UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Complete embedded hypersurfaces in product spaces

Leonel Renzo Ccama Cuyo

Dissertação de Mestrado do Programa de Pós-Graduação em Matemática (PPG-Mat)

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: ______________________

Leonel Renzo Ccama Cuyo

Complete embedded hypersurfaces in product spaces

Dissertation submitted to the Instituto de Ciências Matemáticas e de Computação – ICMC-USP – in accordance with the requirements of the Mathematics Graduate Program, for the degree of Master in Science. *FINAL VERSION*

Concentration Area: Mathematics

Advisor: Prof. Dr. Fernando Manfio

USP – São Carlos
March 2023

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

C111c

Ccama Cuyo, Leonel Renzo
Complete embedded hypersurfaces in product spaces / Leonel Renzo Ccama Cuyo; orientador Fernando Manfio. -- São Carlos, 2023.
62 p.

Dissertação (Mestrado - Programa de Pós-Graduação em Matemática) -- Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2023.

1. Class A. 2. Hadamard-Stoker-type theorem. 3. Complete hypersurfaces in product spaces. I. Manfio, Fernando, orient. II. Título.

Bibliotecários responsáveis pela estrutura de catalogação da publicação de acordo com a AACR2:
Gláucia Maria Saia Cristianini - CRB - 8/4938
Juliana de Souza Moraes - CRB - 8/6176

Leonel Renzo Ccama Cuyo

Hipersuperfícies completas e mergulhadas em espaços produto

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *VERSÃO REVISADA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Fernando Manfio

USP – São Carlos
Março de 2023

This work is dedicated to my relatives, friends, my advisor and professors, colleagues and to everybody who still care about me for their immense support.

ACKNOWLEDGEMENTS

I owe an unpayable debt to Professor Ruy Tojeiro and Professor Ronaldo Freire de Lima for helping me in understanding the most difficult notions, ideas and arguments contained in their scientific papers—I have done little more than rearrange their ideas in a way that seems acceptable for a dissertation. Most of all, I owe a huge debt of gratitude to my advisor, Professor Fernando Manfio, for introducing me to Submanifold Geometry and for supporting me trough all these years by indicating what to study, answering my enquiries, as well as suggesting the way on which to present the content of this work. Beyond that, I would like to thank Arnando Sousa, who was always able to attend to me, correcting my mathematical errors, and providing useful ideas to crash my doubts; Milagros Anculli for allowing me to frequent her place of study, bringing helpful ideas for my research, as well as supporting me emotionally when I feel stuck; Victor Mendoza who was thoughtful with his support by introducing me to his colleagues at the Singularities laboratory, ICMC, and by letting me to study there; Alan Sousa for bringing good ideas and references to help on my research, as well as for correcting my mistakes in the Portuguese version of my abstract for this work; Inácio Rabelo, who corrected my grammar mistakes in Portuguese when I need to express my ideas clearly. Thanks are also due to Paulo Farias, Estela Garcia, Raquel Silva, and Sergio Pinillos, all of them provided great help with doubts on Differential Geometry; Paulo Christo and Igor do Nascimento, who helped me with technical doubts on Algebra. Those of my colleagues and friends here at ICMC—Beatriz Rabelo, João Pissolato, Edmundo de Castro, Amanda Dias, Abraham Rojas, Giovanny Barrera, Gabriel dos Reis, Aires Menani—have helped me directly on indirectly to finish this work, for which I am really grateful.

Finally, I would like to thank my family for their immense support and patience: my mother Nancy Cuyo for supporting me emotionally and economically, and my brother Ricardo Ccama for taking care of my pet dog Zeus while I was dedicated to write this dissertation; without their invaluable help I would never be able to achieve this life goal.

Leonel C. Cuyo

"This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001"

"For me, it is difficult to think of something more impressive than the strong connection between Geometry and Topology—based on some properties we can predict the actual nature of the objects we stick with."

(Leonel R. C. Cuyo)

ABSTRACT

CCAMA CUYO, L. R. **Complete embedded hypersurfaces in product spaces**. 2023. 62 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

The hypersurfaces $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1, 0, 1\}$, where $\mathbb{Q}^n_\varepsilon$ denotes a simply connected space form with curvature ε, which belong to the class $\mathscr{A}$ are those for which $\operatorname{grad} h$ is a principal direction of its shape operator A; here h stands for the height function of f. Fundamental examples of such hypersurfaces are: open subsets of slices of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, products $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface in $\mathbb{Q}^n_\varepsilon$, as well as isometric immersions $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, built up from a parallel family of hypersurfaces in $\mathbb{Q}^n_\varepsilon$ and a smooth function of one real variable. A remarkable fact is that any hypersurface that belongs to the class $\mathscr{A}$ is whether one of this fundamental examples or is locally given by the latter isometric immersions. In this work we present the precise statement behind this fact, as well as prove it. On the other hand, complete hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$ with positive definite second fundamental form, whose height function has at least one critical point, are embedded, homeomorphic to either the unit sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$, and bound a convex set, where $\mathscr{H}$ stands for a general Cartan-Hadamard manifold. That is the essential content of a Hadamard-Stoker-type theorem for complete hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$, theorem which we also present here. In addition, it is shown that these hypersurfaces are rigid among the hypersurfaces with the same extrinsic curvature.

Keywords: Hypersurface; class $\mathscr{A}$; Flat normal bundle; Constant angle; Hadamard-Stoker; Rigidity.

RESUMO

CCAMA CUYO, L. R. **Hipersuperfícies completas e mergulhadas em espaços produto**. 2023. 62 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

As hipersuperfícies $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1, 0, 1\}$, onde $\mathbb{Q}^n_\varepsilon$ denota uma forma espacial simplesmente conexa com curvatura ε, que pertencem à classe $\mathscr{A}$, são aquelas para as quais $\operatorname{grad} h$ é uma direção principal do seu operador forma A, sendo h a função altura de f. Exemplos fundamentais de tais hipersuperfícies são: subconjuntos abertos de slices $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, produtos $M^{n-1} \times \mathbb{R}$, onde M^{n-1} é uma hipersuperfície em $\mathbb{Q}^n_\varepsilon$, além de imersões isométricas $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ que são construídas utilizando uma família paralela de hipersuperfícies em $\mathbb{Q}^n_\varepsilon$ e uma função suave de uma variável real. Um fato notável é que qualquer hipersuperfície pertencente à classe $\mathscr{A}$ deve ser ou um destes exemplos fundamentais ou é localmente dado pelas últimas imersões isométricas. Neste trabalho, apresentamos o enunciado preciso por trás deste fato e o demonstramos. Por outro lado, hipersuperfícies completas em $\mathscr{H}^n \times \mathbb{R}$ com segunda forma fundamental definida positiva cuja função altura tem pelo menos um ponto crítico, além de serem mergulhadas são também homeomorfas à esfera $\mathbb{S}^n$ ou ao espaço Euclidiano $\mathbb{R}^n$, e são a fronteira de um determinado conjunto convexo, onde $\mathscr{H}$ denota uma variedade de Cartan-Hadamard arbitrária. Este é o conteúdo essencial de um teorema do tipo Hadamard-Stoker para hipersuperfícies completas em $\mathscr{H} \times \mathbb{R}$ o qual também é apresentado. Além disso, mostraremos que estas hipersuperfícies são rígidas entre as hipersuperfícies de mesma curvatura extrínseca.

Palavras-chave: Hipersuperfície; Classe $\mathscr{A}$; Fibrado normal plano; Ângulo constante; Hadamard-Stoker; Rigidez.

LIST OF FIGURES

CONTENTS

CHAPTER

1

INTRODUCTION

One fundamental problem in Submanifold theory is determine whether a given Riemannian manifold M^n can be isometrically immersed in other given Riemannian manifold $\tilde{M}^{n+p}$; and if so, whether this isometric immersion is "unique". Once an affirmative answer, which is then referred as a fundamental theorem, for the first question is given, a natural problem to be attacked is that of classifying these isometric immersions. In recent years, after the emergence of a fundamental theorem for hypersurfaces in product spaces (Daniel (2009)), the study of hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$, where $\mathbb{H}^n$ and $\mathbb{S}^n$ stand for the hyperbolic space and the unit sphere, respectively, has caught the attention of several geometers, driving to the appearance of many important classes of such hypersurfaces: rotation hypersurfaces (Dillen, Fastenakels and Veken (2009)), hypersurfaces with constant sectional curvature and constant angle hypersurfaces (Manfio and Tojeiro (2011)), and Einstein hypersurfaces (Leandro, Pina and Santos (2021)).

One of such important classes, which contains all previously mentioned classes, is the one formed by all hypersurfaces for which the gradient of their height function is everywhere a principal direction of their shape operators. This class was first introduced and classified by Tojeiro (2010). He also provided an alternative characterization of the hypersurfaces in this class in terms of flatness of their normal bundles in certain (but natural) flat manifold containing the ambient space. Nowadays, the class includes not only hypersurfaces, but also submanifolds for which, roughly speaking, certain vector field is a principal direction of all of its shape operators.

On the other hand, two classical theorems about embeddedness, convexity and rigidity of surfaces in Euclidean space are the Hadamard-Stoker Theorem and the Cohn-Vossen Rigidity Theorem. Whilst the Hadamard-Stoker Theorem establishes that a complete surface with positive curvature immersed in Euclidean space is embedded, bounds an open convex set, and is homeomorphic to a sphere or a plane, the Cohn-Vossen Rigidity Theorem claims that such surface, but a compact one, is rigid. As time went by, there had appeared some generalisations of these theorems to submanifolds in more general ambient spaces. First, Sacksteder generalised them to nonflat hypersurfaces with nonnegative sectional curvature in Euclidean space. Next, in

1970, do Carmo and Warner extended almost completely the theorems to compact hypersurfaces in hyperbolic space $\mathbb{H}^n$ and the unit sphere $\mathbb{S}^n$ — almost as it was the rigidity property of compact hypersurfaces in $\mathbb{H}^n$ which they conjectured but not proved. Later, in 2019, R.F. de Lima and R.L. de Andrade provided a proof to this conjecture. In 1977, S. Alexander contributed with an Hadamard-Stoker-type theorem for compact hypersurfaces with positive semi-definite second fundamental forms in Cartan-Hadamard manifolds $\mathscr{H}^n$. After that, a few works brought generalisations of the Hadamard-Stoker theorem for hypersurfaces in product spaces $\tilde{M} \times \mathbb{R}$, where $\tilde{M}$ stands for either $\mathscr{H}^n$ or $\mathbb{S}^n$. Recently, A Hadamard-Stoker-type theorem for complete hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$ has been established by R. de Lima.

This work has been divided into three chapters excluding this introductory one, and covers a number of topics as follows. Chapter two contains the basic material of Submanifold theory that will be used throughout this work. First, in Section 2.1, the fundamental formulas associated with an isometric immersion, Gauss and Codazzi formulas, are introduced. Then, in Section 2.2, its basic equations are presented, namely, Gauss, Codazzi and Ricci equations. Next, in Section 2.3, we state the Fundamental Theorem of submanifolds. After that, Section 2.4 aims to see how this fundamental formulas and basic equations look for hypersurfaces. We end Chapter two with the Fundamental Theorem for hypersurfaces in space forms.

Chapter three starts introducing the product spaces $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, where $\mathbb{Q}^n_\varepsilon$ stands for a simply connected space form: what they are, its shape operator, and its Gauss formula. Then, in Section 3.2, it continues looking at hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$: the vector fields $\frac{\partial}{\partial t}$ and its projection T onto the hypersurface, the angle function, the additional two equations that arise relating this new ingredients, and the fundamental theorem for hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$. After that, Section 3.3 is dedicated to the main characters of this chapter, the hypersurfaces in the class $\mathscr{A}$. This section provides some basic examples of such hypersurfaces, and then proves that they are the basic pieces from which any hypersurface in the class $\mathscr{A}$ can be build up. Also, it offers an alternative characterization of the hypersurfaces (belonging to the class $\mathscr{A}$) in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1, 1\}$, in terms of flatness of their normal bundles (see Proposition 3.3). In addition, a geometric interpretation of one of this fundamental pieces is given (cf. Remark 3.11). The chapter ends with a complete description of the constant angle hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$.

Chapter four is dedicated to embeddedness, convexity and rigidity properties of hypersurfaces in $\mathscr{H} \times \mathbb{R}$ and $\mathbb{S} \times \mathbb{R}$. In Section 4.1, some notions associated to such hypersurfaces are collected: Cartan-Hadamard manifolds and their properties, definition of top and bottom ends, notion of horizontal section and its elements, and the definition of normal sections. Section 4.2 treats asymptotic rays in $\mathscr{H} \times \mathbb{R}$, their definition and a fundamental property, as well as the definition of geodesic graphs. In section 4.3, we have compiled some useful results: two important generalisations of the Hadamard-Stoker theorem, namely do Carmo-Warner and Alexander Theorems; the statement of the Soul Conjecture; as well as a lemma that tell us that all hypersurfaces with equal extrinsic curvature share the same shape operator provided that one of them has a shape operator with rank ≥ 3 everywhere. Section 4.4 contains one of the most important results in this chapter, a Hadamard-Stoker-type theorem, roughly speaking, for complete hypersurfaces in $\mathscr{H} \times \mathbb{R}$; it also brings a dual result. Finally, in Section 4.5, we present a result that states that complete hypersurfaces in either $\mathbb{H}^n \times \mathbb{R}$ or $\mathbb{S}^n \times \mathbb{R}$ are rigid, though in the smaller class of all hypersurfaces with the same extrinsic curvature.

CHAPTER

2

THE BASIC EQUATIONS OF A SUBMANIFOLD

In this chapter we recall some basic notions and facts of the theory of submanifolds that are used in this work. The second fundamental form and normal connection of an isometric immersion are introduced by means of the Gauss and Weingarten formulas, and their compatibility equations are also presented. We also recall the fundamental theorem of submanifolds. Then we particularise the theory to the case of hypersurfaces — isometric immersions of codimension one. In this chapter we follow the notation and results of the excellent book "Submanifold theory beyond and introduction" by Dajczer and Tojeiro (2019).

2.1 Gauss and Weingarten Formulas

Let M^n and $\tilde{M}^m$ be connected smooth manifolds. An *immersion* is a smooth map $f : M^n \to \tilde{M}^m$ whose differential $f_*(x)$ is injective for all $x \in M^n$. The *codimension* of f is the number $m-n$. Both the immersion f and its image $f(M^n)$ can be referred as an *(immersed) submanifold* of $\tilde{M}$. Because an immersion is locally an embedding, locally, M and $f(M)$ can be identified as well as f can be regarded as the inclusion map. We shall identify likewise T_xM and $f_*(T_xM)$ for all $x \in M$.

An *isometric immersion* is an immersion $f : M^n \to \tilde{M}^m$ between Riemannian manifolds whose differential $f_*(x)$, at each point $x \in M^n$, satisfies for all $X, Y \in T_xM$

$$\langle X, Y\rangle = \langle f_*X, f_*Y\rangle. \tag{2.1}$$

If $f : M^n \to \tilde{M}^m$ is an immersion from a smooth manifold M to a Riemannian manifold $\tilde{M}$, equation (2.1) defines a Riemannian metric on M, the *metric induced* by f, with respect to which f becomes isometric.

Let $f: M^n \to \tilde{M}^m$ be an isometric immersion. The vector bundle $f^*T\tilde{M}$ over M whose fiber over $x \in M$ is $T_{f(x)}\tilde{M}$ is called the *induced vector bundle*. The *normal space* $N_fM(x)$ of f at x is the orthogonal complement of T_xM in $T_{f(x)}\tilde{M}$. The vector bundle N_fM whose fiber over $x \in M$ is $N_fM(x)$ is known as the *normal bundle* of f.

If E is a vector bundle over a smooth manifold M, we denote the set of all its smooth sections by $\Gamma(E)$. However, when E is the tangent bundle TM of M, we prefer $\mathfrak{X}(M)$ to $\Gamma(TM)$.

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}^m$ induces naturally a unique connection $\hat{\nabla}$ on the induced bundle $f^*T\tilde{M}$ which satisfies for all $Z \in \mathfrak{X}(\tilde{M})$, $x \in M$ and $X \in T_xM$

$$\hat{\nabla}_X(Z \circ f) = \tilde{\nabla}_X Z.$$

Henceforth, we shall identify $\hat{\nabla}$ with $\tilde{\nabla}$, and thus use only the symbol $\tilde{\nabla}$ for both connections.

The induced vector bundle $f^*T\tilde{M}$ is decomposed as $f^*T\tilde{M} = TM \oplus N_fM$. Thus, given $X, Y \in TM$, we can also decompose $\tilde{\nabla}_X Y$ with respect to that decomposition as $(\tilde{\nabla}_X Y)^T + (\tilde{\nabla}_X Y)^\perp$. The tangent component $(\tilde{\nabla}_X Y)^T$ turns out to be the image of $\nabla_X Y$ by f_*, where ∇ is the Levi-Civita connection of M. The normal component $(\tilde{\nabla}_X f_* Y)^\perp$ is the value of a map $\alpha^f : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(N_fM)$, called the *second fundamental form* (or *shape tensor*) of f, given by $\alpha^f(X,Y) = (\tilde{\nabla}_X Y)^\perp$. We omit the superindex of α^f when it is clear of which isometric immersion α is the second fundamental form. Consequently, the first basic formula in submanifold theory, called the *Gauss Formula*, is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y). \tag{2.2}$$

The second fundamental form α of f is symmetric, i.e., $\alpha(X,Y) = \alpha(Y,X)$ for $X, Y \in \mathfrak{X}(M)$, because $\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X,Y]$. It is also $C^\infty(M)$-bilinear, thus the value of $\alpha(X,Y)$ at some $x \in M$ depends only on X_x, Y_x. For that reason, we can think of α as a section of $\mathrm{Hom}^2(TM, TM; N_fM)$, which amounts to considering α, at x, as a symmetric bilinear map $T_xM \times T_xM \to N_fM(x)$ for every $x \in M$.

The *shape operator* of f at $x \in M$ in the direction of $\xi \in N_fM(x)$ is the self-adjoint operator $A_\xi : T_xM \to T_xM$ determined by $\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle$ for all $X, Y \in T_xM$.

For $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$, we have

$$\langle \tilde{\nabla}_X \xi, Y\rangle = -\langle \xi, \tilde{\nabla}_X Y\rangle = \langle -A_\xi X, Y\rangle,$$

that is, the tangent component of $\tilde{\nabla}_X \xi$ is $-A_\xi X$ while the normal component is, by definition, the value of a compatible connection $\nabla^\perp$ on N_fM, called the *normal connection* of f, determined by $\nabla^\perp_X \xi = (\tilde{\nabla}_X \xi)^\perp$. Then, the second basic formula in submanifold theory, known as the *Weingarten formula*, is expressed as

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi. \tag{2.3}$$

Given an orthonormal basis $\{X_1, ..., X_n\}$ for T_xM, the *mean curvature vector of f at x* is the normal vector given by

$$H(x) = \frac{1}{n}\sum_{j=1}^{n} \alpha(X_j, X_j).$$

It follows from its definition that it satisfies for every $\xi \in N_fM(x)$

$$n\langle H(x), \xi\rangle = \mathrm{tr}A_\xi, \tag{2.4}$$

which shows that $H(x)$ does not depend on the choice of an orthonormal basis for T_xM.

An isometric immersion $f : M^n \to \tilde{M}^m$ is said to be *totally geodesic* at $x \in M$ if and only if the second fundamental form $\alpha(x)$ is zero. Similarly, a *totally geodesic* isometric immersion is a isometric immersion for which the second fundamental form α vanishes everywhere.

2.2 Gauss, Codazzi and Ricci equations

In this section, we present the compatibility equations that any isometric immersion satisfies. In order to obtain them, the Gauss and Weingarten formulas must be used. We only introduce these compatibility equations, but for the reader who is interested in studying the proccess that is followed to derive these equations, we refer to Dajczer and Tojeiro (2019).

Let $f : M^n \to \tilde{M}^m$ be an isometric immersion and $X, Y, W, Z \in \mathfrak{X}(M)$. If the Riemannian curvature tensors of M and $\tilde{M}$ are denoted by R and $\tilde{R}$, respectively, the Gauss Equation is then determined by

$$R(X,Y)Z = (\tilde{R}(X,Y)Z)^T + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y, \tag{2.5}$$

or equivalently,

$$\langle R(X,Y)Z, W\rangle = \langle \tilde{R}(X,Y)Z, W\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle. \tag{2.6}$$

Similarly, given $x \in M$ and orthonormal tangent vectors $X, Y \in T_xM$, if the sectional curvatures of M and $\tilde{M}$ at x along the linear subspace spanned by X and Y are designated by $K(X,Y)$ and $\tilde{K}(X,Y)$, respectively, the Gauss Equation is expressed as

$$K(X,Y) = \tilde{K}(X,Y) + \langle \alpha(X,X), \alpha(Y,Y)\rangle - |\alpha(X,Y)|^2. \tag{2.7}$$

Next,we have the Codazzi equation of f

$$(\tilde{R}(X,Y)Z)^\perp = (\nabla^\perp_X \alpha)(Y,Z) - (\nabla^\perp_Y \alpha)(X,Z), \tag{2.8}$$

where

$$(\nabla^\perp_X \alpha)(Y,Z) = \nabla^\perp_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z),$$

is the canonical connection in the bundle $\mathrm{Hom}^2(TM, TM; N_fM)$. It has an equivalent form

$$(\tilde{R}(X,Y)\xi)^T = (\nabla_Y A)(X,\xi) - (\nabla_X A)(Y,\xi), \tag{2.9}$$

where

$$(\nabla_Y A)(X, \xi) = \nabla_Y A_\xi X - A_\xi \nabla_Y X - A_{\nabla_Y^\perp \xi} X,$$

is the canonical connection in $\mathrm{Hom}(TM, N_f M, TM)$.

Associated with the normal connection $\nabla^\perp$ of f in $N_f M$, we have the normal curvature tensor $R^\perp$ on $N_f M$ given for $\xi \in \Gamma(N_f M)$ by

$$R^\perp(X,Y)\xi = \nabla_X^\perp \nabla_Y^\perp \xi - \nabla_Y^\perp \nabla_X^\perp \xi - \nabla_{[X,Y]}^\perp \xi.$$

In this way, the Ricci Equation is written as

$$(\tilde{R}(X,Y)\xi)^\perp = R^\perp(X,Y)\xi - \alpha(X, A_\xi Y) + \alpha(A_\xi X, Y), \tag{2.10}$$

or equivalently, if $\eta \in \Gamma(N_f M)$, as

$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle \tilde{R}(X,Y)\xi, \eta\rangle + \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.11}$$

where

$$[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi.$$

2.3 Fundamental theorem of submanifolds

A complete, connected Riemannian manifold with constant sectional curvature is called a *space form*. A space form is said to be *spherical, euclidean* or *hyperbolic* depending upon whether its sectional curvature is positive, zero or negative. An n-dimensional simply connected space form with sectional curvature c is designated by $\mathbb{Q}_c^n$. Thus $\mathbb{Q}_c^n$ stands for $\mathbb{H}_c^n$, $\mathbb{R}^n$ or $\mathbb{S}_c^n$ depending on whether c is negative, zero or positive; we just write $\mathbb{H}^n$ and $\mathbb{S}^n$ when the sectional curvatures are -1 and 1, respectively.

Suppose $f : M^n \to \tilde{M}^m$ is an isometric immersion. If the ambient manifold $\tilde{M}$ has constant sectional curvature c, the compatibility equations (2.5), (2.8) and (2.10) take simpler forms since the ambient curvature tensor fulfils

$$\tilde{R}(X,Y)Z = c(X \wedge Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y).$$

Thus, the Gauss equation becomes

$$R(X,Y)Z = c(X \wedge Y)Z + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y, \tag{2.12}$$

or equivalently,

$$\langle R(X,Y)Z, W\rangle = c\langle (X \wedge Y)Z, W\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle, \tag{2.13}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

The Codazzi equation is now expressed by

$$(\nabla^{\perp}_X \alpha)(Y,Z) = (\nabla^{\perp}_Y \alpha)(X,Z), \tag{2.14}$$

or equivalently,

$$(\nabla_X A)(Y,\xi) = (\nabla_Y A)(X,\xi), \tag{2.15}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and every $\xi \in \Gamma(N_f M)$.

Also, the Ricci equation changes to

$$R^{\perp}(X,Y)\xi = \alpha(X, A_\xi Y) - \alpha(A_\xi X, Y), \tag{2.16}$$

or equivalently,

$$\langle R^{\perp}(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.17}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Gamma(N_f M)$.

We note that equations (2.13), (2.14) and (2.16) are intrinsic: they involve the metric of M, a Riemannian vector bundle $N_f M$ on M endowed with a compatible connection, and a symmetric section α of $\mathrm{Hom}^2(TM, TM; N_f M)$. Then a natural question is whether any such data satisfying (2.13), (2.14) and (2.16) can be regarded as the data associated with an isometric immersion in some Riemannian manifold with constant sectional curvature. The answer is affirmative and is established more precisely by the following theorem, known as the *Fundamental Theorem of submanifolds*.

Theorem 2.1. Existence: Let M^n be a simply connected Riemannian manifold, let E be a Riemannian vector bundle of rank p over M with compatible connection ∇^E and curvature tensor R^E, and let α^E be a symmetric section of $\mathrm{Hom}^2(TM, TM, E)$. For each $\xi \in \Gamma(E)$ define $A^E_\xi \in \Gamma(End(TM))$ by

$$\langle A^E_\xi X, Y\rangle = \langle \alpha^E(X,Y), \xi\rangle, \forall X, Y \in \mathfrak{X}(M).$$

Assume that $(\nabla^E, \alpha^E, A^E, R^E)$ satisfies (2.13), (2.14) and (2.16). Then there exists an isometric immersion $f : M \longrightarrow \mathbb{Q}^{n+p}_c$ and a vector bundle isometry $\phi : E \to N_f M$ such that

$$\nabla^{\perp}\phi = \phi\nabla^E \quad \text{and} \quad \alpha^f = \phi \circ \alpha^E$$

Uniqueness: Let $f, g : M \to \mathbb{Q}^{n+p}_c$ be isometric immersions of a Riemannian manifold. Assume that there exists a vector bundle isometry $\phi : N_f M \to N_g M$ such that

$$\phi\, {}^f\nabla^{\perp} = {}^g\nabla^{\perp}\phi \quad \text{and} \quad \phi \circ \alpha^f = \alpha^g.$$

Then there exists an isometry $\tau : \mathbb{Q}^{n+p}_c \to \mathbb{Q}^{n+p}_c$ such that

$$\tau \circ f = g \quad \text{and} \quad \tau_*|_{N_f M} = \phi.$$

It is worth mentioning that Theorem 2.1 still holds locally if we do not require M to be simply connected: the isometric immersion f, whose existence is guaranteed by this theorem, is only defined on a neighbourhood of each $x \in M$.

2.4 Hypersurfaces

An isometric immersion $f : M^n \to \tilde{M}^{n+1}$ of codimension one is called a *hypersurface*. Every hypersurface $f : M^n \to \tilde{M}^{n+1}$ is locally oriented by a local unit normal. Also, when M is orientable, there is a global unit normal to f. In addition, in spite of the nature of the unit normal ξ, locally or globally defined, ξ is the only unit normal up to a sign. For this reason, we will only write A for the shape operator of f in the direction of ξ.

For a hypersurface $f : M^n \to \tilde{M}^{n+1}$, the fundamental formulas can be considerably simplified as follow. Given $X, Y \in \mathfrak{X}(M)$ and a unit normal ξ, all defined on some open set $U \subset M$, possibly equal to the whole manifold M, we have

$$\alpha(X,Y) = \langle AX, Y\rangle \xi. \tag{2.18}$$

Therefore, Gauss formula now reads as

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \xi. \tag{2.19}$$

Since $|\xi| = 1$ and $\nabla^{\perp}$ is a compatible connection on $N_f M$, we have $\nabla^{\perp}_X \xi = 0$, and the Weingarten formula for ξ is

$$\tilde{\nabla}_X \xi = -AX. \tag{2.20}$$

For an arbitrary $\psi \in \Gamma(N_f M)$, as $\psi = a\xi$ on U for $a = \langle \psi, \xi\rangle$, the Weingarten formula for it is obtained from (2.20) by multiplying both sides by a; thus we only need to consider (2.20) to work.

The compatibility equations change their appearance for a hypersurface on account of (2.18) as follows. Let $f : M^n \to \tilde{M}^{n+1}$ be a hypersurface with unit normal ξ. The Gauss equation for f is given by

$$(\tilde{R}(X,Y)Z)^T = R(X,Y)Z - (AX \wedge AY)Z, \tag{2.21}$$

or equivalently,

$$\langle R(X,Y)Z, W\rangle = \langle \tilde{R}(X,Y)Z, W\rangle + \langle AX, W\rangle\langle AY, Z\rangle - \langle AX, Z\rangle\langle AY, W\rangle, \tag{2.22}$$

where $X, Y, Z, W \in \mathfrak{X}(M)$. Moreover, for all $x \in M$ and orthonormal vectors $X, Y \in T_x M$, in terms of sectional curvatures, we have

$$K(X,Y) = \tilde{K}(X,Y) + \langle AX, X\rangle\langle AY, Y\rangle - \langle AX, Y\rangle^2. \tag{2.23}$$

On the other hand, the Codazzi equation of f for $X, Y \in \mathfrak{X}(M)$ is determined by

$$(\tilde{R}(X,Y)\xi)^T = (\nabla_Y A)X - (\nabla_X A)Y, \tag{2.24}$$

where

$$(\nabla_Y A)X = \nabla_Y AX - A\nabla_Y X. \tag{2.25}$$

As both sides of the Ricci equation for a hypersurface f vanish identically, there is no need to consider it.

If the ambient space $\tilde{M}$ has constant sectional curvature c, the Gauss and Codazzi equations are

$$R(X,Y)Z = c(X \wedge Y)Z + (AX \wedge AY)Z, \tag{2.26}$$

and

$$(\nabla_Y A)X = (\nabla_X A)Y, \tag{2.27}$$

respectively.

The Fundamental Theorem for submanifolds, Theorem 2.1, also has a simpler version for hypersurfaces in simply connected space forms. Before we give the precise statement, let us point out that given two hypersurfaces in an orientable Riemannian manifold $\tilde{M}$, there exists exactly two vector bundle isometries between their normal bundles (DAJCZER; TOJEIRO, 2019, Section 1.4.1).

Theorem 2.2. Existence: Let M^n be a simply connected Riemannian manifold, and A a symmetric section of $\mathrm{End}(TM)$ satisfying (2.26) and (2.27). Then there exists an isometric immersion $f : M \to \mathbb{Q}_c^{n+1}$, and a unit normal ξ to f so that $A_\xi = A$.
Uniqueness: If $f, g : M \to \mathbb{Q}_c^{n+1}$ are two hypersurfaces such that

$$\alpha^g = \phi \circ \alpha^f,$$

where ϕ is one of two vector bundle isometries between $N_f M$ and $N_g M$, then there exists $\tau \in \mathrm{Iso}(\mathbb{Q}_c^{n+1})$ such that

$$g = \tau \circ f \quad \text{and} \quad \tau_*|_{N_f M} = \phi.$$

CHAPTER

3

HYPERSURFACES IN PRODUCT SPACES

In this chapter, we summarise some basic notions and results about hypersurfaces M^n in product spaces $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$. In particular, we present the fundamental equations of such hypersurfaces, as well as a version of the fundamental theorem for them, due to Daniel (2009) (cf. also Lira, Tojeiro and Vitório (2010)). The main results of this chapter provide a complete description of all hypersurfaces M in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ for which the projection, onto TM, of the unit vector field $\frac{\partial}{\partial t}$ on $T(\mathbb{Q}_\varepsilon^n \times \mathbb{R})$, determined by $\frac{\partial}{\partial t} \equiv 1$, is a principal direction. For $\varepsilon \in \{-1, 1\}$, these are all hypersurfaces in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ that have flat normal bundle when regarded as submanifolds of codimension two in the underlying flat space $\mathbb{E}^{n+2}$, that is $\mathbb{L}^{n+2}$ or $\mathbb{R}^{n+2}$ according as $\varepsilon = -1$ or $\varepsilon = 1$, respectively.

3.1 The product manifold $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$

Let us recall that $\mathbb{Q}_\varepsilon^n$ stands for the unit sphere $\mathbb{S}^n$, the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$, according as $\varepsilon = 1$, $\varepsilon = 0$ or $\varepsilon = -1$, respectively. In order to study hypersurfaces into $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$, for $\varepsilon \in \{-1, 1\}$, our approach is to regard the hypersurface as an isometric immersion of codimension two into $\mathbb{E}^{n+2}$, where $\mathbb{E}^{n+2}$ denotes either the Euclidean space $\mathbb{R}^{n+2}$ or the Lorentz space $\mathbb{L}^{n+2}$, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively. More precisely, let $(x_1, \dots, x_{n+2})$ be the standard coordinates on $\mathbb{E}^{n+2}$ with respect to which the flat metric on $\mathbb{E}^{n+2}$ is written as

$$ds^2 = \varepsilon(dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+2})^2.$$

Think of $\mathbb{E}^{n+1}$ as the hyperplane

$$\mathbb{E}^{n+1} = \{(x_1, \dots, x_{n+2}) \in \mathbb{E}^{n+2} : x_{n+2} = 0\},$$

and $\mathbb{Q}_\varepsilon^n$ as the quadric

$$\mathbb{Q}_\varepsilon^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{E}^{n+1} : \varepsilon x_1^2 + \cdots + x_{n+1}^2 = \varepsilon\},$$

where $x_1 > 0$ if $\varepsilon = -1$. Then, as $\mathbb{Q}^n_\varepsilon$ admits an umbilical inclusion into $\mathbb{E}^{n+1}$, we can consider the canonical inclusion

$$i : \mathbb{Q}^n_\varepsilon \times \mathbb{R} \to \mathbb{E}^{n+1} \times \mathbb{R} = \mathbb{E}^{n+2}.$$

The vector field $\xi = \pi \circ i$ is a unit normal vector field to the inclusion i, called the *outward pointing unit normal* to $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, where $\pi : \mathbb{E}^{n+1} \times \mathbb{R} \to \mathbb{E}^{n+1}$ is the projection onto the first factor.

The projection $\pi_\mathbb{R}$ of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ onto the second factor is called the *height function* of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$. The gradient of $\pi_\mathbb{R}$, which is a unit parallel vector field on $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, will be denoted by $\frac{\partial}{\partial t}$. A simple account shows that it is determined by $\frac{\partial}{\partial t}(x,r) = 1 \in \mathbb{R}$, $\forall (x,r) \in \mathbb{Q}^n_\varepsilon \times \mathbb{R}$.

If D denotes the usual covariant derivative of $\mathbb{E}^{n+2}$, we have

$$D_Z\xi = \pi_* i_* Z = i_*\left(Z - \left\langle Z, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}\right),$$

hence the shape operator of i in the direction of ξ is given by

$$A_\xi Z = -Z + \left\langle Z, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t},$$

for every $Z \in \mathfrak{X}(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$.

Given any vector field $Z \in \mathfrak{X}(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$, it can be expressed as follows

$$Z(p,s) = (Z_1^s(p), Z_2^p(s)),$$

where $Z_1^s \in \mathfrak{X}(\mathbb{Q}^n_\varepsilon)$ and $Z_2^p \in \mathfrak{X}(\mathbb{R})$ for all $(p,s) \in \mathbb{Q}^n_\varepsilon \times \mathbb{R}$. Consequently, the Gauss formula for i can be written as

$$D_X i_* Y = i_* \tilde{\nabla}_X Y - \varepsilon \langle X_1, Y_1 \rangle \xi \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$.

3.2 Hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$

Let $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be an oriented hypersurface with unit normal N. Denote by A the shape operator of f in the direction N, which is given for every $X \in \mathfrak{X}(M)$ by

$$AX = -\tilde{\nabla}_X N.$$

The *height function* h of f is the smooth function in $C^\infty(M)$ determined by

$$h := \pi_\mathbb{R} \circ f,$$

or alternatively by $h := \left\langle f, \frac{\partial}{\partial t}\right\rangle$, where we think of the values of f as points in $\mathbb{E}^{n+2}$ if $\varepsilon \in \{-1, 1\}$. On the other hand, the *angle function* ν of f is the smooth real-valued function on M given for every $x \in M$ by

$$\nu(x) = \left\langle N(x), \frac{\partial}{\partial t}\right\rangle.$$

If $T \in \mathfrak{X}(M)$ represents the projection of $\frac{\partial}{\partial t}$ on TM, $\frac{\partial}{\partial t}$ can be written as sum of its tangent (to $f(M)$) and normal (to $f(M)$) components as

$$\frac{\partial}{\partial t} = T + \nu N. \tag{3.2}$$

Note that T coincides with $\operatorname{grad} h$, as the following computation shows

$$\langle \operatorname{grad} h, X\rangle = (\pi_{\mathbb{R}})_* f_* X = \left\langle \frac{\partial}{\partial t}, f_* X \right\rangle = \langle T, X\rangle, \tag{3.3}$$

for all $x \in M$, $X \in T_x M$.

The Gauss and Codazzi equations of f are expressed by

$$\begin{aligned} \langle R(X,Y)Z,W\rangle = {} & \langle AX,W\rangle\langle AY,Z\rangle - \langle AX,Z\rangle\langle AY,W\rangle \\ & + \varepsilon(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ & + \langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle + \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle \\ & - \langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle - \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle), \end{aligned} \tag{3.4}$$

and

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = \varepsilon\nu(\langle Y,T\rangle X - \langle X,T\rangle Y). \tag{3.5}$$

respectively, where $X, Y, Z, W \in \mathfrak{X}(M)$.

Now, because $\frac{\partial}{\partial t}$ is parallel on $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, it holds

$$0 = \tilde{\nabla}_X (T + \nu N) = \nabla_X T + \langle AX, T\rangle N - \nu AX + X(\nu) N,$$

which yields by taking tangential and normal components the following two equations

$$\nabla_X T = \nu AX, \tag{3.6}$$

and

$$X(\nu) = -\langle AX, T\rangle, \tag{3.7}$$

for all $X \in \mathfrak{X}(M)$.

The Gauss and Codazzi equations above are not sufficient conditions for a Riemannian manifold M^n to be isometrically immersed in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$; even if T and ν are given. The reason for this is simple: the vector field T and the smooth function ν satisfy additionally equations (3.6) and (3.7). In fact, we need to add these two equations to ensure the existence of an isometric immersion $f : M \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$. This is the content of the following result, which amounts to a fundamental theorem for hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, due to Daniel (2009) (see also Lira, Tojeiro and Vitório (2010)).

Theorem 3.1. Existence: Let M be a simply connected n-dimensional Riemannian manifold, and ∇ its Levi-Civita connection. Suppose A is a field of self-adjoint operators $A_y : T_y M \to T_y M$,

$y \in M$, $T \in \mathfrak{X}(M)$, and $\nu \in C^\infty(M)$, such that $\|T\|^2 + \nu^2 = 1$. If (A, T, ν) satisfies (3.4)-(3.7), then there exists an isometric immersion $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ such that the shape operator with respect to the normal associated with f is A and

$$\frac{\partial}{\partial t} = T + \nu N.$$

Uniqueness: Given two isometric immersions $f, g : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ with unit normals N_f and N_g, respectively, such that the projections T_f and T_g of $\partial/\partial t$ onto TM satisfy $T_f = T_g$, if there is a vector bundle isometry $\phi : N_f M \to N_g M$ such that $\alpha^g = \phi \circ \alpha^f$ and $N_g = \phi N_f$, then there exists an isometry $\Phi \in \mathrm{Iso}(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$ with $g = \Phi \circ f$.

3.3 The class $\mathscr{A}$

The main theorems in this section provide a complete description of all hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ for which the vector field T is a principal direction. They constitute a theorem first stated and proved by Tojeiro (2010). Those hypersurfaces belong to a well-known class of submanifolds, called class $\mathscr{A}$, which is made up of all isometric immersions $f : M^m \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ with the property that the vector field T is an eigenvector of all shape operators. It is worth mentioning that products $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface in $\mathbb{Q}^n_\varepsilon$, which correspond to the case $\nu \equiv 0$, are particular examples of hypersurfaces in this class. Another simple examples in this class are: *constant angle* hypersurfaces that are not open subsets of slices, *rotational* hypersurfaces with constant sectional curvature, and hypersurfaces with *constant sectional curvature*.

Given an arbitrary hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1, 1\}$, with a normal vector field N, let us consider it as an isometric immersion of codimension two into $\mathbb{E}^{n+2}$, and restrict ξ to $f(M)$ to obtain a unit normal to M in $\mathbb{E}^{n+2}$. Formally, we identify f with the composition $i \circ f$, where i is the inclusion of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ into $\mathbb{E}^{n+2}$, and ξ with the composition $\xi \circ f$.

Proposition 3.2. The shape operator A_ξ of the composition $i \circ f$ satisfies

$$A_\xi T = -\nu^2 T \quad \text{and} \quad A_\xi X = -X \tag{3.8}$$

for every $X \in \{T\}^\perp$.

Proof. Let $X \in \mathfrak{X}(M)$, then

$$\langle A_\xi T, X\rangle = \langle \alpha(T, X) - \varepsilon\langle T_1, X_1\rangle\xi, \xi\rangle = -\langle T_1, X\rangle.$$

Since

$$T_1 = T - \left\langle T, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t} = T - \|T\|^2(T + \nu N) = \nu^2 T - \|T\|^2 \nu N,$$

we have

$$\langle A_\xi T, X\rangle = -\langle \nu^2 T, X\rangle,$$

which yields the first identity in (3.8). On the other hand, for $X \in \{T\}^{\perp}$ and $Y \in \mathfrak{X}(M)$,

$$\langle A_{\xi}X, Y\rangle = -\left\langle X - \left\langle X, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}, Y\right\rangle = -\langle X, Y\rangle,$$

which yields the remaining identity in (3.8). $\square$

The following proposition establishes that, for $\varepsilon \in \{-1, 1\}$, the hypersurfaces in $\mathbb{Q}^n_{\varepsilon} \times \mathbb{R}$ which have T as a principal direction turn out to be the hypersurfaces in $\mathbb{Q}^n_{\varepsilon} \times \mathbb{R}$ that have flat normal bundle when regarded as submanifolds of codimension two in the underlying flat space $\mathbb{E}^{n+2}$, and for which $T \neq 0$ everywhere.

Proposition 3.3. Under the hypotheses of Proposition 3.2, if T does not vanish at $x \in M^n$, then f has flat normal bundle at $x \in M$ as an isometric immersion into $\mathbb{E}^{n+2}$ if and only if T is a principal direction of f at x.

Proof. From the Ricci equation (2.11) for f, the isometric immersion $i \circ f$ has flat normal bundle at x if and only if any two shape operators commute, or equivalently, there exists an orthonormal basis for T_xM with respect to which the matrix representation of each shape operator of f at x is diagonal. This is equivalent to the fact that the shape operators A and A_{ξ} commute. In fact, an arbitrary shape operator A_{η}, $\eta \in N_fM(x)$, can be written uniquely as $aA + bA_{\xi}$; from which it follows that A_{η} commutes with both A and A_{ξ} if we suppose A and A_{ξ} commute, which implies that A_{η} commutes with any shape operator A_{μ}, $\mu \in N_fM(x)$. Now A and A_{ξ} commute if and only if the eigenspaces of A_{ξ} are invariant under A. One direction is immediate: because of the fact that the matrix representation of A and A_{ξ} are diagonal matrices, the eigenspaces of A_{ξ} are also eigenspaces of A. Conversely, from (3.8), if we set $AT = aT$, it follows that $A_{\xi}AZ = AA_{\xi}Z$ for every $Z = X + \lambda T \in T_xM$, where $X \in \{T(x)\}^{\perp}$. Lastly, the eigenspaces of A_{ξ} are invariant under the action of A as long as T is a principal direction of f at x. In fact, if $AT \in \{T\}$, for every $X \in \{T\}^{\perp}$, it holds $\langle AX, T\rangle = \langle X, AT\rangle = 0$, i.e., $AX \in \{T\}^{\perp}$. $\square$

Given a Riemannian manifold M, a smooth curve β in M is said to be a *pre-geodesic* if its arclength reparametrization is a geodesic in M.

Lemma 3.4. If T is a smooth gradient vector field on a Riemannian manifold M such that T does not vanish anywhere and $\|T\|$ is constant along the orthogonal distribution $\{T\}^{\perp}$ of T, then every integral curve of T is a pre-geodesic of M.

Proof. Since T is a gradient vector field and $\|T\|$ is constant along $\{T\}^{\perp}$, for $X \in \{T\}^{\perp}$, we have

$$\langle \nabla_T T, X\rangle = \langle \nabla_X T, T\rangle = \frac{1}{2}X\langle T, T\rangle = \|T\|(X\|T\|) = 0,$$

that is, $\nabla_T T$ is parallel to T. Let γ be an integral curve of T. Its arclenght function

$$s(r) = \int_a^r \|T\| d\rho$$

is a diffeomorphism as $\frac{ds}{dr} = \|T\| > 0$. If $r(s)$ denotes the inverse function of $s(r)$, then the arclength reparametrization $\gamma(r(s))$ of γ has a velocity vector given by

$$\|T(\gamma(r(s)))\|^{-1}T(\gamma(r(s))). \tag{3.9}$$

Note that

$$\nabla_T\|T\|^{-1}T = \|T\|^{-1}\nabla_T T + T(\|T\|^{-1})T,$$

is parallel to T, and

$$0 = T(\langle\|T\|^{-1}T, \|T\|^{-1}T\rangle) = 2\langle\nabla_T\|T\|^{-1}T, \|T\|^{-1}T\rangle$$

implies that $\nabla_T\|T\|^{-1}T$ is also orthogonal to T, i.e., $\gamma(r(s))$ is a geodesic of M. □

In Theorem 3.9 we will stick with hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ which have the vector field T as a principal direction; the associated orthogonal distribution $\{T\}^\perp$ is integrable on account of the following lemma.

Lemma 3.5. If T is an arbitrary gradient vector field on a Riemannian manifold M which does not vanish everywhere, then $\{T\}^\perp$ is integrable.

Proof. For any $X, Y \in \{T\}^\perp$,

$$\langle\nabla_X T, Y\rangle = \langle\nabla_Y T, X\rangle \quad \text{and} \quad \langle\nabla_X Y, T\rangle = -\langle Y, \nabla_X T\rangle,$$

which yields

$$\langle[X,Y], T\rangle = -\langle Y, \nabla_X T\rangle + \langle X, \nabla_Y T\rangle = 0.$$

□

Now we are in a position to state and prove the following two main theorems of this section. Let $g : M^{n-1} \to \mathbb{Q}^n_\varepsilon$ be a hypersurface which admits a unit normal N, and exp the exponential map of $\mathbb{Q}^n_\varepsilon$. The *parallel hypersurface* $g_s : M \to \mathbb{Q}^n_\varepsilon$, $s \in \mathbb{R}$, of g is determined by

$$g_s(x) = \exp_x(sN(x)),$$

i.e., $g_s(x)$ is the point reached in $\mathbb{Q}^n_\varepsilon$ by traversing a distance $|s|$ along the geodesic in $\mathbb{Q}^n_\varepsilon$ with initial point $g(x)$ and initial velocity vector $sN(x)$ (CECIL; RYAN, 2015, p. 14). Thus, for each $s \in \mathbb{R}$, g_s is given by

$$g_s(x) = C_\varepsilon(s)g(x) + S_\varepsilon(s)N_x,$$

where

$$C_\varepsilon(s) = \begin{cases} \cos s, & \text{if } \varepsilon = 1 \\ 1, & \text{if } \varepsilon = 0 \\ \cosh s, & \text{if } \varepsilon = -1. \end{cases} \quad \text{and} \quad S_\varepsilon(s) = \begin{cases} \sin s, & \text{if } \varepsilon = 1 \\ s, & \text{if } \varepsilon = 0 \\ \sinh s, & \text{if } \varepsilon = -1. \end{cases}$$

Given a smooth function $a : I \to \mathbb{R}$ with $a' > 0$ over the open interval $I \subset \mathbb{R}$, define $f : M^{n-1} \times I \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ by

$$f(x, s) = g_s(x) + a(s)\frac{\partial}{\partial t}. \tag{3.10}$$

Theorem 3.6. The map f in (3.10) defines, at regular points, a hypersurface for which T is a principal direction.

Proof. Smoothness of f follows immediately from its definition. Let $(x, s) \in M \times I$, $X \in T_xM$, and $\overline{\beta}(r) = (\beta(r), s)$ be a smooth curve in $M \times I$ such that $\overline{\beta}(0) = (x, s)$ and $\overline{\beta}'(0) = X$. Thus,

$$f_*X = \frac{d}{dr}\Big|_{r=0}\Big(g_s(\beta(r)) + a(s)\frac{\partial}{\partial t}\Big) = (g_s)_*X. \tag{3.11}$$

Similarly, allow $J \subset \mathbb{R}$ to be an open interval small enough such that $0 \in J$ and $s + r \in I$ for all $r \in J$. Then $\zeta(r) = (x, s + r)$, $r \in J$, is a smooth curve in $M \times I$ such that $\zeta(0) = (x, s)$ and $\zeta'(0) = \frac{\partial}{\partial s}$. Thus

$$f_*\frac{\partial}{\partial s} = -\varepsilon S_\varepsilon(s)g(x) + C_\varepsilon(s)N(x) + a'(s)\frac{\partial}{\partial t}, \tag{3.12}$$

and set

$$N_s(x) = -\varepsilon S_\varepsilon(s)g(x) + C_\varepsilon(s)N(x).$$

Because of equations (3.11) and (3.12), a point $(x, s) \in M \times I$ is regular for f if and only if g_s is regular at x. In fact, let $(x, s) \in M \times I$ be a regular point of f. Suppose there is $0 \neq X \in T_xM$ such that $0 = (g_s)_*X$, yielding a contradiction on account of (3.11). Suppose now g_s is regular at $x \in M$. Assume that there is $0 \neq X + r\frac{\partial}{\partial s} \in T_{(x,s)}(M \times I)$ such that $0 = f_*(X + r\frac{\partial}{\partial s}) = (g_s)_*X + rN_s + ra'(s)\frac{\partial}{\partial t}$, which yields the contradiction that g_s is not regular at x for wether $r = 0$ or $r \neq 0$. Now assume $(x, s) \in M \times I$ is a regular point for f. Then, $N_s(x)$ is a unit normal vector to g_s at x. In fact, the Weingarten formula for g, thought as an isometric immersion with codimension 2 into the $\mathbb{E}^{n+1}$ if $\varepsilon \in \{1, -1\}$,

$$N_* = -g_*A_N,$$

implies in

$$(g_s)_*X = C_\varepsilon(s)g_*X - S_\varepsilon(s)g_*A_NX, \tag{3.13}$$

then

$$\begin{aligned}\langle N_s(x), (g_s)_*X\rangle =& \langle -\varepsilon S_\varepsilon(s)g(x), C_\varepsilon(s)g_*X\rangle + \langle \varepsilon S_\varepsilon(s)g(x), S_\varepsilon(s)g_*A_NX\rangle \\ &+ \langle C_\varepsilon(s)N(x), C_\varepsilon(s)g_*X\rangle - \langle C_\varepsilon(s)N(x), S_\varepsilon(s)g_*A_NX\rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\langle N_s(x), N_s(x)\rangle =& \langle \varepsilon S_\varepsilon(s)g(x), \varepsilon S_\varepsilon(s)g(x)\rangle - \langle \varepsilon S_\varepsilon(s)g(x), C_\varepsilon(s)N(x)\rangle \\ &- \langle C_\varepsilon(s)N(x), \varepsilon S_\varepsilon(s)g(x)\rangle + \langle C_\varepsilon(s)N(x), C_\varepsilon(s)N(x)\rangle = 1.\end{aligned}$$

Also

$$\eta(x, s) = -\frac{a'(s)}{b(s)}N_s(x) + \frac{1}{b(s)}\frac{\partial}{\partial t}, \tag{3.14}$$

where $b(s) = \sqrt{1+a'(s)^2}$, is a unit normal vector to f at (x, s). In fact,

$$\langle \eta(x, s), \eta(x, s)\rangle = \left\langle \frac{a'(s)}{b(s)}N_s(x), \frac{a'(s)}{b(s)}N_s(x)\right\rangle + \frac{1}{b(s)^2} = 1,$$

and

$$\begin{aligned}\left\langle \eta, f_*X + rf_*\frac{\partial}{\partial s}\right\rangle &= \left\langle -\frac{a'(s)}{b(s)}N_s(x) + \frac{1}{b(s)}\frac{\partial}{\partial t}, (g_s)_*X\right\rangle - r\frac{a'(s)}{b(s)}\left\langle N_s(x), f_*\frac{\partial}{\partial s}\right\rangle \\ &\quad + \frac{r}{b(s)}\left\langle \frac{\partial}{\partial t}, f_*\frac{\partial}{\partial s}\right\rangle = 0 - r\frac{a'(s)}{b(s)} + r\frac{a'(s)}{b(s)} = 0.\end{aligned} \tag{3.15}$$

Notice that

$$D_{\frac{\partial}{\partial s}} N_s(x) = -\varepsilon S_\varepsilon(s) D_{\frac{\partial}{\partial s}} g(x) - \varepsilon S'_\varepsilon(s) g(x) + C_\varepsilon(s) D_{\frac{\partial}{\partial s}} N(x) + C'_\varepsilon(s) N(x) = -\varepsilon g_s(x),$$

and (3.1) implies that

$$D_Z\frac{\partial}{\partial t} = \tilde{\nabla}_Z\frac{\partial}{\partial t} - \varepsilon\langle Z_1, 0\rangle\xi = 0 \tag{3.16}$$

for every Z tangent vector to $M^{n-1}\times I$. Thus,

$$D_{\frac{\partial}{\partial s}}\eta = -\left[\frac{a'(s)}{b(s)}\right]' N_s(x) + \varepsilon\frac{a'(s)}{b(s)}g_s(x) + \left[\frac{1}{b(s)}\right]'\frac{\partial}{\partial t}.$$

For that reason,

$$\left\langle D_{\frac{\partial}{\partial s}}\eta, f_*X\right\rangle = -\left\langle \left[\frac{a'(s)}{b(s)}\right]' N_s, (g_s)_*X\right\rangle + \left\langle \left[\frac{1}{b(s)}\right]'\frac{\partial}{\partial t}, (g_s)_*X\right\rangle + \left\langle \varepsilon\frac{a'(s)}{b(s)}g_s, (g_s)_*X\right\rangle = 0,$$

so

$$\left\langle A_\eta\frac{\partial}{\partial s}, X\right\rangle = -\langle D_{\partial/\partial s}\eta, f_*X\rangle = 0,$$

for any $X \in T_xM$, i.e., $\frac{\partial}{\partial s}$ is a principal direction since $\langle f_*X, f_*\frac{\partial}{\partial s}\rangle = 0$. Also, since $\nu = \frac{1}{b(s)}$, we have

$$f_*T = \frac{\partial}{\partial t} - \frac{1}{b(s)}\eta = \frac{a'(s)}{b(s)^2}\left(N_s(x) + a'(s)\frac{\partial}{\partial t}\right) = f_*\left(\frac{a'(s)}{b(s)^2}\frac{\partial}{\partial s}\right),$$

and so

$$T = \frac{a'(s)}{b(s)^2}\frac{\partial}{\partial s} \tag{3.17}$$

is a principal direction. □

Let us now explicit the set of regular points of the map f.

Remark 3.7. The regularity of f at (x, s) is equivalent to the regularity of g_s at x. Thus let us study when the last situation happens. Let $\lambda \in \mathbb{R}$ represents an arbitrary principal curvature of g at x with respect to N_x. If $\varepsilon = 0$ and $\lambda \neq 0$, there is a unique $0 \neq \theta \in \mathbb{R}$ such that $\lambda = 1/\theta$. When $\varepsilon = -1$ and $\lambda \notin [-1, 1]$, there exists a unique $0 \neq \theta \in \mathbb{R}$ such that $\coth\theta = \lambda$. Also, there is exactly one $\theta \in (0, \pi)$ such that $\cot\theta = \lambda$ if $\varepsilon = 1$. Consequently, there exists an strictly increasing finite sequence $\theta_1, \ldots, \theta_{m(\varepsilon)}$ of real numbers such that $\lambda_1, \ldots, \lambda_{m(\varepsilon)}$ given for $i \in \{1, \ldots, m(\varepsilon)\}$ by

$$\lambda_i = \begin{cases} \cot\theta_i & , 0 < \theta_i < \pi, \text{ if } \varepsilon = 1 \\ 1/\theta_i & , \theta_i \neq 0, \text{ if } \varepsilon = 0 \\ \coth\theta_i & , \theta_i \neq 0, \text{ if } \varepsilon = -1. \end{cases}$$

are all the (not neccesarily ordered) principal curvatures of g at x with respect to N_x, distinct from zero if $\varepsilon = 0$, and not belonging to $[-1, 1]$ if $\varepsilon = -1$. If $X \in T_x(M^{n-1})$ belongs to the eigenspace of $A_{N(x)}$ associated with λ_i, then

$$(g_s)_*X = \begin{cases} \frac{\sin(\theta_i - s)}{\sin\theta_i} g_*X & , \text{if } \varepsilon = 1, \\ \frac{\theta_i - s}{\theta_i} g_*X & , \text{if } \varepsilon = 0, \\ \frac{\sinh(\theta_i - s)}{\sinh\theta_i} g_*X & , \text{if } \varepsilon = -1. \end{cases}$$

In fact, equation (3.13) yields $(g_s)_*X = (C_\varepsilon(s) - \lambda_i S_\varepsilon(s))g_*X$, where, for $\varepsilon = -1$, $C_\varepsilon(s) - \lambda_i S_\varepsilon(s) = \cosh s - \coth\theta_i \sinh s = \sinh(\theta_i - s)/\sinh\theta_i$; similarly for $\varepsilon \in \{0, 1\}$. For $\varepsilon = -1$, g_s is an immersion at x as long as

$$0 \neq \frac{\sinh(\theta_i - s)}{\sinh(\theta_i)} \text{ for } i = 1, \ldots, m,$$

i.e., $s \neq \theta_i$ for $i = 1, \ldots, m$. Since the argument for $\varepsilon \in \{0, 1\}$ is similar, we have that g_s is an immersion at x if and only if $s \notin \{\theta_1, \ldots, \theta_m\}$ (respectively, $s \neq \theta_i + \pi\mathbb{Z}$ for $i = 1, \ldots, m$) if $\varepsilon \in \{0, -1\}$ (respectively, $\varepsilon = 1$). Thus, the set of regular points of the smooth map f is given by

$$U = \begin{cases} (x, s) \in M \times \mathbb{R} & : s \notin \{\theta_i + \pi\mathbb{Z} : i = 1, \ldots, m\}, \text{ if } \varepsilon = 1, \\ (x, s) \in M \times \mathbb{R} & : s \notin \{\theta_1, \ldots, \theta_m\}, \text{ if } \varepsilon \in \{-1, 0\}. \end{cases}$$

A geometric picture of the set U is provided by Figure 1, where the dashed lines do not belong to the set U. Therefore, if $V \subset M^{n-1}$, connected open set, is given so that $V \times I \subset U$, g_s is an immersion on V for every $s \in I$, and so is f on $V \times I$.

Remark 3.8. I should mention that the definition of the set U given here differs from that provided in Tojeiro (2010, Remark 6.(i)), though this does not represent any problem as all points belonging to the original set U have been included in the current U. In fact, the initial choice (a component of the current set U) was made with the objective of having a connected set U.

The following theorem shows that the (partial) converse of Theorem 3.6 is true.

Theorem 3.9. Any hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $n \geq 2$, for which T is a principal direction and $\nu \neq 0$ everywhere, is locally of the form (3.10).

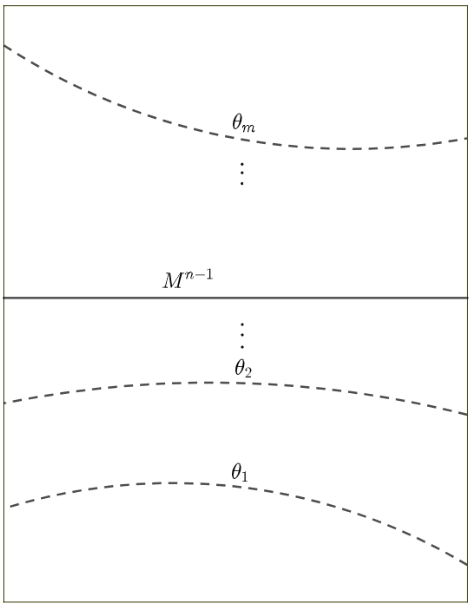

Figure 1 – The set U.

Proof. If $\nu < 0$, we replace the unit nomal η of f by $-\eta$, so that now $\nu > 0$. Thus, we can assume without lose of generality that $\nu > 0$. Thanks to Lemma 3.5, the orthogonal distribution $\{T\}^{\perp}$ is integrable. Also, the distribution $\{T\}$ is integrable as each $S \in \{T\}$ is of the form $S = aT$, $a \in C^{\infty}(M)$. Thus, because $TM = \{T\}^{\perp} \oplus \{T\}$ and Dajczer and Tojeiro (2019, p. -59), there is a diffeomorphism $\psi : M^{n-1} \times I \to M^n$ onto $\psi(M^{n-1} \times I)$, where $I \subset \mathbb{R}$ is an open interval containing 0, such that $\psi(x, \cdot) : I \longrightarrow M^n$ is an integral curve of T for each $x \in M^{n-1}$, and $\psi(\cdot, r) : M^{n-1} \longrightarrow M^n$ is an integral submanifold of $\{T\}^{\perp}$ for each $r \in I$. In particular, $\psi_* X \in \{T\}^{\perp}$ for any $X \in T_x M^{n-1}$. Set $F = f \circ \psi$. Then for every tangent vector $X \in TM^{n-1}$

$$X\left\langle F, \frac{\partial}{\partial t}\right\rangle = \langle f_* \psi_* X, f_* T + \nu\eta\rangle = \langle \psi_* X, T\rangle = 0,$$

and so $\rho(s) = \langle F(x,s), \frac{\partial}{\partial t}\rangle = h(\psi(x,s))$ is a smooth real-valued function on I. For every $X \in \{T\}^{\perp}$, we get $X(\nu) = 0$ from (3.7), then it also holds $X(\|T\|) = 0$ since $\|T\|^2 + \nu^2 = 1$, which implies that every integral curve of T is a pre-geodesic of M by Lemma 3.4. The pre-geodesic $\gamma(r) = \psi(x,r)$, $x \in M^{n-1}$ fixed, has velocity vector $\gamma' = T(\gamma)$ and arc-lenght function $s(r) = \int_0^r \|T(\gamma)\|\, d\rho$, which is a diffeomorphism. If $r(s)$ denotes the inverse function of $s(r)$, then

$$\frac{dr}{ds} = \left(\frac{ds}{dr}(r(s))\right)^{-1} = \|T(\gamma(r(s)))\|^{-1},$$

and the arclenght reparametrization $\gamma(r(s))$ of γ has velocity vector $T(\gamma(r(s)))/\|T(\gamma(r(s)))\|$, which we indicate by $\hat{T}(\gamma(r(s)))$. We claim that $\alpha := \Pi \circ f \circ \psi(x, \cdot) : I \to \mathbb{Q}^n_{\varepsilon}$ is a pre-geodesic of $\mathbb{Q}^n_{\varepsilon}$ for any $x \in M^{n-1}$, where $\Pi : \mathbb{Q}^n_{\varepsilon} \times \mathbb{R} \to \mathbb{Q}^n_{\varepsilon}$ is the projection onto the first factor. In fact, the velocity vector is given by

$$\alpha'(r) = (\Pi_1)_* f_* T(\gamma(r)) = f_* T(\gamma(r)) - \left\langle f_* T(\gamma(r)), \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}.$$

Also

$$\langle f_*T(\gamma(r)), f_*T(\gamma(r))\rangle = \langle T(\gamma(r)), T(\gamma(r))\rangle$$

and

$$\left\langle f_*T(\gamma(r)), \frac{\partial}{\partial t}\right\rangle = \langle T(\gamma(r)),\, T(\gamma(r))\rangle,$$

implies in

$$\|\alpha'(r)\|^2 = \|f_*T(\gamma(r))\|^2 - \left\| \left\langle f_*T(\gamma(r)), \frac{\partial}{\partial t}\right\rangle \right\|^2 = \|T(\gamma(r))\|^2\nu(\gamma(r))^2,$$

that is, $\lambda(r) := \|\alpha'(r)\| = \|T(\gamma(r))\|\nu(\gamma(r)) > 0$, which implies that the arc lenght function of α given by $s(r) = \int_0^r \lambda(\rho)\,d\rho$, is a diffeomorphism. If $r(s)$ denotes the inverse function of $s(r)$, then

$$\frac{dr}{ds} = \left[\frac{ds}{dr}(r(s))\right]^{-1} = (\lambda(r(s)))^{-1}.$$

Thus, the arc lenght reparametrization $\alpha(r(s))$ of α has a velocity vector given by

$$\frac{dr}{ds}\frac{d\alpha}{dr}(r(s)) = (\lambda(r(s)))^{-1}\alpha'(r(s)).$$

For the sake of clarity, we temporarily avoid writing the arguments of $T(\gamma(r(s)))$, $\alpha'(r(s))$, $\nu(\gamma(r(s)))$ and $\lambda(r(s))$, and thus just write T, α', ν and λ, respectively. Now, using (3.1), we have

$$D_{T(\gamma(r))}\lambda^{-1}\alpha' = \tilde{\nabla}_{T(\gamma(r))}\lambda^{-1}\alpha' - \varepsilon\langle\alpha', \lambda^{-1}\alpha'\rangle\xi.$$

Then, for $\varepsilon \in \{-1, 1\}$, if we prove that $D_{T(\gamma(r))}\lambda^{-1}\alpha'$ is in the direction of ξ, we will obtain

$$0 = \tilde{\nabla}_{T(\gamma(r))}\lambda^{-1}\alpha' = \nabla^1_{\alpha'}\lambda^{-1}\alpha',$$

that is, $\alpha(r(s))$ is a geodesic, where ∇^1 is the Levi-Civita connection of $\mathbb{Q}^n_\varepsilon$. Therefore, let us compute

$$D_T\lambda^{-1}\alpha' = D_T\left(f_*(\lambda^{-1}T) - \left\langle f_*(\lambda^{-1}T), \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}\right). \tag{3.18}$$

First, notice that

$$D_T f_*(\lambda^{-1}T) = D_T\,\nu^{-1}f_*\hat{T} = T(\nu^{-1})f_*\hat{T} + \nu^{-1}D_T f_*\hat{T}, \tag{3.19}$$

where $\widehat{T} = T/|T|$. Now, since $\widehat{T}$ is a parallel vector field along γ (see (3.9)),

$$D_T f_*\hat{T} = \nabla_T\hat{T} + \alpha^{i\circ f}(T, \hat{T}) = \langle AT, \hat{T}\rangle\eta + \langle A_\xi T, \hat{T}\rangle\xi = \nu^{-1}T(\|T\|)\eta - \nu^2\|T\|\xi, \tag{3.20}$$

where the third equality holds for (3.8) and the following

$$0 = T(\|T\|^2 + \nu^2) = 2(\|T\|T(\|T\|) + \nu(T\nu)), \tag{3.21}$$

$$\langle AT, \hat{T}\rangle = -\|T\|^{-1}T(\nu) = \nu^{-1}T(\|T\|), \tag{3.22}$$

where we have used (3.7). Now

$$0 = T(\nu\nu^{-1}) = \nu T(\nu^{-1}) + \nu^{-1}T\nu \implies -\nu^{-2}T\nu = T(\nu^{-1}),$$

and (3.21) implies in

$$T(\nu^{-1}) = -\nu^{-3}\nu T(\nu) = \nu^{-3}\|T\|T(\|T\|),$$

which in turn together with (3.19) as well as (3.20) establish

$$\begin{aligned} D_T f_*(\lambda^{-1}T) &= \nu^{-3}\|T\|T(\|T\|)f_*\tilde{T} + \nu^{-2}T(\|T\|)\eta - \nu\|T\|\xi \\ &= \nu^{-3}T(\|T\|)\frac{\partial}{\partial t} - \nu\|T\|\xi. \end{aligned}$$

On the other hand, as $\langle f_*(\lambda^{-1}T), \frac{\partial}{\partial t}\rangle = \nu^{-1}\|T\|$, we have

$$T(\nu^{-1}\|T\|) = T(\nu^{-1})\|T\| + \nu^{-1}T(\|T\|) = \nu^{-3}T(\|T\|)(\|T\|^2+\nu^2) = \nu^{-3}T(\|T\|).$$

Using also (3.16), it follows that

$$D_T(f_*(\lambda^{-1}T) - \langle f_*(\lambda^{-1}T), \frac{\partial}{\partial t}\rangle \frac{\partial}{\partial t}) = -\nu\|T\|\xi, \tag{3.23}$$

and so $\alpha(r(s))$ is a geodesic of $\mathbb{Q}^n_\varepsilon$. We claim that

$$g := \Pi \circ f \circ \psi(\cdot, 0) : M^{n-1} \to \mathbb{Q}^n_\varepsilon,$$

is a hypersurface. In fact, g is smooth and g_* is injective, because $\langle f_*(\psi(\cdot,0))_*X, \partial/\partial t\rangle = 0$ which implies $g_*X = (\pi_1)_*f_*(\psi(\cdot,0))_*X = f_*(\psi(\cdot,0))_*X$; and g is also isometric since $\langle X, Y\rangle = \langle f_*(\psi(\cdot,0))_*X, f_*(\psi(\cdot,0))_*Y\rangle = \langle g_*X, g_*Y\rangle$. Also, the set of velocity vectors

$$\left.\frac{d}{ds}\right|_{s=0}(\Pi \circ f \circ \psi)(x, r(s)), x \in M^{n-1}$$

yield a unit normal to g, so the existence of the parallel hypersurface g_s is guaranteed. Therefore, $g_s(x) = \Pi \circ f \circ \psi(x, r(s))$. Also, set

$$a(s) := \left\langle f \circ \psi(x, r(s)), \frac{\partial}{\partial t} \right\rangle = \rho(r(s)),$$

which is then a smooth function on $J := s(I)$. Thus, we have

$$f(\psi(x, r(s)) = g_s(x) + a(s)\frac{\partial}{\partial t}, \tag{3.24}$$

for any $(x, s) \in M^{n-1} \times J$. From (3.17) we obtain $a'(s) \neq 0$. If $a' < 0$, let $s(\sigma)$ be the diffeomorphism that 'walks' over J from the bigger end point to the smaller end point, so that $ds/d\sigma < 0$. Then, replacing s by $s(\sigma)$ in the expressions of g_s and $a(s)$, we will obtain now $da/d\sigma > 0$. □

Remark 3.10. A fundamental step for the proof of Theorem 3.9 is to prove that $\alpha := \Pi \circ f \circ \psi(x, \cdot)$ is a pre-geodesic of $\mathbb{Q}^n_\varepsilon$. Even though we have only presented the proof to that fact for the cases $\varepsilon \in \{-1, 1\}$, it also holds for $\varepsilon = 0$. In fact, omitting the terms involving ξ from (3.19) to (3.23), it is demonstrated that the covariant derivative in (3.18) is identically zero, i.e., α is likewise a geodesic of $\mathbb{Q}^n_\varepsilon$ for $\varepsilon = 0$.

Remark 3.11. The hypersurface f can be geometrically interpreted: first, assume $\varepsilon \in \{1,-1\}$ and regard g as an isometric immersion with codimension 3 of M^{n-1} into $\mathbb{E}^{n+2}$ by composing g first with the inclusion $i : \mathbb{Q}^n_\varepsilon \hookrightarrow \mathbb{E}^{n+1}$, and then with the inclusion $j : \mathbb{E}^{n+1} \hookrightarrow \mathbb{E}^{n+2}$. Thus, for a given $x \in M^{n-1}$, the normal space $N_g M(x)$ is a Lorentzian or Riemannian vector space of dimension 3, according as $\varepsilon = -1$ or $\varepsilon = 1$, respectively, spanned by $g(x), N(x)$ and $\frac{\partial}{\partial t}$. In fact, for $\varepsilon = -1$, span$\{g(x)\}$ is a 1-dimensional subspace restricted to which the inner product of $N_g M(x)$ is negative definite; and span$\{N_x, \frac{\partial}{\partial t}\}$ is a 2-dimensional subspace of $N_g M(x)$ restricted to which the inner product is positive definite. The analogous statement for $\varepsilon = 1$ is evident. Also, the sections g, N and $\partial/\partial t$ of $N_g M$ are parallel, as the following computations show,

$$\begin{aligned}
&{}^{j\circ i\circ g}\nabla^{\perp}_X N = {}^{i\circ g}\,\nabla^{\perp}_X N = -\langle X, N\rangle g(x) = 0,\\
&{}^{j\circ i\circ g}\nabla^{\perp}_X g = {}^{i\circ g}\,\nabla^{\perp}_X g = 0,\\
&{}^{j\circ i\circ g}\nabla^{\perp}_X \frac{\partial}{\partial t} = 0.
\end{aligned}$$

On the one hand, for each $x \in M$,

$$f(x,s) = C_\varepsilon(s) g(x) + S_\varepsilon(s) N_x + a(s)\frac{\partial}{\partial t} \tag{3.25}$$

can be thought as a smooth curve $f(x,\cdot)$ in a cylinder $\mathbb{Q}^1_\varepsilon \times \mathbb{R} \subset N_g M(x)$—it is enough to notice that $C_\varepsilon(s)g(x) + S_\varepsilon(s)N_x \in \mathbb{Q}^1_\varepsilon \subset \mathbb{E}^2$. On the other hand, for $s \in I$ fixed, $f(\cdot,s)$ can be thought as a curve in $N_g M$. Since, $C_\varepsilon(s)g + S_\varepsilon(s)N + a(s)\frac{\partial}{\partial t}$ is a parallel vector field along $f(\cdot,s)$, f is obtained by parallel transporting the curve $f(x,\cdot)$, $x \in M^{n-1}$, in $N_g M$. Now for $\varepsilon = 0$, regard g as an isometric immersion with codimension 2 of M^{n-1} into $\mathbb{R}^{n+1}$ by composing g with the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}$. In this case, the map f in (3.25) simplifies its aspect to

$$f(x,s) = g(x) + sN(x) + a(s)\frac{\partial}{\partial t},$$

and $N_g M(x)$, $x \in M$, is a Riemannian vector space of dimension 2, spanned by $N(x)$ and $\frac{\partial}{\partial t}$. Also, as before, N and $\frac{\partial}{\partial t}$, sections of $\Gamma(N_g M)$, are parallel. On the one hand, for $x \in M$, $f(x,\cdot)$ can be thought as a curve in the normal space $N_g M(x)$. On the other hand, for $s \in I$, $f(\cdot,s)$ can be thought as a curve in $N_g M$ along of which $g + sN + a(s)\frac{\partial}{\partial t}$ is parallel. Therefore, f is obtained by parallel transporting this curve $f(x,\cdot)$ in $N_g M$.

As a consequence of theorems 3.6 and 3.9, we obtain a complete description of *constant angle hypersurfaces* $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ — hypersurfaces whose angle function is constant.

Corollary 3.12. Given an open interval $I \subset \mathbb{R}$ and $A, B \in \mathbb{R}, A > 0$, set $a(s) = As + B$, $s \in I$. Then the hypersurface given by (3.10), restricted to regular points, has constant angle function. Conversely, any constant angle hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is either an open subset of some slice $\mathbb{Q}^n_\varepsilon \times \{t_0\}$, an open subset of $M^{n-1} \times \mathbb{R}$, where $M^{n-1} \hookrightarrow \mathbb{Q}^n_\varepsilon$ is a hypersurface, or it is locally given by (3.10) with $a(s) = As + B$, $s \in I$, for some $A, B \in \mathbb{R}, A > 0$.

Proof. Let us first show that f has constant angle function. As $a(s) = As$, we have $a'(s) = A$ and the unit normal η to f, determined by (3.14), becomes

$$\eta(x,s) = -\frac{A}{\sqrt{1+A^2}} N_s(x) + \frac{1}{\sqrt{1+A^2}} \frac{\partial}{\partial t},$$

and thus the angle function

$$\theta(x,s) = \left\langle \eta, \frac{\partial}{\partial t} \right\rangle = \frac{1}{\sqrt{1+A^2}},$$

is constant. Conversely, if $\nu = 0$, $f(M^n) \subset M^{n-1} \times \mathbb{R}$ is an open subset, where $M^{n-1} \hookrightarrow \mathbb{Q}^n_\varepsilon$ is a hypersurface. Also, when $\nu = 1$, $f(M^n) \subset \mathbb{Q}^n_\varepsilon \times \{t_0\}$ is an open subset of a slice for some $t_0 \in \mathbb{R}$ (MANFIO; TOJEIRO, 2011, Proposition 2.1). Now, for $\nu \notin \{0,1\}$, $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is a hypersurface with nowhere vanishing angle function and $T \neq 0$ everywhere. In addition, $0 = \langle AT, X \rangle$ holds from (3.7) for any $X \in TM$, which yields $AT = 0 = 0 \cdot T$, i.e., T is a principal direction. Thus, thanks to Theorem 3.9, f is locally given by (3.10), and

$$A := a'(s) = \sqrt{b(s)^2 - 1} = \sqrt{\nu^{-2} - 1}$$

is a positive real number. Consequently, $a(s) = As + B$, for some $B \in \mathbb{R}$. □

CHAPTER

4

EMBEDDEDNESS, CONVEXITY AND RIGIDITY OF HYPERSURFACES IN $\mathscr{H}^n \times \mathbb{R}$ AND $\mathbb{S}^n \times \mathbb{R}$

In this chapter, we present a result, Theorem 4.10, which is a generalisation to hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$, where $\mathscr{H}$ represents an arbitrary Cartan-Hadamard manifold, of the well-known Hadamard-Stoker Theorem for complete surfaces immersed in $\mathbb{R}^3$. Also, it is shown, Theorem 4.13, that these hypersurfaces are rigid among the hypersurfaces with the same extrinsic curvature. Both of these results, as well as another ones in this chapter are due to Lima (2021).

4.1 Preliminaries

A complete, simply connected n-dimensional Riemannian manifold with nonpositive sectional curvature is called a *Cartan-Hadamard manifold*, and denoted by $\mathscr{H}^n$. Euclidean and hyperbolic spaces are simple examples. We shall remember that every Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$. Consequently, given two arbitrary distinct points $x, y \in \mathscr{H}$, there exists a unique geodesic γ_{xy} of $\mathscr{H}$ joining x and y: in fact, because $\exp_x : T_x\mathscr{H} \to \mathscr{H}$ is a diffeomorphism, there is a unique $w \in T_x\mathscr{H}$ such that $y = \exp_x w = \gamma_w(1)$; then it is sufficient to set $\gamma_{xy} := \gamma_w$. A subset C of $\mathscr{H}$ is said to be *convex* when $\gamma_{xy} \subset C$ for any two different points $x, y \in C$.

In Section 3.1 we introduced some ideas for the product manifold $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$: projections on both factors, decomposition of vector fields $Z \in \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ as a pair of horizontal and vertical vector fields (Z_1, Z_2), and the vector field $\frac{\partial}{\partial t}$. Since all of these ideas are amenable to generalisation to arbitrary product manifolds $\tilde{M} \times \mathbb{R}$, we will use them throughout this chapter.

Given $t \in \mathbb{R}$, the embedded submanifold $\tilde{M}_t := \tilde{M} \times \{t\} \subset \tilde{M} \times \mathbb{R}$ will be called the *slice* of $\tilde{M} \times \mathbb{R}$ at level t. Every slice $\tilde{M}_t$ is a totally geodesic submanifold of $\tilde{M} \times \mathbb{R}$ isometric to $\tilde{M}$, so we identify the Levi-Civita connections of both $\tilde{M}$ and an arbitrary slice $\tilde{M}_t$, as well as use the symbol ∇^1 to denote both of them. A geodesic of $\tilde{M} \times \mathbb{R}$ contained in a slice is called *horizontal* while one that is tangent to $\frac{\partial}{\partial t}$ is called *vertical*.

Given a hypersurface $f : M^n \to \tilde{M}^n \times \mathbb{R}$ that admits a unit normal N, for all $X, Y \in \mathfrak{X}(M)$, we have

$$\langle \alpha(X,Y), N \rangle = \langle AX, Y \rangle = -\langle \tilde{\nabla}_X N, Y \rangle = \langle \tilde{\nabla}_X Y, N \rangle. \tag{4.1}$$

We shall remember (Section 3.2) that the vector field T is a gradient vector field. Thus, from (3.2), a point $x \in M$ is a critical point of h as long as $N(x) = \pm\frac{\partial}{\partial t}$, or equivalently $\nu(x) = \pm 1$. It also follows from (3.2) that

$$\tilde{\nabla}_X \operatorname{grad} h = -\nu \tilde{\nabla}_X N - (X\nu) N$$

for $X \in \mathfrak{X}(M)$. Because of this and (4.1), for $X, Y \in \mathfrak{X}(M)$,

$$\operatorname{Hess} h(X,Y) = \langle \tilde{\nabla}_X \operatorname{grad} h, Y \rangle = -\langle \nu \tilde{\nabla}_X N, Y \rangle = \nu \langle \alpha(X,Y), N \rangle. \tag{4.2}$$

Given an open subset $\Omega \subset M$, an integral curve of $\operatorname{grad} h$ in Ω is a smooth curve $\varphi : I \to \Omega$ such that $\varphi' = \operatorname{grad} h(\varphi)$ on I. If $\bar{\Omega}$ contains only one critical point of h, then either

$$\lim_{r \to -\infty} \varphi(r) = x_0 \quad \text{or} \quad \lim_{r \to +\infty} \varphi(r) = x_0$$

according as x_0 is a local minimum or a local maximum, respectively; in the former case it is said that φ is issuing from x_0 and in the latter one that φ is going into x_0 (LIMA, 2021, Section 2.1).

Note that for $X \in TM$,

$$X\nu = \left\langle \tilde{\nabla}_X N, \frac{\partial}{\partial t} \right\rangle = -\langle AX, \operatorname{grad} h \rangle,$$

which in turn implies that

$$\operatorname{grad} \nu = -A \operatorname{grad} h. \tag{4.3}$$

Next, we define the notion of a top or bottom end for a complete hypersurface $f : M^n \to \tilde{M}^n \times \mathbb{R}$.

Definition 4.1. A complete hypersurface $f : M^n \to \tilde{M}^n \times \mathbb{R}$ is said to have a *top end* $E \subset M$ (respectively, *bottom end*) if E is unbounded, and $h(x_k) \to +\infty$ (respectively, $h(x_k) \to -\infty$) for every divergent sequence $(x_k)_{k=1}^{\infty}$ in E.

Another key concept present in the proof of Theorem 4.10 is that of horizontal sections of a hypersurface. If a hypersurface $f : M^n \to \tilde{M}^n \times \mathbb{R}$ is transversal to a slice $\tilde{M}_t$ of $\tilde{M} \times \mathbb{R}$, in which case $f^{-1}(\tilde{M}_t) = h^{-1}(t)$ is an embedded $(n-1)$-dimensional submanifold of M, we call the restriction $f_t := f|_{M_t} : M_t \to \tilde{M}_t$, where M_t is a component of $h^{-1}(t)$, a *horizontal section* of f at level t. It follows immediately that horizontal sections are hypersurfaces of the corresponding slices from the fact that f is a hypersurface. Any horizontal section is also oriented by the unit normal given by

$$\eta = \frac{N - \nu\partial/\partial t}{\|N - \nu\partial/\partial t\|} = \frac{N - \nu\partial/\partial t}{\sqrt{1-\nu^2}}. \tag{4.4}$$

Furthermore, if $x_0 \in M$ is a critical point of the height function of f, a *normal section* for x_0 is a horizontal section $f_t : M_t \to \bar{M}_t$ of f at a level $t > 0$ which satisfies the following two conditions

- M_t is homeomorphic to $\mathbb{S}^{n-1}$ and the boundary of an open set $\Omega_t \subset M$, called *normal region* for x_0, such that the only critical point of h in Ω_t is x_0.
- There is a homeomorphism $\bar{B} \to \bar{\Omega}_t$ taking ∂B into M_t, where $B \subset \mathbb{R}^n$ is an open ball.

The level t is called *normal value* for x_0.

We shall remember that the (vertical) graph $\Gamma(u)$ of a smooth function u defined on some open set $D \subset \tilde{M}$, is an embedded n-dimensional hypersurface of $\tilde{M} \times \mathbb{R}$. It admits a unit normal given via

$$N = \frac{-\operatorname{grad} u + \partial/\partial t}{\sqrt{1 + \|\operatorname{grad} u\|^2}}, \tag{4.5}$$

where $\operatorname{grad} u := \operatorname{grad} u(\pi)$, and $\pi : \tilde{M} \times \mathbb{R} \to \tilde{M}$ is the projection onto the first factor. In fact, it is unit as $\| - \operatorname{grad} u + \partial/\partial t\|^2 = 1 + \|\operatorname{grad} u\|^2$. If $\phi(p) = (p, u(p))$, $p \in D$, are the graph coordinates on $\Gamma(u)$, then for any $Z \in T_{\phi(p)}\Gamma(u)$, there exists a unique $X \in T_p\tilde{M}$ such that $Z = \phi_* X = X + Xu$. Thus,

$$\langle N, Z \rangle = \frac{1}{\sqrt{1 + \|\operatorname{grad} u\|^2}}(-Xu + Xu) = 0,$$

i.e., N is indeed normal to $\Gamma(u)$. And, the angle function is

$$\nu = \frac{1}{\sqrt{1 + \|\operatorname{grad} u\|^2}}. \tag{4.6}$$

Also, from (3.2), (4.5) and (4.6) the horizontal component of $\operatorname{grad} h$ is parallel to $\operatorname{grad} u$:

$$\operatorname{grad} h = \frac{\partial}{\partial t} - \nu N = \frac{1}{1 + \|\operatorname{grad} u\|^2}\operatorname{grad} u + \frac{\|\operatorname{grad} u\|^2}{1 + \|\operatorname{grad} u\|^2}\frac{\partial}{\partial t}.$$

Consequently, the projection $\pi \circ \gamma$ of an integral curve γ of $\operatorname{grad} h$ onto $\tilde{M}$ is parallel to $\operatorname{grad} u$ — in fact, $(\pi \circ \gamma)' = \pi_* \operatorname{grad} h$ — and thus orthogonal to all level sets $\Sigma_t := u^{-1}(t)$, $t \in u(D)$, of u.

In order to consider the right-hand side of the Gauss Equation (2.23), excluding $\tilde{K}(X,Y)$, we define a linear operator A_{XY} on $\operatorname{span}\{X,Y\}$ determined for orthonormal tangent vectors X, Y to M at $x \in M$ by

$$A_{XY} := \pi_{XY} A|_{\operatorname{span}\{X,Y\}},$$

where π_{XY} is the projection of $T_x M$ to $\operatorname{span}\{X, Y\}$.

Proposition 4.2. A_{XY} is a linear operator on span$\{X,Y\}$ with determinant

$$\langle AX,X\rangle\langle AY,Y\rangle - \langle AX,Y\rangle^2.$$

Proof. By Gram-Schmidt algorithm, we can obtain an orthonormal basis $\{X_1,\dots,X_n\}$ for T_xM, where $X_1 = X$ and $X_2 = Y$. Any $Z \in T_xM$ is thus written as $Z = \langle Z, X_1\rangle X_1 + \cdots + \langle Z, X_n\rangle X_n$. Then, π_{XY} is given by

$$\pi_{XY}Z = \langle Z, X\rangle X + \langle Z, Y\rangle Y, \tag{4.7}$$

which implies in π_{XY} being linear, and so is A_{XY}. The matrix representation of A_{XY} with respect to the basis $\{X,Y\}$ is

$$\begin{pmatrix} \langle AX, X\rangle & \langle AY, X\rangle \\ \langle AX, Y\rangle & \langle AY, Y\rangle \end{pmatrix}$$

which yields the remaining assertion. □

Let $\lambda_1,\dots,\lambda_n$ denote the principal curvatures of f at $x \in M$, and let $X_1,\dots,X_n$ be an orthonormal basis of principal directions for T_xM so that $AX_i = \lambda_i X_i$ for every $i \in \{1,\dots,n\}$. It holds

$$\lambda_i \geq c \geq 0 \quad \forall i \in \{1,\dots,n\} \Rightarrow \det A_{XY} \geq c^2 \quad \text{for all orthonormal vectors } X,Y \in T_xM,$$

where the last inequality is strict if the first one is as well (LIMA, 2021, equation 9).

4.2 Asymptotic rays

If $\mathscr{H}^n$ is a Cartan-Hadamard manifold, so is the Riemannian product $\mathscr{H}^n \times \mathbb{R}$. A *geodesic ray* in $\mathscr{H}^n$ is a geodesic of the form $\gamma\colon [0,+\infty) \to \mathscr{H}^n$. Two geodesic rays $\gamma, \sigma\colon [0,+\infty) \to \mathscr{H}^n \times \mathbb{R}$ parametrized by arc length are said to be *asymptotic* if there exists $c \in \mathbb{R}$ such that $d(\gamma(r), \sigma(r)) \leq c$ for all $r \geq 0$, where d represents the Riemannian distance function on $\mathscr{H}^n \times \mathbb{R}$. A remarkable property is that given $p \in \mathscr{H}^n \times \mathbb{R}$ and a geodesic ray γ in $\mathscr{H}^n \times \mathbb{R}$ parametrized by arc length, there exists a unique geodesic ray σ_p parametrized by arc length asymptotic to γ with $\sigma_p(0) = p$ (JOST, 2011, Lemma 5.8.11). Also the map $p \mapsto \sigma_p'(0)$ is a (continuous) vector field on $\mathscr{H}^n \times \mathbb{R}$.

Proposition 4.3. Let $\mathscr{H}^n$ be a Cartan-Hadamard manifold. Then, being asymptotic is an equivalence relation on the class of all geodesic rays parametrized by arc length in $\mathscr{H}$.

Proof. Every geodesic ray γ parametrized by arc length is asymptotic to itself as $d(\gamma(s), \gamma(s)) \equiv 0$. Let γ, σ, and β be geodesic rays parametrized by arc length in $\mathscr{H}$. If γ is asymptotic to σ, there exists $c \in \mathbb{R}$ such that $d(\gamma(s), \sigma(s)) = d(\sigma(s), \gamma(s)) \leq c$ for all $s \geq 0$, so σ is asymptotic to γ. Finally, suppose that there are $c, d \in \mathbb{R}$ satisfying $d(\gamma(s), \sigma(s)) \leq c$ and $d(\sigma(s), \beta(s)) \leq d$ for every $s \geq 0$. By the triangle inequality, we have $d(\gamma(s), \beta(s)) \leq c + d$ for all $s \geq 0$. □

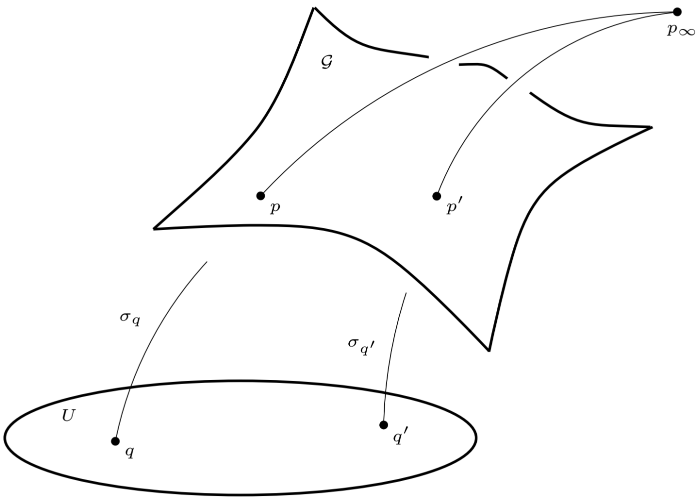

Figure 2 – A geodesic graph in $\mathscr{H} \times \mathbb{R}$.

It is worth mentioning that the set of all equivalence classes is called the *asymptotic boundary* of $\mathscr{H}$, though we will not employ this terminology.

Now, let $\gamma : \mathbb{R} \to \mathscr{H}^n \times \mathbb{R}$ be a geodesic parametrized by arc length. Then

$$\frac{d}{ds}\left\langle \gamma'(s), \frac{\partial}{\partial t} \right\rangle = \left\langle \tilde{\nabla}_{\gamma'}\gamma', \frac{\partial}{\partial t} \right\rangle = 0.$$

that is, the angle between γ' and $\frac{\partial}{\partial t}$ is constant along γ. Thus, any geodesic γ of $\mathscr{H}^n \times \mathbb{R}$ is either horizontal or transversal to all slices of $\mathscr{H}^n \times \mathbb{R}$.

Asymptotic rays allow us to define *geodesic graphs* in $\mathscr{H}^n \times \mathbb{R}$.

Definition 4.4. Let U be a subset of a slice $\mathscr{H}^n_b$ of $\mathscr{H}^n \times \mathbb{R}$. A subset $\mathscr{G} \subset \mathscr{H}^n \times \mathbb{R}$ is said to be a geodesic graph over U if there is a bijection $U \ni q \mapsto p(q) \in \mathscr{G}$ such that

- For each pair $(q, p(q))$, there is a geodesic ray σ_q parametrized by arc length emanating from q and intersecting $\mathscr{G}$ only at p.
- For all $q, q' \in U$, σ_q and $\sigma_{q'}$ are asymptotic.

Figure 2 was taken from Lima (2021) and provides a geometric picture of this definition.

4.3 Helpful facts

In this section we present some useful results that will help us to prove the main results in the following section. First, we have a theorem for hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ obtained by do Carmo and Warner (1970).

Theorem 4.5. Let $f : M^n \to \mathbb{S}^{n+1}$, $n \geq 2$, be a nontotally geodesic hypersurface, where M is a compact, connected, and orientable Riemannian manifold with sectional curvature $K \geq 1$. Then, the following hold:

a. f is an embedding, and M is homeomorphic to $\mathbb{S}^n$.

b. $f(M)$ bounds a closed convex set contained in an open hemisphere of $\mathbb{S}^{n+1}$.

c. f is rigid.

Moreover, the assertion a. and the convexity property in b. still hold if one replaces the sphere $\mathbb{S}^{n+1}$ by the hyperbolic space $\mathbb{H}^{n+1}$, and assume that $K \geq -1$.

Secondly, we have a theorem for compact hypersurfaces in Cartan-Hadamard manifolds established by Alexander (1977).

Theorem 4.6. Let $f : M^n \to \mathscr{H}^{n+1}$, $n \geq 2$, be a compact, connected, and oriented hypersurface in a Hadamard manifold $\mathscr{H}^{n+1}$. If the second fundamental form of f is positive semi-definite, then f is an embedding, M is homeomorphic to $\mathbb{S}^n$, and $f(M)$ bounds an open convex set in $\mathscr{H}^{n+1}$.

Thirdly, here is a fact associated to the *Soul Theorem*, first formulated by Cheeger and Gromoll (1972), and proved by Perelman (1994).

Theorem 4.7. If M^n is a complete noncompact Riemannian manifold with nonnegative sectional curvature, and there exists a point $p \in M$ where all sectional curvatures are positive, then M is diffeomorphic to $\mathbb{R}^n$.

Finally, we have two crucial lemmas for the proof of Theorem 4.10.

Lemma 4.8. Suppose $f : M^n \to \tilde{M}^n \times \mathbb{R}$ is a hypersurface with unit normal N and positive definite (respectively, semi-definite) second fundamental form α. Then, any horizontal section $f_t : M_t \to \tilde{M}_t$ of f has positive definite (respectively, semi-definite) second fundamental form if it is oriented by (4.4).

Proof. Because every slice $\tilde{M}_t$ is totally geodesic, for all $Y \in \mathfrak{X}(\tilde{M}_t)$,

$$\widetilde{\nabla}_Y Y = \nabla^1_Y Y.$$

Now, for every $X \in \mathfrak{X}(M_t)$

$$\begin{aligned} \langle A_\eta X, X \rangle = \langle \alpha^{f_t}(X,X), \eta \rangle = \langle \overline{\nabla}_X X, \eta \rangle &= \frac{1}{\sqrt{1-\theta^2}} \langle \widetilde{\nabla}_X X, N \rangle \\ = \langle \alpha^f(X,X), N \rangle = \langle AX, X \rangle. \end{aligned}$$

Thus, α^{f_t} is either positive definite or positive semi-definite depending on whether α^f is positive definite or positive semi-definite, respectively. □

Let $f : M^n \to \tilde{M}^{n+1}$ be a hypersurface. From the Gauss Equation (2.23) for f, the *extrinsic curvature* $K_{\text{ext}}(f)$ of f at $x \in M$ along the plane span$\{X,Y\}$, where $X, Y \in T_xM$ are orthornormal, is defined via

$$K_{\text{ext}}(f)(X,Y) = K(X,Y) - \tilde{K}(X,Y) = \det A_{XY}.$$

By $\mathscr{C}_{\text{ext}}(f)$ we denote the class of all hypersurfaces $g : M^n \to \tilde{M}^{n+1}$ whose extrinsic curvature coincides with that of f, i.e., for all $x \in M$, and all orthonormal vectors $X, Y \in T_xM$,

$$K_{\text{ext}}(f)(X,Y) = K_{\text{ext}}(g)(X,Y).$$

Notice that $\mathscr{C}_{\text{ext}}(f)$ includes all hypersurfaces $g : M^n \to \tilde{M}^{n+1}$ when $\tilde{M}$ has constant sectional curvature, space forms for instance.

Given a hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, it is said to be *rigid* in $\mathscr{C}_{\text{ext}}(f)$ if for any hypersurface $g : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ in $\mathscr{C}_{\text{ext}}(f)$, there exists $\Phi \in \text{Iso}(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$ such that $g = \Phi \circ f$. On the other hand, if we use (4.7) and suppose f admits a unit normal N, we can rewrite the Gauss equation 3.4 for f as

$$K(X,Y) = \det A_{XY} + \varepsilon(1 - \|\pi_{XY}\,\text{grad}\,h\|^2). \tag{4.8}$$

Lemma 4.9. Let $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $n \geq 3$, $\varepsilon \in \{-1,1\}$, be an oriented hypersurface whose shape operator A_f has rank at least 3 everywhere. If $g : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is a hypersurface in $\mathscr{C}_{\text{ext}}(f)$, there exists a unit normal $N_g \in \Gamma(N_gM)$ such that the shape operator A_g, the height function h^g and the angle function ν_g of g satisfy

$$A_f = A_g, \quad \|\text{grad}\,h^f\| = \|\text{grad}\,h^g\| \quad \text{and} \quad \nu_f^2 = \nu_g^2$$

Proof. Because $g \in \mathscr{C}_{\text{ext}}(f)$, $\tilde{K}(f_*X, f_*Y) = \tilde{K}(g_*X, g_*Y)$ for all $x \in M$ and orthonormal vectors $X, Y \in T_xM$; and thus

$$\langle \tilde{R}(f_*X, f_*Y)f_*Z, f_*W\rangle = \langle \tilde{R}(g_*X, g_*Y)g_*Z, g_*W\rangle,$$

as the sectional curvatures determines the curvature tensor (LEE, 2018, Proposition 8.31). So, from Gauss equations (2.6) for f and g, it follows that

$$\langle \alpha^f(X,W), \alpha^f(Y,Z)\rangle - \langle \alpha^f(X,Z), \alpha^f(Y,W)\rangle = \langle \alpha^g(X,W), \alpha^g(Y,Z)\rangle - \langle \alpha^g(X,Z), \alpha^g(Y,W)\rangle,$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. Since the rank of A_f is at least 3, there exists a vector bundle isometry $\phi : N_fM \to N_gM$ with $\alpha^g = \phi \circ \alpha^f$ (DAJCZER *et al.*, 1990, Proposition 6.10). In particular, $N_g := \phi N_f$ is a unit normal to g. If we set A_g for the shape operator of g in the direction N_g, we have for all $X, Y \in \mathfrak{X}(M)$

$$\langle A_gX, Y\rangle N_g = \alpha^g(X,Y) = \phi(\langle A_fX, Y\rangle N_f) = \langle A_fX, Y\rangle N_g,$$

which implies that $A_f = A_g$ on M. Consequently, from equation (4.8) we obtain

$$\|\pi_{XY}\,\text{grad}\,h^f\| = \|\pi_{XY}\,\text{grad}\,h^g\|$$

for every orthonormal tangent vectors X, Y to M. Given $x \in M$, $X_1, \ldots, X_n$ an orthonormal basis for T_xM, if we write $\operatorname{grad} h^f = a_1X_1 + \cdots + a_nX_n$ and $\operatorname{grad} h^g = b_1X_1 + \cdots + b_nX_n$, the last equation yields $a_i^2 + a_j^2 = b_i^2 + b_j^2$ for all $1 \leq i \neq j \leq n$; which in turn implies that

$$\|\operatorname{grad} h^f\|^2 = a_1^2 + \cdots + a_n^2 = b_1^2 + \cdots + b_n^2 = \|\operatorname{grad} h^g\|^2.$$

Finally, (3.2) implies $\|\operatorname{grad} h^f\|^2 + \nu_f^2 = \nu_g^2 + \|\operatorname{grad} h^g\|^2$, from which we obtain the last claim.

□

4.4 Embeddedness and convexity theorems

In this and the following section we present some results about embeddedness, convexity and rigidity of hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$. These results were first stated and proved by Lima (2021). In order to treat hypersurfaces in $\mathscr{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$ as one group, allow $\tilde{M}^n$ to denote either a Cartan-Hadamard manifold $\mathscr{H}^n$ or the unit sphere $\mathbb{S}^n$. However, when we want to restrict attention to a specific class of hypersurfaces, we deliberately use the corresponding notation for the ambient space, for instance $\mathscr{H}^n \times \mathbb{R}$, rather than the more general notation $\tilde{M}^n \times \mathbb{R}$.

Suppose $f : M^n \to \tilde{M}^n \times \mathbb{R}$ is a complete connected hypersurface oriented by a unit normal N, that its second fundamental form is positive definite, and that its height function h has at least one critical point. It follows that h is a Morse function, i.e., every critical point is nondegenerate: for each critical point x_0 of h, it follows from (4.2) that

$$\operatorname{Hess} h(X,Y) = 0, \forall Y \in T_{x_0}M \Leftrightarrow X = 0 \in T_{x_0}M.$$

Equation (4.2) also implies that the $\operatorname{Hess} h$ at x_0 is positive definite or negative definite according as $\nu = 1$ or $\nu = -1$, respectively. Thus, every critical point is either a strict local minimum or a strict local maximum point of h.

Let us assume that x_0 is a strict local minimum point of h, and that $h(x_0) = 0$. Then, there is a smooth chart $(U, (x_1, \ldots, x_n))$ for M centered at x_0 such that the local representation of h is $x_1^2 + \cdots + x_n^2$. For $t > 0$ small enough, f is transversal to $\tilde{M}_t$; thus if M_t denotes the set $\{(x_1, ..., x_n) \in U; x_1^2 + \cdots + x_n^2 = t\}$, $f_t := f|_{M_t} : M_t \to \tilde{M}_t$ is a horizontal section of f, for instance. Also, M_t is homeomorphic to the sphere $x_1^2 + \cdots + x_n^2 = t$ in $\mathbb{R}^n$, and so homeomorphic to $\mathbb{S}^{n-1}$. In addition, it is also the boundary of the open set

$$\Omega_t := \{(x_1, ..., x_n) \in U; x_1^2 + \cdots + x_n^2 < t\} \subset M,$$

which contains no critical point other than x_0. Restricting the smooth chart (x_i) to $\overline{\Omega_t}$, we obtain a homeomorphism from $\overline{\Omega_t}$ onto $\varphi(\overline{\Omega_t})$, the closed ball $x_1^2 + \cdots + x_n^2 \leq t$ in $\mathbb{R}^n$, whose inverse takes $\partial(\varphi(\overline{\Omega_t}))$ onto M_t. Therefore, $f_t : M_t \to \tilde{M}_t$ is a normal section for x_0.

Next, set $I = (0, t^*) := \{t \in \mathbb{R}^+ : t \text{ is a normal value for } x_0\}$ and $\Omega := \cup_{t \in I} \Omega_t$, where t^* might be either a finite positive real number or $+\infty$. Since we have shown that f_t is a normal section for $t > 0$ near enough to 0, Ω is a nonempty open subset of M; and it is also homeomorphic to $\mathbb{R}^n$. Note that there are three mutually exclusive possibilities for Ω:

i. $\Omega = M$; and thus $\partial\Omega = \emptyset$.

ii. $\Omega \neq M$, and $\partial\Omega$ contains critical points of h.

iii. $\Omega \neq M$, and $\partial\Omega$ contains no critical point of h.

Thanks to Lemma 4.8, each normal section f_t for x_0 has positive definite second fundamental form. In addition, Theorem 4.6 asserts that each normal section $f_t : M_t \to \mathscr{H}_t$ is an embedding, and that $f_t(M_t) \subset \mathscr{H}_t$ is the boundary of an open convex set in $\mathscr{H}_t$, which together with $f_t(M_t)$ is compact as it is contained in a (compact) closed ball of $\mathscr{H}_t$ (LEE, 2018, Lemma 12.16). Do Carmo-Warner theorem yields likewise that every normal section $f_t : M_t \to \mathbb{S}^n_t$ is an embedding, and that $f_t(M_t) \subset \mathbb{S}^n_t$ is the boundary of a compact convex set in $\mathbb{S}^n_t$. Also, as M_t is compact, normal sections $f_t : M_t \to \tilde{M}_t$ are proper embeddings. Since $\overline{\Omega_t}$ is compact and $\tilde{M} \times \mathbb{R}$ is Hausdorff, $f|_{\overline{\Omega_t}}$ is proper, and so is $f|_{\Omega_t}$ as Ω_t is saturated with respect to $f|_{\overline{\Omega_t}}$ (LEE, 2011, Proposition 4.93). It also holds that $f|_{\Omega_t}$ is injective: if $f(x) = f(y)$ for $x, y \in \Omega_t$, we obtain $h(x) = h(y)$ and $x, y \in M_{h(x)}$; thus $x = y$ as $f|_{M_{h(x)}}$ is an embedding. So, for each $t \in I$, $f|_{\Omega_t}$ is a proper embedding (LEE, 2012, Proposition 4.22); and $f(\Omega_t)$ separates $\tilde{M} \times [0, t)$ into two components.

For $\tilde{M} = \mathscr{H}$, one of these components is bounded, which is designated by Λ_t. For $\tilde{M} = \mathbb{S}^n$, both of them are bounded, so we denote by Λ_t the one to which $N(x_0)$ points. Since $N(x_0) = \frac{\partial}{\partial t}$, the unit normal N points towards Λ_t, and so does the mean curvature vector H along Ω_t, as follows from (2.4) if we take $\xi = N$. Also, since $f|_{\Omega_t}$ is an oriented embedded hypersurface that is infinitesimally convex — with positive semi-definite second fundamental form — then it is strictly locally convex (BISHOP, 1974/75), meaning that for each $x \in \Omega_t$, there is a neighbourhood V of $0 \in T_x M$ such that $\exp_{f(x)} V \cap \overline{\Lambda}_t = \{f(x)\}$, where exp is the exponential map of $\tilde{M} \times \mathbb{R}$.

Suppose there are distinct points $p, q \in \Lambda_t$ such that the unique geodesic of $\mathscr{H} \times \mathbb{R}$ joining p and q is not contained in Λ_t. Let $\beta : [0, 1] \to \Lambda_t$ be a smooth curve with $\beta(0) = p$, $\beta(1) = q$ and σ_s the unique geodesic of $\mathscr{H}^n \times \mathbb{R}$ from p to $\beta(s)$ for each $s \in (0, 1]$. Some points of σ_1 do not belong to Λ_t, while all points of σ_s are in Λ_t for $s > 0$ near enough to zero. Thus there is $s_0 \in (0, 1]$ such that σ_{s_0} is tangent to $f(\Omega_t)$, at some point $f(y_0) = \sigma_{s_0}(t_0)$, and contained in Λ_t. Because $f|_{\Omega_t}$ is strictly locally convex, there exists $V \subset T_{y_0} M$ neighbourhood of $0 \in T_{y_0} M$ such that $\exp_{f(y_0)} V \cap \overline{\Lambda}_t = \{f(y_0)\}$. However, for $s > 0$ small enough, $\exp_{f(y_0)} s\sigma'_{s_0}(t_0) \in \exp_{f(y_0)} V \cap \overline{\Lambda}_t$, yielding a contradiction. Consequently, Λ_t is convex.

Theorem 4.10. Let $f : M^n \to \tilde{M}^n \times \mathbb{R}$, $n \geq 3$, be a complete connected oriented hypersurface with positive definite second fundamental form. If its height function h has a critical point, then the following hold

1. f is an embedding, a proper one when $\tilde{M} = \mathscr{H}$, and M is homeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}^n$. In the latter case, f has a bottom or a top end, and $f(M)$ is a geodesic graph over an open subset of a slice of $\mathscr{H} \times \mathbb{R}$ when $\tilde{M} = \mathscr{H}$.

2. $f(M)$ is the boundary of a convex subset of $\mathscr{H} \times \mathbb{R}$.

Proof. We divide the proof considering the three possibilities for Ω.
Case i.
$M(= \Omega)$ is homeomorphic to $\mathbb{R}^n$. Since each $f|_{\Omega_t} : \Omega_t \to \tilde{M}^n \times \mathbb{R}$ is proper, the immersion f is proper; it is also an embedding as it is injective (LEE, 2012, Proposition 4.22). In fact, if $f(x) = f(y)$, then $h(x) = h(y)$, and both x and y belong to $M_{h(x)}$ on which f is an embedding; consequently $x = y$. Also, $f(M) \subset \mathscr{H} \times \mathbb{R}$ is the boundary of the open set $\Lambda := \cup_{t \in I} \Lambda_t$, which is also convex: given $p, q \in \Lambda$, there is $t \in I$ such that $p, q \in \Lambda_t$; and so γ_{pq} is contained in $(\Lambda_t \subset) \Lambda$. In addition, I must be unbounded. In fact, suppose I is bounded. If $f(M) \cap \mathscr{H}_{t^*} \neq \emptyset$, whether f is transversal to $\mathscr{H}_{t^*}$ or not. If it is, by continuity there is some $\ell > t^*$ near enough to t^* such that f is transversal to $\mathscr{H}_\ell$, yielding the contradiction that $t^* \neq \sup_{t \in I} t$. If f is not transversal to $\mathscr{H}_{t^*}$, there is $x \in M$ such that $f_* T_x M = T_{f(x)} \mathscr{H}_{t^*}$; thus x is another critical point of h which belongs to some Ω_t, yielding a contradiction. Therefore, $f(M) \cap \mathscr{H}_{t^*} = \emptyset$; and $\mathscr{H}_{t^*} \subset \overline{\Lambda}$. Let $(p_k)_{k=1}^{\infty}$ be a divergent sequence in $\mathscr{H}_{t^*}$, and $\gamma_k : [0, a_k] \to \overline{\Lambda}$ the geodesic parametrized by arc length of $\mathscr{H} \times \mathbb{R}$ joining $f(x_0)$ to p_k for each $k \in \mathbb{N}$. Since $(\gamma_k'(0))_{k=1}^{\infty}$ is a sequence of unit vectors in $T_{f(x_0)} \mathscr{H} \times \mathbb{R}$, it admits a convergent subsequence. Therefore, we can assume that $(\gamma_k'(0))_{k=0}^{\infty}$ converges to a unit vector $Z_0 \in T_{f(x_0)} \mathscr{H} \times \mathbb{R}$. Let $\gamma : [0, +\infty) \to \mathscr{H} \times \mathbb{R}$ represent the geodesic ray parametrized by arc length with $\gamma(0) = f(x_0)$, $\gamma'(0) = Z_0$. Because, for all $k \in \mathbb{N}$, $\gamma_k([0, a_k]) \subset \overline{\Lambda}$, it follows $\gamma([0, +\infty)) \subset \overline{\Lambda}$. If γ was not horizontal, it would be transversal to $\mathscr{H}_{t^*}$ and not be contained in $\overline{\Lambda}$; thus γ must be horizontal. In fact, since $\gamma(0) = f(x_0)$ and $\overline{\Lambda} \cap \mathscr{H}_0 = \{f(x_0)\}$, it follows that $\gamma \equiv f(x_0)$, yielding a contradiction as well. Consequently, I is unbounded. On the other hand, M is a top end for $f : M^n \to \tilde{M}^n \times \mathbb{R}$: if M was bounded, M would be compact, and so would be $\mathbb{R}^n$; also if there existed a divergent sequence $(p_k)_{k=1}^{\infty}$ in M such that $h(p_k) \nrightarrow +\infty$, we can obtain some Ω_t and a subsequence $(p_{m_k})_{k=1}^{\infty}$ of $(p_k)_{k=1}^{\infty}$ with $f(p_{m_k}) \in \overline{\Lambda_t}$, from which we arrive to a contradiction, proceeding as above. Now, let $(p_k)_{k=1}^{\infty}$ be a divergent sequence in Λ. As we did above, we obtain a geodesic ray parametrized by arc length $\gamma : [0, +\infty) \to \overline{\Lambda}$. Assume there is $t_0 \in (0, +\infty)$ with $\gamma(t_0) \neq f(x_0)$ and $\gamma(t_0) \in \partial\Lambda \subset \mathscr{H} \times \mathbb{R}$, then $\gamma(t_0) = f(y_0)$ for some $y_0 \in M$, and $\gamma'(t_0) = f_* X$ for some $X \in T_{y_0} M$, which yields as before the contradiction that f is not strictly locally convex; thus $\gamma(x_0)$ is the only point of γ in $\partial\Lambda$. Given $x \in M$, there exists a unique geodesic ray σ_x parametrized by arc length with $\sigma_x(0) = f(x)$, asymptotic to γ (see Section 4.2). Like γ, $\sigma_x([0, +\infty)) \subset \overline{\Lambda}$ and $\sigma_x([0, +\infty)) \cap \partial\Lambda = \{f(x)\}$. Because $f(M_{h(x)})$ is the

boundary of a compact convex set in $\mathscr{H}_{h(x)}$, the entire geodesic γ_x containing σ_x cannot be horizontal. Thus it intersects $\mathscr{H}_0$ at some point $q(x)$. Set $U := \{q(x) : x \in M\}$ and consider the map $G : f(M) \ni f(x) \to q(x) \in U$. For $x \in M$, allow $d := d(f(x), \mathscr{H}_0)$ and $\beta(s) := \gamma_x(s-d)$, $s \geq 0$. As $\beta'(s) = \gamma_x{}'(s-d)$, β is a geodesic ray parametrized by arc length that intersects $f(M)$ only at $f(x)$. Let C be the number that claims that σ_x and γ are asymptotic, $\tilde{C} := max_{[0,d]}\, d(\beta(s), \gamma(s))$ and $D := max\{\tilde{C}, C+d\}$, then $d(\beta(s), \gamma(s)) \leq D$, i.e., β is asymptotic to γ, from which we obtain in particular that G is bijective. Consequently, $f(M)$ is a geodesic graph over U. Note that

$$G(f(x)) = \exp_{f(x)}(-d(f(x), \mathscr{H}_0)\sigma_x'(0));$$

in fact, G is a homeomorphism. As a consequence, U is open.

Case ii.

The interval I is bounded; otherwise there would not exist critical points in $\partial\Omega$. If $x_1 \in \partial\Omega$ is a critical point of h, we have $h(x_1) = t^*$. In fact, if $h(x_1) > t^*$, by continuity there is a neighbourhood U of x_1 in M such that $t^* < h(x)$ for all $x \in U$. Nevertheless, because $x_1 \in \partial\Omega$, there exists $x \in U \cap \Omega$ with $h(x) < t^*$. Also, if $h(x_1) < t^*$, x_1 would belong to some normal section of f, and thus not be a critical point. If $N(x_1) = \frac{\partial}{\partial t}$, x_1 would be a strict local minimum point of h; then there would be a neighbourhood $U \subset M$ of x_1 such that $h(x) > t^*$ for every $x_1 \neq x \in U$, which is a contradiction as $x_1 \in \partial\Omega$. Thus $N(x_1) = -\frac{\partial}{\partial t}$, and x_1 is a strict local maximum point of h. In particular, $f(M) \cap \tilde{M}_{t^*} = \{f(x_1)\}$, which implies that $M = \overline{\Omega}$, and so M is compact. As a result, h is a Morse function with only two critical points, hence M is homeomorphic to $\mathbb{S}^n$ (MILNOR, 1973, Theorem 4.1). For $\tilde{M} = \mathscr{H}$, thanks to Alexander theorem, f is a proper embedding and $f(M)$ is the boundary of a convex set in $\mathscr{H} \times \mathbb{R}$. For $\tilde{M} = \mathbb{S}^n$, since f is injective and M is compact, it follows that f is an embedding (LEE, 2012, Proposition 4.22).

Case iii.

We show that this possibility does not actually happen. In fact, first notice that $M_{t^*} := \partial\Omega$ is a connected $(n-1)$-dimensional submanifold of M. As before, $M_{t^*} \subset h^{-1}(t^*)$. Also, since each normal section $f_t : M_t \to \tilde{M}_t$, $t \in (0, t^*)$ is a proper embedding, so is the restriction $f|_{\bar{\Omega}} : \bar{\Omega} \to \tilde{M} \times \mathbb{R}$. If the convex subset $\bar{\Lambda} \subset \mathscr{H} \times \mathbb{R}$ were unbounded, there would exists a divergent sequence $(p_k)_{k=1}^{\infty}$ in $\bar{\Lambda}$, from which we obtain, as before, a geodesic ray parametrized by arc length γ emanating from $f(x_0)$ and contained in $\bar{\Lambda}$. This γ must be horizontal, otherwise it will not be contained in $\bar{\Lambda}$. Nevertheless, $\bar{\Lambda} \cap \mathscr{H}_0 = \{f(x_0)\}$, yielding the contradiction that $\gamma \equiv f(x_0)$. Therefore, $\bar{\Lambda}$ is bounded, and so compact. Since $f|_{\bar{\Omega}} : \bar{\Omega} \to \tilde{M} \times \mathbb{R}$ is a proper embedding, and both $\bar{\Lambda} \subset \mathscr{H} \times \mathbb{R}$ and $\mathbb{S}^n_{t^*}$ are compact, we have that $M_{t^*} \subset M$ is compact. For a given $t \in (0, t^*)$, the flow of $\operatorname{grad} h$ from M_t to M_{t^*} is a homeomorphism (see proof of Theorem 3.1 in Milnor (1973)). Thus, following the flow across M_{t^*}, we arrive to a normal section $M_{\bar{t}}$ for x_0, with $\bar{t} > t^*$ near enough, yielding a contradiction. $\square$

Definition 4.11. Given a hypersurface $f : M^n \to \mathscr{H}^n \times \mathbb{R}$, it is said to be *cylindrically bounded* if there exists a closed geodesic ball $B \subset \mathscr{H}$ such that $f(M) \subset B \times \mathbb{R}$.

The following theorem is a dual to Theorem 4.10, in the sense that here we assume the height function does not have critical points at all.

Theorem 4.12. Suppose $f : M^n \to \tilde{M}^n \times \mathbb{R}$, $n \geq 3$, is a proper connected and oriented hypersurface with positive semi-definite second fundamental form whose height function has no critical points. Suppose also that it is cylindrically bounded only if $\tilde{M} = \mathscr{H}$. Then f is an embedding and $f(M) = \Sigma \times \mathbb{R}$, where $\Sigma \subset \tilde{M} \times \{0\}$ is a submanifold homeomorphic to $\mathbb{S}^{n-1}$ which bounds an open convex set in $\tilde{M} \times \{0\}$.

Proof. Note that $B_t := B \times \{t\}$ and $\mathbb{S}^n_t := \mathbb{S}^n \times \{t\}$ are compact. If $h(M) \subset [a,b] \subset \mathbb{R}$, then $f(M)$ would be contained in either $\mathbb{S}^n \times [a,b]$ or $B \times [a,b]$ according as $\tilde{M}$ is equal to $\mathbb{S}^n$ or $\mathscr{H}$. Then, as f is proper, M would be compact, yielding the contradiction that h has a critical point; thus h is unbounded above and bellow on M. Thus every horizontal section $f_t : M_t \to \tilde{M}_t$ is compact on account of the properness of f and of the fact that $f_t(M_t)$ is contained in either B_t or $\mathbb{S}^n_t$; also $M = \bigcup_{t \in \mathbb{R}} M_t$. Lemma 4.8 ensures that every horizontal section f_t of f has positive semi-definite second fundamental form. Then, Alexander (4.6) and do Carmo-Warner (4.5) theorems apply and guarantee that M_t is homeomorphic to $\mathbb{S}^{n-1}$, f_t is an embedding, and $f_t(M_t)$ bounds an open convex subset of $\tilde{M}_t$. Thus, M is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$; as well as f is an embedding, for it is proper and an injective smooth immersion (LEE, 2012, Proposition 4.22). If $\tilde{M} = \mathscr{H}$, as in the proof of theorem (4.10), the mean convex side Λ of f — one of the regions on which $\mathscr{H} \times \mathbb{R}$ is divided by $f(M)$ and to which the mean curvature vector H points — as well as its closure $\bar{\Lambda}$ turn out to be convex. For $t \in \mathbb{R}$, $x_0 \in M_t$ and a divergent sequence $(x_k)_{k=1}^{\infty}$ in M such that $h(x_k)$ tends to either $+\infty$ or $-\infty$, the geodesic segment γ_k parametrized by arc length of $\mathscr{H} \times \mathbb{R}$ joining $f(x_0)$ to $f(x_k)$ is contained in $\bar{\Lambda}$. Also, the limit geodesic ray γ of $(\gamma_k)_{k=1}^{\infty}$, is contained in $\bar{\Lambda}$. If γ is not vertical, then it reaches the boundary of $B \times \mathbb{R}$ at some point and get out of it, yielding the contradiction that $\gamma \not\subset \bar{\Lambda}$. Thus γ is vertical and tangent to $f(M)$ at $f(x_0)$. Because both $t \in \mathbb{R}$, $x_0 \in M_t$ are arbitrary, $f(M) = f(M_0) \times \mathbb{R}$. If $\tilde{M} = \mathbb{S}^n$, the sectional curvature K of M and the extrinsic curvature $K_{\text{ext}}(f)$ of f are nonnegative as f has positive semi-definite second fundamental form and the sectional curvature of $\mathbb{S} \times \mathbb{R}$ is nonnegative too. Since h does not have critical points, M is not compact. Then, for each $x \in M$, there exists orthonormal vectors $X, Y \in T_x M$ such that $K(X,Y) = 0$; otherwise, by the Soul conjecture, M would be homeomorphic to $\mathbb{R}^n$, yielding the contradiction that $\mathbb{S}^{n-1} \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n$. Next, from (4.8), we have $\|\Pi_{XY} \operatorname{grad} h\| = \det A_{XY} + 1 \geq 1$, which implies $1 = \|\operatorname{grad} h\|$. Consequently, $f(M) = f(M_0) \times \mathbb{R}$. $\square$

4.5 Rigidity theorems

We start this section mentioning two important results on rigidity of hypersurfaces. First, the classic Beez-Killing Theorem, which asserts that hypersurfaces of dimension ≥ 3 in space forms, whose shape operator have rank ≥ 3, are rigid (DAJCZER; TOJEIRO, 2019).

And secondly, H. Rosenberg and R. Tribuzy rigidity result which roughly speaking claims that complete surfaces immersed in homogeneous 3-manifolds, which include the spaces $\mathbb{Q}^2_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1,1\}$, are rigid among the surfaces with the same extrinsic curvature (ROSENBERG; TRIBUZY, 2012). As the focus of this work are hypersurfaces in product spaces, a natural problem to be studied is the rigidity of hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1,1\}$.

Note that, unlike hypersurfaces in space forms $\mathbb{Q}^n_\varepsilon$, Theorem 3.1 tells us that the fact that two hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ have the same shape operator is in general insufficient to guarantee that they are congruent. Consequently, we treat rigidity of hypersurfaces $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$, $\varepsilon \in \{-1,1\}$, on the more restricted class $\mathscr{C}_{\text{ext}}(f)$. More precisely, we have

Theorem 4.13. Under the hypotheses of Theorem 4.10, though for $\tilde{M}$ denoting $\mathbb{Q}^n_\varepsilon$, $\varepsilon \in \{-1,1\}$, the hypersurface $f : M^n \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ is rigid in $\mathscr{C}_{\text{ext}}(f)$.

Proof. Let $g : M \to \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a complete connected hypersurface in $\mathscr{C}_{\text{ext}}(f)$. Because α^f is positive definite, the shape operator A_f of f has rank greater than or equal to three: for each $x \in M$, if there exists $0 \neq X \in T_xM$ such that $A_f X = 0$, then $0 = \langle \alpha^f(X,X), N\rangle$, yielding the contradiction that α^f is not positive definite. Thanks to Lemma 4.9 and its proof, there exists $N_g = \phi N_f \in \Gamma(N_g M)$ such that $A_f = A_g$, $|\operatorname{grad} h^f| = |\operatorname{grad} h^g|$ and $\nu_f^2 = \nu_g^2$, where $\phi : N_f M \to N_g M$ is the vector bundle isometry that satisfies $\alpha^g = \phi \circ \alpha^f$ (cf. proof of Lemma 4.9). The hypersurface g has positive definite second fundamental form since $0 < \langle A_f X, X\rangle = \langle A_g X, X\rangle$ for all $x \in M$ and $0 \neq X \in T_xM$. The set of critical points of h^g is the same as that of h^f:

$$(h^f)_*(x) = 0 \Leftrightarrow \langle \operatorname{grad} h^f(x), Y\rangle = 0, \forall Y \in T_xM \Leftrightarrow \|\operatorname{grad} h^f(x)\| = 0 = \|\operatorname{grad} h^g(x)\|$$
$$\Leftrightarrow \langle \operatorname{grad} h^g(x), Y\rangle = 0, \forall Y \in T_xM \Leftrightarrow (h^g)_*(x) = 0.$$

Thus g shares with f all properties declared in items 1. and 2.) of the Theorem 4.10. Set $A = A_f = A_g$. Given $\varphi : \mathbb{R} \to M$ and integral curve of $\operatorname{grad} h^f$, by (4.3) we have

$$\frac{d}{ds}\nu_f(\varphi(s)) = \langle \operatorname{grad}\nu_f(\varphi(s)), \varphi'(s)\rangle = -\langle A\varphi'(s), \varphi'(s)\rangle \leq 0. \tag{4.9}$$

Similarly for ν_g; thus ν_f and ν_g are decreasing along integral curves of $\operatorname{grad} h^f$ and $\operatorname{grad} h^g$, respectively. Using (4.3), we compute the differential of both sides $\nu_f^2 = \nu_g^2$ for $x \in M, X \in T_xM$

$$\nu_f(\nu_f)_*X = \nu_g(\nu_g)_*X \Rightarrow \langle \nu_f \operatorname{grad}\nu_f, X\rangle = \langle \nu_g \operatorname{grad}\nu_g, X\rangle \Rightarrow \nu_f \operatorname{grad} h^f = \nu_g \operatorname{grad} h^g$$

If x_0 is a strict local minimum point of h^g (and so of h^f), it holds $\nu_f(x_0) = 1 = \nu_g(x_0)$. Because $\nu_f(x) = \pm\nu_g(x)$, there exists a neighbourhood $U \subset M$ of x_0 such that $\nu_f = \nu_g$ on U; otherwise there will exist a sequence $(x_k)_{k=1}^\infty$ in M converging to x_0 with $\nu_f(x_k) = -\nu_g(x_k)$, yielding a contradiction. After shrinking this neighbourhood, if necessary, we have $\nu_f = \nu_g > 0$ on U; consequently $\operatorname{grad} h^f = \operatorname{grad} h^g$ on U as well. Thus the integral curves of $\operatorname{grad} h^f$ and $\operatorname{grad} h^g$ coincide on U. Because of ν_f and ν_g are decreasing along these common integral curves, U can be taken to be the entire M. Then we have

$$A_f = A_g, \quad \nu_f = \nu_g \quad \text{and} \quad \operatorname{grad} h^f = \operatorname{grad} h^g \quad \text{on } M.$$

Finally, since all conditions in the uniqueness part of Theorem 3.1 are satisfied, there exists an isometry Φ such that $g = \Phi \circ f$. □

A hypersurface S in $\tilde{M} \times \mathbb{R}$ is called a *rotational sphere* with axis L if its horizontal sections are translated geodesic spheres of $\tilde{M}$, to suitable heights, centered at L. The following result is a consequence of Theorem 4.10 and also a Hilbert-Liebmann-type theorem for hypersurfaces in product spaces.

Corollary 4.14. Suppose $f : M_c^n \to \mathbb{Q}_\varepsilon^n \times \mathbb{R}$, $\varepsilon \in \{-1, 1\}$, is an isometric immersion of a complete connected oriented $n(\geq 3)$-dimensional Riemannian manifold M_c^n with constant sectional curvature $c > (1+\varepsilon)/2$. Then f is congruent to an embedded rotational sphere.

Proof. Thanks to Myers theorem, M is compact and has finite fundamental group. The maximum value of the sectional curvature of M in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ is $(1+\varepsilon)/2$: in fact, from (4.8) we see that the maximum value of the sectional curvature of M in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ is 1 (resp., 0), and it happens when $\operatorname{grad} h$ is orthogonal (resp., equal to $\frac{\partial}{\partial t}$ and tangent) to $\operatorname{span}\{X, Y\}$ for $\varepsilon = 1$ (resp., $\varepsilon = -1$). Because of this and the hypothesis on c, if λ_i and X_i, $i = 1, \ldots, n$, denote a principal curvature and a associated principal direction, respetively, Gauss equation (2.23) implies $\lambda_i \lambda_j > 0$ for all $1 \leq i \neq j \leq n$. After posibly changing the orientation of f, we obtain that all $\lambda_i > 0$, i.e., the second fundamental form α of f is positive definite. Since M is compact, the height function h of f has a critical point. Therefore, thanks to theorem 4.10, f is an embbeding and M is homeomorphic to the sphere $\mathbb{S}^n$; so $M = \mathbb{S}_c^n$. As in the proof of theorem 4.10, since M is compact, h has just two critical points — a minimum x_0 and a maximum x_1. Let us assume, as before, $h(x_0) = 0$ and note that $\nu(x_0) = +1$, $\nu(x_1) = -1$. By Manfio and Tojeiro (2011, Lemma 3.1), $\operatorname{grad} h$ is a principal direction on $M \setminus \{x_0, x_1\}$. Since N is a normal to any horizontal section M_t of f, we obtain

$$\langle X, \operatorname{grad} h \rangle = \left\langle (f_t)_* X, \frac{\partial}{\partial t} \right\rangle - \nu \langle f_* X, N \rangle = 0$$

for every tangent vector X to M_t, that is, tangent vectors to horizontal sections are orthogonal to $\operatorname{grad} h$. Because of this, equation (4.3) and the fact that $\operatorname{grad} h$ is a principal direction, we obtain $X(\nu) = \langle -A \operatorname{grad} h, X \rangle = 0$ for every tangent vector X to M_t, i.e., the angle function ν is constant along horizontal sections M_t. As $\operatorname{grad} h$ does not vanish on $M \setminus \{x_0, x_1\}$, its integral curves cover this set. Also, these integral curves issue from x_0 and goes into x_1 (see Section 4.1). The angle function ν decreases along them from $+1$ to -1 on account of (4.9). Therefore, there is a point $x \in M \setminus \{x_0, x_1\}$ on which $\nu(x) = 0$, which implies $\nu \equiv 0$ on the horizontal section M_{t^0}, $t^0 := h(x)$, and positive on M_t for $t \in [0, t^0)$. Set $\Sigma := \{f(x) : x \in M, h(x) < t^0\}$ and $D := \pi(\Sigma)$. We claim that $\pi|_\Sigma$ is bijective. In fact, given distinct points $x, y \in f^{-1}(\Sigma) \subset M$, if they belong to the same normal section $M_{h(x)}$ of f, we have $\pi(f(x)) \neq \pi(f(y))$. If they do not, suppose $\pi(f(x)) = \pi(f(y))$. Then, the 2-dimensional subspace $\mathscr{P}$ of $\mathbb{E}^{n+2}$ containing the points $0, \pi(f(x))$ and $f(x)$, intersects Σ on a curve passing throw the points $f(x)$ and $f(y)$ (see Figure 3). There exists a point $f(z)$ on this curve where $\nu = 0$, yielding the contradiction that

$f(z) \notin \Sigma$. Thus, $\pi|_\Sigma$ is injective and $u := \pi_{\mathbb{R}} \circ (\pi|_\Sigma)^{-1} : D \to \mathbb{R}$ is a smooth function for which Σ is its (vertical) graph. The level sets $\Sigma_t := u^{-1}(t) = \pi(f(M_t)), t \in (0, t^0)$ are (homeomorphic to) $(n-1)$-spheres. From equation (4.6), since the angle function is constant along the horizontal sections M_t of f, it follows that $|\nabla u|$ is constant along every level sphere Σ_t. We note that $\pi(f(x_0))$ is the only critical point of u (cf. equation 4.6), thus $D \setminus \{\pi(f(x_0))\} \subset \mathbb{Q}^n_\varepsilon$ is an open set without critical points of u whose boundary contains only one critical point of u, the minimum point $\pi(f(x_0))$ of u. Also, thanks to Lemma 3.4, each integral curve of ∇u is a geodesic of $\mathbb{Q}^n_\varepsilon$, after reparametrizing by arc lenght. Consequently, all geodesics of $\mathbb{Q}^n_\varepsilon$ in U issue from $f(x_0)$ and are orthogonal to all level spheres Σ_t. As a result, each level sphere Σ_t is a geodesic sphere of $\mathbb{Q}^n_\varepsilon$ centered at $f(x_0)$. For $t \in (0, t^0)$, because horizontal sections of f are the translate of those geodesic spheres to suitable heights in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, it follows that Σ is rotational with axis $\pi(f(x_0)) \times \mathbb{R}$, and boundary $f(M_{t^0})$. Similarly, the set Σ' of all points of $f(M)$ at a height strictly larger than t^0 turns out to be rotational with axis $\pi(f(x_1)) \times \mathbb{R}$ and boundary $f(M_{t^0})$. Then, since Σ' and Σ share the same boundary, their axes must coincide and $f(M) = \overline{\Sigma} \cup \overline{\Sigma'}$ is in fact an embedded rotational sphere. $\square$

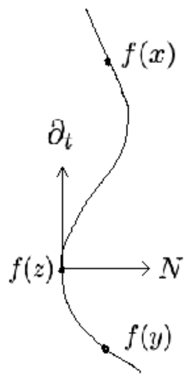

Figure 3 – The instersection curve of Σ and $\mathscr{P}$.

BIBLIOGRAPHY

ALEXANDER, S. Locally convex hypersurfaces of negatively curved spaces. **Proc. Am. Math. Soc.**, v. 64, p. 321–325, 1977. Citation on page 50.

BISHOP, R. L. Infinitesimal convexity implies local convexity. **Indiana Univ. Math. J.**, v. 24, p. 169–172, 1974/75. ISSN 0022-2518. Available: <https://doi.org/10.1512/iumj.1974.24.24014>. Citation on page 53.

CARMO, M. do; WARNER, F. Rigidity and convexity of hypersurfaces in spheres. **J. Dif. Geom.**, v. 4, p. 133–144, 1970. Citation on page 49.

CECIL, T. E.; RYAN, P. J. **Geometry of Hypersurfaces**. [S.l.]: Springer New York, NY, 2015. Citation on page 36.

CHEEGER, J.; GROMOLL, D. On the structure of complete manifolds of nonnegative curvature. **Ann. of Math. (2)**, v. 96, p. 413–443, 1972. ISSN 0003-486X. Available: <https://doi.org/10.2307/1970819>. Citation on page 50.

DAJCZER, M.; ANTONUCCI, M.; OLIVEIRA, G.; LIMA-FILHO, P. **Submanifolds and Isometric Immersions**. [S.l.]: Publish or Perish, 1990. Citation on page 51.

DAJCZER, M.; TOJEIRO, R. **Submanifold Theory Beyond an Introduction**. [S.l.]: Springer, New York, NY, 2019. Citations on pages 23, 25, 29, 40, and 56.

DANIEL, B. Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces. **Trans. Amer. Math. Soc.**, v. 361, n. 12, p. 6255–6282, 2009. ISSN 0002-9947. Available: <https://doi.org/10.1090/S0002-9947-09-04555-3>. Citations on pages 19, 31, and 33.

DILLEN, F.; FASTENAKELS, J.; VEKEN, J. Van der. Rotation hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. **Note Mat.**, v. 29, n. 1, p. 41–54, 2009. ISSN 1123-2536. Citation on page 19.

JOST, J. **Riemannian Geometry and Geometric Analysis**. [S.l.]: Springer Berlin, Heidelberg, 2011. Citation on page 48.

LEANDRO, B.; PINA, R.; SANTOS, J. a. P. dos. Einstein hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. **Bull. Braz. Math. Soc. (N.S.)**, v. 52, n. 3, p. 537–546, 2021. ISSN 1678-7544. Available: <https://doi.org/10.1007/s00574-020-00216-7>. Citation on page 19.

LEE, J. M. **Introduction to Topological Manifolds**. [S.l.]: Springer New York, NY, 2011. Citation on page 53.

____. **Introduction to Smooth Manifolds**. [S.l.]: Springer New York, NY, 2012. Citations on pages 53, 54, 55, and 56.

____. **Introduction to Riemannian Manifolds**. [S.l.]: Springer Cham, 2018. Citations on pages 51 and 53.

LIMA, R. F. de. Embeddedness, convexity, and rigidity of hypersurfaces in product spaces. **Ann. Global Anal. Geom.**, v. 59, n. 3, p. 319–344, 2021. ISSN 0232-704X. Available: <https://doi.org/10.1007/s10455-020-09745-2>. Citations on pages 45, 46, 48, 49, and 52.

LIRA, J. H.; TOJEIRO, R.; VITóRIO, F. A Bonnet theorem for isometric immersions into products of space forms. **Arch. Math. (Basel)**, v. 95, n. 5, p. 469–479, 2010. ISSN 0003-889X. Available: <https://doi.org/10.1007/s00013-010-0183-4>. Citations on pages 31 and 33.

MANFIO, F.; TOJEIRO, R. Hypersurfaces with constant sectional curvature of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. **Illinois J. Math.**, v. 55, n. 1, p. 397–415 (2012), 2011. ISSN 0019-2082. Available: <http://projecteuclid.org/euclid.ijm/1355927042>. Citations on pages 19, 44, and 58.

MILNOR, J. **Morse Theory**. [S.l.]: Annals of Mathematics Studies, 1973. Citation on page 55.

PERELMAN, G. Proof of the soul conjecture of Cheeger and Gromoll. **Journal of Differential Geometry**, Lehigh University, v. 40, n. 1, p. 209 – 212, 1994. Available: <https://doi.org/10.4310/jdg/1214455292>. Citation on page 50.

ROSENBERG, H.; TRIBUZY, R. Rigidity of convex surfaces in the homogeneous spaces. **BULLETIN DES SCIENCES MATHEMATIQUES**, v. 136, n. 8, p. 892–898, DEC 2012. ISSN 0007-4497. Citation on page 57.

TOJEIRO, R. On a class of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. **Bull Braz Math Soc**, v. 41, p. 199–209, 2010. Available: <https://doi.org/10.1007/s00574-010-0009-9>. Citations on pages 19, 34, and 39.

ICMC USP
SÃO CARLOS