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On Hamiltonian systems with critical Sobolev exponents

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Sobre sistemas Hamiltonianos com expoentes críticos de Sobolev

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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To my family and friends.

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"We ourselves feel that what we are doing is just a drop in the ocean. But the ocean would be less because of that missing drop."

Mother Teresa.

ABSTRACT

GUIMARÃES, A. **On Hamiltonian systems with critical Sobolev exponents**. 2022. 57 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

In this thesis we consider lower order perturbations of the critical Lane-Emden system posed on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, inspired by the classical results of Brezis and Nirenberg (BRÉZIS; NIRENBERG, 1983). We solve the problem of finding a positive solution for all dimensions $N \geq 4$. For the critical dimension $N = 3$ we show a new phenomenon, not observed for scalar problems. Namely, there are parts on the critical hyperbola where solutions exist for all 1-homogeneous or subcritical superlinear perturbations and parts where there are no solutions for some of those perturbations.

Keywords: Lane-Emden systems; Critical hyperbola; Critical dimension; Positive solutions.

RESUMO

GUIMARÃES, A. **Sobre sistemas Hamiltonianos com expoentes críticos de Sobolev**. 2022. 57 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Nesta tese consideramos perturbações de ordem inferior do sistema crítico de Lane-Emden em domínios limitados suaves $\Omega \subset \mathbb{R}^N$, com $N \geq 3$, inspirados pelos resultados clássicos de Brézis e Nirenberg (BRÉZIS; NIRENBERG, 1983). Resolvemos o problema de encontrar uma solução positiva para toda dimensão $N \geq 4$. Para a dimensão crítica $N = 3$ mostramos um novo fenômeno, não observado nos problemas escalares. A saber, existem partes na hipérbole crítica onde se têm soluções para toda perturbação homogênea de grau um ou superlinear subcrítica, e partes onde não se têm soluções para algumas destas perturbações.

Palavras-chave: Sistemas de Lane-Emden, Hipérbole Crítica, Dimensão crítica, Solução positiva.

LIST OF SYMBOLS

$o(g(x))$ as $x \to a$ — Any function $f(x)$ such that $\frac{f(x)}{g(x)} \to 0$ as $x \to a$

$L^q(\Omega)$ — Space of measurable functions $v : \Omega \to \mathbb{R}$ s.t. $\int_\Omega |v|^q dx < \infty$

$H_0^1(\Omega)$ — Space of functions $v \in L^2(\Omega)$ s.t. v has first order weak derivatives in $L^2(\Omega)$ and $v = 0$ on $\partial\Omega$

$W^{k,t}(\Omega)$ — Space of functions $v \in L^t(\Omega)$ s.t. v has weak derivatives, up to order k, in $L^t(\Omega)$

$W_0^{k,t}(\Omega)$ — Space of functions $v \in L^t(\Omega)$ s.t. v has weak derivatives, up to order k, in $L^t(\Omega)$ and $v = 0$ on $\partial\Omega$

E_t — $W^{2,\frac{t+1}{t}}(\Omega) \cap W_0^{1,\frac{t+1}{t}}(\Omega)$

$\mathscr{C}_{r,\Omega}$ — $\inf\{\|u\|;\ u \in E_p$ and $|u|_{\frac{r+1}{r}} = 1\}$

$\mathscr{C}_{s,\Omega}$ — $\inf\{\|u\|;\ u \in E_q$ and $|u|_{\frac{s+1}{s}} = 1\}$

S — $\inf_{u \in E_p, |u|_{q+1} = 1} \|u\|$

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CHAPTER

1

INTRODUCTION

In the memorable paper (BRÉZIS; NIRENBERG, 1983) from 1983, Brezis and Nirenberg considered the perturbed Lane-Emden equation with critical growth

$$-\Delta u = \lambda u^t + u^{2^*-1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{1.0.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$, with $1 \leq t < 2^* - 1$. In particular, they discovered a surprising difference between the cases $N \geq 4$ and $N = 3$, the latter named as critical dimension. For the particular case with $t = 1$, namely for

$$-\Delta u = \lambda u + u^{2^*-1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{1.0.2}$$

they proved the existence of a solution for every $0 < \lambda < \lambda_1(\Omega)$, the optimal interval for existence, for $N \geq 4$. In contrast, with $N = 3$, they showed the existence of $0 < \lambda^* < \lambda_1(\Omega)$ such that no solutions exists for $0 < \lambda < \lambda^*$; see (BRÉZIS; NIRENBERG, 1983, Theorem 1.2 and Corollary 1.1). Here $\lambda_1 = \lambda_1(\Omega)$ stands for the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

The notion of critical growth for Hamiltonian systems, as independently introduced by Mitidieri (MITIDIERI, 1993) and van der Vorst (VORST, 1992), soon after considered by several authors, including Clément et al. (CLÉMENT; FIGUEIREDO; MITIDIERI, 1992) and Peletier-van der Vorst (PELETIER; VORST, 1992), is given by the so-called critical hyperbola. In 1998, Hulshof et al. (HULSHOF; MITIDIERI; VORST, 1998) analyzed the version of (1.0.2) in the framework of Hamiltonian systems, namely they considered

$$\begin{cases} -\Delta u = \lambda v + |v|^{p-1} v \text{ in } \Omega, \\ -\Delta v = \mu u + |u|^{q-1} u \text{ in } \Omega, \\ u, v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $N \geq 4$, for (p,q) on the critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.0.3}$$

1.1 Main goals

Fascinating results were proved in (HULSHOF; MITIDIERI; VORST, 1998, Theorem 2), and we think that three important problems were left open:

a) What happens in dimension $N = 3$?

b) What is the meaning of the critical dimension for Hamiltonian elliptic systems?

c) The investigation of the general 1-homogenous perturbation of the critical Lane-Emden system, namely (HS) ahead with $rs = 1$, which includes $r = s = 1$ as a particular case.

Item c) deserves some extra comments, since the most accurate 1-homogenous perturbation to Hamiltonian systems, given below in (HS), is induced by the hyperbola of points (r,s) such that $rs = 1$. Indeed, this hyperbola has been named as the spectral curve for Hamiltonian systems; see (MONTENEGRO, 2000; LEITE; MONTENEGRO, 2019; LEITE; MONTENEGRO, 2020) for linear operators and (SANTOS *et al.*, 2020) in the fully nonlinear scenario. In this thesis we address these three questions and present some results observed in the framework of Hamiltonian systems which are non-existent for scalar problems. In order to accomplish that, consider the following Hamiltonian system

$$\begin{cases} -\Delta u = \lambda |v|^{r-1}v + |v|^{p-1}v \text{ in } \Omega, \\ -\Delta v = \mu |u|^{s-1}u + |u|^{q-1}u \text{ in } \Omega, \\ u, v = 0 \text{ on } \partial\Omega, \end{cases} \tag{HS}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, $\lambda > 0$ and $\mu > 0$. Here (p,q) lies on the critical hyperbola, that is $p > 0$ and $q > 0$ satisfy (1.0.3), and (r,s) is such that

$$0 < r < p, \quad 0 < s < q, \quad rs \geq 1. \tag{1.1.1}$$

Since $\lambda > 0$ and $\mu > 0$, the critical growth system (HS) can be seen as a lower order perturbation of the Lane-Emden critical system

$$-\Delta u = |v|^{p-1}v, \quad -\Delta v = |u|^{q-1}u \quad \text{in } \Omega, \quad u = v = 0 \quad \partial\Omega,$$

as (1.0.1) is a lower order perturbation of the critical Lane-Emden equation

$$-\Delta u = u^{2^*-1}, \ u > 0 \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega.$$

Moreover, condition (1.1.1) on (r,s) for (HS) corresponds to condition $1 \leq t < 2^* - 1$ for (1.0.1).

The main result proved in this work reads as follows.

Theorem 1.1.1. *Let $\lambda > 0$, $\mu > 0$, assume (1.1.1) and in case $rs = 1$ also suppose that $\lambda\mu^r$ is suitably small. If $N \geq 4$ or, $N = 3$ and $p \leq 7/2$ or $p \geq 8$, then (HS) has a classical positive solution.*

The precise condition on the size of $\lambda\mu^r$ (for the case with $rs = 1$) is specified at (2.2.1) and (2.2.2) ahead. Actually, such condition appeared before in (MELO; SANTOS, 2015) and corresponds to the hypothesis $\lambda < \lambda_1$ for equation (1.0.2). Moreover, as proved in (VORST, 1992, Theorem 4.2), if such condition is not verified, then (HS) may have no positive solution in star-shaped domains. Also observe that, in case of $N = 3$, $(7/2, 8)$ and $(8, 7/2)$ are symmetric points on the critical hyperbola (1.0.3).

We call the attention to the fact that, when $\lambda = \mu$, $r = s$, $p = q$, any solution of (HS) is such that $u = v$ (see (SANTOS; NORNBERG; SOAVE, 2021, Example 4.3)), which makes (HS) and (1.0.1) to be equivalent in this case. With this in mind, for $N = 3$, the so-called critical dimension for (1.0.1), we prove the existence of solutions for (p, q) lying on some parts of the critical hyperbola (even if $r = s = 1$), which brings new results when comparing to (HULSHOF; MITIDIERI; VORST, 1998, Theorem 2), where the case $N = 3$ is not considered. Indeed, when setting side by side our results with (HULSHOF; MITIDIERI; VORST, 1998, Theorem 2), our contribution is threefold: we treat the case $N = 3$; for $N = 4$ we do not impose $p \neq 2$ or $p \neq 5$; for $N \geq 3$ we consider the natural 1-homogenous ($rs = 1$) or superlinear ($rs > 1$) perturbations, while (HULSHOF; MITIDIERI; VORST, 1998, Theorem 2) is restricted to the case with $r = s = 1$, $p > 1$ and $q > 1$. In particular, for $N \geq 4$, we cover all the points (p, q) on the critical hyperbola, which includes points with $p < 1$ or $q < 1$ for $N > 4$. Figure 1 ahead illustrates the existence result given by Theorem 1.1.1 for $N \geq 4$.

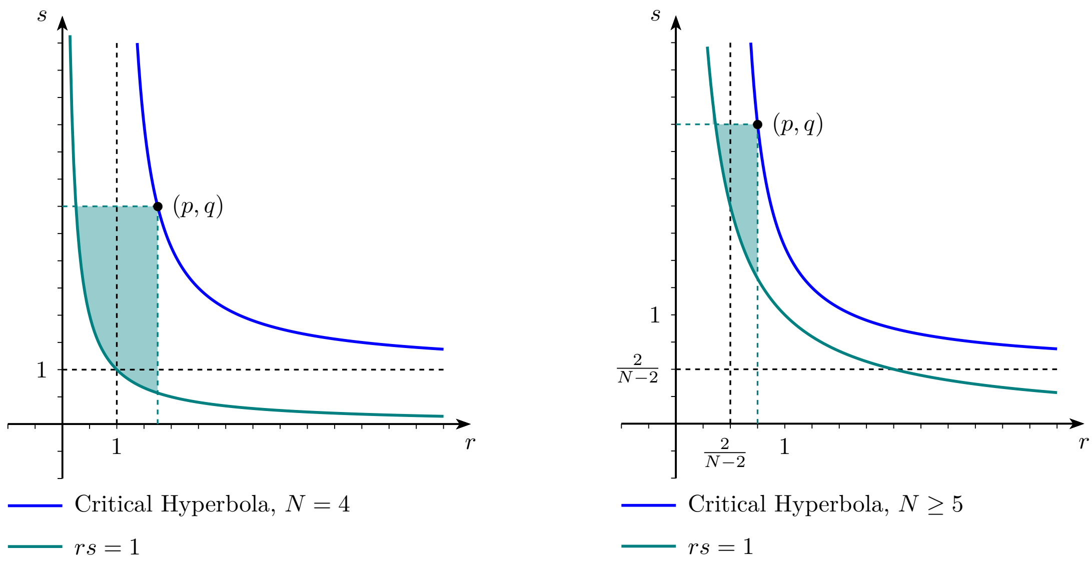

Figure 1 – Given any (p, q) on the critical hyperbola, any (r, s) satisfying (1.1.1) is admissible for finding a positive solution to (HS).

We recall that in the critical dimension $N = 3$, it is not possible to prove the existence of a solution for (1.0.1) in the full range $1 \leq t < 5$. Indeed, as in (BRÉZIS; NIRENBERG, 1983, Corollary 2.3), such existence results is proved only for $3 < t < 5$. This motivates the introduction of the following definition.

Definition 1.1.2. *For $N = 3$, let (p,q) be a point on the critical hyperbola (1.0.3), Ω be a bounded regular domain, and (r,s) satisfying (1.1.1). We say that (p,q) is on a* Critical Region *if, for some Ω and some (r,s), (HS) has no positive solution for some λ and μ small. On the other hand, (p,q) is on a* Noncritical Region *if for all Ω, all (r,s) satisfying (1.1.1), λ and μ suitably small, then (HS) has a positive solution (see Figure 2).*

Finally, we make a link between critical/noncritical regions of the critical hyperbola associated with Hamiltonian systems and the critical dimensions for the biharmonic operator under Navier boundary conditions. We recall that according to (VORST, 1995), the dimensions $N = 5,6,7$ are named as critical for the study of

$$\Delta^2 u = \mu u + u^{\frac{N+4}{N-4}} \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega, \tag{1.1.2}$$

a particular case of (HS) with $\lambda = 0$ and $p = 1$; see also (EDMUNDS; FORTUNATO; JANNELLI, 1990; BERNIS; GRUNAU, 1995) for the case with Dirichlet boundary conditions for the biharmonic and polyharmonic operators, respectively. A first try to understand the phenomenon of critical dimension for Hamiltonian systems was presented in (MELO; SANTOS, 2015). However, the perturbation in (MELO; SANTOS, 2015) makes the problem look like a nonlinear version of the biharmonic equation (1.1.2), as the counterpart of the $p-$Laplacian version for (1.0.1). In the case with $\lambda > 0$ and $\mu > 0$ in (HS), the natural symmetric perturbation of the critical Lane-Emden system, we recover that the only critical dimension is $N = 3$, as it happens to the scalar problem (1.0.1), unveiling the notions of critical and noncritical regions of the critical hyperbola for $N = 3$.

1.2 Open problems and future projects

Once Definition 1.1.2 is posed, it is natural to identity the critical and noncritical of the critical hyperbola for $N = 3$, and this gives rise to the following problems.

Open problems.

1. Find the critical region of the critical hyperbola (1.0.3) for $N = 3$.

2. A simpler problem, but still challenging, is to find the optimal values $7/2 < p_* \leq p^* < 8$ such that (HS) has no solution for any $p_* \leq p \leq p^*$ with $r = s = 1$, $\lambda = \mu$ small, with $\Omega = B(0,1) \subset \mathbb{R}^3$.

For this second question, due to the results in (BRÉZIS; NIRENBERG, 1983, Theorem 1.2) and (3.2.21), we know that $4 \leq p_* \leq 5 \leq p^* \leq 13/2$. Figure 2 illustrates the open problem regarding what should be critical and noncritical regions of the critical hyperbola for $N = 3$.

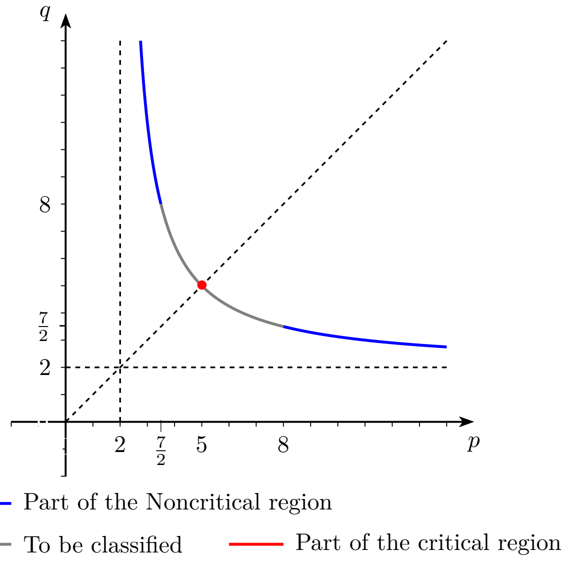

Figure 2 – Critical Hyperbola for $N = 3$

Our work may also serve as motivation for future investigation. In view of the results in (KIM; PISTOIA, 2021, Theorem 1.1), that consider $r = s = 1$, it could be interesting to study blowing up phenomena for system (HS), with $rs = 1$, as $\lambda = \mu \to 0$.

1.3 The structure of this thesis

This thesis is organized as follows.

Chapter 2 is divided in three sections. In the first one, we present the variational approach to treat (HS), namely, writing (HS) as the fourth order equations (P) or (P'). We also define the energy functionals associated with these equations, show that they have the mountain pass geometry and present an upper bound for their mountain pass levels. Section 2.3 is devoted to localize the range where such functionals satisfy the $(PS)_c$-condition and to the proof of Theorem 1.1.1.

Chapter 3 is devoted to the proof of some technical estimates which are crucial for the variational treatment and is divided as follows. In Section 3.1 we prove some identities and inequalities involving the auxiliary functions f_λ^{-1} and $\overline{F}_\lambda$ and their asymptotic behaviours. In Section 3.2 we calculate an upper bound for the mountain pass level c_F of the functional I_F by showing some estimates involving the ground state solutions of the Lane-Emden critical system on $\mathbb{R}^N$ and the auxiliary function f_λ^{-1}.

The most significant classical results used in this thesis are presented in Appendix A, namely the Mountain Pass Theorem, a few convergence results, and two integral inequalities.

CHAPTER

2

EXISTENCE OF SOLUTIONS TO THE SISTEM (HS)

In this chapter, we start by showing the variational approach to treat (HS). After that, in Section 2.2, we prove that the functionals I_F and I_G associated to (P) and (P'), respectively, have the Mountain Pass geometry and compute upper bounds for their mountain pass levels. Finally, in Section 2.3, we localize the range where the functionals I_F and I_G satisfy the $(PS)_c$-condition.

2.1 Variational approach

To deal with (HS), following the same approach as in (MELO; SANTOS, 2015), define

$$\begin{aligned} f_\lambda(t) &= \lambda|t|^{r-1}t + |t|^{p-1}t, \ \ \overline{F}_\lambda(t) = \int_0^t f_\lambda^{-1}(t)dt, \\ g_\mu(t) &= \mu|t|^{s-1}t + |t|^{q-1}t, \ \ \overline{G}_\mu(t) = \int_0^t g_\mu^{-1}(t)dt, \end{aligned} \tag{2.1.1}$$

and rewrite (HS) as one of the fourth-order equations under Navier boundary conditions

$$\begin{cases} \Delta(f_\lambda^{-1}(\Delta u)) = \mu|u|^{s-1}u + |u|^{q-1}u \text{ in } \Omega, \\ u, \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

$$\begin{cases} \Delta(g_\mu^{-1}(\Delta v)) = \lambda|v|^{r-1}v + |v|^{p-1}v \text{ in } \Omega, \\ v, \Delta v = 0 \text{ on } \partial\Omega. \end{cases} \tag{P'}$$

Associated with (P) and (P'), we consider the $C^1(E_p,\mathbb{R})$ and $C^1(E_q,\mathbb{R})$ functionals

$$I_F(u) = \int_\Omega \overline{F}_\lambda(\Delta u)dx - \frac{\mu}{s+1}\int_\Omega |u|^{s+1}dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx,$$

$$I_G(u) = \int_\Omega \overline{G}_\mu(\Delta v)dx - \frac{\lambda}{r+1}\int_\Omega |v|^{r+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx,$$

where $E_t := W^{2,\frac{t+1}{t}}(\Omega) \cap W_0^{1,\frac{t+1}{t}}(\Omega)$ is endowed with the norm $\|u\| = |\Delta u|_{\frac{t+1}{t}}$. Throughout in this work $|w|_\theta$ stand for the $L^\theta(\Omega)$-norm of w.

The variational treatment of (HS) given by studying (P) or (P') is usually called reduction by inversion. This idea has been used by P. L. Lions (LIONS, 1985) and in several other papers, as for example in (CLÉMENT; MITIDIERI, 1997; CLÉMENT; FELMER; MITIDIERI, 1997; HULSHOF; VORST, 1996; BONHEURE; SANTOS; TAVARES, 2014). Here, since the functions f_λ and g_μ are not pure power, and due to the critical growth nature of (HS), we prove in Section 3.1 some sharp estimates on f_λ, whose corresponding versions to g_μ also hold. In order to capture in this inversion the contribution of the term $\lambda|u|^{r-1}u$, to downsize the Mountain Pass level, we compute in Lemma 3.2.1 some integrals on rings involving the ground state solutions of the Lane-Emden critical system on $\mathbb{R}^N$, where terms associated to $u \mapsto \lambda|u|^{r-1}u$ are dominant.

Definition 2.1.1. *We say that $u \in E_p$ is a weak solution of (P) if, and only if, $I'_F(u) = 0$. A function $u \in C^2(\overline{\Omega})$ such that $f_\lambda^{-1}(\Delta u) \in C^2(\Omega)$ is a classical solution of (P) if, and only if, satisfies (P) pointwise. Similarly, we define weak and classical solutions of (P'). Moreover, (u,v) is a classical solution of (HS) if, and only if, $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy (HS) pointwise.*

Lemma 2.1.2. *If u is a weak solution of (P), then it is a classical solution of (P). The converse is also true. Moreover, u is a classical solution of (P) if, and only if, (u,v) is a classical solution of (HS), with $v = f_\lambda^{-1}(-\Delta u)$.*

Proof. We can mimic the proof of (MELO; SANTOS, 2015, Lemma 1), which is based on the arguments in (SANTOS, 2008, Section 4) and (HULSHOF; VORST, 1993, Section 3). □

2.2 Mountain Pass Geometry

Next we show that the functionals I_F and I_G have the Mountain Pass geometry and obtain upper bounds for their Mountain Pass levels. For the cases with $rs = 1$ we introduce the conditions

$$\lambda^{1/r}\mu \le \frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}}\mathscr{C}_{r,\Omega}^{\frac{r+1}{r}}, \tag{2.2.1}$$

$$\lambda\mu^{1/s} \le \frac{(2|\Omega|)^{\frac{s-q}{s(q+1)}}}{2^{\frac{s+1}{s}}}\mathscr{C}_{s,\Omega}^{\frac{s+1}{s}}, \tag{2.2.2}$$

on the size of (λ,μ), where

$$\mathscr{C}_{r,\Omega} = \inf\{\|u\|;\ u \in E_p \text{ and } |u|_{\frac{r+1}{r}} = 1\},\ \ \mathscr{C}_{s,\Omega} = \inf\{\|v\|;\ v \in E_q \text{ and } |v|_{\frac{s+1}{s}} = 1\}.$$

Remark 2.2.1. *Conditions (2.2.1) and (2.2.2) for the case with $rs = 1$ are natural and correspond to the hypothesis on λ and μ in (HULSHOF; MITIDIERI; VORST, 1998, Theorem 2) to treat (HS) with $r = s = 1$, and to the hypothesis $\lambda < \lambda_1$ in (BRÉZIS; NIRENBERG, 1983) to study (1.0.2).*

Proposition 2.2.2. *Let (p,q) and (r,s) be as in (1.0.3) and (1.1.1).*

1. *Then I_F has the Mountain Pass geometry with a local minimum at zero, under the additional condition (2.2.1) when $rs = 1$.*

2. *Then I_G has the Mountain Pass geometry with a local minimum at zero, under the additional condition (2.2.2) when $rs = 1$.*

Proof. Observe that $I_F(0) = 0$ and, from (3.1.2),

$$I_F(u) \leq \frac{p}{p+1}\|u\|^{\frac{p+1}{p}} - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1}, \ \ \forall u \in E_p.$$

Then, $I_F(tu) \to -\infty$ when $t \to \infty$ and $u \neq 0$.

On the other hand, by Lemma 3.1.5,

$$\begin{aligned} I_F(u) &= \int_{|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}} \overline{F}_\lambda(\Delta u)dx + \int_{|\Delta u| > 2\lambda^{\frac{p}{p-r}}} \overline{F}_\lambda(\Delta u)dx - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1} \\ &\geq \frac{1}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1}\int_{|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{r+1}{r}}dx + \frac{1}{2^{\frac{p+1}{p}}}\frac{p}{p+1}\int_{|\Delta u| > 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx \\ &\qquad - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1}. \end{aligned}$$

By Jensen's inequality, for a nonnegative measurable function a and $\alpha > 1$,

$$\int_\omega (a(t))^\alpha dt \geq |\omega|^{1-\alpha}\left(\int_\omega a(t)dt\right)^\alpha. \tag{2.2.3}$$

Since $0 < r < p$, with $\alpha = \frac{r+1}{r}\frac{p}{p+1} > 1$, it follows that

$$\begin{aligned} I_F(u) &\geq \frac{(meas(|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}))^{1-\alpha}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1}\left(\int_{|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx\right)^\alpha \\ &\quad + \frac{1}{2^{\frac{p+1}{p}}}\frac{p}{p+1}\int_{|\Delta u| > 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1} \\ &\geq \frac{|\Omega|^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1}\left(\int_{|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx\right)^\alpha \\ &\quad + \frac{1}{2^{\frac{p+1}{p}}}\frac{p}{p+1}\int_{|\Delta u| > 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1}. \end{aligned}$$

For $u \neq 0$ such that

$$\frac{1}{2^{\frac{p+1}{p}}}\frac{p}{p+1}\|u\|^{-\frac{p-r}{rp}} \geq \frac{|\Omega|^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1} \ \text{ i.e. } \ \|u\| \leq \left(\frac{p(r+1)}{(p+1)r}\right)^{\frac{pr}{p-r}} 2|\Omega|\lambda^{\frac{p}{p-r}},$$

it follows that

$$\begin{aligned} I_F(u) &\geq \frac{|\Omega|^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1}\left[\left(\int_{|\Delta u| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx\right)^\alpha + \left(\int_{|\Delta u| > 2\lambda^{\frac{p}{p-r}}} |\Delta u|^{\frac{p+1}{p}}dx\right)^\alpha\right] \\ &\qquad - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1}. \end{aligned}$$

Since $1-\alpha = -\frac{p-r}{r(p+1)} = -\frac{p}{p+1}\frac{p-r}{pr}$ and $(a+b)^\alpha \le 2^{\alpha-1}(a^\alpha + b^\alpha)$ for $a,b\ge 0$, we infer that

$$I_F(u) \ge \frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\frac{r}{r+1}\|u\|^{\frac{r+1}{r}} - \frac{\mu}{s+1}|u|_{s+1}^{s+1} - \frac{1}{q+1}|u|_{q+1}^{q+1}$$

for all $u\in E$ such that $\|u\| \le \left(\frac{p(r+1)}{(p+1)r}\right)^{\frac{pr}{p-r}} 2|\Omega|\lambda^{\frac{p}{p-r}}$.

If $\frac{r+1}{r} < s+1$, i.e. $\frac{1}{r} < s$, and since $s<q$, it follows that I_F has the Mountain Pass geometry with a local minimum at zero.

On the other hand, if $\frac{r+1}{r} = s+1$, i.e. $\frac{1}{r} = s$, for $u\neq 0$, we infer that

$$\begin{aligned}
(s+1)I_F(u) &\ge \frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}}\|u\|^{\frac{r+1}{r}} - \mu|u|_{\frac{r+1}{r}}^{\frac{r+1}{r}} - \frac{s+1}{q+1}|u|_{q+1}^{q+1}\\
&= \left(\frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}} - \mu|u|_{\frac{r+1}{r}}^{\frac{r+1}{r}}\|u\|^{-\frac{r+1}{r}}\right)\|u\|^{\frac{r+1}{r}} - \frac{s+1}{q+1}|u|_{q+1}^{q+1}\\
&\ge \left(\frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}} - \mu\frac{1}{\mathscr{C}_{r,\Omega}^{\frac{r+1}{r}}}\right)\|u\|^{\frac{r+1}{r}} - \frac{s+1}{q+1}|u|_{q+1}^{q+1}
\end{aligned}$$

and (2.2.1) gives $\frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}\lambda^{1/r}} - \mu\frac{1}{\mathscr{C}_{r,\Omega}^{\frac{r+1}{r}}} > 0$, and again I_F has the Mountain Pass geometry around zero, since $\frac{1}{r} = s < q$. □

Let S be the Sobolev constant for the embedding $E_p \hookrightarrow L^{q+1}(\Omega)$, namely

$$S = \inf_{u\in E_p, |u|_{q+1}=1}\|u\|.$$

Proposition 2.2.3. *Suppose* (1.0.3) *and* (1.1.1), $\mu>0$, $\lambda>0$ *and in the case* $rs=1$ *also assume* (2.2.1). *If* $N\ge 4$ *and* $p\le (N+2)/(N-2)$, *or* $N=3$ *and* $p\le 7/2$, *then the mountain pass level* c_F *of the functional* I_F *is such that* $c_F \in (0, \frac{2}{N}S^{\frac{pN}{2(p+1)}})$.

Proof. See Section 3.2. □

2.3 $(\mathrm{PS})_c$ condition

When treating (P), the main difficulty is the lack of compactness for the embedding $E_p \hookrightarrow L^{q+1}(\Omega)$. Here we localize the levels c for which the $(PS)_c$-condition holds. Throughout this section (1.0.3), (1.1.1), $\mu,\lambda>0$ are assumed, and the main results is the following.

Proposition 2.3.1. *I_F satisfies the $(PS)_c$-condition for all* $c < \frac{2}{N}S^{\frac{pN}{2(p+1)}}$.

We split the proof of this proposition in several lemmas.

Lemma 2.3.2. *Every (PS) sequence for I_F is bounded*

Proof. Let (u_n) be a (PS) sequence for I_F. So, using Corollary 3.1.3 and Lemma 3.1.6, there exists $c \in \mathbb{R}$ and a positive sequence (ε_n) with $\varepsilon_n \to 0$ such that

$$\begin{aligned}
\varepsilon_n \|u_n\| + c \geq I_F(u_n) - \frac{I_F'(u_n)u_n}{s+1} &= \int_\Omega \overline{F}_\lambda(\Delta u_n) - \frac{f_\lambda^{-1}(\Delta u_n)\Delta u_n}{s+1} dx + \frac{(q-s)|u_n|_{q+1}^{q+1}}{(s+1)(q+1)} \\
&\geq \int_{|\Delta u_n| \geq 2\lambda^{\frac{p}{p-r}}} \overline{F}_\lambda(\Delta u_n) - \frac{f_\lambda^{-1}(\Delta u_n)\Delta u_n}{s+1} dx \geq \tau \int_{|\Delta u_n| \geq 2\lambda^{\frac{p}{p-r}}} |\Delta u_n|^{\frac{p+1}{p}} dx \\
&\geq \tau \left(\int_\Omega |\Delta u_n|^{\frac{p+1}{p}} dx - \int_{|\Delta u_n| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta u_n|^{\frac{p+1}{p}} dx \right) \geq \tau \|u_n\|^{\frac{p+1}{p}} - 2^{\frac{p+1}{p}} \lambda^{\frac{p+1}{p-r}} |\Omega|,
\end{aligned}$$

which implies the boundedness of $(\|u_n\|)$. □

To localize the levels where I_F satisfies the (PS) condition the following result, due to P.-L. Lions, is necessary.

Lemma 2.3.3. *Given a bounded sequence (u_n) in E_p, there exists a subsequence, also denoted here by (u_n), such that:*

(i) $u_n \rightharpoonup u$ *in* E_p.

(ii) $u_n \to u$ *a.e. in* Ω *and in* $L^\theta(\Omega)$, *for all* $1 \leq \theta < q+1$.

(iii) $|\Delta u_n|^{\frac{p+1}{p}} \overset{*}{\rightharpoonup} \gamma$ *in the sense of measures on* $\overline{\Omega}$.

(iv) $|u_n|^{q+1} \overset{*}{\rightharpoonup} \nu$ *in the sense of measures on* $\overline{\Omega}$.

(v) There exist an at most countable index set J, a family of points $\{x_j : j \in J\} \subset \overline{\Omega}$ and two sequences $\{\nu_j : j \in J\}, \{\gamma_j : j \in J\} \subset (0, +\infty)$ such that:

$$\nu = |u|^{q+1} + \sum_{j \in J} \nu_j \delta_{x_j}, \ \gamma \geq |\Delta u|^{\frac{p+1}{p}} + \sum_{j \in J} \gamma_j \delta_{x_j},$$

$$S \nu_j^{\frac{p+1}{p} \frac{1}{q+1}} \leq \gamma_j \textit{ for all } j \in J, \textit{ in particular } \sum_{j \in J} \nu_j^{\frac{p+1}{p} \frac{1}{q+1}} < +\infty.$$

(vi) $\nabla u_n \rightharpoonup \nabla u$ *in* $\left(W^{1,\frac{p+1}{p}}(\Omega) \right)^N$.

(vii) $\nabla u_n \to \nabla u$ *a.e. in* Ω *and in* $(L^\sigma(\Omega))^N$, *for all* $1 \leq \sigma < \sigma^*$, *with* $\sigma^* > \frac{p+1}{p}$ *depending on the critical Sobolev embedding of* $W^{1,\frac{p+1}{p}}(\Omega)$.

Proof. See (LIONS, 1985, Lemma I.1) or (SANTOS, 2010, Lemma 3.3). □

An improvement of the previous lemma is given next.

Lemma 2.3.4. *If (u_n) is a (PS)-sequence for I_F, then there exists a subsequence, for short also denoted by (u_n), satisfying (i)-(vii) from Lemma 2.3.3 with the additional fact that J is at most finite.*

Proof. Let $x_j \in \overline{\Omega}$ be a point in the singular support of μ and ν. Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in $B(0,1)$ and $supp(\zeta) \subset B(0,2)$. Moreover for each $\theta > 0$ define $\zeta_\theta(x) := \zeta(\frac{x-x_j}{\theta})$. So there exists constants c_1 and c_2 independent of θ such that

$$|\nabla\zeta_\theta(x)| \le \frac{c_1}{\theta},\ |\Delta\zeta_\theta(x)| \le \frac{c_2}{\theta^2},\ \forall x \in \mathbb{R}^N.$$

By (SANTOS, 2010, Lemma 3.4), $u_n\zeta_\theta \in E_p$, $\forall n \in \mathbb{N}$ *and* $\theta > 0$. Fixing $\theta > 0$, since $(\zeta_\theta u_n)$ is bounded in E_p, $\langle I_F'(u_n), \zeta_\theta u_n\rangle = o(1)$ that is

$$\begin{aligned} o(1) &= \int_\Omega f_\lambda^{-1}(\Delta u_n)\Delta u_n\zeta_\theta dx - \int_\Omega |u_n|^{q+1}\zeta_\theta dx - \mu\int_\Omega |u_n|^{s+1}\zeta_\theta dx \\ &\quad + \int_\Omega f_\lambda^{-1}(\Delta u_n)u_n\Delta\zeta_\theta dx + 2\int_\Omega f_\lambda^{-1}(\Delta u_n)\nabla u_n\nabla\zeta_\theta dx. \end{aligned} \tag{2.3.1}$$

On the other hand, $\zeta_\theta(x) \xrightarrow{\theta\to 0} \delta_{x_j}(x)$, $\forall x \in \Omega$. So from $u_n \rightharpoonup u$ in E_p and $E_p \subset\subset L^{s+1}(\Omega)$

$$\int_\Omega |u_n|^{s+1}\zeta_\theta dx \xrightarrow{n\to\infty} \int_\Omega |u|^{s+1}\zeta_\theta dx \xrightarrow{\theta\to 0} 0.$$

Now, by (3.1.1) and Hölder inequality $\left(\frac{1}{p+1} + \frac{1}{q+1} + \frac{2}{N} = 1\right)$, there exists $C > 0$ independent of n and θ such that

$$\left|\int_\Omega f_\lambda^{-1}(\Delta u_n)u_n\Delta\zeta_\theta dx\right| \le \int_\Omega |\Delta u_n|^{1/p}|u_n||\Delta\zeta_\theta|dx \le C\left(\int_\Omega |u_n|^{q+1}\left|\Delta\zeta\left(\frac{x-x_j}{\theta}\right)\right|^{\frac{q+1}{2}}dx\right)^{\frac{1}{q+1}}.$$

From the definition of the weak* convergence

$$\int_\Omega |u_n|^{q+1}\left|\Delta\zeta\left(\frac{x-x_j}{\theta}\right)\right|^{\frac{q+1}{2}}dx \xrightarrow{n\to\infty} \int_\Omega \left|\Delta\zeta\left(\frac{x-x_j}{\theta}\right)\right|^{\frac{q+1}{2}}d\nu,$$

and since $|\Delta\zeta(\frac{x-x_j}{\theta})|^{(\frac{q+1}{2})} \xrightarrow{\theta\to 0} 0\ \forall x \in \Omega$, by the Lebesgue dominated convergence theorem,

$$\int_\Omega \left|\Delta\zeta\left(\frac{x-x_j}{\theta}\right)\right|^{\frac{q+1}{2}}d\nu \xrightarrow{\theta\to 0} 0.$$

We also have

$$\left|\int_\Omega f_\lambda^{-1}(\Delta u_n)\nabla u_n\nabla\zeta_\theta dx\right| \le C\left(\int_\Omega \left(\frac{1}{\theta}\left|\nabla\zeta\left(\frac{x-x_j}{\theta}\right)\right||\nabla u_n|\right)^{\frac{p+1}{p}}dx\right)^{\frac{p}{p+1}} \quad \text{and}$$

$$\int_\Omega \left(\frac{1}{\theta}\left|\nabla\zeta\left(\frac{x-x_j}{\theta}\right)\right||\nabla u_n|\right)^{\frac{p+1}{p}}dx \xrightarrow{n\to\infty} \int_\Omega \left(\frac{1}{\theta}\left|\nabla\zeta\left(\frac{x-x_j}{\theta}\right)\right||\nabla u|\right)^{\frac{p+1}{p}}dx = O(\theta^{N-\frac{p+1}{p}}).$$

Given $\varepsilon > 0$ let $M(\varepsilon) > 0$ be such that $f_\lambda^{-1}(t)t \geq \frac{1}{1+\varepsilon}|t|^{\frac{p+1}{p}}$ for all $|t| \geq M(\varepsilon)$. Then define

$$A_n := \{x \in B(x_j, 2\theta) \cap \overline{\Omega}; |\Delta u_n(x)| \geq M(\varepsilon)\}, B_n := (\overline{\Omega} \cap B(x_j, 2\theta)) \backslash A_n.$$

Then,

$$\begin{aligned}
\int_{\Omega} f_\lambda^{-1}(\Delta u_n)\Delta u_n \zeta_\theta dx &= \int_{A_n} f_\lambda^{-1}(\Delta u_n)\Delta u_n \zeta_\theta dx + \int_{B_n} f_\lambda^{-1}(\Delta u_n)\Delta u_n \zeta_\theta dx \\
&\geq \frac{1}{1+\varepsilon}\int_{\Omega}|\Delta u_n|^{\frac{p+1}{p}}\zeta_\theta dx + \int_{B_n}(f_\lambda^{-1}(\Delta u_n)\Delta u_n - \frac{1}{1+\varepsilon}|\Delta u_n|^{\frac{p+1}{p}})\zeta_\theta dx \\
= \frac{1}{1+\varepsilon}\int_{\Omega}|\Delta u_n|^{\frac{p+1}{p}}\zeta_\theta dx &+ \int_{B_n}(f_\lambda^{-1}(\Delta u_n)\Delta u_n - \frac{1}{1+\varepsilon}|\Delta u_n|^{\frac{p+1}{p}})\zeta_\theta dx \longrightarrow \frac{1}{1+\varepsilon}\gamma_j
\end{aligned} \tag{2.3.2}$$

by taking the limit as $n \to \infty$ and after as $\theta \to 0$, because

$$\begin{aligned}
\lim_{\theta\to 0}\limsup_{n\to\infty}\left|\int_{B_n}(f_\lambda^{-1}(\Delta u_n)\Delta u_n - \frac{1}{1+\varepsilon}|\Delta u_n|^{\frac{p+1}{p}})\zeta_\theta dx\right| \\
\leq \lim_{\theta\to 0}\limsup_{n\to\infty}\int_{B_n}(f_\lambda^{-1}(M)M + \frac{1}{1+\varepsilon}M^{\frac{p+1}{p}})\zeta_\theta dx = 0.
\end{aligned}$$

Then, from all the above estimates ranging from (2.3.1) to (2.3.2), we infer that

$$0 = \lim_{\theta\to 0} lim_{n\to\infty}\langle I_F'(u_n), u_n\zeta_\theta\rangle \geq \frac{\gamma_j}{1+\varepsilon} - \nu_j$$

which implies that $\nu_j \geq \frac{\gamma_j}{1+\varepsilon}$ for all $\varepsilon > 0$, and hence $\nu_j \geq \gamma_j$. In contrast, since $0 \leq f_\lambda^{-1}(\Delta u_n)\Delta u_n \leq |\Delta u_n|^{\frac{p+1}{p}}$, it follows from all the above estimates ranging from (2.3.1) to (2.3.2) that

$$0 = \lim_{\theta\to 0}\lim_{n\to\infty}\langle I_F'(u_n), u_n\zeta_\theta\rangle \leq \gamma_j - \nu_j,$$

which implies

$$\gamma_j = \nu_j. \tag{2.3.3}$$

Then, from Lemma 2.3.3, $\nu_j \geq S\nu_j^{\frac{p+1}{p}\frac{1}{q+1}}$ and so

$$\nu_j \geq S^{\frac{pN}{2(p+1)}}, \tag{2.3.4}$$

since $\frac{pN}{2(p+1)} = \left(1 - \frac{1}{q+1}\frac{p+1}{p}\right)^{-1}$ and $\gamma_j > 0$. Combining this with $\sum_{j\in J}\nu_j^{\frac{p+1}{p}\frac{1}{q+1}} < +\infty$, it follows that J is at most finite. □

Lemma 2.3.5. *Given a bounded sequence (u_n) in E_p, $K \subset\subset \Omega\backslash\{x_j : j \in J\}$ with $\{x_j : j \in J\}$ from Lemma 2.3.3, then $u_n \to u$ in $L^{q+1}(K)$, up to a subsequence.*

Proof. See the proof of (SANTOS, 2010, Lemma 3.6).

□

Lemma 2.3.6. *If (u_n) is a (PS)-sequence for I_F, and $\{x_j : j \in J\}$ from Lemma 2.3.3, then for every $j \in J$, up to a subsequence,*

$$\lim_{n\to\infty} \int_\Omega \left[\overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n \right] dx \geq \frac{ps-1}{(p+1)(s+1)} \gamma_j.$$

Proof. Consider the even function $H_p(t) := |t|^{-\frac{p+1}{p}} \left(\overline{F}_\lambda(t) - \frac{1}{s+1} f_\lambda^{-1}(t)t \right)$. By Lemma 3.1.1,

$$\lim_{t\to\infty} H_p(t) = \lim_{t\to\infty} \left[\frac{ps-1}{(p+1)(s+1)} \frac{f_\lambda^{-1}(t)}{t^{1/p}} - \frac{p-r}{p+1} \frac{\lambda}{r+1} \frac{|f_\lambda^{-1}(t)|^{r+1}}{t^{\frac{p+1}{p}}} \right] = \frac{ps-1}{(p+1)(s+1)}.$$

Then, given $\varepsilon > 0$ small, there exists $t_0 > 0$ such that

$$H_p(t) > c_\varepsilon > 0 \quad \text{for all } |t| > t_0,$$

with $c_\varepsilon = \frac{ps-1}{(p+1)(s+1)} - \varepsilon$, which implies that

$$\overline{F}_\lambda(t) - \frac{1}{s+1} f_\lambda^{-1}(t)t > c_\varepsilon |t|^{\frac{p+1}{p}} \quad \text{for all } |t| > t_0. \tag{2.3.5}$$

Let A_n, B_n and ζ_θ be like in Lemma 2.3.4, with A_n and B_n associated with t_0. So, from (2.3.5) and Corollary 3.1.3,

$$\begin{aligned} \int_\Omega \overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n dx &\geq c_\varepsilon \left(\int_{A_n} |\Delta u_n|^{\frac{p+1}{p}} \zeta_\theta dx + \int_{B_n} |\Delta u_n|^{\frac{p+1}{p}} \zeta_\theta dx \right) \\ &\quad + \int_{B_n} \left[\overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n - c_\varepsilon |\Delta u_n|^{\frac{p+1}{p}} \right] \zeta_\theta dx \\ &= c_\varepsilon \int_\Omega |\Delta u_n|^{\frac{p+1}{p}} \zeta_\theta dx + \int_{B_n} \left[\overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n - c_\varepsilon |\Delta u_n|^{\frac{p+1}{p}} \right] \zeta_\theta dx. \end{aligned} \tag{2.3.6}$$

On the other hand,

$$\begin{aligned} &\lim_{\theta\to 0} \limsup_{n\to\infty} \left| \int_{B_n} \left[\overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n - c_\varepsilon |\Delta u_n|^{\frac{p+1}{p}} \right] \zeta_\theta dx \right| \\ &\qquad \leq \lim_{\theta\to 0} \limsup_{n\to\infty} \int_\Omega \left[\overline{F}_\lambda(t_0) + \frac{1}{s+1} f_\lambda^{-1}(t_0)t_0 + c_\varepsilon |t_0|^{\frac{p+1}{p}} \right] \zeta_\theta dx = 0, \end{aligned} \tag{2.3.7}$$

and, from Lemma 2.3.3,

$$\lim_{\theta\to 0} \limsup_{n\to\infty} c_\varepsilon \int_\Omega |\Delta u_n|^{\frac{p+1}{p}} \zeta_\theta dx \geq c_\varepsilon \gamma_k. \tag{2.3.8}$$

From (2.3.6), (2.3.7), (2.3.8) and the arbitrariness of $\varepsilon > 0$, we get the desired inequality. □

Lemma 2.3.7. *If (u_n) is a (PS)-sequence for I_F, $K \subset\subset \Omega \backslash \{x_j : j \in J\}$ with $\{x_j : j \in J\}$ from Lemma 2.3.3, then up to a subsequence,*

$$\int_K (f_\lambda^{-1}(\Delta u_n) - f_\lambda^{-1}(\Delta u))(\Delta u_n - \Delta u) dx \xrightarrow{n\to\infty} 0. \tag{2.3.9}$$

Proof. Let $\delta = dist(K,\{x_j : j\in J\})$. For each $\theta\in(0,\delta)$, consider $A_\theta=\{x\in\Omega : dist(x,K)<\theta\}$ and $\xi_\theta\in C_c^\infty(\Omega)$, $0\le\xi_\theta\le 1$, $\xi_\theta\equiv 1$ on $A_{\theta/2}$ and $\xi_\theta\equiv 0$ on $\Omega\backslash A_\theta$. So, by the monotonicity of f_λ^{-1},

$$\begin{aligned}0\le\int_K(f_\lambda^{-1}(\Delta u_n)-f_\lambda^{-1}(\Delta u))(\Delta u_n-\Delta u)dx&\le\int_\Omega(f_\lambda^{-1}(\Delta u_n)-f_\lambda^{-1}(\Delta u))(\Delta u_n-\Delta u)\xi_\theta dx\\&=\int_\Omega f_\lambda^{-1}(\Delta u_n)\Delta u_n\xi_\theta-f_\lambda^{-1}(\Delta u_n)\Delta u\xi_\theta-f_\lambda^{-1}(\Delta u)(\Delta u_n-\Delta u)\xi_\theta dx.\end{aligned}\tag{2.3.10}$$

Fixing $\theta>0$, since $I_F'(u_n)\to 0$ and $(u_n\xi_\theta)$ is bounded in E, then $\langle I_F'(u_n),\xi_\theta u\rangle=o(1)$ and $\langle I_F'(u_n),\xi_\theta u_n\rangle=o(1)$ that is

$$o(1)=\int_\Omega f_\lambda^{-1}(\Delta u_n)(\Delta u\xi_\theta+u\Delta\xi_\theta+2\nabla u\nabla\xi_\theta)dx-\int_\Omega|u_n|^{q-1}u_n\xi_\theta udx-\mu\int_\Omega|u_n|^{s-1}u_n\xi_\theta udx \tag{2.3.11}$$

$$o(1)=\int_\Omega f_\lambda^{-1}(\Delta u_n)(\Delta u_n\xi_\theta+u_n\Delta\xi_\theta+2\nabla u_n\nabla\xi_\theta)dx-\int_\Omega|u_n|^{q+1}\xi_\theta dx-\mu\int_\Omega|u_n|^{s+1}\xi_\theta dx. \tag{2.3.12}$$

From (2.3.10), (2.3.11), (2.3.12), Lemma 2.3.3, Lemma 2.3.5 and (SANTOS, 2010, Lemma 2.4), it follows that

$$\begin{aligned}0\le\int_K(f_\lambda^{-1}(\Delta u_n)-f_\lambda^{-1}(\Delta u))(\Delta u_n-\Delta u)dx&\le\int_\Omega f_\lambda^{-1}(\Delta u_n)\Delta\xi_\theta(u_n-u)dx\\&\quad+2\int_\Omega f_\lambda^{-1}(\Delta u_n)\nabla\xi_\theta\nabla(u_n-u)dx-\int_\Omega f_\lambda^{-1}(\Delta u)(\Delta u_n-\Delta u)\xi_\theta dx+o(1)\\&\le C\left(\int_{A_\theta}|u_n-u|^{q+1}dx\right)^{\frac{1}{q+1}}+C\left(\int_\Omega|\nabla u_n-\nabla u|^{\frac{p+1}{p}}dx\right)^{\frac{p}{p+1}}\\&\quad-\int_\Omega f_\lambda^{-1}(\Delta u)(\Delta u_n-\Delta u)\xi_\theta dx+o(1)=o(1).\end{aligned}$$ □

Lemma 2.3.8. *If (u_n) is a (PS)-sequence for I_F, then $\Delta u_n\xrightarrow{n\to\infty}\Delta u$ a.e. in Ω, up to a subsequence.*

Proof. Let $K\subset\subset\Omega\backslash\{x_j\}_{j\in J}$. By the inverse of the Lebesgue dominated convergence theorem, there exists a subsequence of the integrand in (2.3.9) that converges a.e. in K. Using (DAL MASO; MURAT, 1998, Lemma 6) with

$$X=\mathbb{R},\ \beta_n=f_\lambda^{-1},\ \beta=f_\lambda^{-1},\text{ and }\xi_n=\Delta u_n(x)$$

we get $\Delta u_n\to\Delta u$ *a.e. in* K. Since K is an arbitrary compact subset of $\Omega\backslash\{x_j\}_{j\in J}$, we conclude that $\Delta u_n\to\Delta u$ *a.e. in* Ω. □

Lemma 2.3.9. *If (u_n) is a (PS)-sequence for I_F, then $f_\lambda^{-1}(\Delta u_n)\rightharpoonup f_\lambda^{-1}(\Delta u)$ in $L^{p+1}(\Omega)$, up to a subsequence.*

Proof. Since, up to a subsequence,

$$\begin{cases}\Delta u_n\xrightarrow{n\to\infty}\Delta u\quad\text{a.e. in }\Omega,\\(\Delta u_n)\text{ is bounded in }L^{\frac{p+1}{p}}(\Omega),\text{ and}\\|f_\lambda^{-1}(\Delta u_n)|\le|\Delta u_n|^{1/p},\end{cases}$$

we infer that $f_\lambda^{-1}(\Delta u_n) \xrightarrow{n\to\infty} f_\lambda^{-1}(\Delta u)$ a.e. in Ω, $(f_\lambda^{-1}(\Delta u_n))$ is bounded in $L^{p+1}(\Omega)$, and hence $f_\lambda^{-1}(\Delta u_n) \rightharpoonup f_\lambda^{-1}(\Delta u)$ in $L^{p+1}(\Omega)$. $\square$

Proposition 2.3.10. *If (u_n) is a (PS)-sequence for I_F, then there exist a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u$ in E_p and u is a weak solution of* (P).

Proof. First, since (u_n) is a (PS)-sequence to I_F, $\langle I_F'(u_n), w\rangle \to 0$, for all $w \in E_p$. On the other hand, up to a subsequence,

$$\begin{cases} f_\lambda^{-1}(\Delta u_n) \rightharpoonup f_\lambda^{-1}(\Delta u) \text{ in } L^{p+1}(\Omega), \\ |u_n|^{q-1}u_n \rightharpoonup |u|^{q-1}u \text{ in } L^{\frac{q+1}{q}}(\Omega) \text{ and} \\ |u_n|^{s-1}u_n \to |u|^{s-1}u \text{ in } L^{\frac{s+1}{s}}(\Omega). \end{cases}$$

Thus, for all $w \in E_p$, $\langle I_F'(u_n), w\rangle \to \langle I_F'(u), w\rangle$. Then $\langle I_F'(u), w\rangle = 0$, for all $w \in E_p$, that is, u is a weak solution of (P). $\square$

Proof of Proposition 2.3.1. Let (u_n) be a $(PS)_c$ sequence for I_F with $c < \frac{2}{N} S^{\frac{pN}{2(p+1)}}$. By contradiction, suppose that $J \neq \emptyset$. We can suppose the assertions of Lemma 2.3.4, with $\gamma_j = \nu_j$, $\nu_j \geq S^{\frac{pN}{2(p+1)}}$, for all $j \in J$, by (2.3.3) and (2.3.4). Since (u_n) is bounded in E_p, $\langle I_F'(u_n), u_n\rangle = o(1)$, and Lemmas 2.3.3 and 2.3.6, we infer that

$$\begin{aligned} c &= \lim_{n\to\infty} I_F(u_n) - \frac{1}{s+1}\langle I_F'(u_n), u_n\rangle \\ &= \lim_{n\to\infty} \int_\Omega \overline{F}_\lambda(\Delta u_n) - \frac{1}{s+1} f_\lambda^{-1}(\Delta u_n)\Delta u_n dx + \left(\frac{1}{s+1} - \frac{1}{q+1}\right)\int_\Omega |u_n|^{q+1} dx \\ &\geq \left(\frac{ps-1}{(p+1)(s+1)} + \frac{1}{s+1} - \frac{1}{q+1}\right)\nu_j = \frac{2}{N}\nu_j \geq \frac{2}{N} S^{\frac{pN}{2(p+1)}} \end{aligned}$$

for every $j \in J$, which is a contradiction. Hence, $J = \emptyset$.

Then, from Lemmas 2.3.3 and 2.3.5, since $L^{q+1}(\Omega)$ is uniformly convex, $u_n \to u$ in $L^{q+1}(\Omega)$.

Let $v_n = u_n - u$, thus $v_n \rightharpoonup 0$ in E_p, $\Delta v_n \to 0$ $a.e.$ in Ω and $v_n \to 0$ in $L^{q+1}(\Omega)$. Since

$$|(a+b)f_\lambda^{-1}(a+b) - af_\lambda^{-1}(a)| \leq |a+b|^{\frac{p+1}{p}} + |a|^{\frac{p+1}{p}} \leq 2^p(|b|^{\frac{p+1}{p}} + |a|^{\frac{p+1}{p}}), \quad \forall a, b \in \mathbb{R},$$

from (BRÉZIS; LIEB, 1983, Theorem 2), with $j(t) = tf_\lambda^{-1}(t)$,

$$\begin{aligned} \int_\Omega f_\lambda^{-1}(\Delta u_n)\Delta u_n dx &= \int_\Omega f_\lambda^{-1}(\Delta u + \Delta v_n)(\Delta u + \Delta v_n) dx \\ &= \int_\Omega f_\lambda^{-1}(\Delta u)\Delta u + f_\lambda^{-1}(\Delta v_n)\Delta v_n dx + o(1). \end{aligned}$$

Since u is weak solution of (P),

$$\begin{aligned} o(1) = \langle I_F'(u_n), u_n\rangle &= \int_\Omega f_\lambda^{-1}(\Delta u_n)\Delta u_n - |u_n|^{q+1} - \mu|u_n|^{s+1} dx \\ &= \langle I_F'(u), u\rangle + \int_\Omega f_\lambda^{-1}(\Delta v_n)\Delta v_n dx + o(1) = \int_\Omega f_\lambda^{-1}(\Delta v_n)\Delta v_n dx + o(1), \end{aligned}$$

that is, $f_\lambda^{-1}(\Delta v_n)\Delta v_n \to 0$ in $L^1(\Omega)$. Then, Lemma 3.1.4 and Jensen's inequality (2.2.3) lead to

$$\begin{aligned} 0 \leftarrow \int_\Omega f_\lambda^{-1}(\Delta v_n)\Delta v_n dx &\geq \frac{1}{2^{1/p}} \int_{|\Delta v_n| \geq 2\lambda^{\frac{p}{p-r}}} |\Delta v_n|^{\frac{p+1}{p}} dx + \frac{1}{(2\lambda)^{1/r}} \int_{|\Delta v_n| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta v_n|^{\frac{r+1}{r}} dx \\ &\geq \frac{1}{2^{1/p}} \int_{|\Delta v_n| \geq 2\lambda^{\frac{p}{p-r}}} |\Delta v_n|^{\frac{p+1}{p}} dx + \frac{1}{(2\lambda)^{1/r}} |\Omega|^{1-\alpha} \left(\int_{|\Delta v_n| \leq 2\lambda^{\frac{p}{p-r}}} |\Delta v_n|^{\frac{p+1}{p}} dx \right)^\alpha, \end{aligned}$$

with $\alpha = \frac{p}{p+1}\frac{r+1}{r}$. Therefore $|\Delta v_n|^{\frac{p+1}{p}} \to 0$ in $L^1(\Omega)$, that is, $u_n \to u$ in E_p. □

At this point we have all the tools at hand to prove our main result.

Proof of Theorem 1.1.1. Suppose, without loss of generality, that $p \leq q$. The case $q \leq p$ can be handled similarly, by using I_G instead of I_F. By Propositions 2.2.2, 2.2.3, 2.3.1 and Lemma 2.1.2, the existence of a classical solution is a direct consequence of the Mountain Pass theorem.

Next, we prove that any Mountain Pass solution is signed. Let u be a Mountain Pass solution of (P). So, by Lemma 2.1.2, $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Then, by the classical strong maximum principle for second-order elliptic operators, it is enough to show that Δu does not change sign in Ω. By contradiction, suppose that Δu changes sign in Ω, and let ω be the solution of

$$\begin{cases} -\Delta\omega = |\Delta u| & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

By the Strong Maximum Principle, $\omega > |u|$ in Ω and we infer that

$$\begin{aligned} c_F \leq \max_{t\geq 0} I_F(t\omega) &= \max_{t\geq 0} \left\{ \int_\Omega \overline{F}_\lambda(t\Delta\omega)dx - \frac{\mu}{s+1}t^{s+1} \int_\Omega |\omega|^{s+1}dx - \frac{t^{q+1}}{q+1}\int_\Omega |\omega|^{q+1}dx \right\} \\ &< \max_{t\geq 0} \left\{ \int_\Omega \overline{F}_\lambda(t\Delta u)dx - \frac{\mu}{s+1}t^{s+1} \int_\Omega |u|^{s+1}dx - \frac{t^{q+1}}{q+1}\int_\Omega |u|^{q+1}dx \right\} = \max_{t\geq 0} I_F(tu) = c_F, \end{aligned}$$

which is a contradiction. Hence Δu does not change sign in Ω, and therefore, up to multiplication by -1, $u > 0$ and $-\Delta u > 0$ in Ω. Finally, by Lemma 2.1.2, with $v = f^{-1}(-\Delta u)$, (u, v) is a positive classical solution of (HS). □

CHAPTER

3

TECHNICAL RESULTS

This chapter is devoted to the proof of some technical results used in this thesis. In Section 3.1 we prove some useful properties for the auxiliary functions associated to the problems (P) and (P'), and in Section 3.2 we prove Proposition 2.2.3 by calculating an upper bound for the mountain pass level c_F of the functional I_F.

3.1 Some technical properties of the auxiliary functions

Ahead in this section, where (1.1.1) is assumed, some properties of the functions f_λ^{-1}, $\overline{F}_\lambda$, g_μ^{-1} and $\overline{G}_\mu$, as defined in (2.1.1), are given. Indeed, we can consider f_λ^{-1} and $\overline{F}_\lambda$ and infer the respective properties for the others. We start by showing some useful inequalities. Observe that

$$t^p < f_\lambda(t) \text{ and } \lambda t^r < f_\lambda(t) \text{ for } t > 0$$

and writing $\tau = f_\lambda(t)$ we get

$$f_\lambda^{-1}(\tau) < \tau^{1/p} \text{ and } f_\lambda^{-1}(\tau) < \frac{\tau^{1/r}}{\lambda^{1/r}} \text{ for } t > 0. \tag{3.1.1}$$

So,

$$\overline{F}_\lambda(\tau) \le \frac{p}{p+1}|\tau|^{\frac{p+1}{p}} \text{ and } \overline{F}_\lambda(\tau) \le \frac{r}{r+1}\frac{1}{\lambda^{1/r}}|\tau|^{\frac{r+1}{r}} \quad \forall \tau \in \mathbb{R}. \tag{3.1.2}$$

The next lemmas are used to obtain the geometric condition and upper bounds for the critical level of the Mountain Pass Theorem for the functionals I_F and I_G.

Lemma 3.1.1. $\overline{F}_\lambda(t) = \frac{p}{p+1} f_\lambda^{-1}(t)t - \frac{p-r}{p+1}\frac{\lambda}{r+1}|f_\lambda^{-1}(t)|^{r+1}$ *for all* $t \in \mathbb{R}$

Proof. Set $M(t) := \frac{p}{p+1} f_\lambda^{-1}(t)t - \frac{p-r}{p+1}\frac{\lambda}{r+1}|f_\lambda^{-1}(t)|^{r+1} - \overline{F}_\lambda(t)$. Then $M(0) = 0$, M is even, and for $t > 0$

$$M'(t) = \frac{1}{p+1}\left(-f_\lambda^{-1}(t) + \frac{pt}{\lambda r[f_\lambda^{-1}(t)]^{r-1} + p[f_\lambda^{-1}(t)]^{p-1}} - \frac{\lambda(p-r)\left[f_\lambda^{-1}(t)\right]^r}{\lambda r[f_\lambda^{-1}(t)]^{r-1} + p[f_\lambda^{-1}(t)]^{p-1}}\right)$$
$$= \frac{1}{p+1}\left(\frac{\lambda(p-r)\left[f_\lambda^{-1}(t)\right]^r}{\lambda r[f_\lambda^{-1}(t)]^{r-1} + p[f_\lambda^{-1}(t)]^{p-1}} - \frac{\lambda(p-r)\left[f_\lambda^{-1}(t)\right]^r}{\lambda r[f_\lambda^{-1}(t)]^{r-1} + p[f_\lambda^{-1}(t)]^{p-1}}\right) = 0,$$

which implies the desired identity. □

Lemma 3.1.2. *$\overline{F}_\lambda(t) = \lambda \frac{r}{r+1}|f_\lambda^{-1}(t)|^{r+1} + \frac{p}{p+1}|f_\lambda^{-1}(t)|^{p+1}$ for all $t \in \mathbb{R}$.*

Proof. The argument follows as in the proof of the last Lemma, observing that for all $t > 0$

$$\frac{d}{dt}\left(\lambda \frac{r}{r+1}|f_\lambda^{-1}(t)|^{r+1} + \frac{p}{p+1}|f_\lambda^{-1}(t)|^{p+1}\right) = \frac{\lambda r\left[f_\lambda^{-1}(t)\right]^r + p\left[f_\lambda^{-1}(t)\right]^p}{\lambda r\left[f_\lambda^{-1}(t)\right]^{r-1} + p\left[f_\lambda^{-1}(t)\right]^{p-1}} = f_\lambda^{-1}(t),$$

which coincides with $\frac{d}{dt}\overline{F}_\lambda(t)$. □

Corollary 3.1.3. *$\overline{F}_\lambda(t) \geq \dfrac{f_\lambda^{-1}(t)t}{s+1}$, for all $t \in \mathbb{R}$.*

Proof. Since $f_\lambda^{-1}(t)t = f_\lambda^{-1}(t)f_\lambda(f_\lambda^{-1}(t)) = \lambda|f_\lambda^{-1}(t)|^{r+1} + |f_\lambda^{-1}(t)|^{p+1}$, from Lemma 3.1.2 and (1.1.1), for all $t \in \mathbb{R}$,

$$\overline{F}_\lambda(t) - \frac{f_\lambda^{-1}(t)t}{s+1} = \lambda\left(\frac{r}{r+1} - \frac{1}{s+1}\right)|f_\lambda^{-1}(t)|^{r+1} + \left(\frac{p}{p+1} - \frac{1}{s+1}\right)|f_\lambda^{-1}(t)|^{p+1} \geq 0. \quad \square$$

Lemma 3.1.4.

$$f_\lambda^{-1}(t) \geq \begin{cases} \left(\dfrac{t}{2\lambda}\right)^{1/r}, & \forall\, 0 < t \leq 2\lambda^{\frac{p}{p-r}}, \\ \left(\dfrac{t}{2}\right)^{1/p}, & \forall\, t \geq 2\lambda^{\frac{p}{p-r}}. \end{cases}$$

Proof. Observe that $2\lambda^{\frac{p}{p-r}} = \lambda(\lambda^{\frac{1}{p-r}})^r + (\lambda^{\frac{1}{p-r}})^p = f_\lambda(\lambda^{\frac{1}{p-r}})$ and write $z = f_\lambda^{-1}(t)$.

If $t \leq 2\lambda^{\frac{p}{p-r}}$, applying f_λ^{-1} to this inequality, one gets $z \leq f_\lambda^{-1}(2\lambda^{\frac{p}{p-r}}) = \lambda^{\frac{1}{p-r}}$, that is, $z^p \leq \lambda z^r$, and so $t = z^p + \lambda z^r \leq 2\lambda z^r$, which implies $\left(\frac{t}{2\lambda}\right)^{1/r} \leq f_\lambda^{-1}(t)$.

If $t \geq 2\lambda^{\frac{p}{p-r}}$, then $z \geq \lambda^{\frac{1}{p-r}}$ and $2z^p \geq \lambda z^r + z^p = t$, which implies $f_\lambda^{-1}(t) \geq (t/2)^{1/p}$, as desired. □

Lemma 3.1.5.

$$\overline{F}_\lambda(t) \geq \begin{cases} \dfrac{r}{r+1}\lambda^{-1/r}\left(\dfrac{t}{2}\right)^{\frac{r+1}{r}} + \dfrac{p}{p+1}\left(\dfrac{t}{2\lambda}\right)^{\frac{p+1}{r}}, & \forall\, |t| \leq 2\lambda^{\frac{p}{p-r}}, \\ \dfrac{r}{r+1}\lambda\left(\dfrac{t}{2}\right)^{\frac{r+1}{p}} + \dfrac{p}{p+1}\left(\dfrac{t}{2}\right)^{\frac{p+1}{p}}, & \forall\, |t| \geq 2\lambda^{\frac{p}{p-r}}. \end{cases}$$

Proof. It is a straightforward consequence of Lemmas 3.1.2 and 3.1.4. $\square$

Lemma 3.1.6. *For* $\tau = \frac{ps-1}{2^{\frac{p+1}{p}}(p+1)(s+1)}$,

$$\overline{F}_\lambda(t) - \frac{f_\lambda^{-1}(t)t}{s+1} \geq \tau|t|^{\frac{p+1}{p}} \quad \forall |t| \geq 2\lambda^{\frac{p}{p-r}}.$$

Proof. By Lemma 3.1.4 and the proof of Corollary 3.1.3, for all $|t| \geq 2\lambda^{\frac{p}{p-r}}$,

$$\overline{F}_\lambda(t) - \frac{f_\lambda^{-1}(t)t}{s+1} = \lambda\left(\frac{r}{r+1} - \frac{1}{s+1}\right)|f_\lambda^{-1}(t)|^{r+1} + \left(\frac{p}{p+1} - \frac{1}{s+1}\right)|f_\lambda^{-1}(t)|^{p+1}$$
$$\geq \frac{ps-1}{(p+1)(s+1)}|f_\lambda^{-1}(t)|^{p+1} \geq \frac{ps-1}{(p+1)(s+1)}\left(\frac{|t|}{2}\right)^{\frac{p+1}{p}}. \quad \square$$

Lemma 3.1.7. *For all* $\alpha,\ \beta \in \mathbb{R}$*, there exists* $\theta \in (0,\ 1)$ *such that*

$$0 \leq f_\lambda^{-1}(\alpha+\beta)(\alpha+\beta) \leq f_\lambda^{-1}(\alpha)\alpha + \frac{r+1}{r}|\beta|\,|f_\lambda^{-1}(\alpha+\theta\beta)|.$$

Proof. Consider the function $m(t) = f_\lambda^{-1}(t)t$. Then, m is even, $m'(0) = 0$ and

$$m'(t) = f_\lambda^{-1}(t) + \frac{t f_\lambda^{-1}(t)}{\lambda r\left[f_\lambda^{-1}(t)\right]^r + p\left[f_\lambda^{-1}(t)\right]^p}, \quad \text{for } t > 0.$$

Hence,

$$0 < m'(t) < f_\lambda^{-1}(t) + \frac{t f_\lambda^{-1}(t)}{\lambda r\left[f_\lambda^{-1}(t)\right]^r + r\left[f_\lambda^{-1}(t)\right]^p} = \frac{r+1}{r} f_\lambda^{-1}(t) \ \ \forall\, t > 0,$$

which implies that $|m'(t)| \leq \frac{r+1}{r}|f_\lambda^{-1}(t)|$ for all $t \in \mathbb{R}$. By the mean value theorem, there exists $\theta \in (0,1)$ such that

$$0 \leq f_\lambda^{-1}(\alpha+\beta)(\alpha+\beta) = f_\lambda^{-1}(\alpha)\alpha + m'(\alpha+\theta\beta)\beta \leq f_\lambda^{-1}(\alpha)\alpha + \frac{r+1}{r}|\beta|\,|f_\lambda^{-1}(\alpha+\theta\beta)|. \ \square$$

Lemma 3.1.8.

$$\lim_{t\to\infty} \frac{t^{\frac{p+1}{p}} - f_\lambda^{-1}(t)t}{t^{\frac{r+1}{p}}} = \frac{\lambda}{p}.$$

In particular, given $0 < c < \frac{\lambda}{p}$*, there exists* $t_0 > 0$ *such that*

$$0 \leq f_\lambda^{-1}(t)t \leq |t|^{\frac{p+1}{p}} - c|t|^{\frac{r+1}{p}}, \quad \forall |t| \geq t_0. \tag{3.1.3}$$

Proof. For $t > 0$, writing $t = f_\lambda(\tau)$,

$$\frac{t^{\frac{p+1}{p}} - f_\lambda^{-1}(t)t}{t^{\frac{r+1}{p}}} = \frac{(\lambda\tau^r + \tau^p)^{\frac{p+1}{p}} - \lambda\tau^{r+1} - \tau^{p+1}}{(\lambda\tau^r+\tau^p)^{\frac{r+1}{p}}} = \frac{\left(\lambda\tau^{r-\frac{p(r+1)}{p+1}} + \tau^{p-\frac{p(r+1)}{p+1}}\right)^{\frac{p+1}{p}} - \lambda - \tau^{p-r}}{\tau^{\frac{(r-p)(r+1)}{p}}(\lambda+\tau^{p-r})^{\frac{r+1}{p}}}$$
$$= \frac{(\lambda+\tau^{p-r})\left[\tau^{\frac{r-p}{p}}(\lambda+\tau^{p-r})^{\frac{1}{p}} - 1\right]}{\tau^{\frac{(r-p)(r+1)}{p}}(\lambda+\tau^{p-r})^{\frac{r+1}{p}}} = \frac{\tau^{\frac{r-p}{p}}(\lambda+\tau^{p-r})^{\frac{1}{p}} - 1}{\tau^{\frac{(r-p)(r+1)}{p}}(\lambda+\tau^{p-r})^{\frac{r+1}{p}-1}}.$$

Then, with $y=\tau^{r-p}$, $y \overset{t\to\infty}{\longrightarrow} 0^+$, and it follows that

$$\lim_{t\to\infty}\frac{t^{\frac{p+1}{p}}-f_\lambda^{-1}(t)t}{t^{\frac{r+1}{p}}}=\lim_{y\to 0^+}\frac{y^{1/p}(\lambda+y^{-1})^{1/p}-1}{y^{\frac{r+1}{p}}(\lambda+y^{-1})^{\frac{r+1}{p}-1}}=\lim_{y\to 0^+}\frac{(\lambda y+1)^{1/p}-1}{y(\lambda y+1)^{\frac{r+1}{p}-1}},$$

and applying the L'Hôpital rule,

$$\lim_{t\to\infty}\frac{t^{\frac{p+1}{p}}-f_\lambda^{-1}(t)t}{t^{\frac{r+1}{p}}}=\lim_{y\to 0^+}\frac{\frac{\lambda}{p}(\lambda y+1)^{1/p-1}}{(\lambda y+1)^{\frac{r+1}{p}-1}+\left(\frac{r+1}{p}-1\right)\lambda y(\lambda y+1)^{\frac{r+1}{p}-2}}=\frac{\lambda}{p}. \qquad \square$$

3.2 Upper bound for the Mountain Pass level

Let (p,q) be on the critical hyperbola (1.0.3) and (φ,ψ) be a positive radial solution of the problem

$$-\Delta\varphi=|\psi|^{p-1}\psi,\;\; -\Delta\psi=|\varphi|^{q-1}\varphi,\;\; \text{in } \mathbb{R}^N, \tag{3.2.1}$$

whose qualitative and quantitative properties are described in (HULSHOF; VORST, 1996).

We recall that (φ,ψ) has de following decay at infinity:

$a)$ $\frac{2}{N-2}<p<\frac{N}{N-2}$, $\quad \lim_{t\to\infty} t^{p(N-2)-2}\varphi(t)=b$ and $\lim_{t\to\infty} t^{N-2}\psi(t)=c$,

$b)$ $p=\frac{N}{N-2}$, $\quad \lim_{t\to\infty}\frac{t^{N-2}}{\log t}\varphi(t)=b$ and $\lim_{t\to\infty} t^{N-2}\psi(t)=c$,

$c)$ $\frac{N}{N-2}<p<\frac{N^2+2N-4}{N^2-4N+4}$, $\quad \lim_{t\to\infty} t^{N-2}\varphi(t)=b$ and $\lim_{t\to\infty} t^{N-2}\psi(t)=c$,

$d)$ $p=\frac{N^2+2N-4}{N^2-4N+4}$, $\quad \lim_{t\to\infty} t^{N-2}\varphi(t)=b$ and $\lim_{t\to\infty}\frac{t^{N-2}}{\log t}\psi(t)=c$,

$e)$ $\frac{N^2+2N-4}{N^2-4N+4}<p$, $\quad \lim_{t\to\infty} t^{N-2}\varphi(t)=b$ and $\lim_{t\to\infty} t^{q(N-2)-2}\psi(t)=c$,

where $b>0$ and $c>0$ are constants and $t=|x|$. Fix $a\in\Omega$. Let $\xi_a\in C_c^\infty(\mathbb{R}^N)$ be a function such that $0\le\xi_a(x)\le 1$ for all $x\in\mathbb{R}^N$, $\xi_a\equiv 1$ in $B(a,\rho/2)$, $\xi_a\equiv 0$ in $B(a,\rho)^c$ and $B(a,\rho)\subset\subset\Omega$, $\rho>0$.

Lemma 3.2.1. *Suppose (1.1.1) and let $U_{\delta,a}:=\delta^{\frac{-N}{q+1}}\xi_a(x)\varphi\left(\frac{x-a}{\delta}\right)$, where φ is defined by (3.2.1) and $V_{\delta,a}=|U_{\delta,a}|_{q+1}^{-1}U_{\delta,a}$. Then, for everery $t\in[m,\overline{m}]$, with $m>0$ and $m,\overline{m}$ independent of δ, it holds:*

- *if $N=3$ with $p\le 7/2$, or $N\ge 4$ with $p\le\frac{N+2}{N-2}$, then*

$$\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x)t\Delta V_{\delta,a}(x))dx<t^{\frac{p+1}{p}}S,\quad \text{for } \delta>0 \text{ suitably small}, \tag{3.2.2}$$

- *if $N=3$ with $7/2<p<11$, or $N\ge 4$ with $\frac{N+2}{N-2}<p\le\frac{N^2+2N-4}{N^2-4N+4}$, then*

$$\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x)t\Delta V_{\delta,a}(x))dx<\begin{cases} t^{\frac{p+1}{p}}S+c_1t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}}-c_2t^{\frac{r+1}{p}}\delta^{\frac{N}{q+1}\frac{N}{N-2}}, & \text{if } r<\frac{2}{N-2},\\ t^{\frac{p+1}{p}}S+c_1t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}}-c_2\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|, & \text{if } r=\frac{2}{N-2},\\ t^{\frac{p+1}{p}}S+c_1t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}}-c_2t^{\frac{r+1}{p}}\delta^{\frac{N(p-r)}{p+1}}, & \text{if } r>\frac{2}{N-2},\end{cases} \tag{3.2.3}$$

• *if* $\frac{N^2+2N-4}{N^2-4N+4} < p$, *then*

$$\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x)t\Delta V_{\delta,a}(x))dx < \begin{cases} t^{\frac{p+1}{p}}S + c_1 t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}} - c_2\lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)|, & \textit{if } r+1 = \frac{p+1}{q+1}, \\ t^{\frac{p+1}{p}}S + c_1 t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}} - c_2\lambda\delta^{\frac{Nq}{q+1}}, & \textit{if } r+1 < \frac{p+1}{q+1}, \\ t^{\frac{p+1}{p}}S + c_1 t^{\frac{r+1}{r}}\delta^{\frac{N(r+1)}{r(p+1)}} - c_2\lambda\delta^{\frac{N(p-r)}{p+1}}, & \textit{if } r+1 > \frac{p+1}{q+1}. \end{cases} \tag{3.2.4}$$

Proof. First, observe that

$$\Delta V_\delta = \gamma_{\delta,a}(x) + \sigma_{\delta,a}(x),$$

where

$$\gamma_{\delta,a}(x) := |U_\delta|_{q+1}^{-1}\delta^{\frac{-N}{q+1}}\delta^{-2}\Delta\varphi\left(\frac{x-a}{\delta}\right)$$

and

$$\sigma_{\delta,a}(x) := |U_\delta|_{q+1}^{-1}\delta^{\frac{-N}{q+1}}\left(2\delta^{-1}\nabla\xi_a(x)\nabla\varphi\left(\frac{x-a}{\delta}\right) + \varphi\left(\frac{x-a}{\delta}\right)\Delta\xi_a(x)\right).$$

So,

$$\begin{aligned}\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx = &\\ \int_{B(a,\rho/2)} f_\lambda^{-1}(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))dx &\\ + \int_{\Omega\setminus B(a,\rho/2)} f_\lambda^{-1}(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))dx,&\end{aligned}$$

and since $supp\left(\Delta V_{\delta,a}(x)\right) \subset \overline{B}(a,\rho)$ and $supp(t\sigma_{\delta,a}(x)) \subset \overline{R(a,\rho/2,\rho)}$, where $R(a,\rho/2,\rho) := B(a,\rho)\setminus\overline{B}(a,\rho/2)$, one has

$$\begin{aligned}\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx &= \int_{B(a,\rho/2)} f_\lambda^{-1}(t\gamma_{\delta,a}(x))t\gamma_{\delta,a}(x)dx \\ &+ \int_{R(a,\rho/2,\rho)} f_\lambda^{-1}(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))dx.\end{aligned}$$

Now we split the estimate in two steps, which correspond to the principal part

$$h_{\delta,a} := \int_{B(a,\rho/2)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right)t\gamma_{\delta,a}(x), \tag{3.2.5}$$

and to the residual part

$$j_{\delta,a} := \int_{R(a,\rho/2,\rho)} f_\lambda^{-1}(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))(t\xi_a(x)\gamma_{\delta,a}(x)+t\sigma_{\delta,a}(x))dx. \tag{3.2.6}$$

Step 1. Estimate of (3.2.5)

Using (3.1.1) and the asymptotic behavior of $\Delta\varphi$ as in (HULSHOF; VORST, 1996, Theorem 2) and (SANTOS, 2010, Lemma 6.2), one gets

$$
\begin{aligned}
h_{\delta,a} &= \int_{\mathbb{R}^N} \left|t\gamma_{\delta,a}(x)\right|^{\frac{p+1}{p}} dx + \int_{B(a,\rho/2)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x)dx - \int_{\mathbb{R}^N} \left|t\gamma_{\delta,a}(x)\right|^{\frac{p+1}{p}} dx \\
&= t^{\frac{p+1}{p}} S + \int_{B(a,\rho/2)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x) - |t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}} dx - \int_{\mathbb{R}^N\setminus B(a,\rho/2)} \left|t\gamma_{\delta,a}(x)\right|^{\frac{p+1}{p}} dx.
\end{aligned}
\tag{3.2.7}
$$

The behavior of the last term is already known by (SANTOS, 2010), namely

$$
-\int_{\mathbb{R}^N\setminus B(a,\rho/2)} \left|t\gamma_{\delta,a}(x)\right|^{\frac{p+1}{p}} dx \leq \begin{cases} -t^{\frac{p+1}{p}} C\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q > \frac{N}{N-2}, \\ -t^{\frac{p+1}{p}} C|\log(\delta)|^{p+1}\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q = \frac{N}{N-2}, \\ -t^{\frac{p+1}{p}} C\delta^{qN}, & \text{if } q < \frac{N}{N-2}. \end{cases}
\tag{3.2.8}
$$

Next, we estimate

$$
i_{\delta,a} := \int_{B(a,\rho/2)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x) - |t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}} dx.
\tag{3.2.9}
$$

We consider three parts of the ball $B(a,\rho/2)$, namely the ball $B(a,\delta)$ and the rings $R(a,\delta,\delta^M)$ and $R(a,\delta^M,\rho/2)$, where the number $M<1$ will be defined ahead. This splitting involving rings is key argument to capture the contribution of the term $\lambda|u|^{r-1}r$ to downsize the Mountain Pass level.

Step 1.1. By the behavior of $\Delta\varphi(x)$ it is known that there exists (for δ sufficiently small) $c>0$ such that if $|x-a|<\delta$ then $c<t|\Delta\varphi(\frac{x-a}{\delta})|$, so (3.1.3) and $\frac{N+2(q+1)}{q+1}=\frac{pN}{p+1}$, can be used to infer that

$$
\begin{aligned}
\int_{B(a,\delta)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x) - |t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}} dx &\leq -c\lambda \int_{B(a,\delta)} |t\gamma_{\delta,a}(x)|^{\frac{r+1}{p}} dx \\
&\leq -c'\lambda \int_{B(a,\delta)} \delta^{-N\frac{r+1}{p+1}} dx = -C\lambda\delta^{N\frac{p-r}{p+1}}.
\end{aligned}
$$

Step 1.2. Now focus the attention on the $R(a,\delta,\delta^M)$-term. In this ring, since $1<\frac{|x-a|}{\delta}<\delta^{M-1}$, with $M<1$ to be defined, the asymptotic decay of $\gamma_{\delta,a}(x)$ present in (SANTOS, 2010, Lemma 6.2) can be used, and three cases have to be analized.

Case 1: $q>\frac{N}{N-2}$.
In this case it is known that

$$
\delta^{\frac{-pN}{p+1}}\left|\Delta\varphi\left(\frac{x-a}{\delta}\right)\right| \geq c|x-a|^{-p(N-2)}\delta^{p(N-2)-\frac{pN}{p+1}},
$$

the last term is grater than a constant if $|x-a|\leq\delta^M$ with

$$
(1-M)p(N-2)-\frac{pN}{p+1}=0 \iff M=\frac{p(N-2)-2}{(N-2)(p+1)}=\frac{N}{N-2}\frac{1}{q+1},
$$

observing that $0<M<1$, and $\delta^M<\rho/2$ since $\delta\to 0$. Applying Lemma 3.1.8, it follows that

$$
\begin{aligned}
\int_{R(a,\delta,\delta^M)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x) - |t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}} dx &\leq -c\lambda \int_{R(a,\delta,\delta^M)} \left|t\gamma_{\delta,a}(x)\right|^{\frac{r+1}{p}} dx \\
&\leq -c\lambda \int_{R(a,\delta,\delta^M)} |x-a|^{-(r+1)(N-2)}\delta^{N\frac{r+1}{q+1}} dx = -c\lambda\delta^{N\frac{r+1}{q+1}} \int_\delta^{\delta^M} y^{1-r(N-2)}dy,
\end{aligned}
$$

and hence

$$\int_{R(a,\delta,\delta^M)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right)t\gamma_{\delta,a}(x)-|t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}}dx \le -\begin{cases} c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|, & \text{if } r=\frac{2}{N-2},\\ c\lambda\dfrac{\delta^{\frac{N}{q+1}\frac{N}{N-2}}-\delta^{N\frac{p-r}{p+1}}}{2-r(N-2)}, & \text{if } r\neq\frac{2}{N-2}.\end{cases}$$

Case 2: $q=\frac{N}{N-2}$.
In this case it is known that

$$\delta^{\frac{-pN}{p+1}}\left|\Delta\varphi\left(\frac{x-a}{\delta}\right)\right|\ge c\left(\log\left(\frac{|x-a|}{\delta}\right)+1\right)^p|x-a|^{-p(N-2)}\delta^{p(N-2)-\frac{pN}{p+1}},$$

so one can use $M=\frac{N}{N-2}\frac{1}{q+1}$ and proceed as in Case 1, to obtain the same estimate, which could be even better.

Case 3: $q<\frac{N}{N-2}$.
In this case it is known that

$$\delta^{\frac{-pN}{p+1}}\left|\Delta\varphi\left(\frac{x-a}{\delta}\right)\right|\ge c|x-a|^{-\frac{p(q+1)N}{p+1}}\delta^{\frac{pqN}{p+1}},$$

and this is grater than a constant if $|x-a|\le\delta^M$ with

$$\frac{Mp(q+1)N}{p+1}=\frac{pqN}{p+1}\iff M=\frac{q}{q+1}.$$

So Lemma 3.1.8 can be applied, leading that

$$\begin{aligned}\int_{R(a,\delta,\delta^M)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right)t\gamma_{\delta,a}(x)-|t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}}dx &\le -c\lambda\int_{R(a,\delta,\delta^M)}\left|t\gamma_{\delta,a}(x)\right|^{\frac{r+1}{p}}dx\\ \le -c\lambda\int_{R(a,\delta,\delta^M)}|x-a|^{-\frac{(r+1)(q+1)}{p+1}N}\delta^{\frac{r+1}{p+1}qN}dx &= -c\lambda\delta^{\frac{r+1}{p+1}qN}\int_\delta^{\delta^M}y^{N-1-\frac{(q+1)(r+1)}{p+1}N}dy\end{aligned}$$

and so

$$\int_{R(a,\delta,\delta^M)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right)t\gamma_{\delta,a}(x)-|t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}}dx \le -\begin{cases} c\lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)|, & \text{if } r+1=\frac{p+1}{q+1},\\ c\lambda\dfrac{\delta^{\frac{Nq}{q+1}}-\delta^{N\frac{p-r}{p+1}}}{N-N\frac{(q+1)(r+1)}{p+1}}, & \text{if } r+1\neq\frac{p+1}{q+1}.\end{cases}$$

Step 1.3. Finally we estimate the $R(a,\delta^M,\rho/2)$-term. In this ring, $\left|\frac{x-a}{\delta}\right|>1$, and the asymptotic behavior of $\gamma_{\delta,a}(x)$ present in (SANTOS, 2010, Lemma 6.2) can be used one more time, but in this case $\gamma_{\delta,a}$ becomes small, and then $f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right)t\gamma_{\delta,a}(x)-|t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}}\le -c_\lambda|t\gamma_{\delta,a}(x)|^{\frac{p+1}{p}}$, and one can proceed as in (3.2.8).

At this point, from Steps 1.1, 1.2, and 1.3, we can write the estimates for $i_{\delta,a}$ defined in (3.2.9). But before doing this, note that in all the three cases of Step 1.2, the term $\delta^{\frac{N(p-r)}{p+1}}$ (dominant term in Step 1.1) appear, so it does not need to be repeated.

$$i_{\delta,a} \leq \begin{cases} -c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)| - C\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q > \frac{N}{N-2}, r = \frac{2}{N-2}, \\ -c\lambda\dfrac{\delta^{\frac{N}{q+1}\frac{N}{N-2}} - \delta^{N\frac{p-r}{p+1}}}{2-r(N-2)} - C\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q > \frac{N}{N-2}, r \neq \frac{2}{N-2}, \\ -c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)| - C|\log(\delta)|^{p+1}\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q = \frac{N}{N-2}, r = \frac{2}{N-2}, \\ -c\lambda\frac{\delta^{\frac{N}{q+1}\frac{N}{N-2}} - \delta^{N\frac{p-r}{p+1}}}{2-r(N-2)} - C|\log(\delta)|^{p+1}\delta^{\frac{N(p+1)}{q+1}}, & \text{if } q = \frac{N}{N-2}, r \neq \frac{2}{N-2}, \\ -c\lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)| - C\delta^{qN}, & \text{if } q < \frac{N}{N-2}, r+1 = \frac{p+1}{q+1}, \\ -c\lambda\dfrac{\delta^{\frac{Nq}{q+1}} - \delta^{N\frac{p-r}{p+1}}}{N - N\frac{(q+1)(r+1)}{p+1}} - C\delta^{qN}, & \text{if } q < \frac{N}{N-2}, r+1 \neq \frac{p+1}{q+1}. \end{cases} \tag{3.2.10}$$

Now we summarize all the calculation made in Step 1. To estimate $h_{\delta,a}$, from (3.2.7), we must deal with (3.2.8) and (3.2.10). Observe that the majorante in (3.2.8) also appears in the second terms in (3.2.10). Then, the estimate for (3.2.7) follows from the comparison among the powers of δ in (3.2.10).

For $q > \frac{N}{N-2}$, the terms to be compared are

$$-C\delta^{\frac{N(p+1)}{q+1}} \text{ and } - \begin{cases} c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|, & \text{if } r = \frac{2}{N-2}, \\ c\lambda\dfrac{\delta^{\frac{N}{q+1}\frac{N}{N-2}} - \delta^{N\frac{p-r}{p+1}}}{2-r(N-2)}, & \text{if } r \neq \frac{2}{N-2}, \end{cases} \tag{3.2.11}$$

and the first part is always weaker. Of course $N\frac{p-r}{p+1} > \frac{N}{q+1}\frac{N}{N-2}$ if $r < \frac{2}{N-2}$, and $\frac{N(p+1)}{q+1} > \frac{N}{q+1}\frac{N}{N-2}$ (since $p > \frac{2}{N-2}$ always), so in this case the dominant term is $-c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}$. The same analysis shows that if $r = \frac{2}{N-2}$ the dominant term is $c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|$. If $r > \frac{2}{N-2}$, then $N\frac{p-r}{p+1} < \frac{N}{q+1}\frac{N}{N-2}$ and the dominant term is $-c\delta^{N\frac{p-r}{p+1}}$.

When $q = \frac{N}{N-2}$ the analysis done before gives that the dominant term is

$$- \begin{cases} c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|, & \text{if } r = \frac{2}{N-2}, \\ c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}, & \text{if } r < \frac{2}{N-2}, \\ -t^{\frac{p+1}{p}}C|\log(\delta)|^{p+1}\delta^{\frac{N(p+1)}{q+1}} - C\lambda\delta^{N\frac{p-r}{p+1}} & \text{if } r > \frac{2}{N-2}. \end{cases}$$

Finaly, if $q < \frac{N}{N-2}$ the terms that we have to compare are

$$-t^{\frac{p+1}{p}}C\delta^{qN} \text{ and } - \begin{cases} c\lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)|, & \text{if } r+1 = \frac{p+1}{q+1}, \\ c\lambda\dfrac{\delta^{\frac{Nq}{q+1}} - \delta^{N\frac{p-r}{p+1}}}{N - N\frac{(q+1)(r+1)}{p+1}}, & \text{if } r+1 \neq \frac{p+1}{q+1}. \end{cases}$$

If $r+1 < \frac{p+1}{q+1}$, it is easy to see that $\frac{Nq}{q+1} < N\frac{p-r}{p+1}$ and surely $\frac{Nq}{q+1} < Nq$, so $-c\lambda\delta^{\frac{Nq}{q+1}}$ is the dominant term. The same computation ensures that the dominant term is $-c\lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)|$, if

$r+1=\frac{p+1}{q+1}$. Finally, if $r+1>\frac{p+1}{q+1}$ an analogous computation show that the term $-C\lambda\delta^{N\frac{p-r}{p+1}}$ is the dominant. Then, putting it all together,

$$h_{\delta,a}\leq t^{\frac{p+1}{p}}S-\begin{cases} c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}|\log(\delta)|, & \text{if } q\geq\frac{N}{N-2}, r=\frac{2}{N-2},\\ c\lambda\delta^{\frac{N}{q+1}\frac{N}{N-2}}, & \text{if } q\geq\frac{N}{N-2}, r<\frac{2}{N-2},\\ c\lambda\delta^{\frac{N(p-r)}{p+1}}, & \text{if } q\geq\frac{N}{N-2}, r>\frac{2}{N-2},\\ \lambda\delta^{\frac{Nq}{q+1}}|\log(\delta)|, & \text{if } q<\frac{N}{N-2}, r+1=\frac{p+1}{q+1},\\ c\lambda\delta^{\frac{Nq}{q+1}}, & \text{if } q<\frac{N}{N-2}, r+1<\frac{p+1}{q+1},\\ c\lambda\delta^{\frac{N(p-r)}{p+1}}, & \text{if } q<\frac{N}{N-2}, r+1>\frac{p+1}{q+1}.\end{cases}\tag{3.2.12}$$

Step 2 Estimate of the residual part (3.2.6).

Here $\left|\frac{x-a}{\delta}\right|\geq\frac{\rho}{2\delta}\to\infty$, uniformly with respect to $x\in R(a,\rho/2,\rho)$, as $\delta\to 0$. Then the asymptotic behavior of $\gamma_{\delta,a}$ and $\sigma_{\delta,a}$ present in (SANTOS, 2010, Lemma 6.2) reads

$$\sigma_{\delta,a}(x)\leq\begin{cases} c|U_\delta|_{q+1}^{-1}\delta^{\frac{N}{p+1}}(|x-a|^{-N+1}+|x-a|^{-N+2}), & \text{if } p>\frac{N}{N-2},\\ c|U_\delta|_{q+1}^{-1}\delta^{\frac{N}{p+1}}|\log(\delta)|(|\log|x-a|+1|)(|x-a|^{-N+1}(1+|x-a|)), & \text{if } p=\frac{N}{N-2},\\ c|U_\delta|_{q+1}^{-1}\delta^{\frac{pN}{q+1}}(|x-a|^{-p(N-2)+2}+|x-a|^{-p(N-2)+1}), & \text{if } p<\frac{N}{N-2},\end{cases}$$

and,

$$\gamma_{\delta,a}\leq\begin{cases} c|x-a|^{-p(N-2)}\delta^{\frac{pN}{q+1}}, & \text{if } q>\frac{N}{N-2}\\ c(|\log(|x-a|)|^{p+1}+1)^{\frac{p}{p+1}}|x-a|^{-p(N-2)}|\log\delta|^p\delta^{\frac{pN}{q+1}}, & \text{if } q=\frac{N}{N-2}\\ c|x-a|^{\frac{p(q+1)N}{p+1}}\delta^{\frac{pqN}{p+1}}, & \text{if } q<\frac{N}{N-2}.\end{cases}$$

From this, it follows that $|j_{\delta,a}|$ is bounded from above by

$$|R(a,\rho/2,\rho)|\begin{cases} f_\lambda^{-1}\left(ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{pN}{q+1}}\right)\right)ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{pN}{q+1}}\right), & \text{if } p<\frac{N}{N-2},\\ f_\lambda^{-1}\left(ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}|\log(\delta)|\right)\right)ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}|\log(\delta)|\right), & \text{if } p=\frac{N}{N-2},\\ f_\lambda^{-1}\left(ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}\right)\right)ct\left(\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}\right), & \text{if } p,q>\frac{N}{N-2},\\ f_\lambda^{-1}\left(ct\left(|\log\delta|^p\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}\right)\right)ct\left(|\log\delta|^p\delta^{\frac{pN}{q+1}}+\delta^{\frac{N}{p+1}}\right), & \text{if } q=\frac{N}{N-2},\\ f_\lambda^{-1}\left(ct\left(\delta^{\frac{pqN}{p+1}}+\delta^{\frac{N}{p+1}}\right)\right)ct\left(\delta^{\frac{pqN}{p+1}}+\delta^{\frac{N}{p+1}}\right), & \text{if } q<\frac{N}{N-2},\end{cases}$$

$$\leq|R(a,\rho/2,\rho)|\begin{cases} f_\lambda^{-1}\left(tc\delta^{\frac{pN}{q+1}}\right)tc\delta^{\frac{pN}{q+1}}, & \text{if } p<\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}|\log(\delta)|\right)tc\delta^{\frac{N}{p+1}}|\log(\delta)|, & \text{if } p=\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}\right)tc\delta^{\frac{N}{p+1}}, & \text{if } p,q>\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}\right)tc\delta^{\frac{N}{p+1}}, & \text{if } q=\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}\right)tc\delta^{\frac{N}{p+1}}, & \text{if } q<\frac{N}{N-2},\end{cases}$$

$$=|R(a,\rho/2,\rho)|\begin{cases} f_\lambda^{-1}\left(tc\delta^{\frac{pN}{q+1}}\right)tc\delta^{\frac{pN}{q+1}}, & \text{if } p<\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}|\log(\delta)|\right)tc\delta^{\frac{N}{p+1}}|\log(\delta)|, & \text{if } p=\frac{N}{N-2},\\ f_\lambda^{-1}\left(tc\delta^{\frac{N}{p+1}}\right)tc\delta^{\frac{N}{p+1}}, & \text{if } p>\frac{N}{N-2},\end{cases}$$

and using the asymptotic behavior of f_λ^{-1}, one concludes that

$$|j_{\delta,a}| \leq \begin{cases} c_\lambda \left(t\delta^{\frac{pN}{q+1}}\right)^{\frac{r+1}{r}}, & \text{if } p < \frac{N}{N-2}, \\ c_\lambda \left(t\delta^{\frac{N}{p+1}}|\log(\delta)|\right)^{\frac{r+1}{r}}, & \text{if } p = \frac{N}{N-2}, \\ c_\lambda \left(t\delta^{\frac{N}{p+1}}\right)^{\frac{r+1}{r}}, & \text{if } p > \frac{N}{N-2}. \end{cases} \tag{3.2.13}$$

Step 3: Comparison of the residual parts in (3.2.12) and (3.2.13).

Step 3.1: $p < \frac{N}{N-2}$. First, observe that this implies $q > \frac{N}{N-2}$. To obtain (3.2.2), from the comparison (3.2.11) to obtain (3.2.12), it is enough to verify that $\frac{N(p+1)}{q+1} < \frac{pN}{q+1}\frac{r+1}{r}$, which is equivalent to $p+1 < p\frac{r+1}{r}$, that is $r < p$, which is always the case. Hence, the Lemma is proved in this case.

Step 3.2: $p = \frac{N}{N-2}$. Again this implies $q > \frac{N}{N-2}$, and the procedure to obtain (3.2.2) is identical to that from Step 3.1.

Step 3.3: $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$.

If $r < \frac{2}{N-2}$, to obtain (3.2.2), from (3.2.12) and (3.2.13), one must decide when $\frac{N^2}{(q+1)(N-2)} < \frac{N}{p+1}\frac{r+1}{r}$, that is

$$\frac{N(p+1)}{(q+1)(N-2)} < \frac{r+1}{r} \iff \frac{p(N-2)-2}{N-2} - 1 < \frac{1}{r} \iff r < \frac{1}{p - \frac{N}{N-2}},$$

and this is always true for $N \geq 4$. If $N = 3$, these conditions read

$$3 < p \leq 5, \quad r < 2, \quad \text{and} \quad r < \frac{1}{p-3}$$

which are satisfied with the extra condition $p \leq 7/2$.

If $r = \frac{2}{N-2}$, then $\frac{r+1}{r} = \frac{N}{2}$, to obtain (3.2.2), from (3.2.12) and (3.2.13), one must decide when

$$\frac{p+1}{q+1} \leq \frac{N-2}{2} \iff \frac{p(N-2)-2}{N} \leq \frac{N-2}{2} \iff p \leq \frac{N}{2} + \frac{2}{N-2},$$

and (remember $p \leq \frac{N+2}{N-2}$) this is always true for $N \geq 4$, and with $N = 3$ these conditions read $p \leq 7/2$.

If $r > \frac{2}{N-2}$, to obtain (3.2.2), from (3.2.12) and (3.2.13), one must decide when $\frac{N(p-r)}{p+1} < \frac{N}{p+1}\frac{r+1}{r}$, that is

$$p - r < \frac{r+1}{r} \iff 0 < r^2 + (1-p)r + 1. \tag{3.2.14}$$

Remember that $p \leq \frac{N+2}{N-2}$. Then, for $N > 4$, (3.2.14) is true because such second order polynomial has no real roots. For $N = 4$ and $p < \frac{N+2}{N-2}$, again (3.2.14) has no real root and (3.2.14) is verified. For $N = 4$ and $p = \frac{N+2}{N-2}$, such polynomial has 1 as real root and $r > 1 = 2/(N-2)$, hence (3.2.14) is verified. With $N = 3$, the largest real root of such polynomial is less or equal to 2 for

$p \le 7/2$, hence (3.2.14) is verified because $r > 2 = 2/(N-2)$. This finishes the proof of the lemma. $\square$

Remark 3.2.2. *The estimate of $\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx$ from Lemma 3.2.1 deserves some comments.*

When evaluating $\int_\Omega f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx$, the leading term comes from $\int_{B(0,\rho/2)} f_\lambda^{-1}\left(t\gamma_{\delta,a}(x)\right) t\gamma_{\delta,a}(x)dx$, which carries by itself a (negative) remainder that has to be compared with the residual term $\int_{R(0,\rho/2,\rho)} f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx$. With $N \ge 4$ or $N = 3$ and $p \le 7/2$ the remainder $\int_{R(0,\rho/2,\rho)} f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx$ is smaller than the negative part of the remainder term in $\int_{B(0,\rho/2)} f_\lambda^{-1}(t\Delta V_{\delta,a}(x))t\Delta V_{\delta,a}(x)dx$, which brings down the functional when comparing it to the problem without the perturbation λu^r, and this plays an important role in the results in this work. At this point, it is important to compare the estimates (3.2.2), (3.2.3) and (3.2.4) with (SANTOS, 2010, Eq. (6.4)).

We are almost prepared for the proof of Proposition 2.2.3. Going on this direction, observe that if (1.0.3) and (1.1.1) are satisfied, then

$$\lim_{t\to\infty} I_F(tV_{\delta,a}) = -\infty$$

and the $\max_{t\ge 0} I(tV_{\delta,a})$ is achieved at some $t_\delta > 0$, thus,

$$0 = I_F'(t_\delta V_{\delta,a}) = \int_\Omega f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})\Delta V_{\delta,a}\,dx - t_\delta^s |V_{\delta,a}|_{s+1}^{s+1} - t_\delta^q, \tag{3.2.15}$$

from where we infer that

$$t_\delta^{q+1} = \int_\Omega f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})t_\delta \Delta V_{\delta,a}\,dx - t_\delta^{s+1} |V_{\delta,a}|_{s+1}^{s+1}. \tag{3.2.16}$$

Lemma 3.2.3. *Suppose (1.0.3) and (1.1.1). Then t_δ, as $\delta \to 0$, is bounded form below and above.*

Proof. Suppose by contradiction that $t_\delta \overset{\delta\to 0}{\longrightarrow} 0$. Define $A_\delta := \{x \in \Omega; |t_\delta \Delta V_{\delta,a}(x)| < 1+\lambda\}$ and $B_\delta = \Omega \backslash A_\delta$, small δ give us

$$\begin{aligned} Ct_\delta^{1/r} \le \frac{t_\delta^{1/p}}{(1+\lambda)^{1/p}} \int_{B_\delta} |\Delta V_{\delta,a}(x)|^{\frac{p+1}{p}}dx + \frac{t_\delta^{1/r}}{(1+\lambda)^{1/r}} \int_{A_\delta} |\Delta V_{\delta,a}(x)|^{\frac{r+1}{r}}dx \\ \le \int_\Omega f_\lambda^{-1}(t_\delta \Delta V_{\delta,a}(x))\Delta V_{\delta,a}(x)dx = t_\delta^s o(\delta) + t_\delta^q O(1). \end{aligned}$$

which is a contradiction by the fact that $1/r \le s$ and $1/r < q$, so $t_\delta \nrightarrow 0$. Observe that in the case $rs = 1$ the $o(\delta)$ produces the contradiction.

Now observe that by (3.2.15) and the estimates present in (MELO; SANTOS, 2015)

$$t_\delta^q \le t_\delta^{\frac{1}{p}} S + t_\delta^{\frac{1}{p}} o(\delta) \Longrightarrow t_\delta^{\frac{pq-1}{p}} \le S + o(\delta)$$

that is, $t_\delta \le k < \infty$ for all δ sufficiently small. $\square$

Proof of Proposition 2.2.3. Now, using Lemma 3.1.1 and identity (3.2.16)

$$\begin{aligned}
\max_{t\geq 0} I_F(tV_{\delta,a}) = I_F(t_\delta V_{\delta,a}) &= \int_\Omega \overline{F}_\lambda(t_\delta \Delta V_{\delta,a})dx - \frac{t_\delta^{q+1}}{q+1} - \frac{\mu}{s+1}t_\delta^{s+1}|V_{\delta,a}|_{s+1}^{s+1} \\
&= \frac{p}{p+1}\int_\Omega f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})t_\delta \Delta V_{\delta,a}dx - \lambda\frac{p-r}{p+1}\frac{\left|f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})\right|_{r+1}^{r+1}}{r+1} - \frac{t_\delta^{q+1}}{q+1} - \frac{\mu t_\delta^{s+1}|V_{\delta,a}|_{s+1}^{s+1}}{s+1} \\
&= \frac{2}{N}\int_\Omega f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})t_\delta \Delta V_{\delta,a}dx - \lambda\frac{p-r}{p+1}\frac{\left|f_\lambda^{-1}(t_\delta \Delta V_{\delta,a})\right|_{r+1}^{r+1}}{r+1} - \frac{\mu(q-s)t_\delta^{s+1}|V_{\delta,a}|_{s+1}^{s+1}}{(q+1)(s+1)}.
\end{aligned} \tag{3.2.17}$$

By (3.2.15) and Lemma 3.2.1 one gets

$$t_\delta^q \leq t_\delta^{1/p}S - Ct_\delta^{1/p}\delta^{\frac{N(p+1)}{q+1}} - t_\delta^s\mu|V_{\delta,a}|_{s+1}^{s+1} \quad \Rightarrow \quad t_\delta < S^{\frac{p}{pq-1}}.$$

Combining this with (3.2.17) and Lemma 3.2.1, we infer that

$$\max_{t\geq 0} I_F(tV_{\delta,a}) < \frac{2}{N}t_\delta^{\frac{p+1}{p}}S - \frac{\mu(q-s)}{(q+1)(s+1)}t_\delta^{s+1}|V_{\delta,a}|_{s+1}^{s+1} < \frac{2}{N}S^{\frac{pN}{2(p+1)}},$$

which concludes the proof. □

Remark 3.2.4. *For $N=3$, we mention that the estimates from* (3.2.3) *and* (3.2.4) *can be used to prove the existence of a positive solution to* (HS) *for the pairs (p,q) on the critical hyperbola* (1.0.3) *that are not included in Theorem 1.1.1, namely with $7/2<p<8$, and for some (not all) (r,s) as in* (1.1.1)*. This remark is linked to the condition $3<t<5$ in (BRÉZIS; NIRENBERG, 1983, Corollary 2.3) to prove the existence of a solution to* (1.0.1)*.*

By (3.2.15) and Lemma 3.2.1 one gets

$$t_\delta^q \leq t_\delta^{1/p}S + c_1 t_\delta^{\frac{r+1}{r}}\delta^{\frac{3(r+1)}{r(p+1)}} + i_{\delta,a} - t_\delta^s\mu|V_{\delta,a}|_{s+1}^{s+1}.$$

Combining this with (3.2.17) and Lemma 3.2.1, we infer that

$$\max_{t\geq 0} I_F(tV_{\delta,a}) \leq \frac{2}{3}t_\delta^{\frac{p+1}{p}}S + c_1 t_\delta^{\frac{r+1}{r}}\delta^{\frac{3(r+1)}{r(p+1)}} + i_{\delta,a} - \frac{\mu(q-s)}{(q+1)(s+1)}t_\delta^{s+1}|V_{\delta,a}|_{s+1}^{s+1},$$

and this is smaller than $\frac{2}{3}S^{\frac{3p}{2(p+1)}}$ if, and only if,

$$C\delta^{\frac{3(r+1)}{r(p+1)}} \leq c_1\mu|V_{\delta,a}|_{s+1}^{s+1} - i_{\delta,a} = c_1\mu|V_{\delta,a}|_{s+1}^{s+1} + |i_{\delta,a}|. \tag{3.2.18}$$

To get (3.2.18) it is sufficient to verify, as $\delta\to 0$, that

$$c\delta^{\frac{3(r+1)}{r(p+1)}} < |i_{\delta,a}|, \tag{3.2.19}$$

or

$$c\delta^{\frac{3(r+1)}{r(p+1)}} < c_1\mu|V_{\delta,a}|_{s+1}^{s+1} = \begin{cases} C\mu\delta^{\frac{3(s+1)}{p+1}}, & \text{if } s<2, \\ C\mu\delta^{\frac{9}{p+1}}|\log\delta|, & \text{if } s=2, \\ C\mu\delta^{3-\frac{3(s+1)}{q+1}}, & \text{if } s>2, \end{cases} \tag{3.2.20}$$

where the behavior of $|V_{\delta,a}|_{s+1}$ in (3.2.20) is given in (MELO; SANTOS, 2015, eq. (36)).

To obatin (3.2.19), we keep all the calculation from Step 3.3 for the case with $7/2 < p \leq (N+2)/(N-2) = 5$. Then, we execute similar estimates for $5 < p < 8$. Therefore, for $7/2 < p < 8$, using the residual terms in the first three lines (that is $q \geq \frac{N}{N-2} = 3$) of (3.2.12), which is a refinement of (3.2.10), one can see that (3.2.19) holds if, and only if,

$$\begin{array}{rl} r < \frac{1}{p-3} & \text{if } r < 2, \\ p \leq \frac{7}{2} & \text{if } r = 2, \\ 0 < r^2 + (1-p)r + 1 & \text{if } r > 2, \end{array} \tag{3.2.21}$$

otherwise, the term of (3.2.13) is dominant.

Let us now consider (3.2.20). For $s < 2$ the inequality is equivalent to $\frac{r+1}{r} > s+1$, that is $rs < 1$, which is a contradiction with condition (1.1.1). For $s = 2$ the inequality is equivalent to $\frac{r+1}{r} \geq 3$, that is $r \leq 1/2$, and this together with (1.1.1), gives $r = 1/2$. Finally, for $s > 2$ the inequality is true if

$$\frac{3(r+1)}{r(p+1)} > 3 - \frac{3(s+1)}{q+1} \quad \text{that is,} \quad s+1 > q+1 - \frac{r+1}{r}\frac{q+1}{p+1} = \frac{q+1}{p+1}(p - \frac{1}{r}). \tag{3.2.22}$$

Therefore, given any (p,q) on the critical hyperbola (1.0.3) with $N = 3$, $7/2 < p < 8$, (r,s) as in (1.1.1) with the one of the extra conditions (3.2.21), $(r,s) = (1/2,2)$ or (3.2.22), the mountain pass level of I_F is in the range of compactness and the mountain pass theorem ensures the existence of a solution.

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APPENDIX

A

CLASSICAL RESULTS

A.1 The Mountain Pass Theorem

The primary tool in this work is the Mountain pass theorem due to (AMBROSETTI; RABINOWITZ, 1973). This result says that if a functional defined in a Banach space satisfies the $(PS)_c$-condition and a geometric property, it has a critical point. In order to state this theorem, let us recall the definition of a Palais-Smale sequence.

Definition A.1.1. *Let X be a Banach space, $I : X \to \mathbb{R}$ a $\mathscr{C}^1$ functional, and c a real number. A sequence (u_k) in X is called a Palais-Smale sequence at a level c if*

$$I(u_k) \to c, \text{ and } I'(u_k) \to 0 \text{ in } X^*.$$

The functional I satisfies the $(PS)c$-condition if any Palais-Smale sequence at level c has a convergent subsequence.

Theorem A.1.2 (Mountain Pass)**.** *Let X be a Banach space, $I \in \mathscr{C}^1(X, \mathbb{R})$, $v \in X$ and $r > 0$ such that $\|v\| > r$ and*

$$\inf_{\|u\|=r} I(u) > I(0) \geq I(v).$$

If I satisfies the $(PS)_c$-condition with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma := \{\gamma \in \mathscr{C}([0,1], X); \gamma(0) = 0, I(\gamma(1)) < 0\},$$

then c is a critical value of I.

A.2 Some convergence results

The first result in this section is the Dominated Convergence Theorem that relates almost everywhere and L^p convergences.

Theorem A.2.1 (Lebesgue)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set, $1 \leq p < \infty$, and let (u_k) be a sequence in $L^p(\Omega)$ such that*

1. *$u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$*
2. *there exists $v \in L^p(\Omega)$ such that for all k, $|u_k(x)| \leq v(x)$ a.e. in Ω.*

Then $u \in L^p(\Omega)$ and $u_k \to u$ in the $L^p(\Omega)$ norm, namely $\int_\Omega |u_k - u|^p \, dx \to 0$.

The converse of this theorem holds within the following sense.

Theorem A.2.2. *Let $\Omega \subset \mathbb{R}^N$ be open and let $(u_k) \subset L^p(\Omega)$, $p \in [1, +\infty]$, be a sequence such that $u_k \to u$ in $L^p(\Omega)$ as $k \to \infty$. Then there exist a subsequence $(u_{k_j})_j$ and a function $v \in L^p(\Omega)$ such that*

1. *$u_{k_j}(x) \to u(x)$ a.e. in Ω as $j \to \infty$,*
2. *for all j, $\left|u_{k_j}(x)\right| \leq v(x)$ a.e. in Ω.*

In this thesis, we also use that a bounded sequence in a reflexive space has a subsequence that converges weakly.

Theorem A.2.3. *Assume that E is a reflexive Banach space and let (u_n) be a bounded sequence in E. Then there exists a subsequence (u_{n_k}) that converges in the weak topology $\sigma(E, E^\star)$.*

Due to (BRÉZIS; LIEB, 1983), the following result relates the pointwise convergence and the convergence of integrals in a general space.

Theorem A.2.4 (Brezis-Lieb)**.** *Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$. In addition, let j satisfy the following hypothesis:*

For every sufficiently small $\varepsilon > 0$, there exist two continuous, nonnegative functions φ_ε and ψ_ε such that

$$|j(a+b) - j(a)| \leqslant \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b)$$

for all $a, b \in \mathbb{C}$.

Now let $u_n = u + v_n$ be a sequence of measurable functions from Ω to $\mathbb{C}$ such that:

(i) $v_n \to 0$ *a.e.;*

(ii) $j(u) \in L^1$*;*

(iii) $\int \varphi_\varepsilon(v_n(x)) \, d\mu(x) \leqslant C < \infty$*, for some constant C, independent of ε and n;*

(iv) $\int \psi_\varepsilon(u(x))d\mu(x) < \infty$ *for all* $\varepsilon > 0$.

Then, as $n \to \infty$,

$$\int |j(u+v_n) - j(v_n) - j(u)|\, d\mu \to 0.$$

Finally, we present a convergence lemma due to (DAL MASO; MURAT, 1998) used in the proof of Lemma 2.3.8 that helps to prove the $(PS)_c$-condition for the functionals I_F and I_G.

Lemma A.2.5. *Let* X *be a finite dimensional real Hilbert space with norm* $|\cdot|$ *and scalar product* $(\cdot,\cdot)$. *Let* (β_k) *be a sequence of functions from* X *into* X *which converges uniformly on compact subsets of* X *to a continuous function* β. *Assume that the functions* β_k *are monotone and the* β *is strictly monotone, i.e.*

$$(\beta_k(\zeta) - \beta_k(\xi), \zeta - \xi) \geq 0, \quad (\beta(\zeta) - \beta(\xi), \zeta - \xi) > 0$$

for every k *and for every* $\zeta, \xi \in X$ *with* $\zeta \neq \xi$. *Let* (ξ_k) *be a sequence in* X *and let* ξ *be an element of* X *such that*

$$\lim_{k\to\infty} (\beta_k(\xi_k) - \beta_k(\xi), \xi_k - \xi) = 0.$$

Then (ξ_k) *converges to* ξ *in* X.

A.3 Some Inequalities

The following two lemmas are useful and well-known inequalities that we state for completeness.

Theorem A.3.1 (Jensen's inequality)**.** *Assume* $|\Omega| < \infty$. *Let* $j : \mathbb{R} \to (-\infty, +\infty]$ *be a convex lower semi-continuous function,* $j \not\equiv +\infty$. *Let* $f \in L^1(\Omega)$ *be such that* $f(x) \in D(j)$ *a.e. and* $j(f) \in L^1(\Omega)$. *Prove that*

$$j\left(\frac{1}{|\Omega|}\int_\Omega f dx\right) \leq \frac{1}{|\Omega|}\int_\Omega j(f) dx.$$

Theorem A.3.2 (Hölder's inequality)**.** *Assume that* $f \in L^p$ *and* $g \in L^{p'}$ *with* $1 \leq p \leq \infty$. *Then* $fg \in L^1$ *and*

$$\int |fg| dx \leq |f|_p |g|_{p'}.$$

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