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Robustness of nonuniform and random exponential dichotomies with applications to differential equations

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Robusteza de dicotomias exponenciais, não uniformes e aleatória, com aplicações a equações diferenciais

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP e à Universidad de Sevilla – US, como parte dos requisitos para obtenção dos títulos de Doutor em Ciências – Matemática (ICMC-USP) e PhD (US), de acordo com o convênio acadêmico internacional para dupla titulação de doutorado assinado entre o ICMC-USP e a US. *VERSÃO REVISADA*

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For their endless love, support and encouragement.

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“The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.”

(Henry Poincaré)

ABSTRACT

OLIVEIRA SOUSA, A. N. **Robustness of nonuniform and random exponential dichotomies with applications to differential equations**. 2022. 123 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

In this thesis, we study hyperbolicity for deterministic and random nonautonomous dynamical systems and their applications to differential equations. More precisely, we present results for the following topics: nonuniform hyperbolicity for evolution processes and hyperbolicity for nonautonomous random dynamical systems. Concerning the first one, we study the robustness of the nonuniform exponential dichotomy for continuous and discrete evolution processes. We present an example of an infinite-dimensional differential equation that admits a nonuniform exponential dichotomy and apply the robustness result. Moreover, we study the persistence of nonuniform hyperbolic solutions in semilinear differential equations. Furthermore, we introduce a new concept of nonuniform exponential dichotomy, provide examples, and prove a stability result under perturbations for it. For the second topic, we introduce exponential dichotomies for random and nonautonomous dynamical systems. We prove a robustness result for this notion of hyperbolicity and study its applications to random and nonautonomous differential equations. Among these applications, we study the existence and continuity of random hyperbolic solutions and their associated unstable manifolds. As a consequence, we obtain continuity and topological structural stability for nonautonomous random attractors.

Keywords: exponential dichotomies; evolution processes; nonautonomous random dynamical systems; continuity of attractors; structural stability of attractors; bounded noises.

RESUMO

OLIVEIRA SOUSA, A. N. **Robusteza de dicotomias exponenciais, não uniformes e aleatória, com aplicações a equações diferenciais**. 2022. 123 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Nesta tese, estudamos hiperbolicidade para sistemas dinâmicos não autonomos determínisticos e aleatórios e suas aplicações a equações diferencias. Mais precisamente, apresentamos resultados nos seguintes tópicos: hiperbolicidade não uniforme para processos de evolução e hiperbolicidade para sistemas dinâmicos aleatórios não autônomos. No primiero tópico, estudamos robusteza da dicotomia exponencial não uniforme para processos de evolução contínuous e discretos. Apresentamos uma de equação diferencial em dimensão infinita que admite uma dicotomia exponencial não uniforme e aplicamos o teorema de robusteza. Ademais, estudamos a persistência de soluções hiperbólicas não uniformes em equações diferencias semilineares. Além disso, introduzimos um novo conceito de dicotomia exponencial não uniforme, fornecemos exemplos e provamos um teorema estabilidade sob perturbações. Na segundo tópico introduzimos dicotomias exponencias para sistemas dinâmicos aleatórios e não autônomos. Provamos um resultado de robusteza para essa noção de hiperbolicidade e estudamos suas aplicações a equações diferencias aleatórias e não autônomas. Entre essas aplicações estudamos existência e continuidade de soluções hiperbólicas aleatórias e suas variedades instaveis associadas. Como consequência obtemos continuidade e estabilidade estrutural topológica para atratores aleatórios não autônomos.

Palavras-chave: dicotomias exponencias; processos de evolução; sistemas dinâmicos não autônomos aleatórios; continuidade de atratores; estabilidade estrutural de atratores; ruídos limitados.

RESUMEN

OLIVEIRA SOUSA, A. N. **Robusteza de dicotomias exponenciais, não uniformes e aleatória, com aplicações a equações diferenciais**. 2022. 123 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

En esta tesis, estudiamos la hiperbolicidad para sistemas dinámicos deterministas y aleatorios no autónomos, y sus aplicaciones a ecuaciones diferenciales. Más precisamente, presentamos resultados en los siguientes temas: hiperbolicidad no uniforme para procesos de evolución e hiperbolicidad para sistemas dinámicos aleatorios no autónomos. En el primer tema, estudiamos la robustez de la dicotomía exponencial no uniforme para procesos de evolución continuos y discretos. Presentamos un ejemplo de una ecuación diferencial en dimensión infinita que admite una dicotomía exponencial no uniforme y aplicamos el resultado de robustez. Además, estudiamos la persistencia de soluciones hiperbólicas no uniformes en ecuaciones diferenciales semilineales. Por otro lado, presentamos un nuevo concepto de dicotomía exponencial no uniforme, proporcionamos ejemplos y demostramos un resultado de estabilidad bajo perturbaciones para él. En el segundo tema, presentamos dicotomías exponenciales para sistemas dinámicos aleatorios y no autónomos. Demostramos un resultado de robustez para esta noción de hiperbolicidad y estudiamos sus aplicaciones a ecuaciones diferenciales aleatorias y no autónomas. Entre estas aplicaciones estudiamos la existencia y continuidad de soluciones hiperbólicas aleatorias y sus variedades inestables asociadas. Como consecuencia, obtenemos continuidad y estabilidad estructural topológica para atractores aleatorios no autónomos.

Palabras clave: Dicotomías exponenciales; procesos de evolución; sistemas dinámicos no autonomos aleatorios; continuidad de atractores; estabilidad estructural de atractores; ruidos acotados.

CONTENTS

CHAPTER

1

INTRODUCTION

In the framework of dynamical systems, hyperbolicity plays a fundamental role (see, e.g. (KATOK; HASSELBLATT, 1995; ROBINSON, 1995; SHUB, 1987) and the references therein). It is the key property for most of the results on permanence under perturbations. The permanence, on the other hand is an essential property for dynamical systems that model real life phenomena. That importance is related to the fact that modelling always comes with approximations (due to the empiric nature that it carries) and/or with simplifications (introduced to make models treatable or simply because the complete set of variables that are related to the phenomenon is not known). Therefore, in order that the mathematical model reflects, in some way, the phenomenon modelled, it is essential that its dynamical structures are robust under perturbation. In this thesis, we study robustness of hyperbolicity in two different contexts and we present several applications to infinite-dimensional differential equations. These results are contained in (CARABALLO *et al.*, 2021a; CARABALLO *et al.*, 2021b; CARABALLO *et al.*, 2021c; LANGA; OBAYA; OLIVEIRA-SOUSA, 2021).

First, let us introduce the notion of hyperbolicity. In the discrete case, $x_{n+1} = Bx_n$, *hyperbolic dynamical systems* appear when the spectrum of the bounded linear operator B does not intercept the unit circle in the complex plane. This implies the existence of a *hyperbolic decomposition* of the space, which means that exist two main directions: one where the evolution of the dynamical system decays exponentially and another where it grows exponentially. The set of operators that has such decomposition is an open set in the spaces of bounded linear operators and the operators in this set are called *hyperbolic operators*. In other words, if B is hyperbolic there is a neighborhood of B such that every operator in this neighborhood is hyperbolic. For autonomous differential equations, when A is a bounded linear operator, $\dot{x} = Ax$, by the spectral mapping theorem (KATO, 1995), hyperbolicity is associated with linear operators such that the spectrum does not intersect the imaginary line.

Generally, in nonautonomous differential equations, the notion of hyperbolicity is referred to as *exponential dichotomy*. More precisely, consider the following differential equation in a

Banach space X,

$$\dot{x} = A(t)x, \quad x(s) = x_s \in X. \tag{1.1}$$

Under appropriate conditions, the solutions $x(t,s;x_s)$, $t \geq s$, of this initial value problem define an evolution process $\mathscr{S} := \{S(t,s)\,;\, t \geq s\}$, where $S(t,s)x_s = x(t,s;x_s)$. We say that the evolution process $\mathscr{S}$ admits an *exponential dichotomy* if there exists a family of projections, $\{\Pi^u(t)\,;\, t \in \mathbb{R}\}$ such that for each $t \geq s$ we have that $S(t,s)\Pi^u(s) = \Pi^u(t)S(t,s)$, $S(t,s)$ is an isomorphism from $R(\Pi^u(s))$ onto $R(\Pi^u(t))$, and

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} \leq Ke^{-\alpha(t-s)}, \quad t \geq s; \tag{1.2}$$

$$\|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)} \leq Ke^{\alpha(t-s)}, \quad t < s, \tag{1.3}$$

for some constants $K \geq 1$ and $\alpha > 0$. Note that, since the vector field is changing in time, it is natural to think that for each initial time we have a hyperbolic decomposition that resembles the properties in the autonomous case. There is a long list of works through these last decades about existence of exponential dichotomies and their stability properties, for instance (AULBACH; MINH, 1996; CARVALHO; LANGA, 2007a; CHICONE; LATUSHKIN, 1999; CHOW; LEIVA, 1995a; CHOW; LEIVA, 1996; HALE; ZHANG, 2004; HENRY, 1981; HENRY, 1994; PLISS; SELL, 1999; PöTZSCHE, 2015). In (HENRY, 1981, Section 7.6), see also (CARVALHO; LANGA; ROBINSON, 2013, Chapter 7), a robustness result is proved for exponential dichotomies, in the discrete case, by characterizing dichotomy via admissibility for the associated difference equation and, in the continuous case, by a discretization method. This consists in proving results to "move" between continuous and discrete exponential dichotomies. In this way, one is able to use the robustness of the discrete case to prove a similar result for the continuous case.

In Chapter 2 and Chapter 4 we apply Henry's techniques to study *nonuniform exponential dichotomies for evolution processes* and *exponential dichotomies for nonautonomous random dynamical systems* under perturbation, respectively. In Chapter 3 we introduce a new notion of nonuniform exponential dichotomy and provide a robustness result for it. In Chapter 5 we apply the robustness of exponential dichotomy to study structure stability of attractors for nonautonomous random dynamical systems. Thus we organize this thesis in two main parts: In the first part (Chapter 2 and Chapter 3) we focus on nonuniform hiperbolicity for evolution processes and in the second part (Chapter 4 and Chapter 5) we study exponential dichotomies and applications for nonautonomous random dynamical systems.

A nonuniform exponential dichotomy appears when we allow the constant K in the above definition to be a continuous function $K(s)$ in (1.2) and (1.3). In this case we say that (1.1) admits a *nonuniform exponential dichotomy*, see (BARREIRA; VALLS, 1998) for an introduction. Usually, the nonuniform bound is given by $K(s) = De^{\nu|s|}$ for some $\nu > 0$. As in the uniform case, there are many works concerning issues of existence and robustness for nonuniform exponential dichotomies (ALHALAWA; DRAGICEVIĆ, 2019; BARREIRA; DRAGICEVIĆ; VALLS, 2016;

BARREIRA; DRAGICEVIĆ; VALLS, 2014; BARREIRA; SILVA; VALLS, 2009a; BARREIRA; VALLS, 1998; BARREIRA; VALLS, 2008; BARREIRA; VALLS, 2009; BARREIRA; VALLS, 2015; ZHOU; LU; ZHANG, 2013).

The robustness of nonuniform exponential dichotomy for equation (1.1) can be interpreted as follows: suppose that the associated solution operator (evolution process) admits a nonuniform exponential dichotomy. The problem is to know for which family of bounded linear operators $\{B(t) : t \in \mathbb{R}\}$, the perturbed problem

$$\dot{x} = A(t)x + B(t)x, \tag{1.4}$$

admits a nonuniform exponential dichotomy. In (BARREIRA; VALLS, 2008) the authors studied under which conditions the nonuniform exponential dichotomy is robust in the case of invertible evolution processes. Later, in (ZHOU; LU; ZHANG, 2013) the authors proved a similar result for random difference equations for linear operators without the invertibility requirement. More recently, it was proved that nonuniform exponential dichotomy is robust for continuous evolution processes, also without invertibility, see (BARREIRA; VALLS, 2015). They consider an evolution process that admits a nonuniform exponential dichotomy with some growth rate $\rho(\cdot)$ satisfying some properties. They proved that if $\alpha > 0$ is the exponent and $\nu > 0$ the exponential growth of the bound satisfy $\alpha > 2\nu$ and $B : \mathbb{R} \to \mathscr{L}(X)$ is continuous satisfying $\|B(t)\|_{\mathscr{L}(X)} \leq \delta e^{-3\nu|\rho(t)|}\rho'(t)$, for all $t \in \mathbb{R}$, then the perturbed problem (1.4) admits a ρ-nonuniform exponential dichotomy.

In Chapter 2 we provide an interpretation of the robustness result as *open property*. In fact, if an evolution process $\mathscr{S}$ admits a nonuniform exponential dichotomy, there is an open neighborhood $N(\mathscr{S})$ of $\mathscr{S}$ such that every evolution process in $N(\mathscr{S})$ also admits a nonuniform exponential dichotomy. We prove that if a continuous evolution process admits a nonuniform exponential dichotomy, then each discretization also admits it. Then we use the roughness of the nonuniform exponential dichotomy for discrete evolution processes to obtain that each discretization of the perturbed evolution process also admits a nonuniform exponential dichotomy. Thus, to obtain our robustness result, we have to guarantee that if each discretization of a continuous evolution process $\mathscr{S}$ admits a nonuniform exponential dichotomy, then $\mathscr{S}$ also admits it. Our proof is inspired by the ideas in (HENRY, 1981) and, later, the same technique is applied in Chapter 4 for nonautonomous random dynamical systems.

With this method, we also obtain uniqueness and continuous dependence of projections, and explicit expressions for the bound and exponent of the perturbed evolution process. Besides, since our condition on the exponents is $\alpha > \nu$ we obtain an improvement of (BARREIRA; VALLS, 2015, Theorem 1). We consider only the case $\rho(t) = t$, as the other cases follow by a change of scaling in time. Moreover, we do not assume that the evolution processes are invertible, then it is possible to apply our result to evolutionary differential equations in Banach spaces, as the ones that appears in (CARVALHO; LANGA, 2007a; CARVALHO; LANGA; ROBINSON, 2013; CHOW; LEIVA, 1996; HENRY, 1981).

An important consequence of the robustness result regarding nonlinear evolution processes is the persistence under perturbation of *hyperbolic solutions*, see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 8). More precisely, consider a semilinear differential equation

$$\dot{x} = A(t)x + f(t,x), \ \ x(s) = x_s \in X, \tag{1.5}$$

and suppose that there for each $s \in \mathbb{R}$ and $x_s \in X$ there exists a solution $x(\cdot,s;x_s) : [s,+\infty) \to X$, then there is a nonlinear evolution process $\mathscr{S}_f = \{S_f(t,s) : t \geq s\}$ defined by $S_f(t,s)x = x(t,s;x_s)$. A map $\xi : \mathbb{R} \to X$ is called a *global solution* for $\mathscr{S}_f$ if $S_f(t,s)\xi(s) = \xi(t)$ for every $t \geq s$ and we say that ξ is a *nonuniform hyperbolic solution* if the linearized evolution process over ξ admits a nonuniform exponential dichotomy. In the uniform case it was studied the existence of hyperbolic solutions of $\mathscr{S}_f$ obtained by nonautonomous perturbations of hyperbolic equilibria, see (CARVALHO; LANGA, 2007a). We also applied this method of (CARVALHO; LANGA, 2007b) to the nonautonomous random case in Section 5.1.

Chapter 2 concludes with a result on the persistence of nonuniform hyperbolic solutions under perturbations. In fact, if ξ is a nonuniform hyperbolic solution for $\mathscr{S}_f = \{S_f(t,s) : t \geq s\}$ and g is a map "close" to f, then there exists a nonuniform hyperbolic solution for $\mathscr{S}_g = \{S_g(t,s) : t \geq s\}$ "close" to ξ. Additionally, we also prove that bounded nonuniform hyperbolic solutions are *isolated* in the space of bounded continuous functions $C_b(\mathbb{R},X)$, i.e., if ξ is a nonuniform hyperbolic solution, then there exists a neighborhood of ξ in $C_b(\mathbb{R})$ such that ξ is the only bounded solution for $\mathscr{S}_f$ is this neighborhood. All the results of Chapter 2 can be found in (CARABALLO *et al.*, 2021c). In Section 5.1 we apply similar techniques to study not only existence of hyperbolic solutions of nonautonomous random dynamical systems, but also continuity with respect to a parameter.

In Chapter 3, we propose a new type of nonuniform exponential dichotomy. Let $\{S(t,s) : t \geq s\}$ be a linear evolution process satisfying all the conditions to admit a nonuniform exponential dichotomy, except that (1.2) and (1.3) are changed to

$$\begin{aligned} \|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} \leq& K(t)e^{-\omega(t-s)}, \ \ t \geq s \\ \|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)} \leq& K(t)e^{\omega(t-s)}, \ \ t < s. \end{aligned} \tag{1.6}$$

This means that the bound $K(t)$ appears now depending on the final time t instead of the initial time s and we refer to this notion as **nonuniform exponential dichotomy of type II**, or simply **NEDII**, and to the standard one, presented in Chapter 2 as **nonuniform exponential dichotomy of type I**, or simply **NEDI**. We prove that a NEDII is a different concept of nonuniform hyperbolicity. In fact, we provide examples of evolution processes that admits NEDII and does not admit any NEDI. We also show that NEDI and NEDII are complementary notions of nonuniform hyperbolicity and under certain special conditions it is possible to relate them. One important idea is that there is some *dual correspondence* between them. For instance, if a linear evolution process admits a NEDI, it is expected that the *dual* evolution process admits a NEDII, and vice-versa. The dual evolution process corresponds to the adjoint equation. In (BARREIRA;

VALLS, 1998), they use this type of relation to obtain results for NEDI associated with invertible evolution processes. Another relation is that that NEDI and NEDII are complementary in half lines, $\mathbb{R}^+$ or $\mathbb{R}^-$, with one being more general than the other one depending in which half-line they are defined.

The dual correspondence between NEDI and NEDII allows us to establish a robustness result of NEDII for invertible evolution processes. As an application of the robustness result of Chapter 2 we provide conditions to obtain that NEDII is stable under perturbation. This fact guarantees that NEDII is an reasonable notion of nonuniform hyperbolicity. Furthermore, the robustness result for NEDII can be applied even in situations where we do not know if the NEDI is stable under perturbation. Therefore, one of our goals is to conclude that NEDII is a sensible concept and that to study NEDII is to comprehend better the notion of nonuniform hyperbolicity. The results of Chapter 3 are presented in (LANGA; OBAYA; OLIVEIRA-SOUSA, 2021), where applications of nonuniform exponential dichotomies to the study of the asymptotic behavior of evolution processes are also presented.

The second part of the thesis is concerned with nonautonomous random dynamical systems. Our final goal is to conclude continuity and topological structural stability of attractors. To this aim, in Chapter 4 we first need to establish robustness results of exponential dichotomies for *nonautonomous random dynamical systems* and apply these results in Chapter 5 to study existence of random hyperbolic solutions, which is the first step to study continuity and structural stability of attractors, see (BORTOLAN; CARVALHO; LANGA, 2020) for the nonautonomous deterministic case. Then we study existence and continuity of unstable sets associated to these random hyperbolic solutions and we use these results to tackle the problem of continuity and topological structural stability of nonautonomous random attractors.

To fix some ideas, we consider an autonomous semilinear problem in a Banach space X

$$\dot{y} = Ay + f_0(y), \;\; t > 0, \; y(0) = y \in X, \tag{1.7}$$

and nonautonomous random perturbations of it

$$\dot{y} = Ay + f_\eta(t, \theta_t\omega, y), \; t > \tau, \; y(\tau) = y_\tau \in X, \; \eta \in (0,1], \tag{1.8}$$

where A generates a strongly continuous semigroup $\{e^{At} : t \geq 0\} \subset \mathscr{L}(X)$, $\theta_t : \Omega \to \Omega$ is a random flow defined in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that there exists a hyperbolic equilibrium for (1.7) y_0^*, i.e., y_0^* is such that $f(y_0^*) = -Ay$, and the linearized problem $\dot{z} = Az + f'(y_0^*)z$ admits an exponential dichotomy. Then, we provide conditions to prove *existence and continuity* of “hyperbolic equilibria" for (1.8). In fact, we show that for each small perturbation f_η of (1.7) (see (5.3) for the condition) there exists a global solution of (1.8) ξ_η^* that presents an *hyperbolic behavior*, i.e., the linear nonautonomous random dynamical system generated by

$$\dot{y} = Ay + D_y f_\eta(t, \theta_t\omega, \xi_\eta^*(t, \theta_t\omega))y, \; t \geq \tau,$$

admits an exponential dichotomy, and that these hyperbolic solutions ξ_η^* converges to y_0^*, as $\eta \to 0$.

To prove this result on the existence and continuity of *hyperbolic solutions* for semilinear differential equations, we need first to study permanence of hyperbolicity for linear nonautonomous random dynamical systems. Thus, in Chapter 4 we extend the concept of exponential dichotomies to encompass random and nonautonomous dynamical systems and we provide conditions to guarantee that they persists under perturbation. Accordingly, we consider a small nonautonomous random perturbation $B : \mathbb{R} \times \Omega \to \mathscr{L}(X)$ of an hyperbolic problem $\dot{y} = Ay$, and we provide conditions to prove that $\dot{y} = Ay + B(t, \theta_t \omega)y$ admits exponential dichotomy. Our perturbation $B(t, \omega)$ depends on two parameters, the time t of deterministic nature and another ω varying in a probability set $(\Omega, \mathscr{F}, \mathbb{P})$. This leads us to establish robustness results of exponential dichotomies for *nonautonomous random dynamical systems*, which is a co-cycle (φ, Θ) driven by $\Theta : \mathbb{R} \times \Omega \to \mathbb{R} \times \Omega$. Our proof was inspired by (ZHOU; LU; ZHANG, 2013) (for the discrete case), and (CHOW; LEIVA, 1995b) (for the continuous case), where Henry's techniques are applied to co-cycles.

A natural application of the robustness of hyperbolicity appears in the continuity and topological structural stability of attractors associated to equations (1.7) and (1.8). The proof of continuity is done by proving *upper* and *lower semicontinuity*. On one hand, **upper semicontinuity** means that the perturbed attractors do not become suddenly much larger than the limiting attractor (non-explosion). On the other hand, **lower semicontinuity** means that the perturbed attractors do not become suddenly much smaller than the limiting attractor (non-implosion). For an introduction to the notion of continuity of attractors see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 3).

For nonautonomous (deterministic) dynamical systems the continuity of attractors is very well studied, see for instance (BORTOLAN; CARVALHO; LANGA, 2014; CARVALHO; LANGA, 2007b; CARVALHO; LANGA; ROBINSON, 2009; LANGA *et al.*, 2007). In the nonautonomous random setting, the upper semicontinuity was proved in several examples, see (BATES; LU; WANG, 2014; WANG, 2012b; WANG, 2012a) and the references therein. However, the lower semicontinuity is more difficult to attain due to the fact that one has to prove that the *inner structure* of the limiting attractor is "preserved" under perturbation, in order to ensure that the perturbed attractor occupies a region 'as large as' the region occupied by the limiting attractor. More precisely, the typical conditions one has to assume is that the limiting attractor is the union of the unstable sets of the equilibria and then give conditions to ensure that these equilibria and their unstable sets 'persist' under perturbation, see (ARRIETA; CARVALHO, 2004; BABIN; VISHIK, 1983; BRUSCHI; CARVALHO; CHOLEWA, 2006; HALE; RAUGEL, 1989) for the lower semicontinuity of global attractors, and (CARVALHO; LANGA; ROBINSON, 2009; CARVALHO; LANGA, 2007b; LANGA *et al.*, 2007) for the lower semicontinuity of pullback attractors and (BORTOLAN; CARVALHO; LANGA, 2014) for the lower semicontinuity of uniform attractors. In (CARVALHO; LANGA, 2007b) the authors study the permanence of hyperbolic global solutions and of their corresponding unstable and stable sets, in the nonautonomous setting, and in (CARVALHO; LANGA; ROBINSON, 2009) the authors prove a general result on the lower semicontinuity of pullback attractors allowing the limiting pullback attractor to be given as the closure of a countable (possibly infinity) union of unstable sets of hyperbolic global solutions.

Thus, to prove lower semicontinuity in a nonautonomous random framework we follow this method of proving that the inner structure persists under perturbation. However, this is not expected to happen for general types of noises. Actually, some works provides that the presence of an additive noise destroys the continuity of the attractors (BIANCHI; BLöMKER; YANG, 2016; CRAUEL; FLANDOLI, 1998), see also (CALLAWAY *et al.*, 2017) for an complementary study of such problems. Hence, to obtain our results we will consider small bounded random perturbations as the one introduced in Chapter 4, and then we study the existence and permanence of hyperbolic solutions for (1.8) assuming that the perturbations are uniformly bounded in time. Then, inspired by the results in (CARVALHO; LANGA, 2007b), we study the existence and continuity of the unstable sets associated with this hyperbolic solutions, and we use these results to conclude the lower semicontinuity for the attractors of $\{(\psi_\eta, \Theta) : \eta \in [0,1]\}$, see Theorem 5.3.3. In our proofs, we show how to control the random parameter using techniques of deterministic dynamical systems.

The idea of reproducing the internal structure in the perturbed attractor is not only important to show continuity of attractors, but is also crucial to prove that the dynamics are preserved under perturbation. For instance, in (CARVALHO; LANGA, 2009) the authors provide conditions (permanence of the inner structure) to prove that dynamically gradient semigroups are stable under perturbation. We refer to this property as **topological structural stability**. Gradient dynamical systems were widely studied in the past years, see (ARAGãO-COSTA *et al.*, 2013; BORTOLAN *et al.*, 2020; BORTOLAN; CARVALHO; LANGA, 2014; BORTOLAN; CARVALHO; LANGA, 2020; CARVALHO; LANGA, 2007b; CARABALLO *et al.*, 2010b) for deterministic dynamical systems, and (CARABALLO; LANGA; LIU, 2012; JU; QI; WANG, 2018) for random dynamical systems. In this work, we obtain a result on the topological structural stability for nonautonomous random differential equations, see Theorem 5.4.3. This will be also a consequence of the careful study of the internal structure of these attractors.

We also obtain stronger results on the continuity and topological structural stability of nonautonomous random attractors for the case when the random perturbations are uniformly bounded with respect to the random parameter, see Remark 5.3.6 and Remark 5.4.4 for more details. Moreover, see (BOBRYK, 2021; CARABALLO; LÓPEZ-DE-LA-CRUZ, 2021) for examples of this types of noises.

We finish Chapter 5 with two applications of our abstract results. First, in a family of stochastic differential equations with a nonautonomous multiplicative white noise

$$dy = Bydt + f_0(y)dt + \eta \kappa_t y \circ dW_t,\ t \geq \tau,\ y(\tau) = y_\tau \in X, \tag{1.9}$$

where $\eta \in [0,1]$, and the mapping $\mathbb{R} \ni t \mapsto \kappa_t \in \mathbb{R}$ is a real function that "controls" the growth of the noise in time, see Subsection 5.5.1. Finally, a nonautonomous random perturbation on the damping of a damped wave equation with Dirichlet boundary condition

$$u_{tt} + \beta_\eta(t, \theta_t \omega) u_t - \Delta u = f(u),\ t \geq \tau,\ \eta \in [0,1], \tag{1.10}$$

where $\{\theta_t : \Omega \to \Omega : t \in \mathbb{R}\}$ is a random flow in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and there exists $\beta > 0$ such that β_η converges to β as $\eta \to 0$, see Subsection 5.5.2.

CHAPTER

2

ROBUSTNESS OF NONUNIFORM EXPONENTIAL DICHOTOMY

In this chapter, we study stability properties of nonuniform hyperbolicity for evolution processes associated with differential equations in Banach spaces according to (CARABALLO *et al.*, 2021c). We show that the nonuniform exponential dichotomy is stable under perturbation. Moreover, we provide conditions to obtain uniqueness and continuous dependence of projections associated with nonuniform exponential dichotomies. We also present an example of evolution process in a Banach space that admits nonuniform exponential dichotomy and, for it, we study the permanence of the nonuniform hyperbolicity under perturbation. Finally, we prove persistence of nonuniform hyperbolic solutions for nonlinear evolution processes under perturbations.

2.1 Nonuniform exponential dichotomy: discrete case

In this section, we present some basic facts of nonuniform exponential dichotomy for discrete evolution processes. We establish uniqueness of projections and continuous dependence of projections and present a robustness of the nonuniform exponential dichotomy in the discrete case. We start with the definition of a *discrete evolution process* in a Banach space $(X, \|\cdot\|_X)$ in a particular case where the family of operators are linear bounded operators in X.

Definition 2.1.1. *Let $\mathscr{S} = \{S_{n,m} : n \geq m \text{ with } n,m \in \mathbb{Z}\}$ be a family of bounded linear operators in a Banach space X. We say that $\mathscr{S}$ is a* ***discrete evolution process*** *if*

1. *$S_{n,n} = Id_X$, for all $n \in \mathbb{Z}$;*

2. *$S_{n,m}S_{m,k} = S_{n,k}$, for all $n \geq m \geq k$.*

To simplify the notation, we write $\mathscr{S} = \{S_{n,m} : n \geq m\}$ as an evolution process, whenever is clear we are dealing with discrete ones.

Remark 2.1.2. *Given a discrete evolution process $\mathscr{S} = \{S_{n,m} : n \geq m\}$ it is always possible to associate $\mathscr{S}$ with the family $\{S_n : n \in \mathbb{Z}\}$, where $S_n := S_{n+1,n}$ for all $n \in \mathbb{Z}$. Conversely, for any family of bounded linear operators $\{S_n : n \in \mathbb{Z}\} \subset \mathscr{L}(X)$ define $S_{n,m} := S_{n-1} \cdots S_m$ for $n > m$ and $S_{n,n} := Id_X$ so that $\mathscr{S} = \{S_{n,m} : n \geq m\}$ is discrete evolution process. Therefore, we often refer, indistinctly, to $\{S_n : n \in \mathbb{Z}\}$ or $\{S_{n,m} : n \geq m\}$ as the discrete evolution process.*

Thus it is possible to associate the evolution process $\{S_{n,m} : n \geq m\}$ with the following difference equation

$$x_{n+1} = S_n x_n, \;\; x_n \in X, \;\; n \in \mathbb{Z}, \tag{2.1}$$

where $S_n = S_{n+1,n}$, $n \in \mathbb{Z}$.

Now, we present the definition of *nonuniform exponential dichotomy*.

Definition 2.1.3. *Let $\mathscr{S} = \{S_{n,m} : n \geq m\} \subset \mathscr{L}(X)$ be a discrete evolution process in a Banach space X. We say that $\mathscr{S}$ admits a* ***nonuniform exponential dichotomy*** *if there is a family of continuous projections $\{\Pi^u_n; n \in \mathbb{Z}\}$ in $\mathscr{L}(X)$ such that*

1. *$\Pi^u_n S_{n,m} = S_{n,m}\Pi^u_m$, for $n \geq m$;*
2. *$S_{n,m} : R(\Pi^u_m) \to R(\Pi^u_n)$ is an isomorphism, for $n \geq m$, and we define $S_{m,n}$ as its inverse;*
3. *There exists a function $K : \mathbb{Z} \to [1, +\infty)$ with $K(n) \leq De^{\nu|n|}$, for some $D \geq 1$ and $\nu > 0$, and $\alpha > 0$ such that*

$$\|S_{n,m}\Pi^s_m\|_{\mathscr{L}(X)} \leq K(m)e^{-\alpha(n-m)}, \;\; \forall n \geq m,$$
$$\|S_{n,m}\Pi^u_m\|_{\mathscr{L}(X)} \leq K(m)e^{\alpha(n-m)}, \;\; \forall n \leq m,$$

where $\Pi^s_n := (Id_X - \Pi^u_n)$ for all $n \in \mathbb{Z}$.

In this theory, K and α are usually called the ***bound*** *and the* ***exponent*** *of the exponential dichotomy, respectively.*

We will present a class of perturbations for this equation so that it will be possible to guarantee existence of solutions for the perturbed problem. This study is usually called *admissibility*.

Definition 2.1.4. *Given two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, we say that the pair of spaces $(\mathfrak{Y}, \mathfrak{X})$ is* ***admissible*** *for equation (2.1) if for $\{f_n\}_{n\in\mathbb{Z}} \in \mathfrak{Y}$ there is a solution $\{x_n\}_{n\in\mathbb{Z}} \in \mathfrak{X}$ for the equation*

$$x_{n+1} = S_n x_n + f_n.$$

We now recall the definition of a *Green's function*.

Definition 2.1.5. *Let $\mathscr{S} = \{S_n : n \in \mathbb{Z}\}$ be a discrete evolution process which admits a nonuniform exponential dichotomy with family of projections $\{\Pi_n^u\}_{n\in\mathbb{Z}}$. The* ***Green's function*** *associated to the evolution process $\mathscr{S}$ is given by*

$$G_{n,m} = \begin{cases} S_{n,m}\Pi_m^s, & \text{if } n \geq m, \\ -S_{n,m}\Pi_m^u, & \text{if } n < m. \end{cases}$$

As in the uniform case, the next result shows that it is possible to obtain the solution for

$$x_{n+1} = S_n x_n + f_n, \; n \in \mathbb{Z}. \tag{2.2}$$

using the *Green's function*. A space that appears naturally in this case when dealing with (2.2) is

$$l^\infty_{1/K}(\mathbb{Z}) := \left\{ f : \mathbb{Z} \to X : \sup_{n\in\mathbb{Z}} \left\{ \|f_n\|_X K(n+1) \right\} = M_f < +\infty \right\},$$

where $K : \mathbb{Z} \to \mathbb{R}$ is such that $K(n) \geq 1$ for all $n \in \mathbb{Z}$. Of course, with this notation, $l^\infty_1(\mathbb{Z}) = l^\infty(\mathbb{Z})$.

Theorem 2.1.6. *Assume that the evolution process $\mathscr{S} = \{S_n : n \in \mathbb{Z}\}$ admits a nonuniform exponential dichotomy with bound $K(n) \leq De^{\nu|n|}$ and exponent $\alpha > \nu$. Then the pair $(l^\infty_{1/K}(\mathbb{Z}), l^\infty(\mathbb{Z}))$ is admissible, i.e., if $f \in l^\infty_{1/K}(\mathbb{Z})$ then Equation (2.2) possesses a unique bounded solution given by*

$$x_n = \sum_{-\infty}^{+\infty} G_{n,k+1} f_k, \;\; \forall \, n \in \mathbb{Z}.$$

Proof. First we fix $n \in \mathbb{Z}$, take $m < n$ and write

$$x_n = S_{n,m} x_m + \sum_{k=m}^{n-1} S_{n,k+1} f_k.$$

Then apply Π_n^s in this equation and note that the term $S_{n,m}\Pi_m^s x_m$ satisfies

$$\|S_{n,m}\Pi_m^s x_m\|_X \leq K(m)\, e^{-\alpha(n-m)} \|x_m\|_X.$$

Therefore if $\{x_n\}_{n\in\mathbb{Z}}$ is a bounded sequence, this last term goes to zero when $m \to -\infty$, using that $K(m) \leq e^{\nu|m|}$ and $\nu < \alpha$. Thus, we have that for each $n \in \mathbb{Z}$

$$\Pi_n^s x_n = \sum_{k=-\infty}^{n-1} S_{n,k+1} \Pi_{k+1}^s f_k.$$

Analogously, take now $r > n$ and write

$$x_r = S_{r,n} x_n + \sum_{k=n}^{r} S_{r,k+1} f_k,$$

Then apply the projection Π_r^u and use the inverse operator $S_{n,r}$ to obtain

$$\Pi_n^u x_n = S_{n,r}\Pi_r^u x_r - \sum_{k=n}^{r} S_{n,k+1} \Pi_{k+1}^u f_k,$$

and just notice that

$$\|S_{n,r}\Pi^u_r x_r\|_X \leq K(r)e^{\alpha(n-r)}\|x_r\|_X, \tag{2.3}$$

Again, as $\{x_n\}_{n\in\mathbb{Z}}$ is bounded, and $\nu < \alpha$, this last term goes to zero as $r \to +\infty$. Consequently,

$$\Pi^u_n x_n = -\sum_{k=n}^{+\infty} S_{n,k+1}\Pi^u_{k+1} f_k.$$

Thus, for each $n \in \mathbb{Z}$,

$$x_n = \Pi^s_n x_n + \Pi^u_n x_n = \sum_{-\infty}^{+\infty} G_{n,k+1} f_k.$$

It is easy to see that $x_n = \sum_{-\infty}^{+\infty} G_{n,k+1} f_k$ is a solution for (2.2). Finally, we have that

$$\begin{aligned}\|x_n\|_X &\leq \sum_{k=-\infty}^{n-1} \|S_{n,k+1}\Pi^s_{k+1} f_k\|_X + \sum_{k=n}^{+\infty} \|S_{n,k+1}\Pi^u_{k+1} f_k\|_X \\ &\leq M_f\left(\sum_{k=-\infty}^{n-1} e^{-\alpha(n-k-1)} + \sum_{k=n}^{+\infty} e^{\alpha(n-k-1)}\right).\end{aligned}$$

Therefore, for every $n \in \mathbb{Z}$,

$$\|x_n\|_X \leq M_f \frac{1+e^{-\alpha}}{1-e^{-\alpha}},$$

and existence and uniqueness of a bounded solution for (2.2) is ensured. The proof is complete.

□

As a consequence of Theorem 2.1.6, we obtain uniqueness of the family of projections associated with the nonuniform exponential dichotomy.

Corollary 2.1.7 (Uniqueness of projections). *If $\mathscr{S} = \{S_n : n \in \mathbb{Z}\}$ admits a nonuniform exponential dichotomy with bound $K(n) \leq De^{\nu|n|}$ and exponent $\alpha > \nu$, then the family of projections is uniquely determined.*

Proof. Let $\{\Pi^{u,(i)}_n ; n \in \mathbb{Z}\}$, for $i = 1,2$, projections associated with the evolution process $\mathscr{S}$. Given $x \in X$ and $m \in \mathbb{Z}$ fixed, define $f_n = 0$, for all $n \neq m-1$, and $f_{m-1} = K(m)^{-1}x$. Thus, $f \in l^\infty_{1/K}(\mathbb{Z})$ and from Theorem 2.1.6 there exists a unique solution $\{x_n\}_{n\in\mathbb{Z}}$ for

$$x_{n+1} = S_n x_n + f_n, \ \ n \in \mathbb{Z}.$$

Hence, $x_m = \sum_{-\infty}^{+\infty} G^{(i)}_{m,k+1} f_k = G^{(i)}_{m,m} f_{m-1} = K(m)^{-1}\Pi^{s,(i)}_m x$, for $i = 1,2$. Therefore, $\Pi^{u,(1)}_m = \Pi^{u,(2)}_m$ for all $m \in \mathbb{Z}$. □

Next, we establish a result about continuous dependence of projections.

Theorem 2.1.8 (Continuous dependence of projections)**.** *Suppose that* $\{T_n\}_{n\in\mathbb{Z}}$ *and* $\{S_n\}_{n\in\mathbb{Z}}$ *admit a nonuniform exponential dichotomy with projections* $\{\Pi_n^{u,\mathscr{T}}\}_{n\in\mathbb{Z}}$ *and* $\{\Pi_n^{u,\mathscr{S}}\}_{n\in\mathbb{Z}}$*, exponents* $\alpha_{\mathscr{T}}$ *and* $\alpha_{\mathscr{S}}$*, respectively, and the same bound* $K(n)\le De^{\nu|n|}$*. If* $\nu<\min\{\alpha_{\mathscr{T}},\alpha_{\mathscr{S}}\}$ *and*

$$\sup_{n\in\mathbb{Z}}\left\{K(n+1)\|T_n-S_n\|_{\mathscr{L}(X)}\right\}\le\varepsilon,$$

then

$$\sup_{n\in\mathbb{Z}}\left\{K(n)^{-1}\,\|\Pi_n^{u,\mathscr{T}}-\Pi_n^{u,\mathscr{S}}\|_{\mathscr{L}(X)}\right\}\le\frac{e^{-\alpha_{\mathscr{S}}}+e^{-\alpha_{\mathscr{T}}}}{1-e^{-(\alpha_{\mathscr{S}}+\alpha_{\mathscr{T}})}}\,\varepsilon.$$

Proof. Let $z\in X$ and $m\in\mathbb{Z}$ be fixed and consider

$$f_n=\begin{cases}0, & \text{if } n\neq m-1,\\ K(m)^{-1}z, & \text{if } n=m-1.\end{cases}$$

Thus, by Theorem 2.1.6, there exist bounded solutions $x^k=\{x_n^k\}_{n\in\mathbb{Z}}$ given by $x_n^k:=G_{n,m}^k zK(m)^{-1}$ for $k=\mathscr{T},\mathscr{S}$. Note that, for $n\in\mathbb{Z}$,

$$x_{n+1}^{\mathscr{T}}-S_nx_n^{\mathscr{T}}=T_nx_n^{\mathscr{T}}-S_nx_n^{\mathscr{T}}+f_n$$

and $x_{n+1}^{\mathscr{S}}-S_nx_n^{\mathscr{S}}=f_n$. Then, if $z_n:=x_n^{\mathscr{T}}-x_n^{\mathscr{S}}$ we obtain that $z_{n+1}=S_nz_n+y_n$, where $y_n=(T_n-S_n)x_n^{\mathscr{T}}$ for all $n\in\mathbb{Z}$. Thanks to the boundedness of the sequence $\{x_n^{\mathscr{T}}\}_{n\in\mathbb{Z}}$ and by the hypothesis on T_n-S_n we have that $\{y_nK(n+1)\}_{n\in\mathbb{Z}}$ is bounded, and by Theorem 2.1.6 we have that

$$z_n=\sum_{k=-\infty}^{\infty}G_{n,k+1}^{\mathscr{S}}(T_k-S_k)G_{k,m}^{\mathscr{T}}zK(m)^{-1},$$

and therefore, by the hypothesis on $\mathscr{T}-\mathscr{S}$, we deduce

$$\begin{aligned}\|z_m\|_X &\le \sum_{k=-\infty}^{\infty}K(k+1)e^{-\alpha_{\mathscr{S}}|m-k-1|}\|T_k-S_k\|_{\mathscr{L}(X)}e^{-\alpha_{\mathscr{T}}|k-m|}\|z\|_X\\ &\le \frac{e^{-\alpha_{\mathscr{S}}}+e^{-\alpha_{\mathscr{T}}}}{1-e^{-(\alpha_{\mathscr{S}}+\alpha_{\mathscr{T}})}}\,\varepsilon\,\|z\|_X.\end{aligned}$$

The definition of z in m yields

$$z_m=x_m^{\mathscr{T}}-x_m^{\mathscr{S}}=(G_{m,m}^{\mathscr{T}}-G_{m,m}^{\mathscr{S}})K(m)^{-1}z=(\Pi_m^{u,\mathscr{S}}-\Pi_m^{u,\mathscr{T}})K(m)^{-1}z.$$

Consequently,

$$\|(\Pi_m^{u,\mathscr{S}}-\Pi_m^{u,\mathscr{T}})K(m)^{-1}z\|_X\le\frac{e^{-\alpha_{\mathscr{S}}}+e^{-\alpha_{\mathscr{T}}}}{1-e^{-(\alpha_{\mathscr{S}}+\alpha_{\mathscr{T}})}}\,\varepsilon\,\|z\|_X,$$

which concludes the proof of the theorem. □

Finally, we state a robustness result for discrete evolution processes with nonuniform exponential dichotomies.

Theorem 2.1.9 (Robustness for discrete evolution processes)**.** *Let $\mathscr{S} = \{S_n : n \in \mathbb{Z}\}$, $\mathscr{B} = \{B_n : n \in \mathbb{Z}\} \subset \mathscr{L}(X)$ be discrete evolution processes. Assume that $\mathscr{S}$ admits a nonuniform exponential dichotomy with bound $K(n) \le De^{\nu|n|}$ and exponent $\alpha > \nu$, and that $\mathscr{B}$ satisfies*

$$\|B_k\|_{\mathscr{L}(X)} \le \delta K(k+1)^{-1},\ \forall k \in \mathbb{Z},$$

where $\delta > 0$ is such that $\delta < (1-e^{-\alpha})(1+e^{-\alpha})^{-1}$. Then the perturbed evolution process $\mathscr{T} = \mathscr{S} + \mathscr{B}$ admits a nonuniform exponential dichotomy with exponent

$$\tilde{\alpha} = -\ln(\cosh\alpha - [\cosh^2\alpha - 1 - 2\delta\sinh\alpha]^{1/2}),$$

and bound

$$\tilde{K}(n) = K(n)\left[1 + \frac{\delta}{(1-\rho)(1-e^{-\alpha})}\right]\max[D_1, D_2],$$

where $\rho := \delta(1+e^{-\alpha})(1-e^{-\alpha})^{-1}$, $D_1 := [1-\delta e^{-\alpha}/(1-e^{-\alpha-\tilde{\alpha}})]^{-1}$, $D_2 := [1-\delta e^{-\tilde{\beta}}/(1-e^{-\alpha-\tilde{\beta}})]^{-1}$ and $\tilde{\beta} := \tilde{\alpha} + \ln(1+2\delta\sinh\alpha)$.

The proof of Theorem 2.1.9 follows, step by step, the proof (ZHOU; LU; ZHANG, 2013, Theorem 1) with minimal changes, so it will be omitted. It is important to notice that all the arguments of their proof still hold with the assumption $\alpha > \nu$. In Chapter 4 we present the main ideas of the proof of Zhou *et al.* in a simpler case, see the proof of Theorem 4.2.5.

In the following section we will provide the continuous version of the theorems of this section. We emphasize that one of our goals is to prove a robustness result of nonuniform exponential dichotomy for *continuous evolution processes* with this same condition on the exponents ($\alpha > \nu$).

2.2 Nonuniform exponential dichotomy: continuous case

In this section, we consider evolution processes with parameters in $\mathbb{R}$. We apply Henry's techniques (HENRY, 1981, Section 7.6) to study the nonuniform exponential dichotomies. We prove theorems that allow us to obtain the continuous versions of the results presented in Section 2.1. The main theorem of this section is the robustness for nonuniform exponential dichotomies, namely Theorem 2.2.11. We also provide a version of it made suitable for applications to differential equations, Theorem 2.2.14. In addition, we establish results on uniqueness and continuous dependence of projections associated with nonuniform exponential dichotomy, Corollary 2.2.8 and Theorem 2.2.9, respectively.

We first recall the definition of *evolution process* over a metric space (X,d) with parameters in an interval $\mathbb{J} = \mathbb{R}, \mathbb{R}^+ := \{t \in \mathbb{R} : t \ge 0\}$ or $\mathbb{R}^- := \{t \in \mathbb{R} : t \le 0\}$.

Definition 2.2.1. *Let $\mathscr{S} := \{S(t,s) : X \to X\,; t \ge s,\ t,s \in \mathbb{J}\}$ be a family of continuous operators in a Banach space X with parameters in $\mathbb{J}$. We say that $\mathscr{S}$ is a* ***continuous evolution process*** *in X if*

1. *$S(t,t) = Id_X$, for all $t \in \mathbb{J}$;*

2. *$S(t,s)S(s,\tau) = S(t,\tau)$, for $t \geq s \geq \tau$;*

3. *$\{(t,s) \in \mathbb{J}^2; t \geq s\} \times X \ni (t,s,x) \mapsto S(t,s)x$ is continuous.*

To simplify we usually say that $\mathscr{S} = \{S(t,s) : t \geq s\}$ is an ***evolution process****, whenever is implicit that $\mathscr{S}$ is a continuous evolution process.*

If additionally, the operator $S(t,s)$ is invertible for all $t \geq s$, then we say that $\mathscr{S}$ is an ***invertible evolution process****. In this situation, we write $\mathscr{S} = \{S(t,s) : t,s \in \mathbb{J}\}$, where $S(s,t)$ is the inverse of $S(t,s)$, for $t \geq s$.*

In this chapter, we only consider the the case where the evolution processes are defined in the entire real line $\mathbb{R} = \mathbb{J}$. In Chapter 3 we also consider the cases of semilines $\mathbb{J} = \mathbb{R}^-$ and $\mathbb{J} = \mathbb{R}^+$.

Remark 2.2.2. *Note that the operators $S(t,s) : X \to X$, in the definition above, do not need to be linear. In fact, in Section 2.4, we study permanence of the nonuniform hyperbolic behavior for nonlinear evolution processes.*

We also recall the notion of a *global solution* for an evolution process.

Definition 2.2.3. *Let $\mathscr{S} = \{S(t,s) : t \geq s\}$ be an evolution process. We say that $\xi : \mathbb{R} \to X$ is a* ***global solution*** *for $\mathscr{S}$ if $S(t,s)\xi(s) = \xi(t)$ for every $t \geq s$.*

We say that a global solution ξ is ***backwards bounded*** *if there exists $t_0 \in \mathbb{R}$ such that $\xi(-\infty,t_0] = \{\xi(t) : t \leq t_0\}$ is bounded.*

Now, we present the definition of *nonuniform exponential dichotomy* for linear evolution processes:

Definition 2.2.4. *Let $\mathscr{S} = \{S(t,s); t \geq s\} \subset \mathscr{L}(X)$ be a linear evolution process. We say that $\mathscr{S}$ admits a* ***nonuniform exponential dichotomy*** *if there exists a family of continuous projections $\{\Pi^u(t) : t \in \mathbb{R}\}$ such that*

1. *$\Pi^u(t)S(t,s) = S(t,s)\Pi^u(s)$, for all $t \geq s$;*

2. *$S(t,s) : R(\Pi^u(s)) \to R(\Pi^u(t))$ is an isomorphism, for $t \geq s$, and we define $S(s,t)$ as its inverse;*

3. *There exists a continuous function $K : \mathbb{R} \to [1,+\infty)$ and some constants $\alpha > 0$, $D \geq 1$ and $\nu \geq 0$ such that $K(s) \leq De^{\nu|s|}$ and*

$$\begin{aligned} \|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} &\leq K(s)e^{-\alpha(t-s)}, \ t \geq s; \\ \|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)} &\leq K(s)e^{\alpha(t-s)}, \ t < s, \end{aligned}$$

where $\Pi^s(s) = Id_X - \Pi^u(s)$, *for all* $s \in \mathbb{R}$.

Remark 2.2.5. *This definition also includes uniform exponential dichotomies, when* $t \mapsto K(t)$ *is bounded, and tempered exponential dichotomies, when* $t \mapsto K(t)$ *has a sub-exponential growth, see (BARREIRA; DRAGICEVIĆ; VALLS, 2016; ZHOU; LU; ZHANG, 2013). From now on we assume that* $K(t) = De^{\nu|t|}$, $t \in \mathbb{R}$.

In the following result we study each "discretization at instant t" of an evolution process that admits a nonuniform exponential dichotomy.

Theorem 2.2.6. *Let* $\mathscr{S}$ *be a continuous evolution process that admits a nonuniform exponential dichotomy with bound* $K(t) = De^{\nu|t|}$ *and exponent* $\alpha > 0$. *Then for each* $t \in \mathbb{R}$ *and* $l > 0$ *the discrete evolution process*

$$\{S_{m,n}(t) : m,n \in \mathbb{Z} \text{ with } m \geq n\} := \{S(t+ml, t+nl) : m,n \in \mathbb{Z} \text{ with } m \geq n\}$$

admits a nonuniform exponential dichotomy with bound $\tilde{K}_t(m) := K(t+ml)$ *and exponent* $\tilde{\alpha} = \alpha l$.

Proof. Define, for each $t \in \mathbb{R}$, the family of projections $\{\Pi^u_m(t) = \Pi^u(t+ml) : m \in \mathbb{N}\}$, then

$$\begin{aligned}
\Pi^u_m(t) S_{m,n}(t) &= \Pi^u(t+ml) S(t+ml, t+nl) \\
&= S(t+ml, t+nl) \Pi^u(t+nl) \\
&= S_{m,n}(t) \Pi^u_n(t),
\end{aligned}$$

and the first property is proved. Note that, for $m \geq n$,

$$S_{m,n}(t)|_{R(\Pi^u_n(t))} = S(t+ml, t+nl)|_{R(\Pi^u(t+nl))}$$

and the right hand side of the equation is an isomorphism, so we define the inverse $S_{n,m}(t) : R(\Pi^u(t+ml)) \to R(\Pi^u(t+nl))$.

Finally, for $n \geq m$,

$$\begin{aligned}
\|S_{n,m}(t)(Id_X - \Pi^u_m(t))\|_{\mathscr{L}(X)} &= \|S(t+ml, t+nl)(Id_X - \Pi^u(t+nl))\|_{\mathscr{L}(X)} \\
&\leq K(t+ml) e^{-\alpha l(n-m)},
\end{aligned}$$

and, for $n < m$,

$$\begin{aligned}
\|S_{n,m}(t) \Pi^u_m(t)\|_{\mathscr{L}(X)} &= \|S_{n,m}(t) \Pi^u(t+ml)\|_{\mathscr{L}(X)} \\
&\leq K(t+ml) e^{\alpha l(n-m)}.
\end{aligned}$$

Therefore, $\{S_{n,m}(t) : n \geq m\}$ admits a nonuniform exponential dichotomy with exponent $\tilde{\alpha} = \alpha l$ and bound $\tilde{K}_t(m) = K(t+ml) \leq De^{\nu|t|} e^{\nu l|m|}$, which concludes the proof. □

Remark 2.2.7. *In Theorem 2.2.6, for a fixed $t \in \mathbb{R}$, the discretized evolution process $\{S_n(t) : n \in \mathbb{Z}\}$ appears with a bound K_t dependent of the time t and the exponent $\tilde{\alpha}$ is independent of t. This is an expected difference with the the case of uniform exponential dichotomy, where both, the bound and the exponent of the discretization are independent of t, see (HENRY, 1981).*

Now, as a consequence of Theorem 2.2.6 and Corollary 2.1.7, we obtain the uniqueness of the family of projections.

Corollary 2.2.8 (Uniqueness of the family of projections)**.** *Let $\mathscr{S}$ be an evolution process which admits a nonuniform exponential dichotomy with bound $K(t) = De^{\nu|t|}$, $t \in \mathbb{R}$, and exponent $\alpha > \nu$. Then the family of projections is unique.*

As another application of Theorem 2.2.6, we prove a result on the continuous dependence of projections.

Theorem 2.2.9 (Continuous dependence of projections)**.** *Suppose that $\mathscr{S}$ and $\mathscr{T}$ are linear evolution processes with nonuniform exponential dichotomy with projections $\{\Pi^u_{\mathscr{S}}(t) : t \in \mathbb{R}\}$ and $\{\Pi^u_{\mathscr{T}}(t) : t \in \mathbb{R}\}$ and exponents $\alpha_{\mathscr{T}}, \alpha_{\mathscr{S}}$ and with the same bound $K(t) = De^{\nu|t|}$, for $t \in \mathbb{R}$. If $\nu < \min\{\alpha_{\mathscr{T}}, \alpha_{\mathscr{S}}\}$ and*

$$\sup_{0\le t-s\le 1} \left\{K(t)\|T(t,s) - S(t,s)\|_{\mathscr{L}(X)}\right\} \le \varepsilon, \tag{2.4}$$

then

$$\sup_{t\in\mathbb{R}} \left\{K(t)^{-1}\|\Pi^u_{\mathscr{T}}(t) - \Pi^u_{\mathscr{S}}(t)\|_{\mathscr{L}(X)}\right\} \le \frac{e^{-\alpha_{\mathscr{S}}} + e^{-\alpha_{\mathscr{T}}}}{1 - e^{-(\alpha_{\mathscr{S}}+\alpha_{\mathscr{T}})}}\varepsilon.$$

Proof. Let $t \in \mathbb{R}$, from Theorem 2.2.6 for $l = 1$, $\{T_n(t) = T(t+n+1, t+n) : n \in \mathbb{Z}\}$ and $\{S_n(t) = S(t+n+1, t+n) : n \in \mathbb{Z}\}$ admit a nonuniform exponential dichotomy with exponents $\alpha_{\mathscr{T}}$ and $\alpha_{\mathscr{S}}$ and the same bound $K_t(n) := K(t+n)$. Now, from Theorem 2.1.8 we conclude that

$$K(t+n)^{-1}\|\Pi^u_{\mathscr{T}}(t+n) - \Pi^u_{\mathscr{S}}(t+n)\|_{\mathscr{L}(X)} \le \frac{e^{-\alpha_{\mathscr{S}}} + e^{-\alpha_{\mathscr{T}}}}{1 - e^{-(\alpha_{\mathscr{S}}+\alpha_{\mathscr{T}})}}\varepsilon.$$

Since t is arbitrary and the right-hand side does not depend on t the proof is complete taking $n = 0$. □

Uniqueness and continuous dependence of projections are a simple consequence of Theorem 2.2.6, and of course the results in the discrete case. However, to prove our robustness result, we will need a sort of a reciprocal result of Theorem 2.2.6.

Theorem 2.2.10. *Let $\mathscr{S} = \{S(t,s) : t \ge s\} \subset \mathscr{L}(X)$ be a continuous evolution process. Suppose that*

1. *there exist $l > 0$ and $\nu \ge 0$ such that*

$$L(\nu, l) := \sup_{0\le t-s\le l} \left\{\|S(t,s)\|_{\mathscr{L}(X)}e^{-\nu|t|}\right\} < +\infty,$$

2. *for each $t \in \mathbb{R}$ the discretized process,*

$$\{T_{n,m}(t), n \geq m\} = \{S(t+nl, t+ml), n \geq m\}$$

possesses a nonuniform exponential dichotomy with bound $K_t(\cdot) : \mathbb{Z} \to [1, +\infty)$, with $K_t(m) \leq De^{\nu|t+m|}$ and exponent $\alpha > 0$ independent of t.

If $\nu l < \alpha$, the evolution process $\mathscr{S}$ admits a nonuniform exponential dichotomy with exponent $\hat{\alpha} = (\alpha - \nu l)/l$ and bound

$$\hat{K}(s) = D^2 e^{2\alpha} \max\{L(\nu, l), L(\nu, l)^2\}\, e^{2\nu|s|}.$$

Proof. First, we fix $t \in \mathbb{R}$ and define the linear operator $T_n(t) := T_{n+1,n}(t)$, for each $n \in \mathbb{Z}$. Then for each discrete evolution process $\{T_n(t) : n \in \mathbb{Z}\}$, there exists a family of projections $\{\Pi^u_n(t) : n \in \mathbb{Z}\}$ such that the nonuniform exponential dichotomy conditions are satisfied.

For each fixed $k \in \mathbb{Z}$ we have

$$T_{n+k}(t) = T_n(t+kl), \ \ \forall n \in \mathbb{Z}.$$

Then this linear operator generates the same evolution process with associated projections $\{\Pi^u_{n+k}(t)\}_{n\in\mathbb{Z}}$ and $\{\Pi^u_n(t+kl)\}_{n\in\mathbb{Z}}$. Thus by uniqueness of the projections for the discrete case, namely Corollary 2.1.7, we obtain that for all $n, k \in \mathbb{Z}$,

$$\Pi^u_{n+k}(t) = \Pi^u_n(t+kl).$$

Now, for all $t \in \mathbb{R}$ we define $\Pi^u(t) := \Pi^u_0(t)$. These projections are the candidates to obtain the nonuniform exponential dichotomy.

Let us now prove the boundedness in the case $t \geq s$.

Claim 1: If $t \geq s$, then

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} \leq \hat{K}(s) e^{-\hat{\alpha}(t-s)},$$

where $\Pi^s(s) = Id_X - \Pi^u(s)$, $s \in \mathbb{R}$ and $\hat{K}$ is defined in the statement of the theorem.

Indeed, choose $n \in \mathbb{N}$, such that $nl + s \leq t < (n+1)l + s$, then we write

$$S(t,s)\Pi^s(s) = S(t, s+nl)S(s+nl, s)(Id_X - \Pi^u_0(s)).$$

Thus, by hypothesis,

$$\|S(s+nl, s)(Id_X - \Pi^u_0(s))\|_{\mathscr{L}(X)} = \|T_{n,0}(s)(Id_X - \Pi^u_0(s))\|_{\mathscr{L}(X)} \leq K_s(0)e^{-\alpha n},$$

which implies that

$$\begin{aligned}
\|S(t,s)(Id_X - \Pi^u(s))\|_{\mathscr{L}(X)} &\leq \|S(t, s+nl)\|_{\mathscr{L}(X)} K_s(0) e^{-\alpha n} \\
&= K(s) e^{\alpha(t-nl-s)/l} \|S(t, s+nl)\|_{\mathscr{L}(X)} e^{-\alpha(t-s)/l} \\
&\leq De^{\nu|s|}\, e^{\alpha}\, e^{\nu|t|}\, L(\nu, l)\, e^{-\alpha(t-s)/l},
\end{aligned}$$

where was used the fact that $0 \leq t-s-nl < l$.

Now, note that, if $t \geq s \geq 0$ we have

$$\nu|t| - \alpha(t-s)/l = -(\alpha - \nu l)(t-s)/l + \nu|s|,$$

and, for $s \leq t \leq 0$,

$$\nu|t| - \alpha(t-s)/l = -(\alpha + \nu l)(t-s)/l + \nu|s|,$$

then choose $\hat{\alpha} = (\alpha - \nu l)/l$. Thus, we obtain for $t \geq s \geq 0$ and $s \leq t \leq 0$ that

$$\begin{aligned} \|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} &\leq De^{\nu|s|}\,e^{\alpha}\,e^{\nu|t|}\,L(\nu,l)\,e^{-\alpha(t-s)/l} \\ &\leq DL(\nu,l)\,e^{\alpha}\,e^{2\nu|s|}e^{-\hat{\alpha}(t-s)}. \end{aligned}$$

Finally, for $t \geq 0 \geq s$ we have

$$\begin{aligned} \|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} &= \|S(t,s)\Pi^s(s)^2\|_{\mathscr{L}(X)} \\ &\leq \|S(t,0)\Pi^s(0)\|_{\mathscr{L}(X)}\,\|S(0,s)\Pi^s(s)\|_{\mathscr{L}(X)} \\ &\leq D^2L(\nu,l)^2\,e^{2\alpha}\,e^{2\nu|s|}e^{-\hat{\alpha}(t-s)}. \end{aligned}$$

Therefore, for $t \geq s$,

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} \leq D^2e^{2\alpha}\max\{L(\nu,l),L(\nu,l)^2\}e^{2\nu|s|}e^{-\hat{\alpha}(t-s)}$$

and the first claim is proved.

Now, to prove the other inequality, for $t < s$, we take $n \leq 0$ such that $s+nl \leq t < s+(n+1)l$, and define for $z \in R(\Pi^u(s))$ the linear operator

$$S(t,s)z := S(t,s+nl) \circ [T_{0,n}(s)|_{R(\Pi^u_n(s))}]^{-1}z.$$

In other words,

$$S(t,s)z = S(t,s+nl) \circ T_{n,0}(s)z.$$

Claim 2: If $t < s$, we have

$$\|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)} \leq \hat{K}(s)e^{\hat{\alpha}(t-s)}.$$

Indeed, for $x \in X$ and $s+nl \leq t < s+(n+1)l$, for $n \leq 0$, by hypothesis,

$$\|T_{n,0}(s)\Pi^u_0(s)x\|_X \leq K_s(0)e^{\alpha n}\|x\|_X.$$

Hence, by a similar argument to that in the proof of Claim 1 we obtain that

$$\|S(t,s)\Pi^u(s)x\|_X \leq \|S(t,s+nl)\|_{\mathscr{L}(X)}De^{\nu|s|}e^{\alpha n}\|x\|_X \leq \hat{K}(s)e^{\hat{\alpha}(t-s)}\|x\|_X.$$

Now, to conclude the assertion we take the supremum for $\|x\|_X = 1$.

Claim 3: For all $t_0 \in \mathbb{R}$ we characterize the kernel of $\Pi^u(t_0)$, $N(\Pi^u(t_0)) = \{z \in X : \Pi^u(t_0)z = 0\}$, as

$$N(\Pi^u(t_0)) = \{z \in X : [t_0, +\infty) \ni t \mapsto S(t,t_0)z \text{ is bounded}\}.$$

Let $z \in N(\Pi^u(t_0))$, so by definition $\Pi^u(t_0)z = 0$ and for $t \geq t_0$ we can use Claim 1 to obtain

$$\|S(t,t_0)z\|_X = \|S(t,t_0)(Id_X - \Pi^u(t_0))z\|_X \leq \hat{K}(t_0)e^{-\hat{\alpha}(t-t_0)}\|z\|_X.$$

Therefore, $[t_0, +\infty) \ni t \mapsto S(t,t_0)z$ is bounded.

On the other hand, if $z \notin N(\Pi^u((t_0)))$ and $n > 0$,

$$\begin{aligned} \|\Pi^u(t_0)z\|_X &\leq \|T_{0,n}(t_0)\Pi^u_n(t_0)\|_{\mathscr{L}(X)}\|T_{n,0}(t_0)z\|_X \\ &\leq De^{\nu|t_0|}e^{\nu|n|}e^{-\alpha n}\|S(t_0+nl,t_0)z\|_X. \end{aligned}$$

Thus, we obtain

$$\|\Pi^u(t_0)z\|_X D^{-1}e^{-\nu|t_0|}e^{n(\alpha-\nu)} \leq \|S(t_0+nl,t_0)z\|_X.$$

Consequently, as $\nu < \alpha$ we have that $[t_0, +\infty) \ni t \mapsto S(t,t_0)z$ is not bounded.

Note that the last assertion implies that

$$S(t,t_0)N(\Pi^u(t_0)) \subset N(\Pi^u(t)).$$

Claim 4: The linear operator

$$S(t,t_0) : R(\Pi^u(t_0)) \to X$$

is injective for all $t \geq t_0$.

Indeed, let $z \in R(\Pi^u(t_0))$ with $S(t,t_0)z = 0$. Choose $n \in \mathbb{N}$ so that $t \leq nl + t_0$, then

$$0 = S(t_0+nl,t)0 = S(t_0+nl,t)S(t,t_0)z = T_{n,0}(t_0)z,$$

this implies that $z \in N(T_{n,0}(t_0)|_{R(\Pi^u_0(t_0))}) = \{0\}$.

Claim 5: For all $t_0 \in \mathbb{R}$ the range of $\Pi^u(t_0)$ is

$$R(\Pi^u(t_0)) = \{z \in X : \text{ there exists a backwards bounded solution } \xi \text{ with } \xi(t_0) = z\}.$$

Let $z \in R(\Pi^u(t_0))$ and $t < t_0$, then take $n \in \mathbb{Z}$ such that $t \in [t_0+nl, t_0+(n+1)l]$ and define

$$\xi(t) := S(t,t_0+nl)T_{n,0}(t_0)z = S(t,t_0)z.$$

Now, choose $x \in X$ so that $z = \Pi^u(t_0)x$, thus by Claim 2

$$\|\xi(t)\|_X \leq \hat{K}(t_0)e^{\hat{\alpha}(t-t_0)}\|x\|_X.$$

Thus, ξ is a backward bounded solution such that $\xi(t_0) = z$. Suppose that $z \notin R(\Pi^u(t_0))$ and that there exists $\xi : \mathbb{R} \to X$ a global solution such that $\xi(t_0) = z$. For $n \leq 0$ we can write $z = S(t_0, t_0 + nl)\xi(t_0 + nl)$, thus

$$\begin{aligned} \|(Id_X - \Pi^u(t_0))z\|_X &\leq \|S(t_0, t_0 + nl)(Id_X - \Pi^u(t_0 + nl))\|_{\mathscr{L}(X)} \|\xi(t_0 + nl)\|_X \\ &\leq De^{\nu|t_0|}e^{\nu|n|}e^{\alpha n}\|\xi(t_0 + nl)\|_X. \end{aligned}$$

Therefore,

$$\|(Id_X - \Pi^u(t_0))z\|_X D^{-1} e^{-\nu|t_0|} e^{n(\nu-\alpha)} \leq \|\xi(t_0 + nl)\|_X.$$

Since $\nu < \alpha$, it follows that ξ is not backwards bounded, and the proof of Claim 5 is complete.

Claim 6: $S(t, t_0)R(\Pi^u(t_0)) = R(\Pi^u(t))$.

Indeed, if $z \in R(\Pi^u(t_0))$, then there exists a backwards bounded solution ξ through z in $t = t_0$. Thus, ξ is also a solution through $S(t, t_0)z$ in time t and we see that $S(t, t_0)z \in R(\Pi^u(t))$. On the other hand, if $z \in R(\Pi^u(t))$, there is a backwards bounded solution ξ with $\xi(t) = z$. Therefore, if $n \in \mathbb{Z}$ such that $nl + t \leq t_0 \leq t$, define

$$x = S(t_0, nl + t)S(nl + t, t)z \in R(\Pi^u(t_0)).$$

Therefore, $S(t, t_0)x = z$ and we conclude that $S(t, t_0)|_{R(\Pi^u(t_0))}$ is an isomorphism.

Finally, we prove that the family of projections commutes with the evolution process.

Claim 7: $\Pi^u(t)S(t, s) = S(t, s)\Pi^u(s)$. For $z \in X$, we have that

$$S(t, t_0)z = S(t, t_0)(Id_X - \Pi^u(t_0))z + S(t, t_0)\Pi^u(t_0)z.$$

Now, as $(Id_X - \Pi^u(t_0))z \in N(\Pi^u(t_0))$ and $S(t, t_0)\Pi^u(t_0)z \in R(\Pi^u(t))$, applying $\Pi^u(t)$ we obtain

$$\Pi^u(t)S(t, t_0)z = S(t, t_0)\Pi^u(t_0)z.$$

□

We are ready to present the main result of this section.

Theorem 2.2.11 (Robustness for continuous evolution processes)**.** *Let $\mathscr{S} = \{S(t, s) : t \geq s\} \subset \mathscr{L}(X)$ be an evolution process that admits a nonuniform exponential dichotomy with bound $K(s) = De^{\nu|s|}$ and exponent $\alpha > \nu$. Assume that*

$$L_{\mathscr{S}}(\nu) := \sup_{0 \leq t-s \leq 1} \left\{ e^{-\nu|t|}\|S(t, s)\|_{\mathscr{L}(X)} \right\} < +\infty. \tag{2.5}$$

Then there exists $\varepsilon > 0$ such that if $\mathscr{T} = \{T(t, s) : t \geq s\}$ is an evolution process such that

$$\sup_{0 \leq t-s \leq 1} \left\{ K(t)\|S(t, s) - T(t, s)\|_{\mathscr{L}(X)} \right\} < \varepsilon, \tag{2.6}$$

then $\mathscr{T}$ admits a nonuniform exponential dichotomy with exponent $\hat{\alpha} := \tilde{\alpha} - \nu$ and bound

$$\hat{K}(s) = \tilde{D}^2 e^{2\tilde{\alpha}} \max\{L_{\mathscr{T}}(\nu), L_{\mathscr{T}}(\nu)^2\} e^{2\nu|s|}, \tag{2.7}$$

where $\tilde{D} := D(1+\varepsilon/(1-\rho)(1-e^{-\alpha}))\max\{D_1, D_2\}$, and $\rho, \tilde{\alpha}, D_1$ and D_2 are the same as in Theorem 2.1.9.

Proof. Let $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$, then, by Theorem 2.2.6, the discrete evolution process $\{S_n(t_0) := S(t_0+n+1, t_0+n) : n \in \mathbb{Z}\}$ admits a nonuniform exponential dichotomy with bound $K_t(n) \leq De^{\nu(|t+n|)}$ and exponent $\alpha > 0$. Let $\varepsilon > 0$ be such that $\varepsilon < (1-e^{-\alpha})/(1+e^{-\alpha})$ and $\mathscr{T} = \{T(t,s) : t \geq s\}$ an evolution process that satisfies (2.6). Let $\{T_n(t_0) : n \in \mathbb{Z}\}$ be the discretization of $\mathscr{T}$ at t_0 and define, for each $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$, the linear bounded operator

$$B_n(t_0) := T_n(t_0) - S_n(t_0).$$

Hence, from (2.6), we have that

$$\|B_n(t_0)\|_{\mathscr{L}(X)} < \varepsilon K_{t_0}(n+1)^{-1}.$$

Therefore, by Theorem 2.1.9, the discrete evolution process $T_n(t_0) = S_n(t_0) + B_n(t_0)$ admits a nonuniform exponential dichotomy with exponent

$$\tilde{\alpha} := -\ln(\cosh\alpha - [\cosh^2\alpha - 1 - 2\varepsilon\sinh\alpha]^{1/2}),$$

and bound

$$\tilde{K}_{t_0}(n) := K_{t_0}(n)\left[1 + \frac{\varepsilon}{(1-\rho)(1-e^{-\alpha})}\right]\max[D_1, D_2],$$

where D_1, D_2, ρ are constants that can be found in Theorem 2.1.9.

Since each discretization at time t has the same exponent $\alpha > 0$ we see that ε can be choose independent of t. Thus for each $t \in \mathbb{R}$, the discrete evolution process $\{T_n(t) : n \in \mathbb{Z}\}$ admits nonuniform exponential dichotomy with bound $\tilde{K}_t(n)$ and exponent $\tilde{\alpha}$ defined above. Then Condition 2 of Theorem 2.2.10 holds true for $\mathscr{T}$.

Moreover, from (2.6), $\mathscr{T}$ satisfies

$$\begin{aligned}\|T(t,s)\|_{\mathscr{L}(X)} &\leq \varepsilon K(t)^{-1} + \|S(t,s)\|_{\mathscr{L}(X)} \\ &\leq \varepsilon + \|S(t,s)\|_{\mathscr{L}(X)}, \text{ for } 0 \leq t-s \leq 1\end{aligned}$$

then $\sup_{0\leq t-s\leq 1}\{e^{-\nu|t|}\|T(t,s)\|_{\mathscr{L}(X)}\}$ is finite. Finally, note that it is possible to choose $\varepsilon > 0$ small such that $\tilde{\alpha} > \nu$. Therefore, Theorem 2.2.10 implies that $\mathscr{T}$ admits nonuniform exponential dichotomy with bound $\hat{K}$ defined in (2.7) and exponent $\hat{\alpha} = \tilde{\alpha} - \nu > 0$. □

Remark 2.2.12. *Assumption (2.5) on the growth of $\mathscr{S}$ (analogous to that of (HENRY, 1981, Theorem 7.6.10) with $\nu = 0$, that is, the uniform case) is expected for evolution processes that admit nonuniform exponential dichotomies, see (BARREIRA; VALLS, 1998) or Example 2.11 in Section 2.3.*

Remark 2.2.13. *Theorem 2.2.11 allows us to see the robustness as an open property. In fact, let $\mathfrak{S}_\nu$ be the space of all evolutionary processes that satisfy (2.5) and define a distance in $\mathfrak{S}_\nu$ as*

$$d_\nu(\mathscr{S},\mathscr{T}) := \sup_{0\le t-s\le 1} \left\{ e^{\nu|t|} \|S(t,s)-T(t,s)\|_{\mathscr{L}(X)} \right\}.$$

Then, from Theorem 2.2.11 we see that if $\mathscr{S}\in\mathfrak{S}_\nu$ admits a nonuniform exponential dichotomy with bound $K(t)=De^{\nu|t|}$ and exponent $\alpha>\nu$, then there exists $\varepsilon>0$ such that every evolution process $\mathscr{T}$ in a ε-neighborhood of $\mathscr{S}$ admits a nonuniform exponential dichotomy with bound and exponent given in Theorem 2.2.11.

Now, we present another formulation of Theorem 2.2.11 that allows us to apply the result for differential equations in Banach spaces.

Theorem 2.2.14. *Let $\mathscr{S}=\{S(t,s): t\ge s\}\subset\mathscr{L}(X)$ be an evolution process that admits a nonuniform exponential dichotomy with bound $K(t)=De^{\nu|t|}$, $t\in\mathbb{R}$, and exponent $\alpha>\nu$, and assume that $\mathscr{S}$ satisfies (2.5). Let $\{B(t): t\in\mathbb{R}\}\subset\mathscr{L}(X)$ so that $\mathbb{R}\ni t\mapsto B(t)x$ is continuous for all $x\in X$ and*

$$\|B(t)\|_{\mathscr{L}(X)} < \delta e^{-3\nu|t|}, \;\; \forall t\in\mathbb{R}.$$

Then any evolution process that satisfies the integral equation

$$T(t,s)=S(t,s)+\int_s^t S(t,\tau)B(\tau)T(\tau,s)d\tau\in\mathscr{L}(X), \;\; t\ge s, \tag{2.8}$$

admits a nonuniform exponential dichotomy for suitably small $\delta>0$, with bound and exponent given in Theorem 2.2.11.

Proof. Let $\mathscr{T}=\{T(t,s): t\ge s\}$ be a evolution process satisfying (2.8). Then

$$\|T(t,s)\|_{\mathscr{L}(X)} \le \|S(t,s)\|_{\mathscr{L}(X)} + \int_s^t \|S(t,\tau)\|_{\mathscr{L}(X)}\|B(\tau)\|_{\mathscr{L}(X)}\|T(\tau,s)\|_{\mathscr{L}(X)}d\tau.$$

Thus, fix s and define the function $\phi(t)=e^{-\nu|t|}\|T(t,s)\|_{\mathscr{L}(X)}$, for $t\le s+1$,

$$\phi(t)\le L_{\mathscr{S}}(\nu)+L_{\mathscr{S}}(\nu)\int_s^t \|B(\tau)\|_{\mathscr{L}(X)}e^{\nu|\tau|}\phi(\tau)d\tau$$

By Grönwall's inequality, we obtain that

$$\phi(t)\le L_{\mathscr{S}}(\nu)e^{L_{\mathscr{S}}(\nu)\int_s^t \|B(\tau)\|_{\mathscr{L}(X)}e^{\nu|\tau|}d\tau}, \text{ for } t\le s+1.$$

Therefore,

$$L_{\mathscr{T}}(\nu) := \sup_{0\le t-s\le 1}\left\{e^{-\nu|t|}\|T(t,s)\|_{\mathscr{L}(X)}\right\} < +\infty.$$

Now, for $0\le t-s\le 1$,

$$\begin{aligned}\|S(t,s)-T(t,s)\|_{\mathscr{L}(X)} &\le \int_s^t e^{\nu(|t|+|\tau|)}L_{\mathscr{S}}(\nu)\|B(\tau)\|_{\mathscr{L}(X)}L_{\mathscr{T}}(\nu)d\tau\\ &= L_{\mathscr{T}}(\nu)L_{\mathscr{S}}(\nu)e^{\nu|t|}\int_s^t e^{\nu|\tau|}\|B(\tau)\|_{\mathscr{L}(X)}d\tau.\end{aligned}$$

Then

$$K(t)\|S(t,s)-T(t,s)\|_{\mathscr{L}(X)} \leq L_{\mathscr{T}}(\nu)L_{\mathscr{S}}(\nu)D\delta,$$

and choose $\delta > 0$ suitably small in order to use Theorem 2.2.11 and conclude the proof. □

Theorem 2.2.14 is very useful when dealing with differential equations. In fact, let $\{A(t) : t \in \mathbb{R}\}$ be a family of linear operators, possibly unbounded, and consider

$$\dot{x} = A(t)x, \;\; x(s) = x_s \in X. \tag{2.9}$$

Suppose that, for each $s \in \mathbb{R}$ and $x_s \in X$, there exists a unique solution $x(\cdot,s,x_s) : [s,+\infty) \to X$. Thus there exists an evolution process $\mathscr{S} = \{S(t,s) : t \geq s\}$ defined by $S(t,s)x_s := x(t,s,x_s)$ for each $t \geq s$.

To study robustness of nonuniform exponential dichotomy of problem (2.9), we suppose that $\mathscr{S}$ admits a nonuniform exponential dichotomy with a exponential growth (2.5) and exhibit a class of $\{B(t) : t \in \mathbb{R}\} \subset \mathscr{L}(X)$ so that the perturbed problem

$$\dot{x} = A(t)x + B(t)x, \;\; x(s) = x_s \in X, \tag{2.10}$$

admits a nonuniform exponential dichotomy. In this way, Theorem 2.2.14 ensures that the nonuniform hyperbolicity is preserved for exponentially small perturbations, i.e., if the norm of the perturbation of B does not grow more than $e^{-3\nu|t|}$ for $\nu < \alpha$, then the perturbed problem (2.10) admits a nonuniform exponential dichotomy.

Remark 2.2.15. *In (BARREIRA; VALLS, 2015) a version of Theorem 2.2.14 is proved under different assumptions. They considered a general growth rate $\rho(t)$ for the nonuniform exponential dichotomy and proved that if $\alpha > 2\nu$ and $B : \mathbb{R} \to \mathscr{L}(X)$ is continuous satisfying $\|B(t)\|_{\mathscr{L}(X)} \leq \delta e^{-3\nu|\rho(t)|}\rho'(t)$, for all $t \in \mathbb{R}$, then the perturbed problem (2.10) admits ρ-nonuniform exponential dichotomy. We note that our method does not work for general growth rates $\rho(t)$. We treat the case $\rho(t) = t$, since the other cases can be achieved by a change of scaling in time, and since our condition on the exponents is only $\alpha > \nu$ we obtain an improvement of their robustness result.*

For $A(t)$ bounded and B satisfying $\|B(t)\|_{\mathscr{L}(X)} \leq \delta e^{-2\nu|t|}$, for all t, in (BARREIRA; VALLS, 2008) it was proved a result similar to Theorem 2.2.14. However they assume that the evolution process $\mathscr{S}$ is invertible. In their proof, thanks to invertibility, they can write explicit expressions of the projections for the perturbed evolution process. In applications to partial differential equations, in general, $A(t)$ is not bounded, see Section 2.3.

2.3 An application to infinite-dimensional differential equations

In this section, we show an application of the robustness result in order to obtain examples of evolution processes that admits nonuniform exponential dichotomies. Inspired in an example of (BARREIRA; VALLS, 1998, Proposition 2.3), we provide an evolution process defined on a Banach space that admits a nonuniform exponential dichotomy. Then, we apply Theorem 2.2.14 to study for which class of perturbations the nonuniform hyperbolicity will be preserved.

Let X and Y be two Banach spaces. Suppose that A is a generator of a C^0-semigroup $\{e^{At} : t \geq 0\}$ in X and $B \in \mathscr{L}(Y)$ with $\sigma(A) \subset (-\infty, -\omega)$ and $\sigma(B) \subset (\omega, +\infty)$, for some $\omega > 0$, and there exists $M \geq 1$ such that

$$\begin{aligned} \|e^{A(t-s)}\|_{\mathscr{L}(X)} &\leq Me^{-\omega(t-s)}, t \geq s; \\ \|e^{B(t-s)}\|_{\mathscr{L}(Y)} &\leq Me^{\omega(t-s)}, t < s. \end{aligned}$$

Remark 2.3.1. *Let $\mathscr{C}$ be a generator of a hyperbolic C^0-semigroup $\{e^{\mathscr{C}t} : t \geq 0\}$, i.e., the associated evolution processes $\{e^{\mathscr{C}(t-s)} : t \geq s\}$ admits an uniform exponential dichotomy with a single projection $\Pi^u(t) = Q \in \mathscr{L}(X)$ for every $t \in \mathbb{R}$. Then, there is a decomposition $X = X^u \oplus X^s$ such that $A := \mathscr{C}|_{X^s}$ and $B = \mathscr{C}|_{X^u}$ satisfy the conditions above over X^s and X^u, respectively, see (CARVALHO; LANGA; ROBINSON, 2013; CHOW; LEIVA, 1995a; HENRY, 1981).*

Let $\omega > a > 0$ and define the linear operator in $Z = X \times Y$

$$\mathscr{A}(t) := \begin{bmatrix} A - at\sin(t)Id_X & 0 \\ 0 & B + at\sin(t)Id_Y \end{bmatrix}.$$

Consider the differential equation

$$\dot{z} = \mathscr{A}(t)z, \quad z(s) = z_s \in Z. \tag{2.11}$$

Then, the evolution process associated with problem (2.11) is defined by

$$T(t,s) = (U(t,s), V(t,s))$$

where

$$\begin{aligned} U(t,s) &= e^{A(t-s)} \exp\left\{ -\int_s^t a\tau\sin(\tau)d\tau \right\} \text{ and} \\ V(t,s) &= e^{B(t-s)} \exp\left\{ \int_s^t a\tau\sin(\tau)d\tau \right\} \end{aligned}$$

are evolution processes in X and Y, respectively.

Theorem 2.3.2. *Let $\mathscr{T} = \{T(t,s) : t \geq s\}$ be the evolution process defined above. Then $\mathscr{T}$ admits a nonuniform exponential dichotomy with bound $K(t) = Me^{2a(1+|t|)}$ and exponent $\alpha = \omega - a > 0$.*

Proof. Define the linear operators $\Pi_X^u(t) = P_X$ and $\Pi_Y^u(t) = P_Y$ for all $t \in \mathbb{R}$ where P_X and P_Y are the canonical projections onto X and Y, respectively. Then $T(t,s)\Pi_X^u(s) = U(t,s)$ and $T(t,s)\Pi_Y^u(s) = V(t,s)$ for all $t \geq s$.

In this way we have that P_X commutes with $T(t,s)$, for all $t \geq s$ and since $B \in \mathscr{L}(\mathrm{Y})$ generates a group in Y we have that $V(t,s)$ is an isomorphism over Y. Now, we claim that

$$\begin{aligned} \|U(t,s)\|_{\mathscr{L}(X)} &\leq Me^{2a+(-\omega+a)(t-s)+2a|s|},\ t \geq s \\ \|V(t,s)\|_{\mathscr{L}(X)} &\leq Me^{2a+(\omega+a)(t-s)+2a|s|},\ t \leq s. \end{aligned}$$

Indeed, note that

$$\begin{aligned} \|U(t,s)\|_{\mathscr{L}(X)} &= \exp\left\{-\int_s^t a\tau\sin(\tau)d\tau\right\}\|e^{A(t-s)}\|_{\mathscr{L}(X)} \\ &\leq Me^{-\omega(t-s)+at\cos(t)-as\cos(s)-a\sin(t)+a\sin(s)}, \end{aligned}$$

and write the exponents as following

$$-\omega(t-s)+at\cos(t)-as\cos(s) = -(\omega-a)(t-s)+at(\cos(t)-1)-as(\cos(s)-1).$$

Hence, for $t \geq 0$ and $s > 0$, we see that $at(\cos(t)-1) \leq 0$, and $as(1-\cos(s)) \leq 2as = 2a|s|$. Thus, for $t \geq s \geq 0$, we have that

$$\|U(t,s)\|_{\mathscr{L}(X)} \leq Me^{-(\omega+a)(t-s)+2a|s|+2a}.$$

Now, for $s \leq 0$, $as(1-\cos(s)) \leq 0$. Thus, for $t \geq 0 \geq s$,

$$\|U(t,s)\|_{\mathscr{L}(X)} \leq Me^{(-\omega+a)(t-s)+2a}.$$

Finally, if $s \leq t \leq 0$, we see that

$$-at(1-\cos(t)) \leq 2a|t| \leq 2a|s|.$$

Then

$$\|U(t,s)\|_{\mathscr{L}(X)} \leq Me^{(-\omega+a)(t-s)+2a|s|+2a},\ \text{for } t \geq s. \tag{2.12}$$

Similarly, we obtain that

$$\|V(t,s)\|_{\mathscr{L}(Y)} \leq Me^{2a+2a|s|}e^{(\omega+a)(t-s)},\ \text{for } t < s. \tag{2.13}$$

Therefore, $\mathscr{T}$ admits a nonuniform exponential dichotomy with bound $K(t) = Me^{2a(1+|t|)}$ and exponent $\alpha = \omega - a > 0$. □

Now, apply Theorem 2.2.14 to Equation (2.11).

Theorem 2.3.3. *Consider for each* $\varepsilon > 0$ *an operator* $\mathscr{B}_\varepsilon(t) \in \mathscr{L}(Z)$ *such that* $\|\mathscr{B}_\varepsilon(t)\| \leq \varepsilon e^{-6a|t|}$*, and define*

$$\mathscr{A}_\varepsilon(t) := \mathscr{A}(t) + \mathscr{B}_\varepsilon(t), \quad \forall t \in \mathbb{R}.$$

If $\omega > 3a$*, there exists* $\varepsilon > 0$ *such that the evolution process associated with the problem*

$$\dot{x} = \mathscr{A}_\varepsilon(t)x, \quad x(s) = x_s \in Z. \tag{2.14}$$

admits a nonuniform exponential dichotomy.

Proof. Let us prove first that the evolution problem associated with (2.11) satisfies

$$\sup_{0 \leq t-\tau \leq 1} \{e^{-\nu|t|}\|T(t,\tau)\|_{\mathscr{L}(Z)}\} < +\infty. \tag{2.15}$$

In fact, we have for $t \geq s$ that

$$\|T(t,s)\|_{\mathscr{L}(Z)} \leq \|U(t,s)\|_{\mathscr{L}(X)} + \|V(t,s)\|_{\mathscr{L}(Y)},$$

where U and V are the evolution processes defined in the proof of Theorem 2.3.2. Then it is enough to prove that each evolution process satisfies (2.15) in the corresponding space. From (2.12) we have that

$$e^{-2a|t|}\|U(t,s)\|_{\mathscr{L}(X)} \leq Me^{2a+2a(|s|-|t|)}e^{-(\omega-a)(t-s)} \leq Me^{2a}e^{-(\omega-3a)(t-s)}.$$

Therefore

$$\sup_{0 \leq t-s \leq 1} \{e^{-2a|t|}\|U(t,s)\|_{\mathscr{L}(X)}\} < +\infty, \text{ for all } t \geq s.$$

Now, since $\|e^{B(t-s)}\|_{\mathscr{L}(Y)} \leq \tilde{M}e^{\beta(t-s)}$ for some $\tilde{M} \geq 1$ and $\beta > 0$, for every $t \geq s$, we have that

$$\|V(t,s)\|_{\mathscr{L}(Y)} = \exp\left\{\int_s^t a\tau\sin(\tau)d\tau\right\}\|e^{B(t-s)}\|_{\mathscr{L}(Y)} \leq \tilde{M}e^{4a+2a|t|}e^{(\beta+a)(t-s)},$$

which implies that

$$\sup_{0 \leq t-s \leq 1} \{e^{-2a|t|}\|V(t,s)\|_{\mathscr{L}(Y)}\} < +\infty.$$

Now, from Theorem 2.3.2, $\mathscr{T}$ admits a nonuniform exponential dichotomy where the bound is $K(s) = Me^{2a+2a|s|}$ and exponent $\alpha = \omega - a > 0$. Since $\nu := 2a$ is such that $\alpha > \nu$, we apply Theorem 2.2.14 to conclude that the evolution process generated by (2.14) admits a nonuniform exponential dichotomy. □

Remark 2.3.4. *Note that, in Theorem 2.2.14 the assumption* $\alpha > \nu$ *of Theorem 2.3.3 is expressed by* $\omega > 3a$*. On the other hand, to apply Theorem 1 of (BARREIRA; VALLS, 2015) the hypothesis must be* $\omega > 5a$*, because their condition is* $\alpha > 2\nu$*.*

2.4 Persistence of nonuniform hyperbolic solutions

In this section, we study nonlinear evolution processes associated with a semilinear differential equation. We study persistence of *nonuniform hyperbolic solutions* under perturbation for evolution processes in Banach spaces, Our approach is inspired by the uniform case, see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 8). More precisely, we use *Green's function* to characterize bounded global solutions for semilinear differential equations and conclude that nonuniform hyperbolic solutions are *isolated* in the set of bounded continuous functions, see Theorem 2.4.3. Finally, in Theorem 2.4.4, we provide conditions to prove that nonuniform hyperbolic solutions persist under perturbations.

Consider a semi-linear differential equation

$$\dot{y} = A(t)y + f(t,y), \;\; y(s) = y_s. \tag{2.16}$$

Assume that f is continuous in the first variable and locally Lipschitz in the second and that $\{A(t) : t \in \mathbb{R}\}$ is a family of linear (possibly unbounded) operators associated with an evolution process $\mathscr{T} = \{T(t,s) : t \geq s\} \subset \mathscr{L}(X)$, i.e., for each $s \in \mathbb{R}$ and $x_0 \in X$ the mapping $[s,+\infty) \ni t \to T(t,s)x_0$ is the solution of

$$\dot{x} = A(t)x, \; x(s) = x_0.$$

Then we have a *local mild solution* for problem (2.16), that is, for each $(s,y_s) \in \mathbb{R}\times X$ there exist $\sigma = \sigma(s,y_s) > 0$ and a solution y of the integral equation

$$y(t,s;y_s) = T(t,s)y_s + \int_s^t T(t,\tau)f(\tau,y(\tau,s;y_s))d\tau, \tag{2.17}$$

for all $t \in [s,s+\sigma)$.

If for each $(s,y_s) \in \mathbb{R}\times X$, $\sigma(s,y_s) = +\infty$, we can consider the evolution process $S_f(t,s)y_s = y(t,s;y_s)$. We refer to $\mathscr{S}_f = \{S_f(t,s) : t \geq s\}$ as the evolution process obtained by a non-linear perturbation f of $\mathscr{T}$.

Suppose additionally that $f : \mathbb{R}\times X \to X$ is differentiable with continuous derivatives. Let ξ be a global solution of $\mathscr{S}_f$ (see Definition 2.2.3), and $\mathscr{L}_f = \{L_f(t,s) : t \geq s\}$ is the linearized evolution process of $\mathscr{S}_f$ on ξ. Thus $\mathscr{L}_f$ satisfies

$$L_f(t,s) = T(t,s) + \int_s^t T(t,\tau)D_x f(\tau,\xi(\tau))L_f(\tau,s)d\tau.$$

Definition 2.4.1. *If $\mathscr{L}_f$ admits a nonuniform exponential dichotomy we say that ξ is a **nonuniform hyperbolic solution** for $\mathscr{S}_f$.*

In (BARREIRA; VALLS, 1998) this type of solution is called *nonuniformly hyperbolic trajectory*.

Remark 2.4.2. *Let φ be a global solution for $\mathscr{S}_f$. Then*

$$\varphi(t) = L_f(t,s)\varphi(s) + \int_s^t L_f(t,\tau)[f(\tau,\varphi(\tau)) - D_x f(\tau,\xi(\tau))\varphi(\tau)]d\tau, \;\; t \geq s. \tag{2.18}$$

In particular, the global bounded solution ξ satisfies the integral equation (2.18), see (CARVALHO; LANGA; ROBINSON, 2013, pages 224-225).

The next result allows us to characterize bounded nonuniform hyperbolic solutions.

Theorem 2.4.3. *Assume that there is a global nonuniform hyperbolic solution ξ for $\mathscr{S}_f$ and that $\mathscr{L}_f$ admits a nonuniform exponential dichotomy with bound $K(s) = De^{\nu|s|}$, $s \in \mathbb{R}$, exponent $\alpha > \nu$, and family of projections $\{\Pi^u(t) : t \in \mathbb{R}\}$. If φ is a bounded global solution for $\mathscr{S}_f$, then φ satisfies*

$$\varphi(t) = \int_{-\infty}^{+\infty} G_f(t,\tau)[f(\tau,\varphi(\tau)) - D_x f(\tau,\xi(\tau))\varphi(\tau)]\, d\tau,$$

where G_f is the Green's function associated with the evolution process $\mathscr{L}_f$,

$$G_f(t,s) = \begin{cases} L_f(t,s)(Id_X - \Pi^u(s)), & \text{if } t \geq s, \\ -L_f(t,s)\Pi^u(s) & \text{if } t < s. \end{cases}$$

Moreover, if ξ is a bounded nonuniform hyperbolic solution of $\mathscr{S}_f$ and

$$\rho(\varepsilon) = \sup_{\|x\| \leq \varepsilon} \sup_{t \in \mathbb{R}} \frac{e^{\nu|t|}\, \|f(t,\xi(t)+x) - f(t,\xi(t)) - D_x f(t,\xi(t))x\|}{\|x\|} \to 0, \text{ as } \varepsilon \to 0, \tag{2.19}$$

*then ξ is **isolated** in the set of bounded and continuous functions $C_b(\mathbb{R},X)$, i.e., there is a ε-neighborhood of ξ, namely*

$$N_\varepsilon = \{\varphi \in C_b(\mathbb{R},X) : \sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\|_X < \varepsilon\}, \;\; \varepsilon > 0,$$

such that ξ is the only bounded global solution of $\mathscr{S}_f$ in N_ε.

Proof. If $\tau > t$ we have that

$$\varphi(\tau) = L_f(\tau,t)\varphi(t) + \int_t^\tau L_f(\tau,s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{2.20}$$

Thus, applying $\Pi^u(\tau)$ in the previous equation we obtain

$$\Pi^u(\tau)\varphi(\tau) = L_f(\tau,t)\Pi^u(t)\varphi(t) + \int_t^\tau L_f(\tau,s)\Pi^u(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{2.21}$$

Now, use that $L_f(\tau,t)|_{R(\Pi^u(t))}$ is invertible with inverse $L_f(t,\tau)$ so we obtain

$$L_f(t,\tau)\Pi^u(\tau)\varphi(\tau) = \Pi^u(t)\varphi(t) + \int_t^\tau L_f(t,s)\Pi^u(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{2.22}$$

Since $\mathscr{L}_f$ admits a nonuniform exponential dichotomy with exponent $\alpha > \nu$ and φ is bounded, we obtain that

$$\|L_f(t,\tau)\Pi^u(\tau)\varphi(\tau)\| \leq De^{\nu|\tau|}e^{\alpha(t-\tau)} \sup_{s\in\mathbb{R}} \|\varphi(s)\| \to 0, \text{ as } \tau \to +\infty.$$

Then

$$\Pi^u(t)\varphi(t) = -\int_t^{+\infty} L_f(t,s)\Pi^u(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{2.23}$$

Similarly, for $t > \tau$, as

$$\|L_f(t,\tau)(Id_X - \Pi^u(\tau))\varphi(\tau)\| \leq De^{\nu|\tau|}e^{-\alpha(t-\tau)} \sup_{s\in\mathbb{R}} \|\varphi(s)\| \to 0, \text{ as } \tau \to -\infty,$$

thus

$$(Id_X - \Pi^u(t))\varphi(t) = \int_{-\infty}^t L_f(t,s)(Id_X - \Pi^u(s))[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{2.24}$$

Therefore, the result follows by writing $\varphi(t) = (Id_X - \Pi^u(t))\varphi(t) + \Pi^u(t)\varphi(t)$ and using the previous expressions.

Finally, we prove that ξ is isolated. Let $\varphi \in C_b(\mathbb{R},X)$ be a bounded global solution of $\mathscr{S}_f$ with $\sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\| \leq \varepsilon$. Since ξ and φ are bounded we apply the first part of the proof for each and obtain

$$\varphi(t) - \xi(t) = \int_{-\infty}^{+\infty} G_f(t,\tau)[f(\tau,\varphi(\tau)) - f(\tau,\xi(\tau)) - D_x f(\tau,\xi(\tau))(\varphi(\tau)) - \xi(\tau)]d\tau.$$

Note that, the Green's function $\mathscr{G}_f$ satisfies

$$\|G_f(t,\tau)\|_{\mathscr{L}(X)} \leq De^{\nu|\tau|}e^{-\alpha|t-\tau|}, \text{ for all } t,\tau \in \mathbb{R},$$

this together with condition (2.19) we obtain

$$\sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\| \leq 2D\rho(\varepsilon)\alpha^{-1} \sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\|.$$

For $\varepsilon > 0$ such that $2D\rho(\varepsilon)\alpha^{-1} < 1$ we see that $\varphi(t) = \xi(t)$ for all $t \in \mathbb{R}$. Therefore, ξ is isolated and the proof is complete. □

Now, as an application of Theorem 2.2.11 we prove a result on the *persistence of nonuniform hyperbolic solutions*.

Theorem 2.4.4 (Persistence of nonuniform hyperbolic solutions)**.** *Let $f : \mathbb{R} \times X \to X$ be a continuous map with continuous first derivative with respect to the second variable, $\mathscr{T}$ be a linear evolution processes and $\mathscr{S}_f$ be the evolution process generated by f and $\mathscr{T}$. Assume that*

1. $\mathscr{T}$ satisfies

$$\sup_{0\leq t-s\leq 1} \{e^{-\nu|t|}\|T(t,s)\|_{\mathscr{L}(X)}\} < +\infty, \tag{2.25}$$

2. *there is a global nonuniform hyperbolic solution* ξ *for* $\mathscr{S}_f$*, i.e.,* $\mathscr{L}_f$ *admits a nonuniform exponential dichotomy with bound* $K(s) = De^{\nu|s|}$*, for all* $s \in \mathbb{R}$*, and exponent* $\alpha > \nu$.

3. ξ *is bounded with* $\sup_{t\in\mathbb{R}} \|\xi(t)\| < M$*;*

4. f *satisfies Condition* (2.19)*;*

5. *the derivative of* f *satisfies*

$$\sup_{\|x\|\leq M} \sup_{t\in\mathbb{R}} \{e^{\nu|t|}\|D_x f(t,x)\|_{\mathscr{L}(X)}\} < +\infty;$$

6. $g : \mathbb{R}\times X \to X$ *is differentiable with continuous first derivative with respect to the second variable and satisfies*

$$\sup_{\|x\|_X\leq\varepsilon} \sup_{t\in\mathbb{R}} e^{3\nu|t|}\|D_x g(t,\xi(t)+x) - D_x g(t,\xi(t))\|_{\mathscr{L}(X)} < \frac{\delta}{2}, \text{ and} \tag{2.26}$$

$$\sup_{\substack{t\in\mathbb{R}\\ \|x\|_X\leq M}} e^{3\nu|t|}\{\|f(t,x)-g(t,x)\|_X+\|D_x f(t,x)-D_x g(t,x)\|_{\mathscr{L}(X)}\} < \frac{\varepsilon\alpha}{4D}, \tag{2.27}$$

for $0 < \varepsilon < \varepsilon_0 := \min\{M - \sup_{t\in\mathbb{R}} \|\xi(t)\|_X, 2\delta D\alpha^{-1}\}$ *suitable small, where* $\delta > 0$ *is the same as in Theorem 2.2.14 applied for* $\mathscr{L}_f$.

Then there exists a unique nonuniform hyperbolic solution ψ *for* $\mathscr{S}_g$ *such that*

$$\sup_{t\in\mathbb{R}} \|\xi(t) - \psi(t)\| < \varepsilon.$$

Proof. If y is a global bounded solution for $\mathscr{S}_g$, then, as in Remark 2.4.2, we have that

$$\begin{aligned} y(t) &= L_f(t,s)y(s) + \int_s^t L_f(t,\tau)[g(\tau,y(\tau)) - D_x f(\tau,\xi(\tau))y(\tau)]d\tau, \\ \xi(t) &= L_f(t,s)\xi(s) + \int_s^t L_f(t,\tau)[f(\tau,\xi(\tau)) - D_x f(\tau,\xi(\tau))\xi(\tau)]d\tau. \end{aligned} \tag{2.28}$$

Thus $\phi(t) = y(t) - \xi(t)$ satisfies the following integral equation

$$\phi(t) = L_f(t,s)\phi(s) + \int_s^t L_f(t,\tau)h(\tau,\phi(\tau))d\tau, \tag{2.29}$$

where $h(t,\phi(t)) = g(t,\phi(t)+\xi(t)) - f(t,\xi(t)) - D_x f(t,\xi(t))\phi(t)$.

Then , by Theorem 2.4.3, there exists a bounded solution of (2.29) in

$$B_\varepsilon := \{\phi : \mathbb{R} \to X : \phi \text{ is continuous and } \sup_{t\in\mathbb{R}} \|\phi(t)\| < \varepsilon\},$$

if and only if, the operator

$$(\mathscr{F}\varphi)(t) = \int_{-\infty}^{+\infty} G_f(t,s)h(s,\varphi(s))ds$$

has a fixed point in the space B_ε.

Now, we use the fact that $\mathscr{L}_f$ admits a nonuniform exponential dichotomy to show that $\mathscr{F}$ has a unique fixed point in B_ε, for suitable small $\varepsilon > 0$. In order to use the Banach fixed point Theorem, we have to prove that $\mathscr{F}$ is a contraction and that $\mathscr{F}B_\varepsilon \subset B_\varepsilon$.

For $0 < \varepsilon < \varepsilon_0$ and $\phi \in B_\varepsilon$, we have

$$\begin{aligned}\|(\mathscr{F}\phi)(t)\|_X &\leq D\int_{-\infty}^{+\infty} e^{\nu|s|}e^{-\alpha|t-s|}\|h(s,\phi(s))\|_X ds\\ &\leq 2D\alpha^{-1}\sup_{t\in\mathbb{R}} e^{\nu|t|}\,\|g(t,\xi(t)+\phi(t))-f(t,\xi(t)+\phi(t))\|_X\\ &+ 2D\alpha^{-1}\varepsilon\sup_{\|x\|\leq\varepsilon}\sup_{t\in\mathbb{R}}\frac{e^{\nu|t|}\,\|f(t,\xi(t)+x)-f(t,\xi(t))-D_xf(t,\xi(t))x\|_X}{\|x\|_X}\\ &\leq \varepsilon/2+2\alpha^{-1}D\rho(\varepsilon)\varepsilon.\end{aligned}$$

Thus, choosing $\varepsilon \in (0,\varepsilon_0)$ such that $4\alpha^{-1}D\rho(\varepsilon) < 1$, we see that $\mathscr{F}\phi \in B_\varepsilon$. Now, we show that $\mathscr{F}$ is a contraction. In fact, with similar computations we are able to prove for $\phi_1,\phi_2 \in B_\varepsilon$ that

$$\|(\mathscr{F}\phi_1)(t)-(\mathscr{F}\phi_2)(t)\|_X \leq \frac{1}{2}\sup_{t\in\mathbb{R}}\|\phi_1(t)-\phi_2(t)\|_X.$$

Therefore, there is a unique fixed point ϕ in B_ε and we obtain $\psi = \phi+\xi$ a global solution of $\mathscr{S}_g$.

Finally, we prove that ψ is a nonuniform hyperbolic solution, that means, the linear evolution process $\mathscr{L}_g := \{L_g(t,s) : t \geq s\}$ that satisfies

$$L_g(t,\tau) = T(t,\tau)+\int_\tau^t T(t,s)D_xg(s,\psi(s))L_g(s,\tau)ds$$

admits a nonuniform exponential dichotomy.

To that end, we show that $\mathscr{L}_f$ satisfies conditions of Theorem 2.2.14 and we see $\mathscr{L}_g$ as a small perturbation of $\mathscr{L}_f$. Indeed, since $\mathscr{T}$ satisfies (2.25) and

$$L_f(t,s) = T(t,s)+\int_s^t T(t,\tau)D_xf(\tau,\xi(\tau))L_f(\tau,s)d\tau,$$

from a Grönwall's inequality and assumption (5) we see that

$$\sup_{0\leq t-\tau\leq 1}\{e^{-\nu|t|}\,\|L_f(t,\tau)\|_{\mathscr{L}(X)}\} < +\infty.$$

Finally, note that

$$L_g(t,\tau) = L_f(t,\tau)+\int_\tau^t L_f(t,s)[D_xg(s,\psi(s))-D_xf(s,\xi(s))]L_g(s,\tau)ds.$$

Now, define $B(s) := D_xg(s,\psi(s))-D_xf(s,\xi(s))$ for all $s \in \mathbb{R}$. Since

$$\begin{aligned}\|B(s)\|_{\mathscr{L}(X)} &\leq \|D_xg(s,\psi(s))-D_xg(s,\xi(s))\|_{\mathscr{L}(X)}\\ &\quad+\|D_xg(s,\xi(s))-D_xf(s,\xi(s))\|_{\mathscr{L}(X)},\end{aligned}$$

hypotheses (2.26) and (2.27) imply that $\|B(t)\| \leq \delta e^{-3\nu|t|}$, $t \in \mathbb{R}$. Therefore $B : \mathbb{R} \to \mathscr{L}(X)$ satisfies conditions of Theorem 2.2.14 and we conclude ψ is a nonuniform hyperbolic solution of $\mathscr{S}_g$. $\square$

Remark 2.4.5. *Note that, in Theorem 2.4.4, f and g have to be C^1-close with an exponential weight. In fact, the functions one has to consider are of the form $h : \mathbb{R} \times X \to X$ such that $h(t,x) = e^{-3\nu|t|}h_0(t,x)$ for some h_0 that satisfy the conditions for the uniform case (with $\nu = 0$ see (CARVALHO; LANGA; ROBINSON, 2013, Lemma 8.3)), and this exponential weight has to be considered on the C^1-proximity of the functions as in (2.27).*

Remark 2.4.6. *We point out that the existence of a nonuniform hyperbolic solution can be achieved by an application of Theorem 2.4.4 in the special case where $f = 0$, so $\xi \equiv 0$ is a nonuniform hyperbolic solution of $\dot{y} = A(t)y$.*

CHAPTER

3

NONUNIFORM EXPONENTIAL DICHOTOMY OF TYPE II

In this Chapter, we propose a new type of nonuniform exponential dichotomy. We study the relations between this new concept and the standard one, namely Definition 2.2.4. Moreover, we provide several examples and prove a robustness result for it.

3.1 Nonuniform exponential of type II: introduction

One of our goals is to provide a quantitative analysis of the growth rates of the exponential dichotomies, so now we define again nonuniform exponential dichotomy with different growth rates and in a fixed interval $\mathbb{J}$, to distinguish precisely the exponents and interval where it is defined. The next one we will refer to as *nonuniform exponential dichotomy of type I.*

Definition 3.1.1. *Let $\mathscr{S} = \{S(t,s)\,;\, t \geq s\} \subset \mathscr{L}(X)$ be a linear evolution process in a Banach space $(X, \|\cdot\|_X)$. We say that S admits* ***nonuniform exponential dichotomy of type I on $\mathbb{J}$, or simply NEDI****, if there exists a family of continuous projections $\{\Pi^u(t)\,;\, t \in \mathbb{J}\}$ such that*

1. *$\Pi^u(t)S(t,s) = S(t,s)\Pi^u(s)$, for all $t \geq s$;*

2. *$S(t,s)|_{R(\Pi^u(s))}$ is an isomorphism for all $t \geq s$, and the inverse over $R(\Pi^u(t))$ we denote by $S(s,t)$;*

3. *there exist $M, \alpha, \beta > 0$, and $\delta, \nu \geq 0$ such that*

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)} \leq Me^{\delta|s|}e^{-\alpha(t-s)}, \ \ t \geq s,$$

where $\Pi^s(s) := (I - \Pi^u(s))$ for all $s \in \mathbb{J}$ and

$$\|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)} \leq Me^{\nu|s|}e^{\beta(t-s)}, \ \ t < s.$$

Of course, Definition 2.2.4 and Definition 3.1.1 are equivalent whenever $\mathbb{J}=\mathbb{R}$. Indeed, if $\upsilon=\max\{\delta,\nu\}$ and $\omega=\min\{\alpha,\beta\}$, then $K(t)=Me^{\upsilon|t|}$ and $\omega>0$ are the bound and exponent, respectively.

We present another notion of nonuniform exponential dichotomy with a slight modification over Item 3 of Definition 3.1.1.

Definition 3.1.2. *Let $\mathscr{S}=\{S(t,s)\,;\,t\geq s\}\subset\mathscr{L}(X)$ be a linear evolution process in a Banach space $(X,\|\cdot\|_X)$. We say that S admits* ***nonuniform exponential dichotomy of type II on $\mathbb{J}$, or simply NEDII****, if there exists a family of continuous projections $\{\Pi^u(t)\,;\,t\in\mathbb{J}\}$ such that*

1. *$\Pi^u(t)S(t,s)=S(t,s)\Pi^u(s)$, for all $t\geq s$;*
2. *$S(t,s)|_{R(\Pi^u(s))}$ is an isomorphism for all $t\geq s$ and the inverse over $R(\Pi^u(t))$ we denote by $S(s,t)$;*
3. *there exist $M,\alpha,\beta>0$ and $\nu,\delta\geq 0$ such that*

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)}\leq Me^{\delta|t|}e^{-\alpha(t-s)},\ \ t\geq s \tag{3.1}$$

where $\Pi^s(s):=Id_X-\Pi^u(s)$, for all $s\in\mathbb{J}$, and

$$\|S(t,s)\Pi^u(s)\|_{\mathscr{L}(X)}\leq Me^{\nu|t|}e^{\beta(t-s)},\ \ t<s. \tag{3.2}$$

In this work, we will use the following notations.

Remark 3.1.3. *Let $\mathscr{S}$ be an evolution process that admits a nonuniform exponential dichotomy of type $i\in\{I,II\}$, families of projections Π_i^s and Π_i^u. Then we introduce the following notation*

1. *the* ***stable set at instant*** *t, $X_i^s(t):=\Pi_i^s(t)X$ and the* ***unstable set at the instant*** *t; $X_i^u(t):=\Pi_i^u(t)X$ for all $t\in\mathbb{J}$;*
2. *the* ***stable family*** *$X_i^s:=\{X_i^s(t):t\in\mathbb{J}\}$, and the* ***unstable family*** *$X_i^u:=\{X_i^u(t):t\in\mathbb{J}\}$;*
3. *$X_i^s(\alpha,\delta)=\{X_i^j(t):t\in\mathbb{J}\}$ to mean that over stable family the bound is given by $M^s(t)=Me^{\delta|t|}$ and the exponent by $\alpha>0$, see (3.1);*
4. *$X_i^u(\beta,\nu)=\{X_i^j(t):t\in\mathbb{J}\}$ to mean that over the unstable family the bound is given by $M^u(t)=Me^{\nu|t|}$ and the exponent by $\beta>0$, see (3.2).*

In the case that $\mathbb{J}=\mathbb{R}$, the names “stable" and “unstable" in NEDII have the standard sense of exponential dichotomy, only when $\alpha>\delta$ and $\beta>\nu$. In fact, at this situation, for every $s\in\mathbb{R}$ fixed, we see that $S(t,s)\Pi^s(s)\to 0$ as $t\to+\infty$ and $S(t,s)\Pi^u(s)\to 0$ as $t\to-\infty$. However, there are examples of evolution processes that admits NEDII with $\alpha<\delta$ or $\beta<\nu$ with some interesting properties to be explored. For instance, even in this “pathological situation”, it is possible to obtain applications on the asymptotic behavior for evolution processes.

The following result provides a simpler way to relate both types of nonuniform exponential dichotomies.

Theorem 3.1.4. *Let $\mathscr{S}$ be an evolution process.*

On $\mathbb{R}^+$:

1. *there exists a NEDI with $X_I^s(\alpha,\delta)$, if and only if, there exists a NEDII with $X_{II}^s(\alpha+\delta,\delta)$;*
2. *there exists a NEDI with $X_I^u(\beta+\nu,\nu)$, if and only if, there exists a NEDII with $X_{II}^u(\beta,\nu)$.*

On $\mathbb{R}^-$:

1. *there exists a NEDII with $X_{II}^s(\alpha,\delta)$, if and only if, there exists a NEDI with $X_I^s(\alpha+\delta,\delta)$;*
2. *there exists a NEDII with $X_{II}^u(\beta+\nu,\nu)$, if and only if, there exists a NEDI with $X_I^u(\beta,\nu)$.*

Proof. Note that, if $t,s\in\mathbb{R}^+$ we have

$$-\alpha(t-s)+\delta s=-(\alpha+\delta)(t-s)+\delta t.$$

Hence, for an evolution process $\mathscr{S}$ that admits NEDI with bound on the stable set $M^s(s)=Me^{\delta|s|}$, for some $M,\delta>0$ (the case $\delta=0$ is trivial), and exponent $\alpha>0$ we have that

$$\|S(t,s)\Pi^s(s)\|_{\mathscr{L}(X)}\leq Me^{-\alpha(t-s)+\delta|s|}=Me^{-(\alpha+\delta)(t-s)+\delta|t|},$$

which finishes the proof of Item 1. Similarly, Item 2 follows from the relation

$$\alpha(t-s)+\delta|t|=(\alpha+\delta)(t-s)+\delta|s|,\ \ t,s\in\mathbb{R}^+.$$

The proof on $\mathbb{R}^-$ is similar to the case on $\mathbb{R}^+$. □

Next corollary summarize the relations of Theorem 3.1.4.

Corollary 3.1.5. *Let $\mathscr{S}$ be an evolution process in a semi-line, i.e., $\mathbb{J}=\mathbb{R}^+$ or $\mathbb{R}^-$. If $\mathscr{S}$ admits NEDI (or NEDII) with bound $M(t)=Me^{\upsilon|t|}$ and exponent and $\omega>\upsilon$, then $\mathscr{S}$ admits NEDII (NEDI) with bound $M(t)=Me^{\upsilon|t|}$ and exponent $\omega-\upsilon>0$.*

The analysis as in Corollary 3.1.5 is not optimal, we lose information when unifying the exponents α,β and the growth of the bound or order $e^{\delta|t|}$ or $e^{\nu|t|}$. Note that the same problem occurs when we study the exponents in the whole line. Hence, to provide an "optimal" analysis on the relation of the exponents and the growth of the bound, we sometimes consider different exponents, even in the half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$.

3.2 Examples of NEDII

In this section, we provide examples of scalar evolution processes that admit nonuniform exponential dichotomies (of type I and II). Our goal is to guarantee that NEDII is a new concept and explore the differences with the standard notion.

Proposition 3.2.1. *Let $a,b>0$ and $\mathscr{S}=\{S(t,s):t\geq s\}$ be the evolution process defined by $x(t,s;x_0):=S(t,s)x_0$, where x is the solution for $\dot{x}=-bx-at\sin(t)x$, $t\geq s$ at the initial data $x(s)=x_0\in\mathbb{R}$. We have that*

1. *$\mathscr{S}$ admits a NEDII on $\mathbb{R}^+$ with $X^s_{II}(b+a,2a)$ and $\Pi^u(t)=0$ for all $t\geq 0$.*
2. *$\mathscr{S}$ admits a NEDI on $\mathbb{R}^-$ with $X^s_{I}(b+a,2a)$ and $\Pi^u(t)=0$ for all $t\geq 0$.*

Additionally, if $b>a$, then

1. *$\mathscr{S}$ admits a NEDI on $\mathbb{R}^+$ with $X^s_{I}(b-a,2a)$ and $\Pi^u(t)=0$ for all $t\geq 0$.*
2. *$\mathscr{S}$ admits a NEDII on $\mathbb{R}^-$ with $X^s_{II}(b-a,2a)$ and $\Pi^u(t)=0$ for all $t\geq 0$.*
3. *$\mathscr{S}$ admits NEDI and NEDII on $\mathbb{R}$, with $X^s_{j}(b-a,2a)$, $j=I,II$, and $\Pi^u(t)=0$ for all $t\geq 0$.*

Proof. Note that $S(t,s)x=e^{-b(t-s)+at\cos(t)-as\cos(s)-a\sin(t)+a\sin(s)}x$, $t,s\in\mathbb{R}$ and $x\in\mathbb{R}$. Hence

$$\|S(t,s)\|_{\mathscr{L}(\mathbb{R})}=S(t,s)\,1=e^{-(b+a)(t-s)+at(\cos(t)+1)-as(\cos(s)+1)-a\sin(t)+a\sin(s)}.$$

By similar arguments to those presented in the proof of Theorem 2.3.2, we conclude that

$$S(t,s)\,1\leq e^{2a-(b+a)(t-s)+2at},\ t\geq s\geq 0,$$
$$S(t,s)\,1\leq e^{2a-(b+a)(t-s)+2a|s|},\ s\leq t\leq 0.$$

which finishes the prove of the first two items. The proof of the remaining items follows from Theorem 3.1.4. □

Remark 3.2.2. *Let $\mathscr{S}$ be the evolution process defined in Proposition 3.2.1 Note that, if $b<3a$, $\mathscr{S}$ admits a NEDII on $\mathbb{R}^+$ with $X^s_{II}(\alpha_2,\delta_2)$, where $\alpha_2:=b+a=>2a=:\delta_2$ and a NEDI on $\mathbb{R}^+$ with $X^s_{I}(\alpha_1,\delta_1)$ $\alpha_1:=b-a<2a=:\delta_1$. Hence, in some situations, it is possible to choose NEDII with "better" relation in the exponents than NEDI. Of course, an analogous relation its obtained over $\mathbb{R}^-$, but $\mathscr{S}$ admits NEDI and NEDII, where NEDI has the "better" relation on the exponents for NEDI.*

Next, we provide an example of an evolution process that admits NEDII with two different projections and does not admit any NEDI.

Proposition 3.2.3. *Define* $f_0 : \mathbb{R} \to \mathbb{R}$ *by*

$$f_0(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ -1 & \text{if } t < 0, \end{cases} \tag{3.3}$$

and consider the Caratheodory differential equation

$$\dot{x} = f_0(t)x, \text{ for } t \in \mathbb{R}.$$

Then, the induced evolution process $\mathscr{S}_0 = \{S_0(t,s) : t,s \in \mathbb{R}\}$ *admits a NEDII on* $\mathbb{R}$ *with two different families of projections, and does not admit any NEDI on* $\mathbb{R}$.

Proof. For each $t \in \mathbb{R}$ define the real function $T(t) : \mathbb{R} \to \mathbb{R}$ as

$$T(t)x = \begin{cases} e^t x & \text{if } t \geq 0, \\ e^{-t} x & \text{if } t \leq 0, \end{cases}$$

for each $x \in \mathbb{R}$. Note that $T(t)$ is an homeomorphism on $\mathbb{R}$ and that $S_0(t,s) = T(t)T(s)^{-1}$ for every $t,s \in \mathbb{R}$.

First we show that $\mathscr{S}_0$ admits a NEDII with the $\Pi^u(t) = 0$ for all $t \in \mathbb{R}$, i.e., we prove that $\mathscr{S}_0$ satisfies

$$\|S_0(t,s)\|_{\mathscr{L}(\mathbb{R})} = S_0(t,s)\,1 \leq e^{-(t-s)+2|t|}, \text{ for all } t \geq s. \tag{3.4}$$

Indeed, if $s \leq 0 \leq t$ we are able to write $S_0(t,s)1 = e^{s+t} = e^{-(t-s)+2(t+s)}$. Now, let $t \geq s \geq 0$ then

$$t - s \leq -(t-s) + 2t = -(t-s) + 2|t|.$$

Thus $S_0(t,s)1 = e^{t-s} \leq e^{-(t-s)+2|t|}$, for $t \geq s \geq 0$. Finally, if $s \leq t \leq 0$ then $S_0(t,s)1 = e^{-(t-s)}$ and $\mathscr{S}_0$ satisfies (3.4).

Similarly, it is possible to prove that

$$S_0(t,s)\,1 \leq e^{t-s+2|t|}, \; t \leq s. \tag{3.5}$$

Therefore, $\mathscr{S}_0 = \{S_0(t,s) : t \geq s\}$ admits a NEDII with $\Pi^u(\cdot) = 0$ and $\Pi^u(\cdot) = Id_{\mathbb{R}}$.

Finally, suppose that $\mathscr{S}_0$ admits a NEDI on $\mathbb{R}$, then exists $\{\widetilde{\Pi}^u(t) : t \in \mathbb{R}\}$ a family of projections so that satisfies all the conditions from the Definition 3.1.1. It is straightforward to verify that $\widetilde{\Pi}^u(\cdot)$ must be constant equal to the identity map $Id_{\mathbb{R}}$ or the null operator 0.

Assume that $\widetilde{\Pi}^u = Id_{\mathbb{R}}$. Then there are $\widetilde{M}, \tilde{\beta} > 0$, and $\tilde{\nu} \geq 0$ such that

$$S_0(t,s)1 \leq \widetilde{M} e^{\tilde{\beta}(t-s)+\tilde{\nu}|s|}, \; t \leq s. \tag{3.6}$$

Then, for each $s \in \mathbb{R}$ fixed, $S_0(t,s)1 \to 0$ as $t \to -\infty$, which is a contradiction.

Similarly, we prove that we can not have $\widetilde{\Pi}^u = 0$, and therefore $\mathscr{S}_0$ does not admit any NEDI. □

Proposition 3.2.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that*

1. $\lim_{t\to+\infty} f(t)=1$;
2. $\lim_{t\to-\infty} f(t)=-1$.

Then, the evolution process $\mathscr{S}_f=\{S_f(t,s):t,s\in\mathbb{R}\}$, induced by $\dot{x}=f(t)x$, admits a NEDII on $\mathbb{R}$ with two different projections. Moreover, $\mathscr{S}_f$ does not admits any NEDI on $\mathbb{R}$.

Proof. Let f_0 be the function defined in (3.3). Note that

$$\lim_{|t-s|\to+\infty}\frac{1}{t-s}\int_s^t |f(r)-f_0(r)|dr=0.$$

Then, for any $\varepsilon\in(0,1)$, there exists $K_\varepsilon>0$ such that

$$\int_s^t |f(r)-f_0(r)|dr\le K_\varepsilon+\varepsilon|t-s|,\ \text{for each } t,s\in\mathbb{R},$$

which yields to

$$\int_s^t f(r)dr\le\int_s^t f_0(r)dr+K_\varepsilon+\varepsilon|t-s|,\ \text{for each } t,s\in\mathbb{R}. \tag{3.7}$$

Thus

$$S_f(t,s)1\le M_\varepsilon S_0(t,s)e^{\varepsilon(t-s)},\ t\ge s, \tag{3.8}$$

where $M_\varepsilon=e^{K_\varepsilon}$. Hence, by the proof of Proposition 3.2.3,

$$S_f(t,s)1\le M_\varepsilon e^{-(1-\varepsilon)(t-s)+2|t|},\ t\ge s.$$

Therefore, $\mathscr{S}_f$ admits a NEDII with projections $\Pi^u(t)=0$ for every $t\in\mathbb{R}$.

We now use (3.7) and Proposition 3.2.3, with $t\ge s$ replaced by $t\le s$, to obtain that $\mathscr{S}_f$ admits a NEDII with projections $\Pi^u(t)=Id_{\mathbb{R}}$, for every $t\in\mathbb{R}$.

Let us prove now that $\mathscr{S}_f$ does not admit NEDI. Suppose that $\mathscr{S}_f$ admits a NEDI with projections $\Pi^u(t)$ for $t\in\mathbb{R}$. Then, $\Pi^u(t)$ must be constant equal to the null operator or the identity map. First, assume that $\widetilde{\Pi}^u(t)=0$, for every $t\ge0$. Thus there exist $\widetilde{M},\tilde{\alpha}>0$ and $\tilde{\delta}\ge0$ such that

$$S_f(t,s)1\le\widetilde{M}e^{-\tilde{\alpha}(t-s)+\tilde{\delta}|s|},\ t\ge s. \tag{3.9}$$

By similar arguments used to prove (3.8), it is possible to verify that

$$S_0(t,s)1\le M_\varepsilon S_f(t,s)e^{\varepsilon(t-s)},\ t\ge s. \tag{3.10}$$

Then for $\varepsilon\in(0,\tilde{\alpha})$, inequality (3.10) implies that $\mathscr{S}_0$ admits a NEDI, which is a contradiction with Proposition 3.2.3. By a similar analysis, it is possible to see that if $\mathscr{S}_f$ admits a NEDI with projections $\widetilde{\Pi}^u(t)=Id_{\mathbb{R}}$, for $t\in\mathbb{R}$, then $\mathscr{S}_0$ will also admits a NEDI, which will be a contradiction with Proposition 3.2.3. The proof is complete. □

The next proposition provides an example of an evolution process with NEDII that do not admit any NEDI over a half-line.

Proposition 3.2.5. *Consider the ordinary differential equation*

$$\dot{x} = g(t)x, \text{ for } t \geq 0,$$

where g is the real function defined as

$$g(t) = \begin{cases} 0 & \text{if } t \in (0,1], \\ 1 & \text{if } t \in (n!, (n+1)!], \text{ for } n = 2k, k = 0,1,\cdots, \\ -n & \text{if } t \in (n!, (n+1)!], \text{ for } n = 2k+1, k = 0,1,\cdots. \end{cases}$$

Then there exists an evolution process $\mathscr{S}_g = \{S_g(t,s) : t,s \geq 0\}$ such that:

1. *$\mathscr{S}_g$ admits a NEDII in $\mathbb{R}^+$ with $X_I^s(1,2)$ and projection $\Pi^u(t) = 0$, $t \geq 0$.*

2. *$\mathscr{S}_g$ does not admit any NEDI on $\mathbb{R}^+$.*

Proof. Note that

$$S_g(t,s)\,1 \leq e^{t-s}, \ \ t \geq s \geq 0.$$

Since $t-s \leq -(t-s)+2t$, for $t \geq s \geq 0$, $\mathscr{S}_g$ admits a NEDII with projection $\Pi^u = 0$ and exponents $\alpha = 1$ and $\delta = 2$.

Now, we prove that $\mathscr{S}_g$ does not admit any NEDI. Indeed, if $\mathscr{S}_g$ admits NEDI with projection $\Pi^u(\cdot)$ constant equal to 0 or $Id_{\mathbb{R}}$. Suppose that $\Pi^u(t) = 0$, for each $t \geq 0$. This means that there are $M, \alpha, \delta > 0$ such that

$$S_g(t,s)\,1 \leq Me^{\delta|s|-\alpha(t-s)}, \text{ for all } t \geq s \geq 0. \tag{3.11}$$

For $n = 2k$ for some $k \in \mathbb{N}$, we choose $t_n = (n+1)!$ and $s_n = n!$, thus $t_n - s_n = ns_n$. Thus, from (3.11)

$$S_g(t_n,s_n)\,1 = e^{ns_n} \leq Me^{\delta s_n - \alpha(t_n - s_n)}, \text{ for all even } n.$$

Hence,

$$e^{ns_n(1+\alpha)-\delta s_n} \leq M, \text{ for all even } n \in \mathbb{N}.$$

which is a contradiction, because the sequence on the right-hand side is not bounded. Therefore, $\mathscr{S}_g$ does not admit NEDI with projection $\Pi^u = 0$.

Now, if we assume that $\mathscr{S}_g$ admits a NEDI with projection $\Pi^u := Id_{\mathbb{R}}$, following the same line of arguments above we will obtain a contraction. Therefore, $\mathscr{S}_g$ does not admit any NEDI and the proof is complete. □

3.3 NED for invertible evolution processes

In this section, we study nonuniform exponential dichotomies for invertible evolution processes. We provide a relationship between NEDI and NEDII. As an application we establish a robustness result of NEDII.

Before proving the next result we recall the concept of dual operator of a linear operator in a Banach space. For an arbitrary bounded linear functional $x^* \in X^*$ we write $x^*(x) := \langle x, x^* \rangle \in \mathbb{R}$.

Definition 3.3.1. *Let $A : D(A) \subset X \to X$ be a linear operator such that $D(A)$ is dense in X. The **dual operator** $A^* : D(A^*) \subset X^* \to X^*$ of A is defined by: $D(A^*)$ is the set of $x^* \in X^*$ such that there exists $z^* \in X^*$ such that*

$$\langle Ax, x^* \rangle = \langle x, z^* \rangle,\ x \in D(A). \tag{3.12}$$

For $x^* \in X^*$ we define $A^*x^* = z^*$ as the only element of X^* that satisfies (3.12).

For the next result, we only need to consider the dual operator of an bounded linear operator $A \in \mathscr{L}(X)$. Of course, in this situation, $D(A^*) = X^*$ and $A^* \in \mathscr{L}(X^*)$.

The next result provides a fundamental relation between these two notions of nonuniform exponential dichotomies for invertible evolution processes.

Theorem 3.3.2. *Let $\mathscr{S} = \{S(t,s) : t,s \in \mathbb{J}\} \subset \mathscr{L}(X)$ be an invertible evolution process in a Banach space X. Define the bounded linear operator in the dual space X^**

$$T(t,s) = [S(s,t)]^*, \text{ for all } t,s \in \mathbb{J}.$$

Then $\mathscr{T} := \{T(t,s) : t,s \in \mathbb{J}\}$ defines a invertible evolution process in X^*.

Additionally, if $\mathscr{S}$ admits a NEDI (NEDII) with bound $M(t) = Me^{\upsilon|t|}$, for $t \in \mathbb{J}$, exponent $\omega > 0$, and families of projections Π^u and Π^s, for some $M, \upsilon > 0$. Then $\mathscr{T}$ admits a NEDII (NEDI) with bound $M(t)$ and exponent $\omega > 0$, and family of projections $\widetilde{\Pi}^u = [\Pi^s]^*$ and $\widetilde{\Pi}^s = [\Pi^u]^*$, where

$$[\Pi^k]^* := \{[\Pi^k(t)]^* : t \in \mathbb{J}\},\ \ k = u, s. \tag{3.13}$$

Proof. Lets first show that $\mathscr{T}$ defines an evolution process in X^*. Let $t, s, \tau \in \mathbb{J}$ then

$$T(t,t) = [S(t,t)]^* = [Id_X]^* = Id_{X^*},$$

and also

$$T(t,s)T(s,\tau) = [S(s,t)]^*[S(\tau,s)]^* = [S(\tau,s)S(s,t)]^* = T(t,\tau),$$

where we use duality properties and that $\mathscr{S}$ is an evolution process. Now, let (t_n, s_n, x_n^*) be a sequence in $\mathbb{J}^2 \times X^*$ such that $(t_n, s_n)x_n \to (t, s, x^*)$ as $n \to +\infty$, we will prove that $T(t_n, s_n)x_n^* \to$

$T(t,s)x^*$ as $n \to +\infty$. First, note that

$$\begin{aligned} \|T(t_n,s_n)x_n^* - T(t,s)x^*\|_{\mathscr{L}(X^*)} &= \sup_{\|x\|_X=1} |\langle x, T(t_n,s_n)x_n^*\rangle - \langle x, T(t,s)x^*\rangle| \\ &= \sup_{\|x\|_X=1} |\langle S(s_n,t_n)x, x_n^*\rangle - \langle S(s,t)x, x^*\rangle|. \end{aligned}$$

For any $x \in X$, we have that

$$\begin{aligned} &|\langle S(s_n,t_n)x, x_n^*\rangle - \langle S(s,t)x, x^*\rangle| \\ &\leq |\langle S(s_n,t_n)x - S(s,t)x, x_n^*\rangle| + |\langle S(s,t)x, x_n^* - x^*\rangle| \\ &\leq \|x_n^*\|_{X^*} \|S(s_n,t_n)x - S(s,t)x\|_X + \|S(s,t)x\|_X \|x_n^* - x^*\|_{X^*}. \end{aligned}$$

Since $\{x_n^*\}$ is a bounded sequence in X^*, to obtain that

$$\lim_{n\to+\infty} \|T(t_n,s_n)x_n^* - T(t,s)x^*\|_{\mathscr{L}(X^*)} = 0.$$

Therefore, $\mathscr{T}$ define an invertible evolution process in X^*.

Now, assuming that $\mathscr{S}$ admits a NEDI and we claim that $\mathscr{T}$ admits a NEDII.

Indeed, since $\mathscr{S}$ admits a NEDI, there exists a family of projections $\{\Pi^u(t) : t \in \mathbb{J}\}$ such that satisfies the conditions in Definition 3.1.1 for $\mathscr{S}$.

Define $\widetilde{\Pi}^s(t) := [\Pi^u(t)]^*$ for all $t \in \mathbb{J}$. Then $\{\widetilde{\Pi}^s(t) : t \in \mathbb{J}\}$ is a family of projections on X^* such that

$$T(t,s)\widetilde{\Pi}^s(s) = [S(s,t)]^*[\Pi^u(s)]^* = [\Pi^u(s)S(s,t)]^*, \tag{3.14}$$

Since $\Pi^u(s)S(s,t) = S(s,t)\Pi^u(t)$, we conclude that $T(t,s)\widetilde{\Pi}^s(s)\widetilde{\Pi}^s(t)T(t,s)$.

Moreover,

$$\begin{aligned} \|T(t,s)\widetilde{\Pi}^s(s)\|_{\mathscr{L}(X^*)} &= \|[S(s,t)\Pi^u(t)]^*\|_{\mathscr{L}(X^*)} \\ &= \|S(s,t)\Pi^u(t)\|_{\mathscr{L}(X)} \\ &\leq Me^{\upsilon|t|}e^{\omega(s-t)}, \text{ for } t \geq s, \end{aligned}$$

and, if $\widetilde{\Pi}^u(t) = Id_{X^*} - \widetilde{\Pi}^s(t)$, we obtain that

$$\begin{aligned} \|T(t,s)\widetilde{\Pi}^u(s)\|_{\mathscr{L}(X^*)} &= \|[S(s,t)\Pi^s(t)]^*\|_{\mathscr{L}(X^*)} \\ &\leq Me^{\upsilon|t|}e^{-\omega(s-t)}, \text{ for } t \leq s. \end{aligned}$$

Then, according to the inequalities above, $\mathscr{T}$ admits NEDII on $\mathbb{J}$ with exponent $\omega > 0$ and bound $Me^{\upsilon|t|}$, $t \in \mathbb{R}$.

Finally, if $\mathscr{S}$ admits a NEDII following the same line of arguments of the above proof we conclude that $\mathscr{T}$ admits a NEDI, with the same relations between projections, bound and exponent. □

Now, as a consequence of Theorem 2.2.11 and Theorem 3.3.2 we prove a robustness result for NEDII.

Theorem 3.3.3 (Robustness of NEDII)**.** *Let $\mathscr{S}_1 = \{S_1(t,s) : t,s \in \mathbb{R}\}$ be an invertible evolution process that admits a NEDII with bound $M(t) = Me^{\upsilon|t|}$, $t \in \mathbb{R}$, for some $M, \omega > 0$, and exponent $\omega > \upsilon$. Suppose that $\mathscr{S}_1$ satisfies*

$$\sup_{0\leq t-s\leq 1} \{e^{-\upsilon|t|} \|S_1(t,s)\|_{\mathscr{L}(X)}\} < +\infty. \tag{3.15}$$

Then there exists $\varepsilon > 0$ such that if $\mathscr{S}_2$ is another invertible evolution process such that

$$\sup_{0\leq t-s\leq 1} \{e^{\upsilon|s|} \|S_1(t,s) - S_2(t,s)\|_{\mathscr{L}(X)}\} < \varepsilon. \tag{3.16}$$

Then $\mathscr{T}_2 := \{T_2(t,s) = [S_2(s,t)]^* : t,s \in \mathbb{R}\}$ admits a NEDI with exponent $\hat{\omega} > 0$ and bound $\widehat{M}$ provided in Theorem 2.2.11 for ε small enough.

Additionally, if X is reflexive, then $\mathscr{S}_2$ admits a NEDII with the same bound and exponent of $\mathscr{T}_2$.

Proof. Let $\mathscr{T}_1 = \{T_1(t,s) : t,s \in \mathbb{R}\}$ be the evolution process over X^* defined by $T_1(t,s) := [S_1(s,t)]^*$ for all $t,s \in \mathbb{R}$.

Then, from Theorem 3.3.2, $\mathscr{T}_1$ admits a NEDI with bound $M(t) = Me^{\upsilon|t|}$ and exponent $\omega > \upsilon$. From (3.15), $\mathscr{T}_1$ satisfies

$$\sup_{0\leq t-s\leq 1} \{e^{-\upsilon|t|} \|T_1(t,s)\|_{\mathscr{L}(X^*)}\} = \sup_{0\leq t-s\leq 1} \{e^{-\upsilon|t|} \|S_1(s,t)\|_{\mathscr{L}(X)}\} < +\infty.$$

Therefore, by Theorem 2.2.11, there exists $\varepsilon > 0$ such that if $\mathscr{T} = \{T(t,s) : t,s \in \mathbb{R}\}$ is an evolution process over X^* such that

$$\sup_{0\leq t-s\leq 1} \{e^{\upsilon|s|} \|T_1(t,s) - T(t,s)\|_{\mathscr{L}(X^*)}\} < \varepsilon, \tag{3.17}$$

then $\mathscr{T}$ admits NEDI with exponents $\hat{\omega}$ and bound $\widehat{M}$, given by Theorem 2.2.11.

Let $\mathscr{S}_2$ be an evolution process over X such that satisfies equation (3.16). Hence $T_2(t,s) := [S_2(s,t)]^*$ defines an evolution process $\mathscr{T}_2$ over X^* satisfying (3.17). Thus $\mathscr{T}_2$ admits a NEDI with exponent $\hat{\omega} > 0$ and bound $\widehat{M}$, which finishes the first part of the proof.

Finally, we assume that X is reflexive. Let $J : X \to X^{**}$ be the evaluation map, i.e., J is defined by $x \mapsto Jx \in X^{**}$, where $\langle x^*, Jx(x^{**})\rangle = \langle x, x^*\rangle$, for every $x^* \in X^*$. Since X is reflexive, J is an isometric isomorphism, and $\mathscr{S}_2$ satisfies

$$S_2(t,s) = J^{-1}[S_2(t,s)]^{**}J, \text{ for every } t,s \in \mathbb{R}. \tag{3.18}$$

Now, from the first part of the proof $\mathscr{S}_2^* = \{[S_2(t,s)]^* : t,s \in \mathbb{R}\}$ admits a NEDI. Hence Theorem 3.3.2 implies that $\mathscr{S}_2^{**} = \{[S_2(t,s)]^{**} : t,s \in \mathbb{R}\}$ admits a NEDII with bound $\widehat{M}$, exponent $\hat{\omega} > 0$, and family of projections $\{\widehat{\Pi}^u(t) : t \in \mathbb{R}\}$. Then, it is straightforward to verify that $\mathscr{S}_2$ admits a NEDII with bound $\widehat{M}$ and exponent $\hat{\omega}$, and projections $\Pi^u(t) = J^{-1}\widehat{\Pi}^u(t)J$ for $t \in \mathbb{R}$, and the proof is complete. □

Remark 3.3.4. *There are evolution processes such that has a NEDII with bound $M_2(t) = Me^{\upsilon_2|t|}$ and exponent $\omega_2 > \upsilon_2$ and admits a NEDI bound $K_1(t) = Me^{\upsilon_1|t|}$ and exponent $\omega_1 < \upsilon_1$. For those it is possible apply the robustness result of NEDII, Theorem 3.3.3, and it is not possible to apply the robustness result of NEDI, Theorem 2.2.11, because of the conditions on the exponents $\omega_1 < \delta_1$, see Example 3.3.5.*

Therefore, we established a robustness result of NEDII that can be applied in a situation where the robustness of NEDI, Theorem 2.2.11, cannot be applied, which emphasizes the importance of the nonuniform exponential dichotomy of type II.

Next, we provide an example of an invertible evolution process in $\mathbb{R}$, with the proprieties describe in Remark 3.3.4.

Example 3.3.5. *Let $a,b,c,d > 0$ with $b > a$ and $d > c$. Consider the real function*

$$f(t) = \begin{cases} -b - at\sin(t), & \text{if } t \geq 0, \\ -d - ct\sin(t), & \text{if } t < 0. \end{cases}$$

Let $\mathscr{T} = \{T(t,s) : t,s \in \mathbb{R}\}$ be the evolution process associated to $\dot{x}(t) = f(t)x(t)$. Then, from Example 3.2.1, it is straightforward to verify that $\mathscr{T}$ admits NEDII with $X^s_{II}(\alpha_2,\delta)$ and $\Pi^u_{II} = 0$, and a NEDI with $X^s_I(\alpha_1,\delta)$, and $\Pi^u_I = 0$, where

$$\alpha_2 = \min\{b+a, d-c\},\ \alpha_1 = \min\{b-a, d+c\} \text{ and } \delta = \max\{2a, 2c\}.$$

In particular, $\mathscr{T}$ satisfies condition 3.15.

Note that it is possible to choose a,b,c,d such that $\alpha_2 > \delta$ and $\alpha_1 < \delta$, for instance: $d < 1/2$, $a > 1$, and $b \in (a+1, 3a)$. Thus, for these choices, it is possible to apply the Robustness of NEDII, namely Theorem 3.3.3, and it is not possible to apply the Robustness of NEDI, Theorem 2.2.11. Therefore, in this case, we know for sure that NEDII persists under perturbation and we do not know if NEDI does.

Of course, a symmetric claim holds for NEDI: there are a,b,c,d such that $\alpha_1 > \delta$ and $\alpha_2 < \delta$. Therefore, together, NEDI and NEDII, provides an completely analysis of existence of an nonuniform exponential dichotomy for $\dot{x} = f(t)x$ and whenever this type of nonuniform hyperbolicity is preserved under perturbation.

CHAPTER

4

EXPONENTIAL DICHOTOMIES FOR NONAUTONOMOUS RANDOM DYNAMICAL SYSTEMS

In this chapter, we introduce the notion of *hyperbolicity* for linear nonautonomous random dynamical systems. This concept is expressed precisely by exponential dichotomies and it is introduced in this nonautonomous random setting in (CARABALLO *et al.*, 2021b). The proofs of this chapter also are applications of Henry's method, thus we follow the same line of arguments of Chapter 2, and some of the arguments are included for those readers interested only in the hyperbolicity for nonautonomous random dynamical systems.

4.1 Exponential dichotomy for nonautonomous random dynamical systems

First, we introduce the notion of *nonautonomous random dynamical systems* in a metric space (X,d). We start with *random flow* defined in a probability space.

Definition 4.1.1. *Let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space. We say that a family of maps* $\{\theta_t:\Omega\to\Omega : t\in\mathbb{R}\}$ *is a* ***random flow*** *if*

1. $\theta_0 = Id_\Omega$;
2. $\theta_{t+s}=\theta_t\circ\theta_s$, *for all* $t,s\in\mathbb{R}$;
3. $\theta_t:\Omega\to\Omega$ *is measurable for all* $t\in\mathbb{R}$.

Definition 4.1.2. *Let* $\{\theta_t:\Omega\to\Omega : t\in\mathbb{R}\}$ *be a random flow. Define* $\Theta_t(\tau,\omega):=(t+\tau,\theta_t\omega)$ *for each* $(\tau,\omega)\in\mathbb{R}\times\Omega$, *and* $t\in\mathbb{R}$. *We say that a family of maps* $\{\psi(t,\tau,\omega):X\to X;(t,\tau,\omega)\in$

$\mathbb{R}^+ \times \mathbb{R} \times \Omega\}$ *is a* ***nonautonomous random dynamical system (NRDS) or co-cycle*** *driven by* Θ *if*

1. *the mapping* $\mathbb{R}^+ \times \Omega \times X \ni (t, \omega, x) \mapsto \psi(t, \tau, \omega)x \in X$ *is measurable for each fixed* $\tau \in \mathbb{R}$*;*
2. $\psi(0, \tau, \omega) = Id_X$*, for each* $(\tau, \omega) \in \mathbb{R} \times \Omega$*;*
3. $\psi(t+s, \tau, \omega) = \psi(t, \Theta_s(\tau, \omega)) \circ \psi(s, \tau, \omega)$*, for every* $t, s \geq 0$ *in* $\mathbb{R}$*, and* $(\tau, \omega) \in \mathbb{R} \times \Omega$*;*
4. $\psi(t, \tau, \omega) : X \to X$ *is a continuous map for each* $(t, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega$*.*

We usually denote the pair $(\psi, \Theta)_{(X, \mathbb{R} \times \Omega)}$*, or* (ψ, Θ)*, to denote the co-cycle* ψ *driven by* Θ*.*

Remark 4.1.3. *We will write* $\omega_\tau := (\tau, \omega) \in \mathbb{R} \times \Omega$*, and* $\Theta_t(\omega_\tau) := (t+\tau, \theta_t\omega) = (\theta_t\omega)_{\tau+t}$*.*

Throughout this work we will assume that a nonautonomous random dynamical system (ψ, Θ) satisfies

$$\mathbb{R}^+ \times X \ni (t, x) \mapsto \psi(t, \omega_\tau)x \in X \text{ is continuous, for each } \omega_\tau \in \mathbb{R} \times \Omega. \tag{4.1}$$

This assumption is sensible in the applications, e.g., when the co-cycle is induced by a well-posed stochastic/random differential equation. Hence, we can associate our co-cycle with a family of *evolution processes*.

Remark 4.1.4. *Let* $(\psi, \Theta)_{(X, \mathbb{R} \times \Omega)}$ *be a nonautonomous random dynamical system which satisfies* (4.1)*. Then, for each* $\omega_\tau \in \mathbb{R} \times \Omega$*, we define the following evolution process*

$$\Psi_{\omega_\tau} := \{\psi_{t,s}(\omega_\tau) := \psi(t-s, \Theta_s \omega_\tau)\,;\, t \geq s\}.$$

Recall the definition of *strongly measurable*:

Definition 4.1.5. *Let* Ω *be a measurable space, and* X *a Banach space. A map* $P : \Omega \to \mathscr{L}(X)$ *is said to be* ***strongly measurable*** *if for every* $x \in X$ *the map* $\Omega \ni \omega \mapsto P(\omega)x \in X$ *is measurable.*

Definition 4.1.6. *A map* $D : \mathbb{T} \times \Omega \to \mathbb{R}$ *is said to be* Θ***-invariant*** *if for each* $\omega_\tau \in \mathbb{R} \times \Omega$ *we have that* $D(\Theta_t \omega_\tau) = D(\omega_\tau)$*, for every* $t \in \mathbb{T}$*.*

Now, we define the notion of *exponential dichotomy* for linear nonautonomous random dynamical systems.

Definition 4.1.7. *A nonautonomous random dynamical system* (φ, Θ) *such that* $\varphi(t, \tau, \omega) \in \mathscr{L}(X)$*, for all* $(t, \tau, \omega) \in \mathbb{T}^+ \times \mathbb{T} \times \Omega$*, is said to admit an* ***exponential dichotomy*** *if there exists a* θ*-invariant subset* $\tilde{\Omega}$ *of* Ω *with full measure,* $\mathbb{P}(\tilde{\Omega}) = 1$*, and a family of projections,* $\Pi^s := \{\Pi^s(\omega_\tau) : \omega_\tau \in \mathbb{T} \times \tilde{\Omega}\}$ *such that*

1. *for each* $\tau \in \mathbb{T}$ *the map* $\Pi^s_\tau(\cdot) := \Pi^s(\tau, \cdot) : \tilde{\Omega} \to \mathscr{L}(X)$ *is strongly measurable;*

2. $\Pi^s(\Theta_t\omega_\tau)\varphi(t,\omega_\tau) = \varphi(t,\omega_\tau)\Pi^s(\omega_\tau)$, *for every* $t \in \mathbb{T}^+$ *and* $\omega_\tau \in \mathbb{T}\times\tilde{\Omega}$*;*

3. $\varphi(t,\omega_\tau) : R(\Pi^u(\omega_\tau)) \to R(\Pi^s(\Theta_t\omega_\tau))$ *is an isomorphism, where* $\Pi^u_\tau := Id_X - \Pi^s_\tau$ *for all* $\tau \in \mathbb{T}$*;*

4. *there exist* Θ*-invariant maps* $\alpha : \mathbb{T}\times\Omega \to (0,+\infty)$ *and* $K : \mathbb{T}\times\Omega \to [1,+\infty)$ *such that*

$$\begin{aligned}\|\varphi(t,\omega_\tau)\Pi^s(\omega_\tau)\|_{\mathscr{L}(X)} &\leq K(\omega_\tau)e^{-\alpha(\omega_\tau)t}, \text{ for every } t \geq 0;\\ \|\varphi(t,\omega_\tau)\Pi^u(\omega_\tau)\|_{\mathscr{L}(X)} &\leq K(\omega_\tau)e^{\alpha(\omega_\tau)t}, \text{ for every } t \leq 0,\end{aligned}$$

for every $\omega_\tau \in \mathbb{T}\times\tilde{\Omega}$

In this case, the function K is called a ***bound*** *and* α *an* ***exponent*** *for the exponential dichotomy.*

We refer to the exponential dichotomy as: ***continuous*** *if* $\mathbb{T} := \mathbb{R}$*, and* ***discrete*** *if* $\mathbb{T} := \mathbb{N}$*.*

Remark 4.1.8. *If for each* ω_p *the map* $\mathbb{R} \ni t \to K(\Theta_t\omega_p)$ *is not constant we say that* (φ,Θ) *admits an* ***nonuniform (with respect to*** t***) exponential dichotomy****. In the special case when the mapping* $\mathbb{R} \ni t \to K(\Theta_t\omega_p)$ *is tempered (or sub-exponential) we say that* (φ,Θ) *admits a* ***tempered exponential dichotomy****.*

Remark 4.1.9. *If a co-cycle* (φ,Θ) *admits an exponential dichotomy, then for each fixed* $\omega_p \in \mathbb{T}\times\tilde{\Omega}$ *the associated evolution process* Φ_{ω_p} *also admits it in the sense of (HENRY, 1981, Section 7.6).*

Indeed, let Π^s *be a family of projections associated with the exponential dichotomy, then for each fixed* $\omega_p \in \mathbb{T}\times\tilde{\Omega}$ *define* $P_t(\omega_p) := \Pi^s(\Theta_t\omega_p)$*, for every* $t \in \mathbb{T}$*. Thus*

1. *for every* $t,s \in \mathbb{T}$ *with* $t \geq s$ *we have*

$$\begin{aligned}P_t(\omega_p)\varphi_{t,s}(\omega_p) &= \Pi^s(\Theta_{t-s}\Theta_s\omega_p)\varphi(t-s,\Theta_s\omega_p)\\ &= \varphi(t-s,\Theta_s\omega_p)\Pi^s(\Theta_s\omega_p)\\ &= \varphi_{t,s}(\omega_p)P_s(\omega_p).\end{aligned}$$

2. *if* $Q = Id_X - P$ *and* $\Pi^u = Id_X - \Pi^s$*, then* $\varphi_{t,s}(\omega_p)|_{R(Q_s(\omega_p))} = \varphi(t-s,\Theta_s\omega_p)|_{R(\Pi^u(\Theta_s\omega_p))}$ *is an isomorphism for all* $t \geq s$ *in* $\mathbb{T}$*.*

3. *Finally,*

$$\|\varphi(t-s,\Theta_s\omega_p)\Pi^s(\Theta_s\omega_p)\|_{\mathscr{L}(X)} \leq K(\Theta_s\omega_p)e^{-\alpha(\omega_p)(t-s)}, \text{ for every } t \geq s; \quad (4.2)$$

$$\|\varphi(t-s,\Theta_s\omega_p)\Pi^u(\Theta_s\omega_p)\|_{\mathscr{L}(X)} \leq K(\Theta_s\omega_p)e^{\alpha(\omega_p)(t-s)}, \text{ for every } t \leq s, \quad (4.3)$$

where $K : \mathbb{T}\times\tilde{\Omega} \to [1,+\infty)$ *is the bound of the exponential dichotomy, which is* Θ*-invariant, i.e.,* $K(\Theta_s\omega_p) = K(\omega_p)$*.*

The interpretation of a nonautonomous random dynamical system as a family of evolution processes with a parameter provides also a special function usually called as *Green's function*:

Definition 4.1.10. *Let (φ,Θ) be a co-cycle which admits an exponential dichotomy with family of projections Π^s. A **Green's function** associated to (φ,Θ) and family of projection Π^s is given by*

$$G_{\omega_p}(t,s)=\begin{cases}\varphi_{t,s}(\omega_p)\Pi^s(\Theta_s\omega_p), & \text{if } t\geq s,\\ -\varphi_{t,s}(\omega_p)\Pi^u(\Theta_s\omega_p), & \text{if } t<s,\end{cases}$$

for each ω_p fixed.

4.2 Exponential dichotomy for NRDS: discrete case

In this section, we study a discrete nonautonomous random dynamical system with exponential dichotomy. The goal is to present a summary of results concerning exponential dichotomy for discrete co-cycles driving by flows over non-compact symbols spaces that we are going to need in order to establish robustness results of hyperbolicity for differential equations. We prove that the property of admitting a discrete exponential dichotomy is stable under perturbation (Theorem 4.2.5), a type of admissibility result (Theorem 4.2.2), and uniqueness and continuous dependence of projections (see Corollary 4.2.3 and Theorem 4.3.6, respectively).

The techniques presented in this section are the same used in (HENRY, 1981) for deterministic dynamical systems and in (ZHOU; LU; ZHANG, 2013) for random dynamical systems.

A linear discrete nonautonomous random dynamical systems (φ,Θ) can be associated with nonautonomous random difference equations. In fact, for each $\omega_p\in\mathbb{Z}\times\Omega$ we study

$$x_{n+1}=A(\Theta_n\omega_p)x_n,\ \ x_n\in X \text{ and } n\in\mathbb{Z}, \tag{4.4}$$

where $A:\mathbb{T}\times\Omega\to\mathscr{L}(X)$ and $\varphi(n,\omega_p):=A(\Theta_{n-1}\omega_p)\circ\cdots\circ A(\omega_p)$ for $n>0$ and $\varphi(0,\omega_p)=Id_X$.

Theorem 4.2.1. *Assume that a nonautonomous random dynamical system (φ,Θ) admits an exponential dichotomy with bound K and exponent α. Let $\omega_p\in\mathbb{Z}\times\Omega$ be fixed and f be a sequence in $l^\infty(\mathbb{Z})$. Then*

$$x_{n+1}=A(\Theta_n\omega_p)x_n+f_n,\ \ x_n\in X \text{ and } n\in\mathbb{Z}, \tag{4.5}$$

possesses a unique bounded solution $x(\cdot,\omega_p)$ given by

$$x(n,\omega_p)=\sum_{-\infty}^{+\infty}G_{\omega_p}(n,k+1)f_k,\ \ \forall\, n\in\mathbb{Z}.$$

Proof. Let ω_p be an arbitrary parameter in $\mathbb{Z}\times\tilde{\Omega}$. First fix $n\in\mathbb{Z}$, take $m<n$ and write

$$x_n=\varphi_{n,m}(\omega_p)x_m+\sum_{k=m}^{n-1}\varphi_{n,k+1}(\omega_p)f_k.$$

Then apply $\Pi^s(\Theta_n\omega_p)$ in this equation and note that the term $\varphi_{n,m}(\omega_p)\Pi^s(\Theta_m\omega_p)x_m$ satisfies

$$\|\varphi_{n,m}(\omega_p)\Pi^s(\Theta_m\omega_p)x_m\|_X\le K(\omega_p)\,e^{-\alpha(\omega_p)(n-m)}\|x_m\|_X.$$

Therefore, if $\{x_n\}_{n\in\mathbb{Z}}$ is a bounded sequence, this last term goes to zero when $m\to-\infty$. Thus, we have that for each $n\in\mathbb{Z}$

$$\Pi^s(\Theta_n\omega_p)x_n=\sum_{k=-\infty}^{n-1}\varphi_{n,k+1}(\omega_p)\Pi^s(\Theta_{k+1}\omega_p)f_k.$$

Analogously, take now $r>n$ and write

$$x_r=\varphi_{r,n}(\omega_p)x_n+\sum_{k=n}^{r}\varphi_{n,k+1}(\omega_p)f_k,$$

Then apply the projection $\Pi^u(\Theta_r\omega_p)$ and use the inverse operator $\varphi_{n,r}(\omega_p)$ to obtain

$$\Pi^u(\Theta_n\omega_p)x_n=\varphi_{n,r}(\omega_p)\Pi^u(\Theta_r\omega_p)x_r-\sum_{k=n}^{r}\varphi_{n,k+1}(\omega_p)Q_{k+1}f_k,$$

and now just notice that

$$\|\varphi_{n,r}(\omega_p)\Pi^u(\Theta_r\omega_p)x_r\|_X\le K(\omega_p)e^{\alpha(\omega)(n-r)}\|x_r\|_X. \tag{4.6}$$

Again, as $\{x_n\}_{n\in\mathbb{Z}}$ is bounded, this last term goes to zero as $r\to+\infty$. Consequently,

$$\Pi^u(\Theta_n\omega_p)x_n=-\sum_{k=n}^{+\infty}\varphi_{n,k+1}(\omega_p)\Pi^u(\Theta_{k+1}\omega_p)f_k.$$

Thus, for each $n\in\mathbb{Z}$,

$$x_n=\Pi^s(\Theta_n\omega_p)x_n+\Pi^u(\Theta_n\omega_p)x_n=\sum_{-\infty}^{+\infty}G_{\omega_p}(n,k+1)f_k.$$

Therefore, if x is a bounded solution of (4.5) then $x_n=\sum_{-\infty}^{+\infty}G_{\omega_p}(n,k+1)f_k$. Conversely, it is easy to see that $x_n(\omega_p):=\sum_{-\infty}^{+\infty}G_{\omega_p}(n,k+1)f_k$ is a solution for (4.5) for every ω_p. Finally, we have that

$$\begin{aligned}\|x_n(\omega_p)\|_X &\le \sum_{k=-\infty}^{n-1}\|\varphi_{n,k+1}(\omega_p)\Pi^s(\Theta_{k+1}\omega_p)f_k\|_X+\sum_{k=n}^{+\infty}\|\varphi_{n,k+1}(\omega_p)\Pi^u(\Theta_{k+1}\omega_p)f_k\|_X\\ &\le \|f\|_{l^\infty}K(\omega_p)\left[\sum_{k=-\infty}^{n-1}e^{-\alpha(\omega_p)(n-k-1)}+\sum_{k=n}^{+\infty}e^{\alpha(\omega_p)(n-k-1)}\right].\end{aligned}$$

Therefore, for every $n\in\mathbb{Z}$

$$\|x_n(\omega_p)\|_X\le\|f\|_{l^\infty}K(\omega_p)\frac{1+e^{-\alpha(\omega_p)}}{1-e^{-\alpha(\omega_p)}},$$

and the existence and uniqueness of a bounded solution for (4.5) is ensured. The proof is complete. □

We prove existence of solutions for the non-homogeneous problem

$$x_{n+1} = A(\Theta_n \omega_p)x_n + B(\Theta_n \omega_p)x_n + f_n, \text{ for every } n \in \mathbb{Z}. \tag{4.7}$$

Theorem 4.2.2. *Let (φ, Θ) be a co-cycle generated by $A : \mathbb{Z} \times \Omega \to \mathscr{L}(X)$ and assume that it admits an exponential dichotomy with bound K and exponent α. Then there exists a Θ-invariant map δ with*

$$0 \leq \delta(\omega_p) < \frac{1 - e^{-\alpha(\omega_p)}}{1 + e^{-\alpha(\omega_p)}}, \text{ for each } \omega_p \in \mathbb{Z} \times \Omega,$$

for which, if $B : \mathbb{Z} \times \Omega \to \mathscr{L}(X)$ satisfies

$$\|B(\Theta_k \omega_p)\|_{\mathscr{L}(X)} \leq \delta(\omega_p)K(\omega_p)^{-1}, \forall k \in \mathbb{Z}, \tag{4.8}$$

then, for each ω_p fixed and $\{f_n\} \in l^\infty(\mathbb{Z})$ the difference equation

$$x_{n+1} = A(\Theta_n \omega_p)x_n + B(\Theta_n \omega_p)x_n + f_n, \text{ for every } n \in \mathbb{Z}, \tag{4.9}$$

possesses a unique bounded solution $x(\cdot, \omega_p)$.

Proof. Let $\omega_p \in \mathbb{Z} \times \tilde{\Omega}$ and $f \in l^\infty(\mathbb{Z})$. As a standard procedure, see for instance (HENRY, 1981; ZHOU; LU; ZHANG, 2013), we only need to prove that the operator

$$(\Gamma_f x)(n, \omega_p) := \sum_{k=-\infty}^{+\infty} G_{\omega_p}(n, k+1)(B(\Theta_k \omega_p)x_k + f_k), \quad \forall n \in \mathbb{Z}$$

has a unique fixed point $x(\cdot, \omega_p)$ in $l^\infty(\mathbb{Z})$.

First, let us prove that $\Gamma_f x(\cdot, \omega_p) \in l^\infty(\mathbb{Z})$, for $x \in l^\infty(\mathbb{Z})$.

$$\begin{aligned}
&\|(\Gamma_f x)(n, \omega_p)\|_X \leq \sum_{k=-\infty}^{+\infty} \|G_{\omega_p}(n, k+1)\|_{\mathscr{L}(X)}(\|B(\Theta_k \omega_p)\|_{\mathscr{L}(X)} \|x_k\|_X + \|f_k\|_X) \\
&\leq \sum_{k=-\infty}^{+\infty} K(\Theta_{k+1}\omega_p)e^{-\alpha(\omega_p)|n-1-k|}(\delta(\omega_p)K(\omega)^{-1}\|x_k\|_X + \|f_k\|_X) \\
&\leq \sum_{k=-\infty}^{+\infty} e^{-\alpha(\omega_p)|n-1-k|}(\delta(\omega_p)\|x\|_{l^\infty} + \|f\|_{l^\infty}K(\omega_p)) \\
&\leq \frac{1 + e^{-\alpha(\omega_p)}}{1 - e^{-\alpha(\omega_p)}}(\delta(\omega_p)\|x\|_{l^\infty} + \|f\|_{l^\infty}K(\omega_p)) < +\infty.
\end{aligned}$$

Then, $\Gamma_f(\cdot, \omega_p)(l^\infty(\mathbb{Z})) \subset l^\infty(\mathbb{Z})$. Finally, if $x, y \in l^\infty(\mathbb{Z})$, we have that

$$\begin{aligned}
&\|(\Gamma_f x)(n, \omega_p) - (\Gamma_f y)(n, \omega_p)\|_X \\
&\leq \sum_{k=-\infty}^{+\infty} \|G_{\omega_p}(n, k+1)\|_{\mathscr{L}(X)} \|B(\Theta_k \omega_p\|_{\mathscr{L}(X)}\|x_k - y_k\|_X \\
&\leq \sum_{k=-\infty}^{+\infty} K(\Theta_{k+1}\omega_p)e^{-\alpha(\omega_p)|n-1-k|}\delta(\omega_p)K(\omega_p)^{-1}\|x_k - y_k\|_X \\
&\leq \frac{1 + e^{-\alpha(\omega_p)}}{1 - e^{-\alpha(\omega_p)}}\delta(\omega_p)\|x - y\|_{l^\infty}.
\end{aligned}$$

Therefore,

$$\|\Gamma_f x(\cdot,\omega_p)-\Gamma_f y(\cdot,\omega_p)\|_{l^\infty}\leq \frac{1+e^{-\alpha(\omega_p)}}{1-e^{-\alpha(\omega_p)}}\delta(\omega_p)\|x-y\|_{l^\infty},$$

thus, we choose $\delta(\omega_p)<\frac{1-e^{-\alpha(\omega_p)}}{1+e^{-\alpha(\omega_p)}}$, thus $\Gamma_f(\cdot,\omega_p)$ is a contraction in $l^\infty(\mathbb{Z})$. In this way, we obtain $x\in l^\infty(\mathbb{Z})$ such that $x_n(\omega_p)=(\Gamma_f x)(n,\omega_p)$ for each $n\in\mathbb{Z}$, in other words, $x(\cdot,\omega_p)$ is the only solution for (4.9). □

The following corollary establishes uniqueness for the family of projections.

Corollary 4.2.3. *If (φ,Θ) admits an exponential dichotomy, then the family of projections are uniquely determined.*

Proof. Let $\Pi^{u,(i)}$, for $i=1,2$, be projections associated with an exponential dichotomy of (φ,Θ).

Given $\omega_p\in\mathbb{Z}\times\tilde{\Omega}$ and $z\in X$, define $f_n=0$, for all $n\neq-1$, and $f_{-1}=z$. From Theorem 4.2.2 with $B=0$, there exists $\{x(n,\omega_p):n\in\mathbb{Z}\}$ the unique bounded solution of

$$x_{n+1}(\omega_p)=A(\Theta_n\omega_p)x_n+f_n,\ \ n\in\mathbb{Z}.$$

From the proof of Theorem 4.2.2 (with $B=0$), we know that this solution is given by

$$x_n(\omega_p)=\sum_{k=-\infty}^{+\infty}G^{(i)}_{\omega_p}(n,k+1)f_k,\ \ \text{for } i=1,2, \tag{4.10}$$

where $G^{(i)}$ is the Green's function associated with $\Pi^{u,(i)}$, for $i=1,2$. By uniqueness of the solution, we must have that $x_0(\omega_p)=\sum_{-\infty}^{+\infty}G^{(i)}_{\omega_p}(0,k+1)f_k=G^{(i)}_{\omega_p}(0,0)f_{-1}=\Pi^{u,(i)}(\omega_p)z$, for $i=1,2$. Therefore, $\Pi^{u,(1)}(\omega_p)=\Pi^{u,(2)}(\omega_p)$ for all $\omega_p\in\mathbb{T}\times\tilde{\Omega}$. □

For later, we will need a type of Grönwall's inequality, see (BARREIRA; SILVA; VALLS, 2009b) for a proof.

Lemma 4.2.4. *Let a and D be positive constants and γ,δ nonnegative constants. Suppose that $u:=\{u_n\}_{n\in\mathbb{J}}$ is a nonnegative bounded sequence on $\mathbb{J}=\mathbb{Z}^+_N$ (or $\mathbb{Z}^-_N$), such that*

$$u_n\leq\gamma De^{-a(n-N)}+\delta D\sum_{k=N}^{+\infty}e^{-a|n-k-1|}u_k,\ \ n\in\mathbb{J}=\mathbb{Z}^+_N,$$

$$(\text{ or } u_n\leq\gamma De^{-a(n-N)}+\delta D\sum_{k=-\infty}^{N-1}e^{-a|n-k-1|}u_k,\ \ n\in\mathbb{J}=\mathbb{Z}^-_N,)$$

where $\delta<D^{-1}(1-e^{-a})/(1+e^{-a})$.

Then

$$u_n\leq\frac{\gamma D}{1-\delta De^{-a}/(1-e^{-(a+\tilde{a})})}e^{-\tilde{a}(n-N)},\ \ n\in\mathbb{J}=\mathbb{Z}^+_N,$$

$$(\text{ or } u_n\leq\frac{\gamma D}{1-\delta De^{-\tilde{b}}/(1-e^{-(a+\tilde{b})})}e^{-\tilde{b}(N-n)},\ \ n\in\mathbb{J}=\mathbb{Z}^-_N),$$

where $\tilde{a}:=-\ln(\cosh a-[\cosh^2a-1-2\delta\sinh a]^{1/2})$ and $\tilde{b}:=\tilde{a}+\ln(1+2\delta D\sinh a)$.

Now, we prove a robustness result of exponential dichotomies for nonautonomous random dynamical systems.

Theorem 4.2.5. *Let (ψ,Θ) be a discrete nonautonomous random dynamical system with an exponential dichotomy with bound K and exponent α. There exists a Θ-invariant map with*

$$0 \leq \delta(\omega_p) < \frac{1-e^{-\alpha(\omega_p)}}{1+e^{-\alpha(\omega_p)}}, \text{ for each } \omega_p \in \mathbb{Z}\times\Omega,$$

for which, if (φ,Θ) is a discrete nonautonomous random dynamical system such that

$$\sup_{n\in\mathbb{N}}\{K(\omega_p)\|\psi(1,\Theta_n\omega_p)-\varphi(1,\Theta_n\omega_p)\|_{\mathscr{L}(X)}\} \leq \delta(\omega_p), \tag{4.11}$$

then (φ,Θ) admits an exponential dichotomy with bound

$$M(\omega_p) := K(\omega_p)\left(1+\frac{\delta(\omega_p)}{(1-\rho(\omega_p))(1-e^{-\alpha(\omega_p)})}\right)\max\{D_1(\omega_p),D_2(\omega_p)\},$$

and exponent

$$\tilde{\alpha}(\omega_p) := -\ln(\cosh\alpha(\omega_p)-[\cosh^2\alpha(\omega_p)-1-2\delta(\omega_p)\sinh\alpha(\omega_p)]^{1/2}),$$

where

$$\begin{aligned}
\rho(\omega_p) &:= \delta(\omega_p)(1+e^{-\alpha(\omega_p)})/(1-e^{-\alpha(\omega_p)}),\\
D_1(\omega_p) &:= [1-\delta(\omega_p)e^{-\alpha(\omega_p)}/(1-e^{-\alpha(\omega_p)-\tilde{\alpha}(\omega_p)})]^{-1},\\
D_2(\omega_p) &:= [1-\delta(\omega_p)e^{-\tilde{\beta}(\omega_p)}/(1-e^{-\alpha(\omega_p)-\tilde{\beta}(\omega_p)})]^{-1},\\
\tilde{\beta}(\omega_p) &:= \tilde{\alpha}(\omega_p)+\ln(1+2\delta(\omega_p)\sinh\alpha(\omega_p))
\end{aligned}$$

To prove Theorem 4.2.5 we first prove a lemma that provides a decomposition of the space X associated with the linear nonautonomous random dynamical system generated by the perturbed homogeneous problem (4.7).

Lemma 4.2.6. *Assume conditions of Theorem 4.2.2 are satisfied. Then, for each $\omega_p \in \mathbb{Z}\times\tilde{\Omega}$, X admits a decomposition*

$$X = V^+(\omega_p)\bigoplus V^-(\omega_p).$$

Furthermore, $\psi(n,\omega_p)V^+(\omega_p) \subset V^+(\Theta_n\omega_p)$, $\psi(n,\omega_p)V^-(\omega_p) = V^-(\Theta_n\omega_p)$ and $\psi(n,\omega_p)|_{V^-(\omega_p)}$ is an isomorphism for each $n \geq 0$, where (ψ,Θ) is the co-cycle associated with $A+B$ in problem (4.7).

Proof. Thanks to Theorem 4.2.2, there exists a Θ-invariant measurable map δ such that for each perturbation satisfying (4.8), the pair $(l^\infty(\mathbb{Z}),l^\infty(\mathbb{Z}))$ is admissible for (4.9), for each $\omega_p \in \mathbb{Z}\times\tilde{\Omega}$.

Hence, take (ψ,Θ) a co-cycle satisfying (4.11) and let $B(\omega_p) := \psi(1,\omega_p)-\varphi(1,\omega_p)$ be a linear bounded perturbation and consider problem (4.7). Also, for each ω_p we define the evolution process $\Psi_{\omega_p} := \{\psi_{n,m}(\omega_p) : n \geq m\}$ associated with (ψ,Θ).

Define the following sets

$$\begin{aligned} V^+(\omega_p) &:= \{z \in X\,;\, \sup_{n\in\mathbb{N}} \|\psi(n,\omega_p)z\|_{\mathscr{L}(X)} < +\infty\}, \\ V^-(\omega_p) &:= \{z \in X\,;\ \text{there is a backwards bounded solution for } \Psi_{\omega_p} \text{ through } z\}. \end{aligned}$$

These are the candidates to be the subspaces that provide the desired decomposition. We prove this fact in four steps.

Step 1: $V^+(\omega_p)$ and $V^-(\omega_p)$ are closed subspaces, for each $\omega_p \in \mathbb{Z}\times\tilde{\Omega}$.

First, note that as $\psi(n,\omega_p)$ is a bounded linear operator it follows that for each ω_p fixed we have that $V^+(\omega_p)$ is a subspace of X. We prove now that it is closed. Let $z \in V^+(\omega_p)$, then $x_n(\omega_p) := \psi(n,\omega_p)z$ is a bounded solution of (4.7) for $n\in\mathbb{N}$. Then $x(\cdot,\omega_p)$ satisfies

$$\begin{aligned} x_n(\omega_p) &= \varphi(n,\omega_p)\Pi^s(\omega_p)z + \sum_{k=0}^{n-1}\varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^s(\theta_{k+1}\omega_p)B(\Theta_k\omega_p)x_k(\omega_p) \\ &+ \sum_{k=n}^{+\infty}\varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^u(\theta_{k+1}\omega_p)B(\Theta_k\omega_p)x_k(\omega_p),\ \ n\in\mathbb{N}. \end{aligned}$$

Hence

$$\|x_n(\omega_p)\|_X \le K(\omega_p)e^{-\alpha(\omega_p)n}\|z\|_X + \delta(\omega_p)\sum_{k=0}^{+\infty} e^{-\alpha(\omega_p)|n-1-k|}\|x(k,\omega_p)\|_X,\ \ n\in\mathbb{N}.$$

Since $\delta(\omega_p) \le (1-e^{-\alpha(\omega_p)})/(1+e^{-\alpha(\omega_p)})$, by Lemma 4.2.4, we obtain

$$\|x_n(\omega_p)\|_X \le \frac{K(\omega_p)\|z\|_X}{1-\delta(\omega_p)e^{-\alpha(\omega_p)}/(1-e^{\alpha(\omega_p)+\tilde{\alpha}(\omega_p)})}e^{-\alpha(\omega_p)n},\ \ n\in\mathbb{N}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $V^+(\omega_p)$ such that $z_j \to z$ as $j\to+\infty$. Note that

$$\|\psi(n,\omega_p)z_j\|_X \le K(\omega_p)D_1(\omega_p)e^{-\alpha(\omega_p)n}\|z_j\|_X,\ \ n\in\mathbb{N},$$

Then

$$\|\psi(n,\omega)z\|_X \le D_1(\omega_p)K(\omega_p)e^{-\alpha(\omega_p)n}\|z\|_X,\ \text{for } n\in\mathbb{N},$$

and therefore $z\in V^+(\omega_p)$. Thus $V^+(\omega_p)$ is closed for each $\omega_p\in\mathbb{Z}\times\tilde{\Omega}$.

Now we show that $V^-(\omega_p)$ is a closed subspace. Note that, since $\psi(n,\omega_p)$ is a linear operator for all $n\in\mathbb{N}$, it follows that $V^-(\omega_p)$ is subspace of X. Let us prove that it is closed; it will be very similar to the proof for $V^+(\omega_p)$ and, for this reason, some steps will be omitted. We claim that each $z\in V^-(\omega_p)$ satisfies

$$\|\xi(n)\|_X \le D_2(\omega_p)K(\omega_p)e^{\tilde{\beta}(\omega_p)n}\|z\|_X,\ \ n\le 0, \tag{4.12}$$

where ξ is a backwards bounded solution of $\Psi_{\omega_p} := \{\psi(t-s,\Theta_s\omega_p)\}$ through z, D_2, and $\tilde{\beta}$ are in Theorem 4.2.5. Indeed, note that

$$\begin{aligned} \xi(n) &= \varphi(n,\omega_p)\Pi^u(\omega_p)z - \sum_{k=n}^{-1}\varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^u(\theta_{k+1}\omega_p)B(\Theta_k\omega_p)\xi(k) \\ &+ \sum_{k=-\infty}^{n-1}\varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^s(\theta_{k+1}\omega_p)B(\Theta_k\omega_p)\xi(k). \end{aligned}$$

Thus

$$\|\xi(n)\|_X \leq K(\omega_p)e^{\alpha(\omega_p)n}\|z\|_X + \delta(\omega_p)\sum_{k=-\infty}^{-1} e^{-\alpha(\omega_p)|n-1-k|}\|\xi(k)\|_X, \ \ n \leq 0,$$

and Lemma 4.2.4 implies (4.12). Then, for a sequence $\{z_j\}_{j\in\mathbb{Z}}$ in $V^-(\omega_p)$ such that $z_j \to z$ as $j \to +\infty$, we will show that $z \in V^-(\omega_p)$. In fact, for each $j \in \mathbb{N}$ there exists a backwards bounded solution ξ_j of Ψ_{ω_p} such that $\xi_j(0) = z_j$. Thanks to (4.12) it is possible to construct a backwards solution ξ through z, such that $\xi(n) = \lim_{j\to+\infty}\xi_j(n)$ for each fixed $n \leq 0$.

To see that $z \in V^-(\omega_p)$ we have to show that ξ is bounded. In fact, for all fixed $n \leq 0$ we have

$$\begin{aligned} \|\xi(n)\|_X &\leq \|\xi(n)-\xi_j(n)\|_X + \|\xi_j(n)\|_X \\ &\leq \|\xi(n)-\xi_j(n)\|_X + K(\omega_p)D_2(\omega_p)e^{\tilde{\beta}(\omega_p)n}\|z_j\|_X, \end{aligned}$$

which yields to

$$\|\xi(n)\|_X \leq K(\omega_p)D_2(\omega_p)e^{\tilde{\beta}(\omega_p)n}\|z\|_X, \ \ n \leq 0,$$

and concludes the proof of the first step.

Step 2: $V^+(\omega_p)$ is positively invariant and $V^-(\omega_p)$ is invariant.

Note that if $z \in V^+(\omega_p)$, then $\{\psi(k,\omega_p)z\}_{k\in\mathbb{N}}$ is a bounded sequence, then for a fixed $m \geq 0$, by the co-cycle property

$$\psi(k,\Theta_m\omega_p)\psi(m,\omega_p)z = \psi(m+k,\omega_p)z$$

is an element of the bounded sequence $\{\psi(k,\omega_p)z\}_{k\in\mathbb{N}}$ for each $k>0$, and therefore is also bounded. Thus

$$\psi(m,\omega_p)z \in V^+(\Theta_m\omega_p), \text{ and } \psi(m,\omega_p)V^+(\omega_p) \subset V^+(\Theta_m\omega_p).$$

Let us prove now that $\psi(k,\omega_p)V^-(\omega_p) \subset V^-(\Theta_k\omega_p)$, for all $k \geq 0$. Let $z \in V^-(\omega_p)$ and ξ a backwards bounded solution of Ψ_{ω_p} such that $\xi(0) = z$.

For fixed $k \geq 0$ define $\tilde{z} = \psi(k,\omega_p)z$ and

$$\tilde{\xi}(n) = \begin{cases} \psi(n,\omega_p)z, & \text{if } n \in \{0,1,\cdots,k\} \\ \xi(n), & \text{if } n \leq 0. \end{cases}$$

Notice that $\tilde{\xi}$ is a backwards bounded solution for $\Psi_{\Theta_k\omega_p}$ and $\tilde{\xi}(k) = \tilde{z}$, which implies that $\tilde{z} \in V^-(\Theta_k\omega_p)$.

Now, let $z \in V^-(\Theta_k\omega_p)$, then there is $\hat{\xi}$ a backwards bounded solution for $\Psi_{\theta_k\omega_p}$ such that $\hat{\xi}(0) = z$. In particular

$$z = \psi_{0,-k}(\Theta_k\omega_p)\hat{\xi}(-k) = \psi(k,\omega_p)\hat{\xi}(-k),$$

thus it is enough to prove that $\hat{\xi}(-k) \in V^-(\omega_p)$. Define $\xi(n) := \hat{\xi}(n-k)$ for $n \leq 0$. It is clear that ξ is a backwards bounded solution for Ψ_{ω_p} and through $\hat{\xi}(-k)$. By definition we conclude that $\hat{\xi}(-k) \in V^-(\omega_p)$ and Step 2 is complete.

Step 3: $X = V^+(\omega_p) \oplus V^-(\omega_p)$.

Let $z \in V^+(\omega_p) \cap V^-(\omega_p)$. Then by the definition of these subspaces we see that there is a bounded global solution ξ for Ψ_{ω_p} through z. On the other hand, applying Theorem 4.2.2 with $f = 0$, we know that the only complete bounded solution is the $\xi = 0$, thus by uniqueness we have that $\xi(n) = 0$, for all $n \in \mathbb{Z}$, in particular $z = 0$. Then $V^+(\omega_p) \cap V^-(\omega_p) = \{0\}$.

Let $z \in X$ and define $f_n = 0$ for all $n \neq -1$, and $f_{-1} = z/K(\omega_p)$. Thus by Theorem 4.2.2 there exists $x(\cdot, \omega_p) = \{x_n(\omega_p)\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ the solution of (4.7). This solution satisfies for $n \geq m$

$$x_n(\omega_p) = \psi_{n,m}(\omega_p)x_m(\omega_p) + \sum_{k=m}^{n-1} \psi(n-k-1, \Theta_{k+1}\omega_p)f_k.$$

Rewriting this we obtain

$$\begin{aligned} x_n(\omega_p) &= \psi(n, \omega_p)x_0(\omega_p), \quad n \geq 0, \\ x_0(\omega_p) &= \psi(1, \Theta_{-1}\omega_p)x_{-1}(\omega_p) + z/K(\omega_p), \\ x_n(\omega_p) &= \psi_{n,m}(\omega_p)x_m, \quad m \leq n \leq -1. \end{aligned}$$

Thus, since x is bounded, we see that $x_0(\omega_p) \in V^+(\omega_p)$.

Note that $\psi(1, \Theta_{-1}\omega_p)x_{-1}(\omega_p) \in V^-(\omega_p)$. In fact, define $\xi(0) := x_0(\omega_p) - z/K(\omega_p)$ and $\xi(n) := x_n(\omega_p)$ for $n \leq -1$. Then ξ is a backwards bounded solution through $\psi(1, \Theta_{-1}\omega_p)x_{-1}(\omega_p) = x_0(\omega_p) - z/K(\omega_p)$, which means that $x_0(\omega_p) - z/K(\omega_p) \in V^-(\omega_p)$.

Therefore,

$$z = x_0(\omega_p)K(\omega_p) - (x_0(\omega_p)K(\omega_p) - z) \in V^+(\omega_p) + V^-(\omega_p),$$

which completes the proof of Step 3.

Step 4: $\psi(m, \omega_p)|_{V^-(\omega_p)} : V^-(\omega_p) \to V^-(\Theta_m\omega_p)$ is an isomorphism, for $m \geq 0$.

By Step 2, we already have that $\psi(m, \omega_p)|_{V^-(\omega_p)}$ is surjective, so now we show that is injective. Given $z \in V^-(\omega_p)$, then from the proof of Step 2, we know that there exists a backwards bounded solution $\tilde{\xi}$ of $\Psi_{\Theta_m\omega_p}$ such that $\tilde{\xi}(0) = \psi(m, \omega_p)z$ and $\tilde{\xi}(-m) = z$. Thus, from (4.12)

$$\|\tilde{\xi}(n)\|_X \leq D_2(\omega_p)K(\Theta_m\omega_p)e^{\tilde{\beta}(\omega_p)m}\|\psi(m, \omega_p)z\|_X, \quad n \leq 0.$$

In particular,

$$\|z\|_X \leq D_2(\omega_p)K(\Theta_m\omega_p)e^{-\tilde{\beta}(\omega_p)m}\|\psi(m, \omega_p)z\|_X.$$

Hence, $\psi(m, \omega_p)|_{V^-(\omega_p)}$ is injective, and the proof of the lemma is complete. □

Now, we are ready to prove our result on the robustness of exponential dichotomies for discrete nonautonomous random dynamical systems.

Proof of Theorem 4.2.5. From Lemma 4.2.6, for each $\omega_p \in \mathbb{Z}\times\tilde{\Omega}$, we have $X = V^+(\omega_p)\oplus V^-(\omega_p)$, so define $\widetilde{\Pi}^s(\omega_p)$ as the projection from X onto $V^+(\omega_p)$, and $\widetilde{\Pi}^u(\omega_p) := Id_X - \widetilde{\Pi}^s(\omega_p)$. By the invariance of $V^-(\omega_p)$ and positively invariance of $V^+(\omega_p)$ we have that

$$\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p) = \widetilde{\Pi}^s(\Theta_n\omega_p)\psi(n,\omega_p), \text{ for all } n\in\mathbb{N}.$$

Again, by Lemma 4.2.6, $\psi(n,\omega_p)|_{R(\widetilde{\Pi}^u(\omega_p))}$ is an isomorphism and we define $\psi(-n,\Theta_n\omega_p)$ as the inverse of the map $\psi(n,\omega_p)|_{V^-(\omega_p)}$.

Let $z\in X$ and define $f_n = 0$ for all $n\neq -1$, and $f_{-1} = z/K(\omega_p)$. Then Equation (4.9) has a unique bounded solution $x(\cdot,\omega_p)$, with $x_n(\omega_p)\in V^+(\Theta_n\omega_p)$ for all $n\in\mathbb{N}$, $x_n(\omega_p)\in V^-(\Theta_n\omega_p)$ for all $n\leq -1$, and $x_0(\omega_p) - z/K(\omega_p)\in V^-(\omega_p)$. Note that

$$x_n(\omega_p) = \frac{1}{K(\omega_p)}\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z, \text{ for all } n\in\mathbb{N}, \tag{4.13}$$

and

$$x_m(\omega_p) = \frac{-1}{K(\omega_p)}\psi(m,\omega_p)\widetilde{\Pi}^u(\omega_p)z, \text{ for all } m\leq -1. \tag{4.14}$$

Now, since $x(\cdot,\omega_p)$ is a solution of $x_{n+1} = A(\Theta_n\omega_p)x_n + B(\Theta_n\omega_p)x_n + f_n$, for all $n\in\mathbb{Z}$ we have that

$$x_n(\omega_p) = G_{\omega_p}(n,0)f_{-1} + \sum_{-\infty}^{+\infty} G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)x_k(\omega_p),$$

where G is the Green's function associated with the co-cycle (φ,Θ) and family of projections Π^s. Hence,

$$\begin{aligned}\|x_n(\omega_p)\|_X &\leq e^{-\alpha(\omega_p)|n|}\|z\|_X + \sum_{-\infty}^{+\infty} K(\omega_p)e^{-\alpha(\omega_p)|n-k-1|}\|B(\Theta_k\omega_p)\|_{\mathscr{L}(X)}\|x_k(\omega_p)\|_X \\ &\leq e^{-\alpha(\omega_p)|n|}\|z\|_X + \|x(\cdot,\omega_p)\|_{l^\infty(\mathbb{Z})}\delta(\omega_p)\frac{1+e^{-\alpha(\omega_p)}}{1-e^{-\alpha(\omega_p)}}.\end{aligned}$$

Thus,

$$\|x(\cdot,\omega_p)\|_{l^\infty(\mathbb{Z})} \leq \frac{\|z\|_X}{1-\rho(\omega_p)},$$

where $\rho(\omega_p) = \delta(\omega_p)(1+e^{-\alpha(\omega_p)})/(1-e^{-\alpha(\omega_p)})$. In particular, from (4.13)

$$\|\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z\|_X \leq \frac{K(\omega_p)}{1-\rho(\omega_p)}\|z\|_X,\ n\geq 0, \tag{4.15}$$

and from (4.14)

$$\|\psi(m,\omega_p)\widetilde{\Pi}^u(\omega_p)z\|_X \leq \frac{K(\omega_p)}{1-\rho(\omega_p)}\|z\|_X,\ m\leq -1. \tag{4.16}$$

Now, we use (4.16) to obtain a better estimate than (4.15) for the norm of $\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)$. Since $\widetilde{\Pi}^s(\omega_p)z \in V^+(\omega_p)$, we know that $\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z$ defines a bounded solution for

$$x_{n+1}(\omega_p) = A(\Theta_n\omega_p)x_n(\omega_p) + B(\Theta_n\omega_p)x_n(\omega_p), \quad n \geq 0,$$

and since $\widetilde{\Pi}^u(\omega_p)z \in V^-(\omega_p)$, there exists $\xi : (-\infty,0]\cap\mathbb{Z} \to X$ backwards bounded solution of Ψ_{ω_p} such that $\xi(0) = \widetilde{\Pi}^u(\omega_p)z$. In particular, $\xi(n) := \psi(n,\omega_p)\widetilde{\Pi}^u(\omega_p)z$, for $n \leq -1$. Hence

$$\begin{aligned}\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z &= \varphi(n,\omega_p)\Pi^s(\omega_p)\widetilde{\Pi}^s(\omega_p)z \\ &+ \sum_{k=0}^{+\infty} G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^s(\omega_p)z, \; n \geq 0,\end{aligned}$$

and

$$\begin{aligned}\psi(n,\omega_p)\widetilde{\Pi}^u(\omega_p)z &= \varphi(n,\omega_p)\Pi^u(\omega_p)\widetilde{\Pi}^u(\omega_p)z \\ &+ \sum_{k=-\infty}^{-1} G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^u(\omega_p)z, \; n \leq 0.\end{aligned}$$

For $n = 0$ we obtain

$$\widetilde{\Pi}^u(\omega_p)z = \Pi^u(\omega_p)\widetilde{\Pi}^u(\omega_p)z \tag{4.17}$$

$$+ \sum_{k=-\infty}^{-1} \varphi(-k-1,\Theta_{k+1}\omega_p)\Pi^s(\Theta_{k+1}\omega_p)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^u(\omega_p)z. \tag{4.18}$$

Thus

$$\begin{aligned}\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z &= \varphi(n,\omega_p)\Pi^s(\omega_p)z - \varphi(n,\omega_p)\Pi^s(\omega_p)\widetilde{\Pi}^u(\omega_p)z \\ &+ \sum_{k=0}^{+\infty} G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^s(\omega_p)z, \; n \geq 0,\end{aligned}$$

hence, from (4.17)

$$\begin{aligned}&\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)z = \varphi(n,\omega_p)\Pi^s(\omega_p)z \\ &- \sum_{k=-\infty}^{-1} \varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^s(\Theta_{k+1}\omega_p)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^u(\omega_p)z \\ &+ \sum_{k=0}^{+\infty} G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^s(\omega_p)z, \; n \geq 0.\end{aligned}$$

Thus, from (4.16)

$$\begin{aligned}\|\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)\|_{\mathscr{L}(X)} &\leq K(\omega_p)\left(1 + \frac{\delta(\omega_p)}{(1-\rho(\omega_p))(1-e^{\alpha(\omega_p)})}\right)e^{-\alpha(\omega_p)n} \\ &+ \delta(\omega_p)\sum_{k=0}^{+\infty} e^{-\alpha(\omega_p)|n-k-1|}\|\psi(k,\omega_p)\widetilde{\Pi}^s(\omega_p)\|_{\mathscr{L}(X)}, \; n \geq 0.\end{aligned}$$

Again, from (4.16) we have that

$$\|\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)\|_{\mathscr{L}(X)} \leq K_1(\omega_p) := 1 + K(\omega_p)/(1-\rho(\omega_p)), \; \forall n \geq 0.$$

Hence, $u_n := K_1(\omega_p)^{-1}\|\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)\|_{\mathscr{L}(X)}$ is uniformly bounded for each $n \geq 0$ and ω_p. Thus,

$$u_n \leq \frac{K(\omega_p)}{K_1(\omega_p)}\left[1+\frac{\delta(\omega_p)}{(1-\rho(\omega_p))(1-e^{-\alpha(\omega_p)})}\right]e^{-\alpha(\omega_p)n}+\delta(\omega_p)\sum_{k=0}^{+\infty}e^{-\alpha(\omega_p)|n-k-1|}u_k.$$

Then, by Lemma 4.2.4

$$\|\psi(n,\omega_p)\widetilde{\Pi}^s(\omega_p)\|_{\mathscr{L}(X)} \leq \frac{K(\omega_p)\left[1+\frac{\delta(\omega_p)}{(1-\rho(\omega_p))(1-e^{-\alpha(\omega_p)})}\right]}{1-\delta(\omega_p)e^{-\alpha(\omega_p)}/(1-e^{-\alpha(\omega_p)-\tilde{\alpha}(\omega_p)})}e^{-\tilde{\alpha}(\omega_p)n},\ n\geq 0.$$

Similarly, we use (4.15) and

$$\begin{aligned}&\psi(n,\omega_p)\widetilde{\Pi}^u(\omega_p)z = \varphi(n,\omega_p)\Pi^u(\omega_p)z\\&+\sum_{k=0}^{+\infty}\varphi(n-k-1,\Theta_{k+1}\omega_p)\Pi^u(\Theta_{k+1}\omega_p)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^s(\omega_p)z\\&+\sum_{k=-\infty}^{-1}G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\psi(k,\omega_p)\widetilde{\Pi}^u(\omega_p)z,\ n\leq 0\end{aligned}$$

to obtain

$$\begin{aligned}&\|\psi(n,\omega_p)\widetilde{\Pi}^u(\omega_p)\|_{\mathscr{L}(X)} \leq K(\omega_p)\left(1+\frac{\delta(\omega_p)e^{-\alpha(\omega_p)}}{(1-\rho(\omega_p))(1-e^{\alpha(\omega_p)})}\right)e^{\alpha(\omega_p)n}\\&+\delta(\omega_p)\sum_{k=-\infty}^{-1}e^{-\alpha(\omega_p)|n-k-1|}\|\psi(k,\omega_p)\widetilde{\Pi}^u(\omega_p)\|_{\mathscr{L}(X)},\ n\leq 0;\end{aligned}$$

and from Lemma 4.2.4

$$\|\psi(n,\omega_p)\widetilde{\Pi}^u(\omega_p)\|_{\mathscr{L}(X)} \leq \frac{K(\omega_p)\left[1+\frac{\delta(\omega_p)e^{-\alpha(\omega_p)}}{(1-\rho(\omega_p))(1-e^{-\alpha(\omega_p)})}\right]}{1-\delta(\omega_p)e^{-\tilde{\beta}(\omega_p)}/(1-e^{-\alpha(\omega_p)-\tilde{\beta}(\omega_p)})}e^{\tilde{\beta}(\omega_p)n},\ n\leq 0.$$

Finally, we prove that, for each $p\in\mathbb{Z}$, the operator $\Pi^s(p,\cdot):\Omega\to\mathscr{L}(X)$ is strongly measurable. Note that

$$\widetilde{G}_{\omega_p}(n,0)z = G_{\omega_p}(n,0)z+\sum_{k=-\infty}^{+\infty}G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)\tilde{G}_{\omega_p}(k,0)z, \forall n\in\mathbb{Z},\ z\in X, \tag{4.19}$$

where $\widetilde{G}$ is the Green's function of (ψ,Θ) and family of projection $\widetilde{\Pi}^s$. Let $T(\omega,p,z)$ be the operator defined on $l^\infty(\mathbb{Z})$ by

$$T(\omega,p,z)g_n := G_{\omega_p}(n,0)z+\sum_{k=-\infty}^{+\infty}G_{\omega_p}(n,k+1)B(\Theta_k\omega_p)g_k.$$

Note that $T(\omega,p,z)$ has a unique fixed point $g\in l^\infty(\mathbb{Z})$, and from (4.19) we know that $g=\{\widetilde{G}_{\omega_p}(n,0)z\}_{n\in\mathbb{Z}}$. Moreover, since $T(\cdot,p,z)$ is $(\mathscr{F},\mathscr{B}(X))$-measurable and $\{\widetilde{G}_{\omega_p}(n,0)z\}_{n\in\mathbb{Z}}$ is a limit of the iteration of $T(\cdot,p,z)$ starting at $0 \in l^\infty(\mathbb{Z})$ we conclude that $\{\widetilde{G}_{(p,\cdot)}(n,0)z\}_{n\in\mathbb{Z}}$ is a $(\mathscr{F},\mathscr{B}(X))$-measurable sequence, which implies that $\Pi^s(p,\cdot)$ is strongly measurable.

Therefore (ψ,Θ) admits an exponential dichotomy on a Θ-invariant subset $\tilde{\Omega}$ of full measure with projection $\widetilde{\Pi}^s$, exponent $\tilde{\alpha}$, and bound $\tilde{K}$. □

Our proof of Theorem 4.2.5 follows the same line of arguments of Theorem 1 presented in (ZHOU; LU; ZHANG, 2013). One of the differences is that we use the evolution processes Φ_{ω_p} associated with a given co-cycle φ, see Remark 4.1.4. This simple fact provides fundamental ideas for the proof on the continuous case (Subsection 4.3), and makes the writing much simpler in the discrete case.

Remark 4.2.7. *Actually, the proof of (ZHOU; LU; ZHANG, 2013, Theorem 1) works for any co-cycle defined of a non-compact symbol space. In fact, let φ be a linear co-cycle driving by a flow $\Sigma\times\Omega \ni (\sigma,\omega)\mapsto \Theta_t(\sigma,\omega) = (\theta_t^1\sigma,\theta_t^2\omega)$, where $\theta_t^1:\Sigma\to\Sigma$ is a flow in a metric space Σ and $\theta_t^2:\Omega\to\Omega$ is a random flow, and $t\in\mathbb{Z}$. Then, following the ideas of (ZHOU; LU; ZHANG, 2013), it is possible to provide a suitable definition of tempered exponential dichotomy for a general linear co-cycle $(\varphi,\Theta)_{(X,\Sigma\times\Omega)}$ and to prove a robustness result for it.*

In our case, we choose to deal with the case where the bound K is Θ-invariant because we want to understand the effect of a bounded noise on the hyperbolicity of an autonomous problem, and therefore it is not expected to obtain tempered exponential dichotomies.

Remark 4.2.8. *We provide explicit expressions for the bound $\tilde{K}$ and exponent $\tilde{\alpha}$ for the obtained exponential dichotomy. This can also be done in the continuous case.*

Now, we prove a continuous dependence result of the projections associated with exponential dichotomy for co-cycle.

Theorem 4.2.9 (Continuous dependence of projections)**.** *Suppose that (φ,Θ) and (ψ,Θ) are nonautonomous random dynamical systems and that admit an exponential dichotomy with projections Π^s_φ and Π^s_ψ, exponents α_φ and α_ψ, respectively, with the same bound K. If*

$$\sup_{n\in\mathbb{Z}}\{K(\omega_p)\|\varphi_n(\omega_p)-\psi_n(\omega_p)\|_{\mathscr{L}(X)}\}\leq\varepsilon,$$

then

$$\sup_{n\in\mathbb{Z}}\|\Pi^s_\varphi(\Theta_n\omega_p)-\Pi^s_\psi(\Theta_n\omega_p)\|_{\mathscr{L}(X)}\leq\frac{e^{-\alpha_\psi(\omega_p)}+e^{-\alpha_\varphi(\omega_p)}}{1-e^{-(\alpha_\psi(\omega_p)+\alpha_\varphi(\omega_p))}}\varepsilon.$$

Proof. Let $z\in X$, $\omega\in\tilde{\Omega}$, and $m,p\in\mathbb{Z}$ be fixed and consider

$$f_n(\omega_p)=\begin{cases}0, & \text{if } n\neq m-1,\\ z, & \text{if } n=m-1.\end{cases}$$

Thus by Theorem 4.2.2 for each $\omega_p\in\mathbb{Z}\times\tilde{\Omega}$ there exists a bounded solution $x(\omega_p)=\{x_n(\omega_p)\}_{n\in\mathbb{Z}}$ given by $x_n^j(\omega_p):=G^j_{\omega_p}(n,m)z^{-1}$ for $j=\varphi,\psi$. Note that

$$x^\varphi_{n+1}-\psi_n(\omega_p)x^\varphi_n=\varphi_n(\omega_p)x^\varphi_n-\psi_n(\omega_p)x^\varphi_n+f_n(\omega_p)$$

and $x_{n+1}^{\psi} - \psi_n(\omega_p)x_n^{\psi} = f_n(\omega_p)$. Then, if $z_n := x_n^{\varphi} - x_n^{\psi}$ we obtain that $z_{n+1} = \varphi_n z_n + y_n$, where $y_n := (\varphi_n(\omega_p) - \psi_n(\omega_p)x_n^{\varphi}(\omega_p)$ for all $n \in \mathbb{Z}$. Thanks to the boundedness of the sequence $\{x_n^{\varphi}(\omega_p)\}_{n\in\mathbb{Z}}$ and by the hypotheses on $\varphi_n - \psi_n$ we have that $\{y_n\}_{n\in\mathbb{Z}}$ is bounded and by Theorem 4.2.2 we have that

$$z_n = \sum_{k=-\infty}^{\infty} G_{\omega_p}^{\psi}(n,k+1)(\omega_p)(\varphi_k(\omega_p) - \psi_k(\omega_p))G_{\omega_p}^{\varphi}(k,m)z$$

and therefore, by the hypotheses on $\Psi - \Phi$, we deduce

$$\begin{aligned} \|z_m\|_X &\leq \sum_{k=-\infty}^{\infty} K(\omega_p)e^{-\alpha_\psi(\omega_p)|m-k-1|}e^{-\alpha_\psi(\omega_p)|k-m|}\|\varphi_k(\omega_p) - \psi_k(\omega_p)\|_{\mathscr{L}(X)}\|z\|_X \\ &\leq \frac{e^{-\alpha_\psi(\omega_p)} + e^{-\alpha_\varphi(\omega_p)}}{1 - e^{-(\alpha_\psi(\omega_p)+\alpha_\varphi(\omega_p))}}\,\varepsilon\,\|z\|_X. \end{aligned}$$

The definition of z in m yields

$$z_m = x_m^{\varphi} - x_m^{\psi} = (G_{\omega_p}^{\varphi}(m,m) - G_{\omega_p}^{\psi}(m,m))z = (\Pi_\varphi^s(\Theta_m\omega_p) - \Pi_\psi^s(\Theta_m\omega_p))z.$$

Consequently,

$$\|(\Pi_\varphi^s(\Theta_m\omega_p) - \Pi_\psi^s(\Theta_m\omega_p))z\|_X \leq \frac{e^{-\alpha_\psi(\omega_p)} + e^{-\alpha_\varphi(\omega_p)}}{1 - e^{-(\alpha_\psi(\omega_p)+\alpha_\varphi(\omega_p))}}\,\varepsilon\,\|z\|_X,$$

which concludes the proof of the theorem. □

4.3 Exponential dichotomy: continuous case

We study exponential dichotomies for a continuous nonautonomous random dynamical system (φ,Θ). Our goal is to prove a robustness result for nonautonomous random dynamical systems that possesses a uniform exponential dichotomy. This section follows closely the ideas of (CHOW; LEIVA, 1995b). However, while they consider a driving flow $\theta_t : \Sigma \to \Sigma$ on a compact Hausdorff space Σ in a deterministic context, we deal with a nonautonomous random dynamical systems driven by a flow $\mathbb{R}\times\Omega \ni (\tau,\omega) \mapsto \Theta_t(\tau,\omega) = (t+\tau,\theta_t\omega) \in \mathbb{R}\times\Omega$, where $\theta_t : \Omega \to \Omega$ is a random flow defined on a probability space Ω.

We first prove results to compare existence of exponential dichotomies between continuous and discrete nonautonomous random dynamical systems (Theorem 4.3.1 and Theorem 2.2.10). As applications of these results we obtain a robustness result of exponential dichotomies for continuous nonautonomous random co-cycles (Theorem 4.3.4), and uniqueness and continuous dependence of the family of projections associated with the exponential dichotomy (Corollary 4.3.2 and Theorem 4.3.6, respectively).

We first prove that if a co-cycle possesses an exponential dichotomy, then its discretization also admits an exponential dichotomy.

Theorem 4.3.1. *Let $(\varphi,\Theta)_{(X,\mathbb{R}\times\Omega)}$ be a linear co-cycle that admits an exponential dichotomy with bound K, exponent α and family of projections $\Pi^u := \{\Pi^u(\omega_p) : \omega_p \in \mathbb{R}\times\widetilde{\Omega}\}$, where $\mathbb{P}(\widetilde{\Omega}) = 1$ and $\widetilde{\Omega}$ is θ-invariant. Then for each $\omega_p \in \mathbb{R}\times\widetilde{\Omega}$ the sequence of linear operators $\{\varphi_n(\omega_p) := \varphi(1,\Theta_n\omega_p); n \in \mathbb{Z}\}$ admits an exponential dichotomy with bound $K(\omega_p)$ and exponent $\alpha(\omega_p)$.*

Proof. Let Π^s be a family of projections associated with the exponential dichotomy. Define, for each $n \in \mathbb{Z}$ and ω_p, the projector $P_n(\omega_p) := \Pi^s(\Theta_n\omega_p)$. Then

$$\begin{aligned} P_{n+1}(\omega_p)\varphi_n(\omega_p) &= \Pi^s(\Theta_{n+1}(\omega_p))\varphi(1,\Theta_n(\omega_p)) \\ &= \varphi(1,\Theta_n(\omega_p))\Pi^s(\Theta_n(\omega_p)) \\ &= \varphi_n(\omega_p)P_n(\omega_p), \end{aligned}$$

and the first property is proved. Note that, if $Q = Id_X - P$, we have that

$$\varphi_n(\omega_p)|_{R(Q_n(\omega_p)))} = \varphi(1,\Theta_n(\omega_p))|_{R(\Pi^u(\Theta_n(\omega_p)))}$$

is an isomorphism onto $R(Q_{n+1}(\omega_p))$.

Finally, for $n \geq m$ we see that

$$\|\varphi_{n,m}(\omega_p)P_m(\omega_p)\|_{\mathscr{L}(X)} \leq K(\Theta_m\omega_p)e^{-\alpha(\omega_p)(n-m)},$$

and for $n < m$

$$\|\varphi_{n,m}(\omega_p)Q_m(\omega_p)\|_{\mathscr{L}(X)} \leq K(\Theta_m\omega_p)e^{\alpha(\omega_p)(n-m)},$$

where $\varphi_{n,m}(\omega_p)$ is the inverse of $\varphi_{m,n}(\omega_p)$ over $R(Q_m(\omega_p))$, and the proof follows by the Θ-invariance property of K. □

As a corollary of Theorem 4.3.1 and Corollary 4.2.3 we obtain uniqueness of projectors for the continuous case.

Corollary 4.3.2. *If (φ,Θ) admits an exponential dichotomy, then the family of projections are uniquely determined.*

Now, we provide conditions to prove a kind of converse result of Theorem 4.3.1. If the discretization admits an exponential dichotomy then the continuous co-cycle also possesses it.

Theorem 4.3.3. *Let $(\varphi,\Theta)_{(X,\mathbb{R}\times\Omega)}$ be a co-cycle and for each ω_p consider the associated sequence of operators*

$$\{\varphi_n(\omega_p) := \varphi(1,\Theta_n\omega_p)\}_{n\in\mathbb{Z}}.$$

Suppose that there is a full measure set $\tilde{\Omega}$ such that for each $\omega_p \in \mathbb{R}\times\tilde{\Omega}$

- *we have that*

$$L(\omega_p) := \sup_{0\leq t\leq 1} \|\varphi(t,\omega_p)\|_{\mathscr{L}(X)} < +\infty,$$

satisfies $L(\Theta_t\omega_p) \leq L(\omega_p)$*, for all* $t \in \mathbb{R}$.

- *there exists* Θ*-invariant maps* K, α *such that the sequence* $\{\varphi_n(\omega_p)\}_{n\in\mathbb{Z}}$ *admits an exponential dichotomy with bound* $K(\omega_p)$*, exponent* $\alpha(\omega_p)$*, and family of projections* $\{P_n(\omega_p) : n \in \mathbb{Z}\}$ *such that for each* $(n,p) \in \mathbb{Z}\times\mathbb{R}$ *the map* $P_n(p,\cdot) : \tilde{\Omega} \to \mathscr{L}(X)$ *is strongly measurable.*

Then (φ,Θ) *admits an exponential dichotomy with exponent* α*, and bound*

$$\hat{K}(\omega_p) = K(\omega_p) \sup_{0\leq t\leq 1} \{\|\varphi(t,\omega_p)\|_{\mathscr{L}(X)} e^{\alpha(\omega_p)t}\}.$$

Proof. Let $\{P_n(\omega_p); n\in\mathbb{Z}\}$ be the family of projectors associated with the exponential dichotomy of $\{\varphi_n(\omega_p)\}_{n\in\mathbb{Z}}$. Define $\Pi^s : \mathbb{R}\times\tilde{\Omega} \to \mathscr{L}(X)$ by

$$\Pi^s(\omega_p) := P_0(\omega_p).$$

Thus for each $p \in \mathbb{R}$ fixed $\Pi^s(p,\cdot) : \tilde{\Omega} \to \mathscr{L}(X)$ is strongly measurable.

Claim 1: For each $k \in \mathbb{Z}$ fixed, we have that $P_k(\omega_p) = \Pi^s(\Theta_k\omega_p)$.

Indeed, for each $k \in \mathbb{Z}$ fixed the sequence $\{\varphi_n(\Theta_k\omega_p)\}_{n\in\mathbb{Z}}$ admits an exponential dichotomy with projections $\{P_n(\Theta_k\omega_p); n \in \mathbb{Z}\}$. Note that,

$$\varphi_n(\Theta_k\omega_p) = \varphi(1,\Theta_n(\Theta_k\omega_p)) = \varphi_{n+k}(\omega_p).$$

Then, from Lemma 4.2.3 we have that $P_n(\Theta_k\omega_p) = P_{n+k}(\omega_p)$ for all $n,k \in \mathbb{Z}$. In particular, $P_k(\omega_p) = \Pi^s(\Theta_k\omega_p)$.

Next, we prove that this projector operator is the candidate to obtain the exponential dichotomy.

Claim 2: For all $t \geq 0$ and $\omega_p \in \mathbb{Z}\times\tilde{\Omega}$, we have that

$$\|\varphi(t,\omega_p)\Pi^s(\omega_p)\|_{\mathscr{L}(X)} \leq \hat{K}(\omega_p)e^{-\alpha(\omega_p)t},$$

where $\hat{K}(\omega_p) = K(\omega_p)\sup_{0\leq t\leq 1}\{e^{\alpha(\omega_p)t}\|\varphi(t,\omega_p)\|_{\mathscr{L}(X)}\}$.

Indeed, choose $n \in \mathbb{N}$, such that $n \leq t < n+1$, then we write

$$\varphi(t,\omega_p) = \varphi(t-n,\Theta_n\omega_p)\varphi(n,\omega_p).$$

Therefore

$$\begin{aligned} \|\varphi(t,\omega_p)\Pi^s(\omega_p)\|_{\mathscr{L}(X)} &\leq K(\omega_p)e^{-\alpha(\omega_p)n}\|\varphi(t-n,\Theta_n\omega_p)\|_{\mathscr{L}(X)} \\ &\leq \hat{K}(\omega_p)e^{-\alpha(\omega_p)t}. \end{aligned}$$

Claim 3: Let $x \in R(\Pi^u(\omega_p))$, $t < 0$ and choose $n \leq 0$ such that $n \leq t < n+1$. Define the linear operator

$$\varphi(t,\omega_p)x := \varphi(t-n,\Theta_n\omega_p)\varphi(n,\omega_p)x,$$

where $\varphi(n,\omega_p)$ is the inverse of $\varphi(-n,\Theta_n\omega_p)|_{R(\Pi^u(\Theta_n\omega_p))}$. Then for all $t \leq 0$

$$\|\varphi(t,\omega_p)\Pi^u(\omega_p)\|_{\mathscr{L}(X)} \leq \hat{K}(\omega_p)e^{\alpha(\omega_p)t}.$$

The proof of Claim 3 follows by a similar argument used on the proof of Claim 2.

Claim 4: The range of $\Pi^s(\omega_p)$ is characterized as follows

$$R(\Pi^s(\omega_p)) = \{z \in X\,; [0,+\infty) \ni t \mapsto \varphi(t,\omega_p)z \text{ is bounded }\}.$$

Indeed, if $x \in R(\Pi^s(\omega_p))$ from Claim 2 we have that

$$\|\varphi(t,\omega_p)x\|_X \leq \hat{K}(\omega_p)e^{-\alpha(\omega_p)t}\|x\|_X, \text{ for every } t \geq 0.$$

Thus, it follows that $[0,+\infty) \ni t \mapsto \varphi(t,\omega_p)x$ is bounded. Conversely, suppose that $x \notin R(\Pi^s(\omega_p))$ and define $v = \varphi(n,\omega_p)\Pi^u(\omega_p)x$,, hence

$$\|\varphi(-n,\Theta_n\omega_p)\Pi^u(\Theta_n\omega_p)v\|_X \leq K(\Theta_n\omega_p)e^{-\alpha(\omega_p)n}\|v\|_X,$$

for $n \geq 0$. Thus we obtain

$$\|\Pi^u(\omega_p)x\|_X \leq K(\omega_p)e^{\alpha(\omega_p)n}\|\varphi(n,\omega_p)\Pi^u(\omega_p)x\|_X, \text{ for } n \geq 0.$$

Since $\Pi^u(\omega_p)x \neq 0$ and we obtain that $[0,+\infty) \ni n \mapsto \varphi(n,\omega_p)\Pi^u(\omega_p)x$ is unbounded, then the mapping $[0,+\infty) \ni t \mapsto \varphi(n,\omega_p)x$ is unbounded and complete the proof of Claim 4.

Claim 5: The range of $\Pi^u(\omega_p)$ is characterized as follows: $z \in \Pi^u(\omega_p)$ if and only if there exists a backwards bounded solution ξ^* for the evolution process $\Phi_{\omega_p} = \{\varphi_{t,s}(\omega_p) : t \geq s\}$ such that $\xi^*(0) = z$.

In fact, let $z \in R(\Pi^u(\omega_p))$, and $t < 0$, we define

$$\xi(t) = \varphi(t-n,\Theta_n\omega_p)\varphi(n,\omega_p)z,$$

where $n \leq 0$ is such that $n \leq t < n+1$. Then ξ is a backwards bounded solution for the evolution process $\{\varphi_{t,s}(\omega_p) : t \geq s\}$ through z. In fact, for $t \geq s$

$$\begin{aligned}\varphi_{t,s}(\omega_p)\xi(s) &= \varphi(t-s,\Theta_s\omega_p)\varphi(s-n,\Theta_n\omega_p)\varphi(n,\omega_p)z\\ &= \varphi(t-n,\Theta_n\omega_p)z = \xi(t),\end{aligned}$$

which shows that ξ is a backwards solution. From Claim 3 we have that ξ is bounded and $\xi(0) = z$. Conversely, choose $x \in X$ such that $x \notin R(\Pi^u(\omega_p))$ and suppose that there exists a backwards solution of $\{\varphi_{t,s}(\omega_p) : t \geq s\}$ through x on $t = 0$. Then, for $n \leq 0$,

$$\begin{aligned}\|\Pi^s(\omega_p)x\|_X &= \|\Pi^s(\omega_p)\varphi(-n,\Theta_n\omega_p)\xi(n)\|_X\\ &\leq \|\varphi(-n,\Theta_n\omega_p)\Pi^s(\Theta_n\omega_p)\|_{\mathscr{L}(X)}\|\xi(n)\|_X\\ &\leq K(\omega_p)e^{\alpha(\omega_p)n}\|\xi(n)\|_X.\end{aligned}$$

Since $\Pi^s(\omega_p)x \neq 0$ we see that ξ is unbounded, and the proof is complete.

Now, from Claim 4 and Claim 5, we obtain that:

Claim 6: $R(\Pi^s(\cdot))$ is positively invariant and $R(\Pi^u(\omega_p)$ is invariant, i.e.,

$$\begin{aligned}\varphi(t,\omega_p)R(\Pi^s(\omega_p)) &\subset R(\Pi^s(\Theta_t\omega_p)),\ \text{for all } t\geq 0,\ \text{and}\\ \varphi(t,\omega_p)R(\Pi^u(\omega_p)) &= R(\Pi^u(\Theta_t\omega_p)),\ \text{for all } t\geq 0.\end{aligned}$$

Claim 7: The linear operator $\varphi(t,\omega_p) : R(\Pi^u(\omega_p)) \to X$ is injective.

Indeed, let $z \in R(\Pi^u(\omega_p)$ such that $\varphi(t,\omega_p)z = 0$. Choose $n \in \mathbb{N}$ such that $n \leq t \leq n+1$, then

$$0 = \varphi(n-t,\Theta_t\omega_p)\varphi(t,\omega_p)z = \varphi(n,\omega_p)z.$$

Now, Claim 7 follows by the fact that $\varphi(n,\omega_p)|_{R(\Pi^u(\omega_p))}$ is injective for any integer $n \leq 0$.

Then it follows directly from Claims 6 and 7 that $\varphi(t,\omega_p) : R(\Pi^u(\omega_p)) \to R(\Pi^u(\Theta_t\omega_p))$ is an isomorphism.

Finally, from Claim 6, we obtain that $\varphi(t,\omega_p)\Pi^s(\omega_p) = \Pi^s(\Theta_t\omega_p)\varphi(t,\omega_p)$ for all $t \geq 0$, and the proof of the theorem is complete. □

Now, we state our robustness result for nonautonomous random dynamical systems with an exponential dichotomy.

Theorem 4.3.4. *Let (φ,Θ) be an co-cycle with an exponential dichotomy with bound K and exponent α. Assume that there is a random variable $L : \mathbb{R}\times\Omega \to (0,+\infty)$ such that*

$$L(\omega_p) := \sup_{0\leq t\leq 1}\left\{\|\varphi(t,\omega_p)\|_{\mathscr{L}(X)}\right\} < +\infty,$$

that satisfies $L(\Theta_t\omega_p) \leq L(\omega_p)$, for all $t \in \mathbb{R}$. Then there exists a Θ-invariant map $\delta : \mathbb{Z}\times\Omega \to \mathbb{R}$ with

$$0 < \delta(\omega_p) < \frac{1-e^{-\alpha(\omega_p)}}{1+e^{-\alpha(\omega_p)}},\ \textit{for each } \omega_p \in \mathbb{R}\times\Omega,$$

such that every co-cycle (ψ,Θ) satisfying

$$\sup_{0\leq t\leq 1}\left\{\|\varphi(t,\omega_p)-\psi(t,\omega_p)\|_{\mathscr{L}(X)}\right\} \leq \delta(\omega_p),$$

admits an exponential dichotomy with exponent $\tilde{\alpha}(\omega_p)$ and bound

$$\hat{M}(\omega_p) = M(\omega_p)\sup_{0\leq t\leq 1}\left\{\|\psi(t,\omega_p)\|_{\mathscr{L}(X)}e^{\tilde{\alpha}(\omega_p)t}\right\},$$

where M and $\tilde{\alpha}$ are the bound and exponent of the discretization of (ψ,Θ) given in Theorem 4.2.5.

Proof. First, we consider the discretization of the co-cycle (φ,Θ), i.e., for each ω_p we consider the family of linear operators $\{\varphi_n(\omega_p) := \varphi(1,\omega_p) : n \in \mathbb{Z}\}$. From Theorem 4.3.1, we have that $\{\varphi_n(\omega_p) : n \in \mathbb{Z}\}$ admits an exponential dichotomy with bound $K(\omega_p)$ and exponent $\alpha(\omega_p)$. By Theorem 4.2.5, there exists a Θ-invariant map δ such that if $\{\psi_n(\omega_p)\}_{n\in\mathbb{Z}}$ is a sequence of bounded linear operators which satisfies

$$\sup_{n\in\mathbb{Z}} \|\varphi_n(\omega_p) - \psi_n(\omega_p)\|_{\mathscr{L}(X)} \leq \delta(\Theta_n \omega_p) = \delta(\omega_p),$$

$\{\psi_n(\omega_p)\}_{n\in\mathbb{Z}}$ admits an exponential dichotomy with bound $M(\omega_p)$ and exponent $\tilde{\alpha}(\omega_p)$ (see Theorem 4.2.5). Now, in order to use Theorem 4.3.3 to guarantee that (ψ,Θ) admits an exponential dichotomy it remains only to see that

$$\sup_{0\leq t\leq 1} \|\psi(t,\omega_p)\|_{\mathscr{L}(X)} \leq \delta(\omega_p) + L(\omega_p) < +\infty.$$

Therefore, the hypotheses of Theorem 4.3.3 are satisfied, and the proof is complete. □

Remark 4.3.5. *Note that for each nonautonomous evolution process (φ,Θ) with a uniform exponential dichotomy there exists δ in the previous theorem that depends only on the exponent of exponential dichotomy. When applying Theorem 4.2.5 we obtain explicit functions for the bound and exponent of the perturbations, which is an improvement of the result of robustness in the case of Ω a Hausdorff compact topological space of (CHOW; LEIVA, 1995b).*

To end this subsection we extend the result on the continuous dependence of projections for discrete co-cycle, Theorem 4.2.9, to continuous co-cycle.

Theorem 4.3.6. *Suppose that (φ,Θ) and (ψ,Θ) are nonautonomous random dynamical systems and that they admit an exponential dichotomy with projections Π^s_φ and Π^s_ψ, and exponents α_φ and α_ψ, respectively. If*

$$\sup_{t\in\mathbb{R}} \left\{ K(\omega_p) \|\varphi(t,\omega_p) - \psi(t,\omega_p)\|_{\mathscr{L}(X)} \right\} \leq \varepsilon,$$

then

$$\sup_{t\in\mathbb{R}} \|\Pi^s_\varphi(\Theta_t\omega_p) - \Pi^s_\psi(\Theta_t\omega_p)\|_{\mathscr{L}(X)} \leq \frac{e^{-\alpha_\psi(\omega_p)} + e^{-\alpha_\varphi(\omega_p)}}{1 - e^{-(\alpha_\psi(\omega_p)+\alpha_\varphi(\omega_p))}}\,\varepsilon.$$

Proof. The proof is a consequence of Theorem 4.3.3 and Theorem 4.2.9. □

4.4 Applications to linear nonautonomous random differential equations

In this section, we shall study linear nonautonomous random differential equations on a Banach space X. We provide conditions to guarantee the existence of an exponential dichotomy for a nonautonomous random perturbation of a hyperbolic autonomous problem. The results contained here were inspired by (CHOW; LEIVA, 1995b) in a deterministic context, where the base flow is a group over a Hausdorff compact set.

Remark 4.4.1. *Before we start we remark some facts about nonautonomous random differential equations and generation of nonautonomous random dynamical systems. Let* $(\theta, \Omega, \mathscr{F}, \mathbb{P})$ *be a random flow, and consider the following initial value problem*

$$\dot{y} = f(t, \theta_t \omega, y), \ \ t > \tau \text{ and } y(\tau) = y_0. \tag{4.20}$$

Assume that for almost all $\omega \in \Omega$ *the solutions of* (4.20) *are associated with a nonlinear evolution process* $\mathscr{S}_\omega := \{S_\omega(t,s) : t \geq s\}$. *More precisely, if for every* $\tau \in \mathbb{R}$ *and* $y_0 \in X$ *there exists* $[\tau, +\infty) \ni t \mapsto y(t, \tau, \omega; y_0)$ *a solution for* (4.20)*, then we define* $S_\omega(t, \tau, \omega)u_0 := y(t, \tau, \omega; u_0)$.

Another equivalent way to generate a dynamical system from problem (4.20) *is the following: define* $x(t) := y(t+\tau, \tau, \omega; u_0)$*, for every* $t \geq 0$ *and some fixed* $\tau \in \mathbb{R}$*. Hence we obtain the initial value problem*

$$\dot{x} = f(t+\tau, \theta_{t+\tau}\omega, x), \ \ t > 0 \text{ and } x(0) = x_0.$$

Now, the relation $f(t+\tau, \theta_{t+\tau}\omega) = f(\Theta_t(t, \theta_\tau \omega))$*, where* Θ *is the flow* $\{\Theta_t : \mathbb{R} \times \Omega \to \mathbb{R} \times \Omega\}_{t \in \mathbb{R}}$ *defined as* $\Theta_t(\tau, \omega) = (t+\tau, \theta_t \omega)$*, leads us to consider a nonautonomous random differential with a nonlinearity driven by the flow* Θ *i.e.*

$$\dot{z} = f(\Theta_t(\tau, \omega), z), \ \ t > 0 \text{ and } z(0) = z_0, \text{ for each } (\tau, \omega) \in \mathbb{R} \times \Omega. \tag{4.21}$$

Thus, the solutions of (4.21) *defines a nonautonomous random dynamical system* $\varphi(t, \tau, \omega)z_0 := z(t, (\tau, \omega); z_0)$*. Therefore, we rewrite problem* (4.20) *using formulation of* (4.21) *as follows*

$$\varphi(t, \tau, \omega)y_0 := y(t+\tau, \tau, \theta_{-\tau}\omega; y_0).$$

Or equivalently, for almost all $\omega \in \Omega$ *we have that*

$$\varphi(t, \tau, \omega) = S_{\theta_{-\tau}\omega}(t+\tau, \tau), \ t \geq 0, \ \tau \in \mathbb{R}.$$

Consequently, we study asymptotic behavior of both dynamical systems: the co-cycle $(\varphi, \Theta)_{(X, \mathbb{R}\times\Omega)}$ *generated by* (4.21)*, and the family of evolution processes* $\{\mathscr{S}_\omega ; \, \omega \in \Omega\}$ *associated with* (4.20).

Let A be the generator of a strongly continuous semigroup $\{e^{At} : t \geq 0\}$ and $B : \mathbb{R} \times \Omega \to \mathscr{L}(X)$ be a bounded operator depending on parameters on $\mathbb{R} \times \Omega$. We study the linear problem

$$\dot{x} = Ax + B(\Theta_t \omega_\tau)x, \ \ t > 0 \text{ and } x(0) = x_0. \tag{4.22}$$

where $\omega_\tau := (\tau, \omega)$ and for every $t \in \mathbb{R}$ the map $\Theta_t : \mathbb{R} \times \Omega \to \mathbb{R} \times \Omega$ is defined by $\Theta_t \omega_\tau := (t+\tau, \theta_t \omega)$.

To study equation (4.22), we consider the following family of integral equations

$$x(t,\tau,\omega;x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}B(\Theta_s\omega_\tau)x(s)ds,\ x_0 \in X,\ t \geq 0,\ \omega_\tau \in \mathbb{R}\times\Omega.$$

We have the following result on the robustness of exponential dichotomies for linear nonautonomous random differential equations.

Theorem 4.4.2. *Let (φ,Θ) be a linear nonautonomous random dynamical system with*

$$L(\omega_\tau) := \sup_{t\in\mathbb{R}} \|\varphi(t,\omega_\tau)\|_{\mathscr{L}(X)} < +\infty,\ \text{for each } \omega_\tau \in \mathbb{R}\times\Omega, \tag{4.23}$$

where $\sup_{t\in\mathbb{R}} L(\Theta_t\omega_\tau) \leq L(\omega_\tau)$. Suppose that (φ,Θ) admits an exponential dichotomy with exponent α and bound K. Then there exists a Θ-invariant map $\varepsilon : \mathbb{R}\times\Omega \to (0,+\infty)$ such that for every $B : \mathbb{R}\times\Omega \to \mathscr{L}(X)$ with

$$\sup_{0\leq t\leq 1} \|\int_0^t B(\Theta_s\omega_\tau)xds\|_X < \varepsilon(\omega_\tau)\|x\|_X$$

a nonautonomous random dynamical system satisfying

$$\psi(t,\omega_\tau) = \varphi(t,\omega_\tau) + \int_0^t \varphi(t-s,\Theta_s\omega_\tau)B(\Theta_s\omega_\tau)\psi(s,\omega_\tau)ds \tag{4.24}$$

admits an exponential dichotomy with bound $\hat{M}$ and exponent $\tilde{\alpha}$ provided in Theorem 4.3.4.

Proof. Let (ψ,Θ) be a nonautonomous random dynamical system satisfying (4.24). Then

$$\|\psi(t,\omega_\tau)x\|_X \leq L(\omega_\tau)\|x\|_X + \int_0^t L(\Theta_s\omega_\tau)\|B(\Theta_s\omega_\tau)\|_{\mathscr{L}(X)}\|\psi(s,\omega_\tau)x\|_X\,ds,\ 0\leq t\leq 1.$$

From Grönwall's inequality we obtain $\|\psi(t,\omega_\tau)\|_{\mathscr{L}(X)} \leq L_1(\omega_\tau) := L(\omega_\tau)e^{L(\omega_\tau)\varepsilon(\omega_\tau)}$, for every $0\leq t\leq 1$.

Hence, for every $0\leq t\leq 1$

$$\|\varphi(t,\omega_\tau)x - \psi(t,\omega_\tau)x\|_X \leq \varepsilon(\omega_\tau)L(\omega_\tau)L_1(\omega_\tau)\|x\|_X.$$

Finally, since (φ,Θ) admits an exponential dichotomy, there exists a Θ-invariant measurable map $\delta > 0$ as in Theorem 4.3.4. Therefore, for each ω_τ choose $\varepsilon = \varepsilon(\omega_\tau) > 0$ such that $\varepsilon(\omega_\tau)L(\omega_\tau)L_1(\omega_\tau) < \delta(\omega_\tau)$. Note that, for every $t \in \mathbb{R}$ $L_2(\Theta_t\omega_\tau) := L(\Theta_t\omega_\tau)L_1(\Theta_t\omega_\tau) \leq L_2(\omega_\tau)$, therefore we choose $\varepsilon(\Theta_t\omega_\tau) = \varepsilon(\omega_\tau)$, and the proof is complete. □

Remark 4.4.3. *Theorem 4.4.2 is saying that if problem $\dot{x} = A(\Theta_t\omega_\tau)x$ generates a nonautonomous random dynamical system with an exponential dichotomy, then for the class of bounded linear perturbation B given, in the above theorem, the perturbed nonautonomous random dynamical system generated by problem*

$$\dot{x} = A(\Theta_t\omega_\tau)x + B(\Theta_t\omega_\tau)x,\ x(0) = x_0 \in X,\ t \geq 0, \tag{4.25}$$

admits an exponential dichotomy.

Now, as a corollary we have the following robustness result for an autonomous problem under nonautonomous random perturbation.

Theorem 4.4.4. *Assume that A generates an* ***analytic semigroup*** *$\{e^{At} : t \geq 0\}$, and that the spectrum of A, $\sigma(A)$, does not intersect the imaginary axis and that the set $\sigma^+ := \{\lambda \in \sigma(A) : Re\lambda > 0\}$ is compact. Then there exists a Θ-invariant map $\varepsilon : \mathbb{R} \times \Omega \to (0, +\infty)$ such that, if $B : \mathbb{R} \times \Omega \to \mathscr{L}(X)$ satisfies*

$$\sup_{0 \leq t \leq 1} \| \int_0^t B(\Theta_s \omega_\tau) x ds \|_X < \varepsilon(\omega_\tau) \|x\|_X,$$

then any nonautonomous random dynamical system satisfying

$$\varphi(t, \omega_\tau) = e^{At} + \int_0^t e^{A(t-s)} B(\Theta_s \omega_\tau) \varphi(s, \omega_\tau) \, ds \tag{4.26}$$

admits an exponential dichotomy with bound $\hat{M}$ and exponent $\tilde{\alpha}$ provided in Theorem 4.3.4.

Proof. These assumptions on A implies the existence of an exponential dichotomy for $\{e^{At} : t \geq 0\}$, see (HENRY, 1981). In fact, if γ is a smooth closed simple curve in $\rho(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ oriented counterclockwise and enclosing σ^+ let

$$Q = Q(\sigma^+) = \frac{1}{2\pi i} \int_\gamma (\lambda - A)^{-1} d\lambda,$$

and define $X^+ = QX$, $X^- = (I - Q)X$, and $A^\pm := A_\pm$. At this scenario, A^- generates a strongly continuous semigroup on X^-, $A^+ \in \mathscr{L}(X^+)$, and there are $M \geq 1$, $\beta > 0$ such that

$$\|e^{A^+ t}\|_{\mathscr{L}(X^+)} \leq M e^{\beta t}, \ t \leq 0;$$
$$\|e^{A^- t}\|_{\mathscr{L}(X^-)} \leq M e^{-\beta t}, \ t \geq 0.$$

Now we are ready to apply Theorem 4.4.2 for $\{e^{At} : t \geq 0\}$. □

Remark 4.4.5. *As in (HENRY, 1981), Theorem 4.4.4 can be proved in the parabolic case, when $-A$ is a sectorial operator with $A \in \mathscr{L}(X^\alpha, X)$, for a unbounded perturbation $B : \mathbb{R} \times \Omega \to \mathscr{L}(X^\alpha, X)$, where X^α is a fractional power of X, with $0 \leq \alpha < 1$. In this situation there exists a Θ-invariant ε such that if B satisfies*

$$\|B(\Theta_t \omega_\tau) x\|_X \leq b(\omega_\tau) \|x\|_{X^\alpha},$$
$$\sup_{0 \leq t \leq 1} \| \int_0^t B(\Theta_s \omega_\tau) x ds \|_X < q(\omega_\tau) \|x\|_{X^\alpha},$$

and $q(\omega_\tau)^\delta b(\omega_\tau)^{1-\delta} \leq \varepsilon(\omega_\tau)$ with $0 < \delta < (1-\alpha)/2$, then any nonautonomous random dynamical system satisfying (4.26) admits an exponential dichotomy in X^α.

Remark 4.4.6. *Note that Theorem 4.4.4 can be proved independently of Theorem 4.4.2 and any nonautonomous random dynamical systems $(\varphi_\varepsilon, \Theta)$ satisfying (4.26) will satisfy the hypotheses of Theorem 4.4.2.*

CHAPTER

5

STRUCTURAL STABILITY OF NONAUTONOMOUS RANDOM ATTRACTORS

In this chapter, we study continuity and topological structural stability of attractors for nonautonomous random differential equations obtained by small bounded random perturbations of autonomous semilinear problems. First, we study existence and continuity of random hyperbolic solutions and its associated unstable sets. Then, we use this to establish lower semicontinuity of nonautonomous random attractors and to show that the gradient structure persists under nonautonomous random perturbations. Finally, we apply the abstract results in a stochastic differential equation and in a damped wave equation with a perturbation on the damping.

5.1 Existence and continuity of random hyperbolic solutions

In this section, we study a semilinear problem under a nonautonomous random perturbation. We provide conditions to obtain existence of a bounded random hyperbolic solution for a co-cycle. We consider the semilinear problem on a separable Banach space X

$$\dot{y} = By + f_0(y), \;\; y(0) = y_0 \tag{5.1}$$

and a nonautonomous random perturbation of it

$$\dot{y} = By + f_\eta(\Theta_t \omega_\tau, y), \;\; y(0) = y_0, \tag{5.2}$$

where (θ, Ω) is a random flow, and $\{\Theta_t : t \in \mathbb{R}\}$ is a driving flow given by $\Theta_t(\omega_\tau) = (t + \tau, \theta_t \omega)$ for every $\omega_\tau \in \mathbb{R} \times \Omega$, and $\eta \in (0, 1]$ is a real parameter.

We suppose that $f_\eta(\omega_\tau,\cdot)\in C^1(X)$, for every $\eta\in[0,1]$, $\omega_\tau\in\mathbb{R}\times\Omega$, and that

$$\lim_{\eta\to 0}\sup_{(t,x)\in\mathbb{R}\times B(0,r)}\left\{\|f_\eta(\Theta_t\omega_\tau,x)-f_0(x)\|_X+\|(f_\eta)_x(\Theta_t\omega_\tau,x)-f_0'(x)\|_{\mathscr{L}(X)}\right\}=0, \tag{5.3}$$

for all $r\geq 0$ and $\omega_\tau\in\mathbb{R}\times\Omega$. This ensures local well-posedness and differentiability with respect to the initial conditions of (5.1) and (5.2), for each $\omega_\tau\in\mathbb{R}\times\Omega$. Additionally, we assume global well-posedness, so that (5.1) is associated with a semigroup $\mathscr{T}_0=\{T_0(t):t\geq 0\}$, and that (5.2) is associated with nonautonomous random dynamical system (ψ_η,Θ), for each $\eta\in[0,1]$.

Our goal is to prove that if $\{T(t):t\geq 0\}$ has a hyperbolic equilibrium y_0^*, then there exists a (unique) *random hyperbolic equilibrium* for (ψ_η,Θ) near y_0^*, for $\eta>0$ "small enough". We first prove existence and continuity of global solutions for (5.2) and then show that these solutions exhibit a hyperbolic behavior.

Definition 5.1.1. *Let (ψ,Θ) be a nonautonomous random dynamical system. We say that a map $\zeta:\mathbb{R}\times\Omega\to X$ is a global solution for (ψ,Θ) if*

$$\psi(t,\omega_\tau)\zeta(\omega_\tau)=\zeta(\Theta_t\omega_\tau),\ \textit{for every } t\geq 0.$$

Remark 5.1.2. *Let (ψ,Θ) be a nonautonomous random dynamical system and a global solution ζ. Then, for each ω_τ fixed, the mapping $\mathbb{R}\ni t\mapsto\xi(t,\omega_\tau):=\zeta(\Theta_t\omega_\tau)$ defines a global solution for the evolution process*

$$\{\psi(t-s,\Theta_s\omega_\tau):t\geq s\}.$$

Suppose that $y_0^*\in X$ is a **hyperbolic equilibrium** for (5.1), i.e., the linear operator $A:=B+f_0'(y_0^*)$ generates an autonomous evolution process $\{e^{A(t-s}:t\geq s\}$ that admits an exponential dichotomy.

Define, for some $r>0$ fixed,

$$\lambda(\eta,\omega_\tau):=\sup_{(t,x)\in\mathbb{R}\times B_r(y_0^*)}\left\{\|f_\eta(\Theta_t(\omega_\tau),x)-f_0(x)\|_X+\|(f_\eta)_x(\Theta_t(\omega_\tau),x)-f_0'(x)\|_{\mathscr{L}(X)}\right\}.$$

Hence, from (5.3), we have that

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\lambda(\eta,\Theta_t\omega_\tau)=0. \tag{5.4}$$

Also suppose

$$\rho(\varepsilon):=\sup_{x\in B_r(y_0^*)}\sup_{\|h\|\leq\varepsilon}\left\{\frac{\|f_0(x+h)-f_0(x)-f_0'(x)h\|_X}{\|h\|_X}\right\}\to 0,\ \text{as } \varepsilon\to 0. \tag{5.5}$$

Theorem 5.1.3. *Let y_0^* be a hyperbolic equilibrium for (5.1) and assume that (5.4) and (5.5) hold. Given $\varepsilon>0$ small enough, there exists a Θ-invariant map $\eta_\varepsilon:\mathbb{R}\times\Omega\to(0,1]$ such that there exists*

$$\mathbb{R}\ni t\mapsto\xi_\eta^*(t,\omega_\tau)\in X,\ \textit{for every } \eta\in(0,\eta_\varepsilon(\omega_\tau)],$$

a global solution of $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \geq s\}$ *such that*

$$\sup_{t\in\mathbb{R}} \|\xi^*_\eta(t,\omega_\tau) - y^*_0\|_X < \varepsilon, \text{ for every } \eta \in (0,\eta_\varepsilon(\omega_\tau)].$$

Proof. Let y be a global solution of (5.2). Then, if we define $\phi = y - y^*_0$, it satisfies

$$\dot{\phi} = A\phi + g_\eta(\Theta_t\omega_\tau,\phi),$$

where $g_\eta(\Theta_t\omega_\tau,\phi) = f_\eta(\Theta_t\omega_\tau,y^*_0+\phi) - f_0(y^*_0) - f'_0(y^*_0)\phi$, so that

$$\phi(t) = e^{A(t-\tau)}\phi(\tau) + \int_\tau^t e^{A(t-s)} g_\eta(\Theta_s\omega_\tau,\phi(s))ds \tag{5.6}$$

Hence, if we project Q and $I-Q$ and take limits we obtain

$$\phi(t) = \int_{-\infty}^{+\infty} G_A(t,s)g_\eta(\Theta_s\omega_\tau,\phi(s))ds,$$

where G is the Green's function associated with the semigroup $\{e^{At} : t \geq 0\}$ and projection Q.

Consequently, a complete bounded solution to (5.6) exists in a small neighborhood of $x = 0$, if and only if, the operator

$$\mathscr{I}_{\omega_\tau,\eta}(\phi)(t) = \int_{-\infty}^{+\infty} G_A(t,s)g_\eta(\Theta_s\omega_\tau,\phi(s))ds$$

has a unique fixed point in the set

$$\mathfrak{X}_\varepsilon := \left\{\phi : \mathbb{R} \to X : \sup_{t\in\mathbb{R}} \|\phi(t)\|_X \leq \varepsilon\right\}$$

for a given $\varepsilon > 0$ small. This follows by a fixed point argument for $\mathscr{I}_{\omega_\tau,\eta}$ for each $\omega \in \Omega$ fixed.

Indeed, let $\varepsilon_1 > 0$ such that

$$\|f'_0(y^*_0+h) - f'_0(y^*_0)\|_{\mathscr{L}(X)} < \frac{1}{6M\beta^{-1}}, \text{ for every } \|h\|_X < \varepsilon_1, \tag{5.7}$$

and $\varepsilon_2 \in (0,1/2)$ be such that $\rho_0(\varepsilon) < 1/6M\beta^{-1}$, for every $0 < \varepsilon < \varepsilon_2$, where $M > 1$ is the bound, and $\beta > 0$ is the exponent of the exponential dichotomy of $\{e^{At} : t \geq 0\}$. Define $\varepsilon_0 = \min\{\varepsilon_1,\varepsilon_2/2\}$ and for a given $\omega_\tau \in \mathbb{R}\times\Omega$ fixed and $\varepsilon \in (0,\varepsilon_0)$, define $\eta_\varepsilon(\omega_\tau) > 0$ such that

$$\lambda(\eta,\omega_\tau) < \frac{\varepsilon}{6M\beta^{-1}}, \text{ for every } \eta \in (0,\eta_\varepsilon(\omega_\tau)].$$

Then, it is possible to prove that $\mathscr{I}_{\omega_\tau,\eta}$ maps $\mathfrak{X}_\varepsilon$ into itself. In fact, for $\phi \in \mathfrak{X}_\varepsilon$

$$\begin{aligned}\|g_\eta(\Theta_t\omega_\tau,\phi(t))\|_X &\leq \|f_\eta(\Theta_t\omega_\tau,y^*_0+\phi(t)) - f_0(y^*_0+\phi(t))\|_X + \rho(\varepsilon)\varepsilon \\ &\leq \lambda(\eta,\omega_\tau) + \rho_0(\varepsilon)\varepsilon,\end{aligned}$$

hence

$$\|\mathscr{I}_{\eta,\omega}\phi(t)\|_X \leq 2\beta^{-1}M\|g_\eta(\Theta_t\omega_\tau,\phi(t))\|_X < \varepsilon,$$

and that $\mathscr{I}_{\omega_\tau,\eta}$ is a contraction. Let $\phi_1,\phi_2 \in \mathfrak{X}_\varepsilon$

$$\begin{aligned}\|g_\eta(\Theta_t\omega_\tau,\phi_1(t)) &- g_\eta(\Theta_t\omega_\tau,\phi_2(t))\|_X \\ &\leq \|f_\eta(\Theta_t\omega_\tau,y_0^*+\phi_1(t))-f_\eta(\Theta_t\omega_\tau,y_0^*+\phi_2(t))-f_0'(y_0^*)(\phi_1(t)-\phi_2(t))\|_X \\ &\leq \left[\lambda(\eta,\omega_\tau)+\rho(\varepsilon)+\|f_0'(y_0^*+\phi_1)-f_0'(y_0^*)\|_{\mathscr{L}(X)}\right]\|(\phi_1(t)-\phi_2(t))\|_X.\end{aligned}$$

Then

$$\|\mathscr{I}_{\eta,\omega}\phi_1(t)-\mathscr{I}_{\eta,\omega}\phi_1(t)\|_X \leq \frac{1}{2}\|\phi_1(t)-\phi_2(t)\|_X.$$

Therefore, there exists a fixed point $\phi_\eta^*(\cdot,\omega_\tau)$ of $\mathscr{I}_{\omega_\tau,\eta}$ in $\mathfrak{X}_\varepsilon$, and the global solution of (5.2) is given by $\xi_\eta^*(\cdot,\omega_\tau)=\phi_\eta^*(\cdot,\omega_\tau)+y_0^*$. □

As in (CARVALHO; LANGA, 2007b), these solutions $\{\xi_\eta^*\}$ play the role of an *hyperbolic equilibrium* for (5.2). Given $\varepsilon>0$ define, for each ω_τ fixed and $\eta\in(0,\eta_\varepsilon(\omega_\tau)]$,

$$\zeta_\eta^*(\tau,\omega) := \xi_\eta^*(0,\omega_\tau).$$

Note that, for each ω_τ fixed, there exists $\eta_\varepsilon(\omega_\tau)>0$ such that the mapping $\mathbb{R}\ni t\mapsto \xi_\eta^*(t,\omega_\tau):=\zeta_\eta^*(\Theta_t\omega_\tau)$, $t\in\mathbb{R}$ is a complete solution for

$$\dot{x} = Bx+f_\eta(\Theta_t\omega_\tau,x),\ \eta\in(0,\eta_\varepsilon(\omega_\tau)]. \tag{5.8}$$

Then, to ensure that ξ_η^* exhibits a hyperbolic behavior, we linearized problem (5.8) over ζ_η^* and guarantee that the associated linear nonautonomous random dynamical system admits an exponential dichotomy.

Remark 5.1.4. *Let $\omega_\tau\in\mathbb{R}\times\Omega$ be fixed, $x_\eta(\cdot,\omega_\tau)$ a solution of (5.2) and define $z_\eta(t)=x_\eta(t,\omega_\tau)-\zeta_\eta^*(\Theta_t\omega_\tau)$, for each $t\geq 0$ and $\eta\in(0,\eta_\varepsilon(\omega_\tau)]$. Then*

$$\dot{z} = Az+B_\eta(\Theta_t\omega_\tau)z+h_\eta(\Theta_t\omega_\tau,z), \tag{5.9}$$

where $B_\eta(\Theta_t\omega_\tau)=(f_\eta)_z(\Theta_t\omega_\tau,\zeta_\eta^*(\Theta_t\omega_\tau))-f_0'(y_0^*)$, and

$$h_\eta(\Theta_t\omega_\tau,z) := f_\eta(\Theta_t\omega_\tau,\zeta_\eta^*(\Theta_t\omega_\tau)+z)-f_\eta(\Theta_t\omega_\tau,\zeta_\eta^*(\Theta_t\omega_\tau))-(f_\eta)_z(\Theta_t\omega_\tau,\zeta_\eta^*(\Theta_t\omega_\tau))z.$$

Thus, 0 is a globally defined bounded solution for (5.9) and $h_\eta(\Theta_t\omega_\tau,0)=0$, $(h_\eta)_z(\Theta_t\omega_\tau,0)=0\in\mathscr{L}(X)$.

We consider the linearized problem associated with (5.9)

$$\dot{z} = Az+B_\eta(\Theta_t\omega_\tau)z,\ t\geq 0,\ z(0)=z_0\in X. \tag{5.10}$$

Note that for each $\omega_\tau\in\mathbb{R}\times\Omega$ fixed

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|B_\eta(\Theta_t\omega_\tau)\|_{\mathscr{L}(X)}=0.$$

Then problem (5.10) generates a co-cycle (φ_η, Θ) which satisfies

$$\varphi_\eta(t,\tau,\omega)x_0 = e^{At}x_0 + \int_0^t e^{A(t-s)}B_\eta(\Theta_s\omega_\tau)\varphi_\eta(s,\tau,\omega)x_0 ds.$$

Since $\{e^{At} : t \geq 0\}$ admits exponential dichotomy, from Theorem 4.4.4 we know that the linear co-cycle (φ_η, Θ) admits an exponential dichotomy for each suitable small $\eta > 0$.

Thus ζ^*_η is a global solution that exhibits hyperbolic behavior and it suggests the definition of random hyperbolic solution for (5.2).

Definition 5.1.5. *Let B be a generator of a strongly continuous semigroup, and $f : \mathbb{R}\times\Omega\times X \to X$ such that for each (t,ω) fixed $X \ni x \mapsto f(t,\omega,x)$ is differentiable, and $\zeta : \mathbb{R}\times\Omega \to X$ be a bounded global solution of*

$$\dot{x} = Bx + f(\Theta_t\omega_\tau, x),\ t \geq 0, x(0) = x_0 \in X. \tag{5.11}$$

We say that ζ is a ***random hyperbolic solution*** *of (5.11) if there exists a linear nonautonomous random dynamical system (φ, Θ) satisfying*

$$\varphi(t,\omega_\tau) = e^{Bt} + \int_0^t e^{A(t-s)}D_x f(\Theta_s\omega_\tau, \zeta(\Theta_s\omega_\tau))\varphi(s,\omega_\tau)ds, \text{ for all } \omega_\tau \in \mathbb{R}\times\Omega,$$

and (φ, Θ) admits an exponential dichotomy.

Joining Theorem 5.1.3 and and the argumets presented in Remark 5.1.4 we obtain:

Theorem 5.1.6 (Existence and continuity of hyperbolic solutions)**.** *Let y^*_0 be a hyperbolic equilibrium for (5.1) and assume that (5.3) and (5.5) hold. Given $\varepsilon > 0$ suitable small, there exists a Θ-invariant map $\eta_\varepsilon : \mathbb{R}\times\Omega \to (0,1]$ such that:*

1. *for each $\omega_\tau \in \mathbb{R}\times\Omega$ fixed, given $\eta \in (0, \eta_\varepsilon(\omega_\tau)]$, there exists a global hyperbolic solution $\mathbb{R} \ni t \mapsto \zeta_\eta(t,\omega_\tau)$ of the evolution process $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ satisfying*

$$\sup_{t\in\mathbb{R}} \|\zeta^*_\eta(t,\omega_\tau) - y^*_0\|_X < \varepsilon, \tag{5.12}$$

   and $\zeta_\eta(t,\omega_\tau) = \zeta_\eta(0,\Theta_t\omega_\tau)$, for all $t \in \mathbb{R}$.

2. *for each Θ-invariant function $\bar{\eta} : \mathbb{R}\times\Omega \to [0,1]$ with $\bar{\eta}(\omega_\tau) \leq \eta_\varepsilon(\omega_\tau)$, there exists a random hyperbolic solution $\xi^*_{\bar{\eta}} : \mathbb{R}\times\Omega \to X$ of $(\psi_{\bar{\eta}}, \Theta)$ defined by*

$$\xi^*_{\bar{\eta}}(\omega_\tau) := \zeta^*_{\bar{\eta}(\omega_\tau)}(0,\omega_\tau),$$

   and satisfying (5.12).

Theorem 5.1.6 is the first step to the study of existence and continuity of unstable and stable sets, which are the main tool to conclude lower semicontinuity and topological structural stability of attractors.

Remark 5.1.7. *Theorem 5.1.6[Item 2] provides existence and continuity of random hyperbolic solutions for nonautonomous random perturbations of a autonomous problem. However, this result of persistence can be proved in a general context. In other words, following similar steps, it is possible to prove that random hyperbolic solutions are stable under (random nonautonomous) perturbations.*

Remark 5.1.8. *Suppose that $\{y_1^*,\cdots,y_p^*\}$ is a set of hyperbolic equilibria for (5.1). Then there exists $\varepsilon_0>0$ such that y_i^* is isolated in $B(y_i^*,\varepsilon_0)$ and $B(y_i^*,\varepsilon_0)\cap B(y_j^*,\varepsilon_0)=\emptyset$, $j\neq i$. Theorem 5.1.6 guarantees that for each $i\in\{1,\cdots,p\}$ and $\varepsilon_0'\in(0,\varepsilon_0)$ suitable small fixed, there exits a Θ-invariant function $\eta_{0,i}:\mathbb{R}\times\Omega\to(0,1]$ satisfying the conclusions of Theorem 5.1.6.*

Define $\eta_0(\omega_\tau)=\min_{0\le i\le p}\{\eta_{0,i}(\omega_\tau)\}$, for $\omega_\tau\in\mathbb{R}\times\Omega$. Let ω_τ be fixed, then for each $\eta\in(0,\eta_0(\omega_\tau)]$ there exists $\zeta_{i,\eta}^*(\cdot,\omega_\tau)$ is a hyperbolic solution of $\{\psi_\eta(t-s,\Theta_s\omega_\tau):t\ge s\}$ such that

$$\sup_{t\in\mathbb{R}}\|\zeta_{i,\eta}^*(t,\omega_\tau)-y_i^*\|_X<\varepsilon_0', \text{ for every } i\in\{1,\cdots,p\}.$$

Remark 5.1.9. *In the parabolic case, when A is sectorial, with $A\in\mathscr{L}(X^\delta,X)$, $0<\delta<1$, where X^δ is a fractional power of X, we cannot assume that the nonlinearity $f_0:U\subset X\to X$ is differentiable, see (BORTOLAN et al., 2020). We have to assume that the hyperbolic equilibrium y_0^* is in X^δ and that U is a open neighborhood of y_0^* in X^δ such that $f_0:U\subset X^\delta\to X$ is differentiable with derivative $f'(y_0^*)\in\mathscr{L}(X^\delta,X)$. Also, we have to use a slightly different estimative on the Green's function of $\{e^{At}:t\ge 0\}$*

$$\begin{aligned}\|G_A(t,s)\|_{\mathscr{L}(X,X^\delta)} &\le D(M,\delta)(t-s)^{-\delta}e^{-\beta|t-s|}, 0<t-s\le 1\\ \|G_A(t,s)\|_{\mathscr{L}(X,X^\delta)} &\le D(M,\delta)e^{-\beta|t-s|}, \text{ otherwise},\end{aligned}$$

where $D=D(M,\delta)$ is a constant, see (HENRY, 1981). Under these conditions the proof of existence and continuity of bounded random hyperbolic equilibrium on X^δ is analogous to the argument used in Theorem 5.1.6.

5.2 Existence and continuity of unstable sets

In this section, we study existence and continuity of unstable sets for the hyperbolic solutions obtained in Theorem 5.1.6. Under the same assumptions of Section 5.1, we will apply the techniques of the deterministic case (CARVALHO; LANGA, 2007b) to our problem. The idea here is to revisit the proofs to track the dependence on the parameter $\omega_\tau\in\mathbb{R}\times\Omega$ in the arguments.

First, we recall the definition of the *unstable set* for a global solution ξ of an evolution process, which was introduced in (CARVALHO; LANGA, 2007b).

Definition 5.2.1. *Let $\mathscr{S} = \{S(t,s) : t \geq s\}$ be an evolution process, and $\xi : \mathbb{R} \to X$ be a* ***global solution*** *of $\mathscr{S}$. The* ***unstable set*** *of ξ is defined as*

$$W^u(\xi) = \Big\{(t,z) \in \mathbb{R} \times X : \text{ there is a global solution } \zeta \text{ of } \mathscr{S} \text{ such that } \zeta(t) = z, \text{ and } \lim_{s \to -\infty} \|\zeta(s) - \xi(s)\|_X = 0\Big\}.$$

The section of *$W^u(\xi)$* ***at time*** *$t \in \mathbb{R}$ is denoted by $W^u(\xi)(t) = \{z \in X : (t,z) \in W^u(\xi)\}$.*

Next, we extend the concept of *unstable set* for nonautonomous random dynamical systems.

Definition 5.2.2. *Let (ψ, Θ) be a nonautonomous random dynamical system and $\xi^* : \mathbb{R} \times \Omega \to X$ be a random hyperbolic solution of (ψ, Θ). The* ***unstable set*** *of ξ^* is the family*

$$W^u(\xi^*) = \{W^u(\xi^*; \omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\},$$

where, for each ω_τ, $W^u(\xi^*; \omega_\tau)$ is the unstable set of the hyperbolic solution $t \mapsto \xi^*(\Theta_t \omega_\tau)$ of the evolution process $\Psi_{\omega_\tau} = \{\psi(t-s, \Theta_s \omega_\tau) : t \geq s\}$. ***The section of*** *$W^u(\xi^*; \omega_\tau)$* ***at time*** *$t \in \mathbb{R}$ is denoted by*

$$W^u(\xi^*; \omega_\tau)(t) = \{z \in X : (t,z) \in W^u(\xi^*; \omega_\tau)\}.$$

Let $\delta : \mathbb{R} \times \Omega \to (0, +\infty)$ be a Θ-invariant map, a ***local unstable set*** *is a family $W^{u,\delta}(\xi^*) = \{W^{u,\delta}(\xi^*; \omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$, where*

$$W^{u,\delta}(\xi^*; \omega_\tau) = \Big\{(t,z) \in \mathbb{R} \times X : \text{ there is a global solution } \zeta \text{ of } \Psi_{\omega_\tau} \text{ such that } \zeta(t) = z, \; \|\zeta(s) - \xi^*(\Theta_s \omega_\tau)\|_X \leq \delta(\omega_\tau), \; \forall s \leq t, \text{ and } \lim_{s \to -\infty} \|\zeta(s) - \xi^*(\Theta_s \omega_\tau)\|_X = 0\Big\},$$

and ***the section of*** *$W^{u,\delta}(\xi^*; \omega_\tau)$* ***at time*** *$t$ is defined by*

$$W^{u,\delta}(\xi^*; \omega_\tau)(t) = \{z \in X : (t,z) \in W^{u,\delta}(\xi^*; \omega_\tau)\}.$$

This definition can also be seen as an extension of the linear case, see the definition of $V^-(\omega_p)$ in Lemma 4.2.6.

Before we continue, we recall the definition of attractors for a nonautonomous random dynamical system.

Definition 5.2.3. *Let $K : \Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say that K is* ***measurable*** *if the mapping $\Omega \ni \omega \mapsto d(x, K(\omega))$ is $(\mathscr{F}, \mathscr{B}_\mathbb{R})$-measurable for every fixed $x \in X$.*

In Definition 5.2.3, we used that X is a complete separable metric space, see (CASTAING; VALADIER, 1977, Chapter III).

Definition 5.2.4. *Let $\hat{\mathscr{A}} = \{\mathscr{A}(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$ be a family of nonempty subsets of X. We say that $\hat{\mathscr{A}}$ is a* ***nonautonomous random attractor*** *for (ψ, Θ) if the following conditions are fulfilled:*

1. *$\mathscr{A}(\omega_\tau)$ is compact, for every $\omega_\tau \in \mathbb{R} \times \Omega$;*

2. *the set-valued mapping $\omega \mapsto \mathscr{A}(\tau, \omega)$ is measurable, for each $\tau \in \mathbb{R}$;*

3. *$\hat{\mathscr{A}}$ is invariant, i.e., $\psi(t, \omega_\tau)\mathscr{A}(\omega_\tau) = \mathscr{A}(\Theta_t \omega_\tau)$ for every $t \geq 0$ and $\omega_\tau \in \mathbb{R} \times \Omega$;*

4. *$\hat{\mathscr{A}}$ pullback attracts every bounded subset of X, i.e., for every bounded subset B of X and $\omega_\tau \in \mathbb{R} \times \Omega$,*
$$\lim_{t \to +\infty} dist(\psi(t, \Theta_{-t}\omega_\tau)B, \mathscr{A}(\omega_\tau)) = 0,$$
where $dist(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$ is the usual Hausdorff semi-distance;

5. *$\hat{\mathscr{A}}$ is the minimal closed family that pullback attracts bounded subsets of X, i.e., if $\{F(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$ is a family of closed subsets of X that pullback attracts every bounded subset of X, then $\mathscr{A}(\omega_\tau) \subset F(\omega_\tau)$, for every $\omega_\tau \in \mathbb{R} \times \Omega$.*

For existence of nonautonomous random attractors and applications to differential equations, see (WANG, 2012b) and the references therein.

Since we will associated our co-cycle (ψ, Θ) with a family of evolution processes as in Remark 4.1.4, we recall the notion of *pullback attractors*.

Definition 5.2.5. *Let $\mathscr{S} = \{S(t, s) : t \geq s\}$ be an evolution process in X and $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ be a family of nonempty subsets of X. We say that $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is a* ***pullback attractor*** *for $\mathscr{S}$ if*

1. *$\mathscr{A}(t)$ is compact, for every $t \in \mathbb{R}$;*

2. *$\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is invariant, i.e., $S(t, s)\mathscr{A}(s) = \mathscr{A}(t)$, $\forall t \geq s$;*

3. *$\{\mathscr{A}(t) : t \in \mathbb{R}\}$ pullback attracts every bounded subset of X, i.e., for every bounded subset B of X,*
$$\lim_{s \to -\infty} dist(S(t, s)B, \mathscr{A}(t)) = 0;$$

4. *$\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is the minimal closed family that pullback attracts bounded subsets of X.*

There are several works that deal with existence and continuity (upper and lower semicontinuity) of pullback attractors, we refer the reader to (CARABALLO; ŁUKASZEWICZ; REAL, 2006; CARABALLO *et al.*, 2010a; CARVALHO; LANGA; ROBINSON, 2013; BORTOLAN; CARVALHO; LANGA, 2020), where many other references to earlier results can be found.

Remark 5.2.6. *Let (ψ,Θ) be a nonautonomous random dynamical system with an attractor $\{\mathscr{A}(\omega_\tau):\omega_\tau\in\mathbb{R}\times\Omega\}$. Then, for each ω_τ fixed, the evolution process Ψ_{ω_τ} has a pullback attractor given by $\{A(\Theta_t\omega_\tau):t\in\mathbb{R}\}$.*

The following lemma provides some properties of the unstable set.

Lemma 5.2.7. *Let (ψ,Θ) be a nonautonomous random dynamical system and $\xi^*:\mathbb{R}\times\Omega\to X$ be a random hyperbolic solution of (ψ,Θ).*

For each $\omega_\tau\in\mathbb{R}\times\Omega$ and $t\in\mathbb{R}$,

$$W^u(\xi^*;\omega_\tau)(t)=W^u(\xi^*;\Theta_t\omega_\tau)(0). \tag{5.13}$$

Moreover, if (ψ,Θ) has a nonautonomous random attractor $\{\mathscr{A}(\omega_\tau):\omega_\tau\in\mathbb{R}\}$ and ξ^* is bounded, then

$$W^u(\xi^*;\omega_\tau)(0)\subset\mathscr{A}(\omega_\tau),\ \forall\,\omega_\tau. \tag{5.14}$$

Proof. First we prove (5.13). Let $z\in W^u(\xi^*;\omega_\tau)(t)$, then there exists a global solution $\zeta:\mathbb{R}\to X$ of Ψ_{ω_τ} such that $\zeta(t)=z$ and $\|\zeta(s)-\xi^*(\Theta_s\omega_\tau)\|_X\overset{s\to-\infty}{\longrightarrow}0$. Define, $\tilde{\zeta}(s)=\zeta(t+s)$, $s\in\mathbb{R}$, thus $\tilde{\zeta}$ is a global solution for $\Psi_{\Theta_t\omega_\tau}$ such that $\tilde{\zeta}(0)=z$ and

$$\|\tilde{\zeta}(s)-\xi^*(\Theta_s\Theta_t\omega_\tau)\|_X=\|\zeta(s+t)-\xi^*(\Theta_{s+t}\omega_\tau)\|_X\overset{s\to-\infty}{\longrightarrow}0.$$

Therefore, $z\in W^u(\xi^*,\Theta_t\omega_\tau)(0)$. By similar arguments, we see that

$$W^u(\xi^*,\Theta_t\omega_\tau)(0)\subset W^u(\xi^*,\omega_\tau)(t),$$

which concludes the proof of (5.13).

For the second claim, let $z\in W^u(\xi^*;\omega_\tau)(0)$, then there exists a global solution $\zeta:\mathbb{R}\to X$ of Ψ_{ω_τ} such that $\zeta(0)=z$ and $\|\zeta(s)-\xi^*(\Theta_s\omega_\tau)\|_X\to0$ as $s\to-\infty$. Since $\{\xi^*(\Theta_t\omega_\tau):t\in(-\infty,0]\}$ is bounded, the set $B=\zeta((-\infty,0])$ is bounded and therefore

$$\lim_{s\to-\infty}dist_H(\psi(-s,\Theta_s\omega_\tau)B,\mathscr{A}(\omega_\tau))=0, \tag{5.15}$$

then $d(z,\mathscr{A}(\omega_\tau))=0$ and $z\in\mathscr{A}(\omega_\tau)$ for each ω_τ. The proof is complete. □

Lemma 5.2.7 implies that the attractor contains all the unstable sets of hyperbolic solutions. Later, in Section 5.4, we will give conditions under which the attractor is equal to the union of these unstable sets.

Remark 5.2.8. *Let $\mathscr{S}=\{S(t,s):t\geq s\}$ be an evolution process with a pullback attractor $\{\mathscr{A}(t):t\in\mathbb{R}\}$ such that $\cup_{t\leq0}\mathscr{A}(t)$ is bounded. In this case,*

$$\mathscr{A}(t)=\bigcup\{W^u(\xi)(t):\xi\text{ is a backwards bounded solution}\},\ \forall t\in\mathbb{R}. \tag{5.16}$$

Therefore, it is natural to search for the minimal collection of backwards bounded solutions whose unstable sets form the attractor. Of course many backwards bounded solutions have the same unstable set, and thus it is natural to seek for backward-separated solutions, see (CARVALHO; LANGA; ROBINSON, 2013, Section 3.3) for more details. In Section 5.4, we will provide conditions to obtain that there is a distinguished set of backwards bounded global solutions that forms the nonautonomous random attractor. These conditions rely on the hyperbolicity and it is through this distinguished set that we will be able to address the lower semicontinuity of nonautonomous random attractors.

Next, we prove that the local unstable sets for these hyperbolic solutions are given as graphs, following the same line of arguments presented in (CARVALHO; LANGA, 2007b). In fact, if ξ_η^* is a random hyperbolic solution of ψ_η, we will show that the elements in $W_\eta^{u,\delta}(\xi_\eta^*;\omega_\tau)$ will be those of the form

$$(t,\xi_\eta^*(\Theta_t\omega_\tau)+\Pi_\eta^u(\Theta_t\omega_\tau)z+\Sigma^u(\omega_\tau)(t,\Pi_\eta^u(\Theta_t\omega_\tau)z))\in\mathbb{R}\times X,\ \text{and}\ \|z\|_X\leq\delta(\omega_\tau),$$

where $\delta:\mathbb{R}\times\Omega\to(0,+\infty)$ is a Θ-invariant map, for some Lipschitz map Σ^u. Moreover, we will obtain that as $\eta\to 0$ these local unstable sets "converges" to the unstable sets of the autonomous problem (5.1).

Let ω_τ be fixed, $\eta\in(0,\eta_0(\omega_\tau)]$, and $t\mapsto\xi_\eta^*(\Theta_t\omega_\tau)$ a hyperbolic solution of $\{\psi_\eta(t-s,\Theta_s\omega_\tau):t\geq s\}$ obtained by Theorem 5.1.6. Then, the change of variables $z(t)=y(t)-\xi_\eta^*(\Theta_t\omega_\tau)$ allows us to concentrate on the existence of invariant sets of global hyperbolic solutions around the zero solution of

$$\dot{z}=Az+B_\eta(\Theta_t\omega_\tau)z+h_\eta(\Theta_t\omega_\tau,z),\ \ z(s)=z_0\in X,\tag{5.17}$$

where $A=B+f_0'(y^*)$, $B_\eta(\omega_\tau)=(f_\eta)_x(\omega_\tau,\xi_\eta^*(\omega_\tau))-f_0'(y^*)$ and

$$\begin{aligned}h_\eta(\Theta_t\omega_\tau,z):=&f_\eta(\Theta_t\omega_\tau,\xi_\eta^*(\Theta_t\omega_\tau)+z)-f_\eta(\Theta_t\omega_\tau,\xi_\eta^*(\Theta_t\omega_\tau))\\&-(f_\eta)_z(\Theta_t\omega_\tau,\xi_\eta^*(\Theta_t\omega_\tau))z.\end{aligned}$$

Thus $z=0$ is a globally defined bounded solution for (5.17) where $h_\eta(\omega_\tau,\cdot):X\to X$ differentiable with $h_\eta(\omega_\tau,0)=0$, $(h_\eta)_x(\omega_\tau,0)=0\in\mathscr{L}(X)$, for all $\eta\in[0,\eta_0(\omega_\tau)]$. Furthermore, (5.3) implies that

$$\lim_{\eta\to0}\sup_{(t,x)\in\mathbb{R}\times B(0,r)}\Big\{\|h_\eta(\Theta_t\omega_\tau,x)-h_0(x)\|_X+\|(h_\eta)_x(\Theta_t\omega_\tau,x)-h_0'(x)\|_{\mathscr{L}(X)}\Big\}=0,\tag{5.18}$$

for all $r>0$ and $\omega_\tau\in\mathbb{R}\times\Omega$.

We recall that, η_0 is chosen such that the linear evolution process $\{\varphi_\eta(t-s,\Theta_s\omega_\tau):t\geq s\}$, given by

$$\varphi_\eta(t-s,\Theta_s\omega_\tau)=e^{A(t-s)}+\int_s^t e^{A(t-r)}B_\eta(\Theta_r\omega_\tau)\varphi_\eta(r-s,\Theta_s\omega_\tau)\,dr,\ t\geq s,\tag{5.19}$$

admits an exponential dichotomy with bound M_η, exponent α_η and family of projections $\{\Pi^u_\eta(t) : t \in \mathbb{R}\}$, for every $\eta \in (0, \eta_0(\omega_\tau)]$, see the proof of Theorem 5.1.6. Moreover, for each Θ-invariant function $\bar{\eta} : \mathbb{R}\times\Omega \to [0,1]$, with $\bar{\eta}(\omega_\tau) \leq \eta_0(\omega_\tau)$, the co-cycle $(\varphi_{\bar{\eta}}, \Theta)$ admits an exponential dichotomy with bound $M_{\bar{\eta}}$, exponent $\alpha_{\bar{\eta}}$ and family of projections $\{\Pi^u_{\bar{\eta}}(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$.

If z is a solution of (5.17) we write $z^u(t) = \Pi^u_\eta(t)z(t)$ and $z^s(t) = \Pi^s_\eta(t)z(t)$, $t \in \mathbb{R}$, where $\Pi^u_\eta(t) = Id_X - \Pi^s_\eta(t)$, $t \in \mathbb{R}$. Then z^u and z^s are the solutions of

$$\begin{aligned} \dot{z}^u &= A_\eta(\Theta_t\omega_\tau)z^u + h^u_\eta(\Theta_t\omega_\tau, z^u(t) + z^s(t)),\\ \dot{z}^s &= A_\eta(\Theta_t\omega_\tau)z^s + h^s_\eta(\Theta_t\omega_\tau, z^u(t) + z^s(t)), \end{aligned} \tag{5.20}$$

where $A_\eta(\omega_\tau) = A + B_\eta(\omega_\tau)$, and $h^k_\eta(\omega_\tau,\cdot) = \Pi^k_\eta(\omega_\tau)h_\eta(\omega_\tau,\cdot)$, $\; k = u, s$.

Since, for each ω_τ fixed, $h^k_\eta(\Theta_t\omega_\tau, 0) = 0$, with $(h^k_\eta)_x(\Theta_t\omega_\tau, 0) = 0$ and h^k_η are continuous differentiable in X, uniformly with respect to t, we obtain that given $\rho > 0$ there exists $\delta_0(\omega_\tau) > 0$ such that if $\|z\|_X, \|\tilde{z}\|_X \leq \delta_0(\omega_\tau)$ then

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \|h^k_\eta(\Theta_t\omega_\tau, z)\|_X \leq \rho,\\ &\sup_{t\in\mathbb{R}} \|h^k_\eta(\Theta_t\omega_\tau, z) - h^k_\eta(\Theta_t\omega_\tau, \tilde{z})\| \leq \rho\|z - \tilde{z}\|_X, \quad k = s, u. \end{aligned} \tag{5.21}$$

Note that, from (5.18), it is possible to choose $\delta_0 : \mathbb{R}\times\Omega \to (0, +\infty)$ as a Θ-invariant function. This is one of the main differences to the deterministic case and to work with the Θ-invariance is the key to our further results.

Remark 5.2.9. *For each ω_τ fixed, it is possible to extend $h^u_\eta(\omega_\tau,\cdot), h^s_\eta(\omega_\tau,\cdot)$ outside the ball of radius $\delta_0(\omega_\tau)$ such that this extension satisfies both conditions in (5.21) for all $z, \tilde{z} \in X$, see (CARVALHO; LANGA, 2007b). Therefore, we obtain the existence and continuity of unstable and stable set, as a graph, for h^u_η and h^s_η satisfying (5.21), for all $z, \tilde{z} \in X$, then, using a localization procedure, we conclude existence and continuity of local unstable sets, as a graph, for the case when h^k_η satisfies (5.21) in the ball of radius $\delta(\omega_\tau)$, for each $\omega_\tau \in \mathbb{R}\times\Omega$.*

Assuming that (5.21) holds for all $z, \tilde{z} \in X$, we will obtain that, for all suitably small ρ, the unstable sets are graphs of Lipschitz maps in the class defined next. Given $L > 0$ and a family of projections $\{\Pi^u(s) : s \in \mathbb{R}\}$. Denote by $\mathscr{LB}(L)$ a complete metric space of all bounded and globally Lipschitz continuous functions $\Sigma : \mathbb{R}\times X \to X$ such that $\mathbb{R}\times X \ni (s,z) \mapsto \Sigma(s,z) := \Sigma(s, \Pi^u(s)z) \in \Pi^s(s)X$ and

$$\begin{aligned} &\sup\left\{\|\Sigma(s, \Pi^u(s)z)\|_X ; (s,z) \in \mathbb{R}\times X\right\} \leq L,\\ &\|\Sigma(s, \Pi^u(s)z) - \Sigma(s, \Pi^u(s)\tilde{z})\|_X \leq L\|\Pi^u(s)z - \Pi^u(s)\tilde{z}\|_X, \end{aligned} \tag{5.22}$$

with distance between $\Sigma, \widetilde{\Sigma} \in \mathscr{LB}(L)$ given by

$$|||\Sigma - \tilde{\Sigma}||| := \sup_{(t,z)\in\mathbb{R}\times X} \|\Sigma(t,z) - \tilde{\Sigma}(t,z)\|_X. \tag{5.23}$$

Theorem 5.2.10. *Let $\omega_\tau \in \mathbb{R}\times\Omega$ be fixed, and $\eta \in [0,\eta_0(\omega_\tau)]$. Suppose that $\rho > 0$ is suitable small such that there is $L = L(\rho,\alpha_\eta,M_\eta) > 0$ satisfying*

$$\begin{aligned}&\frac{\rho M_\eta}{\alpha_\eta} \le L, \quad \frac{\rho M_\eta}{\alpha_\eta}(1+L) < 1\\ &\frac{\rho M_\eta^2(1+L)}{\alpha_\eta - \rho M_\eta(1+L)} \le L,\\ &\rho M_\eta + \frac{\rho^2 M_\eta^2(1+L)(1+M_\eta)}{2\alpha_\eta - \rho M_\eta(1+L)} < \frac{\alpha_\eta}{2}.\end{aligned} \tag{5.24}$$

Then, for each $\omega_\tau \in \mathbb{R}\times\Omega$ fixed and $\eta \in (0,\eta_0(\omega_\tau)]$, there exists $\Sigma_\eta^u = \Sigma_{\eta,\omega_\tau}^u \in \mathscr{LB}(L)$, such that the unstable set of the zero solution of (5.17) is given by

$$W_\eta^u(0) = \{(s,z) \in \mathbb{R}\times X : z = \Pi_\eta^u(s)z + \Sigma_\eta^u(s,\Pi_\eta^u(s)z)\}, \tag{5.25}$$

and, for any $r > 0$ and $s \in \mathbb{R}$,

$$\sup_{t\le s}\sup_{\|z\|_X\le r}\{\|\Pi_\eta^u(t)z - \Pi_0^u z\|_X + \|\Sigma_\eta^u(t,\Pi_\eta^u(t)z) - \Sigma_0^u(\Pi_0^u z)\|_X\} \overset{\eta\to 0}{\to} 0. \tag{5.26}$$

Furthermore, if $\zeta(t) = \zeta^u(t)+\zeta^s(t)$, where $\zeta^k(t) = \Pi_\eta^k(t)\zeta(t)$, for $k = u,s$, is a backward-bounded global solution of (5.17), then there is $\gamma > 0$ such that,

$$\|\zeta^s(t) - \Sigma_\eta^u(t,\zeta^u(t))\|_X \le M_\eta e^{-\gamma(t-s)}\|\zeta^s(s) - \Sigma_\eta^u(s,\zeta^u(s))\|_X,\ t \ge s. \tag{5.27}$$

Theorem 5.2.10 follows directly from (CARVALHO; LANGA, 2007b, Theorem 3.1).

From Theorem 5.1.6 and Theorem 5.2.10, we can obtain the existence and continuity of local unstable sets.

Theorem 5.2.11 (Existence and continuity of local unstable set)**.** *Let $\eta \in [0,1]$, and $h_\eta : \mathbb{R}\times\Omega\times X \to X$ by such that for each ω_τ, the mapping $z \mapsto h_\eta(\omega_\tau,z)$ is continuously differentiable. Consider*

$$\dot z = A_\eta(\Theta_t\omega_\tau)z + h_\eta(\Theta_t\omega_\tau,z), \quad \omega_\tau \in \mathbb{R}\times\Omega. \tag{5.28}$$

Assume that $h_\eta(\omega_\tau,0) = 0$, $(h_\eta)_x(\omega_\tau,0) = 0 \in \mathscr{L}(X)$, $h_0 : X \to X$, $A_0(\Theta_t\omega_\tau) = A$, $\{h_\eta\}_{\eta\in[0,1]}$ satisfies (5.18), and that $z_0^* = 0$ is a hyperbolic solution of (5.28) for $\eta = 0$. Then given $\varepsilon_0 > 0$ suitable small, the following hold:

1. *There exist a Θ-invariant function $\eta_0 : \mathbb{R}\times\Omega \to [0,1]$ such that $z_\eta^* = 0$ is a hyperbolic solution of (5.28), for each $\eta \in (0,\eta_0(\omega_\tau)]$. In particular, the linear evolution process $\{\varphi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$, associated to the linear part of (5.28) (corresponding to the linearization of $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$ around $\xi_\eta^*(\Theta_t\omega_\tau)$), admits an exponential dichotomy with family of projections $\{\Pi_\eta^u(s) : s \in \mathbb{R}\}$.*

2. *The families of projections* $\Pi^u_\eta = \{\Pi^u_\eta(s) : s \in \mathbb{R}\}$*,* $\eta \in (0, \eta_0(\omega_\tau)]$ *satisfy*

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \|\Pi^u_\eta(t) - \Pi^u_0\|_{\mathscr{L}(X)} = 0. \tag{5.29}$$

3. *There exist* Θ*-invariant function* $\delta_0 : \mathbb{R} \times \Omega \to (0, +\infty)$ *(independent of* η*) such that for each* ω_τ *and* $\eta \in [0, \eta_0(\omega_\tau)]$*, and a map*

$$\mathbb{R} \times B_X(0, \delta_0(\omega_\tau)) \ni (s, z) \mapsto \Sigma^u_\eta(s, z) := \Sigma^u_\eta(s, \Pi^u_\eta(s)z), \tag{5.30}$$

with the property: given $\delta \in (0, \delta_0(\omega_\tau))$*, there exists* $0 < \delta'' < \delta' < \delta$*,*

$$\begin{aligned} &\{\Pi^u_\eta(s)z + \Sigma^u_\eta(s, \Pi^u_\eta(s)z) \,:\, \|z\|_X \leq \delta''\} \subset \\ &W^{u,\delta'}_\eta(0)(s) \subset \\ &\{\Pi^u_\eta(s)z + \Sigma^u_\eta(s, \Pi^u_\eta(s)z) \,:\, \|z\|_X \leq \delta\}. \end{aligned} \tag{5.31}$$

4. *For each* ω_τ *fixed, the family of graphs of the maps* $\{\Sigma_\eta\}_{\eta \in (0, \eta_0(\omega_\tau)]}$ *behaves continuously at* $\eta = 0$*:*

$$\sup_{t \leq s} \sup_{\|z\| \leq \delta_0(\omega_\tau)} \{\|\Pi^u_\eta(t) - \Pi^u_0\|_{\mathscr{L}(X)} + \|\Sigma^u_\eta(t, \Pi^u_\eta(t)z) - \Sigma^u_0(\Pi^u_0 z)\|_X\} \overset{\eta \to 0}{\to} 0, \ \forall s \in \mathbb{R}. \tag{5.32}$$

Proof. Item 1 is a corollary of Theorem 5.1.6 and Item 2 follows from the continuous dependence of projections, in the sense of (CARVALHO; LANGA; ROBINSON, 2013, Theorem 7.9) for evolution processes.

By hypotheses, let $\rho > 0$ be such that there is L satisfying (5.24), then there exists $\delta_0(\omega_\tau)$ such that (5.21) is satisfied for $z, \bar{z} \in B_X(0, \delta_0(\omega_\tau))$.

According to Remark 5.2.9 and Theorem 5.2.10, by a cut-off procedure, we obtain the desired function $\Sigma^u_\eta : \mathbb{R} \times B_X(0, \delta_0(\omega_\tau)) \to X$, for each $\eta \in (0, \eta_0(\omega_\tau)]$.

Thus, we only need to check (5.31). We claim that given $\delta \in (0, \delta_0(\omega_\tau))$, there exists $\delta' < \delta$ such that any global solution $\zeta : \mathbb{R} \to X$ of $\{\psi_\eta(t - s, \Theta_s \omega_\tau) : t \geq s\}$ on the unstable set such that $\|\zeta(s)\| \leq \delta'$ must satisfy $\|\zeta(t)\| \leq \delta$, for $t \leq s$.

Indeed, from (5.20), $\zeta^u(t) = \Pi^u_\eta(t)\zeta(t)$ satisfies

$$\begin{aligned} \zeta^u(t) = {} & \varphi_\eta(t - s, \Theta_s \omega_\tau)\Pi^u_\eta(s)\zeta_0 \\ & + \int_s^t \varphi_\eta(t - r, \Theta_r \omega_\tau)\Pi^u_\eta(r) h^u_\eta(\Theta_r \omega_\tau, \zeta^u(r) + \Sigma^u_\eta(r, \zeta^u(r)))dr, \ t \leq s. \end{aligned}$$

Since $\{\varphi_\eta(t - s, \Theta_s \omega_\tau) : t \geq s\}$ admits an exponential dichotomy, due to Grönwall's inequality, we obtain

$$\|\zeta^u(t)\|_X \leq M_\eta e^{(\alpha_\eta - \rho M_\eta(1+L))(t-s)} \|\zeta^u(s)\|_X, \ \ t \leq s.$$

Also, since $\|\Sigma_\eta^u(t,\zeta^u(t))\|_X \leq L\|\zeta^u(t)\|_X$, $t\in\mathbb{R}$, we have that

$$\|\zeta(t)\|_X \leq M_\eta^2(1+L)e^{(\alpha_\eta-\rho M_\eta(1+L))(t-s)}\|\zeta(s)\|_X, \ \ t\leq s. \tag{5.33}$$

Then, taking $\delta' = \delta/M_\eta^2(1+L)$, we see that

$$W_\eta^{u,\delta'}(0)(s) \subset \{\Pi_\eta^u(s)z + \Sigma_\eta^u(s,\Pi_\eta^u(s)z) : \|z\|_X \leq \delta\}. \tag{5.34}$$

Finally, by the above argument, we also conclude that there exists $\delta'' \in (0,\delta')$ such that

$$\{\Pi_\eta^u(s)z + \Sigma_\eta^u(s,\Pi_\eta^u(s)z) : \|z\|_X \leq \delta''\} \subset W_\eta^{u,\delta'}(0)(s). \tag{5.35}$$

The proof is complete. □

Remark 5.2.12. *We observe that, as in Theorem 5.1.6[Item 2], using Θ-invariant functions $\bar{\eta}:\mathbb{R}\times\Omega\to(0,1]$ it is possible to conclude existence of local unstable manifolds of the random hyperbolic solutions $\xi_{\bar{\eta}}^*$ for the nonautonomous random dynamical systems $\psi_{\bar{\eta}}$.*

We emphasize that these results on the existence and continuity of local unstable sets are the key to obtain lower semicontinuity and topological structural stability, as we will see in the following sections.

Remark 5.2.13. *We can obtain similar results concerning the existence and continuity of local stable sets following similar arguments to those presented here and (CARVALHO; LANGA, 2007b) for the deterministic case.*

5.3 Continuity of nonautonomous random attractors

In this section, we prove the continuity of attractors in the situation that the perturbed system is nonautonomous random whereas the limiting is an autonomous dynamical system which has an attractor given as union of unstable sets of hyperbolic equilibria.

First, we recall the definition of continuity for family of sets in Banach space X, for an introduction of these notions see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 3). .

Definition 5.3.1. *Let $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ be a family of subsets of a Banach space X. We say that $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is*

*(1) **Upper semicontinuous** at $\eta=0$ if $\lim_{\eta\to0} dist_H(A_\eta,A_0)=0$.*

*(2) **Lower semicontinuous** at $\eta=0$ if $\lim_{\eta\to0} dist_H(A_0,A_\eta)=0$.*

*(3) **Continuous** at $\eta=0$ if it is upper and lower semicontinuous at $\eta=0$.*

*Let Λ be a nonempty set. We say that $\{\mathscr{A}_\eta(\lambda):\lambda\in\Lambda\}_{\eta\in[0,1]}$ is **upper (lower) semicontinuous at** $\eta=0$ if $\{\mathscr{A}_\eta(\lambda)\}_{\eta\in[0,1]}$ is upper (lower) semicontinuous at $\eta=0$, for each $\lambda\in\Lambda$.*

The following lemma shows that continuity can be characterized by the behavior of sequences.

Lemma 5.3.2. *Let $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ be a family of subsets of a Banach space X.*

(1.1) *If $\mathscr{A}_0$ is compact and $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta=0$, then given $\eta_n\to 0^+$ as $n\to+\infty$ and $x_n\in\mathscr{A}_{\eta_n}$, there exists $\{\eta_{n_k}\}_k$ and $x_0\in\mathscr{A}_0$ such that $x_{n_k}\to x_0$ as $k\to\infty$.*

(1.2) *If for every sequences $\eta_n\to 0$ and $x_n\in\mathscr{A}_{\eta_n}$, there is a subsequence $\eta_{n_k k}$ and $x_0\in\mathscr{A}_0$ such that $x_{n_k}\to x_0$ as $k\to+\infty$, then $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta=0$.*

(2.1) *If $\mathscr{A}_0$ is compact and for each $x_0\in\mathscr{A}$ and $\eta_n\to 0$ as $n\to+\infty$ there exist there exists $\{\eta_{n_k}\}_k$ and $x_k\in\mathscr{A}_{\eta_{n_k}}$ such that $x_k\to x_0$ as $k\to+\infty$, then $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta=0$.*

(2.2) *If $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta=0$, then for each $x_0\in\mathscr{A}$ and $\eta_n\to 0$ as $n\to+\infty$ there exist there exists $\{\eta_{n_k}\}_k$ and $x_k\in\mathscr{A}_{\eta_{n_k}}$ such that $x_k\to x_0$ as $k\to+\infty$.*

For the proof see (CARVALHO; LANGA; ROBINSON, 2013, Lemma 3.2) and note that the compactness is only needed in items **(1.1)** and and **(2.1)**.

Now, we present a result on the continuity of attractors, as a consequence of a careful study of their internal structure, presented in the previews sections.

Theorem 5.3.3 (Continuity of nonautonomous random attractors)**.** *Let $\mathscr{T}_0=\{T_0(t):t\geq 0\}$ be the semigroup associated to (5.1) and (ψ_η,Θ) be the nonautonomous dynamical systems associated to (5.2), and assume that condition (5.3) is satisfied. Additionally, suppose that*

(a) *For each $\eta\in[0,1]$, the co-cycle (ψ_η,Θ) has a nonautonomous random attractor $\{\mathscr{A}_\eta(\omega_\tau):\omega_\tau\in\mathbb{R}\times\Omega\}$ and*

$$\overline{\bigcup_{t\in\mathbb{R}}\bigcup_{\eta\in[0,1]}\mathscr{A}_\eta(\Theta_t\omega_\tau)} \text{ is compact, } \forall\,\omega_\tau\in\mathbb{R}\times\Omega;$$

(b) *$\mathscr{T}_0=\{T_0(t):t\geq 0\}$ is a semigroup with global attractor given by*

$$\mathscr{A}_0=\bigcup_{j=1}^{p}W^u(y_j^*), \tag{5.36}$$

for which all the equilibria $\{y_j^*:1\leq j\leq p\}$ are hyperbolic.

Then given $\varepsilon_0>0$ suitable small, there exists a Θ-invariant function $\eta_0:\mathbb{R}\times\Omega\to(0,1]$ such that, for each ω_τ fixed, the following hold:

1. *For any* $\eta \in (0, \eta_0(\omega_\tau)]$ *and* $j \in \{1, \cdots, p\}$, *there exists a hyperbolic solution* $\xi^*_{j,\eta}$ *of* $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ *with*

$$\sup_j \sup_{t\in\mathbb{R}} \|\xi^*_{j,\eta}(\Theta_t\omega_\tau) - y^*_j\|_X < \varepsilon_0, \tag{5.37}$$

where the linearized associated evolution process admits an exponential dichotomy with family of projections $\{\Pi^u_{j,\eta}(s) : s \in \mathbb{R}\}$.

2. *There exists* $\delta_0(\omega_\tau) > 0$, *where* δ_0 *is* Θ*-invariant and independent of* η, *such that for each* ω_τ, $j \in \{1, \cdots, p\}$, *and* $\eta \in [0, \eta_0(\omega_\tau)]$, *there exists a map*

$$\mathbb{R} \times B_X(0, \delta_0(\omega_\tau)) \ni (s,z) \mapsto \Sigma^u_{j,\eta}(s,z) := \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z), \tag{5.38}$$

with the property: given $\delta \in (0, \delta_0(\omega_\tau))$, *there exists* $0 < \delta'' < \delta' < \delta$,

$$\begin{aligned} &\{\xi^*_{j,\eta}(s) + \Pi^u_{j,\eta}(s)z + \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z) \,:\, \|z\|_X \leq \delta''\} \subset \\ &W^{u,\delta'}_{j,\eta}(\xi^*_{j,\eta})(s) \subset \\ &\{\xi^*_{j,\eta}(s) + \Pi^u_{j,\eta}(s)z + \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z) \,:\, \|z\|_X \leq \delta\}; \end{aligned} \tag{5.39}$$

3. *The family of graphs of* $\{\Sigma^u_{j,\eta}\}_{\eta\in[0,\eta_0(\omega_\tau)]}$ *is continuous at* $\eta = 0$ *as in Theorem 5.2.11[Item (4)], for each* $j \in \{1, \cdots, p\}$.

4. *For each* ω_τ, *the family of pullback attractors* $\{\mathscr{A}_\eta(\Theta_t\omega_\tau) : t \in \mathbb{R}\}_{\eta\in[0,\eta_0(\omega_\tau)]}$ *is continuous at* $\eta = 0$.

In particular, we have continuity of nonautonomous random attractors in the following sense: given $\varepsilon > 0$, *there exists a* Θ*-invariant function* $\eta_\varepsilon \leq \eta_0$ *such that, for every* Θ*-invariant function* $\bar{\eta}$, *with* $\bar{\eta} \leq \eta_\varepsilon$, *we have*

$$\sup_{t\in\mathbb{R}} d_H(\mathscr{A}_{\bar{\eta}}(\Theta_t\omega_\tau), \mathscr{A}_0) < \varepsilon, \;\; \forall\, \omega_\tau \in \mathbb{R} \times \Omega, \tag{5.40}$$

where $\{\mathscr{A}_{\bar{\eta}}(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$ *is the nonautonomous random attractor of* $(\psi_{\bar{\eta}}, \Theta)$ *and* $d_H(A,B) = \max\{dist_H(A,B), dist_H(B,A)\}$, *for* $A, B \subset X$.

Proof. Note that, items 1,2 and 3 are consequences of Theorem 5.1.6 and Theorem 5.2.11, thus to conclude the proof we only need to prove Item 4.

Note that, from (5.3), we are able to prove that

$$\lim_{\eta\to 0} \sup_{t\in[0,T]} \sup_{s\in\mathbb{R}} \sup_{\|z\|_X\leq r} \|\psi_\eta(t, \Theta_s\omega_\tau)z - T_0(t)z\|_X \to 0, \tag{5.41}$$

for any $T, r > 0$ and $\omega_\tau \in \mathbb{R} \times \Omega$.

The proof of upper semicontinuity follows from standard arguments using (5.41) and Hypothesis **(a)**, see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 3) for pullback attractors, and (CARABALLO; LANGA, 2003; CARABALLO; LANGA; ROBINSON, 1998; WANG, 2012a) for random attractors.

Now, we prove lower semicontinuity using Lemma 5.3.2. In fact, let $\omega_\tau \in \mathbb{R}\times\Omega$, $t\in\mathbb{R}$, and $x_0 \in \mathscr{A}_0$, we will show that there exist sequences $\eta_k \in (0,\eta_0(\omega_\tau)]$, with $\eta_k \to 0$, and $x_k \in \mathscr{A}_{\eta_k}(\Theta_t\omega_\tau)$ such that $x_k \to x_0$ as $k\to+\infty$.

Indeed, from (5.36), $x_0 \in W^u(y_j^*)$ for some $j\in\{1,\cdots,p\}$. By Item 3 of Theorem 5.2.11, there exist $0<\delta''<\delta'<\delta_0(\omega_\tau)$ such that

$$W_0^{u,\delta''}(y_j^*) \subset \{y_j^* + \Pi_{j,0}^u z + \Sigma_0^u(\Pi_{j,0}^u z) : \|z\|_X \le \delta'\}, \text{ and} \tag{5.42}$$

$$\{\xi_{j,\eta}^*(r) + \Pi_{j,\eta}^u(r)z + \Sigma_{j,\eta}^u(r,\Pi_{j,\eta}^u(r)z) : \|z\|_X \le \delta'\} \subset W_\eta^{u,\delta_0}(\xi_{j,\eta}^*)(r), \tag{5.43}$$

for every $r\in\mathbb{R}$ and $\eta\in(0,\eta_0(\omega_\tau)]$. Thus there exists a global solution $\zeta:\mathbb{R}\to X$ of $\mathscr{T}_0$ such that $\zeta(0)=x_0$ and $\zeta(-s)\in W_0^{u,\delta''}(y_j^*)$, for some $s\ge 0$.

Since $\zeta(-s)\in\{y_j^*+\Pi_{j,0}^u z+\Sigma_{j,0}^u(\Pi_{j,0}^u z),\ \ \|z\|_X\le\delta'\}$, by Theorem 5.2.11[Item 4], there exist $\{\eta_k\}\subset(0,\eta_0(\omega_\tau)]$ and $z_k \in \{\xi_{j,\eta_k}^*(t-s)+\Pi_{j,\eta_k}^u(t-s)z+\Sigma_{j,\eta_k}^u(t-s,\Pi_{j,\eta_k}^u(t-s)z) : \|z\|_X\le\delta'\}$ with $\eta_k\to 0$ and $z_k\to\zeta(-s)$ as $k\to+\infty$.

By (5.43) and Lemma 5.2.7, we see that $x_k=\psi_{\eta_k}(t-(t-s),\Theta_{t-s}\omega_\tau)z_k \in \mathscr{A}_{\eta_k}(\Theta_t\omega_\tau)$, for all $k\in\mathbb{N}$. Then, we use (5.41) and that $\lim_k z_k=\zeta(-s)$, to guarantee that $\lim_k x_k = x_0$, and the proof is complete. □

Remark 5.3.4. *Theorem 5.3.3 can be extended to the case where the limit is nonautonomous. The key steps of the proof will be again the Θ-invariance for the maps involved.*

Remark 5.3.5. *Alternatively, Assumption **(a)** can be replaced by the following two conditions:*

(a.1) *For each $\eta\in[0,1]$, the co-cycle (ψ_η,Θ) has a nonautonomous random attractor $\{\mathscr{A}_\eta(\omega_\tau) : \omega_\tau\in\mathbb{R}\times\Omega\}$ and*

$$\overline{\bigcup_{t\in\mathbb{R}}\bigcup_{\eta\in[0,1]}\mathscr{A}_\eta(\Theta_t\omega_\tau)} \text{ is bounded, } \forall\,\omega_\tau\in\mathbb{R}\times\Omega;$$

(a.2) *The family $\{\psi_\eta,\Theta\}_{\eta\in[0,1]}$ is **collectively asymptotic compact** in X, i.e., for all ω_τ, the sequence*

$$\{\psi_{\eta_n}(t_n,\Theta_{-t_n}\omega_\tau)x_n\} \text{ has a convergent subsequence in } X$$

whenever $\eta_n\to 0$, $t_n\to+\infty$, and $\{x_n\}$ is a bounded sequence in X,

and Theorem 5.3.3 will still hold true. This will be the case when applying this result for damped wave equations, see Subsection 5.5.2.

Remark 5.3.6. *Theorem 5.3.3 is not optimal in the sense that we can not obtain the limit*

$$\sup_{\omega_\tau\in\mathbb{R}\times\Omega} d_H(\mathscr{A}_\eta(\omega_\tau),\mathscr{A}_0)\to 0, \quad \text{as } \eta\to 0. \tag{5.44}$$

To obtain this conclusion one should assume

$$\sup_{\omega_\tau\in\mathbb{R}\times\Omega}\sup_{x\in B(0,r)}\left\{\|f_\eta(\omega_\tau,x)-f_0(x)\|_X+\|(f_\eta)_x(\omega_\tau,x)-f_0'(x)\|_{\mathscr{L}(X)}\right\}\overset{\eta\to 0}{\to}0, \quad (5.45)$$

for all $r\geq 0$, instead of (5.3). In this case, it is possible to obtain the conclusions of Theorem 5.3.3 with $\eta_0>0$ and $\delta_0>0$ independent of ω_τ, and therefore to conclude (5.44). Note that this case is similar to the deterministic case, see (CARVALHO; LANGA; ROBINSON, 2009, Theorem 3.1).

However, in the applications to check condition (5.45) one has to assume that the noise is uniformly bounded as in Remark 5.5.4, see also (BOBRYK, 2021; CARABALLO et al., 2020) for more examples of uniformly bounded noises. On the other hand, in Section 5.5 we provide an example, namely Example 5.5.2, where conditions of Theorem 5.3.3 are checked, but we do not know if its possible to verify (5.45).

Now, that the study of continuity of attractor is complete, the next step is to prove that the gradient structure is preserved under nonautonomous random perturbation.

5.4 Topological structural stability

In this section, we present a result on the topological structural stability of attractors for nonautonomous random dynamical systems. We study co-cycles (ψ_η,Θ) obtained by nonautonomous random perturbations of a gradient semigroup $\{T_0(t): t\geq 0\}$.

First, we recall some basic concepts necessary to define *dynamically gradient evolution processes*.

Definition 5.4.1. *Let $\mathscr{S}=\{S(t,s): t\geq s\}$ be an evolution process with a pullback attractor $\{\mathscr{A}(t): t\in\mathbb{R}\}$ and $\widehat{E}=\{E(t): t\in\mathbb{R}\}$ be an invariant family for $\mathscr{S}$.*

1. *Given a family of open sets $\widehat{U}=\{U(t): t\in\mathbb{R}\}$ such that $\widehat{E}\subset\widehat{U}$ (i.e., $E(t)\subset U(t)$, for every $t\in\mathbb{R}$) we say that $\widehat{E}$ is the* ***maximal invariant*** *in $\widehat{U}$ if given an invariant family $\widehat{F}$ in $\widehat{U}$, then $\widehat{F}\subset\widehat{E}$.*

2. *If there is a $\varepsilon_0>0$ such that $\widehat{E}$ is the maximal invariant family in $\{O_{\varepsilon_0}(E(t)): t\in\mathbb{R}\}$, we say that $\widehat{E}$ is a* ***isolated invariant family****.*

3. *We say that $\{\widehat{E}_1,\cdots,\widehat{E}_p\}$ is a* ***disjoint collection of isolated invariant families*** *if $\widehat{E}_i$ is an isolated invariant family for every $0\leq i\leq p$ and there is $\varepsilon_0>0$ such that $O_{\varepsilon_0}(E_j(t))\cap O_{\varepsilon_0}(E_i(t))=\emptyset$, for $i\neq j$ and every $t\in\mathbb{R}$.*

4. *A* ***homoclinic structure*** *in $\{\widehat{E}_1,\cdots,\widehat{E}_p\}$ is a subcollection $\{\widehat{E}_{l_1},\cdots,\widehat{E}_{l_k}\}$, with $k\leq p$, and a set of global solutions $\{\zeta_1,\cdots,\zeta_k\}$ of (ψ,Θ) in $\mathscr{A}$ which, setting $\widehat{E}_{l_k+1}=\widehat{E}_{k_1}$, satisfy*

$$\lim_{t\to-\infty}d(\zeta_i(t),E_{l_i}(t))=0, \text{ and } \lim_{t\to+\infty}d(\zeta_i(t),E_{l_{i+1}}(t))=0, \quad (5.46)$$

for each $1 \leq i \leq k$, and there exists a $\varepsilon > 0$ such that

$$\sup_{t\in\mathbb{R}} d\big(\,\zeta_i(t), \bigcup_{i=1}^{p} O_\varepsilon(E_{l_i}(t))\,\big) > 0,\ \forall\, 1 \leq i \leq k,\ \textit{and}\ t \in \mathbb{R}\times\Omega. \tag{5.47}$$

Definition 5.4.2. *Let $\mathscr{S} = \{S(t,s) : t \geq s\}$ with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ which contains a disjoint collection of invariant families $\{E_1,\cdots,E_p\}$. We say that $\mathscr{S}$ is a **dynamically gradient evolution process** with respect to $\{\widehat{E}_1,\cdots,\widehat{E}_p\}$ if*

- ***(G1)** If $\zeta : \mathbb{R} \to X$ is a global solution of $\mathscr{S}$ such that $\zeta(t) \in \mathscr{A}(t)$, then there exist $i, j \in \{1,\cdots,p\}$ such that*

$$\lim_{t\to-\infty} d(\zeta(t),E_i(t)) = 0,\ \textit{and}\ \lim_{t\to+\infty} d(\zeta(t),E_j(t)) = 0. \tag{5.48}$$

- ***(G2)** $\{\widehat{E}_1,\cdots,\widehat{E}_p\}$ does not admit any homoclinic structure.*

This notion of dynamically gradient was studied for random dynamical systems in (CARABALLO; LANGA; LIU, 2012; JU; QI; WANG, 2018). For topological structural stability of deterministic autonomous or nonautonomous dynamical systems, see (ARAGãO-COSTA *et al.*, 2013; BORTOLAN *et al.*, 2020; CARVALHO; LANGA, 2009).

Now, we present our result on the topological structural stability for random dynamical systems.

Theorem 5.4.3. *Assume that hypotheses of Theorem 5.3.3 are fulfilled and additionally assume that $\mathscr{T}_0 = \{T_0(t-s) : t \geq s\}$ is a gradient evolution process with respect to $\{y_1^*,\cdots,y_p^*\}$, where y_j^* is hyperbolic for every $1 \leq j \leq p$.*

Then, there exists a Θ-invariant function $\eta_1 : \mathbb{R}\times\Omega \to (0,1)$ such that for each ω_τ fixed the evolution process $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \geq s\}$ is dynamically gradient with respect to $\{\xi_{1,\eta}^*,\cdots,\xi_{p,\eta}^*\}$, $\forall \eta \leq \eta_1(\omega_\tau)$. Consequently,

$$\mathscr{A}_\eta(\Theta_t\omega_\tau) = \bigcup_{j=1}^{p} W_\eta^u(\xi_{j,\eta}^*;\omega_\tau)(t), \forall \eta \in [0,\eta_1(\omega_\tau)]. \tag{5.49}$$

Proof. Let $\omega_\tau \in \mathbb{R}\times\Omega$ be fixed and $\eta \in (0,\eta_0(\omega_\tau)]$. Let us prove the following claim: there exists $\delta' \in (0,\delta_0(\omega_\tau))$ such that, if $\zeta_\eta : \mathbb{R} \to X$ is a global solution in $\{\mathscr{A}_\eta(\Theta_t\omega_\tau) : t \in \mathbb{R}\}$ so that

$$\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X < \delta',\ \ \forall t \leq t_0\ \ (t \geq t_0),\ \text{for some}\ t_0 \in \mathbb{R}, \tag{5.50}$$

then $\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X \overset{t\to-\infty}{\longrightarrow} 0$ $\ (\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X \overset{t\to+\infty}{\longrightarrow} 0)$.

We prove only the backwards case, the proof of the forward case will be similar using the analogous results for the stable sets. First, note that $\tilde{\zeta}(t) = \zeta_\eta(t) - \xi_{j,\eta}^*(t)$, for $t \in \mathbb{R}$, $j \in$

$\{1,\cdots,p\}$, and $\eta \in (0,\eta_0(\omega_\tau)]$, thus we analyze the dynamics around the solution $z=0$ of (5.17). From Theorem 5.2.11[Item 3], there exists $0<\delta'<\delta<\delta_0(\omega_\tau)$ such that

$$\{\Pi^u_{j,\eta}(s)z+\Sigma^u_{j,\eta}(s,\Pi^u_{j,\eta}(s)z) : \|z\|_X \le \delta'\} \subset W^{u,\delta}_\eta(0)(s), \forall s\in\mathbb{R}. \tag{5.51}$$

Thus, (5.50) implies that $\tilde{\zeta}(t)$ is inside the $\delta_0(\omega_\tau)$-neighborhood for all $t\le t_0$.

Hence, from (5.27) applied in the $\delta_0(\omega_\tau)$-neighborhood of $z=0$, we must have that $\tilde{\zeta}(t_0)\in\{\Pi^u_{j,\eta}(t_0)z+\Sigma^u_{j,\eta}(t_0,\Pi^u_{j,\eta}(t_0)z) : \|z\|_X\le\delta'\}$. Therefore, from (5.51), $\tilde{\zeta}(t_0)\in W^{u,\delta}_\eta(0)(t_0)$ and the proof of the claim is complete.

In this way, the proof will be a consequence of (BORTOLAN; CARVALHO; LANGA, 2020, Theorem 8.14). □

Remark 5.4.4. *Note that, if we assume (5.45) in Theorem 5.3.3 instead of (5.3), we obtain $\eta_1>0$, independent of ω_τ, such that $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t\ge s\}$ is a dynamically gradient evolution process with respect to $\{\xi^*_{1,\eta},\cdots,\xi^*_{p,\eta}\}$, $\forall\eta\le\eta_1$. In this case, this notion of dynamically gradient is compatible with the notion that appears in (CARABALLO; LANGA; LIU, 2012, Definition 4.17).*

Remark 5.4.5. *We believe that with the techniques employed in this chapter it is also possible to obtain geometric structural stability, i.e., to show that Morse-Smale is stable under nonautonomous random perturbations and that there will be phase diagram isomorphism between the perturbed attractors and the limiting attractor, as we see in the deterministic case (BORTOLAN; CARVALHO; LANGA, 2020, Chapter 12). This will be pursued in a future work.*

5.5 Applications to differential equations

In this section, we present two applications. We first consider a semilinear differential equation with a small nonautonomous multiplicative white noise, and then we study the effect of a small bounded noise in the damping of a damped wave equation.

5.5.1 Stochastic differential equations

We consider the following family of stochastic differential equations with a nonautonomous multiplicative white noise

$$dy = Bydt + f(y)dt + \eta\kappa_t y\circ dW_t,\ t\ge\tau,\ y(\tau)=y_\tau, \tag{5.52}$$

where B is a generator of a C^0-semigroup $\{e^{Bt} : t\ge 0\}$ on X, the family $\{W_t : t\in\mathbb{R}\}$ is the standard Wiener process, see (ARNOLD, 1998; CARABALLO; HAN, 2016), and $\kappa:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, and $\eta>0$. Equation (5.52) was considered in (CARABALLO *et al.*, 2021b) to study hyperbolicity. Next, we will modify problem (5.52) to see it as a nonautonomous random differential equation satisfying the conditions of our results on the continuity and topological structure stability of attractors.

The canonical sample space of a Wiener process is $\Omega := C_0(\mathbb{R})$ the set of continuous functions over $\mathbb{R}$ which are 0 at 0 equipped with the compact open topology. We denote $\mathscr{F}$ the associated Borel σ-algebra. Let $\mathbb{P}$ be the Wiener probability measure on $\mathscr{F}$ which is given by the distribution of a two-sided Wiener process with trajectories in $C_0(\mathbb{R})$. The flow θ is given by the Wiener shifts

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t),\ t \in \mathbb{R},\ \omega \in \Omega.$$

Lemma 5.5.1. *Consider the following scalar stochastic differential equation*

$$dz_t + zdt = dW_t. \tag{5.53}$$

There exists a θ-invariant subset $\tilde{\Omega} \in \mathscr{F}$ of full measure such that $\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0,\ \omega \in \tilde{\Omega}$ and, for such ω, the random variable given by

$$z^*(\omega) = -\int_{-\infty}^{0} e^s \omega(s) ds$$

is well defined. Moreover, for $\omega \in \tilde{\Omega}$, the mapping $(t,\omega) \mapsto z^*(\theta_t\omega)$ is a stationary solution of (5.53) *with continuous trajectories, and*

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{t} = 0,\ \forall\, \omega \in \tilde{\Omega}. \tag{5.54}$$

For the proof of Lemma 5.5.1 see (CARABALLO; KLOEDEN; SCHMALFUSS, 2004, Lemma 4.1).

Let y be a solution for (5.52) and consider $v(t,\omega) := e^{-\eta\kappa_t z^*(\theta_t\omega)} y(t,\omega)$. Hence, v has to satisfy the following nonautonomous random differential equation

$$\dot{v} = Bv + e^{-\eta\kappa_t z^*(\theta_t\omega)} f(e^{\eta\kappa_t z^*(\theta_t\omega)} v) + \eta[\kappa_t - \dot{\kappa}_t] z^*(\theta_t\omega) v, \tag{5.55}$$

Define $f_\eta(t,\omega,v) := e^{-\eta\kappa_t z^*(\omega)} f(e^{\eta\kappa_t z^*(\omega)} v) + \eta[\kappa_t - \dot{\kappa}_t] z^*(\omega) v$.

Since the mapping $t \mapsto z^*(\theta_t\omega)$ has a sublinear growth, due to (5.54), it is possible to choose a differential real function κ for which there are random variables $m_1, m_2 > 0$ such that

$$m_1(\omega) := \sup_{t\in\mathbb{R}}\{|\kappa_t z^*(\theta_t\omega)|\} < \infty, \text{ and } m_2(\omega) := \sup_{t\in\mathbb{R}}\{|[\kappa_t - \dot{\kappa}_t] z^*(\theta_t\omega)|\} < \infty.$$

Thus, using arguments similar to those of (CARABALLO *et al.*, 2021b, Section 3.3) we prove that the family $\{f_\eta : \eta \in [0,1]\}$ satisfies (5.3).

At this point, one can choose any gradient semigroup associated to $\dot{y} = By + f(y)$ and consider the perturbation $\eta\kappa_t y \circ dW_t$ and apply our results to the modified differential equation (5.55). In particular:

Example 5.5.2. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a smooth real-valued function and $f(x) = -\nabla F(x)$, $x \in \mathbb{R}^N$, and consider*

$$\dot{x} = f(x) + \eta\kappa_t x \circ dW_t,\ t > 0.$$

When $\eta = 0$ this is called a gradient system. Then we obtain the nonautonomous random differential equations

$$\dot{x} = e^{-\eta \kappa_t z^*(\theta_t \omega)} f(e^{\eta \kappa_t z^*(\theta_t \omega)} x) + \eta [\kappa_t - \dot{\kappa}_t] z^*(\theta_t \omega) x, \ \eta \in [0,1]. \tag{5.56}$$

Assume that there exists $R_0, \sigma > 0$ such that

$$f(x) \cdot x < -\sigma, \text{ for all } |x| \geq R_0, \tag{5.57}$$

and that the set $\{x \in \mathbb{R}^N : f(x) = 0\}$ is finite and consist only in hyperbolic equilibria. Then, $\dot{x} = f(x)$ is globally well posed and its associated with a semigroup $\{T_0(t) : t \geq 0\}$, which is gradient with respect to $\{x_1^*, \cdots, x_p^*\}$.

Then, the nonautonomous random dynamical systems associated to (5.56) have attractors $\{A_\eta(\omega_\tau) : \omega_\tau \in \mathbb{R}\}$, and this family of attractors satisfies the conclusions of Theorem 5.3.3 and Theorem 5.4.3.

5.5.2 An application to partial differential equation

Now, we provide an application for a damped wave equation.

Consider the damped wave equation

$$u_{tt} + \beta u_t - \Delta u = f(u), \text{ in } D \tag{5.58}$$

with boundary condition $u = 0$, in ∂D, where D be a bounded smooth domain in $\mathbb{R}^3$, and $\beta \in (0, +\infty)$. For $f : \mathbb{R} \to \mathbb{R}$ we assume that

$$f \in C^2(\mathbb{R}), \ |f''(s)| \leq c(1 + |s|), \tag{5.59}$$

for some $c > 0$. Now, we consider a small random perturbation on the damping,

$$u_{tt} + \beta_\eta(\Theta_t \omega) u_t - \Delta u = f(u), \text{ in } D.$$

where $\beta_\eta(\omega_\tau) := \beta + \eta |\kappa_\tau z^*(\omega)|$, $\eta \in [0,1]$, $\omega_\tau \in \mathbb{R} \times \Omega$, for some κ such that $\sup_{t \in \mathbb{R}} \{|\kappa_t z^*(\theta_t \omega)|\} < \infty$, Thus, there exists two Θ-invariant maps $b_0, b_1 : \mathbb{R} \times \Omega \to (0, +\infty)$ such that $b_0(\omega_\tau) \leq \beta_\eta(\Theta_t \omega_\tau) \leq b_1(\omega_\tau)$, for every $\omega_\tau \in \mathbb{R} \times \Omega$.

The initial data will be taken in the space $X = H_0^1(D) \times L^2(D)$. Hence, we obtain the family of abstract evolutionary equations in X

$$\dot{y} = B_\eta(\Theta_t \omega_\tau) y + F(y), \ \eta \in [0,1] \tag{5.60}$$

where

$$y = \begin{pmatrix} u \\ v \end{pmatrix} \in X, \ B_\eta(\omega_\tau) = \begin{pmatrix} 0 & I \\ -A & -\beta_\eta(\omega_\tau) \end{pmatrix}, \ F(y) = \begin{pmatrix} 0 \\ f^e(u) \end{pmatrix},$$

where $A : D(A) \subset L^2(D) \to L^2(D)$ is $-\Delta$ with Dirichlet boundary condition, with $f^e : H_0^1(D) \to L^2(D)$ is given by $f^e(y_1)(x) = f(y_1(x))$ for $x \in D$. Thus, conditions (5.59) implies local and global well-posedness and that f^e is continuously differentiable, see (ARRIETA; CARVALHO; HALE, 1992) or (CARVALHO; LANGA; ROBINSON, 2013, Chapter 15) for details.

Additionally, condition (5.59) also implies that f^e satisfies

$$\sup_{\|u\|_{H_0^1(D)} \le r} \sup_{\|h\|_{H_0^1} \le \varepsilon} \left\{ \frac{\|f^e(u+h) - f^e(u) - f^{e\prime}(u)h\|_{L^2(D)}}{\|h\|_{H_0^1(D)}} \right\} \to 0, \text{ as } \varepsilon \to 0.$$

for any $r > 0$. This implies that F satisfies (5.5) with $y_0^* = 0$ and for any $r > 0$.

Consider the functional $V : H_0^1(D) \times L^2(D) \to \mathbb{R}$ given by

$$V_0(u,v) = \frac{1}{2}\int_D |\nabla u|^2 + \frac{\beta}{2}\int_D v^2 - \int_D G(u), \tag{5.61}$$

where $G(u)(x) = \int_0^{u(x)} f(s)\,ds$. Thus V_0 is a Lyapunov function relative to the set of equilibria for (5.58), which we assume that is finite. The hyperbolic equilibrium points of (5.58) are of the form $y_0^* = (u_0^*, 0)$ where u_0^* is a solution of $-\Delta u = f(u)$ such that $0 \notin \sigma(-\Delta + D_x f^e(u_0^*) Id_X)$. Thus (5.58) is associated with a gradient semigroup $\{T_0(t) : t \ge 0\}$. See (BRUNOVSKY; RAUGEL, 2003) for conditions to obtain that this type of dynamics is generic on damped wave equations.

For each $y_0 \in X$, $\omega_\tau \in \mathbb{R} \times \Omega$, and $\eta \in [0,1]$ Equation (5.60) possess a unique solution which can be written as

$$\psi_\eta(t,\omega_\tau)y_0 = \varphi_\eta(t,\omega_\tau)y_0 + \phi_\eta(t,\omega_\tau)y_0, \;\; t \ge 0. \tag{5.62}$$

where $\{\varphi_\eta(t,\omega) : t \in [0,+\infty),\, \omega \in \Omega\}$ is the solution operator of (5.60) with $f = 0$, and

$$\phi_\eta(t,\omega_\tau)y_0 = \int_0^t \varphi_\eta(t-s,\Theta_s\omega_\tau) F(\psi_\eta(s,\omega_\tau)y_0) ds. \tag{5.63}$$

Towards the existence of attractors, we have the following lemma.

Lemma 5.5.3. *There exists a bounded subset B (independent of (t,ω)) which pullback attracts at time $\tau \in \mathbb{R}$, for each $\tau \le t$, every bounded subset of X under the action of $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$. In particular, $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$ is strongly pullback dissipative.*

Furthermore, there are $K > 0$ and a Θ-invariant function $\alpha : \mathbb{R} \times \Omega \to (0,+\infty)$, both independent of η, such that

$$\|\varphi_\eta(t,\omega_\tau)\|_{\mathscr{L}(X)} \le K e^{-\alpha(\omega_\tau)t}, \; t \ge 0, \tag{5.64}$$

and $\phi_\eta(t,\omega_\tau)$ is a compact operator for every $(t,\omega_\tau) \in (0,+\infty) \times \mathbb{R} \times \Omega$. In particular, ψ_η is pullback asymptotically compact for each $\eta \in [0,1]$, in the sense of (WANG, 2012b, Definition 2.14).

The proof of Lemma 5.5.3 follows step by step the arguments presented in (CARABALLO *et al.*, 2010a, Section 2.1) (or see (CARVALHO; LANGA; ROBINSON, 2013, Chapter 15) for more detailed proofs), thus it will be omitted. Thus there are nonautonomous random attractors $\{\mathscr{A}_\eta(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$ for (ψ_η, Θ) for all $\eta \in [0,1]$ satisfying Condition **(a.1)** of Remark 5.3.5, see alternatively (WANG, 2012b). Additionally, using arguments similar to those in (CARABALLO *et al.*, 2010a) the family $\{(\psi_\eta, \Theta)\}_{\eta\in[0,1]}$ is collectively pullback asymptotically compact at $\eta = 0$. Therefore, conditions of Remark 5.3.5 are satisfied and it is possible to apply our results to conclude that the family of attractors behaves continuously (using Theorem 5.3.3) and that we have topological structural stability (using Theorem 5.4.3).

Remark 5.5.4. *Instead of considering* $\beta_\eta(\omega_\tau) := \beta + \eta|\kappa_\tau z^*(\omega)|$, *we could have considered the following perturbations*

$$\tilde{\beta}_\eta(\omega) = \beta + \eta\frac{2}{\pi}\arctan\circ z^*(\omega),\ \omega\in\Omega, \eta\in[0,1], \tag{5.65}$$

for $\beta \in (1,+\infty)$. *For this perturbations a condition as* (5.45) *is verify for the symbol space* Ω *instead of* $\mathbb{R}\times\Omega$. *See also (CARABALLO; LÓPEZ-DE-LA-CRUZ, 2021) where the authors study this type of perturbations.*

CHAPTER

6

CONCLUSION: ENGLISH, PORTUGUESE AND SPANISH

6.1 Conclusion

The discretization of evolution processes employed to study nonuniform exponential dichotomies, namely Theorem 2.2.6 and Theorem 2.2.10, allowed us to compare continuous and discrete dynamical systems that exhibit nonuniform hyperbolicity. This method was introduced by (HENRY, 1981) in the case of uniform exponential dichotomies. Through this technique we establish the following results to the nonuniform case: uniqueness of the family of projections (Corollary 2.2.8); continuous dependence of projections (Theorem 2.2.9); and robustness of nonuniform exponential dichotomies (Theorem 2.2.11 and Theorem 3.3.3).

Moreover, it was possible to prove the robustness result with the assumption $\alpha > \nu$, which is the sharpest condition we can get with this technique. We also note that condition (2.5) is not required in (BARREIRA; VALLS, 2015), while is needed when applying this discretization method. Therefore, we obtain an improvement on the exponents of the robustness result of (BARREIRA; VALLS, 2015) at the price of having to assume (2.5). However, it is not a restrictive condition when dealing with evolution processes with nonuniform exponential dichotomies, see Example 2.11 or (BARREIRA; VALLS, 1998).

The continuous dependence of projections and the persistence of hyperbolic solutions play an important role in the study of continuity of local unstable sets and the latter in the study of attractors under perturbation, as we have seen in Chapter 5. However, it is not clear yet how to apply the results of stability of nonuniform hyperbolicity in the theory of attractors under perturbation. On the other hand, the persistence of nonuniform hyperbolic solutions and continuous dependence of projections should be important to study continuity of invariant manifolds associated to the nonuniform hyperbolic solutions. This, in turn, will be crucial in a possible application in the theory of attractors. In (LANGA; OBAYA; OLIVEIRA-SOUSA,

2021) is presented applications of nonuniform exponential dichotomy of type II, studied in Chapter 3, in the existence of pullback and forward attractors for evolution processes.

Combining the ideas of Chapter 2 and Chapter 4 it is possible to prove robustness of nonuniform exponential dichotomies for nonautonomos random dynamical systems. Since in Chapter 4 our goal was to study the effect of a small bounded noise on autonomous problems, it was not expected to obtain nonuniform behavior on our hyperbolicity. We emphasize that to consider a bounded noise was a crucial to prove permanence of the hyperbolicity and it is sensible in real life applications (CARABALLO *et al.*, 2019; CARABALLO *et al.*, 2018; CARR, 2017).

The robustness of exponential dichotomy for nonautonomous random dynamical systems is a fundamental property in the study of stability results for random dynamics. We were able to study hyperbolicity for nonautonomous random differential equations and obtained global solutions that behave as hyperbolic equilibria. As in (BORTOLAN; CARVALHO; LANGA, 2014; CARVALHO; LANGA, 2007b; CARVALHO; LANGA; ROBINSON, 2009) this was an important step in order to obtain continuity and structure stability of attractors on nonautonomous random attractors in Chapter 5. Finally, we note that the results of Chapter 4 and Chapter 5 can also by applied for general non-compact random dynamical systems, see (WANG, 2012b) for a formal definition.

6.2 Conclusão

A discretização de processos de evolução utilizada para estudar dicotomias exponenciais não uniformes, Teorema 2.2.6 e Teorema 2.2.10, nos permitiu comparar sistemas dinâmicos contínuos e discretos que exibem hiperbolicidade não uniforme. Este método foi introduzido por (HENRY, 1981) no caso de dicotomias exponenciais uniformes. Por meio dessa técnica, estabelecemos os seguintes resultados para o caso não uniforme: unicidade da família de projeções (Corolário 2.2.8); dependência continua das projeções (Teorema 2.2.8); robusteza de dicotomias exponenciais (Teorema 2.2.11 e Teorema 3.3.3).

Além disso, foi possível comprovar o resultado de robusteza com a condição $\alpha > \nu$, que é a condição mais fina que podemos obter com esta técnica. Também observamos que a condição (2.5) não é necessária em (BARREIRA; VALLS, 2015), porém é requerida quando empregamos esse método de discretização. Deste modo, obtemos uma melhora do resultado de (BARREIRA; VALLS, 2015) com relação aos expoentes pagando o preço de ter que assumir (2.5). No entanto, (2.5) não é uma condição restritiva quando se trata de processos de evolução com dicotomias exponenciais não uniformes com pode ser observado em (BARREIRA; VALLS, 1998) ou no Exemplo 2.11.

Como vimos no Capítulo 5, a dependência contínua de projeções e a persistência de soluções hiperbólicas desempenham um papel importante no estudo da continuidade de conjuntos instáveis locais e este último no estudo de atratores sob perturbação. No entanto, ainda não está claro como aplicar os resultados da estabilidade da hiperbolicidade não uniforme na teoria dos atratores sob perturbação. Por outro lado, a persistência de soluções hiperbólicas não uniformes e a dependência contínua de projeções devem ser importantes para estudar a continuidade de variedades invariantes associadas às soluções hiperbólicas não uniformes. Notamos que em (LANGA; OBAYA; OLIVEIRA-SOUSA, 2021) são apresentadas aplicações da dicotomia exponencial não uniforme do tipo II, estudada no Capítulo 3, a teoria de existência de atratores pullback and forward para processos de evolução.

Combinando as ideais dos capítulos 2 e 4 é possível provar a robustez de dicotomias exponenciais não uniformes para sistemas dinâmicos aleatórios não uniformes. Porém, dado que nos capítulos 4 e 5 o nosso objetivo era estudar o efeito de um pequeno ruído limitado em problemas autônomos, não era esperado obter um comportamento não uniforme em nossa hiperbolicidade. Enfatizamos que considerar um ruído limitado foi crucial para provar a permanência da hiperbolicidade e que é sensato em aplicações (CARABALLO *et al.*, 2019; CARABALLO *et al.*, 2018; CARR, 2017).

A robustez da dicotomia exponencial para sistemas dinâmicos aleatórios não autônomos é uma propriedade fundamental no estudo de resultados de estabilidade para dinâmica aleatória. Fomos capazes de estudar a hiperbolicidade para equações diferenciais aleatórias não autônomas e obter soluções globais que se comportam como equilíbrios hiperbólicos. Como em (BORTOLAN; CARVALHO; LANGA, 2014; CARVALHO; LANGA, 2007b; CARVALHO; LANGA; ROBINSON, 2009) esse foi um passo importante para compreender a continuidade e estabilidade estrutural de atratores aleatórios não autônomos no Capítulo 5. Finalmente, notamos que os resultados dos capítulos 4 e 5 podem ser aplicados a sistemas dinâmicos não autônomos definidos em espaços de símbolos não compactos, ver (WANG, 2012b) para uma definição formal.

6.3 Conclusión

La discretización de los procesos de evolución empleada para estudiar dicotomías exponenciales no uniformes, a saber, el Teorema 2.2.6 y Teorema 2.2.10, nos permitió comparar sistemas dinámicos continuos y discretos que exhiben hiperbolicidad no uniforme. Este método fue introducido por (HENRY, 1981) en el caso de dicotomías exponenciales uniformes. Mediante esta técnica establecemos los siguientes resultados del caso no uniforme: unicidad de la familia de proyecciones (Corolario 2.2.8); dependencia continua de las proyecciones (Teorema 2.2.8); robustez de dicotomías exponenciales (Teorema 2.2.11 y Teorema 3.3.3).

Además, fue posible probar el resultado de robustez con la condición $\alpha > \nu$, que es la más precisa que podemos obtener con esta técnica. También notamos que la condición (2.5) no se requiere en (BARREIRA; VALLS, 2015), y es necesaria cuando se aplica este método de discretización. Por tanto, obtenemos una mejora en los exponentes del resultado de robustez de

(BARREIRA; VALLS, 2015) a cambio de tener que asumir la condición (2.5). Sin embargo, no es una condición restrictiva cuando se trata de procesos de evolución con dicotomías exponenciales no uniformes, ver Ejemplo 2.11 o (BARREIRA; VALLS, 1998).

Como hemos visto en el Capítulo 5, la dependencia continua de las proyecciones y la persistencia de las soluciones hiperbólicas juegan un papel importante en el estudio de la continuidad de los conjuntos inestables locales y en el estudio de los atractores bajo perturbación. Sin embargo, aún no está claro cómo aplicar los resultados de la estabilidad de la hiperbolicidad no uniforme en la teoría de los atractores bajo perturbación. Por otro lado, la persistencia de soluciones hiperbólicas no uniformes y la dependencia continua de las proyecciones pudieron ser importantes para estudiar la continuidad de las variedades invariantes asociadas a las soluciones hiperbólicas no uniformes. Observamos que, en (LANGA; OBAYA; OLIVEIRA-SOUSA, 2021) se presentan aplicaciones de la dicotomía exponencial no uniforme de tipo II, estudiada en el Capítulo 3, en la existencia de atractores pullback y forward para los procesos de evolución.

Combinando las ideas del Capítulo 2 y del Capítulo 4 es posible probar la robustez de dicotomías exponenciales no uniformes para sistemas dinámicos no autónomos aleatorios. Dado que nuestro objetivo era estudiar el efecto de un pequeño ruido acotado en problemas autónomos, no se esperaba obtener un comportamiento no uniforme en nuestra hiperbolicidad. Destacamos que considerar un ruido acotado fue crucial para demostrar la permanencia de la hiperbolicidad y que es sensato en aplicaciones de la vida real (CARABALLO *et al.*, 2019; CARABALLO *et al.*, 2018; CARR, 2017).

La robustez de la dicotomía exponencial para los sistemas dinámicos aleatorios no autónomos es una propiedad fundamental en el estudio de los resultados de estabilidad para la dinámica aleatoria. Pudimos estudiar la hiperbolicidad para ecuaciones diferenciales aleatorias no autónomas y obtuvimos soluciones globales que se comportan como equilibrios hiperbólicos. Como en (BORTOLAN; CARVALHO; LANGA, 2014; CARVALHO; LANGA, 2007b; CARVALHO; LANGA; ROBINSON, 2009), este fue un paso importante para obtener la continuidad y estabilidad de la estructura de los atractores en atractores aleatorios no autónomos en el Capítulo 5. Finalmente, notamos que los resultados del Capítulo 4 y del Capítulo 5 también se pueden aplicar para sistemas dinámicos aleatorios no compactos generales, consulte (WANG, 2012b) para la definición.

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