset P_1 \subset \cdots \subset P_m = Y$, in X and Y, respectively. Denote by $\{U_i\}_{i=1}^{n}$ and $\{W_j\}_{j=1}^{m}$ the open categorical sets such that

$$O_i \setminus O_{i-1} \subset U_i \qquad \text{and} \qquad P_j \setminus P_{j-1} \subset W_j.$$

Define $Q_0 = \emptyset$, and for $r = 1, \ldots, n+m-1$, define $Q_r \subset X \times Y$ by

$$Q_r = \bigcup_{i+j=r+1} O_i \times P_j = \bigcup_{j=1}^{r} O_j \times P_{r+1-j}$$

in which we define $O_i = \emptyset$, for $i > n$, and $P_j = \emptyset$, for $j > m$.

Note that $Q_{n+m-1} = O_n \times P_m = X \times Y$, and to see that for $r < n+m-1$ one has $Q_r \subset Q_{r+1}$, simply notice that for $j < n$ we have

$$O_j \times P_{r+1-j} \subset O_{j+1} \times P_{r+1-j} \subset Q_{r+1},$$

and for $j = n$, since $r+1-n < m$, we have

$$O_n \times P_{r+1-n} \subset O_n \times P_{r+2-n} \subset Q_{r+1},$$

hence we have a nested sequence of open sets

$$\emptyset = Q_0 \subset Q_1 \subset \cdots \subset Q_{n+m-1} = X \times Y.$$

Our goal now is to show that the previous sequence is a categorical one, so we have to show that each $Q_r \setminus Q_{r-1}$ is contained in an open categorical subset of $X \times Y$. To do so, let us first show that

$$Q_r \setminus Q_{r-1} = \bigcup_{j=1}^{r} (O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j}), \tag{2.2}$$

for all $r = 1, \ldots, m+n-1$.

First suppose that $(x,y) \in (O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j})$, for some $j \in \{1, \ldots, r\}$, then $(x,y) \in O_j \times P_{r-j+1} \subset Q_r$. Since $y \notin P_{r-j}$, from the fact that the P sequence is nested, we conclude that $(x,y) \notin \bigcup_{k=j}^{r} O_k \times P_{r-k}$. Analogously, $x \notin O_{j-1}$ implies $(x,y) \notin \bigcup_{k=1}^{j-1} O_k \times P_{r-k}$, these two unions add up to Q_{r-1}, hence $(x,y) \notin Q_{r-1}$ and $(x,y) \in Q_r \setminus Q_{r-1}$.

To prove the inverse inclusion, suppose $(x,y) \in Q_r \setminus Q_{r-1}$, for some $r = 1, \ldots, m+n-1$. Notice that $x \in O_j \setminus O_{j-1}$, for some $j \in \{1, \ldots, r\}$, and $(x,y) \in O_k \times P_{r-k+1}$, for some $k \geq j$, but then $P_{r-k+1} \subset P_{r-j+1}$ implies that $(x,y) \in O_j \times P_{r-j+1}$. Since $(x,y) \notin Q_{r-1}$, we have $(x,y) \notin O_j \times P_{r-j}$, and since $x \in Q_j$, we must have $y \notin P_{r-j}$, whence $(x,y) \in (O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j})$, thus proving equation 2.2.

Notice that $(O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j}) \subset U_j \times W_{r-j+1}$, which is a categorical set in $X \times Y$, and that if $j \neq k$ we have

$$\overline{(O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j})} \cap (O_k \setminus O_{k-1}) \times (P_{r-k+1} \setminus P_{r-k}) = \emptyset,$$
$$(O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j}) \cap \overline{(O_k \setminus O_{k-1}) \times (P_{r-k+1} \setminus P_{r-k})} = \emptyset,$$

as a consequence of the fact that both the sequences P and O are nested.

Since $X \times Y$ is completely normal, there are pairwise disjoint open sets $A_{1r}, \ldots, A_{rr}$ such that $(O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j}) \subset A_{jr}$. If we take $B_{jr} = A_j \cap (U_j \times W_{r-j+1})$, we still have that $B_{1r}, \ldots, B_{rr}$ are pairwise disjoint and $(O_j \setminus O_{j-1}) \times (P_{r-j+1} \setminus P_{r-j}) \subset B_{jr}$, and since $B_{jr} \subset U_j \times W_{r-j+1}$, we get the additional property that B_{jr} is a categorical open subset of $X \times Y$. Define $Z_r = \bigcup_{j=1}^{r} B_{jr}$, for $r = 1, \ldots, n+m-1$, clearly Z_r is open and, since it is a union of disjoint categorical sets, it must be categorical, once $X \times Y$ is path connected. Furthermore, notice that $Q_r \setminus Q_{r-1} \subset Z_r$, hence $Z_1, \ldots, Z_{m+n-1}$ covers $X \times Y$, and we conclude that $\mathrm{cat}(X \times Y) \leq m+n-1 = \mathrm{cat}(X) + \mathrm{cat}(Y) - 1$. $\square$

Next, we present the second version of the theorem, this time we will make no assumptions about $X \times Y$, instead we shall suppose that both X and Y are normal spaces. Before proving this version of the theorem, let us establish some useful definitions and preliminary results.

Definition 2.2.5 ((CORNEA *et al.*, 2003)). Let X be a topological space, we say that an open covering $\mathscr{U} = \{U_\alpha\}_\alpha$ of X is an ***i*-covering** if any element $x \in X$ is contained in at least i open sets of $\mathscr{U}$. We define the LS i-category of X, $\mathrm{cat}^i(X)$, as the least integer k such that there is an open categorical i-covering of X of cardinality k.

In view of definition 2.2.5, it is obvious that the usual categorical open covering is simply a categorical open 1-covering.

Now, we present a result with the goal of comparing $\operatorname{cat}(X)$ with $\operatorname{cat}^i(X)$, for any integer i.

Lemma 2.2.6. Suppose X is a path-connected normal space and $\{U_1,\dots,U_n\}$ is a categorical open i-covering of X. Then there exists an open set U_{n+1} such that $\{U_1,\dots,U_n,U_{n+1}\}$ is an open categorical $(i+1)$-covering.

Proof. For each $m \in \{1,\dots,n\}$, define $C_m = \{\omega \subset \{1,\dots,n\} \mid |\omega| = i \text{ and } m \notin \omega\}$, in which $|\omega|$ is the cardinality of ω. For $m = 1,\dots,n$, let $F_m \subset X$ be given by

$$F_m = \bigcup_{\omega \in C_m} \left(\bigcap_{j \in \omega} U_j \right).$$

Notice that

$$(X \setminus U_1) \subset F_1,$$

since $U_1,\dots,U_n$ is an i-covering (so any $x \in (X \setminus U_1)$ must be in some intersection of i elements of the covering, excluding U_1). Hence $(X \setminus F_1) \subset U_1$ and F_1 is clearly open, so we have a closed set, $(X \setminus F_1)$, contained in an open set, U_1, thus, by the normality of X, there must be an open set $V_1 \subset X$, with

$$(X \setminus F_1) \subset V_1 \subset \overline{V}_1 \subset U_1.$$

Suppose we have constructed $V_1,\dots,V_{m-1}$ such that for every $k \in \{1,\dots,m-1\}$ we have

$$(X \setminus F_k) \cap \left(\bigcap_{j=1}^{k-1} (X \setminus U_j) \right) \subset V_k \subset \overline{V}_k \subset U_k \cap \left(\bigcap_{j=1}^{k-1} (X \setminus \overline{V}_k) \right),$$

then we know that $(X \setminus U_m) \subset F_m$ (for the same reason as the case when $m = 1$) and $\overline{V}_j \subset U_j$, for all $j = 1,\dots,m-1$, thus we have

$$(X \setminus U_m) \cup \left(\bigcup_{j=1}^{m-1} \overline{V}_j \right) \subset F_m \cup \left(\bigcup_{j=1}^{m-1} U_j \right),$$

and by taking the complement inclusion we have

$$(X \setminus F_m) \cap \left(\bigcap_{j=1}^{m-1} (X \setminus U_j) \right) \subset U_m \cap \left(\bigcap_{j=1}^{m-1} (X \setminus \overline{V}_j) \right),$$

which is again a closed set contained in an open set, and by normality of X there must be an open set V_m such that

$$(X \setminus F_m) \cap \left(\bigcap_{j=1}^{m-1} (X \setminus U_j) \right) \subset V_m \subset \overline{V}_m \subset U_m \cap \left(\bigcap_{j=1}^{m-1} (X \setminus \overline{V}_j) \right).$$

Notice that $V_1,\dots,V_n$ are disjoint, since if $1 \le k < l \le n$ we have, by the previous inclusion, $V_l \subset (X \setminus \overline{V}_j) \subset (X \setminus V_k)$. Also, since $V_j \subset U_j$, we conclude that each V_j is categorical.

Define $U_{n+1} = \bigcup_{j=1}^{n} V_j$, then U_{n+1} is a disjoint union of open categorical sets in a space X which is path-connected, hence U_{n+1} is itself a open categorical set, and $\{U_1,\dots,U_n,U_{n+1}\}$ is an open categorical covering, so the only thing left to prove is that it is an $(i+1)$-covering.

To do so, take $x \in X$, if x is already in $(i+1)$ elements of the original covering $\{U_1,\dots,U_n\}$, there is nothing to be proven, if not, then x is contained in exactly i-elements of the original covering, and we proceed as follows. Let $t \in \{1,\dots,n\}$ be the smallest integer such that $x \in U_t$, then clearly $x \notin F_t$ and for any $j = 1,\dots,t-1$ we have $x \notin U_j$ (since t is the smallest with that property), hence $x \in \bigcap_{j=1}^{t-1}(X \setminus U_j)$, therefore

$$x \in (X \setminus F_t) \cap \left(\bigcap_{j=1}^{t-1} (X \setminus U_j) \right) \subset V_m \subset U_{n+1},$$

whence $\{U_1,\dots,U_{n+1}\}$ is in fact an (i+1)-covering of X. □

An immediate consequence of the lemma 2.2.6 is the following corollary.

Corollary 2.2.7. For X a path-connected normal space, we have that $\operatorname{cat}(X) \le n$ if and only if $\operatorname{cat}^i(X) \le n+i-1$.

Proof. Suppose $\operatorname{cat}(X) \le n$, then there is a categorical open 1-covering of X of cardinality n. By applying lemma 2.2.6 $i-1$ times, we get an open categorical i-covering of cardinality $n+1-i$, hence $\operatorname{cat}^i(X) \le n+i-1$.

Conversely, if $\operatorname{cat}^i(X) \le n+i-1$, then let $\{U_1,\dots,U_n,\dots,U_{n+i-1}\}$ be a categorical open i-covering of X. Then it is not difficult to see that $\{U_1,\dots,U_n\}$ must also cover X, since each $x \in X$ is in i elements of the original covering and we have extracted only $i-1$ elements, namely $U_{n+1},\dots,U_{n+i-1}$, to construct the new covering, hence $\operatorname{cat}(X) \le n$. □

With this in hands, we are finally ready to present the second version of the product inequality for the LS category.

Theorem 2.2.8. If X and Y are path-connected normal spaces, then $\operatorname{cat}(X \times Y) < \operatorname{cat}(X) + \operatorname{cat}(Y)$.

Proof. Suppose $\operatorname{cat}(X) = n$ and $\operatorname{cat}(Y) = m$. By corollary 2.2.7 we have that $\operatorname{cat}^m(X) \le n+m-1$ and $\operatorname{cat}^n(Y) \le n+m-1$. Let $\mathscr{U} = \{U_1,\dots,U_{n+m-1}\}$ be a categorical open m-covering of X and $\mathscr{V} = \{V_1,\dots,V_{n+m-1}\}$ a categorical open n-covering of Y. Define $\mathscr{W} = \{W_1,\dots,W_{n+m-1}\}$, in which $W_j = U_j \times V_j$. Notice that each W_j is a categorical open set in

$X \times Y$. To show that $\mathscr{W}$ covers $X \times Y$, let $(x,y) \in X \times Y$, then, since $\mathscr{U}$ is an m-covering of X, there are m indices $\{j_1,\dots,j_m\} \subset \{1,\dots,n+m-1\}$ such that $x \in U_{j_i}$, for all $i = 1,\dots,m$. Analogously, there are indices $\{k_1,\dots,k_n\} \subset \{1,\dots,n+m-1\}$ such that $y \in U_{k_i}$, for $i = 1,\dots,n$. There must be an intersection between these two sets of indices, otherwise we would have $n+m$ distinct indices in a set of cardinality $n+m-1$. So if we denote by j the index such that $x \in U_j$ and $y \in V_j$, we obviously have $(x,y) \in W_j$. Whence $\mathscr{W}$ is a categorical open covering of $X \times Y$ and $\operatorname{cat}(X \times Y) \leq |\mathscr{W}| = n+m-1 = \operatorname{cat}(X) + \operatorname{cat}(Y) - 1$. $\square$

Since the product of CW complexes (respectively metric spaces) is always a CW complex (respectively a metric space), and since CW complexes and metric spaces are always completely normal, both theorems 2.2.4 and 2.2.8 imply the following corollary.

Corollary 2.2.9. If $X_1,\dots,X_n$ is a collection of CW complexes (or metric spaces), then

$$\operatorname{cat}(X_1 \times \cdots \times X_n) \leq \operatorname{cat}(X_1) + \cdots + \operatorname{cat}(X_n) - n + 1.$$

Example 2.2.10. As a generalization of example 2.1.12, we may define the spaces $T^n_m = \prod_{i=1}^n S^m$, and as in example 2.1.12 we have that $T^{n+1}_m = S^m \times T^n_m$, so by the Künneth theorem we know that if K is a field, then $H^{mn}(T^n_m;K) = K$ with the cross product

$$\times : H^m(S^m;K) \otimes H^{m(n-1)}(T^{n-1}_m;K) \to H^{nm}(T^n_m;K)$$

being an isomorphism, whence there is a nonzero element in $H^n(T^n_m;K)$ given by $u_1 \smile \cdots \smile u_n$, and $\operatorname{cup}_K(T^n_m) = n+1 \leq \operatorname{cat}(T^n_m)$. In this case, the dimension inequality is not really helpful, since the dimension of T^n_m is nm, we conclude that $n+1 \leq \operatorname{cat}(T^n_m) \leq nm+1$, which only determines the category in the case where $m = 1$, which is exactly example 2.1.12. In the general case, we may use the product inequality

$$\operatorname{cat}(T^n_m) = \operatorname{cat}(S^m \times \cdots \times S^m) \leq \operatorname{cat}(S^m) + \cdots + \operatorname{cat}(S^m) - n + 1 = 2n - n + 1 = n + 1,$$

so we conclude that $\operatorname{cat}(T^n_m) = n+1$, for all positive integers n and m.

2.3 Whitehead's Formulation

In this section, we will present an alternative formulation of LS category, due to George W. Whitehead.

Definition 2.3.1 ((CORNEA *et al.*, 2003))**.** The **fat wedge** of order k of a based space $(X,*_X)$ is given by

$$T^k(X) = \left\{(x_1,\dots,x_k) \in X^k \ : \ \text{at least one } x_j \text{ is the basepoint } *_X\right\}.$$

Let us denote by $j_k : T^k(X) \to X^k$ the canonical inclusion, and let $\Delta : X \to X^k$ be the diagonal map.

Definition 2.3.2 ((CORNEA *et al.*, 2003), Whitehead's formulation)**.** The Whitehead category of X, $\mathrm{cat}^{Wh}(X)$ is the smallest integer n (or ∞) such that the diagonal map $\Delta : X \to X^n$ factors through the fat wedge $T^n(X)$ up to homotopy, in other words, there is a map Δ' such that the following diagram homotopy commutes

$$\begin{tikzcd} & T^n(X) \arrow[d, "j_n"] \\ X \arrow[ur, "\Delta'"] \arrow[r, "\Delta"'] & X^n \end{tikzcd}$$

We will prove that this definition coincides with the open covering definition of category, for the particular case of normal, path-connected spaces with non-degenerate basepoint. And since we generally deal with spaces with the homotopy type of a connected CW complex, we can consider both category definitions as equivalent.

First, we prove the following lemma.

Lemma 2.3.3 ((CORNEA *et al.*, 2003))**.** Let X be a normal topological space. In this case, $\mathrm{cat}(X) \leq n$ if and only if there is an open covering $\{V_1, \ldots, V_n\}$, such that there are homotopies $H_j : X \times I \to X$, with $H_j|_{X\times\{0\}} = id_X$ and $H(v, 1) = *_j$, for all $v \in V_j$, in which $*_j$ is a fixed point associated with V_j.

Proof. The "if" part is simple since the condition stated above implies the existence of an open categorical covering with n open subsets. Let us prove the "only if" part of the statement.

Suppose X is normal and $\mathrm{cat}(X) \leq n$, then by definition there exists a categorical open covering $U_1, \ldots, U_n$ of X, with homopies $G_j : U_j \times I \to X$ starting at the inclusion $U_j \hookrightarrow X$ and ending as a constant map $G_j(u, 1) = *_j$.

Since X is normal, there are open coverings $\{V_j\}_{j=1}^n$ and $\{W_j\}_{j=1}^n$ of X such that

$$V_j \subset \overline{V}_j \subset W_j \subset \overline{W}_j \subset U_j.$$

$\overline{V}_j$ and $X \setminus W_j$ are two disjoint closed subsets of a normal space, hence there is a continuous function $\lambda_j : X \to I$, such that $\lambda_j(\overline{V}_j) = 1$ and $\lambda_j(X \setminus W) = 0$.

With this, we can define the homotopy $H_j : X \times I \to X$, given by

$$H_j(x,t) = \begin{cases} x, \text{ if } x \in X \setminus W_j; \\ G_j(x, t\lambda_j(x)), \text{ if } x \in \overline{W_j}. \end{cases}$$

H_j is continuous on two closed subsets $X \setminus W_j \times I$ and $\overline{W}_j \times I$, which cover $X \times I$, and on the intersection $(X \setminus W_j \times I) \cap (\overline{W}_j \times I) = \overline{W}_j \setminus W_j \times I$ the two definitions coincide, hence H_j is itself continuous. It is not difficult to see that $H_j(x,0)=x$, for all $x\in X$, and $H_j(v,1)=G_j(v,1)=*_j$, for all $v\in V_j$. So we have the desired homotopy. $\square$

Theorem 2.3.4 ((CORNEA *et al.*, 2003)). For X a normal path-connected topological space with non-degenerate basepoint, we have that the Whitehead's definition of category coincides with the usual open categorical covering definition, that is $\mathrm{cat}^{Wh}(X)=\mathrm{cat}(X)$.

Proof. Suppose $\mathrm{cat}^{\mathrm{Wh}}(X)=n$, then there is a map $\Delta' : X\to T^n(X)$, such that the diagram

$$\begin{tikzcd} & T^n(X) \arrow[d, "j"] \\ X \arrow[ur, "\Delta'"] \arrow[r, "\Delta"] & X^n \end{tikzcd}$$

homotopy commutes, in other words, there is a homotopy $H: X\times I\to X^n$ such that $H(x,0)=\Delta(x)$ and $H(x,1)=j\Delta'(x)$, for all $x\in X$.

Let $p_i : X^n\to X$ be the ith projection onto X. Then we have $p_iH : X\times I\to X$ with $p_iH(x,0)=x$ and $p_iH(x,1)=p_ij\Delta'(x)$, for all $x\in X$.

Let N be an open neighborhood of the basepoint $*$ which contracts to $*$ (such a neighborhood exists since $*$ is non-degenerate), and define $U_i=(p_ij\Delta)^{-1}(N)$.

By definition we have that $T^n(X)=\bigcup_i p_i^{-1}(*)$ and $j\Delta'\subset T^n(X)$, hence

$$(j\Delta')^{-1}\left(\bigcup_i p_i^{-1}(*)\right)=(j\Delta')^{-1}(T^n(X))=X,$$

and

$$\begin{aligned}\bigcup_i U_i&=\bigcup_i (p_ij\Delta')^{-1}(N)=\bigcup_i (j\Delta')^{-1}(p_i^{-1}(N))\\ &=(j\Delta')^{-1}\left(\bigcup_i p_i^{-1}(N)\right)\supset (j\Delta')^{-1}\left(\bigcup_i p_i^{-1}(*)\right)=(j\Delta')^{-1}(T^n(X))=X,\end{aligned}$$

therefore, $U_1,\dots,U_n$ is an open covering of X. It remains to be shown that this covering is categorical.

Let $G: N\times I\to X$ be the contracting homotopy for N, contracting to $*$, then we may define the contracting homotopies $L_i : U_i\times I\to X$ by

$$L_i(u,t)=\begin{cases} p_iH(u,2t), & 0\le t\le 1/2;\\ G(p_ij\Delta(u),2t-1), & 1/2\le t\le 1;\end{cases}$$

hence each U_i is categorical and $\mathrm{cat}(X)\le n=\mathrm{cat}^{\mathrm{Wh}}(X)$.

Conversely, suppose $\mathrm{cat}(X)=n$, with $\{U_1,\dots,U_n\}$ a categorical covering as in lemma 2.3.3, i.e., there are homotopies $H_i : X\times I\to X$ such that $H_i(x,0)=x$, for all $x\in X$, and $H_i(u,1)=*$, for all $u\in U_i$.

Define $H : X \times I \to X^n$ by $H(x,t) = (H_1(x,t), \dots, H_n(x,t))$, for all $x \in X$ and $t \in I$. Notice that, for all $x \in X$, $H(x,0) = \Delta(x)$ and $H(x,1) \in T^n(X)$, since x must be in some U_i, which implies $H_i(x,1) = *$. Define $\Delta' : X \to T^n(X)$ by $\Delta'(x) = H(x,1)$. Then $j\Delta' \simeq \Delta$ (via the homotopy given by H), in which $j : T^n(X) \hookrightarrow X^n$ is the canonical inclusion, hence, by Whitehead's definition of category, we conclude that $\mathrm{cat}^{\mathrm{Wh}}(X) \leq n = \mathrm{cat}(X)$, therefore $\mathrm{cat}(X) = \mathrm{cat}^{\mathrm{Wh}}(X)$. $\square$

One particularly interesting class of spaces for which theorem 2.3.4 is valid is the class of path-connected CW complex. Using Whitehead's definition of category we can obtain a more restrictive upper bound than the dimension of the space for this class of spaces.

Theorem 2.3.5 ((CORNEA *et al.*, 2003))**.** If X is an $(n-1)$-connected $(n \geq 1)$ CW complex of dimension $N < \infty$, then

$$\mathrm{cat}(X) \leq \frac{\dim(X)}{n} + 1.$$

Proof. Since X is $(n-1)$-connected, we may assume its CW structure is given by a single 0-cell (the basepoint) and no other cells of dimension lower than n, i.e., $X = \{x_0\} \cup X_n \cup \dots \cup X_N$, in which X_k denotes the k-skeleton of X (to see why X has such a CW structure refer to proposition 4.15 in (HATCHER, 2002)).

Given this CW structure of X, we see that the $(n(k+1)-1)$-skeleton of the fat wedge $T^{k+1}(X)$ is equal to the $(n(k+1)-1)$-skeleton of X^{k+1}, since the difference between these two spaces only appears in dimension equal to or greater than $n(k+1)$.

Choose k to be the integer such that $nk \leq N < n(k+1)$. By the cellular approximation theorem (1.6.20) we have that the diagonal map $\Delta : X \to X^{k+1}$ is homotopic to a cellular map $\tilde{\Delta} : X \to X^{k+1}$, and since $\dim(X) = N$, we have that the image of Δ is contained in the N-skeleton of X^{k+1}, which is a subset of $T^{k+1}(X)$, hence we may define $\Delta' : X \to T^{k+1}(X)$ such that $j\Delta' = \tilde{\Delta} \simeq \Delta$, in which $j : T^{k+1}(X) \hookrightarrow X^{k+1}$ is the canonical inclusion, hence, by Whitehead's definition of category, we conclude that

$$\mathrm{cat}(X) \leq k+1 \leq \frac{N}{n} + 1 = \frac{\dim(X)}{n} + 1.$$

$\square$

As a final remark let us show that Whitehead's definition of category is a homotopy invariant.

Theorem 2.3.6. If X and Y are based spaces of the same homotopy type, then $\mathrm{cat}^{\mathrm{Wh}}(X) = \mathrm{cat}^{\mathrm{Wh}}(Y)$.

Proof. First notice that if X and Y are of the same homotopy type, with homotopy equivalence $f : X \to Y$ and homotopy inverse $g : Y \to X$, then the map $f^n : X^n \to Y^n$ given by $f^n(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$, for all $(x_1, \dots, x_n) \in X^n$, is a homotopy equivalence with homotopy inverse $g^n : Y^n \to X^n$ defined in an analogous way. If $F : X \times I \to X$ is a homotopy such that $F(x,0) = gf(x)$ and $F(x,1) = x$, for all $x \in X$, then $F^n : X^n \times I \to X^n$ given by $F^n(x_1, \dots, x_n, t) = (F(x_1,t), \dots, F(x_n,t))$, for all $(x_1, \dots, x_n, t) \in X^n \times I$, is a homotopy with $F^n(x_1, \dots, x_n, 0) = g^n f^n(x_1, \dots, x_n)$ and $F^n(x_1, \dots, x_n, 1) = (x_1, \dots, x_n)$, for all $(x_1, \dots, x_n) \in X^n$. Furthermore, since we are dealing with based spaces, we clearly have $f(T^n(X)) \subset T^n(Y)$ and $F(T^n(X) \times I) \subset T^n(X)$, hence we may restrict f^n and g^n to define homotopy equivalences $f' : T^n(X) \to T^n(Y)$ and $g' : T^n(Y) \to T^n(X)$, so that $j_Y f' = f^n|_{T^n(X)}$, in which $j_Y : T^n(Y) \hookrightarrow Y^n$ is the canonical inclusion.

Now suppose $\mathrm{cat}^{\mathrm{Wh}}(X) = n$, and let $\Delta'_X : X \to T^n(X)$ be a map such that the diagram

$$\begin{tikzcd} & T^n(X) \arrow[d, hook, "j_X"] \\ X \arrow[ur, "\Delta'_X"] \arrow[r, "\Delta_X"] & X^n \end{tikzcd}$$

homotopy commutes.

Define $\Delta'_Y : Y \to T^n(Y)$ by $\Delta'_Y = f' \Delta'_X g$. Notice that $\Delta_Y f = f^n \Delta_X$, hence $g^n \Delta_Y f = g^n f^n \Delta_X \simeq \Delta_X$, and that $j_Y f' = f^n|_{T^n(X)}$ implies that $g^n(j_Y f') \simeq j_X$.

We have

$$g^n(j_Y \Delta'_Y) = g^n(j_Y f' \Delta'_X g) = (g^n j_Y f')\Delta'_X g \simeq j_X \Delta'_X g \simeq \Delta_X g \simeq g^n \Delta_Y f g \simeq g^n \Delta_Y,$$

hence

$$j_Y \Delta'_Y \simeq f^n g^n (j_Y \Delta'_Y) \simeq f^n g^n \Delta_Y \simeq \Delta_Y,$$

and in conclusion $\mathrm{cat}^{\mathrm{Wh}}(Y) \leq n = \mathrm{cat}^{\mathrm{Wh}}(X)$. The proof of the opposite inequality is completely analogous. □

2.4 Ganea's formulation

The formulation of category presented in this section is due to Tudor Ganea, and like Whitehead's formulation, it is very important from the point of view of homotopy theory.

Definition 2.4.1 ((CORNEA *et al.*, 2003))**.** Define $\tilde{G}_n(X)$ to be the homotopy pullback in the diagram

$$\begin{tikzcd} \tilde{G}_n(X) \arrow[r] \arrow[d, "\tilde{p}_n"] & T^n(X) \arrow[d, "j_n"] \\ X \arrow[r, "\Delta"] & X^n \end{tikzcd}$$

we can take $\tilde{p}_n$ to be a fibration, which is called the **nth Ganea's fibration**.

Proposition 2.4.2 ((CORNEA *et al.*, 2003))**.** Given $\tilde{p}_n : \tilde{G}_n(X) \to X$ the nth Ganea fibration of a space X, there exists a section $s : X \to \tilde{G}_n(X)$ of $\tilde{p}_n$ if and only if $\mathrm{cat}^{\mathrm{Wh}}(X) \leq n$.

Proof. Remember, from proposition 1.4.5, that for a fibration as $\tilde{p}_n$, the existence of a section is equivalent to the existence of a section up to homotopy, i.e., a map $s : X \to \tilde{G}_n(X)$ such that $\tilde{p}_n s \simeq id_X$.

With this in hands, suppose $\mathrm{cat}^{\mathrm{Wh}}(X) \leq n$. Let $\Delta' : X \to T^n(X)$ be the map such that $j_n \Delta' \simeq \Delta$ (as in definition 2.3.2). Then the unbroken arrow diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "s"] \arrow[drr, bend left, "\Delta'"] \arrow[ddr, bend right, "id_X"'] & & \\
& \tilde{G}_n(X) \arrow[r, "\delta"] \arrow[d, "\tilde{p}_n"] & T^n(X) \arrow[d, "j_n"] \\
& X \arrow[r, "\Delta"] & X^n
\end{tikzcd}$$

commutes homotopically, and by the homotopy pullback property there exists a map $s : X \to \tilde{G}_n(X)$, so that the diagram homotopy commutes. Hence $\tilde{p}_n s \simeq id_X$, and by our initial remark there is a section of $\tilde{p}_n$.

Conversely, suppose $s : X \to \tilde{G}_n(X)$ is a section of $\tilde{p}_n$, and define $\Delta' = \delta s$, then $j_n \Delta' = j_n \delta s \simeq \Delta \tilde{p}_n s = \Delta$, whence $\mathrm{cat}^{\mathrm{Wh}}(X) \leq n$. □

It is evident, by proposition 2.4.2 and theorem 2.3.4, that we may define the category of a path-connected normal space with non-degenerate basepoint as being the smallest integer such that a section of $\tilde{p}_n$ exists. This will indeed be the final formulation, but before we do so let us introduce a useful construction for obtaining the Ganea spaces and fibrations, which is the way Ganea himself originally did.

Definition 2.4.3 ((CORNEA *et al.*, 2003), The Fibre-Cofibre Construction)**.** Let X be a topological space with basepoint $*$, we shall build the Ganea fibrations $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$ inductively.

First consider $F_1(X) \xrightarrow{i_1} G_1(X) \xrightarrow{p_1} X$ to be the fibration $\Omega X \xhookrightarrow{i_1} PX \xrightarrow{p_1} X$ in which ΩX is the loop space of X, PX is the space of all paths starting on $*$, i_1 is simply the inclusion map and p_1 maps each path to its final point, $p_1(\gamma) = \gamma(1)$.

Suppose we have defined the fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$, let us use it to construct the $n+1$ case. Let $C(i_n) = G_n(X) \cup C(F_n(X))$ be the mapping cone of i_n. Define $q_n : C(i_n) \to X$ to be the extension of p_n by mapping $C(F_n(X))$ into $*$.

Now we can turn q_n into a fibration $p_{n+1} : G_{n+1}(X) \to X$, written explicitly we have

$$G_{n+1}(X) \doteq \left\{ (x, \gamma) \in C(i_n) \times X^I \ : \ q_n(x) = \gamma(0) \right\},$$

and $p_{n+1}(x,\gamma) = \gamma(1)$, for all $x \in C(i_n)$, and we have the homotopy commutative diagram

$$\begin{tikzcd} C(i_n) \arrow[r, "h_n"] \arrow[dr, "q_n"'] & G_{n+1}(X) \arrow[d, "p_{n+1}"] \\ & X \end{tikzcd}$$

with h_n defined by $h_n(x) = (x, \gamma_{q_n(x)})$, in which γ_α is the constant path at α. Then h_n is a homotopy equivalence, and we have the more complete homotopy commutative diagram

$$\begin{tikzcd} G_n(X) \arrow[r, hook] \arrow[dr, "p_n"'] & C(i_n) \arrow[r, "\simeq"] \arrow[d, "q_n"] & G_{n+1}(X) \arrow[dl, "p_{n+1}"] \\ & X & \end{tikzcd}$$

Hence inductively we build the following diagram of Ganea fibrations

$$\begin{tikzcd} \Omega X = F_1(X) \arrow[r] \arrow[d, "i_1"] & F_2(X) \arrow[r] \arrow[d, "i_2"] & F_3(X) \arrow[r] \arrow[d, "i_3"] & \cdots \arrow[r] & F_n(X) \arrow[r] \arrow[d, "i_n"] & \cdots \\ PX = G_1(X) \arrow[r] \arrow[d, "p_1"] & G_2(X) \arrow[r] \arrow[d, "p_2"] & G_3(X) \arrow[r] \arrow[d, "p_3"] & \cdots \arrow[r] & G_n(X) \arrow[r] \arrow[d, "p_n"] & \cdots \\ X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & \cdots \arrow[r, equal] & X \arrow[r, equal] & \cdots \end{tikzcd}$$

The previous definition is functorial, and we can see this inductively: for the case $n = 1$, if $f : X \to Y$ is a based map, then $G_1(f) : G_1(X) = PX \to G_1(Y) = PY$ defined by $G_1(f)(\gamma) = f \circ \gamma$ is a based map, this makes G_1 into a covariant functor, and the following diagram commutes

$$\begin{tikzcd} G_1(X) \arrow[r, "G_1(f)"] \arrow[d, "p_1^X"] & G_1(Y) \arrow[d, "p_1^Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, suppose we have defined the covariant functor G_n, and let us use it to contruct G_{n+1}. Remember that

$$G_{n+1}(X) = \left\{(x,\gamma) \in G_n(X) \cup C(F_n(X)) \times X^I \ : \ q_n(x) = \gamma(0)\right\}.$$

Given $f : X \to Y$ we have $G_n(f) : G_n(X) \to G_n(Y)$ and we define $G_{n+1}(f) : G_{n+1}(X) \to G_{n+1}(Y)$ by

$$G_{n+1}(f)(x,\gamma) = \begin{cases} (G_n(f)(x), f\circ\gamma), & \text{if } x \in G_n(X); \\ ([G_n(f)(i_n(z)),t], f\circ\gamma), & \text{if } x = [z,t] \in C(F_n(X)). \end{cases}$$

G_{n+1} is well defined, and since G_n is a covariant functor, so is G_{n+1}. Summing this all up, we have the following proposition.

Proposition 2.4.4 ((CORNEA *et al.*, 2003))**.** G_n is a covariant functor for each $n \in \mathbb{N}$ and the following diagram commutes

$$\begin{tikzcd} G_n(X) \arrow[r, "G_n(f)"] \arrow[d, "p_n^X"] & G_n(Y) \arrow[d, "p_n^Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 2.4.5 ((CORNEA *et al.*, 2003) theorem 1.63)**.** For all n, we have that $G_n(X)$ and $\tilde{G}_n(X)$ have the same homotopy type, and the homotopy equivalence satisfies the following homotopy commutative diagram

$$\begin{tikzcd} G_n(X) \arrow[rr, "\simeq"] \arrow[dr, "p_n"'] & & \tilde{G}_n(X) \arrow[dl, "\tilde{p}_n"] \\ & X & \end{tikzcd}$$

Now we can give the new definition of category.

Definition 2.4.6 ((CORNEA *et al.*, 2003))**.** The category of a path-connected normal space X with non-degenerate basepoint, $\mathrm{cat}(X)$, is the smallest integer n such that there is a section for the nth Ganea fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$.

2.5 LS category lower bounds

The Toomer invariant

In this section, we shall introduce the Toomer invariant of a topological space, which is an important lower bound for category.

Definition 2.5.1 ((CORNEA *et al.*, 2003))**.** Let R be a commutivative ring, X a topological space and $p_n : G_n(X) \to X$ be the nth Ganea fibration. The **Toomer invariant of X with coefficients in R**, $e_R(X)$ is the least integer $k \geq 0$ such that $H_*(p_k) : H_*(G_n(X)) \to H_*(X)$ is an epimorphism.

Proposition 2.5.2 ((CORNEA *et al.*, 2003))**.** For path-connected normal spaces with non-degenerate basepoint the Toomer invariant is a lower bound for the LS category.

Proof. If $cat(X) \leq n$, by Ganea's definition, there is a section $s : X \to G_n(X)$ of p_n, i.e., $p_n \circ s = id_X$, which implies $H_*(p_n) \circ H_*(s) = id_{H_*(X)}$, hence $H_*(p_n)$ is an epimorphism and we have $e_R(X) \leq n$. □

Category Weight

Definition 2.5.3 ((CORNEA *et al.*, 2003))**.** Let X be a CW-complex and $u \in H^*(X;R)$, $u \neq 0$, a cohomology class. Then the **category weight** of u, $\text{wgt}(u)$, is the greatest integer k (or ∞) such that $H^*(p_{k-1})(u) = 0$, in which p_{k-1} is the $(k-1)$ Ganea fibration.

If $u \in H^*(X;R)$ is the zero class $(u = 0)$, we usually write $\text{wgt}(u) = \infty$, since $H^*(p_{k-1})(u) = 0$, for all k.

Proposition 2.5.4. For X a CW-complex and u a non-zero cohomology class, if $H^*(p_n)(u) \neq 0$, then $H^*(p_{n+1})(u) \neq 0$. And if $H^*(p_n)(u) = 0$, then $H^*(p_{n-1})(u) = 0$.

Proof. Both claims are easily shown with the following homotopy commutative diagram

$$\begin{tikzcd} G_n(X) \arrow[rr, "f_n"] \arrow[dr, "p_n"'] & & G_{n+1}(X) \arrow[dl, "p_{n+1}"] \\ & X & \end{tikzcd}$$

from which we get $H^*(f_n) \circ H^*(p_{n+1}) = H^*(p_n)$. So, $H^*(p_n)(u) \neq 0$ implies $H^*(p_{n+1})(u) \neq 0$ and $H^*(p_{n+1})(u) = 0 \implies H^*(p_n)(u) = 0$. □

Proposition 2.5.4 shows that category weight can also be defined to be the smallest integer k such that $H^*(p_k)(u) \neq 0$.

Proposition 2.5.5 ((CORNEA *et al.*, 2003))**.** If X is a path-connected CW complex, then $\text{cat}(X) \geq \text{wgt}(u)$, for any cohomology class $u \neq 0$.

Proof. Suppose $\text{cat}(X) \leq n$, then there is a section s for the nth Ganea fibration p_n, therefore p_n is surjective and $H^*(p_n)$ is injective, whence $H^*(p_n)(u) \neq 0$, and we conclude that $\text{wgt}(u) \leq n$. □

CHAPTER

3

TOPOLOGICAL COMPLEXITY

3.1 The Motion Planning Problem

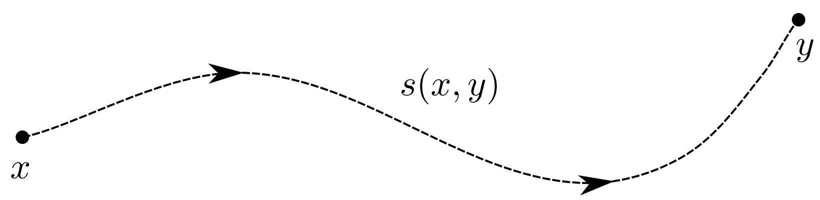
In this section, we will introduce what is called in robotics the **Motion Planning Problem**. The main reference on the topic is (LATOMBE, 1991).

For many decades, one big problem in robotics has been the development of autonomous robots, in short, these are robots that can perform highly complex tasks without detailed guidance on how to proceed on each step.

As a simple example, we have a robot that can move freely on a 2D surface. In the non-autonomous case, a person would have to control the robot's every move using a controlling system. In the alternative autonomous case, the robot would be programmed to choose paths on the surface automatically when given the task "go from point A to point B".

For any given mechanical system, we can describe every configuration of it as a point in a topological space X. In this scenario, the task of obtaining a motion planner on X reduces to obtaining a function $s : X \times X \to X^I$ from the Cartesian product $X \times X$ to the space of all paths in X, X^I.

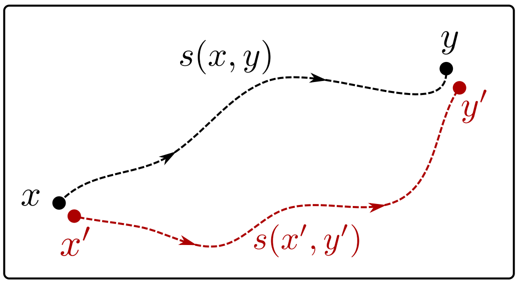
Figure 7 – Representation of a motion planning function $s : X \times X \to X^I$.

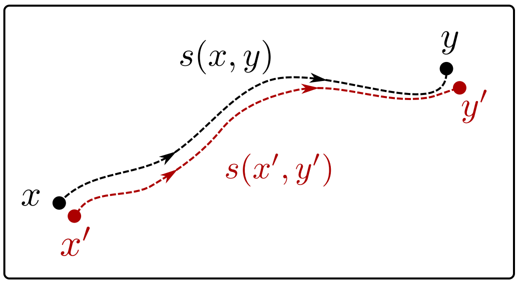


Source: Elaborated by the author.

If X is path-connected (as is expected for any space of configurations of a mechanical system), then we can always construct a function s as mentioned above. The difficulties arise when we require s to be continuous (considering the open-compact topology in X^I). This is a valid requirement for practical reasons since the continuity, in this case, implies that small perturbations in the initial and final points $(x,y) \in X \times X$ do not result in completely different paths in X^I.

Figure 8 – Two motion planning functions, on the left, a discontinuous one, and on the right, the desired continuous version, with small variations resulting in little change between the paths $s(x,y)$ and $s(x',y')$.





Source: Elaborated by the author.

For a more formal definition of a motion planner, first define the path fibration $\pi : X^I \to X \times X$ such that $\pi(\gamma) = (\gamma(0), \gamma(1))$, for all $\gamma \in X^I$, and we have the following definition.

Definition 3.1.1 ((FARBER, 2003)). In the conditions previously described, a **motion planner** in X is a section of the fibration π.

As we can see from the next theorem, it is a really strict class of spaces that assume a continuous motion planner as previously described.

Theorem 3.1.2 ((FARBER, 2003)). Let X be a topological space, then there exists a continuous motion planning function $s : X \times X \to X^I$, if and only if X is contractible.

Proof. First, suppose that there is a continuous motion planning function $s : X \times X \to X^I$, then fix a point $x_0 \in X$ and define the homotopy

$$\begin{aligned} F : X \times I &\to X \\ (x,t) &\mapsto s(x,x_0)(t), \end{aligned}$$

for all $x \in X$ and $t \in I$, which clearly contracts X into the point x_0.

Conversely, suppose X is contractible, then there exists a homotopy $F : X \times I \to X$ which contracts X into a single point x_0. With this, we can build the motion planner $s : X \times X \to X^I$ defined by

$$s(x,y)(t) = \begin{cases} F(x,2t), & 0 \le t \le 1/2; \\ F(y,2-2t), & 1/2 \le t \le 1. \end{cases}$$

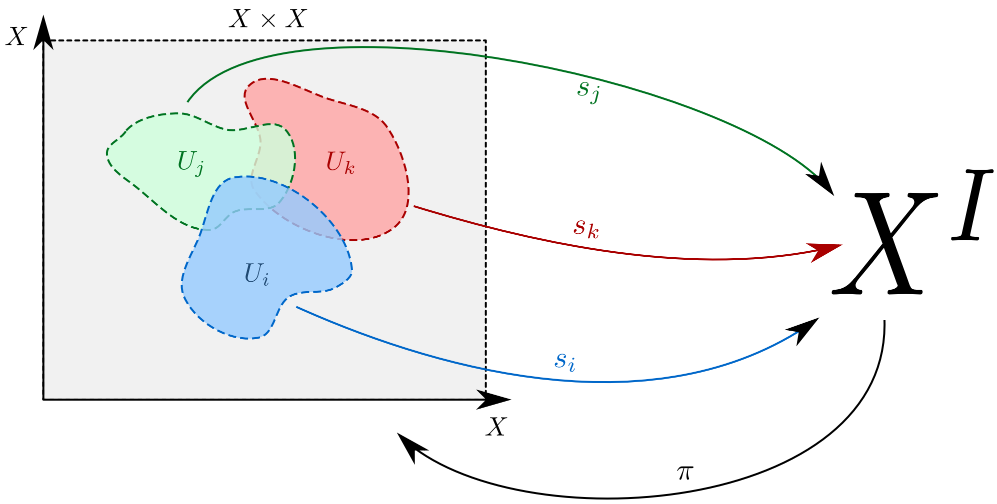
□

3.2 Topological Complexity

Inspired by the motion planning problem, Michael Farber introduced in 2003 a topological invariant (later it will be shown that it is even a homotopy invariant) with the idea of measuring how "complex" it would be to develop a motion planning algorithm for a given mechanical system. Since then, much work has been done, and many mathematicians are studying and developing the theory around this new invariant called Topological Complexity.

Definition 3.2.1 ((FARBER, 2003))**.** Given a topological space X, we say that the **Topological Complexity** of X, $\mathrm{TC}(X)$, is the smallest integer k such that there exist $U_1,\ldots,U_k$ an open covering of $X\times X$ and local sections $s_i : U_i \to X^I$ of the path fibration π, for all $i = 1,\ldots,n$.

Figure 9 – Representation of the decomposition of $X\times X$, as in definition 3.2.1.



Source: Elaborated by the author.

Notice that the basic idea in definition 3.2.1 is to decompose the space $X\times X$ into open subspaces in each of which we can find a section of π, that is, a local continuous motion planning function.

Clearly, for contractible spaces we will have $\mathrm{TC}(X) = 1$. This follows immediately from theorem 3.1.2, and means that we do not need to decompose the space into smaller parts to find a motion planning algorithm.

Remark 3.2.2. Notice that in definition 3.2.1, the Topological Complexity measures exactly how many open subspaces are needed to cover the whole space with local sections of π, it is important to mention that some authors use the so-called "normalized version" of topological complexity, which is one less than the original definition. There is no huge advantage in using one or the other definition, but in all that follows we shall use the original definition.

As mentioned earlier the Topological Complexity only equals 1 when the space is contractible. Now we see an even more general result, not only for $TC(X) = 1$, but for any case, it only depends on the homotopy type of the space.

Theorem 3.2.3 ((FARBER, 2003))**.** The topological complexity of X, $\mathrm{TC}(X)$, depends only on its homotopy type.

Proof. Suppose X and Y are of the same homotopy type, with $f : X \to Y$ a homotopy equivalence and $g : Y \to X$ its homotopy inverse. Let $\mathrm{TC}(X) = k$, and consider an open covering $U_1, \dots, U_k$ of $X \times X$ with local sections $s_i : U_i \to X^I$. By defining $V_i = (g \times g)^{-1}(U_i)$, we have an open covering of $Y \times Y$, and we can explicitly define local sections on each V_i. To do so, first consider $F : Y \times I \to Y$ to be a homotopy starting on id_Y and ending on $f \circ g$, then we define the section $r_i : V_i \to PY$ as

$$r_i(A,B)(t) = \begin{cases} F(A,3t), & 0 \leq t \leq 1/3; \\ f(s_i(g(A),g(B))(3t-1)), & 1/3 \leq t \leq 2/3; \\ F(B,3-3t), & 2/3 \leq t \leq 1. \end{cases}$$

Hence, we found k local sections of the path fibration in Y, therefore $\mathrm{TC}(Y) \leq k = TC(X)$. The opposite inequality is completely analogous, whence $TC(Y) = TC(X)$ □

Lemma 3.2.4 ((FARBER, 2003))**.** For any topological space X we have

$$\mathrm{cat}(X) \leq \mathrm{TC}(X).$$

Proof. Suppose $\mathrm{TC}(X) = k$, and let $U_1, \dots, U_k$ be the open covering in $X \times X$ with local sections, $s_i : U_i \to X^I$, of the path fibration. Fix a point $a \in X$, and define $f : X \to X \times X$ by $f(x) = (a,x)$, for all $x \in X$. This function is clearly continuous, hence $V_i = f^{-1}(U_i)$ is open in X. Moreover, since $U_1, \dots, U_k$ is an open covering of $X \times X$, $V_1, \dots, V_k$ is an open covering of X. Finally, notice that each V_i is categorical, to see that (if $V_i \neq \emptyset$), consider the homotopy $F : V_i \times I \to X$ given by $F(x,t) = s_i(x,a)(t)$, for all $x \in V_i$ and $t \in I$. Whence we conclude that $\mathrm{cat}(X) \leq k = \mathrm{TC}(X)$. □

Lemma 3.2.5. For any path-connected topological space X we have $\mathrm{TC}(X) \leq \mathrm{cat}(X \times X)$

Proof. Suppose $U \subset X \times X$ is an open categorical subset of $X \times X$. Let $F : U \times I \to X \times X$ be a contracting homotopy, i.e., $F(u,0) = u$ and $F(u,1) = (a,a)$, for some $a \in X$ and for all $u \in U$. Notice that we may choose any point $a \in X$, since X is path connected. Then we can define $s : U_i \to X^I$ by

$$s(u)(t) = \begin{cases} F_1(u, 2t), & \text{for } 0 \leq t \leq 1/2; \\ F_2(u, 2-2t), & \text{for } 1/2 \leq t \leq 1; \end{cases}$$

in which $F = (F_1, F_2)$. Clearly, s is a local section of the path fibration $\pi : X^I \to X \times X$. Hence, if $\operatorname{cat}(X \times X) = n$ and $U_1, \ldots, U_n$ is a categorical covering of $X \times X$, then this covering also satisfies the topological complexity's conditions, hence $\mathrm{TC}(X) \leq n = \operatorname{cat}(X \times X)$. □

Theorem 3.2.6 ((FARBER, 2003))**.** If X is a path-connected, paracompact and locally contractible space, such that $X \times X$ is completely normal (in particular, if X is a path connected, locally contractible CW complex or metric space), then

$$\operatorname{cat}(X) \leq \mathrm{TC}(X) \leq 2\operatorname{cat}(X) - 1,$$

and

$$\mathrm{TC}(X) \leq 2\dim(X) + 1.$$

Proof. We already proved in lemma 3.2.4 that $\operatorname{cat}(X) \leq TC(X)$. And by combining lemma 3.2.5 and theorem 2.2.4 we get

$$\mathrm{TC}(X) \leq \operatorname{cat}(X \times X) \leq 2\operatorname{cat}(X) - 1.$$

The other inequality follows from the fact that $\operatorname{cat}(X) \leq \dim(X) + 1$ (see theorem 2.1.4). □

Theorem 3.2.7 ((FARBER, 2004))**.** If X is an $(n-1)$-connected CW complex. Then

$$\mathrm{TC}(X) \leq \frac{2\dim(X)}{n} + 1.$$

Proof. In the conditions of the theorem we have that $\operatorname{cat}(X) \leq \frac{\dim(X)}{n} + 1$ (see theorem 2.3.5), this together with the inequality $\mathrm{TC}(X) \leq 2\operatorname{cat}(X) - 1$ proves the theorem. □

Corollary 3.2.8 ((FARBER, 2004))**.** If X is a simply connected CW complex, then

$$\mathrm{TC}(X) \leq \dim(X) + 1.$$

Proof. Simply connected means 1-connected, so $n = 2$ in the previous theorem gives the desired result. □

Theorem 3.2.9 ((FARBER, 2003))**.** If X and Y are metrizable path-connected spaces, then

$$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1.$$

Proof. Denote $\mathrm{TC}(X) = m$ and $\mathrm{TC}(Y) = n$,let $U_1, \ldots, U_m$ be an open covering of $X \times X$ such that there exists a continuous motion planning $s_i : U_i \to X^I$, for each $i = 1, \ldots, m$. Since $X \times X$ is metrizable, it is paracompact, hence there is a partition of unity $f_i : X \times X \to [0,1]$, $i = 1, \ldots, m$, subordinate to the open covering $\{U_i\}_i$.

Analogously, let $V_1, \ldots, V_n$ be an open covering of $Y \times Y$, $\sigma_j : V_j \to PY$, $j = 1, \ldots, n$ a continuous motion planning, and $f_j : Y \times Y \to [0,1]$ a partition of unity subordinate to the covering.

Given a pair $S \times T \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ we define

$$W(S,T) = \left\{ (x,y,z,w) \in (X \times Y) \times (X \times Y) \ : \begin{array}{c} f_i(x,z)g_j(y,w) > f_{i'}(x,z)g_{j'}(y,w), \\ \forall\ (i,j) \in S \times T \text{ and } (i',j') \notin S \times T \end{array} \right\}$$

Notice that:

(a) Every $W(S,T)$ is open.

This is indeed true, since each $f_i g_j - f_{i'} g_{j'}$ is a continuous function, we have that $W(S,T)$ is a finite intersection of open sets

$$W(S,T) = \bigcap_{\substack{(i,j) \in S \times T \\ (i',j') \notin S \times T}} (f_i g_j - f_{i'} g_{j'})^{-1}((0,\infty))$$

and therefore it is itself open.

(b) If $S \times T \not\subseteq S' \times T'$ and $S \times T \not\supseteq S' \times T'$, then $W(S,T) \cap W(S',T') = \emptyset$.

Let $(i,j) \in S \times T \setminus S' \times T'$ and $(i',j') \in S' \times T' \setminus S \times T$. If $(x,y,z,w) \in W(S,T)$, then $f_i(x,z)g_j(y,w) > f_{i'}(x,z)g_{j'}(y,w)$, thus $(x,y,z,w) \notin W(S',T')$, whence $W(S,T) \cap W(S',T') = \emptyset$.

(c) If $(i,j) \in S \times T$, then $W(S,T) \subset U_i \times V_j$, and there is a continuous motion planning function $\rho_{ij} : U_i \times V_j \to P(X \times Y)$, which can be restricted to $W(S,T)$.

Notice that we are considering the natural homeomorphism between $(X \times Y) \times (X \times Y)$ and $(X \times X) \times (Y \times Y)$, so

$$U_i \times V_j \doteq \{(x,y,z,w) \in (X \times Y) \times (X \times Y) \mid (x,z) \in U_i,\ (y,w) \in V_j\}.$$

Suppose $(i,j) \in S \times T$, if $(x,y,z,w) \in W(S,T)$, then $f_i(x,z)g_j(y,w) > 0$, which implies $(x,z) \in U_i$ and $(y,w) \in V_j$, hence $W(S,T) \subset U_i \times V_j$.

The continuous motion planning on $U_i \times V_j$ is defined as

$$\begin{aligned} \rho_{ij} : U_i \times V_j &\longrightarrow P(X \times Y) \\ (x,y,z,w) &\longmapsto \rho_{ij}(x,y,z,w) : [0,1] \to X \times Y \\ &\qquad\qquad\qquad\qquad t \ \mapsto (s_i(x,z)(t), \sigma_j(y,w)(t)) \end{aligned}$$

(d) The sets $W(S,T)$ cover $(X\times Y)\times(X\times Y)$.

Let $(x,y,z,w)\in(X\times Y)\times(X\times Y)$ be any point, and consider $S\times T\subset\{1,\ldots,m\}\times\{1,\ldots,n\}$ such that S consists of the indices i for which $f_i(x,z)=\max_k\{f_k(x,z)\}$, and T consists of the indices j such that $g_j(y,w)=\max_l\{g_l(y,w)\}$, then we clearly have $(x,y,z,w)\in W(S,T)$.

Now, define the open set

$$W_k=\bigcup_{|S|+|T|=k}W(S,T),$$

for $k=2,3,\ldots,n+m$. Notice that, from item (c), we have that $W(S,T)\cap W(S',T')=\emptyset$, whenever $|S|+|T|=k=|S'|+|T'|$, whence there exists a continuous motion planning on each W_k, and we conclude that $\mathrm{TC}(X\times Y)\leq m+n-1$.

□

Since the Cartesian product of metrizable path-connected spaces is still metrizable path-connected, we get, as a consequence of theorem 3.2.9, the following corollary.

Corollary 3.2.10. If $X_1,\ldots,X_n$ are metrizable path-connected spaces, then

$$\mathrm{TC}(X_1\times X_2\times\cdots\times X_n)\leq\mathrm{TC}(X_1)+\mathrm{TC}(X_2)+\cdots+\mathrm{TC}(X_n)-n+1.$$

3.3 Schwarz Genus and a Cohomological Bound for TC

In what follows, we shall deduce a sophisticated relation between the topological complexity of a space X and its cohomology ring $H^*(X\times X;R)$. This relation was first shown by Albert Schwarz while dealing with a numerical invariant nowadays called the Schwarz Genus of a fibration, which is a generalization of Farber's Topological Complexity.

Definition 3.3.1 ((SCHWARZ, 1966))**.** Given a fibration $\mathfrak{B}=(E,B,F,p)$, define the genus of $\mathfrak{B}$, $g(\mathfrak{B})$ or $g(p)$, to be the smallest integer k such that there is an open covering $U_1,\ldots,U_k$ of the base B with local sections $s_i:U_i\to E$ of p defined for all i.

Remark 3.3.2. Notice that the Topological Complexity is simply the Schwarz Genus of the path fibration as defined in example 1.4.3.

One might have already noticed the similarities between topological complexity and the Lusternik-Schnirelmann category just by looking at both definitions. In fact, both concepts are specific cases of the Schwarz Genus. For the Topological Complexity, we have already seen that it is simply the genus of the path fibration. The following proposition shows that the LS category is also the genus of a specific fibration.

Proposition 3.3.3. Let (X,x_0) be a path connected based space, consider the space

$$P_0X = \{\gamma \in X^I \mid \gamma(0) = x_0\}$$

and define the fibration $p : P_0X \to X$ by $p(\gamma) = \gamma(1)$, then $\mathrm{cat}(X) = g(p)$.

Proof. Suppose $U \subset X$ is a subset for which there is a local section of p, i.e., there is a map $s : U \to P_0X$ such that $s(u)(0) = x_0$ and $s(u)(1) = u$, for all $u \in U$. So we can define $F : U \times I \to X$ by $F(u,t) = s(u)(t)$, for all $u \in U$ and $t \in I$, for which we have $F(u,0) = x_0$ and $F(u,1) = u$, for all $u \in U$, hence U is categorical.

Conversely, suppose $U \subset X$ is categorical, then there is $F : U \times I \to X$ such that $F(u,0) = x_0$ and $F(u,1) = u$, for all $u \in U$, thus we can define a local section of p by $s : U \to P_0X$ given by $s(u)(t) = F(u,t)$, for all $u \in U$ and $t \in I$. □

Remark 3.3.4. Notice that, if $\mathfrak{B} = (E,B,F,p)$ is a fibration of genus 1, $g(\mathfrak{B}) = 1$, this means that there is a global section for p, which we will denote by $s : B \to E$. Since we have $ps = id_B$, the functoriality of cohomology implies that $s_*^* p_*^* = id_{H^*(B;R)}$, in which R is any commutative ring. This means that p_*^* is injective, in other words, $\ker(p_*^*) = 0$.

From remark 3.3.4, we understand that if a fibration $p : E \to B$ has genus 1, then its cohomology induced homomorphism has a trivial kernel. In what follows we shall prove a more general statement, namely that, if $g(p) = n$, then $(\ker(p_*^*))^n = 0$, in which

$$(\ker(p_*^*))^n = \{u_1 \smile u_2 \smile \cdots \smile u_n \mid u_i \in \ker(p_*^*)\}.$$

Definition 3.3.5 ((SCHWARZ, 1966))**.** Let R be a commutative ring and $\mathfrak{B} = (E,B,F,p)$ a fibration. The **length of the fibration** $\mathfrak{B}$ with coefficients in R, denoted $l_R(\mathfrak{B})$ or $l_R(p)$, is the greatest integer n for which $(\ker(p_*^*))^n \neq 0$.

From remark 3.3.4 when $g(\mathfrak{B}) = 1$, we have $1 = g(\mathfrak{B}) > l_R(\mathfrak{B}) = 0$. We wish to show that this inequality between the genus and the length of a fibration is valid in general. For this purpose, we first prove the succeeding lemma.

Lemma 3.3.6 ((SCHWARZ, 1966))**.** Let R be a PID, $\mathfrak{B} = (E,B,F,p)$ a fibration, $\{B_1,\ldots,B_n\}$ an open covering of the base B, and $\mathfrak{B}_i = (p^{-1}(B_i),B_i,F_i,p_i)$ the restriction fibrations, for $i = 1,\ldots,n$. Then

$$l_R(\mathfrak{B}) < n + \sum_{i=1}^{n} l_R(\mathfrak{B}_i).$$

Proof. First of all, notice that each $\mathfrak{B}_i$ is in fact a fibration, by what we have proven in proposition 1.4.2.

Let $\lambda_i : B_i \hookrightarrow B$ be the canonical inclusions, and $s_i \doteq l_R(\mathfrak{B}_i)$, for $i = 1,\ldots n$. And denote $M \doteq n + \sum_{i=1}^{n} s_i$.

By contradiction, let us assume that

$$l_R(\mathfrak{B}) \geq n + \sum_{i=1}^{n} s_i = M,$$

so there is a non trivial element in $(\ker(p_*^*))^M$, in other words, there are cohomology classes $u_j \in \ker(p_*^*) \subset H^*(B;R)$, for $j = 1, \ldots, M$, such that $u_1 \smile \cdots \smile u_M \neq 0$.

Consider the following cohomology classes

$$\begin{aligned}
v_1 &= u_1 \smile u_2 \smile \cdots \smile u_{1+s_1} \\
v_2 &= u_{s_1+2} \smile \cdots \smile u_{(s_1+2)+s_2} \\
&\vdots \\
v_i &= u_{s_1+\cdots+s_{i-1}+i} \smile \cdots \smile u_{(s_1+\cdots+s_{i-1}+i)+s_i} \\
&\vdots \\
v_n &= u_{s_1+\cdots+s_{n-1}+n} \smile \cdots \smile u_{(s_1+\cdots+s_{n-1}+n)+s_n}
\end{aligned}$$

We clearly have $\lambda_{i*}^*(v_i) = 0$, for $i = 1, \ldots, n$, otherwise we would have a non zero product,

$$\lambda_{i*}^*(v_i) = \lambda_{i*}^*(u_{s_1+\cdots+s_{i-1}+i}) \smile \cdots \smile \lambda_{i*}^*(u_{(s_1+\cdots+s_{i-1}+i)+s_i}),$$

of $s_i + 1$ elements in $\ker(p_{i*}^*)$, notice that they are in fact in $\ker(p_{i*}^*)$, since $p_*^*(u_j) = 0$ and $p_{i*}^*\lambda_{i*}^* = (\lambda_i p_i)_*^*$, and $\lambda_i p_i$ is simply a restriction of p to $p^{-1}(U_i)$, so by proposition 1.3.4 we have $p_{i*}^*\lambda_{i*}^*(u_j) = 0$. Hence, if $\lambda_{i*}^*(v_i) \neq 0$, this would contradict the fact that $s_i = l_R(\mathfrak{B}_i)$.

Remember that, in proposition 1.3.11, we saw that for an open covering, such as $\mathscr{U} = \{B_1, \ldots, B_n\}$, we have that $H_n^{\mathscr{U}}(B;R)$ is isomorphic to the usual $H_n(B;R)$, which means we can consider the chain complex $C^{\mathscr{U}}(B;R)$ with only the singular simplexes that have its image entirely contained in one of the elements B_i of the base $\mathscr{U}$. If we dualize such a chain complex (by the Hom(_,R) functor) we can produce cohomology modules, which by the Universal Coefficients Theorem are isomorphic to the usual cohomology. So when talking about cohomology classes, we may always restrict ourselves to classes $[\varphi]$, with $\varphi : C_n^{\mathscr{U}} \to R$, this will be useful in the remainder of the proof.

Since $\lambda_i : B_i \hookrightarrow B$ is the inclusion, we have $\lambda_{i*}^* : H^*(B;R) \to H^*(B_i;R)$, and if $\lambda_{i*}^*([\varphi]) = 0$, for some k-cocycle φ, then $[\varphi\lambda_{i\#}] = [\varphi|_{C_k(B_i;R)}] = 0$, which means $\varphi|_{C_k(B_i;R)} = \delta\psi$ for some $\psi \in C_{k-1}(B_i;R)$.

Notice that since $\lambda_{i*}^*(v_i) = 0$, if we define $v = v_1 \smile \cdots \smile v_n$, we will have $\lambda_{i*}^*(v) = 0$, for all $i = 1, \ldots n$, since by proposition 1.3.28 we have

$$\begin{aligned}
\lambda_{i*}^*(v) &= \lambda_{i*}^*(v_1) \smile \lambda_{i*}^*(v_2) \smile \cdots \smile \lambda_{i*}^*(v_i) \smile \cdots \smile \lambda_{i*}^*(v_n) \\
&= \lambda_{i*}^*(v_1) \smile \lambda_{i*}^*(v_2) \smile \cdots \smile 0 \smile \cdots \smile \lambda_{i*}^*(v_n) \\
&= 0
\end{aligned}$$

Now let us show that $\lambda_{i*}^*(v) = 0$, for all i, implies that $v = 0$. Suppose $v \in H^k(B;R)$, and let $v = [\varphi]$, in which $\varphi \in (C_k^{\mathscr{U}}(B;R))^*$, so φ is an R-homomorphism $\varphi : C_k^{\mathscr{U}}(B;R) \to R$, whence

$$0 = \lambda_{i*}^*([\varphi]) = [\lambda_i^\#(\varphi)] = [\varphi\lambda_{i\#}] = [\varphi|_{C_k(B_i;R)}],$$

for the last equality we are using the fact that $\lambda_i : B_i \hookrightarrow B$ is an inclusion, hence $\lambda_{i\#} : C_k(B_i;R) \to C_k^{\mathscr{U}}(B;R)$ is also an inclusion. For a cohomology class like $[\varphi|_{C_k(B_i;R)}]$ to be zero, means that

$$\varphi|_{C_k(B_i;R)} = \delta\psi_i = \psi_i\partial,$$

for some R-homomorphism $\psi_i : C_{k-1}(B_i;R) \to R$.

Notice that $C_{k-1}^{\mathscr{U}}(B;R)$ can be viewed as a direct sum $\bigoplus_i C_{k-1}(B_i;R)$, hence the collection of homomorphisms $\{\psi_i\}_i$ defines a new homomorphism

$$\psi : C_{k-1}^{\mathscr{U}}(B;R) \to R,$$

such that $\psi|_{C_{k-1}(B_i;R)} = \psi_i$. Now we have

$$(\delta\psi)|_{C_k(B_i;R)} = (\psi\partial)|_{C_k(B_i;R)} = \psi|_{C_{k-1}(B_i;R)}\partial|_{C_k(B_i;R)} = \psi_i\partial|_{C_k(B_i;R)} = \delta\psi_i = \varphi|_{C_k(B_i;R)},$$

whence $\varphi = \delta\psi$, therefore $[\varphi] = 0$, and we conclude that

$$0 = [\varphi] = v = v_1 \smile \cdots \smile v_n = u_1 \smile \cdots \smile u_M,$$

contradicting our initial hypothesis. □

Theorem 3.3.7 ((SCHWARZ, 1966))**.** Let R be a PID and $\mathfrak{B} = (E,B,F,p)$ a fibration, then $g(\mathfrak{B}) > l_R(\mathfrak{B})$.

Proof. If $g(\mathfrak{B}) = \infty$ there is nothing to be proven. Suppose $g(\mathfrak{B}) = n < \infty$, and let $\{B_1,\ldots,B_n\}$ be an open covering of B such that there are local sections of p in each B_i, i.e., there are continuous functions $s_i : B_i :\to E$ such that the composition $ps_i : B_i \hookrightarrow B$ becomes the canonical inclusion.

Let $\mathfrak{B}_i = (p^{-1}(B_i), B_i, F_i, p_i)$ be the restriction fibrations (as in proposition 1.4.2), for $i = 1,\ldots,n$, and notice that the section $s_i : B_i \to E$ must have all its image in $p^{-1}(B_i)$, hence it restricts to a map $r_i : B_i \to p^{-1}(B_i)$, so that $p_i r_i = id_{B_i}$, so for the induced maps in cohomology we have $s_{i*}^* p_{i*}^* = id_{B_{i*}}^* = id_{H^*(B_i;R)}$, hence p_{i*}^* is injective, in other words, $\ker(p_{i*}^*) = 0$, which implies that $l_R(\mathfrak{B}_i) = 0$.

Finally, by lemma 3.3.6 we get

$$l_R(\mathfrak{B}) < n + \sum_{i=1}^n l_R(\mathfrak{B}_i) = n = g(\mathfrak{B}).$$

□

Now we can translate what we have proven to some results regarding the Topological Complexity. Since the Topological Complexity is simply the Schwarz genus of the path fibration $\pi : X^I \to X \times X$, we get the following corollary, as a direct consequence of theorem 3.3.7.

Corollary 3.3.8. Given a space X and a PID R, we have $TC(X) = g(\pi) > l_R(\pi)$.

Now, let us analyze what happens when we take the cohomology's coefficient ring to be a field K.

First, recall that in this case, all the homology and cohomology modules are free, so the Tor factor in the Künneth Theorem vanishes, and we end up with the cross product being an isomorphism

$$\begin{aligned} \mu : H^*(X;K) \otimes H^*(X;K) &\to H^*(X \times X;K) \\ (u,v) &\mapsto u \times v. \end{aligned}$$

As sown previously, the relation between the cup product and the cross product is given by $u \smile v = \Delta_*^*(u \times v)$ (see section 1.3), represented in the following commutative diagram

$$\begin{tikzcd} H^*(X;K) \otimes H^*(X;K) \arrow[r, "\mu"] \arrow[dr, "\smile"'] & H^*(X \times X;K) \arrow[d, "\Delta_*^*"] \\ & H^*(X;K) \end{tikzcd}$$

Notice that if we define $\alpha : X \to X^I$ by $\alpha(x)(t) = x$, for all $x \in X$ and $t \in I$ (i.e, the constant path at each point x), then we get the following commutative diagram, and its cohomology induced version

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\Delta"'] & X^I \arrow[d, "\pi"] \\ & X \times X \end{tikzcd} \qquad \begin{tikzcd} H^*(X;K) & H^*(X^I;K) \arrow[l, "\alpha_*^*"'] \\ & H^*(X \times X;K) \arrow[u, "\pi_*^*"] \arrow[ul, "\Delta_*^*"] \end{tikzcd}$$

thus we get the following commutative diagram

$$\begin{tikzcd} H^*(X;K) \otimes H^*(X;K) \arrow[r, "\mu"] \arrow[ddr, "\smile"'] & H^*(X \times X;K) \arrow[d, "\pi_*^*"] \\ & H^*(X^I;K) \arrow[d, "\alpha_*^*"] \\ & H^*(X;K) \end{tikzcd}$$

We already know that μ is an isomorphism, now we want to show that α_*^* is also one, which will imply that $\ker(\pi_*^*) \approx \ker(\smile)$.

To see that α_*^* must be an isomorphism, notice that $\alpha : X \to X^I$ is a homotopy equivalence with homotopy inverse $\beta : X^I \to X$ given by $\beta(\gamma) = \gamma(0)$, for all $\gamma \in X^I$. We clearly have $\beta\alpha = id_X$, and to show that $\alpha\beta \simeq id_{X^I}$ consider the homotopy $F : X^I \times I \to X^I$ given by

$$F(\gamma,s)(t) = \begin{cases} \gamma(0), \text{ for } 0 \leq t \leq 1-s; \\ \gamma(t-1+s), \text{ for } 1-s \leq t \leq 1. \end{cases}$$

To see that this is indeed a continuous function, remember that the sets

$$V(K,U) = \{\gamma \in X^I \mid \gamma(K) \subset U\},$$

with $K \subset I$ compact and $U \subset X$ open, form a subbase of the compact-open topology in X^I, hence a basic open set in X^I is a finite intersection, $W = \bigcap_{i=1}^m V(K_i,U_i)$, of such sets. It is easy to see by the definition of F that $F(\gamma,s)(K_i) \subset \gamma(K_i)$, for any $K_i \subset X$, hence

$$F\left(\left(\bigcap_i V(K_i,U_i)\right) \times I\right) \subset \bigcap_i V(K_i,U_i),$$

and since $(\bigcap_i V(K_i,U_i)) \times I$ is open in $X^I \times I$, we conclude that F is continuous. Finally, notice that $F(\gamma,0) = \alpha\beta(\gamma)$ and $F(\gamma,1) = \gamma$, for all $\gamma \in X^I$, hence $\alpha\beta \simeq id_{X^I}$, and α is in fact a homotopy equivalence, which implies that its cohomology induced homomorphism, α_*^*, is an isomorphism, and by the diagram above we conclude that

$$\ker(\pi_*^*) = \mu(\ker(\smile)).$$

Remember that μ is a ring homomorphism, where the product in $H^*(X;K) \otimes H^*(X;K)$ is defined as

$$(u_1 \otimes v_1).(u_2 \otimes v_2) = (-1)^{|v_1||u_2|}(u_1 \smile u_2) \otimes (v_1 \smile v_2),$$

so if we define

$$(\ker(\smile))^k = \{(u_1 \otimes v_1)\dots(v_k \otimes u_k) \mid (v_i \otimes u_i) \in \ker(\smile)\},$$

we clearly have

$$\mu((\ker(\smile))^k) = (\mu(\ker(\smile)))^k = (\ker(\pi_*^*))^k.$$

Definition 3.3.9 ((FARBER, 2003))**.** Using the same notation as above, we call $\ker(\smile) \subset H^*(X;K) \otimes H^*(X;K)$ the **ideal of zero divisors** of $H^*(X;K)$. The greatest integer k such that $(\ker(\smile))^k \neq 0$ is called the **zero divisors cup length** of $H^*(X;K)$, and is denoted by $\mathrm{zcl}_K(X)$.

From the discussion above we immediately get the following proposition.

Proposition 3.3.10. For any space X and field K, one has $\mathrm{zcl}_K(X) = l_K(\pi)$, in which $\pi : X^I \to X \times X$ is the path fibration.

Proof. This follows from the equation we obtained earlier $\mu((\ker(\smile))^k) = (\ker(\pi_*^*))^k$, since μ is an isomorphism, this equation clearly implies that $(\ker(\smile))^k = 0 \iff (\ker(\pi_*^*))^k = 0$, whence $\mathrm{zcl}_K(X) = l_K(\pi)$. □

Finally, the proposition above together with corollary 3.3.8 produces the following theorem.

Theorem 3.3.11 ((FARBER, 2003))**.** Given X a topological space and K a field, we have $\mathrm{TC}(X) > \mathrm{zcl}_K(X)$.

Example 3.3.12 ((FARBER, 2003))**.** We know that the n-sphere's cohomology ring $H^*(S^n;K)$ has two generators, let us denote them by $1 \in H^0(S^n;K)$ and $u \in H^n(S^n;K)$, in which 1 is the class of the cocycle $\varphi_1 : C_0(X;K) \to K$ such that $\varphi_1(\sigma) = 1_K$ for any 0-cycle σ, hence the class 1 is the multiplicative identity of the cohomology ring $H^*(S^n;K)$. If we define $a = 1 \otimes u - u \otimes 1 \in H^*(S^n;K) \otimes H^*(S^n;K)$, it is easy to see that a is a zero divisor since $\smile (a) = 1 \smile u - u \smile 1 = u - u = 0$. Another zero divisor is $b = u \otimes u$, since $u \smile u \in H^{2n}(S^n;K)$. We may compute $a.a = (1 \otimes u - u \otimes 1).(1 \otimes u - u \otimes 1)$, in which we get the following four terms:

$$
\begin{aligned}
-(1 \otimes u).(u \otimes 1) &= -(-1)^{n^2} u \otimes u = -(-1)^n u \otimes u = (-1)^{n-1} u \otimes u;\\
-(u \otimes 1).(1 \otimes u) &= -(-1)^0 u \otimes u = -u \otimes u;\\
(1 \otimes u).(1 \otimes u) &= (-1)^0 1 \otimes (u \smile u) = 0;\\
(u \otimes 1)(u \otimes 1) &= (-1)^0 (u \smile u) \otimes 1 = 0.
\end{aligned}
$$

The two last ones are zero since $u \smile u \in H^{2n}(S^n;K) = 0$, hence

$$a.a = ((-1)^{n-1} - 1) u \otimes u,$$

so if n is even we get $a.a = -2b$, and if K is a field of characteristic different from 2, for example $\mathbb{Q}$, we get

$$TC(S^n) > zcl_{\mathbb{Q}}(S^n) \geq \begin{cases} 2, & \text{if n is even;} \\ 1, & \text{if n is odd.} \end{cases}$$

Theorem 3.3.13 ((FARBER, 2003))**.** The topological complexity of the n-sphere S^n is given by

$$\mathrm{TC}(S^n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Notice that $2 = \mathrm{cat}(S^n) \leq \mathrm{TC}(S^n) \leq 2\mathrm{cat}(S^n) - 1 = 3$, and by example 3.3.12 we know that for n even $\mathrm{TC}(S^n) > 2$, so this case is totally determined. Now, we only need to show that $\mathrm{TC}(S^n) = 2$ for n odd. We shall do this explicitly by constructing an open covering of $S^n \times S^n$, given by

$$U = \{(x,y) \in S^n \times S^n \mid x \neq -y\},$$

and

$$V = \{(x,y) \in S^n \times S^n \mid x \neq y\}.$$

It is easy to see that $\{U,V\}$ is an open covering of $S^n \times S^n$, We wish to show that there are continuous motion planning functions $s : U \to (S^n)^I$ and $r : V \to (S^n)^I$, i.e., two local sections of the path fibration $\pi : (S^n)^I \to S^n \times S^n$. It is easy to construct s, since for any $(x,y) \in U$ we have $x \neq -y$, we know that there must be a shortest path on S^n from x to y, so we define $s(x,y)$ to be this path with constant speed. For r we define the path $r(x,y)$ in two steps, since $x \neq y$ there is a shortest path between x and $-y$, in the first step $r(x,y)$ follows this path with constant speed. Since n is odd, there is a non vanishing continuous tangent vector field $F : S^n \to \mathbb{R}^{n+1}$. In the second part of the path $r(x,y)$, we move from $-y$ to y via the spherical arc

$$-\cos(\pi t).y + \sin(\pi t).\frac{F(y)}{|F(y)|}, \text{ for } t \in [0,1].$$

So, the covering $\{U,V\}$ proves that $\mathrm{TC}(S^n) \leq 2$, for n odd, thus completing the proof. □

The Topological Complexity of a Robot Arm

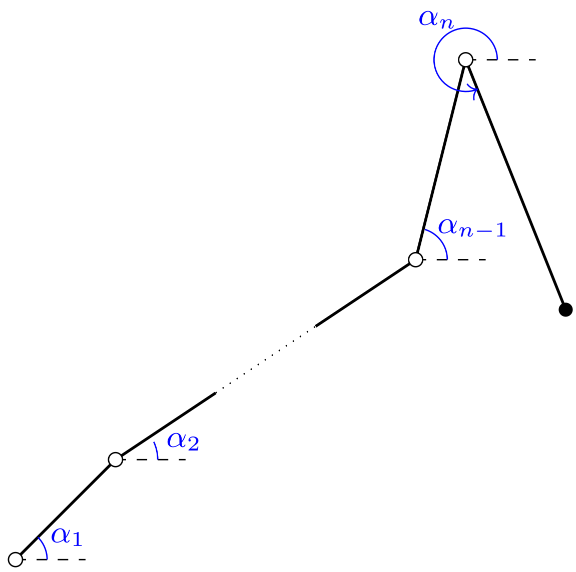
As an application of the theory developed until here, we will compute the topological complexity of a robot arm. We may think of a robot arm as a collection of rigid bars connected by flexible joints as in figure 10, with the first bar fixed, so that all possible configurations of this mechanical system are determined by the angles α_i (in the planar case), hence the configurations space is $T^n = \prod_{i=1}^{n} S^1$. It is easy to see that in the spatial case (in which the joints can move in any direction in a 3 dimensional space) we have that the configurations space is given by $T_2^n = \prod_{i=1}^{n} S^2$.

The problem of determining the topological complexity of a robot arm is fully resolved once we prove the following theorem.

Theorem 3.3.14 ((FARBER, 2003))**.** For n and m positive integers we have

$$\mathrm{TC}(T_m^n) = \begin{cases} n+1, & \text{if } m \text{ is odd;} \\ 2n+1, & \text{if } m \text{ is even.} \end{cases}$$

Figure 10 – Representation of a robot arm structure.



Source: Elaborated by the author.

Proof. Remember, from example 2.2.10, that $\mathrm{cat}(T^n_m) = n+1$, for any positive integers n and m, so theorem 3.2.6 implies that

$$n+1 \leq \mathrm{TC}(T^n_m) \leq 2n+1,$$

and by corollary 3.2.10 we have

$$TC(T^n_m) \leq (TC(S^m)-1)n+1.$$

From theorem 3.3.13 we conclude that for m odd, $\mathrm{TC}(T^n_m) = n+1$.

Now, suppose m is even. Let $u_i \in H^m(T^n_m;\mathbb{Q})$ be given by $u_i = p_{i*}^*(u)$, in which u is the generator of $H^m(S^m;\mathbb{Q})$ and $p_i : T^n_m \to S^m$ is the projection on the ith factor. Since p_i has a right inverse (the canonical inclusion), we conclude that p_{i*}^* has a left inverse, hence p_{i*}^* is injective, and therefore u_i is non zero.

Now, define $a_i = 1 \otimes u_i - u_i \otimes 1$, for $i = 1, \dots n$, elements in the ideal of zero divisors of $H^*(T^n_m;\mathbb{Q}) \otimes H^*(T^n_m;\mathbb{Q})$.

Notice that $u_i \smile u_i = p_{i*}^*(u) \smile p_{i*}^*(u) = p_{i*}^*(u \smile u) = 0$, since $u \smile u \in H^{2m}(S^m;\mathbb{Q}) = 0$. By the discussion in example 2.2.10, we know that $u_1 \smile \cdots \smile u_n \neq 0$.

For m even we get

$$a_i a_i = 2(1 \otimes (u_i \smile u_i) - u_i \otimes u_i) = -2a_i \otimes a_i,$$

since $u_i \smile u_i = 0$, and since $(u_i \otimes u_i)(u_{i+1} \otimes u_{i+1}) = (-1)^{|u_i||u_{i+1}|}(u_i \smile u_{i+1}) \otimes (u_i \otimes u_{i+1})$, we get

$$\prod_{i=1}^{n} a_i a_i = (-1)^N 2^n (u_1 \smile \cdots \smile u_n) \otimes (u_1 \smile \cdots \smile u_n) \neq 0,$$

in which N is an integer. Hence, $\mathrm{zcl}_{\mathbb{Q}}(T^n_m) = 2n$, and by theorem 3.3.11 we conclude that $\mathrm{TC}(T^n_m) = 2n+1$, for m even. $\square$

Theorem 3.3.14 shows that for a robot arm the topological complexity is proportional to the number of joints n, in the planar case being $n+1$ and in the spatial case $2n+1$.

TC of Topological Groups and Rigid Body Motion Planning

The possible movements of a rigid body in a 2-dimensional space are: translation by any vector in $\mathbb{R}^2$ and rotation around the axis perpendicular to the plane. Mathematically speaking, this configuration space is simply $\mathbb{R}^2 \times SO(2)$, which is usually called the Special Euclidean group, denoted by $SE(2)$.

This can be generalized to $SE(n) = \mathbb{R}^n \times SO(n)$, which described the movement of a rigid body in an n-dimensional space. To compute the topological complexity of a rigid body we will make use of the following lemma.

Lemma 3.3.15 ((FARBER, 2004))**.** If X is a path-connected topological group (in particular, if X is a connected Lie Group), then $\mathrm{cat}(X) = \mathrm{TC}(X)$.

Proof. Suppose X is a path-connected topological group, we already know from the general case (lemma 3.2.4) that $\mathrm{cat}(X) \leq \mathrm{TC}(X)$, so we only need to prove the reverse inequality.

Suppose $\mathrm{cat}(X) = m$ and let $U_1, \ldots, U_m$ be an open covering of X with homotopies $F : U_i \times I \to X$, such that $F_i(u,0) = u$ and $F(u,1) = e$, for all $u \in U_i$, in which e is the group identity of X.

Define $f : X \times X \to X$ given by $f(x,y) = xy^{-1}$. Since X is a topological group, f is a continuous map, and we can define $W_i = f^{-1}(U_i)$, for $i = 1, \ldots, m$, which clearly forms an open covering of $X \times X$.

Define $s_i : W_i \to X^I$ by $s(x,y)(t) = (F(xy^{-1},t))y$, for all $(x,y) \in W_i$ and $t \in I$. This is clearly a local section of the path fibration in X, hence $TC(X) \leq m = \mathrm{cat}(X)$. $\square$

Since $\mathbb{R}^n$ is contractible, and both TC and cat are homotopy invariants we have that $\mathrm{TC}(SE(n)) = \mathrm{TC}(SO(n)) = \mathrm{cat}(SO(n))$, the last equality being a consequence of lemma 3.3.15 and the well known fact that $SO(n)$ is a Lie group.

2-dimensional case (n=2): In this case, we know that $SO(2) = S^1$, from which we conclude that $\mathrm{TC}(SE(2)) = \mathrm{cat}(S^1) = 2$.

3-dimensional case (n=3): First, let us recall that the n-dimensional real projective, $\mathbb{RP}^n$, is defined as the space of all 1 dimensional subspaces of $\mathbb{R}^{n+1}$, explicitly

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{(x \sim \lambda x)} = \frac{S^n}{x \sim -x} = \frac{D^n}{\sim},$$

in the last case the equivalence relation is given by $x = -x$, for $x \in \partial D^n = S^{n-1}$.

One can see that $SO(3) = \mathbb{RP}^3$. The idea behind the proof is to consider the 3-dimensional disk $D^3_\pi = \{x \in \mathbb{R}^3 \mid \|x\| \leq \pi\}$ of radius π, and notice that each point $x \in D^3_\pi$ describes a rotation along the axis crossing x and the origin with angle given by $\|x\|$ (with $-x$ being the rotation by the same angle in the opposite direction), and since rotations by angles of π and $-\pi$ yield the same result, we can identify the antipodal boundary points, thus getting a space homeomorphic to $D^3/\sim$, as described above.

Thus, we have $\mathrm{cat}(SO(3)) = \mathrm{cat}(\mathbb{RP}^3)$. The LS category of real projective spaces are well known. First, notice that the dimension inequality implies $\mathrm{cat}(\mathbb{RP}^n) \leq \dim(\mathbb{RP}^n) + 1 = n+1$. It is well known that $H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 1$ (i.e., $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is a generator of the cohomology module of order 1) (see theorem 3.19 in (HATCHER, 2002)). So, it is quite obvious that the least integer k for which $(\tilde{H}^*(\mathbb{RP}^n; \mathbb{Z}_2))^k = 0$ is $k = n+1$, in other words, $\mathrm{cup}_{\mathbb{Z}_2}(\mathbb{RP}^n) = n+1 \leq cat(\mathbb{RP}^n) \leq n+1$. With this, we conclude that

$$\mathrm{TC}(SE(3)) = \mathrm{cat}(SE(3)) = \mathrm{cat}(SO(3)) = \mathrm{cat}(\mathbb{RP}^3) = 4.$$

One might wonder whether we are able to compute the category of $SO(n)$, for any n, and the answer is that at the moment this is still an open question, the latest results are given in (IWASE; MIMURA; NISHIMOTO, 2005) and (IWASE; KIKUCHI; MIYAUCHI, 2007), in which the category of $SO(n)$ is computed up to $n = 10$, yielding

$$\begin{aligned} &cat(SO(4)) = 5; && cat(SO(8)) = 13; \\ &cat(SO(5)) = 9; && cat(SO(9)) = 21; \\ &cat(SO(6)) = 10; && cat(SO(10)) = 22. \\ &cat(SO(7)) = 12; && \end{aligned}$$

3.4 Order of Instability

In this section, we will introduce the concept of order of instability of a motion planner, the main reference for this section is (FARBER, 2004).

An alternative useful definition of a motion planner is to consider the subspaces covering $X \times X$ to be Euclidean neighborhood retracts (ENR) instead of open subsets. More precisely, we have the following definition.

Definition 3.4.1 ((FARBER, 2004))**.** Given a topological space X, an **ENR motion planner** is a covering of $X \times X$ by subspaces $F_1, \ldots, F_k$, such that

1. Each F_i is an ENR.
2. $F_i \cap F_j = \emptyset$, if $i \neq j$.
3. There is a local section $s_i : F_i \to X^I$ of the path fibration π for each i.

By using this definition for each pair of points (A, B) there is only one possible path going from A to B, since all the ENRs are pairwise disjoint.

Example 3.4.2 ((FARBER, 2004))**.** Suppose X is a finite dimensional polyhedron (with a CW structure). We can explicitely construct an ENR motion planner on it.

To do so, let n be the dimension of X, and let X^k be the k-skeleton of X. Define $S_k = X^k \setminus X^{k-1}$ and

$$F_j = \cup_{i+k=j} S_i \times S_k.$$

Then $F_0, \ldots, F_{2n}$ is a covering of $X \times X$. We clearly have $F_j \cap F_l = \emptyset$, if $j \neq l$.

Notice that each S_k is homeomorphic to the union of disjoint open sets in $\mathbb{R}^k$, hence it is itself open and therefore an ENR. Now, we easily conclude, using lemmas 1.7.4 and 1.7.2, that each F_j is an ENR.

What is left to show is that there are local sections of the path fibration in each F_j.

We show this explicitly. First, fix a point in the interior of each cell of X, and fix also paths between each pair of those points. Now, given a point $(A, B) \in F_j$, we have $A \in S_i$ for some i, and $B \in S_k$, for some k, such that $i + k = j$. Thus, A lies in an open i-cell, e^i, and B in an open k-cell, e^k.

Let $c_i \in e^i$ and $c_k \in e^k$ be the points fixed in these cells, and let σ denote the fixed path from c_i to c_k. The path from A to B will be defined as: first go from A to c_i following the straight line in the i-cell, then proceed through σ, and finally go from c_k to B following the straighline in the k-cell. This clearly defines a continuous motion planner on each F_j.

Corollary 3.4.3 ((FARBER, 2004))**.** For any n-dimensional Polyhedron it is possible to find an ENR motion planner by decomposing the space into $2n+1$ subspaces.

In the case of 1-dimensional polyhedra (graphs), corollary 3.4.3 is saying that we can get a motion planning algorithm with 3 local rules. One rule describes motions between vertices, the other between vertices and points in open 1-cells, and the last one describes motions between points in open 1-cells.

Definition 3.4.4 ((FARBER, 2004))**.** Let X be a topological space with an ENR motion planner given by $F_1, \ldots, F_k$ and $s_i : F_k \to X^I$. Given a point $(A,B) \in X \times X$, the **order of instability** of the motion planner at (A,B) is the largest integer r such that any open neighborhood of (A,B) instersects r distinct elements of the covering $F_1, \ldots, F_k$. The **order of instablity** of the motion planner is defined as the maximum order of instability at a point of $X \times X$.

An equivalent way of defining the order of instability of a motion planner, using the notation from definition 3.4.4, is by saying that the order of instability at $(A,B) \in X \times X$ equals the biggest integer r such that

$$(A,B) \in \overline{F}_{i_1} \cap \cdots \cap \overline{F}_{i_r}, \qquad \text{for some} \;\; 1 \leq i_1 < i_2 < \cdots < i_r \leq k.$$

Equivalently we can define the order of instability of the motion planner as the largest r sucht that

$$\overline{F}_{i_1} \cap \cdots \cap \overline{F}_{i_r} \neq \emptyset, \qquad \text{for some} \;\; 1 \leq i_1 < i_2 < \cdots < i_r \leq k.$$

The order of instability represents an important measure of how "unstable" a motion planner is. For example, if a certain motion planner has order of instability r, it means that there is a point $(A,B) \in X \times X$ for which small perturbations can cause it to be in r different ENRs from the motion planner, in other words, small perturbations can cause big changes in the path followed, which can be a problem for computational reasons, so an ideal motion planner has the smallest order of instability possible.

Lemma 3.4.5 ((FARBER, 2004))**.** Let X be a path-connected metrizable space, and $U_1, \ldots, U_k$ an open covering of $X \times X$ with local sections, $s_i : U_i \to X^I$, of the path fibration. If for some integer $1 \leq r \leq k$ we have

$$U_{i_1} \cap \cdots \cap U_{i_r} = \emptyset,$$

for any set of indexes $1 \leq i_1 < i_2 < \cdots < i_r \leq k$, then $\mathrm{TC}(X) < r$ (in particular if $U_1 \cap \cdots \cap U_k = \emptyset$, then $\mathrm{TC}(X) < k$).

Proof. Since X is a metric space, it is Hausforff and paracompact, hence any open covering of X has a partition of unity subordinate to it.

Let $U_1, \ldots U_k$ be an open covering of $X \times X$ as described above, and let $f_1, \ldots, f_k : X \times X \to \mathbb{R}$ be a partition of unity subordinate to this covering.

Given any subset of indices $S \subset \{1, \ldots, k\}$ define

$$W(S) = \{(x,y) \in X \times X \mid f_i(x,y) > 0, \text{ if } i \in S \text{ and } f_i(x,y) > f_j(x,y), \text{ if } i \in S \text{ and } j \notin S\},$$

and notice that:

1. $W(S) \subset X \times X$ is an open set, since it is a finite intersection of open subsets like ${f_i}^{-1}((0,\infty))$ and $(f_i - f_j)^{-1}((0,\infty))$.

2. If $|S| = |S'|$ and $S \neq S'$, then $W(S) \cap W(S') = \emptyset$.

3. If $i \in S$, then $W(S) \subset U_i$. This is obvious, since for any $(x,y) \in W(S)$ we have $f_i(x,y) > 0$, hence $(x,y) \in \operatorname{supp}(f_i) \subset U_i$.

4. $\bigcup_{|S|<r} W(S) = X \times X$. Indeed, if $(x,y) \in X \times X$, let $i_1, \ldots, i_t$ be all the indices such that $f_{i_l}(x,y) > 0$, then $(x,y) \in W(i_1, \ldots, i_t)$. Furthermore, we must have $t < r$, otherwise we would have that (x,y) is in the support of at least r functions f_i meaning that (x,y) would be in at least r elements of the covering $U_1, \ldots, U_k$, contradicting our initial hypothesis that any intersection of r elements of the covering is empty.

Define

$$W_j = \bigcup_{|S|=j} W(S),$$

for $j = 1, \ldots, r-1$, then, from items (1) and (4), above we have that $W_1, \ldots, W_{r-1}$ is an open covering of $X \times X$. From (2) and (3) we know that W_j is a disjoint union of open subsets in each of which there is a section of the path fibration, whence we may define a section in W_j, and we conclude that $\mathrm{TC}(X) \leq r-1$. □

To simplify notation on the next theorem, let us define, for now, the **order of instability of a space** X, denoted $\mathrm{OI}(X)$, to be the smallest possible order of instability of an ENR motion planner in X. Le us also define the ENR topological complexity, denoted $\mathrm{TC}^{\mathrm{ENR}}(X)$ to be the smallest number of ENR subsets of X necessary to construct an ENR motion planner.

Theorem 3.4.6 ((FARBER, 2004))**.** For X a connected C^∞-smooth manifold we have $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{ENR}}(X) = \mathrm{OI}(X)$

Proof. Suppose X is a connected C^∞-smooth manifold. We will start by proving that $\mathrm{TC}(X) \leq \mathrm{TC}^{\mathrm{ENR}}(X)$. To do so, suppose $F_1, \ldots, F_k$ is an ENR motion planner of X, with local sections of the path fibration given by $s_i : F_i \to X^I$. To prove that $k \geq \mathrm{TC}(X)$, we shall show that there are open neighborhoods of each F_i in which we can extend the local sections.

We know that $X \times X$ is a smooth manifold (theorem 1.2.11) and that $X \times X \subset \mathbb{R}^n$ for some n (theorem 1.2.12), hence $X \times X$ is also locally contractible and locally compact, once it is a manifold, and by theorem 1.7.7 we conclude that $X \times X$ is an ENR.

So, we have that both F_i and $X \times X$ are ENRs, thus we may apply lemma 1.7.6, from which we get that there is an open subset U_i, and a retraction $r : U_i \to F$, such that

$F_i \subset U_i \subset X \times X$, and $kr : U_i \to X \times X$ is homotopic to the inclusion $j : U_i \hookrightarrow X \times X$, in which $k : F_i \hookrightarrow X \times X$ is the canonical inclusion.

Let $H : U_i \times I \to X \times X$ be a homotopy with $H(x,0) = j(x)$ and $H(x,1) = kr(x)$, for all $x \in U_i$.

Notice that if $(x,y) \in U_i \subset X \times X$, then $H(x,y,t) = (\gamma(x,y,t), \delta(x,y,t))$, with $\gamma, \delta : U_i \times I \to X$ continuous functions. We have

$$(\gamma(x,y,0), \delta(x,y,0)) = (x,y),$$

and $(\gamma(x,y,1), \delta(x,y,1)) \in F_i$, for all $(x,y) \in U_i$.

So, we may define $s_i' : U_i \to X^I$ by

$$s_i'(x,y)(t) = \begin{cases} \gamma(x,y,3t), \ 0 \leq t \leq 1/3; \\ s_i(\gamma(x,y,1), \delta(x,y,1))(3t-1), \ 1/3 \leq t \leq 2/3; \\ \delta(x,y,3-3t), \ 2/3 \leq t \leq 1. \end{cases}$$

Thus, $s'(x,y)$ is formed by going from x to $\gamma(x,y,1)$ via γ, then s_i connects $\gamma(x,y,1)$ and $\delta(x,y,1)$, and finally it goes from $\delta(x,y,1)$ to y via the reverse path of δ, all these functions are continuous, hence s_i' is continuous. We conclude that there is a covering of X by open sets $U_1, \dots, U_k$, with $F_i \subset U_i$, such that there are local sections of the path fibration in each of those sets, hence $\mathrm{TC}(X) \leq k$.

Conversely, suppose $\mathrm{TC}(X) = k$, we will show that there must be an ENR motion planner with k elements. Let $U_1, \dots, U_k$ be an open covering of $X \times X$ with local sections of the path fibration given by $s_i : U_i \to X^I$. Since $X \times X$ is a smooth manifold, theorem 1.2.14 guarantees that there exists a smooth partition of unity $f_i : X \times X \to \mathbb{R}$, $i = 1, \dots, n$, subordinate to $U_1, \dots, U_n$.

Let c_i, $i = 1, \dots k$, be regular values for the respective f_i such that $0 < c_i < 1$ and $c_1 + \cdots + c_k = 1$. These numbers exist, since by Sard's theorem (1.2.13) we know that the set of critical values of each f_i, C_i, has measure zero in $\mathbb{R}$, hence C_i must be a discrete set of points and the same is true for $C_1 \times \cdots \times C_k \subset \mathbb{R}^k$, so it can not contain the subset $\{(x_1, \dots, x_k) \subset R^k \mid x_1 + \cdots + x_k = 1, \ 0 < x_i < 1\}$, which is clearly not discrete, whence we can find $c_1, \dots, c_k$ as desired.

Define

$$V_i = \left\{(x,y) \in X \times X \mid f_i(x,y) \geq c_i \text{ and } f_j(x,y) < c_j, \text{ for all } j < i\right\}.$$

Notice that

1. $V_i \subset U_i$, once $\mathrm{supp}(f_i) \subset U_i$ and $c_i > 0$. Hence we may restrict the local sections and define $s_i' = s_i|_{V_i}$, which are local sections of the path fibration on V_i.

2. $V_i \cap V_j = \emptyset$, if $i \neq j$

3. $V_1 \cup \cdots \cup V_k = X \times X$, since for any $(x,y) \in X \times X$ we have $\sum_i f_i(x,y) = 1$ and $\sum_i c_i = 1$, with $0 \leq f_i(x,y) \leq 1$ and $0 < c_i < 1$, there must be some index i for which $f_i(x,y) \geq c_i$, and we can also choose i to be the least of those indexes, which clearly implies $(x,y) \in V_i$.

4. Since $V_i = {f_1}^{-1}((-\infty, c_1)) \cap \cdots \cap {f_{i-1}}^{-1}((-\infty, c_{i-1})) \cap {f_i}^{-i}([c_1, \infty))$, we have that V_i is a manifold, whence it is an ENR.

With this we proved that $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{ENR}}(X)$, and now we proceed to prove the second part, namely $\mathrm{TC}(X) = \mathrm{OI}(X)$.

From what we have proven, we can already infer that $\mathrm{OI}(X) \leq TC(X)$, since for an ENR motion planner with $\mathrm{TC}(X)$ local sections (which we have just shown that exists) the order of instability is lower than or equal to $\mathrm{TC}(X)$.

The only thing left to show is that $\mathrm{TC}(X) \leq \mathrm{OI}(X)$. To do so, suppose $F_1, \ldots, F_k \subset X \times X$, with $s_i : F_i \to X^I$, is an ENR motion planner of X with order of instability r. By definition $\overline{F}_{i_1} \cap \cdots \cap \overline{F}_{i_{r+1}} = \emptyset$, for any collection of $r+1$ distinct indices $1 \leq i_1 < \cdots < i_{r+1} \leq k$.

For each index $i = 1, \ldots, k$ define a continuous non negative function $f_i : X \times X \to R$, such that $f_i^{-1}(0) = \overline{F}_i$ (the existence of such functions is guaranteed by theorem 1.2.15).

Define the continuous function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \min\{f_{i_1}(x,y) + \cdots + f_{i_{r+1}}(x,y) \mid 1 \leq i_1 < \cdots < i_{r+1} \leq k\},$$

and notice that $\phi(x,y) > 0$, for all $(x,y) \in X \times X$, otherwise there would be an element (x,y) in an intersection $\overline{F}_{i_1} \cap \cdots \cap \overline{F}_{i_{r+1}}$.

Consider the set

$$U_i = \{(x,y) \in X \times X \mid (r+1) f_i(x,y) < \phi(x,y)\},$$

which is clearly open since $U_i = (\phi - (r+1) f_i)^{-1}((0,\infty))$, and notice that $U_1, \ldots, U_k$ forms an open covering of $X \times X$, since $\overline{F}_i \subset U_i$, for all i.

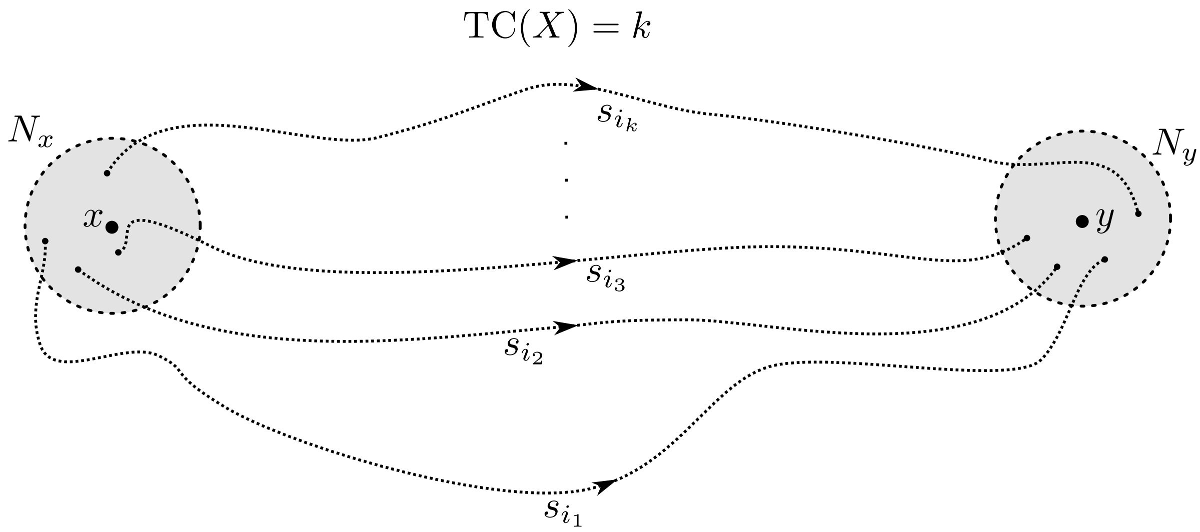
Note that $U_{i_1} \cap \cdots \cap U_{i_{r+1}} = \emptyset$, for any choice of indices $0 \leq i_1 < \cdots < i_{r+1} \leq k$, otherwise we would have $f_{i_1}(x,y) + \cdots + f_{i_{r+1}}(x,y) < \phi(x,y)$, contradicting the definition of ϕ.

In the first part of the proof (when showing that $\mathrm{TC}(X) = \mathrm{TC}^{\mathrm{ENR}}(X)$) we showed that there is an open covering $U'_1, \ldots, U'_k$ of $X \times X$, with $F_i \subset U'_i$ and local sections $s'_i : U'_i \to X^I$ of the path fibration.

Define $V_i = U_i \cap U'_i \supset F_i$ with local sections $s'_i|_{V_i}$. Then we clearly have that $V_1, \ldots, V_k$ is an open covering of $X \times X$, and $V_{i_1} \cap \cdots \cap V_{i_{r+1}} = \emptyset$, for all $1 \leq i_1 < \cdots < i_{r+1} \leq k$, hence, by lemma 3.4.5, we have that $\mathrm{TC}(X) \leq r$, which concludes the proof. □

What we just proved shows that the topological complexity determines, in some sense, a lower bound for how unstable any motion planning algorithm will be in a given space. More precisely, for a space X and any motion planning algorithm in X, given by local sections $s_1, \ldots, s_n$ of the path fibration, we will be able to find pair of points $(x, y) \in X \times X$ for which small variations can produce $\mathrm{TC}(X)$ different paths, and this will be true regardless of how close you stay to the originally chosen points, as shown in figure 11.

Figure 11 – There are points x and y in X such that for any given neighborhoods N_x and N_y of x and y, respectively, there are $TC(X) = k$ different paths given by k different local sections of the chosen motion planning algorithm connecting the two neighborhoods.



Source: Elaborated by the author.

CHAPTER

4

THE FIBREWISE METHOD

4.1 Fibrewise Topology

Fibrewise Spaces

Definition 4.1.1 ((CRABB; JAMES, 2012))**.** Consider a topological space B, which we will call the **base space**. A **fibrewise space** over B is just a topological space X with a continuous map $p : X \to B$, called the **projection**. In the case of p being a fibration, we usually refer to X as a **fibrant**.

Remark 4.1.2. Usually, one writes only X for a fibrewise space, but we should always think of the pair (X, p) as being the fibrewise space, since altering the projection p yields a different space.

Definition 4.1.3 ((CRABB; JAMES, 2012))**.** Let (X, p) be a fibrewise space over B. For each $b \in B$ we define the **fibre of** b as the subset $X_b = p^{-1}(b)$ of X.

Definition 4.1.4 ((CRABB; JAMES, 2012))**.** Let (X, p) and (Y, q) be two fibrewise spaces over B. A **fibrewise map** from X to Y is a continuous function $f : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Now we can introduce the **category of fibrewise spaces**, which has as its objects all the fibrewise spaces, and as its morphisms all the fibrewise maps. An equivalence in this category is called a **fibrewise topological equivalence** or a **fibrewise homeomorphism**.

Proposition 4.1.5. A map $f : X \to Y$ between fibrewise spaces (X,p), (Y,q) over B is a fibrewise map if and only if $f(X_b) \subset Y_b$, for all $b \in B$.

Proof. First, let us assume that $f : X \to Y$ is a fibrewise map. Given $b \in B$ and $y \in f(X_b)$ we have that there exists $x \in X_b$ such that $y = f(x)$. Notice that $x \in X_b$ implies $p(x) = b$, and $p(x) = q \circ f$, whence $q(y) = p(x) = b$, and we conclude that $y \in Y_b$.

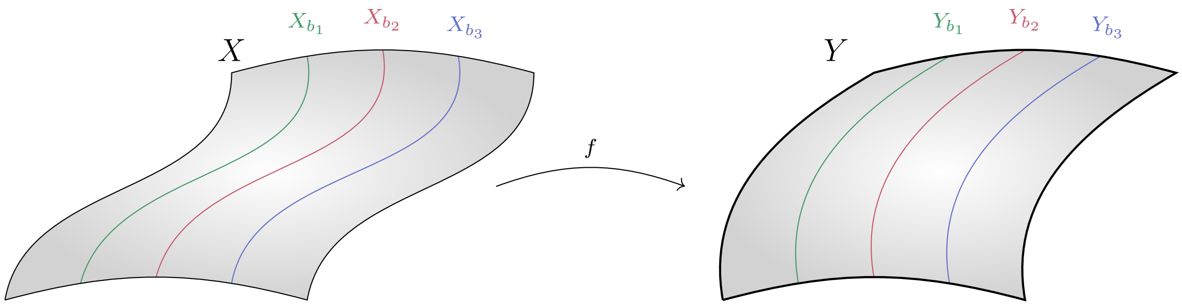
Conversely, assume that $f(X_b) \subset Y_b$, for all $b \in B$. For any $x \in X$ we have $f(X_{p(x)}) \subset Y_{p(x)}$, hence there exists an element $y \in Y_{p(x)}$ such that $f(x) = y$, which implies $q \circ f(x) = q(y) = p(x)$. Since this is the case for any $x \in X$, we conclude that $q \circ f = p$, and f is a fibrewise map. □

Notice that if $f : X \to Y$ is a fibrewise map between fibrewise spaces over B, one can naturally define a fibrewise map between $X_{B'}$ and $Y_{B'}$, in which $B' \subset B$ (here we are considering $X_{B'} \doteq p^{-1}(B')$). This map is defined in the following manner

$$\begin{aligned} f_{B'} : X_{B'} &\longrightarrow Y_{B'} \\ x &\longmapsto f(x), \end{aligned}$$

so a fibrewise map $f : X \to Y$ defines many continuous maps between the fibres $f_b : X_b \to Y_b$.

Figure 12 – For a map $f : X \to Y$ to be fibrewise, apart from being continuous, it essentially needs to map each fibre of X (e.g. $X_{b_1}, X_{b_2}, X_{b_3}$), into the respective fibre of Y $(Y_{b_1}, Y_{b_2}, Y_{b_3})$.



Source: Elaborated by the author.

Notice that if the fibrewise map $f : X \to Y$ is a fibrewise homeomorphism, then each map $f_b : X_b \to Y_b$ is a homeomorphism, i.e., all the fibers are homeomorphic. This is not a sufficient condition for two spaces to be fibrewise homeomorphic, take for example the case in which $X = \mathbb{R}_d$ (the real line with the discrete topology), $Y = \mathbb{R}$, $B = \mathbb{R}$. We consider $X = \mathbb{R}_d$ and $Y = \mathbb{R}$ to be fibrewise spaces over $\mathbb{R}$, with the projections being the identity map. Then all the respective fibers are homeomorphic $X_b = \{b\} = Y_b$, but X and Y are not fibrewise homeomorphic (they are not even homeomorphic in the usual sense).

Definition 4.1.6 ((CRABB; JAMES, 2012))**.** Given $f : X \to Y$ a fibrewise map between the fibrewise spaces over B (X,p) and (Y,q). If there is a section $t : B \to Y$ of $q : Y \to B$, such that $f = t \circ p$, then we say f is **fibrewise constant**.

If $f : X \to Y$ is a fibrewise constant map then it is constant on each fibre X_b, since if $x_1, x_2 \in X_b$, then $f(x_1) = t(p(x_1)) = t(b) = t(p(x_2)) = f(x_2)$. But the converse is not true, there are fibrewise maps which are constant on each fibre, but not fibrewise constant. One example is to take $\mathbb{R}_d$ over $\mathbb{R}$, then the identity map $i : \mathbb{R}_d \to \mathbb{R}$ is constant on each fibre, but it is not fibrewise constant, since there is no section $s : \mathbb{R} \to \mathbb{R}_d$.

Definition 4.1.7 ((CRABB; JAMES, 2012))**.** Given a family of fibrewise spaces $\{X_j\}_j$ over B, we define the **fibrewise product** $\prod_B X_j$ to be a subspace of the direct product $\prod X_j$ as a topological space, such that it is a fibrewise space over B satisfying the condition: for any fibrewise space X over B if we randomly choose fibrewise maps $f_j : X \to X_j$ for each index j, then the map

$$\begin{aligned} f : X &\longrightarrow \prod_B X_j \\ x &\longmapsto \{f_j(x)\}_j \end{aligned} \tag{4.1}$$

is fibrewise, and all the fibrewise maps from X into $\prod_B X_j$ correspond precisely to the families $\{f_j\}_j$, in the sense that each family gives us a different fibrewise map.

In the following proposition, we show that the fibrewise product always exists and is unique.

Proposition 4.1.8. Following the notation of definition 4.1.7, let us show that

$$\prod_B X_j = A \doteq \left\{ \{x_j\}_j \in \prod X_j \;:\; p_i(x_i) = p_j(x_j), \text{ for all } i, j \right\},$$

and that the projection is defined as

$$\begin{aligned} p : \prod_B X_j &\longrightarrow B \\ \{x_j\}_j &\longmapsto p_j(x_j). \end{aligned}$$

Proof. First, let us show that the suggested space satisfies the condition of definition 4.1.7. Let X be any fibrewise topological space over B and consider fibrewise functions $f_j : X \to X_j$, for each j. We wish to show that the function defined in 4.1, with A in the place of $\prod_B X_j$, is a fibrewise map. From general topology, we know it will be continuous, since each factor is continuous, so we only need to show the fibrewise condition.

Let us denote by $q : X \to B$ the projection of X. We want to show that $p \circ f = q$. Remember that $f = \{f_j\}_j$, and for each j we have $p_j \circ f_j = q$.

Take $x \in X$, then $p_j(f_j(x)) = q(x)$, for all j, hence $\{f_j(x)\}_j \in A$. And from the way we defined p we have $p \circ f(x) = p(\{f_j(x)\}_j) = p_j(f_j(x)) = q(x)$, hereby concluding that f is a fibrewise map.

Now, let us show that each family of fibrewise maps $\{f_j\}_j$ defines a different map. It is easier to show the counter-positive, so suppose the families $\{f_j\}_j$ and $\{g_j\}_j$ define the same fibrewise map f from X to A, but this naturally means that $f_j(x) = g_j(x)$, for all $x \in X$ and j.

To see that A is the only space satisfying the conditions from definition 4.1.7, notice that if any point from $\left(\prod X_j\right) \setminus A$ was included there would be no way of defining a projection. And if we consider a subspace $\tilde{A} \subset \prod X_j$ so that $A \setminus \tilde{A} \neq \emptyset$, then given $\{a_j\}_j \in A \setminus \tilde{A}$, we can construct a family of fibrewise maps $\{f_j\}_j$ from the fibrewise space $\{*\}$, with its projection given by $q_*(*) = p_j(a_j)$, to the space X_j, each function defined by $f_j(*) = a_j$. Then the function $f = \{f_j\}_j$ falls out of $\tilde{A}$, as we wanted to show. $\square$

There is one more way we can visualize the fibrewise product. From proposition 4.1.8, we see that this product is basically the union of the product of the corresponding fibres of $\{X_j\}_j$. More precisely,

$$\prod_B X_j = \bigcup_{b \in B} \left(\prod (X_j)_b\right),$$

hence it satisfies the following property

$$\left(\prod_B X_j\right)_b = \prod (X_j)_b.$$

In case we have only two fibrewise spaces X and Y over B, we usually denote their fibrewise product by $X \times_B Y$.

Fibrewise Pointed Spaces

Definition 4.1.9 ((CRABB; JAMES, 2012))**.** A **fibrewise pointed space** over a base space B is a triple (X,p,s) in which (X,p) is a fibrewise space over B and $s : B \to X$ is a section of p (i.e., $p \circ s = id_B$).

Definition 4.1.10 ((CRABB; JAMES, 2012))**.** Given two fibrewise pointed spaces, (X,p,s) and (Y,q,t), we define a **fibrewise pointed map** between X and Y, to be any map $f : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} & B \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

The fibrewise pointed spaces constitute a category with its morphism being the fibrewise pointed maps.

Remark 4.1.11. Just like we defined the concept of fibrewise constant map in the previous section, we can define the concept of **fibrewise pointed constant map**, and in this case, it turns out to be a really simple definition, since for two fibrewise pointed spaces (X, p_X, B, s_X) and (Y, p_Y, B, s_Y) there is exactly one fibrewise pointed constant map, namely the map $s_Y \circ p_X$. This is due to the fact that if we defined, as in the previous section, that a fibrewise pointed map $f : X \to Y$ is fibrewise pointed constant if there exists t a section of p_Y such that

$$f = t \circ p_X,$$

then we have

$$s_Y = f \circ s_X = t \circ (p_X \circ s_X) = t \circ id_B = t,$$

hence $f = s_Y \circ p_X$.

4.2 Fibrewise Homotopy Theory

Using the ideas of Fibrewise Topology, it is possible to introduce a homotopy theory for this kind of space. In this section, we shall give a short introduction to this topic, and later on, we will use it to introduce the fibrewise version of the Lusternik-Schnirelmann Category.

In all the definitions that follow we can replace "fibrewise" with "fibrewise pointed" obtaining the pointed version of all these concepts.

Fibrewise homotopy

Definition 4.2.1 ((CRABB; JAMES, 2012))**.** If $f : X \to Y$ and $g : X \to Y$ are two fibrewise maps of fibrewise spaces over B, then a **fibrewise homotopy** between f and g is a homotopy f_t in the usual sense, such that $f_t : X \to Y$ is a fibrewise map, for each $t \in I$. If this is the case, we say that f and g are **fibrewise homotopic**, usually denoted by $f \simeq_B g$.

Fibrewise homotopic is easily shown to be an equivalence relation between fibrewise maps, and the set of all equivalence classes is denoted by $\pi_B[X;Y]$.

Definition 4.2.2 ((CRABB; JAMES, 2012))**.** Given fibrewise spaces X and Y over B, a fibrewise map $\phi : X \to Y$ is called a **fibrewise homotopy equivalence** if there exists a fibrewise map $\psi : Y \to X$ such that $\psi \circ \phi \simeq_B id_X$ and $\phi \circ \psi \simeq_B id_Y$. If this is the case, we say that X and Y have the same **fibrewise homotopy type**.

Definition 4.2.3 ((CRABB; JAMES, 2012))**.** A fibrewise map $f : X \to Y$ is said to be **fibrewise null-homotopic** if it is fibrewise homotopic to a fibrewise constant map.

Definition 4.2.4 ((CRABB; JAMES, 2012)). Given X a fibrewise space over B, and $A \subset X$. We say that A is a **fibrewise retract** of X if there exists a fibrewise map $r : X \to A$ such that $r|_A = id_A$, in which case r is called a **fibrewise retraction**.

Definition 4.2.5. A fibrewise space X over B is said to be fibrewise compressible to a subspace $D \subset X$ if there is a fibrewise homotopy $F_t : A \to X$, $t \in I$, such that F_0 is the inclusion map $A \hookrightarrow X$ and $F_1(A) \subset D$.

Fibrewise Cofibration and Fibration

Definition 4.2.6 ((CRABB; JAMES, 2012)). Given fibrewise spaces X and A over B, and a fibrewise map $u : A \to X$, we say that u has the **fibrewise homotopy extension property** if for every fibrewise space E over B, with a fibrewise map $f : X \to E$ and a fibrewise homotopy $g_t : A \to E$ such that the following diagram

$$\begin{tikzcd} A \arrow[d, "u"] \arrow[dr, "g_0"] & \\ X \arrow[r, "f"] & E \end{tikzcd}$$

commutes, there exists a fibrewise homotopy $f_t : X \to A$ such that $f_0 = f$ and the diagram

$$\begin{tikzcd} A \arrow[d, "u"] \arrow[dr, "g_t"] & \\ X \arrow[r, "f_t"] & E \end{tikzcd}$$

commutes, for all $t \in I$. In this case, $u : A \to X$ is called a **fibrewise cofibration** and the pair (X, u) is called a **fibrewise cofibre space**. If u is an inclusion map then the pair (X, A) is called a **fibrewise cofibred pair**.

Proposition 4.2.7 ((CRABB; JAMES, 2012) Proposition 4.1). If X is a fibrewise space and $A \subset X$ is a closed subset with $u : A \to X$ being the inclusion map, then u is a fibrewise cofibration if and only if $(X \times \{0\}) \cup (A \times I)$ is a fibrewise retract of $X \times I$.

Definition 4.2.8 ((CRABB; JAMES, 2012)). Consider X and E fibrewise spaces over B, then a fibrewise map $p : E \to X$ is said to have the **fibrewise homotopy lifting property**, if for any fibrewise space A over B with a fibrewise map $f : A \to E$ and a fibrewise homotopy $g_t : A \to X$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g_0"'] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes, there exists a fibrewise homotopy $h_t : A \to E$ such that $h_0 = f$ and the diagram

$$\begin{tikzcd} A \arrow[r, "h_t"] \arrow[dr, "g_t"'] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes, for all $t \in I$. In this case, $p : E \to X$ is called a **fibrewise fibration** and the pair (E, p) is called a **fibrewise fibre space** over X.

Definition 4.2.9. Let (X, p, s) be a fibrewise pointed space over B, then X is said to be a **fibrewise well-pointed space** over B if the section $s : B \to X$ is a fibrewise cofibration.

Remark 4.2.10. Notice that a fibrewise pointed space is similar to a based space, but instead of a basepoint $x_0 \in X$ we have $s(B) \in X$, so if we consider the case where B is a single point, then we are back at the case of a based space. With this in mind, the previously described condition of a fibrewise well-pointed space is similar to considering a based space with a non-degenerate basepoint.

The fibrewise Strøm Structure

In section 1.4, we briefly discussed the Strøm Structure, now we shall adapt this concept to the fibrewise case.

Definition 4.2.11. A **fibrewise pair** is a pair of spaces (X,A) such that both (X, p_X, B) and (A, p_A, B) are fibrewise spaces with $A \subset X$ and $p_A = p_X|_A$.

Definition 4.2.12. A **fibrewise pointed pair** is a pair of spaces (X,A) such that both (X, p_X, B, s_X) and (A, p_A, B, s_A) are fibrewise pointed spaces with $A \subset X$, $s_X(X) \subset A$, $p_A = p_X|_A$ and $s_A(b) = s_X(b)$, for all $b \in B$.

Definition 4.2.13. If (X,A) is a fibrewise pair, then a **Fibrewise Strøm Structure** on (X,A) is a pair (α, h) of a map $\alpha : X \to I$ and a fibrewise homotopy $h : X \times I \to X$ relative to A starting at the identity map, such that

(i) $\alpha(a) = 0$, if and only if $a \in A$;

(ii) $h(x,t) \in A$, if $t > \alpha(x)$.

Proposition 4.2.14. Let (X,A) be a closed fibrewise pair. Then, (X,A) is fibrewise cofibred if and only if $(X \times \{0\}) \cup (A \times I)$ is a fibrewise retract of $X \times I$.

Proof. First, suppose $u : A \hookrightarrow X$ is a cofibration. In the homotopy extension property, take $E = (X \times \{0\}) \cup (A \times I)$ to be the fibrewise space over B with projection given by $p_E(x,t) = p_X(x)$, for all $(x,t) \in E$. Define the fibrewise map $f : X \to E$ given by $f(x) = (x, 0)$, for all $x \in X$, and the fibrewise homotopy $G : A \times I \to E$ given by $G(a,t) = (a,t)$, for all $(a,t) \in A \times I$. Then the extension (via the fibrewise homotopy extension property) $F : X \times I \to E$ is a fibrewise retraction of $X \times I$ onto $(X \times \{0\}) \cup (A \times I)$.

Conversely, suppose there is a fibrewise retraction,

$$r : X \times I \to (X \times \{0\}) \cup (A \times I),$$

given a fibrewise map $f: X \to E$ and a homotopy $G: A \times I \to E$ (as in the homotopy extension property). Then, similarly to the proof of lemma 1.4.27, we may define a continuous map (since A is closed) given by $H : (X \times \{0\}) \cup A \times I \to E$, such that $H(x,0) = f(x)$, for all $x \in X$, and $H(a,t) = G(a,t)$, for all $a \in A$ and $t \in I$. Then $H \circ r$ is the desired fibrewise homotopy. □

Proposition 4.2.15. If (X,A) is a closed fibrewise pair. Then (X,A) is fibrewise cofibred if and only if (X,A) admits a fibrewise Strøm Structure.

Proof. If (X,A) is a fibrewise cofibred pair, then from proposition 4.2.14 there is a retraction $r: X \times I \to (X \times \{0\}) \cup (A \times I)$. Since I is compact we can define map $\alpha : X \to I$ by

$$\alpha(x) = \sup_{t \in I} |\pi_2 \circ r(x,t) - t|,$$

and $h: X \times I \to X$ by

$$h(x,t) = \pi_1 \circ r(x,t), \text{ for all } (x,t) \in X \times I,$$

in which we are considering $\pi_1 : X \times I \to X$ and $\pi_2 : X \times I \to I$ to be the projections on the first and second coordinates, respectively.

We claim that (α, h) is a fibrewise Strøm Structure on (X,A). Indeed, notice that both α and h are continuous functions, hence h is a fibrewise homotopy, and, for all $a \in A$, we have $h(a,t) = \pi_1 \circ r(a,t) = \pi_1(a,t) = a$, whence h is relative to A. We clearly have that $\alpha(a) = 0$, for all $a \in A$. Finally, if $t > \alpha(x)$ we must have $\pi_2 \circ r(x,t) \neq 0$, hence $r(x,t) \in A \times I$, therefore $h(x,t) = \pi_1 \circ r(x,t) \in A$, for all $(x,t) \in X \times I$. Thus proving that (α, h) is in fact a Strøm Structure.

Conversely, if (α, h) is a Strøm Structure on (X,A), then a fibrewise retraction $r: X \times I \to (X \times \{0\}) \cup A \times I$ is defined by

$$r(x,t) = \begin{cases} (h(x,t),0), \ t \leq \alpha(x); \\ (h(x,t,t-\alpha(x))), \ t \geq \alpha(x). \end{cases}$$

hence, by proposition 4.2.14, (X,A) is a cofibred pair. □

Recall definition 1.4.22 of NDR pair, given a fibrewise pair (X,A) we can define a **fibrewise NDR pair** by simply adding the condition that h must be a fibrewise homotopy.

4.3 Monoidal and Symmetric Topological Complexity

When defining topological complexity there were no remarks made about the local sections of the path fibration, everything was aloud as long as we managed to decompose the space into open subsets with local sections. From a more practical point of view, when thinking of a realistic setup, we may wish to add some straightforward requirements for the local sections.

First of all, we did not ask for the path from a point $x \in X$ to itself to be the constant path, but in a practical motion planning algorithm, this condition is rather obvious. To include this condition, we introduce a new numerical invariant as follows.

Definition 4.3.1 ((IWASE; SAKAI, 2010))**.** Given a topological space X, the **monoidal topological complexity**, $\mathrm{TC}^M(X)$, is the smallest integer k, such that there exists an open covering $U_1, \ldots, U_k$ of $X \times X$, with $\Delta(X) \subset U_i$, for all $i = 1, \ldots, k$, and each U_i admits a continuous section $s_i : U_i \to PX$ for the path fibration, satisfying $s_i(x,x) = c_x$ (the constant path at x).

One may think of this extra condition as an initial step to taking distance minimizing paths, as the constant path is indeed the smallest path between a point and itself. We clearly have $\mathrm{TC}(X) \leq \mathrm{TC}^M(X)$, so it might be the case that for distance minimizing algorithms the complexity (or the order of instability) becomes even greater than $\mathrm{TC}(X)$. In the following section, we shall explore the relations between $\mathrm{TC}(X)$ and $\mathrm{TC}^M(X)$ by using a concept called fibrewise LS category, but first, let us introduce another type of topological complexity.

Another logical step for building a motion planning algorithm is to consider the path from $x \in X$ to $y \in X$ to be the inverse path of y to x, in other words, if s_i is a local section we wish to build an algorithm in which $s_i(x,y)(t) = s_i(y,x)(1-t)$, for all $t \in I$.

Before defining this new numerical invariant, we need to introduce a fibration, which will replace the path fibration in this case. Consider the space

$$P'X = \{\gamma \in X^I \mid \gamma(0) \neq \gamma(1)\},$$

and consider the group actions of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on $P'X$ and $X \times X \setminus \Delta(X)$ given by $1\gamma = \gamma^{-1}$ ($\gamma^{-1}(t) = \gamma(1-t)$, for all $t \in I$) and $1.(x,y) = (y,x)$, respectively. Then we have the following fibration

$$\begin{aligned} P'X/\mathbb{Z}_2 &\longrightarrow (X \times X \setminus \Delta(X))/\mathbb{Z}_2 \\ \overline{\gamma} &\longmapsto \overline{(\gamma(0), \gamma(1))} \end{aligned}$$

which we will call the **symmetric path fibration**.

Definition 4.3.2 ((FARBER, 2006))**.** The **symmetric topological complexity** of a space X, $\mathrm{TC}^S(X)$, is the Schwarz genus of its symmetric path fibration.

Remark 4.3.3. Notice that if $\mathrm{TC}^S(X) = k$ and $V_1, \ldots, V_k$ is an open covering of $(X \times X \setminus \Delta(X))/\mathbb{Z}_2$, with local sections $s_i : V_i \to P'X/\mathbb{Z}_2$ of the symmetric path fibration, then for each element $\{(x,y),(y,x)\} \in V_i$ we are obtaining an element $\{\gamma,\gamma^{-1}\}$, with $\gamma(0) = x$ and $\gamma(1) = y$. Hence, we get a motion planning algorithm as wanted, with the path $s_i(x,y)$ being the reverse path of $s_i(y,x)$.

4.4 The Fibrewise Lusternik-Schnirelmann Category

In this section, we present the concept of Fibrewise Lusternik-Schnirelmann Category and some important results about this numerical invariant. Our main reference for this section is (IWASE; SAKAI, 2010).

Before starting the discussion, let us remark that if (X, p_X, B, s_X) is a fibrewise pointed space, then a **fibrewise pointed subspace** of X is a subspace $A \subset X$ such that $s_X(B) \subset A$, in this way we get that $(A, p_X|_A, B, s_X)$ is a fibrewise pointed space.

Let us begin with a useful lemma.

Lemma 4.4.1. Given a fibrewise pointed space (X, p_X, B, s_X) and a fibrewise pointed subspace $U \subset X$ (that means $s_X(B) \subset U$). Then, the inclusion $U \hookrightarrow X$ is fibrewise pointed null-homotopic in X if and only if U is fibrewise compressible to $s_x(B) \subset X$ by a fibrewise pointed homotopy.

Proof. Suppose that $U \subset X$ is fibrewise pointed null-homotopic, then there is a fibrewise pointed homotopy $H_t : U \to X$, $t \in I$, such that H_0 is the canonical inclusion and H_1 is a fibrewise pointed constant map. Since the homotopy is fibrewise pointed we have that the diagram

$$\begin{tikzcd} & B \arrow[dl, "s_X"'] \arrow[dr, "s_X"] & \\ U \arrow[rr, "H_t"] \arrow[dr, "p_U"'] & & X \arrow[dl, "p_X"] \\ & B & \end{tikzcd}$$

commutes, for all $t \in I$, in which $p_U = p_X|_U$. We can see that $H_1 s_X = s_X$, which implies $s_X(B) \subset H_1(U)$. Furthermore, since H_1 is a fibrewise pointed constant map, we know from remark 4.1.11 that $H_1 = s_X p_U$, hence $H_1(U) \subset s_X(B)$, therefore $H_1(U) = s_X(B)$ and H_t is a fibrewise pointed homotopy compressing U to $s_X(B)$.

Conversely, suppose $H_t : U \to X$ is a fibrewise pointed homotopy compressing U to $s_X(B)$, which means that $H_1(U) \subset s_X(B)$. As in the previous case, $H_1 s_X = s_X$ implies that $s_X(B) \subset H_1(U)$, and we get $H_1(U) = s_X(B)$. Since $p_X H_1 = p_U$, we have that, for $x \in U_b$, $H_1(x) = s_X(b) = s_X p_U(x)$, hence $H_1 = s_X p_U$ is a fibrewise pointed constant map. $\square$

Now we introduce the concepts of Fibrewise pointed and unpointed LS category.

Definition 4.4.2 ((IWASE; SAKAI, 2010))**.** The **Fibrewise Pointed LS category** of a fibrewise pointed space X over B, usually written as $\text{cat}_B^B(X)$, is the minimal number $m \geq 0$, such that there exists an open covering $U_1, \ldots, U_m$ of X with $s_x(B) \subset U_i$, for all i, and each inclusion $U_i \hookrightarrow X$ is fibrewise pointed null-homotopic in X. If no such m exists we write $\text{cat}_B^B(X) = \infty$.

Definition 4.4.3 ((IWASE; SAKAI, 2010))**.** The **Fibrewise Unpointed LS category** of a fibrewise pointed space X over B, usually written as $\text{cat}_B^*(X)$, is the least positive integer m, such that there exists an open covering $U_1, \ldots, U_m$ of X with each U_i fibrewise compressible into $s_X(B)$ in X by a fibrewise homotopy. If there is no such m we say $\text{cat}_B^*(X) = \infty$.

Remark 4.4.4. In (IWASE; SAKAI, 2010), both definitions 4.4.2 and 4.4.3 differ slightly from what we presented here since Iwase and Sakai used the "normalized" definition, in which one considers coverings with one element more than was introduced here, in that case, the value of the category becomes one less than in the "non-normalized" version. We chose this version mainly for consistency, since for both Topological complexity and LS category we are using the "non-normalized" version.

Now, inspired by the previously shown inequality $TC(X) \leq \text{cat}(X \times X)$ (see lemma 3.2.5), we will analyze what the relation is between these new kinds of category and topological complexity.

From now on, for any topological space X we define the fibrewise pointed space $d(X) = (X \times X, p_X, X, s_X)$, in which $p_X : X \times X \to X$ is the projection on the second coordinate, and $s_X = \Delta : X \to X \times X$ is the diagonal map.

Theorem 4.4.5 ((IWASE; SAKAI, 2010))**.** For X a path connected space and $d(X) = (X \times X, p_X, X, s_X)$, as described above, we have

1. $\text{TC}(X) = \text{cat}_X^*(X \times X)$.
2. $\text{TC}^M(X) = \text{cat}_X^X(X \times X)$.

Proof. Let us start by proving equality number 2. Assume $\text{TC}^M(X) = m$. Then, there is an open covering $U_1, \ldots, U_m$ ($\Delta(X) \subset U_i$) and local sections $s_i : U_i \to X^I$ of the path fibration $\pi : X^I \to X \times X$, satisfying $s_i(x,x) = c_x$, for all $x \in X$.

Remember that $s_X = \Delta$, hence $s_X(X) \subset U_i$ implies that each U_i is a fibrewise pointed subspace of $d(X)$, we wish to show that they are fibrewise pointed nullhomotopic. For that, consider the homotopy

$$\begin{aligned} H_t : U_i &\to X \times X \\ (a,b) &\mapsto (s_i(a,b)(t), b), \end{aligned}$$

clearly each H_t is a fibrewise pointed map, since $H_t s_X = H_t \Delta = \Delta = s_X$ and $p_X H_t = p_{U_i}$, once the second coordinate is fixed. Thus we have a fibrewise pointed homotopy, with H_0 being the includion map and $H_1(U_i) = \Delta(X) = s_X(X)$, hence, by lemma 4.4.1, we conclude that H_1 is a fibrewise constant map and $\mathrm{cat}_X^X(X \times X) \leq m = TC(X)$.

For the inverse inequality suppose that $\mathrm{cat}_X^X(X) = m$, then there is an open covering $U_1, \ldots, U_m$ of $X \times X$ with each $U_i \hookrightarrow X \times X$ fibewise pointed nullhomotopic, which, by lemma 4.4.1, is equivalent to U_i being fibrewise pointed compressible to $s_X(X) = \Delta(X)$. So, there is a fibrewise pointed homotopy $H_t : U_i \to X \times X$ from $U_i \hookrightarrow X \times X$ to a map $H_1 : U_i \to X \times X$ such that $H_1(U_i) \subset \Delta(X)$. Since the homotopy is fibrewise pointed, it must satisfy the following conditions

1. $p_X H_t(a,b) = p_{U_i}(a,b) = b$, for all $(a,b) \in U_i$ and $t \in I$;
2. $H_t s_{U_i}(x) = s_X(x)$, for all $x \in X$ and $t \in I$.

Condition (1) implies that $H_t(a,b) = (F_t(a,b), b)$, for all $(a,b) \in U_i$ and $t \in I$, in which $F_t : U_i \to X$ is a homotopy. Condition (2) implies that $H_t(x,x) = (x,x)$, for all $x \in X$ and $t \in I$, hence $F_t(x,x) = x$, for all $x \in X$ and $t \in I$.

Now we define

$$\begin{aligned} s_i : U_i &\to X^I \\ (a,b) &\mapsto F_t(a,b), \end{aligned}$$

which is clearly a section of the path fibration, since $F_0(a,b) = a$ and $F_1(a,b) = b$. Furthermore, condition (2) implies that $s_i(a,a)(t)$ is the constant path, hence $U_1, \ldots, U_m$ is an open covering satisfying the conditions for monoidal TC, thus $\mathrm{TC}^M(X) \leq m = \mathrm{cat}_X^X(X \times X)$, and we finally conclude that $\mathrm{TC}^M(X) = \mathrm{cat}_X^X(X \times X)$.

In an analogous way, we prove the first equality in the theorem. Suppose that $\mathrm{TC}(X) = m$, and let $U_1, \ldots, U_m$ be an open covering of X with local sections of the path fibration given by $s_i : U_i \to X^I$. Then we may define the homotopy $H_t : U_i \to X \times X$ by

$$H_t(a,b) = (s_i(a,b)(t), b),$$

for all $(a,b) \in U_i$ and $t \in I$. Notice that the second coordinate stays fixed by H_t, hence it is a fibrewise homotopy ($p_X H_t = p_{U_i}$), and since $H_1(U_i) \subset s_X(X) = \Delta(X)$, we conclude that $\mathrm{cat}_X^*(X \times X) \leq m = \mathrm{TC}(X)$.

For the opposite inequality, suppose $\mathrm{cat}_X^*(X \times X) = m$, and let $U_1, \ldots, U_m$ be an open covering of $X \times X$, with each U_i fibrewise compressible to $\Delta(X)$. Let $H_t : U_i \to X \times X$ be a fibrewise homotopy from the inclusion $U_i \hookrightarrow X \times X$ to a map $H_1 : U_i \to X \times X$ such that $H_1(U_i) \subset \Delta(X)$. Since H_t is fibrewise, we have $p_X H_t = p_{U_i}$, which implies, similarly to the previous case, that $H_t(a,t) = (F_t(a,b),b)$, for some homotopy $F_t : U_i \to X$. Then we can define the map $s_i : U_i \to X^I$ as $s_i(a,b) = F_t(a,b)$, which is clearly a continuous section of the path fibration. Therefore $\mathrm{TC}(X) \leq m = \mathrm{cat}^*_X(X)$ and we conclude that $\mathrm{TC}(X) = \mathrm{cat}^*_X(X)$. $\square$

We can use theorem 4.4.5 to try to find a relation between the monoidal and the standard Topological Complexity, since now we know that this is equivalent to finding a relation between the fibrewise pointed and unpointed LS categories.

Theorem 4.4.6 ((IWASE; SAKAI, 2010))**.** For X a fibrewise well-pointed space over B we have

$$\mathrm{cat}^*_B(X) \leq \mathrm{cat}^B_B(X) \leq \mathrm{cat}^*_B(X) + 1.$$

Proof. The inequality $\mathrm{cat}^*_B(X) \leq \mathrm{cat}^B_B(X)$ is immediate from the definitions of pointed and unpointed fibrewise LS category. So, we only need to prove that $\mathrm{cat}^B_B(X) \leq \mathrm{cat}^*_B(X) + 1$.

First, remember that the well-pointed condition means that $(X, s_X(B))$ is a fibrewise cofibred pair, and from proposition 4.2.15 we know that this is equivalent to saying that there is a Strøm structure on $(X, s_X(B))$, i.e., there is a map $u : X \to I$ and a fibrewise homotopy $h : X \times I \to X$ relative to $s_X(B)$, such that $u^{-1}(0) = s_X(B)$, $h(x,0) = x$, for all $x \in X$, and $h(x,1) \in s_X(B)$, if $u(x) < 1$.

Notice that, since h is relative to $s_X(B)$, we have $hs_X(b) = s_X(b)$, which implies that h is actually a fibrewise **pointed** homotopy. Furthermore, if $u(x) < 1$, we have $h(x,1) = s_x(b)$ for some $b \in B$, and from the fibrewise condition we have $p_X(x) = p_X h(x,1) = p_X s_X(b) = b$, hence $h(x,1) = s_X p_X(x)$, for $u(x) < 1$, which is a fibrewise pointed constant map. In other words, if $U = u^{-1}([0,1))$, we have that the inclusion $U \hookrightarrow X$ is fibrewise pointed nullhomotopic, via the fibrewise pointed homotopy $h|_{U \times I}$.

Suppose $\mathrm{cat}^*_B(X) = m$ and let $U_1, \ldots, U_m$ be an open covering of X with each U_i fibrewise compressible to $s_X(B)$, via a fibrewise homotopy $H_i : U_i \times I \to X$.

Define the open sets

$$V = u^{-1}([0,1/3)) \quad \text{and} \quad U_i' = U_i \setminus u^{-1}([0,1/2]),\ i = 1, \ldots, m,$$

and

$$V_i = U_i' \cup V,\ i = 1, \ldots, m \quad \text{and} \quad V_{m+1} = u^{-1}([0,2/3)).$$

Notice that V and U_i' are disjoint sets and we can define the homotopy $G_i : V_i \times I \to X$ to be

$$G_i(x,t) = \begin{cases} h(x,t), & \text{if } x \in V; \\ H_i(x,t), & \text{if } x \in U_i'. \end{cases}$$

for $i = 1, \ldots, m$, and there is no problem with the continuity of the function above, once we have that there are disjoint closed sets, $C = u^{-1}([0, 1/3])$ and $D = u^{-1}([1/2, 1])$, with $V \subset C$ and $U_i' \subset D$.

For the case $m+1$, we simply define $G_{m+1} = h|_{V_{m+1} \times I}$, and for this case it is immediate that V_{m+1} is fibrewise pointed nullhomotopic, via the fibrewise pointed homotopy G_{m+1}.

Since h and H_i are both fibrewise, we conclude that G_i is fibrewise. Furthermore, we have that $G_i(s_X(b), t) = h(s_X(b), t) = s_X(b)$, hence G_i is a fibrewise pointed homotopy. Lastly, the map $h(\_, 1)$ is fibrewise pointed constant in V and $H_i(\_, 1)$ is fibrewise constant, hence $G(\_, 1)$ is fibrewise constant, and since it is also a pointed map, it must be a fibrewise pointed constant map, hence V_i is fibrewise pointed null-homotopic, via the fibrewise pointed homotopy G_i.

Now, we have a set $\{V_1, \ldots, V_m, V_{m+1}\}$ of open fibrewise pointed nullhomotopic sets, it only remains to prove that this is a covering of X. To see this, notice that $U_i \subset V_i \cup V_{m+1}$, for $i = 1, \ldots, m$, since $U_1, \ldots, U_m$ covers X, we have that $V_1, \ldots V_{m+1}$ is also a covering, and finally we conclude that $\mathrm{cat}_B^B(X) \leq m + 1 = \mathrm{cat}_B^*(X) + 1$. $\square$

Theorems 4.4.5 and 4.4.6 show that for a path-connected space X, whenever $d(X) = (X \times X, p_X, X, s_X)$ is a fibrewise well-pointed space, we have $\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1$. As shown in (IWASE; SAKAI, 2010), it is sufficient to suppose that X has the homotopy type of a locally finite simplicial complex, which yields the following corollary.

Corollary 4.4.7 ((IWASE; SAKAI, 2010))**.** If X has the homotopy type of a locally finite simplicial complex, then $\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1$.

APPENDIX

A

R-MODULES AND HOMOLOGICAL ALGEBRA

A.1 R-modules

Basic definitions and results

Usually, one defines left modules and right modules over a ring R, but if R is a commutative ring these two notions become the same in some sense (every right R-module is equivalent to a left R-module and vice versa). Since we mainly deal with modules over commutative rings R (in many cases Principal Ideal Domains), we will only present the concept of R-modules for this case.

Definition A.1.1 ((LANG, 2002))**.** Let $(R,+,\cdot)$ be a commutative ring with unity. Then, an **R-module** is an abelian group $(M,+)$ together with an operation $\alpha : R\times M \to M$, usually denoted by $\alpha(r,x) = rx$, satisfying, for all $x,y \in M$ and $r,s \in R$, the following conditions

1. $r(sx) = (rs)x$;
2. $r(x+y) = rx+ry$;
3. $(r+s)x = rx+sx$;
4. $1x = x$.

A **submodule** of M is any subset $N \subset M$, such that N is also a module, in other words, N is a subgroup of M and $rn \in N$, for all $r \in R$ and $n \in N$. If $A \subset M$ is any subset of M, not necessarily a submodule, then the submodule of M generated by A, denoted $\langle A\rangle$ is the smallest submodule containing A, more concretely $\langle A\rangle = \bigcap_{A\subset N} N$, in which N are submodules of M, an equivalent way to put it is $\langle A\rangle = \{r_1a_1 + \cdots + r_ka_k \mid k \in \mathbb{N},\ r_i \in R,\ a_i \in A\}$ if $A \neq \emptyset$ and $\langle \emptyset\rangle = \{0\}$.

Definition A.1.2 ((LANG, 2002))**.** If N is a submodule of M, we define the quotient R-module M/N to be the quotient group M/N (which always exists, since M is abelian) with the operation from R defined by $r(m+N) = (rm)+N$, in which $m+N$ denotes the equivalence class of m in M/N, which can also be denoted by $[m]$ or $[m]_N$ in some cases.

Definition A.1.3. If M is an R-module, we say that $x \in M$ is a **torsion element** if there is a regular element $r \in R$ (i.e., $sr = 0 \iff s = 0 \in R$) such that $rx = 0$. The set $T(M) = \{x \in M \mid x \text{ is a torsion element}\}$ is a submodule of M, called the **torsion submodule** of M, if $T(M) = 0$ we say that M is a **torsionfree** module.

Notice that in the case of abelian groups ($\mathbb{Z}$-modules), definition A.1.3 is saying that the torsion elements are the group elements of finite order.

Definition A.1.4 ((LANG, 2002))**.** If M and N are R-modules, a function $f : M \to N$ is said to be an **R-homomorphism** if $f(rx+y) = rf(x)+f(y)$, for all $r \in R$ and $x,y \in M$. If, in addition, f is surjective, injective or bijective we call it an **R-epimorphism**, **R-monomorphism**, or **R-isomorphism**, respectively. If an R-isomorphism exists, we say that M and N are isomorphic, and we denote this by $M \approx N$

Theorem A.1.5 ((LANG, 2002))**.** If $f : M \to N$ is an R-homomorphism, then $\mathrm{Im}(f) \approx M/ker(f)$

Lemma A.1.6. Let $\phi : M \to M'$ be an R-homomorphism. Let $Q \subset N \subset M$ and $Q' \subset N' \subset M'$ be submodules such that $\phi(N) \subset N'$ and $\phi(Q) \subset Q'$, then the function $\overline{\phi} : N/Q \to N'/Q'$ given by $\overline{\phi}(x+Q) = \phi(x)+Q'$, is an R-homomorphism. In addition, if $\phi(N) = N'$, then $\overline{\phi}$ is surjective.

The proof of lemma A.1.6 is pretty straightforward, simply notice that $\overline{\phi}$ is well defined, after that it is not difficult to show it is a homomorphism by using the fact that ϕ is one.

Definition A.1.7. An R-module M is said to be **free** if it has a subset $B \subset M$, called the **basis** of M, satisfying the following equivalent conditions

1. Every $x \in M$ can be written in a unique way as $x = r_1b_1 + \ldots r_nb_n$, with $r_i \in R$ and $b_i \in B$.

2. Every $x \in M$ can be written as $x = r_1b_1 + \ldots r_nb_n$, with $r_i \in R$ and $b_i \in B$, and if $s_i \in R$ and $b'_i \in B$ are such that $s_1b'_1, \ldots s_kb'_k = 0$, then $s_1 = \cdots = s_k = 0$ (we usually call this the **linear independence** property on B).

3. Any function $f : B \to N$ of B into an R-module N, can be uniquely extended to a homomorphism $\overline{f} : M \to N$, i.e., $\overline{f}|_B = f$.

It may not be clear that the three statements in definition A.1.7 are equivalent, for that reason let us present a brief argument here. The fact that $1 \iff 2$ is not difficult to see, so we will only show the slightly more difficult statement 2 $\iff$ 3. If statement 2 is true, let $f : B \to N$ be a map, in which N is another R-module. Notice that $\overline{f} : M \to N$ given by $\overline{f}(\sum_i r_i b_i) = \sum_i r_i f(b_i)$, $r_i \in R$, $b_i \in B$, is the only homomorphism extending f, since the representations $\sum_i r_i b_i$ are unique. Conversely, suppose statement 3 is true, and consider the inclusion $i : B \hookrightarrow M$, then there is a unique homomorphism $\bar{i} : M \to M$ extending i, hence this has to be the identity $\bar{i} = id_M$. This implies that $\langle B \rangle = M$, otherwise we could define a homomorphism which is the identity on $\langle B \rangle$ and 0 on $M \setminus \langle B \rangle$, contradicting the uniqueness of the extension $\bar{i}$. To prove the linear independence of B, let $b_i \in B$ and $r_i \in R$ be such that $\sum_i r_i b_i = 0$, for each i define the function $g_i : B \to R$ such that g_i is zero everywhere, except for $g_i(b_i) = 1$, let $\overline{g}_i : M \to R$ be the unique homomorphism extending g_i, we have $r_j = \overline{g}_j(\sum_i r_i b_i) = \overline{g}_j(0) = 0$, for all j, thus proving the equivalence of 2 and 3.

Proposition A.1.8. Let M be a free R-module with basis B, if $C \subset B$, then $\langle C \rangle$ is free and $M/\langle C \rangle$ is also free.

Proof. If B is a basis for M, then $\langle C \rangle$ is clearly free with basis C. If $C = B$ then $M/\langle C \rangle = 0$ is obviously free. Suppose C is a proper subset of B, we claim that $\tilde{B} \doteq \{[b] \mid b \in B \setminus C\}$ is a basis for $M/\langle C \rangle$. First, notice that if $[m] \in M/\langle C \rangle$, then $m \in M$ and we have $m = \sum_{b \in B} r_b b$, hence $[m] = \sum_{b \in B \setminus C} r_b [b]$. Furthermore, if $r_1[b_1] + \cdots + r_n[b_n] = 0$, with $b_i \in B \setminus C$, then $[r_1 b_1 + \cdots + r_n b_n] = 0$, which implies $(r_1 b_1 + \cdots + r_n b_n) \in \langle C \rangle$, hence $r_1 b_1 + \cdots + r_n b_n = s_1 c_1 + \cdots + s_m c_m$, for some $c_i \in C$ and $s_i \in R$, since all b_i and c_j are in B, we must have $r_1 = \cdots = r_n = s_1 = \cdots = s_m = 0$, whence $\tilde{B}$ is indeed a basis for $M/\langle C \rangle$. □

Definition A.1.9 ((VICK, 2012)). For any set S, one can define the **free R-module generated by S** to be the set

$$F(S) = \{f : S \to R \mid f \text{ is a function}\},$$

with the group operation defined by $(f+g)(s) = f(s) + g(s)$, for all $f, g \in F(S)$ and $s \in S$, and the operations by R given by $(rf)(s) = r(f(s))$, for $r \in R$, $f \in F(S)$ and $s \in S$. This is clearly an $R-module$ with basis the set of functions $f_s : S \to R$ such that $f_s(s) = 1$ and $f_s(s') = 0$, for all $s' \neq s$.

We may represent an element of $f \in F(S)$ by what is called a **formal sum**. Since $f = r_1 f_{s_1} + \cdots + r_k f_{s_k}$, we write for simplicity f as the formal sum $f = r_1 s_1 + \cdots + r_k s_k$. With this notation, any $f \in F(S)$ is of the form $f = \sum_{s \in S} r_s s$, with $r_s \neq 0$ for finitely many $s \in S$. This representation is easier to use when dealing with elements of $F(S)$, in fact if $f, g \in F(S)$, then $f = \sum_{s \in S} r_s s$ and $g = \sum_{s \in S} t_s s$, with $r_s, t_s \in R$, and we have $f + g = \sum_{s \in S}(r_s + t_s)s$, in addition, if $\lambda \in R$ then $\lambda f = \sum_{s \in S}(\lambda r_s)s$.

Proposition A.1.10. If R is an Integral Domain and M is a finitely generated R-module, then M is torsion-free if and only if it is isomorphic to a submodule of a free R-module.

Proof. The "if" part is very straightforward, suppose M is isomorphic to a submodule N of a free module F. Notice that any free module F is torsionfree, in fact if $x \in F$ and $r \in R$ with $r \neq 0$, then if $rx = 0$, let $x = \sum_i r_i b_i$, in which $\{b_i\}_i$ is a basis for F, then $\sum_i r r_i b_i = 0$, hence $rr_i = 0$, for all i, but since R is an integral domain we must have $r_i = 0$, for all i, whence $x = 0$. Since N is a submodule of F, it must also be torsionfree, and so is M.

Conversely, let $M = \langle b_1, \dots, b_n \rangle$ be a torsionfree R-module. Let $\{a_1, \dots, a_s\}$ be a subset of $\{b_1, \dots, b_n\}$, which is R-linear independent and is maximal for this property, meaning that if $a_{s+1}, \dots, a_n$ are the remaining elements of $\{b_1, \dots, b_n\}$, then $\{a_1, \dots, a_s, a_i\}$ is not R-linear independent, for any $s+1 \leq i \leq n$ (such a subset exists by a simple argument using Zorn's lemma, and by noticing that $\{b_i\}$ is R-linear independent, since M is torsionfree). If $s = n$, we conclude that M itself is free and we are done. If $s < n$, we know that $F = \langle a_1, \dots, a_s \rangle$ is a free module, and there exists $r_i \in R$, $r_i \neq 0$, for $s+1 \leq i \leq n$ such that $r_i a_i \in F$, let $r = r_{s+1} r_{s+2} \dots r_n$, then $r \neq 0$, since R is an integral domain, and $ra_i \in F$, for all i, hence $rM \subset F$. Thus, we can define the homomorphism $\varphi : M \to F$ given by $\varphi(m) = rm$, since M is torsionfree we clearly have $\ker(\varphi) = 0$, hence $M \approx \varphi(M) \subset F$, as we wanted. $\square$

Theorem A.1.11. If R is a Principal Ideal Domain (PID), then any submodule of a free R-module is itself free.

Proof. Suppose R is a Principal Ideal Domain.

Simple case: First, let us show that the statement of the theorem is valid for R (as an R-module). It is easy to see that any ring is a module over itself, and it is always free, generated by the multiplicative identity, namely $R = \langle 1 \rangle$. Clearly, the submodules of R coincide with its ideals, so the simplest one is $\{0\}$, which is clearly free. The other possible submodules are the ideals $\langle r \rangle$, with $r \neq 0$. Notice that $\{r\}$ is a basis for $\langle r \rangle$, since it generates it, and if $sr = 0$, for some $s \in R$, we must have $s = 0$, since R is a domain (there are no zero divisors). So, we conclude that all submodules of R are free, and they are either $\{0\}$ or generated by a single element.

General case: Let F be a free module over R, and $E \subset F$ a submodule. Our goal is to prove that E is free, we will do so by using Zorn's Lemma.

First, let $B \subset F$ be a basis for F, and for $C \subset B$ denote $E_C \doteq E \cap \langle C \rangle$. Define the set

$$\mathscr{F} \doteq \{(E_C, B_C) \mid E_C \text{ is free, and generated by } B_C,\ C \subset B\},$$

which is clearly non-empty since $(E_\emptyset, \emptyset) \in \mathscr{F}$, and we can define the partial order

$$(E_C, B_C) \leq (E_D, B_D) \iff C \subset D \text{ and } B_C \subset B_D.$$

Let $\mathscr{E} \subset \mathscr{F}$ be a non-empty totally ordered subset (also called a **chain**). We claim that if $\overline{C} \doteq \bigcup_{(E_C, B_C) \in \mathscr{E}} C$ and $B_{\overline{C}} \doteq \bigcup_{(E_C, B_C) \in \mathscr{E}} B_C$, then $(E_{\overline{C}}, B_{\overline{C}})$ is in $\mathscr{F}$ and is an upper bound for $\mathscr{E}$. It is clear that $(E_{\overline{C}}, B_{\overline{C}})$ is an upper bound for $\mathscr{E}$, so we only have to show that $(E_{\overline{C}}, B_{\overline{C}})$ is in fact an element of $\mathscr{F}$.

It is easy to see that $\overline{C} \subset B$. Let us show that $B_{\overline{C}}$ is a basis of $E_{\overline{C}}$. If $c \in E_{\overline{C}} = E \cap \langle \overline{C} \rangle$, then it can be written as $x = r_1 c_1 + \cdots + r_k c_k$, with $c_1, \ldots, c_k \in \bigcup_{(E_C, B_C) \in \mathscr{E}} C$, and since $\{C\}_{(E_C, B_C) \in \mathscr{E}}$ is totally ordered, there is $C_o \in \{C\}_{(E_C, B_C) \in \mathscr{E}}$ such that $c_1, \ldots, c_k \in C_o$, hence $x \in E_{C_o}$, thus $x = s_1 b_1 + \cdots + s_m b_m$, $s_i \in R$, $b_i \in B_{C_o} \subset B_{\overline{C}}$, so $B_{\overline{C}}$ does in fact generate $E_{\overline{C}}$. Finally, if $s_1, \ldots, s_m \in R$ and $b_1, \ldots, b_m \in B_{\overline{C}}$ are such that $s_1 b_1 + \cdots + s_m b_m = 0$, then there is C_o such that $b_1, \ldots, b_m \in B_{C_o}$, and since this is a basis we conclude $s_1 = \cdots = s_m = 0$, whence $B_{\overline{C}}$ is in fact a basis.

So, we have concluded that every chain in $\mathscr{F}$ has an upper bound in $\mathscr{F}$, thus Zorn's lemma implies that there is a maximal element in $\mathscr{F}$, let us denote this element by (E_{C_M}, B_{C_M}). Our goal is to show that $C_M = B$, which will imply that $E_{C_M} = E \cap \langle C_M \rangle = E \cap \langle B \rangle = E \cap F = E$, thus showing that E is free.

By contradiction, suppose $C_M \neq B$, then let $\omega \in B \setminus C_M$, and consider $D = C_M \cup \{\omega\} \subset B$. We claim that E_D is a free submodule of F. Since F is free with basis B, and $D \subset B$, we know that $\langle D \rangle$ is also free, so there is a homomorphism $\varphi : \langle D \rangle \to R$ such that $\varphi(C_M) = 0$ and $\varphi(\omega) = 1$. Clearly φ is surjective, with $\ker \varphi = \langle C_M \rangle$, hence $\langle D \rangle / \langle C_M \rangle \approx R$, by the first isomorphism theorem. Notice that $E_{C_M} = E_D \cap \langle C_M \rangle$, so the second isomorphism theorem implies $E_D / E_{C_M} = E_D / (E_D \cap \langle C_M \rangle) \approx (E_D + \langle C_M \rangle) / \langle C_M \rangle$, which is a submodule of $\langle D \rangle / \langle C_M \rangle \approx R$, so from the result for the simple case we know that E_D / E_{C_M} is a free R-module, which is either $\{0\}$ or is generated by a single element.

If $E_D / E_{C_M} = \{0\}$, then $E_D = E_{C_M}$, which implies $(E_D, B_{C_M}) \in \mathscr{F}$, but this is a contradiction since we clearly have $(E_{C_M}, B_{C_M}) \leq (E_D, B_{C_M})$.

In the other hand, if E_D / E_{C_M} has a basis of a single non-zero element, let us say $[a] \in E_D / E_{C_M}$, then we claim that $B_D \doteq B_{C_M} \cup \{a\}$ is a basis for E_D. Indeed, if $r_0 u + r_1 c_1, \ldots, r_n c_n = 0$, with $r_i \in R$ and $c_i \in C_M$, then $r_0 [u] = [r_0 u + r_1 c_1, \ldots, r_{n-1} c_n] = [0]$ implies $r_0 = 0$, which then implies $r_1 = \cdots = r_n = 0$. On top of that, if $z \in E_D$, then $[z] = s_0 [u]$ for some $s_0 \in R$, which implies $(z - s_0 u) \in E_{C_M}$, so there are $s_1 \ldots s_n \in R$ and $c_1 \ldots c_n \in C_M$ such that $(z - s_0 u) = s_0 u + s_1 c_1 + \cdots + s_n cn$, hence B_D is in fact a basis of E_D, and again we have a contradiction. Whence, we finally conclude that C_M has to be B, which implies that E is a free submodule. $\square$

The simple case in the previous proof is actually stronger, we have the following result.

Proposition A.1.12. All submodules of R (as an R-module) are free if and only if R is a PID.

Proof. If R is a PID we have already proven in the previous theorem that R as an R-

module has the property stated in the proposition. Suppose all submodules of R are free, then if $ab = 0$ for $a, b \in R$ non-zero elements, then $a = 0$ or $b = 0$, otherwise we would have the submodule $\langle b \rangle$, which we are assuming to be free, but $ab = 0$ with $a \neq 0$ contradicts this fact, hence R is in fact a domain. By contradiction, suppose there was a submodule of $I \subset R$ which is not a principal ideal since I is free there is a basis $B \subset I$ of I, with at least two elements in it, let $a, b \in B$ be two different elements, then $ab + (-b)a = 0$, since R is commutative, which contradicts the fact that B is a basis. Whence R is a Principal Ideal Domain (PID). □

In view of proposition A.1.12, we can construct simple examples of free modules with non-free submodules. For instance, consider $\mathbb{Z}_6$ as a module over itself, then $\{0, 3\}$ is a non-free submodule, since the only possible basis is $\{3\}$, but $2.3 = 0$ implies it is not a basis.

Corollary A.1.13. If R is a Principal Ideal Domain (**PID**) and M is a finitely generated torsionfree R-module, then M is free.

Proof. By proposition A.1.10 we know that M is isomorphic to a submodule N of a free R-module F, but by theorem A.1.11 N is also free, hence M must be free. □

Tensor product of R-modules

Definition A.1.14. Given R-modules M, N and P, we say that a map $f : M \times N \to P$ is R-bilinear if $f|_{M\times\{n\}}$ and $f|_{\{m\}\times N}$ are R-homomorphisms for any $n \in N$ and $m \in M$. In other words, for all $m_1,\ m_2 \in M$, $n_1,\ n_2 \in N$ and $r, s \in R$ we must have

$$f(rm_1 + m_2, sn_1 + n_2) = rsf(m_1, n_1) + rf(m_1, n_2) + sf(m_2, n_1) + f(m_2, n_2).$$

Theorem A.1.15 ((DAVIS; KIRK, 2001))**.** Given R-modules M and N there is a unique R-module $M \otimes_R N$, called the **tensor product** of M and N, and a R-bilinear map $\pi : M \times N \to M \otimes_R N$, such that for any R-bilinear map $f : M \times N \to P$ there exists a unique R-homomorphism $\overline{f} : M \otimes N \to P$ such that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\pi"'] & P \\ M \otimes_R N \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Proof. Let us first prove the existence of the tensor product of M and N. We will explicitly construct an R-module satisfying the condition specified in the theorem.

First consider $F(M \times N)$ to be the free R-module generated by $M \times N$. The idea is that we want to take a quotient of $F(M \times N)$ in a way that the map from $M \times N$ to $F(M \times N)/\sim$ taking (m, n) to the equivalence class $[(m, n)]$ is R-bilinear, so we are basically asking for the equivalence relation $(rm,n) \sim r(m,n) \sim (m,rn)$, $(m+m',n) \sim (m,n)+(m',n)$ and $(m,n+n') \sim (m,n)+(m,n')$. In other words, let $L(M\times N)$ be the submodule of $F(M\times N)$ generated by elements of the following kind:

1. $(rm,n)-r(m,n)$;
2. $(m,rn)-r(m,n)$;
3. $(m+m',n)-(m,n)+(m',n)$;
4. $(m,n+n')-(m,n)+(m,n')$.

Then we define $M\otimes_R N = F(M\times N)/L(M\times N)$ and $\pi : M\times N \to M\otimes_R N$ is simply $\pi(m,n) = [(m,n)]$, usually denoted $m\otimes n = \pi(m,n)$. Clearly π is an R-bilinear map, in other words, we have:

1. $(rm)\otimes n = r(m\otimes n) = m\otimes(rn)$;
2. $(m+m')\otimes n = m\otimes n + m'\otimes n$;
3. $m\otimes(n+n') = m\otimes n + m\otimes n'$,

for all $r\in R$, $m,m'\in M$ and $n,n'\in N$. From item 1 above, we conclude that a general element of $M\otimes_R N$ is of the form $\sum_{i=1}^n m_i\otimes n_i$ (this represents the equivalence class of the element $\sum_{i=1}^n (m_i,n_i)$), but this representation is by no means unique, notice that even when $n=1$ we can have two or more ways of writing the same element, namely $(rm)\otimes n = m\otimes(rn)$.

Now, suppose there is an R-bilninear map $f : M\times N \to P$, we want to show that there is a unique R-homomorphism $\overline{f} : M\otimes N \to P$ such that $\overline{f}\pi = f$. Uniqueness is simple, suppose there is such an R-homomorphism, then $\overline{f}(\sum_{i=1}^n m_i\otimes n_i) = \sum_{i=1}^n \overline{f}(m_i\otimes n_i) = \sum_{i=1}^n \overline{f}\pi(m_i,n_i) = \sum_{i=1}^n f(m_i,n_i)$. To show the existence, consider the R-homomorphism induced by f on $F(M\times N)$, denote it by $\hat{f} : F(M\times N) \to P$. Since f is R-bilinear, we have $\hat{f}(L(M\times N)) = 0$, hence by lemma A.1.6 it induces an R-homomorphism $\overline{f} : M\otimes_R N \to P$ given by $\overline{f}(\sum_i m_i\otimes n_i) = \hat{f}(\sum_i (m_i,n_i)) = \sum_i f(m_i,n_i)$. Hence, $\overline{f}\pi(m,n) = \overline{f}(m\otimes n) = f(m,n)$, for all $(m,n)\in M\times N$. Thus, ending the proof of the existence of the tensor product.

To prove uniqueness of the tensor product, suppose T and T' are two R-modules satisfying the conditions for being the tensor product of M and N, with R-bilinear maps $\pi : M\times N \to T$ and $\pi' : M\times N \to T'$. By definition of the tensor product, there must be R-homomorphisms $\overline{\pi} : T' \to T$ and $\overline{\pi'} : T \to T'$ such that $\overline{\pi'}\pi = \pi'$ and $\overline{\pi}\pi' = \pi$, but then $(\overline{\pi}\overline{\pi'})\pi = \pi$. But the R-homomorphism satisfying this equality is unique, hence $\overline{\pi}\overline{\pi'} = id_T$, analogously $\overline{\pi'}\overline{\pi} = id_{T'}$, so we conclude that $\overline{\pi}$ is an isomorphism between T' and T, which proves uniqueness of the tensor product. □

Notice that, by definition, any R-bilinear map $f : M\times N \to P$ induces an R-homomorphism $\overline{f} : M\otimes_R N \to P$ given by $\overline{f}(\sum_i m_i\otimes n_i) = \sum_i f(m,n)$. With this in mind, in many situations we simply say "consider the homomorphism $m\otimes n \mapsto f(m,n)$", meaning the R-homomorphism induced by the R-bilinear map f.

In the next proposition, we present the most important basic properties of the tensor product.

Proposition A.1.16 ((DAVIS; KIRK, 2001))**.** If M, N, P, Q are R-modules, then:

1. $M \otimes_R N \approx N \otimes_R M$;
2. $M \otimes_R R \approx R \otimes_R M \approx M$;
3. $(M \otimes_R N) \otimes_R P \approx M \otimes_R (N \otimes_R P)$;
4. $(\bigoplus_\alpha M_\alpha) \otimes_R N \approx \bigoplus_\alpha (M_\alpha \otimes_R N)$;
5. $\left(\bigoplus_\alpha M_\alpha\right) \otimes_R (\bigoplus_\beta N_\beta) \approx \bigoplus_{\alpha,\beta} (M_\alpha \otimes_R N_\beta)$.

Proof. **1.** Consider the R-homomorphisms $A \otimes_R B \ni a \otimes b \mapsto b \otimes a \in B \otimes_R A$, it is clearly an isomorphism, with inverse $B \otimes_R A \ni b \otimes a \mapsto a \otimes b \in A \otimes_R B$.
2. Consider the R-homomorphism $R \otimes_R M \ni r \otimes m \mapsto rm \in M$, and as a candidate for inverse consider the map $M \ni m \mapsto 1 \otimes m$, this is in fact an R-homomorphism, since $1 \otimes (rm + m') = r(1 \otimes m) + 1 \otimes m'$, it is clearly the inverse of the R-homomorphism taken earlier, hence we have an isomorphism.
3. Notice that a general element of $(M \otimes_R N) \otimes_R P$ is of the form $\sum_j [(\sum_j m_{ij} \otimes n_{ij}) \otimes p_j]$, and by the bilinearity of the tensor product, this can always be written as an element of the form $\sum_i (m_i \otimes n_i) \otimes p_i$. Let $\overline{f}$ be the homomorphism induced by the R-bilinear application $f : (M \otimes_R N) \times P \to M \otimes_R (N \otimes_R P)$ given by $f(\sum_i m_i \otimes n_i, p) = \sum_i m_i \otimes (n_i \otimes p)$. It is not difficult to see that $\overline{f}(\sum_i (m_i \otimes n_i) \otimes p_i) = \sum_i m_i \otimes (n_i \otimes p_i)$, and one can analogously build a homomorphism $\overline{g} : M \otimes_R (N \otimes_R P) \to (M \otimes_R N) \otimes_R P$ such that $\overline{g}(\sum_i m_i \otimes (n_i \otimes p_i) = \sum_i (m_i \otimes n_i) \otimes p_i$, then clearly $\overline{f}$ is an isomorphism with inverse $\overline{g}$.
4. Consider the map $f : (\bigoplus_\alpha M_\alpha) \times N \to \bigoplus_\alpha (M_\alpha \otimes_R N)$ given by $f((m_\alpha)_\alpha, n) = (m_\alpha \otimes n)_\alpha$, since it is R-bilinear, it induces a homomorphism $\overline{f} : (\bigoplus_\alpha M_\alpha) \otimes_R N \to \bigoplus (M_\alpha \otimes_R N)$ such that $\overline{f}((m_\alpha)_\alpha \otimes n) = (m_\alpha \otimes n)_\alpha$. Let us show that $\overline{f}$ is a isomorphism by constructing its inverse homomorphism. For each α let $i_\alpha : M_\alpha \hookrightarrow \bigoplus_\alpha M_\alpha$ be the canonical inclusion, and define the maps $g_\alpha : M_\alpha \times N \to (\bigoplus_\alpha M_\alpha) \otimes_R N$ given by $g_\alpha(m_\alpha, n) = i_\alpha(m_\alpha) \otimes n$, since this are R-bilinear maps they induce R-homomorphisms $\overline{g}_\alpha : M \otimes_R N \to (\bigoplus_\alpha M_\alpha) \otimes_R N$ such that $\overline{g}_\alpha(m_\alpha \otimes n) = i_\alpha(m_\alpha) \otimes n$. With this we can define a new R-homomorphism $\overline{g} : \bigoplus_\alpha M_\alpha \otimes_R N \to (\bigoplus) \alpha M_\alpha) \otimes_R N$ given by $\overline{g}((m_\alpha \otimes n_\alpha)_\alpha) = \sum_\alpha \overline{g}_\alpha(m_\alpha \otimes n_\alpha)$. It is not difficult to see that $\overline{g}$ is the inverse of $\overline{f}$ by a simple computation. Finally, item **5** is a simple consequence of item 4. □

Property 3 in proposition A.1.16 induces us to simply write $M \otimes_R N \otimes_R P$ for the R-module $(M \otimes_R N) \otimes_R P$. An ***R*-Multilinear map** is a map $f : M_1 \times \cdots \times M_n \to N$, in which M_j and N are R-modules, such that for any $j = 1, \ldots, n$, and any element $a = (a_1, \ldots, a_{j-1},\ a_{j+1}, \ldots, a_n) \in M_1 \times \cdots \times M_{j-1} \times M_{j+1} \times \cdots \times M_n$. If we define the inclusion

$$\begin{aligned} i_j^a : M_j &\to M_1 \times \cdots \times M_n \\ m_j &\mapsto (a_1, \ldots, a_{j-1}, m_j, a_{j+1}, \ldots, a_n), \end{aligned}$$

then fi_j^a is an R-homomorphism. Using this definition, it is possible to introduce the tensor product $M_1 \otimes_R \cdots \otimes_R M_n$ similarly to what was done for the case $n = 2$, and this new definition would be equivalent to taking the iterated definition $M_1 \otimes_R \cdots \otimes_R M_{j-1} \otimes_R M_j = (M_1 \otimes_R \cdots \otimes_R M_{j-1}) \otimes_R M_j$ for $j \geq 2$.

Proposition A.1.17 ((DAVIS; KIRK, 2001))**.** If $f : M \to P$ and $g : N \to Q$ are R-homomorphisms, there is an R-homomorphism, denoted $f \otimes g : M \otimes_R N \to P \otimes_R Q$, given by $f \otimes g(m \otimes n) = f(m) \otimes g(n)$.

Proof. Simply notice that $M \times N \ni (m,n) \mapsto f(m) \otimes g(n) \in P \otimes_R Q$ is a bilinear map, so it induces a unique homomorphism $f \otimes g : M \otimes_R N \to P \otimes_R Q$. □

Proposition A.1.18. The tensor product with a fixed R-module N is a covariant functor from the category of R-modules into itself. It takes any R-module M to $M \otimes_R N$ and any R-homomorphism $f : M \to P$ to $f \otimes id : M \otimes_R N \to P \otimes_R N$.

Proof. One can easily see that $(f \otimes id)(g \otimes id) = fg \otimes id$ and if $id : M \to M$ is the identity of M then $id \otimes id : M \otimes_R N \to M \otimes_R N$ is just $id_{M \otimes_R N}$. □

Lemma A.1.19 ((SPANIER, 1989))**.** The tensor product of two R-epimorphisms is an R-epimorphism.

Proof. Let $f : M \to P$ and $g : N \to Q$ be R-epimorphisms, then any element of $P \otimes_R Q$ can be written as $\sum_i f(m_i) \otimes g(n_i)$, whence $f \otimes g$ is surjective. □

Lemma A.1.20 ((SPANIER, 1989))**.** For two R-epimorphisms $f : M \to P$ and $g : N \to Q$, the kernel $\ker(f \otimes g)$ is generated by elements $m \otimes n$ such that $m \in \ker(f)$ or $n \in \ker(g)$

Proof. Let $A \subset M \otimes N$ be the submodule generated by elements $m \otimes n$ such that $m \in \ker(f)$ or $n \in \ker(g)$. Let $\lambda : M \otimes_R N \to (M \otimes_R N)/A$ be the projection onto the quotient module. Since f and g are surjective, we can define the R-bilinear map

$$\begin{aligned} \varphi : P \times Q &\to (M \otimes_R N)/A \\ (p,q) &\mapsto \lambda(m \otimes n) \end{aligned}$$

in which $f(m) = p$ and $g(n) = q$. This is well defined, since if $f(m') = f(m) = p$, then $(m - m') \in \ker(f)$, which implies $(m - m') \otimes n \in A$, hence

$$0 = \lambda((m - m') \otimes n) = \lambda(m \otimes n - m' \otimes n) = \lambda(m \otimes n) - \lambda(m' \otimes n),$$

and analogously if $g(n') = g(n) = q$. The R-bilinearity follows from the fact that if $f(m) = p$ and $f(m') = p'$, then $f(m + rm') = p + rp'$, hence

$$\varphi(p + rp', q) = \lambda((m + rm') \otimes n) = \lambda(m \otimes n) + r\lambda(m' \otimes n) = \varphi(p,q) + r\varphi(p',q),$$

and analogously for the second variable. This bilinear map induces a unique homomorphism

$$\begin{aligned}\overline{\varphi}: P\otimes_R Q &\to (M\otimes_R N)/A\\ p\otimes q &\mapsto \lambda(m\otimes n),\end{aligned}$$

with $f(m)=p$ and $g(n)=q$. It is clear from the way $\overline{\varphi}$ was defined that the following diagram commutes

$$\begin{tikzcd} M\otimes_R N \arrow[r, "f\otimes g"] \arrow[dr, "\lambda"'] & P\otimes_R Q \arrow[d, "\overline{\varphi}"] \\ & (M\otimes_R N)/A \end{tikzcd}$$

hence $\ker(f\otimes g)\subset A$, and since the reverse inclusion is obvious, we have $\ker(f\otimes g)=A$. □

Lemma A.1.21 ((SPANIER, 1989))**.** Given an exact sequence of R-modules

$$M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0\ ,$$

then for any R-module Q the following sequence is exact

$$M\otimes_R Q \xrightarrow{f\otimes id} N\otimes_R Q \xrightarrow{g\otimes id} P\otimes_R Q \longrightarrow 0\ .$$

Proof. Since g and id are epimorphisms, by lemma A.1.19 $g\otimes id$ is an epimorphism, and by lemma A.1.20 we know that $\ker(g\otimes id)$ is generated by elements $n\otimes q$ such that $n\in\ker(g)=\mathrm{Im}(f)$, but this are the same elements that generate $\mathrm{Im}(f\otimes id)$, hence the sequence is exact. □

Proposition A.1.22. If $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is a short exact split sequence of R-modules, then $0 \longrightarrow M\otimes_R Q \xrightarrow{f\otimes id} N\otimes_R Q \xrightarrow{g\otimes id} P\otimes_R Q \longrightarrow 0$ is a short exact split sequence.

Proof. The statement follows by lemma A.1.21 and the fact that if $h: N\to M$ is a left inverse for f, then $h\otimes id: N\otimes_R Q\to M\otimes_R Q$ is a left inverse for $f\otimes id$. □

Proposition A.1.23. If $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is a short exact sequence of R-modules, and F is a free R-module, then $0 \longrightarrow M\otimes_R F \xrightarrow{f\otimes id} N\otimes_R F \xrightarrow{g\otimes id} P\otimes_R F \longrightarrow 0$ is a short exact sequence.

Proof. From lemma A.1.21 the only thing left to prove is that $f\otimes id$ is injective. Let B be a basis of F, then

$$\begin{aligned}F &\to \bigoplus_{b\in B} R\\ \sum_{b\in B} r_b b &\mapsto (r_b)_b,\end{aligned}$$

is an R-isomorphism, and we can define the composition of isomorphisms

$$M \otimes_R F \to M \otimes_R \bigoplus_{b \in B} R \to \bigoplus_{b \in R} M \otimes_R R \to \bigoplus_{b \in B} M$$

$$a \otimes \left(\sum_b r_b b \right) \mapsto \quad a \otimes (r_b)_b \mapsto \quad (a \otimes r_b)_b \mapsto (r_b a)_b,$$

and we denote this isomorphism by ψ_M. It is easy to verify that the diagram

$$\begin{tikzcd} M \otimes_R F \arrow[r, "f \otimes id"] \arrow[d, "\psi_M"] & N \otimes_R F \arrow[d, "\psi_N"] \\ \bigoplus_b M \arrow[r, "\oplus_b f"] & \bigoplus_b N \end{tikzcd}$$

commutes and since f is injective we have that $\oplus_b f$ is also injective, which implies the injectivity of $f \otimes id$, since ψ_M and ψ_N are isomorphisms. □

A.2 Homological Algebra

In this appendix, we present several results from Homological Algebra used in the main text.

Definition A.2.1 ((WEIBEL, 1995))**.** A chain complex C is a family of R-modules $\{C_n\}_{n \in \mathbb{Z}}$ and R-homomorphisms $\{\partial_n : C_n \to C_{n-1}\}$

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots ,$$

such that $\partial_n \partial_{n+1} = 0$, for all $n \in \mathbb{Z}$.

For simplicity, we will omit the indexes and write simply ∂ for every ∂_n. The condition $\partial\partial = 0$ is equivalent to $\mathrm{Im}(\partial) \subset \ker(\partial) \subset (C_n)$. We usually denote $B_n = \mathrm{Im}(\partial) \subset C_n$ and $Z_n = \ker(\partial) \subset C_n$.

Definition A.2.2 ((WEIBEL, 1995))**.** The n **homology module** of the chain complex C is the quotient R-module $H_n(C) \doteq Z_n(C)/B_n(C)$

Definition A.2.3 ((LANG, 2002))**.** Given two chain complexes C and C', a **chain map** $f : C \to C'$ is a collection of maps $f_n : C_n \to C'_n$ such that all squares in the following diagram commute

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'"] & C'_n \arrow[r, "\partial'"] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

As for the map ∂, we may omit the indexes for a chain map, and write simply $f : C_n \to C_{n-1}$, since it is clear from the domain which f_n we are considering.

Theorem A.2.4 ((LANG, 2002))**.** The chain complexes as described above form a category, called the **category of chain complexes of R-modules**, the objects of which are chain complexes C, and the morphisms are chain maps $f : C \to C'$.

Theorem A.2.5 ((LANG, 2002))**.** A chain map $f : C \to C'$ satisfies $f(Z_n) \subset Z'_n$ and $f(B_n) \subset B'_n$, hence it naturally induces homomorphisms $f_* : H_n(C) \to H_n(C')$. If $g : C' \to C''$ is another chain map, then $gf : C \to C''$ is also a chain map and $(gf)_* = g_* f_*$.

Corollary A.2.6 ((WEIBEL, 1995))**.** H_n describes a covariant functor from the category of chain complexes of R-modules to the category of R-modules, for all n. It takes a chain complex C to $H_n(C)$ and a chain map $f : C \to C'$ to $H_n(f) \doteq f_* : H_n(C) \to H_n(C')$, this is usually called the **n-homology functor**.

Definition A.2.7 ((WEIBEL, 1995))**.** A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-modules and R-homomorphisms is said to be an **exact sequence** if $\mathrm{Im}(f) = \ker(g)$. More generally, a sequence $\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$ is said to be exact if all triples $A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1}$ are exact. A **short exact sequence** is an exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

Notice that in a short exact sequence as described above f has to be injective and g has to be surjective. Furthermore, if we had an exact sequence $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$, then h is an isomorphism.

Lemma A.2.8 ((HATCHER, 2002))**.** For a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ the following statements are equivalent.

1. There is an R-homomorphism which is a left inverse of f.

2. There is an R-homomorphism which is a right inverse of g.

3. There is an R-isomorphism $\psi : B \to A \oplus C$ such that the following diagram commutes

$$\begin{tikzcd} & & B \arrow[dd, "\psi"] \arrow[dr, "g"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "i"'] & & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, "p"'] & & \end{tikzcd}$$

in which i and p are the canonical inclusion and projection homomorphisms. A sequence satisfying this conditions is called a **split exact sequence**.

Lemma A.2.9. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R modules. If C is free, then the sequence is split exact.

Proof. Simply let $D \subset C$ be a basis for C, and define the R-homomorphism $h : C \to B$, such that for each $d \in D$, $h(d)$ is an element of $g^{-1}(\{d\})$, which is nonempty since g is surjective. Then, h is clearly a right inverse for g, hence the sequence splits. $\square$

Proposition A.2.10. If the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[d, "g"] \\
0 \arrow[r] & D \arrow[r, "\delta"] & E \arrow[r, "\varepsilon"] & F
\end{tikzcd}$$

of R-modules is such that each row is exact and it commutes, then there is a unique homomorphism $h : A \to D$ for which the diagram still commutes. Furthermore, if f is an isomorphism and g a monomorphism, then h is an isomorphism.

Proof. Notice that for any $a \in A$ we have $\varepsilon f \alpha(a) = g\beta\alpha(a) = 0$. Therefore $f\alpha(a) \in \ker(\varepsilon) = \mathrm{Im}(\delta)$, and since δ is injective there is a unique element $d \in D$ such that $\delta(d) = f\alpha(a)$, define the map $h : A \to D$ by $h(a) = d$ such that $\delta(d) = f\alpha(a)$, this map clearly makes the diagram commute and $h(ra + a') = rh(a) + h(a')$, hence h is a homomorphism. Suppose now that f is an isomorphism and g is a monomorphism. If $h(a) = h(a')$, then $0 = \delta h(a - a') = f\alpha(a - a')$ and since both α and f are injective we have $a = a'$, thus h is injective. If we take any $d \in D$, then since f is surjective there is $b \in B$ with $f(b) = \delta(d)$, then $g\beta(b) = \varepsilon f(b) = \varepsilon\delta(d) = 0$, and since g is injective we must have $\beta(b) = 0$, so by exactness of the first row there is $\alpha \in A$ such that $\alpha(a) = b$, whence $h(a) = d$, and we conclude that h is an isomorphism. $\square$

Definition A.2.11 ((WEIBEL, 1995))**.** If we fix N an R-module, for each R-module M we can define $\mathrm{Hom}_R(M,N)$ to be the R-module of R-homomorphisms from M to N. Given an R-homomorphism $f : M \to M'$ we define the R-homomorphism $f^* : \mathrm{Hom}_R(M',N) \to \mathrm{Hom}_R(M,N)$ given by $f^*(\varphi) = \varphi f$.

Theorem A.2.12 ((HATCHER, 2002))**.** If we fix an R-module N , then $\mathrm{Hom}_R(\cdot,N)$ is a contravariant functor from the R-module category to itself, defined as in definition A.2.11.

Definition A.2.13 ((HATCHER, 2002))**.** Given chain complex C and an R-module N, we can apply the $\mathrm{Hom}_R(\cdot,N)$ functor to obtain a new chain complex $C^* = \mathrm{Hom}_R(C,N) \doteq \{\mathrm{Hom}_R(Cn,N)\}_{n\in\mathbb{Z}}$, with maps $\delta = \partial^* : \mathrm{Hom}_R(Cn,N) \to \mathrm{Hom}_R(C_{n+1},N)$, since $\delta(\varphi) = \varphi\partial$ we clearly have $\delta\delta = 0$, and we can write this new chain complex as

$$\cdots \longrightarrow C^*_{n-1} \xrightarrow{\delta} C^*_n \xrightarrow{\delta} C^*_{n+1} \longrightarrow \cdots .$$

This complex, in which the index is increasing, is usually called a **cochain complex**.

As for the case of chain complexes, we have $\operatorname{Im}(\delta) \subset \ker(\delta) \subset C_n^*$, we usually denote $B^n(C) \doteq \operatorname{Im}(\delta)$ and $Z^n(C) \doteq \ker(\delta)$, and define the n **cohomology R-module** of C as the quotient module $H^n(C) = Z^n(C)/B^n(C)$.

A **cochain map** will be simply the dual notion of a chain map, i.e., just invert all arrows in the chain map definition. If $f : C \to C'$ is a chain map, then by applying the $\operatorname{Hom}_R(\cdot, N)$ functor we get a cochain map $f^* : C'* \to C*$, and we have $f^*(Z^n(C')) \subset Z^n(C)$ and $f^*(B^n(C')) \subset B^n(C)$, so by lemma A.1.6 we have a well defined R-homomorphism $f^*_* : H^n(C') \to H^n(C)$. It is not difficult to check that if $f : C \to C'$ and $g : C' \to C''$ are chain maps, then $(gf)^*_* = f^*_* g^*_*$, so H^n is a contravariant functor from the category of chain complexes of R-modules to the category of R-modules, it is usually referred to as the n **cohomology functor**.

Definition A.2.14 ((WEIBEL, 1995))**.** A sequence $C \xrightarrow{f} C' \xrightarrow{g} C''$ of chain complexes and chain maps is said to be an **exact sequence of chain complexes** if $\operatorname{Im}(f) = \ker(g)$, meaning that this is true in all levels $C_n \xrightarrow{f} C'_n \xrightarrow{g} C''_n$. More generally, a sequence $\cdots \longrightarrow C \xrightarrow{f} C' \xrightarrow{g} C'' \longrightarrow \cdots$ is said to be exact if all triples $C \xrightarrow{f} C' \xrightarrow{g} C''$ are exact. A **short exact sequence of chain complexes** is an exact sequence of chain complexes of the form $0 \longrightarrow C \xrightarrow{f} C' \xrightarrow{g} C'' \longrightarrow 0$.

Lemma A.2.15 (**Snake Lemma** (WEIBEL, 1995))**.** If the following is a commutative diagram of R-modules with exact rows

$$\begin{tikzcd}
 & C_n \arrow[r, "f"] \arrow[d, "\partial"] & C'_n \arrow[r, "g"] \arrow[d, "\partial'"] & C''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f"] & C'_{n-1} \arrow[r, "g"] & C''_{n-1} &
\end{tikzcd}$$

then there is an exact sequence

$$\ker(\partial) \xrightarrow{f_o} \ker(\partial') \xrightarrow{g_o} \ker(\partial'') \xrightarrow{\alpha} \operatorname{Coker}(\partial) \xrightarrow{\tilde{f}} \operatorname{Coker}(\partial') \xrightarrow{\tilde{g}} \operatorname{Coker}(\partial'') ,$$

in which α is called the **attaching map**.

Since the proof of the Snake Lemma is not provided in the reference, it will be shown here.

Proof. The maps f_o, g_o, $\tilde{f}$ and $\tilde{g}$ are defined in the natural way, f_o and g_o are restrictions of f and g respectively (with the range also restricted, which works since $f(\ker(\partial)) \subset \ker(\partial')$, and analogously for g), and since $f(\operatorname{Im}(\partial)) \subset \operatorname{Im}(\partial')$, we have by lemma A.1.6 a well defined homomorphism $\tilde{f}$, analogously for $\tilde{g}$. The map α will be defined latter in the proof, first let us analyze the exactness of the sequence in the two extremities, namely we want to show that $\ker(g_o) = \operatorname{Im}(f_o)$ and $\ker(\tilde{g}) = \operatorname{Im}(\tilde{f})$.

For the fist equality ($\ker(g_o) = \mathrm{Im}(f_o)$), let $c'_n \in \ker(g_o) \subset \ker(\partial')$, then $c'_n \in \ker(g) = \mathrm{Im}(f)$, so there is a $c_n \in C_n$ such that $f(c_n) = c'_n$. So we have $f\partial(c_n) = \partial' f(c_n) = \partial'(c'_n) = 0$, since f from the bottom row is injective, we have $\partial(c_n) = 0$, hence $c_n \in \ker(\partial)$, and since $f_o(c_n) = f(c_n) = c'_n$, we have $c'_n \in \mathrm{Im}(f_o)$. Conversely, if $c'_n \in \mathrm{Im}(f_o)$, then there is $c_n \in \ker(\partial)$ such that $f_o(c_n) = c'_n$, and we have $g_o f_o(c_n) = gf(c_n) = 0$, thus concluding the equality $\ker(g_o) = \mathrm{Im}(f_o)$.

For the second equality ($\ker(\tilde{g}) = \mathrm{Im}(\tilde{f})$), let $(c' + \mathrm{Im}(\partial')) \in \ker(\tilde{g})$, then $g(c') \in \mathrm{Im}(\partial'')$, so pick $c''_n \in C''_n$ so that $\partial''(c''_n) = g(c')$, and since g in the upper row is surjective there is $c'_n \in C'_n$ such that $g(c'_n) = c''_n$, whence $g\partial'(c'_n) = \partial'' g(c') = \partial''(c''_n) = g(c')$. Thus $(c' - \partial'(c'_n)) \in \ker(g)$, from exactness we know there is an element $c \in C_{n-1}$ such that $f(c) = c' - \partial'(c'_n)$, then clearly $\tilde{f}(c + \mathrm{Im}(\partial)) = c' + \mathrm{Im}(\partial')$. It is clear that $gf = 0$ implies $\tilde{g}\tilde{f} = 0$, thus concluding the veracity of the equality $\ker(\tilde{g}) = \mathrm{Im}(\tilde{f})$.

Let us now finally define the attaching map $\alpha : \ker(\partial'') \to \mathrm{Coker}(\partial)$:

1. For $c'' \in \ker(\partial'')$, since g is surjective there is a $c' \in C'_n$ such that $g(c') = c''$.
2. We have $g\partial'(c') = \partial'' g(c') = \partial''(c'') = 0$, hence $\partial'(c') \in \ker(g)$, by exactness there is an element $c \in C_{n-1}$ such that $f(c) = \partial'(c')$.

We define $\alpha(c'') = c + \mathrm{Im}(\partial)$.

We still need to analyze if α is really a homomorphism, its definition is a bit confusing, but this is not a difficult task. First, let us see that α is indeed a well defined map. Suppose that for $c'' \in \ker(\partial'')$ we have $c'_1,\ c'_2 \in C'_n$ and $c_1,\ c_2 \in C_{n-1}$ such that $g(c'_1) = g(c'_2) = c''$ and $f(c_1) = \partial'(c'_1)$, $f(c_2) = \partial'(c'_2)$. Notice that $(c'_1 - c'_2) \in \ker(g) = \mathrm{Im}(f)$, let $a \in C_n$ be such that $f(a) = c'_1 - c'_2$, then $f(c_1 - c_2) = \partial'(c'_1 - c'_2) = \partial' f(a) = f\partial(a)$, since f is injective we conclude $c_1 - c_2 = \partial(a)$, hence $(c_1 + \mathrm{Im}(\partial)) = (c_2 + \mathrm{Im}(\partial))$, and the map is indeed well defined.

To show it is an R-homomorphism consider $c''_1, c''_2 \in C''_n$ and $r \in R$. Suppose $\alpha(c''_i) = c_i + \mathrm{Im}(\partial)$, with $c'_i \in C'_n$ such that $g(c'_i) = c''_i$ and $f(c_i) = \partial'(c'_i)$, for $i = 1,2$. Then, clearly $g(rc'_1 + c'_2) = rc''_1 + c''_2$ and $f(rc_1 + c_2) = \partial(rc'_1 + c'_2)$, hence $\alpha(rc''_1 + c''_2) = (rc_1 + c_2 + \mathrm{Im}(\partial)) = r(c_1 + \mathrm{Im}(\partial)) + (c_2 + \mathrm{Im}(\partial)) = r\alpha(c''_1) + \alpha(c''_2)$.

Now, to finish the proof, we need to verify exactness, we have already shown that $\mathrm{Im}(f_o) = \ker(g_o)$ and $\mathrm{Im}(\tilde{f}) = \ker(\tilde{g})$, so there are two other equalities left to prove. Let us start with $\mathrm{Im}(g_o) = \ker(\alpha)$. If $c'' \in \mathrm{Im}(g_o)$, then there is $c' \in \ker(\partial')$ such that $g_o(c') = g(c') = c''$, notice that $\partial'(c') = 0 = f(0)$ implies $\alpha(c'') = 0$. Conversely, if $c'' \in \ker(\alpha)$, then there are $c' \in C'_n$ and $c \in C_n$ such that $g(c') = c''$ and $f(c) = \partial'(c')$, so that $\alpha(c'') = c + \mathrm{Im}(\partial) = 0$, which means $c \in \mathrm{Im}(\partial)$, let $a \in C_n$ be so that $\partial(a) = c$. Then $\partial'(c') = f(c) = f\partial(a) = \partial' f(a)$, thus $c' - f(a) \in \ker(\partial')$ and $g_o(c' - f(a)) = g(c' - f(a)) = g(c') - gf(a) = g(c') = c''$, since $gf = 0$.

Let us now prove the last equality $\mathrm{Im}(\alpha) = \ker(\tilde{f})$. First, if $(c + \mathrm{Im}(\partial)) \in \mathrm{Im}(\alpha)$, then let $c' \in C'_n$ and $c'' \in \ker(\partial'')$ be such that $g(c') = c''$ and $f(c) = \partial'(c')$, in other words, $\alpha(c'') = c + \mathrm{Im}(\partial)$. Then $\tilde{f}(c + \mathrm{Im}(\partial)) = f(c) + \mathrm{Im}(\partial') = \partial'(c') + \mathrm{Im}(\partial') = 0$. Conversely, if $(c + \mathrm{Im}(\partial)) \in \ker(\tilde{f})$, then $f(c) \in \mathrm{Im}(\partial')$, let $c' \in C'_n$ be such that $\partial'(c') = f(c)$, then by definition of α we have $\alpha(g(c')) = c + \mathrm{Im}(\partial)$, notice that $g(c') \in \ker(\partial'')$, since $\partial'' g(c') = g\partial'(c') = gf(c) = 0$. □

Theorem A.2.16 ((WEIBEL, 1995)). If $0 \longrightarrow C \xrightarrow{f} C' \xrightarrow{g} C'' \longrightarrow 0$ is a short exact sequence of chain complexes, then there is a long exact sequence associated to it, namely

$$\cdots \longrightarrow H_n(C) \xrightarrow{f_*} H_n(C') \xrightarrow{g_*} H_n(C'') \xrightarrow{\alpha} H_{n-1}(C) \longrightarrow \cdots ,$$

which is called the **long exact sequence of homology**.

Proof. It follows from the snake lemma that the rows are exact in the following commutative diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Coker}(\partial) = \frac{C_n}{B_n} & \xrightarrow{\tilde{f}} & \mathrm{Coker}(\partial') = \frac{C'_n}{B'_n} & \xrightarrow{\tilde{g}} & \mathrm{Coker}(\partial'') = \frac{C''_n}{B''_n} & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle\tilde{\partial}} & & \downarrow{\scriptstyle\tilde{\partial}'} & & \downarrow{\scriptstyle\tilde{\partial}''} & \\
0 & \longrightarrow & \ker(\partial) = Z_{n-1} & \xrightarrow{f_o} & \ker(\partial') = Z'_{n-1} & \xrightarrow{g_o} & \ker(\partial'') = Z''_{n-1} &
\end{array}$$

with $\tilde{\partial}$ defined in the natural way $\tilde{\partial}(c + B_n) = \partial(c)$, this is well defined since $\partial(B_n) = 0$ and $\mathrm{Im}(\partial) \subset Z_{n-1}$, analogously for ∂' and ∂''. It is clear that $\mathrm{Im}(\tilde{\partial}) = \mathrm{Im}(\partial) = B_{n-1}$. We also have $\tilde{\partial}(c + B_n) = \partial(c) = 0$ if an only if $c \in Z_n$, hence $\ker(\tilde{\partial}) = Z_n/B_n$.

Since $\ker(\tilde{\partial}) = H_n(C)$ and $\mathrm{Coker}(\tilde{\partial}) = H_{n-1}(C)$, and the analogous is valid for ∂' and ∂'', we have by the snake lemma the following exact sequence

$$H_n(C) \xrightarrow{f_*} H_n(C') \xrightarrow{g_*} H_n(C'') \xrightarrow{\alpha} H_{n-1}(C) \xrightarrow{f_*} H_{n-1}(C') \xrightarrow{g_*} H_{n-1}(C'') .$$

By gluing these sequences, for all n, we get the long exact sequence from the theorem. □

Let us analyze what the attaching map α becomes in theorem A.2.16. If $(z'' + B''_n) \in H_n(C'')$, then there is $(c' + B'_n) \in C'_n/B'_n$ such that $\tilde{g}(c' + B'_n) = g(c') + B''_n = z'' + B''_n$, and there is $z \in Z_{n-1}$ such that $f_o(z) = f(z) = \tilde{\partial}'(c' + B'_n) = \partial'(c')$, and we have $\alpha(z'' + B''_n) = z + B_{n-1}$. Notice that a completely equivalent **definition of the connecting homomorphism** α is: $\alpha(z'' + B''_n) = z + B_n$, with $z'' \in Z''_n$ and $z \in Z_{n-1}$, if there is $c' \in C'_n$ such that $g(c') = z''$ and $f(z) = \partial'(c')$.

In a suitable scenario, one can easily introduce the dual notion of the long exact sequence of homology. From now on, by **dual of a sequence** of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ we mean the sequence $C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$ generated by the functor $\mathrm{Hom}_R(\cdot, N)$.

Corollary A.2.17. If $0 \longrightarrow C''^{*} \xrightarrow{g^*} C'^{*} \xrightarrow{f^*} C^* \longrightarrow 0$ is a short exact sequence of cochain complexes, then we have the following long exact sequence

$$\cdots \longrightarrow H^n(C'') \xrightarrow{g^*_*} H^n(C') \xrightarrow{f^*_*} H^n(C) \xrightarrow{\beta} H^{n+1}(C) \longrightarrow \cdots,$$

which is called the **long exact sequence of cohomology**.

Corollary A.2.17 follows directly from theorem A.2.16. We can construct a slightly stronger assertion in view of the following lemma.

Lemma A.2.18. The dual of an exact sequence $A \to B \to C \to 0$ is exact.

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence, then the dual sequence generated by $\mathrm{Hom}_R(\cdot, N)$, namely $0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$, is clearly a chain complex, that is, $\{0\} \subset \ker(g*)$ and $\mathrm{Im}(g^*) \subset \ker(f^*)$, since $0 = (gf)^* = f^*g^*$, so the only thing left to do is prove the opposite inclusions.

To see that $\{0\} \supset \ker(g^*)$, take $\varphi : C \to N$ in the kernel of g^*, i.e., $g^*(\varphi) = \varphi g = 0$, and since g is surjective, we have $\{0\} = \varphi g(B) = \varphi(C)$, hence $\phi = 0$.

To show that $\mathrm{Im}(g^*) \supset \ker(f^*)$, take $\psi : B \to N$ in the kernel of f^*, i.e., $f^*(\psi) = \psi f = 0$, which implies $\mathrm{Im}(f) = \ker(g) \subset \ker(\psi)$, this means that $g(a) = g(b)$ implies $\psi(a) = \psi(b)$, for any $a, b \in B$. Since g is surjective, we can define for each $c \in C$, $b_c \in B$ such that $g(b_c) = c$. With this, we can introduce the map $\varphi : C \to N$ given by $\varphi(c) = \psi(b_c)$. We claim that φ is an R-homomorphism. To see that, take $r \in R$ and $c_1, c_2 \in C$, then $\varphi(rc_1 + c_2) = \psi(b_{rc_1+c_2})$, we also have $g(b_{rc_1+c_2}) = rc_1 + c_2 = rg(b_{c_1}) + g(b_{c_2}) = g(rb_{c_1} + b_{c_2})$, this implies that $\psi(b_{rc_1+c_2}) = \psi(rb_{c_1} + b_{c_2}) = r\psi(b_{c_1}) + \psi(b_{c_2}) = r\varphi(c_1) + \varphi(c_2)$. Finally, we claim that $g^*(\phi) = \psi$, this is easily shown since from the definition of ϕ we have $\phi g(b) = \psi(b_{g(b)})$, and $g(b_{g(b)}) = g(b)$ implies $\psi(b_{g(b)}) = \psi(b)$, hence we conclude $\mathrm{Im}(g^*) \supset \ker(f^*)$.

□

Lemma A.2.19. The dual of a split exact sequence is a split exact sequence.

Proof. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence, and let $0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \longrightarrow 0$ be its dual sequence. From lemma A.2.18, the only thing left to prove is that f^* is surjective and the sequence splits. Notice that if h is a left inverse for f, then h^* is a right inverse for f^*, since $(hf)^* = f^*h^*$, and since the original sequence splits this finishes our proof. □

Corollary A.2.20. If $0 \longrightarrow C \xrightarrow{f} C' \xrightarrow{g} C'' \longrightarrow 0$ is a short exact sequence of chain complexes, such that the dual $0 \longrightarrow C''^{*} \xrightarrow{g^*} C'^{*} \xrightarrow{f^*} C^* \longrightarrow 0$ is still short exact, then there is a long exact sequence

$$\cdots \longrightarrow H^n(C'') \xrightarrow{g_*^*} H^n(C') \xrightarrow{f_*^*} H^n(C) \xrightarrow{\beta} H^{n+1}(C'') \longrightarrow \cdots ,$$

called the **long exact sequence of cohomology**. In particular, this is true if the original sequence splits.

Definition A.2.21 ((WEIBEL, 1995)). If $f,g : C \to C'$ are two chain maps, we say that f and g are **chain homotopic** if there is a map $T : C_n \to C'_{n+1}$ such that $f - g = T\partial + \partial T$, for all levels n. In this case, T is called a **chain homotopy** between f and g.

Theorem A.2.22 ((WEIBEL, 1995)). Two chain homotopic maps $f,g : C \to C'$ induce the same maps on homology and cohomology.

Proof. Let T be a chain homotopy from f to g, i.e., $f - g = T\partial + \partial T$. If we take a cycle $z \in Z_n$, then $f(z) - g(z) = T\partial(z) + \partial T(z) = \partial T(z)$, hence $f(z) - g(z) \in B'_n$, which implies $f(z) + B'_n = g(z) + B'(n)$, whence $f_* = g_*$. For cohomology we use an analogous argument, by simply noticing that the dual chain maps $f^*, g^* : C'^* \to C^*$ satisfy the relation $f^* - g^* = \delta T^* + T^*\delta$, hence $f_*^* = g_*^*$. □

Proposition A.2.23. If R is a commutative ring with unit, then for any R-module M the R-module $\mathrm{Hom}_R(R,M)$ is isomorphic to M by the natural isomorphism

$$\begin{aligned} \Psi : M &\longrightarrow \mathrm{Hom}_R(R,M) \\ m &\longmapsto (f_m : R \to M), \end{aligned}$$

in which $f_m(1) = m$.

Ext and Tor Functors

Definition A.2.24 ((SPANIER, 1989)). Given any chain complex of R-modules, $C = \{C_n, \partial_n\}_n$, and an R module, M, we can define the chain complex $C \otimes_R M$ by

$$\cdots \longrightarrow C_{n+1} \otimes_R M \xrightarrow{\partial \otimes id} C_n \otimes_R M \xrightarrow{\partial \otimes id} C_{n-1} \otimes_R M \longrightarrow \cdots ,$$

and the homology R-modules $H_n(C \otimes_R M)$ are called the **homology R-modules of C with coefficients in M**, denoted $H_n(C;M)$.

Notice that in definition A.2.24, $\{C_n \otimes M, \partial \otimes id\}_n$ is in fact a chain complex since $(\partial \otimes id)(\partial \otimes id) = \partial\partial \otimes id = 0 \otimes id = 0$.

Definition A.2.25 ((HATCHER, 2002)). Given M an R-module, a **resolution** of M is an exact sequence F given by

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0 ,$$

additionally if all F_i are free then F is called a **free resolution**.

Notice that given a free resolution F of M, and an R-module N we can obtain the following chain complexes

$$\cdots \longrightarrow F_2 \otimes_R N \xrightarrow{f_2 \otimes id} F_1 \otimes_R N \xrightarrow{f_1 \otimes id} F_0 \otimes_R N \xrightarrow{f_0 \otimes id} M \otimes_R N \longrightarrow 0 \ ,$$

and

$$0 \longrightarrow M^* \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1 \xrightarrow{f_2^*} F_2 \longrightarrow \cdots \ .$$

Let us momentarily write $H_n(F;N) = \ker(f_n \otimes id)/\operatorname{Im}(f_{n+1} \otimes id)$ and $H^n(F;N) = \ker(f_{n+1}^*)/\operatorname{Im}(f_n^*)$, for $n = 0, 1, \ldots$, in what follows we will show that these R-modules are independent of the free resolution F, and will introduce a special notation for them.

Theorem A.2.26 ((HATCHER, 2002))**.** Given R-modules M and M', with free resolutions F and F', if $\varphi : M \to M'$ is an R-homomorphism, then there is a chain map $\alpha : F \to F'$ extending φ, i.e., every square commutes in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & M' \arrow[r] & 0
\end{tikzcd}$$

if $\beta : F \to F'$ is another chain map extending φ, then α and β are chain homotopic. As a consequence, if $\varphi : M \to M'$ is an isomorphism, and N an R-module we have $H_n(F;N) \approx H_n(F';N)$ and $H^n(F;N) \approx H^n(F';N)$

Proof. Let us fix the notation B_i and B_i' are basis sets for F_i and F_i'. The proof of the first part is done by induction. For the zeroth step define the R-homomorphism $\alpha_0 : F_0 \to F_0'$ which sends basis elements $b \in B_0$ to some element in ${f_0'}^{-1}(\{\varphi f_0(b)\})$, which is nonempty since f_0' is surjective. If α_{n-1} is defined, define $\alpha_n : F_n \to F_n'$ as the R-homomorphism taking basis elements $b \in B_n$ to an element of the set ${f_n'}^{-1}(\alpha_{n-1} f_n(b))$, which is nonempty since $\alpha_{n-1} f_n(F_n) \in \ker(f_{n-1}') = \operatorname{Im}(f_n')$.

Now suppose $\beta : F \to F'$ is another chain map extending φ, then $\gamma = \alpha - \beta$ is a chain map extending the trivial homomorphism $0 : M \to M'$. We want to define R-homomorphisms $T_n : F_n \to F_{n+1}'$ such that $\gamma_n = f_{n+1}' T_n + T_{n-1} f_n$, this is also done inductively. Clearly $T_{-1} = 0$, so for $n = 0$ the equation becomes $f_1' T_0 = \gamma_0$. Consider the homomorphisms $T_0 : F_0 \to F_1'$ which takes a basis element $b \in B_0$ to some element in ${f_1'}^{-1}(\{\gamma_0(b)\})$, which is nonempty since $\gamma_0(b) \in \ker(f_0') = \operatorname{Im}(f_1')$. If T_{n-1} has being defined, consider $T_n : F_n \to F_{n+1}'$ to be the R-homomorphism taking a basis element $b \in B_n$ to an element of ${f_{n+1}'}^{-1}(\{\gamma_n(b) - T_{n-1} f_n(b)\})$, to see that this set is nonempty we use the fact that $\gamma_{n-1} = f_n' T_{n-1} + T_{n-2} f_{n-1}$ and $\gamma_{n-1} f_n = f_n' \gamma_n$, and apply this to

$$\begin{aligned}
f_n'(\gamma_n - T_{n-1} f_n) &= f_n' \gamma_n - f_n' T_{n-1} f_n = \gamma_{n-1} f_n - f_n' T_{n-1} f_n \\
&= (f_n' T_{n-1} + T_{n-2} f_{n-1}) f_n - f_n' T_{n-1} f_n \\
&= T_{n-2} f_{n-1} f_n = 0,
\end{aligned}$$

hence $\mathrm{Im}(\gamma_n - T_{n-1}f_n) \subset \ker(f'_n) = \mathrm{Im}(f'_{n+1})$. In conclusion, we have that α and β are chain homotopic, since $\alpha_n - \beta_n = f'_{n+1}T_n + T_{n-1}f_n$.

Finally, suppose φ is an isomorphism with $\psi : M' \to M$ its inverse. Let $\alpha : F \to F'$ and $\beta : F' \to F$ be chain maps extending φ and ψ, respectively. It is easy to see that the composition $\beta\alpha : F \to F$ defines a chain map extending the identity $id : M \to M$. From what was proven earlier, $\beta\alpha$ has to be chain homotopic to the identity chain map $id_F : F \to F$, which is simply the identity map at every level n. By theorem A.2.22, we have that the induced maps in homology and cohomology by $\beta\alpha$ are equal to the identity map, hence $(\alpha_n)_* : H_n(F;N) \to H_n(F';N)$ and $(\alpha_n)^*_* : H^n(F';N) \to H^n(F;N)$ are isomorphisms, for any R-module N. □

Definition A.2.27 ((HATCHER, 2002))**.** Given R-modules M and N, and F a free resolution of M

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0\ ,$$

we have the chain complexes

$$\cdots \longrightarrow F_2 \otimes_R N \xrightarrow{f_2 \otimes id} F_1 \otimes_R N \xrightarrow{f_1 \otimes id} F_0 \otimes_R N \xrightarrow{f_0 \otimes id} M \otimes_R N \longrightarrow 0\ ,$$

and

$$0 \longrightarrow M^* \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1 \xrightarrow{f_2^*} F_2 \longrightarrow \cdots\ ,$$

and we define the R-modules

$$\mathbf{Tor}_n(M,N) = \frac{\ker(f_n \otimes id)}{\mathrm{Im}(f_{n+1} \otimes id)} \qquad \text{and} \qquad \mathbf{Ext}_n(M,N) = \frac{\ker(f^*_{n+1})}{\mathrm{Im}(f^*_n)},$$

for $n = 1, 2, \ldots$. Notice that since $F_1 \to F_0 \to M \to 0$ is exact, both $F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$ and $0 \to M^* \to F_0^* \to F_1^*$ are exact (by lemmas A.2.18 and A.1.21), so if we used the same definition as above for $n = 0$, we would always have $\mathrm{Tor}_0(M,N) = \mathrm{Ext}_0(M,N) = 0$. This is usually referred to as the reduced Tor and Ext groups, sometimes written $\tilde{\mathrm{Tor}}_n$ and $\tilde{\mathrm{Ext}}_n$, with the unreduced version being the homology R-modules of the chain complexes

$$\cdots \longrightarrow F_2 \otimes_R N \xrightarrow{f_2 \otimes id} F_1 \otimes_R N \xrightarrow{f_1 \otimes id} F_0 \otimes_R N \longrightarrow 0\ ,$$

and

$$0 \longrightarrow F_0^* \xrightarrow{f_1^*} F_1 \xrightarrow{f_2^*} F_2 \longrightarrow \cdots\ ,$$

so we have (by exactness)

$$\mathrm{Tor}_0(M,N) = (F_0 \otimes_R N)/\mathrm{Im}(f_1 \otimes id) = (F_0 \otimes_R N)/\ker(f_0 \otimes id) \approx M \otimes_R N,$$

and

$$\mathrm{Ext}_0(M,N) = \ker(f_1^*) = \mathrm{Im}(f_0^*) \approx M^* = \mathrm{Hom}_R(M;N).$$

From theorem A.2.26, we know that $\text{Tor}_n(M,N)$ and $\text{Ext}_n(M,N)$, as defined previously, are independent from the free resolution F, hence they are well defined.

Remark A.2.28. For any R-module M, there is a free resolution constructed in the following way. Take $F_0 = F(M)$ (remember that $F(S)$ is the free R-module generated by the set S), and consider the homomorphism $f_0 : F_0 \to M$ given by $f_0(m) = m$, which is clearly surjective. Now, by induction define $F_n = F(\ker(f_{n-1})$ and $f_n : F_n \to F_{n-1}$ to be the homomorphism given by $f_n(b) = b$, for all $b \in \ker(f_{n-1})$, with this we have $\text{Im}(f_n) = \ker(f_{n-1})$

$$
\begin{array}{ccccccccccccc}
 & & F(\ker(f_{n-1})) & & F(\ker(f_{n-2})) & & & & F(\ker(f_0)) & & F(M) & & \\
 & & \| & & \| & & & & \| & & \| & & \\
\cdots & \longrightarrow & F_n & \xrightarrow{f_n} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M \longrightarrow 0
\end{array}
$$

If R is a PID, we can get a simpler free resolution for M, namely

$$
\begin{array}{ccccccc}
 & & \ker(f_0) & & F(M) & & \\
 & & \| & & \| & & \\
0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M \longrightarrow 0
\end{array}
$$

in which f_0 is the same as in the general case, and f_1 is the inclusion homomorphism. Notice that for any R-module this is an exact sequence, but we can only guarantee it is a free resolution if R is a PID, since from proposition A.1.12 we know that $\ker(f_0)$ is a free module. Notice that this free resolution implies that for any R-module M, with R being a PID, $\text{Tor}_n(M,N) = \text{Ext}_n(M,N) = 0$, for all $n \geq 2$.

Remark A.2.29. In many situations in algebraic topology we will only be interested in $\text{Tor}_1(M,N)$ and $\text{Ext}_1(M,N)$, therefore we may write $\text{Tor}(M,N)$ and $\text{Ext}(M,N)$ without any index when referring to them.

Next, we present the most important properties of Tor_n and Ext_n, these properties are really important when trying to compute these R-modules.

Proposition A.2.30 ((HATCHER, 2002))**.** If M, N, P and Q are R-modules and $\{M_\alpha\}_\alpha$ is a family of R-modules we have:

(T1) $\text{Tor}_n(M,N) \approx \text{Tor}_n(N,M)$, if R is a PID;

(T2) $\text{Tor}_n\left(\bigoplus_\alpha M_\alpha, N\right) = \bigoplus_\alpha \text{Tor}_n(M_\alpha, N)$;

(T3) $\text{Tor}_n(M,N) = 0$, for $n \geq 1$, if M or N is free;

(T3') $\text{Tor}_n(M,N) = 0$, for $n \geq 1$, if M or N is torsionfree and R is a PID;

(T4) If R is a PID, then $\text{Tor}_n(M,N) = \text{Tor}_n(T(M),N)$, in which $T(M)$ is the torsion submodule of M;

(T5) $\text{Tor}_1(\mathbb{Z}_n, M) \approx \ker(\phi_n)$, in which we are considering $\mathbb{Z}$-modules, and $\phi_n : M \to M$ given by $\phi_n(m) = nm$;

(T6) Any short exact sequence $0 \to N \to P \to Q \to 0$ induces a long exact sequence

$$\cdots \longrightarrow \text{Tor}_n(M,N) \longrightarrow \text{Tor}_n(M,P) \longrightarrow \text{Tor}_n(M,Q) \longrightarrow \text{Tor}_{n-1}(M,N) \longrightarrow \cdots$$

$$\cdots \longrightarrow \text{Tor}(M,N) \longrightarrow \text{Tor}(M,P) \longrightarrow \text{Tor}(M,Q) \longrightarrow$$

$$\longrightarrow M \otimes_R N \longrightarrow M \otimes_R P \longrightarrow M \otimes_R Q \longrightarrow 0$$

(E1) $\text{Ext}_n\left(\bigoplus_\alpha M_\alpha, N\right) = \bigoplus_\alpha \text{Ext}_n(M_\alpha, N)$;

(E2) $\text{Ext}_n(M,N) = 0$ if M is free

(E3) $\text{Ext}_1(\mathbb{Z}_n, M) = M/nM$, for $\mathbb{Z}$ modules.

Proof. (T2) and (E1) are a consequence of the fact that if $F_\alpha = \{F_\alpha^i, f_\alpha^i : F_\alpha^i \to F_\alpha^{i-1}\}_i$ is a free resolution for each M_α, then the direct sum $\bigoplus_\alpha F_\alpha = \left\{\bigoplus_\alpha F_\alpha^i, \oplus_\alpha f_\alpha^i\right\}_i$ is a free resolution of $\bigoplus_\alpha M_\alpha$.

For (T5) and (E3) consider the free resolution of the $\mathbb{Z}$-module $\mathbb{Z}_n$ given by $0 \longrightarrow \mathbb{Z} \xrightarrow{f_n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$, in which $f_n(z) = nz$, for some $n \in \mathbb{Z}_{>0}$, by applying the definitions of Ext_1 and Tor_1 to this free resolution, we obtain the desired results.

To prove (E2) and (T3) suppose M is free, then $0 \to M \to M \to 0$ is a free resolution of M, which clearly implies $\text{Ext}_n(M,N) = \text{Tor}_n(M,N) = 0$, for $n \geq 1$.

The property (T6) is basically a consequence of the long exact sequence of homology (Theorem A.2.16) . Let $\dots F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. If $0 \to N \to P \to Q \to 0$ is a short exact sequence, then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes_R N & \longrightarrow & F_1 \otimes P & \longrightarrow & F_1 \otimes Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes_R N & \longrightarrow & F_0 \otimes P & \longrightarrow & F_0 \otimes Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and from proposition A.1.23 we have that each line in the diagram above is a short exact sequence, hence theorem A.2.16 implies the existence of a long exact sequence as in statement (T6).

Now, we use (T6) to prove (T1). If R is a PID, we have shown that an R-module N assumes a free resolution $0 \to F_1 \to F_0 \to N \to 0$, which in particular is a short exact sequence. Thus, applying (T6), we get an exact sequence $0 \to \text{Tor}(M,N) \to M \otimes_R F_1 \to$

$M \otimes F_0 \to M \otimes N \to 0$, here since F_1 and F_0 are free we have $\mathrm{Tor}(F_1, N) = \mathrm{Tor}(F_0, N) = 0$. From the definition of $\mathrm{Tor}(N, M)$, we have the exact sequence $0 \to \mathrm{Tor}(N, M) \to F_1 \otimes_R M \to F_0 \otimes_R M \to N \otimes_R M \to 0$. The natural isomorphisms $M \otimes_R F_i \approx F_i \otimes_R M$ give us a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Tor}(M,N) & \longrightarrow & M \otimes_R F_1 & \longrightarrow & M \otimes_R F_0 & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\
 & & & & \wr\!\!\downarrow & & \wr\!\!\downarrow & & \wr\!\!\downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}(N,M) & \longrightarrow & F_1 \otimes_R M & \longrightarrow & F_0 \otimes_R M & \longrightarrow & N \otimes_R M & \longrightarrow & 0
\end{array}$$

and by proposition A.2.10 we conclude that $\mathrm{Tor}(M, N) \approx \mathrm{Tor}(N, M)$.

Now we prove (T3'). Since R is a PID, we have a free resolution $0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow M \longrightarrow 0$, which implies $\mathrm{Tor}_n(M, N) = 0$, for $n \geq 2$, so the only thing left to prove is that $\mathrm{Tor}(M, N) = 0$. Suppose M is torsion free (the case where N is torsion free will be analogous, since we have proven that $\mathrm{Tor}(M, N) \approx \mathrm{Tor}(N, M)$), we have $\mathrm{Tor}(M, N) = \ker(f_1 \otimes id)$, so we basically want to show that $\ker(f_1 \otimes id) = 0$. If $\sum_i x_i \otimes n_i \in \ker(f_1 \otimes id)$, then $\sum_i f(x_i) \otimes n_i = 0$, which means this sum reduces to zero after a finite number of bilinear operations, using the tensor product properties. There is a finite number of elements of N involved in this operations, these elements generate the submodule $N_0 \subset N$, hence $\sum_i x_i \otimes n_i$ lies in the kernel of $f_1 \otimes id : F_1 \otimes_R N_0 \to F_0 \otimes_R N_0$, and since N_0 is a finitely generated torsionfree module, it is free by corollary A.1.13. Hence, $\mathrm{Tor}(M, N_0) = 0$, which implies that the kernel mentioned is trivial, hence $\sum_i x_i \otimes n_i = 0$, and we conclude that $\mathrm{Tor}(M, N) = 0$.

$\square$

BIBLIOGRAPHY

ARKOWITZ, M. **Introduction to homotopy theory**. [S.l.]: Springer Science & Business Media, 2011. Citations on pages 46, 48, 49, 50, 51, 60, 62, 64, and 77.

CORNEA, O.; LUPTON, G.; OPREA, J.; TANRÉ, D. **Lusternik-Schnirelmann category**. [S.l.]: American Mathematical Society, 2003. Citations on pages 24, 71, 74, 77, 79, 80, 81, 82, 85, 86, 87, 88, 89, 90, 92, and 93.

CRABB, M. C.; JAMES, I. M. **Fibrewise homotopy theory**. [S.l.]: Springer Science & Business Media, 2012. Citations on pages 119, 120, 121, 122, 123, and 124.

DAVIS, J. F.; KIRK, P. **Lecture notes in algebraic topology**. [S.l.]: American Mathematical Society, 2001. Citations on pages 45, 138, 140, and 141.

DOLD, A. **Lectures on algebraic topology**. [S.l.]: Springer Science & Business Media, 2012. Citations on pages 69 and 70.

DOWKER, C. H. Topology of metric complexes. **American Journal of Mathematics**, v. 74, n. 3, p. 555–577, 1952. Citation on page 66.

DUGUNDJI, J. **Topology**. Boston: Allyn and Bacon, 1966. Citation on page 24.

EULER, L. Solutio problematis ad geometriam situs pertinentis. **Comment. Acad. Sci. U. Petrop 8**, p. 128–140, 1736. Citation on page 19.

FARBER, M. Topological complexity of motion planning. **Discrete and Computational Geometry**, v. 29, n. 2, p. 211–221, 2003. Citations on pages 19, 20, 73, 96, 97, 98, 99, 106, 107, and 108.

_____. Instabilities of robot motion. **Topology and its Applications**, 2004. Citations on pages 99, 110, 111, 112, 113, and 114.

_____. Topology of robot motion planning. In: **Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology**. [S.l.]: Kluwer Academic Publishers, 2006. p. 185–230. Citation on page 127.

_____. **Invitation to topological robotics**. [S.l.]: European Mathematical Society, 2008. ISBN 9783037190548. Citation on page 19.

FOX, R. H. On the Lusternik-Schnirelmann category. **The Annals of Mathematics**, JSTOR, v. 42, n. 2, p. 333, 1941. Citation on page 21.

HATCHER, A. **Algebraic topology**. [S.l.]: Cambridge University Press, 2002. Citations on pages 19, 20, 26, 28, 29, 32, 33, 34, 36, 41, 42, 43, 44, 45, 48, 49, 51, 52, 65, 66, 67, 68, 69, 76, 88, 111, 144, 145, 150, 151, 152, and 153.

IWASE, N.; KIKUCHI, K.; MIYAUCHI, T. On Lusternik-Schnirelmann category of SO(10). **arXiv preprint**, 2007. Citation on page 111.

IWASE, N.; MIMURA, M.; NISHIMOTO, T. Lusternik-Schnirelmann category of non-simply connected compact simple lie groups. **Topology and its Applications**, Elsevier BV, v. 150, n. 1-3, p. 111–123, may 2005. Citation on page 111.

IWASE, N.; SAKAI, M. Topological complexity is a fibrewise L–S category. **Topology and its Applications**, p. 10–21, 2010. Citations on pages 21, 22, 127, 128, 129, 131, and 132.

_____. Erratum to "topological complexity is a fibrewise L–S category". **Topology and its Applications**, p. 2810–2813, 2012. Citation on page 21.

JAMES, I. M. On category, in the sense of Lusternik-Schnirelmann. **Topology**, v. 17, p. 331–348, 1978. Citations on pages 71, 72, and 73.

_____. **General topology and homotopy theory**. [S.l.]: Springer Science & Business Media, 2012. Citation on page 48.

LANG, S. **Algebra**. Third edition. [S.l.]: Springer-Verlag New York, 2002. Citations on pages 133, 134, 143, and 144.

LATOMBE, J.-C. **Robot motion planning**. [S.l.]: Springer Science & Business Media, 1991. Citations on pages 19 and 95.

LEE, J. M. **Introduction to topological manifolds**. [S.l.]: Springer-Verlag, New York, 2011. Citations on pages 24, 25, 64, and 65.

_____. **Introduction to smooth manifolds**. [S.l.]: Springer New York, 2012. Citations on pages 25 and 26.

LISTING, J. B. Vorstudien zur topologie. **Vandenhoeck und Ruprecht, Göttingen**, 1848. Citation on page 19.

LUNDELL, A. T.; WEINGRAM, S. **The topology of CW Complexes**. [S.l.]: Springer New York, 1969. Citation on page 65.

LUSTERNIK, L.; SCHNIRELMANN, L. **Méthodes topologiques dans les problèmes variationnels**. [S.l.]: Hermann cie, 1934. Citation on page 21.

MUNKRES, J. **Elements of algebraic topology**. [S.l.]: CRC Press, 1984. Citations on pages 59 and 68.

_____. **Topology**. second edition. [S.l.]: Prentice Hall, 2000. Citations on pages 23, 24, 25, and 57.

NAGATA, J. **Modern dimension theory**. revised and extend edition. [S.l.]: Helderman Verlag Berlin, 1981. Citation on page 59.

PEARS, A. R. **Dimension theory of general spaces**. [S.l.]: Cambridge University Press, 1975. Citation on page 58.

POINCARÉ, H. Analysis situs. **Journal de l'École Polytechnique.**, v. 2, p. 1–123, 1895. Citation on page 19.

SCHWARZ, A. S. The genus of a fiber space. **American Mathematical Society Translations**, v. 55, p. 49–140, 1966. Citations on pages 20, 21, 101, 102, and 104.

SPANIER, E. H. **Algebraic topology**. [S.l.]: Springer Science & Business Media, 1989. Citations on pages 141, 142, and 150.

TU, L. W. **An introduction to manifolds**. Second edition. [S.l.]: Springer Science & Business Media, 2011. Citation on page 24.

VICK, J. W. **Homology theory: an introduction to algebraic topology**. Second. [S.l.]: Springer Science & Business Media, 2012. Citations on pages 68 and 135.

WEIBEL, C. A. **An introduction to homological algebra**. [S.l.]: Cambridge University Press, 1995. Citations on pages 143, 144, 145, 146, 148, and 150.

WHITEHEAD, G. W. **Elements of homotopy theory**. [S.l.]: Springer-Verlag New York Inc., 1978. Citations on pages 19, 52, 53, and 65.

INDEX

ICMC USP
SÃO CARLOS