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Resolubility of linear Cauchy problems on Fréchet spaces and a delayed Kaldor's model

Alex Pereira da Silva

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Resolubilidade de problemas lineares de Cauchy em espaços de Fréchet e um modelo de Kaldor com retardo

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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"Grande coisa é haver recebido do céu uma partícula da sabedoria,
o dom de achar as relações das coisas,
a faculdade de as comparar e o talento de concluir!"
(Memórias Póstumas de Brás Cubas, Machado de Assis)

ABSTRACT

SILVA, A. P. **Resolubility of linear Cauchy problems on Fréchet spaces and a delayed Kaldor's model**. 2019. 116 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

The long-run aim of this thesis is to solve delay differential equations with infinite delay of the type
$$\frac{d}{dt}u(t) = Au(t) + \int_{-\infty}^{t} u(s)k(t-s)\,ds + f\big(t, u(t)\big),$$
on Fréchet spaces under an extended theory of groups of linear operators; where A is a linear operator, $k(s) \geqslant 0$ satisfies $\int_0^\infty k(s)\,ds = 1$ and f is a nonlinear map. In order to pursue such a goal we study a discrete delay model which explains the natural economic fluctuations considering how economic stability is affected by the role of the fiscal and monetary policies and a possible government inefficiency concerning its fiscal policy decision-making. On the other hand, we start to develop such an extended theory by considering linear Cauchy problems associated to a continuous linear operator on Fréchet spaces, for which we establish necessary and sufficient conditions for generation of a uniformly continuous group which provides the unique solution. Further consequences arises by considering pseudodifferential operators with constant coefficients defined on a particular Fréchet space of distributions, namely $\mathscr{F}L^2_{loc}$, and special attention is given to the distributional solution of the heat equation on $\mathscr{F}L^2_{loc}$ for all time, which extends the standard solution on Hilbert spaces for positive time.

Keywords: Pseudodifferential operators, Fréchet spaces, Linear Cauchy problems, Delay differential equations, Kaldor's model.

RESUMO

SILVA, A. P. **Resolubilidade de problemas lineares de Cauchy em espaços de Fréchet e um modelo de Kaldor com retardo**. 2019. 116 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

O objetivo a longo prazo desta tese é resolver equações diferenciais da forma

$$\frac{d}{dt}u(t) = Au(t) + \int_{-\infty}^{t} u(s)k(t-s)\,ds + f\big(t, u(t)\big),$$

em espaços de Fréchet estendendo a teoria de grupos de operadores lineares; sendo A um operador linear, $k(s) \geqslant 0$ tal que $\int_0^\infty k(s)\,ds = 1$ e f uma função não linear. Perseguindo tal fim, estudamos um modelo com retardo que explica as flutuações naturais da economia considerando como a estabilidade econômica é afetada pela atuação do governo, suas políticas fiscal e monetária e uma possível ineficiência do governo no que diz respeito à sua tomada de decisão na política fiscal. Por outro lado, damos início a referida extensão da teoria de grupos ao considerar problemas de Cauchy lineares associados a operadores lineares contínuos em espaços de Fréchet, para os quais estabelecemos condições necessárias e suficientes para a geração de um grupo uniformemente contínuo em tal espaço que fornece a única solução do problema. Consequências adicionais surgem quando se considera operadores pseudodiferenciais com coeficientes constantes definidos em um particular espaço de Fréchet de distribuições, a saber $\mathscr{F}L^2_{loc}$, e uma atenção especial é dada à solução distribucional da equação do calor em $\mathscr{F}L^2_{loc}$ para todo tempo, a qual estende a solução usual em espaços de Hilbert para tempo positivo.

Palavras-chave: Operadores pseudodiferenciais, Espaços de Fréchet, Problemas de Cauchy lineares, Equações diferenciais com retardo, Modelo de Kaldor.

RESUMEN

SILVA, A. P. **Resolubilidade de problemas lineares de Cauchy em espaços de Fréchet e um modelo de Kaldor com retardo**. 2019. 116 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

El objetivo a largo plazo de esta tesis doctoral es resolver ecuaciones diferenciales de la forma

$$\frac{d}{dt}u(t) = Au(t) + \int_{-\infty}^{t} u(s)k(t-s)\,ds + f\big(t,u(t)\big),$$

en espacios de Fréchet extendiendo la teoria de grupos de operadores lineales; siendo A un operador linear, $k(s) \geqslant 0$ satisface $\int_0^\infty k(s)\,ds = 1$ y f una función no linear. Persiguiendo tal fin, estudiamos un modelo con retraso que enseña las fluctuaciones de la economía considerando como la estabilidad económica es afectada por la actuación del gobierno, sus políticas fiscal y monetaria y una posible ineficiencia del gobierno en lo que se refiere a la su toma de decisión. Por otro lado, damos inicio a la referida extensión de la teoría de grupos puesto que consideramos problemas de Cauchy lineales asociados a operadores lineales contínuos en espacios de Fréchet, para los cuales establecimos condiciones necessarias y suficientes para la generación de un grupo uniformemente contínuo en tal espacio y que proporcione la única solución del problema. Consequencias adicionales surgen cuando se considera operadores pseudodiferenciales con coeficientes constantes definidos en un particular espacio de Fréchet de distribuciones, a saber $\mathscr{F}L^2_{loc}$, y una atención especial es dada a la solución distribucional de la ecuación del calor en $\mathscr{F}L^2_{loc}$ para todo el tiempo, la cual extiende la solución usual en espacios de Hilbert para tiempo positivo.

Palabras clave: Operadores pseudodiferenciales, Espacios de Fréchet, Problemas de Cauchy lineales, Ecuaciones diferenciales con retraso, Modelo de Kaldor.

LIST OF SYMBOLS

$\mathbb{N}$ — the set of all natural numbers, that is, $\{1,2,3,\ldots\}$.

$\mathbb{Q}$ — the set of all rational numbers.

$\mathbb{Z}_+$ — the set of all nonnegative integer numbers, that is, $\{0,1,2,3,\ldots\}$.

$\mathbb{Z}_+^N$ — $\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$ (with N factors).

$\mathbb{R}$ — the set of all real numbers.

$\mathbb{R}_+$ — the set of all nonnegative real numbers.

$\mathbb{R}^N$ — $\mathbb{R} \times \cdots \times \mathbb{R}$ (with N factors).

$\xi \cdot x$ or $\langle \xi, x \rangle$ — the inner product of vectors x and y in $\mathbb{R}^N$.

$\mathbb{R}_+^N$ — $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ (with N factors).

$\mathbb{C}$ — the set of all complex numbers.

$\Re\lambda$ — the real part of a complex number λ.

$\Im\lambda$ — the imaginary part of a complex number λ.

$\mathbb{C}^N$ — $\mathbb{C} \times \cdots \times \mathbb{C}$ (with N factors), which can and will be identified with $\mathbb{R}^N + i\mathbb{R}^N$.

$\mathbb{K}$ — denotes either $\mathbb{R}$ or $\mathbb{C}$.

(X,τ) — a space X equipped with the topology τ.

$\overline{A}$ — the closure of a subset A of a given topological space (X,τ).

(X,ρ) — a space X equipped with the metric ρ.

$B_X(x_0,r)$ — the set of all points x of a given metric space (X,ρ) such that $\rho(x,x_0) < r$.

$B_X[x_0,r]$ — the closure of $B_X(x_0,r)$.

$B(x_0,r)$ — denotes $B_X(x_0,r)$ whenever $X = \mathbb{R}^N$ is clear.

$B[x_0,r]$ — denotes $B_X[x_0,r]$ whenever $X = \mathbb{R}^N$ is clear.

B_n — denotes $B_X(0,n)$ whenever $X = \mathbb{R}^N$ is clear, with $n \in \mathbb{N}$.

X' or X^* — the space of all continuous linear functionals $\phi : X \to \mathbb{K}$, given a TVS X.

$\big(X,(p_\alpha)_{\alpha\in A}\big)$ — a vector space $(X,+,\cdot)$ equipped with the seminorms $p_\alpha, \alpha \in A$.

$\big(X,(p_n)_{n\in\mathbb{N}}\big)$ — a Fréchet space equipped with the seminorms $p_n, n \in \mathbb{N}$.

$(X, \|\cdot\|_X)$ — a Banach space equipped with the norm $\|\cdot\|_X$.

$\mathscr{L}(X,Y)$ — the space of all bounded linear maps from X to Y.

$\mathscr{L}(X)$ — the space of all bounded linear maps from X to itself.

$\mathscr{L}_{sc}(X)$ — the space of all strongly compatible linear maps from $\big(X,(p_n)_n\big)$ to itself; see page 71.

Id_X or 1 — the identity (linear) operator on X.

$R(n;A)$ — the resolvent operator $(n\,\mathrm{Id}_X - A)^{-1} \in \mathscr{L}(X)$, with $n \in \mathbb{N}$; see Section 2.3 .

$T(\cdot)$ — a group or semigroup of linear bounded operators $T(t) : X \to X$, with $t \in \mathbb{R}$ or $t \geqslant 0$.

$L^p(\mathbb{R}^N)$ — the space of all Lebesgue measurable functions such that $\displaystyle\int_{\mathbb{R}^N} |f|^p$ is finite.

$L^p_{loc}(\mathbb{R}^N)$ — the space of all functions f such that $\displaystyle\int_{B_n} |f|^p$ is finite, for every $n \in \mathbb{N}$.

$C_0(\mathbb{R}^N)$ — the space of all continuous functions on $\mathbb{R}^N$ such that $f(x) \to 0$ as $|x| \to \infty$.

$C\big([0,1],\mathbb{R}^N\big)$ — the space of all continuous functions mapping $[0,1]$ into $\mathbb{R}^N$.

$BC(\mathbb{R}^N)$ — the space of all uniformly bounded continuous functions defined on $\mathbb{R}^N$.

$C^k(\mathbb{R}^N)$ — the space of all functions on $\mathbb{R}^N$ with continuous derivatives up to order k on $\mathbb{R}^N$.

$C^\infty(\mathbb{R}^N)$ — the space of all infinitely differentiable functions on $\mathbb{R}^N$.

$C^\infty_c(\mathbb{R}^N)$ — the space of all infinitely differentiable functions which has compact support.

$\mathcal{S}(\mathbb{R}^N)$ — denotes the Schwartz space on$\mathbb{R}^N$.

$\mathscr{F}f$ or $\widehat{f}$ — the Fourier transform of a function f.

$\mathscr{F}^{-1}f$ or $\check{f}$ — the inverse Fourier transform of a function f.

$\mathcal{D}'(\mathbb{R}^N)$ or simply $\mathcal{D}'$ — the space of all distributions on $\mathbb{R}^N$.

$\mathcal{S}'(\mathbb{R}^N)$ or simply $\mathcal{S}'$ — the space of all tempered distributions on $\mathbb{R}^N$.

$\mathcal{E}'(\mathbb{R}^N)$ or simply $\mathcal{E}'$ — the space of all compactly supported distributions on $\mathbb{R}^N$.

$\mathscr{F}L^2_{loc}(\mathbb{R}^N)$ or simply $\mathscr{F}L^2_{loc}$ — a Fréchet space of distributions; see page 78.

$H^s(\mathbb{R}^N)$ or simply H^s — the Hilbert Sobolev space of order s on $\mathbb{R}^N$, with $s \geqslant 0$.

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CHAPTER

1

INTRODUCTION

"La mathematica è l'alfabeto in cui Dio è scritto l'Universo"
Galileo Galilei (GALILEI; SOSIO, 1992)

When it comes to motivate nonmathematicians to study mathematics, one of the greatest difficulties concerns the fact that the main argument, namely modeling, requires a lot of mathematical tools in order to turn a model into something really useful. This happens because reality is not simple. From phenomena observation, one interprets variations and reactions and measures quantities, which unavoidably carries errors of measure (and sometimes even small errors cause great divergent results); and every model deliberately dismisses some aspects of the phenomenon so it can be treated. Besides it is important to become aware that everything is finite and nothing is continuous even though our perception of time, quantities and variations leads us to believe otherwise. And yet to model phenomena by assuming they are equipped with continuity, for instance, is the best we can do and fortunately it fits pretty well the real situation.

Invariably, description and prediction are fundamental concerns for science. Aware of the limitations listed above, assuming that finitely many quantities (which can be denoted by $u = (u_1, \ldots, u_N)$) provide, at an instant of time t, a reliable representation of the state of a given system, if one wants to determine the subsequent behavior, one may also assume that the rate of change of such a vector $u(t)$ depends only on t, $u(t)$ itself and interactions with some external forces. This is the so called principle of causality and is the core of a modeling by ordinary differential equations. However, wondering a little further, one sees that it is a first approximation of the real world: the present state is expected to be a consequence of past states as well.

There are several ways under which the history of a phenomenon can be added to the modeling. In order to become familiar with some of them, let us examine different versions of a very famous model in biology - namely the Lotka-Volterra model - which describes the interaction between two species in a closed environment: one species (the predator) preys on the other species (the prey), while the prey lives on a different source of food. If u_1 and u_2 denote the prey and predator populations respectively, then the prey-predator equations read as follows:

$$\begin{aligned} \frac{du_1}{dt}(t) &= u_1\big(a_1 - \alpha_1 u_2\big) \\ \frac{du_2}{dt}(t) &= u_2\big(-a_2 + \alpha_2 u_1\big) \end{aligned}, \tag{1.1}$$

which are explained by the reasonings below:

i. the prey population is the total food supply for the predators;

ii. in the absence of predators, the prey population grows at a rate $a_1 > 0$ which is proportional to the current population;

iii. in the absence of prey, the predator population declines at a rate $a_2 > 0$ which is proportional to the current population;

iv. proportionally to the number of predator/prey encounters, prey population decreases at a rate $\alpha_1 > 0$ whereas predator population increases at a rate $\alpha_2 > 0$.

Under equations (1.1), periodic fluctuations arise naturally as a consequence of the species interaction itself instead of being a result of external circumstances, as seasons or human interference; which was validated under controlled experimental conditions. By assumption **i.**, in the absence of the predator the prey population grows indefinitely. Since this does not fit reality, we could consider the carrying capacity of prey population by imposing for instance a logistic growth to it. And thus the first equation is replaced by $u_1'(t) = u_1(t)\big(a_1 - bu_1(t) - \alpha_1 u_2(t)\big)$, for some positive constant b. Besides, as pointed out by Hutchinson (HUTCHINSON, 1948), the predator population varies today based on how many preys were consumed a certain time before, that is, we should consider a delay in assimilation of consumed prey; which leads to the following discrete delay predator-prey model:

$$\begin{aligned} \frac{du_1}{dt}(t) &= u_1(t)\big(a_1 - bu_1(t) - \alpha_1 u_2(t)\big) \\ \frac{du_2}{dt}(t) &= u_2(t)\big(-a_2 + \alpha_2 u_1(t-\tau)\big) \end{aligned}. \tag{1.2}$$

A model such as (1.2) does incorporate a dependence on its past history: its second equation states that the density of the prey population at time $t-\tau$ affects the growth of the predator population at time t. However it would be more realistic to assume that the density dependence is distributed over an interval in the past rather than concentrated at a single time instant. Hence a delayed distributed differential Lotka-Volterra model arises:

$$\begin{aligned} &\frac{du_1}{dt}(t) = u_1(t)\big(a_1 - bu_1(t) - \alpha_1 u_2(t)\big) \\ &\frac{du_2}{dt}(t) = u_2(t)\left(-a_2 + \alpha_2 u_1(t) + \int_{-\infty}^{t} u_1(s)k(t-s)\,ds\right) \end{aligned}. \tag{1.3}$$

where $[0,\infty) \ni s \ni\mapsto k(s) \in [0,\infty)$ is assumed to satisfy $\int_0^\infty k(s)\,ds = 1$. The fraction $k(t-s)$ of prey eaten at time $t-s$ is assumed to be translated into predator biomass at time t. Naturally, the solution of (1.3) is expected to lie in space $C\big(\mathbb{R},\mathbb{R}^2\big)$ at least, which is not normable. This means that its topology is not equivalent to no topology which comes from a norm on $C\big(\mathbb{R},\mathbb{R}^2\big)$. Actually it is a Fréchet space, as we shall study in Chapter 2.

Concerning these differential equations, improvements were gradually obtained as we considered the following scenarios: no delay; fixed delay; and distributed delay, which can be an infinite one depending on the support of the kernel $s \mapsto k(s)$. The text we are about to read in the form of chapters consists of two quite different approaches linked by a common aim: to better understand how the world around us behaves by writing it down under mathematical language, as Galileo suggested five centuries ago. Ideally we want to solve delay differential equations with infinite delay of the type, such as

$$\frac{du}{dt}(t) = Au(t) + \int_{-\infty}^{t} u(s)k(t-s)\,ds + f\big(t,u(t)\big), \tag{1.4}$$

under an extended theory of (semi)groups on Fréchet spaces, where A is a linear operator and f is a map which represents the nonlinear perturbations of the phenomenon. By doing so, we free the solution of evolution problems from the geometry usually implicitly imposed by its formulation on Banach spaces. More precisely, it is well known that every Hilbert space is isometrically isomorphic to some ℓ^2 space[1], from which one sees that there is only one geometry for infinite dimensional Hilbert spaces. As for separable Banach spaces, we know that they always can be embedded in some (separable) Hilbert space.

On the one hand, Chapter 3 introduces the theory of groups of bounded operators on Fréchet spaces which is needed in order to solve the linear Cauchy problem

$$\begin{cases} u'(t) = a(D)u, t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

on a Fréchet space X, provided that $a(D)\colon X \to X$ is a bounded linear operator. Such an approach extend the usual one on Banach spaces, where the unique solution is provided by the map $e^{t\,a(D)}$. It is the first step towards a general theory to solve semilinear Cauchy problems on Fréchet spaces and a fortiori (1.4). To do so, we establish the concept of

[1] if $(e_\alpha)_{\alpha\in A}$ is an orthonormal basis for a Hilbert space H, where A is not necessarily countable, then H is isometrically isomorphic to $\ell^2(A)$; see Proposition 5.30 of (FOLLAND, 1999)

"strongly compatible operators" with which the exponential map $e^{t\,a(D)}$, $t \in \mathbb{R}$, is a well defined bounded linear operator on the phase space X and yields the (unique) solution we seek. Such a compatibility provides a natural way to topologyze the space of all bounded strongly compatible operators, under which the convergence of the exponential operator $e^{t\,a(D)}$ holds, instead of a pointwisely convergence, as in (YOSIDA, 1980; CHOE, 1985). Besides, we study a special Fréchet space of distributions and linear Cauchy problems on it. The main application concerns the heat equation which in this setting admits a distribution backwards solution which extends the classical solution obtained by analytic semigroup theory on Hilbert spaces.

On the other hand, in Chapter 4 we deal with an economic model of the form

$$u'(t) = f\big(u(t), \alpha u(t-\tau)\big) \text{ in } \mathbb{R}^4, \tag{1.5}$$

which is a nonlinear approach to explain the natural economic fluctuations. We study the existence of a positive equilibrium point and its local stability with or without delay time $\tau > 0$. Besides, we analyze how such a stability switches as τ and the parameter α vary. Even though our aim is to set this model under a formulation with infinite delay, this first step was necessary and it is already an improvement on (TAKEUCHI; YAMAMURA, 2004), whose authors deal with a simplified version of (1.5) in $\mathbb{R}^3$ with fixed delay.

Therefore every chapter contributes with our aim under different perspectives. Chapter 4 deals with a simple application of delay differential equations aiming a more sophisticated version for which there is no available group theory on Fréchet spaces yet; whereas Chapter 3 starts out such an abstract theory with some outstanding applications already. Additionally, Chapter 2 comes to smooth the reading of this thesis by introducing some topics on semigroup theory on locally convex spaces, functional differential equations and macroeconomic theory.

What was initially proposed to be my thesis project strongly differs from what this final version actually is. Also, not everything we wanted to obtain from the study of the extended Kaldor's model proposed in (TAKEUCHI; YAMAMURA, 2004) was possible and many other questions have arisen with not necessarily answers we could give so far. This is quite natural when it comes to scientific research. In professor Carvalho's words, "to pursue a thesis is like to enter into a room with the lights off touching what you find on the way, trying to find out what it is and how it works without reading instructions; and times the lights flash so you can have a glance of the room - that is the advisor in action". At last, he also once said to me that "if you knew from the beginning what you were going to achieve with your thesis, it would not be a thesis; it would be an exercise".

CHAPTER

2

PRELIMINARIES

With this initial chapter we intend to smooth the reading of the thesis by fixing some notations, by providing some results and by discussing some topics which we believe are not as well known as others. However, the reader is expected to be familiar with the following topics:

(i) basic concepts of **set theory** (Chapter 0 of (FOLLAND, 1999) is enough);

(ii) **topological spaces** (we recommend (WILLARD, 2004), although Chapter 4 of (FOLLAND, 1999) is enough most of the time);

(iii) functional analysis in **Banach spaces** (while (FOLLAND, 1999) provides a good compact course of it, (BREZIS, 2010) is more complete);

(iv) **Lebesgue integration** theory (see (FOLLAND, 1999; STEIN; SHAKARCHI, 2009));

(v) **analysis** in euclidean spaces (it suffices to check (LOOMIS; STERNBERG, 2014)); and

(vi) **ordinary differential equations** (we use (HALE, 2009) as standard reference, but the reader can start with (ROBINSON, 2001)).

We start the chapter by studying the spaces we aim to set as natural phase space when formulating evolution problems and modeling phenomena, namely, locally convex spaces. Thus Section 2.1 contains the basic background on the theory of topological vector spaces with which we shall acquire a powerful set of tools from Functional Analysis on Fréchet spaces. In order to propose a distributional formulation to evolution problems, we present a brief review of Fourier analysis in Section 2.2. Besides, since we seek to propose an alternative approach to the generation of uniformly bounded groups on Fréchet spaces, it is appropriate to compare it with the already existing theory on locally convex spaces. Although the theory of semigroups of linear operators on Banach spaces is very well known, the reader may not be familiar with its extension to locally convex spaces. Hence the purpose of Section 2.3 is to provide this content. About Section 2.4, many results to ordinary differential equations can be extended to retarded delay differential equations, which are necessary to carry the reasoning out in Chapter 4. Meanwhile we shall explore in Section 2.5 how mathematics has been used in modeling economics. And in order to Kaldor's business cycle model can be properly appreciated, we give a brief introduction to macroeconomic theory, by discussing important principles of it, such as: liquidity preference; aggregate demand and aggregate supply; monetary and fiscal policies; and growth of the national output. For such a discussion, the reader is expected to be familiar with the nomenclatures only.

Every function in this text is assumed to be complex valued, unless otherwise specified. In Sections 2.4 and 2.5 functions are assumed to $\mathbb{R}^N$-valued or simply $\mathbb{R}$-valued. Suppose $X = (X,+,\cdot)$ is a vector space equipped with a topology (in Section 2.1, we shall impose some conditions in order to be a useful topology), we shall denote its dual space - the space of continuous linear functionals $\phi : X \to \mathbb{K}$ - by X' or X^*. We shall not deal with the algebraic dual space of X and therefore we avoid confusion since unfortunately both notations are commonly used in both senses in functional analysis textbooks. The evaluation of a given $\phi \in X'$ on a vector $x \in X$ is denoted by $\langle \phi, x \rangle$.

2.1 Locally convex spaces

In this section we collect some definitions and results with special focus on locally convex spaces (hereafter LCSs). While the reader is expected to be familiar with basic point set topology and functional analysis for Banach spaces, there are some twists that may or may not be familiar, which are crucial when dealing with LCSs. The main references are (FOLLAND, 1999; NARICI; BECKENSTEIN, 2010; OSBORNE, 2013; REED; SIMON, 1980; RUDIN, 1991; WILLARD, 2004; YOSIDA, 1980).

There are basically three subjects to be discussed: nets; seminorms; and local convexity. A net is basically a generalized sequence in which the natural numbers are replaced by a directed set and plays a central role when it comes to study LCS topologies (although the concept of filters would be a proper substitute). Also, neighborhoods are not assumed to be open sets, which is a crucial point for most results to be proved; and they will often be treated in terms of seminorms. Finally, concept of "locally convex" is the basic property needed to obtain important results such as the Hahn-Banach theorems and the Banach-Steinhaus theorem.

We start the discussion of LCSs by highlighting the difference between continuity and sequential continuity. For a function $f : \mathbb{R} \to \mathbb{R}$, one may define continuity using sequential convergence - that is, f is said to be continuous at x_0 if $f(x_n) \to f(x_0)$ as $n \to \infty$, whenever $x_n \to x_0$ as $n \to \infty$ - which is referred as **sequential continuity**.

Sequential convergence is able to describe only those topologies in which the number of basic neighborhoods around each point is no greater than the number of terms in the sequences. In other words, sequences describe the topology of a space X whenever X is a first-countable space[1]. By describing a topology with sequences, we mean obtaining characterizations of basic topological properties from sequences: let $X = (X, \tau_X)$ and $Y = (Y, \tau_Y)$ be first-countable spaces, then

a. $U \subset X$ is τ_X-open if and only if $(x_n)_n$ is eventually in U, whenever $x_n \to x \in U$;

b. $F \subset X$ is τ_X-closed if and only if $x \in F$ whenever $(x_n)_n \subset F$ and $x_n \to x$ as $n \to \infty$; and

c. a function $f\colon X \to Y$ is continuous if and only if f is sequentially continuous.

As pointed out in (FOLLAND, 1999), considering the pointwise topology, the sequential closure of the set of all continuous functions $C(\mathbb{R}, \mathbb{C})$ is a proper subset of the set $\mathbb{C}^{\mathbb{R}}$ of all functions $f\colon \mathbb{R} \to \mathbb{C}$. On the other hand, $C(\mathbb{R}, \mathbb{C})$ is a dense subset of $\mathbb{C}^{\mathbb{R}}$.

A successful generalization of the notion of sequence lies in retaining the essential ordering of the natural numbers. We replace the linearity of the order on $\mathbb{N}$ by some other way of giving a definite "positive orientation" to our ordered sets. The following definition has stood the test of time. A set Λ is said to be a **directed set** if it is equipped with a binary relation $\preceq$ such that

- $\lambda \preceq \lambda$;
- if $\lambda_1 \preceq \lambda_2$ and $\lambda_2 \preceq \lambda_3$ then $\lambda_1 \preceq \lambda_3$; and
- if $\lambda_1, \lambda_2 \in \Lambda$ then there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \preceq \lambda_3$ and $\lambda_2 \preceq \lambda_3$.

The relation $\preceq$ is sometimes referred to as a direction on Λ and is said to direct Λ. If $\lambda_1 \preceq \lambda_2$, we may also write $\lambda_2 \succeq \lambda_1$.

A **net** in a set X is a mapping $\lambda \mapsto x_\lambda$ from a directed Λ into X. We shall denote it by $(x_\lambda)_{\lambda \in \Lambda}$ or simply (x_λ) if Λ is understood; and we say that it is indexed by Λ. A net $(x_\lambda)_{\lambda \in \Lambda}$ in topological space X **converges** to $x \in X$ if, for every neighborhood U of x, there exists $\lambda_0 = \lambda_0(U) \in \Lambda$ such that $x_\lambda \in U$ whenever $\lambda \succeq \lambda_0$; we write

$$x = \lim_{\lambda \in \Lambda} x_\lambda \quad \text{or} \quad x = \lim_{\lambda} x_\lambda \quad \text{or} \quad x_\lambda \xrightarrow[\lambda \in \Lambda]{X} x \quad \text{or simply} \quad x_\lambda \to x;$$

and we say that (x_λ) is **eventually** in every neighborhood of x. Since the notion of a net is a generalization of the notion of a sequence (with $\mathbb{N}$ being replaced by Λ), this is consistent with standard terminology for sequences.

[1] See (WILLARD, 2004) for a more extensive discussion about it.

Example 2.1. We provide some simple examples of directed sets and nets in the following:

- the set $\mathbb{N}$ of natural numbers is a directed set when given its usual order. Thus every sequence $(x_n)_{n\in\mathbb{N}}$ is a net;

- the set $\mathscr{N}_x$ of all neighborhoods of a point x in a topological space X, with $U \preceq V$ whenever $U \supset V$ is a directed set and we say that $\mathscr{N}_x$ is directed by reverse inclusion. It provides a fundamental connection between the concepts of nets on X and the topological properties of X;

- given $a \in \mathbb{R}$, the set $\mathbb{R}\setminus\{a\}$ is a directed set provided that $x \preceq y$ whenever $|x-a| > |y-a|$; from whence one may see how nets notion occurs in defining limits of real variables.

- the collection $\mathscr{P}$ of all finite partitions of the closed interval $[a,b]$ into closed subintervals is a directed set when ordered by the relation $P_1 \preceq P_2$ whenever P_2 refines P_1. Given a real-valued function f on $[a,b]$, we can define a net $S_L\colon \mathscr{P} \to \mathbb{R}$ by letting $S_L(P)$ be the lower Riemann sum of f over the partition P; likewise, we can define $S_U\colon \mathscr{P} \to \mathbb{R}$ by letting $S_U(P)$ be the upper Riemann sum of f over P. Convergence of both of these nets to the number c simply means

$$\int_a^b f(x)\,dx = c.$$

  This example is historically important; it is what first led Moore and Smith to the concept of a net.

- the Cartesian product $\Lambda_1 \times \Lambda_2$ of two directed sets is always directed by the order relation $(\lambda_1,\lambda_2) \preceq (\lambda_1',\lambda_2')$ whenever $\lambda_1 \preceq_{\Lambda_1} \lambda_1'$ and $\lambda_2 \preceq_{\Lambda_2} \lambda_2'$. It is the main tool to a diagonal argument when it comes to prove that a topological space X is Hausdorff if and only if every net in X converges to at most one point.

Now we introduce the notion of subnets, which performs much the same functions as subsequences, although with some reservations. Like a subsequence, a subnet involves a reparametrization of a net, but the manner in which this happens is much more general; so general, in fact, that a subnet of a sequence need not be a subsequence.

A **subnet** of a net $(x_\lambda)_{\lambda\in\Lambda}$ is a net $(z_\gamma)_{\gamma\in\Gamma}$ together with a map $\Gamma \ni \gamma \mapsto \lambda_\gamma \in \Lambda$ such that

- for every $\lambda_0 \in \Lambda$ there exists $\gamma_0 \in \Gamma$ such that $\lambda_\gamma \succeq \lambda_0$ whenever $\gamma \succeq \gamma_0$; and

- $z_\gamma = x_{\lambda_\gamma}$.

Clearly if a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point x then so does any subnet $(x_{\lambda_\gamma})_{\gamma\in\Gamma}$. At last, a point $y \in X$ is a **cluster point** of (x_λ) if for every neighborhood U of y and $\lambda \in \Lambda$ there exists $\lambda_0 \succeq \lambda$ such that $x_{\lambda_0} \in U$.

Theorem 2.2. Let X and Y be topological spaces, let $f\colon X \to Y$ be a function, $x \in X$ and $E \subset X$.

a. a net has a cluster point x if and only if it has a subnet which converges to x;

b. $x \in \bar{E}$ if and only if x is the limit of some net in E;

c. f is continuous at x if and only if $f(x_\lambda) \xrightarrow{Y} f(x)$ whenever $x_\lambda \xrightarrow{X} x$;

d. X is compact if and only if every net has a cluster point; and

e. A net (x_λ) in a product $X = \prod_{j\in J} X_j$ converges to x if and only if $\pi_j(x_\lambda) \to \pi_j(x)$, for every canonical projection $\pi_j\colon X \to X_j$.

As a consequence, one may prove the Tychonoff theorem with very little effort, see (FOLLAND, 1999; WILLARD, 2004). It is frequently useful to consider topologies on vector spaces other than those defined by norms, the only crucial requirement being that the topology should be well behaved with respect to the vector operations. Precisely, a **topological vector space** X over $\mathbb{R}$ or $\mathbb{C}$ (hereafter TVS) is a vector space, which is also a topological space, in which the vector space operations are continuous. Letting $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$ we require the maps

$$X \times X \ni (x,y) \mapsto x+y \in X \text{ (addition)}$$

and

$$\mathbb{K} \times X \ni (r,x) \mapsto rx \in X \text{ (scalar multiplication)}$$

to be continuous.

Topological vector spaces enjoy some nice topological properties earned from the vector structure. For instance, a TVS is T_0 if and only if it is T_1 if and only if it is T_2 if and only if it is T_3 if and only if $\{0\}$ is closed if and only if $\bigcap_{B\in\mathscr{B}} B = \{0\}$, for a fixed neighborhood base $\mathscr{B}$ at 0.

It is time to keep in mind how important Hausdorff condition is, since we want nets to have unique limits. We shall define the meaning of “Cauchy” here (it is slightly subtle) and define completeness as “Cauchy $\Rightarrow$ Convergent”, as expected. For metric spaces, a sequence $(x_n)_n$ converges to x when the terms x_n get close to x; and the sequence $(x_n)_n$ is Cauchy when the terms x_n get close to each other. To motivate how that translates into for TVSs, consider a convergent sequence $x_n \to x$ in a TVS X. If B is a neighborhood of 0 then for n sufficiently large, $x_n \in x+B$, that is $x_n - x \in B$. Aha! That B is fixed; the group operation says (in a uniform sense) that a point x is close to another point y when $y-x \in B$.

A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is said to be a **Cauchy net** if the net $(x_\lambda - x_\gamma)_{\Lambda\times\Lambda}$ is eventually in every neighborhood of 0. And X is said to be **complete** if every Cauchy net in X is convergent. One fact we have gotten used to when working with metric spaces is that sequences are "enough". When it comes to TVSs, sequential completeness implies completeness if the space is first countable Hausdorff. It is noteworthy that every LCS X can be embedded in a complete LCS, in which X forms a dense subset. See (NARICI; BECKENSTEIN, 2010), Theorem 5.11.5; or (TREVES, 2016), Theorem 5.2.

Most TVSs that arise in practice are locally convex and Hausdorff. A locally convex topological vector space will be called a **locally convex space** (abbreviated LCS in the literature). Of course, the adverb "locally" in "locally convex" means exactly what an adverb should: there exists a base for the topology consisting of convex sets, that is, sets A such that if $x,y \in A$ then $tx+(1-t)y \in A$, for $0<t<1$.

Locally convex spaces are often defined in terms of seminorms. Namely, if we are given a family of seminorms on X, the "semiballs" that they define can be used to generate a topology in the same way that the balls defined by a norm generate the topology on a normed vector space. Let us make it precise.

A **seminorm** on a vector space X is a function $X \ni x \mapsto p(x) \in [0,\infty)$ such that

- $p(rx) = |r|p(x)$, for every $r \in \mathbb{K}$ and $x \in X$; and
- $p(x+y) \leqslant p(x)+p(y)$, for every $x,y \in X$. (the triangle inequality)

The first property clearly implies that $p(0)=0$. A seminorm such that $p(x)=0$ only when $x=0$ is called a **norm**.

Theorem 2.3. Let $(p_\alpha)_{\alpha\in A}$ be a family of seminorms on a vector space X and let τ denote the topology generated by the sets

$$B_\alpha(x,\varepsilon) := \{y \in X \colon p_\alpha(y-x) < \varepsilon\},$$

with $\alpha \in A, x \in X$ and $\varepsilon > 0$.

a. for every $x \in X$, the sets $B_\alpha(x,\varepsilon)$ form a neighborhood subbase at x;

b. a net (x_λ) in X converges to x if and only if $p_\alpha(x_\lambda - x) \to 0$, for every $\alpha \in A$;

c. (X,τ) is a locally convex space;

d. (X,τ) is Hausdorff if and only if there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$ whenever $x \neq 0$; in this case, $(p_\alpha)_{\alpha\in A}$ is said to be a **separating family of seminorms**; and

e. if (X,τ) is Hausdorff and A is countable then (X,τ) is metrizable with a translation invariant metric ρ (that is, $\rho(x,y)=\rho(x+z,y+z)$ for all $x,y,z\in X$).

Some authors uses this construction as the definition of "locally convex space", see (REED; SIMON, 1980). We shall write $X=\big(X,(p_\alpha)_{\alpha\in A}\big)$ to denote the vector space X equipped with the topology generated by the seminorms $p_\alpha, \alpha\in A$. Such a construction is not overly restrictive: every LCS has a neighborhood base $\mathscr{B}$ at 0 consisting of convex balanced sets (which means sets B with the following property: for every $x\in B$ and $r\in\mathbb{K}$ such that $|r|\leqslant 1$, $cx\in B$); then one can take their Minkowski functionals p_B defined by $p_B(x):=\inf\{t>0\colon t^{-1}x\in B\}$, which are seminorms thanks to the nice properties of $\mathscr{B}$; and apply this construction to $(p_B)_{B\in\mathscr{B}}$. The resulting topology is equivalent to the original one. This situation does happen even more often when A is countable.

Given a TVS $X=(X,\tau)$, a family of seminorms $(p_\alpha)_{\alpha\in A}$ on X is said to be a **fundamental family of seminorms** (for X) if the topology they generate is equivalent to the original one, namely τ.

Clearly the main class of maps between TVSs we are interested in is the class of linear maps, which enjoy nice properties thanks to the vector space structure.

Proposition 2.4. Let $X=\big(X,(p_\alpha)_{\alpha\in A}\big)$ and $Y=\big(Y,(q_\beta)_{\beta\in B}\big)$ be TVSs. A linear map $T\colon X\to Y$ is continuous if and only if for each $\beta\in B$ there exist finitely many indices $\alpha\in\tilde{A}\subset A$ and $C=C(\beta,T)>0$ such that

$$q_\beta(Tx)\leqslant C\sum_{\alpha\in\tilde{A}}p_\alpha(x).$$

The notion of boundedness for a subset B of a Banach space $(X,\|\cdot\|_X)$ is very clear and it is equivalent to the following statement: for every neighborhood V of 0 in X there exists a scalar number $t_0=t_0(B,V)>0$ such that $B\subset tV$ whenever $t\geqslant t_0$. If B is a subset of a TVS, this is the standard definition of B being a **bounded** set. In terms of seminorms, a subset B of a TVS $X=\big(X,(p_\alpha)_\alpha\big)$ is bounded if $p_\alpha(B):=\{p_\alpha(x)\in\mathbb{R}\colon x\in B\}$ is a bounded set, for every α. Besides, a linear operator $T\colon X\to Y$ between Banach spaces is continuous if and only if it is bounded, in the sense that $T(B)\subset Y$ is bounded whenever $B\subset X$ is bounded. This is not quite the same for general TVSs.

Theorem 2.5. Let $T\colon X\to Y$ be a linear transformation between two TVSs and consider the following possible properties of T:

a. T is continuous;

b. T is bounded;

c. if $x_n\to 0$ then $\{Tx_n\colon n\in\mathbb{N}\}\subset Y$ is bounded; and

d. T is sequentially continuous at 0, that is, $Tx_n \to 0$ whenever $x_n \to 0$.

Then the implications **a.** $\Rightarrow$ **b.** $\Rightarrow$ **c.** hold. If in addition X is metrizable then **c.** $\Rightarrow$ **d.** $\Rightarrow$ **a.**, so that all four properties are equivalent.

Complete metric spaces were introduced along with the definition of metric spaces by Maurice Fréchet (pronounced as mɔʀis fʀeʃɛ, French IPA) in his doctoral thesis (FRÉCHET, 1906) and vigorously pursued by a host of Polish mathematicians in the 1920's. Fréchet and Stefan Banach (pronounced as stɛfan banax, Polish IPA) were contemporaneous mathematicians. Fréchet was the first to use the term "Banach space". Banach repaid this favor by coining the term "Fréchet space" for complete metrizable TVS. Local convexity condition was later appended by Bourbaki (pronounced as nikɔla buʀbaki, French IPA).

Proposition 2.6. Let X be a TVS. The following are equivalent:

a. X is locally convex, complete and metrizable with a translation invariant metric;

b. X is locally convex, complete and metrizable;

c. X is complete and its topology is generated by a countable separating family of seminorms; and

d. X is a Hausdorff complete space which admits a neighborhood base at 0 consisting of countably many convex balanced absorbing sets.

A TVS X is called a **Fréchet space** if possesses the properties (a)-(d) of Proposition 2.6. Given a fundamental family of seminorms $(p_j)_{j\in\mathbb{N}}$ for a Fréchet space X, one can consider the seminorms $q_k, k \in \mathbb{N}$, defined by

$$q_k(x) := p_1(x) + \cdots + p_k(x), x \in X,$$

which turn out to be an equivalent fundamental family of seminorms for X; and we say that $(q_k)_{k\in\mathbb{N}}$ is a **saturated family of seminorms**, see (NARICI; BECKENSTEIN, 2010). They have the property that $q_k \leqslant q_l$ whenever $k \leqslant l$. In this case, by Proposition 2.4, a linear functional f on X is continuous if and only if $|f(x)| \leqslant C q_k(x), x \in X$, for some seminorm q_k and constant $C > 0$. Also, boundedness can be reformulated in terms of the seminorms of a fundamental family for X, instead of all (continuous) seminorms on X.

Example 2.7.

- clearly every Banach space is a Fréchet space.
- the space $C(\mathbb{R}^N;\mathbb{C})$ of all continuous functions $f\colon \mathbb{R}^N \to \mathbb{C}$ equipped with the family of seminorms

$$p_n(f) := \sup_{|x|\leqslant n} |f(x)|, n \in \mathbb{N},$$

is a Fréchet space. When it comes to study delay differential equations with infinite delay, the space $C\big((-\infty,0],\mathbb{C}\big)$ is of particular interest, see (WALTHER, 2016a; WALTHER, 2016b).

Usually, it is said that this Fréchet space convergence is the uniform convergence on compact sets. As the reader may deduce, the same idea is true for $C(H;\mathbb{C})$, where H is a locally compact σ-compact Hausdorff space.

Proof. It is not hard to check that every p_n defines a norm on $C\big(\mathbb{R}^N;\mathbb{C}\big)$ and since a function is null if and only if it is null in every ball $\overline{B_n}$, the family $(p_n)_{n\in\mathbb{N}}$ is separating. By Theorem 2.3, $C\big(\mathbb{R}^N;\mathbb{C}\big)$ is a Hausdorff metrizable LCS with a translation invariant metric ρ. Therefore one just have to check its completeness, which is straightforward since the limit of a sequence of continuous functions on $\overline{B_n}$ is again a continuous function on $\overline{B_n}$. □

- the space of all holomorphic functions in the complex plane is a Fréchet space when equipped with the seminorms p_n of $C(\mathbb{C};\mathbb{C})$.

- C^∞, the space of infinitely differentiable complex-valued functions on $\mathbb{R}^N$, together with the seminorms
$$C^\infty \ni f \mapsto \max_{|x|,|\alpha|\leqslant n}\left|\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|, n\in\mathbb{N},$$
is a Fréchet space (as well as every space C^k).

- now we consider the Schwartz space $\mathcal{S}=\mathcal{S}\big(\mathbb{R}^N;\mathbb{C}\big)$: the class of those smooth functions which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. Formally, we set
$$\mathcal{S} := \{f\in C^\infty \colon \|f\|_{(n,\alpha)} < \infty \text{ for every } n \text{ and } \alpha\},$$
where, for $n\in\mathbb{Z}_+$ and $\alpha\in\mathbb{Z}_+^N$, the map
$$f\mapsto \|f\|_{(n,\alpha)} := \sup_{x\in\mathbb{R}^N}(1+|x|)^n|\partial^\alpha f(x)|$$
defines a seminorm on $\mathcal{S}$. Sometimes it is said that $\mathcal{S}$ consists of all rapidly decreasing functions on $\mathbb{R}^N$. It is not hard to see that every compactly supported C^∞ function is a Schwartz function as well as the map $x\mapsto p(x)e^{-|x|^2}$, for a given polynomial $p\colon\mathbb{R}^N\to\mathbb{C}$.

It is a separable Fréchet space on which the Fourier transform is a topological isomorphism (by the Fourier Inversion theorem 2.26).

Proof. The only nontrivial points are completeness and separability. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}$ i.e. $\|f_j-f_k\|_{(n,\alpha)}$ for all n,α. In particular, for each α the sequence $\left(\partial^\alpha f_k\right)_{k\in\mathbb{N}}$ converges uniformly to a function g_α. If e_j denotes the vector $(0,\ldots,1,\ldots,0)$ with the 1 in the jth position then letting $k\to\infty$ in

$$f_k(x+te_j)-f_k(x)=\int_0^t \partial_j f_k(x+se_j)\,ds$$

we obtain

$$g_0(x+te_j)-g_0(x)=\int_0^t g_{e_j}(x+se_j)\,ds.$$

The fundamental theorem of calculus implies that $g_{e_j}=\partial_j g_0$ and an induction on $|\alpha|$ then yields $g_\alpha=\partial^\alpha g_0$ for all α. It is easy to check that $\|f_k-g_0\|_{(n,\alpha)}\to 0$.

On the separability, since $C_c^\infty \overset{d}{\hookrightarrow} \mathcal{S} \overset{d}{\hookrightarrow} L^p$ for $1\leqslant p<\infty$ - where $\overset{d}{\hookrightarrow}$ denotes dense continuous inclusions -, one concludes that $\mathcal{S}$ is separable. See (CORDARO, 1999), page 100, or (FOLLAND, 1999), Proposition 9.9. □

Besides, if $f,g\in C^\infty$ then the Leibniz formula,

$$\partial^\alpha(fg)=\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\left(\partial^{\alpha-\gamma}f\right)\left(\partial^\gamma g\right),$$

implies that $f\in\mathcal{S}$ if and only if $x^\beta\partial^\alpha f$ is bounded for every multiindex α,β if and only if $\partial^\alpha(x^\beta f)$ is bounded for every multiindex α,β.

- for every finite $p\geqslant 1$, $L^p_{loc}=L^p_{loc}\left(\mathbb{R}^N;\mathbb{C}\right)$ is a Fréchet space with the seminorms

$$f\mapsto\left(\int_{|x|\leqslant n}|f(x)|^p\,dx\right)^{1/p},n\in\mathbb{N}.$$

Remark 2.8. There is a certain duality between separability and metrizability, as it occurs with Banach spaces. Let X be a LCS.

- If X is separable and K is an equicontinuous subset of X' then K is metrizable with respect to $\sigma(X',X)$ topology.
- If X is a separable metrizable space then $\left(X',\sigma(X',X)\right)$ is separable.
- The inductive limit of countably many separable spaces is separable.

To check the definition of the weak topology $\sigma(X',X)$ when X is a LCS, see for instance (OSBORNE, 2013).

The primary reason why LCSs are so useful is that there are guaranteed to be plenty of continuous linear functionals; and the Hahn-Banach theorem provides them.

Theorem 2.9 (The Hahn-Banach theorem)**.** Let Y be a subspace of a real vector space X and let $p\colon X \to [0,\infty)$ be a sublinear functional on X.

If $f\colon Y \to \mathbb{R}$ is a linear functional which is dominated by p on Y then f extends to a linear functional $F\colon X \to \mathbb{R}$ which is dominated by p on X. If, in addition, X is a TVS and p is continuous then F is continuous as well.

From the result below, it follows the main separation theorems for convex sets in a Hausdorff LCS X. In particular, the dual space of X separates points; and it also separates points from subspaces.

Theorem 2.10 (The Hahn-Banach theorem - geometric form)**.** Let C_1 and C_2 be two disjoint nonempty convex subsets of an LCS X. If C_1 is closed and C_2 is compact then there are a real number r_0 and a continuous linear functional $F\colon X \to \mathbb{R}$ for which

$$F(x) < r_0 < F(y) \text{ for every } x \in C_1 \text{ and } y \in C_2.$$

The local convexity and Hausdorff conditions provide the minimal requirements under which three classic results hold; namely, the Uniform Boundedness Property; the Open Mapping theorem; and the Closed Graph theorem.

Theorem 2.11 (The Banach-Steinhaus theorem)**.** Let X and Y be Hausdorff LCSs, let $\mathscr{H} \subset \mathscr{L}(X,Y)$ and consider the following three conditions on $\mathscr{H}$:

(i) $\mathscr{H}$ is equicontinuous.

(ii) $\mathscr{H}$ is bounded for the bounded convergence.

(iii) $\mathscr{H}$ is bounded for the pointwise convergence.

Then

a. **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** always.

b. if X is sequentially complete then **(iii)** $\Rightarrow$ **(ii)**.

c. if X is Fréchet then **(i)**-**(iii)** are all equivalent.

Technically, the term “uniform boundedness theorem” applies best to part **b.** while “Banach-Steinhaus theorem” applies best to part **c.**. As stated here, Theorem 2.11 is more general than the classical Banach-Steinhaus theorem but there seems to be some disagreement in the literature as to exactly what result should be called the “Banach-Steinhaus theorem” and what should be called the “uniform boundedness theorem”.

Remark 2.12. A family $\mathscr{H} \subset \mathscr{L}(X,Y)$ is said to be **equicontinuous** if for every neighborhood V of 0 in Y there exists a neighborhood U of 0 in X such that $T(U) \subset V$ for every $T \in \mathscr{H}$. We say that $\mathscr{H}$ is **bounded for the bounded convergence** if for every bounded set $B \subset X$ the set

$$\bigcup_{T \in \mathscr{H}} T(B)$$

is bounded in Y; and $\mathscr{H}$ is **bounded for the pointwise convergence** if for every $x \in X$ the set $\{Tx \colon T \in \mathscr{H}\}$ is bounded in Y.

Clearly these concepts may be reformulated in terms of seminorms. We may write $X = \big(X, (p_\alpha)_{\alpha \in A}\big)$ and $Y = \big(Y, (q_\beta)_{\beta \in B}\big)$. Hence $\mathscr{H}$ is equicontinuous if for every $\alpha \in A$ there exist $\beta = \beta(\alpha) \in B$ and a constant $C = C(\alpha) > 0$ such that $q_\beta(Tx) \leqslant C p_\alpha(x)$ for every $T \in \mathscr{H}$ and $x \in X$. Similarly for the other concepts.

As a consequence the limit operator of a pointwisely convergent pointwisely bounded net of operators in $\mathscr{L}(X,Y)$ is a continuous linear operator as well, provided that X is minimally "good". In case you are curious, we should require X to be a barreled TVS and Y to be a LCS. A TVS is said to be **barreled** if every absorbing balanced convex closed subset is a neighborhood of 0. Fortunately every Fréchet space is barreled and some results we present actually require the space to be only barreled and sometimes something else not too strong such as being Hausdorff. If one deals only with sequences, things turn out to be slightly simpler.

As a curiosity, we present the Arzelà-Ascoli theorem for TVSs, probably a surprise to some readers as it was for us. It essentially states that $\mathscr{H} \subset \big(C(X,\mathbb{K}), \tau_K\big)$ is compact if and only if it is equicontinuous, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and τ_K denotes the topology of uniform convergence on compact sets (which is also called the compact-open topology).

Theorem 2.13 (The Arzelà-Ascoli theorem for TVSs)**.** Let X be a TVS.

- **a.** If $\mathscr{H}$ is an equicontinuous subset of $\big(C(X,\mathbb{K}), \tau_K\big)$ and is bounded for the pointwise convergence then $\mathscr{H}$ is precompact.
- **b.** if X is locally compact and $\mathscr{H}$ is a precompact subset of $\big(C(X,\mathbb{K}), \tau_K\big)$ then $\mathscr{H}$ is equicontinuous and bounded for the pointwise convergence.

There are several versions of the Open Mapping theorem for TVSs, involving different weaker assumptions on the TVSs and a weaker concept of a mapping "being open". Given two TVSs X and Y, a linear map $A \colon X \to Y$ is **almost/nearly open** if $\overline{A(U)}$ is a neighborhood of 0 in Y whenever U is a neighborhood of 0 in X.

Theorem 2.14 (The Open Mapping theorem)**.** Let X be a Fréchet space and let Y be a Hausdorff LCS. If $A \colon X \to Y$ is a continuous nearly open linear map then A is open (and onto).

Theorem 2.15. Let X and Y be Fréchet spaces and let $A\colon X \to Y$ be a linear map.

a. (Closed Graph theorem) if A is closed then it is continuous.

b. (Open Mapping theorem) if A is closed and onto then it is open.

Actually, to obtain the Closed Graph theorem, it suffices to assume that X is either a Baire TVS or a barreled TVS; and that Y is a complete pseudometrizable LCS. Besides, since the idea of the proof of the Open Mapping theorem is to apply the Closed Graph theorem to $\tilde{A}^{-1}$, where $\tilde{A}\colon X/A^{-1}(0) \to Y$ is defined by $\tilde{A}\big(x + A^{-1}(0)\big) := Ax$, one sees that it suffices to assume that the codomain of A is either a Baire TVS or a barreled TVS; and that its domain is a complete pseudometrizable LCS.

Remark 2.16. By the category theorem ((RUDIN, 1991), Theorem 2.2), every complete metrizable TVS is a Baire space. Since local convexity is so essential as we have seen, Proposition 2.6 implies that the best examples of Baire TVSs are Fréchet spaces.

Of course, the category theorem also states that locally compact Hausdorff spaces are Baire spaces. But as the well known Riesz theorem claims, every locally compact TVS has finite dimension; and we would go back to euclidean geometry.

2.2 Elements of Fourier analysis and distribution theory

We shall deal with functions and distributions defined on $\mathbb{R}^N$ instead on an open subset of $\mathbb{R}^N$, since the latter will not be necessary. Hence we simplify the notation by not mentioning the space $\mathbb{R}^N$; for instance, we write L^1 instead of $L^1\big(\mathbb{R}^N\big)$ or $L^1\big(\mathbb{R}^N, \mathbb{C}\big)$.

Theorem 2.17 (The Dominated Convergence theorem)**.** Let (f_n) be a sequence of L^1 functions such that $f_n \to f$ a.e. and $|f_n|$ are dominated by some nonnegative L^1 function then $f \in L^1$ and $\displaystyle\int f = \lim_n \int f_n$.

Theorem 2.18 (The Fubini-Tonelli theorem)**.** Suppose that $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. Given a measurable function $f\colon X \times Y \to \mathbb{C}$, if either f is nonnegative or $f \in L^1$ then

$$\int f\, d(\mu \times \nu) = \int \left(\int f\, d\mu \right) d\nu = \int \left(\int f\, d\nu \right) d\mu$$

holds.

Theorem 2.19. Let $f\colon \mathbb{R}^N \to \mathbb{C}$ be a measurable function and let c and s denote positive constants.

a. if $|f(x)| \leqslant c|x|^{-s}$ on $B := B(0,1)$ for some $s < N$ then $f \in L^1(B)$;

b. if $|f(x)| \geqslant c|x|^{-n}$ on B then $f \notin L^1(B)$;

c. if $|f(x)| \leqslant c|x|^{-s}$ on B^c for some $s > N$ then $f \in L^1(B^c)$; and

d. if $|f(x)| \geqslant c|x|^{-n}$ on B^c then $f \notin L^1(B^c)$.

Theorem 2.20 (The Lebesgue Differentiation theorem)**.** If $f \in L^1_{loc}$ then

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy \; = f(x), \text{ for almost every } x.$$

Corollary 2.21. If $f \in L^1_{loc}$ satisfies $\int f\phi = 0$ for every $\phi \in C_c^\infty$ then $f = 0$.

Theorem 2.22 (The Riesz Representation theorem)**.** Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in \left(L^p\right)'$ then there exists a unique function $f \in L^q$ such that $\|f\|_{L^q} = \|u\|_{(L^p)'}$ and

$$\langle u, \phi \rangle = \int f\phi \text{ for every } \phi \in L^p.$$

Hence L^q is isometrically isometric to $\left(L^p\right)'$.

Theorem 2.23. Let $1 < p < \infty$, $\phi \in L^1$ with $\int \phi = a$ and set $\phi_t(x) := t^{-N}\phi(t^{-1}x)$.

a. if $f \in L^p$ then $f * \phi_t \to af$ in the L^p norm as $t \to 0$;

b. If f is bounded and uniformly continuous then $f * \phi_t \to af$ uniformly as $t \to 0$; and

c. if $f \in L^\infty$ and f is continuous in an open set U then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

Proposition 2.24. C_c^∞ (and hence $\mathcal{S}$) is dense in L^p, $1 \leqslant p \leqslant \infty$.

We begin by defining the **Fourier transform** of a function $f \in L^1$ by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} f(x)\,dx,$$

where $\xi \cdot x = \langle \xi, x \rangle$ denotes the $\mathbb{R}^N$ inner product of ξ and x. As in (FOLLAND, 1999), we use the notation $\mathscr{F}$ for the Fourier transform only in certain situations where it is needed for clarity. Clearly $\|f\|_u \leqslant \|f\|_{L^1}$ and $\xi \mapsto \widehat{f}(\xi)$ is continuous by the Dominated Convergence theorem; thus

$$\mathscr{F} : L^1 \to BC$$

is linear and continuous.

Theorem 2.25. Let $f, g \in L^1$ and let τ_z stand for the translation map: $\tau_z f(x) := f(x - z)$.

a. $\mathscr{F}(\tau_z f)(\xi) = e^{-2\pi i \xi \cdot z}(\mathscr{F} f)(\xi)$ and $\left(\tau_z \mathscr{F} f\right)(\xi) = \mathscr{F}\left(e^{2\pi i \langle z, \cdot \rangle} f(\cdot)\right)(\xi)$;

b. if T is an invertible linear transformation on $\mathbb{R}^N$ and $S = (T^*)^{-1}$ is its inverse transpose then

$$\mathscr{F}(f \circ T) = |\det T|^{-1} (\mathscr{F} f) \circ S;$$

c. $\mathscr{F}(f * g) = (\mathscr{F} f)(\mathscr{F} g)$;

d. if $x^\alpha f \in L^1$ whenever $|\alpha| \leqslant k$ then $\widehat{f} \in C^k$ and

$$\partial^\alpha \big(\mathscr{F} f\big)(\xi) = \mathscr{F}\big((-2\pi i x)^\alpha f\big)(\xi);$$

e. if $f \in C^k$; $\partial^\alpha f \in L^1$ whenever $|\alpha| \leqslant k$; and $\partial^\alpha f \in C_0$ whenever $|\alpha| \leqslant k-1$, then

$$\mathscr{F}\big(\partial^\alpha f\big)(\xi) = (2\pi i \xi)^\alpha \big(\mathscr{F} f\big)(\xi);$$

and

f. (The Riemann-Lebesgue Lemma) $\mathscr{F}(L^1) \subset C_0$.

In the result above, C_0 denotes the space of continuous functions $f \colon \mathbb{R}^N \to \mathbb{C}$ such that $f(x) \to 0$ as $|x| \to \infty$.

We are now ready to invert the Fourier transform. If $f \in L^1$, we define

$$\check{f}(x) := \widehat{f}(-x) = \int_{\mathbb{R}^N} e^{2\pi i \xi \cdot x} f(\xi)\, d\xi.$$

Theorem 2.26 (The Fourier Inversion theorem)**.** If $f, \widehat{f} \in L^1$ then f agrees almost everywhere with a continuous function f_0 and

$$(\widehat{f})^{\vee} = (\check{f})^{\wedge} = f_0.$$

Consequently, $\mathscr{F}$ is an isomorphism of $\mathcal{S}$ onto itself.

Theorem 2.27 (The Plancherel theorem)**.** The Fourier transform $\mathscr{F}$ is a unitary isomorphism on L^2, that is, an isomorphism such that

$$\big(\mathscr{F} f, \mathscr{F} g\big)_{L^2} = (f, g)_{L^2}, \text{ for every } f, g \in L^2.$$

Nearly all the spaces routinely used in analysis are one of four types: Banach spaces, Fréchet spaces, LF-spaces, or the dual spaces of Fréchet spaces or LF-spaces. The main example of an LF-space is precisely the test functions space on $\mathbb{R}^N$: C_c^∞, which is so important that it is conceivable that LF-spaces, as a class of locally convex spaces, would be defined even if C_c^∞ were the only example. Let $B_n := \{x \in \mathbb{R}^N : |x| \leqslant n\}$; then

$$X_n := \{\phi \in C_c^\infty : \operatorname{supp} \phi \subset B_n\}, n \in \mathbb{N},$$

is a Fréchet space with the following three properties (which turn out to define LF-spaces):

- $X_n \subset X_{n+1}$;
- the topology X_{n+1} induces on X_n is its Fréchet topology; and
- $X_n \neq X$ for every n.

The base for the LF-topology on C_c^∞ is given by all convex balanced subsets $B \subset C_c^\infty$ such that $B \cap X_n$ is a neighborhood of 0 in the Fréchet topology of X_n, for every n. Hence,

- a sequence (ϕ_n) in C_c^∞ converges to 0 in C_c^∞ if there exists a compact subset $K \subset \mathbb{R}^N$ such that every ϕ_n belongs to $C_c^\infty(K)$ and $\phi_n \xrightarrow[n\to\infty]{C_c^\infty(K)} 0$;
- let Y be an LCS and $T\colon C_c^\infty \to Y$ be a linear operator. Since every $C_c^\infty(B_n)$ is metrizable, T is continuous if and only if $T\big|_{C_c^\infty(B_n)}$ is sequentially continuous at 0, for every n.

The name "LF-space" comes from the fact that those spaces can be seen as a limit of Fréchet spaces, in some sense. The reader may promptly infer about the definition of LB-spaces and take as an example the space of all polynomials of one real variable. LF-spaces are complete Hausdorff LCSs but never (pseudo)metrizable and hence they define a strictly larger class of that one formed by Fréchet spaces. Besides LF-spaces provide a good example of complete meager TVSs; see (NARICI; BECKENSTEIN, 2010).

Given a function $f \in L^1_{loc}$, the map $\phi \mapsto \int \phi f$ is a well-defined linear functional on C_c^∞ and the pointwise values of f can be recovered a.e. from it by the following result, which is an application of the Lebesgue differentiation theorem:

Theorem 2.28. Suppose $|\phi(x)| \leqslant c(1+|x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$ and $\int \phi = 1$. For $x \in \mathbb{R}^N$ and $t > 0$, let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. If$f \in L^1_{loc}$ then, for every x in the Lebesgue set of f,

$$\lim_{t\to 0} f * \phi_t(x) = f(x);$$

in particular, for almost every x and for every x at which f is continuous.

This means that we may abandon the classical definition of function as a map that assigns to each point of $\mathbb{R}^N$ a vector value. Instead of dealing with the pointwise values of f, we can consider the family of integrals $\int f\phi$ as ϕ ranges over C_c^∞ when it comes to recognize it as a function. But there are many linear functionals on C_c^∞ that are not of the form $\phi \mapsto \int \phi f$ and this provides a notion of "generalized functions".

A **distribution** on $\mathbb{R}^N$ is a continuous linear functional u on C_c^∞ and we write

$$\begin{array}{rccc} u\colon & C_c^\infty & \to & \mathbb{C} \\ & \phi & \mapsto & \langle u, \phi \rangle \end{array}.$$

In other words, a distribution on $\mathbb{R}^N$ is a linear functional $u\colon C_c^\infty \to \mathbb{C}$ with the property that for every compact subset $K \subset \mathbb{R}^N$ there exist a positive constant $c = c(K,u)$ and a non-negative integer $k = k(K,u)$ such that

$$|\langle u, \phi \rangle| \leqslant c \sum_{|\alpha| \leqslant k} \sup_{x \in \mathbb{R}^N} |\partial^\alpha \phi(x)|, \text{ for } \phi \in C_c^\infty(K, \mathbb{C}).$$

The space of all distributions on $\mathbb{R}^N$ is denoted by $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^N)$, which is equipped with the $\bigstar$-weak topology, that is, the topology of the pointwise convergence. The standard notation $\mathcal{D}'$ for the space of distributions comes from Schwartz's notation $\mathcal{D}$ for C_c^∞, which is also quite common. We say that two distributions $u, v \in \mathcal{D}'$ are equal if $\langle u - v, \phi \rangle = 0$, for every $\phi \in C_c^\infty$.

Example 2.29.

- every function $f \in L^p_{loc}$ defines a distribution by setting
$$\langle f, \phi \rangle := \int_{\mathbb{R}^N} f(x)\phi(x)\,dx, \text{ for every } \phi \in C_c^\infty;$$
- the point mass at the origin, namely $\phi \mapsto \phi(0)$, is also a distribution - which plays a central role in distribution theory and it is denoted by δ; and
- if μ is a Radon measure on $\mathbb{R}^N$ then $\phi \mapsto \int \phi\, d\mu$ is a distribution.

The theory of distributions frees differential calculus from certain difficulties that arise because nondifferentiable functions exist. We define the **derivative** $\partial^\alpha u$ of a distribution u by

$$\langle \partial^\alpha u, \phi \rangle := (-1)^\alpha \langle u, \partial^\alpha \phi \rangle,$$

which is motivated by the formula $\int (\partial^\alpha f)\phi = (-1)^{|\alpha|} \int f(\partial^\alpha \phi)$, for $f \in C^{|\alpha|}$ and $\phi \in C_c^\infty$.

Hence the usual formal rules of calculus hold and every distribution u has partial derivatives which are again distributions; in particular, u is infinitely differentiable. Besides, if $u \in \mathcal{D}'$ is actually differentiable then this new notion of derivative coincides with the usual one. Similarly, this procedure motivates the following definitions for a given distribution u.

- **(Multiplication by smooth functions)** If $\psi \in C^\infty$ then ψu is the distribution given by
$$\langle \psi u, \phi \rangle := \langle u, \psi \phi \rangle, \text{ for } \phi \in C_c^\infty.$$
- **(Translation)** Let τ_z stand for the translation map (as in Theorem 2.25). For $\phi \in C_c^\infty$, we set
$$\langle \tau_z u, \phi \rangle := \langle u, \tau_{-z} \phi \rangle.$$
Usually the point mass at z, $\tau_z \delta$, is denoted by δ_z.

- **(Composition with linear maps)** If T is an invertible operator on $\mathbb{R}^N$ then

$$\langle Tu, \phi\rangle := |\det T|^{-1}\langle u, \phi \circ T^{-1}\rangle, \text{ for } \phi \in C_c^\infty,$$

defines a distribution.

- **(Convolution with test functions)** We use the following notation for the reflexion map: $\tilde{\phi}(x) = \phi(-x)$. If $\psi \in C_c^\infty$ then the **convolution** $u * \psi$ is the distribution defined by

$$\langle u * \psi, \phi\rangle := \langle u, \phi * \tilde{\psi}\rangle, \text{ for } \phi \in C_c^\infty,$$

which is motivated by the formula $\int (f * \psi)\phi = \int f(\phi * \tilde{\psi})$, for a given function $f \in L^1_{loc}$. Or equivalently $u * \psi$ is the distribution defined by the C^∞ function $x \mapsto \langle u, \tau_x \tilde{\psi}\rangle$. Such a definition comes from the formula $f * \psi(x) = \int f(\tau_x \tilde{\psi})$, for a given function $f \in L^1_{loc}$; and it naturally carries out the property:

$$\partial^\alpha (u * \psi) = (\partial^\alpha u) * \psi = u * (\partial^\alpha \psi).$$

As a consequence, one can prove that C_c^∞ is dense in $\mathcal{D}'$ in the topology of $\mathcal{D}'$.

The **support** of a distribution u is denoted by $\operatorname{supp} u$ and is defined as follows: $x \notin \operatorname{supp} u$ if there exists an open neighborhood U of x such that $\langle u, \phi\rangle = 0$, for every $\phi \in C_c^\infty(U)$. The space of all **compactly supported distributions** is denoted by $\mathcal{E}'$ and can be identified with the dual space of C^∞. More precisely, if $u \in \mathcal{E}'$ then it extends uniquely to a continuous linear functional on C^∞; conversely, if u is a continuous linear functional on C^∞ then $u\big|_{C_c^\infty} \in \mathcal{E}'$. The linear operations discussed above preserve the class $\mathcal{E}'$ and additional convolution properties allow us to define $u * v$ and $v * u$ as distributions, provided that $u \in \mathcal{D}'$ and $v \in \mathcal{E}'$. Actually, $u * v = v * u$. See (RUDIN, 1991; SCHWARTZ, 1966).

For a function $f \colon \mathbb{R}^N \to \mathbb{R}$, the formula $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$ provides a connection between smoothness and decay: f is smooth if and only if $\widehat{f}$ rapidly decays to 0 as $|\xi| \to \infty$. If one can apply the inverse Fourier transform then also f rapidly decays to 0 as $|x| \to \infty$ if and only if $\widehat{f}$ is smooth. Thus some ordinary differential equations can be transformed into algebraic equations, which are easier to deal with and to solve. The same idea holds for L^2 functions with weak derivatives or even distributions with their distributional derivatives and the result that formally establish such a connection is the Paley-Wiener-Schwartz theorem (for L^2 functions and for compactly supported distributions). It relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform. The theorem is named for Raymond Paley (1907-1933) and Norbert Wiener (1894-1964), who originally applied to square-integrable functions, see (STEIN; WEISS, 1971); the first version using distributions was due to Laurent Schwartz (1915-2002). It basically states that a distribution is compactly supported if and only if its Fourier transform can be extended to an analytic function whose growth has a particular exponential boundedness.

Proposition 2.30. If $u \in \mathcal{E}'$ then the distribution $\widehat{u}$ is actually a function on $\mathbb{R}^N$, namely,

$$\widehat{u}(\xi) = \big\langle u, \exp\big(-2\pi i \langle \xi, \cdot \rangle\big)\big\rangle, \xi \in \mathbb{R}^N,$$

which can be extended to an analytic function on $\mathbb{C}^N$.

Theorem 2.31 (The Paley-Wiener-Schwartz theorem)**.** An analytic function $V : \mathbb{C}^N \to \mathbb{C}$ is the Fourier transform of a compactly supported distribution u if and only if

$$|V(z)| \leqslant c(1+|z|)^L \exp(R|y|), \text{ for every } z = x+iy \in \mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N,$$

for some choice of constants $c, R > 0$ and $L \in \mathbb{N}$, which depend on u.

He have extended several linear operations to distributions in $\mathcal{D}'$ or in $\mathcal{E}'$ such as differentiation and convolution (with a fixed appropriately supported distribution v). But the Fourier transform $\mathscr{F}$ is a notable omission in that list. It happens that $\mathscr{F}$ does not map C_c^∞ into itself and hence $\mathscr{F} : \mathcal{D}' \to \mathcal{D}'$ is not well defined. Actually, if $\phi \in C_c^\infty$ then $\widehat{\phi}$ cannot vanish on any nonempty open set unless $\phi \equiv 0$. In fact, let $\widehat{\phi} = 0$ on a neighborhood of ξ_0 of $\mathbb{R}^N$, which we may assume to be 0 (otherwise we would deal with $\psi(x) = e^{-2\pi i \xi_0 \cdot x}$). Since $\operatorname{supp}\phi$ is compact, ϕ has a Maclaurin representation and we can integrate term by term to obtain

$$\widehat{\phi}(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \int (-2\pi i \xi \cdot x)^k \phi(x)\, dx = \sum_{\alpha} \frac{\xi^\alpha}{\alpha!} \int (-2\pi i x)^\alpha \phi(x)\, dx = \sum_{\alpha} \frac{\xi^\alpha}{\alpha!} \partial^\alpha \widehat{\phi}(0) = 0$$

by assumption, from whence $\widehat{\phi} = 0$ and then $\phi = 0$.

On the other hand, the Schwartz class $\mathcal{S}$ is a slightly larger space of test functions such that $C_c^\infty \overset{d}{\hookrightarrow} \mathcal{S} \overset{d}{\hookrightarrow} C^\infty$ and $\mathscr{F} : \mathcal{S} \to \mathcal{S}$ is a topological isomorphism. The dual of this Fréchet space provides the suitable distributional setting to define the Fourier transform.

A **tempered distribution** is a continuous linear functional u on $\mathcal{S}$ and hence we write $u \in \mathcal{S}'$. Since $C_c^\infty \overset{d}{\hookrightarrow} \mathcal{S}$, we may and shall, identify $\mathcal{S}'$ with the set of distributions that extend continuously from C_c^∞ to $\mathcal{S}$.

Example 2.32.

- every compactly supported distribution is tempered: $\mathcal{E}' \subset \mathcal{S}'$; and
- if $f \in L^1_{loc}$ with $\|(1+|x|)^N f\|_{L^1} < \infty$ then $\phi \mapsto \int \phi f$ is a tempered distribution. And similarly for L^p_{loc} functions.
- every L^p function, $1 \leqslant p \leqslant \infty$, is a tempered distribution; so is every polynomial and more generally every measurable function which is dominated by some polynomial.

The reader is invited to check that the operations of differentiation, translation and composition with linear transformations all map $\mathcal{S}$ and $\mathcal{S}'$ into themselves. However the map $u \mapsto \psi u$ preserves $\mathcal{S}$ and $\mathcal{S}'$ if ψ and all its derivative have at most polynomial growth at infinity:

$$\left|\partial^\alpha \psi(x)\right| \leqslant C_\alpha (1+|x|)^{n(\alpha)} \text{ for every } \alpha,$$

which are called **slowly increasing functions**. Clearly every polynomial is slowly increasing.

At last, inspired on the equality $\int \widehat{f} g = \int f \widehat{g}$ for $f, g \in L^1$, the Fourier transform $\widehat{u}$ of a tempered distribution u is defined by

$$\langle \widehat{u}, \phi \rangle := \langle u, \widehat{\phi} \rangle \text{ for } \phi \in \mathcal{S},$$

whence $\widehat{u}$ is again a tempered distribution and we immediately obtain analogous properties of those in Proposition 2.25. Moreover, we can extend the inverse Fourier transform on $\mathcal{S}$ to $\mathcal{S}'$ by setting $\langle \check{u}, \phi \rangle := \langle u, \check{\phi} \rangle$, $\phi \in \mathcal{S}$; thus $\mathscr{F}$ is a topological isomorphism on $\mathcal{S}'$ as well. This definition and the continuous inclusion $\mathcal{E}' \hookrightarrow \mathcal{S}'$ justify that Proposition 2.30 is well stated.

When one says that the Fourier transform is well behaved on L^2, there are at least one main reason to agree with it: by the Plancherel theorem, $\mathscr{F}$ is a unitary isomorphism on L^2 which converts differentiation into multiplication by the coordinate functions. As a consequence, we have the following: let $k \in \mathbb{N}$ and $f \in L^2$, its distributional derivatives $\partial^\alpha f$ are actually L^2 functions for every $|\alpha| \leqslant k$ if and only if $(2\pi i \xi)^\alpha \widehat{f}$ belongs to L^2 for every $|\alpha| \leqslant k$ if and only if $(1+4\pi^2|\xi|^2)^{k/2}\widehat{f}$ belongs to L^2, because

$$c_1\left(1+4\pi^2|\xi|^2\right)^k \leqslant \sum_{|\alpha|\leqslant k} |\xi^\alpha|^2 \leqslant c_2\left(1+4\pi^2|\xi|^2\right)^k,$$

for some positive constants c_1 and c_2, which depend on k and N.

The space H^k of all L^2 functions which satisfy this condition can be equipped with one of the following equivalent norms:

$$f \mapsto \left(\sum_{|\alpha|\leqslant k} \left\|\partial^\alpha f\right\|_{L^2}^2\right)^{1/2} \quad \text{and} \quad f \mapsto \left\|(1+4\pi^2|\xi|^2)^{k/2}\widehat{f}\right\|_{L^2}.$$

The advantage of the latter norm is that it makes sense for any $k \in \mathbb{R}$. Besides, for any $s \in \mathbb{R}$, the map $\xi \mapsto (1+4\pi^2|\xi|^2)^{s/2}$ is a slowly increasing function, consequently a tempered distribution which turns out to be a Fourier multiplier used to define the Bessel potentials $\Lambda_s \colon \mathcal{S}' \to \mathcal{S}'$ by setting

$$\Lambda_s u := \left((1+4\pi^2|\xi|^2)^{s/2}\widehat{u}\right)^{\vee} \text{ for every } u \in \mathcal{S}'.$$

These facts collected provide the background to define the generalized **Sobolev spaces** $H^s = H^s(\mathbb{R}^N;\mathbb{C})$ by

$$H^s := \{u \in \mathcal{S}' \colon \Lambda_s u \in L^2\}, s \in \mathbb{R},$$

which is a Hilbert space with the inner product $(\cdot,\cdot)_{H^s}$ defined by

$$(u,v)_{H^s} := \int_{\mathbb{R}^N} \Lambda_s u \overline{\Lambda_s v},$$

Readers who wish to learn more about Fourier multipliers and generalized Sobolev spaces $W^{s,p} = W^{s,p}(\mathbb{R}^N;\mathbb{C})$, with $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, (actually their definition is quite guessable at this point), can find a brief introduction in (ADAMS; FOURNIER, 2003) and detailed treatments in (BERGH; LOFSTROM, 2012; PEETRE; DEPT, 1976; TRIEBEL, 1978).

2.3 Semigroups of linear operators on LCSs

Although we shall not use the entire theory of the equicontinuous C_0-semigroups on LCSs, we shall present it anyway since it may be not known by many readers who deal with semigroups in Banach spaces. It is crucial to understand the generalization we obtained in Chapter 3.

To avoid confusion, in this section we shall write $(X, \|\cdot\|_X)$ to denote Banach spaces and we shall write $\big(X, (p_\alpha)_{\alpha\in A}\big)$ or simply X to denote **sequentially complete Hausdorff LCSs**.

The theory of semigroups of bounded linear operators in a Banach space is concerned with the problem of determining the most general bounded linear operator valued function $T(t)$, $t \geqslant 0$, which satisfies the equations

$$T(t+s) = T(t) \circ T(s) \text{ and } T(0) = Id_X. \tag{2.1}$$

The problem was investigated by E. Hille [2] and K. Yosida [5] independently of each other around 1948. Thanks to the notion of the infinitesimal generator A of $T(t)$, which is pointwisely defined by the limit of $t^{-1}\big(T(t) - Id_X\big)$ as $t \to 0^+$ whenever it exists, they discussed the generation of $T(t)$ in terms of A and obtained a characterization of the infinitesimal generator in terms of its spectral properties.

Proposition 2.33 (Hille)**.** Let $\big(T(t)\big)_{t\geqslant 0}$ be a family of bounded linear operators in $(X, \|\cdot\|_X)$ such that $T(t+s) = T(t) \circ T(s)$ for every $t, s > 0$. If $(0,\infty) \ni t \mapsto \log \|T(t)\|_{\mathscr{L}(X)}$ is bounded from above on every interval $(0,\tau)$, $\tau > 0$, then

$$\lim_{t\to\infty} t^{-1} \log \|T(t)\|_{\mathscr{L}(X)} = \inf_{t>0} t^{-1} \log \|T(t)\|_{\mathscr{L}(X)}.$$

Proof. Set $f(t) := \log \|T(t)\|_{\mathscr{L}(X)}$. It is easily checked that $f(t+s) \leqslant f(t) + f(s)$.

First assume that $\beta := \inf_{t>0} t^{-1} f(t)$ is finite. Given $\varepsilon > 0$, take $t_\varepsilon > 0$ such that $f(t_\varepsilon) \leqslant (\beta + \varepsilon) t_\varepsilon$. Now let $t > t_\varepsilon$ and choose a nonnegative integer $n = n(t)$ which satisfies $nt_\varepsilon \leqslant t < (n+1)t_\varepsilon$, then $f(t) \leqslant f(nt_\varepsilon) + f(t - nt_\varepsilon) \leqslant nt_\varepsilon \dfrac{f(t_\varepsilon)}{t_\varepsilon} + f(t - nt_\varepsilon)$, whence

$$\beta \leqslant \frac{f(t)}{t} \leqslant \frac{n(t)t_\varepsilon}{t}(\beta + \varepsilon) + \frac{f\big(t - n(t)t_\varepsilon\big)}{t}.$$

By construction, $t^{-1} n(t) t_\varepsilon \leqslant 1$; and $t - n(t)t_\varepsilon \in (0, t_\varepsilon)$ so that $f(t - n(t)t_\varepsilon)$ is bounded from above as $t \to \infty$. Thus $\lim_{t\to\infty} t^{-1} f(t) = \beta$; the argument is similar for $\beta = -\infty$.

□

A family $\big(T(t)\big)_{t\geqslant 0}$ of bounded linear operators in X is said to be a **semigroup** on X if it satisfies (2.1); we write $T(\cdot)$ for short. If in addition $T(\cdot)$ satisfies

$$x = \lim_{t\to 0^+} T(t)x \text{ for every } x \in X,$$

we say that $T(\cdot)$ is a **strongly continuous semigroup** or a **semigroup of class C_0** or simply a **C_0-semigroup**.

By Proposition 2.33, every C_0-semigroup $T(\cdot)$ on $(X, \|\cdot\|_X)$ has an **exponential boundedness**, in the sense that

$$\|T(t)\|_{\mathscr{L}(X)} \leqslant M \exp(\beta t) \text{ for every } t \geqslant 0,$$

for some real constants $M > 0$ and β. We sometimes say that $T(\cdot)$ is β-exponentially bounded. Indeed, let $M := \sup_{0\leqslant t\leqslant \tau} \|T(t)\|_{\mathscr{L}(X)} < \infty$ - which is possible for some $\tau > 0$, by the Banach-Steinhaus theorem - and let $\beta := \tau^{-1} \log \|T(\tau)\|_{\mathscr{L}(X)}$, that is $\|T(\tau)\|_{\mathscr{L}(X)} = \exp(\beta\tau)$. Given $t \geqslant 0$, we may write $t = n\tau + r$, where $0 \leqslant r < \tau$ and $n = 0, 1, 2 \ldots$ and then

$$\|T(t)\|_{\mathscr{L}(X)} \leqslant \|T(\tau)\|^n_{\mathscr{L}(X)} \|T(r)\|_{\mathscr{L}(X)} = M \exp(n\beta\tau) \leqslant M \exp(|\beta|\tau) \exp\big(\beta(n\tau + r)\big).$$

Moreover, by setting $S(t) := e^{-\beta t} T(t)$, the family of operators $S(\cdot)$ defines a C_0-semigroup on X such that $\|S(t)\|_{\mathscr{L}(X)} \leqslant M$ for every $t \geqslant 0$. Under the association between T and S, we shall often assume without loss of generality that $\beta = 0$. In this case, if M is no greater than 1 then $T(\cdot)$ is called a **contraction C_0-semigroup**.

Theorem 2.34. A semigroup $T(\cdot)$ on $\big(X, \|\cdot\|_X\big)$ is strongly continuous if and only if it is weakly continuous; that is,

$$[0, \infty) \ni t \mapsto \langle x^*, T(t)x \rangle \in \mathbb{C}$$

is continuous, for every $x \in X$ and every $x^* \in X^*$.

This result is quite surprising and the reader may see its proof in (YOSIDA, 1980), Theorem IX.1, or in (ENGEL *et al.*, 2006), Theorem 5.8. We provide a sketch of the proof though: first, one can check by the Banach-Steinhauss theorem that $\|T(t)\|_{\mathscr{L}(X)}$ is bounded for every $t > 0$ small enough; by setting $x(t) := T(t)x_0$, owing to the fact that the set M of all finite $\mathbb{Q}$-linear combinations of vectors $x(s)$, with $s \in \mathbb{Q}$, is a dense set in $\{x(t)\colon t \geqslant 0\}$, one obtains

$$\limsup_{t\to 0^+} \|x(t) - x_0\|_X \leqslant \left(1 + \sup_{[0,1]} \|T(t)\|_{\mathscr{L}(X)}\right) \|x_n - x_0\|_X$$

and the proof is complete, since $\inf_{x_n \in M} \|x_n - x_0\|_X = 0$.

From now on let $X = \big(X, (p_\alpha)_{\alpha\in A}\big)$ be a sequentially complete Hausdorff LCS equipped with the topology generated by the family of seminorms $(p_\alpha)_{\alpha\in A}$.

An **equicontinuous C_0-semigroup** on X is a family $\big(T(t)\big)_{t\geqslant 0}$ of bounded operators on X that satisfies (2.1) and are equicontinuous in t in the sense of the Banach-Steinhaus theorem for Hausdorff LCSs; that is, for every $\alpha \in A$ there exist $\beta = \beta(\alpha) \in A$ and a positive constant $c = c(\alpha)$ such that

$$\sup_{t\geqslant 0} p_\alpha\big(T(t)x\big) \leqslant c\, p_\beta(x) \text{ for every } x \in X. \tag{2.2}$$

Keeping in mind that the aim is to solve linear Cauchy problems in X, such as

$$\begin{cases} u_t = Au, t \geqslant 0 \\ u(0) = u_0 \end{cases}, \tag{2.3}$$

the definition below becomes natural by interpreting $u(t; u_0) := T(t)u_0$ as the evolution of the problem starting at u_0. In other words, the linear operator A can be seen as the time derivative of the solution C_0-semigroup $T(\cdot)$ on X.

Given an equicontinuous C_0-semigroup $T(\cdot)$ on X, we define its **infinitesimal generator** A by

$$Ax := \lim_{t\to 0^+} \frac{T(t) - I}{t}(x), \tag{2.4}$$

that is, A is the linear operator defined on $D(A)$, the set of points $x \in X$ for such the limit above exists and for $x \in D(A)$, Ax is defined by (2.4). At least the null vector is in $D(A)$; but actually it is a much larger subspace of X, as we shall see in Theorem 2.35.

Theorem 2.35. Let $T(\cdot)$ be an equicontinuous C_0-semigroup on X and let A be its infinitesimal generator.

a. A is a densely defined sequentially closed linear operator;

b. for every $x \in D(A)$, the map $[0, \infty) \ni t \mapsto T(t)x \in D(A)$ is well defined, it is continuously differentiable and it satisfies

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax \text{ for every } t > 0;$$

c. if $\lambda \in \mathbb{C}$ has positive real part then the resolvent operator $(\lambda - A)^{-1}$ exists, it is continuous and is given by

$$(\lambda - A)^{-1}x = R(\lambda;A)x := \int_0^\infty \exp(-\lambda s)T(s)x\,ds \text{ for every } x \in X.$$

d. For every $x \in X$,

$$x = \lim_{\lambda\to\infty} \lambda(\lambda I_X - A)^{-1}x$$

and

$$AR(\lambda;A)x = \big(\lambda R(\lambda;A) - I_X\big)x. \tag{2.5}$$

If, in addition, $x \in D(A)$ then A commutes with $R(\lambda;A)$ in (2.5) and in particular the Yosida approximation,

$$Ax = \lim_{\lambda\to\infty} \lambda A(\lambda I_X - A)^{-1}x,$$

holds as well.

e. if B is the infinitesimal generator of an equicontinuous C_0-semigroup $S(\cdot)$ on X and $A = B$ then $T(t) = S(t)$ for every $t \geqslant 0$.

Proof. **a.** For every $n \in \mathbb{N}$, let $C_n \colon X \to X$ be the linear operator defined by the Laplace transform of $t \mapsto T(t)$ multiplied by n, ie,

$$C_n x := \int_0^\infty ne^{-ns}T(s)x\,ds, \text{ for every } x \in X,$$

in the sense of the Riemann integral, which is obtained by using the seminorms p_α on X in place of the absolute value of a number. That every C_n is a bounded linear operator on X and that the improper integral is well defined follows from the sequential completeness of X and the inequality below:

$$p_\alpha(C_n x) \leqslant \lim_{N\to\infty} \int_0^N ne^{-ns} p_\beta\big(T(s)x\big)\,ds \leqslant c(\alpha)p_{\beta(\alpha)} \lim_{N\to\infty} \int_0^N ne^{-ns}\,ds = c(\alpha)p_{\beta(\alpha)},$$

where $c(\alpha) > 0$ and $\beta(\alpha) \in A$ were taken as in the definition of equicontinuity of $T(\cdot)$.

We claim that $C_n(X) \subset D(A)$ for every $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} C_n x = x \text{ for every } x \in X, \tag{2.6}$$

whence $\bigcup_{n=1}^{\infty} C_n(X)$ and $D(A)$ are dense in X. Given $x \in X$, we have

$$\begin{aligned} h^{-1}\big(T(h) - I_X\big)C_n x &= \frac{e^{nh}-1}{h}\int_h^\infty ne^{-ns}T(s)x\,ds - h^{-1}\int_0^h e^{-ns}T(s)x\,ds \\ &= \frac{e^{nh}-1}{h}\left[C_n x - \int_0^h ne^{-ns}T(s)x\,ds\right] - h^{-1}\int_0^h ne^{-ns}T(s)x\,ds, \end{aligned}$$

where the second term on the right tends to nx as $h \to 0^+$, by the continuity of $s \mapsto ne^{-ns}T(s)x$. Indeed, given $\varepsilon > 0$ and $\alpha \in A$, for $h > 0$ small enough,

$$p_\alpha\left(h^{-1}\int_0^h ne^{-ns}T(s)x\,ds - nx\right) \leqslant h^{-1}\int_0^h np_\alpha\big(e^{-ns}T(s)x - x\big) \leqslant n\varepsilon.$$

Similarly, one may prove that the first term on the right tends to nC_nx as $h \to 0^+$. Hence $C_nx \in D(A)$ and $AC_nx = n(C_n - I_X)x$ for every $x \in X$.

Now, given $\varepsilon > 0$ and $\alpha \in A$, we may write

$$\begin{aligned} p_\alpha(C_nx - x) &= p_\alpha\left(\int_0^\infty ne^{-ns}\big(T(s)x - x\big)\,ds\right) \\ &\leqslant \int_0^\delta ne^{-ns}p_\alpha\big(T(s)x - x\big)\,ds + \int_\delta^\infty ne^{-ns}\Big(p_\alpha\big(T(s)x\big) + p_\alpha(x)\Big)\,ds \end{aligned}$$

where $\delta > 0$ was chosen so that $p_\alpha\big(T(s)x - x\big) < \varepsilon$ whenever $0 \leqslant s < \delta$.

Since $\int_0^\infty ne^{-ns}\,ds = 1$, the first term on the right is dominated by ε; and since $\big(T(s)x\big)_{s\geqslant 0}$ is equicontinuous in s, the second term on the right goes to 0 as $n \to \infty$.

We shall prove **b.** and **c.** and a fortiori conclude that A is a sequentially closed operator.

b. If $x \in D(A)$ then

$$T(t)Ax = \lim_{h\to 0^+}\frac{T(t+h) - T(t)}{h}x = \lim_{h\to 0^+}\frac{T(h) - I_X}{h}T(t)x$$

so that $T(t)x \in D(A)$, $T(t)Ax = AT(t)x$ and $t \mapsto T(t)x$ is right differentiable, whenever $x \in D(A)$. It suffices to prove that the left derivative exists everywhere and it coincides with the right derivative. To do so, we invoke the following result:

Lemma: if one of the Dini derivatives $\overline{D}^+f, \underline{D}^+f, \overline{D}^-f$ and $\underline{D}^-f$ of a continuous real-valued function f is finite and continuous then f is continuously differentiable and its derivative is the Dini derivative.

The reader may see its proof in (YOSIDA, 1980), page 239. Given $x \in D(A)$, let $\phi \in X'$ be on continuous functional on X, then the map $t \mapsto f(t) := \langle \phi, T(t)x\rangle$, is continuous and satisfies

$$\frac{d^+}{dt}f(t) = \langle \phi, AT(t)x\rangle = \langle \phi, T(t)Ax\rangle,$$

whence f is differentiable in t and

$$\langle \phi, T(t)x\rangle - \langle \phi, x\rangle = \int_0^t \frac{d^+}{ds}f(s)\,ds = \left\langle \phi, \int_0^t T(s)Ax\,ds\right\rangle.$$

Since $\phi \in X'$ is arbitrary, $T(t)x - x = \int_0^t T(s)Ax\,ds$ for every $x \in D(A)$; and then by the continuity of $s \mapsto T(s)Ax$, we see that

$$\frac{d}{dt}T(t)x := \lim_{h\to 0}\frac{T(t+h) - T(t)}{h}x = T(t)Ax.$$

c.,d. First we claim that, for every $n \in \mathbb{N}$, $(nI - A)^{-1}$ exists. Because if not there would exist $x_0 \neq 0$ such that $Ax_0 = nx_0$. By the geometric form of the Hahn-Banach theorem (Theorem 2.10), take $\phi \in X'$ such that $\phi(x_0) = 1$ and then set $f(t) := \langle \phi, T(t)x_0 \rangle$, which is differentiable by item **b.**, since $x_0 \in D(A)$. On the one hand, the equicontinuity of $\big(T(\cdot)\big)_{t \geqslant 0}$ implies that $f(t)$ is bounded in t. But on the other, the unique solution of the differential equation

$$\frac{df}{dt}(t) = \langle \phi, T(t)Ax_0 \rangle = \langle \phi, T(t)nx_0 \rangle = nf(t),$$

with initial condition $f(0) = \phi(x_0) = 1$, is $f(t) = e^{nt}$, a contradiction. Thus the inverse $(nI_X - A)^{-1}$ must exist, for every $n \in \mathbb{N}$.

Besides, $AC_n x = n(C_n - I_X)x$ implies that $(nI_X - A)C_n x = nx$ for every $x \in X$, whence $nI_X - A$ maps $C_n(X) \subset D(A)$ onto X in a one-to-one way. Owing to the fact that $(nI_X - A)^{-1}$ does exist, $nI_X - A$ must map $D(A)$ onto X in a one-to-one way, consequently $C_n(X) = D(A)$ and $(nI_X - A)^{-1} = n^{-1}C_n \in \mathscr{L}(X)$.

We are ready to prove that the graph of A, $\{(x, Ax) \colon x \in D(A)\}$, is a sequentially closed subset of $X \times X$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $D(A)$ such that $x_n \to x \in X$ and $Ax_n \to y \in X$ then

$$x = \lim_n (I_X - A)^{-1}(I_X - A)x_n = (I_X - A)^{-1}(x - y)$$

so that $x \in D(A)$ and $(I_X - A)x = x - y$; and we are done.

Now, if $\lambda \in \mathbb{C}$ has positive real part then

$$R(\lambda; A)x := \int_0^\infty e^{-\lambda s} T(s)x\, ds, \text{ for } x \in X,$$

defines a bounded linear operator on X. Fixed $\tau \in \mathbb{R}$, it is easily checked that $A - i\tau I_X$ is the infinitesimal generator of $\big(e^{-i\tau t}T(t)\big)_{t \geqslant 0}$, which is an equicontinuous C_0-semigroup on X as well as $T(\cdot)$. Arguing as before, one concludes that $R(s + i\tau; A)$ provides the resolvent operator $\big((s + i\tau)I_X - A\big)^{-1}$ of $\big(e^{-i\tau t}T(t)\big)_{t \geqslant 0}$, whenever $s > 0$. Also, one should notice that $D(A) = (\lambda I_X - A)^{-1}(X)$ and

$$\lim_{\lambda \to \infty} \lambda(\lambda I_X - A)^{-1}x = x \text{ for every } x \in X.$$

Besides, by composing $(\lambda I_X - A)^{-1} = R(\lambda; A)$ and $(\lambda I_X - A)$ in different orders, one obtains that

$$A(\lambda I_X - A)^{-1}x = \big(\lambda(\lambda I_X - A)^{-1} - I_X\big)x \text{ for every } x \in X;$$

and additionally $AR(\lambda; A)x = R(\lambda; A)Ax$ holds whenever $x \in D(A)$.

e. If $x \in D(A) = D(B)$ then the map $t \mapsto T(t-s)S(s)x$ is differentiable and

$$\frac{d}{ds}T(t-s)S(s)x = -AT(t-s)S(s)x + T(t-s)BS(s)x$$

which is zero because A commutes with $T(\cdot)$ and $A = B$. And since $T(t-s)S(s)x$ has the same value at $s = 0$ and $s = t$, namely $T(t)x = S(t)x$, one sees that $T(t) = S(t)$ in $D(A)$, which is dense in X; and the proof is complete.

□

It is noteworthy that the same arguments in the proof holds for nets provided that X is complete rather than sequentially complete. Doing so, one obtains that the infinitesimal generator of an equicontinuous C_0-semigroup is a closed operator instead of a sequentially closed operator, for instance. But clearly X "being complete" is a stronger requirement than X "being sequentially complete". As the reader promptly sees the right half plane of the complex plane is always in the resolvent set of the infinitesimal generator of an equicontinuous C_0-semigroup. This means that such a uniform continuity has implicitly a geometric spectral condition, which was manageable when dealing with Banach spaces by multiplying by an exponential factor $e^{\beta t}, t \geqslant 0$.

Let us examine a few examples.

Example 2.36. We define the linear operator $T(t)$ on the Banach space $BC(\mathbb{R},\mathbb{R})$, for $t \geqslant 0$, by

$$\big(T(t)\phi\big)(s) := \phi(t+s) \text{ for every } \phi \in BC.$$

Clearly the condition (2.1) holds. Since every function ϕ is uniformly continuous on $\mathbb{R}$ and $\|T(t)\phi\|_u = \|\phi\|_u$, it follows that $T(t)\phi \to \phi$ as $t \to 0^+$ for every ϕ and hence $T(\cdot)$ is an equicontinuous contraction C_0-semigroup on BC.

Now we turn to its infinitesimal generator A. We claim that $D(A)$ consists of those functions $\phi \in BC$ which are differentiable with ϕ' belongs to BC; and that $A\phi(s) = \phi'(s)$ for every $\phi \in D(A)$.

Let $J := (1-A)^{-1} \in \mathscr{L}(BC)$. If $\phi \in D(A) = J(BC)$ then for some $\psi \in BC$ we have $\phi(t) := (J\psi)(t) = \int_t^\infty e^{-(s-t)}\psi(s)\,ds$, which is differentiable and

$$\phi'(t) = -\psi(t) + \phi(t) = (J - I_{BC})\psi(t) = AJ\psi(t) = A\phi(t),$$

because $AJ = -(1-J)$; from whence $D(A) \subset \{\phi \in BC \colon \phi \text{ is differentiable and } \phi' \in BC\}$. Conversely, let $\phi \colon \mathbb{R} \to \mathbb{C}$ be such that $\phi, \phi' \in BC$; we shall verify that $\phi \in D(A)$. To do so, we define ψ by setting

$$\phi'(t) - \phi(t) = -\psi(t),$$

and we define $\tilde{\phi} := J\psi \in D(A)$; whence $\tilde{\phi}'(t) - \tilde{\phi}(t) = -\psi(t)$ (proceeding as before). Note that $\big(\phi - \tilde{\phi}\big)'(t) = \big(\phi - \tilde{\phi}\big)(t)$ so that $(\phi - \tilde{\phi})(t) = ce^t$ but $\phi - \tilde{\phi}$ is supposed to be uniformly bounded, so $c = 0$ necessarily. Thus $\phi = \tilde{\phi} \in D(A)$ and $A\phi(t) = \phi'(t)$.

Example 2.37. Let $T(t) \colon BC(\mathbb{R},\mathbb{C}) \to BC(\mathbb{R},\mathbb{R})$, $t > 0$, be defined by

$$\big(T(t)\phi\big)(s) := \int_{\mathbb{R}} G_t(s-r)\phi(r)\,dr,$$

where G_t denotes the Gaussian probability density,

$$G_t(r) = \frac{1}{\sqrt{2\pi t}} e^{-r^2/2t} \text{ for } r \in \mathbb{R} \text{ and } t > 0;$$

and we set $T(0) := I_{BC}$. The family $\big(T(t)\big)_{t\geqslant 0}$ is an equicontinuous semigroup because $\|T(t)\phi\|_u \leqslant \|\phi\|_u \|G_t\|_{L^1} = \|\phi\|_u$ and

$$\frac{1}{\sqrt{2\pi(t+\tilde{t})}} e^{-r^2/2(t+\tilde{t})} = \frac{1}{\sqrt{2\pi t}} \frac{1}{\sqrt{2\pi \tilde{t}}} \int_{\mathbb{R}} e^{-(r-\tilde{r})^2/2t} e^{-\tilde{r}^2/2\tilde{t}} \, d\tilde{r},$$

which is a well known formula which can be deduced by using the Fourier transform. Now, to prove that $T(\cdot)\phi \to \phi$ as $t \to 0^+$, we argue as follows: given $\varepsilon > 0$ let $\delta > 0$ be such that $|\phi(s) - \phi(\tilde{s})| < \varepsilon$ whenever $|s - \tilde{s}| < \delta$, so

$$\begin{aligned}
|(T(t)\phi)(s) - \phi(s)| &= \left| \int_{\mathbb{R}} G_t(s-r)\big(\phi(r) - \phi(s)\big)\, dr \right| \\
&= \left| \int_{\mathbb{R}} G_1(r)\big(\phi(s - r\sqrt{t}) - \phi(s)\big)\, dr \right| \; \big(\text{by replacing } r \text{ by } (s-r)/\sqrt{t}\big) \\
&\leqslant \varepsilon \left| \int_{|r\sqrt{t}|<\delta} G_1(r)\, dr \right| + \left| \int_{|r\sqrt{t}|\geqslant\delta} G_1(r)\big(\phi(s - r\sqrt{t}) - \phi(s)\big)\, dr \right| \\
&\leqslant \varepsilon + 2\|\phi\|_u \int_{|r\sqrt{t}|\geqslant\delta} G_1(r)\, dr \to \varepsilon \text{ as } t \to 0^+,
\end{aligned}$$

because $\int G_1(r)\, dr = 1$. Hence $\|(T(t)\phi) - \phi\|_u \to 0^+$ and $T(\cdot)$ is an equicontinuous C_0-semigroup on BC. About its infinitesimal generator A, let $J := (1-A)^{-1} \in \mathscr{L}(BC)$ and $\phi \in D(A) = J(BC)$ then for some $\psi \in BC$ we have

$$\begin{aligned}
\phi(t) = J\psi(t) &= \int_{\mathbb{R}} \psi(r) \left(\int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-s-(t-r)^2/2s}\, ds \right) dr \\
&= \int_{\mathbb{R}} \psi(r) \left(\int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-s^2-(t-r)^2/2s^2}\, ds \right) dr \; \big(\text{by replacing } s \text{ by } s^2\big) \\
&= \int_{\mathbb{R}} \psi(r) \frac{1}{\sqrt{2}} e^{-\sqrt{2}|t-r|}\, dr,
\end{aligned}$$

where the last equality holds because $\int_0^\infty e^{-\left(s^2+c^2/s^2\right)}\, ds = \frac{\sqrt{\pi}}{2} e^{-2c}$, for $c > 0$. Besides

$$\phi'(t) = \int_t^\infty \psi(r) e^{-\sqrt{2}(r-t)}\, dr - \int_{-\infty}^t \psi(r) e^{-\sqrt{2}(t-r)}\, dr$$

and

$$\begin{aligned}
\phi''(t) &= -\psi(t) - \psi(t) + \sqrt{2} \int_t^\infty \psi(r) e^{-\sqrt{2}(r-t)}\, dr + \sqrt{2} \int_{-\infty}^t \psi(r) e^{-\sqrt{2}(t-r)}\, dr \\
&= 2\big(-\psi(t) + \phi(t)\big).
\end{aligned}$$

Also, $A\phi = AJ\psi = (J - I_{BC})\psi = \phi - \psi$ so that $A\phi = \phi''/2$ for every $\phi \in D(A)$. On the other hand, let ϕ and ϕ'' belong to BC, define ψ and $\tilde{\phi}$ by $\phi'' - 2\phi = -2\psi$ and

$\tilde{\phi} := J\psi \in D(A)$, whence $\tilde{\phi}'' - 2\tilde{\phi} = -2\psi$. This implies that $\left(\phi - \tilde{\phi}\right)(t) = c_1 e^{\sqrt{2}t} + c_2 e^{\sqrt{2}t}$, which cannot be bounded unless $c_1 = c_2 = 0$. Thus $\phi = \tilde{\phi} \in D(A)$ and $A\phi = \phi''/2$.

Therefore, the differential operator $\frac{1}{2}\frac{d^2}{dt^2}$ is the infinitesimal generator of the equicontinuos C_0-semigroup associated to the Gaussian kernel on BC.

In both examples, $BC(\mathbb{R},\mathbb{R})$ can be replaced by $L^p(\mathbb{R},\mathbb{R})$ with few adaptations.

Theorem 2.38. Let $A \in \mathscr{L}(X)$ be such that $\left(A^k\right)_{k\in\mathbb{N}}$ is an equicontinuous family in k.

a. for every $x \in X$, the series

$$\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} x$$

converges in X, for every $t \geqslant 0$.

b. for every $t \geqslant 0$, the map $x \mapsto \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} x$ is a continuous linear operator; which we shall denote by $\exp(tA)$ or e^{tA}.

c. if $B \in \mathscr{L}(X)$ is such that $\left(B^k\right)_{k\in\mathbb{N}}$ is an equicontinuous family in k then

$$\exp\left(t(A+B)\right) = \exp(tA)\exp(tB) \text{ for every } t \geqslant 0,$$

whenever $AB = BA$.

d. for every $x \in X$,

$$\lim_{t\to 0^+} \frac{\exp(tA) - I}{t} x = Ax$$

and hence

$$\frac{d}{dt} e^{tA} x = \exp(tA)Ax = A\exp(tA)x \text{ for every } t \geqslant 0.$$

Proof. **a.,b.** By the equicontinuity of $\left(A^k\right)_{k\in\mathbb{N}}$, if $x \in X$ and $\alpha \in A$ then

$$p_\alpha\left(\sum_{k=m}^{n} \frac{(tA)^k}{k!} x\right) \leqslant \sum_{k=m}^{n} \frac{(t)^k}{k!} p_\alpha(A^k x) \leqslant c(\alpha) p_{\beta(\alpha)}(x) \sum_{k=m}^{n} \frac{(t)^k}{k!} \to 0 \text{ as } n,m \to \infty,$$

so that $\left(\sum_{k=0}^{n} \frac{(tA)^k}{k!} x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, therefore convergent and we denote its limit by $\exp(tA)x$ or $e^{tA}x$. Besides, by putting $m = 0$ in the expression above, one sees that $p_\alpha\left(e^{tA}x\right) \leqslant e^t c(\alpha) p_{\beta(\alpha)}(x)$ for $t \geqslant 0$, so that $x \mapsto e^{tA}x$ defines a bounded linear operator on X.

c. Given $x \in X$ and $\alpha \in A$, we have

$$\begin{aligned} p_\alpha\big((A+B)^N x\big) &\leqslant \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} p_\alpha\big(A^{N-n}B^n x\big) \\ &\leqslant \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} c(\alpha) p_{\beta(\alpha)}\big(B^n x\big) \\ &\leqslant 2^N c(\alpha) \sup_{n\in\mathbb{N}} p_{\beta(\alpha)}\big(B^n x\big) \end{aligned}$$

from whence it follows that $\big(2^{-N}(A+B)^N\big)_{N\in\mathbb{N}}$ is equicontinuous in N and hence we can define $e^{t(A+B)}$. Owing the fact that $AB = BA$, we rearrange the series $\sum_{k=0}^{\infty} \frac{\big(t(A+B)\big)^k}{k!} x$ to obtain $\left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(tB)^k}{k!} x\right)$ as in the case of numerical series.

d. Finally, for $x \in X$ and $\alpha \in A$,

$$p_\alpha\left(\frac{e^{hA} - I_X}{h} x - Ax\right) \leqslant \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} p_\alpha\big(A^n x\big) \leqslant c(\alpha) p_{\beta(\alpha)}(x) \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} \to 0 \text{ as } h \to 0^+.$$

□

We shall prove the following fundamental result concerning the representation (2.7) below by the Yosida approximations; and the characterization of equicontinuous C_0-semigroups in terms of the corresponding infinitesimal generators.

Theorem 2.39 (The Hille-Yosida generation theorem)**.** Let $A\colon D(A) \subset X \to X$ be a linear operator and let $R(n;A)$ denote its resolvent operator $(nI - A)^{-1} \in \mathscr{L}(X)$, whenever it exists, with $n \in \mathbb{N}$. Consider the following statements:

(i) A is the infinitesimal generator of an equicontinuous C_0-semigroup $T(\cdot)$; and

(ii) $\big((I - n^{-1}A)^{-m}\big)_{n,m}$ is an equicontinuous family in $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$.

Then

a. **(i)** $\Rightarrow$ **(ii)** and

$$T(t)x = \lim_{n\to\infty} \exp\big(tA(I_X - n^{-1}A)^{-1}\big)x \text{ for every } x \in X, \tag{2.7}$$

with uniform convergence in t on every compact interval of t.

b. if A is a densely defined closed operator and the resolvent $R(n;A)$ exists for every $n \in \mathbb{N}$ then **(ii)** $\Rightarrow$ **(i)**.

Proof. **a.** It is not hard to check that the resolvent equality

$$R(\mu;A)-R(\lambda;A)=(\lambda-\mu)R(\lambda;A)R(\mu;A)$$

holds whenever $\lambda,\mu\in\mathbb{C}$ have positive real part, from whence we deduce that

$$\frac{dR(\lambda;A)}{d\lambda}=\lim\frac{R(\mu;A)-R(\lambda;A)}{\mu-\lambda}=-R(\lambda;A)^2$$

and moreover $\{\lambda\in\mathbb{C}\colon \Re\lambda>0\}\ni\lambda\mapsto R(\lambda;A)\in\mathscr{L}(X)$ is infinitely differentiable with

$$\frac{d^mR(\lambda;A)}{d\lambda^m}=(-1)^m m!R(\lambda;A)^{m+1}=\int_0^\infty(-s)^m e^{-\lambda s}T(s)x\,ds$$

for every $m=0,1,2,\ldots$. The differentiation under the integral sign is possible because $\big(T(t)x\big)_{t\geqslant 0}$ is equicontinuous in t and because $\int_0^\infty(-s)^m e^{-\lambda s}\,ds=\frac{m!}{\lambda^{m+1}}$ if $\Re\lambda>0$. Now we prove the equicontinuity of $\big((I-n^{-1}A)^{-m}\big)$: given $\alpha\in A$, $n\in\mathbb{N}$ and $m\in\mathbb{Z}_+$, we have

$$p_\alpha\Big(\big(nR(n;A)\big)^{m+1}x\Big)\leqslant\frac{n^{m+1}}{m!}\int_0^\infty(-s)^m e^{-ns}p_\alpha\big(T(s)x\big)\,ds\leqslant\sup_{s\geqslant 0}p_\alpha\big(T(s)x\big)$$

and since $T(\cdot)$ is equicontinuous and $nR(n;A)=(I_X-n^{-1}A)^{-1}$.

b. On the other hand let $J_n:=(I_X-n^{-1}A)^{-1}$ so that $AJ_nx=J_nAx=n(J_n-I_X)x$ whenever $x\in D(A)$. Since $\big(J_n(Ax)\big)_{n\in\mathbb{N}}$ is equibounded in $n\in\mathbb{N}$,

$$J_nx-x=n^{-1}J_nAx\to 0\text{ as }n\to\infty,$$

which holds for $x\in X$ as well, owing the fact that $D(A)$ is dense in X and $\big(J_n\big)_{n\in\mathbb{N}}$ is equicontinuous in $n\in\mathbb{N}$. For every $t\geqslant 0$ and $n\in\mathbb{N}$, we set $T_n(t)\colon X\to X$ by

$$T_n(t)x:=\exp(-nt)\exp(ntJ_n)x=\exp\big(tn(J_n-I_X)\big)x=\exp(tAJ_n)x\text{ for }x\in X,$$

which is well defined because $\big(J_n^m\big)_{n\in\mathbb{N},m\in\mathbb{Z}_+}$ is equicontinuous in n and m:

$$p_\alpha\big(\exp(ntJ_n)x\big)\leqslant\sum_{m=0}^\infty\frac{(nt)^m}{m!}p_\alpha(J_n^mx)\leqslant\exp(nt)c(\alpha)p_{\beta(\alpha)}(x).$$

As a consequence, $\big(T_n(t)\big)_{n,t}$ is an equicontinuous family in $n\in\mathbb{N}$ and $t\geqslant 0$. Applying Theorem 2.38, item **d.**, to $AJ_n\in\mathscr{L}(X)$ we obtain that $\frac{d}{dt}T_n(t)x=AJ_nT_n(t)x=T_n(t)AJ_nx$ and then for $x\in D(A)$ and $\alpha\in A$

$$\begin{aligned}p_\alpha\big(T_n(t)x-T_m(t)x\big)&=p_\alpha\left(\int_0^t\frac{d}{ds}T_m(t-s)T_n(s)x\,ds\right)\\&=p_\alpha\left(\int_0^t T_m(t-s)T_n(s)\big(AJ_n-AJ_m\big)x\,ds\right)\\&\leqslant\int_0^t c(\alpha)p_{\beta(\alpha)}\big((J_n-J_m)Ax\big)\,ds\to 0\text{ as }n,m\to\infty,\end{aligned}$$

where the convergence in t is uniform on every compact interval of t. Since $D(A)$ is dense in X, the operator $T(t)\colon X \to X$, given by

$$T(t)x := \lim_{n\to\infty} T_n(t)x \text{ for } x \in X,$$

is a well defined bounded linear operator, $\big(T(t)\big)_{t\geqslant 0}$ is an equicontinuous family in t and $[0,\infty) \ni t \mapsto T(t)x$ is a continuous map.

We claim that $\big(T(t)\big)_{t\geqslant 0}$ satisfies the semigroup property (2.1) and A is precisely its infinitesimal generator. Clearly $T(0) = I_X$ and $T_n(s+t) = T_n(s)T_n(t)$, for $s,t \geqslant 0$ and $n \in \mathbb{N}$. By writing $x(t) := T(t)x$ and $x_n(t) := T_n(t)x$ to simplify the notation, we have

$$\begin{aligned}
p_\alpha\big(x(t+s) - x(t)x(s)\big) &\leqslant p_\alpha\big(x(t+s) - x_n(t+s)\big) + p_\alpha\big(x_n(t+s) - x_n(t)x_n(s)\big) \\
&\quad + p_\alpha\big(x_n(t)x_n(s) - x_n(t)x(s)\big) + p_\alpha\big(x_n(t)x(s) - x(t)x(s)\big) \\
&\leqslant p_\alpha\big(x(t+s) - x_n(t+s)\big) + c(\alpha)p_{\beta(\alpha)}\big(x_n(s) - x(s)\big) \\
&\quad + p_\alpha\Big(\big(x_n(t) - x(t)\big)x(s)\Big) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

so that $p_\alpha\big(x(t+s) - x(t)x(s)\big) = 0$ for every $\alpha \in A$, from which it follows that $T(t+s) = T(t)T(s)$ for every $s,t \geqslant 0$.

Let $B\colon D(B) \subset X \to X$ denote the infinitesimal generator of $T(\cdot)$. By Theorem 2.35, item **e.**, it suffices to prove that $A = B$. As the reader may readily verify, $\lim_{n\to\infty} T_n(t)AJ_nx = T(t)Ax$ for every $x \in D(A)$, which implies that B is an extension of A. Indeed if $x \in D(A)$ then

$$\begin{aligned}
h^{-1}\big(T(h)x - x\big) &= h^{-1} \lim_{n\to\infty} \big(T_n(h)x - x\big) \\
&= h^{-1} \lim_{n\to\infty} \int_0^t T_n(s)AJ_nx\,ds \\
&= h^{-1} \int_0^t \Big(\lim_{n\to\infty} T_n(s)AJ_nx\Big)\,ds \\
&= h^{-1} \int_0^t T(s)Ax\,ds \to Ax \text{ as } h \to 0^+
\end{aligned}$$

so that the limit $\lim_{h\to 0^+} h^{-1}\big(T(h)x - x\big)$ exists and its equal to Ax. In other words, if $x \in D(A)$ then $x \in D(B)$ and $Ax = Bx$.

Fix $n \in \mathbb{N}$. On the one hand, since B is the infinitesimal generator of $T(\cdot)$, $nI_X - B$ maps $D(B)$ onto X in a one-to-one way. On the other, by hypothesis the resolvent $R(n;A)$ exists so that $nI_X - A$ also maps $D(A)$ onto X in a one-to-one way. Thus A and B must coincide; and the proof is complete.

□

If $X = (X, \|\cdot\|_X)$ is a Banach space then **(ii)** is equivalent to $\|(1-n^{-1}A)^{-m}\|_{\mathscr{L}(X)} \leqslant c$ for every $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, for some positive constant c which does not depend on n and m. For the case of contraction semigroups, c is taken to be 1.

2.4 Functional differential equations

When one wonders about how to explain observed processes from biology, physics, engineering, economics and chemistry, one often concludes that the present state is a consequence of past states and their evolution as interacting with some external forces. As a consequence, modelings where the principle of causality - that is, the principle that events from the past and those from the present are independent - is assumed to hold ought to be taken as only a first approximation of the true situation. When a model does not incorporate a dependence on its past history, it generally consists of so-called ordinary differential equations (hereafter **ODE**s) or partial differential equations (hereafter **PDE**s). Models incorporating past history generally include delay differential equations (hereafter **DDE**s) or functional differential equations (hereafter **FDE**s).

In Chapter 4, we shall consider an extended version of the economic Kaldor model in $\mathbb{R}^4$ which takes the form

$$\frac{du(t)}{dt} = u'(t) = f\big(u(t), \alpha u(t-\tau)\big), \tag{2.8}$$

with $\tau \geqslant 0$ and $\alpha > 0$ is an economic parameter which measures how strong is the government fiscal policy; therefore it is an autonomous delayed differential equation in $\mathbb{R}^4$ (TAKEUCHI; YAMAMURA, 2004). Originally, the model was formulated as an ODE in $\mathbb{R}^2$ without the government role and it explains the economic fluctuations as a natural phenomenon which arises in economy; see (KALDOR, 1940). The extended version however includes the government policies and the money market; and, when it comes to put such policies in practice, the model now considers the fact that government takes time to recognize an opportunity to interfere, to plan its action and to finally implement it, which is precisely where the role of the delay time lays.

Our aim here is to provide the theory necessary to obtain existence and uniqueness of a solution of (2.8) which depends continuously on the economic parameters involved; keeping in mind that we need to analyze how the stability of the equilibrium point jointly responds to α and τ. Although the equation to be treated in Kaldor's model is autonomous, in this section we shall state the results to the more general equation, since there is nearly no difficulty in proving such results in this setting.

Let Ω be an open subset of $\mathbb{R} \times \mathscr{C}$, where $\mathscr{C} := \Big(C\big([-\tau,0],\mathbb{R}^N\big), \|\cdot\|_u\Big)$ denotes the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^N$ with the topology of uniform convergence, which is given by the norm

$$\|\phi\|_u := \sup_{-\tau \leqslant t \leqslant 0} |\phi(t)| \quad \text{for every } \phi \in \mathscr{C}.$$

Besides, let $u_t \in \mathscr{C}$ be defined by $u_t(s) := u(s+t)$, for some continuous function u which is suitably defined on a neighborhood of $[-\tau,0]$. We shall then consider problems of the form

$$u'(t) = f(t, u_t), \tag{2.9}$$

which is called a **retarded delay differential equation** (hereafter RDDE). It is noteworthy that the problem (2.9) is well posed if one specifies a function defined on $[-\tau, 0]$ as an initial data. For a precise discussion on well-posedness, see (HALE; LUNEL, 1993). The first step is to establish when (2.9) admits a unique solution which depends continuously on the initial data.

Theorem 2.40. Suppose that $f\colon \Omega \subset \mathbb{R} \times \mathscr{C} \to \mathbb{R}^N$ is continuous.

a. for every $(t_0, \phi) \in \Omega$ there is a solution of (2.9) passing through (t_0, ϕ);

b. if f is Lipschitz in ϕ in every compact subset of Ω then for every $(t_0, \phi) \in \Omega$ there is a unique solution of (2.9) passing through (t_0, ϕ);

Besides, let $u_k = u_k(t_k, \phi_k, f_k)$ be the solution of $\dot{v}(t) = f_k(t, v_t)$ through $(t_k, \phi_k) \in \Omega$, for every $k \in \mathbb{Z}_+$, where $u_0 =: u$ exists and is unique on $[t_0 - \tau, T]$.

c. if $(t_k, \phi_k) \to (t_0, \phi_0)$ and $\|f_k - f_0\|_u \to 0$, as $k \to \infty$, in some properly chosen neighborhood $V \subset \Omega$ then every solution u_k exists on $[t_k - \tau, T]$ for k large enough and $u_k \to u_0$ uniformly on $[t_0 - \tau, T]$.

Remark 2.41.
- One may take V as an ε-neighborhood of the compact set $\{(t, u_0) : t \in [t_0, T]\} \subset \Omega$, such that f_0 is bounded on V.
- by $u_k \to u_0$ uniformly on $[t_0 - \tau, T]$, one should read the following: for any $\varepsilon > 0$, there exists $\tilde{k} = \tilde{k}(\varepsilon)$ such that $x_k(t)$ is defined on $[t_0 - \tau + \varepsilon, T]$ whenever $k \geqslant \tilde{k}$ and $u_k \to u_0$ uniformly on $[t_0 - \tau + \varepsilon, T]$.
- the proof of Theorem 2.40 is carried out by analogy with the result for ordinary differential equations, therefore it is a direct application of Schauder fixed-point theorem.

For now, suppose that $f\colon \mathbb{R} \times \mathscr{C} \to \mathbb{R}^N$ satisfies $f(t, 0) = 0$ for every $t \in \mathbb{R}$ then the solution $u = 0$ of (2.9) is said to be

(i) **stable** if for any $s \in \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(s, \varepsilon)$ such that $u_t(s, \phi) \in B(0, \varepsilon)$ for $t \geqslant s$, whenever $\phi \in B(0, \delta)$;

(ii) **asymptotically stable** if it is stable and there exists $r_0 = r_0(s) > 0$ such that $u(s, \phi)(t) \to 0$ as $t \to \infty$, whenever $\phi \in B(0, r_0)$;

(iii) **uniformly stable** if the number δ in the definition is independent of s; and

(iv) **uniformly asymptotically stable** if it is uniformly stable and there exists $r_0 > 0$ such that for every $\eta > 0$ there exists an instant of time $t_0(\eta) > 0$ such that $u_t(s,\phi) \in B(0,\eta)$ for $t \geqslant s + t_0(\eta)$ and $s \in \mathbb{R}$, whenever $\phi \in B(0,r_0)$.

In general, if $u(t)$ is any solution of (2.9) then u is said to be **stable** if the solution $v = 0$ of the equation

$$v'(t) = f(t, v_t + u_t) - f(t, u_t)$$

is stable. Similarly, one may define the other concepts of stability for u. As the reader may already infer, f was assumed to be defined on the entire space $\mathbb{R} \times \mathscr{C}$ instead of on an open subset of it just to avoid some notational inconveniences.

A great deal about delay differential equations can be learned by a study of its simplest representative, namely a differential equation with a fixed delay, for a real function $t \mapsto u(t)$:

$$u'(t) = -\alpha u(t-1), \tag{2.10}$$

where α is a real number. We may interpret (2.10) as a system governed by feedback with a time lag for which the feedback is negative with respect to the zero solution whenever $\alpha > 0$ and it is negative whenever $\alpha < 0$. If the initial data u_0 is a continuous real function then a repeated integration argument on the intervals $[n-1, n]$ results in a unique continuous solution $u\colon [-1,\infty) \to \mathbb{R}$ which is differentiable for $t > 0$, it has a right derivative at $t = 0$ and it satisfies (2.10) for every $t \geqslant 0$. It is an unpleasant fact that even for this simple linear equation, the stability of the trivial equilibrium requires an analysis of the roots of a transcendental equation. Proceeding exactly as for ODEs, we seek (complex) values of λ such that $u(t) = \exp(\lambda t)$ is a solution of (2.10), which happens if and only if $\lambda \in \mathbb{C}$ is a root of the **characteristic equation**

$$\lambda + \alpha e^{-\lambda} = 0,$$

and in this case we say that λ is a **characteristic root** of (2.10). Owing to the fact that $\lambda \mapsto h(\lambda) := \lambda + \alpha e^{-\lambda}$ is an analytic function, we have the following lemma.

Lemma 2.42. **a.** The set of characteristic roots has no accumulation point in $\mathbb{C}$ and consequently is a countable (possibly finite) set;

b. If there exist infinitely many characteristic roots λ_n then $|\lambda_n| \to \infty$ as $n \to \infty$.

c. There are only finitely many characteristic roots with positive real part.

d. Every characteristic root λ has finite order, that is, there exists $n \in \mathbb{N}$ such that $h(\lambda) = h'(\lambda) = \cdots = h^{(n-1)}(\lambda) = 0$ and $h^{(n)}(\lambda) \neq 0$.

e. If λ is a characteristic root, so is its conjugate $\overline{\lambda}$.

In contrast to scalar autonomous ODEs, oscillatory solutions $u(t)$ of (2.10) do exist and are associated to non-real characteristic roots $\lambda \in \mathbb{C}$. We summarize some well known properties of the zero solution of (2.10) depending on α; see (SMITH, 2010) for example.

a. if $\alpha < 0$ then $u = 0$ is unstable.

b. if $0 < \alpha < \pi/2$ then $u = 0$ is asymptotically stable.

c. if $\alpha = \pi/2$ then $u(t) = \sin(t\pi/2)$ and $u(t) = \cos(t\pi/2)$ are solutions, hence periodic solutions.

d. if $\alpha > \pi/2$ then $u = 0$ is unstable.

e. every solution of (2.10) is oscillatory[2] if and only if $\alpha > e^{-1}$.

More generally, consider the following autonomous linear RDDE

$$u'(t) = Lu_t, \tag{2.11}$$

where $L\colon \mathscr{C} \to \mathbb{R}^N$ is continuous, then its characteristic equation is given by

$$h(\lambda) := \det\Big(\lambda I - L\big(\exp(\lambda\cdot)I\big)\Big) = 0, \lambda \in \mathbb{C}. \tag{2.12}$$

Based on our experience with ODEs we have the right to expect that the trivial solution is asymptotically stable if all roots λ of the characteristic equation have negative real part and that it is unstable if there is a root with positive real part. Indeed, this is the case.

Theorem 2.43. **a.** the zero solution of (2.11) is uniformly asymptotically stable if every characteristic root has negative real part;

b. the zero solution of (2.11) is unstable if $\Re\lambda > 0$ for some characteristic root λ; and

c. if some characteristic root is a nonsimple pure complex root then (2.11) is unstable.

As the reader already knows, the local stability of an autonomous nonlinear system of ODEs can be determined by studying its correspondent linearized system. More precisely, the eigenvalues location in the complex plane dictates whether the equilibrium point of the linearized system is stable or not; and hence we conclude about the local stability of the original system. RDDEs herds such a nice property although stochastic (funcional) differential equations do not.

[2] A solution $u(t)$ of (2.10) is said to be oscillatory if it has arbitrarily large zeros.

Theorem 2.44 ((KUANG, 1993), Theorem 2.4.2)**.** Consider the perturbed equation

$$u'(t) = Lu_t + g(u_t), \tag{2.13}$$

where $L\colon \mathscr{C} \to \mathbb{R}^N$ is linear continuous. Suppose that $g\colon \mathscr{C} \to \mathbb{R}^N$ is continuous together with its first derivative g_ϕ, with $g(0) = 0$ and $g_\phi(0) = 0$.

a. if the zero solution of (2.11) is uniformly asymptotically stable then the zero solution of (2.13) is also uniformly asymptotically stable; and

b. if $\Re\lambda > 0$ for some characteristic root λ then the zero solution of (2.13) is unstable.

In other words, the local stability of the trivial solution (i.e., the zero solution) of (2.13) depends on the locations of the characteristic roots of the associated linearized equation. The characteristic equation is a function of the delay time τ and hence so are its characteristic roots. Consequently, as the length of τ changes, the location of the characteristic roots changes in the complex plane and thus the stability of the trivial solution may also change, by Theorem 2.43. Such a phenomenon is referred to as **stability switch**. On the one hand, for a scalar delay equation

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} u(t) + \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} u(t-\tau) = 0,$$

where $a_n \neq 0$ and $n \geqslant m$, the characteristic equation takes the form $Q_0(\lambda) + Q_\tau(\lambda)e^{-\lambda\tau} = 0$, where Q_0 and Q_λ are the polynomials associated to the coefficients a_k and b_k, respectively. On the other hand, for systems with one delay τ their characteristic equation may be written in the same but Q_0 may not be a polynomial if the system has several delays.

Theorem 2.45 ((KUANG, 1993), Theorem 3.4.1)**.** Consider the equations

$$Q_0(\lambda) + e^{-\lambda\tau} Q_\tau(\lambda) = 0, \text{ for } \lambda \in \mathbb{C}, \tag{2.14}$$

and

$$F(y) := |Q_0(iy)|^2 - |Q_\tau(iy)|^2 = 0, \text{ for } y \in \mathbb{R}. \tag{2.15}$$

Suppose that $\lambda \mapsto Q_0(\lambda), Q_\tau(\lambda)$ are analytic functions for $\Re\lambda > 0$ and that

(i) there is no common pure imaginary roots of Q_0 and Q_τ;

(ii) $\overline{Q_0(-iy)} = Q_0(iy)$ and $\overline{Q_\tau(-iy)} = Q_\tau(iy)$, for every $y \in \mathbb{R}$;

(iii) $\lambda = 0$ is not a root for (2.14);

(iv) $\limsup\limits_{\substack{|\lambda|\to\infty \\ \Re\lambda \geqslant 0}} \left| \dfrac{Q_\tau(\lambda)}{Q_0(\lambda)} \right| < 1$; and

(v) the equation (2.15) admits only finitely many real roots.

Then

a. if $F(y) = 0$ has no positive roots then no stability switch occurs.

b. if $F(y) = 0$ has at least one positive root and each of them is simple then, as τ increases, a finite number of stability switches occurs and eventually u^* becomes unstable.

We shall present part of the proof in Chapter 4 in order to jointly analyze how the stability of the equilibrium point of the extended Kaldor model switches as τ and an important economic parameter vary.

At last, we present the classical Routh-Hurwitz criteria to determine whether all the roots of a polynomial have negative real parts. See (GANTMACHER; BRENNER, 2005).

Theorem 2.46 (The Routh-Hurwitz criteria)**.** Consider the polynomial

$$Q(\lambda) := a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + \lambda^n,$$

where $a_i \in \mathbb{R}$, for $i = 1, \cdots, n$. All roots of Q have negative real parts if and only if $D_1, \ldots, D_n > 0$, where

$$D_j := \begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2j-1} & a_{2j-2} & a_{2j-3} & a_{2j-4} & a_{2j-5} & a_{2j-6} & \cdots & a_j \end{vmatrix}.$$

However, we shall apply it for two cases, where the degree of the polynomial is either three or four.

Corollary 2.47. Consider the polynomials

$$Q_0(\lambda) := a_4 + a_3\lambda + a_2\lambda^2 + \cdots + a_1\lambda^3 + \lambda^4$$

and

$$Q_\tau(\lambda) := b_3 + b_2\lambda + b_1\lambda^2 + \lambda^3,$$

where $a_i, b_j \in \mathbb{R}$, for $i = 1,2,3,4$ and $j = 1,2,3$. Then

a. all roots of Q_0 have negative real parts if and only if $a_1, a_3, a_4 > 0$ and $a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0$.

b. all roots of Q_τ have negative real parts if and only if $b_1, b_3 > 0$ and $b_1b_2 - b_3 > 0$.

2.5 Mathematization of economics

In James Tobin's words, a contemporaneous economist who defends government intervention to stabilize output and avoid recessions, "the question of growth is nothing new but a new disguise for an age-old issue, one which has always intrigued and preoccupied economics: the present versus the future".

As stated in (MANKIW, 2003), thanks to rising incomes, material standards of living have improved substantially over time for most families in most countries. Academics (BARRO; Sala-i-Martin, 2004; Sala-i-Martin, 2006; MANKIW, 2003) and organizations such as World Bank agree that the economic growth and poverty reduction have a positive correlation and hence predicting the former is an important goal to be pursued. Economists have realized throughout the last century how imperative mathematics is for such a goal and one of the main reasons for it is the breakdown of many economies, especially under the pressure of high inflation and the major influence of inflationary expectations, which directed attention to dynamics instead of a comparative statics approach. By its very nature, dynamics involves time derivatives or difference equations, where time is considered in discrete units.

We shall present some basic aspects of economics in order to properly discuss and justify the Kaldor model (which is an ODE in $\mathbb{R}^2$) and its extended version (which leads to an autonomous DDE in $\mathbb{R}^4$), together with the assumptions to obtain stability; which we shall state to be reasonable from an economic point of view. Most of the text below was extracted from (MANKIW, 2003) with few modifications if any.

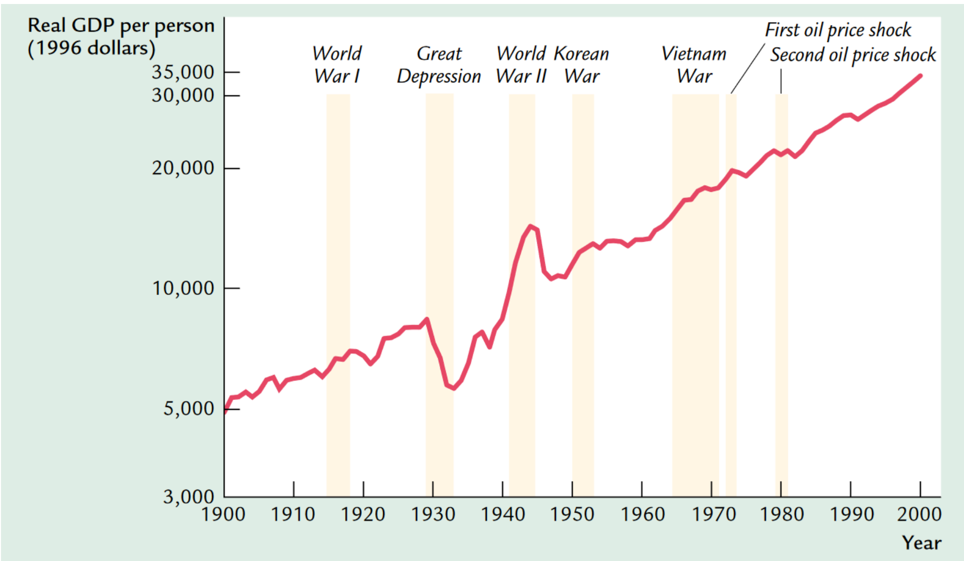


Figure 1 – Real GDP per person in the U.S. Economy in the last century.

Why have some countries experienced rapid growth in incomes over the past century while others stay mired in poverty? Why do some countries have high rates of inflation while others maintain stable prices? Why do all countries experience recessions and depressions - recurrent periods of falling incomes and rising unemployment - and how can government policy reduce the frequency and severity of these episodes? Macroeconomics, the study of the economy as a whole, attempts to answer these and many related questions. The macroeconomist's ability to predict the future course of economic events is no better than the meteorologist's ability to predict next month's weather. But quite a lot about how the economy works is known and this knowledge is useful both for explaining economic events and for formulating economic policy.

Three macroeconomic variables are especially important when it comes to measure the performance of an economy: real gross domestic product (GDP), the inflation rate and the unemployment rate. Real GDP measures the total income of everyone in the economy (adjusted for the level of prices). The inflation rate measures how fast prices are rising. The unemployment rate measures the fraction of the labor force that is out of work. Macroeconomists study how these variables are determined, why they change overtime and how they interact with one another.

Figure 1 shows that real GDP per person tends to grow over time and that this normal growth is sometimes interrupted by periods of declining income, called **recessions** or **depressions**. As it often happens when it comes to macroeconomics graphs, real GDP is plotted here on a logarithmic scale so that equal distances on the vertical axis represent equal percentage changes. For instance, the distance between 5,000 and 10,000 (a 100 percent change) is the same as the distance between 10,000 and 20,000 (a 100 percent change). For such a graph, data source is U.S. Bureau of the Census (Historical Statistics of the United States: Colonial Times to 1970) and U.S. Department of Commerce.

On the other hand, microeconomics is the study of how firms and individuals make decisions and how these decision makers interact. Because macroeconomic events arise from many microeconomic interactions, macroeconomy theory uses many of the tools of microeconomics, whence we assume the following 10 principles (see (MANKIW, 2011; MARSHALL, 2013)):

- people face **tradeoffs**:

  Suppose you have an amount of resources that are finite (which is quite often, right?). When one pursues a goal which depends on these resources, one has to weigh the cost of such a goal in terms of theses resources and also to compare it with other goals that consume these resources to decide which option is better (in some sense). This is a tradeoff.

  Example: Consider when parents have to decide how to spend their family income. They can buy food, clothing or a family vacation. Or they can save some of the family income for retirement or the children's college education. When they choose to spend an extra dollar on one of these goods, they have one less dollar to spend on some other good. As society, individuals face a central tradeoff between efficiency and equity. Efficiency means that society is getting the most it can from its scarce resources. Equity means that the benefits of those resources are distributed fairly among society's members. In other words, efficiency refers to the size of the economic pie and equity refers to how the pie is divided. Often, when government policies are being designed, these two goals conflict.

- the **cost** of something is what you give up to get it:

  If one wonders about the decision to go to college, one promptly thinks of its benefits (for instance, intellectual enrichment and a lifetime of better job opportunities) and its cost. But what is the cost? You might be tempted to add up the money you spend on tuition, books, room and board. Yet this total does not truly represent what you give up to spend a year in college. If you were not going to college, you still would need a place to sleep and food to eat. Indeed, the cost of room and board at your school might be less than the rent and food expenses that you would pay living on your own. In this case, the savings on room and board are a benefit of going to college.

  But the major problem with this calculation of costs is that it ignores the largest cost of going to college: your time. When you spend a year listening to lectures, reading textbooks and writing papers, you cannot spend that time working at a job. The opportunity cost of an item is what you give up to get that item.

- rational people think at the **margin**:

  In many situations, people make the best decisions by thinking at the margin. Suppose, for instance, that you asked a friend for advice about how many years to stay in school. If he were to compare for you the lifestyle of a person with a Ph.D. to that of a grade school dropout, you might complain that this comparison is not helpful for your decision. You have some education already and most likely are deciding whether to spend an extra year or two in school. To make this decision,you need to know the additional benefits that an extra year in school would offer (higher wages throughout life and the sheer joy of learning) and the additional costs that you would incur (tuition and the forgone wages while you are in school). By comparing these marginal benefits and marginal costs, you can evaluate whether the extra year is worthwhile.

- people respond to **incentives**:

  Because people make decisions by comparing costs and benefits, their behavior may change when the costs or benefits change. That is, people respond to incentives.

- **trade** can make everyone better off:

  Trade between the United States and Japan is not like a sports contest, where one side wins and the other side loses. Your family would not be better off isolating itself from all other families. If it did, your family would need to grow its own food, make its own clothes and build its own home. Countries as well as families benefit from the ability to trade with one another because trade allows them to specialize in what they do best and to enjoy a greater variety of goods and services. Here "to do best" may mean a lot of things: for instance, to produce a high quality good using a not widely known technology or a better input if we think about minerals and their derivatives; to provide a faster medical attendance due to better trained employees and so on. The Japanese as well as the Americans and the French and the Egyptians and the Brazilians are as much partners in the world economy as they are competitors among them.

- **markets** are usually a good way to organize economic activity:

  Today, most countries are trying to develop a market where economy is organized by the decisions of millions of firms and households - this is called a market economy. Firms decide whom to hire and what to make. Households decide which firms to work for and what to buy with their incomes. These firms and households interact in the marketplace, where prices and self-interest guide their decisions. In his 1776 book *An Inquiry into the Nature and Causes of the Wealth of Nations*, economist Adam Smith made the most famous observation in all of economics: Households and firms interacting in markets act as if they are guided by an "invisible hand" that leads them to desirable market outcomes. It is the most basic statement that the price is a consequence of demand and supply forces, when in a fair market economy.

  Because households and firms look at prices when deciding what to buy and sell, they unknowingly take into account the social benefits and costs of their actions. As a result, prices guide these individual decision makers to reach outcomes that, in many cases, maximize the welfare of society as a whole. It also explains the even greater harm caused by policies that directly control prices and it explains the failure of communism, since in communist countries, prices were not determined in the marketplace but were dictated by central planners. These planners lacked the information that gets reflected in prices when prices are free to respond to market forces (demand and supply).

- **governments** can sometimes improve market outcomes:

  There are two broad reasons for a government to intervene in the economy: to promote efficiency and to promote equity. That is, most policies aim either to enlarge the economic pie or to change how the pie is divided.

If a chemical factory does not bear the entire cost of the smoke it emits, it will likely emit too much. Here, the government can raise economic well-being through environmental regulation. When a scientist makes an important discovery, he produces a valuable resource that other people can use. In this case, the government can raise economic well-being by subsidizing basic research, as in fact it does. Also, if a single economic actor (or small group of actors) has a substantial influence on market prices then a market failure is in course. For example, suppose that everyone in town needs water but there is only one well; as a consequence, its owner has an excessive market power - in this case a monopoly - over the sale of water. The well owner is not subject to the rigorous competition and hence regulating the price that the monopolist charges can potentially enhance economic efficiency.

The invisible hand is even less able to ensure that economic prosperity is distributed fairly. A market economy rewards people according to their ability to produce things that other people are willing to pay for. A goal of many public policies is to achieve a more equitable distribution of economic well-being.

- a country's **standard of living** depends on its ability to produce goods and services:

  Almost all variation in living standards is attributable to differences in countries' productivity - that is, the amount of goods and services produced from each hour of a worker's time. To boost living standards, policymakers need to raise productivity by ensuring that workers are well educated, have the tools needed to produce goods and services and have access to the best available technology.

- **prices** rise when the government prints too much money:

  In Germany in January 1921, a daily newspaper cost 0.30 marks. Less than two years later, in November 1922, the same newspaper cost 70,000,000 marks. All other prices in the economy rose by similar amounts. This episode is one of history's most spectacular examples of inflation, an increase in the overall level of prices in the economy.

  What causes inflation? In almost all cases of large or persistent inflation, the culprit turns out to be the same - growth in the quantity of money. Taking money as an ordinary good, the reasoning is as follows: when a government creates large quantities of the nation's money (supply rising), the value of the money falls and hence you will need more money to buy a particular good.

- society faces a short-run tradeoff between **inflation** and **unemployment**:

  A simple explanation is that it arises because some prices are slow to adjust. Suppose, for example, that the government reduces the quantity of money in the economy. In the long run, the only result of this policy change will be a fall in the overall level of prices. Yet not all prices will adjust immediately. It may take several years before all

> firms issue new catalogs, all unions make wage concessions and all restaurants print new menus. That is, prices are said to be sticky in the short run. Because prices are sticky, various types of government policy have short-run effects that differ from their long-run effects. When the government reduces the quantity of money, for instance, it reduces the amount that people spend. Lower spending, together with prices that are stuck too high, reduces the quantity of goods and services that firms sell. Lower sales, in turn, cause firms to lay off workers. Thus, the reduction in the quantity of money raises unemployment temporarily until prices have fully adjusted to the change.
>
> The curve that illustrates this tradeoff between inflation and unemployment is called the Phillips curve, after the economist who first examined this relationship. The Phillips curve remains a controversial topic among economists, but most of them today accept the idea that there is a short-run tradeoff between inflation and unemployment. By changing the amount that the government spends, the amount it taxes and the amount of money it prints, policymakers can, in the short run, influence the combination of inflation and unemployment that the economy experiences. These instruments define what are called monetary and fiscal policies; and since they are potentially so powerful, how policymakers should use these instruments to control the economy, if at all, is a subject of continuing debate.

Even the most sophisticated economic analysis is built using the ten principles introduced here. The model whose stability we shall study aims to explain the economic fluctuations in terms of national GDP. Economic models illustrate the relationships among the variables in mathematical terms. They are useful because they help us to dispense with irrelevant details and to focus on important connections.

At last we present the essence of one of the most important macroeconomic models - the **IS-LM model**, which was introduced in a classic article by the Nobel-Prize-winning economist John R. Hicks; see (HICKS, 1937). In (KEYNES, 1964), Keynes proposed that an economy's total income was, in the short run, determined largely by the desire to spend by households, firms and the government. The more people want to spend, the more goods and services firms can sell. The more firms can sell, the more output they will choose to produce and the more workers they will choose to hire. Thus the problem during recessions and depressions, according to Keynes, was inadequate spending.

Under IS-LM perspective, there are only two markets which determines economy dynamics **in short run**: the goods market and the money market. IS stands for *investment* and *saving*, and the IS curve represents what's going on in the market for goods and services. LM stands for "liquidity" and "money", and the LM curve represents what's happening to the supply and demand for money. Recall that money supply is entirely controlled by the government, since it prints every banknote in circulation and consequently determines the available money quantity. Thus IS modeling concerns how consumers decide to spend or to save their money meanwhile LM modeling concerns the supply and demand for real money balances, which determine what interest rate prevails in the economy. The reader may think on interest rate as the price of money therefore.

The equilibrium of the economy is the point at which the IS curve and the LM curve cross. This point gives the interest rate r and the level of income Y that satisfy conditions for equilibrium in both the goods market and the money market, for given values of government spending, taxes, the money supply and the price level. Instead of examine every aspect of the model, we choose to state some of its conclusions:

1. a decrease in government purchases or an increase in taxes reduces income.
2. a higher interest rate lowers planned investment and this in turn lowers national income. The downward-sloping IS curve summarizes this negative relationship between the interest rate r and income Y.
3. increases in the money supply lower the interest rate.
4. a higher level of income raises the demand for real money balances and this in turn raises the interest rate. The upward-sloping LM curve summarizes this positive relationship between income and the interest rate.
5. the IS-LM model combines the elements of the Keynesian cross and the elements of the theory of liquidity preference (a dollar in the future is less valuable than a dollar today). The IS curve shows the points that satisfy equilibrium in the goods market and the LM curve shows the points that satisfy equilibrium in the money market. The intersection of the IS and LM curves shows the interest rate and income that satisfy equilibrium in both markets.

CHAPTER

3

GROUPS OF BOUNDED OPERATORS ON FRÉCHET SPACES

We consider the linear Cauchy problem

$$\begin{cases} u_t = a(D)u, t \in \mathbb{R} \\ u(0) = u_0 \end{cases}, \tag{3.1}$$

where $a(D)\colon X \to X$ is continuous operator on a Fréchet space X. Imposing a condition - which is neither stronger nor weaker than the equicontinuity of the powers of $a(D)$ - we present necessary and sufficient conditions for generation of a uniformly continuous group on X which provides the unique solution of (3.1). As a consequence, if $a(D)$ is a pseudodifferential operator with constant coefficients defined on $\mathscr{F}L^2_{loc}$ - a particular Fréchet space of distributions - then we also provide necessary and sufficient conditions so that the restriction $\{e^{t\,a(D)}\}_{t\geqslant 0}$ is a well defined semigroup on L^2 and $\mathcal{E}'$. We conclude that the solution of the heat equation on $\mathscr{F}L^2_{loc}$ for all $t \in \mathbb{R}$ extends the standard solution on Hilbert spaces for $t \geqslant 0$.

3.1 Introduction

If A is a pseudodifferential operator, e.g., $A = \frac{d}{dx}$, one may consider the Cauchy problem associated to it, namely

$$\begin{cases} u_t = Au, t \in I \\ u(0) = u_0 \end{cases}, \tag{3.2}$$

and try to solve it for a certain class of functions u_0 and a fixed interval of time I. By modeling biological, physical and economic phenomena, evolution problems such as (3.2) naturally arise from partial differential equations (hereafter, PDEs) by interpreting $(t,x) \mapsto u(t,x)$ as a vector-valued mapping $t \mapsto u(t,\cdot)$, let us say, $u(t,\cdot) \in L^2(\mathbb{R}^N)$.

A function $\mathbb{R} \times \mathbb{R}^N \ni (t,x) \mapsto u(t,x) \equiv \big(u(t)\big)(x) \in X$ is said to be a solution of (3.2) if it is differentiable on the temporal variable t, it satisfies $\frac{d}{dt}u(t,x) = Au(t,x)$ for every (t,x) and it satisfies the so-called initial condition, that is, $u(0,x) = u_0(x)$ for every $x \in \mathbb{R}^N$, for a given function $u_0 \colon \mathbb{R}^N \to X$.

The main approach consists in dealing with a closed linear operator $A \colon D(A) \subset X \to X$, which is densely defined on a Banach space X. On the one hand, this setting has provided a rich theory over the last fifty years with which (3.2) can be solved by a strongly continuous semigroup $\big(T(t) \colon X \to X\big)_{t \geqslant 0}$ whenever some spectral conditions on A are fulfilled. On the other hand, many well-known topological vector spaces arising in PDEs analysis are not normable - such as $C^1\big((-\infty,0]\big)$ in equations with infinite delay. In this setting, a natural trade-off arises: to lose good topological properties on the space but to obtain better properties on the operators. For instance, every linear differential operator with constant coefficients is bounded on the Schwartz space.

Let X be a Hausdorff locally convex space (hereafter, HLCS). The map

$$t \mapsto \exp(tA)u_0 := \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n\right) u_0, \text{ for } u_0 \in X, \tag{3.3}$$

provides the unique solution of (3.2) on X for $t \geqslant 0$, whenever X is sequentially complete and $(A^n)_{n \in \mathbb{N}}$ is an equicontinuous family of bounded operators defined on X, see Section 2.3.

The generation of a C_0-semigroup on HLCSs as (3.3) has already been dealt by other authors adding hypothesis on the generator or on the phase space X. In (CHOE, 1985; LEMLE; WU, 2011), the C_0-semigroup is assumed to be quasi-equicontinuous and in (KRAAIJ, 2016) it is assumed to be locally equicontinuous and X is equipped with an auxiliary norm. Others treat the question in some particular Fréchet spaces, such as done by Dembart (DEMBART, 1974) (who considered the phase space as the space of the continuous functions defined on $[a,b]$ into a fixed topological vector space E) and by Frerick et al. (FRERICK *et al.*, 2013) (by setting $X = \mathbb{K}^{\mathbb{N}}$, that is, the collection of scalar sequences).

Hence we felt confident to point out some results about the generation of uniformly continuous groups (whose definition invokes a stronger convergence rather than a pointwise one) on Fréchet spaces. We extend for instance the main generation result recently obtained in (GOLIńSKA; WEGNER, 2015), where the authors do not establish the necessity implication for the exponential map convergence in the topology of bounded convergence. Besides, we provide further applications to linear Cauchy problems in which $A = a(D)$ is a pseudodifferential operator on $\mathscr{F}L^2_{loc}$; remarkably, we can extend the analytic semigroup generated by the heat operator $-(1-\Delta)$ on L^2 to the group $\left(e^{-t(1-\Delta)}\right)_{t \in \mathbb{R}}$ on $\mathscr{F}L^2_{loc}$ and thus obtain a distributional solution of the heat equation backwards in time.

This chapter is organized as follows. We establish in Section 3.2 the generation theorem (Theorem 3.4) on abstract Fréchet spaces for bounded linear operators which enjoy a simple compatibility property with respect to the Fréchet topology. In Section 3.3, we construct a large Fréchet space which consists of distributions, namely $\mathscr{F}L^2_{loc}$, and which also contains L^2 and the space of compactly supported distributions, $\mathcal{E}'(\mathbb{R}^N)$, as topological vector subspaces. Section 3.4 is utterly concerned with applying the results of Section 3.2 to evolution problems in $\mathscr{F}L^2_{loc}$ including criteria for regularization process of the solution and positive invariance on L^2. Some conclusions and open problems we keep in sight are presented in Section 3.5.

3.2 Strongly compatible operators and generation theorem

Let $X = \big(X, (p_j)_{j\in\mathbb{N}}\big)$ be a Fréchet space and $\mathscr{L}(X)$ be the space of all bounded linear operators on X. We shall require the appropriate compatibility between the operator A and the Fréchet topology on X so that its exponential operator is well defined and provides the solution of the associated Cauchy problem.

Definition 3.1. An operator $A \in \mathscr{L}(X)$ is said to be strongly compatible with $(p_j)_{j\in\mathbb{N}}$ and we write $A \in \mathscr{L}_{sc}(X)$ if, for every $j \in \mathbb{N}$,

$$p_j^X(A) := \sup_{p_j(x)=1} p_j(Ax) < \infty. \tag{3.4}$$

If $X = (X, \|\cdot\|_X)$ is a Banach space then $T \in \mathscr{L}(X)$ if and only if $T \in \mathscr{L}_{sc}(X)$. Note that the identity operator on X is always strongly compatible whichever is the choice of seminorms. Since one may not know all continuous seminorms on X explicitly, such a dependence on (p_j) is convenient, in the sense that one just has to compute (3.4) for some well known fundamental family of seminorms on X. The condition (3.4) is not obvious, since the set $\{x \in X : p_j(x) = 1\}$ is not bounded, in general. Recall that a set is bounded if and only if it is bounded with respect to every seminorm p_j.

Proposition 3.2. The countable family of seminorms $\big(p_j^X\big)_{j\in\mathbb{N}}$ defines a Fréchet topology on $\mathscr{L}_{sc}(X)$.

Proof. First we claim that if $A \in \mathscr{L}_{sc}(X)$ then

$$p_j(Ax) \leqslant p_j^X(A)p_j(x) \tag{3.5}$$

for every $x \in X$ and every j; and consequently, $p_j^X(A^n) \leqslant p_j^X(A)^n$, for every $n \in \mathbb{N}$.

Indeed, for a fixed $j \in \mathbb{N}$, if $x \in X$ satisfies $p_j(Ax) \neq 0$ then $x_0 = \dfrac{1}{p_j(x)}x$ satisfies $p_j(Ax_0) \leqslant p_j^X(A)$, from whence we deduce that $p_j(Ax) \leqslant p_j^X(A)\,p_j(x)$, for every $x \in X$.

Moreover, since $p_j(A^n x) \leqslant p_j^X(A)\, p_j(A^{n-1}x)$, for every $x \in X$ and every natural number $n \geqslant 2$, the result then follows by induction.

Besides, (3.5) implies that

(P1) $p_j(Ax) = 0$ whenever $p_j(x) = 0$, for every j; and

(P2) $\left(p_j^X\right)_{j\in\mathbb{N}}$ is a separating family of seminorms, because $p_j^X(A) = 0$ for every j implies by (3.5) that $p_j(Ax) = 0$ for every j; and since $(p_j)_{j\in\mathbb{N}}$ is separating, $Ax = 0$ for every $x \in X$, that is, $A = 0$ in $\mathscr{L}(X)$.

As a consequence, the topology it generates on $\mathscr{L}_{sc}(X)$ is Hausdorff and metrizable, by Theorem 2.3.

About the completeness, suppose that $(A_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to $\left(p_j^X\right)_{j\in\mathbb{N}}$. By (3.5), we get that $(A_k x)_k$ is a Cauchy sequence in X, for every $x \in X$, so that the map

$$X \ni x \mapsto Ax := \lim_{k\to\infty} A_k x \in X$$

is a well defined linear operator on X. One has only to notice that

$$p_j(Ax) = \lim_{k\to\infty} p_j(A_k x) \leqslant \overline{\lim_{k\to\infty}}\, p_j^X(A_k) p_j(x)$$

to conclude that $A \in \mathscr{L}_{sc}(X)$. Therefore $\left(\mathscr{L}_{sc}(X), \left(p_j^X\right)_{j\in\mathbb{N}}\right)$ is a Fréchet space.

□

In (COSTA, 2019), an operator which satisfies property **(P1) is said to be compatible with $(p_j)_j$, which turns out to be a natural condition to obtain hyperbolicity.** It is a simple exercise to verify that if $A \in \mathscr{L}(X)$ is compatible with $(p_j)_{j\in\mathbb{N}}$ then one of the expressions below

$$\sup_{p_j(x)=1} p_j(Ax), \quad \sup_{p_j(x)<1} p_j(Ax) \text{ and } \sup_{p_j(x)\leqslant 1} p_j(Ax)$$

is finite if and only if all three are; and in this case, they all coincide.

Definition 3.3. A family $\{T(t) : t \in \mathbb{R}\} \subset \mathscr{L}(X)$ is said to be a **C_0-group** on X if it satisfies the conditions $T(0) = \mathrm{id}_X$, $T(t+s) = T(t)T(s)$ and $T(\tau)x \xrightarrow[\tau\to 0]{X} x$, for every $s,t \in \mathbb{R}$ and $x \in X$. In this case, we shall write $T(\cdot)$ instead of $\{T(t) : t \in \mathbb{R}\}$.

Its **infinitesimal generator** $A\colon D(A) \subset X \to X$ is defined by

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t},$$

where $x \in D(A)$ if and only if the limit above exists.

In addition, if $T(t) \to I$ in $\mathscr{L}_{sc}(X)$ as $t \to 0$, $T(\cdot) \subset \mathscr{L}_{sc}(X)$ is said to be a **uniformly continuous group** on X.

If the parameter t of the family of bounded operators $T(t)$ is now indexed on the interval $[0,\infty)$ in the definition above then by replacing $\lim\limits_{t\to 0}$ by $\lim\limits_{t\to 0^+}$ we get the concept of C_0-semigroups, its infinitesimal generator and the concept of uniformly continuous semigroup.

Theorem 3.4. Let $A\colon D(A)\subset X\to X$ be a linear operator. The following are equivalent:

a. A is everywhere defined and it is strongly compatible with $\big(X,(p_j)_{j\in\mathbb{N}}\big)$;

b. A is the infinitesimal generator of a uniformly continuous group $T(\cdot)$ on $\big(X,(p_j)_{j\in\mathbb{N}}\big)$; in which case it is given by

$$T(t)=e^{tA}:=\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}\in\mathscr{L}_{sc}(X),\ \text{for every } t\in\mathbb{R},$$

where the convergence is with respect to the $\mathscr{L}_{sc}(X)$-topology.

Proof. Let $A\in\mathscr{L}_{sc}(X)$ and $S_N:=\sum\limits_{n=0}^{N}\frac{(tA)^n}{n!}\in\mathscr{L}_{sc}(X)$. Given $\varepsilon>0$,

$$p_j^X(S_N-S_M)\leqslant\sum_{n=M+1}^{N}\frac{1}{n!}\big(t\,p_j^X(A)\big)^n<\varepsilon,$$

for N,M large enough. Clearly e^{0A} is the identity operator on X. Also, since $\sum\limits_{n=0}^{\infty}\frac{(tA)^n}{n!}$ is absolutely convergent, we conclude that $e^{(s+t)A}=e^{sA}e^{tA}$ for all $t,s\in\mathbb{R}$, by the classical formula for product of series. Besides,

$$p_j^X(e^{tA}-\mathrm{id}_X)=p_j^X\left(\sum_{n=1}^{\infty}\frac{(tA)^n}{n!}\right)\leqslant\sum_{n=1}^{\infty}\frac{\big(t\,p_j^X(A)\big)^n}{n!}=e^{t\,p_j^X(A)}-1,$$

whence $\{e^{tA}:t\in\mathbb{R}\}$ is a uniformly continuous group on X.

By the definition of an infinitesimal generator, if $x\in X$ and $t\neq 0$ we have

$$p_j\left(\frac{e^{tA}x-x}{t}-Ax\right)\leqslant\frac{1}{t}\sum_{n=2}^{\infty}\frac{\big(t\,p_j^X(A)\big)^n}{n!}p_j(x)=\left(\frac{e^{t\,p_j^X(A)}-1}{t}-p_j^X(A)\right)p_j(x),$$

hence A is the infinitesimal generator of $\{e^{tA}:t\in\mathbb{R}\}$.

As for its reciprocal, consider the normed spaces $X_j:=\big(X/p_j^{-1}(\{0\}),\|\cdot\|_j\big)$, where

$$\|[x]_j\|_j:=\inf_{p_j(z)=0}p_j(x-z),\ \text{for } [x]_j\in X/p_j^{-1}\big(\{0\}\big).$$

It suffices to prove the result assuming that every X_j is a Banach space; otherwise one could deal with the completions of X_j. We may assume that $p_j\leqslant p_{j+1}$ for every j, so that $X_1 \subset X_2 \subset \cdots \subset X$. Let $T_j(t)\colon X_j \to X_j$ be defined by $T_j(t)[x]_j := [T(t)x]_j$.

Claim 1: $\{T_j(t) : t \in \mathbb{R}\}$ is a uniformly continuous group on X_j, for every j.

We may write $\|T_j(t)[x]_j\|_j = \inf_{p_j(z)=0} p_j\Big(T(t)x - T(t)z - \big(z - T(t)z\big)\Big)$, which is dominated by

$$\inf_{p_j(z)=0} \left\{p_j^X\big(T(t)\big)p_j(x-z) + p_j(z) + p_j\big(T(t)z\big)\right\} = p_j^X\big(T(t)\big)\|[x]_j\|_j,$$

since $T(t)$ is strongly compatible with $(p_k)_{k\in\mathbb{N}}$; so that $T_j(t) \in \mathscr{L}(X_j)$.

It is clear that $T_j(0)$ is the identity operator on X_j. Also, for $t,s \in \mathbb{R}$, we get that $T_j(t)\big(T_j(s)[x]_j\big) = [T(t)\circ T(s)x]_j = T_j(t+s)[x]_j$ and

$$\|T_j(t) - \mathrm{id}_{X_j}\|_{\mathscr{L}(X_j)} = \sup_{\|[x]_j\|_j=1}\ \inf_{p_j(z)=0} p_j\big(T(t)x - x - z\big)$$

is dominated by $\displaystyle\sup_{\|[x]_j\|_j=1}\ \inf_{p_j(z)=0} p_j^X\big(T(t)-\mathrm{id}_X\big)p_j(x-z) = p_j^X\big(T(t)-\mathrm{id}_X\big) \xrightarrow[t\to 0]{\mathbb{R}} 0.$

The maps $x \mapsto \sigma_j(x) := [x]_j$ and $[x]_{j+1} \mapsto \pi_j\big([x]_{j+1}\big) := [x]_j$ are continuous and, by construction, we get $\big(T_j(t)\circ\pi_j\big)([x]_{j+1}) = \big(\pi_j \circ T_{j+1}(t)\big)([x]_{j+1})$.

It is natural to seek the infinitesimal generator of $T(\cdot)$ using the infinitesimal generators A_j of $T_j(\cdot)$, wondering whether exists a linear operator $A\colon X \to X$ such that every $A_j\colon X_j \to X_j$ is just the projection of A on X_j induced by σ_j; that is, $[Ax]_j = A_j[x]_j$ holds for every j and $x \in X$. Well, this is the case.

Claim 2: there exists a unique linear operator $A\colon X \to X$ that turns

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "\sigma_j"'] & X \arrow[d, "\sigma_j"] \\ X_j \arrow[r, "A_j"'] & X_j \end{tikzcd} \tag{3.6}$$

into a commutative diagram, for every $j \in \mathbb{N}$. Besides, $A \in \mathscr{L}_{sc}(X)$ and it is the infinitesimal generator of $T(\cdot)$.

Fix $x \in X$. Since every σ_j is surjective, we obtain a sequence $(z_j)_j$ in X, which depends on x, such that $\sigma_j(z_j) = A_j \circ \sigma_j(x)$ for every $j \in \mathbb{N}$, then

$$\sigma_j(z_j) = A_j\circ\sigma_j(x) = \pi_j\big(A_{j+1}\circ\sigma_{j+1}(x)\big) = \pi_j\big(\sigma_{j+1}(z_{j+1})\big) = \sigma_j(z_{j+1})$$

so that $p_l(z_j - z_k) = 0$ whenever $j,k \geqslant l$. Hence we define a linear operator $A\colon X \to X$ by setting $Ax := \lim\limits_{j\to\infty} z_j$, which satisfies

$$\sigma_j(Ax) = \sigma_j\Big(\lim_{k\to\infty} z_k\Big) = \lim_{\substack{k\to\infty\\ k\geqslant j}} \sigma_j(z_k - z_j) + \sigma_j(z_j) = \sigma_j(z_j) = \big(A_j\circ\sigma_j\big)(x).$$

Since $(p_j)_j$ is a separating family of seminorms, $x \mapsto Ax$ is well defined and it is the unique linear operator that turns (3.6) into a commutative diagram. Moreover,

$$\begin{aligned}
\sup_{p_j(x)\leqslant 1} p_j(Ax) &= \sup_{p_j(x)\leqslant 1}\left\{\inf_{p_j(z)=0} p_j(Ax)-p_j(z)\right\} \\
&\leqslant \sup_{p_j(x)\leqslant 1}\left\{\inf_{p_j(z)=0} p_j(Ax-z)\right\} = \sup_{p_j(x)\leqslant 1} \|[Ax]_j\|_j \\
&\leqslant \sup_{\|[x]_j\|_j\leqslant 1} \|A_j[x]_j\|_j < \infty
\end{aligned}$$

and the last inequality holds because $\|[x]_j\|_j \leqslant p_j(x)$. Hence $A \in \mathscr{L}_{sc}(X)$.

It is not hard to see that these projections σ_j have a handy property: if $[x_\lambda]_j \xrightarrow[X_j]{\lambda\in\Lambda} [0]_j$ for every j, then $(x_\lambda)_{\lambda\in\Lambda}$ is convergent in X and $x_\lambda \xrightarrow[X]{\lambda\in\Lambda} 0$.

Finally, given $x \in X$, for every $j \in \mathbb{N}$ we have

$$\left[Ax - \frac{T(t)x-x}{t}\right]_j = [Ax]_j - \frac{[T(t)x]_j-[x]_j}{t} = A_j[x]_j - \frac{T_j(t)[x]_j-[x]_j}{t},$$

so that the net $\dfrac{T(t)x-x}{t}$ converges to Ax, for every $x\in X$; and we conclude that

$$T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = e^{tA}, \text{ for every } t\in\mathbb{R}.$$

□

As in Banach spaces, groups with the same infinitesimal generator coincide.

Proposition 3.5. If $T(\cdot)$ and $S(\cdot)$ are uniformly continuous groups on X with

$$\lim_{t\to 0}\frac{T(t)-\mathrm{id}_X}{t} = A = \lim_{t\to 0}\frac{S(t)-\mathrm{id}_X}{t} \text{ in } \mathscr{L}_{sc}(X),$$

then $T(t) = S(t)$ for every $t\in\mathbb{R}$.

Proof. We shall prove that given $\tau,\varepsilon>0$, we get $p_j^X\big(T(t)-S(t)\big)\leqslant\varepsilon$ for every $0\leqslant t\leqslant\tau$ and $j\in\mathbb{N}$. By continuity of the map $t\mapsto p_j^X\big(S(t)\big)p_j^X\big(T(t)\big)$, we get that $p_j^X\big(T(t)\big)\,p_j^X\big(S(s)\big)\leqslant c = c(j,\tau,S,T)>0$, whenever $0\leqslant s,t\leqslant\tau$.

By hypothesis, there exists a positive constant $\delta=\delta(j,\tau,\varepsilon,S,T)>0$ such that $h^{-1}p_j^X\big(T(h)-S(h)\big)\leqslant\dfrac{\varepsilon}{\tau c}$ for $0\leqslant h\leqslant\delta$. For $0\leqslant t\leqslant\tau$, take $n\in\mathbb{N}$ such that $t/n<\delta$, so $p_j^X\big(T(t)-S(t)\big) = p_j^X\left(T\left(n\frac{t}{n}\right)-S\left(n\frac{t}{n}\right)\right)$ is dominated by

$$\sum_{k=0}^{n-1} p_j^X\left(T\left(\frac{(n-k-1)t}{n}\right)\right)p_j^X\left(T\left(\frac{t}{n}\right)-S\left(\frac{t}{n}\right)\right)p_j^X\left(S\left(\frac{kt}{n}\right)\right)$$

and such an expression is dominated by $nc\dfrac{\varepsilon}{\tau c}\dfrac{t}{n}\leqslant\varepsilon$.

□

Corollary 3.6. If $T(\cdot)$ is a uniformly continuous group on X then

a. there exists a unique operator A in $\mathscr{L}_{sc}(X)$ such that $T(t) = e^{tA}$;

b. the operator A in part a) is the infinitesimal generator of $T(\cdot)$;

c. there exist nonnegative numbers ω_j such that $p_j^X\big(T(t)\big) \leqslant \exp(\omega_j t)$, for every $t \in \mathbb{R}$; and

d. the map $\mathbb{R} \ni t \mapsto T(t) \in \mathscr{L}_{sc}(X)$ is differentiable and

$$\frac{dT(t)}{dt} = A \circ T(t) = T(t) \circ A, \text{ for every } t.$$

Consequently, the Cauchy problem

$$\begin{cases} T'(t) = AT(t), \; t \in \mathbb{R} \\ T(0) = id_X \end{cases}$$

possesses a unique solution whenever $A \in \mathscr{L}_{sc}(X)$, by Gronwall's inequality.

Remark 3.7. Let X be a sequentially complete HLCS. Several remarks are in order:

1) in (YOSIDA, 1980), the generation result reads as follows: Let $B \in \mathscr{L}(X)$. If for every continuous seminorm p on X there exists a continous seminorm $q = q(p)$ on X such that

$$\sup_{k \in \mathbb{N}} p\big(B^k x\big) \leqslant q(x), \text{ for every } x \in X, \tag{3.7}$$

then the map

$$X \ni x \mapsto \sum_{k=0}^{\infty} \frac{tB^k}{k!} x, \text{ for every } t \geqslant 0,$$

is well defined and it is continuous.

If (3.7) holds, $\{B^k\}_{k \in \mathbb{N}}$ is said to be an equicontinuous family with respect to k. It is a result of generation of a C_0-semigroup with pointwise convergence. On the other hand, by Proposition 3.2, if $A \in \mathscr{L}_{sc}(X)$ then

$$p_j(A^k x) \leqslant [p_j^X(A)]^k p_j(x), \text{ for every } x \in X \text{ and for every } j, k \in \mathbb{N},$$

which might not have the uniformity of (3.7) on k. And since nothing is assumed over other seminorms but $(p_j)_j$, the conditions are not comparable. In fact, let $X = C(\mathbb{R})$ with the seminorms $p_j(\phi) := \sup_{|\xi| \leq j} |\phi(\xi)|$, $j \in \mathbb{N}$. If $A\colon X \to X$ is given by $A\phi = a\phi$, where $a(\xi) = |\xi|^2, \xi \in \mathbb{R}$, then $A^k \phi = |\xi|^{2k}\phi$, for any $k \in \mathbb{N}$, so we may choose ϕ_0 such that $p_j(A^k \phi_0) = j^{2k}$, for any $k \in \mathbb{N}$. Thus given j we cannot find a continuous seminorm q such that $p_j(A^k \phi_0) \leq q(\phi_0)$ for all $k \in \mathbb{N}$.

Moreover, Theorem 3.4 provides a complete characterization of uniformly continuous group, **with convergence in the space of operators**. It is also noteworthy that the reciprocal of Yosida's result requires that $B\colon D(B) \subset X \to X$ is known to be densely defined and that the resolvent $(n\,id_X - B)^{-1} \in \mathscr{L}(X)$ exists for every $n \in \mathbb{N}$.

2) In (BABALOLA, 1974), a C_0-semigroup $\{S(s) : s \geqslant 0\}$ in X is called a $(C_0, 1)$-semigroup if for every continuous seminorm p on X there exist a positive number σ_p and a continuous seminorm $q = q(p)$ on X so that

$$p\big(S(s)x\big) \leqslant e^{\sigma_p s} q(x), \text{ for every } x \in X \text{ and every } s \geqslant 0. \tag{3.8}$$

The author then establishes results about the generation of $(C_0, 1)$-semigroups (instead of uniformly bounded groups) and the perturbation of infinitesimal generators. Again (3.8) is not comparable with (3.4), since $A \in \mathscr{L}_{sc}(X)$ implies that $p_j\big(T(t)x\big) \leqslant p_j^X\big(T(t)\big) p_j(x)$, by Theorem 3.4.

3) Let Γ be a fundamental family of seminorms on X and let $\mathscr{L}_b(X)$ be the space $\mathscr{L}(X)$ equipped with the topology of uniform convergence on the bounded subsets of X, which is stronger than the $\mathscr{L}_{sc}(X)$-topology. The main result in (GOLIńSKA; WEGNER, 2015) reads as follows: suppose that there exists $\mu > 0$ with the property that for every $p \in \Gamma$ there exist $q = q(\mu, p) \in \Gamma$ and $M = M(\mu, p) \geqslant 0$ such that

$$p(A^k x) \leqslant M \mu^k q(x), \text{ for every } x \in X \text{ and } k \in \mathbb{N}. \tag{3.9}$$

Then A generates a uniformly continuous semigroup which is given by the exponential series expansion, with convergence in $\mathscr{L}_b(X)$.

If (3.4) holds then (3.9) holds by setting $\mu = p_j^X(A), q = p$ and $M = 1$. Although (3.4) is a stronger condition, it is certainly easier to compute with and it is used to topologyze $\mathscr{L}_{sc}(X)$ appropriately so that $\exp(A)$ is well defined as the exponential $\mathscr{L}_{sc}(X)$-series expansion. Besides, we obtained a complete characterization in Theorem 3.4 which is not achieved in (GOLIńSKA; WEGNER, 2015).

4) In (FRERICK *et al.*, 2013), the authors study the generation of C_0-semigroups on quojections and its main application concerns the following question from (CONEJERO, 2007): If $T(\cdot)$ is a C_0-semigroup on $\mathbb{K}^{\mathbb{N}}$, is there a bounded operator A such that $T(\cdot)$ is represented pointwisely by the exponential series expansion of A? That is not our aim.

5) Other authors also deal with C_0-semigroup generation under weak assumptions, such as the semigroup $T(\cdot)$ to be quasi-equicontinuous (in the sense that there exists a constant $\beta \geqslant 0$ such that $\big(e^{-\beta t} T(t)\big)_{t \geqslant 0}$ is an equicontinuous family with respect to t) in (CHOE, 1985; LEMLE; WU, 2011); or the semigroup $T(\cdot)$ to be locally equicontinuous (in the sense that for every $t \geqslant 0$ and every continuous seminorm p there exists a continuous seminorm $q = q(t,p)$ such that $p\big(T(s)x\big) \leqslant q(x)$ for every $0 \leqslant s \leqslant t$ and $x \in X$) as in (KRAAIJ, 2016).

3.3 A Fréchet space of distributions

Doubtlessly, geometric intuition has been a good guide to solving many differential problems by seeking solution on Hilbert spaces - the more general topological vector spaces which enjoy euclidean geometric properties. Recall that every Hilbert space is isometrically isomorphic to some ℓ^2 sequences space, see (FOLLAND, 1999), Proposition 5.30. In practice, this means that there is no other geometry in infinite dimensional Hilbert spaces rather than that one ℓ^2 has. Hence, by formulating a (Cauchy) problem on a Hilbert space, one promptly requires the solution to obey such a (unique) Hilbert geometry. We are convinced that such a formulation has restrained our understanding of many phenomena. About the Banach case, it is known that every separable Banach space can be embedded in some (separable) Hilbert space, and separability is a common property when it comes to applications.

Thus we chose to deal with weaker topologies to obtain a continuous solution of linear Cauchy problems associated to certain pseudodifferential operators and as a consequence we can study the meaning of the heat equation solution backwards in time. The first generalization of Banach spaces that comes to mind are the Fréchet spaces, where the norm topology is replaced by the topology generated by a separating family of seminorms, which therefore actually behaves as a norm.

Recall that if ϕ is a Schwartz function (for which we write $\phi \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$), its Fourier transform is given by

$$\big(\mathscr{F}\phi\big)(\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\,\phi(x)\,dx,$$

whereas $\phi \mapsto \check{\phi}$ stands for the inverse Fourier transform. Let $\mathcal{S}'(\mathbb{R}^N, \mathbb{C})$ be equipped with the $\bigstar$-weak topology. See Section 2.1 or (FOLLAND, 1999; HÖRMANDER, 1980).

Definition 3.8. A pseudodifferential operator $a(D)\colon \mathcal{S}\big(\mathbb{R}^N\big) \to \mathcal{S}\big(\mathbb{R}^N\big)$ of order m on $\mathbb{R}^N$ with constant coefficients (or constant coefficients m-ΨDO, for short) is a linear map given by

$$\big(a(D)\phi\big)(x) := \big(a\widehat{\phi}\big)^{\vee}(x), x \in \mathbb{R}^N,$$

for every $\phi \in \mathcal{S}\big(\mathbb{R}^N\big)$, where $a \in C^\infty(\mathbb{R}^N)$ satisfies the property that for all multiindex α there is a constant $c_\alpha > 0$ such that $\big|\partial^\alpha a(\xi)\big| \leqslant c_\alpha(1+|\xi|)^{m-|\alpha|}$, for every $\xi \in \mathbb{R}^N$.

We shall now present a remarkable Fréchet space, namely $\mathscr{F}L^2_{loc}$, which was introduced by Treves (TREVES, 1976) and whose elements are distributions of $\mathcal{D}'$. Also, owing to the fact that every distribution u with compact support (and we write $u \in \mathcal{E}'$) and every L^2 function belong to $\mathscr{F}L^2_{loc}$, some good properties of the Fourier transform on L^2 are preserved and the Paley-Wiener-Schwartz theorem can be extensively invoked.

Let $\mathscr{F}L^2_{loc}$ be the completion of the metric space (E,d) constructed as follows: let

$$E := \mathscr{F}^{-1}\left(\mathcal{S}' \cap L^2_{loc}\right)$$

be endowed with the topology generated by the seminorms $\mathfrak{p}_j(u) := \|\widehat{u}\|_{L^2(B_j)}$ for $u \in E$ and $j \in \mathbb{N}$, which turns out to be a separating family, whence the function

$$d(u,v) := \sum_{j=1}^{\infty} 2^{-j} \frac{\mathfrak{p}_j(u-v)}{1+\mathfrak{p}_j(u-v)}$$

defines a metric on E. It is not hard to see that its topology as a complete metric space is equivalent to the one generated by the extended seminorms $\mathfrak{p}_j \colon \mathscr{F}L^2_{loc} \to [0,\infty)$ and hence $\mathscr{F}L^2_{loc} = \left(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\right)$ is a Fréchet space.

We provide further properties.

Proposition 3.9. **a.** the Fourier transform $\mathscr{F} \colon \mathscr{F}L^2_{loc} \to L^2_{loc}$ is well defined and it is continuous;

b. every element of $\mathscr{F}L^2_{loc}$ is a distribution $\mathcal{D}'\left(\mathbb{R}^N\right)$. Hence, $\mathscr{F}L^2_{loc}$ is a Fréchet space of distributions;

c. L^2 and $\mathcal{E}'$ are topological subspaces of $\mathscr{F}L^2_{loc}$ and $\left(L^2, \|\cdot\|_{L^2}\right) \hookrightarrow \mathscr{F}L^2_{loc}$. In particular, every Sobolev space $\left(H^s, \|\cdot\|_s\right)$ is continuously embedded on $\mathscr{F}L^2_{loc}$, $s \geqslant 0$;

d. every constant coefficients m-ΨDO $a(D)$ induces a strongly compatible operator on $\left(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\right)$ by setting

$$a(D)[u] := \left[a(D)u\right], \text{ for } [u] \in \mathscr{F}L^2_{loc}.$$

Consequently, $\mathbb{R} \ni t \mapsto e^{a(D)t}u_0 \in \mathscr{F}L^2_{loc}$ provides the unique solution of

$$\begin{cases} u_t = a(D)u, t \in \mathbb{R} \\ u(0) = u_0 \in \mathscr{F}L^2_{loc} \end{cases},$$

e. $\left(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\right)$ is a quojection.

Proof. Let $[u] \in \mathscr{F}L^2_{loc}$. If $(u_l)_{l\in\mathbb{N}} \in [u]$ then $(\widehat{u_l})_{l\in\mathbb{N}}$ is a Cauchy sequence in L^2_{loc}, whence there exists a unique $w \in L^2_{loc}$ such that $\widehat{u_l} \overset{l\to\infty}{\longrightarrow} w$ in L^2_{loc} and we set $\widehat{[u]} := w$, which does not depend on the choice of $(u_l)_l$ in $[u]$. Thus we define

$$C_c^\infty \ni \phi \mapsto \langle [u], \phi \rangle := \int_{\mathbb{R}^N} \widehat{[u]}(\xi)\phi(\xi)\,d\xi \ \in \mathbb{C}.$$

Let K be a compact subset of $\mathbb{R}^N$. If $\operatorname{supp}\phi \subset K \subset B(0,j)$ then

$$|\langle [u],\phi\rangle| \leqslant \|\widehat{[u]}\|_{L^2(B(0,j))}\,\|\phi\|_{L^2(K)} \leqslant |B(0,j)|^{1/2}\mathfrak{p}_j([u])\,\sup_K|\phi|,$$

whence $[u]$ is actually a distribution and one can write u instead of $[u]$.

By Paley-Wiener-Schwartz theorem we obtain the inclusion $\mathcal{E}' \subset E$ and by Plancherel theorem the embedding $\left(L^2,\|\cdot\|_{L^2}\right) \hookrightarrow E$; and (c) follows.

Let $a(D)$ be a constant coefficients m-ΨDO and $[u] \in \mathscr{F}L^2_{loc}$. If $|\xi| \leqslant j$ then

$$\mathscr{F}\big(a(D)u\big)(\xi) = \lim_{\substack{l\to\infty\\ L^2(B(0,j))}} \mathscr{F}\big(a(D)u_l\big)(\xi) = \lim_{\substack{l\to\infty\\ L^2(B(0,j))}} a(\xi)\,\widehat{u_l}(\xi) = a(\xi)\,\widehat{[u]}(\xi),$$

where $\widehat{[u]}$ is the limit in $L^2\big(B(0,j)\big)$ of the Fourier transform of some sequence $(u_l)_{l\in\mathbb{N}} \in [u]$, by definition. That $a(D)$ is strongly compatible with $\mathfrak{p}_j$ follows from

$$\mathfrak{p}_j\big(a(D)[u]\big) = \left(\int_{|\xi|\leqslant j} |a(\xi)|^2 \left|\widehat{[u]}(\xi)\right|^2 d\xi\right)^{1/2} \leqslant \|a\|_{L^\infty(B(0,j))}\,\mathfrak{p}_j\big([u]\big).$$

As for (e), let $X_j = \mathscr{F}L^2_{loc}/\mathfrak{p}_j^{-1}(\{0\})$, so we have to prove that every X_j is a Banach space.

We claim that every $X_j \equiv L^2\big(B(0,j)\big)$. By definition, if $[u] \in \mathscr{F}L^2_{loc}$ then

$$\big[[u]\big]_j := \big\{[u]+[v] : \mathfrak{p}_j([v]) = 0\big\} = \left\{[f] \in \mathscr{F}L^2_{loc} : \widehat{[f]}(\xi) = \widehat{[u]}(\xi)\ a.e.\ |\xi| \le j\right\}$$

and $\left\|\big[[u]\big]_j\right\|_{X_j} = \left\|\widehat{[u]}\right\|_{L^2(B(0,j))}$, therefore one can identify $\big[[u]\big]_j$ with $\widehat{[u]}\Big|_{B(0,j)}$.

□

As pointed out in (TREVES, 1976), $\mathscr{F}L^2_{loc}$ is a reflexive Fréchet space, whose dual space, $\mathscr{F}L^2_c$, is the inductive limit of a sequence of Hilbert spaces. The space L^2_c denotes those L^2 functions with compact support whereas $\mathscr{F}L^2_c$ is constructed by analogy with $\mathscr{F}L^2_{loc}$. To prove such a statement, one just has to use the Riesz representation theorem for $L^2(B_j)$ spaces and the Open Mapping theorem for LCSs.

3.4 Some applications to PDEs

Since $\mathcal{E}'\big(\mathbb{R}^N\big)$ is a subspace of $\mathscr{F}L^2_{loc}$, one may wonder whether a semigroup $\{e^{ta(D)} : t \geqslant 0\}$ in $\mathscr{F}L^2_{loc}$ lets $\mathcal{E}'\big(\mathbb{R}^N\big)$ invariant. We provide a sufficient condition for every N and the equivalence only for $N=1$. Besides, owing to the fact that $\big(L^2,\|\cdot\|_{L^2}\big) \hookrightarrow \mathscr{F}L^2_{loc}$, we provide a complete characterization of the semigroups $\{e^{ta(D)} : t \geqslant 0\}$ in $\mathscr{F}L^2_{loc}$ which let $L^2\big(\mathbb{R}^N\big)$ invariant, for any N.

If $z \in \mathbb{C}^N$ we shall write $z = \xi + i\eta$, with $\xi = \Re z$ and $\eta = \Im z$ in $\mathbb{R}^N$.

Theorem 3.10. Let $a(D)$ be a constant coefficients m-ΨDO, with $m>0$, let $\xi \mapsto a(\xi)$ be its symbol and $\{e^{ta(D)} : t \in \mathbb{R}\}$ be the group generated by $a(D)$ on $\mathscr{F}L^2_{loc}(\mathbb{R}^N)$, according to Proposition 3.9 (d).

a. Suppose that $a(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$. If $m=1$ and $\Re\, a_\alpha = 0$ whenever $|\alpha|=1$ then

$$e^{ta(D)}\big(\mathcal{E}'(\mathbb{R}^N)\big) \subset \mathcal{E}'(\mathbb{R}^N), \text{ for every } t \in \mathbb{R}. \tag{3.10}$$

In addition, if $N=1$ then the converse holds.

b. If there exists $C>0$ such that $\Re a(\xi) \leq -C|\xi|^m$ whenever $|\xi|$ is large enough, then we have the following regularization effect:

$$e^{ta(D)}\big(\mathcal{E}'(\mathbb{R}^N)\big) \subset \mathcal{S}(\mathbb{R}^N), \text{ for all } t>0. \tag{3.11}$$

c. We have the following positive invariance:

$$e^{ta(D)}\big(L^2(\mathbb{R}^N)\big) \subset L^2(\mathbb{R}^N), \text{ for all } t \geqslant 0 \tag{3.12}$$

if and only if

$$\sup_{\xi\in\mathbb{R}^N} e^{t\,\Re\, a(\xi)} < \infty, \text{ for all } t \geqslant 0. \tag{3.13}$$

d. if $\Re\, a(\xi) \leqslant 0$ whenever $|\xi|$ is large enough, then (3.12) holds.

Proof. **a.** If $|\alpha|=1$ then $a_\alpha = ib_j$ with $b_j \in \mathbb{R}$ for every $j=1,2,\dots n$ and

$$a(D) = \sum_{|\alpha|\leq 1} a_\alpha D^\alpha = a_0 + \sum_{j=1}^n ib_j(2\pi i)^{-1}\frac{\partial}{\partial x_j},$$

so that $\xi \mapsto a(\xi) = a_0 + \sum_{j=1}^n ib_j\xi_j$. Let $u \in \mathcal{E}'(\mathbb{R}^N)$, $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$, then

$$\mathscr{F}\left(e^{ta(D)}u\right)(\xi) = e^{ta(\xi)}\widehat{u}(\xi) = e^{ta_0}\left[e^{2\pi i\left(\frac{tb}{2\pi}\right)\cdot\xi}\widehat{u}(\xi)\right] = e^{ta_0}\mathscr{F}\left(\tau_{(tb/2\pi)}u\right)(\xi),$$

where $b=(b_1,\cdots,b_n)$ and τ_h stands for the translation by $h \in \mathbb{R}^N$. Hence (3.10) holds, $e^{ta(D)} : \mathcal{E}'(\mathbb{R}^N) \to \mathcal{E}'(\mathbb{R}^N)$ is well defined and it coincides with e^{ta_0} times the translation by $tb/2\pi$.

Now setting $N=1$, suppose that (3.10) holds and let $u \in \mathcal{E}'(\mathbb{R})$.

On the one hand, by Paley-Wiener-Schwartz theorem $e^{ta(D)}u \in \mathcal{E}'(\mathbb{R})$ if and only if $\mathscr{F}\left(e^{ta(D)}u\right) : \mathbb{R} \to \mathbb{C}$ has an analytic extension $V = V_{(t,u)} : \mathbb{C} \to \mathbb{C}$ and there exist constants $C = C_{(t,u)}, R = R_{(t,u)} > 0$ and $L = L_{(t,u)} \in \mathbb{N}$ such that

$$|V(z)| \leqslant C(1+|z|)^L e^{R|\Im\, z|}, \text{ for every } z \in \mathbb{C}.$$

On the other hand, since $\mathbb{C} \ni z \mapsto a(z) \in \mathbb{C}$ is a polynomial, we deduce that $\mathbb{R} \ni \xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\widehat{u}(\xi)$ admits a unique analytic extension, which is given by

$$\mathbb{C} \ni z \mapsto e^{ta(z)}\widehat{u}(z) = V(z) \in \mathbb{C}.$$

Besides, $\big|\mathscr{F}\big(e^{ta(D)}u\big)\big| = e^{t\,\Re a(z)}|\widehat{u}(z)|$ holds, for every $t \in \mathbb{R}$ and $z \in \mathbb{C}$. Combining this with the estimate of V, setting $u = \delta$ and $t = 1$, we obtain

$$e^{\Re a(z)} \leqslant C(1+|z|)^L e^{R|\Im z|}, \text{ for every } z \in \mathbb{C}, \tag{3.14}$$

which holds if and only if the order m of $a(z)$ is equal to 1. Otherwise, $z \mapsto \Re a(z)$ is a polynomial with degree $\geqslant 2$ and onto $\mathbb{R}$, which contradicts (3.14). Hence we write $a(\xi) = a_0 + a_1\xi$, for some $a_0, a_1 \in \mathbb{C}$, and we claim that $\Re a_1 = 0$; from which the result follows. By (3.14)

$$e^{\Re a(\xi)} = e^{\Re a_0}e^{\Re a_1 \xi} \leq C(1+|\xi|), \text{ for every } \xi \in \mathbb{R},$$

which cannot be true for all $\xi \in \mathbb{R}$ if $\Re a_1 \neq 0$.

b. Let $u \in \mathcal{E}'\big(\mathbb{R}^N\big)$ and $t \geqslant 0$, then $\mathbb{R}^N \ni \xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\widehat{u}(\xi)$ is a C^∞ function. By Paley-Wiener-Schwartz theorem

$$|\mathscr{F}\big(e^{ta(D)}u\big)(\xi)| = e^{t\Re a(\xi)}|\widehat{u}(\xi)| \leq C_u(1+|\xi|)^{L_u}e^{t\Re a(\xi)}$$

and by hypothesis the right side of this inequality vanishes at infinity faster than any power of $|\xi|$. Now it is easy to deduce that $\xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi)$ is a Schwartz function; and (b) follows.

c. For $u \in L^2\big(\mathbb{R}^N\big)$, $e^{ta(D)}u \in L^2\big(\mathbb{R}^N\big)$ if and only if $\mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\widehat{u}$ belongs to $L^2\big(\mathbb{R}^N\big)$ if and only if $|e^{ta(\xi)}\widehat{u}(\xi)| = e^{t\,\Re a(\xi)}|\widehat{u}(\xi)|$ belongs to $L^2\big(\mathbb{R}^N\big)$.

Suppose that (3.13) holds. Let $u \in L^2\big(\mathbb{R}^N\big)$ and $t \geqslant 0$ then

$$\int_{\mathbb{R}^N} e^{2t\,\Re a(\xi)}|\widehat{u}(\xi)|^2\,d\xi \leqslant \left(\sup_{\xi\in\mathbb{R}^N} e^{2t\,\Re a(\xi)}\right)\|\widehat{u}\|_{L^2},$$

so that (3.12) is true. Conversely, suppose that (3.13) does not hold. Take $t \geqslant 0$, a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ and a collection of disjoint balls $B_n := B(\xi_n; r_n)$ such that $|\xi_n| \to \infty$, $e^{2t\,\Re a(\xi_n)} \geqslant 2^n/n$ and $e^{2t\,\Re a(\xi)} \geqslant 2^n/2n$ for every $\xi \in B_n$, for every $n \in \mathbb{N}$. Let f_n be defined by

$$\mathbb{R}^N \ni \xi \mapsto f_n(\xi) := \frac{2^{-n/2}}{|B_n|^{1/2}}\chi_{B_n}(\xi),$$

then the function $f := \sum_n f_n$ belongs to $L^2\big(\mathbb{R}^N\big)$, since

$$\int_{\mathbb{R}^N} f^2(\xi)\,d\xi = \int_{\mathbb{R}^N}\left(\sum_{n=1}^\infty f_n^2(\xi)\right)d\xi = \sum_{n=1}^\infty \int_{\mathbb{R}^N} f_n^2(\xi)\,d\xi = \sum_{n=1}^\infty \frac{1}{2^n} < \infty.$$

On the other hand, $e^{ta(D)}\check{f}$ does not belong to $L^2(\mathbb{R}^N)$, because

$$\int_{\mathbb{R}^N} e^{2t\Re a(\xi)}|f(\xi)|^2 d\xi = \sum_{n=1}^{\infty}\int_{B_n} e^{2t\Re a(\xi)} f_n^2(\xi)\, d\xi \geqslant \sum_{n=1}^{\infty}\frac{1}{2n} = \infty.$$

Hence (3.12) does not hold for $u := \check{f} \in L^2(\mathbb{R}^N)$, from which (c) and (d) follow.

□

Hence in order to obtain a C_0-semigroup on $L^2(\mathbb{R}^N)$ which is generated by a linear differential operator with constant coefficients, we may replace the spectral conditions of Hille-Yosida theorem (on Banach spaces) by the condition (3.13).

Proposition 3.11. Consider the heat equation in $\mathbb{R}^N$:

$$\begin{cases} u_t + u = \Delta u, t > 0 \\ u(0) = u_0 \end{cases}. \tag{3.15}$$

If $u_0 \in \mathscr{F}L^2_{loc}$ the evolution problem (3.15) can be solved for every $t \in \mathbb{R}$ in a distributional sense. Moreover, if $u_0 \in L^2$ then such a solution extends the solution given by the analytic semigroup generated by $-(1-\Delta)$ on L^2 forwards in time.

Proof. On the one hand, $A := 1-\Delta\colon H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N,\mathbb{C}) \to L^2(\mathbb{R}^N,\mathbb{C})$, as a linear operator in L^2, is a sectorial operator with $\Re\sigma(A) > 0$, whence $-(1-\Delta)$ generates an analytic semigroup on L^2 indicated by $\{e^{-At} : t \geq 0\}$. Besides, the fractional power spaces associated to A are the usual Sobolev spaces H^s, characterized by Bessel potentials: $H^s = \left\{u \in \mathcal{S}' : (1+4\pi^2|\xi|^2)^{s/2}\widehat{u} \in L^2\right\}$.

On the other hand, the map $\xi \mapsto a(\xi) := -(1+4\pi^2|\xi|^2)$ is the symbol of the 2-ΨDO operator $a(D) := -(1-\Delta) : \mathscr{F}L^2_{loc} \to \mathscr{F}L^2_{loc}$, which generates a continuous group on $\mathscr{F}L^2_{loc}$, denoted by $\{e^{a(D)t} : t \in \mathbb{R}\}$. Also, by Corollary 3.10, $\{e^{a(D)t}\colon L^2 \to L^2\}_{t\geqslant 0}$ is a continuous semigroup.

Thus we obtained two semigroups in L^2 generated by the heat operator under two different approaches of generation. However we claim that they are the same semigroup and therefore the group on $\mathscr{F}L^2_{loc}$ extends the analytic semigroup defined on L^2. Let $t > 0$ and $u_0 \in L^2$. First, $\mathscr{F}\left(e^{-tA}u_0\right) = e^{-(1+4\pi^2|\xi|^2)t}\widehat{u_0}$, as in (HENRY, 1981), page 34; besides by the definition of $e^{a(D)t}$, we may apply the Fourier transform on it to obtain $\mathscr{F}\left(e^{a(D)t}u_0\right) = e^{ta(\xi)}\widehat{u_0}$. Since they are elements of L^2, we conclude by Plancherel theorem that both semigroups coincide on L^2.

□

The amusing consequence of it is that the heat equation (3.15) can be solved backwards in time for any initial data $u_0 \in L^2 \subset \mathscr{F}L^2_{loc}$, in a distributional sense. Essentially, for $u_0 \in L^2$, the regularity of $e^{-tA}u$ has three stages indexed by the time parameter: for $t<0$, $e^{-t(1-\Delta)}u_0 \in \mathscr{F}L^2_{loc}$, that is, the solution backwards belongs to a space of very low regularity; if $t=0$, there is nothing to add, u_0 belongs to L^2; and for $t>0$, $e^{-t(1-\Delta)}u \in \bigcap_{s\in\mathbb{R}} H^s \subset C^\infty$, which is the regularization effect forward in time promoted by this sectorial operator.

The exponential factor in $\mathscr{F}\left(e^{-t(1-\Delta)}u_0\right) = e^{-(1+4\pi^2|\xi|^2)t}\widehat{u_0}$ explains how the regularity of (3.15) responds to the time parameter, since

$$\int_{\mathbb{R}^N} e^{-2t(1+4\pi^2|\xi|^2)}(1+|\xi|)^{2M} d\xi < \infty, \text{ for } t>0 \text{ and } M\in\mathbb{N},$$

and

$$\lim_{|\xi|\to\infty} e^{-2t(1+4\pi^2|\xi|^2)}(1+|\xi|)^{2M} = \infty, \text{ for } t<0 \text{ and } M\in\mathbb{N}.$$

The key point here is that the fractional power spaces associated to $1-\Delta$ are completely characterized by a property which essentially connects Fourier analysis with the usual Hilbert spaces.

Example 3.12 (**The *i* derivative operator on** $\mathbb{R}$)**.** If $A = i\frac{d}{dx} : H^1 \subset L^2 \to L^2$ then we cannot solve (3.2) using the mainstream approach of Banach spaces because A does not fulfill the spectral conditions of Hille-Yosida theorem. On the other hand, since $a(\xi) = -2\pi\xi$ is its symbol, it generates a semigroup $\{e^{itd/dx} : t \geqslant 0\}$ on L^2 by Theorem 3.10 (c), which provides its unique solution on L^2.

Example 3.13 (**The positive power of the Laplace operator on** $\mathbb{R}^n$)**.** Let $\alpha > 0$. The symbol $\xi \mapsto a(\xi) = -(4\pi^2|\xi|^2)^\alpha$ is associated to the operator $-(-\Delta)^\alpha$ so that $e^{-t(-\Delta)^\alpha}u \in \mathcal{S}(\mathbb{R}^n)$, for every $t>0$, whenever $u \in \mathcal{E}'(\mathbb{R}^n)$, by Theorem 3.10 (b).

In particular, the solution of the Cauchy problem

$$\begin{cases} u_t = -(-\Delta)^\alpha u, \ t\in\mathbb{R} \\ u(0) = \delta, \end{cases}$$

belongs to $\mathcal{S}(\mathbb{R}^N)$ for every $t>0$, where δ stands for Dirac δ-distribution.

Example 3.14 (**The derivative operator on** $\mathbb{R}$)**.** Consider the Cauchy problem

$$\begin{cases} u_t = u_x, t\in\mathbb{R} \\ u(0) = u_0 \in C^\infty \end{cases}. \tag{3.16}$$

Under the mainstream approach, one may impose three restrictions in order to solve (3.16): i) $t \geqslant 0$; ii) u_0 has a derivative u_0' and both are uniformly bounded continuous functions (we write $u_0, u_0' \in C_b(\mathbb{R},\mathbb{C})$); and iii) restricting the domain of $A = \frac{d}{dx}$ so that it is a closed densely defined operator on $C_b(\mathbb{R},\mathbb{C})$. Thus, the C_0-semigroup generated by $\frac{d}{dx}$ is the translation semigroup, see (PAZY, 1983).

On the other hand, we can solve (3.16) in the (Fréchet) phase space $C^\infty(\mathbb{R},\mathbb{C})$ without any further assumptions; besides the group $\frac{d}{dx}$ generates extends the C_0-semigroup above. Let $C^\infty_{\exp}$ be the set of all functions $\phi \in C^\infty$ such that, for every $m \in \mathbb{Z}_+$ and $j \in \mathbb{N}$, there exists a constant $M = M(\phi, m, j) > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{|x|\leqslant j} \left| M^{-n} \frac{d^{n+m}}{dx^{n+m}} \phi(x) \right| < \infty.$$

Proposition 3.15. Every $\phi \in C^\infty_{\exp}$ is a real analytic function. Moreover,

a. $C^\infty_{\exp}$ is a dense subspace of $C^\infty(\mathbb{R})$;

b. the partial sums $S_N := \sum_{n=0}^{N} \frac{t^n}{n!} \frac{d^n}{dx^n}\phi$ converges in $C^\infty(\mathbb{R})$ to a function in $C^\infty_{\exp}$, for every $\phi \in C^\infty_{\exp}$ and $t \in \mathbb{R}$; its limits is denoted by $e^{t\frac{d}{dx}}\phi$;

c. $e^{t\frac{d}{dx}} : C^\infty_{\exp} \to C^\infty_{\exp}$ is well defined, it is a bounded linear operator and hence by density $e^{t\frac{d}{dx}} \in \mathscr{L}\big(C^\infty(\mathbb{R})\big)$;

d. the family of operators $\{e^{t\frac{d}{dx}} : t \in \mathbb{R}\}$ is a uniformly continuous group on $C^\infty(\mathbb{R})$ such that

$$\left(e^{t\frac{d}{dx}}\phi\right)(s) = \phi(s+t), \text{ for every } s \in \mathbb{R}.$$

Proof. Since $x \mapsto e^{-x^2}$ belongs to $C^\infty_{\exp}$, one may argue with it as a mollifying function to obtain the proof.

□

3.5 Final Comments

If X is a Fréchet space and $A\colon X \to X$ is strongly compatible with it then the operator $\exp(tA)$ is strongly compatible as well and solves the Cauchy problem

$$\begin{cases} u_t = Au, t \in \mathbb{R} \\ u(0) = u_0 \in X \end{cases}.$$

We have established criteria to identify whether the semigroup generated by a constant coefficients m-ΨDO defined on $\mathscr{F}L^2_{loc}$ acts on L^2 and $\mathcal{E}'$; and we analyzed the regularization of initial data backwards and forwards by the solution group of the heat equation on $\mathscr{F}L^2_{loc}$, which extends the standard solution on Hilbert spaces for positive times. This explains partially the regularization process which the exponential of the Laplacian operator performs.

The strong connection with the mainstream approach and the results achieved have convinced us that we may consider hyperbolicity (see (COSTA, 2019)), non-autonomous linear operators $A = A(t)$, generation of analytic semigroups and semilinear problems as well. Besides, it is not clear how the $\mathcal{E}'$ equipped with its original topology is related to its topology as a subspace of $\mathscr{F}L^2_{loc}$.

Our future aims concern all these subjects.

CHAPTER

4

AN EXTENDED KALDOR'S MODEL WITH DELAYED FISCAL POLICY

This chapter is devoted to analyze the stability of the economy according to an extended version of Kaldor's economic growth model, which was formulated by Takeuchi and Yamamura (TAKEUCHI; YAMAMURA, 2004) and considers the role of the government and the effect of its both monetary and fiscal policies. We improve the treatment given by these authors by considering the full version of the model, that is, with four variables instead of three - namely, national income, capacity of production, bonds value and money supply. Besides, we study how a possible government inefficiency concerning its fiscal policy decision-making can affect the economic stability; which is achieved by introducing a time delay which represents the time between the recognition of an opportunity of implementing a fiscal policy measure and the actual implementation.

In the Section 4.1, after a brief discussion about this extended version, its improvements, its restrictions and some assumptions, we prove the existence and uniqueness of a positive equilibrium point. We then establish sufficient conditions so that such a point is asymptotically stable, which can be done with or without delay time on the fiscal policy implementation, as we shall see in Section 4.2. For further conclusions, we run simulations and analyze the effect of the fiscal policy strength and the delay time size over the long-term stability in the model. See Section 4.3. All results, some conclusions and future considerations are included in Section 4.4.

4.1 An extended Kaldor's business cycle model

We shall briefly describe the original version of Kaldor's model (KALDOR, 1940) and some of its reformulations (CHANG; SMYTH, 1971; GANDOLFO, 1997; KADDAR; ALAOUI, 2008; ICHIMURA, 1955; KRAWIEC; SZYDLOWSKI, 1999). The reader is expected to have some familiarity with macroeconomics theory; at least the contents we presented in Section 2.5. Most of economic quantities depends on time although we do not write it down explicitly.

Under the Keynesian theory[1], **investment** (denoted by I) and **savings** (denoted by S) depend positively on the **(national) income/GDP** (denoted by Y). By definition, S is the portion of income Y that is not consumed, for which we write $S = Y - C$, where C clearly stands for (national) **consumption**. Since the production adjusts in order to satisfy the demand for goods, the equilibrium is achieved precisely when $I = S$. Indeed, if $I > S$ then $C + I > Y$ which means that aggregate demand for goods exceeds supply, which stimulates production to increase. On the other hand, if $I < S$ then supply exceeds demand, whence production decreases. Thus there are only two extreme scenarios by considering a linear dependence on Y, as it was frequently the case in early 20th century. If we write

$$\begin{aligned} S(Y) &= S_0 + a_0 Y \\ I(Y) &= I_0 + a_1 Y \end{aligned},$$

then either $a_0 > a_1$ or $a_1 < a_0$. In the first case, the equilibrium point (where $(Y, I) = (Y, S)$) is stable; and in the latter it is unstable. In other words, economic activity is either completely stable or completely unstable, which does not fit reality. The case $a_0 = a_1$ is unimportant since it would imply that the economy is static, which again does not fit the reality. This modeling can be improved by allowing the curves $Y \mapsto S(Y)$ and $Y \mapsto I(Y)$ to actually be curves.

In the early forties, Nicholas Kaldor (pronounced as nɪkələs kəldoʊr, English IPA), a Cambridge economist in the post-war period, developed a business cycle model which explained the natural fluctuations of the economy as we have observed in Fig. 1. He was one of the first economists to propose a nonlinear formulation for investment I and saving S as functions of income Y and capital stock (denoted by K), which was a tremendous improvement over classical linear models and it is invariably present over the last decades: (BLINDER; SOLOW, 1973; CHANG; SMYTH, 1971; CESARE; SPORTELLI, 2005; GANDOLFO, 1997; ICHIMURA, 1955; KRAWIEC; SZYDLOWSKI, 1999; MATSUMOTO; SZIDAROVSZKY; ASADA, 2016; MATSUMOTO; SZIDAROVSZKY, 2016; MIRCEA; NEAMTU; OPRIS, 2011; TAKEUCHI; YAMAMURA, 2004). In the sixties, Goodwin (GOODWIN, 1967) proposed a model inspired on Lotka-Volterra equations with the same aim, which also has been studied over the years, as in (GABISCH; LORENZ, 1989; MATSUMOTO; MERLONE; SZIDAROVSZKY, 2018; MATSUMOTO; NAKAYAMA; SZIDAROVSZKY, 2018).

[1] (KEYNES, 1964): Regarded widely as the cornerstone of Keynesian thought, by challenging the established classical economics and introducing new concepts right after the Great Depression.

Originally, Kaldor's model was studied with graphic techniques and the first rigorous mathematical study is due to (ICHIMURA, 1955) and later (CHANG; SMYTH, 1971), under which we can enumerate the following assumptions:

K1) $Y, K \mapsto I(Y,K), S(Y,K)$ are nonlinear functions;

K2) $Y'(t) = \alpha\big(I(t) - S(t)\big)$, with $\alpha > 0$;

K3) $K'(t) = I(Y,K)$ so that the variation of capital stock is simply the investment;

K4) $\dfrac{\partial I}{\partial K} < 0 < \dfrac{\partial I}{\partial Y}, \dfrac{\partial S}{\partial K}, \dfrac{\partial S}{\partial Y}$;

K5) $\dfrac{\partial I}{\partial Y} > \dfrac{\partial S}{\partial Y}$ at "normal" level of production; and

K6) $\dfrac{\partial I}{\partial Y} < \dfrac{\partial S}{\partial Y}$ if the economy faces either a recession (too low levels of production) or a too strong growth (high levels of production).

Here, K denotes the capital stocks: the total value of the buildings, machines, warehouses and so on which are used to produce goods and services. Thus **K1)** states that the decision to invest or not (which means to save money) depends on the income Y and on the already available capacity of production K, which is pretty obvious. The key point is the nonlinear dependence.

The reader probably might have noticed that Y is sometimes called GDP and (national) income. They are the same: the quantity of money one obtains by producing goods and offering services. It is noteworthy that income is a nomenclature from microeconomics (therefore the income a person or a family acquires) while GDP comes from macroeconomics (which deals with economies instead of individuals). Such a quantity dictates how rich a country is and it grows as long as the nation is capable of accumulating richness, which is possible basically by either increasing the capacity of production or decreasing the amount of money kept under the mattress; and **K2)** express it.

About the investment assumption **K3)**, it is reasonable to state that the variation of capital stock is a consequence of the actual investment, which is assumed to be equal to the planned investment, so that $K'(t) = I(t)$. That I and S depend positively on the income Y is a typical assumption of Keynesian theory: as richer a nation is more money are available to invest and to save. On the other hand, imposing that $I_K < 0 < S_K$ is quite natural: as the capital stock increases, the necessity to increase production get smaller and smaller, and the best opportunities to invest are taken, remaining only those investments that are not that profitable (hence $I_K < 0$); also as a consequence, as production expands, the prices fall and this is crucial for consumers because now less money is required to maintain the usual consume, that is, there will be more money available to be saved (hence $S_K > 0$). Clearly, I_K stands for the partial derivative of I with respect to K; and analogously for the others.

Expectations are a key point when it comes to modeling economy. Differently from the natural sciences, dynamics enters macroeconomics in such a way that the present depends not only on the past but also on the future, owing to the fact that economic agents have expectations (or beliefs) about the future which strongly influence their decision on the present. At the moment there is no generally accepted way of modeling expectations and every one of them has its strengths and weaknesses.

Under normal levels of production, there is the expectation that economy will keep growing and this confidence promotes more investment than savings. However, if a recession is in course then one believes that its investments are not worthwhile and that the best thing one can do is to keep its money under the mattress; and if the production is taken to be saturated then there will be an expectation that the economic growth is coming to an end, whence the people rather save than invest. Such a reasoning is expressed in assumptions **K5)** and **K6)**.

When one recalls that $S = Y - C$, Kaldor's model reads as follows:

$$\begin{cases} Y'(t) = \alpha\big(C(t) + I(t) - Y(t)\big) \\ K'(t) = I\big(Y(t), K(t)\big) \end{cases}, \tag{4.1}$$

which is an autonomous ODE in $\mathbb{R}^2$, as we promised. A qualitative analysis of local stability leads to the existence of a limit cycle; arguing with either the Poincaré-Bendixson theorem as in (CHANG; SMYTH, 1971) or with the Hopf bifurcation theorem as in (GANDOLFO, 1997). In (ICHIMURA, 1955), Liénard techinques are used to guarantee the oscillations. As a consequence, national income and capacity of production suffer alternating cycles of increasing and decreasing that characterize the economic activity.

A simplifying implicit assumption is that the economy is closed, that is, there is no trade with other nations. If this were not the case, one would have a similar formulation which aggregates the economy of all nations taken into account; and then the model would explains the evolution of the global economy instead of a particular one.

Several different delay formulations of (4.1) have been considered in the last years. For instance, substituting its second equation by

$$K'(t) = I\big(Y(t - \tau_0), K(t - \tau_1)\big) - \delta K(t),$$

one obtains a formulation that considers the gestation lag of investment and the depreciation effect thanks to the positive parameter δ. The case $\tau_0 > 0 = \tau_1$ was firstly studied in (KRAWIEC; SZYDLOWSKI, 1999), - where the model is thereafter called the Kaldor-Kalecki model, referring to (KALDOR, 1940; KALECKI, 1935) - and more recently in (MATSUMOTO; SZIDAROVSZKY; ASADA, 2016), where the authors proved that the dynamic behavior is affected quantitatively by the investment delay but not qualitatively; the case $\tau_0 = \tau_1 > 0$ was considered by (KADDAR; ALAOUI, 2008), and also by (MIRCEA; NEAMTU; OPRIS, 2011), adding a noise perturbation. In 2009, Zhou and Li (ZHOU; LI, 2009) analyzed a combination of IS-LM model and Kaldor's model with two time delays in the capital accumulation processes.

Following (WOLFSTETTER, 1982), Takeuchi and Yamamura (TAKEUCHI; YAMAMURA, 2004) added the government and a delay time on its fiscal policy to the model, which were important elements missing, as pointed out in (BLINDER; SOLOW, 1973; MATSUMOTO, 2008). Such a formulation in $\mathbb{R}^4$ consists on an adaptation on the equations in (KALDOR, 1940), a government budget constraint and a monetary market equation. To make this precise, we introduce some economic quantities:

(i) the aggregate value of bonds varies with time, so we write $t \mapsto B(t)$. Every bond is assumed to be a consol, that is, a bond with a fixed income security and no maturity date;

(ii) money supply $M(t)$ - which is entirely controlled by the government - together with money demand $t \mapsto L\big(Y(t), M(t)\big)$ are the forces of the money market, in the sense that

$$M'(t) = L\big(Y(t), M(t)\big) - M(t); \tag{4.2}$$

(iii) the price level $t \mapsto p\big(Y(t)\big)$ is an index that corrects the real value of bonds and the money power throughout the time;

(iv) the tax revenue is

$$t \mapsto T(t) = T\big(Y(t), B(t)\big) = \theta\left(Y(t) + \frac{B(t)}{p\big(Y(t)\big)}\right) - T_0,$$

where $0 < \theta < 1$ is the tax rate over the income and the profits on the bonds, and $T_0 > 0$;

(v) government expenditure is

$$t \mapsto G(t) = G_0 + \beta\big(Y^* - Y(t-\tau)\big),$$

where G_0 is the fixed spending and $\beta > 0$ measures how the expenditure responds to the excess (or lack) of national income, assuming that the government always know the equilibrium national income Y^*. The constant delay $\tau \geqslant 0$ represents the policy lag, since it naturally takes time to recognize opportunities to implement a stabilization policy and to actually put it in practice;

(vi) the interest rate of the bonds is $t \mapsto r\big(Y(t), M(t)\big)$ and it is basically the money price;

Hence the government budget constraint reads as follows:

$$\frac{M'(t)}{p(Y)} + \frac{B'(t)}{r(Y,M)p(Y)} = G(t) + \frac{B(t)}{p(Y)} - T(Y,B), \tag{4.3}$$

which equates the changes in the stocks of bonds and money to the government deficit, since it is assumed that selling bonds and printing banknotes finance the government deficit. Besides,

(vii) the national consumption is

$$C(t) = C_0 + c_1\left(Y(t) + \frac{B(t)}{p(Y)} - T(t)\right) + c_2\left(\frac{B(t)}{r(Y,M)p(Y)} + \frac{M(t)}{p(Y)}\right),$$

where $0 < c_1, c_2 < 1$ are the marginal propensity to consume the available income and the available wealth respectively and $C_0 > 0$ is the minimal (basically vital) consumption; and

(viii) the (nonlinear) investment function $t \mapsto I\big(Y(t), K(t), M(t)\big)$ represents the amount of money spent on buying goods for future use, which should provide more money.

By considering the money depreciation over time, the variables B and M have to be corrected by the price level, whence the values $Y, C, I, T, G, K, B/p$ and M/p are measured in real terms (let us say, euro or dollar).

An extended version of Kaldor's model in $\mathbb{R}^4$ arises by adding (4.2) and (4.3) to the original formulation (4.1) together with the adapted consumption and investment functions and the government expenditure; it reads as follows:

$$\begin{cases} Y'(t) = \alpha\big(C(t) + I(t) + G(t) - Y(t)\big) \\ K'(t) = I\big(Y(t), K(t), M(t)\big) \\ \dfrac{M'(t)}{p(Y)} + \dfrac{B'(t)}{r(Y,M)p(Y)} = G(t) + \dfrac{B(t)}{p(Y)} - T(Y,B) \\ M'(t) = L(Y,M) - M(t) \end{cases} \tag{4.4}$$

On the one hand, fiscal policy refers to the mechanism of increasing or decreasing the expenditure G, which directly affects the economic activity, stimulating it or discouraging it, respectively. The government pursues such a policy by adjusting the parameter β, which is assumed to be positive, since the Kaldor model is essentially a Keynesian one. One could consider that fiscal policy includes the alteration of taxation levels as well, which is achieved by adjusting the parameter $0 < \theta < 1$. But we do not consider this way because the tax rate θ is predetermined and nearly unchangeable by political reasons. The immediate consequence of such an assumption is that we do not analyze the stability of the equilibrium point with respect to this parameter.

On the other hand, monetary policy refers to the fact that is the government who effectively prints every banknote in circulation and consequently determines the available money quantity, which affects the price level and the interest rate and consequently investment and national production. These two policies together allow the government to promote economic stability or, unfortunately, instability.

As in (BLINDER; SOLOW, 1973), Takeuchi and Yamamura considered two extreme scenarios (both lead to an $\mathbb{R}^3$ formulation): money finance case by setting $B' \equiv 0$ in (4.4); and bond finance case by setting $M' \equiv 0$. In the former, the government controls the money supply but bonds offer keeps constant $B = \overline{B}$; and in the latter, the government controls the bonds supply in order to finance its deficit but it cannot adjust its money supply ($M = \overline{M}$). And then the model stability is analyzed under these two settings with or without delay time τ. However such scenarios separately do not fit the practical government activity, therefore we take a step forward by analyzing the model (4.4) in $\mathbb{R}^4$ with its full budget constraint and with or without delay time.

By setting $u = (u_1, u_2, u_3, u_4) \equiv (Y, K, B, M)$ in (4.4), we obtain

$$\begin{cases} u_1'(t) = \alpha\Big(-\big(1-(1-\theta)c_1\big)u_1(t) - \beta u_1(t-\tau) + I(u) \\ \qquad + \dfrac{\big((1-\theta)c_1 + c_2/r(u)\big)u_3(t) + c_2 u_4(t)}{p(u)} \\ \qquad + C_0 + c_1 T_0 + G_0 + \beta Y^* \Big) \\ u_2'(t) = I(u) \\ u_3'(t) = r(u)p(u)\Big(-\theta u_1(t) - \beta u_1(t-\tau) \\ \qquad + \dfrac{(1-\theta)u_3(t) + u_4(t) - L(u)}{p(u)} \\ \qquad + G_0 + \beta Y^* + T_0 \Big) \\ u_4'(t) = L(u) - u_4(t) \end{cases} . \tag{4.5}$$

All the functions are assumed to be as smooth as necessary. Additionally, consider the following assumptions for every $u \in \mathbb{R}^4_+$:

(A1) $L(u)\Big|_{u_4=0} > 0$, $\lim\limits_{u_4 \to \infty} L(u) < 0$ and $\dfrac{\partial L}{\partial u_4}(u) \leqslant 0 < \dfrac{\partial L}{\partial u_1}(u)$;

(A2) $I(u)\Big|_{u_2=0} > 0$, $\lim\limits_{u_2 \to \infty} I(u) < 0$ and $\dfrac{\partial I}{\partial u_2}(u) < 0 < \dfrac{\partial I}{\partial u_1}(u), \dfrac{\partial I}{\partial u_4}(u)$;

(A3) $p(u) > 0$ and $\dfrac{dp}{du_1}(u) > 0$; and

(A4) $0 < r(u) < 1$ and $\dfrac{\partial r}{\partial u_4}(u) < 0 < \dfrac{\partial r}{\partial u_1}(u)$.

Under these assumptions, we can prove the existence and the uniqueness of a positive equilibrium point.

Assume the government establishes some equilibrium income $Y^* > 0$ as target and pursues it. By **(A1)**, the right-hand side of the last equation in (4.5) applied for $u_1 = Y^*$ is a function of u_4, namely $u_4 \mapsto L(Y^*, u_4) - u_4$, such that it is positive for $u_4 = 0$ and it becomes negative as u_4 increases, since $\frac{\partial L}{\partial u_4}(u) \leqslant 0$ and $\lim\limits_{u_4 \to \infty} L(u) < 0$. Thus we obtain a unique value $u_4 = M^* > 0$ for which that expression is null.

Setting $u_1 = Y^*$ and $u_4 = M^*$ in the second equation of (4.4), thanks to **(A2)**, we may argue as before to obtain a unique value $u_2 = K^* > 0$ such that $I(Y^*, K^*, M^*) = 0$. Now we set $u_1 = Y^*, u_2 = K^*$ and $u_4 = M^*$ in the first and third equations of (4.4). Their right-hand sides vanish if and only if

$$\begin{cases} 0 = C(Y^*, B, M^*) + G_0 - Y^* \\ 0 = G_0 + \dfrac{B}{p(Y^*, M^*)} - T(Y^*, B), \end{cases} \tag{4.6}$$

which is a linear system on the variables B and G_0. So there exists a unique positive equilibrium point $u^* = (Y^*, K^*, B^*, M^*)$ if and only if the government can fix a compatible value $G_0 > 0$ so that the system above admits a unique positive solution $B = B^*$. It is noteworthy that u^* does not depend on β. Also, about the conditions **(A3)** and **(A4)**, we just have used the fact that the functions p and r are positive.

Therefore, we have proved the following result.

Lemma 4.1. Suppose that the conditions **(A1)-(A4)** hold. Given $Y^* > 0$, if (4.6) admits a unique positive solution (B^*, G_0) then (4.5) admits a unique positive equilibrium point u^* associated to Y^*, which does not depend on β.

In (TAKEUCHI; YAMAMURA, 2004), under suitable additional technical assumptions one has to deal with expressions where either it is possible to extract a unique positive Y^* from one of the equations and then $M^* > 0$ from other so that $I(Y^*, K, M^*) = 0$ provides a unique $K^* > 0$; or by imposing a lower bound to Y^* it is possible to obtain $B^* > 0$ as function of Y^* and the remaining argumentation follows analogously.

However we do not have such a scenario, that is, it is not possible to determine a unique $Y^* > 0$ since every of the four equations of (4.4) depends nontrivially on at least two variables. Thus we assume that the government is able to establish a national income $Y^* > 0$ as target and a compatible expenditure value $G_0 > 0$. Doing so, one can obtain a unique associated equilibrium point $u^* = (Y^*, K^*, B^*, M^*)$ in $\mathbb{R}^4_+$ as we did. In our opinion, such a setting is realistic because governments pursue annual growth rates - consequently they reconsider future values of Y^* as the economy grows - and they adjust their expenditures and policies accordingly. The reader should recognize now why Lemma 4.1 requires (B^*, G_0) to be positive.

Remark 4.2 (About the assumptions). It is natural to expect that the richer a nation the more money it demands; and clearly the money demand $L(u)$ decreases as more money $u_4 = M$ is provided, whence the derivative assumptions of **(A1)** are reasonable from the economic point of view.

As the infrastructure of a nation improves together with its capacity of production - K_2 increasing - the best opportunities of investment disappear; such a phenomenon is expressed by $\frac{\partial I}{\partial u_2} < 0$. Besides, investment is stimulated by economic activity and it essentially requires money, whence we unsurprisingly required $\frac{\partial I}{\partial u_1}$ and $\frac{\partial I}{\partial u_4}$ to be positive.

In capitalist economies, prices rising is an intrinsic reaction to the economic growth, justifying $\frac{dp}{du_1}$ is assumed to be positive. The only point that the government should be concerned about is to keep the associated inflation under control. Clearly $p(u)$ must be positive since it is associated with a weighted mean of all prices practiced in the markets.

The interest rate $r(u)$ is a percentage that defines the remuneration over the money loaned by investors to the government. The reader knows this financial operation by bonds. The more money is available, the smaller is the necessity of the government to be financed by third parties and hence it offers lower remunerations to investors, that is, $\frac{\partial r}{\partial u_4} < 0$.

Moreover, $\frac{\partial r}{\partial u_1} > 0$ follows from liquidity preference theory, as in the IS-LM model, which basically states that one dollar today is worth more than one dollar tomorrow. The logic is the following: greater income implies greater money demand which increases the price of money, that is, the interest rate r.

The government cannot print banknotes as it pleases it because it would promote a scenario of hyperinflation very hard to handle with and which would immediately cause loss of a prime function of money: store of value. In such an extreme situation, no one wants an additional one dollar bill: money demand is negative! This is expressed by $\lim_{u_4 \to \infty} L(u) < 0$. Finally, investment refers to the gain of production capacity while depreciation refers to its loss due whether to wear and tear or to obsolete technology. If the production capacity is too high, there is no new investment projects for some time until the point where there is inevitably depreciation; and $\lim_{u_2 \to \infty} I(u) < 0$ expresses it.

4.2 Local stability of Kaldor's model

Now we analyze the local stability of Kaldor's model (first without delay and later with it) by considering its linearization, as in (HALE; LUNEL, 1993; KUANG, 1993), for the nontrivial equilibrium point u^* obtained in Lemma 4.1.

4.2.1 Model without delay time

We shall evaluate the Jacobian matrix for the differential system (4.5) on u^*, omitting the argument of functions and its derivatives or even the symbol *. For instance, we simply write r_1 to denote $\dfrac{\partial r}{\partial u_1}(u^*)$. This minor abuse of notation rarely causes problems and it will be very handy for the expressions to come, which will require several renamings.

By (4.6), if $G^* = (\theta Y^* - G_0 - T_0)/(1-\theta)$ then $B^* = p^* G^*$ and the Jacobian matrix evaluated at u^* is given by

$$J = \begin{pmatrix} F_{11}(\beta) & \alpha I_2 & F_{13} & F_{14} \\ I_1 & I_2 & 0 & I_4 \\ F_{31}(\beta) & 0 & F_{33} & r^* F_{44} \\ L_1 & 0 & 0 & -F_{44} \end{pmatrix}, \tag{4.7}$$

where

$$\begin{aligned} k_1 &= \alpha\big(1-(1-\theta)c_1\big) & k_2 &= \alpha c_2 \\ b_{11} &= k_1 + k_2\left[\frac{r_1 G^*}{(r^*)^2} + \frac{\dot p^* L}{(p^*)^2}\right] + \dot p^* F_{13} G^* & b_{31} &= r^*\big(L_1 + \theta p^* + (1-\theta)\dot p^* G^*\big) \\ F_{11}(\beta) &:= -\alpha\beta + \alpha I_1 - b_{11} & F_{31}(\beta) &:= -p^* r^* \beta - b_{31} \\ F_{13} &= \frac{k_2 + (\alpha - k_1) r^*}{p^* r^*} & F_{14} &= \alpha I_4 + k_2\left[\frac{1}{p^*} - \frac{r_4 G^*}{(r^*)^2}\right] \\ 0 < F_{33} &= (1-\theta) r^* < 1 & F_{44} &= 1 - L_4. \end{aligned}$$

Furthermore, we write

$$\begin{aligned} \mu &= p^* r^* F_{13} - \alpha F_{33} & \nu &= F_{44} - I_2 \\ \sigma &= \nu - F_{33} & \Gamma &= k_2\left[\frac{1}{p^*} - \frac{r_4 G^*}{(r^*)^2}\right] \end{aligned}$$

so that $k_1, k_2, b_{11}, F_{13}, b_{31}, F_{14}, \Gamma, F_{44}, \nu > 0$. The characteristic equation is

$$\lambda^4 + a_1(\beta)\lambda^3 + a_2(\beta)\lambda^2 + a_3(\beta)\lambda + a_4(\beta) = 0,$$

where $a_j \equiv a_j(\beta) := a_{j0} + a_{j1}\beta$, for $j = 1,2,3,4$, are given by

- $a_{10} = b_{11} - \alpha I_1 + \sigma$,
- $a_{11} = \alpha$,
- $a_{20} = -I_2(\alpha I_1 + I_2) + \nu\left(-I_2 - F_{33} - \frac{F_{14} L_1}{\nu}\right) + \sigma(b_{11} - \alpha I_1) + F_{13} b_{31}$,
- $a_{21} = \alpha\nu + \mu$,
- $a_{30} = F_{13} F_{44}\big(b_{31} - r^* L_1\big) - b_{11} F_{33} I_2 F_{44}\left[\frac{1}{F_{33}}\left(1 - \frac{\Gamma L_1}{b_{11} F_{44}}\right) + \frac{1}{I_2} - \frac{1}{F_{44}}\right] + F_{33}\big(F_{14} L_1 + F_{44}(\alpha I_1 + I_2)\big) - b_{31} I_2 F_{13}$,

- $a_{31} = -\alpha I_2 F_{44} + \mu\nu$,
- $a_{40} = -I_2 F_{13} F_{44} \big(b_{31} - r^* L_1\big) + b_{11} F_{33} I_2 F_{44} \left[1 - \dfrac{\Gamma L_1}{b_{11} F_{44}}\right]$ and
- $a_{41} = -\mu I_2 F_{44}$.

We shall analyze whether or not the stability of u^* is sensitive with respect to how strong the fiscal policy is, that is, with respect to the parameter $\beta > 0$. By Routh-Hurwitz criteria, u^* is asymptotically stable if and only if $a_1, a_3, a_4 > 0$ and $a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0$.

Lemma 4.3. Suppose that

(H1) $c_2 > (1-\theta)(1-c_1) r^*$ and $\sigma > 0$;

(H2) $F_{33} + F_{14} L_1/\nu < -I_2 < \alpha I_1 < b_{11}$; and

(H3) $\dfrac{1}{F_{33}} \left(1 - \dfrac{\Gamma L_1}{b_{11} F_{44}}\right) + \dfrac{1}{I_2} - \dfrac{1}{F_{44}} > 0.$

Then

a. if $a_{40} \geqslant 0$ then $a_j > 0, j = 1,2,3,4$ for every $\beta > 0$;

b. if $a_{40} < 0$ then $a_j > 0, j = 1,2,3,4$ for every $\beta > -a_{40}/a_{41}$.

Moreover, $a_{41} > 0$ if and only if $c_2 > (1-\theta)(1-c_1) r^*$.

Proof. It is not hard to see that $\mu = \alpha\big(c_2 - (1-\theta)(1-c_1) r^*\big)$, which is positive by **(H1)** and then $a_{21} > 0$. Also, by **(H2)**

$$a_{10} = \underbrace{b_{11} - \alpha I_1}_{>0} \underbrace{- I_2 - F_{33}}_{>0} + \underbrace{F_{44}}_{>0} > 0$$

and

$$a_{20} = -I_2 \underbrace{(\alpha I_1 + I_2)}_{>0} + \nu \underbrace{\left(-I_2 - F_{33} - \frac{F_{14} L_1}{\nu}\right)}_{>0} + \sigma \underbrace{(b_{11} - \alpha I_1)}_{>0} + F_{13} b_{31} > 0,$$

whence $a_1(\beta), a_2(\beta) > 0$ for every $\beta > 0$. Clearly a_{30} and a_{31} are sums of positive terms since $I_2 < 0$; whence $a_3(\beta) > 0$ for every $\beta > 0$. Although $a_{41} > 0$ by **(H1)**, a_{40} is a sum of a positive term and a negative one, whence instead of controlling its sign we consider the sign of $a_4(\beta)$ for both cases as stated.

□

Remark 4.4. Since ν is a sum of two (possibly large) positive numbers and F_{33} is a product of two numbers which lie in $(0,1)$, it is not restrictive to assume that $\sigma = \nu - F_{33} > 0$ in **(H1)**.

As the reader may promptly realize, even for $a_j(\beta)$, which depends linearly on β, the main challenge is renaming, rearranging and noticing conveniently expressions and hypotheses in order to guarantee the positive sign of large sums and then to fulfill the Routh-Hurwitz conditions. The main theorem below deals with the sign of $p_{RH}(\beta) = a_1a_2a_3 - a_3^2 - a_1^2a_4$, which is a cubic function of β and which has over 500 terms if it is fully expanded. Although computing systems, such as Wolfram Mathematica, are very handy for symbolic expressions, they are not able to assimilate the concept of 'convenient rearrangements' and hence we must deal with some hard parts by ourselves.

We may write $p_{RH}(\beta) = Q_0 + Q_1\beta + Q_2\beta^2 + Q_3\beta^3$, where

$$Q_0 = \tfrac{a_{30}}{2}(a_{10}a_{20} - 2a_{30}) + \tfrac{a_{10}}{2}(a_{20}a_{30} - 2a_{10}a_{40}),$$

$$Q_1 = a_{31}\underbrace{(a_{10}a_{20} - 2a_{30})}_{\textbf{(E-1.1)}} + \alpha\underbrace{(a_{20}a_{30} - 2a_{10}a_{40})}_{\textbf{(E-1.2)}} + a_{10}\underbrace{(a_{21}a_{30} - a_{10}a_{41})}_{\textbf{(E-1.3)}},$$

$$Q_2 = a_{10}\underbrace{(a_{21}a_{31} - 2\alpha a_{41})}_{\textbf{(E-2.1)}} + a_{31}\underbrace{(\alpha a_{20} - a_{31})}_{\textbf{(E-2.2)}} + \alpha\underbrace{(a_{21}a_{30} - \alpha a_{40})}_{\textbf{(E-2.3)}} \text{ and}$$

$$Q_3 = \alpha(a_{21}a_{31} - \alpha a_{41}).$$

Theorem 4.5. Suppose that **(E-1.1)**, **(E-1.2)**, **(E-1.3)** and **(E-2.2)** are positive. Under the assumptions of Lemma 4.3, we have

a. if $a_{40} \geqslant 0$ then u^* is asymptotically stable for every $\beta > 0$.

b. if $a_{40} < 0$ then u^* is asymptotically stable for every $\beta > -a_{40}/a_{41}$.

Proof. Note that Q_0 is a linear combination of **(E-1.1)** and **(E-1.2)** with positive weights (under the assumptions of Lemma 4.3) and that if **(E-2.1)** is positive then Q_3 is positive as well. Hence it is sufficient to control the sign of the expressions on Q_1 and Q_2 in order to obtain $p_{RH}(\beta) > 0$ possibly adding a restriction on β.

On the one hand, if **(H1)** holds then

$$a_{21}a_{31} - 2\alpha a_{41} = \mu^2\nu - \alpha^2\nu I_2F_{44} + \alpha\mu(I_2^2 + F_{44}^2 - I_2F_{44})$$

is a sum of positive terms and **(E-2.1)** is positive, which immediately implies that p_{RH} is positive for $\beta > 0$ large enough. Also, if **(H1)** holds then **(E-2.3)** is positive independently of the sign of a_{40}:

$$\begin{aligned} a_{21}a_{30} - \alpha a_{40} &> (\mu + \alpha\nu)F_{13}F_{44}\big(b_{31} - r^*L_1\big) + \alpha I_2F_{13}F_{44}\big(b_{31} - r^*L_1\big) \\ &= F_{13}F_{44}\big(b_{31} - r^*L_1\big)(\mu + \alpha F_{44} - \alpha I_2 + \alpha I_2) > 0. \end{aligned}$$

On the other hand, let us deal with **(E-2.2)**. From all possible assumptions, the cleanest is $\alpha a_{20} - a_{31} > 0$, but we could pursue others conditions. For instance, it is easy to see that $\alpha a_{20} - a_{31}$ is greater than

$$\alpha\sigma\left(b_{11} - \alpha I_1 - \frac{I_2}{\sigma}(\alpha I_1 + I_2 - F_{44})\right) + \alpha\nu\left(-I_2 - F_{33} - \frac{F_{14}L_1}{\nu} - \frac{\mu}{\alpha}\right),$$

which is positive if each term is; note that these two conditions are slightly stronger than **(H2)**, since $I_2/\sigma, \mu/\alpha \in (0,1)$. Or yet, $\alpha a_{20} - a_{31}$ is greater than

$$\alpha\nu\left(-I_2 - F_{33} - \frac{F_{14}L_1}{\nu} + I_2\frac{F_{44}}{\nu} - \frac{\mu}{\alpha}\right)$$

and asking this expression to be positive is again a slightly stronger condition than **(H2)**, since $F_{44}/\nu, \mu/\alpha \in (0,1)$. In both cases, the new conditions are considerably larger though. Similarly one can obtain conditions for **(E-1.1)**, **(E-1.2)**, **(E-1.3)** and **(E-2.2)** to be positive, where **(E-1.1)** is the one which demands more effort since it has a longer expression to be dealt with. However we abide by the cleanest assumptions sparing the reader the gruesome estimates and their details; and the proof is complete. □

Remark 4.6. Actually we proved that if **(H1)** holds then $Q_3 > 0$ which implies that a strong fiscal policy (that is, a scenario where $\beta > 0$ is large enough) always promotes a long-term stable economy, as long as the government does not delay its implementation (since we are dealing with the model without delay so far).

4.2.2 Model with delay time in fiscal policy

Invariably economic dynamics involves human behavior, which is a decisive factor to be taken into account. It basically refers to the capacity of making decisions after recognizing opportunities and evaluating available resources. Such an aspect can be added appropriately to an economic model by formulating it with delay; that is, instead of considering differential equations where the variables react instantly to external forces independently of the past, a delay formulation does take into account the fact that the past is important when comes to making decisions. A formulation with a nonconstant delay function $t \mapsto \tau(t)$ or considering the government expenditure as function of a weighted average of the national income, let us say $\beta\int_{-\tau(t)}^{0}\big(Y^* - Y(s)\big)f(s)\,ds$, provides a more realistic modeling. We shall analyze the constant delay case.

In (4.5), delay time τ models the government capacity of recognizing, formulating and implementing fiscal policies. To obtain such a fixed value τ, one may evaluate the mean policy lag of a nation considering a given period of time. The associated linearized model evaluated at u^* is given by $u'(t) = J_0 u(t) + J_\tau u(t-\tau)$, where

$$J_0 = \begin{pmatrix} \alpha I_1 - b_{11} & I_2 & F_{13} & F_{14} \\ I_1 & I_2 & 0 & I_4 \\ -b_{31} & 0 & F_{33} & r^*F_{44} \\ L_1 & 0 & 0 & -F_{44} \end{pmatrix} \text{ and } J_\tau = \begin{pmatrix} -\alpha\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p^*r^*\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $J = J_0 + J_\tau$ and its characteristic equation can be written as

$$Q_0(\lambda) + e^{-\lambda\tau} Q_\tau(\lambda) = 0, \tag{4.8}$$

where

$$Q_0(\lambda) := \lambda^4 + a_{10}\lambda^3 + a_{20}\lambda^2 + a_{30}\lambda + a_{40} \text{ and}$$
$$Q_\tau(\lambda;\beta) \equiv Q_\tau(\lambda) := \left(a_{11}\lambda^3 + a_{21}\lambda^2 + a_{31}\lambda + a_{41}\right)\beta.$$

By Theorem 4.5, the equilibrium point u^* is locally stable for every $\beta > 0$, whenever $\tau = 0$. For $\tau > 0$, we know that u^* is locally asymptotically stable if and only if every root of (4.8) has negative real part, see (HALE; LUNEL, 1993; KUANG, 1993). Also instability is equivalent to the existence of at least one root with positive real part.

Remark 4.7. Under the assumptions of Lemma 4.3, if the additional assumption

(H4) $a_{40} > 0$

holds then the real part of every root of Q_0 is negative, by Routh-Hurwitz criteria.

From now on, we assume that **(H1)-(H4)** hold. We shall study how the local stability of u^* responds to fiscal policy strength, $\beta > 0$, and time lag, τ. First we apply a stability switch result to the delayed model (4.4), namely Theorem 2.45. For convenience though, we state it again below. For a complex number z, we write its real and imaginary parts as $\Re z$ and $\Im z$, respectively.

Theorem 4.8. Consider the equations (4.8) on λ and

$$F(y) := |Q_0(iy)|^2 - |Q_\tau(iy)|^2 = 0, \text{ for } y \in \mathbb{R}. \tag{4.9}$$

Suppose that $\lambda \mapsto Q_0(\lambda), Q_\tau(\lambda)$ are analytic functions for $\Re\lambda > 0$ and that

(i) there is no common pure imaginary roots of Q_0 and Q_τ;

(ii) $\overline{Q_0(-iy)} = Q_0(iy)$ and $\overline{Q_\tau(-iy)} = Q_\tau(iy)$, for every $y \in \mathbb{R}$;

(iii) $\lambda = 0$ is not a root for (4.8);

(iv) $\limsup\limits_{\substack{|\lambda|\to\infty \\ \Re\lambda \geqslant 0}} \left|\dfrac{Q_\tau(\lambda)}{Q_0(\lambda)}\right| < 1$; and

(v) the equation (4.9) admits only finitely many real roots.

Then

a. if $F(y) = 0$ has no positive roots then no stability switch occurs.

b. if $F(y) = 0$ has at least one positive root and each of them is simple then, as τ increases, a finite number of stability switches occurs and eventually u^* becomes unstable.

The assumption **(i)** holds by Remark 4.7 and **(iii)** holds because $a_4(\beta) > 0$ for every positive β. Since Q_0 and Q_λ are polynomials with real coefficients and $\deg Q_0 > \deg Q_\tau$, assumptions **(ii)**, **(iv)** and **(v)** hold. Although F clearly depends on β, we shall omit such a dependence from time to time. Thus we shall analyze the stability of u^* by studying the positive roots of $F(y) = 0$. Setting $z = y^2$, the function F can be written as

$$F(z;\beta) \equiv F(z) = z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4,$$

where $b_j \equiv b_j(\beta)$, $j = 1,2,3,4$, are given by

$$\begin{aligned}
b_1(\beta) &= a_{10}^2 - 2a_{20} - a_{11}^2\beta^2,\\
b_2(\beta) &= a_{20}^2 - 2a_{10}a_{30} + 2a_{40} + (2a_{11}a_{31} - a_{21}^2)\beta^2,\\
b_3(\beta) &= a_{30}^2 - 2a_{20}a_{40} + (2a_{21}a_{41} - a_{31}^2)\beta^2 \text{ and}\\
b_4(\beta) &= a_{40}^2 - a_{41}^2\beta^2.
\end{aligned}$$

Clearly, $F(y) = 0$ has no positive roots whenever $F(z) = 0$ has no positive roots. Actually, they have the same number of positive simple roots.

It is noteworthy that $b_4(\beta) < 0$ if and only if $\beta > a_{40}/a_{41}$, and in this case the number of positive roots of F can be 1, 2 or 3 only. As we shall see with simulations in Section 4.3, under a weak fiscal policy scenario - more precisely, for $0 < \beta < a_{40}/a_{41}$ -, the government efficiency on implementing it does not harm the economic stability because $z \mapsto F(z)$ has no positive zeros. On the other hand, a more careful analysis is required if $\beta > a_{40}/a_{41}$.

Now we discuss the relationship between the parameters β and τ. First note that a pure complex number $\lambda = iy$, where $y > 0$, is a root of (4.8) if and only if y is a positive root of (4.9).

We shall show only the sufficiency in order to fix some notations. Note that $|Q_0(iy)/Q_\tau(iy)| = 1$ so that there exists a unique value $\phi(y) \in [0, 2\pi)$ such that $-e^{-i\phi(y)} = Q_0(iy)/Q_\tau(iy)$ and hence $\lambda = iy$ is a root of (4.8) whenever τ is of the form $(\phi(y) + 2n\pi)/y$, with $n \in \mathbb{Z}_+$.

The reader should promptly see that, by hypothesis **(i)** of Theorem 2.45, $Q_\tau(iy)$ cannot be zero, whenever iy is a root of (4.8) or y is a root of (4.9). If we write $Q_l(iy) = Q_{l,\Re}(y) + iQ_{l,\Im}(y)$, $l = 0, \tau$, then after some computations we see that $\phi(y) \in [0, 2\pi)$ is the angle that satisfies the equations

$$\begin{cases} \cos\big(\phi(y)\big) = -\dfrac{Q_{0,\Re}(y)Q_{\tau,\Re}(y)+Q_{0,\Im}(y)Q_{\tau,\Im}(y)}{|Q_\tau(y)|^2} =: -\dfrac{A(y)}{|Q_\tau(y)|^2} \\ \sin\big(\phi(y)\big) = \dfrac{-Q_{0,\Re}(y)Q_{\tau,\Im}(y)+Q_{0,\Im}(y)Q_{\tau,\Re}(y)}{|Q_\tau(y)|^2} =: \dfrac{B(y)}{|Q_\tau(y)|^2} \end{cases},$$

so that $(0,\infty) \ni y \mapsto \phi(y) \in (0,2\pi)$ is defined by

$$\phi(y) = \begin{cases} \arctan\left(\frac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi),\sin(\phi)>0 \\ \pi/2, & \text{if } \cos(\phi)=0 \text{ and } \sin(\phi)=1 \\ \pi+\arctan\left(\frac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi)<0 \\ 3\pi/2, & \text{if } \cos(\phi)=0 \text{ and } \sin(\phi)=-1 \\ 2\pi+\arctan\left(\frac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi)>0 \text{ and } \sin(\phi)<0 \end{cases}.$$

We regard the root of (4.8) as a function of τ by writing $\tau \mapsto \lambda(\tau) = x(\tau)+iy(\tau)$ and then we study the sign of the derivative of $\Re\lambda(\tau)$ at the points where $\lambda(\tau)$ is purely imaginary, which are precisely where a stability switch may occur, since $\lambda=0$ is not a root of (4.8). Arguing as in the proof of Theorem 2.45, we see that $\lambda(\tau)$ is differentiable at $\tau=\tau^*$ whenever $\lambda(\tau^*)$ is a simple root. And if, in addition, $\lambda(\tau^*)=iy(\tau^*)$ then we explicitly obtain

$$\left(\frac{d\lambda(\tau)}{d\tau}\Big|_{\tau=\tau^*}\right)^{-1} = \left[-\frac{Q_0'(\lambda)}{\lambda Q_0(\lambda)}+\frac{Q_\tau'(\lambda)}{\lambda Q_\tau(\lambda)}-\frac{\tau}{\lambda}\right]\Big|_{\tau=\tau^*},$$

so we can determine the direction of motion of $x(\tau)$ as τ passes through τ^* according to

$$S := \operatorname{sign}\frac{d\Re\lambda(\tau)}{d\tau}\Big|_{\tau=\tau^*} = \operatorname{sign}\frac{dF(y)}{dy}\Big|_{y=y(\tau^*)}. \tag{4.10}$$

Lemma 4.9 ((KUANG, 1993), Theorem 3.4.1)**.** If y^* is a simple positive root of (4.9) then there exists a pair of simple conjugate pure imaginary roots $\lambda(\tau^*)=\pm iy(\tau^*)$ of (4.8) at $\tau^*=\phi(y^*)/y^*$ which crosses the imaginary axis according to (4.10). More precisely,

a. if $S>0$ then $\lambda(\tau)$ crosses the imaginary axis at $\tau=\tau^*$ from left to right, that is, u^* becomes unstable; and

b. if $S<0$ then $\lambda(\tau)$ crosses the imaginary axis at $\tau=\tau^*$ from right to left, that is, u^* becomes stable.

If $y_1^* > \cdots > y_m^* > 0$ are the simple positive roots of (4.9) then, by making explicit the dependence on β, we write

$$S_{i,n}(\tau) := \tau - \frac{\phi\big(y_i^*(\beta)\big)+2n\pi}{y_i^*(\beta)},$$

for $i=1,\dots,m$ and $n\in\mathbb{Z}_+$. It is an auxiliary function whose zero is the value τ at which $\lambda(\tau)$ crosses the imaginary axis. These are the tools needed to study numerically the stability switch.

4.3 Numerical simulations

First, we emphasize that u^* does not depend on β but its stability may. By running simulations in Wolfram Mathematica 11.3, we shall analyze several aspects: how the number of positive simple roots of $F(z;\beta)=0$ changes as β varies; how the convergence of the solution responds to greater values of τ, with eventual instability; the sensitiveness of hypotheses with respect to economic parameters; the stability region in the $\beta\tau$-plane; and so on. For $u=(u_1,u_2,u_3,u_4)\in\mathbb{R}^4_+$, set

$$\begin{array}{lll} \alpha = 0.40 & C_0 = 10 & p(u) = 0.4u_1 + 10 \\ c_1 = 0.40 & T_0 = 20 & r(u) = \dfrac{1+u_1}{1+u_1+5u_4} \\ c_2 = 0.15 & \theta = 0.35 & L(u) = 5u_1 - u_4 + 50 \end{array}$$

As for the investment function, we consider two different formulations, namely,

$$I(u) = -0.25u_2 + 5r(u)u_4 + 100 \tag{4.11}$$

and

$$\mathfrak{I}(u) = \eta\tilde{I}(u) + (1-\eta)I(u), \tag{4.12}$$

where $\eta\in(0,1)$ is fixed and

$$\tilde{I}(u_1,u_2) := 25\exp\left(\frac{-\log 2}{(\frac{15}{1000}u_1 + 10^{-5})^2}\right) + \frac{u_1}{100} + 5\frac{320^3}{(u_2+1)^3}.$$

The first one satisfies the original assumptions of Kaldor's paper (KALDOR, 1940) concerning the nonlinearity of I with respect to $u_1 = Y$ but not with respect to $u_2 = K$. For numerical simulations and to verify the sufficient assumptions, the linear dependence on u_2 is convenient though. The second one is an adaption of the investment function which appears in (MATSUMOTO; SZIDAROVSZKY; ASADA, 2016) and it completely satisfies Kaldor's assumptions over the shape of I curve with respect to $u_1 = Y$ and $u_2 = K$.

We shall consider two subsections for the simulations accordingly to the choice of investment function.

Remark 4.10. The delay equations demand a function $\psi\colon[-\tau,0]\to\mathbb{R}^4$ as initial data. In the simulations, we considered exponential functions with a slow increasing rate, for instance $t\mapsto\exp(0.02t)25$ for $u_1(t)$. Not that an economy increases indefinitely exponentially on time, but in short-run it is reasonable that a nation has an economic growth of 2% and that is precisely the point.

4.3.1 Investment function given by (4.11)

The economic assumptions **(A1)-(A4)** and the technical hypotheses **(H1)-(H3)** are satisfied. If the government pursues the national income $Y^* = 100$ then it must fix

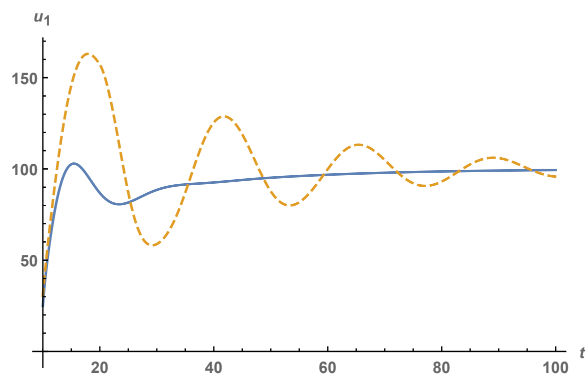
$G_0 = G(u^*) = 0.51$ to obtain $u^* = (100.00, 776.36, 1114.68, 275.00)$ as the unique positive equilibrium point of (4.4), which is asymptotically stable for every $\beta > 0$, by Lemma 4.3 and Theorem 4.5. In other words, in such a scenario, assuming that the government instantly applies its fiscal policies then no matter how strong they are, the economy is always stable.

Remark 4.11. We have not been able to establish general necessary conditions to existence of a positive equilibrium point. However, we know that a_{41} is positive if and only if $c_2 > (1-\theta)(1-c_1)r^*$, see Lemma 4.3; whence it is a necessary condition to fulfill Routh-Hurwitz criteria. This very same condition was already required by (TAKEUCHI; YAMAMURA, 2004) in Theorem 3.2 to obtain the stability of the equilibrium point.

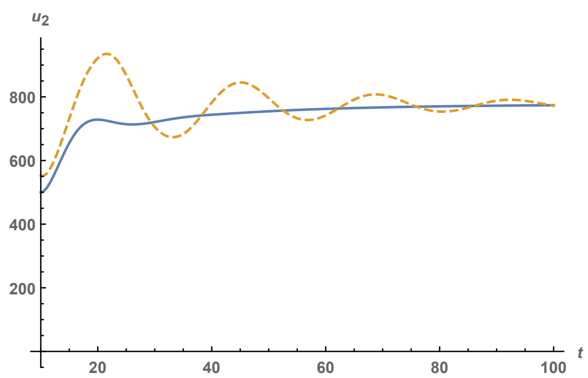
First, we set $\beta = 0.40$, then the eigenvalues of the associated Jacobian matrix at u^* are

$$-2.2112, -0.0415 \text{ and } -0.2097 \pm 0.3409i$$

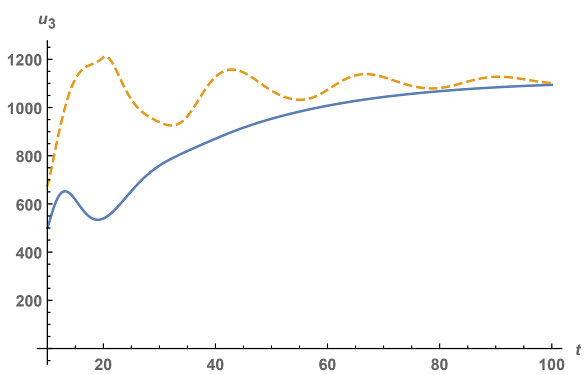
and we obtain the graphs in Fig. 2 comparing the numerical solutions for the model with or without delay of $\tau = 10$, represented by the dashed and continuous lines, respectively.



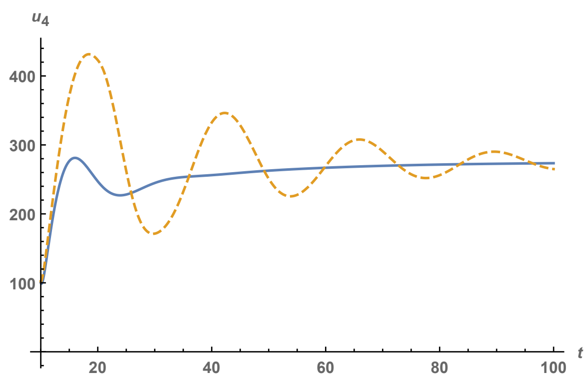
(a) The national income $u_1 = Y$.



(b) The capital stock $u_2 = K$.



(c) The governments bonds $u_3 = B$.



(d) The money supply $u_4 = M$.

Figure 2 – The evolution in time with and without delay.

We have $T(u^*) = 22.80$, which says that in equilibrium the government revenue represents about 20% of the national richness, a compatible idea with the capitalist philosophy about a moderate size for the state accounts. Also, it is reasonable that an interest rate of $r^* = 6.84\%$ in equilibrium promotes high values in the bonds market, that is, $B^* = 1114.68$. In Fig. 3, we compare the numerical solutions of u_1 associated to $\tau = 0$ (the continuous line), $\tau = 10$ (the dotted line) and $\tau = 25$ (the dashed line).

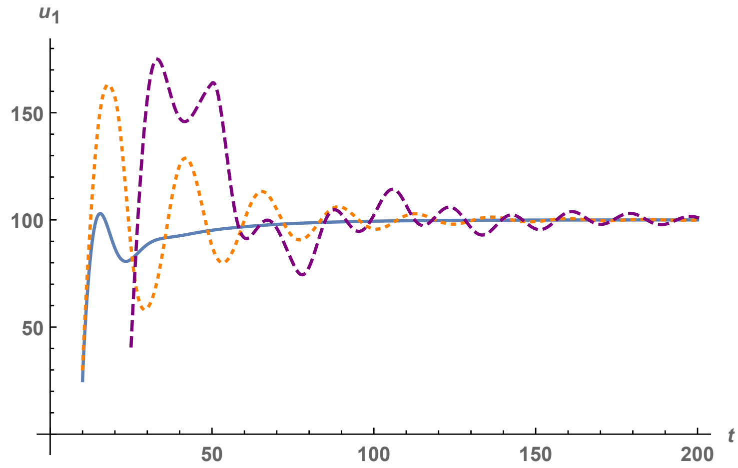
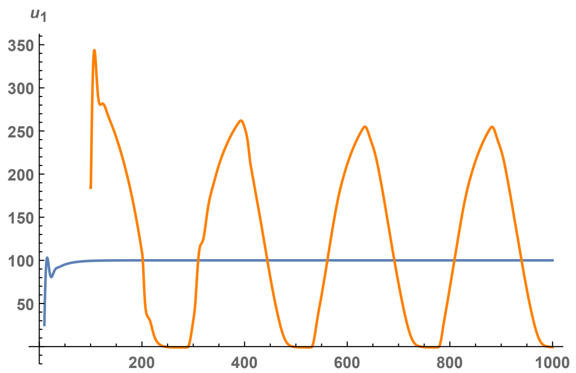
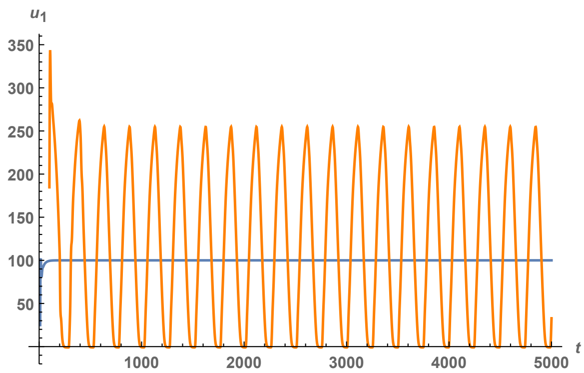


Figure 3 – The evolution of $u_1(t)$ under different values of τ.

As τ increases the solution associated to its delay equation becomes more erratic but $u_1^* = 100$ still is asymptotically stable for $\tau = 25$; actually even for $\tau = 50$. The evolution of u_1 for $\tau = 100$ is showed in Fig. 4.



(a) Until $t = 1000$.



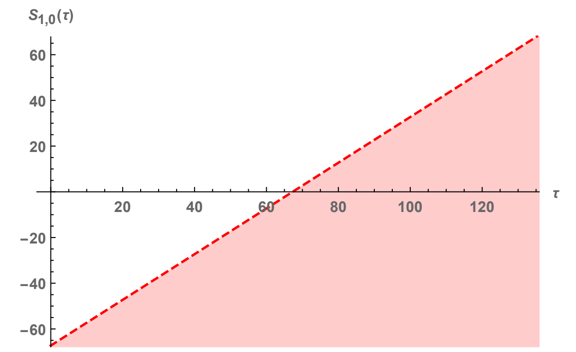
(b) Until $t = 5000$.

Figure 4 – The evolution of u_1 with $\beta = 0.40$ and $\tau = 100$.

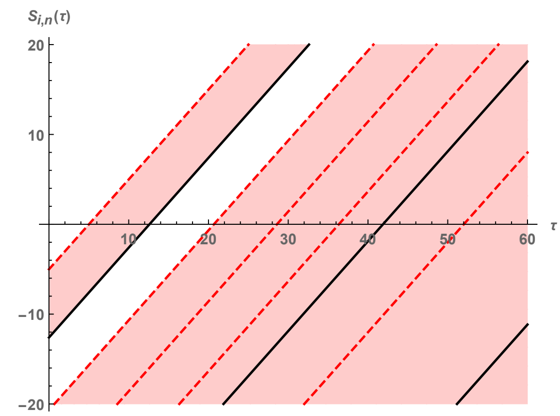
However, since $a_{40} = 0.0048$, Theorem 2.45 holds and then we shall study how τ affects the stability of (4.4). If $N(\beta)$ denotes the number of positive simple roots of $F(y;\beta) = 0$ then

$$N(\beta) = \begin{cases} 0, & \text{if } 0 < \beta < 0.1964 \\ 1, & \text{if } 0.1964 < \beta < 0.6071 \text{ or } \beta > 0.7790 \\ 3, & \text{if } 0.6071 < \beta < 0.7790 \end{cases}.$$

The unique positive simple root of $F(y;0.40)=0$ is $y^*=0.035472$ and the derivative $\frac{d}{dy}F(y;0.40)$ is always positive for $y>0$, whence the crossing the imaginary axis is always to the right half-plane, that is, stability switch occurs only toward instability. Actually, Fig. 5a points out that the stability switch already occurs toward instability at $\tau=67.28$.



(a) For $\beta=0.40$, u^* becomes unstable at $\tau=67.28$.



(b) For $\beta=0.70$, u^* becomes unstable as τ enters in some shaded area and it becomes stable as τ leaves some shaded area.

Figure 5 – The auxiliary function $S_{i,n}$ and the stability switch of u^*.

On the other hand, the switch stability analysis for $\beta=0.70$ is far more involving since now there are three positive simple roots of $F(y;0.70)=0$; namely, $y_1^*=0.4010, y_2^*=0.2146$ and $y_3^*=0.0878$.

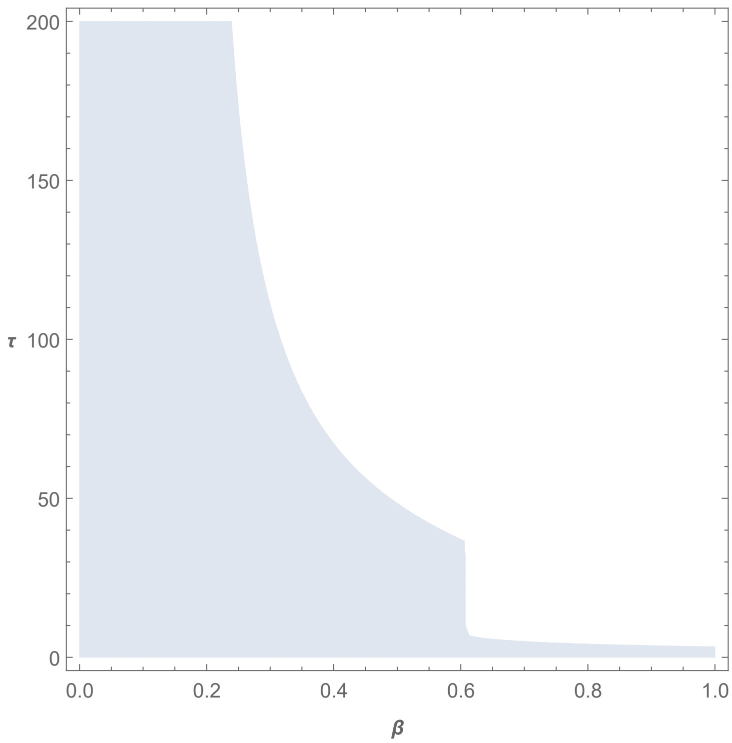


Figure 6 – Stability of u^* with respect to β and τ.

Besides, $\frac{d}{dy}F(y;0.70)$ is negative if $0.1599 < y < 0.3432$ and it is positive otherwise; whence it follows that $\frac{d}{dy}F(y_1^*;0.70)$ and $\frac{d}{dy}F(y_3^*;0.70)$ are positive but $\frac{d}{dy}F(y_2^*;0.70) < 0$. Thus the crossing at iy_1^* and iy_3^* must be to the right half-plane; and the crossing at iy_2^* must be to the left half-plane.

Recall that (4.8) admits infinitely many complex roots $\lambda = \lambda(\tau)$, which depend on τ. Let $\tau_{i,n}$ be the zero of $S_{i,n}$. In Fig. 5b, the interceptions of the lines with the τ-axis are at $\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{3,0} < \tau_{1,2} < \tau_{2,1} < \tau_{1,3} < \dots$ and we see how the stability of u^* changes as τ increases. At $\tau = \tau_{1,0} = 5.01$, one of the roots of (4.8) crosses to the right half-plane and then the stability switch occurs toward instability; at the second value $\tau = \tau_{2,0} = 12.60$, such a root crosses back to the left half-plane and the switch occurs toward stability. As one may note, as τ increases, passing by $\tau_{1,1}, \tau_{3,0}$ and $\tau_{1,2}$, three roots of (4.8) cross to the right half-plane but only one of them crosses back to the left half-plane at $\tau_{2,1} = 41.88$ and the instability persists thereafter because the number of roots crossing to the right half-plane exceeds the number of those crossing back. Finally, by setting $\beta = 0.15$, we have that $b_4(0.15) > 0$ and all roots of $F(y) = 0$ are complex, consequently u^* is (locally) asymptotically stable for every time delay $\tau > 0$.

All such conclusions are summarized in Fig. 6, which shows the relationship between the stability of the equilibrium point u^* and the parameters β and τ. The boundary of the region consists of the pairs (β, τ) such that $\tau = \tau(\beta)$ is the smallest delay time at which a stability switch occurs toward instability. In the shaded region, u^* is asymptotically stable and out of it, there are three possibilities. No stability switch occurs if $\beta < a_{40}/a_{41} = 0.1964$, because $N(\beta) = 0$. If $0.6071 < \beta < 0.7790$ then finitely many stability switches occur as τ increases, as we discussed above, but u^* eventually becomes (locally) unstable. If β is greater than 0.1964 but is not in $(0.6071, 0.7790)$, we have instability for every τ such that (β, τ) lays outside the shaded region.

4.3.2 Investment function given by (4.12)

Let $\eta = 0.4, \alpha = 0.2$ and consider the investment function in (4.12). In this setting, the economic assumptions **(A1)-(A4)** and the technical hypotheses **(H1)-(H4)** are satisfied. Although **(E-1.1)**,**(E-1.2)** and **(E-1.3)** are positive, unfortunately **(E-2.2)** is negative (even after several attempts with different values of parameters) and hence we cannot apply Theorem 4.5. However we can verify numerically that the maximum value of the real part of the eigenvalues of the Jacobian matrix at the equilibrium point is negative for every $0 < \beta < 1$. Thus Kaldor's model (4.4), with $\tau = 0$, is locally asymptotically stable even without fulfilling the conditions of our results, making explicit the fact that they are not necessary. Furthermore, we can verify that Theorem 2.45 holds because, for every $0 < \beta < 1$, there is no common pure imaginary roots of Q_0 and Q_τ and $\lambda = 0$ is not a root for (4.8).

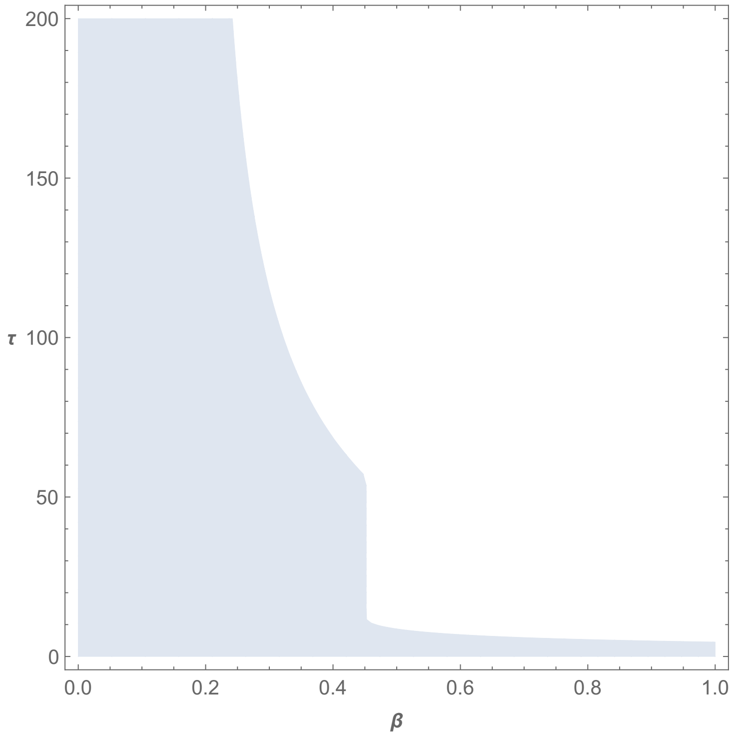
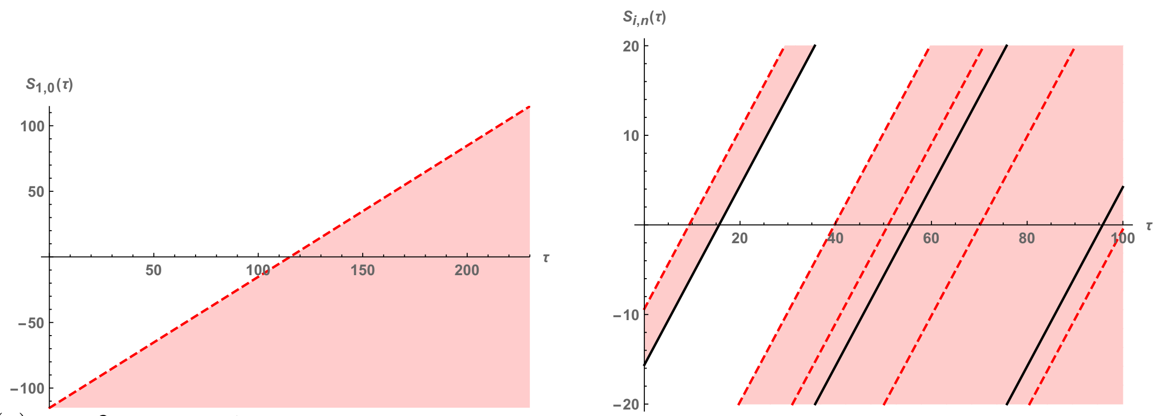


Figure 7 – Stability of u^* with respect to β and τ.

Proceeding as before, we have that

$$N(\beta) = \begin{cases} 0, & \text{if } 0 < \beta < 0.1964 \\ 1, & \text{if } 0.1964 < \beta < 0.4523 \text{ or } \beta > 0.5980 \\ 3, & \text{if } 0.4523 < \beta < 0.5980 \end{cases}$$

and the region of stability of u^* in the $\beta\tau$-plane is given by Fig. 7.



(a) For $\beta = 0.30$, u^* becomes unstable at $\tau = 115.19$.

(b) For $\beta = 0.48$: u^* becomes unstable as τ passes by some dashed (red) line and it becomes stable as τ passes by some continuous (black) line.

Figure 8 – The stability switch of u^*.

Under a moderate fiscal policy, $\beta = 0.30$, the government inefficiency does not harm the economic stability until $\tau_{1,0} = 115.19$, see Fig.8a. On the other hand, if the fiscal policy is slightly stronger, let us say $\beta = 0.48$, several switch stabilities occur as τ increases. More precisely, at $\tau_{1,0} = 9.36$, the stability switch occurs toward instability; at the second value $\tau_{2,0} = 15.63$, the switch occurs toward stability and so on, depending on whether $\tau = \tau_{2,n}$ or $\tau = \tau_{1,n}, \tau_{3,n}$. See Fig. 8b, where $\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{3,0} < \tau_{2,1} < \tau_{1,2} < \tau_{2,2} < \tau_{1,3} < \ldots$.

4.4 Conclusions

We consider an extended version of the classical Kaldor's economic growth model adding the government role to the economic dynamics: monetary and fiscal policies and the government budget constraint are taken into account, leading to a differential system in $\mathbb{R}^4$, with or without a delay time on the fiscal policy. An analysis of the model stated in (4.5) is itself an improvement over (TAKEUCHI; YAMAMURA, 2004) who turned it into two simpler versions in $\mathbb{R}^3$ by imposing either $B' = 0$ or $M' = 0$, that is, extreme scenarios where either the government is incapable to manage its bonds supply or it is incapable to establish its money supply.

Firstly we have proved the existence and uniqueness of a positive equilibrium point under reasonable economic assumptions (which represent an improvement over those technical ones required by (TAKEUCHI; YAMAMURA, 2004)). Secondly we have established sufficient conditions under which (4.5), with $\tau = 0$, is locally asymptotically stable with a possible restriction over the fiscal policy strength. Under a simple additional assumption, namely **(H4)**, we have applied a classical stability switch result to study how the fiscal policy delay time may lead to an unstable economic scenario.

β	**Strength of the fiscal policy**	**First value τ under which u^* is unstable**	**Conclusion**
0.15	weak	∞	the economy is always stable
0.40	moderate	67.28	an inefficient government can lead the economy to instability
0.70	strong	5.01	the economic stability is very sensitive to the government efficiency

Table 1 – The effects of the fiscal policy on the economy.

In Section 4.3 we have run simulations with two different investment functions, splitting it into two subsections. On the one hand, all assumptions needed for the results we have presented are satisfied by the investment function (4.11) and by the other functions and parameters. Table 1 summarizes the results of Subsection 4.3.1.

Here government efficiency refers to the time efficiency on recognizing opportunities to implement a fiscal policy, formulating it and then implementing it. Curiously, if the fiscal policy is very strong, let us say $\beta = 0.9$, the conclusion are quite the same as those for $\beta = 0.4$.

On the other hand, in Subsection 4.3.2, the investment function is a convex combination of the previous one with the investment function suggested by (MATSUMOTO; SZIDAROVSZKY; ASADA, 2016). Although we cannot apply Theorem 4.5 for $\alpha = 0.2$, we were able to verify numerically that the equilibrium point is always locally asymptotically stable for $0 < \beta < 1$ and $\tau = 0$; and that the switch stability theorem holds as well.

The less simplifications are imposed and the more relevant aspects are considered, the more realistic a model is. For instance, one should expect that the government capacity of recognizing, formulating and implementing fiscal policies varies with time, that is, it is more reasonable to assume a delay function $t \mapsto \tau(t)$ instead of a fixed delay time. Also the economy intrinsically carries a volatility which comes from the human behavior factor and which can be appropriately added to the model by considering certain economic parameters random. For instance, $0 < c_1, c_2 < 1$ dictate how big is the portion of the income that will be spent, which are associated with the (microeconomic) perception whether or not the economy prospers and it will continue to do so. And as we have discussed in Section 4.1, one could aggregate a delayed investment formulation of Kaldor-Kalecki's model suitably adapted; as in (MATSUMOTO; SZIDAROVSZKY; ASADA, 2016). Besides, a question of structural stability arises. Comparing (4.1) and (4.4), one may wonder if the limit cycle structure of (4.1) is present in the extended model. More precisely, is it possible to obtain the original $\mathbb{R}^2$ dynamics from (4.4) by deforming it appropriately?

Our future aims concerns these subjects and other related ones.

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