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On symbol correspondences for spin and quark systems

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Sobre correspondências de símbolos para sistemas de spin e de quark

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *VERSÃO REVISADA*

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Este trabalho é dedicado a quem faz ciência e arte.

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"One cannot attempt to arrest this thought at any stage of its unceasing development without closing one's mind to the understanding of its further progress."

(Leon Rosenfeld)

RESUMO

ALCÂNTARA, P. A. S. **Sobre correspondências de símbolos para sistemas de spin e de quark**. 2022. 176 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Neste trabalho, investigamos o princípio de correspondência aplicado a sistemas mecânicos simétricos por grupos de Lie compactos por meio de *correspondências de símbolos*. Na primeira parte, consideramos sistemas simétricos por $SU(2)$, os chamados *sistemas de spin*: verificamos algumas condições nas quais correspondências de símbolos apresentam localização de estados puros no limite semiclássico, o que fornece um novo critério para emergência da dinâmica clássica; após isso, introduzimos a noção de *quantização sequencial* a partir da qual obtemos uma nova caracterização de localização assintótica. Na segunda parte, damos início ao estudo de correspondências de símbolos para sistemas simétricos por $SU(3)$, os *sistemas de quark*: caracterizamos as correspondências de símbolos para tais sistemas e obtemos algumas propriedades e fórmulas para os produtos torcidos induzidos nos espaços de símbolos.

Palavras-chave: Física-matemática, Princípio de correspondência, Teoria da representação, Correspondências de símbolos, Quantização.

ABSTRACT

ALCÂNTARA, P. A. S. **On symbol correspondences for spin and quark systems**. 2022. 176 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

In this work, we investigate the correspondence principle applied to mechanical systems symmetric under a compact Lie group by means of *symbol correspondences*. In the first part, we consider systems symmetric under $SU(2)$, the so-called *spin systems*: we verify conditions under which symbol correspondences present localization of pure states at the semiclassical limit, which provides a new criterion for the emergence of classical dynamics; then, we introduce the notion of *sequential quantization* from which we get a new characterization of asymptotic localization. In the second part, we begin the study of symbol correspondences for systems symmetric under $SU(3)$, the *quark systems*: we characterize the symbol correspondences for such systems and obtain some properties and formulas for twisted products induced on the spaces of symbols.

Keywords: Mathematical physics, Correspondence principle, Representation theory, Symbol correspondences, Quantization.

CONTENTS

GENERAL INTRODUCTION

This Master's thesis is comprised of two parts, united within a general framework, but completely independent of each other.

The first part is a continuation of the study of symbol correspondences for mechanical systems which are symmetric by $SU(2)$, the so-called "spin systems", whose point of departure is the study carried out in the book *Symbol correspondences for spin systems* (RIOS; STRAUME, 2014). This first part started as an undergraduate research project and continued during my first year in the Master's Program.

More specifically, Chapter 2 in Part I provides a summary of many of the main definitions and results in (RIOS; STRAUME, 2014), but Chapters 3–5 in Part I present original research material. Of these, the main result in Chapter 3 and most of the results in Chapter 4 were obtained in preliminary form during my undergraduate research and, together with an expanded version of Chapter 2, comprised my Bachelor's thesis (TCC). On the other hand, almost all of Chapter 5 was worked out during my first year in the Master's Program, when we also polished and expanded Chapters 3 and 4 and wrote Chapters 1 and 6. The whole Part I of this Master's thesis, including Appendices A and B, was posted in the present form (with minor style adaptations) on 17 February 2021 in the arXiv (2004.03929 v4) and was submitted for publication shortly afterwards.

The second part of this Master's thesis starts a comprehensive study of symbol correspondences for mechanical systems which are symmetric by $SU(3)$, the so-called "quark systems". The whole of Part II consists of original research material, presenting original results (as far as we know), and was entirely worked out during my Master's Program, more intensely during its second year. We plan on posting this Part II in the arXiv soon and submitting it for publication afterwards.

What binds the two parts together is the problem of quantization/dequantization of mechanical systems that are symmetric under the action of a compact Lie group G. There are several notions of quantization and dequantization in the literature, but the usage of symmetry by a compact Lie group to define the mechanical systems of interest leads us to a general framework, as briefly described below, which relies mainly on well known concepts in Differential Geometry, Representation Theory and Harmonic Analysis.

First, it is known that coadjoint orbits of G are models of homogeneous symplectic G-manifolds, and any G-orbit is diffeomorphic to a quotient space G/H for $H \subset G$ an isotropy subgroup of a point on the orbit. So each coadjoint orbit of G (or, equivalently, suitable coset space) is a classical phase space defining its Poisson algebra on $C^\infty_{\mathbb{C}}(G/H)$.

Secondly, we have Schur's Lemma and two distinct facts valid for compact groups: every unitary representation is completely reducible and every irreducible representation is finite dimensional and unitary. Hence, the irreducible representations of G are finite dimensional quantum systems whose dynamics are given by matrix algebras.

Finally, we use Peter-Weyl Theorem to obtain an orthonormal basis for $L^2(G)$ consisting of normalized complex conjugate coefficients of irreducible representations of G, conventionally called "harmonic functions". Restricting to harmonic functions right invariant under H, we find a basis for $L^2(G/H)$. Since complex conjugate coefficients of an irreducible representation ρ of G span a direct sum of representations all equivalent to ρ, this basis provides a decomposition of $L^2(G/H)$ into irreducible representations of G.

Therefore, when mapping the operator space of an irreducible representation of G to $C^\infty_{\mathbb{C}}(G/H)$ in a G-equivariant way, we get that each irreducible subrepresentation of the operator space generates an isomorphic irreducible representation within $C^\infty_{\mathbb{C}}(G/H)$, thus a subrepresentation of $L^2(G/H)$ spanned by a finite set of harmonic functions.[1]

In Part I of this Master's thesis, we further explore this framework for the case $G = SU(2)$ and $H = S(U(1)\times U(1)) \simeq U(1)$, as started in (RIOS; STRAUME, 2014), now focusing on asymptotic questions. In Part II, we start to apply this framework for the case $G = SU(3)$ with H being isomorphic to either $U(2)$ or $U(1)\times U(1)$. In the first case, $S(U(1)\times U(2)) = H \simeq U(2)$, we refer to a pure-quark system, while generic quark systems refer to the cases where $S(U(1)\times U(1)\times U(1)) = H \simeq U(1)\times U(1)$.

[1] For more details, see (FOLLAND, 2016; HUMPHREYS, 1973; KIRILLOV, 2004; LEE, 2012; NACHBIN, 1965) and references therein.

Part I

Spin systems

CHAPTER

1

INTRODUCTION TO PART I

Quantum or classical mechanical systems which are symmetric under $SU(2)$ are called spin systems. While the Poisson algebra of the classical spin system is infinite dimensional (smooth functions on the 2-sphere S^2), the operator algebra of a quantum spin-j system is finite dimensional (linear operators on $\mathbb{C}^{n+1}$, where $n = 2j \in \mathbb{N}$). Thus, one is naturally led to ask if, or under which conditions, the Poisson algebra of the classical spin system emerges as the asymptotic limit of the operator algebra of quantum spin-j systems, as $j \to \infty$. But in full generality, this question can only be well posed if we first specify a sequence of injective $SU(2)$-equivariant linear maps from linear operators on $\mathbb{C}^{n+1}$ to smooth functions on S^2 (satisfying a few other properties), a so-called sequence of *symbol correspondences*.

This question has been addressed and answered in the research monograph (RIOS; STRAUME, 2014). An important conclusion of (RIOS; STRAUME, 2014) is that, given a generic sequence of symbol correspondences, the sequence of spin-j quantum systems do not approach the classical spin system as j grows indefinitely. Actually, for spin systems, a necessary and sufficient condition was found in (RIOS; STRAUME, 2014) for the asymptotic emergence of the classical Poisson algebra from a sequence of *twisted algebras* of the corresponding symbols. Symbol correspondence sequences with this property are said to be of Poisson or anti-Poisson type and the necessary and sufficient condition for such was stated in (RIOS; STRAUME, 2014) in numerical terms, as a non-generic asymptotic condition on the sequence of *characteristic numbers* of the respective symbol correspondences, as follows.

For $n = 2j \in \mathbb{N}$,[1] a *quantum spin-j system* is a complex Hilbert space $\mathcal{H}_j \simeq \mathbb{C}^{n+1}$ together with an irreducible unitary representation $\varphi_j : SU(2) \to U(n+1)$. Then, a *spin-$j$*

[1] In this text, we adopt the usual convention $\mathbb{N} = \{1, 2, 3, \cdots\}$ and denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

symbol correspondence W^j is a linear injective and $SO(3)$-equivariant map[2]

$$W^j : M_{\mathbb{C}}(n+1) \to C^\infty_{\mathbb{C}}(S^2) \ , \ \ P \mapsto W^j_P \ , \tag{1.1}$$

such that $W^j_{P^\dagger} = \overline{W^j_P}$ and $W^j_I = 1$, cf. Definition 2.2.1. By decomposing the operator space into $SO(3)$-invariant subspaces (indexed by $l,\ \ 0 \le l \le n$), we establish a standard basis $\{\boldsymbol{e}^j(l,m)\}_{-l\le m\le l\le n}$ of $M_{\mathbb{C}}(n+1)$, cf. Theorem 2.1.2, so that

$$W^j : \sqrt{n+1}\boldsymbol{e}^j(l,m) \mapsto c^n_l Y^m_l \ , \tag{1.2}$$

where Y^m_l are the spherical harmonics, and the n nonzero real numbers[3]

$$c^n_l \in \mathbb{R}^* \ , \ \ \ 1 \le l \le n = 2j \ , \tag{1.3}$$

are the *characteristic numbers* of W^j, cf. Theorem 2.2.1.

Given a symbol correspondence W^j, the operator product is "imported" as a product of functions on the sphere, the *twisted product of symbols*, $\star^n_{\vec{c}}$, defined by

$$W^j_P \star^n_{\vec{c}} W^j_Q = W^j_{PQ} \ , \tag{1.4}$$

where $\vec{c} = (c^n_1, \cdots, c^n_n)$ denotes the n-tuple of characteristic numbers of W^j.

Now, the standard area form on homogeneous S^2 is a $SO(3)$-invariant symplectic form, then, a *symbol correspondence sequence* $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$ is of *Poisson type* if the asymptotic $n \to \infty$ limit of the twisted product of symbols always coincides with their pointwise product and if, to first order in $1/n$, the asymptotic limit of the *twisted commutator* of symbols equals $\sqrt{-1}$ times their Poisson bracket (being of *anti-Poisson type* if it equals $-\sqrt{-1}$ times their Poisson bracket).

In fact, care is needed with this asymptotic limit because, although the image of any operator by (1.1)-(1.2) is a smooth function, the limit of images of operators in a sequence may not belong to $C^\infty_{\mathbb{C}}(S^2)$, and this is intrinsically related to the subject of this work. But this subtlety can be circumvented in order to well define symbol correspondence sequences of (anti-)Poisson type, cf. Definition 2.3.3.

Then it was shown in (RIOS; STRAUME, 2014) that, for any symbol correspondence sequence $\boldsymbol{W}$,

$$\begin{aligned} \text{Poisson type} \iff \lim_{n\to\infty} c^n_l = 1 \ , \ \forall l \in \mathbb{N} \ , \\ \text{anti-Poisson type} \iff \lim_{n\to\infty} c^n_l = (-1)^l \ , \ \forall l \in \mathbb{N} \ , \end{aligned} \tag{1.5}$$

cf. Theorem 2.3.2 below (RIOS; STRAUME, 2014, Theorem 8.2.21). Thus, being of (anti-)Poisson type is a nongeneric condition for a symbol correspondence sequence because the sequence of symbol products, defined by (1.2)-(1.4), clearly depends (and only depends)

[2] The action of $SU(2)$ on both spaces, $M_{\mathbb{C}}(n+1)$ and $C^\infty_{\mathbb{C}}(S^2)$, is effectively an $SO(3)$ action.
[3] $W^j_I \equiv 1 \iff c^n_0 \equiv 1$.

on the bi-sequence $(c_l^n)_{l\leq n\in\mathbb{N}}$ of characteristic numbers, but condition (1.5) is obviously too stringent for a generic bi-sequence $(c_l^n)_{l\leq n\in\mathbb{N}}$ satisfying just (1.3).

Furthermore, as also shown in (RIOS; STRAUME, 2014), this asymptotic condition fails to be generic even in each of two very important subsets of symbol correspondence sequences: the subset consisting of sequences of *isometric* (Stratonovich-Weyl) correspondences, for which the map (1.2) is an isometry[4], and the subset of sequences of *mapping-positive* (coherent-state) correspondences, for which the map (1.2) is positive.

For isometric correspondences there is the additional requirement: $|c_l^n| \equiv 1$, but because the signs can generically be anything, condition (1.5) is still too stringent. The requirements on the c_l^n's for a symbol correspondence to be mapping-positive are less strict, hence condition (1.5) is more stringent on this subset. In fact, these two subsets, isometric and mapping-positive, were conjectured in (RIOS; STRAUME, 2014) to be disjoint. In this work, this important conjecture is proved in Theorem 3.0.2.

However, given the relevance of singling out the symbol correspondence sequences of Poisson or anti-Poisson type, we were still missing a geometrical or "physically intuitive" interpretation for when a sequence of symbol correspondences takes the sequence of spin-j quantum systems to the classical spin system in the asymptotic $j\to\infty$ limit. In other words, we wanted a better answer to the question:

$$\text{How can we "see" condition (1.5)?} \tag{1.6}$$

This question is answered in this work by looking at the asymptotic localization of the symbols of pure J_3-invariant states[5].

For a quantum spin-j system, if $\vec{J} = (J_1, J_2, J_3)$ is the operator of angular momentum, a *pure J_3-invariant state* is a vector $\boldsymbol{u}(j,m) \in \mathcal{H}_j \simeq \mathbb{C}^{n+1}$ satisfying $J_3\boldsymbol{u}(j,m) = m\boldsymbol{u}(j,m)$, $-j \leq m \leq j$, and these comprise the standard basis of the spin-j system, indexed from highest ($m = j$) to lowest ($m = -j$) weight, so that a pure J_3-state can also be seen as a projector onto the k^{th} subspace, $\Pi_k \in M_{\mathbb{C}}(n+1)$, $1 \leq k \leq n+1$, with $k = j - m + 1$, and the convex hull of all projectors Π_k constitute the set of all J_3-invariant states (pure and mixed).

Considering the sequence of quantum spin-j systems, for a natural sequence

$$(k_n)_{n\in\mathbb{N}} \ \text{ satisfying } \ k_n \in \mathbb{N},\ 1 \leq k_n \leq n+1,$$

[4] With respect to the normalized inner products on both spaces, cf. Definition 2.2.2.

[5] For affine quantum systems, i.e. ordinary quantum mechanics with Hilbert space $L^2(\mathbb{R}^k)$, it's well known that Ehrenfest's theorem fails to hold in general for pure states which are not sufficiently localized in $\mathbb{R}^k$ (cf. e.g. (COHEN-TANNOUDJI; DIU; LALOE, 1977)), thus the notion that emergence of Poisson mechanics from quantum mechanics should be related to (asymptotic) localization of pure states is somewhat "physically intuitive". Because we look at J_3-invariant pure states in the case of spin systems, the analogy with affine systems is perhaps clearer if there we think of localization of momentum eigenstates.

the sequence $(\Pi_{k_n})_{n\in\mathbb{N}}$ of projectors is said to be r-*convergent* if

$$k_n/n \to r \in [0,1] \ , \ \text{ as } \ n\to\infty \ ,$$

and in this case, given a sequence of symbol correspondences $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$, the sequence $(\rho^j_{k_n})_{n\in\mathbb{N}}$ of quasi-probability distributions on $[-1,1]$, given by

$$\rho^j_{k_n} = \frac{n+1}{2} W^j_{\Pi_{k_n}} \ \text{ restricted to } z\text{-axis} \ ,$$

is said to be an r-*convergent* Π-*distribution sequence*, cf. Definition 4.0.1. Then, a symbol correspondence sequence $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$ *localizes* (resp. *anti-localizes*) *classically* if every r-convergent Π-distribution sequence localizes classically at

$$z_0 = 1-2r \ \ (\text{resp. } \ z_0 = 2r-1) \ \in \ [-1,1] \ .$$

In the above sentence, the word 'every' is important and by *classical localization at* z_0 we mean that the sequence of quasi-probability distributions converges, as distribution, to Dirac's $\delta(z-z_0)$ distribution on $C^\infty_{\mathbb{C}}([-1,1])$, in the sense that

$$\lim_{n\to\infty}\int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz = f(z_0) \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ , \tag{1.7}$$

cf. Definitions 4.0.2-4.0.3. Hence, for all practical purposes, if a symbol correspondence sequence (anti-)localizes classically, for every $r\in[0,1]$ we can "see" the symbols of any r-convergent Π-distribution sequence "concentrating" asymptotically on the parallel of colatitude $\varphi = \arccos(z_0)$ on the sphere (the "classical picture").

Then, Corollaries 4.2.2.2, 4.2.2.3 and 4.2.4.1 state that a sequence of mapping-positive symbol correspondences, or else a sequence of isometric symbol correspondences, is of (anti-)Poisson type if and only if the given symbol correspondence sequence (anti-)localizes classically. In fact, Corollary 4.2.2.2 presents one of the main results of this part of the work, providing a partial answer to question (1.6) by stating that, for any symbol correspondence sequence $\boldsymbol{W}$,

$$\text{classical (anti-)localization} \ \Rightarrow \ \text{(anti-)Poisson type.} \tag{1.8}$$

However, the converse is not true in general and thus such a classical localization condition is a stronger condition than asymptotic Poisson emergence, for general symbol correspondence sequences. This is another main result of this part of the work, which is presented in Theorem 4.2.5. Before, Theorem 4.2.3 presented yet another important result, providing sufficient conditions for classical (anti-)localization of a symbol correspondence sequence of (anti-)Poisson type, as:

$$\text{direction } (\Leftarrow) \text{ in (1.8) holds for a polynomial growth with } l \text{ of } |c^n_l| \ , \tag{1.9}$$

cf. (4.16) for the more explicit bounds. Accordingly, the characteristic numbers of mapping-positive symbol correspondence sequences and of isometric symbol correspondence sequences, both satisfy the polynomial bounds in (4.16).

Hence, the standard (resp. alternate) Berezin symbol correspondence sequence, being mapping-positive, and the standard (resp. alternate) Stratonovich-Weyl symbol correspondence sequence, being isometric, both being of Poisson (resp. anti-Poisson) type, both localize (resp. anti-localize) classically.

Now, for general symbol correspondences there is a well defined notion of duality, cf. Definition 2.2.7, and the dual of a mapping-positive symbol correspondence is a *positive-dual* symbol correspondence, cf. Definition 3.0.1. But it turns out that, in general, the characteristic numbers of positive-dual symbol correspondence sequences of (anti-)Poisson type do not satisfy the polynomial bounds in (4.16).

However, there are important examples of positive-dual symbol correspondence sequences of (anti-)Poisson type, specially the standard (resp. alternate) Toeplitz corresspondence sequence, dual of the standard (resp. alternate) Berezin correspondence sequence, for which their $|c_l^n|$ have well behaved exponential growth with l, allowing us to consider some weaker notions of asymptotic localization of a symbol correspondence sequence, encompassing these special examples.

For such a generalization, called *μ-analitical (anti-)localization* of $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$, the r-convergent sequences of quasi-probability distributions $(\rho^j_{k_n})_{n\in\mathbb{N}}$ converge to Dirac's $\delta(z-z_0)$ distribution on $\mathcal{A}_\mu([-1,1]) \subset C^\infty_{\mathbb{C}}([-1,1])$, the space of complex-analytic extensions to the interior of the Bernstein ellipse $\partial\mathcal{E}_\mu$, with foci ± 1 and sum of semi-axis equal to μ, for some $\mu > 1$, cf. (4.30) and Definition 4.2.3.

This part is organized as follows.

In Chapter 2 we present a summary of the main constructions and results of (RIOS; STRAUME, 2014) which are necessary for understanding the questions addressed in this part. Thus, after reviewing the main definitions and properties of quantum and classical spin systems, we review the main definitions and results on symbol correspondences for spin systems and their associated twisted products of symbols. In order to make Part I minimally self-contained, this chapter is not too short.

In Chapter 3 we prove the splitting Theorem 3.0.2, which states that the subsets of isometric (Stratonovich-Weyl) correspondences, mapping-positive (coherent-state) correspondences, and positive-dual correspondences, are mutually disjoint.

In Chapter 4 we address the questions on localization, presenting the main definitions of asymptotic localization and proving the main results alluded to above. In order to develop a better understanding of the issues involved, before moving on to the general case, we start by studying these questions of asymptotic localization in the case of mapping-positive correspondences, and this leads to another interesting result of this part of the work, which states that a mapping-positive symbol correspondence sequence localizes (resp. anti-localizes) classically if and only if

$$\lim_{n\to\infty} c_1^n = 1 \text{ (resp. } = -1) \ , \ \lim_{n\to\infty} c_2^n = 1 \ , \tag{1.10}$$

cf. Theorem 4.1.4. Hence, from (1.8) we have that, for mapping-positive symbol correspondence sequences, the Poisson algebra of smooth functions emerges as an asymptotic limit of the twisted algebra of symbols if and only if (1.10) is satisfied, a much weaker condition than (1.5) for general correspondences. In addition we have that, for mapping-positive and for positive-dual symbol correspondence sequences, (1.10) implies the r.h.s. of (1.5), a rather strong implication, cf. Corollary 4.2.2.4.

Up to Chapter 4, we shall have been investigating the localization property for spin systems from the approach which is inverse of quantization, namely, dequantization and (semi)classical limit (same approach used in (RIOS; STRAUME, 2014)). But it is also possible to investigate asymptotic localization of symbol correspondences for spin systems from the quantization approach.

Now, there are various approaches to quantization of a Poisson manifold, but the most common one, deformation quantization, was shown by Rieffel (over three decades ago) to be ill suited for producing a $SO(3)$-invariant "star product" on S^2 that actually converges (RIEFFEL, 1989) (see also (HAWKINS, 2008)). This problem disappears when defining a sequence of twisted products of the form (1.4), for $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$ of (anti-)Poisson type (cf. (RIOS; STRAUME, 2014, Chap. 9)), and now we can also take advantage of having symbol correspondence sequences in order to define *sequential quantizations* of smooth functions on S^2, as developed in Chapter 5 and outlined below.

Given a symbol correspondence sequence of (anti-)Poisson type $\boldsymbol{W} = (W^j)_{n\in\mathbb{N}}$, for any $f \in C^\infty_{\mathbb{C}}(S^2)$ we define its W*-quantization* as the sequence of operators

$$\mathbf{F}^w = (F_n^w)_{n\in\mathbb{N}}, \ F_n^w = [W^j]^{-1}(f) \ , \tag{1.11}$$

with a similar definition for

$$\widetilde{\mathbf{F}}^w = (\widetilde{F}_n^w)_{n\in\mathbb{N}} \ , \ \widetilde{F}_n^w = [\widetilde{W}^j]^{-1}(f) \ , \tag{1.12}$$

the $\widetilde{W}$*-quantization* of f, which is obtained from the dual symbol correspondence sequence $\widetilde{\boldsymbol{W}} = (\widetilde{W}^j)_{n\in\mathbb{N}}$, cf. Definitions 2.2.7 and 5.1.3.

On the other hand, given an operator sequence $\mathbf{F} = (F_n)_{n\in\mathbb{N}}$, for each j we have an action $F_n : \mathcal{H}_j \to \mathcal{H}_j$ and we want to make sense of such an action in the limit $j \to \infty$. For this, we develop the notion of a *ground Hilbert space* and operators on this asymptotic limit of a nested sequence of Hilbert spaces, as follows.

Let $\mathfrak{H} = (\mathcal{H}_j, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$ denote a sequence of $(n+1)$-dimensional complex Hilbert spaces, where $\langle\cdot|\cdot\rangle_j$ is the inner product on $\mathcal{H}_j$, so that $\mathbf{F} : \mathfrak{H} \to \mathfrak{H}$ as above. We say that $\mathfrak{H}^<$ is a nested sequence of Hilbert spaces if there are *nesting maps* $\iota_j^{j'} : \mathcal{H}_j \to \mathcal{H}_{j'}$ whenever $j \leq j'$, which are isometric inclusions. This allows us to define a *nested norm* $||\cdot||$ on $\mathfrak{H}^<$ and then the notion of a *convergent state sequence* $\Phi \in \mathfrak{H}^<_\infty$, $\Phi = (\phi^j)_{2j\in\mathbb{N}}$, $\phi^j \in \mathcal{H}_j$ (convergent in the sense of Cauchy w.r.t. $||\cdot||$), and a convergent operator sequence $\mathbf{F} : \mathfrak{H}^<_\infty \to \mathfrak{H}^<_\infty$, cf. Definitions 5.2.2-5.2.3.

With the notion of a *well-nested basis sequence* $\mathfrak{E} = (\{\mathbf{e}_k^j\}_{1\leq k\leq 2j+1})_{2j\in\mathbb{N}}$, where each $\{\mathbf{e}_k^j\}_{1\leq k\leq 2j+1}$ is a basis for $\mathcal{H}_j$, cf. Definition 5.2.4, we can identify the ground Hilbert space $\mathcal{H}$ of $\mathfrak{H}^<_\infty$ with the space of complex ℓ^2-sequences spanned by the *grounding basis* $\mathcal{E} = \{\mathbf{e}_k\}_{k\in\mathbb{N}}$, where $\mathbf{e}_k = \lim_{j\to\infty} \mathbf{e}_k^j$, $\forall k \in \mathbb{N}$, since $(\mathbf{e}_k^j)_{2j\in\mathbb{N}} \in \mathfrak{H}^<_\infty$, for each $k \in \mathbb{N}$, cf. Definition 5.2.5 and Theorem 5.2.1.

This latter theorem also shows that $(\mathfrak{E}, \mathcal{E})$ provides a canonical isomorphism between $\mathcal{H}$ and the set of equivalence classes of convergent state sequences, where, for $\Phi = (\phi_j)_{2j\in\mathbb{N}}, \Phi' = (\phi'_j)_{2j\in\mathbb{N}} \in \mathfrak{H}^<_\infty$,

$$\Phi \approx \Phi' \iff \lim_{j\to\infty} \phi_j = \lim_{j\to\infty} \phi'_j \quad \therefore \ \mathcal{H} \simeq \mathfrak{H}^<_\infty/\approx \ . \tag{1.13}$$

Then, Definition 5.2.7 imports this equivalence relation to convergent operator sequences,

$$\mathbf{F} \approx \mathbf{F}' \iff \mathbf{F}(\Phi) \approx \mathbf{F}'(\Phi) \ , \ \forall \Phi \in \mathfrak{H}^<_\infty \ ,$$

so that finally Theorem 5.2.5 identifies an operator on $\mathcal{H}$ with the equivalence class of a convergent operator sequence, that is,

$$\mathbf{F} : \mathfrak{H}^<_\infty \to \mathfrak{H}^<_\infty \iff [\mathbf{F}] = \mathcal{F} : \mathcal{H} \to \mathcal{H} \ . \tag{1.14}$$

These definitions and theorems are illustrated in Examples 2 and 3. In the first one, the nested sequence of Hilbert spaces $\mathcal{H}_j \hookrightarrow \mathcal{H}_{j'}$ is identified with a sequence of holomorphic polynomials on $\mathbb{C}P^1 \simeq S^2$ of degrees $2j = n \leq n' = 2j'$. Then, the ground Hilbert space is a convergent subspace of the ring of formal power series in one complex variable, which can be thought of as a space of modified holomorphic functions on $\mathbb{C}P^1$. In the second example, we identify $\mathcal{H}_j \hookrightarrow \mathcal{H}_{j'}$ with modified complex Fourier polinomials of degrees $j \leq j'$ on $\mathbb{R} \mod 2\pi$, the latter identified with a closed geodesic $\mathcal{X} \subset S^2$, and the ground Hilbert space is the space of complex functions on $\mathcal{X} \subset S^2$ defined via modified complex Fourier series.

Now, for some operator sequences $\mathbf{F} = (F_n)_{n\in\mathbb{N}}$, $F_n \in M_{\mathbb{C}}(n+1)$, one has a natural definition of its asymptotic norm $||\mathbf{F}||_\infty$ which is independent of the ground Hilbert space $\mathcal{H}$ on which $\mathbf{F}$ acts asymptotically via (1.14), in contrast with the usual operator norm for $\mathcal{F} : \mathcal{H} \to \mathcal{H}$. And in fact, for any operator sequence $\mathbf{F}$, one has a natural definition of $\mathbf{F}$

being *upper-bounded*, which is independent of the sequence of Hilbert spaces on which it acts, cf. Definition 5.1.2. Then, Proposition 5.4.1 states that, given a well-nested sequence of Hilbert spaces $(\boldsymbol{\mathfrak{H}}^{<}, \boldsymbol{\mathfrak{L}})$, an operator sequence $\mathbf{F}$ is a convergent operator sequence $\mathbf{F} : \boldsymbol{\mathfrak{H}}^{<}_{\infty} \to \boldsymbol{\mathfrak{H}}^{<}_{\infty}$, inducing an operator on the ground Hilbert space, $\mathcal{F} = [\mathbf{F}] : \mathcal{H} \to \mathcal{H}$, only if $\mathbf{F}$ is upper bounded.

In this setting, Example 4 show that if a symbol correspondence sequence $\boldsymbol{W}$ is of (anti-)Poisson type, generally their W- and $\widetilde{W}$-quantized functions may not define asymptotic operators on a ground Hilbert space. Then, Theorems 5.4.4 and 5.4.5 present relations between classical (resp. μ-analytical) localization of $\boldsymbol{W}$ and equality of the L^2-norm of f with the asymptotic norms $||\cdot||_{\infty}$ of $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$, cf. (1.11)-(1.12), for (J_3-invariant) functions $f \in C^{\infty}_{\mathbb{C}}([-1,1])$ (resp. $\mathcal{A}_{\mu}([-1,1])$). Thus, many of the results in section 5.2 provide relations between asymptotic localization of a symbol correspondence sequence $\boldsymbol{W}$ and the possibility that their W- and $\widetilde{W}$-quantized functions define asymptotic operators on a ground Hilbert space.

Finally, in the quantization setting there is a natural definition of a symbol correspondence sequence of (anti-)Poisson-type possessing classical (anti-)expectation, which means that, for any r-convergent $(\Pi_{k_n})_{n\in\mathbb{N}}$ sequence,

$$\exists \lim_{n\to\infty} \langle \Pi_{k_n} | \widetilde{F}^w_n \rangle \in \mathbb{C} \ , \ \forall f \in C^{\infty}_{\mathbb{C}}(S^2) \ , \tag{1.15}$$

cf. (1.12), where $\langle\cdot|\cdot\rangle$ is the usual Hilbert-Schmidt inner product of operators. Then, Theorem 5.4.8 shows the equivalence between classical (anti-)expectation of (anti-)Poisson-type $\boldsymbol{W}$ and its classical (anti-)localization. But at first sight, it looks surprising that the mere existence of the limit in (1.15) should be equivalent to the specific form of the limit in (1.7). Thus, showing their equivalence is the last main result of this part of the work.

We close, in Chapter 6, with some final thoughts on why asymptotic Poisson emergence is generally a weaker property than the classical localization property.

For completeness, in the appendix we provide a pedestrian proof of a well-known result in probability theory which is used for Theorem 4.1.4, and a proof of a well-known formula, Edmonds formula, used to prove Theorems 4.2.2 and 4.2.3.

CHAPTER

2

REVIEW OF SOME BASIC DEFINITIONS AND RESULTS

This chapter follows closely to the more detailed and extended treatment in (RIOS; STRAUME, 2014)[1].

2.1 Quantum and classical spin systems

Let $SU(2)$ be the special unitary subgroup of $GL_2(\mathbb{C})$, satisfying: $\det g = 1$ and $gg^\dagger = g^\dagger g = e$ for all $g \in SU(2)$. From these properties, $SU(2) \simeq S^3 = \{z_1, z_2 \in \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}$, hence it is compact and simply connected. The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is generated by the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ satisfying the commutation relations

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \ ,$$

where ϵ_{abc} is totally antisymmetric, and is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of the special orthogonal group $SO(3)$. For any isomorphism $d\psi : \mathfrak{su}(2) \to \mathfrak{so}(3)$, the homomorphism $\psi : SU(2) \to SO(3)$ has kernel $\mathbb{Z}_2$, thus $SO(3) = SU(2)/\mathbb{Z}_2$.

Since $SU(2)$ is compact, every one of its irreducible unitary representations is finite dimensional (cf. (NACHBIN, 1961)), commonly labeled by a half-integer number j, where $2j+1 \in \mathbb{N}$ is its dimension, so we will denote it by φ_j. A representation φ_j of $SU(2)$ is

[1] Some of the definitions and results summarized in this chapter are not present in the preliminary version of (RIOS; STRAUME, 2014) that was first posted in the arXiv (which could not be later updated there due to copyright restrictions). Although (RIOS; STRAUME, 2014) presents a full treatment of general symbol correspondences for spin system, various excellent treatments of some special symbol correspondences can be found elsewhere, cf. e.g. (BEREZIN, 1975), (VARILLY; GRACIA-BONDIA, 1989) and the review (KLIMOV; ROMERO; GUISE, 2017) and references therein.

also a representation of $SO(3)$ if and only if j is an integer. Now, for $n = 2j \in \mathbb{N}$, let $M_{\mathbb{C}}(n+1)$ denote the algebra of $(n+1)$-square complex matrices.

Definition 2.1.1 ((RIOS; STRAUME, 2014))**.** *A* quantum spin-j system *is a complex Hilbert space $\mathcal{H}_j \simeq \mathbb{C}^{n+1}$ with an irreducible unitary representation $\varphi_j : SU(2) \to G \subset U(n+1)$, G isomorphic to $SU(2)$ or $SO(3)$ according to whether j is strictly half-integer or is integer. The* operator algebra *of the spin-j system*[2] *is $\mathcal{B}(\mathcal{H}_j) \simeq M_{\mathbb{C}}(n+1)$.*

The spin operators $J_k = J_k^j = d\varphi_j(\sigma_k/2) \in \mathcal{B}(\mathcal{H}_j)$ correspond to the x, y, and z components of the total angular momentum or spin[3]. The usual approach for a spin-j system is to diagonalize the operator $J_3 \equiv J_z$, which has eigenvalues $m = -j, -j+1, ..., j-1, j$. We denote the vectors of an orthonormal basis of $\mathcal{H}_j$ comprised of eigenvectors of J_3 by $\boldsymbol{u}(j,m)$. Then, we define the ladder operators $J_\pm = J_1 \pm iJ_2$ and the norm-square operator $J^2 = J_1^2 + J_2^2 + J_3^2 = J_\mp J_\pm + J_3(J_3 \pm I)$, where I is the identity. The commutation relations are

$$[J_a, J_b] = i\epsilon_{abc}J_c \ , \quad [J_+, J_-] = 2J_3 \ , \quad [J_3, J_\pm] = \pm J_\pm \ , \tag{2.1}$$

thus $J^2 = j(j+1)I$, and these operators act as

$$\begin{aligned} J_3(\boldsymbol{u}(j,m)) &= m(\boldsymbol{u}(j,m)) \ , \\ J_+(\boldsymbol{u}(j,m)) &= \alpha_{j,m}\boldsymbol{u}(j,m+1) \ , \\ J_-(\boldsymbol{u}(j,m)) &= \beta_{j,m}\boldsymbol{u}(j,m-1) \ , \end{aligned} \tag{2.2}$$

where $\alpha_{j,m}$ and $\beta_{j,m}$ are non zero constants, except for $\alpha_{j,j} = \beta_{j,-j} = 0$. To (almost completely) eliminate the freedom of choosing an individual phase factor for each $\boldsymbol{u}(j,m)$, we choose a highest weight vector $\boldsymbol{u}(j,j)$ and fix all the other phases so that the constants $\beta_{j,m}$ are nonnegative real numbers. Thus, for such a basis, called *standard*, there is just one free phase on the choice of $\boldsymbol{u}(j,j)$ and

$$\alpha_{j,m} = \sqrt{(j-m)(j+m+1)} \ , \quad \beta_{j,m} = \sqrt{(j+m)(j-m+1)} \ . \tag{2.3}$$

To extend the representation $\varphi_j(g)$ acting on $\mathcal{H}_j$ to a representation $\Phi_j(g)$ acting on the operator space $M_{\mathbb{C}}(n+1) \simeq \mathcal{B}(\mathcal{H}_j) = Hom(\mathcal{H}_j, \mathcal{H}_j) \simeq \mathcal{H}_j \otimes \mathcal{H}_j^*$, we use the dual representation of $\varphi_j(g)$ acting on $\mathcal{H}_j^*$, denoted by $\check{\varphi}_j(g)$. Via the inner-product identification $\mathcal{H}_j^* \leftrightarrow \mathcal{H}_j$, we have that $\check{\varphi}_j(g) \leftrightarrow \varphi_j(g)^{-1}$, $\forall g \in SU(2)$. For the spin operators, $\check{J}_1 \leftrightarrow -J_1$, $\check{J}_2 \leftrightarrow J_2$ and $\check{J}_3 \leftrightarrow -J_3$, so $\check{J}_+ \leftrightarrow -J_-$ and $\check{J}_- \leftrightarrow -J_+$. Hence, a standard basis of $\mathcal{H}_j^*$ is formed by vectors

$$\check{\boldsymbol{u}}(j,m) \leftrightarrow (-1)^{j+m}\boldsymbol{u}(j,-m) \ , \tag{2.4}$$

[2] One often finds in the literature the term *spin-j system* referring to a lattice of n spin-1/2 particles, but such lattices have spatial degrees of freedom which are fully neglected in our definition.

[3] We are taking $\hbar = 1$ (equivalent to a rescaling of units).

where $\leftrightarrow$ denotes that the l.h.s. is identified as the dual of the r.h.s.

Thus, $\boldsymbol{u}(j,m_1)\otimes\check{\boldsymbol{u}}(j,m_2) = (-1)^{j+m_2}\mathcal{E}_{j-m_1+1,j+m_2+1}$, where $\mathcal{E}_{k,l}\in M_{\mathbb{C}}(n+1)$ is the one-element matrix $[\mathcal{E}_{k,l}]_{p,q}=\delta_{k,p}\delta_{l,q}$. Hence, $\varphi_j(g)\otimes\check{\varphi}_j(g)\leftrightarrow\varphi_j(g)\otimes\varphi_j(g)^{-1}$,

$$\Phi_j(g):\mathcal{B}(\mathcal{H}_j)\to\mathcal{B}(\mathcal{H}_j)\ ,\ P\mapsto P^g=\varphi_j(g)P\varphi_j(g)^{-1}\ ,\ \forall g\in SU(2), \tag{2.5}$$

and the spin operators $\boldsymbol{J}_k = d\Phi_j(\sigma_k/2)$ acting on $\mathcal{B}(\mathcal{H}_j)$ can be identified with $J_k = d\varphi_j(\sigma_k/2)$ acting on $P\in\mathcal{B}(\mathcal{H}_j)$ via the commutator, that is,

$$P\ \mapsto\ [J_k,P]=:J_k(P)\ . \tag{2.6}$$

Theorem 2.1.1 (*cf. e.g.* (RIOS; STRAUME, 2014))**.** *The Clebsch-Gordan series for $\mathcal{B}(\mathcal{H}_j)$ is*

$$\varphi_j\otimes\check{\varphi}_j=\bigoplus_{l=0}^{n}\varphi_l\ .$$

Thus, the induced action (2.5) of $SU(2)$ on $\mathcal{B}(\mathcal{H}_j)$ is effectively an $SO(3)$ action. For each φ_l, we find a standard basis of vectors $\boldsymbol{e}^j(l,m)$ as we did for $\mathcal{H}_j$. The orthonormal basis $\{\boldsymbol{u}(j,m_1)\otimes\check{\boldsymbol{u}}(j,m_2)\}$ of $\mathcal{B}(\mathcal{H}_j)$ is called *uncoupled*, whereas the basis consisting of $\boldsymbol{e}^j(l,m)$, where $\{\boldsymbol{e}^j(l,m), -l\le m\le l\}$ is a basis of the $(2l+1)$-dimensional $SO(3)$-invariant subspace φ_l, is called *coupled*[4]. This basis satisfies

$$\begin{aligned}&[J_+,\boldsymbol{e}^j(l,m)]=\alpha_{l,m}\boldsymbol{e}^j(l,m+1)\ ,\ [J_-,\boldsymbol{e}^j(l,m)]=\beta_{l,m}\boldsymbol{e}^j(l,m-1)\ ,\\ &[J_3,\boldsymbol{e}^j(l,m)]=m\boldsymbol{e}^j(l,m)\ ,\ \sum_{k=1}^{3}[J_k,[J_k,\boldsymbol{e}^j(l,m)]]=l(l+1)\boldsymbol{e}^j(l,m)\ ,\end{aligned} \tag{2.7}$$

being also orthonormal w.r.t. the Hilbert-Schmidt inner product on $M_{\mathbb{C}}(n+1)$.

Theorem 2.1.2 ((RIOS; STRAUME, 2014))**.** *The coupled standard basis vectors $\boldsymbol{e}^j(l,m)$ of $\mathcal{B}(\mathcal{H}_j)$ satisfy*

$$\boldsymbol{e}^j(l,-m)=(-1)^m\boldsymbol{e}^j(l,m)^T \tag{2.8}$$

and are given explicitly, for $0\le m\le l$, by

$$\boldsymbol{e}^j(l,m)\ =\ \frac{(-1)^l}{\nu^n_{l,m}}\sum_{k=0}^{l-m}(-1)^k\binom{l-m}{k}J_-^{l-m-k}J_+^lJ_-^k\ , \tag{2.9}$$

$$\nu^n_{l,m}\ =\ \frac{l!}{\sqrt{2l+1}}\sqrt{\frac{(n+l+1)!}{(n-l)!}}\sqrt{\frac{(l-m)!}{(l+m)!}}\ . \tag{2.10}$$

The coefficients of the change of orthonormal basis

$$\boldsymbol{u}(j,m_1)\otimes\check{\boldsymbol{u}}(j,m_2)=\sum_{l=0}^{n}\sum_{m=-l}^{l}C^{\,j,\ j,\ l}_{m_1,m_2,m}\boldsymbol{e}^j(l,m) \tag{2.11}$$

4 Taking $\mathcal{H}_j^*$ as another spin-j system, this is equivalent to addition (actually subtraction) of spin. In Dirac's notation, we can write $\boldsymbol{u}(j,m_1)\otimes\check{\boldsymbol{u}}(j,m_2)=|j,m_1,j,m_2\rangle$ and $\boldsymbol{e}^j(l,m)=|(j,j)l,m\rangle$.

are called Clebsch-Gordan coefficients and for $SU(2)$ they have been extensively studied[5], cf. e.g. (BIEDENHARN; LOUCK, 1984; VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988). They are unique up to a phase and we adopt the usual convention in which all Clebsch-Gordan coefficients are real.

We now turn to the classical spin system.

Definition 2.1.2 ((RIOS; STRAUME, 2014)**.** *The* classical spin system *is the homogeneous 2-sphere with its Poisson algebra $\{C^\infty_{\mathbb{C}}(S^2), \omega\}$, where the symplectic form is the usual area form with local expression $\omega = \sin\varphi \, d\varphi \wedge d\theta$ in spherical coordinates w.r.t. the north pole.*

We refer to (RIOS; STRAUME, 2014) for a detailed justification of the above definition, but here we point out that every quantum spin-j system is a mechanical system with one degree of freedom, which is consistent with its classical phase space being the 2-dimensional symplectic manifold which is the generic coadjoint orbit of $SU(2)$.

Now, just as for the quantum operator spaces, we can find an orthonormal basis for $C^\infty_{\mathbb{C}}(S^2)$ by decomposing the action of $SO(3)$ on $C^\infty_{\mathbb{C}}(S^2)$ in $(2l+1)$-dimensional invariant subspaces which are spanned by the standard basis of spherical harmonics:

$$Y_l^m(\boldsymbol{n}) = \sqrt{2l+1}\sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\varphi)e^{im\theta}, \tag{2.12}$$

where (φ, θ) are spherical polar coordinates of $\boldsymbol{n} \in S^2$ w.r.t. the north pole $\boldsymbol{n}_0$, in other words, the colatitude and longitude on S^2, and P_l^m are the associated Legendre polynomials on $[-1, 1]$. As in (2.7), Y_l^m satisfies

$$J_+(Y_l^m) = \alpha_{l,m} Y_l^{m+1} \ , \ J_-(Y_l^m) = \beta_{l,m} Y_l^{m-1} \ , \ J_3(Y_l^m) = mY_l^m \ , \tag{2.13}$$

where $J_\pm = J_1 \pm iJ_2$ and $J_k = iL_k$, for the generators L_k of the Lie algebra $\mathfrak{so}(3)$ acting on $C^\infty_{\mathbb{C}}(S^2)$ via $L_1 = z\partial_y - y\partial_z$, $L_2 = x\partial_z - z\partial_x$, $L_3 = y\partial_x - x\partial_y$, with (x, y, z) denoting the cartesian coordinates of the unit sphere $S^2 \subset \mathbb{R}^3$. Accordingly, for $J^2 = J_1^2 + J_2^2 + J_3^2$, Y_l^m also satisfies

$$J^2(Y_l^m) = l(l+1)Y_l^m \quad , \quad \langle Y_l^m | Y_{l'}^{m'} \rangle = \delta_{l,l'}\delta_{m,m'} \ , \tag{2.14}$$

where $\langle\cdot|\cdot\rangle$ is the normalized inner product on $C^\infty_{\mathbb{C}}(S^2)$, cf. (2.16) below.

[5] In particular, the Clebsch-Gordan coefficients in (2.11) vanish when $m \neq m_1 + m_2$ so that the summation in m in (2.11) is actually moot, only the term with $m = m_1 + m_2$ survives in its r.h.s.

2.2 Symbol correspondences

By a spin-j symbol correspondence, we mean a map that associates to every spin-j operator a unique function on the 2-sphere, called its symbol, satisfying very basic and natural properties, as follows.

Definition 2.2.1. *A map* $W^j : \mathcal{B}(\mathcal{H}_j) \to C^{\infty}_{\mathbb{C}}(S^2)$, $P \mapsto W^j[P] = W^j_P$ *, is a* symbol correspondence for a spin-j system *if,* $\forall P \in \mathcal{B}(\mathcal{H}_j)$, $\forall g \in SO(3)$*, it satisfies*

i) Linearity and injectivity;

ii) Equivariance: $W^j_{P^g} = (W^j_P)^g$;

iii) Reality: $W^j_{P^\dagger} = \overline{W^j_P}$;

iv) Normalization: $\frac{1}{4\pi}\int_{S^2} W^j_P dS = \frac{1}{n+1} tr(P)$.

If injectivity fails but all other properties hold, W^j *is called a* pre-symbol map.

Remark 2.2.1. *In (ii), on the r.h.s. the action on functions is the one induced by the standard* $SO(3)$ *action on the unit sphere* $S^2 \subset \mathbb{R}^3$*, and on the l.h.s. the action on operators* ($P \mapsto P^g$) *is given by (2.5) for any of the two choices of lifting* $g \in SO(3)$ *to* $\tilde{g} \in SU(2)$*. From (iii), hermitian operators are mapped to real functions. Condition (iv) is necessary to assure that* $I \mapsto 1$ *(constant function 1).*

The above properties for a spin-j symbol correspondence were first set out by Stratonovich (STRATONOVICH, 1956) who imposed a more strict property which implies (iv), as follows:

Definition 2.2.2. *A* Stratonovich-Weyl correspondence *is a symbol correspondence that is an isometry with respect to the normalized inner products, that is, it satisfies*

$$v)\ Isometry: \qquad \langle P|Q\rangle_j = \langle W^j_P | W^j_Q\rangle \ ,$$

where

$$\langle P|Q\rangle_j = \frac{1}{n+1}\langle P|Q\rangle = \frac{1}{n+1} tr(P^\dagger Q) \ , \tag{2.15}$$

$$\langle W^j_P | W^j_Q\rangle = \frac{1}{4\pi}\int_{S^2} \overline{W^j_P} W^j_Q dS \ . \tag{2.16}$$

But as can be seen from various examples, the first one set out by Berezin (BEREZIN, 1975), this isometry condition is too strict to be imposed on general symbol correspondences.

Schur's lemma implies that a symbol correspondence is an isomorphism between the subspaces spanned by $\boldsymbol{e}^j(l,m)$ and Y_l^m for fixed l. So we can turn the injectivity requirement into bijectivity by taking $W^j : \mathcal{B}(\mathcal{H}_j) \to Poly_{\mathbb{C}}(S^2)_{\leq n}$, where

$$Poly_{\mathbb{C}}(S^2)_{\leq n}$$

is the space of complex polynomials on S^2 of proper degree less than or equal to n.

Any symbol correspondence can be characterized as follows:

Theorem 2.2.1 ((RIOS; STRAUME, 2014)). *A map $W^j : \mathcal{B}(\mathcal{H}_j) \to C^\infty_{\mathbb{C}}(S^2)$ is a symbol correspondence if and only if there is a diagonal matrix $K \in M_{\mathbb{C}}(n+1)$ such that $tr(K) = 1$ and*

$$W^j_P(\boldsymbol{n}) = tr(PK(\boldsymbol{n})), \tag{2.17}$$

where $K(\boldsymbol{n}) = K^g$ (cf. (2.5)) for $\boldsymbol{n} = g\boldsymbol{n}_0$, $\boldsymbol{n}_0$ the north pole on S^2. K is given by

$$K = \frac{1}{n+1}I + \sum_{l=1}^{n} c_l^n \sqrt{\frac{2l+1}{n+1}}\boldsymbol{e}^j(l,0) \ , \tag{2.18}$$

with $\boldsymbol{e}^j(l,0)$ given by (2.9), where $c_l^n \in \mathbb{R}^*$, for $l = 1, ..., n$. In particular,

$$W^j : \sqrt{n+1}\boldsymbol{e}^j(l,m) \mapsto c_l^n Y_l^m \in Poly_{\mathbb{C}}(S^2)_{\leq n} \ . \tag{2.19}$$

Definition 2.2.3 ((RIOS; STRAUME, 2014)). *The diagonal matrix K as above is called the* operator kernel *and the n nonzero real numbers c_l^n as above are called the* characteristic numbers *of the spin-j symbol correspondence.*

The following theorem, first obtained in an equivalent form by Gracia-Bondia and Varilly (VARILLY; GRACIA-BONDIA, 1989), follows immediately from (2.19).

Theorem 2.2.2. *A symbol correspondence is a Stratonovich-Weyl correspondence if and only if all of its characteristic numbers have unitary norm, that is,*

$$|c_l^n| = 1 \ , \ 1 \leq \forall l \leq n \ . \tag{2.20}$$

In particular, we have the following more special cases:

Definition 2.2.4 ((RIOS; STRAUME, 2014)). *The* standard Stratonovich-Weyl correspondence *is the symbol correspondence with all characteristic numbers equal to 1, i.e. given by $\varepsilon_l^n = 1$, $1 \leq \forall l \leq n$. The* alternate Stratonovich-Weyl correspondence *is the symbol correspondence with characteristic numbers given by $\varepsilon_{l-}^n = (-1)^l$, $1 \leq \forall l \leq n$.*

Definition 2.2.5. *A* Berezin correspondence[6] *is a symbol correspondence defined via (2.17) by an operator kernel which is a projector $\Pi_k = \mathcal{E}_{k,k}$.*

[6] The original definition by Berezin (BEREZIN, 1975) uses only the first projector, but this more general form was, as far as we know, set out by Marc Rieffel (lectures at Berkeley, 2002).

Now, the projectors Π_k decompose as

$$\Pi_k = \frac{1}{n+1} I + (-1)^{k+1} \sum_{l=1}^{n} C_{m,-m,\ 0}^{\,j,\ \ j,\ l} \boldsymbol{e}^j(l,0), \tag{2.21}$$

where $m = j-k+1$ (*cf.* (RIOS; STRAUME, 2014)). Since some Clebsch-Gordan coefficient on the decomposition may vanish, not all Π_k define symbol correspondences for all $n \in \mathbb{N}$.

However, let $h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ denote the inner product that is conjugate linear in the first variable and consider the maps:

$$\Phi_j : \mathbb{C}^2 \to \mathbb{C}^{n+1},\ (z_1, z_2) \mapsto \left(z_1^n, ..., \sqrt{\binom{n}{k}} z_1^{n-k} z_2^k, ..., z_2^n\right) ,$$

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2,\ (z_1, z_2) \mapsto (-\overline{z}_2, \overline{z}_1) \quad ; \quad \pi : \mathbb{C}^2 \to \mathbb{R}^3,\ (z_1, z_2) \mapsto \boldsymbol{n} = (x, y, z) ,$$
$$x + iy = 2\overline{z}_1 z_2 \ ,\ \ z = |z_1|^2 - |z_2|^2 \ .$$

On $S^3 \subset \mathbb{C}^2$, $\pi : S^3 \to S^2$ is the *Hopf map* and $\Phi_j : S^3 \to S^{2n+1}$. Then we have:

Theorem 2.2.3 ((RIOS; STRAUME, 2014))**.** *For every $n \in \mathbb{N}$, consider the maps*

$$B^j, B^{j-} : M_{\mathbb{C}}(n+1) \to Poly_{\mathbb{C}}(S^2)_{\leq n}$$
$$B^j : P \mapsto B^j_P(\boldsymbol{n}) = h(\Phi_j(z_1, z_2), P\Phi_j(z_1, z_2)),$$
$$B^{j-} : P \mapsto B^{j-}_P(\boldsymbol{n}) = h(\Phi^-_j(z_1, z_2), P\Phi^-_j(z_1, z_2)),$$

where $\Phi^-_j = \Phi_j \circ \sigma$, $\boldsymbol{n} = \pi(z_1, z_2)$, $(z_1, z_2) \in S^3$. Then, each of the maps B^j, B^{j-} satisfies (i)-(iv) in Definition 2.2.1 and is therefore a symbol correspondence. The operator kernel for B^j is Π_1 and the one for B^{j-} is Π_{n+1}. Their characteristic numbers are denoted by b^n_l and b^n_{l-}, respectively, and are given by

$$b^n_l = \sqrt{\frac{n+1}{2l+1}} C^{\,j,\ j,\ l}_{j,-j,0} = \frac{n!\sqrt{n+1}}{\sqrt{(n+l+1)!(n-l)!}} \ ,\ \ b^n_{l-} = (-1)^l b^n_l \ . \tag{2.22}$$

Definition 2.2.6 ((RIOS; STRAUME, 2014))**.** *The map B^j defines the* standard Berezin correspondence *(cf. (BEREZIN, 1975)) and the map B^{j-} defines the* alternate Berezin correspondence.

Definition 2.2.7. *If W^j is a spin-j symbol correspondence, its* dual *is the spin-j symbol correspondence $\widetilde{W}^j$ satisfying, $\forall P, Q \in \mathcal{B}(\mathcal{H}_j)$,*

$$\langle P|Q\rangle_j = \langle \widetilde{W}^j_P | W^j_Q \rangle = \langle W^j_P | \widetilde{W}^j_Q \rangle \ . \tag{2.23}$$

Theorem 2.2.4 ((RIOS; STRAUME, 2014))**.** *If $\forall P \in \mathcal{B}(\mathcal{H}^j)$, $W^j : P \mapsto W^j_P$ is the spin-j symbol correspondence defined by the operator kernel K via (2.17), with characteristic numbers c^n_l, then its dual is the symbol correspondence $\widetilde{W}^j : P \mapsto \widetilde{W}^j_P$ defined implicitly by*

$$P = \frac{n+1}{4\pi} \int_{S^2} \widetilde{W}^j_P(\boldsymbol{n}) K(\boldsymbol{n}) dS$$

and its characteristic numbers are

$$\tilde{c}^n_l = 1/c^n_l \ . \tag{2.24}$$

Definition 2.2.8 ((RIOS; STRAUME, 2014)). *The* standard Toeplitz correspondence[7] *is the symbol correspondence dual to the standard Berezin correspondence. Likewise, the* alternate Toeplitz correspondence *is dual to the alternate Berezin correspondence.*

Corollary 2.2.4.1. *The standard (resp. alternate) Toeplitz correspondence is well defined* $\forall n \in \mathbb{N}$ *and has characteristic numbers* $t_l^n = 1/b_l^n$ *(resp.* $t_{l-}^n = (-1)^l/b_l^n$*).*

Remark 2.2.2. *It follows from Definitions 2.2.2 and 2.2.7 that Stratonovich-Weyl (isometric) correspondences are self-dual, and vice-versa. But from (2.22), standard and alternate Berezin and Toeplitz correspondences are not isometric.*

Note also from (2.22) that all $b_l^n > 0$*. Symbol correspondences with all* $c_l^n > 0$ *are called* characteristic-positive*. In particular, the standard Stratonovich-Weyl correspondence is the only characteristic-positive isometric correspondence.*

Berezin correspondences are particular cases of a more general class:

Definition 2.2.9. *A symbol correspondence is* mapping-positive *if it maps positive (resp. positive-definite) operators to positive (resp. strictly-positive) functions.*

Theorem 2.2.5 ((RIOS; STRAUME, 2014)). *A symbol correspondence is mapping-positive if and only if its operator kernel is a convex combination of projectors* Π_k*, for* $k = 1, ..., n+1$*.*

As explained in (RIOS; STRAUME, 2014), from the above theorem a synonym for mapping-positive correspondence is *coherent-state* correspondence. Like for Berezin correspondences, in general not all convex combinations of projectors define an operator kernel for which the pre-symbol map is injective. But we have the following:

Theorem 2.2.6 ((RIOS; STRAUME, 2014)). *For every* $n \in \mathbb{N}$*, the positive pre-symbol map* $M_{\mathbb{C}}(n+1) \to Poly_{\mathbb{C}}(S^2)_{\leq n}$ *defined via (2.17) by the operator kernel*

$$S_{1/2} = \frac{1}{2}\left(\Pi_{\lfloor j+1/2 \rfloor} + \Pi_{\lfloor j+1 \rfloor}\right) \ , \tag{2.25}$$

where $\lfloor x \rfloor$ *denotes integer part of* x*, is bijective, i.e., the operator kernel* $K = S_{1/2}$ *as above defines a mapping-positive symbol correspondence for every* $n \in \mathbb{N}$*.*

Definition 2.2.10 ((RIOS; STRAUME, 2014)). *For every* $n = 2j \in \mathbb{N}$*, the* upper-middle-state *symbol correspondence is the one defined by the operator kernel* $S_{1/2}$ *in (2.25).*

The expressions for the characteristic numbers p_l^n of this symbol correspondence are rather long and can be found in (RIOS; STRAUME, 2014, Proposition 6.2.54). The correspondence with characteristic numbers $p_{l-}^n = (-1)^l p_l^n$, also well defined $\forall n \in \mathbb{N}$, is called the *lower-middle-state* correspondence. Neither of these two is characteristic-positive.

[7] The standard Berezin and standard Toeplitz correspondences are also commonly known in the literature as Berezin's covariant and contravariant correspondences, respectively (cf. (BEREZIN, 1975)).

2.3 Twisted products and symbol correspondence sequences

The operator space of a spin-j system has a natural algebraic structure defined by the usual matrix product. Via any given symbol correspondence, $Poly_{\mathbb{C}}(S^2)_{\leq n} \subset C^\infty_{\mathbb{C}}(S^2)$ imports this algebraic structure.

Definition 2.3.1. *Given a symbol correspondence W^j such that $\vec{c} = (c_1^n, \cdots, c_n^n)$ is the n-tuple of its characteristic numbers, the* twisted product of symbols

$$\star_{\vec{c}}^n : Poly_{\mathbb{C}}(S^2)_{\leq n} \times Poly_{\mathbb{C}}(S^2)_{\leq n} \to Poly_{\mathbb{C}}(S^2)_{\leq n}$$

is defined $\forall P, Q \in \mathcal{B}(\mathcal{H}_j)$ by

$$W_P^j \star_{\vec{c}}^n W_Q^j = W_{PQ}^j \ .$$

Theorem 2.3.1. *For any symbol correspondence W^j, the twisted product of symbols defines a $SO(3)$-invariant associative unital star algebra on $Poly_{\mathbb{C}}(S^2)_{\leq n}$.*

We refer to (RIOS; STRAUME, 2014) for detailed extensive formulas for twisted products of general and some special symbol correspondences, as well as various of their properties,

For each correspondence W^j, the space of symbols $Poly_{\mathbb{C}}(S^2)_{\leq n} \subset C^\infty_{\mathbb{C}}(S^2)$ with usual addition and induced twisted product $\star_{\vec{c}}^n$ is called the *$\vec{c}$-twisted j-algebra* of smooth functions on the sphere. For a given spin number j, all $\vec{c}$-twisted j-algebras are isomorphic, since they are all isomorphic to the operator algebra $M_{\mathbb{C}}(n+1)$.

But now we introduce the following:

Definition 2.3.2 ((RIOS; STRAUME, 2014))**.** *Let $\Delta^+(\mathbb{N}^2) = \{(n, l) \in \mathbb{N}^2 : n \geq l > 0\}$. For any function $\mathcal{C} : \Delta^+(\mathbb{N}^2) \to \mathbb{R}^*$, we denote by $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{2j=n\in\mathbb{N}}$ the* sequence of symbol correspondences *with characteristic numbers $c_l^n = \mathcal{C}(n, l)$, $c_0^n = 1, \forall n \in \mathbb{N}$. And we denote by*

$$\boldsymbol{W}_{\mathcal{C}}(S^2, \star) = ((Poly_{\mathbb{C}}(S^2)_{\leq n}, \star_{\vec{c}}^n))_{n\in\mathbb{N}}$$

the associated sequence of twisted algebras *of smooth functions on the sphere. In particular, we say that $\boldsymbol{W}_{\mathcal{C}}$ is of* limiting type *if $\exists \lim_{n\to\infty} \mathcal{C}(n, l) = c_l^\infty \in \mathbb{R}$, $\forall l \in \mathbb{N}$.*

Therefore, we are interested in whether and how the Poisson algebra of smooth functions on the sphere emerges asymptotically from a sequence of twisted algebras associated to a given sequence of symbol correspondences.

Definition 2.3.3 ((RIOS; STRAUME, 2014))**.** *A sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}}$ is of* Poisson type *if, $\forall l_1, l_2 \in \mathbb{N}$, its twisted products satisfy* *i)* $\lim_{n\to\infty}(Y_{l_1}^{m_1} \star_{\vec{c}}^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star_{\vec{c}}^n Y_{l_1}^{m_1}) = 0$,

ii) $\lim_{n\to\infty}(Y_{l_1}^{m_1} \star_{\vec{c}}^n Y_{l_2}^{m_2} + Y_{l_2}^{m_2} \star_{\vec{c}}^n Y_{l_1}^{m_1}) = 2Y_{l_1}^{m_1}Y_{l_2}^{m_2}$,

iii) $\lim_{n\to\infty}(n[Y_{l_1}^{m_1} \star_{\vec{c}}^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star_{\vec{c}}^n Y_{l_1}^{m_1}]) = 2i\{Y_{l_1}^{m_1}, Y_{l_2}^{m_2}\}$.

And it is of anti-Poisson type *if the third property is replaced by*

iii') $\lim_{n\to\infty}(n[Y_{l_1}^{m_1} \star_{\vec{c}}^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star_{\vec{c}}^n Y_{l_1}^{m_1}]) = -2i\{Y_{l_1}^{m_1}, Y_{l_2}^{m_2}\}$.

The convergences taken uniformly.

Now, there is a simple numerical criterion to know when the above definition is satisfied by a sequence of symbol correspondences.

Theorem 2.3.2 ((RIOS; STRAUME, 2014))**.** *A sequence of symbol correspondences* $\boldsymbol{W}_{\mathcal{C}}$ *is of Poisson (resp. anti-Poisson) type if and only if it is of limiting type and its characteristic numbers satisfy* $c_l^\infty = 1$ *(resp.* $c_l^\infty = (-1)^l$*),* $\forall l \in \mathbb{N}$.

Thus, generic sequences of symbol correspondences are not of Poisson or anti-Poisson type, nor even of limiting type, and this is also the case for generic sequences of Stratonovich-Weyl (isometric) symbol correspondences, cf. (RIOS; STRAUME, 2014).

Corollary 2.3.2.1 ((RIOS; STRAUME, 2014))**.** *The sequence of standard Stratonovich-Weyl correspondences and the sequences of standard Berezin and standard Toeplitz correspondences are of Poisson type. The sequences of alternate Stratonovich-Weyl, alternate Berezin and alternate Toeplitz correspondences are all of anti-Poisson type. The sequence of upper-middle-state correspondences is of limiting type but not of Poisson or anti-Poisson type. Likewise for the sequence of lower-middle-state correspondences. The sequences of their dual correspondences are not even of limiting type.*

The symbol correspondence sequences of Poisson or anti-Poisson type are the ones for which Poisson dynamics emerges asymptotically from quantum dynamics in the limit of high spin number $j \to \infty$. In this work, we are interested in obtaining a more intuitive criterion which can replace the numerical criterion of Theorem 2.3.2. This will be the subject of Chapter 4, further below.

CHAPTER

3

A SPLITTING OF THE SET OF SPIN-j SYMBOL CORRESPONDENCES

The mapping-positive property for a correspondence implies in many nice analytical properties for the symbols and, in particular, the corresponding symbols of (pure or mixed) states are nonnegative functions which, upon suitable renormalization, can be seen as probability densities on S^2. However, no Stratonovich-Weyl (*i.e.*, isometric) correspondence possesses this nice mapping-positive property. This statement was stated as a conjecture in (RIOS; STRAUME, 2014). Here we shall give its proof. In fact, we present below a more general result, which follows from introducing:

Definition 3.0.1. *If W^j is mapping-positive symbol correspondence, cf. Definition 2.2.9, then its dual $\widetilde{W}^j$ is called a* positive-dual *symbol correspondence.*

Thus, the standard and alternate Toeplitz correspondences and the duals of upper and lower-middle state correspondences are positive-dual correspondences.

Just like mapping-positive correspondences map positive(-definite) operators to (strictly-)positive functions, the positive-dual correspondences do the reverse:

Proposition 3.0.1. *If W^j is positive-dual, then $[W^j]^{-1} : Poly_{\mathbb{C}}(S^2)_{\leq n} \to M_{\mathbb{C}}(n+1)$ maps (strictly-)positive functions to positive(-definite) operators.*

Proof. This follows from (2.23) and the positivity of $\widetilde{W}^j$. Let $f \in Poly_{\mathbb{R}^+}(S^2)$ be positive and $F = [W^j]^{-1}(f)$. Since f is real, F is Hermitian and there is a basis $\{e_1, ..., e_{n+1}\}$ that diagonalizes F. For Π_{e_k} the projector onto the e_k subspace, the eigenvalues of F are $\langle \Pi_{e_k} | F \rangle$. Now, $\langle \Pi_{e_k} | F \rangle = (n+1) \langle \widetilde{W}^j_{\Pi_{e_k}} | f \rangle$, *cf.* (2.23). But Π_{e_k} is a positive operator, so $\widetilde{W}^j_{\Pi_{e_k}}$ is a (nontrivial) positive function, since $\widetilde{W}^j$ is mapping-positive. Therefore, $\langle \widetilde{W}^j_{\Pi_{e_k}} | f \rangle \geq 0$ and the eigenvalues of F are all nonnegative. If f is strictly-positive, every $\langle \widetilde{W}^j_{\Pi_{e_k}} | f \rangle > 0$

because for each k there is an open $\mathcal{B}_k \subset S^2$ s.t. $\widetilde{W}^j_{\Pi_{e_k}}$ and f are both strictly positive on $\mathcal{B}_k$. $\square$

Now, let us denote by $\mathcal{S}^j$ the set of all spin-j symbol correspondences and by $\mathcal{S}^j_{=}$, $\mathcal{S}^j_{<}$ and $\mathcal{S}^j_{>}$ the subsets of isometric, mapping-positive and positive-dual spin-j symbol correspondences, respectively. We then have the following splitting:

Theorem 3.0.2. *For any $n = 2j \in \mathbb{N}$, the subsets $\mathcal{S}^j_{=}$, $\mathcal{S}^j_{<}$ and $\mathcal{S}^j_{>}$ are mutually disjoint.*

Proof. Let c^n_l, $l = 1, ..., n$, be the characteristic numbers of a mapping-positive correspondence $W^j \in \mathcal{S}^j_{<}$. From Theorem 2.2.5 and eqs. (2.18) and (2.21), we have

$$c^n_l = \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} (-1)^{k+1} a_k C^{j,\ \ \ j,\ l}_{m,-m,\ 0} \ , \tag{3.1}$$

where $m = j-k+1$, $a_k \geq 0$, for $1 \leq k \leq n+1$, and $\sum_{k=1}^{n+1} a_k = 1$. Since the Clebsch-Gordan coefficients are coefficients of a unitary transformation of basis, they satisfy $|C^{j,\ \ \ j,\ l}_{m,-m,\ 0}| \leq 1$. Hence,

$$|c^n_l| = \left| \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} (-1)^{k+1} a_k C^{j,\ \ \ j,\ l}_{m,-m,\ 0} \right| \leq \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} a_k \left| C^{j,\ \ \ j,\ l}_{m,-m,\ 0} \right| \leq \sqrt{\frac{n+1}{2l+1}}.$$

That is,

$$|c^n_l| \leq \sqrt{\frac{n+1}{2l+1}} \ , \quad 1 \leq \forall l \leq n \ . \tag{3.2}$$

In particular,

$$|c^n_l| < 1 \ , \quad j < \forall l \leq n = 2j \ . \tag{3.3}$$

From Theorem 2.2.4, eq. (2.24), the characteristic numbers $\tilde{c}^n_l$ of its dual $\widetilde{W}^j$ satisfy

$$|\tilde{c}^n_l| > 1 \ , \quad j < \forall l \leq n = 2j \ . \tag{3.4}$$

Thus, from (2.20), (3.3) and (3.4), the three subsets are mutually disjoint. $\square$

Remark 3.0.1. *We must bear in mind, however, that the union $\mathcal{U}^j = \mathcal{S}^j_{<} \cup \mathcal{S}^j_{=} \cup \mathcal{S}^j_{>}$ is a proper subset*[1] *of $\mathcal{S}^j$. Therefore, to complete this splitting of $\mathcal{S}^j$ we must add the complementary subset $\mathcal{S}^j \setminus \mathcal{U}^j$.*

Now, for any $n = 2j \in \mathbb{N}$, the set $\mathcal{S}^j$ of all spin-j symbol correspondences defines a groupoid $\mathcal{D}^j$, the dequantization groupoid of the spin-j system, cf. (RIOS; STRAUME, 2014, Definition 7.1.19, Proposition 7.2.20), which is isomorphic to the pair groupoid over $\mathcal{S}^j$. Therefore, each of the pair groupoids over $\mathcal{S}^j_{=}$, or $\mathcal{S}^j_{<}$, or $\mathcal{S}^j_{>}$ is isomorphic to a subgroupoid of the dequantization groupoid, these being mutually disjoint.

[1] This is easy to see for $j \geq 3/2$, just take some $k \in \mathbb{N}$ s.t. $j < k < 2j$ and take c^n_l to satisfy (3.3) for $j < l \leq k$ and satisfy (3.4) for $k < l \leq 2j$.

CHAPTER

4

ASYMPTOTIC LOCALIZATION OF CORRESPONDENCE SEQUENCES

We now come to a main purpose of this part: establishing a more intuitive or geometric criterion for a given sequence of symbol correspondences to be of Poisson or anti-Poisson type, complementing the algebraic criterion of Theorem 2.3.2.

Motivated by a physical view of the projectors Π_k, $k = 1, ..., n+1$, which are J_3-invariant pure states of a spin-j system, given a symbol correspondence sequence we ask whether, or in what way, some kind of asymptotic ($n \to \infty$) localization of the symbols of projectors could be related to the asymptotic emergence of the Poisson algebra from the given sequence of twisted algebras.

Let W^j be a symbol correspondence with characteristic numbers c_l^n. From Theorem 2.2.1 and expressions (2.12) and (2.21), we have

$$W^j_{\Pi_k}(\boldsymbol{n}) = \frac{1}{n+1} + \frac{(-1)^{k-1}}{\sqrt{n+1}} \sum_{l=1}^{n} c_l^n C^{j,\ j,\ l}_{m,-m,\ 0} \sqrt{2l+1} P_l(\cos\varphi) \ , \tag{4.1}$$

where $P_l = P_l^0$ are the Legendre polynomials and $m = j - k + 1$.

Note that these symbols are invariant by rotations around the z-axis, $z = \cos\varphi$, according to our general convention of diagonalizing the z-spin operator J_3.

Remark 4.0.1. *Let W^{j-} be the alternate correspondence of W^j, that is, if c_l^n are the characteristic numbers of W^j, then $c_{l-}^n = (-1)^l c_l^n$ are the characteristic numbers of W^{j-}. Then, cf. (RIOS; STRAUME, 2014, equations (4.31) and (6.12)), $W^j_{\Pi_k}(z) = W^{j-}_{\Pi_k}(-z)$, and thus they are the reflection of each other with respect to the equator.*

But the same reflection relates $(m, -m)$-pairs of symbols of projectors under the same correspondence, as follows: recalling that Π_k is related to the J_3-eigenvalue m by $k = j - m + 1$, let k^- correspond to the eigenvalue $-m$, i.e., $k^- = j + m + 1$. Then, *because Clebsch-Gordan coefficients satisfy* $C^{j,\ j,\ l}_{m_1,m_2,m} = (-1)^{2j-l}C^{j,\ j,\ l}_{-m_1,-m_2,-m}$ *(cf. e.g. (BIEDENHARN; LOUCK, 1984)), from (4.1) we have that* $W^j_{\Pi_k}(z) = W^j_{\Pi_{k^-}}(-z)$*. That is,* $\forall z \in [-1,1]$,

$$W^j_{\Pi_{k^-}}(z) = W^{j-}_{\Pi_k}(z) = W^j_{\Pi_k}(-z) \ . \tag{4.2}$$

We now introduce the main definitions that will be useful for our purposes. First:

Definition 4.0.1. *Let* $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ *be a sequence of symbol correspondences (cf. Definition 2.3.2) and let* $(k_n)_{n\in\mathbb{N}}$ *be a sequence of natural numbers satisfying* $1 \le k_n \le n+1$*, so that* $\Pi_{k_n} \in \mathcal{B}(\mathcal{H}_j)$ *for every* $n \in \mathbb{N}$*.*

A Π sequence *is a sequence of projectors* $(\Pi_{k_n})_{n\in\mathbb{N}}$*. Its corresponding* Π-symbol sequence *is the sequence of their symbols* $(W^j_{\Pi_{k_n}})_{n\in\mathbb{N}}$*. Its associated* Π-distribution sequence *is the sequence* $(\rho^j_{k_n})_{n\in\mathbb{N}}$ *of quasiprobability distributions on* $[-1,1]$*, where* $\rho^j_{k_n} = \frac{n+1}{2}W^j_{\Pi_{k_n}}$ *(restricted to the* z*-axis by* J_3*-invariance) so that* $\int_{-1}^1 \rho^j_{k_n}(z)dz = 1$*.*

These sequences are said to be r-convergent *if* $k_n/n \to r \in [0,1]$ *, as* $n \to \infty$*.*

The characteristic numbers of the sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}}$ will also be referred to as the characteristic numbers of its Π-symbol and Π-distribution sequences. If ρ^j_k is an element of a Π-distribution sequence with characteristic numbers c^n_l, from (4.1) we get explicitly that

$$\rho^j_k(z) = \frac{1}{2} + \frac{(-1)^{k-1}\sqrt{n+1}}{2}\sum_{l=1}^{n} c^n_l C^{j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{2l+1}P_l(z) \ . \tag{4.3}$$

Now, if $\Pi_{k_n} = \Pi_1$, $\forall n \in \mathbb{N}$, then $r = 0$. Similarly, if $\Pi_{k_n} = \Pi_{n+1}$, $\forall n \in \mathbb{N}$, then $r = 1$. Classically, we could imagine that the symbols for the first sequence would "localize" at the north pole $z_0 = 1$ and the latter at the south pole $z_0 = -1$. More generally, if $k_n/n \to r$, the "standard classical picture" would be a symbol "localized" at the parallel of colatitude $\varphi = \arccos(1-2r)$, or equivalently, a z-axis quasiprobabilty distribution "localized" at $z_0 = 1-2r$.

In order to be more precise, we present our second main definition:

Definition 4.0.2. *A* Π*-distribution sequence* $(\rho^j_{k_n})_{n\in\mathbb{N}}$ *is said to* localize classically *at* $z_0 \in [-1,1]$ *if it converges, as distribution, to Dirac's* $\delta(z-z_0)$ *distribution on* $C^\infty_{\mathbb{C}}([-1,1])$*, that is,*

$$\lim_{n\to\infty}\int_{-1}^1 f(z)\rho^j_{k_n}(z)dz = f(z_0) \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ . \tag{4.4}$$

Remark 4.0.2. *The term* classical *in the above definition refers not only to the asymptotic limit* $n \to \infty$*, but also to the fact that* $C^\infty_{\mathbb{C}}([-1,1])$ *is isomorphic to the space of* J_3*-invariant functions of the classical spin system, cf. Definition 2.1.2.*

On the other hand, if $f \in C^\infty_{\mathbb{C}}(S^2)$ is not J_3-invariant, then its S^1-average

$$\bar{f} \in C^\infty_{\mathbb{C}}([-1,1]) \ , \quad \bar{f}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z,\theta)d\theta \ , \tag{4.5}$$

is J_3-invariant, so we can extend Definition 4.0.2 to general classical functions via[1]

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{S^2} f(z,\theta)\rho^j_{k_n}(z)dzd\theta = \bar{f}(z_0) \ , \ \forall f \in C^\infty_{\mathbb{C}}(S^2) \ . \tag{4.6}$$

Thus, "thinking classically", we could expect or imagine that every r-convergent Π-distribution sequence would localize classically at $z_0 = 1-2r$, $\forall r \in [0,1]$. But in view of Remark 4.0.1, the "classical picture" could also be reflected on the equator.

This leads to our third main definition:

Definition 4.0.3. *A sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ is said to* localize classically *if every r-convergent Π-distribution sequence localizes classically at $z_0 = 1-2r$, $\forall r \in [0,1]$. Likewise, $\boldsymbol{W}_{\mathcal{C}}$ is said to* anti-localize classically *if every r-convergent Π-distribution sequence localizes classically at $z_0 = 2r-1$, $\forall r \in [0,1]$.*

4.1 Classical localization of mapping-positive correspondence sequences

In order to first bypass the more complicated general case, we start by taking a special look at Π-distribution sequences that are, in fact, sequences of probability distributions. By Definition 2.2.9, this is assured by considering:

Definition 4.1.1. *A Π-symbol sequence is* positive *if it is constructed from a sequence of mapping-positive correspondences. Likewise for its Π-distribution sequence.*

As examples, we present the following explicit expressions (very useful for numerical computations) for the standard and alternate Berezin Π-symbol sequences:

Proposition 4.1.1. *The standard/alternate Berezin Π-symbol sequences, $(B^j_{\Pi_{k_n}})_{n\in\mathbb{N}}$ and $(B^{j-}_{\Pi_{k_n}})_{n\in\mathbb{N}}$, are given by*

$$B^j_{\Pi_{k_n}}(z) = \binom{n}{k_n - 1}\frac{(1+z)^{n-k_n+1}(1-z)^{k_n-1}}{2^n} \ , \ B^{j-}_{\Pi_{k_n}}(z) = B^j_{\Pi_{k_n}}(-z) \ .$$

Proof. From Theorem 2.2.3, taking $P = \Pi_{k_n}$, we obtain an expression in terms of $|z_1|^2$ and $|z_2|^2$. Using that $|z_1|^2 + |z_2|^2 = 1$ and the Hopf map, $z = |z_1|^2 - |z_2|^2$, we get the expression for $B^j_{\Pi_{k_n}}$. The second expression is a particular case of (4.2). □

[1] Of course, (4.4) and (4.6) are equivalent because $\frac{1}{2\pi}\int_{S^2} f(z,\theta)\rho^j_{k_n}(z)dzd\theta = \int_{-1}^{1}\bar{f}(z)\rho^j_{k_n}(z)dz$.

The lemma below follows straightforwardly from Chebyshev's inequality. It is well-known but not so easy to find in exactly these terms in the literature, thus, for completeness, we present a proof of this lemma in the Appendix A.

For $(\rho_n)_{n\in\mathbb{N}}$ a sequence of probability distributions on $[-1,1]$, denote by E_n the expected value operator defined by ρ_n and let $\mu_n = E_n(z)$ and $\sigma_n^2 = E_n((z-\mu_n)^2) = E_n(z^2) - \mu_n^2$ denote the mean and variance of ρ_n, respectively.

Lemma 4.1.2. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability distributions on $[-1,1]$ with mean values $(\mu_n)_{n\in\mathbb{N}}$ and variances $(\sigma_n^2)_{n\in\mathbb{N}}$. Then (ρ_n) converges, as distribution, to Dirac's $\delta(z-\mu)$ distribution on $C^0_{\mathbb{C}}([-1,1])$, that is,*

$$\lim_{n\to\infty}\int_{-1}^{1} f(z)\rho_n(z)dz = f(\mu)\ ,\ \ \forall f\in C^0_{\mathbb{C}}([-1,1])\ , \tag{4.7}$$

if and only if $\mu_n\to\mu$ and $\sigma_n^2\to 0$.

Now, let $(\rho^j_{k_n})_{n\in\mathbb{N}}$ be a positive Π-distribution sequence with characteristic numbers c_l^n. To compute μ_n and σ_n^2, we must integrate $\int_{-1}^1 z\rho^j_{k_n}dz$ and $\int_{-1}^1 z^2\rho^j_{k_n}dz$. Therefore, we need expressions for Clebsh-Gordan coefficients of the form $C^{j,\ \ j,\ 1}_{m,-m,\ 0}$ and $C^{j,\ \ j,\ 2}_{m,-m,\ 0}$. From (VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988),

$$C^{j,\ \ j,\ 1}_{m,-m,\ 0} = (-1)^{k+1}2(j-k+1)\sqrt{\frac{3}{(n+2)(n+1)n}},$$

$$C^{j,\ \ j,\ 2}_{m,-m,\ 0} = (-1)^{k-1}\sqrt{5}\frac{(n-k+1)(n-k)-4(k-1)(n-k+1)+(k-1)(k-2)}{\sqrt{(n-1)n(n+1)(n+2)(n+3)}},$$

where, as usual, $m=j-k+1$. Hence, we easily get:

Lemma 4.1.3. *For any positive Π-distribution sequence $(\rho^j_{k_n})_{n\in\mathbb{N}}$, we have that*

$$\mu_n = c_1^n\frac{n-2(k_n-1)}{\sqrt{n(n+2)}}, \tag{4.8}$$

$$\begin{aligned}\sigma_n^2 = &\frac{2c_2^n[(n-k_n+1)(n-k_n)-4(k_n-1)(n-k_n+1)+(k_n-1)(k_n-2)]}{3\sqrt{(n-1)n(n+2)(n+3)}}\\ &+\frac{1}{3}-\frac{(c_1^n)^2(n-2(k_n-1))^2}{n(n+2)}.\end{aligned} \tag{4.9}$$

From Lemmas 4.1.2-4.1.3 we obtain in a rather straightforward way:

Theorem 4.1.4. *A mapping-positive symbol correspondence sequence localizes (resp. anti-localizes) classically if and only if*

$$\lim_{n\to\infty} c_1^n = 1\ \textit{(resp.}\ = -1)\ ,\ \lim_{n\to\infty} c_2^n = 1\ . \tag{4.10}$$

Combined with Theorem 2.3.2 we immediately have:

Corollary 4.1.4.1. *If a sequence of mapping-positive symbol correspondences is of Poisson (resp. anti-Poisson) type, then it localizes (resp. anti-localizes) classically.*

In particular, the standard (resp. alternate) Berezin symbol correspondence sequence localizes (resp. anti-localizes) classically. But from (RIOS; STRAUME, 2014, Proposition 6.2.54), the upper-middle-state and lower-middle-state symbol correspondence sequences do not satisfy the conditions of Theorem 4.1.4 and in fact they do not localize or anti-localize classically, in accordance with the fact that they are not of Poisson or anti-Poisson type. The failure of these mapping-positive symbol correspondence sequences to (anti-)localize classically can be inferred from numerical computations for finite but growing values of n. It also follows from Corollary 4.2.2.2 below.

Remark 4.1.1. *As seen from above, for mapping-positive symbol correspondence sequences the condition for their classical (anti-)localization is a condition on the asymptotic limit of their first two characteristic numbers, only. Thus, for mapping-positive symbol correspondence sequences, (anti-)Poisson emergence seems to be a stronger condition than classical (anti-)localization, cf. Theorem 2.3.2. But we shall see below that this is not so (cf. Corollaries 4.2.2.3-4.2.2.4) and that, in fact, the opposite is the case for general symbol correspondence sequences.*

4.2 Asymptotic localization of general symbol correspondence sequences

In order to expand our study to the general case, we now use Edmonds formula. Proofs of this formula are found in the literature, but we have not found a proof treating the fully general case, so we present one in the Appendix B.

Lemma 4.2.1 (Edmonds formula, cf. (EDMONDS, 1955))**.** *Let $(k_n)_{n\in\mathbb{N}}$ be as in Definition 4.0.1 and let $m = j - k_n + 1$ (depending implicitly on n). If $k_n/n \to r \in [0,1]$, then*

$$\lim_{n\to\infty} (-1)^{k_n-1} C^{j,\ \ j,\ l}_{m,-m,\ 0} \sqrt{\frac{n+1}{2l+1}} = P_l(1-2r)\ ,\ \forall l \in \mathbb{N}\ , \tag{4.11}$$

where P_l is the l^{th} Legendre polynomial.

We then have:

Theorem 4.2.2. *A sequence of symbol correspondences $\boldsymbol{W}_C$ is of Poisson type if and only if $\forall r \in [0,1]$ its r-convergent Π-distribution sequences satisfy*

$$\lim_{n\to\infty} \int_{-1}^{1} P_l(z)\rho^j_{k_n}(z)dz = P_l(1-2r), \quad \forall l \in \mathbb{N}. \tag{4.12}$$

And $\boldsymbol{W}_{\mathcal{C}}$ *is of anti-Poisson type if and only if* $\forall r \in [0,1]$ *its* r*-convergent* Π*-distribution sequences satisfy*

$$\lim_{n\to\infty}\int_{-1}^{1} P_l(z)\rho^j_{k_n}(z)dz = P_l(2r-1), \quad \forall l \in \mathbb{N}. \tag{4.13}$$

Proof. From (4.3) and Lemma 4.2.1, we have

$$\lim_{n\to\infty}\int_{-1}^{1} P_l(z)\rho^j_{k_n}(z)dz = \lim_{n\to\infty}(-1)^{k_n-1}c^n_l C^{j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{\frac{n+1}{2l+1}}\ ,\ \forall l \in \mathbb{N}\ . \tag{4.14}$$

Assuming Poisson, $\lim_{n\to\infty} c^n_l = 1, \forall l \in \mathbb{N}$, then (4.12) follows immediately from Edmonds formula. On the other hand, for all $l \in \mathbb{N}$ there exists $r \in [0,1]$ such that $P_l(1-2r) \neq 0$. So from Edmonds formula, (4.12) holds only if $\lim_{n\to\infty} c^n_l = 1, \forall l \in \mathbb{N}$. Similarly for the anti-Poisson case, using that $P_l(-z) = (-1)^l P_l(z)$. □

In view of the above theorem, we introduce the following definition:

Definition 4.2.1. *A* Π*-distribution sequence* $(\rho^j_{k_n})_{n\in\mathbb{N}}$ *is said to* localize polynomially *at* $z_0 \in [-1,1]$ *if*

$$\lim_{n\to\infty}\int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz = f(z_0)\ ,\ \forall f \in Poly_{\mathbb{C}}([-1,1])\ . \tag{4.15}$$

Then, a sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}}$ *is said to* localize polynomially*, resp.* anti-localize polynomially*, if every* r*-convergent* Π*-distribution sequence localizes polynomially at* $1-2r$*, resp. at* $2r-1$*,* $\forall r \in [0,1]$.

And then Theorem 4.2.2 can be rewritten as

Corollary 4.2.2.1. *A sequence of symbol correspondences is of Poisson (resp. anti-Poisson) type if and only if it localizes (resp. anti-localizes) polynomially.*

And because classical localization (cf. Definitions 4.0.2 and 4.0.3) implies polynomial localization (cf. Definition 4.2.1), we immediately have

Corollary 4.2.2.2. *A sequence of symbol correspondences is of Poisson (resp. anti-Poisson) type if it localizes (resp. anti-localizes) classically.*

From Theorem 4.1.4 and Corollary 4.2.2.2 we have a new criterion to complement Theorem 2.3.2 in the case of mapping-positive symbol correspondence sequences:

Corollary 4.2.2.3. *A mapping-positive symbol correspondence sequence is of Poisson (resp. anti-Poisson) type if and only if it localizes (resp. anti-localizes) classically.*

And in addition, we have the following strong implication[2]:

Corollary 4.2.2.4. *For mapping-positive symbol correspondence sequences and for positive-dual symbol correspondence sequences,*

$$\lim_{n\to\infty} c_1^n = 1 \ (resp. \ = -1) \ , \ \lim_{n\to\infty} c_2^n = 1 \ ,$$

implies

$$\lim_{n\to\infty} c_l^n = 1 \ (resp. \ = (-1)^l) \ , \ \forall l \in \mathbb{N} \ .$$

Proof. For the mapping-positive case, this follows straightforwardly from Corollary 4.2.2.3 and Theorems 2.3.2 and 4.1.4. For the positive-dual case, we just recall that every positive-dual correspondence is the dual of a mapping-positive correspondence whose characteristic numbers are related by (2.24). □

Since polynomials are dense in $C^\infty_{\mathbb{C}}([-1,1])$, one might expect that (4.12), or (4.13), could imply classical (anti-)localization in the sense of Definition 4.0.2. But there is no such implication, so we now investigate the converse of Corollary 4.2.2.2.

Theorem 4.2.3. *If a sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ is of (anti-)Poisson type and if there exist $d \in \mathbb{N}_0$ and $K_d > 0$ such that*

$$|c_l^n| \le K_d \prod_{t=1}^{d} (2(l-t)+1) \ , \ n \ge \forall l > d+1 \ , \tag{4.16}$$

with the product assumed to be 1 if $d = 0$, then $\boldsymbol{W}_{\mathcal{C}}$ (anti-)localizes classically.

Proof. For any $f \in C^\infty_{\mathbb{C}}([-1,1])$, we have that

$$f = \lim_{p\to\infty} \sum_{l=0}^{p} a_l P_l \ , \tag{4.17}$$

where the convergence is absolute and uniform. Then, from (4.3) we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz &= \lim_{n\to\infty}\lim_{p\to\infty} \sum_{l=0}^{p} \int_{-1}^{1} a_l P_l(z)\rho^j_{k_n}(z)dz \\
&= a_0 + \lim_{n\to\infty}\lim_{p\to\infty} \sum_{l=1}^{p} a_l c_l^n (-1)^{k_n-1} C^{j,j,l}_{m,-m,0}\sqrt{\frac{n+1}{2l+1}} \\
&= a_0 + \lim_{n\to\infty} \sum_{l=1}^{n} a_l c_l^n (-1)^{k_n-1} C^{j,j,l}_{m,-m,0}\sqrt{\frac{n+1}{2l+1}} \ .
\end{aligned} \tag{4.18}$$

[2] In the simplest cases, standard/alternate Berezin and Toeplitz correspondence sequences, it is not too difficult to see that this implication also follows from (3.1) and some recurrence relations for Clebsch-Gordan coefficients, cf. e.g. (VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988).

In order to apply Edmonds formula, we must be able to take the limit $n \to \infty$ inside the summation in the last line of (4.18). For simplicity, we now write

$$\alpha_l^n = a_l c_l^n (-1)^{k_n - 1} C^{j,j,l}_{m,-m,0} \sqrt{\frac{n+1}{2l+1}} \ . \tag{4.19}$$

From the C-G symmetry $C^{j,j,l}_{m,-m,0} = \sqrt{\frac{2l+1}{n+1}} C^{l,j,j}_{0,-m,-m}$ plus unitarity, we have

$$|C^{j,j,l}_{m,-m,0}| \leq \sqrt{\frac{2l+1}{n+1}} \ , \tag{4.20}$$

as well as $|C^{j,j,l}_{m,-m,0}| \leq 1$. In any case,

$$|\alpha_l^n| \leq |a_l c_l^n| \ .$$

We remove the dependence on n using the (anti-)Poisson hypothesis, which implies $|c_l^\infty| = 1$. Thus, for any fixed l, the sequence $\{|c_l^n|\}_{n \geq l}$ is convergent.

Hence, for every $l \in \mathbb{N}$, there exists $B_l > 0$ s.t.

$$|c_l^n| \leq B_l \ , \ \forall n \geq l \quad \Rightarrow \quad |\alpha_l^n| \leq B_l |a_l| \ , \ \forall n \geq l \ . \tag{4.21}$$

But by hypothesis, there exist $d \in \mathbb{N}_0$ and $K_d > 0$ such that

$$B_l \leq K_d \prod_{t=1}^{d} (2(l-t)+1) \ , \ \forall l > d+1 \ . \tag{4.22}$$

We now use the following lemma, which is a corollary of (WANG, 2018, Theorem 2.2).

Lemma 4.2.4 ((WANG, 2018)). *For any $f \in C^{k+1}_{\mathbb{C}}([-1,1])$, $k \in \mathbb{N}_0$, there exists $A_k > 0$ s.t.*

$$|a_l| \leq \frac{A_k}{\sqrt{2(l-k)-1}} \prod_{t=1}^{k} \frac{1}{2(l-t)+1} \ , \ \forall l \geq k+1 \ . \tag{4.23}$$

Thus, given d as in the hypothesis of the theorem, take $k = d+1$ in Lemma 4.2.4, since $f \in C^\infty_{\mathbb{C}}([-1,1]) \subset C^{d+2}_{\mathbb{C}}([-1,1])$. From (4.21)-(4.23),

$$|\alpha_l^n| \leq \frac{A_{d+1} K_d}{(2(l-d)-1)^{3/2}} = M_l^d \ , \ \forall n \geq l \geq d+2 \ . \tag{4.24}$$

Returning to (4.18), we write

$$\lim_{n\to\infty} \int_{-1}^{1} f(z) \rho^j_{k_n}(z) dz = a_0 + \lim_{n\to\infty} \sum_{l=1}^{d+1} \alpha_l^n + \lim_{n\to\infty} \sum_{l=d+2}^{n} \alpha_l^n \ .$$

The first limit is trivially interchanged with the finite sum. For the second limit, we apply Tannery's Theorem, since $|\alpha_l^n| \leq M_l^d$, $\forall n \geq l \geq d+2$, and $\sum_{l=d+2}^{\infty} M_l^d < \infty$, cf. (4.24). Then, from (4.17)-(4.19), using Edmonds formula (4.11) and the (anti-) Poisson hypothesis, we conclude the thesis. $\square$

As a direct corollary, we have the important special case:

Corollary 4.2.4.1. *The sequence of standard (resp. alternate) Stratonovich-Weyl symbol correspondences localizes (resp. anti-localizes) classically.*

Remark 4.2.1. *Note from (3.2) that mapping-positive symbol correspondence sequences satisfy equation (4.16), thus Corollary 4.1.4.1 of Theorem 4.1.4 can also be seen as a direct corollary of Theorem 4.2.3.*

In Theorem 4.2.3, the bounds (4.16) on the characteristic numbers are necessary for applying Tannery's theorem and Edmonds formula to (4.18), but these are sufficient, in principle not necessary means of assuring classical (anti-)localization (cf. also Theorem 5.4.8 further below and its discussion in the concluding chapter).

Therefore, one could still ask whether (anti-)Poisson condition implies classical (anti-)localization in general. But in this respect we have the following:

Theorem 4.2.5. *For general sequences of symbol correspondences, the classical (anti-)localization property is in fact stronger than the (anti-)Poisson property.*

Proof. The proof consists in exhibiting a set of symbol correspondence sequences of (anti-)Poisson type that fail to (anti-)localize classically.

For any $f \in C^\infty_{\mathbb{C}}([-1,1])$ with Legendre series (4.17) such that $a_l \neq 0$, $\forall l \in \mathbb{N}$, consider the sequence of symbol correspondences $\boldsymbol{W}_{\mathcal{C}}$ with characteristic numbers

$$c_l^n = \begin{cases} (2l+1)/a_l, & \forall n = l \\ 1, & \text{otherwise} \end{cases} . \tag{4.25}$$

For every $l \in \mathbb{N}$ and $n > l$, we have $c_l^n = 1$. Thus, $c_l^\infty = 1$, $\forall l \in \mathbb{N}$, which means that the symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ is of Poisson type. Its r-convergent Π-distribution sequences satisfy

$$\int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz - f(1-2r) = (-1)^{k_n-1} C^{j,j,n}_{m,-m,0}\sqrt{n+1}\left(\sqrt{2n+1} - \frac{a_n}{\sqrt{2n+1}}\right)$$
$$+ a_0 + \sum_{l=1}^{n} a_l (-1)^{k_n-1} C^{j,j,l}_{m,-m,0}\sqrt{\frac{n+1}{2l+1}} - f(1-2r) .$$

Hence,

$$\left|\int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz - f(1-2r)\right| \geq \left|\left|C^{j,j,n}_{m,-m,0}\sqrt{n+1}\left(\sqrt{2n+1} - \frac{a_n}{\sqrt{2n+1}}\right)\right| - R_n\right| ,$$

where

$$R_n = \left|a_0 + \sum_{l=1}^{n} a_l (-1)^{k_n-1} C^{j,j,l}_{m,-m,0}\sqrt{\frac{n+1}{2l+1}} - f(1-2r)\right| .$$

From (VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988), we have that

$$C^{j,j,n}_{m,-m,0} = \frac{(n!)^2}{(j+m)!(j-m)!\sqrt{(2n)!}} \tag{4.26}$$

and from (4.18) and Corollary 4.2.4.1, we have that $R_n \to 0$ as $n \to \infty$, $\forall r \in [0,1]$. But for $r = 1/2$ we can choose $m = 0$ whenever j is integer, thus, under these assumptions, by Stirling approximation (B.3), we have from (4.26) and (4.23) that

$$\left| C^{j,j,n}_{m,-m,0}\sqrt{n+1}\left(\sqrt{2n+1} - \frac{a_n}{\sqrt{2n+1}}\right)\right| \sim \frac{2n}{(\pi n)^{1/4}}$$

for large integers j. Hence, such r-convergent Π-distribution sequence, for $r = 1/2$, constructed from a symbol correspondence sequence of Poisson type, fails to converge pointwise on f at 0. Therefore, $\boldsymbol{W}_{\mathcal{C}}$ does not localize classically. By considering the symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ with characteristic numbers

$$c^n_l = \begin{cases} (2l+1)/a_l, & \forall n = l \\ (-1)^l, & \text{otherwise} \end{cases} ,$$

we have that $\boldsymbol{W}_{\mathcal{C}}$ is of anti-Poisson type but does not anti-localize classically[3]. □

On the other hand, one might think that the more well-known and amply-used symbol correspondence sequences would satisfy the conditions of Theorem 4.2.3. However, although the bounds (4.16) do not seem too strong, we actually have:

Proposition 4.2.6. *The polynomial bounds (4.16) are not satisfied for the standard and the alternate Toeplitz correspondence sequences.*

Proof. Recall that the characteristic numbers of the standard and alternate Toeplitz correspondences are $t^n_l = 1/b^n_l$, with b^n_l given by (2.22), and $t^n_{l-} = (-1)^l t^n_l$. We first verify that for any fixed $l \in \mathbb{N}$ the sequence $\{t^n_l\}_{n \geq l}$ is decreasing:

$$\frac{t^{n+1}_l}{t^n_l} = \frac{\frac{1}{(n+1)!}\sqrt{\frac{(n+l+2)!(n-l+1)!}{n+2}}}{\frac{1}{n!}\sqrt{\frac{(n+l+1)!(n-l)!}{n+1}}}$$
$$= \sqrt{\frac{(n+l+2)(n-l+1)}{(n+2)(n+1)}} = \sqrt{1 - \frac{l(l+1)}{(n+2)(n+1)}} \; < \; 1 \; .$$

Therefore,

$$t^n_l \leq t^l_l \; , \; \forall n \geq l \; . \tag{4.27}$$

[3] Note that $\boldsymbol{W}_{\mathcal{C}}$ built as above is just one of infinite possibilities and, in particular, it would have been sufficient to define $c^l_l = \sqrt{2l+1}/a_l$, for instance. However, the construction we chose above will be useful for defining an example that clarifies another property, later on (cf. Example 4).

However, computing explicitly the limiting value of

$$t_l^l = \frac{1}{l!}\sqrt{\frac{(2l+1)!}{l+1}} = \sqrt{\frac{(2l+1)!}{l!(l+1)!}} \ , \tag{4.28}$$

using Stirling formula (B.3), we obtain

$$t_l^l = |t_{l-}^l| = T_l \sim \frac{2^{l+1/2}}{(l\pi)^{1/4}} \ , \text{ as } l \to \infty \ . \tag{4.29}$$

Thus, T_l increases exponentially and cannot be bounded by any polynomial. □

Motivated by these examples, in particular equations (4.27)-(4.29), we introduce a one-parameter family of weaker asymptotic localizations, as follows.

For every $\mu > 1$, let $\mathcal{E}_\mu \subset \mathbb{C}$ denote the closed interior of the Bernstein ellipse

$$\partial\mathcal{E}_\mu = \left\{ z \in \mathbb{C} \mid z = (u + u^{-1})/2, \ u = \mu e^{i\phi}, \ \phi \in [-\pi, \pi] \right\} \ , \tag{4.30}$$

with foci at ± 1 and sum of major and minor semi-axis equal to μ. Denote by

$$\mathcal{A}_\mu([-1,1]) \subset C^\infty_{\mathbb{C}}([-1,1])$$

the subspace of all smooth complex functions on $[-1,1]$ which admit holomorphic extensions to $\mathcal{E}_\mu \subset \mathbb{C}$. Note that

$$1 < \mu_1 < \mu_2 \ \Rightarrow \ \mathcal{E}_{\mu_1} \subset \mathcal{E}_{\mu_2} \ \Rightarrow \ \mathcal{A}_{\mu_1}([-1,1]) \supset \mathcal{A}_{\mu_2}([-1,1]) \ , \tag{4.31}$$

thus $f \in \mathcal{A}_\infty([-1,1])$ if f is the restriction to $[-1,1] \subset \mathbb{C}$ of an entire function. In particular,

$$Poly_{\mathbb{C}}([-1,1]) \subsetneq \mathcal{A}_\infty([-1,1]) = \bigcap_{\mu>1} \mathcal{A}_\mu([-1,1]) \ .$$

Definition 4.2.2. *We shall say that f is* μ-analytic on $[-1,1]$ *if $f \in \mathcal{A}_\mu([-1,1])$.*

We now introduce:

Definition 4.2.3. *A Π-distribution sequence $(\rho^j_{k_n})_{n\in\mathbb{N}}$ is said to* localize μ-analytically *at $z_0 \in [-1,1]$ if*

$$\lim_{n\to\infty} \int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz = f(z_0) \ , \ \forall f \in \mathcal{A}_\mu([-1,1]) \ . \tag{4.32}$$

Then, a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ is said to localize μ-analytically, *resp.* anti-localize μ-analytically, *if every r-convergent Π-distribution sequence localizes μ-analytically at $1-2r$, resp. at $2r-1$, $\forall r \in [0,1]$.*

From (4.31), if $\boldsymbol{W}_{\mathcal{C}}$ localizes μ_1-analytically, then it localizes μ_2-analytically, $\forall \mu_2 > \mu_1$. But the converse does not necessarily hold. Then we have:

Theorem 4.2.7. *The sequence of standard (resp. alternate) Toeplitz symbol correspondences localizes (resp. anti-localizes) μ-analytically, $\forall \mu > 2$.*

Proof. The proof follows similarly to the proof of Theorem 4.2.3, using (4.27)-(4.29) and the fact that, for any $f \in \mathcal{A}_\mu([-1,1])$, we have from (XIANG, 2012, Corollary 2.1) that f can be written as (4.17) with

$$|a_l| \leq \frac{2M\sqrt{l}}{\mu^{l-1}(\mu^2-1)} \ , \ \forall l \geq 1 \ , \tag{4.33}$$

where $|f(z)| \leq M$, for $z \in \mathcal{E}_\mu$. □

On the other hand, inspired by the previous considerations, we can think in the opposite direction and ask whether a symbol correspondence sequence can localize in a broader sense than the classical one. This leads to the following:

Definition 4.2.4. *Assume that a Π-distribution sequence $(\rho^j_{k_n})_{n\in\mathbb{N}}$ satisfies*

$$\lim_{n\to\infty} \int_{-1}^{1} f(z)\rho^j_{k_n}(z)dz = f(z_0) \ , \ \forall f \in \mathcal{F}_\mathbb{C}([-1,1]) \ , \tag{4.34}$$

where $\mathcal{F}_\mathbb{C}([-1,1])$ is a space of complex functions on $[-1,1]$ detailed below.

In each case, $(\rho^j_{k_n})$ is said to:

localize k-differentiably *at z_0, if* $\mathcal{F}_\mathbb{C}([-1,1]) = C^k_\mathbb{C}([-1,1])$.

(if $k=0$, it localizes continuously*, and it* localizes differentiably *if $k=1$).*

localize absolute-continuously *at z_0, if* $\mathcal{F}_\mathbb{C}([-1,1]) = AC_\mathbb{C}([-1,1])$.

localize α-Hölder continuously *at z_0, if* $\mathcal{F}_\mathbb{C}([-1,1]) = C^{0,\alpha}_\mathbb{C}([-1,1])$.

localize α-Hölder k-differentiably *at z_0, if* $\mathcal{F}_\mathbb{C}([-1,1]) = C^{k,\alpha}_\mathbb{C}([-1,1])$.

Then, a symbol correspondence sequence $\boldsymbol{W}_\mathcal{C}$ (anti-)localizes k-differentiably *(*continuously, differentiably*), if every r-convergent Π-distribution sequence localizes k-differentiably (*continuously, differentiably*) at $1-2r$ (resp. $2r-1$), $\forall r \in [0,1]$.*

And similarly for when $\boldsymbol{W}_\mathcal{C}$ is said to (anti-)localize absolute-continuously*, or* α-Hölder continuously*, or* α-Hölder k-differentiably.

Now, using Lemma 4.1.2, we immediately generalize Theorem 4.1.4 to continuous (anti-)localization, and then, using Theorem 2.3.2 and Corollary 4.2.2.4, we obtain:

Corollary 4.2.7.1. *A mapping-positive symbol correspondence sequence (anti-)localizes continuously if and only if it is of (anti-)Poisson type.*

But for general symbol correspondence sequences of (anti-)Poisson type we need additional conditions on their characteristic numbers to guarantee (anti-)localization in each of the senses above, as already seen for classical (anti-)localization.

We are not aware of sufficient bounds which apply for every case, but the corollary below follows from Lemma 4.2.4 and the fact that the Legendre series of f converges uniformly to f if $f \in C^k_{\mathbb{C}}([-1,1])$ for $k \geq 1$, cf. (ATKINSON; HAN, 2010).

Corollary 4.2.7.2. *Under the same hypothesis of Theorem 4.2.3, $\boldsymbol{W}_{\mathcal{C}}$ (anti-)localizes k-differentiably, for $k = d + 2$.*

Then, taking $d = 0$ in (4.16), we also have:

Corollary 4.2.7.3. *The standard (resp. alternate) Stratonovich-Weyl symbol correspondence sequence localizes (resp. anti-localizes) 2-differentiably.*

When trying to extend Corollary 4.2.7.3 beyond 2-differential localization, for the standard and alternate S-W symbol correspondence sequences, we encounter two difficulties. The first one is that for a general $f \in C^0_{\mathbb{C}}([-1,1])$, its Legendre series may not converge to f uniformly and this is used in the proof of Theorem 4.2.3. This problem can be avoided if we restrict to $f \in C^{0,\alpha}_{\mathbb{C}}([-1,1])$, $1/2 < \alpha \leq 1$, or to $f \in C^1_{\mathbb{C}}([-1,1])$. But then, upper bounds for the Legendre coefficients of functions in these spaces are not known to us.

CHAPTER

5

SEQUENTIAL QUANTIZATIONS AND ASYMPTOTIC LOCALIZATION

So far we have been investigating the (semi)classical limit of quantum spin systems via their dequantizations. In this Chapter we start investigations in the opposite direction, i.e. from the classical spin system to quantum spin systems, often referred to as quantization. One of our main purposes, at this point, is to find relations between properties of quantized functions and asymptotic localization of symbol correspondence sequences.

5.1 Sequential quantizations of the 2-sphere

We start with some basic definitions. Denote by

$$\mathfrak{M} \;=\; \{\mathbf{F} = (F_n)_{n\in\mathbb{N}},\; F_n \in M_{\mathbb{C}}(n+1)\}$$

the set of all sequences of spin-j operators. In this set, we can define the natural operations of *sum*, *product* and *multiplication by a scalar*, as follows:

Definition 5.1.1. *For any $\mathbf{F}, \mathbf{G} \in \mathfrak{M}$, their* sum *is given by*

$$\mathbf{F} + \mathbf{G} \;=\; (F_n + G_n)_{n\in\mathbb{N}} \;\in\; \mathfrak{M} \tag{5.1}$$

and their product *is given by*

$$\mathbf{F}\mathbf{G} \;=\; (F_n G_n)_{n\in\mathbb{N}} \;\in\; \mathfrak{M}\;, \tag{5.2}$$

where $F_n G_n$ is the product in $M_{\mathbb{C}}(n+1)$, with $\mathbf{I} = (I_n)_{n\in\mathbb{N}} \in \mathfrak{M}$ as the unit.

The multiplication of an operator sequence $\mathbf{F}$ by $a \in \mathbb{C}$, $a\mathbf{F} = (aF_n)_{n\in\mathbb{N}}$, extends naturally to multiplication by a scalar *as being the multiplication of $\mathbf{F}$ by a sequence of complex numbers $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, $a_n \in \mathbb{C}$, given by*

$$\mathbf{a}\mathbf{F} \;=\; (a_n F_n)_{n\in\mathbb{N}} \;\in\; \mathfrak{M}\;, \tag{5.3}$$

with the set of scalars

$$\mathfrak{C} \;=\; \{\mathbf{a} = (a_n)_{n\in\mathbb{N}},\ a_n \in \mathbb{C}\}$$

forming a commutative ring under the natural operations

$$\mathbf{a} + \mathbf{b} = (a_n + b_n)_{n\in\mathbb{N}} \quad , \quad \mathbf{ab} = (a_n b_n)_{n\in\mathbb{N}} \ . \tag{5.4}$$

Remark 5.1.1. *Therefore, the set of operator sequences $\mathfrak{M}$ has the structure of a unital associative (noncommutative) algebra over the commutative ring $\mathfrak{C}$ of scalars. However, $\mathfrak{C}$ is not a field because there are nontrivial zero divisors.*

Recall the normalized inner product $\langle\cdot|\cdot\rangle_j : M_{\mathbb{C}}(n+1) \times M_{\mathbb{C}}(n+1) \to \mathbb{C}$,

$$\langle F_n|G_n\rangle_j = \frac{1}{n+1} tr(F_n^\dagger G_n) \ , \qquad cf.\ (2.15), \tag{5.5}$$

satisfying $\langle I_n, I_n\rangle_j = 1$ and let the norm $||\cdot|| : M_{\mathbb{C}}(n+1) \to \mathbb{R}^+$ be defined as

$$||F_n|| = \sqrt{\langle F_n|F_n\rangle_j} \ . \tag{5.6}$$

Definition 5.1.2. *For any $\mathbf{F}, \mathbf{G} \in \mathfrak{M}$, their* normalized inner product *is given by*

$$\langle \mathbf{F}|\mathbf{G}\rangle = (\langle F_n|G_n\rangle_j)_{n\in\mathbb{N}} \in \mathfrak{C} \ . \tag{5.7}$$

For any $\mathbf{F} \in \mathfrak{M}$, its norm $||\mathbf{F}|| \in \mathfrak{R}^+$, *where $\mathfrak{R}^+ = \{\mathbf{a} = (a_n)_{n\in\mathbb{N}},\ a_n \in \mathbb{R}^+\} \subset \mathfrak{C}$, is given by*

$$||\mathbf{F}|| = (||F_n||)_{n\in\mathbb{N}} \ , \ ||\mathbf{F}||^2 = \langle \mathbf{F}|\mathbf{F}\rangle \ . \tag{5.8}$$

Its lower asymptotic norm *and* upper asymptotic norm *are given by*

$$||\mathbf{F}||_< = \liminf_{n\to\infty} ||F_n|| \in \mathbb{R}^+ \cup \{\infty\} \ , \ ||\mathbf{F}||_> = \limsup_{n\to\infty} ||F_n|| \in \mathbb{R}^+ \cup \{\infty\}. \tag{5.9}$$

When these are equal, the asymptotic norm *is denoted by*

$$||\mathbf{F}||_\infty = \lim_{n\to\infty} ||F_n|| \ \in \ \mathbb{R}^+ \cup \{\infty\} \ . \tag{5.10}$$

$\mathbf{F} \in \mathfrak{M}$ *is* upper bounded *if* $\exists\ ||\mathbf{F}||_> \in \mathbb{R}$, *and is* properly bounded *if* $\exists\ ||\mathbf{F}||_\infty \in \mathbb{R}$.

For any $\mathbf{F} \in \mathfrak{M}$, its trace $\mathrm{tr}(\mathbf{F})$ *is given by*

$$\mathrm{tr}(\mathbf{F}) = (\mathrm{tr}(F_n))_{n\in\mathbb{N}} \in \mathfrak{C} \ . \tag{5.11}$$

Its asymptotic trace*, when it exists, is given/denoted by*

$$\mathrm{tr}_\infty(\mathbf{F}) = \lim_{n\to\infty} \mathrm{tr}(F_n) \in \mathbb{C} \ . \tag{5.12}$$

$\mathbf{F} \in \mathfrak{M}$ *is* trace-class *if* $\exists\ tr_\infty(|\mathbf{F}|) \in \mathbb{R}$, *for* $|\mathbf{F}| = (|F_n| = \sqrt{F_n^* F_n})$.

In this last section, we were particularly interested in investigating sequences of operators for which, given a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$, its sequence of symbols converge to a J_3-invariant classical function $f \in C^\infty([-1,1])$ or to a J_3-invariant μ-analytic function $f \in \mathcal{A}_\mu([-1,1])$. In this case, if f is the (uniform) limit of a sequence $(f_n)_{n\in\mathbb{N}}$ of J_3-invariant symbols of the form

$$f_n = \sum_{l=0}^{n} \chi_l^n P_l \ , \ \chi_l^n \in \mathbb{C} \ , \tag{5.13}$$

then for simplicity we can consider sequences of J_3-invariant operators which can be written in the form $\mathbf{F}^W = (F_n^W)_{n\in\mathbb{N}}$, for $F_n^W \in M_{\mathbb{C}}(n+1)$ given by

$$F_n^W = \sum_{l=0}^{n} \frac{\chi_l^n}{c_l^n\sqrt{2l+1}} \widehat{\mathbf{e}}^j(l,0) \ , \tag{5.14}$$

where

$$\widehat{\mathbf{e}}^j(l,m) = \sqrt{n+1}\mathbf{e}^j(l,m) \tag{5.15}$$

are orthonormal basis vectors of $M_{\mathbb{C}}(n+1)$ w.r.t. the normalized inner product $\langle\cdot|\cdot\rangle_j$ on $M_{\mathbb{C}}(n+1)$, with $I_n = \widehat{\mathbf{e}}^j(0,0)$, that is,

$$\langle \widehat{\mathbf{e}}^j(l,m)|\widehat{\mathbf{e}}^j(l',m')\rangle_j = \delta_{l,l'}\delta_{m,m'} \ . \tag{5.16}$$

Furthermore, from the Legendre series for f,

$$f = \lim_{n\to\infty} \sum_{l=0}^{n} a_l P_l \ , \tag{5.17}$$

we have that

$$\lim_{n\to\infty} \chi_l^n = a_l = \frac{2l+1}{2} \int_{-1}^{1} f(z)P_l(z)dz \ , \ \forall l \in \mathbb{N}. \tag{5.18}$$

In view of all this, we now introduce the following general definition:

Definition 5.1.3. *For any classical function $f \in C^\infty_{\mathbb{C}}(S^2)$, with harmonic series*

$$f = \lim_{n\to\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} a_l^m Y_l^m \ , \quad a_l^m = \langle Y_l^m|f\rangle \in \mathbb{C} \ , \tag{5.19}$$

*given a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ with characteristic numbers c_l^n, the W-*pseudoquantization *of f is the operator sequence $\mathbf{F}^w = (F_n^w)_{n\in\mathbb{N}} \in \mathfrak{M}$ given by*

$$F_n^w = [W^j]^{-1}(f) = \sum_{l=0}^{n} \sum_{m=-l}^{l} \frac{a_l^m}{c_l^n} \widehat{\mathbf{e}}^j(l,m) \ , \tag{5.20}$$

*and the $\widetilde{W}$-*pseudoquantization *of f is the operator sequence $\widetilde{\mathbf{F}}^w = (\widetilde{F}_n^w)_{n\in\mathbb{N}} \in \mathfrak{M}$,*

$$\widetilde{F}_n^w = [\widetilde{W}^j]^{-1}(f) = \sum_{l=0}^{n} \sum_{m=-l}^{l} a_l^m c_l^n \widehat{\mathbf{e}}^j(l,m) \ . \tag{5.21}$$

If $\boldsymbol{W}_{\mathcal{C}}$ is of (anti-)Poisson type, then $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$ are called, respectively, the W-quantization *and the* $\widetilde{W}$-quantization *of f.*

Remark 5.1.2. *As discussed in (RIOS; STRAUME, 2014, Chapter 9), applying the term* quantization *to the Poisson algebra* $\{C^\infty_{\mathbb{C}}(S^2), \omega\}$ *requires that* $\mathbf{W}_{\mathcal{C}}$ *be of (anti-)Poisson type.*

Still, nothing has been explicitly said about the sequence of spaces which are acted upon by the operator sequences $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$ defined above for $f \in C^\infty_{\mathbb{C}}(S^2)$. In order to explore concrete representations of the sequential quantizations $\mathbf{F}^w, \widetilde{\mathbf{F}}^w \in \mathfrak{M}$, we start with the following proposition, which was set out by Bargmann (BARGMANN, 1962).

Definition 5.1.4 ((BARGMANN, 1961; BARGMANN, 1962; SEGAL, 1962))**.** *Let* $\mathcal{H}_{ol2}$ *denote the space of holomorphic functions in 2 complex variables. We shall denote by* $\mathcal{H}^{2\mu}_{ol2}$ *the Hilbert space of bi-holomorphic functions that are* L^2*-integrable with respect to the inner product*

$$\langle \phi | \psi \rangle_{\mathcal{H}^{2\mu}_{ol2}} = \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} \overline{\phi(u)} \psi(u) d\mu(u) \ , \tag{5.22}$$

where $u = (u_1, u_2) \in \mathbb{C}^2$ *and*

$$d\mu(u) = e^{-u\overline{u}} du d\overline{u} = d\mu(u_1, u_2) \ , \tag{5.23}$$

with $u\overline{u} = u_1\overline{u}_1 + u_2\overline{u}_2$, $dud\overline{u} = du_1 \wedge du_2 \wedge d\overline{u}_1 \wedge d\overline{u}_2$.

Proposition 5.1.1 ((BARGMANN, 1962))**.** *The inner product defined by (5.22)-(5.23) is invariant under the standard action of* $SU(2)$ *on* $\mathbb{C}^2$*. Under this action, we have the splitting:*

$$\mathcal{H}^{2\mu}_{ol2} = \bigoplus_{2j=0}^{\infty} \mathcal{H}^j_{om2} \ , \tag{5.24}$$

where $\mathcal{H}^j_{om2}$ *is the space of homogeneous polynomials of degree* $n = 2j$ *in 2 complex variables, so that each* $\mathcal{H}^j_{om2}$*, of dimension* $n+1$*, defines a spin-*j *system, in other words, the* $SU(2)$ *action on* $\mathcal{H}^{2\mu}_{ol2}$ *splits into irreducible unitary representations*

$$\varphi_j : SU(2) \to U(n+1) \ \textit{acting on each}\ \mathcal{H}^j_{om2} \ .$$

Furthermore, in each $\mathcal{H}^j_{om2}$ *we define the restricted inner product* $\langle \cdot | \cdot \rangle_{\mathcal{H}^j_{om2}}$ *induced from (5.22)-(5.23), that is,* $\forall \phi_j, \psi_j \in \mathcal{H}^j_{om}$*,*

$$\langle \phi_j | \psi_j \rangle_{\mathcal{H}^j_{om}} = \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} \overline{\phi_j(u)} \psi_j(u) d\mu(u) = \langle \phi_j | \psi_j \rangle_{\mathcal{H}^2_{ol2}} \ , \tag{5.25}$$

with $d\mu(u)$ *still given by (5.23). It follows that the set*

$$\left\{ \mathbf{u}(j,m) = \frac{u_1^{j-m} u_2^{j+m}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j \le m \le j} \tag{5.26}$$

forms a standard basis for the respective spin-j *system, that is,* $\langle \mathbf{u}(j,m) | \mathbf{u}(j,m') \rangle = \delta_{m,m'}$ *, where* $\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_{\mathcal{H}^j_{om2}}$ *, and* $\mathbf{u}(j,m)$ *satisfies (2.2)-(2.3).*

Example 1. *In view of the above, any operator $T^j_\alpha : \mathcal{H}^j_{om2} \to \mathcal{H}^j_{om2}$, $\psi_j \mapsto \phi_j$ is determined by an integral kernel $\mathcal{K}^j_\alpha$ via*

$$\phi_j(u) = \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} \mathcal{K}^j_\alpha(u,v)\psi_j(v)d\mu(v) \tag{5.27}$$

which from (5.25)-(2.2) has the form

$$\mathcal{K}^j_\alpha(u,v) \;=\; \sum_{m,m'=-j}^{j} \frac{\kappa(\alpha)^j_{m,m'} u_1^{j-m} u_2^{j+m} \overline{v}_1^{j-m'} \overline{v}_2^{j+m'}}{\sqrt{(j-m)!(j+m)!(j-m')!(j+m')!}} \ , \tag{5.28}$$

$$\kappa(\alpha)^j_{m,m'} = (-1)^{j-m'} M(\alpha)^j_{m,-m'} \ , \tag{5.29}$$

cf. (2.4), where $M(\alpha)^j_{m,m'}$ are the entries in the matrix

$$\sum_{m,m'=-j}^{j} M(\alpha)^j_{m,m'} \boldsymbol{u}(j,m) \otimes \check{\boldsymbol{u}}(j,m') \ \in \ M_{\mathbb{C}}(n+1) \ . \tag{5.30}$$

Then, for $f \in C^\infty_{\mathbb{C}}(S^2)$ and $\boldsymbol{W}_{\mathcal{C}}$ of (anti-)Poisson type, the coefficients $\widetilde{\kappa}^w(f)^j_{m,m'}$ in (5.28) of the integral kernel $\widetilde{\mathcal{K}}^j_w[f]$ in the $\widetilde{W}$-quantization of f are given by

$$\widetilde{\kappa}^w(f)^j_{m,m'} = \sqrt{n+1}(-1)^{j-m'} \sum_{l=0}^{n} \sum_{\bar{m}=-l}^{l} c^n_l \langle Y^{\bar{m}}_l | f\rangle C^{j,\ \ j,\ \ l}_{m,-m',\bar{m}} \ , \tag{5.31}$$

cf. (5.15), (5.19), (5.21), (2.11), with $\langle\cdot|\cdot\rangle$ given by (2.16). Similarly, from (5.20), for the coefficients $\kappa^w(f)^j_{m,m'}$ and the integral kernel $\mathcal{K}^j_w[f]$ in the W-quantization sequence of f, substituting $c^n_l \leftrightarrow 1/c^n_l$ in (5.31). Thus, $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$ define sequences of kernels $\mathbf{K}_w[f] = (\mathcal{K}^j_w[f])_{n\in\mathbb{N}}$ and $\widetilde{\mathbf{K}}_w[f] = (\widetilde{\mathcal{K}}^j_w[f])_{n\in\mathbb{N}}$, determining sequences of integral operators acting via (5.27) on the sequence of Hilbert spaces $\mathcal{H}^j_{om2}$.

Now, because $\langle\cdot|\cdot\rangle_{\mathcal{H}^j_{om2}}$ is the restriction to $\mathcal{H}^j_{om2} \subset \mathcal{H}^{2\mu}_{ol2}$ of the inner product (5.22) on $\mathcal{H}^{2\mu}_{ol2}$, it follows that a countable orthonormal basis for $\mathcal{H}^{2\mu}_{ol2}$ is given by

$$\left\{ \mathbf{u}(j,m) = \frac{u_1^{j-m} u_2^{j+m}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j\le m\le j,\ 2j\in\mathbb{N}_0} .$$

However, in this Bargmann representation it is not so easy to picture the limit $j \to \infty$. From one perspective, the one described above, the Hilbert space $\mathcal{H}^j_{om2}$ tends, as $j \to \infty$, to the space of homogeneous holomorphic polynomials of infinite degree in 2 complex variables, which is meaningless. More meaningful would be to consider the direct sum (5.24) and consider the Hilbert space $\mathcal{H}^{2\mu}_{ol2}$ of L^2-bi-holomorphic functions, which could also be seen as a sequence of "partial sums"

$$(\mathcal{H}^{2,k}_{ol2})_{k\in\mathbb{N}_0} \ , \quad \mathcal{H}^{2,k}_{ol2} = \bigoplus_{2j=0}^{k} \mathcal{H}^j_{om2} \ .$$

However, for each $k \in \mathbb{N}_0$, $\dim(\mathcal{H}^{2,k}_{ol2}) = (k+1)(k+2)/2$ thus, for $k > 0$, each $\mathcal{H}^{2,k}_{ol2}$ is "too big" for a spin-j system, defining instead a reducible unitary representation of $SU(2)$ consisting of the direct sum of spin-j systems for all $2j \leq k$, including the 1-dimensional trivial spin system $2j = 0$. In other words, the separable Hilbert space $\mathcal{H}^{2\mu}_{ol2}$ is "too big" to be the $j \to \infty$ limit Hilbert space of spin-j systems.

Therefore, we need a more tailored approach, to be developed below.

5.2 Ground Hilbert spaces and asymptotic operators

Let $\mathfrak{H}$ be a sequence of complex Hilbert spaces,

$$\mathfrak{H} = (\mathcal{H}^j)_{2j=n\in\mathbb{N}} \ , \quad \dim_{\mathbb{C}}(\mathcal{H}^j) = n+1 \ \Leftrightarrow \ \mathcal{H}^j \simeq \mathbb{C}^{n+1} \ , \tag{5.32}$$

each $\mathcal{H}^j$ with its inner product $\langle\cdot|\cdot\rangle_j$, conjugate linear in the first entry so that, if $\{\mathbf{e}^j_k\}_{1\leq k\leq n+1}$ is an orthonormal basis w.r.t. $\langle\cdot|\cdot\rangle_j$ on $\mathcal{H}^j$, for $\phi^j, \psi^j \in \mathcal{H}^j$ we have:

$$\phi^j = \sum_{k=1}^{n+1} \alpha^j_k \mathbf{e}^j_k \ , \ \psi^j = \sum_{k=1}^{n+1} \beta^j_k \mathbf{e}^j_k \quad \Rightarrow \quad \langle\phi^j|\psi^j\rangle_j = \sum_{k=1}^{n+1} \overline{\alpha}^j_k \beta^j_k \ . \tag{5.33}$$

Definition 5.2.1. *For any* $\Phi = (\phi^j)_{2j=n\in\mathbb{N}}$ *and* $\mathfrak{H} = (\mathcal{H}^j, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$ *, we denote*

$$\Phi = (\phi^j)_{2j=n\in\mathbb{N}} \in \mathfrak{H} \quad \Longleftrightarrow \quad \phi^j \in \mathcal{H}^j \ , \ \forall n = 2j \in \mathbb{N} \ . \tag{5.34}$$

For any $\Phi = (\phi^j)_{2j=n\in\mathbb{N}}, \Psi = (\psi^j)_{2j=n\in\mathbb{N}} \in \mathfrak{H}$ *, their* sum *is defined naturally by*

$$\Phi + \Psi = (\phi^j + \psi^j)_{2j=n\in\mathbb{N}} \ \in \ \mathfrak{H} \ . \tag{5.35}$$

Also, $\mathfrak{H}$ *is a bi-module of the commutative ring of scalars* $\mathfrak{C}$*, for the* multiplication by scalar *defined naturally, for every* $\mathbf{a} \in \mathfrak{C}$*,* $\Phi \in \mathfrak{H}$ *, by*

$$\mathbf{a}\Phi = \Phi\mathbf{a} = (a_n\phi^j)_{2j=n\in\mathbb{N}} \ \in \ \mathfrak{H} \tag{5.36}$$

and $\mathfrak{H}$ *is a left module for the algebra of operator sequences* $\mathfrak{M}$ *with* sequential action $\mathfrak{M} \times \mathfrak{H} \to \mathfrak{H}$ *given by*

$$(\mathbf{F}, \Phi) \ \mapsto \quad \mathbf{F}(\Phi) = (F_n(\phi^j))_{2j=n\in\mathbb{N}} \ , \tag{5.37}$$

where $F_n(\phi^j)$ *is the usual action* $(\mathcal{B}^j \times \mathcal{H}^j \to \mathcal{H}^j) \simeq (M_{\mathbb{C}}(n+1) \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1})$.

Furthermore, we define the inner product *and* norm *on* $\mathfrak{H}$ *, respectively by*

$$\begin{aligned} \langle\Phi|\Psi\rangle &= (\langle\phi^j|\psi^j\rangle_j)_{2j=n\in\mathbb{N}} \quad \in \ \mathfrak{C} \ , & (5.38) \\ ||\Phi||^2 &= \langle\Phi|\Phi\rangle = (\langle\phi^j|\phi^j\rangle_j)_{2j=n\in\mathbb{N}} = (||\phi^j||^2)_{2j=n\in\mathbb{N}} \quad \in \ \mathfrak{R}^+. & (5.39) \end{aligned}$$

Then, for any $\mathbf{F} \in \mathfrak{M}$*, the* $\mathfrak{H}$-operator norm $||\mathbf{F}||_{op} \in \mathfrak{R}^+$ *is given by*

$$||\mathbf{F}||_{op} = (||F_n||_{op}) \ , \quad ||F_n||_{op} = \sup_{\phi^j \in \mathcal{H}_j \setminus \{0\}} ||F_n(\phi^j)||/||\phi^j|| \ , \tag{5.40}$$

and its asymptotic $\mathfrak{H}$-operator norm, *when it makes sense, is given/denoted by*

$$||\mathbf{F}||_{op}^{\infty} = \lim_{n\to\infty} ||F_n||_{op} \in \mathbb{R}^+ \cup \{\infty\} \ . \tag{5.41}$$

If $\|\mathbf{F}\|_{op}$ *is bounded,* $\mathbf{F}$ *is called* $\mathfrak{H}$-bounded, *otherwise it is called* $\mathfrak{H}$-unbounded.

Definition 5.2.2. *We say that* $\mathfrak{H}^< = (\mathcal{H}^j, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$ *is a* nested sequence of Hilbert spaces *if, for every* $j \le j'$, *there exists an injective linear* nesting map $\iota_j^{j'} : \mathcal{H}^j \to \mathcal{H}^{j'}$ *which is also an isometry, i.e. such that* $\langle\cdot|\cdot\rangle_j = (\iota_j^{j'})^* \langle\cdot|\cdot\rangle_{j'}$.

It follows that, for any $j \le j'$, $\mathcal{H}^{j'} = \iota_j^{j'}(\mathcal{H}^j) \oplus (\iota_j^{j'}(\mathcal{H}^j))^\perp$, where $\perp$ is with respect to $\langle\cdot|\cdot\rangle_{j'}$. Thus, $\forall\phi^{j'} \in \mathcal{H}^{j'}$ we define

$$\phi^{j'} = \hat{\phi}_j^{j'} + \check{\phi}_j^{j'} \ , \quad \hat{\phi}_j^{j'} = P_{j,j'}(\phi^{j'}) \ , \ \check{\phi}_j^{j'} = P_{j,j'}^\perp(\phi^{j'}) \ , \tag{5.42}$$

where $P_{j,j'}, P_{j,j'}^\perp$ are the projectors $P_{j,j'} : \mathcal{H}^{j'} \to \mathrm{Im}(\iota_j^{j'})$, $P_{j,j'}^\perp : \mathcal{H}^{j'} \to \mathrm{Im}(\iota_j^{j'})^\perp$, and also, $\forall\psi^j \in \mathcal{H}^j$ we define

$$\psi_j^{j'} = \iota_j^{j'}(\psi^j) \in \mathcal{H}^{j'} \implies \hat{\psi}_j^{j'} = \psi_j^{j'} \ , \ \check{\psi}_j^{j'} = 0 \ . \tag{5.43}$$

Definition 5.2.3. *Let* $\mathfrak{H}^< = (\mathcal{H}^j, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$ *be a nested sequence of Hilbert spaces. We say that* $\mathbf{F} = (F_n)_{n\in\mathbb{N}} \in \mathfrak{M}$ *is* rigid *if* $\mathbf{F} : \mathfrak{H}^< \to \mathfrak{H}^<$ *given by (5.37) also satisfies*

$$F_{n'} \circ \iota_j^{j'} = \iota_j^{j'} \circ F_n \ , \ \forall n, n' \ \textit{with} \ \ n = 2j \le 2j' = n' \ , \tag{5.44}$$

and we denote by $\mathfrak{M}^<$ *the set of all rigid operator sequences,* $\mathfrak{M}^< \subset \mathfrak{M}$.

Furthermore, on $\mathfrak{H}^< = (\mathcal{H}^j, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$, *the* nested norm $\|\cdot\|$ *is defined for any* $\psi^j \in \mathcal{H}^j$, $\phi^{j'} \in \mathcal{H}^{j'}$, *with* $j \le j'$, *by*

$$\left\|\phi^{j'} - \psi^j\right\| := \left\|\phi^{j'} - \psi_j^{j'}\right\| \ , \tag{5.45}$$

where the norm on the r.h.s. of (5.45) is taken w.r.t. $\langle\cdot|\cdot\rangle_{j'}$, *cf. (5.6). Similarly for* $j' \le j$. *Then, the* space of convergent state sequences *is*

$$\mathfrak{H}_\infty^< = \left\{ \Phi = (\phi^j)_{2j=n\in\mathbb{N}} \in \mathfrak{H}^< \ \middle| \ \exists \lim_{j\to\infty} \phi^j \right\} \ , \tag{5.46}$$

where convergence of $\Phi = (\phi^j)_{2j=n\in\mathbb{N}} \in \mathfrak{H}^<$ *is in the sense of Cauchy w.r.t. the nested norm on* $\mathfrak{H}^<$ *given by (5.45).*

Accordingly, the set of convergent operator sequences *is defined as*

$$\mathfrak{M}_\infty = \left\{ \mathbf{F} \in \mathfrak{M} \ \middle| \ \mathbf{F} : \mathfrak{H}_\infty^< \to \mathfrak{H}_\infty^< \right\} \ . \tag{5.47}$$

Here we use notation (5.34) and redefine all operations in Definition 5.2.1 as natural restrictions to $\mathfrak{H}_\infty^<$ *and* $\mathfrak{M}_\infty$, *so that, in particular,* $\mathfrak{M}_\infty \times \mathfrak{H}_\infty^< \to \mathfrak{H}_\infty^<$ *is still given by (5.37),* $\forall \mathbf{F} \in \mathfrak{M}_\infty$.

Remark 5.2.1. *The sets $\mathfrak{H}^<_\infty$ and $\mathfrak{M}_\infty$ are the relevant ones for studying the asymptotic limit $j\to\infty$. On the other hand, apart from the identity and some projector sequences, in general not many operator sequences are rigid. But in Example 3, the sequence $(J_3^j)_{j\in\mathbb{N}}$ is rigid, where $J_3^j = d\varphi_j(iL_3)$, $L_3\in\mathfrak{so}(3)$, for $(\varphi_j)_{j\in\mathbb{Z}}$ a sequence of representations $\varphi_j : SO(3)\to U(n+1)$. Hence, in this case, it is also rigid the sequence $(\varphi_j(g))_{j\in\mathbb{N}}$, $\forall g = g(\theta)\in S^1 = \{\exp(\theta L_3) : \theta\in\mathbb{R} \bmod 2\pi\}\subset SO(3)$.*

Now, given an orthonormal basis $\{\mathbf{e}^j_k\}_{1\le k\le 2j+1}$ for $\mathcal{H}^j\subset\mathfrak{H}^<_\infty$ there is a canonical way to choose an orthonormal basis for $\mathcal{H}^{j+1/2}\subset\mathfrak{H}^<_\infty$, up to a phase, by taking

$$\begin{aligned} &\mathbf{e}^{j+1/2}_k = \iota^{j+1/2}_j(\mathbf{e}^j_k)\ ,\ 1\le k\le 2j+1\ ,\\ &\mathbf{e}^{j+1/2}_{2j+2}\in(\iota^{j+1/2}_j(\mathcal{H}^j))^\perp\ ,\ \left\|\mathbf{e}^{j+1/2}_{2j+2}\right\| = 1\ . \end{aligned} \tag{5.48}$$

Thus, starting with a standard basis for $\mathcal{H}^{1/2}\subset\mathfrak{H}^<_\infty$, we can inductively define an orthonormal basis for every $\mathcal{H}^j\subset\mathfrak{H}^<_\infty$, up to $2j-1$ choices of phases.

Definition 5.2.4. *We say that $\mathfrak{H}^<_\infty$ is* well-nested *if there exists a canonical choice of phase for $\mathbf{e}^{j+1/2}_{2j+2}$ in (5.48), for every $2j = n\in\mathbb{N}$.*

The unique sequence of orthonormal basis $\mathfrak{E} = (\mathcal{E}^j)_{2j\in\mathbb{N}} = (\{\mathbf{e}^j_k\}_{1\le k\le 2j+1})_{2j\in\mathbb{N}}$ for $\mathfrak{H}^<_\infty$ which is obtained inductively from a standard basis $\mathcal{E}^{1/2}$ for $\mathcal{H}^{1/2}\subset\mathfrak{H}^<_\infty$ using the canonical phase choice is called a well-nested *basis sequence*[1] *for $\mathfrak{H}^<_\infty$.*

From now on, we shall assume that $\mathfrak{H}^<_\infty$ is well-nested, with a well-nested basis sequence $\mathfrak{E} = (\mathcal{E}^j)_{2j\in\mathbb{N}} = (\{\mathbf{e}^j_k\}_{1\le k\le 2j+1})_{2j\in\mathbb{N}}$.

Definition 5.2.5. *The* ground Hilbert space $\mathcal{H}$ *for well-nested $(\mathfrak{H}^<_\infty,\mathfrak{E})$ is the space of complex ℓ^2-sequences spanned by the* grounding basis[2] *$\mathcal{E} = \{\mathbf{e}_k\}_{k\in\mathbb{N}} = \mathcal{E}^\infty$,*

$$\mathbf{e}_k := \lim_{j\to\infty}\mathbf{e}^j_k\ ,\ \forall k\in\mathbb{N}\ , \tag{5.49}$$

where each limit in (5.49) is taken in the sense of Definition 5.2.3, with inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{H}$ being conjugate-linear in the first entry and satisfying

$$\langle\mathbf{e}_k|\mathbf{e}_{k'}\rangle = \delta_{k,k'}\ ,\ \forall k,k'\in\mathbb{N}\ . \tag{5.50}$$

That is, $\mathcal{E}$ provides the identification $\mathcal{H}\ni\phi \iff (\alpha_k)_{k\in\mathbb{N}}\in\ell^2$, $\alpha_k\in\mathbb{C}$, via

$$\phi\in\mathcal{H} \iff \phi = \sum_{k=1}^\infty \alpha_k\mathbf{e}_k\ ,\ \langle\phi|\phi\rangle = \sum_{k=1}^\infty|\alpha_k|^2 < \infty\ . \tag{5.51}$$

[1] Alternatively, given a sequence of representations $\varphi_j : SU(2)\to U(n+1)$ defining a sequence of standard basis $\mathcal{E}^j$ for $\mathcal{H}^j$, the well-nested basis sequence is defined by specifying the nesting maps $\iota^{j'}_j : \mathcal{E}^j\to\mathcal{E}^{j'}$ plus a unique choice of "overall constant" for each j, that is, a function $\eta:\mathbb{N}\to\mathbb{C}^*$, cf. e.g. (5.75) for $\eta(2j) = \nu_j$ in Example 2 and (5.112) for $\eta(2j) = \rho_j$ in Example 3.

[2] From $\mathbb{N}\leftrightarrow\mathbb{Z}$, sometimes it may be useful to describe this countable basis as $\mathcal{E} = \{\mathbf{e}_m\}_{m\in\mathbb{Z}}$ or as $\mathcal{E} = \{\mathbf{e}_m\}_{2m\in\mathbb{Z}}$, with $\mathbf{e}_m = \lim_{j\to\infty}\mathbf{e}^j_m$ for $\{\mathbf{e}^j_m\}_{-j\le m\le j}$ orthonormal basis of $\mathcal{H}^j$.

Noting that, $\forall j \in \mathbb{N}$, $\langle\cdot|\cdot\rangle_j$ is conjugate-linear in the first entry and satisfies

$$\langle \mathbf{e}^j_k | \mathbf{e}^j_{k'} \rangle_j = \delta_{k,k'} \ , \ 1 \leq \forall k, k' \leq 2j+1 \ , \tag{5.52}$$

we can take (5.50) to be the consistency condition between (5.49) and (5.52).

Theorem 5.2.1. *Let* $(\mathfrak{H}^<_\infty, \mathfrak{E})$ *be well-nested and let* $[\Phi]$ *denote the equivalence class of* $\Phi \in \mathfrak{H}^<_\infty$*, under the equivalence relation* $(\phi^j) = \Phi \approx \widetilde{\Phi} = (\widetilde{\phi}^j) \in \mathfrak{H}^<_\infty$ *given by*

$$\Phi \approx \widetilde{\Phi} \quad \Longleftrightarrow \quad \lim_{j\to\infty} \phi^j = \lim_{j\to\infty} \widetilde{\phi}^j \ . \tag{5.53}$$

Then, $\mathcal{H}$ *is isomorphic to* $\mathfrak{H}^<_\infty / \approx$ *and* $\mathfrak{E}$ *provides a canonical isomorphism*

$$\mathcal{H} \ = \ \{\phi = \lim_{j\to\infty} \phi^j \equiv [\Phi] \mid \Phi = (\phi^j) \in [\Phi]\} \ . \tag{5.54}$$

Furthermore, $\mathfrak{E}$ *determines the sequence of isometries* $\Gamma : (\mathfrak{H}^<_\infty, \mathfrak{E}) \to (\mathcal{H}, \mathcal{E})$*, with* $\Gamma = (\gamma_j)_{2j\in\mathbb{N}}$ *given by*

$$\gamma_j : (\mathcal{H}^j, \mathcal{E}^j) \to (\mathcal{H}, \mathcal{E}) \ , \quad \phi^j = \sum_{k=1}^{2j+1} \alpha^j_k \mathbf{e}^j_k \ \mapsto \ \sum_{k=1}^{2j+1} \alpha^j_k \mathbf{e}_k = \gamma_j(\phi^j) \ . \tag{5.55}$$

Proof. First, for any $k \in \mathbb{N}$, let $(\hat{\mathbf{e}}^j_k)_{2j\in\mathbb{N}} \in \mathfrak{H}^<$ be such that $\hat{\mathbf{e}}^j_k = 0$, for $2j+1 < k$, $\hat{\mathbf{e}}^j_k = \mathbf{e}^j_k$, otherwise. Then, clearly, $(\hat{\mathbf{e}}^j_k)_{2j\in\mathbb{N}} \in \mathfrak{H}^<_\infty$, hence, for any $k \in \mathbb{N}$, we have the canonical identification $\mathbf{e}_k = [(\hat{\mathbf{e}}^j_k)_{2j\in\mathbb{N}}] \in \mathfrak{H}^<_\infty / \approx$.

Now, define $L : \mathcal{H} \to \mathfrak{H}^<_\infty$ and $\Lambda : \mathcal{H} \to \mathfrak{H}^<_\infty / \approx$, by

$$\phi = \sum_{k=1}^{\infty} \alpha_k \, \mathbf{e}_k \ \mapsto \ L\phi = (\phi^j) \ , \ \phi^j = \sum_{k=1}^{n+1} \alpha_k \, \mathbf{e}^j_k \in \mathcal{H}^j \tag{5.56}$$

$$\Lambda\phi \ = \ [L\phi] \ . \tag{5.57}$$

It is clear that Λ is linear, we will show it is a bijection.

For $\phi, \psi \in \mathcal{H}$, let $L\phi = (\phi^j)$ and $L\psi = (\psi^j)$ be as in (5.56). If $\phi \neq \psi$, then $\|\phi - \psi\| > 0$, where the norm $\|\cdot\|$ in $\mathcal{H}$ is the $\langle\cdot|\cdot\rangle$ norm, and because $\langle \mathbf{e}^j_k | \mathbf{e}^j_{k'} \rangle = \langle \mathbf{e}_k | \mathbf{e}_{k'} \rangle$, $\forall 2j+1 \geq k, k'$, there exists j_0 such that $\left\| \phi^{j'} - \psi^j \right\| > 0$, $\forall j', j \geq j_0$. Thus, $\Lambda\phi \neq \Lambda\psi$ and Λ is an injection.

Now, let $\widetilde{\Phi} = (\widetilde{\phi}^j) \in \mathfrak{H}^<_\infty$ be a general element, with

$$\widetilde{\phi}^j = \sum_{k=1}^{n+1} \alpha^j_k \, \mathbf{e}^j_k \ . \tag{5.58}$$

For any $\epsilon > 0$ there is $2j_0 = n_0 \in \mathbb{N}$ such that

$$\left\| \widetilde{\phi}^{j'} - \widetilde{\phi}^j \right\|^2 = \sum_{k=1}^{2j+1} \left| \alpha^{j'}_k - \alpha^j_k \right|^2 + \sum_{k=2j+2}^{2j'+1} \left| \alpha^{j'}_k \right|^2 < \epsilon \tag{5.59}$$

for every $j' > j \geq j_0$. Thus, for any $k \in \mathbb{N}$, $(\alpha_k^j)_{2j\in\mathbb{N}}$ is a Cauchy sequence of complex numbers and, hence, there exists $(\alpha_k)_{k\in\mathbb{N}} \in \mathfrak{C}$ satisfying

$$\lim_{j\to\infty} \alpha_k^j = \alpha_k \ . \tag{5.60}$$

From (5.59), we get

$$\sum_{k=2j_0+2}^{2j+1} |\alpha_k|^2 = \lim_{j'\to\infty} \sum_{k=2j_0+2}^{2j+1} \left|\alpha_k^{j'}\right|^2 \leq \limsup_{j'\to\infty} \sum_{k=2j_0+2}^{2j'+1} \left|\alpha_k^{j'}\right|^2 \leq \epsilon \ ,$$

for any $j > j_0$, thus

$$\sum_{k=2j_0+2}^{\infty} |\alpha_k|^2 \leq \epsilon \ . \tag{5.61}$$

This implies that $(s_n)_{n\in\mathbb{N}} \in \mathfrak{R}^+$, with $s_n = \sum_{k=1}^{n+1} |\alpha_k|^2$, is a Cauchy sequence, which means that $(\alpha_k)_{k\in\mathbb{N}}$ is a complex ℓ^2-sequence, and thus

$$\phi = \sum_{k=1}^{\infty} \alpha_k \, \mathbf{e}_k \in \mathcal{H} \ .$$

We then need to show that $L\phi$ given by (5.56) is equivalent to $\widetilde{\Phi}$ given by (5.58), with the relation between α_k^j and α_k given by (5.60). Again, from (5.59),

$$\left\|\phi^j - \widetilde{\phi}^j\right\|^2 = \sum_{k=1}^{2j+1} \left|\alpha_k - \alpha_k^j\right|^2 = \lim_{j'\to\infty} \sum_{k=1}^{2j+1} \left|\alpha_k^{j'} - \alpha_k^j\right|^2 \leq \epsilon \tag{5.62}$$

for $j \geq j_0$, therefore $L\phi \approx \tilde{\Phi}$. Thus, Λ is a surjection, hence it is a bijection.

In particular, we have $\Lambda \mathbf{e}_k = [(\hat{\mathbf{e}}_k^j)_{2j\in\mathbb{N}}]$, as should be. Also, (5.61) and (5.62) show that $\lim_{j\to\infty} \widetilde{\phi}^j = \phi$, so (5.54) holds.

To finish, we show that the family of nesting maps $\{\iota_j^{j'}\}$ induces the sequence of isometries Γ. Fix j_0. For any $\phi^{j_0} \in \mathcal{H}^{j_0}$, take $\Phi = (\phi^j)$, where

$$\phi^j = \begin{cases} \iota_{j_0}^j(\phi^{j_0}) \ , & j \geq j_0 \\ 0 \ , & j < j_0 \end{cases} .$$

Then, $\|\phi_j - \phi_{j_0}\| = 0$ for every $j \geq j_0$. So $\tilde{\gamma}_{j_0} : \mathcal{H}^{j_0} \to \mathfrak{H}_\infty^< / \approx \, : \phi^{j_0} \mapsto [\Phi]$ is well defined. So we make $\gamma_{j_0} = \Lambda^{-1} \circ \tilde{\gamma}_{j_0}$. A straightforward calculation gives $\gamma_{j_0}(\mathbf{e}_k^{j_0}) = \mathbf{e}_k$ for any $\mathbf{e}_k^{j_0} \in \mathcal{E}^{j_0}$, and this proves that (5.55) holds. $\square$

Corollary 5.2.1.1. *For any $\Phi \in \mathfrak{H}^<$, we have that $\Phi \in \mathfrak{H}_\infty^<$ if and only if $\Gamma(\Phi)$ is convergent in $\mathcal{H}$. Also, for $\phi = \lim_{n\to\infty} \Gamma(\Phi)$, we have $[\Phi] \equiv \phi$, cf. (5.56)-(5.57).*

We now investigate the relations between operator sequences $\mathbf{F}$ and operators on the ground Hilbert space $\mathcal{H}$. First we have the following natural definitions.

Definition 5.2.6. *Let $(\mathcal{H}, \mathcal{E})$ be the ground Hilbert space for a well-nested $(\mathfrak{H}_\infty^<, \mathfrak{E})$.*

Given any $\mathbf{F} = (F_n) \in \mathfrak{M}$, for $\mathfrak{M} \times \mathfrak{H}^< \to \mathfrak{H}^<$ as in (5.37), the Γ-induced operator sequence *$(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the sequence of operators $\mathcal{F}_n : \mathcal{H} \to \mathcal{H}$ defined by*

$$\mathcal{F}_n(\phi) := \mathcal{F}_n \circ \gamma_j(\phi^j) = \gamma_j \circ F_n(\phi^j) \ , \quad \textit{for} \quad \phi^j \in L\phi \ , \quad n = 2j \ . \tag{5.63}$$

We also recall that, for any $\mathcal{F} : \mathcal{H} \to \mathcal{H}$, its $\mathcal{H}$-operator-norm *$||\mathcal{F}||_{op}$ is given by*

$$||\mathcal{F}||_{op} = \sup_{\phi \in \mathcal{H} \backslash \{0\}} \|\mathcal{F}(\phi)\| / \|\phi\| \ \in \ \mathbb{R}^+ \cup \{\infty\} \ , \tag{5.64}$$

and $||\mathcal{F}||_{op} < \infty$ means that $\mathcal{F}$ is bounded, *or* $\mathcal{H}$-bounded.

Proposition 5.2.2. *Let $\mathbf{F} = (F_n)_{n\in\mathbb{N}} \in \mathfrak{M}$. If its Γ-induced operator sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ converges in $\mathcal{H}$-operator-norm, then $\mathbf{F} \in \mathfrak{M}_\infty$.*

Proof. If $\Phi = (\phi^j) \in \mathfrak{H}_\infty^<$, we have

$$\begin{aligned} \left\| \mathcal{F}_{n'} \circ \gamma_{j'}(\phi^{j'}) - \mathcal{F}_n \circ \gamma_j(\phi^j) \right\| &\leq \left\| \mathcal{F}_{n'} \circ \gamma_{j'}(\phi^{j'}) - \mathcal{F}_n \circ \gamma_{j'}(\phi^{j'}) \right\| \\ &\quad + \left\| \mathcal{F}_n \circ \gamma_{j'}(\phi^{j'}) - \mathcal{F}_n \circ \gamma_j(\phi^j) \right\| \\ &\leq \|\mathcal{F}_{n'} - \mathcal{F}_n\|_{op} \left\| \phi^{j'} \right\| \\ &\quad + \|\mathcal{F}_n\|_{op} \left\| \gamma_{j'}(\phi^{j'}) - \gamma_j(\phi^j) \right\| \end{aligned}$$

for $n' = 2j'$ and $n = 2j$. Let $\phi \in \mathcal{H}$ and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be such that $\gamma_j(\phi^j) \to \phi$ and $\mathcal{F}_n \to \mathcal{F}$. There is $n_1 = 2j_1 \in \mathbb{N}$ for which

$$n = 2j > n_1 \implies \left\| \phi^j \right\| \leq 2\|\phi\| \ , \ \|\mathcal{F}_n\|_{op} \leq 2\|\mathcal{F}\|_{op} \ .$$

In addition, given any $\epsilon > 0$ there is $n_2 = 2j_2 \in \mathbb{N}$ for which

$$n' = 2j', n = 2j > n_2 \implies \|\mathcal{F}_{n'} - \mathcal{F}_n\|_{op} \leq \frac{\epsilon}{4\|\phi\|} \ , \ \left\| \gamma_{j'}(\phi^{j'}) - \gamma_j(\phi^j) \right\| \leq \frac{\epsilon}{4\|\mathcal{F}\|_{op}} \ .$$

Then, for $n_0 = 2j_0 = \max\{n_1, n_2\}$, we have

$$n' = 2j', n = 2j > n_0 \implies \left\| \mathcal{F}_{n'} \circ \gamma_{j'}(\phi^{j'}) - \mathcal{F}_n \circ \gamma_j(\phi^j) \right\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

Thus, $(\mathcal{F}_n \circ \gamma_j(\phi^j))$ is a Cauchy sequence and $(F_n(\phi^j)) \in \mathfrak{H}_\infty^<$, by Corollary 5.2.1.1. □

Proposition 5.2.3. *If $\mathbf{F} \in \mathfrak{M}_\infty$, then $\mathbf{F}$ is $\mathfrak{H}$-bounded.*

Proof. Suppose that $\mathbf{F}$ is $\mathfrak{H}$-unbounded and assume without loss of generality that $\|F_n\|_{op} \neq 0$ for all $n \in \mathbb{N}$ and $\|F_n\|_{op} \to \infty$. Let $(\phi^j) \in \mathfrak{H}^<$ be such that $\|\phi^j\| = 1$ and $\|F_n\|_{op} = \|F_n(\phi^j)\|$. Now,

$$\psi^j = \frac{1}{\sqrt{\|F_n\|_{op}}} \phi^j$$

defines a sequence $\Psi = (\psi^j) \in \mathfrak{H}^{<}_{\infty}$ such that $[\Psi] \equiv 0$, but

$$\left\| F_n(\psi^j) \right\| = \frac{\|F_n(\phi^j)\|}{\sqrt{\|F_n\|_{op}}} = \sqrt{\|F_n\|_{op}} \to \infty \ ,$$

in contradiction with the fact that $\mathbf{F} \in \mathfrak{M}_{\infty}$. □

Proposition 5.2.4. *If $\Phi, \Psi \in \mathfrak{H}^{<}_{\infty}$ and $\mathbf{F} \in \mathfrak{M}_{\infty}$, then $\mathbf{F}(\Phi) \approx \mathbf{F}(\Psi)$ if $\Phi \approx \Psi$.*

Proof. It is sufficient to show that if $\Phi \in \mathfrak{H}^{<}_{\infty}$ satisfies $[\Phi] \equiv 0$ and $\mathbf{F} \in \mathfrak{M}_{\infty}$, then $\mathbf{F}(\Phi) \approx \Phi$. Let $\Phi = (\phi^j)$ and $\mathbf{F} = (F_n)$. Since $\|F_n(\phi^j)\| \leq \|F_n\|_{op}\|\phi^j\|$ and $\|\phi^j\| \to 0$, by Proprosition 5.2.3, we get $\|F_n(\phi^j)\| \to 0$. □

Definition 5.2.7. *Two operator sequences $\mathbf{F}, \mathbf{F}' \in \mathfrak{M}_{\infty}$ are* equivalent*, denoted $\mathbf{F} \approx \mathbf{F}'$, if $\mathbf{F}(\Phi) \approx \mathbf{F}'(\Phi)$ for every $\Phi \in \mathfrak{H}^{<}_{\infty}$. Let us denote the space of equivalent classes of convergent operator sequences by $\mathcal{B} = \mathfrak{M}_{\infty}/\approx$.*

Then, given any $[\mathbf{F}] \in \mathcal{B}$, the induced operator *$\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\phi \mapsto \mathcal{F}(\phi) \equiv [\mathbf{F}'(\Phi)] \ , \ \textit{for} \ \ \phi \equiv [\Phi] \ , \ \Phi \in \mathfrak{H}^{<}_{\infty} \ , \tag{5.65}$$

for any $\mathbf{F}' \in [\mathbf{F}] \in \mathcal{B}$.

The following shows that the operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is well defined, according to Definitions 5.2.5, 5.2.6, 5.2.7, Theorem 5.2.1 and Propositions 5.2.2, 5.2.3, 5.2.4.

Theorem 5.2.5. *For any $[\mathbf{F}] \in \mathcal{B}$, the induced operator $\mathcal{F}$ on $\mathcal{H}$, given by (5.65), is bounded and coincides, for any $\mathbf{F}' \in [\mathbf{F}]$, with the pointwise limit of the induced sequence of operators $(\mathcal{F}'_n)$ on $\mathcal{H}$, that is, $(\mathcal{F}'_n(\phi)) \to \mathcal{F}(\phi)$ for every $\phi \in \mathcal{H}$, with $\mathcal{F}'_n$ given by (5.63).*

Proof. Given $\phi \in \mathcal{H}$, let $L\phi = (\phi^j) = \Phi$ be as in (5.56), then

$$\phi = \sum_{k=1}^{\infty} \alpha_k \, \mathbf{e}_k \implies \gamma_j(\phi^j) = \sum_{k=1}^{n+1} \alpha_k \, \mathbf{e}_k \ ,$$

so $\phi \equiv [\Phi]$ and $\mathcal{F}'_n(\phi) = \mathcal{F}'_n(\gamma_j(\phi^j))$. We also have $\mathcal{F}(\phi) \equiv [\mathbf{F}'(\Phi)]$, then

$$\mathcal{F}(\phi) = \lim_{n\to\infty} \gamma_j \circ F'_n(\phi^j) = \lim_{n\to\infty} \mathcal{F}'_n(\gamma_j(\phi^j)) = \lim_{n\to\infty} \mathcal{F}'_n(\phi) \ ,$$

cf. (5.53)-(5.54) and (5.63). That is, $(\mathcal{F}'_n)$ is a pointwise convergent sequence to $\mathcal{F}$.

Now, for any $\mathbf{F}' \in [\mathbf{F}] \in \mathcal{B}$, since $\mathcal{F}$ is the pointwise limit of the operator sequence $(\mathcal{F}'_n)$ induced by $\mathbf{F}'$ then, because each $\mathcal{F}'_n$ is bounded and $\mathbf{F}'$ is $\mathfrak{H}$-bounded, we get that $\mathcal{F}$ is also $\mathcal{H}$-bounded and $\mathcal{F}(\phi) \in \mathcal{H}$. □

In view of Theorem 5.2.5 above and Proposition 5.2.2, we single out:

Definition 5.2.8. *We say that* $\mathbf{F} = (F_n)_{n\in\mathbb{N}} \in \mathfrak{M}_\infty$ *is a* strongly convergent operator sequence *if its* Γ*-induced operator sequence* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *converges in* $\mathcal{H}$*-operator-norm. We denote by* $\hat{\mathfrak{M}}_\infty$ *the set of all strongly convergent operator sequences.*

Proposition 5.2.6. $\hat{\mathfrak{M}}_\infty$ *is a proper subset of* $\mathfrak{M}_\infty$.

Proof. Take, for instance, the operator sequence $\mathbf{F} = (F_n)_{n\in\mathbb{N}}$ determined by

$$F_n(\mathbf{e}_k^j) = \sqrt{\frac{k}{n+1}}\mathbf{e}_k^j \ , \ \forall \mathbf{e}_k^j \in \mathcal{E}^j \ , \ \forall n = 2j \in \mathbb{N} \ ,$$

where $\mathfrak{E} = (\mathcal{E}^j)_{2j\in\mathbb{N}}$ is a well-nested basis for $\mathfrak{H}^<$.

It is straightforward to see that $\mathbf{F} \in \mathfrak{M}_\infty$ but $\mathbf{F} \notin \hat{\mathfrak{M}}_\infty$. □

Finally, we also have the following results. First:

Proposition 5.2.7. *If* $\mathbf{F} = (F_n) \in \mathfrak{M}_\infty$*, then* $\mathbf{F}^\dagger = (F_n^\dagger) \in \mathfrak{M}_\infty$ *and the operator induced by* $[\mathbf{F}^\dagger]$ *is the Hermitian conjugate of the operator induced by* $[\mathbf{F}]$.

Proof. Let $(\mathcal{F}_n)$ and $(\mathcal{F}_n^\dagger)$ be the Γ-induced operator sequences of $\mathbf{F}$ and $\mathbf{F}^\dagger$, respectively. Also, let $\mathcal{F}$ be the operator induced by $[\mathbf{F}]$. Given $\Phi = (\phi^j) \in \mathfrak{H}_\infty^<$ and $\psi \in \mathcal{H}$, let $\phi = \lim_{n\to\infty} \gamma_j(\phi^j)$. Then

$$\left\langle \mathcal{F}_n^\dagger \circ \gamma_j(\phi^j) \middle| \psi \right\rangle = \left\langle \gamma_j(\phi^j) \middle| \mathcal{F}_n(\psi) \right\rangle \to \langle \phi | \mathcal{F}(\psi) \rangle \ . \tag{5.66}$$

Since ψ is any element of $\mathcal{H}$, we conclude that $(\mathcal{F}_n^\dagger \circ \gamma_j(\phi^j))$ converges to $\mathcal{F}^\dagger(\phi)$. □

Definition 5.2.9. *For any operator* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$*, its* trace*, if it exists, is*

$$\mathrm{tr}(\mathcal{F}) = \sum_{k=1}^{\infty} \langle \mathbf{e}_k | \mathcal{F}(\mathbf{e}_k) \rangle \ , \tag{5.67}$$

where $\mathcal{E} = \{\mathbf{e}_k\}_{k\in\mathbb{N}}$ *is a countable orthonormal basis for* $\mathcal{H}$. *And* $\mathcal{F}$ *is* trace-class *if* $\exists\ \mathrm{tr}(|\mathcal{F}|) \in \mathbb{R}^+$*, for* $|\mathcal{F}| = \sqrt{\mathcal{F}^\dagger \mathcal{F}}$.

We recall that the set of trace-class operators forms an ideal in the algebra of operators on an infinite-dimensional Hilbert space. Then, we have:

Proposition 5.2.8. *For* $[\mathbf{F}] \subset \mathfrak{M}_\infty$*, if* $\mathbf{F}'$ *is trace-class, for any* $\mathbf{F}' \in [\mathbf{F}]$*, then the induced operator* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *is trace-class and*

$$\mathrm{tr}(|\mathcal{F}|) \leq \mathrm{tr}_\infty(|\mathbf{F}|) \ . \tag{5.68}$$

Proof. As before, for $\mathbf{F}' \in [\mathbf{F}]$, take the induced operator sequence $(\mathcal{F}'_n)$. Then,

$$\begin{aligned}\operatorname{tr}(|\mathcal{F}|) &= \sum_{k=1}^{\infty} \langle \mathbf{e}_k | |\mathcal{F}|(\mathbf{e}_k)\rangle = \lim_{n\to\infty} \sum_{k=1}^{n+1} \langle \mathbf{e}_k | |\mathcal{F}|(\mathbf{e}_k)\rangle \\ &= \lim_{n\to\infty} \lim_{n'\to\infty} \sum_{k=1}^{n+1} \langle \mathbf{e}_k | |\mathcal{F}'_{n'}|(\mathbf{e}_k)\rangle \\ &\leq \lim_{n'\to\infty} \operatorname{tr}(|\mathcal{F}'_{n'}|) = \lim_{n'\to\infty} \operatorname{tr}(|F'_{n'}|) \ .\end{aligned} \tag{5.69}$$

So if $\mathbf{F}'$ is trace-class, then $\operatorname{tr}(|\mathcal{F}|) \leq \operatorname{tr}_\infty(|\mathbf{F}'|)$. □

The converse of the last proposition is not true. Take, for example,

$$\mathbf{F} = (F_n) \ , \ F_n = \frac{I_n}{\sqrt{n+1}} = |F_n| \ . \tag{5.70}$$

Trivially, $\|F_n\|_{op} = 1/\sqrt{n+1}$, thus we have $(F_n) \in \mathfrak{M}_\infty$ by Proposition 5.2.2. Let $\mathcal{F}$ be the operator induced by $[\mathbf{F}]$. For any $\phi \in \mathcal{H}$, from Proposition 5.2.5, we have

$$\|\mathcal{F}(\phi)\| = \lim_{n\to\infty} \|\mathcal{F}_n(\phi)\| \leq \lim_{n\to\infty} \|\mathcal{F}_n\|_{op} \|\phi\| = 0 \ .$$

Thus, $\mathcal{F}$ is identically null. So $|\mathcal{F}|$ is also identically null and $\operatorname{tr}(|\mathcal{F}|) = 0$. However, from (5.70), $\operatorname{tr}(|F_n|) = \sqrt{n+1}$, thus $\operatorname{tr}_\infty(|\mathbf{F}|) = \operatorname{tr}_\infty(\mathbf{F}) = \infty$.

5.3 Concrete examples of asymptotic spin quantization

We now provide two concrete examples that illustrate some of the previous definitions and results.

Example 2. *S^2 is a Kähler manifold, $S^2 \simeq \mathbb{C}P^1$. Then, recalling Definition 5.1.4 and Proposition 5.1.1, for each j we can pass from $\mathcal{H}^j_{om2}$ to $Poly(\mathbb{C})^h_{\leq n}$, the space of holomorphic polynomials of degree $\leq n$ in one complex variable, via identification*

$$id^j_1 : \ \mathcal{H}^j_{om2} \ni u_1^{j-m} u_2^{j+m} \longleftrightarrow \nu_j z^{j-m} \in Poly(\mathbb{C})^h_{\leq n} \ , \tag{5.71}$$

where $\nu_j \in \mathbb{C}$ is a constant, for each j, and $z \in \mathbb{C}$ is a holomorphic coordinate of $\mathbb{C}P^1$, the latter seen in this coordinate system as the flattened plane, with $SU(2)$ now acting on $z \simeq u_1/u_2 \in \mathbb{C}$ via Möbius transformations.

Thus, for each $\phi^j \in Poly(\mathbb{C})^h_{\leq n}$ there is a unique "lift" $\widetilde{\phi}^j \in \mathcal{H}^j_{om2}$ satisfying

$$\phi^j(z) = \widetilde{\phi}^j(u_1, u_2) \ ,$$

via the identification id^j_1 given by (5.71). From this, it follows that, for each j, we can define a $SU(2)$-invariant inner product $\langle\cdot|\cdot\rangle^j_{Poly(\mathbb{C})^h_{\leq n}}$ via (5.71) by

$$\langle \phi^j | \psi^j \rangle^j_{Poly(\mathbb{C})^h_{\leq n}} = \langle \widetilde{\phi^j} | \widetilde{\psi^j} \rangle_{\mathcal{H}^j_{om2}} \ . \tag{5.72}$$

However, just as the identification (5.71) is j-dependent, so is the inner product on the l.h.s. of (5.72), therefore this product cannot be written simply as the restriction to $Poly(\mathbb{C})^h_{\leq n}$ of a j-invariant inner product on the space $\mathcal{H}_{ol1}$ of holomorphic functions on $\mathbb{C}$. The inner product (5.72) can also be written in integral form,

$$\langle \phi^j | \psi^j \rangle^j_{Poly(\mathbb{C})^h_{\leq n}} \;=\; \int_{\mathbb{C}} \overline{\phi^j(z)} \psi^j(z) d\mu^j(z) \ , \tag{5.73}$$

but, in contrast with (5.25), now $d\mu^j(z)$ depends explicitly on j.

Anyway, from (5.72), the set

$$\left\{ \delta^m_j(z) \equiv \frac{\nu_j z^{j-m}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j\leq m\leq j} \ , \tag{5.74}$$

forms an orthonormal basis for

$$id^j_1(\mathcal{H}^j_{om2}, \langle\cdot|\cdot\rangle) =: (\mathcal{H}^j_z, \langle\cdot|\cdot\rangle_j) := (Poly(\mathbb{C})^h_{\leq n}, \langle\cdot|\cdot\rangle^j_{Poly(\mathbb{C})^h_{\leq n}}) \ .$$

In principle, for each j the choice of (j-dependent) constant $\nu_j \in \mathbb{C}$ is arbitrary. But there is a canonical way to choose ν_j for every j, by looking at the function 1. Setting $m = j$ in (5.74) we obtain

$$1 \equiv \frac{\nu_j}{\sqrt{(2j)!}} \implies \nu_j = \sqrt{(2j)!} \quad \textit{as a canonical choice} \ . \tag{5.75}$$

With the canonical choice (5.75), identifying $p = j - m$, we set

$$\mathcal{E}^j = \left\{ \boldsymbol{u}(j,m) \equiv \frac{z^{j-m}\sqrt{(2j)!}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j\leq m\leq j} \Leftrightarrow \quad \left\{ \boldsymbol{u}(j,m) \equiv \sqrt{\binom{n}{p}} z^p =: \boldsymbol{u}^n_p \right\}_{0\leq p\leq n} \tag{5.76}$$

as the standard basis for $(\mathcal{H}^j_z, \langle\cdot|\cdot\rangle_j)$ satisfying (2.2)-(2.3), so that

$$\langle \phi^j | \psi^j \rangle^j_{Poly(\mathbb{C})^h_{\leq n}} \equiv \langle \phi^j | \psi^j \rangle_j \;=\; \sum_{p=0}^{n} \overline{\alpha}^j_p \beta^j_p \ , \tag{5.77}$$

$$\phi^j(z) = \sum_{p=0}^{n} \alpha^j_p \sqrt{\binom{n}{p}} z^p \quad , \quad \psi^j(z) = \sum_{p=0}^{n} \beta^j_p \sqrt{\binom{n}{p}} z^p \ , \tag{5.78}$$

and we have the identification

$$J^j_3 \;\longleftrightarrow\; j - z\frac{\partial}{\partial z} : \mathcal{H}^j_z \to \mathcal{H}^j_z \ , \ \forall n = 2j \in \mathbb{N} \ . \tag{5.79}$$

We now note that the sequence of Hilbert spaces $\mathfrak{H}^< = (\mathcal{H}^j_z, \langle\cdot|\cdot\rangle_j)_{2j\in\mathbb{N}}$ is a nested sequence, with nesting maps $\iota^{j'}_j : \mathcal{H}^j_z \to \mathcal{H}^{j'}_z$ determined for $j \leq j'$ by

$$\iota^{j'}_j(\boldsymbol{u}^n_p) = \boldsymbol{u}^{n'}_p , \tag{5.80}$$

and the canonical choice (5.75) defines a well-nested basis sequence $\mathfrak{E} = (\mathcal{E}^j)_{2j\in\mathbb{N}}$*, with* $\mathcal{E}^j$ *given by (5.76).*

Thus, from Theorem 5.2.1, if $(\phi^j)_{2j\in\mathbb{N}} \in \mathfrak{H}^<_\infty$ *is given as in (5.78), then* $(\alpha_p)_{p\in\mathbb{N}_0}$*, with* $\alpha_p = \lim_{j\to\infty}\alpha^j_p \in \mathbb{C}$*, is a* ℓ^2*-sequence and* $\phi = \lim_{j\to\infty}\phi^j \in \mathcal{H}_z$*, where* $\mathcal{H}_z$ *is the ground Hilbert space of* $\mathfrak{H}^<_\infty$*. In this case, from Theorem 5.2.5, an operator sequence* $\mathbf{T} \in \mathfrak{M}_\infty$ *takes the* ℓ^2*-sequence* $(\alpha_p)_{p\in\mathbb{N}_0}$ *to an* ℓ^2*-sequence* $(\beta_p)_{p\in\mathbb{N}_0}$*, with* $\beta_p = \lim_{j\to\infty}\beta^j_p \in \mathbb{C}$*, for* β^j_i *as in (5.78), and defines an operator* $\mathcal{T} : \mathcal{H}_z \to \mathcal{H}_z$*.*

But the inner product on each $\mathcal{H}^j_z$ *can also be written in the integral form (5.73) and one obtains straightforwardly that the measure/integrator is given by*

$$d\mu^j(z) = \frac{n+1}{2\pi i}\frac{d\overline{z}\wedge dz}{(1+z\overline{z})^{n+2}} \ , \tag{5.81}$$

which yields the set (5.76) as an orthonormal basis for $\mathcal{H}^j_z$ *with respect to the inner product* $\langle\cdot|\cdot\rangle_j$ *given by (5.73) and (5.77)-(5.81), cf. also (BEREZIN, 1975).*

Thus, from the above and (5.56)-(5.57), we can also characterize the ground Hilbert space $\mathcal{H}_z$ *as the subspace of formal power series in* z*,* $\mathbb{C}[[z]]$ *, satisfying*[3]

$$\psi \in \mathcal{H}_z \subset \mathbb{C}[[z]] \iff \exists \lim_{j\to\infty}\int_{\mathbb{C}}|\psi^j(z)|^2 d\mu^j(z) \ , \ (\psi^j)_{2j\in\mathbb{N}} = L\psi \ , \ cf. \ (5.56). \tag{5.82}$$

Then, repeating the steps in Example 1, using (5.73)-(5.81) we can write any operator $T^j_\alpha : \mathcal{H}^j_z \to \mathcal{H}^j_z$*,* $\psi_j \mapsto \phi_j$ *as determined by an integral kernel* $\mathcal{K}^j_\alpha$ *via*

$$\phi_j(z_1) = \int_{\mathbb{C}}\mathcal{K}^j_\alpha(z_1,z_2)\psi_j(z_2)d\mu^j(z_2) \ , \tag{5.83}$$

with $d\mu^j(z)$ *given by (5.81), which thus has the form, cf. (2.4) and (5.30),*

$$\mathcal{K}^j_\alpha(z_1,z_2) \ = \sum_{p_1,p_2=0}^{n}(-1)^{p_2}M(\alpha)^j_{j-p_1,p_2-j}\sqrt{\binom{n}{p_1}}\sqrt{\binom{n}{p_2}}z_1^{p_1}\overline{z}_2^{p_2} \ ,$$

so that, for $f \in C^\infty_{\mathbb{C}}(S^2)$*, with* $\boldsymbol{W}_\mathcal{C}$ *of (anti-)Poisson type, the* W*-quantization of* f*,* $\mathbf{F}^w$*, determines a sequence of integral operators* $\mathbf{T}_w[f] = (T^j_w[f])_{2j=n\in\mathbb{N}}$ *acting on the well-nested sequence of Hilbert spaces* $\mathfrak{H}^< = (\mathcal{H}^j_z, \langle\cdot|\cdot\rangle_j)_{2j=n\in\mathbb{N}}$ *via (5.83) by the sequence of integral kernels* $\mathbf{K}_w[f] = (\mathcal{K}^j_w[f])_{n\in\mathbb{N}}$*, where*

$$\mathcal{K}^j_w[f](z_1,z_2) = \sqrt{n+1}\sum_{p_1,p_2,l=0}^{n}(-1)^{p_2}\sqrt{\binom{n}{p_1}}\sqrt{\binom{n}{p_2}}C^{j, \quad j, \quad l}_{j-p_1,p_2-j,m}\frac{\langle Y^m_l|f\rangle}{c^n_l}z_1^{p_1}\overline{z}_2^{p_2} \tag{5.84}$$

cf. (5.15), (5.19)-(5.21), (2.11), with $\langle\cdot|\cdot\rangle$ *given by (2.16). And similarly for* $(\widetilde{\mathcal{K}}^j_w[f])_{2j\in\mathbb{N}}$ *from* $\widetilde{\mathbf{F}}^w$*, the* $\widetilde{W}$*-quantization of* f*, replacing* $c^n_l \leftrightarrow 1/c^n_l$ *in (5.84).*

[3] It is necessary to restrict to $\psi^j \in Poly(\mathbb{C})^h_{\leq n} \subset \mathbb{C}[[z]]$ in (5.82), with $(\psi^j)_{2j\in\mathbb{N}} = L\psi$, because the inner product defined by (5.73) and (5.81) is ill-defined or divergent on $\mathbb{C}[[z]] \setminus Poly(\mathbb{C})^h_{\leq n}$.

Therefore, for $\psi \in \mathcal{H}_z$ as in (5.82), setting $(\psi^j)_{2j\in\mathbb{N}} = L\psi$ in (5.83), cf. (5.56), if $\boldsymbol{W}_{\mathcal{C}}$ is such that $\mathbf{T}_w[f] = (T_w^j[f])_{2j\in\mathbb{N}} \in \mathfrak{M}_\infty$, then $\mathcal{T}_w[f] : \psi \mapsto \phi \in \mathcal{H}_z$, where

$$\phi(z_1) = \lim_{j\to\infty} \int_{\mathbb{C}} \mathcal{K}_w^j[f](z_1, z_2)\psi^j(z_2)d\mu^j(z_2) \ , \tag{5.85}$$

and similarly for $\widetilde{\mathcal{T}}_w[f]$ with $\widetilde{\mathcal{K}}_w^j[f]$ in (5.85).

Naïvely, one could have thought of defining integral operators on the space of L^2-holomorphic functions on $\mathbb{C}$, with respect to the inner product on $\mathcal{H}_{ol1}$ defined by analogy with the inner product on $\mathcal{H}_{ol2}$, as

$$\langle\phi|\psi\rangle_{\mathcal{H}_{ol1}^{2\mu}} = \int_{\mathbb{C}} \overline{\phi(z)}\psi(z)d\mu(z) \ , \quad d\mu(z) = e^{-z\overline{z}}\frac{d\overline{z}\wedge dz}{2\pi i} \ . \tag{5.86}$$

However, this inner product is not $SU(2)$-invariant, for the $SU(2)$-action on $\mathcal{H}_{ol1}^{2\mu}$ induced from the $SU(2)$-Möbius-action on $z \in \mathbb{C}$. This can also be seen from the fact that an orthonormal basis for $\mathcal{H}_{ol1}^{2\mu}$ w.r.t. the above inner product is given by

$$\{\mathbf{v}_p = z^p/\sqrt{p!}\}_{p\in\mathbb{N}_0} \ . \tag{5.87}$$

Therefore, for any $2j = n > 1$, the set $\{\mathbf{e}_p = z^p/\sqrt{p!}\}_{0\le p\le n}$ does not constitute a standard basis for the spin-j system. In fact, from (5.76) and (5.87), we have that

$$\boldsymbol{u}_p^n = \sqrt{\frac{n!}{(n-p)!}}\mathbf{v}_p$$

so that, from (2.2)-(2.3), for $J_3^j = d\varphi_j(\sigma_3/2)$, $J_\pm^j = d\varphi_j(\sigma_\pm/2)$, where $\varphi_j : SU(2) \to U(n+1)$ is the irreducible representation on $\widetilde{\mathcal{H}^j} \subset \mathcal{H}_{ol1}$,

$$J_3^j(\boldsymbol{u}_p^n) = (j-p)\boldsymbol{u}_p^n \iff J_3^j(\mathbf{v}_p) = (j-p)\mathbf{v}_p$$

but, cf. (2.2)-(2.3), with $p = j - m$,

$$\begin{aligned} J_+^j(\boldsymbol{u}_p^n) &= \sqrt{p(n-p+1)}\boldsymbol{u}_{p-1}^n \implies J_+^j(\mathbf{v}_p) = \sqrt{p}\mathbf{v}_{p-1} \ , \\ J_-^j(\boldsymbol{u}_p^n) &= \sqrt{(n-p)(p+1)}\boldsymbol{u}_{p+1}^n \implies J_-^j(\mathbf{v}_p) = (n-p)\sqrt{(p+1)}\mathbf{v}_{p+1} \ . \end{aligned} \tag{5.88}$$

Thus, while both are basis formed by eigenvectors of J_3^j, for any j, we see from (5.88) that the basis $\{\mathbf{v}_p = z^p/\sqrt{p!}\}_{0\le p\le n}$ is not $SU(2)$-equivariant, for any $2j = n > 1$.

Example 3. *It is common to think of the quantization of a function f on a real symplectic manifold $\mathcal{M}$ as defining a differential or an integral operator on a Hilbert space of functions on a real Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$. For spin systems, any simple closed curve on S^2 is a Lagrangian submanifold, but, by symmetry considerations, it is natural to focus on the closed geodesics.*

Thus, let $\mathcal{X} \subset S^2$ be a closed geodesic, which we can take to be the equator of the coordinate system (φ, θ) on S^2, so that θ parametrizes $\mathcal{X}$. We now construct three well-nested sequences of Hilbert spaces adapted to $\mathcal{X} \subset S^2$, as follows.

In the first construction, we trivially "repeat" Example 2, looking at restrictions of functions on $\mathbb{C}P^1$ to $\mathcal{X} \simeq S^1 = \{z = e^{i\theta}\} \subset \mathbb{C}P^1 \simeq S^2$ via identification

$$\widetilde{id}_2^j : \mathcal{H}_z^j \equiv Poly(\mathbb{C})_{\leq n}^h \ni \nu_j z^{j-m} \longleftrightarrow \nu_j e^{i(j-m)\theta} \in \mathcal{H}_{e^{i\theta}}^j \ , \tag{5.89}$$

so that all expressions are imported directly from Example 2 and then everything gets well defined on $\mathcal{X} \subset S^2$, but we don't get anything new and we loose the integral expressions of Example 2. This is $SU(2)$-invariant, but uninteresting.

The second construction is quite more interesting, but now we shall restrict to the subsequence of $SU(2)$ representations with integer j's, in other words, this second construction is restricted to the sequence of $SO(3)$ representations.

Then, for each $j \in \mathbb{N}$, let $\mathcal{H}^j$ be the $(2j+1)$-dimensional complex vector space spanned by $\{e^{im\theta}\}_{-j \leq m \leq j}$. To be more explicit, we shall denote it by $\mathcal{H}_\theta^j$. In order to turn $(\mathcal{H}_\theta^j)_{j\in\mathbb{N}}$ into a well-nested sequence of Hilbert spaces, recalling identification (5.71) of Example 2, we start by identifying, for each $j \in \mathbb{N}$,

$$id_2^j : \mathcal{H}_z^j \equiv Poly(\mathbb{C})_{\leq n}^h \ni \nu_j z^{j-m} \longleftrightarrow \rho_j e^{im\theta} \in \mathcal{H}_\theta^j \ , \tag{5.90}$$

where $\rho_j \in \mathbb{C}$ is a constant for each j. Then, from (5.74), a $SO(3)$-invariant inner product $\langle\cdot|\cdot\rangle_j$ on $\mathcal{H}_\theta^j$ is determined by setting

$$\left\{ \omega_j^m = \frac{\rho_j e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j \leq m \leq j} \tag{5.91}$$

as an orthonormal basis for $\mathcal{H}_\theta^j$, so that $\langle \omega_j^m | \omega_j^{m'} \rangle_j = \delta_{m,m'}$.

In order to complete the construction of a well-nested basis sequence $\mathfrak{E}$ for the nested sequence $\mathfrak{H}^< = (\mathcal{H}_\theta^j, \langle\cdot|\cdot\rangle_j)_{2j\in\mathbb{N}}$, we need to define the nesting maps and specify a canonical choice for all ρ_j's. For the choice of ρ_j's, we can proceed as in Example 2 and look at the constant function 1 on $\mathcal{X} \subset S^2$,

$$(j \in \mathbb{N},\ m = 0):\ 1 \equiv \frac{\rho_j}{j!} \implies \rho_j = j! \ \ , \textit{ as a canonical choice} \ . \tag{5.92}$$

Thus, with the canonical choice (5.92) for ρ_j in (5.91),

$$\mathcal{E}^j = \left\{ \boldsymbol{u}(j,m) = \frac{j! e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j \leq m \leq j} \tag{5.93}$$

is a standard basis for $\mathcal{H}_\theta^j$ satisfying (2.2)-(2.3).

The nesting maps $\iota_j^{j'} : \mathcal{H}_\theta^j \to \mathcal{H}_\theta^{j'}$ *must be consistent with* id_2^j *given by (5.90) and nesting maps* $\check{\iota}_j^{j'} : \mathcal{H}_z^j \to \mathcal{H}_z^{j'}$*, that is,* $\forall j \leq j'$ *they must satisfy* $\iota_j^{j'} \circ id_2^j = id_2^{j'} \circ \check{\iota}_j^{j'}$*. The first tentative is to fix* $\check{\iota}_j^{j'}$ *as determined by (5.80). With this choice, the nesting maps* $\tilde{\iota}_j^{j'} : \mathcal{H}_\theta^j \to \mathcal{H}_\theta^{j'}$ *satisfy* $\tilde{\iota}_j^{j'}(\boldsymbol{u}(j,m)) = \boldsymbol{u}(j, m + j' - j)$*. However, these nesting maps are not well suited to the choice (5.92) for* ρ_j*. Because, in Example 2, for every* j *the constant function* 1 *was associated to the highest-weight vector, and the nesting maps (5.80) took highest weight to highest weight. Now, the constant function* 1 *is associated to the middle-weight vector, for every* j*, but the nesting maps* $\tilde{\iota}_j^{j'}$ *induced from (5.80) do not take middle-weight to middle-weight.*

So, we define new nesting maps $\iota_j^{j'} : \mathcal{H}_\theta^j \to \mathcal{H}_\theta^{j'}$*, determined for* $j \leq j'$ *by*

$$\iota_j^{j'}(\boldsymbol{u}(j,m)) = \boldsymbol{u}(j', m) \ . \tag{5.94}$$

Since both j *and* j' *are integers, (5.94) is well defined, but it induces new nesting maps* $\check{\iota}_j^{j'} : \mathcal{H}_z^j \to \mathcal{H}_z^{j'}$ *on the subsequence* $(\mathcal{H}_z^j)_{j\in\mathbb{N}}$ *via* $\iota_j^{j'} \circ id_2^j = id_2^{j'} \circ \check{\iota}_j^{j'}$*.*

Thus, with the identification (5.90), the nesting defined by (5.94) and the canonical choice (5.92), our well-nested basis sequence for $\mathfrak{H}^< = (\mathcal{H}_\theta^j, \langle\cdot|\cdot\rangle_j)_{j\in\mathbb{N}}$ *is given by* $\mathfrak{E} = (\mathcal{E}^j)_{j\in\mathbb{N}}$*, where each* $\mathcal{E}^j$ *is given by (5.93).*

In this way, a sequence $\Phi = (\phi^j)_{j\in\mathbb{N}} \in \mathfrak{H}^<$ *can be identified with a sequence of* $SO(3)$*-equivariant Fourier polynomials of degree* j *on* $\mathcal{X} \subset S^2$*, with*

$$\phi^j(\theta) \ = \ j! \sum_{m=-j}^{j} \frac{\alpha_m^j e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \ = \ \phi^j(\theta + 2\pi) \ \in \ \mathcal{H}_\theta^j \ . \tag{5.95}$$

Accordingly, we shall call α_m^j *in (5.95) the modified Fourier coefficients of* ϕ^j*, so that, if* α_m^j, β_m^j *are modified Fourier coefficients of* $\phi^j, \psi^j \in \mathcal{H}_\theta^j$ *as in (5.95), then*

$$\langle \phi^j | \psi^j \rangle_j = \sum_{m=-j}^{j} \overline{\alpha}_m^j \beta_m^j \ . \tag{5.96}$$

Furthermore, it follows from (5.93) and (5.95) that the operator J_3 *can be identified in this representation as*

$$J_3 \longleftrightarrow -i\frac{\partial}{\partial\theta} : \mathcal{H}_\theta^j \to \mathcal{H}_\theta^j \ , \ \forall j \in \mathbb{N} \ . \tag{5.97}$$

Note that J_3 *is* j*-invariant (compare with (5.79)), but recall that the sequence* $(J_3^j)_{j\in\mathbb{N}}$ *is a rigid operator sequence for the nesting maps (5.94), cf. Remark 5.2.1.*

However, in contrast with Example 2, now it is not so easy to write the inner product given by (5.95)-(5.96) in terms of an integral[4] *on* $\mathcal{X} \simeq \mathbb{R} \bmod 2\pi$*. Thus, we will now restrict ourselves to a purely discrete description.*

[4] The standard definition $\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\phi}_j(\theta)\psi_j(\theta)d\theta$ is not $SO(3)$-invariant, similarly to (5.86).

Then, repeating the steps in Example 2, now without integral descriptions, given $f \in C^\infty_{\mathbb{C}}(S^2)$, with $\boldsymbol{W}_{\mathcal{C}}$ of (anti-)Poisson type, the W-quantization of f, $\mathbf{F}^w$, determines a sequence of operators $\mathbf{T}_w[f] = (T^j_w[f])_{j\in\mathbb{N}}$ acting on the well-nested sequence of Hilbert spaces $\mathfrak{H}^< = (\mathcal{H}^j_\theta, \langle\cdot|\cdot\rangle_j)_{j\in\mathbb{N}}$ as

$$T^j_w[f] : \mathcal{H}^j_\theta \to \mathcal{H}^j_\theta \ , \ \phi^j \mapsto \psi^j \ ,$$

where, for $\phi^j(\theta)$ as in (5.95),

$$\psi^j(\theta) = j!\sqrt{2j+1}\sum_{l=0}^{2j}\sum_{m,m'=-j}^{j}\frac{\langle Y_l^{\bar m}|f\rangle}{c_l^{2j}}C^{j,\ \ j,\ \ l}_{m,-m',\bar m}\frac{(-1)^{j-m'}\alpha^j_{m'}e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \tag{5.98}$$

and similarly for the $\widetilde{W}$-quantization of f, replacing $c^n_l \leftrightarrow 1/c^n_l$ in (5.98).

Put another way, $T^j_w[f]$ takes the modified Fourier coefficients α^j_m of ϕ^j to the modified Fourier coefficients β^j_m of ψ^j, cf. (5.95), where

$$\beta^j_m = \sqrt{n+1}\sum_{l=0}^{n}\sum_{m'=-j}^{j}\frac{\langle Y_l^{\bar m}|f\rangle}{c_l^{n}}C^{j,\ \ j,\ \ l}_{m,-m',\bar m}(-1)^{j-m'}\alpha^j_{m'} \ . \tag{5.99}$$

From Theorem 5.2.1, if $(\phi^j)_{j\in\mathbb{N}} \in \mathfrak{H}^<_\infty$ is given as in (5.95) and $\mathcal{H}_\theta$ denotes the ground Hilbert space of $\mathfrak{H}^<_\infty$, then we have the identification[5]:

$$\phi = \lim_{j\to\infty}\phi^j \in \mathcal{H}_\theta \iff (\alpha_m)_{m\in\mathbb{Z}} \ , \ \sum_{m=-\infty}^{\infty}|\alpha_m|^2 \ < \infty \ , \tag{5.100}$$

where, cf. (5.94),

$$\alpha_m = \lim_{j\to\infty}\alpha^j_m \in \mathbb{C} \ , \ \forall m \in \mathbb{Z} \ , \tag{5.101}$$

so that $\phi \equiv [L\phi = (\widetilde{\phi}^j)_{j\in\mathbb{N}}] = \lim_{j\to\infty}\widetilde{\phi}^j$ is given explicitly by

$$\phi \ = \ \lim_{j\to\infty} j!\sum_{m=-j}^{j}\frac{\alpha_m e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \ \in \ \mathcal{H}_\theta \ . \tag{5.102}$$

Then, if $\mathbf{T}_w[f] \in \mathfrak{M}_\infty$, from Theorem 5.2.5 we have that

$$\mathbf{T}_w[f] \implies \mathcal{T}_w[f] : \mathcal{H}_\theta \to \mathcal{H}_\theta \ , \ \phi \mapsto \psi \ ,$$

where $\psi \equiv [L\psi = (\widetilde{\psi}^j)_{j\in\mathbb{N}}] = \lim_{j\to\infty}\widetilde{\psi}^j$ is given by

$$\psi \ = \ \lim_{j\to\infty} j!\sum_{m=-j}^{j}\frac{\beta_m e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \ \in \ \mathcal{H}_\theta \ , \tag{5.103}$$

with, cf. (5.99),

$$\beta_m = \lim_{j\to\infty}\sqrt{n+1}\sum_{l=0}^{n}\sum_{m'=-j}^{j}\frac{\langle Y_l^{\bar m}|f\rangle}{c_l^{n}}C^{j,\ \ j,\ \ l}_{m,-m',\bar m}(-1)^{j-m'}\alpha_{m'} \ , \tag{5.104}$$

[5] Slightly abusing nomenclature, we also refer to the ordered set $(\alpha_m)_{m\in\mathbb{Z}}$ as a sequence.

satisfying

$$\sum_{m=-\infty}^{\infty} |\beta_m|^2 \ < \infty \ , \quad \textit{so that} \quad (\beta_m)_{m\in\mathbb{Z}} \ \longleftrightarrow \ \psi \in \mathcal{H}_\theta \ . \tag{5.105}$$

And similarly for $\widetilde{\mathcal{T}}_w[f] : \mathcal{H}_\theta \to \mathcal{H}_\theta$, *by substituting* $c_l^n \leftrightarrow 1/c_l^n$ *in (5.104).*

Now, ϕ *and* ψ *satisfying (5.102)-(5.103) converge in nested-*$\mathfrak{Z}$*-norm. Therefore, using Tannery, if* $(\alpha_m)_{m\in\mathbb{Z}}$ *satisfies (5.100) and* $\mathbf{T}_w[f] \in \mathfrak{M}_\infty$ *then*

$$\lim_{j\to\infty} \ (j!)^2 \sum_{m=-j}^{j} \frac{|\beta_m|^2}{(j-m)!(j+m)!} \ = \ \lim_{j\to\infty} \sum_{m=-j}^{j} |\beta_m|^2 \ , \tag{5.106}$$

with a similar equation holding for $(\alpha_m)_{m\in\mathbb{Z}}$.

On the other hand, functional convergence of ϕ *and* ψ *given by (5.102)-(5.103) is assured if the sequences* $(\alpha_m)_{m\in\mathbb{Z}}$ *and* $(\beta_m)_{m\in\mathbb{Z}}$ *are both* ℓ^1*-sequences, that is, with absolutely convergent series, and then it follows that,*

$$\infty \ > \ \lim_{j\to\infty} \ j! \sum_{m=-j}^{j} \frac{|\alpha_m|}{\sqrt{(j-m)!(j+m)!}} \ \Longleftrightarrow \ \sum_{m=-\infty}^{\infty} |\alpha_m| \ < \ \infty \tag{5.107}$$

and likewise for $(\beta_m)_{m\in\mathbb{Z}}$, *in which case we have that,* $\forall\theta \in \mathbb{R} \bmod 2\pi$,

$$\lim_{j\to\infty} \ j! \sum_{m=-j}^{j} \frac{\gamma_m e^{im\theta}}{\sqrt{(j-m)!(j+m)!}} \ = \ \lim_{j\to\infty} \sum_{m=-j}^{j} \gamma_m e^{im\theta} \ , \tag{5.108}$$

where $(\gamma_m)_{m\in\mathbb{Z}}$ *stands for both* $(\alpha_m)_{m\in\mathbb{Z}}$ *and* $(\beta_m)_{m\in\mathbb{Z}}$.

In the third construction, we extend the second construction, from the sequence of $SO(3)$ *representation to the full sequence of* $SU(2)$ *representations, as follows.*

For j *half-integer, we adjust (5.90)-(5.91) in order to accommodate the constant function* 1 *so that it is associated to* $m = 1/2, -1/2, 1/2, -1/2, \cdots$ *Hence,*

$$id_3^j : \mathcal{H}_z^j \equiv Poly(\mathbb{C})_{\le n}^h \ni \nu_j z^{j-m} \ \longleftrightarrow \ \rho_j e^{i(m-\epsilon_j)\theta} \in \mathcal{H}_\theta^j \ , \tag{5.109}$$

where

$$\epsilon_j = \frac{\sin(j\pi)}{2} \ , \ 2j \in \mathbb{N} \ . \tag{5.110}$$

Then,

$$\mathcal{E}^j = \left\{ \boldsymbol{u}(j,m) = \frac{\rho_j e^{i(m-\epsilon_j)\theta}}{\sqrt{(j-m)!(j+m)!}} \right\}_{-j\le m\le j} \tag{5.111}$$

is our standard basis for $\mathcal{H}_\theta^j$, $2j \in \mathbb{N}$, *where*

$$\rho_j = \begin{cases} j! & , \ j \in \mathbb{N} \\ (j-1/2)!\sqrt{j+1/2} & , \ j = k - 1/2 \ , \ k \in \mathbb{N} \end{cases} \tag{5.112}$$

is the canonical choice so that $1 \longleftrightarrow (m_j)_{2j\in\mathbb{N}} = (1/2, 0, -1/2, 0, 1/2, 0, -1/2, 0, \cdots)$. *Accordingly, the nesting maps are now determined by*

$$\iota_j^{j'}(\boldsymbol{u}(j,m)) = \boldsymbol{u}(j', m + \epsilon_{j'} - \epsilon_j) \ . \tag{5.113}$$

In this way, our well-nested sequence of Hilbert spaces $(\mathfrak{H}^<, \mathfrak{E})$, *where* $\mathfrak{H} = (\mathcal{H}_\theta^j)_{2j\in\mathbb{N}}$, $\mathfrak{E} = (\mathcal{E}^j)_{2j\in\mathbb{N}}$, *is now defined by (5.110)-(5.113).*

Now, the sequence of spin operators (J_3^j) *is not rigid anymore, since it has an explicitly dependence on* j. *This can also be seen from (2.2), which implies*

$$\begin{aligned} \iota_j^{j'} \circ J_3^j(\mathbf{u}(j,m)) &= m\, \mathbf{u}(j', m + \epsilon_{j'} - \epsilon_j) \\ &\neq (m + \epsilon_{j'} - \epsilon_j)\, \mathbf{u}(j', m + \epsilon_{j'} - \epsilon_j) = J_3^{j'} \circ \iota_j^{j'}(\mathbf{u}(j,m)) \end{aligned} \tag{5.114}$$

And now, a sequence $\Phi = (\phi^j)_{2j\in\mathbb{N}} \in \mathfrak{H}^<$ *can be identified with a sequence of complex Fourier polynomials*

$$\phi^j \ = \ \rho_j \sum_{m=-j}^{j} \frac{\alpha_{m-\epsilon_j}^j e^{i(m-\epsilon_j)\theta}}{\sqrt{(j-m)!(j+m)!}} \in \mathcal{H}_\theta^j \ , \tag{5.115}$$

where $\alpha_{m-\epsilon_j}^j$ *are the modified Fourier coefficients, and then we identify the* W*-quantization* $\mathbf{F}^w$ *of* $f \in C_{\mathbb{C}}^\infty(S^2)$ *via* $\boldsymbol{W}_{\mathcal{C}}$ *of Poisson type with a sequence of operators* $\mathbf{T}_w[f] = (T_w^j[f])_{2j\in\mathbb{N}}$ *acting on* $\mathfrak{H}^< = (\mathcal{H}_\theta^j, \langle\cdot|\cdot\rangle_j)_{2j\in\mathbb{N}}$, *so that*

$$\psi^j \ = \ \rho_j \sum_{m=-j}^{j} \frac{\beta_{m-\epsilon_j}^j e^{i(m-\epsilon_j)\theta}}{\sqrt{(j-m)!(j+m)!}} \in \mathcal{H}_\theta^j \ , \tag{5.116}$$

for $\psi^j = T_w^j[f](\phi^j) = \psi^j$ *and* $\phi^j \in \mathcal{H}_\theta^j$ *as in (5.115), where*

$$\beta_{m-\epsilon_j}^j \ = \ \sqrt{n+1} \sum_{l=0}^{n} \sum_{m'=-j}^{j} \frac{\langle Y_l^{\bar{m}} | f \rangle}{c_l^n} C_{m,-m',\bar{m}}^{\,j,\ \ j,\ \ l} (-1)^{j-m'} \alpha_{m'-\epsilon_j}^j \ . \tag{5.117}$$

From Theorem 5.2.1, if $\Phi \in \mathfrak{H}_\infty^<$, *then we have the identification (5.100), that is,*

$$\phi = \lim_{j\to\infty} \phi^j \in \mathcal{H}_\theta \iff (\alpha_m)_{m\in\mathbb{Z}} \ , \ \sum_{m=-\infty}^{\infty} |\alpha_m|^2 < \infty \ ,$$

where now

$$\alpha_m = \lim_{j\to\infty} \alpha_{m-\epsilon_j}^j \in \mathbb{C} \ , \quad \forall m \in \mathbb{Z} \ . \tag{5.118}$$

So $\phi \equiv [L\phi = (\widetilde{\phi}^j)] = \lim_{j\to\infty} \widetilde{\phi}^j$ *is given by*

$$\phi \ = \ \lim_{j\to\infty} \ \rho_j \sum_{m=-j-\epsilon_j}^{j-\epsilon_j} \frac{\alpha_m e^{im\theta}}{\sqrt{(j-m-\epsilon_j)!(j+m+\epsilon_j)!}} \ . \tag{5.119}$$

If $\mathbf{T}_w[f] \in \mathfrak{M}_\infty$*, the* Γ*-induced operator* $\mathcal{T}_w[f] : \mathcal{H}_\theta \to \mathcal{H}_\theta$*, maps* $\phi \mapsto \psi$*, for* $\psi \equiv [\Psi = (\psi^j)]$*. Using again* $\psi \equiv [L\psi = (\widetilde{\psi}^j)] = \lim_{j\to\infty} \widetilde{\psi}^j$*, we get*

$$\psi \;=\; \lim_{j\to\infty} \; \rho_j \sum_{m=-j-\epsilon_j}^{j-\epsilon_j} \frac{\beta_m e^{im\theta}}{\sqrt{(j-m-\epsilon_j)!(j+m+\epsilon_j)!}} \;, \tag{5.120}$$

where $(\beta_m)_{m\in\mathbb{Z}}$ *satisfying (5.105) are given by*

$$\beta_m = \lim_{j\to\infty} \sqrt{n+1} \sum_{l=0}^{n} \sum_{m'=-j-\epsilon_j}^{j-\epsilon_j} \frac{\langle Y_l^{\bar m} | f\rangle}{c_l^n} C^{\,j,\;\; j,\;\; l}_{m-\epsilon_j, -m'+\epsilon_j, \bar m} (-1)^{j-m'} \alpha_{m'} \;. \tag{5.121}$$

Again, it is important to emphasize that (5.119) and (5.120) always converge in the nested-$\mathfrak{H}$*-norm. Thus, again from (5.100), (5.105) and Tannery's theorem,*

$$\lim_{j\to\infty} \; \rho_j^2 \sum_{m=-j-\epsilon_j}^{j+\epsilon_j} \frac{|\beta_m|^2}{(j-m-\epsilon_j)!(j+m+\epsilon_j)!} \;=\; \lim_{j\to\infty} \sum_{m=-j}^{j} |\beta_m|^2 \;, \tag{5.122}$$

with a similar equation holding for $(\alpha_m)_{m\in\mathbb{Z}}$*.*

However, now the study of functional convergence of (5.119) and (5.120), i.e. the study of whether ϕ *and* ψ *are actually well defined functions on the whole or a.e. on* $\mathcal{X} \subset S^2$*, can be simplified by taking* $\mathfrak{H}^<$ *as three disjoint subsequences:*

$$\mathfrak{H}_1^< = (\mathcal{H}_\theta^j)_{2j\equiv 1 \pmod 4} \;,\; \mathfrak{H}_2^< = (\mathcal{H}_\theta^j)_{j\in\mathbb{N}} \;,\; \mathfrak{H}_3^< = (\mathcal{H}_\theta^j)_{2j\equiv 3 \pmod 4} \;.$$

But note that we could have defined the extension $SO(3) \hookrightarrow SU(2)$ *of this quantization by choosing* $\epsilon'_j = -\epsilon_j$ *so that* $1 \longleftrightarrow (m_j)_{2j\in\mathbb{N}} = (-1/2, 0, 1/2, 0, \cdots)$*, in which case the roles of* $\mathfrak{H}_1^<$ *and* $\mathfrak{H}_3^<$ *would be interchanged. Hence, we could also define the* $SU(2)$ *quantization by taking the average of these two choices.*

Anyway, it is not hard to see that all choices are asymptotically equivalent and all subsequences converge equally, as functions on $\mathcal{X} \subset S^2$*, when both* $(\alpha_m)_{m\in\mathbb{Z}}$ *and* $(\beta_m)_{m\in\mathbb{Z}}$ *are complex* ℓ^1*-sequences, in which case we have that,* $\forall \theta \in \mathbb{R} \bmod 2\pi$*,*

$$\lim_{j\to\infty} \; \rho_j \sum_{m=-j-\epsilon_j}^{j+\epsilon_j} \frac{\gamma_m e^{im\theta}}{\sqrt{(j-m-\epsilon_j)!(j+m+\epsilon_j)!}} \;=\; \lim_{j\to\infty} \sum_{m=-j}^{j} \gamma_m e^{im\theta} \;, \tag{5.123}$$

with $(\gamma_m)_{m\in\mathbb{Z}}$ *standing for both* $(\alpha_m)_{m\in\mathbb{Z}}$ *and* $(\beta_m)_{m\in\mathbb{Z}}$*.*

We emphasize that the theory developed in Section 5.2 asserts that $(\beta_m)_{m\in\mathbb{Z}} \in \ell^2$ *when* $(\alpha_m)_{m\in\mathbb{Z}} \in \ell^2$*, if* $\mathbf{F} \in \mathfrak{M}_\infty$ *. But when the series of* $(\alpha_m)_{m\in\mathbb{Z}}$ *is also absolutely convergent, that is, if* $(\alpha_m)_{m\in\mathbb{Z}} \in \ell^1$*, then further hypotheses on* $\mathbf{F}$ *may generally be needed for* $(\beta_m)_{m\in\mathbb{Z}}$ *to also be a complex* ℓ^1*-sequence. We shall not investigate these further hypotheses here. We shall also defer further investigations on the functional convergence of (5.102)-(5.103) and (5.119)-(5.120) in the more subtle cases when* $(\alpha_m)_{m\in\mathbb{Z}}$ *and* $(\beta_m)_{m\in\mathbb{Z}}$ *are complex* ℓ^2*-sequences but not complex* ℓ^1*-sequences.*

5.4 Quantized functions and asymptotic localization

The general theory developed in Section 5.2 and exemplified in Section 5.3 asserts that the operator sequences $\mathbf{T} \notin \mathfrak{M}_\infty$ are not suited to the limit $j \to \infty$, cf. Theorem 5.2.5.

Here, it is important to clarify that the limit $j \to \infty$ is asymptotic and the existence of a ground Hilbert space and operators therein could be seen as providing consistency requirements for a sequential quantization of S^2, but recalling that we only have a quantum spin system when $2j \in \mathbb{N}$. Thus, for instance, in (5.106) of Example 3, the equality generally becomes an inequality if the limit $j \to \infty$ is not taken on both sides, that is, we have an inequality for any $j \in \mathbb{N}$. Similarly in (5.108), as well as in (5.122)-(5.123) for any $2j \in \mathbb{N}$. And this is very important to consider when working out expansions in j^{-1} to the asymptotic expressions.

Nonetheless, the asymptotic consistency is fundamental for quantization, so we now start investigating the conditions for a function $f \in C^\infty_{\mathbb{C}}(S^2)$ to define operator sequences $\mathbf{F}^w \simeq \mathbf{T}_w[f] \in \mathfrak{M}_\infty$ and/or $\widetilde{\mathbf{F}}^w \simeq \widetilde{\mathbf{T}}_w[f] \in \mathfrak{M}_\infty$, as the ones in Examples 2-3, in terms of a given correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$. For this, we shall use:

Proposition 5.4.1. *Let $(\mathfrak{H}^<_\infty, \mathfrak{E})$ be well-nested with ground $\mathcal{H}$. If $\mathbf{F} \in \mathfrak{M}_\infty$, then $\mathbf{F}$ is upper bounded (cf. Definition 5.1.2).*

Proof. Let $\mathfrak{E} = (\mathbf{e}_k)_{k\in\mathbb{N}}$, $\mathbf{e}_k = \lim_{j\to\infty} \mathbf{e}^j_k$, $\forall k \in \mathbb{N}$. For $F_n \in \mathbf{F}$, we have

$$\|F_n\|^2 = \frac{1}{n+1}\sum_{k=1}^{n+1}\left\langle \mathbf{e}^j_k \middle| F_n^* F_n(\mathbf{e}^j_k)\right\rangle = \frac{1}{n+1}\sum_{k=1}^{n+1}\left\| F_n(\mathbf{e}^j_k)\right\|^2 \le \|F_n\|^2_{op} \tag{5.124}$$

By Proposition 5.2.3, $\mathbf{F} \in \mathfrak{M}_\infty \implies \mathbf{F}$ is $\mathfrak{H}$-bounded, thus $\mathbf{F}$ is upper bounded. □

Now, given a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$, from equations (5.19)-(5.21) in Definition 5.1.3, we see that the W- and $\widetilde{W}$-(pseudo)quantizations of f are defined via the spherical harmonic series of f. Thus, in a weaker sense, for any $f \in L^2_{\mathbb{C}}(S^2)$ we can define $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$. This includes all continuous functions and much more.

However, in a stronger sense, which is tied up with the questions of asymptotic localization, we need uniform convergence to f of the spherical harmonic series of f. And this is guaranteed if $f \in C^{k,\alpha}_{\mathbb{C}}(S^2)$, with $k + \alpha > 1/2$, cf. e.g. (ATKINSON; HAN, 2010). Thus, in this strong sense, Definition 5.1.3 can in principle be extended to differentiable functions or α-Hölder continuous functions, with $1/2 < \alpha \le 1$.

Anyway, such extensions shall not be studied here and all results will only be stated for classical functions, that is, $f \in C^\infty_{\mathbb{C}}(S^2)$. So, we now start investigating the conditions for such a W- or $\widetilde{W}$-quantized function to be upper bounded, in terms of the properties of the symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$.

But for our investigations related to asymptotic localization, it will often be simpler to restrict our attention to J_3-invariant functions and operators (most generalizations to non-J_3-invariant cases being rather straightforward). Thus, if $(a_l)_{l\in\mathbb{N}_0}$ is the sequence of Legendre coefficients of a J_3-invariant classical function $f \in C^\infty_{\mathbb{C}}([-1,1])$, with $a_l \in \mathbb{C}$ as in (5.17)-(5.18), then $\mathbf{F}^w = (F^w_n)_{n\in\mathbb{N}}$ and $\widetilde{\mathbf{F}}^w = (\widetilde{F}^w_n)_{n\in\mathbb{N}}$ are given by (cf. equations (5.13)-(5.21) and Definition 5.1.3):

$$F^w_n = [W^j]^{-1}(f) = \sum_{l=0}^{n} \frac{a_l}{c^n_l\sqrt{2l+1}}\widehat{\mathbf{e}}^j(l,0) \ , \tag{5.125}$$

$$\widetilde{F}^w_n = [\widetilde{W}^j]^{-1}(f) = \sum_{l=0}^{n} \frac{a_l c^n_l}{\sqrt{2l+1}}\widehat{\mathbf{e}}^j(l,0) \ . \tag{5.126}$$

Thus, from (5.16) we have that

$$||\mathbf{F}^w||^2_< = \liminf_{n\to\infty} \sum_{l=0}^{n} \frac{1}{2l+1}\left|\frac{a_l}{c^n_l}\right|^2 \quad , \quad ||\mathbf{F}^w||^2_> = \limsup_{n\to\infty} \sum_{l=0}^{n} \frac{1}{2l+1}\left|\frac{a_l}{c^n_l}\right|^2 \tag{5.127}$$

$$||\widetilde{\mathbf{F}}^w||^2_< = \liminf_{n\to\infty} \sum_{l=0}^{n} \frac{|a_l c^n_l|^2}{2l+1} \quad , \quad ||\widetilde{\mathbf{F}}^w||^2_> = \limsup_{n\to\infty} \sum_{l=0}^{n} \frac{|a_l c^n_l|^2}{2l+1} \tag{5.128}$$

with obvious generalizations of (5.127) and (5.128) when $\mathbf{F}^w$ and $\widetilde{\mathbf{F}}^w$ are given in the general form (5.20) and (5.21), using (5.16).

On the other hand, the L^2-norm $||\cdot|| : C^\infty_{\mathbb{C}}([-1,1]) \to \mathbb{R}^+$, $f \mapsto ||f||$, is given by

$$||f||^2 = \frac{1}{2}\int_{-1}^{1} |f(z)|^2 dz \ , \tag{5.129}$$

being the L^2-norm on $C^\infty_{\mathbb{C}}(S^2)$ restricted to J_3-invariant functions. Thus, because

$$\frac{1}{2}\int_{-1}^{1} P_l(z)P_{l'}(z)dz = \frac{\delta_{l,l'}}{2l+1} \ ,$$

in terms of the Legendre coefficients of f, cf. (5.17), we have that

$$||f||^2 = \lim_{n\to\infty} \sum_{l=0}^{n} \frac{|a_l|^2}{2l+1} \ . \tag{5.130}$$

For starters, we do not assume $\boldsymbol{W}_{\mathcal{C}}$ is of (anti-)Poisson type. We then have:

Proposition 5.4.2. *If $\boldsymbol{W}_{\mathcal{C}}$ is an isometric symbol correspondence sequence, then*

$$||\mathbf{F}^w||_\infty = ||f|| = ||\widetilde{\mathbf{F}}^w||_\infty \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ . \tag{5.131}$$

If $\boldsymbol{W}_{\mathcal{C}}$ is a positive-dual symbol correspondence sequence, then

$$||\mathbf{F}^w||_> \leq ||f|| \leq ||\widetilde{\mathbf{F}}^w||_< \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ . \tag{5.132}$$

Equivalently, $\boldsymbol{W}_{\mathcal{C}}$ is a mapping-positive symbol correspondence sequence, then

$$||\widetilde{\mathbf{F}}^w||_> \leq ||f|| \leq ||\mathbf{F}^w||_< \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ . \tag{5.133}$$

Proof. For isometric symbol correspondence sequences, (5.131) is obvious from (5.127), (5.128) and (5.130). For mapping-positive symbol correspondence sequences, recalling (3.1) and (4.20), plus the fact that $\sum_{k=1}^{n+1} a_k = 1$ in (3.1), we see that $|c_l^n| \leq 1, 0 \leq \forall l \leq n, \forall n \in \mathbb{N}$. So, $\forall f \in C^\infty_\mathbb{C}([-1,1])$, from (5.127)-(5.128) and (5.130) we get (5.133). Equivalence between (5.132) and (5.133) is obvious from the definitions. □

We note that if $\boldsymbol{W}_\mathcal{C}$ is just mapping-positive (or equivalently just positive-dual), the inequalities in (5.133) or (5.132) can be very strict and, in fact, some of the asymptotic operator norms can be 0 or ∞, as shown by the example of the upper-middle-state symbol correspondence sequence. Because, for this correspondence, we recall from (RIOS; STRAUME, 2014, eq.(6.57)) that its characteristic numbers p_l^n satisfy $\lim_{n\to\infty} p_l^n = 0$ for every $l = 2k+1$ odd. Hence, if f is an odd function, $||\widetilde{\mathbf{F}}^w||_< = ||\widetilde{\mathbf{F}}^w||_> = 0$, and if f is not an even function, $||\mathbf{F}^w||_< = ||\mathbf{F}^w||_> = \infty$.

Thus, in order to have more control, we recall:

Definition 5.4.1 ((RIOS; STRAUME, 2014))**.** *A symbol correspondence sequence* $\boldsymbol{W}_\mathcal{C}$ *is of* quasi-classical *type if*

$$\lim_{n\to\infty} |c_l^n| = 1 \ , \ \forall l \in \mathbb{N} \ . \tag{5.134}$$

Of course, every symbol correspondence sequence of (anti-)Poisson type is of quasi-classical type, but the converse does not hold in general, cf. Theorem 2.3.2.

Then we have:

Proposition 5.4.3. *If* $\boldsymbol{W}_\mathcal{C}$ *is positive-dual and also of quasi-classical type, then*

$$||\mathbf{F}^w||_\infty = ||f|| \ , \ \forall f \in C^\infty_\mathbb{C}([-1,1]) \ . \tag{5.135}$$

Equivalently, if $\boldsymbol{W}_\mathcal{C}$ *is mapping-positive and also of quasi-classical type, then*

$$||\widetilde{\mathbf{F}}^w||_\infty = ||f|| \ , \ \forall f \in C^\infty_\mathbb{C}([-1,1]) \ . \tag{5.136}$$

Proof. Because the two statements are equivalent, we prove the mapping-positive case. Then, from the first inequality in (5.133) and the fact that $||f|| < \infty$, *cf.* (5.129)-(5.130), we apply Tannery's theorem to (5.128) to get from the quasi-classical condition (5.134) that $||\widetilde{\mathbf{F}}^w||_< = ||\widetilde{\mathbf{F}}^w||_> = ||f||$. □

Remark 5.4.1. *Propositions 5.4.2 and 5.4.3 can be written in the general form, with* $C^\infty_\mathbb{C}(S^2)$ *replacing* $C^\infty_\mathbb{C}([-1,1])$ *in their statements.*

In principle, however, we cannot guarantee equality for the remaining inequality in (5.132), resp. (5.133), if we restrict to generic positive-dual, resp. mapping-positive symbol correspondence sequences of quasi-classical or even of (anti-)Poisson type. But in this respect, we have the following result for the more special cases:

Theorem 5.4.4. *For the standard and alternate Berezin correspondence sequences, or equivalently for the standard and alternate Toeplitz correspondence sequences,*

$$||\mathbf{F}^w||_\infty = ||f|| = ||\widetilde{\mathbf{F}}^w||_\infty \ , \ \forall f \in \mathcal{A}_\mu([-1,1]) \ , \ \forall \mu > 2 \ . \tag{5.137}$$

Proof. As in the proof of Proposition 5.4.3, here it is sufficient to prove for just one of the correspondence sequences of the statement. Take the standard Toeplitz correspondence sequence. Since $\mathcal{A}_\mu([-1,1]) \subset C^\infty_{\mathbb{C}}([-1,1])$ for all $\mu > 1$, from Proposition 5.4.3 we have that $||\mathbf{F}^w||_\infty = ||f||$, $\forall f \in \mathcal{A}_\mu([-1,1])$, $\forall \mu > 2$. On the other hand, following the proof of Theorem 4.2.7 we conclude that, if $f \in \mathcal{A}_\mu([-1,1])$ for any $\mu > 2$, then we can apply Tannery's theorem to (5.128) and then the quasi-classical condition (5.134) implies that $||\widetilde{\mathbf{F}}^w||_< = ||\widetilde{\mathbf{F}}^w||_> = ||f||$. □

In the same vein, if $\boldsymbol{W}_\mathcal{C}$ is a symbol correspondence sequence of (anti-)Poisson type, but without further conditions, it may happen that, for some $f \in C^\infty_{\mathbb{C}}([-1,1])$, the asymptotic operator norm of its W-quantization, or of its $\widetilde{W}$-quantization, blows up to infinity, as the following example shows.

Example 4. *Take the sequence of symbol correspondences dual to $\boldsymbol{W}_\mathcal{C}$ in the proof of Theorem 4.2.5 (cf. (4.25)), that is, for any $f \in C^\infty_{\mathbb{C}}([-1,1])$ with Legendre coefficients $a_l \neq 0$, $\forall l \in \mathbb{N}$, set*

$$c_l^n = \begin{cases} a_l/(2l+1), & \forall n = l \\ 1, & \text{otherwise} \end{cases} \ . \tag{5.138}$$

This sequence of symbol correspondences is also of Poisson type, but, for $n > m$, the norm of the W-quantization of f satisfies

$$||F_n^w||^2 - ||F_m^w||^2 = 2(n-m) + \sum_{l=m+1}^{n-1} \frac{|a_l|^2}{2l+1} > 2(n-m) > 2 \ . \tag{5.139}$$

Thus, $||\mathbf{F}^w||$ is not a Cauchy sequence. Being increasing, $||\mathbf{F}^w||_< = ||\mathbf{F}^w||_> = \infty$.

Of course, by taking the symbol correspondence sequence $\boldsymbol{W}_\mathcal{C}$ as in the proof of Theorem 4.2.5 (cf. (4.25)), we have that $\boldsymbol{W}_\mathcal{C}$ is of Poisson type but $||\widetilde{\mathbf{F}}^w||_\infty = \infty$.

Therefore, if $\boldsymbol{W}_\mathcal{C}$ is just of (anti-)Poisson type, equality (5.131) does not hold in general. But recalling Theorem 4.2.3 and Lemma 4.2.4, we have:

Theorem 5.4.5. *If a symbol correspondence sequence $\boldsymbol{W}_\mathcal{C}$ is of quasi-classical type and if there exist $d_1, d_2 \in \mathbb{N}_0$ and $K_{d_1}, K_{d_2} > 0$ such that*

$$\frac{1}{K_{d_1} \prod_{t=0}^{d_1}(2(l-t)+1)} \leq |c_l^n| \leq K_{d_2} \prod_{t=0}^{d_2}(2(l-t)+1) \ , \ n \geq \forall l > d+1 \ , \tag{5.140}$$

where $d = max\{d_1, d_2\}$, then (5.131) holds, that is,

$$||\mathbf{F}^w||_\infty = ||f|| = ||\widetilde{\mathbf{F}}^w||_\infty \ , \ \forall f \in C^\infty_{\mathbb{C}}([-1,1]) \ . \tag{5.141}$$

Proof. Following the proof of Theorem 4.2.3, from Lemma 4.2.4, the inequality (5.140) implies that, if $f \in C^\infty_{\mathbb{C}}([-1,1])$, then we can apply Tannery's theorem to (5.127) and (5.128), hence the quasi-classical condition (5.134) implies (5.131). $\square$

In particular, of course, the above theorem provides a sufficient condition on a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ of (anti-)Poisson type to guarantee that the L^2-norm of every classical function agrees with the asymptotic norms of its W-quantized and $\widetilde{W}$-quantized operator sequences.

Therefore, Theorems 5.4.4 and 5.4.5 provide connections between the conditions for equality of classical and asymptotic operator norms, and the conditions for the asymptotic localization of symbol correspondence sequences, as expressed in Theorems 4.2.3 and 4.2.7. But the relations between localization of symbol correspondence sequences and properties of quantized functions can be further explored.

First, we have the following proposition, complementing Propositions 5.4.2-5.4.3.

Proposition 5.4.6. *Let $\boldsymbol{W}_{\mathcal{C}} = (W^j)$ and $\widetilde{\boldsymbol{W}}_{\mathcal{C}} = (\widetilde{W}^j)$ be a symbol correspondence sequence and its dual. For every operator sequence $\mathbf{F} = (F_n)_{n\in\mathbb{N}}$, $F_n \in M_{\mathbb{C}}(n+1)$,*

$$||\mathbf{F}||_\infty \in \mathbb{R} \ \Rightarrow \lim_{n\to\infty} ||W^j_{F_n}|| \in \mathbb{R} \tag{5.142}$$

if and only if, for every sequence of polynomials $(f_n)_{n\in\mathbb{N}}$, $f_n \in Poly_{\mathbb{C}}(S^2)_{\leq n}$,

$$\lim_{n\to\infty} ||f_n|| \in \mathbb{R} \Rightarrow ||\widetilde{\mathbf{F}}^w||_\infty \in \mathbb{R} \ , \tag{5.143}$$

where $\widetilde{\mathbf{F}}^w = (\widetilde{F}^w_n)$ is the operator sequence given by $\widetilde{F}^w_n = [\widetilde{W}^j]^{-1}(f_n)$.

Proof. To simplify notation, we show for J_3-invariant operators and functions, the generalization being straightforward. Given a sequence of J_3-invariant operators $\mathbf{F} = (F_n)$, we can write

$$F_n = \sum_{l=0}^{n} \frac{\chi^n_l}{\sqrt{2l+1}} \widehat{\mathbf{e}}^j(l,0) \ ,$$

where $\chi^n_l \in \mathbb{C}$. Recalling that $c^n_0 = 1$, we get

$$W^j_{F_n} = \sum_{l=0}^{n} \chi^n_l c^n_l P_l \ .$$

Thus,

$$||F_n|| = \sum_{l=0}^{n} \frac{|\chi^n_l|^2}{2l+1} \quad \text{and} \quad ||W^j_{F_n}|| = \sum_{l=0}^{n} \frac{|\chi^n_l c^n_l|^2}{2l+1} \ . \tag{5.144}$$

Analogously, given a sequence of J_3-invariant polynomials (f_n), we can write

$$f_n = \sum_{l=0}^{n} \chi^n_l P_l \ ,$$

from which we get

$$\widetilde{F}_n^w = \sum_{l=0}^{n} \frac{\chi_l^n c_l^n}{\sqrt{2l+1}} \widehat{\mathbf{e}}^j(l,0) \ .$$

So we have

$$||f_n|| = \sum_{l=0}^{n} \frac{|\chi_l^n|^2}{2l+1} \quad \text{and} \quad ||\widetilde{F}_n^w|| = \sum_{l=0}^{n} \frac{|\chi_l^n c_l^n|^2}{2l+1} \ . \tag{5.145}$$

From (5.144) and (5.145), we have the enunciated equivalence. □

Therefore, in a rather strong sense we can say that, if a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ is "more appropriate" for dequantization, then its dual is "more appropriate" for (pseudo)quantization, and vice-versa. This falls in line with Definition 2.2.9 and Proposition 3.0.1, that is:

Proposition 5.4.7. *If $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ is a mapping-positive symbol correspondence sequence, then, for every strictly positive classical function $f \in C^\infty_{\mathbb{R}^+}(S^2)$, its $\widetilde{W}$-(pseudo)quantization $\widetilde{\mathbf{F}}^w = (\widetilde{F}_n^w)_{n\in\mathbb{N}}$, $\widetilde{F}_n^w = [\widetilde{W}^j]^{-1}(f)$, is a sequence of positive-definite operators $\forall n \geq n_0$, for some $n_0 \in \mathbb{N}$.*

Proof. Let f_n be the n^{th} partial sum of the decomposition of f in spherical harmonics, cf. (5.19). Because $(f_n) \to f$ in the sup norm and f is strictly positive, $\exists n_0 \in \mathbb{N}$ such that f_n is strictly positive, $\forall n \geq n_0$. But each f_n, $n \geq n_0$, is a polynomial function, and because each W^j is mapping-positive, $\widetilde{W}^j$ is positive dual, thus from Proposition 3.0.1, $[\widetilde{W}^j]^{-1}(f) = [\widetilde{W}^j]^{-1}(f_n)$ is positive-definite, cf. (5.21). □

We thus introduce the following definitions:

Definition 5.4.2. *We shall say that an operator sequence $\mathbf{F} = (F_n)_{n\in\mathbb{N}}$ possesses* asymptotic expectation *if, for every r-convergent Π-sequence $(\Pi_{k_n})_{n\in\mathbb{N}}$, the expectation sequence $(\langle \Pi_{k_n} | F_n \rangle)_{n\in\mathbb{N}}$ also converges.*

Remark 5.4.2. *Note that the above definition uses the usual Hilbert-Schmidt inner product $\langle\cdot|\cdot\rangle$ and* not *the normalized product $\langle\cdot|\cdot\rangle_j = \frac{1}{n+1}\langle\cdot|\cdot\rangle$ given by (5.5).*

Definition 5.4.3. *We say that a symbol correspondence sequence of (anti-)Poisson type $\boldsymbol{W}_{\mathcal{C}} = (W^j)_{n\in\mathbb{N}}$ possesses* classical (anti-)expectation *if the $\widetilde{W}$-quantization $\widetilde{\mathbf{F}}^w$ of any classical function $f \in C^\infty_{\mathbb{C}}(S^2)$ possesses asymptotic expectation.*

And then we have:

Theorem 5.4.8. *A symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ (anti-)localizes classically (cf. Definitions 4.0.2-4.0.3) if and only if it possesses classical (anti-)expectation.*

Proof. To simplify notation, assume $f = \bar{f}$, where $\bar{f}$ denotes the S^1-average of f, as in (4.5). The generalization to $f \neq \bar{f}$ is straightforward from (4.5)-(4.6).

Suppose that $\boldsymbol{W}_{\mathcal{C}}$ possesses classical expectation. Then, it is of Poisson type and, given any r-convergent Π sequence,

$$\exists \lim_{n\to\infty} \langle \Pi_{k_n} | \widetilde{F}^w_n \rangle = \lim_{n\to\infty} \int_{-1}^{1} W^j_{F^w_n}(z) \rho^j_{k_n}(z) dz \in \mathbb{C},\ \forall f \in C^\infty_{\mathbb{C}}([-1,1]), \tag{5.146}$$

cf. (2.23), where $\rho^j_{k_n} = \frac{n+1}{2} W^j_{\Pi_{k_n}}$, with $\mathbf{F}^w = (F^w_n)$ and $\widetilde{\mathbf{F}}^w = (\widetilde{F}^w_n)$ being the W-quantization and $\widetilde{W}$-quantization of f, respectively. We shall denote $W^j_{F_n} \equiv f_n$, which is the n^{th} partial sum of the Legendre series of f. From (5.146) we have

$$\lim_{n\to\infty} \int_{-1}^{1} f(z) \rho^j_{k_n}(z) dz = \lim_{n\to\infty} \int_{-1}^{1} f_n(z) \rho^j_{k_n}(z) dz \in \mathbb{C},\ \forall f \in C^\infty_{\mathbb{C}}([-1,1]). \tag{5.147}$$

Now, the space $C^\infty_{\mathbb{C}}(S^2)$ is a Fréchet space for the topology $\mathcal{E}$ generated by the collection of seminorms $\{||\cdot||_m\}_{m\in\mathbb{N}_0}$, where $||g||_m = \sup_{u\in S^2}\{| - \Delta^{m/2} g(u)|\}$, Δ the Laplace operator on S^2. Likewise for $C^\infty_{\mathbb{C}}([-1,1])$, by restricting to J_3-invariant functions, and we also have (cf. e.g. (MORIMOTO, 1991, Proposition 2.47 and Corollary 2.49)):

Lemma 5.4.9. *For any $f \in C^\infty_{\mathbb{C}}([-1,1])$, $(f_n) \to f$ in the topology $\mathcal{E}$.*

Furthermore, we have the following lemma (cf. e.g. (RUDIN, 1991, Theorem 2.8)):

Lemma 5.4.10. *The quasi-probability distribution ρ on $C^\infty_{\mathbb{C}}([-1,1])$, defined by*

$$\int_{-1}^{1} f(z)\rho(z) dz \coloneqq \lim_{n\to\infty} \int_{-1}^{1} f(z) \rho^j_{k_n} dz\ , \tag{5.148}$$

is continuous in the topology $\mathcal{E}$.

On the other hand, from the Poisson condition we have that

$$\int_{-1}^{1} f_n(z)\rho(z) dz = f_n(1-2r)\ ,\ \forall n \in \mathbb{N}\ , \tag{5.149}$$

cf. Definition 4.2.1, eqs. (4.15) and (5.148), and Corollary 4.2.2.1.

Thus, from (5.149) and Lemmas 5.4.9-5.4.10,

$$\int_{-1}^{1} f(z)\rho(z) dz = f(1-2r)\ ,\ \forall f \in C^\infty_{\mathbb{C}}([-1,1])\ .$$

If we suppose anti-Poisson condition, the r.h.s. of (5.149) must be $f_n(2r-1)$, so the result of the expression above changes to $f(2r-1)$, which means that the sequence of correspondences anti-localizes classically.

Conversely, assuming classical (anti-)localization of $\boldsymbol{W}_{\mathcal{C}}$, asymptotic expectation of the $\widetilde{W}$-quantization of a classical function follows trivially from definitions. $\square$

Remark 5.4.3. *For $\mathcal{E}$ in Lemmas 5.4.9-5.4.10, we could also have used the simpler collection of seminorms $\{||\cdot||_m\}_{m\in\mathbb{N}_0}$, where $||g||_m = \sup_{z\in[-1,1]}\{|g^{(m)}(z)|\}$.*

We now comment on possible extensions/restrictions of Theorem 5.4.8.

First, we should realize that a fundamental hypothesis of Theorem 5.4.8 is, besides (anti-)Poisson, the convergence of equation (4.18) $\forall f \in C^\infty_{\mathbb{C}}([-1,1])$. But this hypothesis can be restricted to the case $\forall f \in \mathcal{A}_\mu([-1,1])$ or extended to the cases $\forall f \in C^k_{\mathbb{C}}([-1,1])$, $k \geq 1$, or $\forall f \in C^{0,\alpha}_{\mathbb{C}}([-1,1])$, $1/2 < \alpha \leq 1$, because in all these cases we have that the Legendre series of f converges uniformly to f.

However, we are not sure if Lemma 5.4.10 holds true for $\mathcal{A}_\mu([-1,1])$ with the topology of holomorphic functions, and it is known that Lemma 5.4.9 does not hold true for $C^{0,\alpha}_{\mathbb{C}}([-1,1])$ with the topology given by its natural norm.

Finally, the difficulty in generalizing Theorem 5.4.8 to $C^k_{\mathbb{C}}(S^2)$ stems from the fact that Lemma 5.4.9 cannot be generalized to the natural topology $\mathcal{E}_k$ that makes $C^k_{\mathbb{C}}([-1,1])$ a Fréchet space because, for any $f \in C^k_{\mathbb{C}}([-1,1])$, one can guarantee $f_n \to f$ in all seminorms $||\cdot||_m$, $0 \leq m \leq k-2$, but $\mathcal{E}_k$ is the topology on $C^k_{\mathbb{C}}([-1,1])$ that is generated by the collection of seminorms $\{||\cdot||_m\}_{0\leq m\leq k}$.

CHAPTER

6

DISCUSSION OF PART I

If a symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ localizes (resp. anti-localizes) classically, then it is of Poisson (resp. anti-Poisson) type, cf. Corollary 4.2.2.2, but the converse does not hold in general, cf. Theorem 4.2.5. A sufficient condition for the converse of Corollary 4.2.2.2 to hold is that the characteristic numbers of $\boldsymbol{W}_{\mathcal{C}}$ satisfy certain bounds, cf. (4.16), but these polynomial bounds are not satisfied for general symbol correspondence sequences of Poisson (resp. anti-Poisson) type, as per the standard (resp. alternate) Toeplitz correspondence sequence, cf. Proposition 4.2.6.

Motivated by these examples, we thus defined a notion of μ-analytic localization, $\mu > 1$, which amounts to Π-distribution sequences converging to Dirac's distribution on an appropriate subspace $\mathcal{A}_\mu([-1,1]) \subset C^\infty_{\mathbb{C}}([-1,1])$, cf. Definition 4.2.3, and in this way we have that the standard (resp. alternate) Toeplitz symbol correspondence sequence localizes (resp. anti-localizes) μ-analytically, for every $\mu > 2$.

However, we should have in mind that the specific polynomial bounds on the characteristic numbers stated in Theorem 4.2.3, cf. (4.16), were assumed in view of the sharpest known bounds on the coefficients of Legendre expansions of functions in $C^\infty_{\mathbb{C}}([-1,1])$, cf. (WANG, 2018). If new sharper bounds for these coefficients can be found, then rougher bounds on the characteristic numbers could be assumed. In the same vein, if new bounds sharper than (4.33) can be found for the Legendre coefficients of functions in $\mathcal{A}_\mu([-1,1])$, we might still ask whether the standard (alternate) Toeplitz correspondence sequence (anti-)localizes μ-analytically for some $\mu < 2$.

In order to approach these questions from the inverse direction, going from the classical system to quantum systems, we had to develop the theory of sequential quantizations of S^2, particularly the notion of a (well-)nested sequence of Hilbert spaces $\mathfrak{H}^<$, the notion of a convergent state sequence $\Phi \in \mathfrak{H}^<_\infty$ and the notion of a ground Hilbert space $\mathcal{H}$, all of which were made explicit in Examples 2 and 3.

We saw that an operator sequence $\mathbf{F} : \mathfrak{H}^{<} \to \mathfrak{H}^{<}$ defines an asymptotic operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ only if $\mathbf{F}$ is upper bounded, cf. Proposition 5.4.1. Then, we found that just being of (anti-)Poisson type is not sufficient for a symbol correspondence sequence to define upper bounded quantized functions, *cf.* Example 4. In this setting, Theorems 5.4.4 and 5.4.5 imply that the L^2-norm of any classical function is equal to the asymptotic norms of its W-quantization and/or $\widetilde{W}$-quantization when $\boldsymbol{W}_{\mathcal{C}}$ satisfies similar sufficient conditions for its asymptotic (anti-)localization.

Furthermore, the quantization approach provided an equivalent characterization for classical (anti-)localization of a symbol correspondence sequence in terms of $\boldsymbol{W}_{\mathcal{C}}$ possessing the natural property of classical (anti-)expectation, *cf.* Definition 5.4.2 and Theorem 5.4.8. But a closer look at the hypothesis of Theorem 5.4.8 reveals that classical (anti-)localization of $\boldsymbol{W}_{\mathcal{C}}$ of (anti-)Poisson type is equivalent to having every r-convergent sequence $(\rho^{j}_{k_n})_{n\in\mathbb{N}}$ converge to an element in the dual of $C^{\infty}_{\mathbb{C}}([-1,1])$. Hence, for Theorem 5.4.8 we only assume (anti-)Poisson condition for the c^n_l's and the convergence of equation (4.18) and this hypothesis seems weaker than the one for Theorem 4.2.3 because, there, the hypothesis of the polynomial bounds for the c^n_l's that is expressed by equation (4.16) guarantees absolute convergence of (4.18). It turns out, however, that the polynomial bounds assumed in Theorem 4.2.3 are too vaguely stated to be able to distinguish between absolute and conditional convergence of (4.18), so that, in practice, it seems hard to use Theorem 5.4.8 in order to sharpen the conditions on the c^n_l's.

Now, in order to have a better feeling of the subtleties involved in relating the (anti-)Poisson condition to classical (anti-)localization, recall from Definition 2.3.3 that the (anti-)Poisson condition refers to the asymptotic limit of weakly oscillatory symbols ($j \to \infty$ keeping finite l's). Thus, the (anti-)Poisson condition in practice refers to the $j \to \infty$ asymptotics of smooth functions with an arbitrary l-cutoff. Such functions, when J_3-invariant, lie in $Poly_{\mathbb{C}}([-1,1])$ and Corollary 4.2.2.1 states that the (anti-)Poisson condition is equivalent to polynomial (anti-)localization, cf. Definition 4.2.1. However, classical (anti-)localization of a symbol correspondence sequence requires all its r-convergent Π-distribution sequences converging to Dirac's distributions on $C^{\infty}_{\mathbb{C}}([-1,1])$, cf. Definition 4.0.2, and generally this is related to the much subtler asymptotics of highly oscillatory symbols ($j, l \to \infty$), as well[1].

In summary, classical (anti-)localization of a general symbol correspondence sequence $\boldsymbol{W}_{\mathcal{C}}$ (equivalent to classical expectation for $\boldsymbol{W}_{\mathcal{C}}$) is a stronger property than $\boldsymbol{W}_{\mathcal{C}}$ being of (anti-)Poisson type. This is not the case for sequences of mapping-positive or isometric symbol correspondences, for which these two properties are equivalent, because (3.2) and (2.20) impose sufficient bounds on the "weights" of highly oscillatory ($l \to \infty$)

[1] We refer to (RIOS; STRAUME, 2014) for discussions of various subtleties associated to highly oscillatory symbols, in particular subtleties associated to the asymptotic limit of twisted products of such symbols.

components of their symbols, cf. (1.2) and (1.9). But as the Toeplitz examples suggest, although a positive-dual $\boldsymbol{W}_{\mathcal{C}}$ of (anti-)Poisson type may not have such bounding weights (4.16), this class of symbol correspondence sequence may still satisfy weaker forms of asymptotic localization (μ-analytical).

Finally, we could probably say that this work consists in a first precise study, albeit still very limited in scope, of some subtleties involved in the semiclassical asymptotics of highly oscillatory symbols, in the context of spin systems[2]

[2] In the context of affine mechanical systems, studies of subtleties in the semiclassical asymptotics of highly oscillatory symbols actually abound. Here we just mention the pioneering work of Berry on semiclassical Wigner functions of pure states (BERRY, 1977) and some studies of their singularities (CRAIZER; DOMITRZ; RIOS, 2020; DOMITRZ; MANOEL; RIOS, 2013; DOMITRZ; RIOS, 2014).

Part II

Quark systems

CHAPTER

7

INTRODUCTION TO PART II

Inspired by the treatment of symbol correspondences between quantum and classical mechanical systems symmetric under $SU(2)$, as presented in (RIOS; STRAUME, 2014), we try to expand our knowledge about the correspondence principle for systems symmetric under a compact Lie group by studying mechanical systems with $SU(3)$-symmetry[1]. Since $SU(3)$ is the symmetry group of the strong force, we call such systems *quark systems*.

The first remarkable difference between the $SU(2)$ and the $SU(3)$ cases is that in the former case there is just one classical system, namely the Poisson algebra of smooth functions on $\mathbb{C}P^1 \simeq S^2$, whereas in the latter case there are two types of symplectic phase space: the complex projective plane $\mathbb{C}P^2$ and a fiber bundle $\mathcal{E}$ over $\mathbb{C}P^2$ with fibers $\mathbb{C}P^1$, denoted $\mathbb{C}P^1 \hookrightarrow \mathcal{E} \to \mathbb{C}P^2$. The quantum systems of interest here are the Hilbert spaces $\mathcal{H}_{p,q}$ with an irreducible representation $Q(p,q)$ of $SU(3)$, for p and q non negative integers.

Then the main question posed for this second part of this Master's thesis can be addressed in the following way: when/how is it possible to injectively map the space of operators on $\mathcal{H}_{p,q}$ to the space of smooth functions on $\mathbb{C}P^2$, or $\mathcal{E}$, in an $SU(3)$-equivariant way which also ensures that quantum observables give rise to classical observables?

To answer the question above we define *symbol correspondences*[2] in the spirit of what is already done in literature, especially in (RIOS; STRAUME, 2014), as linear injective maps $W : \mathcal{B}(\mathcal{H}_{p,q}) \to C^\infty_{\mathbb{C}}(\mathcal{O})$, where $\mathcal{O}$ is either $\mathbb{C}P^2$ or $\mathcal{E}$, satisfying a few extra properties: (i) any such map W is $SU(3)$-equivariant, (ii) the image of any Hermitian operator is a real function and (iii) the normalization condition

$$\int_{\mathcal{O}} W_A(x)dx = \frac{1}{\dim Q(p,q)} \operatorname{tr}(A) \tag{7.1}$$

[1] Initial efforts in this direction can be found in (KLIMOV; GUISE, 2010; KLIMOV; ROMERO; GUISE, 2017; MARTINS; KLIMOV; GUISE, 2019).

[2] It may be fair to call them *symplectic symbol correspondences* because we are working only with symplectic manifolds. For $SU(3)$, one could also try to work with a Poisson manifold irregularly foliated by symplectic leaves, trying to define *Poissonian symbol correspondences*.

applies to every operator $A \in \mathcal{B}(\mathcal{H}_{p,q})$, with respect to a normalized left invariant integral on $\mathcal{O}$. Condition (ii) encodes that W maps observables to observables and condition (iii) means it preserves expected values.

It turns out that for $\mathcal{O} = \mathbb{C}P^2$ we can only define symbol correspondences for irreducible representations of type $Q(p,0)$ or $Q(0,q)$. We refer to the classical and quantum systems associated to $\mathbb{C}P^2$ and Hilbert spaces $\mathcal{H}_{p,0}$ or $\mathcal{H}_{0,q}$ as *pure-quark systems*, since the pertinent irreducible representations of $SU(3)$ emerge from systems of p quarks only, or q antiquarks only.[3] Characterization of symbol correspondences for pure-quark systems is very similar to what is known for spin systems. In particular, the correspondences for $\mathcal{H}_{p,0}$, or $\mathcal{H}_{0,p}$, are unequivocally determined by an ordered set of nonzero real numbers

$$c_n \in \mathbb{R}^* \ , \quad 1 \leq n \leq p \ , \tag{7.2}$$

called *characteristic numbers*, so that their moduli space is $(\mathbb{R}^*)^p$, cf. Proposition 9.3.1 and Corollary 9.3.1.1.

However, when $\mathcal{O}$ is the fiber bundle $\mathcal{E}$, we can define symbol correspondences for any $\mathcal{H}_{p,q}$, i.e. any irreducible representation ρ of class $Q(p,q)$ of $SU(3)$, so we refer to the classical and quantum systems associated to $\mathcal{E}$ and $\mathcal{H}_{p,q}$ as *generic quark systems*[4]. Here we see some new phenomena occurring as, for instance, the characterization of symbol correspondences via full rank complex matrices, called *characteristic matrices*, cf. Proposition 10.3.1, so that the moduli space of symbol correspondences for generic quark systems is a product of non compact Stiefel manifolds, cf. Corollary 10.3.1.1.

Just as it happens for spin systems, for both pure-quark and generic quark systems, Propositions 9.3.2 and 10.3.2 show that a symbol correspondence W can be realized as expectation over a Hermitian operator with unitary trace K, called *operator kernel*, via

$$W_A(gx_0) = \mathrm{tr}(AK^g) \tag{7.3}$$

for any $g \in SU(3)$, where $x_0 \in \mathcal{O}$ is a suitable point and K^g denotes the action of g on K,

$$SU(3) \ni g : K \mapsto \rho(g)K\rho(g)^{-1} = K^g \ ,$$

where ρ is the irreducible $SU(3)$-representation on the respective quantum system. So one can interpret general symbol correspondences as expectation values over pseudo-states. Furthermore, if the operator kernel is also a positive operator, i.e. if K is precisely a state, the correspondence maps positive(-definite) operators to (strictly-)positive functions and is called a *mapping-positive correspondence*.

[3] One can also interpret such pure-quark systems as solution spaces for the three dimensional isotropic oscillator, cf. Remark 9.2.2.

[4] We emphasize that quantum pure-quark systems are special cases of quantum generic quark systems.

Adaptations of a proof from (RIOS; STRAUME, 2014) show that the projection on the lowest weight vector of $\mathcal{H}_{p,0}$ is an operator kernel as well as the projection on highest weight vector of $\mathcal{H}_{0,p}$, so the symbol correspondences they generate are called *Berezin correspondences*, cf. Propositions 9.3.9 and 9.3.10 and Definitions 9.3.7 and 10.3.7.

Berezin correspondences also present examples of a natural relation between correspondences for dual quark systems (a concealed feature for spin systems since these are self dual systems). Accordingly, in (RIOS; STRAUME, 2014) the authors defined *alternate correspondences*, whilst here we shall use the term *antipodal correspondences*, cf. Definitions 9.3.8 and 10.3.8, Propositions 9.3.13 and 10.3.7, Remarks 9.3.2 and 9.3.3.

In this work, we also start to study noncommutative products on some finite dimensional subspaces of $C^\infty_\mathbb{C}(\mathcal{O})$, which are induced by the product of operators on $\mathcal{H}_{p,q}$ via symbol correspondences, what are called *twisted products of symbols*. In Propositions 9.4.3, 9.4.4, 10.4.3 and 10.4.4 we give some expressions for explicit computations of twisted products. And in Propositions 9.4.7 and 10.4.7 we show that antipodal correspondences induce a "reverse symbolic dynamics" via twisted product, cf. Remark 9.4.2.

This Part II is organized as follows.

In Chapter 8 we present general facts concerning the Lie group $SU(3)$ that are necessary for the construction of symbol correspondences as, in particular, the characterization of its irreducible unitary representations, in Section 8.1, for which we make use of the Gelfand-Tsetlin technology, and the description of $\mathbb{C}P^2$ and $\mathcal{E}$ as coadjoint orbits of $SU(3)$, in Section 8.3.

In Chapters 9 and 10 we work out the description of symbol correspondences for pure-quark systems and generic quark systems, respectively. The first sections of both chapters present the constructions of harmonic functions on the classical phase spaces, from which we identify the pertinent irreducible representations of $SU(3)$. The last two sections of each chapter are devoted to symbol correspondences and their twisted products.

Chapter 11 presents a brief discussion of the results obtained in this second part of this Master's thesis, with indication of some topics for future investigations.

CHAPTER

8

ON THE LIE GROUP $SU(3)$

Let $SU(3)$ be the special unitary subgroup of $GL_3(\mathbb{C})$, satisfying: $\det g = 1$ and $gg^\dagger = g^\dagger g = e$ for all $g \in SU(3)$. As a manifold, $SU(3)$ can be seen as a fiber bundle over S^5 whose fibers are $S^3 \simeq SU(2)$ (see discussion in section 8.3), hence it is a compact Lie group of real dimension 8. The Lie algebra of $SU(3)$, denoted by $\mathfrak{su}(3)$, can be generated by $i\lambda_k$, for $k = 1, ..., 8$, where

$$
\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;,\; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \;,\; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \;,\; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}
\tag{8.1}
$$

are Hermitian matrices, known as *Gell-Mann matrices*, satisfying

$$\mathrm{tr}(\lambda_a \lambda_b) = 2\delta_{ab} \;, \tag{8.2}$$

and

$$[\lambda_a, \lambda_b] = 2i \sum_{c=1}^{8} f^{abc} \lambda_c \tag{8.3}$$

with f^{abc} totally antisymmetric and completely determined by the values in Table 1 – see e.g. (GREINER; MüLLER, 1994). It follows that $SU(3)$ is not only compact, but also a simple Lie group.

abc	123	147	156	246	257	345	367	458	678
f^{abc}	1	$1/2$	$-1/2$	$1/2$	$1/2$	$1/2$	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/2$

Table 1 – Structure constants for Gell-Mann matrices.

8.1 Irreducible unitary representations

In order to describe irreducible representations of $SU(3)$, it is useful to take the complexification of $\mathfrak{su}(3)$. Thus, we will work with complex linear combination of the so-called F-*spin* operators:

$$F_k = \frac{1}{2}\lambda_k \ . \tag{8.4}$$

As expected, due to their names, using the F-spin operators we get a Lie bracket structure for $SU(3)$ more similar to the canonical one for $SU(2)$. By defining

$$\begin{aligned} T_\pm = F_1 \pm iF_2 \ , \ T_3 = F_3 \ , \\ V_\pm = F_4 \pm iF_5 \ , U_\pm = F_6 \pm iF_7 \ , \\ Y = \frac{2}{\sqrt{3}}F_8 \ , \end{aligned} \tag{8.5}$$

one can easily verify that

$$T_\pm^\dagger = T_\mp \ , \ U_\pm^\dagger = U_\mp \ , \ V_\pm^\dagger = V_\mp \ , \ T_3^\dagger = T_3 \ , \ Y^\dagger = Y \tag{8.6}$$

and

$$[Y, T_3] = 0 \ . \tag{8.7}$$

For Y, we also have

$$[Y, T_\pm] = 0 \quad , \quad [Y, U_\pm] = \pm U_\pm \quad , \quad [Y, V_\pm] = \pm V_\pm \ . \tag{8.8}$$

For the other operators, the following holds:

$$[T_3, T_\pm] = \pm T_\pm \quad , \quad [T_3, U_\pm] = \mp\frac{1}{2}U_\pm \quad , \quad [T_3, V_\pm] = \pm\frac{1}{2}V_\pm \quad , \quad [T_+, T_-] = 2T_3 \ , \tag{8.9}$$

$$[U_+, U_-] = \frac{3}{2}Y - T_3 \quad , \quad [V_+, V_-] = \frac{3}{2}Y + T_3 \ . \tag{8.10}$$

$$[T_+, U_+] = V_+ \quad , \quad [T_+, U_-] = 0 \quad , \quad [T_+, V_+] = 0 \quad , \quad [T_+, V_-] = -U_- \tag{8.11}$$

$$[U_+, V_+] = 0 \quad , \quad [U_+, V_-] = T_- \ . \tag{8.12}$$

From (8.7) and (8.10), we are compelled to define

$$U_3 = \frac{3}{4}Y - \frac{1}{2}T_3 \quad , \quad V_3 = \frac{3}{4}Y + \frac{1}{2}T_3 \ , \tag{8.13}$$

so we get[1]

$$\begin{aligned} &[U_3, U_\pm] = \pm U_\pm \quad , \quad [U_3, V_\pm] = \pm\frac{1}{2} V_\pm \\ &[V_3, V_\pm] = \pm V_\pm \quad , \quad [V_3, U_\pm] = \pm\frac{1}{2} U_\pm \ , \end{aligned} \tag{8.14}$$

$$[T_3, U_3] = [U_3, V_3] = [V_3, T_3] = 0 \ . \tag{8.15}$$

The remaining commutation relations can be computed using the Hermitian conjugate of the above expressions. We thus conclude that $SU(3)$ is of rank 2 and a straightforward calculation gives us

$$\mathrm{tr}(T_3 T_3) = \frac{1}{2} \ , \quad \mathrm{tr}(YY) \ = \frac{2}{3} \ , \quad \mathrm{tr}(T_3 Y) = 0 \ , \tag{8.16}$$

so $\{iT_3, iY\}$ is an orthogonal, but not normal basis of the Cartan subalgebra of $\mathfrak{su}(3)$.

Equations (8.14)-(8.15) together with equations (8.8)-(8.9), show that the root system of $SU(3)$ is composed by three root systems of $SU(2)$, with the same length, framing a regular hexagon as in Figure 1.

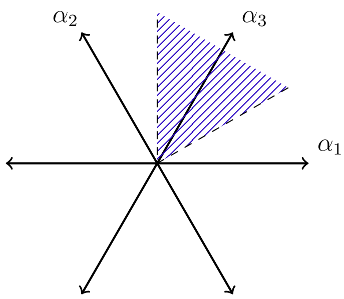


Figure 1 – Root diagram of $\mathfrak{su}(3)$.

The roots α_1, α_2 and α_3 are associated to the ladder operators T_+, U_+ and V_+, respectively. We choose the fundamental Weyl chamber as the blue hatched one, enclosed by the dashed lines, so that $\{\alpha_1, \alpha_2, \alpha_3\}$ is the set of positive roots and $\{\alpha_1, \alpha_2\}$ is the set of simple roots. Let ω_1 and ω_2 be the fundamental weights satisfying

$$2\frac{\langle \omega_j | \alpha_k \rangle}{\|\alpha_k\|^2} = \delta_{j,k} \ , \quad j, k \in \{1, 2\} \ , \tag{8.17}$$

where $\langle \, | \, \rangle$ is the canonical Euclidean inner product on the root space. Writing the fundamental weights as linear combination of the simple roots $\{\alpha_1, \alpha_2\}$, and using that

$$\langle \alpha_1 | \alpha_2 \rangle = -\|\alpha_1\| \|\alpha_2\|/2 \quad , \quad \|\alpha_1\| = \|\alpha_2\| \ , \tag{8.18}$$

the orthogonality relations $\langle \omega_1 | \alpha_2 \rangle = \langle \omega_2 | \alpha_1 \rangle = 0$ imply

$$\omega_1 = a\,(2\alpha_1 + \alpha_2) \qquad , \qquad \omega_2 = b\,(\alpha_1 + 2\alpha_2) \ .$$

[1] Among the infinitely many $SU(2)$ subgroups of $SU(3)$, the ones associated to $\{T_3, T_\pm\}$, $\{U_3, U_\pm\}$ and $\{V_3, V_\pm\}$ are singled out as the three *standard* $SU(2)$ subgroups of $SU(3)$.

The remaining relations given by (8.17) imply $a = b = 1/3$, so we finally get

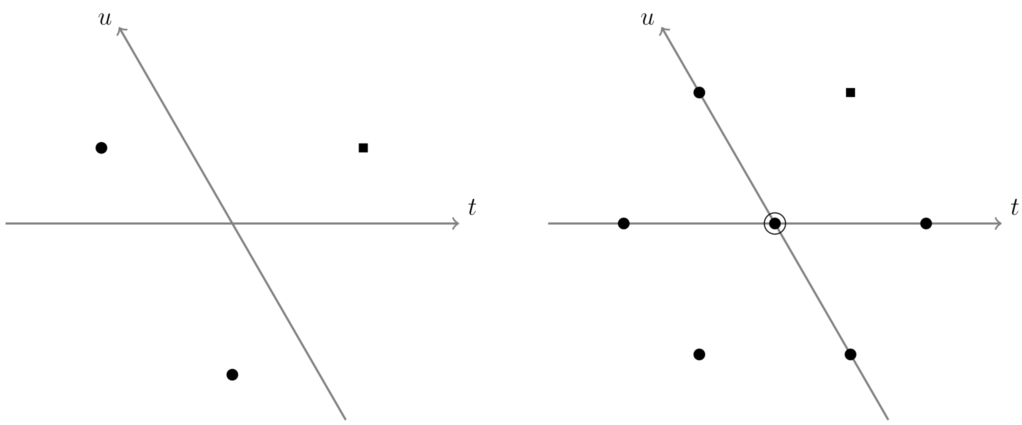
$$w_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) \qquad , \qquad w_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) \ . \tag{8.19}$$

Thus, we label the classes of irreducible unitary representations of $SU(3)$ by two nonnegative integers p and q, denoting each class by $Q(p,q)$, where $(p,q) \equiv p\omega_1 + q\omega_2$ is the highest weight of the representation. In addition, we shall often refer to an unitary irreducible representation class $Q(p,q)$ in a less specific way simply as an unitary irreducible representation $Q(p,q)$, or just as a representation $Q(p,q)$. Accordingly, for a representation $Q(p,q)$, p and q are the maximal integers such that $(T_-)^p$ and $(U_-)^q$ can be applied to the highest weight vector before vanishing, and an orthonormal basis of weight vectors for the representation $Q(p,q)$, where[2]

$$\dim Q(p,q) = \frac{(p+1)(q+1)(p+q+2)}{2} \ , \tag{8.20}$$

is obtained via linear combinations of the action of $(T_-)^a(U_-)^b(T_-)^c$ on the highest weight vector.

Remark 8.1.1. *The weights of $Q(p,q)$ can be placed on a plane diagram that is very useful for seeing the action of the step operators. In the examples below, the highest weights are highlighted as square dots and the multiplicities of a weight are represented by rings around the weight. Also, the axes t and u express the eigenvalues for the operators T_3 and U_3, respectively.*



(a) Representation $Q(1,0)$. (b) Representation $Q(1,1)$.

Figure 2 – Examples of weight diagram for irreducible representations of $SU(3)$.

Now, let ρ be an unitary irreducible representation of class $Q(p,q)$ acting on a complex Hilbert space $(\mathcal{H}, \langle\,|\,\rangle)$ of dimension $\dim Q(p,q)$. Then, we use the fact that

[2] Cf. Weyl dimensionality formula (HUMPHREYS, 1973).

the weights of $Q(p,q)$ can be labeled by (ν_1,ν_2,ν_3) and I satisfying the Gelfand-Tsetlin pattern[3]

$$\begin{gathered} 0 \le r_- \le q \le r_+ \le p+q \ , \ r_- \le \nu_1 \le r_+ \ , \\ \nu_2 = (r_+ + r_-) - \nu_1 \ , \ \nu_3 = p + 2q - (r_+ + r_-) \ , \\ I = \frac{1}{2}(r_+ - r_-) \ , \end{gathered} \tag{8.21}$$

where r_+ and r_- are integers. The quantities

$$t = (\nu_1 - \nu_2)/2 \ , \quad u = (\nu_2 - \nu_3)/2 \ , \quad v = (\nu_1 - \nu_3)/2 \tag{8.22}$$

are the eigenvalues[4] for the operators T_3, U_3 and V_3, respectively, and the index I, which is the spin number of the subrepresentation of $SU(2)$ related to $\{T_3, T_\pm\}$, is usually referred to as the *isospin*, in the physics literature. In this way, $T_\pm$ changes $(\nu_1,\nu_2,\nu_3) \mapsto (\nu_1 \pm 1, \nu_2 \mp 1, \nu_3)$, $U_\pm$ changes $(\nu_1,\nu_2,\nu_3) \mapsto (\nu_1, \nu_2 \pm 1, \nu_3 \mp 1)$ and $V_\pm$ changes $(\nu_1,\nu_2,\nu_3) \mapsto (\nu_1 \pm 1, \nu_2, \nu_3 \mp 1)$.

In particular, the highest weight is given by

$$\nu_1 = p + q \tag{8.23}$$

and

$$r_+ = p + q \ , \quad r_- = q \ , \tag{8.24}$$

so that

$$\nu_2 = q \ , \quad \nu_3 = 0 \ , \quad I = p/2 \ . \tag{8.25}$$

Definition 8.1.1 (cf. e.g. (BAIRD; BIEDENHARN, 1963))**.** *A* Gelfand-Tsetlin basis $\{\boldsymbol{e}((p,q);\boldsymbol{\nu},I)\}$*, or simply a* GT basis*, of a* $SU(3)$*-representation of class* $Q(p,q)$ *is an orthonormal basis indexed by the isospin* I *(the spin number of* $\{T_3, T_\pm\}$*) and*

$$\boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3) \ , \tag{8.26}$$

as specified above, cf. (8.21)-(8.22), constructed by fixing a highest weight vector

$$\boldsymbol{e}((p,q);(p+q,q,0),p/2)$$

and choosing the other basis vectors such that

$$T_- \boldsymbol{e}((p,q);(\nu_1,\nu_2,\nu_3),I) = \sqrt{(I+t)(I-t+1)}\, \boldsymbol{e}((p,q);(\nu_1-1,\nu_2+1,\nu_3),I) \ , \tag{8.27}$$

[3] The same method is used in (BAIRD; BIEDENHARN, 1963). For a general description of Gelfand-Tsetlin pattern, see (LOUCK, 1970; ZHELOBENKO, 1973).

[4] Henceforth the weights from which we are taking the eigenvalues will be specified or will be clear from the presence or absence of subscript and superscript in t, u and v.

$$
\begin{aligned}
U_{-}\boldsymbol{e}((p,q);(\nu_1,\nu_2,\nu_3),I) = \\
&\sqrt{\frac{(I-t)(p+q-I+t-\nu_1+1)(I-t+\nu_1-q)(I-t+\nu_1+1)}{2I(2I+1)}} \\
&\times \boldsymbol{e}((p,q);(\nu_1,\nu_2-1,\nu_3+1),I-1/2) \\
&+\sqrt{\frac{(I+t+1)(p+q-\nu_1+I+t+2)(q-\nu_1+I+t+1)(\nu_1-I-t)}{(2I+2)(2I+1)}} \\
&\times \boldsymbol{e}((p,q);(\nu_1,\nu_2-1,\nu_3+1),I+1/2)\ .
\end{aligned}
\tag{8.28}
$$

Considering the anti-isomorphism $\mathcal{H}^* \leftrightarrow \mathcal{H}$ via inner product, for the dual representation $\check{\rho} \leftrightarrow \rho$, we get that $\check{T}_3 \leftrightarrow -T_3$, $\check{U}_3 \leftrightarrow -U_3$, $\check{V}_3 \leftrightarrow -V_3$, $\check{T}_\pm \leftrightarrow -T_\mp$, $\check{U}_\pm \leftrightarrow -U_\mp$ and $\check{V}_\pm \leftrightarrow -V_\mp$. Thus, the states of $\check{\rho}$ are related to the states of ρ by

$$\check{I} = I\ ;\quad \check{t} = -t\ ,\ \check{u} = -u\ , \tag{8.29}$$

which implies

$$\check{\nu}_1 = p+q-\nu_1 \quad , \quad \check{\nu}_2 = p+q-\nu_2 \quad , \quad \check{\nu}_3 = p+q-\nu_3\ . \tag{8.30}$$

Therefore, the highest weight of the representation $\check{\rho}$ on $\mathcal{H}^*$ is $-\mu$, where μ is the lowest weight of ρ. Since the lowest weight is the image of the highest weight by the longest element of the Weyl group, we find out that $(q,p) = q\omega_1 + p\omega_2$ is the highest weight of $\check{\rho}$, that is, $\check{\rho}$ is of class $Q(q,p)$, so that

$$Q(p,q)^* = Q(q,p)\ . \tag{8.31}$$

Notation 1. *In the light of this dualization symmetry, we introduce the simplifying notation*[5]*:*

$$\boldsymbol{p} = (p,q) \in \mathbb{N}_0 \times \mathbb{N}_0 \leftrightarrow \check{\boldsymbol{p}} = (q,p)\ ,\quad \textit{with}\quad |\boldsymbol{p}| = |\check{\boldsymbol{p}}| = p+q\ . \tag{8.32}$$

Then, to better state the relations in (8.30), we define

$$\Delta^{|\boldsymbol{p}|}_{\boldsymbol{\nu},\boldsymbol{\mu}} \equiv \Delta^{p+q}_{\boldsymbol{\nu},\boldsymbol{\mu}} := \begin{cases} 1 \ , & \textit{if} \quad \boldsymbol{\nu}+\boldsymbol{\mu} = (p+q,p+q,p+q) = (|\boldsymbol{p}|,|\boldsymbol{p}|,|\boldsymbol{p}|) \\ 0 \ , & \textit{otherwise} \end{cases}\ . \tag{8.33}$$

Definition 8.1.2. *Given a Gelfand-Tsetlin basis $\{\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\}$ of a $SU(3)$-representation of class $Q(\boldsymbol{p}) \equiv Q(p,q)$, the* induced Gelfand-Tsetlin basis of the dual representation *of class $Q(p,q)^* = Q(q,p) \equiv Q(\check{\boldsymbol{p}})$ is the basis comprised by the vectors*

$$\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\check{\boldsymbol{\nu}},I) = (-1)^{2(t_\nu+u_\nu)}\boldsymbol{e}^*(\boldsymbol{p};\boldsymbol{\nu},I)\ , \tag{8.34}$$

[5] We shall use the convention which identifies the set of natural numbers as $\mathbb{N} = \{1,2,3,\cdots\}$ and denote $\mathbb{N}_0 = \{0,1,2,3,\cdots\}$.

where $\boldsymbol{e}^*(\boldsymbol{p};\boldsymbol{\nu},I) \in Q(\breve{\boldsymbol{p}})$ is the Hermitian dual of $\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)$, that is, the dual of $\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)$ via Hermitian inner product on $Q(\boldsymbol{p})$, and where $t_{\boldsymbol{\nu}}$ and $u_{\boldsymbol{\nu}}$ stand as in (8.22), with $\boldsymbol{\nu}$ and $\breve{\boldsymbol{\nu}}$ satisfying the duality relations (8.30), that is, using (8.33),

$$\textit{duality:}\ \ \boldsymbol{\nu} \leftrightarrow \breve{\boldsymbol{\nu}} \iff \Delta^{|\boldsymbol{p}|}_{\boldsymbol{\nu},\breve{\boldsymbol{\nu}}} = 1\ . \tag{8.35}$$

Remark 8.1.2. *A GT basis $\{\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\}$ and the induced GT basis $\breve{\boldsymbol{e}}(\breve{\boldsymbol{p}};\breve{\boldsymbol{\nu}},I)$ are related by an involution. Considering the natural isomorphism between a finite dimensional vector space and its double dual, the dual GT basis induced by $\breve{\boldsymbol{e}}(\breve{\boldsymbol{p}};\breve{\boldsymbol{\nu}},I)$ is precisely $\{\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\}$, that is,*

$$\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I) = (-1)^{2(t+u)}\breve{\boldsymbol{e}}^*(\breve{\boldsymbol{p}};\breve{\boldsymbol{\nu}},I)\ . \tag{8.36}$$

This contrasts with standard convention for irreducible representations of $SU(2)$, since for a $SU(2)$-representation with spin number j, there is a phase $(-1)^{2j}$ between a standard basis and the basis of the double dual space induced by the basis of the dual space, c.f. (RIOS; STRAUME, 2014).

Definition 8.1.3. *The* Wigner D-functions *(in the GT basis) of an irreducible unitary $SU(3)$-representation ρ of class $Q(\boldsymbol{p}) \equiv Q(p,q)$ are the functions*

$$D^{\boldsymbol{p}}_{\boldsymbol{\nu}I,\boldsymbol{\mu}J}(g) = \langle\, \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\,|\,\rho(g)\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu},J)\,\rangle\ . \tag{8.37}$$

Using the conjugate symmetry of the inner product and the relation $\langle v|w\rangle = \langle w^*|v^*\rangle$ between inner products of $\mathcal{H}$ and $\mathcal{H}^*$, we get

$$\begin{aligned}\overline{D^{\boldsymbol{p}}_{\boldsymbol{\nu}I,\boldsymbol{\mu}J}}(g) &= \langle\, \rho(g)\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu},J)\,|\,\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\,\rangle\\ &= \langle \boldsymbol{e}^*(\boldsymbol{p};\boldsymbol{\nu},I)|\breve{\rho}(g)\boldsymbol{e}^*(\boldsymbol{p};\boldsymbol{\mu},J)\rangle\\ &= (-1)^{2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}}+t_{\boldsymbol{\mu}}+u_{\boldsymbol{\mu}})}\,\langle\, \breve{\boldsymbol{e}}(\breve{\boldsymbol{p}};\breve{\boldsymbol{\nu}},I)\,|\,\breve{\rho}(g)\breve{\boldsymbol{e}}(\breve{\boldsymbol{p}};\breve{\boldsymbol{\mu}},J)\,\rangle\\ &= (-1)^{2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}}+t_{\boldsymbol{\mu}}+u_{\boldsymbol{\mu}})}\,D^{\breve{\boldsymbol{p}}}_{\breve{\boldsymbol{\nu}}I,\breve{\boldsymbol{\mu}}J}(g)\end{aligned} \tag{8.38}$$

for $\Delta^{|\boldsymbol{p}|}_{\boldsymbol{\nu},\breve{\boldsymbol{\nu}}} = \Delta^{|\boldsymbol{p}|}_{\boldsymbol{\mu},\breve{\boldsymbol{\mu}}} = 1$.

8.2 Clebsch-Gordan series and space of operators

An irreducible unitary representation ρ of class $Q(\boldsymbol{p})$ on $\mathcal{H}$ extends to an unitary representation (with respect to the trace inner product) on $\mathcal{B}(\mathcal{H})$ via adjoint action: for $A \in \mathcal{B}(\mathcal{H})$ and $g \in SU(3)$, the action of g on A is

$$A^g = \rho(g)A\rho(g)^{-1}\ . \tag{8.39}$$

Now, we recall that $\mathcal{B}(\mathcal{H})$ is naturally isomorphic to $\mathcal{H}\otimes\mathcal{H}^*$ in a manner that (8.39) matches the representation $\rho\otimes\breve{\rho}$ of class $Q(\boldsymbol{p})\otimes Q(\breve{\boldsymbol{p}})$ on $\mathcal{H}\otimes\mathcal{H}^*$.

The decomposition of a tensor product of irreducible unitary $SU(3)$-representations into a direct sum of irreducible unitary $SU(3)$-representations is known as the *Clebsch-Gordan series* of $SU(3)$.

Theorem 8.2.1 (see e.g. (COLEMAN, 1964)). *The Clebsch-Gordan series of $SU(3)$ are given by*

$$Q(p_1, q_1) \otimes Q(p_2, q_2) = \bigoplus_{n=0}^{\min(p_1,q_2)} \bigoplus_{m=0}^{\min(p_2,q_1)} Q(p_1 - n, p_2 - m; q_1 - m, q_2 - n) \ , \tag{8.40}$$

where

$$\begin{aligned} Q(r_1, r_2; s_1, s_2) = & Q(r_1 + r_2, s_1 + s_2) \oplus \left(\bigoplus_{k=1}^{\min(r_1,r_2)} Q(r_1 + r_2 - 2k, s_1 + s_2 + k) \right) \\ & \oplus \left(\bigoplus_{k=1}^{\min(s_1,s_2)} Q(r_1 + r_2 + k, s_1 + s_2 - 2k) \right) \ . \end{aligned} \tag{8.41}$$

Corollary 8.2.1.1. *For $p_1 = q_2 = p$ and $q_1 = p_2 = q$, the Clebsch-Gordan series assumes the form*

$$\begin{aligned} Q(p, q) \otimes Q(q, p) = \bigoplus_{n=0}^{p} \bigoplus_{m=0}^{q} \Bigg\{ & Q(p + q - n - m, p + q - n - m) \\ & \oplus \left[\bigoplus_{k=1}^{\min(p-n,q-m)} \Big(Q(p + q - n - m - 2k, p + q - n - m + k) \right. \\ & \left. \oplus\, Q(p + q - n - m + k, p + q - n - m - 2k) \Big) \right] \Bigg\} \ . \end{aligned} \tag{8.42}$$

Note that an irreducible unitary representation of class $Q(\boldsymbol{a})$ may appear more than once in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$ and of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$, in general.

Notation 2. *We shall denote the multiplicity of $Q(\boldsymbol{a}) = Q(a, b)$ in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$ by*

$$\mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a}) \ . \tag{8.43}$$

To distinguish multiple appearances of the same class of representation in a Clebsch-Gordan series, we will write

$$(\boldsymbol{a}; \sigma) = (a, b; \sigma) \ , \quad Q(\boldsymbol{a}; \sigma) \equiv Q(a, b; \sigma) \ , \tag{8.44}$$

where the index σ counts the multiplicity starting from 1 to $\mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a})$.

We thus provide two basis for $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$.

Definition 8.2.1. *An* uncoupled GT basis *of the tensor product representation $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$ is a basis comprised by the tensor product of GT basis $\{\boldsymbol{e}(\boldsymbol{p}_1; \boldsymbol{\nu}_1, I_1)\}$ of $Q(\boldsymbol{p}_1)$ and $\{\boldsymbol{e}(\boldsymbol{p}_2; \boldsymbol{\nu}_2, I_2)\}$ of $Q(\boldsymbol{p}_2)$.*

Definition 8.2.2. *A* coupled GT basis *of the tensor product representation $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$ is the union of GT basis $\{\boldsymbol{e}(\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\}$ of each $Q(\boldsymbol{a};\sigma)\equiv Q(a,b;\sigma)$ in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$.*

Remark 8.2.1. *For the sake of good reading, we will not write the labels of the representations of the tensor product on the elements of a coupled basis, but it will be clear from the context. Also, unless specified otherwise, from now on we shall always refer to the uncoupled and coupled basis of the tensor product as meaning their respective GT basis, and likewise for the Clebsch-Gordan coefficients.*

Both these basis are orthonormal, so they are related by an unitary transformation.

Definition 8.2.3. *The* Clebsch-Gordan coefficients *(in the GT basis) are the entries of the transformation that relate a coupled and an uncoupled GT basis of $Q(p_1,q_1)\otimes Q(p_2,q_2)$:*

$$C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}=\langle\, \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\,|\,\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\,\rangle\ , \tag{8.45}$$

where $\langle\cdot|\cdot\rangle$ is the $SU(3)$-invariant inner product induced by the ones on each representation of the tensor product.

We are able to fix a relative phase between a coupled and an uncoupled basis so that all Clebsch-Gordan are real. Usually, one chooses some set of Clebsch-Gordan coefficients to be positive and the remaining coefficients are completely determined via the action of the step operators on the basis vectors – cf. e.g. (SWART, 1963). In the next section, we shall return to this problem. What is important now is that we take Clebsch-Gordan coefficients as real numbers, so that we have

$$\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)=\sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\ , \tag{8.46}$$

$$\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)=\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\ , \tag{8.47}$$

$$\begin{aligned}\sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1' I_1',\boldsymbol{\nu}_2' I_2',\ \boldsymbol{\nu} I}&=\delta_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_1'}\delta_{\boldsymbol{\nu}_2,\boldsymbol{\nu}_2'}\delta_{I_1,I_1'}\delta_{I_2,I_2'}\ ,\\ \sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma_1)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{b};\sigma_2)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu}' I'}&=\delta_{\boldsymbol{a},\boldsymbol{b}}\delta_{\sigma_1,\sigma_2}\delta_{\boldsymbol{\nu},\boldsymbol{\nu}'}\delta_{I,I'}\ .\end{aligned} \tag{8.48}$$

Remark 8.2.2. *We also point out that, from the way the GT basis for $SU(3)$ representations were constructed using $SU(2)$ subrepresentations, the $SU(3)$ Clebsh-Gordan coefficientes in the GT basis are related to the $SU(2)$ Clebsh-Gordan coefficientes by*

$$C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}=C^{I_1,\ I_2,\ I}_{t_{\nu_1},t_{\nu_2},t_{\nu}}C^{\boldsymbol{p}_1,\ \ \boldsymbol{p}_2,\ \ (\boldsymbol{a};\sigma)}_{u_{\nu_1}I_1,u_{\nu_2}I_2,\ u_{\nu}I}\ , \tag{8.49}$$

where the second coefficient on the r.h.s. is called isoscalar factor*, and this provides explicit equations for the $SU(3)$ Clebsh-Gordan coefficientes in the GT basis in terms of explicit equations for the $SU(2)$ Clebsh-Gordan coefficientes, as found in (RIOS; STRAUME, 2014) and (VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988), for instance.*

From decompositions (8.46)-(8.47), we obtain some sufficient conditions for the Clebsch-Gordan coefficients to be zero. Since $\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)$ and $\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)$ are basis vectors of $SU(2)$-representations with spin numbers I_1 and I_2, their tensor product is a vector of the tensor product of the $SU(2)$-representations they belong to, that is, the Clebsch-Gordan coefficients are zero if I_1, I_2 and I do not satisfy the triangle inequality. Also, using superscripts to identify the $SU(3)$-representations, the operators T_3 and U_3 in $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$ have the form

$$\begin{aligned} \bigoplus_{(\boldsymbol{a};\sigma)} T_3^{(\boldsymbol{a};\sigma)} &= T_3^{\boldsymbol{p}_1}\otimes \mathbb{1} + \mathbb{1}\otimes T_3^{\boldsymbol{p}_2} \ , \\ \bigoplus_{(\boldsymbol{a};\sigma)} U_3^{(\boldsymbol{a};\sigma)} &= U_3^{\boldsymbol{p}_1}\otimes \mathbb{1} + \mathbb{1}\otimes U_3^{\boldsymbol{p}_2} \ , \end{aligned} \tag{8.50}$$

where $\mathbb{1}$ is the identity operator. Thus the Clebsch-Gordan coefficients are zero if $t\neq t_1+t_2$ or $u\neq u_1+u_2$, for t, t_1, t_2, u, u_1 and u_2 being the eigenvalues of T_3 and U_3 related to the weights $\boldsymbol{\nu}$, $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. To summarize, let $\delta(x,y,z)$ be equal to 1 if x, y and z satisfy the triangle inequality, or 0 otherwise, and let

$$\nabla_{\boldsymbol{\nu},\boldsymbol{\mu}} := \delta_{t_{\boldsymbol{\nu}},t_{\boldsymbol{\mu}}}\delta_{u_{\boldsymbol{\nu}},u_{\boldsymbol{\mu}}} \ , \tag{8.51}$$

where $\delta_{m,n}$ is the Kronecker delta. Then,

$$C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu}I} \neq 0 \qquad \Longrightarrow \qquad \begin{cases} \nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = 1 \\ \delta(I_1,I_2,I) = 1 \end{cases} . \tag{8.52}$$

That established, we proceed to establish the decomposition of the pointwise product of Wigner D-functions and the decomposition of the operator product.

Lemma 8.2.2. *The pointwise product of Wigner D-functions of $SU(3)$ can be decomposed into a sum of the form*

$$D^{\boldsymbol{p}_1}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_1' I_1'} D^{\boldsymbol{p}_2}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}_2' I_2'} = \sum_{(\boldsymbol{a};\sigma)} \sum_{\substack{\boldsymbol{\nu},I \\ \boldsymbol{\nu}',I'}} C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu}I} C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1' I_1',\boldsymbol{\nu}_2' I_2',\ \boldsymbol{\nu}'I'} D^{\boldsymbol{a}}_{\boldsymbol{\nu}I,\boldsymbol{\nu}'I'} \ , \tag{8.53}$$

where the summations are restricted to $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = \nabla_{\boldsymbol{\nu}_1'+\boldsymbol{\nu}_2',\boldsymbol{\nu}'} = 1$, $\delta(I_1,I_2,I)=\delta(I_1',I_2',I')=1$ and $(\boldsymbol{a};\sigma)$ such that $Q(\boldsymbol{a};\sigma)$ is in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$.

Proof. The proof is analogous to the $SU(2)$ case. Let $g \in SU(3)$. From (8.46) and (8.52), we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} D^{\boldsymbol{p}_1}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}'_1,I'_1}(g) D^{\boldsymbol{p}_2}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}'_2,I'_2}(g)\, \boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes \boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2) = \\ \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu}',I'}} C^{\boldsymbol{p}_1,\ \ \boldsymbol{p}_2,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}'_1 I'_1,\boldsymbol{\nu}'_2 I'_2,\ \boldsymbol{\nu}' I'} \sum_{\boldsymbol{\nu},I} D^{\boldsymbol{a}}_{\boldsymbol{\nu} I,\boldsymbol{\nu}' I'}(g) \boldsymbol{e}((\boldsymbol{a};\sigma),\boldsymbol{\nu},I)\ ,\end{aligned} \tag{8.54}$$

where the sum over $(\boldsymbol{a};\sigma)$, $\boldsymbol{\nu}'$ and I' satisfies the statement. Now, we use (8.47) and (8.52) to obtain

$$\begin{aligned}&\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} D^{\boldsymbol{p}_1}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}'_1 I'_1}(g) D^{\boldsymbol{p}_2}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}'_2 I'_2}(g) \boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes \boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2) = \\ &\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} \sum_{(\boldsymbol{a};\sigma)} \sum_{\substack{\boldsymbol{\nu},I\\ \boldsymbol{\nu}',I'}} C^{\boldsymbol{p}_1,\ \ \boldsymbol{p}_2,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}'_1 I'_1,\boldsymbol{\nu}'_2 I'_2,\ \boldsymbol{\nu}' I'} C^{\boldsymbol{p}_1,\ \ \boldsymbol{p}_2,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I} D^{\boldsymbol{a}}_{\boldsymbol{\nu} I,\boldsymbol{\nu}' I'}(g) \boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes \boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\ ,\end{aligned} \tag{8.55}$$

where $\boldsymbol{\nu}$, $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$, I, I_1 and I_2 are related as in the statement. The decomposition in a basis is unique, so this finishes the proof. □

Again, let $\mathcal{H}$ be a Hilbert space with an irreducible $SU(3)$-representation of class $Q(\boldsymbol{p})$. Also, let $\{\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\}$ be a GT basis of such space and $\{\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\check{\boldsymbol{\nu}},\check{I})\}$ be the induced GT basis of its dual space. The trivial representation within $\mathcal{B}(\mathcal{H})$ is spanned by the normalized operator

$$\begin{aligned}\frac{1}{\sqrt{\dim Q(\boldsymbol{p})}}\mathbb{1} &= \frac{1}{\sqrt{\dim Q(\boldsymbol{p})}} \sum_{\boldsymbol{\nu},I} \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\otimes \boldsymbol{e}^*(\boldsymbol{p};\boldsymbol{\nu},I) \\ &= \frac{1}{\sqrt{\dim Q(\boldsymbol{p})}} \sum_{\boldsymbol{\nu},I} (-1)^{2(t+u)} e(\boldsymbol{p};\nu,I)\otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}};\check{\boldsymbol{\nu}},I)\ .\end{aligned} \tag{8.56}$$

For operators $A, R \in \mathcal{B}(\mathcal{H})$,

$$A = \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\nu}',I') \quad , \quad R = \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu}',J')\otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu},J)\ , \tag{8.57}$$

their product is given by

$$AR = \delta_{I',J'} \Delta^{|\boldsymbol{p}|}_{\boldsymbol{\nu}',\boldsymbol{\mu}'} (-1)^{2(t_{\boldsymbol{\nu}'}+u_{\boldsymbol{\nu}'})} \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu},J)\ , \tag{8.58}$$

where $t_{\boldsymbol{\nu}'} = (\nu'_1-\nu'_2)/2$ and $u_{\boldsymbol{\nu}'} = (\nu'_2-\nu'_3)/2$ for $\boldsymbol{\nu}' = (\nu'_1,\nu'_2,\nu'_3)$, cf. Definition 8.1.2.

Lemma 8.2.3. *Let $(\boldsymbol{a}_1;\sigma_1)$ and $(\boldsymbol{a}_2;\sigma_2)$ label $SU(3)$-representations in the Clebsch-Gordan series of $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$. The operator product of elements of a coupled basis of $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$ can be decomposed into*

$$\boldsymbol{e}((\boldsymbol{a}_1;\sigma_1);\boldsymbol{\nu}_1,I_1)\boldsymbol{e}((\boldsymbol{a}_2;\sigma_2);\boldsymbol{\nu}_2,I_2) = \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}} \mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I} \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\ , \tag{8.59}$$

with summation over $(\boldsymbol{a};\sigma)$ *restricted to* $Q(\boldsymbol{a};\sigma)$ *in the Clebsch-Gordan series of* $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$*, and*

$$\mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I} = \sum_{\substack{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3\\ J_1,J_2,J_3}} (-1)^{2(t_{\boldsymbol{\mu}_2}+u_{\boldsymbol{\mu}_2})} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_1;\sigma_1)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_2 J_2,\ \boldsymbol{\nu}_1 I_1} \times C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_2;\sigma_2)}_{\check{\boldsymbol{\mu}}_2 J_2,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu}_2 I_2} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu} I}\ . \tag{8.60}$$

In particular, $\mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I}\neq 0$ *implies that* $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}}=1$ *and* $I\leq I_1+I_2+|\boldsymbol{p}|$*.*

Proof. Using (8.47), we write

$$\boldsymbol{e}((\boldsymbol{a}_1;\sigma_1);\boldsymbol{\nu}_1,I_1)=\sum_{\substack{\boldsymbol{\mu}_1,J_1\\ \boldsymbol{\mu}_2,J_2}} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_1;\sigma_1)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_2 J_2,\ \boldsymbol{\nu}_1 I_1}\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu}_1,J_1)\otimes\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu}_2,J_2) \tag{8.61}$$

and

$$\boldsymbol{e}((\boldsymbol{a}_2;\sigma_2);\boldsymbol{\nu}_2,I_2)=\sum_{\substack{\boldsymbol{\mu}_3,J_3\\ \boldsymbol{\mu}_4,J_4}} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_2;\sigma_2)}_{\boldsymbol{\mu}_4 J_4,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu}_2 I_2}\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu}_4,J_4)\otimes\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu}_3,J_3)\ . \tag{8.62}$$

From (8.58), we have

$$\boldsymbol{e}((\boldsymbol{a}_1;\sigma_1);\boldsymbol{\nu}_1,I_1)\boldsymbol{e}((\boldsymbol{a}_2;\sigma_2);\boldsymbol{\nu}_2,I_2)=\sum_{\substack{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3\\ J_1,J_2,J_3}} (-1)^{2(t_{\boldsymbol{\mu}_2}+u_{\boldsymbol{\mu}_2})} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_1;\sigma_1)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_2 J_2,\ \boldsymbol{\nu}_1 I_1} \times C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_2;\sigma_2)}_{\check{\boldsymbol{\mu}}_2 J_2,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu}_2 I_2}\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\mu}_1,J_1)\otimes\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu}_3,J_3)\ . \tag{8.63}$$

Now, from (8.46) and (8.52),

$$\boldsymbol{e}((\boldsymbol{a}_1;\sigma_1);\boldsymbol{\nu}_1,I_1)\boldsymbol{e}((\boldsymbol{a}_2;\sigma_2);\boldsymbol{\nu}_2,I_2)=\sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}}\sum_{\substack{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3\\ J_1,J_2,J_3}} (-1)^{2(t_{\boldsymbol{\mu}_2}+u_{\boldsymbol{\mu}_2})} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_1;\sigma_1)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_2 J_2,\ \boldsymbol{\nu}_1 I_1} \times C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a}_2;\sigma_2)}_{\check{\boldsymbol{\mu}}_2 J_2,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu}_2 I_2} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\mu}_1 J_1,\boldsymbol{\mu}_3 J_3,\ \boldsymbol{\nu} I}\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\ , \tag{8.64}$$

where $\nabla_{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2,\boldsymbol{\nu}_1}=\nabla_{\check{\boldsymbol{\mu}}_2+\boldsymbol{\mu}_3,\boldsymbol{\nu}_2}=\nabla_{\boldsymbol{\mu}_1+\boldsymbol{\mu}_3,\boldsymbol{\nu}}=1$, $J_1\leq I_1+J_2$, $J_3\leq I_2+J_2$ and $I\leq J_1+J_3$. It follows that $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}}=1$ and, considering that $J_2\leq|\boldsymbol{p}|/2$, which can be inferred from (8.21), we have $I\leq I_1+I_2+|\boldsymbol{p}|$. □

8.2.1 Mixed Casimir operators and symmetric CG coefficients

We have avoided until now the problem of specifying a decomposition for degenerate representations in general Clebsch-Gordan series $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$. If $Q(\boldsymbol{a})$ is such that $\mathfrak{m}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a})>1$ (cf. Notation 2), there is no unique way to decompose

$$\bigoplus_{\sigma=1}^{\mathfrak{m}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a})} Q(\boldsymbol{a};\sigma)\ \subset\ Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2) \tag{8.65}$$

into irreducible representations of class $Q(\boldsymbol{a})$. To fix a unique convention for such a decomposition, we shall use the method of mixed Casimir operators, based on (CHEW; SHARP, 1966; CHEW; SHARP, 1967; PLUHAR; SMIRNOV; TOLSTOY, 1986) and described below. The envisaged decomposition provides more symmetric Clebsch-Gordan coefficients.

Let $\mathcal{H}$ be a Hilbert space carrying an irreducible representation of class $Q(p,q)$ and let A_{jk}, for $j,k \in \{1,2,3\}$, be the generators satisfying:

$$A_{12} = T_+ \ , \quad A_{23} = U_+ \ , \quad T_3 = \frac{1}{2}(A_{11} - A_{22}) \ , \quad U_3 = \frac{1}{2}(A_{22} - A_{33}) \ , \tag{8.66}$$

$$A^\dagger_{jk} = A_{kj} \ , \quad A_{11} + A_{22} + A_{33} = 0 \ , \tag{8.67}$$

$$[A_{jk}, A_{lm}] = \delta_{l,k} A_{jm} - \delta_{j,m} A_{lk} \ . \tag{8.68}$$

Then, $Q(q,p)$ is generated by the operators

$$\breve{A}_{jk} = -A_{kj} \ . \tag{8.69}$$

For $Q(p,q)$, the *quadratic Casimir operator* is

$$C_2 := \frac{1}{2} \sum_{j,k=1}^{3} A_{jk} A_{kj} \tag{8.70}$$

and the *cubic Casimir operator* is

$$C_3 := \sum_{j,k,l=1}^{3} A_{jk} A_{kl} A_{lj} \ , \tag{8.71}$$

so that

$$\begin{aligned} C_2 &= \frac{1}{3}[(p+q)(p+q+3) - pq]\mathbb{1} \ , \\ C_3 - 3C_2 &= \frac{1}{9}(p-q)(p+2q+3)(2p+q+3)\mathbb{1} \ , \end{aligned} \tag{8.72}$$

cf. (SHARP; BAEYER, 1966).

Now, for $x \in \{1,2,3\}$, consider the Hilbert space $\mathcal{H}_x$ carrying the representation $Q(\boldsymbol{p}_x) = Q(p_x, q_x)$ and the triple tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. Also, let $A^{(x)}_{jk}$, $C^{(x)}_2$ and $C^{(x)}_3$ be operators relative to $Q(\boldsymbol{p}_x)$ as defined above. For simplicity, given an operator $A^{(x)}$ on $\mathcal{H}_x$, we just write $A^{(x)}$ to denote its tensor product with the identity operator. Thus,

$$A^{(x)}_{jk} A^{(y)}_{lm} = A^{(y)}_{lm} A^{(x)}_{jk} \tag{8.73}$$

for $x \neq y$. The *mixed Casimir operators* are defined by

$$C^{xy}_2 := \sum_{j,k} A^{(x)}_{jk} A^{(y)}_{kj} = \sum_{j,k} A^{(y)}_{jk} A^{(x)}_{kj} =: C^{yx}_2 \ , \tag{8.74}$$

$$C_3^{xyz} := \sum_{j,k,l} A_{jk}^{(x)} A_{kl}^{(y)} A_{lj}^{(z)} \ , \tag{8.75}$$

for indices not all equal. If x, y, z are all distinct, equation (8.73) implies

$$\sum_{j,k,l} A_{jk}^{(x)} A_{kl}^{(y)} A_{lj}^{(z)} = \sum_{j,k,l} A_{kl}^{(y)} A_{lj}^{(z)} A_{jk}^{(x)} = \sum_{j,k,l} A_{lj}^{(z)} A_{jk}^{(x)} A_{kl}^{(y)} \ , \tag{8.76}$$

so

$$C_3^{xyz} = C_3^{yzx} = C_3^{zxy} \ , \tag{8.77}$$

and

$$C_3^{xyy} = \sum_{j,k,l} A_{jk}^{(x)} A_{kl}^{(y)} A_{lj}^{(y)} = \sum_{j,k,l} A_{kl}^{(y)} A_{lj}^{(y)} A_{jk}^{(x)} = C_3^{yyx} \ , \tag{8.78}$$

but

$$\begin{aligned} C_3^{xyx} &= \sum_{j,k,l} A_{jk}^{(x)} A_{kl}^{(y)} A_{lj}^{(x)} = \sum_{j,k,l} A_{kl}^{(y)} A_{jk}^{(x)} A_{lj}^{(x)} \\ &= \sum_{j,k,l} A_{kl}^{(y)} \left(\left[A_{jk}^{(x)}, A_{lj}^{(x)} \right] + A_{lj}^{(x)} A_{jk}^{(x)} \right) \\ &= \sum_{j,k,l} A_{kl}^{(y)} \left(\delta_{l,k} A_{jj}^{(x)} - A_{lk}^{(x)} + A_{lj}^{(x)} A_{jk}^{(x)} \right) \\ &= C_3^{yxx} - C_2^{yx} \ . \end{aligned} \tag{8.79}$$

Such operators arise, for example, from the Casimir operators on $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2) \otimes Q(\boldsymbol{p}_3)$:

$$\frac{1}{2} \sum_{j,k} \left(A_{jk}^{(1)} + A_{jk}^{(2)} + A_{jk}^{(3)} \right) \left(A_{kj}^{(1)} + A_{kj}^{(2)} + A_{kj}^{(2)} \right) = C_2^{(1)} + C_2^{(2)} + C_2^{12} + C_2^{23} + C_2^{31} \ , \tag{8.80}$$

$$\begin{aligned} &\sum_{j,k,l} \left(A_{jk}^{(1)} + A_{jk}^{(2)} + A_{jk}^{(3)} \right) \left(A_{kl}^{(1)} + A_{kl}^{(2)} + A_{kl}^{(3)} \right) \left(A_{lj}^{(1)} + A_{lj}^{(2)} + A_{lj}^{(3)} \right) \\ &\quad = C_3^{(1)} + C_3^{(2)} + C_3^{(3)} + C_3^{112} + C_3^{113} + C_3^{121} + C_3^{122} + C_3^{123} + C_3^{131} + C_3^{132} \\ &\quad\quad + C_3^{133} + C_3^{211} + C_3^{212} + C_3^{213} + C_3^{221} + C_3^{223} + C_3^{231} + C_3^{232} + C_3^{233} \\ &\quad\quad + C_3^{311} + C_3^{312} + C_3^{313} + C_3^{321} + C_3^{322} + C_3^{323} + C_3^{331} + C_3^{332} \ . \end{aligned} \tag{8.81}$$

From the mixed cubic Casimir operators, we set

$$\mathbf{S}_{xy} := \frac{1}{2}(C_3^{xyy} - C_3^{yxx}) \tag{8.82}$$

and

$$\mathbf{S}_{xyz} := \frac{1}{3}(\mathbf{S}_{xy} + \mathbf{S}_{yz} + \mathbf{S}_{zx}) \ . \tag{8.83}$$

Lemma 8.2.4. *The operators $\mathbf{S}_{xy}$ and $\mathbf{S}_{xyz}$ are anti-symmetric under odd permutation of the indices, that is,*

$$\mathbf{S}_{xy} = -\mathbf{S}_{yx} \ , \tag{8.84}$$

$$\mathbf{S}_{xyz} = -\mathbf{S}_{yxz} = -\mathbf{S}_{xzy} \ . \tag{8.85}$$

Proof. We have that

$$\mathbf{S}_{xy} = \frac{1}{2}(C_3^{xyy} - C_3^{yxx}) = -\frac{1}{2}(C_3^{yxx} - C_3^{xyy}) = -\mathbf{S}_{yx} \ ,$$

so

$$\mathbf{S}_{xyz} = \frac{1}{3}(\mathbf{S}_{xy} + \mathbf{S}_{yz} + \mathbf{S}_{zx}) = -\frac{1}{3}(\mathbf{S}_{yx} + \mathbf{S}_{zy} + \mathbf{S}_{xz}) \ .$$

Thus, $\mathbf{S}_{xyz} = -\mathbf{S}_{yxz} = -\mathbf{S}_{xzy}$. □

For representations $Q(\breve{\boldsymbol{p}}_x)$ generated by the operators

$$\breve{A}^{(x)}_{jk} = -A^{(x)}_{kj} \ , \tag{8.86}$$

cf. (8.69), we have

$$\breve{C}_2^{xy} = \sum_{j,k} \breve{A}^{(x)}_{jk} \breve{A}^{(y)}_{kj} = \sum_{j,k} A^{(x)}_{kj} A^{(y)}_{jk} = C_2^{xy} \ , \tag{8.87}$$

$$\breve{C}_3^{xyz} = \sum_{j,k,l} \breve{A}^{(x)}_{jk} \breve{A}^{(y)}_{kl} \breve{A}^{(z)}_{lj} = -\sum_{j,k,l} A^{(x)}_{kj} A^{(y)}_{lk} A^{(z)}_{jl} \ . \tag{8.88}$$

In particular,

$$\begin{aligned}
\breve{C}_3^{xyy} &= -\sum_{j,k,l} A^{(x)}_{kj} A^{(y)}_{lk} A^{(y)}_{jl} \\
&= -\sum_{j,k,l} A^{(x)}_{kj} \left(\left[A^{(y)}_{lk}, A^{(y)}_{jl} \right] + A^{(y)}_{jl} A^{(y)}_{lk} \right) \\
&= -\sum_{j,k,l} A^{(x)}_{kj} (\delta_{j,k} A^{(y)}_{ll} - A^{(y)}_{jk} + A^{(y)}_{jl} A^{(y)}_{lk}) \\
&= C_2^{xy} - C_3^{xyy}
\end{aligned} \tag{8.89}$$

Therefore, we have:

Lemma 8.2.5. *The operators* $\mathbf{S}_{xy}$ *and* $\mathbf{S}_{xyz}$ *are anti-symmetric under dualization, that is,*

$$\breve{\mathbf{S}}_{xy} = -\mathbf{S}_{xy} \ , \tag{8.90}$$

$$\breve{\mathbf{S}}_{xyz} = -\mathbf{S}_{xyz} \ . \tag{8.91}$$

Proof. From (8.74), (8.82) and (8.89), we have that

$$\breve{\mathbf{S}}_{xy} = \frac{1}{2}(\breve{C}_3^{xyy} - \breve{C}_3^{yxx}) = -\frac{1}{2}(C_3^{xyy} - C_3^{yxx}) = -\mathbf{S}_{xy} \ .$$

From (8.90), we have

$$\breve{\mathbf{S}}_{xyz} = \frac{1}{3}(\breve{\mathbf{S}}_{xy} + \breve{\mathbf{S}}_{yz} + \breve{\mathbf{S}}_{zx}) = -\frac{1}{3}(\mathbf{S}_{xy} + \mathbf{S}_{yz} + \mathbf{S}_{zx}) = -\mathbf{S}_{xyz} \ .$$

□

Now, let $\mathcal{H}^0 \equiv \mathcal{H}^0_{123}$ be a maximal subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ where $SU(3)$ acts trivially. That means

$$A^{(1)}_{jk} + A^{(2)}_{jk} + A^{(3)}_{jk} = 0 \tag{8.92}$$

on $\mathcal{H}^0$, for all $j, k \in \{1, 2, 3\}$.

Lemma 8.2.6. *$\mathcal{H}^0_{123}$ is not null if and only if there is a representation of class $Q(\check{\boldsymbol{p}}_3)$ in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)$.*

Proof. Considering the Clebsch-Gordan series

$$Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2) = \bigoplus_{(\boldsymbol{a};\sigma)} Q(\boldsymbol{a};\sigma) \ , \tag{8.93}$$

we have

$$Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)\otimes Q(\boldsymbol{p}_3) = \bigoplus_{(\boldsymbol{a};\sigma)} Q(\boldsymbol{a};\sigma)\otimes Q(\boldsymbol{p}_3) \ . \tag{8.94}$$

Note that $k \geq 1$ in (8.41), so there exists a factor of class $Q(0,0)$ in the CG series of $Q(a,b)\otimes Q(p_3,q_3)$ if and only if there is $0 \leq n \leq \min(a,q_3)$ and $0 \leq m \leq \min(p_3,b)$ satisfying

$$a - n + p_3 - m = b - m + q_3 - n = 0 \ . \tag{8.95}$$

The only possible solution is $n = a = q_3$ and $m = p_3 = b$. Thus, $\mathcal{H}^0$ is not null if and only if there is a representation of class $Q(\boldsymbol{a}) = Q(\check{\boldsymbol{p}}_3)$ in the l.h.s. of (8.93). □

Lemma 8.2.7. *On $\mathcal{H}^0$, the following holds:*

$$\mathbf{S}_{123} = \mathbf{S}_{12} - \frac{1}{3}C_3^{(1)} + \frac{1}{3}C_3^{(2)} \ . \tag{8.96}$$

Proof. From (8.92),

$$\begin{aligned} C_3^{233} &= \sum_{j,k,l} A^{(2)}_{jk}\left(A^{(1)}_{kl} + A^{(2)}_{kl}\right)\left(A^{(1)}_{lj} + A^{(2)}_{lj}\right) \\ &= C_3^{211} + C_3^{212} + C_3^{221} + C_3^{(2)} \ . \end{aligned} \tag{8.97}$$

For $x = 1$ and $y = 2$, equation (8.78) reads

$$C_3^{221} = C_3^{122} \tag{8.98}$$

and likewise, for $x = 2$ and $y = 1$,

$$C_3^{211} = C_3^{112} \ . \tag{8.99}$$

Also,

$$C_3^{212} = C_3^{122} - C_2^{12} \ , \tag{8.100}$$

cf. (8.79), and

$$C_3^{233} = C_3^{211} + 2C_3^{122} + C_3^{(2)} - C_2^{12} \ . \tag{8.101}$$

For C_3^{322} we have

$$\begin{aligned} C_3^{322} &= -\sum_{j,k,l}(A^{(1)}_{jk} + A^{(2)}_{jk})A^{(2)}_{kl}A^{(2)}_{lj} \\ &= -C_3^{122} - C_3^{(2)} \ . \end{aligned} \tag{8.102}$$

Then

$$\mathbf{S}_{23} = \frac{1}{2}(C_3^{211} + 3C_3^{122} + 2C_3^{(2)} - C_2^{12}) \ . \tag{8.103}$$

By formal identification of the formulas, we get in a straightforward way

$$\mathbf{S}_{31} = -\mathbf{S}_{13} = -\frac{1}{2}(C_3^{122} + 3C_3^{211} + 2C_3^{(1)} - C_2^{21}) \ , \tag{8.104}$$

so

$$\mathbf{S}_{23} + \mathbf{S}_{31} = C_3^{122} - C_3^{211} - C_3^{(1)} + C_3^{(2)} = 2\,\mathbf{S}_{12} - C_3^{(1)} + C_3^{(2)} \ , \tag{8.105}$$

where we used $C_2^{12} = C_2^{21}$. Therefore,

$$\mathbf{S}_{123} = \frac{1}{3}(\mathbf{S}_{12} + \mathbf{S}_{23} + \mathbf{S}_{31}) = \mathbf{S}_{12} - \frac{1}{3}C_3^{(1)} + \frac{1}{3}C_3^{(2)} \ , \tag{8.106}$$

on $\mathcal{H}^0$. □

Notation 3. *In view of the previous lemma, for $\mathcal{H}^0 \neq 0$, we shall denote*

$$\mathbf{S}_{123}^0 := \mathbf{S}_{123}|_{\mathcal{H}^0} \ . \tag{8.107}$$

Lemma 8.2.8. *The operators $\mathbf{S}_{12}$ and $\mathbf{S}_{123}^0$ are Hermitian and $SU(3)$-invariant.*

Proof. By straightforward calculation, we have

$$(C_3^{xyy})^\dagger = \sum_{j,k,l} \left(A_{jk}^{(x)} A_{kl}^{(y)} A_{lj}^{(y)}\right)^\dagger = \sum_{j,k,l} A_{jl}^{(y)} A_{lk}^{(y)} A_{kj}^{(x)} = C_3^{yyx} \ , \tag{8.108}$$

so $(C_3^{122})^\dagger = C_3^{122}$ and $(C_3^{211})^\dagger = C_3^{211}$, which implies

$$(\mathbf{S}_{12})^\dagger = \frac{1}{2}(C_3^{122} - C_3^{211})^\dagger = \frac{1}{2}(C_3^{122} - C_3^{211}) = \mathbf{S}_{12} \ . \tag{8.109}$$

For the $SU(3)$-invariance, we will need the following equalities:

$$A_{cd}^{(x)} A_{jk}^{(x)} = \left[A_{cd}^{(x)}, A_{jk}^{(x)}\right] + A_{jk}^{(x)} A_{cd}^{(x)} = \delta_{j,d} A_{ck}^{(x)} - \delta_{c,k} A_{jd}^{(x)} + A_{jk}^{(x)} A_{cd}^{(x)} \ , \tag{8.110}$$

$$\begin{aligned} A_{cd}^{(x)} A_{kl}^{(x)} A_{lj}^{(x)} &= \left(\left[A_{cd}^{(x)}, A_{kl}^{(x)}\right] + A_{kl}^{(x)} A_{cd}^{(x)}\right) A_{lj}^{(x)} \\ &= \left(\delta_{k,d} A_{cl}^{(x)} - \delta_{c,l} A_{kd}^{(x)} + A_{kl}^{(x)} A_{cd}^{(x)}\right) A_{lj}^{(x)} \\ &= \left(\delta_{k,d} A_{cl}^{(x)} - \delta_{c,l} A_{kd}^{(x)}\right) A_{lj}^{(x)} \\ &\quad + A_{kl}^{(x)} \left(\left[A_{cd}^{(x)}, A_{lj}^{(x)}\right] + A_{lj}^{(x)} A_{cd}^{(x)}\right) \\ &= \left(\delta_{k,d} A_{cl}^{(x)} - \delta_{c,l} A_{kd}^{(x)}\right) A_{lj}^{(x)} \\ &\quad + A_{kl}^{(x)} \left(\delta_{l,d} A_{cj}^{(x)} - \delta_{c,j} A_{ld}^{(x)}\right) \\ &\quad + A_{kl}^{(x)} A_{lj}^{(x)} A_{cd}^{(x)} \ . \end{aligned} \tag{8.111}$$

With this, we obtain

$$A_{cd}^{(x)} C_3^{xyy} = \sum_{k,l} A_{ck}^{(x)} A_{kl}^{(y)} A_{ld}^{(y)} - \sum_{j,l} A_{jd}^{(x)} A_{cl}^{(y)} A_{lj}^{(y)} + C_3^{xyy} A_{cd}^{(x)} \ , \tag{8.112}$$

$$A^{(x)}_{cd} C^{yxx}_3 = \sum_{j,l} A^{(y)}_{jd} A^{(x)}_{cl} A^{(x)}_{lj} - \sum_{k,l} A^{(y)}_{ck} A^{(x)}_{kl} A^{(x)}_{ld} + C^{yxx}_3 A^{(x)}_{cd} \ , \tag{8.113}$$

Thus,

$$\left[A^{(1)}_{cd}, C^{122}_3 - C^{211}_3\right] = \left[A^{(2)}_{cd}, C^{211}_3 - C^{122}_3\right] \implies \left[A^{(1)}_{cd} + A^{(2)}_{cd}, \mathbf{S}_{12}\right] = 0 \ . \tag{8.114}$$

Hence, $\mathbf{S}_{12}$ is Hermitian and $SU(3)$-invariant. The result for $\mathbf{S}^0_{123}$ follows immediately from Lemma 8.2.7. □

In this way, we decompose degenerate representations in the Clebsch-Gordan series of $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$ via diagonalization of the operator

$$\mathbf{S} := \mathbf{S}_{12} - \frac{1}{3} C^{(1)}_3 + \frac{1}{3} C^{(2)}_3 \ , \tag{8.115}$$

satisfying

$$\mathbf{S}|_{\mathcal{H}^0_{123}} = \mathbf{S}^0_{123} \ , \tag{8.116}$$

cf. (8.107). Because $\mathbf{S}$ is built from Casimir operators, the eigenvalues s_{123} of $\mathbf{S}^0_{123}$ depend only on the representations comprising $\mathcal{H}^0_{123} \subset \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

Thus, for

$$\bigoplus_{\sigma=1}^{\mathfrak{m}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a})} Q(\boldsymbol{a};\sigma) \subset Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2) \ , \tag{8.117}$$

with $Q(\check{\boldsymbol{a}}) = Q(\boldsymbol{p}_3)$, cf. Lemma 8.2.6, we can define a function

$$s_{\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a}} : \{1, ..., \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a})\} \to \mathbb{R} \ , \ \sigma \mapsto s_{\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a}}(\sigma) \ , \tag{8.118}$$

which indexes the eigenvalues of $\mathbf{S}^0_{123}$ in this case. For simplicity, we shall denote

$$s_{\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{a}} \equiv s_{\boldsymbol{a}} \tag{8.119}$$

and, henceforth, *we shall adopt the convention that $s_{\boldsymbol{a}}$ is an increasing function of the multiplicity counting index*, that is,

$$\{1, ..., \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a})\} \ni \sigma \mapsto s_{\boldsymbol{a}}(\sigma) \in \mathbb{R} \ , \quad s_{\boldsymbol{a}}(\sigma) \leq s_{\boldsymbol{a}}(\sigma + 1) \ . \tag{8.120}$$

In other words, *the eigenvalues of $\mathbf{S}^0_{123}$ are indexed by increasing order.*

Remark 8.2.3. *In (CHEW; SHARP, 1966) and (PLUHAR; SMIRNOV; TOLSTOY, 1986), the authors compute the entries of matrices equivalent to $\mathbf{S}$ (they differ only by a scalar factor or by a multiple of the identity) restricted to a subspace of $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$ generated by the highest states of a same subrepresentation class. From the matrix entries, it is possible to infer the existence of a cyclic vector, which implies that the eigenvalues of $\mathbf{S}^0_{123}$ are all distinct, i.e., the function $s_{\boldsymbol{a}}$ of (8.118)-(8.120) is strictly increasing,*

$$s_{\boldsymbol{a}}(\sigma) < s_{\boldsymbol{a}}(\sigma + 1) \ , \tag{8.121}$$

and thus defines an injection $\sigma \mapsto s_{\boldsymbol{a}}(\sigma)$, so that the eigenvalues of $\mathbf{S}$ *can be used to label degenerate representations, rather than our counting index σ. This approach is implemented in (PLUHAR; SMIRNOV; TOLSTOY, 1986; PLUHAR; WEIGERT; HOLAN, 1986), for instance. However, so far we have been unable to fully reproduce the computations of the referred papers, so we maintain the use of the counting index σ to distinguish representations of the same class in $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$, but now adopting convention (8.120).*

Notation 4. *For $\bigoplus_\sigma Q(\boldsymbol{a};\sigma)$ in the CG series of $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$, the following involution in the set of multiplicity indices will be relevant:*

$$\{1, ..., \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a})\} \ni \sigma \mapsto \check{\sigma} = \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{a}) - \sigma + 1 \ . \tag{8.122}$$

Remark 8.2.4. *The involution (8.122) is at par with the involution $s \mapsto -s$ which is consequence of (8.84)-(8.85) and (8.90)-(8.91), taking into account the convention (8.120).*

From (8.56) and Lemmas 8.2.4-8.2.8, considering the convention in (8.120), we can choose coupled basis $\{\boldsymbol{e}_{12}((\check{\boldsymbol{p}}_3;\sigma);\boldsymbol{\nu},I)\}$, $\{\boldsymbol{e}_{21}((\check{\boldsymbol{p}}_3;\sigma);\boldsymbol{\nu},I)\}$, $\{\boldsymbol{e}_{13}((\check{\boldsymbol{p}}_2;\sigma);\boldsymbol{\nu},I)\}$ and $\{\check{\boldsymbol{e}}_{12}((\boldsymbol{p}_3;\sigma);\boldsymbol{\nu},I)\}$ for $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2)$, $Q(\boldsymbol{p}_2) \otimes Q(\boldsymbol{p}_1)$, $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_3)$ and $Q(\check{\boldsymbol{p}}_1) \otimes Q(\check{\boldsymbol{p}}_2)$, respectively, such that

$$1 \leq \sigma \leq \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_2; \check{\boldsymbol{p}}_3) = \mathfrak{m}(\boldsymbol{p}_2, \boldsymbol{p}_1; \check{\boldsymbol{p}}_3) = \mathfrak{m}(\boldsymbol{p}_1, \boldsymbol{p}_3; \check{\boldsymbol{p}}_2) = \mathfrak{m}(\check{\boldsymbol{p}}_1, \check{\boldsymbol{p}}_2; \boldsymbol{p}_3) \tag{8.123}$$

and $\mathcal{H}^0$ is spanned by

$$\begin{aligned}
&\sum_{\boldsymbol{\nu},I} \frac{(-1)^{2(t+u)}}{\sqrt{\dim Q(\boldsymbol{p}_3)}} \boldsymbol{e}_{12}((\check{\boldsymbol{p}}_3;\sigma);\boldsymbol{\nu},I) \otimes \boldsymbol{e}(\boldsymbol{p}_3;\check{\boldsymbol{\nu}},I) \\
&\quad = (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|} \sum_{\boldsymbol{\nu},I} \frac{(-1)^{2(t+u)}}{\sqrt{\dim Q(\boldsymbol{p}_3)}} \boldsymbol{e}_{21}((\check{\boldsymbol{p}}_3;\check{\sigma});\boldsymbol{\nu},I) \otimes \boldsymbol{e}(\boldsymbol{p}_3;\check{\boldsymbol{\nu}},I) \\
&\quad = (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|} \sum_{\boldsymbol{\nu},I} \frac{(-1)^{2(t+u)}}{\sqrt{\dim Q(\boldsymbol{p}_2)}} \boldsymbol{e}_{13}((\check{\boldsymbol{p}}_2;\check{\sigma});\boldsymbol{\nu},I) \otimes \boldsymbol{e}(\boldsymbol{p}_2;\check{\boldsymbol{\nu}},I) \\
&\quad = (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|} \sum_{\boldsymbol{\nu},I} \frac{(-1)^{2(t+u)}}{\sqrt{\dim Q(\check{\boldsymbol{p}}_3)}} \check{\boldsymbol{e}}_{12}((\boldsymbol{p}_3;\check{\sigma});\boldsymbol{\nu},I) \otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}}_3;\check{\boldsymbol{\nu}},I) \ ,
\end{aligned} \tag{8.124}$$

where we have made use of (8.122). As consequence, now the Clebsch-Gordan coefficients satisfy a bigger set of symmetry relations, as follows.

Proposition 8.2.9. *For $SU(3)$ representations of class $Q(\boldsymbol{p}_1)$ and $Q(\boldsymbol{p}_2)$, the Clebsch-Gordan coefficients for the Clebsch-Gordan series $Q(\boldsymbol{p}_1) \otimes Q(\boldsymbol{p}_2) = \bigoplus Q(\boldsymbol{a};\sigma)$ satisfy*

$$\begin{aligned}
C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1, \boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I} &= (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{a}|} C^{\boldsymbol{p}_2,\ \boldsymbol{p}_1,\ (\boldsymbol{a};\check{\sigma})}_{\boldsymbol{\nu}_2 I_2, \boldsymbol{\nu}_1 I_1,\ \boldsymbol{\nu} I} \\
&= (-1)^{|\boldsymbol{p}_1|-2(t_{\nu_1}+u_{\nu_1})} \sqrt{\frac{\dim Q(\boldsymbol{a})}{\dim Q(\boldsymbol{p}_2)}}\, C^{\boldsymbol{p}_1,\ \check{\boldsymbol{a}},\ (\check{\boldsymbol{p}}_2;\check{\sigma})}_{\boldsymbol{\nu}_1 I_1, \check{\boldsymbol{\nu}} I,\ \check{\boldsymbol{\nu}}_2 I_2} \\
&= (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{a}|} C^{\check{\boldsymbol{p}}_1,\ \check{\boldsymbol{p}}_2,\ (\check{\boldsymbol{a}};\check{\sigma})}_{\check{\boldsymbol{\nu}}_1 I_1, \check{\boldsymbol{\nu}}_2 I_2,\ \check{\boldsymbol{\nu}} I} \ .
\end{aligned} \tag{8.125}$$

Proof. By writing (8.124) with $\boldsymbol{a} = \check{\boldsymbol{p}}_3$ and taking the inner product with suitable uncoupled basis, the result follows straightforwardly from the symmetries of $\mathbf{S}$ satisfying Lemma 8.2.7, cf. (8.84)-(8.85) and (8.90)-(8.91), using (8.120) and (8.122). $\square$

We highlight that each generator A_{jk} can be realized as a real matrix on a GT basis, so $\mathbf{S}$ can be seen as a symmetric real matrix acting on

$$\mathrm{Span}_{\mathbb{R}}\{\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu},I)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\mu},J)\}\ , \tag{8.126}$$

which means that the elements of any basis of the previous paragraph can be constructed as real linear combinations of the respective uncoupled basis. With this convention, all Clebsch-Gordan coefficients are still real, and (8.48) and (8.52) still hold.

In particular, the Hermitian conjugate $\dagger$ of operators in $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$ satisfies

$$\boldsymbol{e}^{\dagger}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) = (-1)^{2(t+u)}\boldsymbol{e}((\check{\boldsymbol{a}};\sigma);\check{\boldsymbol{\nu}},I) \tag{8.127}$$

and the phases are chosen such that

$$\boldsymbol{e}((0,0);(0,0,0),0) = \frac{(-1)^{|\boldsymbol{p}|}}{\sqrt{\dim Q(\boldsymbol{p})}}\mathbb{1}\ , \tag{8.128}$$

this being the only element with non vanishing trace.

Besides Hermitian conjugate, the adjoint

$$* : Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}}) \to Q(\check{\boldsymbol{p}})\otimes Q(\boldsymbol{p}) : A\mapsto A^*\ , \tag{8.129}$$

is given in the uncoupled basis by

$$\begin{aligned} * : \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\otimes\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu},J) &= (-1)^{2(t_\mu+u_\mu)}\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\otimes\boldsymbol{e}^*(\boldsymbol{p};\check{\boldsymbol{\mu}},J) \\ &\mapsto (-1)^{2(t_\mu+u_\mu)}\boldsymbol{e}^*(\boldsymbol{p};\check{\boldsymbol{\mu}},J)\otimes\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I) \\ &= \check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\mu},J)\otimes\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu},I)\ , \end{aligned} \tag{8.130}$$

cf. (8.34). Thus, for the coupled basis of $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$, *cf.* (8.47), the adjoint is given by

$$\begin{aligned} * : \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) = &\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}_1,I_1)\otimes\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\nu}_2,I_2) \\ \mapsto &\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} C^{\boldsymbol{p},\ \ \check{\boldsymbol{p}},\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I}\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\nu}_2,I_2)\otimes\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}_1,I_1) \\ = &(-1)^{|\boldsymbol{a}|}\sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} C^{\check{\boldsymbol{p}},\ \ \boldsymbol{p},\ \ (\boldsymbol{a};\check{\sigma})}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}_1 I_1,\ \boldsymbol{\nu} I}\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\nu}_2,I_2)\otimes\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}_1,I_1)\ . \end{aligned} \tag{8.131}$$

In the light of Remark 8.1.2, from the above calculation, we identify

$$\check{\boldsymbol{e}}((\boldsymbol{a};\check{\sigma});\boldsymbol{\nu},I) := \sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2}} C^{\check{\boldsymbol{p}},\ \ \boldsymbol{p},\ \ (\boldsymbol{a};\check{\sigma})}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}_1 I_1,\ \boldsymbol{\nu} I}\check{\boldsymbol{e}}(\check{\boldsymbol{p}};\boldsymbol{\nu}_2,I_2)\otimes\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}_1,I_1)\ , \tag{8.132}$$

so that

$$\boldsymbol{e}^*((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) = (-1)^{|\boldsymbol{a}|}\check{\boldsymbol{e}}((\boldsymbol{a};\check{\sigma});\boldsymbol{\nu},I)\ . \tag{8.133}$$

Definition 8.2.4. *Given a coupled basis* $\{\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I\}$ *of* $Q(\boldsymbol{p})\otimes Q(\check{\boldsymbol{p}})$*, the* induced coupled basis *of* $Q(\check{\boldsymbol{p}})\otimes Q(\boldsymbol{p})$ *is the basis* $\{\check{\boldsymbol{e}}((\boldsymbol{a};\check{\sigma});\boldsymbol{\nu},I)\}$ *satisfying (8.132)-(8.133).*

8.2.2 Wigner symbols

For $SU(2)$, the Clebsch-Gordan coefficients can be substituted by other coefficients with better symmetry properties, the so-called *Wigner* $3jm$*-symbols* (see for instance (RIOS; STRAUME, 2014; VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988; WIGNER, 1959)). The references (DEROME; SHARP, 1965; BUTLER, 1975) address such symmetry problem for general compact groups, and generalized Wigner $3jm$-symbols are defined. Here, we follow the conventions in (PLUHAR; WEIGERT; HOLAN, 1986).

Definition 8.2.5. *The* Wigner coupling symbol *is the coefficient denoted by the round brackets below:*

$$\begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}, I \end{pmatrix} = \frac{(-1)^{|\boldsymbol{a}|+2(t_\nu+u_\nu)}}{\sqrt{\dim Q(\check{\boldsymbol{a}})}} C^{\boldsymbol{p}_1,\ \boldsymbol{p}_2,\ (\check{\boldsymbol{a}};\sigma)}_{\boldsymbol{\nu}_1 I_1, \boldsymbol{\nu}_2 I_2,\ \check{\boldsymbol{\nu}} I} \ . \tag{8.134}$$

Thus, from Proposition 8.2.9, we have

$$\begin{aligned} \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}, I \end{pmatrix} &= (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{a}|} \begin{pmatrix} \boldsymbol{p}_2 & \boldsymbol{p}_1 & (\boldsymbol{a};\check{\sigma}) \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}, I \end{pmatrix} \\ &= (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{a}|} \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{a} & (\boldsymbol{p}_2;\check{\sigma}) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}, I & \boldsymbol{\nu}_2, I_2 \end{pmatrix} \\ &= (-1)^{|\boldsymbol{p}_1|+|\boldsymbol{p}_2|+|\boldsymbol{a}|} \begin{pmatrix} \check{\boldsymbol{p}}_1 & \check{\boldsymbol{p}}_2 & (\check{\boldsymbol{a}};\check{\sigma}) \\ \check{\boldsymbol{\nu}}_1, I_1 & \check{\boldsymbol{\nu}}_2, I_2 & \check{\boldsymbol{\nu}}, I \end{pmatrix} . \end{aligned} \tag{8.135}$$

Another useful property of Wigner coupling symbols arises when we study the recoupling problem of a triple tensor product $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)\otimes Q(\boldsymbol{p}_3)$, whose decomposition can be realized in any order. Consider, for example, the Clebsch-Gordan series

$$Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2) = \bigoplus_{(\boldsymbol{a}_{12};\sigma_{12})} Q(\boldsymbol{a}_{12};\sigma_{12}) \ , \quad Q(\boldsymbol{a}_{12})\otimes Q(\boldsymbol{p}_3) = \bigoplus_{(\boldsymbol{a};\sigma)} Q(\boldsymbol{a};\sigma) \ . \tag{8.136}$$

With this, we get a basis $\{\boldsymbol{e}((\boldsymbol{a}_{12};\sigma_{12}),(\boldsymbol{a};\sigma);\boldsymbol{\mu},J)\}$ for $Q(\boldsymbol{p}_1)\otimes Q(\boldsymbol{p}_2)\otimes Q(\boldsymbol{p}_3)$ satisfying

$$\begin{aligned} \boldsymbol{e}((\boldsymbol{a}_{12};\sigma_{12}),(\boldsymbol{a};\sigma);\boldsymbol{\mu},J) = & \sum_{\substack{\boldsymbol{\mu}_{12},J_{12} \\ \boldsymbol{\nu}_3,I_3}} \sum_{\substack{\boldsymbol{\nu}_1,I_1 \\ \boldsymbol{\nu}_2,I_2}} C^{\boldsymbol{a}_{12},\ \ \boldsymbol{p}_3,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\mu}_{12}J_{12},\boldsymbol{\nu}_3 I_3,\ \boldsymbol{\mu}J} C^{\boldsymbol{p}_1,\ \ \boldsymbol{p}_2,\ \ (\boldsymbol{a}_{12};\sigma_{12})}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\mu}_{12}J_{12}} \\ & \times \boldsymbol{e}(\boldsymbol{p}_1);\boldsymbol{\nu}_1,I_1)\otimes \boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\otimes \boldsymbol{e}(\boldsymbol{p}_3;\boldsymbol{\nu}_3,I_3) \\ = & \sqrt{\dim Q(\boldsymbol{a}) \dim Q(\boldsymbol{a}_{12})} \sum_{\substack{\boldsymbol{\mu}_{12},J_{12} \\ \boldsymbol{\nu}_3,I_3}} \sum_{\substack{\boldsymbol{\nu}_1,I_1 \\ \boldsymbol{\nu}_2,I_2}} (-1)^{|\boldsymbol{a}|+|\boldsymbol{a}_{12}|+2(t_\mu+u_\mu+t_{\mu_{12}}+u_{\mu_{12}})} \\ & \times \begin{pmatrix} \boldsymbol{a}_{12} & \boldsymbol{p}_3 & (\check{\boldsymbol{a}};\sigma) \\ \boldsymbol{\mu}_{12}, J_{12} & \boldsymbol{\nu}_3, I_3 & \check{\boldsymbol{\mu}}, J \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\check{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \check{\boldsymbol{\mu}}_{12}, J_{12} \end{pmatrix} \\ & \times \boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes \boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\otimes \boldsymbol{e}(\boldsymbol{p}_3;\boldsymbol{\nu}_3,I_3) \ . \end{aligned} \tag{8.137}$$

On the other hand, if we use

$$Q(\boldsymbol{p}_2)\otimes Q(\boldsymbol{p}_3)=\bigoplus_{(\boldsymbol{a}_{23};\sigma_{23})}Q(\boldsymbol{a}_{23};\sigma_{23})\ ,\quad Q(\boldsymbol{a}_{23})\otimes Q(\boldsymbol{p}_1)=\bigoplus_{(\boldsymbol{a}';\sigma')}Q(\boldsymbol{a}';\sigma')\ ,\tag{8.138}$$

we get a basis $\{e((\boldsymbol{a}_{23};\sigma_{23}),(\boldsymbol{a}';\sigma');\boldsymbol{\mu}',J')\}$ satisfying

$$\begin{aligned}\boldsymbol{e}((\boldsymbol{a}_{23};\sigma_{23}),(\boldsymbol{a}';\sigma');\boldsymbol{\mu}',J')&=\sum_{\substack{\boldsymbol{\mu}_{23},J_{23}\\ \boldsymbol{\nu}_1,I_1}}\sum_{\substack{\boldsymbol{\nu}_2,I_2\\ \boldsymbol{\nu}_3,I_3}}C^{\boldsymbol{a}_{23},\quad \boldsymbol{p}_1,\quad (\boldsymbol{a}';\sigma')}_{\boldsymbol{\mu}_{23}J_{23},\boldsymbol{\nu}_1I_1,\ \boldsymbol{\mu}'J'}C^{\boldsymbol{p}_2,\quad \boldsymbol{p}_3,\quad (\boldsymbol{a}_{23};\sigma_{23})}_{\boldsymbol{\nu}_2I_2,\boldsymbol{\nu}_3I_3,\ \boldsymbol{\mu}_{23}J_{23}}\\ &\qquad\times\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\otimes\boldsymbol{e}(\boldsymbol{p}_3;\boldsymbol{\nu}_3,I_3)\\ &=\sqrt{\dim Q(\boldsymbol{a}')\dim Q(\boldsymbol{a}_{23})}\sum_{\substack{\boldsymbol{\mu}_{23},J_{23}\\ \boldsymbol{\nu}_1,I_1}}\sum_{\substack{\boldsymbol{\nu}_2,I_2\\ \boldsymbol{\nu}_3,I_3}}(-1)^{|\boldsymbol{a}'|+|\boldsymbol{a}_{23}|+2(t_{\boldsymbol{\mu}'}+u_{\boldsymbol{\mu}'}+t_{\boldsymbol{\mu}_{23}}+u_{\boldsymbol{\mu}_{23}})}\\ &\quad\times\begin{pmatrix}\boldsymbol{a}_{23} & \boldsymbol{p}_1 & (\check{\boldsymbol{a}}';\sigma')\\ \boldsymbol{\mu}_{23},J_{23} & \boldsymbol{\nu}_1,I_1 & \check{\boldsymbol{\mu}}',J'\end{pmatrix}\begin{pmatrix}\boldsymbol{p}_2 & \boldsymbol{p}_3 & (\check{\boldsymbol{a}}_{23};\sigma_{23})\\ \boldsymbol{\nu}_2,I_2 & \boldsymbol{\nu}_3,I_3 & \check{\boldsymbol{\mu}}_{23},J_{23}\end{pmatrix}\\ &\quad\times\boldsymbol{e}(\boldsymbol{p}_1;\boldsymbol{\nu}_1,I_1)\otimes\boldsymbol{e}(\boldsymbol{p}_2;\boldsymbol{\nu}_2,I_2)\otimes\boldsymbol{e}(\boldsymbol{p}_3;\boldsymbol{\nu}_3,I_3)\ .\end{aligned}\tag{8.139}$$

Of course, there is an unitary transformation relating these basis and

$$\langle\boldsymbol{e}((\boldsymbol{a}_{12};\sigma_{12}),(\boldsymbol{a};\sigma);\boldsymbol{\mu},J)|\boldsymbol{e}((\boldsymbol{a}_{23};\sigma_{23}),(\boldsymbol{a}';\sigma');\boldsymbol{\mu}',J')\rangle\neq 0\implies\begin{cases}\boldsymbol{a}=\boldsymbol{a}'\\ (\boldsymbol{\mu},J)=(\boldsymbol{\mu}',J')\end{cases}.\tag{8.140}$$

Also, the coefficients of the form

$$\langle\boldsymbol{e}((\boldsymbol{a}_{12};\sigma_{12}),(\boldsymbol{a};\sigma);\boldsymbol{\mu},J)|\boldsymbol{e}((\boldsymbol{a}_{23};\sigma_{23}),(\boldsymbol{a};\sigma');\boldsymbol{\mu},J)\rangle\tag{8.141}$$

do not depend on the weight since the vectors in the inner product can be generated from the highest weight vectors of their respective representations by applying the ladder operators and we can wright a highest weight vector of one basis as a linear combination of highest weight vectors of the other basis for equivalent representations.

Definition 8.2.6. *The* Wigner recoupling symbol *is the coefficient denoted by the curly* *brackets below:*

$$\begin{Bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{p}_3 & (\boldsymbol{a};\sigma,\sigma') & (\boldsymbol{a}_{23};\sigma_{23}) \end{Bmatrix} = \frac{(-1)^{|\boldsymbol{a}_{23}|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|}}{\sqrt{\dim Q(\boldsymbol{a}_{12})Q(\boldsymbol{a}_{23})}} \sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2\\ \boldsymbol{\nu}_3,I_3}} \sum_{\substack{\boldsymbol{\mu}_{12},J_{12}\\ \boldsymbol{\mu}_{23},J_{23}}} C^{\boldsymbol{a}_{12},\quad \boldsymbol{p}_3,\quad (\boldsymbol{a};\sigma)}_{\boldsymbol{\mu}_{12}J_{12},\boldsymbol{\nu}_3 I_3,\ \boldsymbol{\mu}J}$$

$$\times C^{\boldsymbol{p}_1,\quad \boldsymbol{p}_2,\quad (\boldsymbol{a}_{12};\sigma_{12})}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\mu}_{12}J_{12}} C^{\boldsymbol{a}_{23},\quad \boldsymbol{p}_1,\quad (\boldsymbol{a};\breve{\sigma}')}_{\boldsymbol{\mu}_{23}J_{23},\boldsymbol{\nu}_1 I_1,\ \boldsymbol{\mu}J} C^{\boldsymbol{p}_2,\quad \boldsymbol{p}_3,\quad (\boldsymbol{a}_{23};\sigma_{23})}_{\boldsymbol{\nu}_2 I_2,\boldsymbol{\nu}_3 I_3,\ \boldsymbol{\mu}_{23}J_{23}}$$

$$= \sum_{\substack{\boldsymbol{\nu}_1,I_1\\ \boldsymbol{\nu}_2,I_2\\ \boldsymbol{\nu}_3,I_3}} \sum_{\substack{\boldsymbol{\mu}_{12},J_{12}\\ \boldsymbol{\mu}_{23},J_{23}\\ \boldsymbol{\mu},J}} (-1)^{|\boldsymbol{p}_2|+|\boldsymbol{p}_3|+|\boldsymbol{a}_{12}|+2(t_{\boldsymbol{\mu}_{12}}+u_{\boldsymbol{\mu}_{12}}+t_{\boldsymbol{\mu}_{23}}+u_{\boldsymbol{\mu}_{23}})}$$
$$\times \begin{pmatrix} \boldsymbol{a}_{12} & \boldsymbol{p}_3 & (\breve{\boldsymbol{a}};\sigma) \\ \boldsymbol{\mu}_{12},J_{12} & \boldsymbol{\nu}_3,I_3 & \breve{\boldsymbol{\mu}},J \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{\nu}_1,I_1 & \boldsymbol{\nu}_2,I_2 & \breve{\boldsymbol{\mu}}_{12},J_{12} \end{pmatrix}$$
$$\times \begin{pmatrix} \boldsymbol{a}_{23} & \boldsymbol{p}_1 & (\breve{\boldsymbol{a}};\breve{\sigma}') \\ \boldsymbol{\mu}_{23},J_{23} & \boldsymbol{\nu}_1,I_1 & \breve{\boldsymbol{\mu}},J \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_2 & \boldsymbol{p}_3 & (\breve{\boldsymbol{a}}_{23};\sigma_{23}) \\ \boldsymbol{\nu}_2,I_2 & \boldsymbol{\nu}_3,I_3 & \breve{\boldsymbol{\mu}}_{23},J_{23} \end{pmatrix}. \tag{8.142}$$

With this definition, we have the following equation:

$$\boldsymbol{e}((\boldsymbol{a}_{23};\sigma_{23}),(\boldsymbol{a};\breve{\sigma}');\boldsymbol{\mu},J) = \sum_{\substack{(\boldsymbol{a}_{12},\sigma_{12})\\ \sigma}} (-1)^{|\boldsymbol{a}_{23}|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|} \sqrt{\dim Q(\boldsymbol{a}_{12}) \dim Q(\boldsymbol{a}_{23})}$$
$$\times \begin{Bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{p}_3 & (\boldsymbol{a};\sigma,\sigma') & (\boldsymbol{a}_{23};\sigma_{23}) \end{Bmatrix} \boldsymbol{e}((\boldsymbol{a}_{12};\sigma_{12}),(\boldsymbol{a};\sigma);\boldsymbol{\mu},J)\ . \tag{8.143}$$

Then, by (8.137) and (8.139), we have *Wigner's identity*[6]:

$$\sum_{\boldsymbol{\mu}_{23},J_{23}} (-1)^{2(t_{\boldsymbol{\nu}_1}+u_{\boldsymbol{\nu}_1})} \begin{pmatrix} \boldsymbol{a}_{23} & \boldsymbol{p}_1 & (\breve{\boldsymbol{a}};\breve{\sigma}') \\ \boldsymbol{\mu}_{23},J_{23} & \boldsymbol{\nu}_1,I_1 & \breve{\boldsymbol{\mu}},J \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_2 & \boldsymbol{p}_3 & (\breve{\boldsymbol{a}}_{23};\sigma_{23}) \\ \boldsymbol{\nu}_2,I_2 & \boldsymbol{\nu}_3,I_3 & \breve{\boldsymbol{\mu}}_{23},J_{23} \end{pmatrix}$$
$$= \sum_{\boldsymbol{\mu}_{12},J_{12}} \sum_{\substack{(\boldsymbol{a}_{12};\sigma_{12})\\ \sigma}} (-1)^{|\boldsymbol{a}_{12}|+|\boldsymbol{p}_2|+|\boldsymbol{p}_3|+2(t_{\boldsymbol{\nu}_3}+u_{\boldsymbol{\nu}_3})} \dim Q(\boldsymbol{a}_{12})$$
$$\times \begin{Bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{p}_3 & (\boldsymbol{a};\sigma,\sigma') & (\boldsymbol{a}_{23};\sigma_{23}) \end{Bmatrix} \begin{pmatrix} \boldsymbol{a}_{12} & \boldsymbol{p}_3 & (\breve{\boldsymbol{a}};\sigma) \\ \boldsymbol{\mu}_{12},J_{12} & \boldsymbol{\nu}_3,I_3 & \breve{\boldsymbol{\mu}},J \end{pmatrix}$$
$$\times \begin{pmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{\nu}_1,I_1 & \boldsymbol{\nu}_2,I_2 & \breve{\boldsymbol{\mu}}_{12},J_{12} \end{pmatrix}. \tag{8.144}$$

[6] An equivalent formula is deduced for $SU(2)$ in (WIGNER, 1959) and for a general compact group in (BUTLER, 1975).

And using (8.135), we obtain

$$\begin{aligned}
\begin{Bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & (\breve{\boldsymbol{a}}_{12};\sigma_{12}) \\ \boldsymbol{p}_3 & (\boldsymbol{a};\sigma,\sigma') & (\boldsymbol{a}_{23};\sigma_{23}) \end{Bmatrix} &= \begin{Bmatrix} \breve{\boldsymbol{p}}_2 & \breve{\boldsymbol{p}}_1 & (\boldsymbol{a}_{12};\sigma_{12}) \\ \boldsymbol{a} & (\boldsymbol{p}_3;\sigma,\sigma_{23}) & (\boldsymbol{a}_{23};\sigma') \end{Bmatrix} \\
&= \begin{Bmatrix} \breve{\boldsymbol{p}}_1 & \boldsymbol{a}_{12} & (\breve{\boldsymbol{p}}_2;\sigma_{12}) \\ \boldsymbol{p}_3 & (\boldsymbol{a}_{23};\sigma_{23},\sigma') & (\boldsymbol{a};\sigma) \end{Bmatrix} \\
&= \begin{Bmatrix} \boldsymbol{p}_3 & \breve{\boldsymbol{a}} & (\boldsymbol{a}_{12};\sigma') \\ \boldsymbol{p}_1 & (\breve{\boldsymbol{p}}_2;\sigma_{12},\sigma_{23}) & (\breve{\boldsymbol{a}}_{23};\sigma) \end{Bmatrix} \\
&= \begin{Bmatrix} \breve{\boldsymbol{p}}_1 & \breve{\boldsymbol{p}}_2 & (\boldsymbol{a}_{12};\breve{\sigma}_{12}) \\ \breve{\boldsymbol{p}}_3 & (\breve{\boldsymbol{a}};\breve{\sigma},\breve{\sigma}') & (\breve{\boldsymbol{a}}_{23};\breve{\sigma}_{23}) \end{Bmatrix} .
\end{aligned} \tag{8.145}$$

Thus, using the Wigner coupling and recoupling symbols, we get a more symmetric way of writing the coefficients $\mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I}$, as follows.

Proposition 8.2.10. *The coefficients* $\mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I}$ *in (8.59)-(8.60) are given by*

$$\begin{aligned}
\mathcal{M}[\boldsymbol{p}]^{(\boldsymbol{a}_1;\sigma_1),(\boldsymbol{a}_2;\sigma_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I} &= \sqrt{\dim Q(\boldsymbol{a}_1)\dim Q(\boldsymbol{a}_2)\dim Q(\boldsymbol{a})} \sum_{\sigma'} (-1)^{|\boldsymbol{p}|+2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})} \\
&\times \begin{Bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{p} & (\boldsymbol{p};\sigma,\sigma_1) & (\boldsymbol{p};\sigma_2) \end{Bmatrix} \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \breve{\boldsymbol{\nu}}, I \end{pmatrix} .
\end{aligned} \tag{8.146}$$

Proof. From (8.144), we have

$$\begin{aligned}
&\sum_{\boldsymbol{\mu}_3, J_3} (-1)^{2(t_{\boldsymbol{\nu}_1}+u_{\boldsymbol{\nu}_1})} \begin{pmatrix} \boldsymbol{p} & \boldsymbol{a}_1 & (\breve{\boldsymbol{p}};\breve{\sigma}_1) \\ \boldsymbol{\mu}_3, J_3 & \boldsymbol{\nu}_1, I_1 & \breve{\boldsymbol{\mu}}_2, J_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_2 & \boldsymbol{p} & (\breve{\boldsymbol{p}};\sigma_2) \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\mu}_1, J_1 & \breve{\boldsymbol{\mu}}_3, J_3 \end{pmatrix} \\
&= \sum_{\boldsymbol{\nu}', I'} \sum_{\substack{(\boldsymbol{a}';\sigma') \\ \sigma''}} (-1)^{|\boldsymbol{a}'|+|\boldsymbol{a}_2|+|\boldsymbol{p}|+2(t_{\boldsymbol{\mu}_1}+u_{\boldsymbol{\mu}_1})} \dim Q(\boldsymbol{a}') \\
&\quad \times \begin{Bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}}';\sigma') \\ \boldsymbol{p} & (\boldsymbol{p};\sigma'',\sigma_1) & (\boldsymbol{p};\sigma_2) \end{Bmatrix} \begin{pmatrix} \boldsymbol{a}' & \boldsymbol{p} & (\breve{\boldsymbol{p}};\sigma'') \\ \boldsymbol{\nu}', I' & \boldsymbol{\mu}_1, J_1 & \breve{\boldsymbol{\mu}}_2, J_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}}';\sigma') \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \breve{\boldsymbol{\nu}}', I' \end{pmatrix} .
\end{aligned} \tag{8.147}$$

Multiplying both sides by

$$(-1)^{2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}}+t_{\boldsymbol{\mu}_2}+u_{\boldsymbol{\mu}_2})} \begin{pmatrix} \boldsymbol{p} & \breve{\boldsymbol{p}} & (\boldsymbol{a};\sigma) \\ \boldsymbol{\mu}_1, J_1 & \breve{\boldsymbol{\mu}}_2, J_2 & \boldsymbol{\nu}, I \end{pmatrix} \tag{8.148}$$

and performing a summation over $\boldsymbol{\mu}_1$, J_1, $\boldsymbol{\mu}_2$ and J_2, by (8.135) and (8.48), we obtain

$$\sum_{\substack{\boldsymbol{\mu}_1,J_1\\ \boldsymbol{\mu}_2,J_2\\ \boldsymbol{\mu}_3,J_3}} (-1)^{2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}}+t_{\boldsymbol{\mu}_3}+u_{\boldsymbol{\mu}_3})} \begin{pmatrix} \boldsymbol{p} & \breve{\boldsymbol{p}} & (\boldsymbol{a};\sigma) \\ \boldsymbol{\mu}_1 & \breve{\boldsymbol{\mu}}_2 & \boldsymbol{\nu}, I \end{pmatrix} \begin{pmatrix} \boldsymbol{p} & \boldsymbol{a}_1 & (\breve{\boldsymbol{p}};\breve{\sigma}_1) \\ \boldsymbol{\mu}_3 & \boldsymbol{\nu}_1, I_1 & \breve{\boldsymbol{\mu}}_2 \end{pmatrix}$$
$$\times \begin{pmatrix} \boldsymbol{a}_2 & \boldsymbol{p} & (\breve{\boldsymbol{p}};\sigma_2) \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\mu}_1 & \breve{\boldsymbol{\mu}}_3 \end{pmatrix} \tag{8.149}$$
$$= \sum_{\sigma'} (-1)^{|\boldsymbol{a}|+|\boldsymbol{a}_2|+|\boldsymbol{p}|} \begin{Bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{p} & (\boldsymbol{p};\sigma,\sigma_1) & (\boldsymbol{p};\sigma_2) \end{Bmatrix} \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \breve{\boldsymbol{\nu}}, I \end{pmatrix} .$$

Again by (8.135), we get what we want. □

As a corollary, using (8.52), we get an improvement in Lemma 8.2.3 with the restriction $\delta(I_1, I_2, I) = 1$ to summation (8.59). We will state this together with a more symmetric way to write the product of operators by means of the next definition:

Definition 8.2.7. *The* Wigner product symbol *is the coefficient denoted by the square brackets below:*

$$\begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}, I \end{bmatrix} [\boldsymbol{p}] = \sqrt{\dim Q(\boldsymbol{a}_1) \dim Q(\boldsymbol{a}_2) \dim Q(\boldsymbol{a})}$$
$$\times \sum_{\sigma'} \begin{Bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{p} & (\boldsymbol{p};\sigma,\sigma_1) & (\boldsymbol{p};\sigma_2) \end{Bmatrix} \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & (\breve{\boldsymbol{a}};\sigma') \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}, I \end{pmatrix} . \tag{8.150}$$

From (8.52), we have

$$\begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}, I \end{bmatrix} [\boldsymbol{p}] \neq 0 \quad \Longrightarrow \quad \begin{cases} \nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\breve{\boldsymbol{\nu}}} = 1 \\ \delta(I_1, I_2, I) = 1 \end{cases} . \tag{8.151}$$

Also, symmetries (8.135) and (8.145) imply

$$\begin{aligned} \begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a}_3;\sigma_3) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 \end{bmatrix} [\boldsymbol{p}] &= \begin{bmatrix} (\boldsymbol{a}_3;\sigma_3) & (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) \\ \boldsymbol{\nu}_3, I_3 & \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 \end{bmatrix} [\boldsymbol{p}] \\ &= \begin{bmatrix} (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a}_3;\sigma_3) & (\boldsymbol{a}_1;\sigma_1) \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 & \boldsymbol{\nu}_1, I_1 \end{bmatrix} [\boldsymbol{p}] \\ &= \begin{bmatrix} (\breve{\boldsymbol{a}}_2;\sigma_2) & (\breve{\boldsymbol{a}}_1;\sigma_1) & (\breve{\boldsymbol{a}}_3;\sigma_3) \\ \breve{\boldsymbol{\nu}}_2, I_2 & \breve{\boldsymbol{\nu}}_1, I_1 & \breve{\boldsymbol{\nu}}_3, I_3 \end{bmatrix} [\boldsymbol{p}] \\ &= (-1)^{\sum_{k=1}^3 |\boldsymbol{a}_k|} \begin{bmatrix} (\breve{\boldsymbol{a}}_1;\breve{\sigma}_1) & (\breve{\boldsymbol{a}}_2;\breve{\sigma}_2) & (\breve{\boldsymbol{a}}_3;\breve{\sigma}_3) \\ \breve{\boldsymbol{\nu}}_1, I_1 & \breve{\boldsymbol{\nu}}_2, I_2 & \breve{\boldsymbol{\nu}}_3, I_3 \end{bmatrix} [\breve{\boldsymbol{p}}] \\ &= (-1)^{\sum_{k=1}^3 |\boldsymbol{a}_k|} \begin{bmatrix} (\boldsymbol{a}_2;\breve{\sigma}_2) & (\boldsymbol{a}_1;\breve{\sigma}_1) & (\boldsymbol{a}_3;\breve{\sigma}_3) \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_3, I_3 \end{bmatrix} [\breve{\boldsymbol{p}}] . \end{aligned} \tag{8.152}$$

Corollary 8.2.10.1. *The operator product of elements of a coupled basis of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$ can be decomposed into*

$$\boldsymbol{e}((\boldsymbol{a}_1;\sigma_1);\boldsymbol{\nu}_1,I_1)\boldsymbol{e}((\boldsymbol{a}_2;\sigma_2);\boldsymbol{\nu}_2,I_2) = \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}} (-1)^{|\boldsymbol{p}|+2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})} \begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \check{\boldsymbol{\nu}}, I \end{bmatrix}[\boldsymbol{p}] \times \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) \tag{8.153}$$

where summation over $(\boldsymbol{a};\sigma)$ is restricted to $Q(\boldsymbol{a};\sigma)$ in the Clebsch-Gordan series of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$, and summations over $\boldsymbol{\nu}$ and I can be simplified using (8.151).

In particular, if we calculate the product of any $\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)$ by the identity operator, recalling (8.128), we obtain

$$\begin{aligned} \begin{bmatrix} (0,0) & (\boldsymbol{a};\sigma) & (\boldsymbol{a}';\sigma') \\ (0,0,0),0 & \boldsymbol{\nu}, I & \boldsymbol{\mu}, J \end{bmatrix}[\boldsymbol{p}] &= \begin{bmatrix} (\boldsymbol{a};\sigma) & (0,0) & (\boldsymbol{a}';\sigma') \\ \boldsymbol{\nu}, I & (0,0,0),0 & \boldsymbol{\mu}, J \end{bmatrix}[\boldsymbol{p}] \\ &= \frac{(-1)^{2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})}}{\sqrt{\dim Q(\boldsymbol{p})}} \delta_{\boldsymbol{\nu},\check{\boldsymbol{\mu}}}\delta_{I,J}\delta_{\boldsymbol{a},\boldsymbol{a}'}\delta_{\sigma,\sigma'} \ . \end{aligned} \tag{8.154}$$

8.3 Coadjoint orbits

Being a simple compact Lie group, $SU(3)$ has equivalent adjoint and coadjoint actions. That is, the coadjoint and adjoint orbits are isomorphic[7]. We identify the root diagram of $\mathfrak{su}(3)$ with the Cartan subalgebra generated by iT_3 and iU_3 by making $\alpha_1 \equiv 2i\,T_3$ and $\alpha_2 \equiv 2i\,U_3$. Then, we obtain

$$w_1 \equiv \frac{i}{2}\lambda_3 + \frac{i}{2\sqrt{3}}\lambda_8 = \frac{i}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad w_2 = \frac{i}{\sqrt{3}}\lambda_8 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ . \tag{8.155}$$

It is well known that each orbit intersects the closed positive Weyl chamber

$$\overline{C} = \{x\,w_2 + y\,w_1 : x, y \geq 0\} \tag{8.156}$$

in precisely one single point (see e.g. (BOTT *et al.*, 1980)). Let $\mathcal{O}_{x,y}$ be the orbit passing through

$$\xi_{x,y} = x\,w_2 + y\,w_1 \in \overline{C}\setminus\{\boldsymbol{0}\} \ . \tag{8.157}$$

From the commutation relations, it is clear that, if $x, y > 0$, then the isotropy subgroup of $\xi_{x,0}$ is

$$H := \left\{ \begin{pmatrix} U & 0 \\ 0 & \det(U)^{-1} \end{pmatrix} \ , \ U \in U(2) \right\} \simeq S\big(U(2)\times U(1)\big) \simeq U(2) \ , \tag{8.158}$$

[7] A general discussion of coadjoint orbits of semisimple Lie groups can be found in (BERNATSKA; HOLOD, 2012)).

whereas the isotropy subgroup of $\xi_{0,y}$ is

$$H' := g_0 H g_0^{-1} = g_0 H g_0 = \left\{ \begin{pmatrix} \det(U)^{-1} & 0 \\ 0 & U \end{pmatrix} \; , \quad U \in U(2) \right\} \simeq S\Big(U(1) \times U(2)\Big) \simeq U(2) \; , \tag{8.159}$$

for

$$g_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in SU(3) \; . \tag{8.160}$$

On the other hand, the isotropy subgroup of $\xi_{x,y}$ is the maximal torus[8]

$$T := \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} : \theta_1 + \theta_2 + \theta_3 = 0 \right\} \simeq S\Big(U(1) \times U(1) \times U(1)\Big) \simeq U(1) \times U(1) \; . \tag{8.161}$$

Therefore, we have two types of non trivial (co)adjoint orbits:

$$\mathcal{O}_{x,0} \simeq SU(3)/H \simeq SU(3)/H' \simeq \mathcal{O}_{0,y}$$

and

$$\mathcal{O}_{x,y} \simeq SU(3)/T \simeq \mathcal{O}_{y,x} \; ,$$

for $x, y > 0$.

For a better realization of such orbits, we recall the complex projective space $\mathbb{C}P^2$: the quotient of $\mathbb{C}^3 \backslash \{\mathbf{0}\}$ by the equivalence relation

$$z \sim z' \iff z = a\, z' \; , \quad a \in \mathbb{C}^* \; . \tag{8.162}$$

To construct $\mathbb{C}P^2$ using this equivalence relation[9], we can look only to the unitary vectors of $\mathbb{C}^3$, reducing our analysis to the $SU(3)$-homogeneous space $S^5 = \{z \in \mathbb{C}^3 : ||z|| = 1\}$. Since the point $(0, 0, 1) \in S^5$ has

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \; : \; U \in SU(2) \right\} \subset SU(3) \tag{8.163}$$

as isotropy subgroup, we have[10] $S^5 \simeq SU(3)/SU(2)$. Also note that, $\forall z \in S^5$, $e^{i\theta} z \sim z$. So $\mathbb{C}P^2 \simeq S^5/S^1$ and the isotropy subgroup of the equivalence class $[0 : 0 : 1] \in \mathbb{C}P^2$ is H, i.e., $SU(3)/H \simeq \mathbb{C}P^2$. By similar argument we get $U(2)/(U(1) \times U(1)) \simeq \mathbb{C}P^1$, so $SU(3)/T \simeq \mathcal{E}$, where $\mathcal{E}$ is the total space of a fiber bundle $\pi : \mathcal{E} \to \mathbb{C}P^2$ with fiber $\mathbb{C}P^1$, which we denote by $\mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi)$, that is,

$$\mathcal{E} = \mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi) \; := \; \mathbb{C}P^1 \hookrightarrow \mathcal{E} \xrightarrow{\pi} \mathbb{C}P^2 \; . \tag{8.164}$$

[8] It is a matter of simple calculation to verify that $T' = g_0 T g_0 = T$.
[9] This construction is presented in (ALEXANIAN *et al.*, 2002).
[10] $SU(2) \simeq S^3$ hence $SU(3)$ is a 3-sphere bundle over S^5.

Thus,

$$\mathcal{O}_{x,y} \simeq \begin{cases} \mathbb{C}P^2 \ , & \text{if} \ \ x = 0 \ \ \text{or} \ \ y = 0 \\ \mathcal{E} \ , & \text{if} \ \ x, y > 0 \end{cases} \quad . \tag{8.165}$$

The orbits $\mathcal{O}_{x,y}$ and $\mathcal{O}_{y,x}$ are related by the involution $\iota = -id$ on $\mathfrak{su}(3)$. Indeed, $\iota \circ Ad_g = Ad_g \circ \iota$ trivially holds for every $g \in SU(3)$ and

$$\iota(x\, w_2 + y\, w_1) = -x\, w_2 - y\, w_1 = Ad_{g_0}(y\, w_2 + x\, w_1) \ , \tag{8.166}$$

so $\iota(\mathcal{O}_{x,y}) = \mathcal{O}_{y,x}$. Thus, ι is an involution on $\mathcal{O}_{x,x}$.

Let $\mathbf{x}_0 = [0 : 0 : 1] \in \mathbb{C}P^2$, whose isotropy subgroup is H, so that the isotropy subgroup of $g_0\mathbf{x}_0 = [1 : 0 : 0]$ is H', and let $\mathbf{z}_0 \in \pi^{-1}(\mathbf{x}_0) \subset \mathcal{E}$ be a point with T as isotropy subgroup. Then consider the equivariant diffeomorphisms

$$\begin{aligned} \psi_{x,0} : \mathcal{O}_{x,0} \to \mathbb{C}P^2 : Ad_g\xi_{x,0} \mapsto g\, \mathbf{x}_0 \ , \\ \psi_{0,y} : \mathcal{O}_{0,y} \to \mathbb{C}P^2 : Ad_g\xi_{0,y} \mapsto g g_0\, \mathbf{x}_0 \ , \\ \psi_{x,y} : \mathcal{O}_{x,y} \to \mathcal{E} : Ad_g\xi_{x,y} \mapsto g\, \mathbf{z}_0 \ , \end{aligned} \tag{8.167}$$

for $x, y > 0$ still holding. Therefore,

$$\psi_{x,0} \circ \iota \circ \psi_{0,x}^{-1} : \mathbb{C}P^2 \to \mathbb{C}P^2 \tag{8.168}$$

is the identity map, and

$$\alpha := \psi_{x,y} \circ \iota \circ \psi_{y,x}^{-1} : \mathcal{E} \to \mathcal{E} \tag{8.169}$$

is an $SU(3)$-equivariant involution. Of course, there is an equivalent involution on each $\mathcal{O}_{x,y} \simeq \mathcal{E}$, namely

$$\alpha_{x,y} := \psi_{x,y}^{-1} \circ \psi_{y,x} \circ \iota : \mathcal{O}_{x,y} \to \mathcal{O}_{x,y} \ , \tag{8.170}$$

which reduces to ι for $x = y$.

Every (co)adjoint orbit of $SU(3)$ is a symplectic manifold, that is, every $\mathcal{O}_{x,y}$ carries a $SU(3)$-invariant symplectic form, the so-called Kirillov-Kostant-Souriau form[11], cf. eg. (KIRILLOV, 2004, Ch. I, sec. 2)). Furthermore, the $SU(3)$-invariant symplectic form on $\mathcal{O}_{x,y}$ induces a normalized left invariant integral on the orbit $\mathcal{O}_{x,y}$ such that any other left invariant integral differs from it by a scalar factor. Then, we can fix this factor for $\mathcal{O}_{x,y}$ so that the lift $\tilde{f} \in C(SU(3))$ of any $f \in C(\mathcal{O}_{x,y})$ satisfies

$$\int_{SU(3)} \tilde{f}(g)\, dg = \int_{\mathcal{O}_{x,y}} f(x)\, dx \tag{8.171}$$

for the Haar integral on $SU(3)$ (see (FOLLAND, 2016, (2.49) Theorem)). With no danger of confusion, we may denote the $SU(3)$-invariant inner product in $L^2(\mathcal{O}_{x,y})$ with respect to such integral simply by $\langle\,|\,\rangle$, that is, for any $f_1, f_2 \in L^2(\mathcal{O}_{x,y})$,

$$\langle f_1 | f_2 \rangle = \int_{\mathcal{O}_{x,y}} \overline{f_1}(x) f_2(x)\, dx \ . \tag{8.172}$$

[11] Actually first identified by Sophus Lie.

Throughout this text, we consider $\mathbb{C}P^2$ and $\mathcal{E}$ equipped with the aforementioned symplectic forms and normalized left invariant integrals.

In what follows, we shall identify as a classical quark system, the Poisson algebra of smooth functions on $\mathcal{O}$. When $\mathcal{O} = \mathbb{C}P^2$, we shall refer to a *classical pure-quark system*. When $\mathcal{O} = \mathcal{E}$, we shall refer to a *generic classical quark system*.

CHAPTER

9

PURE-QUARK SYSTEMS

We start to properly attack our main problem by focusing on the simpler possible phase space for a classical quark system: $\mathbb{C}P^2$. First, we describe the set of harmonic functions on $\mathbb{C}P^2$, which imposes a restriction on the classes of irreducible representations of $SU(3)$ with possible correspondences to smooth functions on $\mathbb{C}P^2$. Then, we proceed to describe the relevant $SU(3)$-representations for this case, as quantum quark systems. To finish, we work out the characterization of all correspondences from such quantum quark systems to the classical quark system of interest and describe the induced twisted products of symbols. The construction and characterization of symbol correspondences in this section is very close to what is done for spin systems in (RIOS; STRAUME, 2014). Accordingly, proofs of some propositions are identical to the $SU(2)$ case. The quantum and classical systems in correspondence, in this chapter, are both called "pure-quark system" and this name is explained right after Definition 9.2.1.

9.1 Classical pure-quark system

Definition 9.1.1. *The* classical pure-quark system *consists of* $\mathbb{C}P^2$ *equipped with its* $SU(3)$*-invariant symplectic form, together with its Poisson algebra on* $C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*.*

Since $\mathbb{C}P^2 \simeq SU(3)/H$, where $H \simeq U(2)$, cf. (8.158), we look for representations $Q(p,q)$ with weights satisfying $t = u = I = 0$ (cf. (8.21)-(8.22)) to determine the harmonic functions on $\mathbb{C}P^2$.

Proposition 9.1.1. *The representations of* $SU(3)$ *with non null vectors fixed by* $H \simeq U(2)$ *are the representations* $Q(n,n)$*. The space fixed by* H *is spanned by* $\boldsymbol{e}((n,n);(n,n,n),0)$*.*

Proof. From $t = u = 0$, we get $\nu_1 = \nu_2 = \nu_3 = \nu$. From the constraints (8.21), we get $r_+ = r_-$, which implies $r_+ = q = \nu = r_-$ that, in turn, implies $\nu = p$. Thus, putting

$n = p = q$, we finish the proof. $\square$

Definition 9.1.2. *The* $\mathbb{C}P^2$ harmonics *are the functions* $X^n_{\boldsymbol{\nu},I} : \mathbb{C}P^2 \to \mathbb{C}$, *such that*

$$X^n_{\boldsymbol{\nu},I}(g\mathbf{x}_0) = (n+1)^{3/2}\overline{D^{(n,n)}_{\boldsymbol{\nu}I,(n,n,n)0}(g)} \ , \tag{9.1}$$

for $\mathbf{x}_0 = [0:0:1] \in \mathbb{C}P^2$, $g \in SU(3)$ *and* $D^{(n,n)}_{\boldsymbol{\nu}I,(n,n,n)0}$ *a Wigner D-function as in Definition 8.1.3.*

The factor $(n+1)^{3/2}$ in the definition of $\mathbb{C}P^2$ harmonics is the square root of the dimension of the representation $Q(n,n)$ and is used to ensure normalization according to Schur's Orthogonality Relations, so that we have

$$\left\langle X^n_{\boldsymbol{\nu},I} \middle| X^m_{\boldsymbol{\mu},J} \right\rangle = \delta_{n,m}\delta_{\boldsymbol{\nu},\boldsymbol{\mu}}\delta_{I,J} \tag{9.2}$$

with respect to the inner product described in section 8.3, cf. (8.171)-(8.172).

We note that

$$X^0_{(0,0,0),0} \equiv 1 \tag{9.3}$$

and, cf. (8.38),

$$\overline{X^n_{\boldsymbol{\nu},I}} = (-1)^{2(t+u)} X^n_{\check{\boldsymbol{\nu}},I} \ , \ \text{for} \ \ \Delta^{2n}_{\boldsymbol{\nu},\check{\boldsymbol{\nu}}} = 1 \ . \tag{9.4}$$

Remark 9.1.1. *Fixed* $x > 0$, *the diffeomorphism* $\psi_{x,0}$ *can be used to carry* $\mathbb{C}P^2$ *harmonics to* $\mathcal{O}_{x,0}$ *by means of the composition* $X^n_{\boldsymbol{\nu},I} \circ \psi_{x,0}$, *cf. (8.167). Equivalently,* $X^n_{\boldsymbol{\nu},I} \circ \psi_{0,x}$ *are the* $\mathbb{C}P^2$ *harmonics carried to the orbit* $\mathcal{O}_{0,x}$. *Consequently, we have a set of harmonic functions on* $\mathcal{O}_{x,0}$ *related to a set of harmonic functions on* $\mathcal{O}_{0,x}$ *by the map* ι, *cf. (8.168).*

Next, we derive the decomposition of the pointwise product of two $\mathbb{C}P^2$ harmonics from Lemma 8.2.2.

Proposition 9.1.2. *The decomposition of the pointwise product of* $\mathbb{C}P^2$ *harmonics is given by*

$$X^{n_1}_{\boldsymbol{\nu}_1,I_1} X^{n_2}_{\boldsymbol{\nu}_2,I_2} = \sum_{\substack{(n,n;\sigma)\\ \boldsymbol{\nu},I}} \left(\frac{(n_1+1)(n_2+1)}{n+1}\right)^{3/2} C^{(n_1,n_1),(n_2,n_2),(n,n;\sigma)}_{\boldsymbol{\nu}_1 I_1,\ \ \boldsymbol{\nu}_2 I_2,\ \ \boldsymbol{\nu} I} C^{(n_1,n_1),(n_2,n_2),(n,n;\sigma)}_{\boldsymbol{n}_1 0,\ \ \boldsymbol{n}_2 0,\ \ \boldsymbol{n} 0} X^n_{\boldsymbol{\nu},I} \ , \tag{9.5}$$

where $\boldsymbol{n}_j = (n_j, n_j, n_j)$ *and* $\boldsymbol{n} = (n,n,n)$, *and summation is restricted to* $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = 1$, $\delta(I_1, I_2, I) = 1$ *and* $Q(n,n;\sigma)$ *in the Clebsch-Gordan series of* $Q(n_1,n_1) \otimes Q(n_2,n_2)$; *in particular,* $|n_1 - n_2| \leq n \leq n_1 + n_2$.

Proof. With a little abuse of notation, we write

$$X^{n_j}_{\boldsymbol{\nu}_j,I_j} = (n_j+1)^{3/2}\overline{D^{(n_j,n_j)}_{\boldsymbol{\nu}_j I_j,\boldsymbol{n}_j 0}} \ , \tag{9.6}$$

and apply Lemma 8.2.2 to get

$$X^{n_1}_{\boldsymbol{\nu}_1,I_1} X^{n_2}_{\boldsymbol{\nu}_2,I_2} = \sum_{(\boldsymbol{a};\sigma)} \sum_{\substack{\boldsymbol{\nu},I \\ \boldsymbol{\mu},0}} ((n_1+1)(n_2+1))^{3/2} \, C^{(n_1,n_1),(n_2,n_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1, \quad \boldsymbol{\nu}_2 I_2, \quad \boldsymbol{\nu} I} \, C^{(n_1,n_1),(n_2,n_2),(\boldsymbol{a};\sigma)}_{\boldsymbol{n}_1 0, \quad \boldsymbol{n}_2 0, \quad \boldsymbol{\mu} 0} \, \overline{D^{\boldsymbol{a}}_{\boldsymbol{\nu} I,\boldsymbol{\mu} 0}} \ , \tag{9.7}$$

where $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = \nabla_{\boldsymbol{n}_1+\boldsymbol{n}_2,\boldsymbol{\mu}} = 1$ and $\delta(I_1, I_2, I) = 1$, so $\boldsymbol{\mu} = (\mu,\mu,\mu)$. But $\boldsymbol{e}(\boldsymbol{a};(\mu,\mu,\mu),0)$ only exists if $\boldsymbol{a} = (\mu,\mu)$. Thus, we set $\boldsymbol{a} = (n,n)$ and $\boldsymbol{\mu} = \boldsymbol{n} = (n,n,n)$. The restriction over n follows from Theorem 8.2.1. □

Remark 9.1.2. *The fact that $\mathbb{C}P^2$ is a symplectic manifold plays no part in the decomposition of the pointwise product of $\mathbb{C}P^2$ harmonics. Accordingly, the next step in the study of the classical pure-quark system amounts to decomposing the Poisson bracket of $\mathbb{C}P^2$ harmonics. However, this problem is considerably harder and is deferred to a later study.*

9.2 Quantum pure-quark systems

From Proposition 9.1.1, we look for representations $Q(p,q)$ such that the tensor product $Q(p,q) \otimes Q(q,p)$ splits into representations of the form $Q(n,n)$, without multiplicities. From Corollary 8.2.1.1, we have that $Q(p,0)$ and $Q(0,p)$ are the only ones that satisfy these requirements. These are special cases of quantum quark systems, for which we make the following definition:

Definition 9.2.1. *Let*[1] $\boldsymbol{p} \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$ *with* $|\boldsymbol{p}| = p$. *A* quantum pure-quark system *is a complex Hilbert space* $\mathcal{H}_{\boldsymbol{p}} \simeq \mathbb{C}^d$, *where*

$$d = \frac{(p+1)(p+2)}{2} \ ,$$

with an irreducible unitary $SU(3)$-representation of class $Q(\boldsymbol{p})$ together with its operator algebra $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$.

In the context of quark systems, such spaces arise in description of systems of p identical quarks only (or p identical antiquarks only), as we shall see below. We call them pure-quark systems because the number of antiquarks (or quarks) is zero for such systems.

An explicit way to construct such representations is by the so-called tensor method. Consider the basic representation ρ_1 of class $Q(1,0)$ on $\mathcal{H}_{1,0} \simeq \mathbb{C}^3$, and let the canonical basis $\{e_1, e_2, e_3\}$ matches a GT basis, where each vector correspond to the respective weight on the diagram of Figure 3.

Then, the tensor product space $\mathcal{H} = \mathcal{H}_{1,0} \otimes ... \otimes \mathcal{H}_{1,0}$ with p copies of $\mathcal{H}_{1,0}$ carries a $SU(3)$-representation ρ given by

$$\rho(g)\, w_1 \otimes ... \otimes w_p = (\rho_1(g) w_1) \otimes ... \otimes (\rho_1(g) w_p) \ . \tag{9.8}$$

[1] We are ignoring the trivial representation $Q(0,0)$.

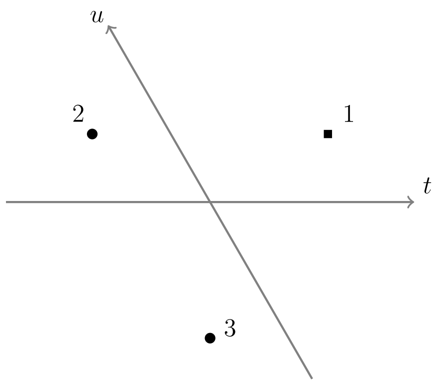


Figure 3 – Each e_k corresponds to the k-th weight.

Now, let $\mathcal{H}_{p,0} = Sym^p(\mathcal{H}_{1,0}) \subset \mathcal{H}$ be the subspace of totally symmetric tensors, that is,

$$\sum_{i_1,\dots,i_p=1}^{3} c_{i_1,\dots,i_p}\, e_{i_1} \otimes \dots \otimes e_{i_p} \in \mathcal{H}_p \tag{9.9}$$

if and only if $c_{i_{f(1)},\dots,i_{f(p)}} = c_{i_1,\dots,i_p}$ for every permutation $f \in S_p$. It is immediate that $\mathcal{H}_{p,0}$ is an invariant subspace.

We can get a basis for $\mathcal{H}_{p,0}$ by means of symmetrization. For $e_{i_1} \otimes \dots \otimes e_{i_p} \in \mathcal{H}$, let j, k and l be the numbers of occurrence of index 1, 2 and 3, respectively, and take

$$e_{j,k,l} = \binom{p}{j,k,l}^{-1/2} \sum_{f \in S_p} e_{i_{f(1)}} \otimes \dots \otimes e_{i_{f(p)}} \in \mathcal{H}_{p,0}\ , \tag{9.10}$$

where

$$\binom{p}{j,k,l} = \frac{p!}{j!\,k!\,l!} \tag{9.11}$$

is the respective coefficient of the trinomial expansion of order p. The set $\{e_{j,k,l} : j+k+l = p\}$ is an orthonormal basis of $\mathcal{H}_{p,0}$ considering the inner product induced by $\mathcal{H}_{1,0}$ on $\mathcal{H}$. Starting with the element $e_{p,0,0}$, we can obtain the basis $\{e_{j,k,l} : j+k+l = p\}$ by recursively applying the ladder operators T_- and U_- and normalizing the result. As can be seen from the diagram above, $e_{j,k,l} = \mu_{j,k,l}(U_-)^l(T_-)^{k+l}e_{p,0,0}$, where $\mu_{j,k,l} > 0$. Since $\dim \mathcal{H}_{p,0} = (p+1)(p+2)/2$ and $e_{p,0,0}$ is a highest weight vector[2] with eigenvalues $p/2$ for T_3 and 0 for U_3, we conclude that the $SU(3)$-representation on $\mathcal{H}_{p,0}$ is an irreducible representation of class $Q(p,0)$. In particular, the map

$$\mathcal{H}_1 \to \mathcal{H}_{p,0} : w = (z_1,z_2,z_3) \mapsto w \otimes \dots \otimes w = \sum_{j+k+l=1} \sqrt{\binom{p}{j,k,l}}\, z_1^j z_2^k z_3^l\, e_{j,k,l} \tag{9.12}$$

is equivariant.

[2] Also, $e_{0,0,p}$ is a lowest weight vector.

An equivalent procedure starting with $\mathcal{H}_{0,1} = \mathcal{H}^*_{1,0}$ gives us the space $\mathcal{H}_{0,p} = \mathcal{H}^*_{p,0}$ with a representation of class $Q(0,p)$ and the equivariant map

$$\mathcal{H}_{0,1} \to \mathcal{H}_{0,p} : w^* = (z_1, z_2, z_3) \mapsto w^* \otimes ... \otimes w^* = \sum_{j+k+l=1} \sqrt{\binom{p}{j,k,l}} z_1^j z_2^k z_3^l \, \check{e}_{j,k,l} \ , \quad (9.13)$$

where $\{\check{e}_{j,k,l} : j+k+l = p\}$ is the basis induced by[3] $\{\check{e}_1 = -e_3^*, \check{e}_2 = e_2^*, \check{e}_3 = -e_1^*\}$ in the same way that $\{e_{j,k,l} : j+k+l = p\}$ is induced by $\{e_1, e_2, e_3\}$, cf. Definition 8.1.2.

In Physics, the space of colors (resp. flavors) of a quark is precisely the representation $Q(1,0)$, with $e_1 \equiv$ *red* (resp. *up quark*), $e_2 \equiv$ *blue* (resp. *down quark*) and $e_3 \equiv$ *green* (resp. *strange quark*). Thus, $Q(p,0)$ is the totally symmetric part of a system of p quarks. Analogously, $Q(0,q)$ is the totally symmetric part of a system of q antiquarks since the representation $Q(0,1)$ describes an antiquark, $\check{e}_1 \equiv$ *antigreen* (*strange antiquark*), $\check{e}_2 \equiv$ antiblue (*down antiquark*) and $\check{e}_3 \equiv$ antired (*up antiquark*).

Remark 9.2.1. *Because the operator space of $Q(p,0)$, or $Q(0,p)$, have the maximal isotropy subgroup $H \simeq U(2)$, these representations are also called* symmetric representations*, in the literature. So, pure-quark systems could also be referred to as symmetric quark systems.*

Remark 9.2.2. *There is another interpretation of the representation $Q(p,0)$ as a quantum system that matches our classical phase space $\mathbb{C}P^2$ in a enlightening way. A three-dimensional isotropic harmonic oscillator is a quantum system which is governed by a Hamiltonian of the form*

$$H = \sum_{i=1}^{3} \left(\frac{1}{2m} P_i^2 + \frac{1}{2} m\omega^2 X_i^2 \right) \ , \quad (9.14)$$

where P_i and X_i are the component operators of momentum and position, for some positive parameters m and ω. It has degenerate energy levels[4]

$$E = \left(n_1 + n_2 + n_3 + \frac{3}{2} \right) \omega \quad (9.15)$$

with $SU(3)$-symmetry given by representations $Q(p,0)$ for $p = n_1 + n_2 + n_3$ (MURGAN; ZENDER, 2018). Thus, a quantum pure-quark system $\mathcal{H}_{p,0}$ can be interpreted as the solution of a quantum isotropic harmonic oscillator of energy $E_p = p + 3/2$, by setting $\omega = 1$.

For the classical three-dimensional isotropic harmonic oscillator of same parameters m and ω, the phase space is $T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ and the Hamiltonian is

$$h = \sum_{i=1}^{3} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega x_i^2 \right) \ , \quad (9.16)$$

[3] Again, $\check{e}_{p,0,0}$ is a highest weight vector and $\check{e}_{0,0,p}$ is a lowest weight vector.
[4] We set $\hbar = 1$ during all this work.

where p_i and x_i are the components of momentum and position, in complete analogy to (9.14). By appropriately rescaling, a region of given fixed energy is identified with $S^5 \subset \mathbb{R}^6$. The solution passing through a point of S^5 is the orbit of the point under an $SO(2)$-action, where $SO(2)$ acts via rotations on each $\mathbb{R}^2$ determined by the pairs (x_i, p_i). But the action of $SO(2)$ on $\mathbb{R}^2$ is equivalent to the action of $U(1)$ on $\mathbb{C}$, so the set of solutions of a classical isotropic three-dimensional harmonic oscillator is identified with $S^5/S^1 = \mathbb{C}P^2$.

If $Q(\boldsymbol{p}) = Q(p, 0)$, the Gelfand-Tsetlin pattern (8.21) is reduced to

$$\begin{gathered} 0 \leq r \leq p \ , \\ 0 \leq \nu_1 \leq r \ , \ \nu_2 = r - \nu_1 \ , \ \nu_3 = p - r \ , \\ I = \frac{r}{2} \ . \end{gathered} \tag{9.17}$$

A similar simplification can be applied to $Q(\boldsymbol{p}) = Q(0, p)$:

$$\begin{gathered} 0 \leq r \leq p \ , \\ r \leq \nu_1 \leq p \ , \ \nu_2 = p + r - \nu_1 \ , \ \nu_3 = p - r \ , \\ I = \frac{p - r}{2} \ . \end{gathered} \tag{9.18}$$

In both cases, the isospin I is determined by ν_3, so we simplify the notation for $\boldsymbol{p} \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$ as

$$\boldsymbol{e}(\boldsymbol{p}; \boldsymbol{\nu}) := \boldsymbol{e}(\boldsymbol{p}; \boldsymbol{\nu}, I) \ , \ \check{\boldsymbol{e}}(\check{\boldsymbol{p}}; \boldsymbol{\nu}) := \check{\boldsymbol{e}}(\check{\boldsymbol{p}}; \boldsymbol{\nu}, I) \ . \tag{9.19}$$

To clear even more the notation, we will denote the elements of a coupled basis of $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ that lies in the $Q(n, n)$-invariant subspace by $\boldsymbol{e}(n; \boldsymbol{\nu}, I)$. Consequently, the notation for the Clebsch-Gordan coefficients can be simplified to

$$C^{\boldsymbol{p},\ \check{\boldsymbol{p}},\ n}_{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu} I} := C^{\boldsymbol{p},\quad \check{\boldsymbol{p}},\quad n}_{\boldsymbol{\nu}_1 I_1, \boldsymbol{\nu}_2 I_2, \boldsymbol{\nu} I} := C^{\boldsymbol{p},\quad \check{\boldsymbol{p}},\quad (n,n)}_{\boldsymbol{\nu}_1 I_1, \boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I} \ . \tag{9.20}$$

Applying the same simplification to the Wigner product symbol, expressions (8.151) and (8.152) now read as

$$\begin{bmatrix} n_1 & n_2 & n_3 \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 \end{bmatrix} [\boldsymbol{p}] \neq 0 \quad \Longrightarrow \quad \begin{cases} \nabla_{\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2, \check{\boldsymbol{\nu}}_3} = 1 \\ \delta(I_1, I_2, I_3) = 1 \end{cases} \ , \tag{9.21}$$

$$
\begin{aligned}
\begin{bmatrix} n_1 & n_2 & n_3 \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 \end{bmatrix}[\boldsymbol{p}] &= \begin{bmatrix} n_3 & n_1 & n_2 \\ \boldsymbol{\nu}_3, I_3 & \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 \end{bmatrix}[\boldsymbol{p}] \\
&= \begin{bmatrix} n_2 & n_3 & n_1 \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 & \boldsymbol{\nu}_1, I_1 \end{bmatrix}[\boldsymbol{p}] \\
&= \begin{bmatrix} n_2 & n_1 & n_3 \\ \check{\boldsymbol{\nu}}_2, I_2 & \check{\boldsymbol{\nu}}_1, I_1 & \check{\boldsymbol{\nu}}_3, I_3 \end{bmatrix}[\boldsymbol{p}] \qquad (9.22) \\
&= \begin{bmatrix} n_1 & n_2 & n_3 \\ \check{\boldsymbol{\nu}}_1, I_1 & \check{\boldsymbol{\nu}}_2, I_2 & \check{\boldsymbol{\nu}}_3, I_3 \end{bmatrix}[\check{\boldsymbol{p}}] \\
&= \begin{bmatrix} n_2 & n_1 & n_3 \\ \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_3, I_3 \end{bmatrix}[\check{\boldsymbol{p}}] \ .
\end{aligned}
$$

Therefore, Corollary 8.2.10.1 takes the special form:

Proposition 9.2.1. *For a quantum pure-quark system $\mathcal{H}_{\boldsymbol{p}}$, $|\boldsymbol{p}| = p$, the product of elements of a coupled basis of the space of operators $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ decomposes in the form*

$$
\boldsymbol{e}(n_1; \boldsymbol{\nu}_1, I_1)\boldsymbol{e}(n_2; \boldsymbol{\nu}_2, I_2) = \sum_{n=0}^{p} \sum_{\boldsymbol{\nu}, I} (-1)^{p+2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})} \begin{bmatrix} n_1 & n_2 & n_3 \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \check{\boldsymbol{\nu}}, I \end{bmatrix}[\boldsymbol{p}]\, \boldsymbol{e}(n; \boldsymbol{\nu}, I) \quad (9.23)
$$

for $0 \le n_1, n_2 \le p$, where summations over $\boldsymbol{\nu}$ and I can be restricted to $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2, \boldsymbol{\nu}} = 1$ and $\delta(I_1, I_2, I) = 1$ due to (9.21).

We also identify the operator algebra $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ with the matrix algebra $M_{\mathbb{C}}(d)$ by means of an uncoupled basis of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$. So let $\boldsymbol{\nu}$ and $\check{\boldsymbol{\nu}}$ be such that $\Delta^{|\boldsymbol{p}|}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = 1$, the operator $\boldsymbol{e}(\boldsymbol{p}, \boldsymbol{\nu}) \otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}}, \check{\boldsymbol{\nu}})$ is a diagonal matrix and its decomposition in the coupled basis can be written as

$$
\boldsymbol{e}(\boldsymbol{p}; \boldsymbol{\nu}) \otimes \check{\boldsymbol{e}}(\check{\boldsymbol{p}}; \check{\boldsymbol{\nu}}) = \sum_{n=0}^{p} \sum_{I=0}^{n} C^{\boldsymbol{p}, \check{\boldsymbol{p}}, n}_{\boldsymbol{\nu}, \check{\boldsymbol{\nu}}, \boldsymbol{n} I} \boldsymbol{e}(n; (n, n, n), I) \ . \quad (9.24)
$$

That is, any diagonal matrix is a linear combination of $\{\boldsymbol{e}(n, (n, n, n), I)\}$. Since the cardinality of this set is $(p+1)(p+2)/2$, it is the set of diagonal matrices of a coupled basis. Clebsch-Gordan coefficients being all real implies that such matrices are also real.

9.3 Symbol correspondences for pure-quark systems

Let $\boldsymbol{p} \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$ with $|\boldsymbol{p}| = p$.

Definition 9.3.1. *A* symbol correspondence for a pure-quark system $(\mathcal{H}_{\boldsymbol{p}}, Q(\boldsymbol{p}))$, *also referred to simply as a* symbol correspondence *or just as a* correspondence, *is an injective linear map $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^{\infty}_{\mathbb{C}}(\mathbb{C}P^2) : A \mapsto W_A$ that satisfies, for any $A \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$,*

i) Equivariance: $\forall g \in SU(3)$, $W_{A^g} = (W_A)^g$;

ii) Reality: $W_{A^\dagger} = \overline{W_A}$ *;*

iii) Normalization: $\int_{\mathbb{C}P^2} W_A(\mathbf{x})d\mathbf{x} = \dfrac{2}{(p+1)(p+2)}\,\mathrm{tr}(A)$.

Remark 9.3.1. *If one replace $\mathbb{C}P^2$ on the definition of a symbol correspondence for a pure quark system by the orbit $\mathcal{O}_{x,0}$ or $\mathcal{O}_{0,x}$, for $x>0$, the definition remains essentially the same. Given any symbol correspondence $W:\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$, the map $W':\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{O}_{x,0})$, $W'_A = W_A \circ \psi_{x,0}$, satisfies the desired properties and $W_A = W'_A \circ \psi_{x,0}^{-1}$. Using $\psi_{0,x}$, we get symbol correspondences with codomain $C^\infty_{\mathbb{C}}(\mathcal{O}_{0,x})$. Conveniently, one can define symbol correspondences as maps $W:\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{O}_{\boldsymbol{p}})$.*

A symbol correspondence is an injective equivariant linear map, thus Schur's Lemma leads us to a simple characterization of these correspondences for a pure-quark system:

Proposition 9.3.1. *A linear map $W:\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ is a symbol correspondence if and only if it maps*

$$W:\quad \sqrt{\frac{(p+1)(p+2)}{2}}\boldsymbol{e}(n;\boldsymbol{\nu},I) \quad\mapsto\quad c_n X^n_{\boldsymbol{\nu},I} \tag{9.25}$$

for $(c_1,...,c_p)\in(\mathbb{R}^*)^p$ and $c_0 = (-1)^p$.

Proof. By the Schur's Lemma, W is equivariant and injective if and only if it provides a mapping of the form (9.25) with $c_n \neq 0$ for every $n\in\{0,...,p\}$. Since $\boldsymbol{e}^\dagger(n,\boldsymbol{\nu},I) = (-1)^{2(t+u)}\boldsymbol{e}(n,\check{\boldsymbol{\nu}},I)$ and $\overline{X^n_{\boldsymbol{\nu},I}} = (-1)^{2(t+u)}X^n_{\check{\boldsymbol{\nu}},I}$, reality holds if and only if the constants c_n are all real numbers. To finish, we have

$$(-1)^p\sqrt{\frac{(p+1)(p+2)}{2}}\boldsymbol{e}(0;(0,0,0),0) = \mathbb{1} \tag{9.26}$$

and $X^0_{\mathbf{0},0} = 1$, then the normalization condition is satisfied if and only if $c_0 = (-1)^p$. □

Corollary 9.3.1.1. *The moduli space of correspondences for a pure-quark system $\mathcal{H}_{\boldsymbol{p}}$ is $(\mathbb{R}^*)^p$.*

But there is another way to construct symbol correspondences. Again, let $\mathbf{x}_0 = [0:0:1]\in\mathbb{C}P^2$. Given an operator $K\in\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ fixed by H, consider the operator-valued function $\mathbb{C}P^2 \to \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) : \mathbf{x}\mapsto K(\mathbf{x}) = K^g$, where $g\in SU(3)$ is such that $\mathbf{x} = g\mathbf{x}_0$.

Proposition 9.3.2. *A map $W:\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2) : A\mapsto W_A$ is a symbol correspondence satisfying (9.25) if and only if*

$$W_A(\mathbf{x}) = \mathrm{tr}(AK(\mathbf{x}))\ , \tag{9.27}$$

that is,

$$W_A(g\mathbf{x}_0) = \mathrm{tr}(AK^g)\ , \tag{9.28}$$

for $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ of the form

$$K = \frac{2}{(p+1)(p+2)}\mathbb{1} + \sum_{n=1}^{p} c_n \sqrt{\frac{2(n+1)^3}{(p+1)(p+2)}}\boldsymbol{e}(n;(n,n,n),0) \ . \tag{9.29}$$

In particular, K is a diagonal matrix with real entries and unitary trace.

Proof. Suppose W is a symbol correspondence given by (9.25). The map $A \mapsto W_A(\mathbf{x}_0)$ is a linear functional, then there exists $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ such that (9.28) holds for the identity of $SU(3)$. From equivariance, we have $W_A(g^{-1}\mathbf{x}_0) = (W_A)^g(\mathbf{x}_0) = W_{A^g}(\mathbf{x}_0) = \mathrm{tr}(A^g K) = \mathrm{tr}(AK^{g^{-1}})$ for very $g \in SU(3)$. Since $\mathbf{x}_0$ is fixed by H, we have $\mathrm{tr}(AK^g) = \mathrm{tr}(AK)$ for every $g \in H$ and every operator A. Thus, we can write, cf. Proposition 9.1.1,

$$K = \sum_{n=0}^{p} k_n \boldsymbol{e}(n;(n,n,n),0) \ , \tag{9.30}$$

that is, K is a diagonal matrix. Taking $A = \boldsymbol{e}(n;\boldsymbol{\nu},I) = (-1)^{2(t+u)}\boldsymbol{e}^\dagger(n;\check{\boldsymbol{\nu}},I)$, we have

$$W_A(g\mathbf{x}_0) = \mathrm{tr}(AK^g) = k_n(-1)^{2(t+u)} D^{(n,n)}_{\check{\boldsymbol{\nu}}I,(n,n,n)0}(g) = k_n \overline{D^{(n,n)}_{\boldsymbol{\nu}I,(n,n,n)0}} \ , \tag{9.31}$$

cf (8.38). Then,

$$W : \boldsymbol{e}(n,\boldsymbol{\nu},I) \mapsto \frac{k_n}{(n+1)^{3/2}} X^n_{\boldsymbol{\nu},I} \ . \tag{9.32}$$

It follows from Proposition 9.3.1 that

$$k_n = c_n \sqrt{\frac{2(n+1)^3}{(p+1)(p+2)}} \ . \tag{9.33}$$

On the other hand, for K given by (9.29), equations (9.30)-(9.32) imply that (9.28) defines a symbol correspondence given by (9.25). □

These results are completely analogous to the case of spin systems, cf. (RIOS; STRAUME, 2014), so we come with the next definition.

Definition 9.3.2. *An* operator kernel *$K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ is an operator that induces a symbol correspondence via (9.28). That is, K is given by (9.29) with non-zero real numbers (c_n) called* characteristic numbers *of both the operator kernel and the symbol correspondence.*

If $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ is an operator kernel, it is diagonal with real entries, thus it is a linear combination of projections of the form

$$K = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \Pi_{\boldsymbol{\nu}} \ , \tag{9.34}$$

for real coefficients $a_{\boldsymbol{\nu}}$. We can separate the summation into different values of isospin:

$$K = \sum_{j=0}^{p} K_j \ , \quad K_j = \sum_{\boldsymbol{\nu} \in j/2} a_{\boldsymbol{\nu}} \Pi_{\boldsymbol{\nu}} \ , \tag{9.35}$$

where $\boldsymbol{\nu} \in j/2$ means the weight $\boldsymbol{\nu}$ has isospin $j/2$, cf. (9.17)-(9.18).

Proposition 9.3.3. *If $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ is an operator kernel, then*

$$K = \sum_{j=0}^{p} a_j \sum_{\boldsymbol{\nu} \in j/2} \Pi_{\boldsymbol{\nu}} \tag{9.36}$$

where the coefficients a_j are real numbers satisfying

$$\sum_{j=0}^{p} a_j (j+1) = 1 \ . \tag{9.37}$$

Proof. Every operator kernel is fixed by H of (8.158), so K must be fixed also by the $SU(2)$ of (8.163). Decomposing K as in (9.35), we have that a component K_j is an operator on a representation $j/2$ of $SU(2)$, therefore it must commute with T_3 and $T_\pm$, which implies each K_j is a multiple of the identity operator on the subspace spanned by the states with same isospin $j/2$, that is,

$$K_j = a_j \sum_{\boldsymbol{\nu} \in j/2} \Pi_{\boldsymbol{\nu}} \ , \tag{9.38}$$

where a_j is real. To finish, $\mathrm{tr}(K) = 1$ implies (9.37). □

We can also use operator kernels to construct symbol correspondences in an implicit way, as follows.

Proposition 9.3.4. *Let K be an operator kernel with characteristic numbers (c_n). The equation*

$$A = \frac{(p+1)(p+2)}{2} \int_{\mathbb{C}P^2} \widetilde{W}_A(\mathbf{x}) K(\mathbf{x}) d\mathbf{x} \ . \tag{9.39}$$

defines a symbol correspondence $\widetilde{W}$ with characteristic numbers $(\widetilde{c}_n)$ given by $\widetilde{c}_n = 1/c_n$.

Proof. We have

$$\begin{aligned}
\int_{\mathbb{C}P^2} X^n_{\boldsymbol{\nu},I}(\mathbf{x}) K(\mathbf{x}) d\mathbf{x} &= \int_{SU(3)} X^n_{\boldsymbol{\nu},I}(g\mathbf{x}_0) K^g dg \\
&= \sum_{n',\boldsymbol{\mu},J} k_{n'} \int_{SU(3)} X^n_{\boldsymbol{\nu},I}(g\mathbf{x}_0) D^{(n',n')}_{\boldsymbol{\mu}J,n'0}(g\mathbf{x}_0) dg \, \boldsymbol{e}(n';\boldsymbol{\mu},J) \\
&= \sum_{n',\boldsymbol{\mu},J} \frac{k_{n'}}{(n'+1)^{3/2}} \left\langle X^{n'}_{\boldsymbol{\mu},J} \middle| X^n_{\boldsymbol{\nu},I} \right\rangle \boldsymbol{e}(n;\boldsymbol{\mu},J) \\
&= \frac{k_n}{(n+1)^{3/2}} \boldsymbol{e}(n;\boldsymbol{\nu},I) \ ,
\end{aligned} \tag{9.40}$$

where

$$k_n = c_n \sqrt{\frac{2(n+1)^3}{(p+1)(p+2)}} \ . \tag{9.41}$$

Thus,

$$\frac{(p+1)(p+2)}{2} \int_{\mathbb{C}P^2} \frac{1}{c_n} X^n_{\boldsymbol{\nu},I}(\mathbf{x}) K(\mathbf{x}) d\mathbf{x} = \sqrt{\frac{(p+1)(p+2)}{2}} \boldsymbol{e}(n;\boldsymbol{\nu},I) \ . \tag{9.42}$$

By Proposition 9.3.1, $\widetilde{W}$ is a symbol correspondence with characteristic numbers $(\widetilde{c}_n)$ satisfying $\widetilde{c}_n = 1/c_n$. □

Actually, there is a duality relation between the symbol correspondences defined by the same operator kernel via (9.27) and via (9.39) considering the normalized inner product

$$\langle A|R\rangle_{\boldsymbol{p}} = \frac{2}{(p+1)(p+2)}\langle A|R\rangle = \frac{2}{(p+1)(p+2)}\operatorname{tr}\Big(A^\dagger R\Big) \tag{9.43}$$

for all $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$.

Definition 9.3.3. *Given a symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$, its* dual correspondence *is the symbol correspondence $\widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ that satisfies*

$$\langle A|R\rangle_{\boldsymbol{p}} = \Big\langle \widetilde{W}_A \Big| W_R \Big\rangle = \Big\langle W_A \Big| \widetilde{W}_R \Big\rangle \ . \tag{9.44}$$

The operators kernels of W and $\widetilde{W}$ are also said to be dual *to each other.*

Proposition 9.3.5. *Let K be an operator kernel. The symbol correspondences defined by K via (9.27) and via (9.39) are dual to each other, that is, for any symbol correspondence with characteristic numbers (c_n), its dual correspondence has characteristic numbers $(1/c_n)$.*

Proof. Given any two operators A and R, if we write $A^\dagger$ as in (9.39) and use the reality property, we get

$$\frac{2}{(p+1)(p+2)}\operatorname{tr}\Big(A^\dagger R\Big) = \int_{\mathbb{C}P^2} \overline{\widetilde{W}_A(\mathbf{x})}\operatorname{tr}(RK(\mathbf{x}))d\mathbf{x} = \int_{\mathbb{C}P^2} \overline{\widetilde{W}_A(\mathbf{x})} W_R(\mathbf{x})d\mathbf{x} \ , \tag{9.45}$$

that is, $\langle A|R\rangle_{\boldsymbol{p}} = \Big\langle \widetilde{W}_A \Big| W_R \Big\rangle$. Writing R as in (9.39), we get $\langle A|R\rangle_{\boldsymbol{p}} = \Big\langle W_A \Big| \widetilde{W}_R \Big\rangle$. □

In particular, a symbol correspondence W is an isometry if and only if it is self-dual.

Definition 9.3.4. *A symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ is a* Stratonovich-Weyl correspondence *if it is an isometry, that is,*

$$\langle A|R\rangle_{\boldsymbol{p}} = \langle W_A|W_R\rangle \tag{9.46}$$

for all $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$.

Proposition 9.3.6. *A symbol correspondence is a Stratonovich-Weyl correspondence if and only if its characteristic numbers (c_n) satisfy $|c_n| = 1$.*

Proof. The proof follows from Proposition 9.3.5. □

Although isometry is a very nice property for a symbol correspondence, there is another property for symbol correspondences that is very reasonable, from a physical perspective. Recall that a Hermitian operator with only nonnegative eigenvalues is called *positive*, or *positive-definite* if all of its eigenvalues are positive, and a real function that takes only nonnegative values is called *positive*, or *strictly-positive* if it only takes positive values.

Definition 9.3.5. *A symbol correspondence for a pure-quark system is* mapping-positive *if it maps positive(-definite) operators to (strictly-)positive functions. A symbol correspondence for a pure-quark system which is dual to a mapping-positive correspondence is a* positive-dual *correspondence.*

In Proposition 9.3.2, we characterize all symbol correspondences as expected values with respect to K^g, where K is an operator kernel, that is, an H-invariant operator satisfying equation (9.29) with $c_n \in \mathbb{R}^*$. From (9.34)-(9.37), K can be identified with an H-invariant *pseudo-state*, since, from Physics, we have the following:

Definition 9.3.6. *An operator on a complex Hilbert space is a* state *if it is a positive operator with unitary trace.*

On the other hand, from (9.34)-(9.37), because the real numbers a_r for a general operator kernel K can be negative, generally K has unitary trace but is not a positive operator and we can see it as providing pseudo-probabilities, since an operator kernel which is actually a state should provide actual probabilities. In fact, we have:

Proposition 9.3.7. *A symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ with operator kernel K is mapping-positive if and only if K is a state, that is, K given by (9.29), with $c_n \in \mathbb{R}^*$, is in the convex hull of $\{\Pi_{\boldsymbol{\nu}}\}$, that is, $K \in Conv\{\Pi_{\boldsymbol{\nu}}\}$.*

Proof. We assume that K is an operator kernel, thus K is given by (9.29) with $c_n \in \mathbb{R}^*$. Now, let K be decomposed as in (9.36). Suppose $K \in Conv\{\Pi_{\boldsymbol{\nu}}\}$, so that the coefficients a_j in the summation are all nonnegative. An operator $A \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ is positive if and only if $A = R^\dagger R$ for some $R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$, and A is positive-definite if and only if R is an automorphism. Thus, for any $g \in SU(3)$ and $\widetilde{R} = R\rho(g)$,

$$
\begin{aligned}
W_A(g\mathbf{x}_0) &= \sum_{j=0}^{p} a_j \sum_{\boldsymbol{\nu}\in j/2} \mathrm{tr}\Big(R^\dagger R\rho(g)\Pi_{\boldsymbol{\nu}}\rho(g)^\dagger\Big) = \sum_{j=0}^{p} a_j \sum_{\boldsymbol{\nu}\in j/2} \mathrm{tr}\Big(R\rho(g)\Pi_{\boldsymbol{\nu}}\rho(g)^\dagger R^\dagger\Big) \\
&= \sum_{j=0}^{p} a_j \sum_{\boldsymbol{\nu}\in j/2} \mathrm{tr}\Big(\widetilde{R}\Pi_{\boldsymbol{\nu}}\widetilde{R}^\dagger\Big) = \sum_{j=0}^{p} a_j \sum_{\boldsymbol{\nu}\in j/2} \Big\|\widetilde{R}(\boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}))\Big\|^2 \\
&\geq 0 \;,
\end{aligned}
\tag{9.47}
$$

where the inequality is strict if $\widetilde{R}$ is an automorphism, which is true if R is an automorphism, that is, if A is positive-definite.

Now, suppose $K \notin Conv\{\Pi_{\boldsymbol{\nu}}\}$. Then, there is a negative coefficient a_j so that any projection $\Pi_{\boldsymbol{\nu}}$ with $\boldsymbol{\nu} \in j/2$ is a positive operator satisfying $W_{\Pi_{\boldsymbol{\nu}}}(\mathbf{x}_0) = \mathrm{tr}(K\Pi_{\boldsymbol{\nu}}) < 0$. □

For a pure-quark system $\mathcal{H}_{\boldsymbol{p}}$, let $\mathcal{S}^{\boldsymbol{p}}_{=}$, $\mathcal{S}^{\boldsymbol{p}}_{<}$ and $\mathcal{S}^{\boldsymbol{p}}_{>}$ be the sets of Stratonovich-Weyl, mapping-positive and positive-dual correspondences, respectively.

Proposition 9.3.8. *The sets $\mathcal{S}^{\boldsymbol{p}}_{=}$, $\mathcal{S}^{\boldsymbol{p}}_{<}$ and $\mathcal{S}^{\boldsymbol{p}}_{>}$ are mutually disjoint.*

Proof. If $W \in \mathcal{S}^{\boldsymbol{p}}_{<}$, then Proposition 9.3.7 implies that its operator kernel K is given by

$$K = \sum_{\boldsymbol{\nu}} (-1)^{2(t+u)} a_{\boldsymbol{\nu}} \boldsymbol{e}(\boldsymbol{p};\boldsymbol{\nu}) \otimes \breve{\boldsymbol{e}}(\breve{\boldsymbol{p}};\breve{\boldsymbol{\nu}}) \ , \tag{9.48}$$

where $a_{\boldsymbol{\nu}}$ are coefficients of a convex combination, that is, they are non negative and sum up to 1. From (9.29), its characteristic numbers are

$$c_n = \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} \sum_{\boldsymbol{\nu}} (-1)^{2(t+u)} a_{\boldsymbol{\nu}} C^{\boldsymbol{p},\breve{\boldsymbol{p}},n}_{\boldsymbol{\nu},\breve{\boldsymbol{\nu}},(n,n,n)0} \ . \tag{9.49}$$

Since CG coefficients are coefficients of an unitary transformation, their absolute values are bounded above by 1, so

$$|c_n| \leq \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} = \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} \ . \tag{9.50}$$

Thus,

$$|c_p| < 1 \tag{9.51}$$

and characteristic numbers $(\widetilde{c}_n)$ of the correspondence $\widetilde{W} \in \mathcal{S}^{\boldsymbol{p}}_{>}$ dual to $W \in \mathcal{S}^{\boldsymbol{p}}_{<}$ satisfy

$$|\widetilde{c}_p| = \frac{1}{|c_p|} > 1 \ , \tag{9.52}$$

cf. Proposition 9.3.5. From Proposition 9.3.6 and inequalities (9.51)-(9.52), we conclude the statement. □

To verify the existence of a mapping-positive correspondence, we consider the defining representation ρ_1 of $SU(3)$ on $\mathbb{C}^3$, and invoke the canonical projection

$$\tilde{\pi} : S^5 \to \mathbb{C}P^2 \ ,$$

inside $\mathbb{C}^3 \to \mathbb{C}P^2$, and

$$\begin{aligned} \Phi_p : \mathbb{C}^3 &\to \mathbb{C}^{(p+1)(p+2)/2} \\ (z_1, z_2, z_3) &\mapsto (z_1^p, \ldots, \sqrt{\binom{p}{j,k,l}} z_1^j z_2^k z_3^l, \ldots, z_3^p) \ , \end{aligned} \tag{9.53}$$

where $\mathbb{C}^{(p+1)(p+2)/2} \simeq \mathcal{H}_{p,0}$ is endowed with a $SU(3)$-representation ρ_p of class $Q(p,0)$, cf. (9.12).

Proposition 9.3.9. *The map $B : \mathcal{B}(\mathcal{H}_{p,0}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2) : A \mapsto B_A$, with*

$$B_A(\mathbf{x}) = \langle \Phi_p(\mathbf{n}) | A \Phi_p(\mathbf{n}) \rangle$$

for $\mathbf{x} \in \mathbb{C}P^2$ *and* $\mathbf{n} \in S^5$ *related by* $\tilde{\pi}(\mathbf{n}) = \mathbf{x}$*, is a mapping-positive symbol correspondence whose operator kernel is the projection* $\Pi_{(0,0,p)}$ *onto the lowest weight vector and whose characteristic numbers are*

$$b_n = (-1)^p \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} C^{(p,0),\ \ (0,p),\ \ n}_{(0,0,p),(p,p,0),(n,n,n),0} \ . \tag{9.54}$$

Proof. First of all, we need to verify that B_A is a well defined function for every $A \in \mathcal{B}(\mathcal{H}_{p,0})$. For $\mathbf{n}, \mathbf{n}' \in S^5$, we have $\tilde{\pi}(\mathbf{n}) = \tilde{\pi}(\mathbf{n}')$ if and only if $\mathbf{n}' = e^{i\theta}\mathbf{n}$, but $\Phi_p(e^{i\theta}\mathbf{n}) = e^{ip\theta}\Phi_p(\mathbf{n})$, so

$$\left\langle \Phi_p(e^{i\theta}\mathbf{n}) \middle| A\Phi_p(e^{i\theta}\mathbf{n}) \right\rangle = \left\langle e^{ip\theta}\Phi_p(\mathbf{n}) \middle| e^{ip\theta} A\Phi_p(\mathbf{n}) \right\rangle = \langle \Phi_p(\mathbf{n}) | A\Phi_p(\mathbf{n}) \rangle \ . \tag{9.55}$$

This proves that B_A is well defined. It is also smooth, since $\tilde{\pi}$ is a surjective submersion and $B_A \circ \tilde{\pi}$ is a composition of smooth functions of S^5.

The linearity of B is trivial. The equivariance follows straightforwardly from the equivariance of Φ_p. For any $g \in SU(3)$,

$$\begin{aligned} B_{A^g}(\mathbf{x}) &= \langle \Phi_p(\mathbf{n}) | A^g \Phi_p(\mathbf{n}) \rangle = \left\langle \Phi_p(\mathbf{n}) \middle| \rho_p(g) A \rho_p(g)^{-1} \Phi_p(\mathbf{n}) \right\rangle \\ &= \left\langle \rho_p(g)^{-1}\Phi_p(\mathbf{n}) \middle| A\rho_p(g)^{-1}\Phi_p(\mathbf{n}) \right\rangle = \left\langle \Phi_p(\rho_1(g)^{-1}\mathbf{n}) \middle| A\Phi_p(\rho_1(g)^{-1}\mathbf{n}) \right\rangle \\ &= B_A(g^{-1}\mathbf{x}) = (B_A)^g(\mathbf{x}) \ , \end{aligned} \tag{9.56}$$

cf. (8.39). Equivariance implies that $\ker B$ is an invariant subspace, and we use that to prove B is injective by means of contradiction. Suppose B is not injective, then $\ker B$ contains an irreducible representation of the form $Q(n,n)$, so the diagonal matrix $\boldsymbol{e}(n, \boldsymbol{n}, I)$ lies in $\ker B$, that is, there exists a non-zero diagonal matrix $D = diag(d_{p,0,0}, ..., d_{j,k,l}, ..., d_{0,0,p}) \in \ker B$. Thus, we have

$$B_D(\mathbf{x}) = \langle \Phi_p(\mathbf{n}) | D\Phi_p(\mathbf{n}) \rangle = \sum_{j+k+l=1} \binom{p}{j,k,l} d_{j,k,l} \, |z_1|^{2j} |z_2|^{2k} |z_3|^{2l} = 0 \tag{9.57}$$

for every $\mathbf{n} = (z_1, z_2, z_3) \in S^5$. It follows that the polynomials

$$\sum_{j+k+l=1} \binom{p}{j,k,l} \mathrm{Re}(d_{j,k,l}) \, x_1^{2j} x_2^{2k} x_3^{2l} \ , \quad \sum_{j+k+l=1} \binom{p}{j,k,l} \mathrm{Im}(d_{j,k,l}) \, x_1^{2j} x_2^{2k} x_3^{2l} \in \mathbb{R}[x_1, x_2, x_3]$$

are identically zero, so D must be a zero matrix, and this is the desired contradiction. Therefore, B is injective.

We know that

$$\begin{aligned} \Pi_{(0,0,p)} &= \boldsymbol{e}((p,0);(0,0,p)) \otimes \boldsymbol{e}^*((p,0);(0,0,p)) \\ &= (-1)^p \, \boldsymbol{e}((p,0);(0,0,p)) \otimes \check{\boldsymbol{e}}((0,p);(p,p,0)) \\ &= (-1)^p \sum_{n=0}^{p} C^{(p,0),\ \ (0,p),\ \ n}_{(0,0,p),(p,p,0),(n,n,n),0} \boldsymbol{e}(n;(n,n,n),0) \\ &= \frac{2}{(p+1)(p+2)} \mathbb{1} + (-1)^p \sum_{n=1}^{p} C^{(p,0),\ \ (0,p),\ \ n}_{(0,0,p),(p,p,0),(n,n,n),0} \, \boldsymbol{e}(n;(n,n,n),0) \ , \end{aligned} \tag{9.58}$$

where the last equality follows from $\mathrm{tr}\Big(\Pi_{(0,0,p)}\Big) = 1$. Since the Clebsch-Gordan coefficients are real and B is an injective map, it is sufficient to show $B_A(g\mathbf{x}_0) = \mathrm{tr}\Big(A\Pi^g_{(0,0,p)}\Big)$. We have that $\mathbf{n}_0 = (0,0,1) \in S^5$ satisfies $\pi(\mathbf{n}_0) = \mathbf{x}_0$ and $\Phi_p(\mathbf{n}_0) = (0,0,...,1)$, so

$$B_A(\mathbf{x}_0) = \langle \Phi_p(\mathbf{n}_0)|A\Phi_p(\mathbf{n}_0)\rangle = \mathrm{tr}\Big(A\Pi_{(0,0,p)}\Big) \tag{9.59}$$

and

$$B_A(g\mathbf{x}_0) = (B_A)^{g^{-1}}(\mathbf{x}_0) = B_{A^{g^{-1}}}(\mathbf{x}_0) = \mathrm{tr}\Big(A^{g^{-1}}\Pi_{(0,0,p)}\Big) = \mathrm{tr}\Big(A\Pi^g_{(0,0,p)}\Big) \ . \tag{9.60}$$

This proves that B is a symbol correspondence sequence with operator kernel $\Pi_{(0,0,p)}$. Proposition 9.3.7 implies that it is a mapping-positive correspondence. □

One could expect that the projection $\Pi_{(p,0,0)} \in \mathcal{B}(\mathcal{H}_{p,0})$ onto the highest weight vector is also an operator kernel. This is not the case since $\Pi_{(p,0,0)}$ is not H-invariant, cf. Proposition 9.3.3 and Proposition 9.3.11. However, this is just a matter of convention, a choice of $U(2)$ subgroup by which we impose invariance. Recalling (9.10), we have $\rho_p(g_0)e_{j,k,l} = (-1)^p e_{l,k,j}$ for g_0 as in (8.160), so that $\Pi_{(p,0,0)} = \Pi^{g_0}_{(0,0,p)}$ and $\Pi_{(p,0,0)}$ is fixed by $H' = g_0 H g_0$ as given in (8.159). If we set $\mathbf{x}'_0 = g_0\mathbf{x}_0 = [1:0:0]$, then we can construct a symbol correspondence $A \mapsto B'_A$ using $\Pi_{(p,0,0)}$:

$$B'_A(g\mathbf{x}'_0) = \mathrm{tr}\Big(A\Pi^g_{(p,0,0)}\Big) \ . \tag{9.61}$$

But

$$B'_A(g\mathbf{x}'_0) = \mathrm{tr}\Big(A\Pi^g_{(p,0,0)}\Big) = \mathrm{tr}\Big(A\Pi^{gg_0}_{(0,0,p)}\Big) = B_A(gg_0\mathbf{x}_0) = B_A(g\mathbf{x}'_0) \ \ , \tag{9.62}$$

that is, B' and B are the same map.

Notwithstanding, a minor adaptation of the argument of the previous proposition proves that $\Pi_{(p,p,0)} \in \mathcal{B}(\mathcal{H}_{0,p})$, the projection onto the highest weight vector, is also an operator kernel. Now, consider $\check{\rho}_1$ and $\check{\rho}_p$ the dual representations of ρ_1 and ρ_p, respectively, so that the maps

$$\omega : \mathbb{C}^3 \to \mathbb{C}^3 : (z_1, z_2, z_3) \mapsto (-\overline{z_3}, \overline{z_2}, -\overline{z_1}) \tag{9.63}$$

and Φ_p are both equivariant in the following sense:

$$\begin{aligned} \check{\rho}_p(g) \circ \omega &= \omega \circ \rho_p(g) \ , \\ \check{\rho}_p(g) \circ \Phi_p &= \Phi_p \circ \check{\rho}_p(g) \ , \end{aligned} \tag{9.64}$$

cf. (9.13).

Proposition 9.3.10. *The map $B^- : \mathcal{B}(\mathcal{H}_{0,p}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2) : A \mapsto B^-_A$, with*

$$B^-_A(\mathbf{x}) = \langle \Phi_p \circ \omega(\mathbf{n})|A\Phi_p \circ \omega(\mathbf{n})\rangle$$

for $\mathbf{x} \in \mathbb{C}P^2$ *and* $\mathbf{n} \in S^5 \subset \mathbb{C}^3$ *related by* $\pi(\mathbf{n}) = \mathbf{x}$*, is a mapping-positive symbol correspondence whose operator kernel is the projection* $\Pi_{(p,p,0)}$ *onto the highest weight vector and whose characteristic numbers are*

$$b_{n-} = (-1)^p \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} C^{(0,p),\ \ (p,0),\ \ n}_{(p,p,0),(0,0,p),(n,n,n),0}\ . \tag{9.65}$$

Proof. The proof goes just as the proof of the previous proposition (linearity is, again, obvious), we just highlight that the following holds: $\Phi_p \circ \omega(e^{i\theta}\mathbf{n}) = e^{-ip\theta}\Phi_p \circ \omega(\mathbf{n})$,

$$\begin{aligned}
\Pi_{(p,p,0)} &= \breve{\boldsymbol{e}}((0,p);(p,p,0)) \otimes \breve{\boldsymbol{e}}^*((0,p);(p,p,0)) \\
&= (-1)^p\, \breve{\boldsymbol{e}}((0,p);(p,p,0)) \otimes \boldsymbol{e}((p,0);(0,0,p)) \\
&= (-1)^p \sum_{n=0}^{p} C^{(0,p),\ \ (p,0),\ \ n}_{(p,p,0),(0,0,p),(n,n,n),0} \boldsymbol{e}(n;(n,n,n),0) \\
&= \frac{2}{(p+1)(p+2)}\mathbb{1} + (-1)^p \sum_{n=1}^{p} C^{(0,p),\ \ (p,0),\ \ n}_{(p,p,0),(0,0,p),(n,n,n),0} \boldsymbol{e}(n;(n,n,n),0)
\end{aligned} \tag{9.66}$$

and $\mathbf{n}_0 = (0,0,1) \in S^5$ satisfies $\Phi_p \circ \omega(\mathbf{n}_0) = (1,...,0,0)$. □

Proposition 9.3.11. *The operator* $\Pi_{(0,0,p)} \in \mathcal{B}(\mathcal{H}_{p,0})$ *is the unique operator kernel among the projections onto weight vectors. The same is true for* $\Pi_{(p,p,0)} \in \mathcal{B}(\mathcal{H}_{0,p})$*.*

Proof. From equations (9.34) and (9.36), if $K = \Pi_{\boldsymbol{\nu}} \in \mathcal{B}(\mathcal{H}_{p,0})$ is an operator kernel, then (9.37) implies that $j = 0$, that is, $\boldsymbol{\nu}$ has null isospin. Using (9.17), we get that $\boldsymbol{\nu} = (0,0,p)$. The proof for $\mathcal{H}_{0,p}$ is completely analogous, but we use (9.18) to conclude $\boldsymbol{\nu} = (p,p,0)$. □

Definition 9.3.7. *For any* $p \in \mathbb{N}$*, the symbol correspondences* $B : \mathcal{B}(\mathcal{H}_{p,0}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*, with operator kernel* $\Pi_{(0,0,p)}$*, and* $B^- : \mathcal{B}(\mathcal{H}_{0,p}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*, with operator kernel* $\Pi_{(p,p,0)}$*, are called* Berezin correspondences.

Proposition 9.3.12. *The Berezin correspondences for* $\mathcal{H}_{p,0}$ *and* $\mathcal{H}_{0,p}$ *have the same characteristic numbers.*

Proof. It follows from Proposition 8.2.9. □

Remark 9.3.2. *Our proof of Proposition 9.3.9 is, in essence, the same given in (RIOS; STRAUME, 2014) for spin systems, where the symmetry group is* $SU(2)$ *and the orbit is* $\mathbb{C}P^1 \simeq S^2$*. However, in that book, an involution on the space of symbol correspondences for spin systems is defined in this context. Let* $\{u_1, u_2\}$ *be a spin standard basis and* $\{\breve{u}_1 = -u_2^*, \breve{u}_2 = u_1^*\}$ *be the dual spin standard basis of* $\mathbb{C}^2$*, so that* $\omega_s(z_1,z_2) = (-\overline{z_2}, \overline{z_1})$ *is the dualization via inner product. Then* ω_s *commutes with the action of* $SU(2)$ *and induces an involution on* $\mathbb{C}P^1$ *that coincides with the antipodal map* α_s*, that is, for the Hopf map* $\pi_s : S^3 \to S^2$*, we have* $\pi_s \circ \omega_s = \alpha_s \circ \pi_s$*. If* W^s *is a symbol correspondence for a spin-*j *system, we have that* $A \mapsto W^s_A \circ \alpha_s$ *is also a symbol correspondence, known as the* alternate correspondence *of* W^s*. This is used to prove that both projections onto lowest and highest weight vectors of the same representation are correspondences. The name "alternate" is due to the fact that symbol correspondences for a spin-*j* system are characterized by a set of non-zero real numbers* $\{c_l : l = 1, ..., 2j\}$*, where each* l *stands for a representation of* $SU(2)$ *in the decomposition of the operator algebra into irreducible representations, just as we have for pure quark systems, and to alternate a correspondence is equivalent to alternate the signs of such numbers, that is, to perform the change* $c_l \mapsto (-1)^l c_l$*.*

One may try to reproduce the same argument for pure-quark systems, but this is not possible since the dualization $\omega(z_1, z_2, z_3) = (-\overline{z_3}, \overline{z_2}, -\overline{z_1})$ *on* $\mathbb{C}^3$ *does not commute with the action of* $SU(3)$*. In fact, for the defining representation* ρ_1 *of* $SU(3)$ *and its dual* $\check{\rho}_1$*, we have that* $\rho_1(g) \circ \omega = \omega \circ \check{\rho}_1(g)$ *and* $\check{\rho}_1(g) \circ \omega = \omega \circ \rho_1(g)$ *for every* $g \in SU(3)$*. Thus, in general, a symbol correspondence* W *for a pure-quark system does not satisfy* $W_{A^g} \circ \tilde{\alpha} = (W_A \circ \tilde{\alpha})^g$*, where* $\tilde{\alpha}([z_1 : z_2 : z_3]) = [-\overline{z_3} : \overline{z_2} : -\overline{z_1}]$ *is the involution on* $\mathbb{C}P^2$ *such that* $\tilde{\alpha} \circ \pi = \pi \circ \omega$*. Take, for instance,* $\mathbf{z}_0 = (0, 0, 1)$*, where* $\pi(\mathbf{z}_0) = [0 : 0 : 1] = \mathbf{x}_0$*, and*

$$g = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{9.67}$$

so

$$\begin{aligned} g\,\tilde{\alpha}(\mathbf{x}_0) &= g\,\tilde{\alpha} \circ \pi(\mathbf{z}_0) = g\,\pi \circ \omega(\mathbf{z}_0) \\ &= \pi(\rho_1(g)\omega(\mathbf{z}_0)) = \pi \circ \omega(\check{\rho}_1(g)\mathbf{z}_0) \\ &= \pi \circ \omega(0, -1/\sqrt{2}, 1/\sqrt{2}) = \tilde{\alpha} \circ \pi(0, -1/\sqrt{2}, 1/\sqrt{2}) \\ &\neq \tilde{\alpha}(g\mathbf{x}_0) \ . \end{aligned} \tag{9.68}$$

At this point one may think to define alternate correspondences for pure-quark systems by applying $c_n \mapsto (-1)^n c_n$ *to the characteristic numbers. However, this is not the correct notion of alternation of correspondences for pure-quark systems as we show below.*

We now establish a solid link between Berezin correspondences for $\mathcal{H}_{\boldsymbol{p}}$ and $\mathcal{H}_{\check{\boldsymbol{p}}}$.

Definition 9.3.8. *For a symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*, its* antipodal correspondence *is the symbol correspondence* $\breve{W} : \mathcal{B}(\mathcal{H}_{\check{\boldsymbol{p}}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *given by*

$$\breve{W}_{A^*} = W_A \ , \tag{9.69}$$

cf. (8.129)-(8.131).

If one defines symbol correspondences from $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ to $C^\infty_{\mathbb{C}}(\mathcal{O}_{\boldsymbol{p}})$ as pointed out in Remark 9.3.1, two antipodal correspondences W and $\breve{W}$ are related by

$$\breve{W}_{A^*} = W_A \circ \iota \ ,$$

thus the name antipodal.

Remark 9.3.3. *It turns out that Definition 9.3.8 is completely analogous to alternation, for spin systems. The classes of irreducible representations of* $SU(2)$ *are self-dual and their symbol correspondences satisfy* $W^s_{A^*} = W^s_A \circ \alpha_s$*, maintaining the same notation of Remark 9.3.2. But, now, for pure-quark systems the irreducible representations are not self-dual* ($\mathcal{H}_{\boldsymbol{p}} \neq \mathcal{H}_{\check{\boldsymbol{p}}}$) *and the alternation of characteristic numbers does not occur anymore.*

Proposition 9.3.13. *Two pure-quark symbol correspondences* $W_1 : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *and* $W_2 : \mathcal{B}(\mathcal{H}_{\check{\boldsymbol{p}}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *are antipodal to each other if and only if their characteristic numbers are equal.*

Proof. The result follows from (8.131) and Proposition 9.3.1. □

Corollary 9.3.13.1. *The Berezin correspondences for* $\mathcal{H}_{\boldsymbol{p}}$ *and* $\mathcal{H}_{\check{\boldsymbol{p}}}$ *are antipodal to each other.*

Corollary 9.3.13.2. *A symbol correspondence for a pure-quark system is Stratonovich-Weyl correspondence if and only if its antipodal correspondence is a Stratonovich-Weyl correspondence.*

9.4 Twisted product for pure-quark system

Again, let $\boldsymbol{p} \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$ with $|\boldsymbol{p}| = p$.

It is obvious from Proposition 9.3.1 that the images of all symbol correspondences for $\mathcal{H}_{\boldsymbol{p}}$ and $\mathcal{H}_{\check{\boldsymbol{p}}}$ are the same space, namely, the space spanned by the $\mathbb{C}P^2$ harmonics $X^n_{\boldsymbol{\nu},I}$ with $0 \leq n \leq p$. Such space shall be denoted by $\mathcal{X}_p$, that is,

$$\mathcal{X}_p = \mathrm{Span}_{\mathbb{C}}\{X^n_{\boldsymbol{\nu},I}\}_{0\leq n\leq p} \ .$$

Now, we translate the operator algebra $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ to $\mathcal{X}_p$ using a symbol correspondence.

Definition 9.4.1. *Given a symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*, the* twisted product *of symbols induced by* W *is the binary operation* $\star$ *on* $\mathcal{X}_p$ *given by*

$$W_A \star W_R = W_{AR} \ , \tag{9.70}$$

for any $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$*. The algebra* $(\mathcal{X}_p, \star)$ *is called a* twisted $\boldsymbol{p}$-algebra*.*

Proposition 9.4.1. *Any twisted* $\boldsymbol{p}$*-algebra* $(\mathcal{X}_p, \star)$ *is*

i) $SU(3)$-equivariant: $(f_1 \star f_2)^g = f_1^g \star f_2^g$*;*

ii) Associative: $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$*;*

iii) Unital: $1 \star f = f \star 1 = f$*;*

iv) A $*$-algebra: $\overline{f_1 \star f_2} = \overline{f_2} \star \overline{f_1}$;

where $f_1, f_2, f_3, f \in \mathcal{X}_p$, $g \in SU(3)$ *and* $1 \in \mathcal{X}_p$ *is the constant function equal to* 1 *on* $\mathbb{C}P^2$, *cf. (9.3).*

Proof. The operator space $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ is a $SU(3)$-equivariant unital associative $*$-algebra with respect to Hermitian conjugate, where $\mathbb{1}$ is the identity. Since any symbol correspondence W for $\mathcal{H}_p$ is a $SU(3)$-equivariant linear isomorphism between $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ and $\mathcal{X}_p$ satisfying reality and $W_{\mathbb{1}} = 1$, the statement is true. □

Proposition 9.4.2. *Fixed* $\boldsymbol{p} \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$, *any two twisted* $\boldsymbol{p}$*-algebras are naturally isomorphic, and any twisted* $\boldsymbol{p}$*-algebra is naturally anti-isomorphic to any* $\check{\boldsymbol{p}}$*-algebra.*

Proof. Let $W_1, W_2 : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ be symbol correspondences. Then $W_1 \circ W_2^{-1} : \mathcal{X}_p \to \mathcal{X}_p$ is an isomorphism because each W_j is an isomorphism onto $\mathcal{X}_p$. If, now, we suppose $W_2 : \mathcal{B}(\mathcal{H}_{\check{\boldsymbol{p}}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$, then $W_1 \circ * \circ W_2^{-1} : \mathcal{X}_p \to \mathcal{X}_p$ is an anti-isomorphism since the adjoint map $*$ is an anti-isomorphism and, again, each W_j is an isomorphism onto $\mathcal{X}_p$. □

Twisted products of $\mathbb{C}P^2$ harmonics can be easily computed and determine the twisted product for all functions in $\mathcal{X}_p$ by bilinearity of the product.

Proposition 9.4.3. *If* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *is a symbol correspondence with characteristic numbers* (c_n), *then the induced twister product is given by*

$$X^{n_1}_{\boldsymbol{\nu}_1, I_1} \star X^{n_2}_{\boldsymbol{\nu}_2, I_2} = \sqrt{\frac{(p+1)(p+2)}{2}} \sum_{n=0}^{p} \sum_{\boldsymbol{\nu}, I} (-1)^{p+2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})} \begin{bmatrix} n_1 & n_2 & n \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \check{\boldsymbol{\nu}}, I \end{bmatrix} [\boldsymbol{p}] \frac{c_n}{c_{n_1} c_{n_2}} X^n_{\boldsymbol{\nu}, I} \tag{9.71}$$

for $0 \leq n_1, n_2 \leq p$, *where summations over* $\boldsymbol{\nu}$ *and* I *can be restricted to* $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2, \boldsymbol{\nu}} = 1$ *and* $\delta(I_1, I_2, I) = 1$ *due to (9.21).*

Proof. The result follows from Propositions 9.2.1 and 9.3.1. □

Any twisted product on $\mathcal{X}_p$ admits an integral formulation, which supposedly allows one to compute it without decomposition of functions in the basis of $\mathbb{C}P^2$ harmonics.

Proposition 9.4.4. *If* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *is a symbol correspondence with operator kernel* K *and characteristic numbers* (c_n), *then the induced twister product is given by*

$$f_1 \star f_2(\mathbf{x}) = \int_{\mathbb{C}P^2 \times \mathbb{C}P^2} f_1(\mathbf{x}_1) f_2(\mathbf{x}_2) \mathbb{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \, d\mathbf{x}_1 d\mathbf{x}_2 \tag{9.72}$$

for any $f_1, f_2 \in \mathcal{X}_p$*, where*

$$
\begin{aligned}
\mathbb{L}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= \left(\frac{(p+1)(p+2)}{2}\right)^2 \operatorname{tr}\Big(\widetilde{K}(\mathbf{x}_1)\widetilde{K}(\mathbf{x}_2)K(\mathbf{x}_3)\Big) \\
&= (-1)^p\sqrt{\frac{(p+1)(p+2)}{2}}\sum_{n_j=0}^{p}\sum_{\boldsymbol{\nu}_j,I_j}\begin{bmatrix} n_1 & n_2 & n_3 \\ \boldsymbol{\nu}_1,I_1 & \boldsymbol{\nu}_2,I_2 & \boldsymbol{\nu}_3,I_3\end{bmatrix}[\boldsymbol{p}]\frac{c_{n_3}}{c_{n_1}c_{n_2}} \\
&\qquad \times \overline{X^{n_1}_{\boldsymbol{\nu}_1,I_1}}(\mathbf{x}_1)\overline{X^{n_2}_{\boldsymbol{\nu}_2,I_2}}(\mathbf{x}_2)\overline{X^{n_3}_{\boldsymbol{\nu}_3,I_3}}(\mathbf{x}_3)
\end{aligned} \tag{9.73}
$$

for $\widetilde{K}$ *being the operator kernel dual to* K*, that is, with characteristic numbers* $(1/c^n)$*.*

Proof. Let $A_1, A_2 \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ so that $f_1 = W_{A_1}$ and $f_2 = W_{A_2}$. Then, by Proposition 9.3.5,

$$
\begin{aligned}
&\left(\frac{(p+1)(p+2)}{2}\right)^2\int_{\mathbb{C}P^2\times\mathbb{C}P^2} f_1(\mathbf{x}_1)f_2(\mathbf{x}_2)\operatorname{tr}\Big(\widetilde{K}(\mathbf{x}_1)\widetilde{K}(\mathbf{x}_2)K(\mathbf{x})\Big)\,d\mathbf{x}_1 d\mathbf{x}_2 \\
&\quad = \operatorname{tr}\left(\left(\frac{(p+1)(p+2)}{2}\right)^2\int_{\mathbb{C}P^2\times\mathbb{C}P^2} f_1(\mathbf{x}_1)f_2(\mathbf{x}_2)\widetilde{K}(\mathbf{x}_1)\widetilde{K}(\mathbf{x}_2)\,d\mathbf{x}_1 d\mathbf{x}_2\, K(\mathbf{x})\right) \\
&\quad = \operatorname{tr}(A_1A_2K(\mathbf{x})) = W_{A_1A_2}(\mathbf{x}) = f_1\star f_2(\mathbf{x})\ .
\end{aligned} \tag{9.74}
$$

It is worth to highlight that $\widetilde{K}(\mathbf{x}_1)$, $\widetilde{K}(\mathbf{x}_2)$ and $K(\mathbf{x}_3)$ can be expanded in the coupled basis using Wigner D-functions so that $\mathbb{L}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ is a linear combinations of $\overline{X^{n_1}_{\boldsymbol{\nu}_1,I_1}}(\mathbf{x}_1)\overline{X^{n_2}_{\boldsymbol{\nu}_2,I_2}}(\mathbf{x}_2)\overline{X^{n_3}_{\boldsymbol{\nu}_3,I_3}}(\mathbf{x}_3)$. That said, Proposition 9.4.3 implies the second equality in (9.73). □

Definition 9.4.2. *The* integral trikernel $\mathbb{L} \in C^\infty_{\mathbb{C}}(\mathbb{C}P^2\times\mathbb{C}P^2\times\mathbb{C}P^2)$ *of a twisted product induced by a symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ *is the function given by (9.73) so that the twisted product is given by (9.72).*

Obviously, the integral in (9.73) is well defined for any pair of smooth functions on $\mathbb{C}P^2$, so it leads to a product on $C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$.

Proposition 9.4.5. *Let* $\mathbb{L}$ *be the integral trikernel of a twisted product* $\star$ *induced by a symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*. Then the binary operation* $\bullet$ *on* $C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*,*

$$
f_1\bullet f_2(\mathbf{x}) = \int_{\mathbb{C}P^2\times\mathbb{C}P^2} f_1(\mathbf{x}_1)f_2(\mathbf{x}_2)\mathbb{L}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x})\,d\mathbf{x}_1 d\mathbf{x}_2 \tag{9.75}
$$

for any $f_1, f_2 \in C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$*, defines a* $SU(3)$*-equivariant associative* $*$*-algebra with respect to complex conjugation. In particular, if* $f_1, f_2 \in \mathcal{X}_p$*, we have* $f_1\bullet f_2 = f_1\star f_2$*. But, if either* f_1 *or* f_2 *is orthogonal to* $\mathcal{X}_p$*, we have* $f_1\bullet f_2 = 0$*.*

Proof. Linearity of integral implies the product is bilinear, hence it defines an algebra. By definition, it is clear that $f_1\bullet f_2 = f_1\star f_2$ if $f_1, f_2 \in \mathcal{X}_p$. Now, suppose f_j is orthogonal to $\mathcal{X}_p$. Thus, it is orthogonal to every $X^n_{\boldsymbol{\nu},I}$ with $n \le p$, which implies the integral over $\mathbf{x}_j$

in (9.75) results in 0, so $f_1 \bullet f_2 = 0$. Since any $f \in C^\infty_{\mathbb{C}}(\mathbb{C}P^2)$ can be decomposed into $f = f_\parallel + f_\perp$, where $f_\parallel \in \mathcal{X}_p$ and $f_\perp$ is orthogonal to $\mathcal{X}_p$, the $SU(3)$-equivariant, associative and $*$-algebra properties of $\star$ extends to $\bullet$. □

For a product $\bullet$ as in the previous proposition, the constant function 1 is not the identity anymore, now it gives an orthogonal projection $C^\infty_{\mathbb{C}}(\mathbb{C}P^2) \to \mathcal{X}_p : f \mapsto 1 \bullet f = f \bullet 1$.

Notation 5. *Before stating general properties of integral trikernels, we establish a convention to denote the* reproducing kernel *on $\mathcal{X}_p$:*

$$\mathcal{R}_p(\mathbf{x}_1, \mathbf{x}_2) = \sum_{n=0}^{p} \sum_{\nu, I} \overline{X^n_{\nu,I}}(\mathbf{x}_1) X^n_{\nu,I}(\mathbf{x}_2) = \mathcal{R}_p(\mathbf{x}_2, \mathbf{x}_1) \ , \tag{9.76}$$

satisfying

$$\int_{\mathbb{C}P^2} f(\mathbf{x}_1) \mathcal{R}_p(\mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x}_1 = f(\mathbf{x}_2) \ , \quad \forall\, f \in \mathcal{X}_p \ . \tag{9.77}$$

Proposition 9.4.6. *Let $\mathbb{L}$ be an integral trikernel of a twisted product $\star$ on $\mathcal{X}_p$. Then, for every $g \in SU(3)$ and every $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{C}P^2$,*

i) $\mathbb{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbb{L}(g\mathbf{x}_1, g\mathbf{x}_2, g\mathbf{x}_3)$*;*

ii) $\int_{\mathbb{C}P^2} \mathbb{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \mathbb{L}(\mathbf{x}, \mathbf{x}_3, \mathbf{x}_4)\, d\mathbf{x} = \int_{\mathbb{C}P^2} \mathbb{L}(\mathbf{x}_1, \mathbf{x}, \mathbf{x}_4) \mathbb{L}(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x})\, d\mathbf{x}$*;*

iii) $\int_{\mathbb{C}P^2} \mathbb{L}(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x} = \int_{\mathbb{C}P^2} \mathbb{L}(\mathbf{x}_1, \mathbf{x}, \mathbf{x}_2)\, d\mathbf{x} = \mathcal{R}_p(\mathbf{x}_1, \mathbf{x}_2)$*;*

iv) $\overline{\mathbb{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)} = \mathbb{L}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3)$*.*

Proof. Let $f_1, f_2 \in \mathcal{X}_p$. Writing the equality $(f_1)^g \star (f_2)^g = (f_1 \star f_2)^g$ in the integral form, we get that $SU(3)$-equivariance of $\star$ is equivalent to property (i). In the same vein, we conclude that each property of this statement is equivalent to the property of Proposition 9.4.1 with same number. □

Remark 9.4.1. *Although the integral formulation of a twisted product on $\mathcal{X}_p$ is supposed to circumvent the necessity of decomposing symbols (elements of $\mathcal{X}_p$) in the basis of $\mathbb{C}P^2$ harmonics, the formula (9.73) for an integral trikernel uses these harmonics explicitly. In (RIOS; STRAUME, 2014), new formulas for integral trikernels of spin systems were obtained using $SU(2)$-invariant 2-points and 3-points functions on $\mathbb{C}P^1$, but a similar exercise in the case of pure-quark systems is much harder and is deferred for later.*

We finish with a relation between twisted algebras induced by antipodal correspondences.

Proposition 9.4.7. *The twisted products $\star$ and $\check{\star}$ induced by a symbol correspondence and its antipodal correspondence satisfy*

$$f_1 \star f_2 = f_2 \,\check{\star}\, f_1 \ . \tag{9.78}$$

Proof. The proof follows from Proposition 9.4.3 and (9.22). □

Corollary 9.4.7.1. *For $\star$ and $\check{\star}$ as in the previous proposition, their integral trikernels $\mathbb{L}$ and $\check{\mathbb{L}}$ satisfy*

$$\mathbb{L}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \check{\mathbb{L}}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) \ . \tag{9.79}$$

Remark 9.4.2. *We already mentioned that the notion of antipodal correspondences for quark systems is analogous to alternation for spin systems considering the appropriate characterization. In addition to the previous discussion, we present a related phenomenon encoded in Proposition 9.4.7 which also happens for spin systems. The commutator $[\,,\,]_\star$ of a twisted product $\star$ satisfies*

$$[f_1, f_2]_\star = [f_2, f_1]_{\check{\star}} \ , \tag{9.80}$$

where $\check{\star}$ is the twisted product induced by the antipodal correspondence. In this way, $\check{\star}$ can be seen as defining the reverse symbolic dynamics of the one defined by $\star$. In fact, recalling Heisenberg's equation for an operator F subject to a Hamiltonian H,

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} \ , \tag{9.81}$$

if F has no explicit temporal dependence, then under a symbol correspondence W its symbol f satisfies

$$\frac{df}{dt} = [f, h]_\star \ , \tag{9.82}$$

where $h = W_H$. It follows that if we set H^* as the Hamiltonian of the dual space, the symbolic dynamics of F^* under $\breve{W}$ is given by

$$\frac{df}{dt} = [f, h]_{\check{\star}} = -[f, h]_\star \ . \tag{9.83}$$

CHAPTER

10

GENERIC QUARK SYSTEMS

In this chapter, we begin a study of correspondences for generic quark systems, that is, representations of generic class $Q(p,q)$ and generic coadjoint orbit $\mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi)$. Although we basically reproduce what we have done for $Q(p,0)$ (or $Q(0,q)$) and $\mathbb{C}P^2$, some new phenomena appear when compared to the previous case.

10.1 Classical generic quark system

Definition 10.1.1. *The* generic classical quark system *is the symplectic total space $\mathcal{E}$ of the fiber bundle $\mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi)$, with base $\mathbb{C}P^2$, fiber $\mathbb{C}P^1$ and projection π,*

$$\mathbb{C}P^1 \hookrightarrow \mathcal{E} \xrightarrow{\pi} \mathbb{C}P^2 \ , \tag{10.1}$$

together with its Poisson algebra on $C^\infty_{\mathbb{C}}(\mathcal{E})$.

We have $\mathcal{E} \simeq SU(3)/T$, where T is the maximal torus (8.161) of $SU(3)$. So we look for representations with weights satisfying $t = u = 0$, cf. (8.21)-(8.22).

Proposition 10.1.1. *The representations of $SU(3)$ with non null vectors fixed by T are of the form $Q(a,b)$ for $a \equiv b \pmod 3$. For*

$$k = |a-b|/3 \ ,$$

the space fixed by T is spanned by the set

$$\{\boldsymbol{e}((a,b); \boldsymbol{\nu}_{(a,b)}, I_\gamma) : \gamma = 1, ..., \min\{a,b\}+1\} \ , \tag{10.2}$$

where

$$\boldsymbol{\nu}_{(a,b)} = \begin{cases} (a+2k, a+2k, a+2k) \ , & \text{if} \quad \min\{a,b\} = a \\ (b+k, b+k, b+k) \ , & \text{if} \quad \min\{a,b\} = b \end{cases} \tag{10.3}$$

and

$$I_\gamma = \gamma - 1 + k \ . \tag{10.4}$$

Proof. From (8.21)-(8.22), we get that $\nu_1 = \nu_2 = \nu_3 = \nu \in \mathbb{N}_0$, with

$$2\nu = r_+ + r_- \tag{10.5}$$

and

$$3\nu = a + 2b \ , \tag{10.6}$$

for

$$r_- \leq b \leq r_+ \leq a + b \ , \quad r_- \leq \nu \leq a + b \ .$$

From (10.6), $a + 2b \equiv 0 \pmod 3$, which implies $a \equiv b \pmod 3$.

For representations of class $Q(a, a+3k)$, with $a, k \in \mathbb{N}_0$, we have that

$$\nu = a + 2k \tag{10.7}$$

and the GT states fixed by T are given by r_+ and r_- satisfying

$$\begin{cases} r_+ + r_- = 2a + 4k \\ 0 \leq r_- \leq a + 3k \leq r_+ \leq 2a + 3k \end{cases} . \tag{10.8}$$

The system has $a + 1$ solutions:

$$\begin{cases} r_+ = 2a + 3k \ , \quad r_- = k \\ \vdots \\ r_+ = a + 3k \ , \quad r_- = a + k \end{cases} \tag{10.9}$$

so that

$$\{\boldsymbol{e}((a, a+3k); (a+2k, a+2k, a+2k), I) : I = k, ..., a+k\} \tag{10.10}$$

spans the subspace fixed by T.

For a representation of class $Q(b+3k, b)$, since it is dual to $Q(b, b+3k)$, the subspace fixed by T is spanned by the dualization of the set $\{\boldsymbol{e}((b, b+3k), (b+2k, b+2k, b+2k), I) : I = k, ..., b+k\}$, which implies that

$$\nu = b + k$$

and is given by the solutions of

$$\begin{cases} r_+ + r_- = 2b + 2k \\ 0 \leq r_- \leq a \leq r_+ \leq 2b + 3k \end{cases} . \tag{10.11}$$

The same reasoning of the previous case leads us to

$$\begin{cases} r_+ = 2b + 2k \ , \quad r_- = 0 \\ \vdots \\ r_+ = b + 2k \ , \quad r_- = b \end{cases} . \tag{10.12}$$

That is,

$$\{e((b+3k, b); (b+k, b+k, b+k), I) : I = k, ..., b+k\} \tag{10.13}$$

spans the subspace fixed by T.

To finish, we order the I-multiplicities by crescent I in both cases (10.10)-(10.13) by setting $I_\gamma = \gamma + k - 1$, where $1 \le \gamma \le \min\{a, b\} + 1$. □

Definition 10.1.2. *The $\mathcal{E}$* harmonics *are the functions*

$$Z^{(a,b,\gamma)}_{\boldsymbol{\nu},I}(g\mathbf{z}_0) = \sqrt{\frac{(a+1)(b+1)(a+b+2)}{2}}\, \overline{D^{(a,b)}_{\boldsymbol{\nu}I,\boldsymbol{\nu}_{(a,b)}I_\gamma}(g)} \tag{10.14}$$

on $\mathcal{E}$, for $\mathbf{z}_0 \in \pi^{-1}([0:0:1]) \subset \mathcal{E}$ with T as isotropy subgroup, $g \in SU(3)$, $a \equiv b \pmod 3$, $(\boldsymbol{\nu}_{(a,b)}, I_\gamma)$ as in Proposition 10.1.1 and $D^{(a,b)}_{\boldsymbol{\nu}I,\boldsymbol{\nu}_{(a,b)}I_\gamma}$ a Wigner D-function as in Definition 8.1.3.

Just as in definition of $\mathbb{C}P^2$ harmonics, the factor $\sqrt{(a+1)(b+1)(a+b+2)/2}$ is the square root of the dimension of the representation $Q(a, b)$ and is used to ensure normalization according to Schur's Orthogonality Relations, so that

$$\left\langle Z^{(a,b,\gamma)}_{\boldsymbol{\nu},I} \middle| Z^{(c,d,\zeta)}_{\boldsymbol{\mu},J} \right\rangle = \delta_{a,c}\delta_{b,d}\delta_{\gamma,\zeta}\delta_{\boldsymbol{\nu},\boldsymbol{\mu}}\delta_{I,J} \ . \tag{10.15}$$

Remark 10.1.1. *Analogously to Remark 9.1.1, for any $x, y > 0$, we can take the generic harmonics as functions on $\mathcal{O}_{x,y}$ via the compositions $Z^{(a,b,\gamma)}_{\boldsymbol{\nu},I} \circ \psi_{x,y}$ so that the harmonic functions on $\mathcal{O}_{x,y}$ are related to the ones on $\mathcal{O}_{y,x}$ by $\alpha_{x,y} \circ \iota$, cf. (8.170). Besides that, the involution $\alpha_{x,y}$ generates another, but somewhat equivalent, set of harmonic functions on $\mathcal{O}_{x,y}$, just as α does to $\mathcal{E}$:*

$$\widetilde{Z}^{(a,b,\gamma)}_{\boldsymbol{\nu},I}(g\boldsymbol{z}_0) = Z^{(a,b,\gamma)}_{\boldsymbol{\nu},I}(gg_0\mathbf{z}_0) \ , \tag{10.16}$$

cf. (8.169).

As expected, we have

$$Z^{(0,0)}_{(0,0,0),0} \equiv 1 \tag{10.17}$$

and, cf. (8.38),

$$\overline{Z^{(a,b,\gamma)}_{\boldsymbol{\nu},I}} = (-1)^{2(t+u)} Z^{(b,a,\gamma)}_{\breve{\boldsymbol{\nu}},I} \ , \ \text{for} \ \ \Delta^{a+b}_{\boldsymbol{\nu},\breve{\boldsymbol{\nu}}} = 1 \ . \tag{10.18}$$

Proposition 10.1.2. *The decomposition of pointwise product of $\mathcal{E}$ harmonics is given by*

$$Z^{(\boldsymbol{a}_1,\gamma_1)}_{\boldsymbol{\nu}_1,I_1} Z^{(\boldsymbol{a}_2,\gamma_2)}_{\boldsymbol{\nu}_2,I_2} = \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I,\gamma}} \sqrt{\frac{\dim Q(\boldsymbol{a}_1)\dim Q(\boldsymbol{a}_2)}{\dim Q(\boldsymbol{a})}} C^{\boldsymbol{a}_1,\ \ \boldsymbol{a}_2,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I} C^{\boldsymbol{a}_1,\ \ \ \ \boldsymbol{a}_2,\ \ \ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_{\boldsymbol{a}_1} I_{\gamma_1},\boldsymbol{\nu}_{\boldsymbol{a}_2} I_{\gamma_2},\ \boldsymbol{\nu}_{\boldsymbol{a}} I_\gamma} Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I}\ , \tag{10.19}$$

for $(\boldsymbol{\nu}_{\boldsymbol{a}_j}, I_{\gamma_j})$ and $(\boldsymbol{\nu}_{\boldsymbol{a}}, I_\gamma)$ as in Proposition 10.1.1, and summation restricted to $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = 1$, $\delta(I_1, I_2, I) = \delta(I_{\gamma_1}, I_{\gamma_2}, I_\gamma) = 1$ and $Q(\boldsymbol{a};\sigma)$ in the Clebsch-Gordan series of $Q(\boldsymbol{a}_1)\otimes Q(\boldsymbol{a}_2)$.

Proof. With a little abuse of notation, again,

$$Z^{(\boldsymbol{a}_j,\gamma_j)}_{\boldsymbol{\nu}_j,I_j} = \sqrt{\dim Q(\boldsymbol{a}_j)}\ \overline{D^{\boldsymbol{a}_j}_{\boldsymbol{\nu}_j I_j,\boldsymbol{\nu}_{\boldsymbol{a}_j} I_{\gamma_j}}} \tag{10.20}$$

and Lemma 8.2.2 give us

$$Z^{(\boldsymbol{a}_1,\gamma_1)}_{\boldsymbol{\nu}_1,I_1} Z^{(\boldsymbol{a}_2,\gamma_2)}_{\boldsymbol{\nu}_2,I_2} = \sum_{(\boldsymbol{a};\sigma)} \sum_{\substack{\boldsymbol{\nu},I\\ \boldsymbol{\mu},J}} \sqrt{\dim Q(\boldsymbol{a}_1)\dim Q(\boldsymbol{a}_2)}\, C^{\boldsymbol{a}_1,\ \ \boldsymbol{a}_2,\ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_1 I_1,\boldsymbol{\nu}_2 I_2,\ \boldsymbol{\nu} I} C^{\boldsymbol{a}_1,\ \ \ \ \boldsymbol{a}_2,\ \ \ \ (\boldsymbol{a};\sigma)}_{\boldsymbol{\nu}_{\boldsymbol{a}_1} I_{\gamma_1},\boldsymbol{\nu}_{\boldsymbol{a}_2} I_{\gamma_2},\ \boldsymbol{\mu} J} \overline{D^{\boldsymbol{a}}_{\boldsymbol{\nu} I,\boldsymbol{\mu} J}}\ , \tag{10.21}$$

where $\nabla_{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2,\boldsymbol{\nu}} = \nabla_{\boldsymbol{\nu}_{\boldsymbol{a}_1}+\boldsymbol{\nu}_{\boldsymbol{a}_2},\boldsymbol{\mu}} = 1$ and $\delta(I_1, I_2, I) = \delta(I_{\gamma_1}, I_{\gamma_2}, J) = 1$, so $\boldsymbol{\mu} = (\mu,\mu,\mu)$. But $\boldsymbol{e}(\boldsymbol{a};(\mu,\mu,\mu),J)$ only exists if $\boldsymbol{a}$ and $(\boldsymbol{\mu}, J)$ are as in Proposition 10.1.1. Thus, we set $\boldsymbol{\mu} = \boldsymbol{\nu}_{\boldsymbol{a}}$ and $J = I_\gamma$. ∎

Remark 10.1.2. *As in the decomposition of the pointwise product of $\mathbb{C}P^2$ harmonics, the decomposition of the pointwise product of $\mathcal{E}$ harmonics follows directly as a special case of Lemma 8.2.2 and does not "see" the symplectic structure on $\mathcal{E}$. Thus, as in the pure-quark case, the next step is to decompose the Poisson bracket of $\mathcal{E}$ harmonics, but this is a much harder problem that is deferred to a later study.*

10.2 Quantum generic quark system

Now, we want representations $Q(p,q)$ such that $Q(p,q)\otimes Q(q,p)$ splits only into representations of the form $Q(a,b)$, $a\equiv b \pmod 3$, with multiplicity less than or equal to $\min\{a,b\}+1$. From Corollary 8.2.1.1, if we suppose, without loss of generality, that $\min\{a,b\} = a$, then the occurrences of $Q(a,b)\oplus Q(b,a)$ are given by the solutions of

$$\begin{cases} a = p+q-n-m-2k \\ 0 \le n \le p-k \\ 0 \le m \le q-k \end{cases}\ , \tag{10.22}$$

where $b = a + 3k$. Of course, we can also assume without loss of generality that $p \geq q$. If $a + k \leq q$, then we have $a + 1$ solutions:

$$\begin{cases} n = p - k - a \ , \quad m = q - k \\ n = p - k - a + 1 \ , \quad m = q - k - 1 \\ \vdots \\ n = p - k \ , \quad m = q - a - k \end{cases} . \tag{10.23}$$

Otherwise, we need to eliminate some lines of the above solutions, which means $Q(a, b) \oplus Q(b, a)$ have multiplicity less than $\min\{a, b\} + 1$. Then, we are able to define:

Definition 10.2.1. *Let*[1] $(p, q) \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})$. *A* quantum generic quark system *is a complex Hilbert space* $\mathcal{H}_{p,q} \simeq \mathbb{C}^d$, *where*

$$d = \frac{(p+1)(q+1)(p+q+2)}{2} \ ,$$

with an irreducible unitary $SU(3)$*-representation of class* $Q(p, q)$ *together with its operator algebra* $\mathcal{B}(\mathcal{H}_{p,q})$.

If $p \geq q$, *the pair* (p, q) *and the system* $\mathcal{H}_{p,q}$ *are called* material. *If* $p > q$, *they are called* baryonic *and if* $p = q$ *they are called* mesonic. *Alternatively, if* $p < q$, *they are called antibaryonic.*

Remark 10.2.1. *The name* material *refers to a system composed of a larger (or equal) number of quarks than antiquarks. From Theorem 8.2.1,*

$$Q(p, 0) \otimes Q(0, q) = \bigoplus_{n=0}^{\min\{p,q\}} Q(p - n, q - n) \ , \tag{10.24}$$

so a generic representation $Q(p, q)$ *is the invariant space of* $Q(p, 0) \otimes Q(0, q)$ *where the product of the highest weight vectors lives in. That means material quark systems can be constructed from systems with number of quarks greater than or equal to number of antiquarks.*

The names baryonic *and* mesonic *make reference to systems with positive and null* baryon number, *respectively. We recall that a system of* p *quarks and* q *antiquarks has baryon number*

$$B = \frac{1}{3}(p - q) \ . \tag{10.25}$$

If $B > 0$, *the system is classified as a baryon; if* $B < 0$ *it is an antibaryon; if* $B = 0$, *we have a meson.*

In particular, quantum generic quark systems encompass quantum pure-quark systems as special cases. But now, all forms of (8.151) and (8.152) are relevant to us, and we cannot further simplify Corollary 8.2.10.1 as we did for pure-quark systems.

[1] Again, we are ignoring the trivial representation $Q(0, 0)$.

10.3 Symbol correspondences for generic quark systems

Let $\boldsymbol{p} \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})$.

The following is completely analogous to Definition 9.3.1.

Definition 10.3.1. *A* symbol correspondence for a generic quark system $(\mathcal{H}_{\boldsymbol{p}}, Q(\boldsymbol{p}))$, *also referred to simply as a* symbol correspondence *or just as a* correspondence, *is an injective linear map* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E}) : P \mapsto W_P$ *that satisfies, for any* $A \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$,

i) Equivariance: $\forall g \in SU(3)$, $W_{A^g} = (W_A)^g$;

ii) Reality: $W_{A^\dagger} = \overline{W_A}$;

iii) Normalization: $\int_{\mathcal{E}} W_A(\mathbf{z})d\mathbf{z} = \dfrac{1}{\dim Q(\boldsymbol{p})}\operatorname{tr}(A)$.

Remark 10.3.1. *In the spirit of Remark 9.3.1, one can replace* $\mathcal{E}$ *by* $\mathcal{O}_{x,y}$ *for any* $x, y > 0$ *using the diffeomorphism* $\psi_{x,y}$.

Notation 6. *Recalling the notations*

$$\boldsymbol{a} = (a, b) \ , \quad \boldsymbol{p} = (p, q) \iff \check{\boldsymbol{p}} = (q, p) \ ,$$

from now on, we shall use the notation

$$m(\boldsymbol{a}) = m(a, b) = \min\{a, b\} + 1 \ . \tag{10.26}$$

and simplify the notation $\mathfrak{m}(\boldsymbol{p}, \check{\boldsymbol{p}}; \boldsymbol{a})$ *for the multiplicity of* $Q(\boldsymbol{a}) = Q(a, b)$ *in the Clebsch-Gordan series of* $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}}) = Q(p, q) \otimes Q(q, p)$ *by setting*

$$\mathfrak{m}(\boldsymbol{p}; \boldsymbol{a}) := \mathfrak{m}(\boldsymbol{p}, \check{\boldsymbol{p}}; \boldsymbol{a}) \ . \tag{10.27}$$

Finally, we set

$$\mathcal{B}(\boldsymbol{p}; \boldsymbol{a}) = \bigoplus_{\sigma=1}^{\mathfrak{m}(\boldsymbol{p};\boldsymbol{a})} Q(\boldsymbol{a}; \sigma) \subset \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \ . \tag{10.28}$$

Theorem 10.3.1. *A linear map* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E}) : A \mapsto W_A$ *is a symbol correspondence if and only if, for each* $Q(\boldsymbol{a}; \sigma)$ *in* $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$, *it maps*

$$W : \sqrt{\dim Q(\boldsymbol{p})}\, \boldsymbol{e}((\boldsymbol{a}; \sigma); \boldsymbol{\nu}, I) \mapsto \sum_{\gamma=1}^{m(\boldsymbol{a})} c^\sigma_\gamma(\boldsymbol{a}) Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I} =: Z\mathfrak{C}(\boldsymbol{a})^\sigma_{\boldsymbol{\nu},I} \quad , \tag{10.29}$$

where $c^\sigma_\gamma(\boldsymbol{a})$ *is the* $\gamma \times \sigma$ *entry of a complex matrix of order* $m(\boldsymbol{a}) \times \mathfrak{m}(\boldsymbol{p}; \boldsymbol{a})$ *denoted by* $\mathfrak{C}(\boldsymbol{a})$, *that is,*

$$\mathfrak{C}(\boldsymbol{a}) = [c^\sigma_\gamma(\boldsymbol{a})] \ , \tag{10.30}$$

with $\mathfrak{C}(\boldsymbol{a})$ *being of full rank and satisfying* $\overline{\mathfrak{C}(\boldsymbol{a})} = \mathfrak{C}(\check{\boldsymbol{a}})$ *and* $\mathfrak{C}(0, 0) = (-1)^{|\boldsymbol{p}|}$.

Proof. Since W is injective and equivariant, the image of $Q(\boldsymbol{a};\sigma)$ is a representation isomorphic to $Q(\boldsymbol{a})$. Let $\{f((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\}$ be a GT basis of the image of $Q(\boldsymbol{a};\sigma)$, so that

$$W : \sqrt{\dim Q(\boldsymbol{p})}\,\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) \mapsto \alpha_{(\boldsymbol{a};\sigma)} f((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) \ , \quad \alpha_{(\boldsymbol{a};\sigma)} \neq 0 \ . \tag{10.31}$$

Because the multiplicity of $Q(\boldsymbol{a})$ in $C^\infty_{\mathbb{C}}(\mathcal{E})$ is $m(\boldsymbol{a})$, cf. (10.2), we must have

$$f((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) = \sum_{\gamma=1}^{m(\boldsymbol{a})} \beta_\gamma^{(\boldsymbol{a};\sigma)} Z_{\boldsymbol{\nu},I}^{(\boldsymbol{a},\gamma)} \ , \tag{10.32}$$

where $Z_{\boldsymbol{\nu},I}^{(\boldsymbol{a},\gamma)}$ are the $\mathcal{E}$ harmonics, cf. Definition 10.1.2, which implies in particular that

$$\sum_{\gamma=1}^{m(\boldsymbol{a})} \left|\beta_\gamma^{(\boldsymbol{a};\sigma)}\right|^2 = \|f((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\|^2 = 1 \ . \tag{10.33}$$

Let

$$c_\gamma^\sigma(\boldsymbol{a}) = \alpha_{(\boldsymbol{a};\sigma)}\beta_\gamma^{(\boldsymbol{a};\sigma)} \ . \tag{10.34}$$

The injection hypothesis implies that the union of basis

$$\bigcup_{\sigma=1}^{\mathfrak{m}(\boldsymbol{p};\boldsymbol{a})} \{f((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)\}$$

is a linearly independent set, hence $\{(\beta_1^{(\boldsymbol{a};\sigma)},...,\beta_{m(\boldsymbol{a})}^{(\boldsymbol{a};\sigma)}) : \sigma = 1,...,\mathfrak{m}(\boldsymbol{p};\boldsymbol{a})\}$ is a linearly independent set in $\mathbb{C}^{m(\boldsymbol{a})}$, cf. (10.32). This means that $\{(c_1^\sigma(\boldsymbol{a}),...,c_{m(\boldsymbol{a})}^\sigma(\boldsymbol{a})) : \sigma = 1,...,\mathfrak{m}(\boldsymbol{p};\boldsymbol{a})\}$ is a linearly independent set too, cf. (10.34), so the complex matrix $\mathfrak{C}(\boldsymbol{a})$ whose $\gamma\times\sigma$ entry is $c_\gamma^\sigma(\boldsymbol{a})$ is of full rank.

We have that $\boldsymbol{e}^\dagger((\boldsymbol{a};\sigma);\boldsymbol{\nu},I) = (-1)^{2(t+u)}\boldsymbol{e}((\check{\boldsymbol{a}};\sigma);\check{\boldsymbol{\nu}},I)$, cf. (8.127), and also that $\overline{Z_{\boldsymbol{\nu},I}^{(\boldsymbol{a},\gamma)}} = (-1)^{2(t+u)}Z_{\check{\boldsymbol{\nu}},I}^{(\check{\boldsymbol{a}},\gamma)}$, cf. (10.18), so the reality condition implies

$$\overline{c_\gamma^\sigma(\boldsymbol{a})} = c_\gamma^\sigma(\check{\boldsymbol{a}}) \ , \tag{10.35}$$

or in a concise form, the matrices $\mathfrak{C}(\boldsymbol{a})$ satisfy $\overline{\mathfrak{C}(\boldsymbol{a})} = \mathfrak{C}(\check{\boldsymbol{a}})$.

The normalization property implies

$$W : (-1)^{|\boldsymbol{p}|}\sqrt{\dim Q(p,q)}\boldsymbol{e}((0,0);(0,0,0),0) \mapsto Z_{(0,0,0),0}^{(0,0)} \ , \tag{10.36}$$

cf. (8.128) and (10.17), hence $\mathfrak{C}(0,0) = (-1)^{|\boldsymbol{p}|}$.

It is more straightforward to prove the converse, that is, to check that a map with these properties is a symbol correspondence, so we leave this to the reader. □

Corollary 10.3.1.1. *The moduli space $\mathfrak{S}_{\boldsymbol{p}}$ of correspondences for a generic quark system $\mathcal{H}_{\boldsymbol{p}}$ can be described as*

$$\mathfrak{S}_{\boldsymbol{p}} \ = \ \left(\prod_{a=0}^{|\boldsymbol{p}|} V_{\mathfrak{m}(\boldsymbol{p};a,a)}(\mathbb{R}^{a+1})\right) \times \left(\prod_{a<b} V_{\mathfrak{m}(\boldsymbol{p};a,b)}(\mathbb{C}^{a+1})\right) \ , \tag{10.37}$$

where $V_k(\mathbb{K}^n) = GL_n(\mathbb{K})/GL_{n,k}(\mathbb{K})$, for $GL_{n,k}(\mathbb{K}) \subset GL_n(\mathbb{K})$ a maximal subgroup that fixes a k-dimensional subspace, is a non compact Stiefel manifold.

The matrices $\mathfrak{C}(\boldsymbol{a})$ are matrix representations of the maps $W|_{\mathcal{B}(\boldsymbol{p};\boldsymbol{a})}$ with respect to a coupled basis of $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ and the $\mathcal{E}$ harmonics. They are analogous to characteristic numbers of symbol correspondences for pure-quark system: in the latter case, the domain and codomain of a symbol correspondence are multiplicity free and have only representations $Q(n,n)$, so it provides a 1×1 real matrix indexed by n. The moduli space in that case is a product of $V_1(\mathbb{R}) = \mathbb{R}^*$.

We now prove a proposition analogous to Proposition 9.3.2.

As usual, let $\mathbf{z}_0 \in \pi^{-1}([0:0:1]) \subset \mathcal{E}$ be a point with T as isotropy subgroup. Now, for an operator $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ fixed by T, we have $\mathcal{E} \to \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) : \mathbf{z} \mapsto K(\mathbf{z}) = K^g$, where $g \in SU(3)$ is such that $\mathbf{z} = g\mathbf{z}_0$.

Proposition 10.3.2. *A map $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E}) : A \mapsto W_A$ is a symbol correspondence satisfying (10.29) if and only if*

$$W_A(\mathbf{z}) = \mathrm{tr}(AK(\mathbf{z})) \ , \tag{10.38}$$

that is,

$$W_A(g\mathbf{z}_0) = \mathrm{tr}(AK^g) \ , \tag{10.39}$$

for $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ of the form

$$K = \frac{1}{\dim Q(\boldsymbol{p})}\mathbb{1} + \sum_{(\boldsymbol{a};\sigma)} \sum_{\gamma=1}^{m(\boldsymbol{a})} \overline{c^\sigma_\gamma(\boldsymbol{a})} \sqrt{\frac{\dim Q(\boldsymbol{a})}{\dim Q(\boldsymbol{p})}}\, \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}_{\boldsymbol{a}}, I_\gamma) \ , \tag{10.40}$$

with $c^\sigma_\gamma(\boldsymbol{a}) = [\mathfrak{C}(\boldsymbol{a})]^\sigma_\gamma$ as in Theorem 10.3.1, where the summation is over all $(\boldsymbol{a};\sigma)$ in the CG series of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$. In particular, K is Hermitian with unitary trace.

Proof. Assuming W is a symbol correspondence, we can reproduce the proof of Proposition 9.3.2 to conclude $W_A(g\mathbf{z}_0) = \mathrm{tr}(AK^g)$, where K is a linear combination of the vectors fixed by T, so

$$K = \sum_{\substack{(\boldsymbol{a};\sigma)\\ \gamma}} k^{(\boldsymbol{a};\sigma)}_\gamma \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}_{\boldsymbol{a}}, I_\gamma) \ , \tag{10.41}$$

cf. Proposition 10.1.1.

For $A = \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}, I) = (-1)^{2(t+u)}\boldsymbol{e}^\dagger((\check{\boldsymbol{a}};\sigma);\check{\boldsymbol{\nu}}, I)$ we get

$$W_A(g\mathbf{z}_0) = \mathrm{tr}(AK^g) = \sum_{\gamma=1}^{m(\boldsymbol{a})} k^{(\check{\boldsymbol{a}};\sigma)}_\gamma (-1)^{2(t+u)} D^{\check{\boldsymbol{a}}}_{\check{\boldsymbol{\nu}}I,\boldsymbol{\nu}_{\check{\boldsymbol{a}}}I_\gamma}(g) = \sum_{\gamma=1}^{m(\boldsymbol{a})} k^{(\check{\boldsymbol{a}};\sigma)}_\gamma \overline{D^{\boldsymbol{a}}_{\boldsymbol{\nu}I,\boldsymbol{\nu}_{\boldsymbol{a}}I_\gamma}} \ , \tag{10.42}$$

cf (8.38). Then,

$$W_A = \sum_{\gamma=1}^{m(\boldsymbol{a})} \frac{k^{(\check{\boldsymbol{a}};\sigma)}_\gamma}{\sqrt{\dim Q(\boldsymbol{a})}} Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I} \ . \tag{10.43}$$

It follows from Theorem 10.3.1 that

$$k_\gamma^{(\check{\boldsymbol{a}};\sigma)} = c_\gamma^\sigma(\boldsymbol{a})\sqrt{\frac{\dim Q(\boldsymbol{a})}{\dim Q(\boldsymbol{p})}} \iff k_\gamma^{(\boldsymbol{a};\sigma)} = c_\gamma^\sigma(\check{\boldsymbol{a}})\sqrt{\frac{\dim Q(\check{\boldsymbol{a}})}{\dim Q(\boldsymbol{p})}}\,. \tag{10.44}$$

Using (10.35) and $\dim Q(\boldsymbol{a}) = \dim Q(\check{\boldsymbol{a}})$, we obtain (10.40).

The Hermitian property of K follows from (10.35) and (8.127) plus the fact that, if $Q(\boldsymbol{a};\sigma)$ is in the CG series of $Q(\boldsymbol{p}) \otimes Q(\check{\boldsymbol{p}})$, then so is $Q(\check{\boldsymbol{a}};\sigma)$. Unitary trace for K is immediate from the fact that every $\boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}_{\boldsymbol{a}}, I_\gamma)$ is traceless (orthogonal to $\mathbb{1}$).

The converse is, again, analogous to Proposition 9.3.2, and it is rather straightforward to verify that, for K given by (10.40), equations (10.41)-(10.43) imply that (10.39) defines a symbol correspondence given by (10.29). $\square$

Definition 10.3.2. *Any $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ that induces a symbol correspondence via (10.39) is an* operator kernel*. Thus, K is given by (10.40), where the numbers $(c_\gamma^\sigma(\boldsymbol{a}))$ are called* characteristic parameters *and the matrices $\mathfrak{C}(\boldsymbol{a})$ with $c_\gamma^\sigma(\boldsymbol{a})$ in the $\gamma \times \sigma$ entry, cf. (10.30), are the* characteristic matrices *of both the operator kernel and the symbol correspondence.*

Remark 10.3.2. *It is worth to explicitly point out that an operator kernel K of a symbol correspondence for a pure quark system $\mathcal{H}_{p,0}$ (or, equivalently, $\mathcal{H}_{0,p}$) is also an operator kernel of a symbol correspondence for $\mathcal{H}_{p,0}$ taken as a generic quark system. If K has characteristic numbers (c_n), it has characteristic parameters $(c_\gamma^1(n,n))$ given by $c_\gamma^1(n,n) = c_n\delta_{\gamma,1}$.*

Now, in the case of generic quark systems, we cannot unambiguously define symbol correspondences in a implicit way as we did in Proposition 9.3.4 because characteristic matrices may have more than one left inverse.

Proposition 10.3.3. *Let $K \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ be an operator kernel with characteristic matrices $\mathfrak{C}(\boldsymbol{a})$. A symbol correspondence $\widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ satisfies*

$$A = \dim Q(\boldsymbol{p}) \int_{\mathcal{E}} \widetilde{W}_A(\mathbf{z}) K(\mathbf{z}) d\mathbf{z} \tag{10.45}$$

if and only if it has characteristic matrices $\widetilde{\mathfrak{C}}(\boldsymbol{a})$ such that $(\widetilde{\mathfrak{C}}(\boldsymbol{a}))^\dagger \mathfrak{C}(\boldsymbol{a}) = \mathbb{1}$.

Proof. By straightforward calculation, we get

$$\begin{aligned}
\int_{\mathcal{E}} Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I}(\mathbf{z})K(\mathbf{z})d\mathbf{z} &= \int_{SU(3)} Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I}(g\mathbf{z}_0)K^g dg \\
&= \sum_{\substack{(\boldsymbol{a}';\sigma')\\ \gamma' \\ \boldsymbol{\mu},J}} k^{(\boldsymbol{a}';\sigma')}_{\gamma'} \int_{SU(3)} Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I}(g\mathbf{z}_0) D^{\boldsymbol{a}'}_{\boldsymbol{\mu}J,\boldsymbol{\nu}_{\boldsymbol{a}'}I_{\gamma'}}(g)dg\, \boldsymbol{e}((\boldsymbol{a}';\sigma');\boldsymbol{\mu},J) \\
&= \sum_{\substack{(\boldsymbol{a}';\sigma')\\ \gamma' \\ \boldsymbol{\mu},J}} \frac{k^{(\boldsymbol{a}';\sigma')}_{\gamma'}}{\sqrt{\dim Q(\boldsymbol{a}')}} \left\langle Z^{(\boldsymbol{a}',\gamma')}_{\boldsymbol{\mu},J} \middle| Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I} \right\rangle \boldsymbol{e}((\boldsymbol{a}';\sigma');\boldsymbol{\mu},J) \\
&= \sum_{\sigma'=1}^{\mathfrak{m}(\boldsymbol{p};\boldsymbol{a})} \frac{k^{(\boldsymbol{a};\sigma')}_{\gamma}}{\sqrt{\dim Q(\boldsymbol{a})}} \boldsymbol{e}((\boldsymbol{a};\sigma');\boldsymbol{\nu},I) \ ,
\end{aligned} \tag{10.46}$$

where

$$k^{(\boldsymbol{a}';\sigma')}_{\gamma'} = \overline{c^{\sigma'}_{\gamma'}(\boldsymbol{a}')} \sqrt{\frac{\dim Q(\boldsymbol{a}')}{\dim Q(\boldsymbol{p})}} \ , \tag{10.47}$$

cf. (10.44). So, for $c^{\sigma}_{\gamma}(\boldsymbol{a})$ and $\tilde{c}^{\sigma}_{\gamma}(\boldsymbol{a})$ being the characteristic parameters associated to $\mathfrak{C}(\boldsymbol{a})$ and $\widetilde{\mathfrak{C}}(\boldsymbol{a})$, respectively, we have, cf. (10.29),

$$\dim Q(\boldsymbol{p}) \int_{\mathcal{E}} Z\widetilde{\mathfrak{C}}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I}(\mathbf{z})K(\mathbf{z})d\mathbf{z} = \sqrt{\dim Q(\boldsymbol{p})} \sum_{\gamma,\sigma'} \tilde{c}^{\sigma}_{\gamma}(\boldsymbol{a}) \overline{c^{\sigma'}_{\gamma}(\boldsymbol{a})}\, \boldsymbol{e}((\boldsymbol{a};\sigma');\boldsymbol{\nu},I) \ . \tag{10.48}$$

Hence, (10.45) holds for $A = \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu},I)$ if and only if

$$\sum_{\gamma=1}^{m(\boldsymbol{a})} \tilde{c}^{\sigma}_{\gamma}(\boldsymbol{a}) \overline{c^{\sigma'}_{\gamma}(\boldsymbol{a})} = \delta_{\sigma,\sigma'} \ , \tag{10.49}$$

which means $(\mathfrak{C}(\boldsymbol{a}))^{\dagger}\widetilde{\mathfrak{C}}(\boldsymbol{a}) = \mathbb{1}$, or equivalently $(\widetilde{\mathfrak{C}}(\boldsymbol{a}))^{\dagger}\mathfrak{C}(\boldsymbol{a}) = \mathbb{1}$. □

Now, let

$$\langle A|R\rangle_{\boldsymbol{p}} = \frac{1}{\dim Q(\boldsymbol{p})} \langle A|R\rangle = \frac{1}{\dim Q(\boldsymbol{p})} \operatorname{tr}\left(A^{\dagger}R\right) \tag{10.50}$$

be the normalized inner product in $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ and $\|\cdot\|_{\boldsymbol{p}}$ be the induced norm.

Definition 10.3.3. *Two symbol correspondences $W, \widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^{\infty}(\mathcal{E})$ satisfying*

$$\langle A|R\rangle_{\boldsymbol{p}} = \left\langle \widetilde{W}_A \middle| W_R \right\rangle = \left\langle W_A \middle| \widetilde{W}_R \right\rangle \tag{10.51}$$

for every $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ are said to be dual correspondences. *The operators kernels of W and $\widetilde{W}$ are also said to be* dual *to each other.*

Proposition 10.3.4. *Two symbol correspondences $W, \widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^{\infty}(\mathcal{E})$ with characteristic matrices $\mathfrak{C}(\boldsymbol{a})$ and $\widetilde{\mathfrak{C}}(\boldsymbol{a})$ are dual to each other if and only if $(\widetilde{\mathfrak{C}}(\boldsymbol{a}))^{\dagger}\mathfrak{C}(\boldsymbol{a}) = \mathbb{1}$.*

Proof. The proof follows analogous to the proof of Proposition 9.3.5 by writing the operators using (10.45) and symbols using (10.39). □

Remark 10.3.3. *Duality is no longer* $1 \leftrightarrow 1$*. Consider, for instance, the correspondences* $W, \widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ *defined respectively by the characteristic parameters* $c^\sigma_\gamma(\boldsymbol{a}) = \delta_{\gamma,\sigma}$ *and* $\tilde{c}^\sigma_\gamma(\boldsymbol{a}) = \delta_{\gamma,\sigma} + \delta_{\gamma,m(\boldsymbol{a})}\delta_{\sigma,1}$*. Then, both* $\widetilde{W}$ *and* W *itself are dual to* W*.*

The correspondence W of the previous remark is obviously an isometry. In addition to such special cases of correspondences which are isometric, now we also have correspondences given by a direct sum of conformal maps.

Definition 10.3.4. *A symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ *is a* Stratonovich-Weyl correspondence *if it is an isometry, that is,*

$$\langle A|R\rangle_{\boldsymbol{p}} = \langle W_A|W_R\rangle \tag{10.52}$$

for all $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$*. If* W *preserves angles for each* $\mathcal{B}(\boldsymbol{p};\boldsymbol{a})$*, that is,*

$$\frac{\langle A|R\rangle_{\boldsymbol{p}}}{\|A\|_{\boldsymbol{p}}\|R\|_{\boldsymbol{p}}} = \frac{\langle W_A|W_R\rangle}{\|W_A\|\|W_R\|} \tag{10.53}$$

for all non null $A, R \in \mathcal{B}(\boldsymbol{p};\boldsymbol{a})$ *and every* $\mathcal{B}(\boldsymbol{p};\boldsymbol{a}) \subset \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$*, cf. (10.28), then* W *shall be called a* semi-conformal correspondence*.*

Proposition 10.3.5. *A symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ *is a Stratonovich-Weyl correspondence if and only if its characteristic matrices are unitary matrices, that is, they satisfy* $(\mathfrak{C}(\boldsymbol{a}))^\dagger\mathfrak{C}(\boldsymbol{a}) = \mathbb{1}$*. Furthermore,* W *is a semi-conformal correspondence if and only if its characteristic matrices are conformal matrices, that is,* $(\mathfrak{C}(\boldsymbol{a}))^\dagger\mathfrak{C}(\boldsymbol{a}) = \alpha(\boldsymbol{a})\,\mathbb{1}$ *for* $\alpha(\boldsymbol{a}) > 0$*, where* $\alpha(\boldsymbol{a}) = \alpha(\check{\boldsymbol{a}})$ *and* $\alpha(0,0) = 1$*.*

Proof. Proposition 10.3.4 implies that W is its own dual if and only if $(\mathfrak{C}(\boldsymbol{a}))^\dagger\mathfrak{C}(\boldsymbol{a}) = \mathbb{1}$ holds, which proves the first part of the statement. For the second part, we use that a linear map is conformal if and only if it is a positive real multiple of an unitary linear map, thus W is a semi-conformal correspondence if and only if there is $\alpha(\boldsymbol{a}) > 0$ for each $\mathcal{B}(\boldsymbol{p};\boldsymbol{a})$ such that $\alpha(\boldsymbol{a})^{-1/2}W|_{\mathcal{B}(\boldsymbol{p};\boldsymbol{a})}$ is an unitary map, and this is true if and only if the characteristic matrices of W satisfy $(\mathfrak{C}(\boldsymbol{a}))^\dagger\mathfrak{C}(\boldsymbol{a}) = \alpha(\boldsymbol{a})\,\mathbb{1}$. The equations $\alpha(0,0) = 1$ and $\alpha(\boldsymbol{a}) = \alpha(\check{\boldsymbol{a}})$ follows from $\overline{\mathfrak{C}(\boldsymbol{a})} = \mathfrak{C}(\check{\boldsymbol{a}})$ and $\mathfrak{C}(0,0) = (-1)^{|\boldsymbol{p}|}$. □

Remark 10.3.4. *A symbol correspondence* W *is an actual conformal map if and only if* $W = \sqrt{\alpha}\,W'$ *for* $\alpha > 0$ *and some Stratonovich-Weyl correspondence* W'*. Since* $W_{\mathbb{1}} = W'_{\mathbb{1}}$*, we must have* $\alpha = 1$*, so the only actual conformal correspondences are the isometric ones.*

We also point out that, for pure-quark systems (likewise for spin systems), every symbol correspondence is a semi-conformal correspondence, with $\alpha(\boldsymbol{a}) = \alpha(n,n) = c_n^2$*.*

Propositions 10.3.3 – 10.3.5 illustrate how characteristic matrices encode all the information about symbol correspondences for generic quark systems in the same vein of characteristic numbers for pure-quark system. The existence of multiple dual representations can be explained by the existence of invariant subspaces $Q(\boldsymbol{a})$ with higher degeneracy within $C^\infty_{\mathbb{C}}(\mathcal{E})$ than within $\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$, or equivalently by the existence of multiple left inverses of characteristic matrices.

In general, there is no natural way to choose a dual correspondence among all possibilities. For semi-conformal symbol correspondences, however, we make the following definition:

Definition 10.3.5. *Let $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ be a semi-conformal correspondence with characteristic matrices $\mathfrak{C}(\boldsymbol{a})$ satisfying $(\mathfrak{C}(\boldsymbol{a}))^\dagger \mathfrak{C}(\boldsymbol{a}) = \alpha(\boldsymbol{a})\,\mathbb{1}$. Its* canonical dual correspondence *is the symbol correspondence $\widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ with characteristic matrices*

$$\widetilde{\mathfrak{C}}(\boldsymbol{a}) = \frac{1}{\alpha(\boldsymbol{a})}\mathfrak{C}(\boldsymbol{a}) \ . \tag{10.54}$$

Thus, Stratonovich-Weyl correspondences for generic quark systems are their own canonical dual correspondences.

Just as for pure-quark systems, positive operator kernels provide special symbol correspondences for generic quark systems.

Definition 10.3.6. *A symbol correspondence W for a generic quark system is* mapping-positive *if it maps positive(-definite) operators to (strictly-)positive functions. If $\widetilde{W}$ is dual to a mapping-positive correspondence, then $\widetilde{W}$ is a* positive-dual *correspondence.*

Proposition 10.3.6. *A symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with operator kernel K is mapping-positive if and only if K is a state, that is, K is a positive operator.*

Proof. Suppose K is a positive operator, so $K = R^\dagger R$ for some $R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$, and let $A = M^\dagger M \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$ be a positive operator. Then, for any $g \in SU(3)$ and $\widetilde{M} = M\rho(g)$,

$$W_A(g\mathbf{z}_0) = \mathrm{tr}\Big(M^\dagger M \rho(g) K \rho(g)^\dagger\Big) = \mathrm{tr}\Big(M\rho(g)K\rho(g)^\dagger M^\dagger\Big) = \mathrm{tr}\Big(\widetilde{M}R^\dagger R\widetilde{M}^\dagger\Big) \geq 0 \ , \tag{10.55}$$

where we used that $\widetilde{M}R^\dagger R\widetilde{M}^\dagger$ is a positive operator. Since K is non null, R is also non null, so there exist $w_0 \in \mathcal{H}_{\boldsymbol{p}}$ such that $\|R(w_0)\|^2 > 0$. If A is positive-definite, $\widetilde{M}^\dagger$ is an automorphism and we can set $w = (\widetilde{M}^\dagger)^{-1}(w_0)$ so that $\|w\| > 0$ and

$$\begin{aligned} W_A(\mathbf{z}) = \mathrm{tr}\Big(\widetilde{M}R^\dagger R\widetilde{M}^\dagger\Big) &\geq \frac{\Big\langle w \Big| \widetilde{M}R^\dagger R\widetilde{M}^\dagger(w)\Big\rangle}{\|w\|^2} = \frac{\Big\langle R\widetilde{M}^\dagger(w)\Big| R\widetilde{M}^\dagger(w)\Big\rangle}{\|w\|^2} \\ &= \frac{\Big\|R\widetilde{M}^\dagger(w)\Big\|^2}{\|w\|^2} = \frac{\|R(w_0)\|^2}{\|w\|^2} \\ &> 0 \ . \end{aligned} \tag{10.56}$$

Now, suppose K is not positive. Then, K has a negative eigenvalue. Let Π be the projection onto an eigenspace of K associated to a negative eigenvalue. We have that $\mathrm{tr}(\Pi K) < 0$. □

In the light of Remark 10.3.2, $\Pi_{(0,0,p)} \in \mathcal{B}(\mathcal{H}_{p,0})$ and $\Pi_{(p,p,0)} \in \mathcal{B}(\mathcal{H}_{0,p})$ are operator kernels of mapping-positive correspondences for generic quark systems.

Definition 10.3.7. *For any $p \in \mathbb{N}$, the mapping-positive correspondences $\mathcal{B}(\mathcal{H}_{p,0}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$, with operator kernel $\Pi_{(0,0,p)}$, and $\mathcal{B}(\mathcal{H}_{0,p}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$, with operator kernel $\Pi_{(p,p,0)}$, are* Berezin correspondences *for generic quark systems.*

Remark 10.3.5. *The Berezin correspondences of Definition 10.3.7 are rather trivial examples, in the sense that the symbols on $\mathbb{C}P^1 \to \mathcal{E} \to \mathbb{C}P^2$ are constant extensions of functions on $\mathbb{C}P^2$. Thus, it would be interesting if we could exhibit some nontrivial examples of mapping-positive correspondences for generic quark systems.*

In fact, we would like to emphasize that we still don't know whether a result similar to Proposition 9.3.8 holds for generic quark systems.

Also, for pure-quark systems every correspondence is semi-conformal, cf. Remark 10.3.4, and for generic quark system every Stratonovich-Weyl correspondence is semi-conformal, cf. Proposition 10.3.5. We still don't know if there is any general relation between mapping-positive (or positive-dual) correspondences and semi-conformal correspondences, for generic quark systems.

To finish this section, we define antipodal correspondences for generic quark systems and show some of their general properties.

Definition 10.3.8. *For a symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$, its* antipodal correspondence *is the symbol correspondence $\breve{W} : \mathcal{B}(\mathcal{H}_{\breve{\boldsymbol{p}}}) \to C^\infty(\mathcal{E})$ given by*

$$\breve{W}_{A^*} = W_A \ , \tag{10.57}$$

cf. (8.129)-(8.131).

Proposition 10.3.7. *The symbol correspondences $\breve{W} : \mathcal{B}(\mathcal{H}_{\breve{\boldsymbol{p}}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with characteristic parameters $(\breve{c}^\sigma_\gamma(\boldsymbol{a}))$ is antipodal to the symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with characteristic parameters $(c^\sigma_\gamma(\boldsymbol{a}))$ if and only if*

$$c^\sigma_\gamma(\boldsymbol{a}) = (-1)^{|\boldsymbol{a}|}\breve{c}^{\breve{\sigma}}_\gamma(\boldsymbol{a}) \ . \tag{10.58}$$

Proof. The result follows from (8.133) and Proposition 10.3.1. □

Recalling (8.122), we have from (10.58) that the characteristic matrices of a symbol correspondence and of its antipodal differ just by a constant $(-1)^{|\boldsymbol{a}|}$ factor and the reverse ordering of columns. We then have the following:

Corollary 10.3.7.1. *For generic quark systems, a symbol correspondence and its antipodal have the same image in $C^\infty_{\mathbb{C}}(\mathcal{E})$.*

Proposition 10.3.8. *If $W, \widetilde{W} : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ are symbol correspondences dual to each other, then their respective antipodal correspondences are also dual to each other.*

Proof. Let $(c^\sigma_\gamma(\boldsymbol{a}))$ and $(\tilde{c}^\sigma_\gamma(\boldsymbol{a}))$ be the characteristic parameters of W and $\widetilde{W}$, respectively. The result follows from

$$\sum_{\gamma=1}^{m(\boldsymbol{a})} \left((-1)^{|\boldsymbol{a}|}\,\overline{\tilde{c}^\sigma_\gamma(\boldsymbol{a})}\right)\left((-1)^{|\boldsymbol{a}|}\,c^{\sigma'}_\gamma(\boldsymbol{a})\right) = \sum_{\gamma=1}^{m(\boldsymbol{a})} \overline{\tilde{c}^\sigma_\gamma(\boldsymbol{a})}c^{\sigma'}_\gamma(\boldsymbol{a}) = \delta_{\sigma,\sigma'} \ . \tag{10.59}$$

□

Corollary 10.3.8.1. *A symbol correspondence for a generic quark system is a semi-conformal (resp. Stratonovich-Weyl) correspondence if and only if its antipodal correspondence is also a semi-conformal (resp. Stratonovich-Weyl) correspondence.*

10.4 Twisted products for generic quark systems

Again, let $\boldsymbol{p} \in (\mathbb{N}\times\mathbb{N}_0)\cup(\mathbb{N}_0\times\mathbb{N})$.

Now, we may have symbol correspondences $W_1, W_2 : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with different image sets, that is, such that $W_1(\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})) \neq W_2(\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}))$. To see this, consider $W_1 \neq W_2 : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ determined by characteristic matrices $\mathfrak{C}[1](\boldsymbol{a})$ and $\mathfrak{C}[2](\boldsymbol{a})$ with respective characteristic parameters $c[1]^\sigma_\gamma(\boldsymbol{a})$ and $c[2]^\sigma_\gamma(\boldsymbol{a})$. Since $\mathfrak{m}(\boldsymbol{p};|\boldsymbol{p}|,|\boldsymbol{p}|)=1$, for $Q(\boldsymbol{a}) = Q(|\boldsymbol{p}|,|\boldsymbol{p}|)$, we can drop the index σ for the characteristic parameters $c[1]_\gamma(|\boldsymbol{p}|,|\boldsymbol{p}|)$ and $c[2]_\gamma(|\boldsymbol{p}|,|\boldsymbol{p}|)$. Then, for $\mathfrak{C}[1](|\boldsymbol{p}|,|\boldsymbol{p}|)$ and $\mathfrak{C}[2](|\boldsymbol{p}|,|\boldsymbol{p}|)$ such that

$$\mathfrak{C}[1](|\boldsymbol{p}|,|\boldsymbol{p}|)\cdot\mathfrak{C}[2](|\boldsymbol{p}|,|\boldsymbol{p}|) = \sum_{\gamma=1}^{|\boldsymbol{p}|+1} c[1]_\gamma(|\boldsymbol{p}|,|\boldsymbol{p}|)c[2]_\gamma(|\boldsymbol{p}|,|\boldsymbol{p}|) = 0 \ ,$$

we have that

$$W_1(Q(|\boldsymbol{p}|,|\boldsymbol{p}|)) = \mathrm{span}\,\{Z\mathfrak{C}[1](|\boldsymbol{p}|,|\boldsymbol{p}|)_{\boldsymbol{\nu},I}\}$$

is orthogonal[2] to

$$W_2(Q(|\boldsymbol{p}|,|\boldsymbol{p}|)) = \mathrm{span}\,\{Z\mathfrak{C}[2](|\boldsymbol{p}|,|\boldsymbol{p}|)_{\boldsymbol{\nu},I}\}\,,$$

hence $W_1(\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})) \neq W_2(\mathcal{B}(\mathcal{H}_{\boldsymbol{p}}))$.

In view of the this fact, for any symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$, we shall denote by $\mathcal{S}_{\boldsymbol{p}}(W)$ the image of W, that is,

$$\mathcal{S}_{\boldsymbol{p}}(W) = W(\mathcal{B}(\mathcal{H}_{\boldsymbol{p}})) \subset C^\infty_{\mathbb{C}}(\mathcal{E}) \ . \tag{10.60}$$

2 We recall that the constant function 1 on $\mathcal{E}$ is in the image of $Q(0,0)$, thus $1 \notin W(Q(|\boldsymbol{p}|,|\boldsymbol{p}|))$, for any $\boldsymbol{p} \in (\mathbb{N}\times\mathbb{N}_0)\cup(\mathbb{N}_0\times\mathbb{N})$ and for any correspondence W.

Definition 10.4.1. *Given a symbol correspondence* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$*, the* twisted product *of symbols induced by* W *is the binary operation* $\star$ *on* $\mathcal{S}_{\boldsymbol{p}}(W)$ *given by*

$$W_A \star W_R = W_{AR} \tag{10.61}$$

for any $A, R \in \mathcal{B}(\mathcal{H}_{\boldsymbol{p}})$*. The algebra* $(\mathcal{S}_{\boldsymbol{p}}(W), \star)$ *is called a* twisted $\boldsymbol{p}$-algebra*.*

The proofs of the two next propositions are exactly as the proofs of Propositions 9.4.1 and 9.4.2, respectively.

Proposition 10.4.1. *Any twisted* $\boldsymbol{p}$*-algebra* $(\mathcal{S}_{\boldsymbol{p}}(W), \star)$ *is*

i) $SU(3)$-equivariant: $(f_1 \star f_2)^g = f_1^g \star f_2^g$;

ii) Associative: $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$;

iii) Unital: $1 \star f = f \star 1 = f$;

iv) A $*$-algebra: $\overline{f_1 \star f_2} = \overline{f_2} \star \overline{f_1}$;

where $f_1, f_2, f_3, f \in \mathcal{S}_{\boldsymbol{p}}(W)$*,* $g \in SU(3)$ *and* $1 \in \mathcal{S}_{\boldsymbol{p}}(W)$ *is the constant function equal to* 1 *on* $\mathcal{E}$*.*

Proposition 10.4.2. *Fixed* $\boldsymbol{p} \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})$*, any two twisted* $\boldsymbol{p}$*-algebras are naturally isomorphic, and any* $\boldsymbol{p}$*-algebra is naturally anti-isomorphic to any* $\check{\boldsymbol{p}}$*-algebra.*

Proof. Although we may have correspondences $W_1, W_2 : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with different images, we still have that each W_j is an isomorphism onto its image, so $W_1 \circ W_2^{-1} : \mathcal{S}_{\boldsymbol{p}}(W_2) \to \mathcal{S}_{\boldsymbol{p}}(W_1)$ is an isomorphism. If $W_2 : \mathcal{B}(\mathcal{H}_{\check{\boldsymbol{p}}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$, then $W_1 \circ * \circ W_2^{-1} : \mathcal{S}_{\check{\boldsymbol{p}}}(W_2) \to \mathcal{S}_{\boldsymbol{p}}(W_1)$ is an anti-isomorphism because the adjoint map $*$ is an anti-isomorphism and, again, each W_j is an isomorphism onto its image. $\square$

We cannot decompose twisted products for generic quark systems into the harmonic basis as we did for pure-quark systems because, in general, $\mathcal{S}_{\boldsymbol{p}}(W)$ is not spanned by the generic harmonics $Z^{(\boldsymbol{a},\gamma)}_{\boldsymbol{\nu},I}$, but by the linear combinations expressed in (10.29). Thus, a procedure equivalent to the one executed in Proposition 9.4.3 leads to the following:

Proposition 10.4.3. *If* $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ *is a symbol correspondence with characteristic matrices* $\mathfrak{C}(\boldsymbol{a})$*, then the induced twisted product is given by*

$$\begin{aligned} &Z\mathfrak{C}(\boldsymbol{a}_1)^{\sigma_1}_{\boldsymbol{\nu}_1,I_1} \star Z\mathfrak{C}(\boldsymbol{a}_2)^{\sigma_2}_{\boldsymbol{\nu}_2,I_2} \\ &\quad = \sqrt{\dim Q(\boldsymbol{p})} \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu},I}} (-1)^{|\boldsymbol{p}|+2(t_{\boldsymbol{\nu}}+u_{\boldsymbol{\nu}})} \begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a};\sigma) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \check{\boldsymbol{\nu}}, I \end{bmatrix} [\boldsymbol{p}]\, Z\mathfrak{C}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I} \ , \end{aligned} \tag{10.62}$$

where $Z\mathfrak{C}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I}$ *is given by (10.29) and where summations over* $\boldsymbol{\nu}$ *and* I *can be simplified using (8.151).*

Proof. The proof follows from Corollary 8.2.10.1 and Proposition 10.3.1. □

In that light, integral formulation of twisted products may be more useful for the generic case.

Proposition 10.4.4. *If $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ is a symbol correspondence with operator kernel K and characteristic matrices $\mathfrak{C}(\boldsymbol{a})$, then the induced twister product is given by*

$$f_1 \star f_2(\mathbf{z}) = \int_{\mathcal{E}\times\mathcal{E}} f_1(\mathbf{z}_1) f_2(\mathbf{z}_2) \mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z})\, d\mathbf{z}_1 d\mathbf{z}_2 \tag{10.63}$$

for any $f_1, f_2 \in \mathcal{S}_{\boldsymbol{p}}(W)$, where

$$\begin{aligned}\mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) &= (\dim Q(\boldsymbol{p}))^2 \operatorname{tr}\Big(\widetilde{K}(\mathbf{z}_1)\widetilde{K}(\mathbf{z}_2)K(\mathbf{z}_3)\Big)\\ &= (-1)^{|\boldsymbol{p}|}\sqrt{\dim Q(\boldsymbol{p})} \sum_{\substack{(\boldsymbol{a}_j;\sigma_j)\\ \boldsymbol{\nu}_j, I_j}} \begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a}_3;\sigma_3) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 \end{bmatrix}[\boldsymbol{p}] \\ &\quad \times \overline{Z\widetilde{\mathfrak{C}}(\boldsymbol{a}_1)^{\sigma_1}_{\boldsymbol{\nu}_1, I_1}}(\mathbf{z}_1)\, \overline{Z\widetilde{\mathfrak{C}}(\boldsymbol{a}_2)^{\sigma_2}_{\boldsymbol{\nu}_2, I_2}}(\mathbf{z}_2)\, \overline{Z\mathfrak{C}(\check{\boldsymbol{a}}_3)^{\sigma_3}_{\boldsymbol{\nu}_3, I_3}}(\mathbf{z}_3)\end{aligned} \tag{10.64}$$

for $\widetilde{\mathfrak{C}}(\boldsymbol{a})$ being the characteristic matrices of an operator kernel $\widetilde{K}$ dual to K.

Proof. The proof follows analogously to Proposition 9.4.4, but now the second equality comes from Proposition 10.4.3.

We emphasize that, although the expression (10.64) for integral trikernels depends explicitly on the choice of dual representation, the twisted product given by (10.63) does not have such dependence. By definition,

$$\begin{aligned}\int_{\mathcal{E}} \overline{Z\widetilde{\mathfrak{C}}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I}}(\mathbf{z}) Z\mathfrak{C}(\boldsymbol{a}')^{\sigma'}_{\boldsymbol{\nu}',I'}(\mathbf{z})\, d\mathbf{z} &= \Big\langle Z\widetilde{\mathfrak{C}}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I} \Big| Z\mathfrak{C}(\boldsymbol{a}')^{\sigma'}_{\boldsymbol{\nu}',I'}\Big\rangle \\ &= \dim Q(\boldsymbol{p})\, \langle \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}, I) | \boldsymbol{e}((\boldsymbol{a}';\sigma');\boldsymbol{\nu}', I')\rangle_{\boldsymbol{p}} \\ &= \langle \boldsymbol{e}((\boldsymbol{a};\sigma);\boldsymbol{\nu}, I) | \boldsymbol{e}((\boldsymbol{a}';\sigma');\boldsymbol{\nu}', I')\rangle\end{aligned} \tag{10.65}$$

no matter which dual correspondence is used. □

Definition 10.4.2. *An* integral trikernel $\mathbb{L} \in C^\infty(\mathcal{E}\times\mathcal{E}\times\mathcal{E})$ *of a twisted product induced by a symbol correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty(\mathcal{E})$ is a function of the form (10.64) so that the twisted product is given by (10.63). If W is a semi-conformal correspondence, the integral trikernel constructed using its canonical dual correspondence is the* canonical integral trikernel*.*

One may use (10.63) to expand a twisted product on $\mathcal{S}_{\boldsymbol{p}}(W)$ induced by some symbol correspondence W to a product $\bullet$ on all $C^\infty_{\mathbb{C}}(\mathcal{E})$ in the same way we did for pure-quark systems in Proposition 9.4.5, but integral trikernels are not unique, so such expansions are not unique too. In addition, the product $\bullet$ in general fails to vanish for functions orthogonal to $\mathcal{S}_{\boldsymbol{p}}(W)$ since we may find a symbol correspondence $\widetilde{W}$ dual to W with $\mathcal{S}_{\boldsymbol{p}}(\widetilde{W}) \neq \mathcal{S}_{\boldsymbol{p}}(W)$ as exemplified in Remark 10.3.3. However, for semi-conformal correspondences and canonical integral trikernels, we have the following:

Proposition 10.4.5. *Let $\mathbb{L}$ be the canonical integral trikernel of a twisted product $\star$ induced by a semi-conformal correspondence $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ with characteristic matrices satisfying $(\mathfrak{C}(\boldsymbol{a}))^\dagger \mathfrak{C}(\boldsymbol{a}) = \alpha(\boldsymbol{a})\, \mathbb{1}$. The binary operation $\bullet$ on $C^\infty_{\mathbb{C}}(\mathcal{E})$ given by*

$$f_1 \bullet f_2(\mathbf{z}) = \int_{\mathcal{E}\times\mathcal{E}} f_1(\mathbf{z}_1) f_2(\mathbf{z}_2) \mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z})\, d\mathbf{z}_1 d\mathbf{z}_2 \tag{10.66}$$

for any $f_1, f_2 \in C^\infty_{\mathbb{C}}(\mathcal{E})$, defines a $SU(3)$-equivariant associative $*$-algebra with respect to complex conjugation. In particular, if $f_1, f_2 \in \mathcal{S}_{\boldsymbol{p}}(W)$, we have $f_1 \bullet f_2 = f_1 \star f_2$. But, if either f_1 or f_2 is orthogonal to $\mathcal{S}_{\boldsymbol{p}}(W)$, then $f_1 \bullet f_2 = 0$.

Proof. The proof follows from the same arguments applied to Proposition 9.4.5, but now it is needed to point out that

$$\begin{aligned}\mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = (-1)^{|\boldsymbol{p}|}\sqrt{\dim Q(\boldsymbol{p})} \sum_{\substack{(\boldsymbol{a}_j;\sigma_j)\\ \boldsymbol{\nu}_j, I_j}} &\begin{bmatrix} (\boldsymbol{a}_1;\sigma_1) & (\boldsymbol{a}_2;\sigma_2) & (\boldsymbol{a}_3;\sigma_3) \\ \boldsymbol{\nu}_1, I_1 & \boldsymbol{\nu}_2, I_2 & \boldsymbol{\nu}_3, I_3 \end{bmatrix} [\boldsymbol{p}] \\ \times \frac{1}{\alpha(\boldsymbol{a})^2} &\overline{Z\mathfrak{C}(\boldsymbol{a}_1)^{\sigma_1}_{\boldsymbol{\nu}_1, I_1}(\mathbf{z}_1)}\, \overline{Z\mathfrak{C}(\boldsymbol{a}_2)^{\sigma_2}_{\boldsymbol{\nu}_2, I_2}(\mathbf{z}_2)}\, \overline{Z\mathfrak{C}(\check{\boldsymbol{a}}_3)^{\sigma_3}_{\boldsymbol{\nu}_3, I_3}(\mathbf{z}_3)}\ .\end{aligned} \tag{10.67}$$

Thus, if $f_j \in C^\infty_{\mathbb{C}}(\mathcal{E})$ is orthogonal to $\mathcal{S}_{\boldsymbol{p}}(W)$, it is orthogonal to every $Z\mathfrak{C}(\boldsymbol{a})^\sigma_{\boldsymbol{\nu}, I}$ and this implies that the integral over $\mathbf{z}_j$ in (10.66) vanishes. $\square$

Analogously to pure-quark systems, a product $\bullet$ satisfying the previous proposition is such that $C^\infty_{\mathbb{C}}(\mathcal{E}) \to \mathcal{S}_{\boldsymbol{p}}(W) : f \mapsto 1 \bullet f = f \bullet 1$ is an orthogonal projection.

Proposition 10.4.6. *Let $\mathbb{L}$ be an integral trikernel of a twisted product $\star$ induced by $W : \mathcal{B}(\mathcal{H}_{\boldsymbol{p}}) \to C^\infty_{\mathbb{C}}(\mathcal{E})$ as in (10.64). Then, for every $g \in SU(3)$ and every $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4 \in \mathcal{E}$,*

i) $\mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = \mathbb{L}(g\mathbf{z}_1, g\mathbf{z}_2, g\mathbf{z}_3)$;

ii) $\int_{\mathcal{E}} \mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}) \mathbb{L}(\mathbf{z}, \mathbf{z}_3, \mathbf{z}_4)\, d\mathbf{z} = \int_{\mathcal{E}} \mathbb{L}(\mathbf{z}_1, \mathbf{z}, \mathbf{z}_4) \mathbb{L}(\mathbf{z}_2, \mathbf{z}_3, \mathbf{z})\, d\mathbf{z}$;

iii) $\int_{\mathcal{E}} \mathbb{L}(\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2)\, d\mathbf{z} = \int_{\mathcal{E}} \mathbb{L}(\mathbf{z}_1, \mathbf{z}, \mathbf{z}_2)\, d\mathbf{z} = \mathcal{R}^W_{\boldsymbol{p}}(\mathbf{z}_1, \mathbf{z}_2)$, *where*

$$\mathcal{R}^W_{\boldsymbol{p}}(\mathbf{z}_1, \mathbf{z}_2) = \sum_{\substack{(\boldsymbol{a};\sigma)\\ \boldsymbol{\nu}, I}} \overline{Z\widetilde{\mathfrak{C}}(\boldsymbol{a})^\sigma_{\boldsymbol{\nu}, I}(\mathbf{z}_1)}\, Z\mathfrak{C}(\boldsymbol{a})^\sigma_{\boldsymbol{\nu}, I}(\mathbf{z}_2)\ , \tag{10.68}$$

satisfies

$$\int_{\mathcal{E}} f(\mathbf{z}_1) \mathcal{R}^W_{\boldsymbol{p}}(\mathbf{z}_1, \mathbf{z}_2)\, d\mathbf{z}_1 = f(\mathbf{z}_2) \tag{10.69}$$

for every $f \in \mathcal{S}_{\boldsymbol{p}}(W)$;

iv) $\overline{\mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)} = \mathbb{L}(\mathbf{z}_2, \mathbf{z}_1, \mathbf{z}_3)$.

Proof. Adapting the proof of Proposition 9.4.6, we get that each property of this statement is equivalent to the property of Proposition 10.4.1 with same number. It is just worth to highlight that the expression for $\mathcal{R}_{\boldsymbol{p}}^W$ comes from (10.64), orthonormality of $\mathcal{E}$ harmonics and (8.154). □

Remark 10.4.1. *In general, $\mathcal{R}_{\boldsymbol{p}}^W(\mathbf{z}_1, \mathbf{z}_2) \neq \mathcal{R}_{\boldsymbol{p}}^W(\mathbf{z}_2, \mathbf{z}_1)$. But if W is a semi-conformal correspondence and $\mathbb{L}$ is its canonical integral trikernel, then (iii) is satisfied with*

$$\mathcal{R}_{\boldsymbol{p}}^W(\mathbf{z}_1, \mathbf{z}_2) = \sum_{\substack{(\boldsymbol{a};\sigma) \\ \boldsymbol{\nu},I}} \frac{1}{\alpha(\boldsymbol{a})} \overline{Z\mathfrak{C}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I}}(\mathbf{z}_1)\, Z\mathfrak{C}(\boldsymbol{a})^{\sigma}_{\boldsymbol{\nu},I}(\mathbf{z}_2) = \mathcal{R}_{\boldsymbol{p}}^W(\mathbf{z}_2, \mathbf{z}_1) \ , \tag{10.70}$$

so that $\mathcal{R}_{\boldsymbol{p}}^W$ is the reproducing kernel *on $\mathcal{S}_{\boldsymbol{p}}(W)$.*

The phenomenon described in Remark 9.4.2 also occurs to generic quark systems.

Proposition 10.4.7. *The twisted products $\star$ and $\check{\star}$ induced by a symbol correspondence and its antipodal correspondence satisfy*

$$f_1 \star f_2 = f_2 \,\check{\star}\, f_1 \ . \tag{10.71}$$

Proof. The proof follows from Proposition 9.4.3 and (8.152). □

Corollary 10.4.7.1. *For $\star$ and $\check{\star}$ as in the previous proposition, we can choose integral trikernels $\mathbb{L}$ and $\check{\mathbb{L}}$ satisfying*

$$\mathbb{L}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = \check{\mathbb{L}}(\mathbf{z}_2, \mathbf{z}_1, \mathbf{z}_3) \ . \tag{10.72}$$

CHAPTER 11

DISCUSSION OF PART II

In this part of the work, we achieved a complete characterization of symbol correspondences for quark systems, considering as quantum systems the irreducible representations of $SU(3)$ and as phase spaces the coadjoint orbits of $SU(3)$.

The problem explored here is settled on general facts common to the case of spin systems, so we see some replicated features as, for example, the realization of symbol correspondences as expectation values over *operators kernel*, which is a special "pseudo-state", that is, an Hermitian operator with unitary trace. Then, the more restricted cases of mapping-positives correspondences are the ones which are generated as expectation values over an "actual state", that is, an operator kernel that is also a positive operator.

In particular, correspondences for pure-quark systems show little formal distinction to what is known for spin systems, being also determined by a set of non zero real numbers, the *characteristic numbers*. Notwithstanding, a remarkable difference occurs due to the breakdown of the self dual property for representations $Q(p,0)$ and $Q(0,p)$ in the case of quantum pure-quark systems: antipodal correspondences are defined for pure-quark systems dual to each other and have the same characteristic numbers.

However, correspondences for generic quark system present some new features originated from the degeneracy of representations within both the quantum operator space $\mathcal{B}(\mathcal{H}_{p,q})$ and the classical function space $C^\infty_{\mathbb{C}}(\mathcal{E})$. Then, the characterization of correspondences for generic quark systems, in the same vein of what is done to pure-quark systems, are given not in terms of characteristic numbers, but in terms of *characteristic matrices*. As consequence, there are multiple correspondences linked by a dual relation and, in addition to isometric (*Stratonovich-Weyl*) correspondences, we have the more general definition of *semi-conformal correspondences* as special cases of symbol correspondences, alongside the special cases of mapping-positive and positive-dual correspondences.

Future work shall be dedicated to the problem of asymptotic behavior of symbol correspondences and verifying the conditions under which the Poisson algebras of classical systems emerge as limit of operator algebras of quantum systems. In this respect, the first problem at hand is to decompose the Poisson bracket of $\mathbb{C}P^2$-harmonics, or $\mathcal{E}$-harmonics, similarly to what has been done in the case of spin systems and spherical harmonics.

Due to the similarity with spin systems, we hope that the asymptotic analysis of pure-quark systems shall be more feasible. On the other hand, generic quark systems seem to be quite more subtle for asymptotic analysis, since the relevant quantum systems are labeled by two indices, $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$, so the construction of sequences of symbol correspondences, as done for spin systems, may involve some arbitrary choices (in principle, we would have to deal with bi-sequences of correspondences and study the asymptotic limit $d \to \infty$, where $d = d(p, q)$ is the dimension of $Q(p, q)$). Furthermore, different correspondences for the same generic quark systems may have different images, so it might be the case that we could generate sequences (or bi-sequences) of symbol correspondences whose images never reach some harmonic function $f \in C^\infty_{\mathbb{C}}(\mathcal{E})$. Nonetheless, we hope that the factorization obtained in Proposition 8.2.10 may be useful in a similar approach to the one performed for spin systems in (RIOS; STRAUME, 2014).

Another direction to be followed is the study of symbol correspondences from quantum quark systems to $SU(3)$-invariant Poisson manifolds, particularly $S^7 \subset \mathbb{R}^8 \simeq \mathfrak{su}(3)$. The 7-sphere can be split as $S^7 \simeq M \cup N$, where M is the disjoint union of two copies of $\mathbb{C}P^2$ and N is an uncountable disjoint union of $\mathcal{E}$ copies. In other words, $\mathcal{E}$ and $\mathbb{C}P^2$ are isomorphic to the symplectic leaves of S^7, with $\mathcal{E}$ isomorphic to the regular leaves of this irregular foliation of S^7. Then, in this respect, the first problem at hand is to understand how to "glue" the harmonic functions of $\mathbb{C}P^2$ and $\mathcal{E}$ in order to obtain $SU(3)$-equivariant smooth functions on S^7.

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APPENDIX

A

PROOF OF LEMMA 4.1.2

This is a basic well-known result for which we could not find a sufficiently pedestrian proof in the literature, so here we provide one, for completeness and reader's convenience.

First, if $\rho_n \to \delta(z-\mu)$ on $C^0_{\mathbb{C}}([-1,1])$, then trivially $\mu_n \to \mu$ and $\sigma_n^2 \to 0$. Now, let's suppose $\mu_n \to \mu$ and $\sigma_n^2 \to 0$.

Given $f \in C^0_{\mathbb{C}}([-1,1])$, let $||f||_0$ be its sup-norm. For any $\eta > 0$, let $\epsilon > 0$ be s.t.

$$|z-\mu| < \epsilon \implies |f(z) - f(\mu)| < \eta/3 \ . \tag{A.1}$$

First, we have that

$$\begin{aligned} \left|\int_{-1}^{1} f(z)\rho_n(z)dz - f(\mu)\right| &\leq \left|\int_{|z-\mu|\geq\epsilon} f(z)\rho_n(z)dz\right| + \left|\int_{|z-\mu|<\epsilon} f(z)\rho_n(z)dz - f(\mu)\right| \\ &\leq ||f||_0 \int_{|z-\mu|\geq\epsilon} \rho_n(z)dz + \left|\int_{|z-\mu|<\epsilon} f(z)\rho_n(z)dz - f(\mu)\right| \end{aligned} \tag{A.2}$$

where the domains of integration on the r.h.s. are contained in $[-1,1]$.

By hypothesis $\mu_n \to \mu$, there exists $n_0 \in \mathbb{N}$ such that, $\forall n > n_0$, $|\mu_n - \mu| < \epsilon/2$. But if $|z-\mu_n| < \epsilon/2$ for $n > n_0$, then $|z-\mu| \leq |z-\mu_n| + |\mu_n - \mu| < \epsilon$. Thus, $\forall n > n_0$, $|z-\mu| \geq \epsilon \implies |z-\mu_n| \geq \epsilon/2$ and, by Chebyshev's inequality,

$$n > n_0 \implies \int_{|z-\mu|\geq\epsilon} \rho_n(z)dz \ \leq \ \int_{|z-\mu_n|\geq\epsilon/2} \rho_n(z)dz \ \leq \ \frac{4\sigma_n^2}{\epsilon^2} \ .$$

Therefore, by hypothesis $\sigma_n^2 \to 0$, there exists $n_1 \geq n_0$ such that

$$n > n_1 \quad \implies \quad ||f||_0 \int_{|z-\mu|\geq\epsilon} \rho_n(z)dz \ < \ \eta/3 \ , \tag{A.3}$$

$$\implies \quad 1 \ \geq \ \int_{|z-\mu|<\epsilon} \rho_n(z)dz \ \geq \ 1 - \frac{\eta}{3||f||_0} \ . \tag{A.4}$$

On the other hand, from (A.1) we get

$$\big(f(\mu) - \eta/3\big) \int_{|z-\mu|<\epsilon} \rho_n(z)dz < \int_{|z-\mu|<\epsilon} f(z)\rho_n(z)dz < \big(f(\mu) + \eta/3\big) \int_{|z-\mu|<\epsilon} \rho_n(z)dz$$

which from (A.4) implies that, for $n > n_1$,

$$\left(f(\mu) - \eta/3\right)\left(1 - \frac{\eta}{3||f||_0}\right) < \int_{|z-\mu|<\epsilon} f(z)\rho_n(z)dz \ < \ f(\mu) + \eta/3 \ ,$$

and therefore

$$n > n_1 \quad \Longrightarrow \quad \left|\int_{|z-\mu|<\epsilon} f(z)\rho_n(z)dz - f(\mu)\right| \ < \ 2\eta/3 \ . \tag{A.5}$$

Thus, from (A.2), (A.3) and (A.5), we have (4.7), that is,

$$\forall \eta > 0, \ \exists n_1 \in \mathbb{N} \quad \text{s.t.} \quad n > n_1 \implies \left|\int_{-1}^{1} f(z)\rho_n(z)dz - f(\mu)\right| < \eta \ .$$

APPENDIX

B

PROOF OF LEMMA 4.2.1

We could not find a complete proof of Edmonds formula in the literature, so here we provide one. At first, we follow Brussard and Tolhoek (BRUSSARD; TOLHOEK, 1957) and Flude (FLUDE, 1998) to prove that

$$\lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} = (-1)^{l-\tau} d^l_{\mu,\tau}(\theta), \tag{B.1}$$

where $d^l_{\mu,\tau}$ is the Wigner (small) d-function and $\theta \in [0,\pi]$ is such that $\cos\theta = \lim_{j\to\infty}(m/j)$, holds for fixed l, μ, τ and if either (i) $j - |m| \to \infty$ or (ii) $j - |m| \to 0$.

For (i), we start with (cf. (BIEDENHARN; LOUCK, 1984; RIOS; STRAUME, 2014; VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988)):

$$\begin{aligned} C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} = &\sqrt{\frac{2j+1}{2j-\tau+l+1}}\sqrt{(l-\tau)!(l+\tau)!(l+\mu)!(l-\mu)!} \\ &\times \sum_z \frac{(-1)^z\sqrt{\Omega_1\Omega_2\Omega_3}}{z!(l-\tau-z)!(l-\mu-z)!(\tau+\mu+z)!} \end{aligned} \tag{B.2}$$

where

$$\Omega_1 = \frac{(2j-\tau-l)!}{(2j-\tau+l)!},\quad \Omega_2 = \frac{(j-\tau-m+\mu)!(j-m)!}{[(j-l-m+\mu+z)!]^2},\quad \Omega_3 = \frac{(j-\tau+m-\mu)!(j+m)!}{[(j-\tau+m-\mu-z)!]^2}\ .$$

The summation over z is finite, so we can take the limit of each term. Thus,

$$\Omega_1\Omega_2\Omega_3 \sim \left(\frac{j-m}{2j}\right)^{2l-\tau-\mu-2z}\left(\frac{j+m}{2j}\right)^{\tau+\mu+2z},$$

where in the expressions for Ω_1, Ω_2, Ω_3, we have used the Stirling approximation

$$n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n}\ . \tag{B.3}$$

Supposing m/j converges, we can take a $\theta \in [0,\pi]$ such that $\cos\theta = \lim_{j\to\infty}(m/j)$, so that $\sin\frac{\theta}{2} = \sqrt{\frac{j-m}{2j}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{j+m}{2j}}$. Then, we get (*cf.* (VARSHALOVICH;

MOSKALEV; KHERSONSKII, 1988) for the exact expression of the Wigner d-function)

$$\begin{aligned}\lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} &= \sqrt{(l-\tau)!(l+\tau)!(l+\mu)!(l-\mu)!}\\ &\times \sum_z \frac{(-1)^z \left(\sin\frac{\theta}{2}\right)^{2l-\tau-\mu-2z}\left(\cos\frac{\theta}{2}\right)^{\tau+\mu+2z}}{z!(l-\tau-z)!(l-\mu-z)!(\tau+\mu+z)!}\\ &= (-1)^{l-\tau} d^l_{\mu,\tau}(\theta)\ .\end{aligned}$$

Now, we must note that $j-|m|$ is a sequence of integer numbers, so the convergence condition (ii) implies that there is a j_0 such that for all $j > j_0$ we have $j-|m| = 0 \Rightarrow m = \pm j$. For $m = j$, we have $\mu \geq |\tau|$ and $m/j \to 1 \Rightarrow \theta = 0 \Rightarrow d^l_{\mu,\tau} = \delta_{\mu,\tau}$; for $m = -j$, we have $\mu \leq |\tau|$ and $m/j \to -1 \Rightarrow \theta = \pi \Rightarrow d^l_{\mu,\tau} = (-1)^{l-\tau}\delta_{-\mu,\tau}$. Writing, again, the exact formula

$$C^{j,\ j-\tau,\ l}_{j,-j\pm\mu,\pm\mu} = \sqrt{\frac{2l+1}{2j-\tau+l+1}}\sqrt{\frac{(l-\tau)!(l\pm\mu)!}{(l+\tau)!(-\tau\pm\mu)!(l\mp\mu)!}}\sqrt{\frac{(2j)!(2j-\tau\mp\mu)!}{(2j-\tau+l)!(2j-\tau-l)!}}$$

and then using (B.3) in the last factor, plus the symmetry properties of Clebsch-Gordan coefficients (*cf.* (BIEDENHARN; LOUCK, 1984; RIOS; STRAUME, 2014; VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988)), we conclude that

$$\lim_{j\to\infty} C^{l,j-\tau,\ j}_{\mu,j-\mu,\ j} = (-1)^{l-\tau}\delta_{\mu,\tau} \quad , \quad \lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,-j-\mu,\ -j} = \delta_{-\mu,\tau} \quad .$$

Now, we expand condition (ii) to condition (iii) $j-|m| < a$ for some $a \in \mathbb{N}$ when $\tau = \mu = 0$. We first assume $j-|m| \to a \in \mathbb{N}_0$. From (VARSHALOVICH; MOSKALEV; KHERSONSKII, 1988), we have the following recursive relation:

$$\begin{aligned}C^{\,l,\qquad j,\qquad j}_{0,\pm(j-a-1),\pm(j-a-1)} = \sqrt{\frac{l(l+1)}{(2j-a)(a+1)}}C^{\,l,\qquad j,\ \ j}_{\pm 1,\pm(j-a-1),\pm(j-a)}\\ + C^{\,l,\quad j,\ \ j}_{0,\pm(j-a),\pm(j-a)}\end{aligned} \tag{B.4}$$

Assuming the coefficients in the rhs of (B.4) satisfy the already proved (B.1), we obtain

$$\lim_{j\to\infty} C^{\,l,\qquad j,\qquad j}_{0,\pm(j-a-1),\pm(j-a-1)} = (-1)^l d^l_{0,0}(\theta)$$

where $\theta = 0$ for plus sign and $\theta = \pi$ for minus sign. The hypothesis holds for $a = 0$, so, by induction, the formula holds for all $a \in \mathbb{N}_0$. Finally, $j-|m|$ does not need to converge to $a \in \mathbb{N}_0$, it is sufficient for it to remain finite, because all the Clebsch-Gordan coefficients converge to the same value.

Putting together all the results, we have proved that, if $\lim_{j\to\infty}(m/j) = \cos\theta = 1-2r$, then

$$\lim_{n\to\infty}(-1)^{k_n-1}C^{\,j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{\frac{n+1}{2l+1}} = \lim_{n\to\infty}(-1)^l C^{l,\ j,\ j}_{0,m,m} = d^l_{0,0}(\theta) = P_l(1-2r)\ ,$$

which is Edmonds formula (4.11).

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