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On exponential dichotomy and frameworks for rigorous computation for differential equations

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This work is dedicated to Kátia and Heitor,
with whom every single day
is a piece of happiness.

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"The only people who achieve much are those who want knowledge so badly that they seek it while the conditions are still unfavorable. Favorable conditions never come."

(C. S. Lewis)

RESUMO

NAKASSIMA, G. K. **Sobre dicotomia exponencial e abordagens para computação rigorosa para equações diferenciais**. 2023. 142 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

Neste trabalho mostramos pesquisas realizadas em três diferentes tópicos: robustez da estabilidade assintótica e dicotomia exponencial para uma classe de equações diferenciais em espaços de Banach; uma abordagem baseada em wavelets para métodos numéricos para equações diferenciais com validação automática a-posteriori, utilizando o método dos polinômios radiais; e o algoritmo Generalized Combinatorial Marching Hypercubes para geração de variedades em altas dimensões, com técnicas combinatórias para maior eficiência computacional.

Palavras-chave: Robustez da estabilidade, Dicotomia exponencial, Métodos computacionais rigorosos, Wavelet de Haar, Combinatorial Marching Hypercubes.

ABSTRACT

NAKASSIMA, G. K. **On exponential dichotomy and frameworks for rigorous computation for differential equations**. 2023. 142 p. Tese (Doutorado em Ciências – Ciências de Computação e Matemática Computacional) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2023.

In this work we show research results on three different topics: robustness of asymptotic stability and exponential dichotomy for a class of differential equations in Banach spaces; a wavelet-based approach for a-posteriori self-validating numerical methods for differential equations, using the radii polynomial method; and the Generalized Combinatorial Marching Hypercubes algorithm for generation of manifolds in high dimensions, using combinatorial techniques to improve computational efficiency.

Keywords: Robustness of stability, Exponential dichotomy, Rigorous computational methods, Haar wavelets, Combinatorial Marching Hypercubes.

CONTENTS

CHAPTER

1

INTRODUCTION

In this work we show results related to differential equations and dynamical systems, from both theoretical and computational points of view. We show research papers for three different topics:

- Robustness of the asymptotic stability and exponential dichotomy for a class of ordinary differential equations in Banach spaces (RODRIGUES; SOLÀ-MORALES; NAKASSIMA, 2020; RODRIGUES; CARABALLO; NAKASSIMA, 2022);
- Validated numerics for differential equations combining Haar wavelets with the radii polynomial approach (NAKASSIMA; GAMEIRO, Submitted for publication);
- A Generalized Combinatorial Marching Hypercubes (GCMH) algorithm, which extends the classic Marching Cubes algorithm for isossurface generation to higher dimensions in a computationally efficient way(CASTELO *et al.*, Submitted for publication).

The first two papers explore more theoretical results, related to structural robustness of dynamical systems. The third paper presents a method for computer-assisted proofs for theorems on differential equations; for example, existence of special structures such as periodic orbits. The fourth paper presents a numerical method to compute approximations to solution manifolds. We aim to use this manifold approximation as a basis for computer-assisted proof methods for multi-parameter differential equations (for example, adapting the algorithm in the third paper); this is work in progress.

We briefly summarize the main ideas and contributions of each paper; more details can be found in the papers themselves.

1.1 Robustness of asymptotic stability and exponential dichotomy

These two papers (RODRIGUES; SOLÀ-MORALES; NAKASSIMA, 2020; RODRIGUES; CARABALLO; NAKASSIMA, 2022) explore robustness characteristics of differential equations in Banach spaces. Given a Banach space $\mathbb{X}$ (not necessarily finite-dimensional) and the space $L(\mathbb{X})$ of all bounded linear operators from $\mathbb{X}$ to itself, we consider the system

$$\dot{x} = A(t)x \quad , \quad A(t) \in L(\mathbb{X}) \text{ for all } t \in \mathbb{R}$$

and we disturb it as follows:

$$\dot{x} = A(t)x + B(t)x \quad , \quad A(t),\, B(t) \in L(\mathbb{X}) \text{ for all } t \in \mathbb{R}$$

It is assumed that $A(t)$ and $B(t)$ belong to a class of functions called Generalized Almost Periodic ($\mathscr{GAP}$) functions, which as the name suggests are a generalization of almost periodic functions (FINK, 2006). This $\mathscr{GAP}$ class is presented in the first paper.

The main contribution of these works is to extend results known for finite-dimensional systems (see e.g. (COPPEL, 1965; COPPEL, 1978; HENRY, 1981)) to infinite-dimensional spaces. As an example of application, many partial differential equations can be recast as abstract differential equations in Banach spaces, by defining the partial derivative as a linear operator. This allows us to use techniques from functional analysis to study such equations without explicitly finding their solutions (EVANS, 2010). However, infinite-dimensional spaces can behave differently than finite dimensional ones; for example, a classical result is that the unit ball in a finite-dimensional space is always compact, but it might not be in an infinite-dimensional space (BREZIS, 2010). Thus the techniques for the extension of the results need to be adapted.

The first paper discuss robustness characteristics of asymptotic stability of differential equations after perturbation. It is shown that asymptotic stability is preserved under integrally small perturbations, that is, if $\left|\int_{t_1}^{t_2} B(t)\, dt\right| \leq \delta$ when $|t_2 - t_1| \leq h$. An important observation is that $B(t)$ does not necessarily need to be small, provided it "oscillates" rapidly. This is made precise in the papers for the $\mathscr{GAP}$ class of functions; in the usual periodic setting, it means that the amplitude of the oscillation can be rather large as long as it has a high enough frequency. Also, when $A(t)$ is an unbounded operator, we still obtain that the perturbed system is asymptotically stable if certain additional conditions are met.

Moreover, we show two striking examples where an asymptotically unstable system is stabilized, one in two dimensions and another in infinite dimensions; this shows that asymptotic instability is *not* robust. This example was based on an example by Kakutani (RICKART, 1960). These results may have impact in areas such as control theory.

The second paper extends the first by studying the robustness of exponential dichotomy under similar perturbations. We show that, if the first system exhibits an exponential dichotomy, the perturbed system also exhibits an exponential dichotomy when the perturbation is integrally small. When $A(t)$ is unbounded, we also obtain an exponential dichotomy if additional conditions similar to the first paper are met. We apply those results to examples in infinite dimensions in both cases.

1.2 Rigorous computation using wavelets

In this paper (NAKASSIMA; GAMEIRO, Submitted for publication) we present a framework for a rigorous computational method using Haar wavelets by combining two techniques: the *radii polynomial approach* and the *Haar wavelet method*.

Consider an initial value problem

$$\dot{x} = f(x) \quad , \quad x(0) = x_0.$$

If $\dot{x} \in L^2([0,1])$, we can expand the derivative $\dot{x}(t)$ in a Haar wavelet series, and thus the solution $x(t)$ can be expanded using the integral of the wavelets $w(t)$:

$$\dot{x}(t) = \sum_{i=1}^{\infty} c_i \psi_i(t) \quad , \quad x(t) = x_0 + \sum_{i=0}^{\infty} c_i w_i(t)$$

and $\mathbf{c} := (c_i)_{i=1}^{\infty} \in \ell^2(\mathbb{R})$. Thus the diferential equation turns into a functional equation

$$F(\mathbf{c}) = 0$$

where $F : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$. Thus, we can find a numerical, finite-dimensional solution $\bar{\mathbf{c}}$ by truncating the above equation and applying an usual numerical method, such as Newton's method. This is known as the *Haar wavelet method* (CHEN; HSIAO, 1997; LEPIK, 2006; MAJAK *et al.*, 2015; MEHANDIRATTA; MEHRA; LEUGERING, 2020).

Our contribution is to develop an a-posteriori verification method which rigorously proves the existence of a true solution in a neighborhood of the numerical one. This is done by employing the so-called *radii polynomial approach* (NAKAO; PLUM; WATANABE, 2019; LESSARD; REINHARDT, 2014; FIGUERAS *et al.*, 2017; REINHARDT; JAMES, 2019). The general idea is as follows. We recast the problem of finding zeros of F into a problem of finding fixed points of the Newton-like map $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ given by

$$T(\mathbf{c}) := \mathbf{c} - AF(\mathbf{c})$$

where $A : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ is a linear operator, taken as an approximation to $DF(\bar{\mathbf{c}})^{-1}$. In this way, finding a fixed point for T is equivalent to finding a zero for F. Then we find bounds for T and DT which are used to construct (finitely many) radii polynomials $p_k(r)$. If we can find $r > 0$ such that $p_k(r) < 0$ for all k, then the verification is successful, and there is a true solution near the numerical one. This is all verified rigorously with the use of interval arithmetic (MOORE; KEARLOTT; CLOUD, 2009).

For our method, we choose the Haar wavelet because of its simplicity. Since its integral can be calculated analytically, we do not need to rely on numerical algorithms or lookup tables in order to relate it to the Haar wavelet themselves. Not only this eliminates a source of numerical error, but also helps finding recursive formulas for the nonlinearity, which are exploited to find sharper bounds for the terms. The Newton-like map thus becomes similar to the one in (BERG; GROOTHEDDE; WILLIAMS, 2015).

This work was motivated by the fact that most other methods use other bases, such as Taylor (REINHARDT; JAMES, 2019), Fourier (FIGUERAS *et al.*, 2017) or Chebyshev (LESSARD; REINHARDT, 2014) series, and thus rely on stricter smoothness assumptions in order to be applicable. Since wavelets are bases for L^2 spaces (HERNÁNDEZ; WEISS, 1996), our method can deal with much less smooth problems.

We show the capabilities and applicability of the method to some example cases, including a discontinuous one and the Lorenz system.

The method has some drawbacks for its capabilities, such as increased complexity and computational power needed. Nevertheless, we believe these can be mitigated in future works, for example by choosing more suitable wavelets to each problem. Also, this work only dealt with first-order equations and quadratic nonlinearities; we believe that similar estimates may be found for higher-order nonlinearities and derivatives. Lastly, the method can be adapted to other applications, such as rigorously finding invariant structures in other problems or continuation methods.

1.3 A Generalized Combinatorial Marching Hypercubes method

In this paper (CASTELO *et al.*, Submitted for publication) we present the Generalized Combinatorial Marching Hypercubes (GCMH) method for generating isomanifold approximations, that is, an approximation to the set

$$\mathscr{M} := \{x \in \mathbb{R}^n : f(x) = 0\}$$

for a given map $f : \mathbb{R}^n \to \mathbb{R}^k$. This paper builds upon a previous generalization (CASTELO; MOUTINHO BUENO; GAMEIRO, 2022), which extended the original Marching Cubes method from (LORENSEN; CLINE, 1987) — which originally only generates isosurfaces for $f : \mathbb{R}^3 \to \mathbb{R}$, that is, surfaces in three dimensions — to functions $f : \mathbb{R}^n \to \mathbb{R}$. The main contribution is extending the method to any dimension and co-dimension in a computationally efficient way, that is, for general functions $f : \mathbb{R}^n \to \mathbb{R}^k$.

In our method, we first generate a grid of n-dimensional hypercubes in $\mathbb{R}^n$. Then, for a given hypercube, we search for manifold vertices on the k-dimensional faces of the hypercube (that is, points of intersection between the manifold and the faces) using a simplicial methods called the CFK triangulation (COXETER, 1934; FREUDENTHAL, 1942; KUHN, 1968). By using combinatorial techniques to label the faces of a given hypercube, we can connect the manifold vertices into edges and higher-dimensional cells. As with all Marching Cubes methods (NEWMAN; YI, 2006), this process can lead to ambiguities; we solve those via a path-following algorithm applying a CFK triangulation step to the faces of dimension $k+1$.

Most other Marching Cubes methods rely on lookup tables (NEWMAN; YI, 2006), which rapidly increase in size as the dimensions grow. Furthermore, while simplex-based methods (ALLGOWER; GEORG, 1980; CASTELO *et al.*, 2006; SCHWAHA; HEINZL, 2010) are also able to compute such approximations to zero-level sets, the number of simplices also grow rapidly as dimensions increase. The use of combinatorial techniques to build the higher-dimensional cell complexes — organized in a structure called "combinatorial skeleton" — greatly reduces computational complexity compared to both approaches. This structure is similar to the one presented in (CASTELO; MOUTINHO BUENO; GAMEIRO, 2022). With this technique, the number of calculations for the GMCH method grow at a much smaller rate as dimensions increase, making it more suitable for higher dimensional problems.

Similarly to previous work (CASTELO; MOUTINHO BUENO; GAMEIRO, 2022), we also present an extension called the Generalized Combinatorial Continuation Hypercube (GCCH) method. Instead of traversing the entire grid as the GCMH algorithm does, the GCCH algorithm uses a starting point and, after each hypercube is processed as in GCMH, it identifies which adjacent hypercubes should be analyzed (if any) and puts them in a list of hypercubes to be processed. This continues until the list is exhausted. While this method can be considerably faster than the GCMH method, it relies on having a starting point and may not find all connected components of $\mathscr{M}$ inside the computational domain with a single starting point.

We apply both algorithms to generate approximations for various manifolds, such as the complex cosine function $z = \cos(w)$ and the Klein Bottle. It also performs much faster than previous algorithms. We also show that the GCCH performs significantly faster, especially when $n-k$ is small. However, the aforementioned drawbacks still apply.

We conclude with some ideas for future work. The two main improvements would be parallelization of the algorithm (especially considering that the GCMH processes one hypercube at a time without input from previous steps, and therefore is highly parallelizable) and an adaptive step. We also consider the application of this method to validated numerics; for example, using the methods in Chapter 3 to validate the numerical approximation obtained from the GCMH and GCCH algorithms.

CHAPTER

2

ROBUSTNESS OF ASYMPTOTIC STABILITY AND EXPONENTIAL DICHOTOMY

Full papers available at:

RODRIGUES, H. M.; SOLÀ-MORALES, J.; NAKASSIMA, G. K. Stability problems in nonautonomous linear differential equations in infinite dimensions. **Communications on Pure and Applied Analysis**, v. 19, n. 6, p. 3189–3207, 2020. ISSN 1534-0392. Available at: <https://www.aimsciences.org/article/doi/10.3934/cpaa.2020138>.

RODRIGUES, H.M.; CARABALLO, T.; NAKASSIMA, G. K. Robustness of exponential dichotomy in a class of generalised almost periodic linear differential equations in infinite dimensional Banach spaces. **Journal of Dynamics and Differential Equations** v. 34, 2841–2865 (2022). Available at: <https://link.springer.com/article/10.1007/s10884-020-09854-3>.

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STABILITY PROBLEMS IN NONAUTONOMOUS LINEAR DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONS

Dedicated to Professor Tomás Caraballo on occasion of his 60th Birthday

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Abstract. In this paper we study the robustness of the stability in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Some applications are obtained to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the stability is robust. These results extend to infinite dimensions some results given in Coppel [3]. Based in Rodrigues [11] and in Kloeden & Rodrigues [10] we introduce a class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable to model these oscillating perturbations. We also present an infinite dimensional example of the previous results.

As counterparts, we show first in another example that it is possible to stabilize an unstable system by using a perturbation with a large period and a small mean value, and finally we give an example where we stabilize an unstable linear ODE with a small perturbation in infinite dimensions by using some ideas developed in Rodrigues & Solà-Morales [21] after an example due to Kakutani (see [13]).

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1. **Introduction.** In several papers of some of us we have been extending or analysing in infinite dimensions some results that were known for finite-dimensional problems. This was the case of Kloeden & Rodrigues [10], Rodrigues [11], Rodrigues & Ruas [16], Rodrigues & Solà-Morales [17, 18, 19, 20], Rodrigues, Caraballo & Gameiro [14] and Rodrigues, Teixeira & Gameiro [15].

Following this philosophy, in this paper we study the relation between the stability properties of a system $\dot{x} = A(t)x$ of ordinary differential equations in an infinite dimensional Banach space $\mathbb{X}$ and a perturbed system $\dot{y} = A(t)y + B(t)y$, where $B(t)$ is supposed to be small in some sense. We suppose first that $A(t)$ and $B(t)$ are bounded operators, continuous and uniformly bounded with respect to $t \in \mathbb{R}$, that the first system is asymptotically stable and that $B(t)$ is integrally small in an arbitrary interval of length bounded by $h > 0$. We establish conditions on the smallness of $B(t)$ in such a way that the perturbed system will also be asymptotically stable. This is established in Theorem 2.1. Then we extend to some cases where $A : \mathcal{D} \to \mathbb{X}$ is unbounded and generates a C^0- semigroup $T(t),\ t \geq 0$. This is established in Theorem 5.1.

In Daleckii & Krein [4] page 178 and in Carvalho et al. [1] similar results are presented about robustness of stability but with a stronger assumption, given by $\frac{1}{\tau_0} \int_t^{t+\tau_0} \|B(\tau)\| d\tau \ \leq \delta$, for some $\tau_0 > 0$, for every $t \in \mathbb{R}$ for sufficiently small δ. One observes that the smallness condition is imposed with the norm inside the integral and in our case the norm appears outside the integral and this makes a significant difference, as it is shown in Theorem 2.1.

Then we introduce in Section 3 a class of functions that we call Generalised Almost Periodic Functions , that contains the usual almost periodic functions. In fact part of it was introduced in Kloeden & Rodrigues [10], where the authors studied perturbations of an hyperbolic equilibrium. This class of Generalized Almost Periodic Functions ($\mathcal{GAP}$) is suitable to define the concept of mean value, as it will be shown, which will be used in this paper.

This new class of functions has some important advantages compared with the usual almost periodic functions, namely, if we perturb an almost periodic function of a variable t with a local perturbation in t, then the perturbed function will no longer be almost periodic. Therefore, the usual class is not robust with respect to this kind of perturbations. It is also not robust with respect to some more general perturbations, like chaotic functions. We understand that the class $\mathcal{GAP}$ is one of the natural classes for our perturbation $B(t)$ belong to.

As an application of Theorem 2.1 we study a system of the form $\dot{y} = A(t)y + B(\omega t)y$ and prove that if $\omega > 0$ is sufficiently large the stability is preserved. When $B(t)$ is periodic the result says that for sufficiently small periods and large oscillations the stability is preserved. The function $B(t)$ does not need to be small and if we have a linear perturbation with large oscillations the stability is preserved. This is shown in Theorem 3.9. In the periodic case the perturbation will have a very small period. In Section 4 we present an example in the infinite-dimensional space ℓ_2 where we show that the stability is preserved. These results extend to infinite dimensions some results of Coppel [3].

Then in Theorem 5.1 we extend the above results to the case where we have an unbounded infinitesimal generator. Henry [8] proves similar results with different applications, but using a different method where he passes from the continuous case to a discrete case and then recovers the results for the continuous problem. Our method follows more the method of Coppel [3] (finite dimension).

As a counterpart of the previous results on the robustness of the stability, the last two sections are devoted to show, by means of examples, that instability is not so difficult to break. In Section 7 we present a two dimensional example where we show that it is possible to stabilise an unstable system with a periodic perturbation with large period and small mean value.

Finally in Section 8 using some ideas developed in Rodrigues & Solà-Morales [21] and in an example of Kakutani [13], we give an example in infinite dimensions where we stabilize an unstable linear system using a linear perturbation $B(t)$ that tends to zero as t tends to infinity.

These two last examples seem to be new in the literature, to our knowledge.

2. **Robustness of Stability.** This section is devoted to state and prove the following Theorem. It extends to infinite dimensional Banach spaces a result of W. A. Coppel [3], Proposition 6, p.6. We think that the key point is the three-terms integration by parts that appears in the beginning of the proof. This integration by parts shows also how the condition of $B(t)$ being integrally small appears along the proof.

Theorem 2.1. *Let $\mathbb{X}$ be a Banach space and $A,\ B:\ \mathbb{R}\to\ L(\mathbb{X})$ be continuous functions such that $\|A(t)\|\le M$ and $\|B(t)\|\le M$ for every $t\in\mathbb{R}$.*

Consider the equations:

$$\dot{x} = A(t)x, \tag{2.1}$$

$$\dot{y} = A(t)y + B(t)y. \tag{2.2}$$

Let $T(t,s)=X(t)X^{-1}(s)$ the evolution operator of (2.1). Suppose that $\|T(t,s)\|\le Ke^{\alpha(t-s)}$ for $t\ge s$, $t,s\in\ \mathbb{R}$, where $\alpha\in\mathbb{R}$ and $K\ge 1$.

Let $\delta,\ h$ be two positive numbers. If $\|\int_{t_1}^{t_2}B(t)dt\|\le\delta$ for $|t_2-t_1|\le h$, and $t_1,t_2\in\mathbb{R}$, then the evolution operator $S(t,s)=Y(t)Y^{-1}(s)$ of (2.2) satisfies the inequality:

$$\|S(t,s)\|\le(1+\delta)Ke^{\beta(t-s)}\ \textit{for}\ t\ge s,\ \ t,s\in\mathbb{R},$$

where $\beta=\alpha+3MK\delta+\frac{\log((1+\delta)K)}{h}$.

If α is negative, h is sufficiently large and δ sufficiently small in such a way that $\beta<0$ then it follows that system (2.2) is asymptotically stable.

Proof. By the variation of constants formula

$$S(t,s)=T(t,s)+\int_s^t T(t,u)B(u)S(u,s)du,\ \ t\ge s.$$

If we let $C(u)=\int_t^u B(\tau)d\tau$,

$$\begin{aligned}\int_s^t T(t,u)B(u)S(u,s)du &= \int_s^t T(t,u)\frac{d}{du}\int_t^u B(\tau)d\tau S(u,s)du\\ &= \int_s^t T(t,u)\frac{d}{du}C(u)S(u,s)du.\end{aligned}$$

Taking derivatives,

$$\begin{aligned}\frac{d}{du}[T(t,u)C(u)S(u,s)] = &-T(t,u)A(u)C(u)S(u,s)+T(t,u)B(u)S(u,s)\\ &+T(t,u)C(u)(A(u)+B(u))S(u,s).\end{aligned}$$

Integrating the above equation gives the three-terms integration by parts we commented above. Then we obtain

$$\begin{aligned}&\int_s^t \frac{d}{du}[T(t,u)C(u)S(u,s)]du\\ &= -\int_s^t T(t,u)A(u)C(u)S(u,s)du + \int_s^t T(t,u)B(u)S(u,s)du\\ &\quad + \int_s^t T(t,u)C(u)A(u)S(u,s)du + \int_s^t T(t,u)C(u)B(u)S(u,s)du.\end{aligned}$$

And so,

$$\begin{aligned}&-T(t,s)C(s)\\ &= -\int_s^t T(t,u)A(u)C(u)S(u,s)du + \int_s^t T(t,u)B(u)S(u,s)du\\ &\quad + \int_s^t T(t,u)C(u)A(u)S(u,s)du + \int_s^t T(t,u)C(u)B(u)S(u,s)du.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_s^t T(t,u)B(u)S(u,s)du\\ &= -T(t,s)C(s) + \int_s^t T(t,u)A(u)C(u)S(u,s)du\\ &\quad - \int_s^t T(t,u)C(u)A(u)S(u,s)du - \int_s^t T(t,u)C(u)B(u)S(u,s)du.\end{aligned}$$

Therefore,

$$\begin{aligned}S(t,s) =& T(t,s) + \int_s^t T(t,u)B(u)S(u,s)du\\ =& T(t,s)(I - C(s)) + \int_s^t T(t,u)A(u)C(u)S(u,s)du\\ & - \int_s^t T(t,u)C(u)A(u)S(u,s)du - \int_s^t T(t,u)C(u)B(u)S(u,s)du.\end{aligned}$$

We first suppose that $s \le t \le s+h$ and estimate $|S(t,s)|$. Let $s \le u \le s+h$. Suppose

$$|C(u)| \le |\int_t^u B(\tau)d\tau| \le \delta.$$

Therefore,

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)} + 3MK\delta \int_s^t e^{-\alpha(t-u)}|S(u,s)|du.$$

and so, using Gronwall's inequality it follows that in an arbitrary interval of length h, say for $s \le t \le s+h$ we have

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)}e^{3MK\delta(t-s)} = K(1+\delta)e^{(\alpha+3MK\delta)(t-s)}.$$

For $t \ge s$ there exists $n \in \mathbb{N}$, $n = n(t,s)$ such that $s + nh \le t \le s + (n+1)h$ and so

$$|S(t, s+nh)| \le K(1+\delta)e^{(\alpha+3MK\delta)(t-s-nh)}.$$

We are going to prove by induction that for $s+nh \le t \le s+(n+1)h$

$$|S(t,s)| \le [K(1+\delta)]^{n+1} e^{(\alpha+3KM\delta)(t-s)}.$$

The case $n=0$ has already been proved. But $S(s+nh,s) = S(s+nh, s+(n-1)h)\cdots S(s+h,s)$ and so

$$|S(s+nh,s)| \le [K(1+\delta)]^{n} e^{(\alpha+3KM\delta)nh}.$$

Therefore for $s+nh \le t \le s+(n+1)h$,

$$\begin{aligned} |S(t,s)| \le & |S(t,s+nh)||S(s+nh,s)| \\ \le & K(1+\delta)e^{(\alpha+3KM\delta)(t-s-nh)}[K(1+\delta)]^n e^{(\alpha+3KM\delta)nh} \\ = & [K(1+\delta)]^{n+1} e^{(\alpha+3KM\delta)(t-s)}. \end{aligned}$$

Therefore for $s+nh \le t \le s+(n+1)h$, we have

$$|S(t,s)| \le [K(1+\delta)]^{n+1} e^{((\alpha+3KM\delta)(t-s)}.$$

Let $\gamma \doteq \frac{\ln((1+\delta)K)}{h}$. Since $t \ge s+nh$, we have

$$[(1+\delta)K]^n = e^{\gamma nh} \le e^{\gamma(t-s)}.$$

Therefore,

$$|S(t,s)| \le K(1+\delta)e^{(\alpha+3KM\delta+\frac{\ln((1+\delta)K)}{h})(t-s)}.$$

□

3. **The space of generalised almost periodic functions.** In this section we introduce the class that we call Generalised Almost Periodic Functions that extends the usual concept of almost periodicity. As we said in the Introduction Section, this new class is more robust with respect to perturbations and it is a natural class for our function $B(t)$ to belong, as it will appear in Theorem 3.9 and its corollary.

Let $(\mathbb{X}, |\cdot|)$ be a Banach space and recall the definition of an almost periodic function [5].

Definition 3.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost periodic if for every sequence (α'_n) there exists a subsequence (α_n) such that the $\lim_{n\to\infty} f(t+\alpha_n)$ exists uniformly in $\mathbb{R}$.

Now let $BUC(\mathbb{R}, L(\mathbb{X})$ denote the space of bounded and uniformly continuous functions $A : \mathbb{R} \to L(\mathbb{X})$, which is a Banach space with the supremum norm $\|A\| \doteq \sup_{t\in\mathbb{R}} |A(t)|$, and define

$$\begin{aligned} \mathcal{F} \doteq \{A \in BUC(\mathbb{R}, L(\mathbb{X})) : & A \text{ is uniformly continuous} \\ & \text{with precompact range } \mathcal{R}(A)\}. \end{aligned}$$

The class $\mathcal{F}$ is quite large and includes both periodic and almost periodic functions as well as other nonrecurrent functions.

Proposition 1. *Let $A(t) \in L(\mathbb{X})$ be almost periodic. Then $A \in \mathcal{F}$.*

Proof. The proof is trivial. □

Theorem 3.2. *$\mathcal{F}$ is a closed subspace of $BUC(\mathbb{R}, L(\mathbb{X}))$ and hence a Banach space.*

Proof. This proof can be found in Kloeden-Rodrigues [10]. □

Lemma 3.3. *Let $\sup_{t\in\mathbb{R}} |A(t)| \le M$, If there exists $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt$ for some $a \in \mathbb{R}$ then it is independent of a.*

Proof. Let $a \in \mathbb{R}$.

$$\left|\frac{1}{T}\int_a^{a+T} A(t)dt - \frac{1}{T}\int_0^T A(t)dt\right| = \left|\frac{1}{T}\left[\int_a^{a+T} A(t)dt - \int_0^T A(t)dt\right]\right|,$$
$$\left|\frac{1}{T}\left[\int_a^0 A(t)dt + \int_T^{a+T} A(t)dt\right]\right| \le \frac{2M|a|}{T} \to 0, \text{ as } T \to \infty.$$

□

Then we define:

Definition 3.4. We say that $A \in \mathcal{F}$ is a generalized almost periodic function if there exists the limit $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt$ in $L(\mathbb{X})$, that is, there exists $\mathbf{A} \in L(\mathbb{X})$ such that, given $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon) > 0$ such that $|\frac{1}{T}\int_a^{a+T} A(t)dt - \mathbf{A}| < \varepsilon$ for every $T \ge T_0$ uniformly with respect to $a \in \mathbb{R}$.

Definition 3.5. We define the class of generalized almost periodic functions as

$$\mathcal{GAP} = \{A \in \mathcal{F} : \ A \text{ is a generalized almost periodic function}\}.$$

Lemma 3.6. *$\mathcal{GAP}$ is a closed subspace of $\mathcal{F}$.*

Proof: Let $A_n \in \mathcal{GAP}$, $A_n \to A$ in $\mathfrak{F}$. We must prove that $A \in \mathcal{GAP}$. Given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $\|A_{n_0} - A\| = \sup_{t\in\mathbb{R}} |A_{n_0}(t) - A(t)| < \varepsilon$.

Since there exists the $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A_{n_0}(t)dt = \mathbf{A}_{n_0}$, there exists $T_0 = T_0(\varepsilon)$ such that

$$T_1,\ T_2 > T_0 \ \Rightarrow \left|\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt\right| < \varepsilon,\ \forall a \in \mathbb{R}.$$

Then $T_1,\ T_2 > T_0 \Rightarrow$

$$\begin{aligned}
&\left|\frac{1}{T_2}\int_a^{a+T_2} A(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A(t)dt\right| \\
&\le \left|\frac{1}{T_2}\int_a^{a+T_2} A(t)dt - \frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt\right| \\
&\quad + \left|\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt\right| \\
&\quad + \left|\frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A(t)dt\right| \\
&\le \frac{1}{T_2}\int_a^{a+T_2} |A(t) - A_{n_0}(t)|dt + \left|\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt\right| \\
&\quad + \frac{1}{T_1}\int_a^{a+T_1} |A(t) - A_{n_0}(t)|dt \le 3\varepsilon.
\end{aligned}$$

Using Cauchy Criterion we conclude that there exists

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt = \mathbf{A} \in L(\mathbb{X}),\ \forall a \in \mathbb{R}.$$

This implies that $A \in \mathcal{GAP}$. □

Definition 3.7. For $A \in \mathcal{GAP}$ we define the mean value of A as:

$$\mathcal{M}(A) \doteq \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt \in L(\mathbb{X}).$$

Lemma 3.8. *The function $\mathcal{M} : \mathcal{GAP} \to L(\mathbb{X})$ is a uniformly continuous function.*

Proof. Let $A, B \in \mathcal{GAP}$. Then

$$\begin{aligned}|\mathcal{M}(A) - \mathcal{M}(B)| &= \left|\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt - \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} B(t)dt\right| \\ &= \left|\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} [A(t) - B(t)]dt\right| \le \sup_{t\in\mathbb{R}} |A(t) - B(t)| = \|A - B\|.\end{aligned}$$

□

Let $\mathcal{O} = \{A \in \ \mathcal{GAP} : \mathcal{M}(A) = \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt = 0, \ \forall \ a \in \mathbb{R}\}$,

Corollary 1. *$\mathcal{O}$ is a closed subspace of $\mathcal{GAP}$.*

Proof. Since $\mathcal{M}(A)$ is a continuous function, the set $\mathcal{O} = \mathcal{M}^{-1}\{\{0\}\}$ is closed set. □

Corollary 2. *Any function $A \in \mathcal{GAP}$ can be written as $A = A_0 + B$, where $A_0 = \mathcal{M}(A)$ and $B \in \mathcal{O}$.*

The next theorem shows that stability is preserved if the linear perturbation has sufficiently large frequency:

Theorem 3.9. *Let $A,\ B :\ \mathbb{R} \to\ L(\mathbb{X})$ be continuous functions such that $\|A(t)\| \le M$ and $\|B(t)\| \le M$ for every $t \in \mathbb{R}$. Suppose that $B(t)$ is a generalized almost periodic function with mean value zero ($\mathcal{GAP}$). Consider the equations:*

$$\dot{x} = A(t)x, \tag{3.1}$$

$$\dot{x} = A(t)x + B(\omega t)x. \tag{3.2}$$

Let $T(t, s)$ be the evolution operator of (3.1). Suppose that $\|T(t, s)\| \le Ke^{-\alpha(t-s)}$ for $t \ge s$, $t, s \in\ \mathbb{R}$, where $\alpha > 0$ and $K > 1$. Then there exists $\tilde{K}$ and $\omega_0 > 0$ such that for $\omega > \omega_0$

$$|S_\omega(t, s)| \le \tilde{K}e^{\frac{-\alpha}{2}(t-s)},\ t \ge s,$$

where $S_\omega(t, s)$ indicates the evolution operator of (3.2).

Proof. We are going to show that for any $h > 0$, $\delta > 0$ there exists $\omega_0 = \omega_0(h, \delta) > 0$ such that if $\omega > \omega_0$ then

$$\left|\int_{t_1}^{t_2} B(\omega t)dt\right| \le \delta \text{ for } |t_2 - t_1| \le h.$$

Let us consider first the case $|t_2 - t_1| \le \frac{\delta}{M}$. Since $|B(t)| \le M$ for every $t \in \mathbb{R}$, we have

$$\left|\int_{t_1}^{t_2} B(\omega t)dt\right| \le \left|\int_{t_1}^{t_2} |B(\omega t)|dt\right| \le M|t_2 - t_1| \le M\frac{\delta}{M} = \delta.$$

To complete the proof we consider now the case $h \ge |t_2 - t_1| \ge \frac{\delta}{M}$.

Since $B(t)$ has mean value zero, there exists $T_0 = T_0(\frac{\delta}{h}) > 0$ such that

$$T \geq T_0 \;\Rightarrow \left| \frac{1}{T} \int_s^{s+T} B(t)dt \right| \leq \frac{\delta}{h} \text{ for all } s \in \mathbb{R}.$$

By a change of variables,

$$\int_{t_1}^{t_2} B(\omega t)dt = \frac{1}{\omega} \int_{\omega t_1}^{\omega t_2} B(u)du,$$

and so for $\frac{\delta}{M} \leq |t_2 - t_1| \leq h$,

$$\left| \int_{t_1}^{t_2} B(\omega t)dt \right| = \frac{1}{\omega |t_2 - t_1|} \left| \int_{\omega t_1}^{\omega t_2} B(u)du \right| |t_2 - t_1| \leq \frac{1}{|\omega t_2 - \omega t_1|} \left| \int_{\omega t_1}^{\omega t_2} B(u)du \right| \; h.$$

If we take $\omega_0 \doteq \frac{MT_0}{\delta}$, we have for $\omega \geq \omega_0$,

$$|\omega t_2 - \omega t_1| \geq \omega_0 |t_2 - t_1| \geq \frac{MT_0}{\delta} \frac{\delta}{M} = T_0.$$

Therefore,

$$\left| \int_{t_1}^{t_2} B(\omega t)dt \right| = \frac{1}{|\omega t_2 - \omega t_1|} \left| \int_{\omega t_1}^{\omega t_2} B(u)du \right| \; h \leq \frac{\delta}{h} \; h = \delta.$$

The result follows from Theorem 3.9 for δ sufficiently small. □

Consider now $A \in \mathcal{GAP}$. Then we have $A(t) = A_0 + B(t)$, where $A_0 = \mathcal{M}(A)$ and $\mathcal{M}(B) = 0$. We suppose that $|A_0| \leq M$ and $|B(t)| \leq M$ for every $t \in \mathbb{R}$. Consider the equations:

$$\dot{x} = A_0 x, \tag{3.3}$$

$$\dot{x} = A_0 x + B(\omega t)x. \tag{3.4}$$

Let $T(t) \doteq e^{A_0 t}$ be the semigroup generated by (3.3) and $S_\omega(t,s)$ be the evolution operator of (3.4).

As a consequence of Theorem 3.9 it follows that if $\sigma(A_0) \subset \{\lambda \in \mathbb{C} : Re(\lambda) < -\alpha\}$ we will have:

Corollary 3. *Suppose $|T(t)| \leq Ke^{-\alpha t}$ for $t \geq 0$, $K \geq 1$. Then there exist $\widetilde{\alpha} < \alpha$, $\widetilde{K} > K$, $\omega_0 > 0$, such that for $\omega > \omega_0$ we have*

$$S_\omega(t,s) \leq \tilde{K} e^{-\widetilde{\alpha}(t-s)}, \forall t \geq s.$$

4. **An infinite dimensional example.** In this section we will construct a true infinite dimensional example to apply the results of the previous section. We are going to use some results of the paper Rodrigues and Solà-Morales [19]. Consider the space $\mathbb{X} = \ell_2$. We consider the operator $J \in \mathcal{L}(X)$ given by the infinite dimensional Jordan matrix:

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix}. \tag{4.1}$$

As it is proved in Rodrigues and Solà-Morales [19] the spectrum of J is the closed unity circle of the complex plane. Now we take $0 < a < 1$ and we define the operator:

$$L := \begin{pmatrix} a & 0 \\ 0 & \nu J + aI \end{pmatrix} = aI + \nu \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} = a\left(I - \left(\frac{-\nu}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}\right). \tag{4.2}$$

If we let

$$D = \left(\frac{-\nu}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix},$$

we have that

$$L = a(I - D).$$

From the same paper above it follows that the spectrum of L is the closed disc $B_\nu(a)$ with center in a and radius ν. Then we take $0 < \nu < \min\{a, 1-a\}$. Then we let $A := \log L = (\log a)I + \log(I - D)$.

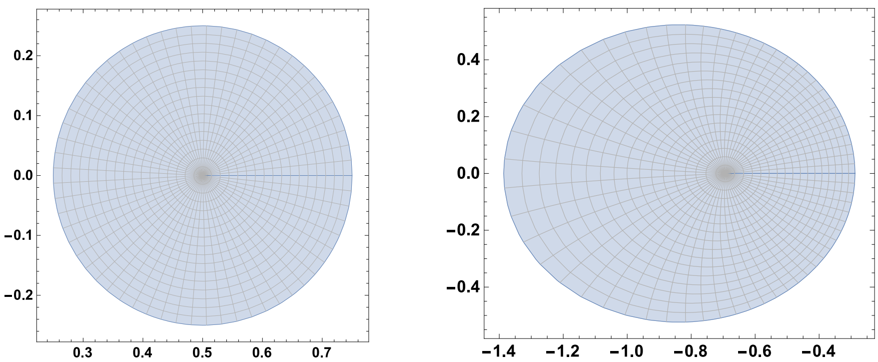

FIGURE 1. Left: The spectrum of L given by $\sigma(L) = B_\nu(a)$. Right: The spectrum of A given by $\sigma(A) = \log(\sigma(L))$, with $a = 1/2$ and $\nu = 1/4$.

But

$$\log(I - D) = -(D + \frac{D^2}{2} + \cdots \frac{D^n}{n} \cdots).$$

Therefore

$$\|\log(I - D)\| \le \frac{\nu}{a} + \frac{(\frac{\nu}{a})^2}{2} + \cdots \frac{(\frac{\nu}{a})^n}{n} + \cdots = -\log(1 - \frac{\nu}{a}).$$

Let $\nu > 0$ sufficiently small such that $0 < -\log(1 - \frac{\nu}{a}) < \frac{a}{2}$. Then it follows that

$$\|e^{At}\| \le e^{(-at - \log(1 - \frac{\nu}{a})t)} \le e^{-\frac{a}{2}t}, \quad t \ge 0.$$

In the space $\mathbb{X} = \ell_2$. We consider the operator $A \in \mathcal{L}(X)$ given above. In FIGURE 1 we show the spectrum of L and the spectrum of A.

Corollary 4. *Consider now the systems:*

$$\dot{x} = Ax, \tag{4.3}$$

$$\dot{y} = Ay + B(\omega t)y. \tag{4.4}$$

where $B \in \mathcal{G}AP$ with mean value zero. Let $M > 0$ be such that $|A| \le M$ and $\sup_{t \in \mathbb{R}} |B(t)| \le M$.

Let $S_\omega(t,s) = Y(t)Y^{-1}(s)$ be the evolution operator associated to system (4.4), where $Y(t)$ is the solution with initial condition $Y(0) = I$, where I indicates the Identity operator.

Then there exists $\tilde{K}$, $\tilde{\alpha}$ and $\omega_0 > 0$ such that for $\omega > \omega_0$

$$|S_\omega(t,s)| \leq \tilde{K} e^{-\tilde{\alpha}(t-s)}, \ t \geq s.$$

Proof. Follows from Theorem 3.9 . □

Next we will present a simple example where the perturbation $B(t)$ belongs to $\mathcal{G}AP$ but it is not almost periodic.

Example 4.1. Let $b : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, bounded with mean value zero. Let

$$B(t) \doteq \begin{pmatrix} 0 & 0 & 0 & \cdots \\ b(t) & 0 & 0 & \cdots \\ 0 & b(t) & 0 & \cdots \\ 0 & 0 & b(t) & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} = b(t) \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix}. \tag{4.5}$$

Then $B \in \mathcal{GAP}$ and has mean value zero. Let $d(t) \doteq \sqrt{1-t^2}$ if $-1 \leq t \leq 1$, $d(t) = 0$ if $t \in (\infty, 0) \cup (1, \infty)$. In the special case that we take $b(t) \doteq d(t) + \cos t$, $B(t)$ is not almost periodic.

Therefore we can apply Corollary 4 if we take $b(\omega t) = d(\omega t) + \cos(\omega t)$ and then we can take $B(\omega t)$ as above.

5. **A case where the infinitesimal generator is unbounded.** Consider the equations:

$$\dot{x} = Ax \tag{5.1}$$

$$\dot{y} = Ay + B(t)y. \tag{5.2}$$

We suppose that $\mathcal{D}$ is dense in $\mathbb{X}$ and $A : \mathcal{D} \to \mathbb{X}$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $T(t)$, such that $|T(t)| \leq Ke^{\alpha t}, \ t \geq 0, \ K \geq 1, \alpha \in \mathbb{R}$.

Now we will analyse some smallness conditions on the perturbation $B(t)$, such that equation (5.2) is also asymptotically stable in the case $\alpha < 0$. The case when $B(t)$ is uniformly small is studied in Kloeden-Rodrigues [10] without leaving the continuous case. Similar results are obtained by Carvalho et al [1], but they first find the result for the discrete case.

Similar results to the next theorem are treated by Carvalho et al [1] and Dalekii-Krein [4] but they use the stronger assumption that $\int_\tau^t |B(t)|$ is small, with the norm inside the integral and in the first one they prove via a discretiztion method. Similar results are obtained by Henry [8] in Thorem 7.6.11, pag. 238, where he also considers first the discrete case, and requires that $B(t)$ is uniformly small and integrally small.

Our result is an extension of a classical result of Coppel [3] for the infinite dimensional case, and A being an unbounded operator.

We will follow the steps of Theorem 2.1 where we imposed that $|B(t)| \leq M$ for every $t \in \mathbb{R}$ and that $|\int_t^u B(\tau)d\tau| \leq \delta$ for $t \leq u \leq t+h$. We also assume that the range of $B(t)$ is contained in the domain of A.

Theorem 5.1. *We assume besides the above assumptions on A and $T(t)$, that $B : \mathbb{R} \to L(\mathbb{X})$ is a continuous function and such that for each $t \in \mathbb{R}$ $AB(t)$ is a bounded operator and $B(t)A$ can be extended to the whole space as a bounded operator. For each $t \in \mathbb{R}$ let $C_t(u) \doteq \int_t^u B(\tau)d\tau$, for $|t-u| \leq h$, where h is a positive real number. We suppose that there are positive numbers M and δ such that*

$$|C_t(u)B(u)| \leq M\delta,\ |C_t(u)A| \leq M\delta, \text{ and } |AC_t(u)| \leq M\delta,\ \text{for } |u-t| \leq h.$$

Let $S(t,s)$ be the evolution operator associated to system (5.2). Then

$$\|S(t,s)\| \leq (1+\delta)Ke^{\beta(t-s)} \text{ for } t \geq s,\ t,s \in \mathbb{R},$$

where

$$\beta = \alpha + 3MK\delta + \frac{\log((1+\delta)K)}{h}.$$

If α is negative, h is sufficiently large and δ sufficiently small in such a way that $\beta < 0$ then it follows that system (5.2) is asymptotically stable.

Proof. The proof follows the ideas of Theorem 2.1. By the variation of constants formula

$$S(t,s) = T(t-s) + \int_s^t T(t-u)B(u)S(u,s)du,\ t \geq s.$$

$$\begin{aligned}\int_s^t T(t-u)B(u)S(u,s)du &= \int_s^t T(t-u)\frac{d}{du}\int_t^u B(\tau)d\tau S(u,s)du \\ &= \int_s^t T(t-u)\frac{d}{du}C_t(u)S(u,s)du.\end{aligned}$$

Taking derivatives,

$$\begin{aligned}\frac{d}{du}[T(t-u)C_t(u)S(u,s)] = &- T(t-u)AC_t(u)S(u,s) + T(t-u)B(u)S(u,s) \\ &+ T(t-u)C_t(u)(A+B(u))S(u,s).\end{aligned}$$

Integrating the above equation, we obtain

$$\begin{aligned}&\int_s^t \frac{d}{du}[T(t-u)C_t(u)S(u,s)]du \\ &= -\int_s^t T(t-u)AC_t(u)S(u,s)du + \int_s^t T(t-u)B(u)S(u,s)du \\ &\quad + \int_s^t T(t-u)C_t(u)AS(u,s)du + \int_s^t T(t-u)C_t(u)B(u)S(u,s)du.\end{aligned}$$

And so,

$$\begin{aligned}-T(t-s)C_t(s) = &-\int_s^t T(t-u)AC_t(u)S(u,s)du + \int_s^t T(t-u)B(u)S(u,s)du \\ &+\int_s^t T(t-u)C_t(u)AS(u,s)du + \int_s^t T(t-u)C_t(u)B(u)S(u,s)du.\end{aligned}$$

Therefore,

$$\int_s^t T(t-u)B(u)S(u,s)du = -T(t-s)C_t(s) + \int_s^t T(t-u)AC_t(u)S(u,s)du$$

$$- \int_s^t T(t-u)C({}_t u)AS(u,s)du - \int_s^t T(t-u)C_t(u)B(u)S(u,s)du.$$

Therefore,

$$\begin{aligned} S(t,s) =& T(t-s) + \int_s^t T(t-u)B(u)S(u,s)du \\ =& T(t-s)(I - C_t(s)) + \int_s^t T(t-u)AC_t(u)S(u,s)du \\ &- \int_s^t T(t-u)C_t(u)AS(u,s)du - \int_s^t T(t-u)C_t(u)B(u)S(u,s)du. \end{aligned}$$

We first suppose that $s \le t \le s+h$ and estimate $|S(t,s)|$.

If $0 \le |u-t| \le h$ then

$$|C_t(u)B(u)| \le |\int_t^u B(\tau)d\tau B(u)| \le M\delta, \ |C_t(u)A| \le M\delta$$

and $|AC_t(u)| \le M\delta$. Therefore,

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)} + 3MK\delta \int_s^t \ e^{-\alpha(t-u)}|S(u,s)|du.$$

and so using Gronwall's inequality it follows that, in an arbitrary interval of length h, say for $s \le t \le s+h$ we have

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)}e^{3MK\delta(t-s)} = K(1+\delta)e^{(\alpha+3MK\delta)(t-s)}.$$

For $t \ge s$ there exists $n \in \mathbb{N}$, $n = n(t,s)$ such that $s+nh \le t \le s+(n+1)h$ and so

$$|S(t,s+nh)| \le K(1+\delta)e^{(\alpha+3MK\delta)(t-s-nh)}.$$

We are going to prove by induction that for $s+nh \le t \le s+(n+1)h$,

$$|S(t,s)| \le [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}.$$

The case $n = 0$ has already been proved. But $S(s+nh,s) = S(s+nh,s+(n-1)h)\cdots S(s+h,s)$ and so

$$|S(s+nh,s)| \le [K(1+\delta)]^n e^{(\alpha+3KM\delta)nh}.$$

Therefore for $s+nh \le t \le s+(n+1)h$,

$$\begin{aligned} |S(t,s)| &\le |S(t,s+nh)||S(s+nh,s)| \\ &\le K(1+\delta)e^{(-\alpha-3KM\delta)(t-s-nh)}[K(1+\delta)]^n e^{(\alpha+3KM\delta)nh} \\ &= [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}. \end{aligned}$$

Therefore for $s+nh \le t \le s+(n+1)h$, we have

$$|S(t,s)| \le [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}.$$

Let $\gamma \doteq \frac{\ln((1+\delta)K)}{h}$. Since $t \ge s+nh$, we have

$$[(1+\delta)K]^n = e^{\gamma nh} \le e^{\gamma(t-s)}.$$

Therefore,

$$|S(t,s)| \le K(1+\delta)e^{(\alpha-+3KM\delta+\frac{\ln((1+\delta)K)}{h})(t-s)}.$$

$\square$

6. **Applications of Section 5.** Consider the following result from Henry [8] pg. 30.

Theorem 6.1. *Suppose A is a closed operator in the Banach space $\mathbb{X}$ and suppose that σ_1 is a bounded spectral set of A, and $\sigma_2 = \sigma(A) - \sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral set. Let E_1, E_2 be the projections associated with these spectral sets, and $X_j = E_j(\mathbb{X})$, $j = 1, 2$. Then $\mathbb{X} = X_1 \oplus X_2$, the X_j are invariant under A, and if A_j is the restriction of A to X_j, then*

$$A_1 : X_1 \to X_1 \text{ is bounded}, \sigma(A_1) = \sigma_1, \quad \mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.$$

With our techniques what we can get is the next result:

Theorem 6.2. *Let h and δ be positive real numbers. Suppose that $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ a generator of a C_0-semigroup $T(t)$, $t \geq 0$, $B(t) \in L(\mathbb{X})$ and $|B(t)| \leq M$ for every $t \in \mathbb{R}$. Suppose we can decompose $\sigma(A) \doteq \sigma_1 \cup \sigma_2$, where σ_1 is a bounded spectral set and $\sigma_2 = \sigma(A) - \sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral set. Suppose there is a smooth curve Γ, oriented positively, that contains σ_1 in its interior and σ_2 is in the exterior of Γ. Consider the projection $P_1 \doteq \frac{-1}{2\pi i} \oint_\Gamma (\lambda - A)^{-1} d\lambda$ that projects $\mathbb{X}$ in the subspace X_1 associated to the spectral set σ_1. Let $P_2 \doteq I - P_1$. $|T(t)P_1| \leq Ke^{-\alpha t}$ and $|T(t)P_2| \leq Ke^{-\mu t}$, for $t \geq 0$, where $\mu > \alpha$. Then AP_1 is a bounded operator and $P_1 A = AP_1$ and so $P_1 A$ is also a bounded operator.*

The above decomposition is chosen in such a way that $|P_2 B(t)| \leq M\delta$ for every $t \in \mathbb{R}$.

In analogy with the bounded case if $C_t(u) \doteq \int_t^u B(\tau) d\tau$, we suppose that

$$|P_1 C_t(u) B| \leq M\delta, \ |P_1 A C_t(u)| \leq M\delta \text{ and } |P_1 C_t(u) A| \leq M\delta, \text{ for } t \leq u \leq t + h. \tag{6.1}$$

Consider the equations:

$$\dot{x} = Ax, \tag{6.2}$$
$$\dot{y} = Ay + B(t)y. \tag{6.3}$$

If the above assumptions are satisfied if δ is sufficiently small, h is sufficiently large and (6.2) is asymptotically stable then system (6.3) is also asymptotically stable.

Proof. The proof follows the ideas of Theorem 5.1. □

Remark 1. The decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ and the smallness conditions (6.1) are satisfied if A is at least a sectorial operator and if $B(t)$ commutes with P_1.

7. **Stabilising unstable systems under small periodic perturbations, with large period.** In contrast with the results of the previous sections on the robustness of the stability, this section and the next one are devoted to show, by means of examples, that instability is not so difficult to break.

The next example is in $\mathbb{X} = \mathbb{R}^2$ and it shows that it is possible to stabilise an unstable system under a small (in mean value) periodic perturbation.

Let $0 < \alpha < \beta$ and $\delta < T$. Let

$$A \doteq \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad R \doteq \begin{pmatrix} 0 & \frac{\pi}{2\delta} \\ -\frac{\pi}{2\delta} & 0 \end{pmatrix}.$$

Let $D(t)$ the T-periodic operator given by

$$D(t) = -A + R, \ T - \delta \leq t < T, \ D(t) = 0 \ t \in \mathbb{R} - [T - \delta, T). \tag{7.1}$$

Consider the systems:

$$\dot{x} = Ax, \tag{7.2}$$

$$\dot{y} = Ay + D(t)y. \tag{7.3}$$

First we observe that $\lim_{T\to\infty} \frac{1}{T}\int_0^T D(s)ds = 0$, that is $B(t)$ has zero mean value, but has large period. Next we are going to prove, using Floquet Theorem that system (7.3) is uniformly asymptotically stable. For the system $\dot{x} = A(t)x$, where $A(t)$ is continuous and T-periodic, we will use Floquet Theorem even if $A(t)$ is not continuous, according to the comment in [7] page 118.

Consider the matrix solution $X(t)$ of (7.2) such that $X(0) = I$ the identity matrix. Then it is given by

$$X(t) = e^{At} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix}.$$

If we let $R \doteq \begin{pmatrix} 0 & \frac{\pi}{2\delta} \\ -\frac{\pi}{2\delta} & 0 \end{pmatrix}$, then we have the rotation matrix:

$$e^{Rt} = \begin{pmatrix} \cos\left(\dfrac{\pi t}{2\delta}\right) & \sin\left(\dfrac{\pi t}{2\delta}\right) \\ -\sin\left(\dfrac{\pi t}{2\delta}\right) & \cos\left(\dfrac{\pi t}{2\delta}\right) \end{pmatrix}.$$

Since

$$X(T-\delta) = e^{A(t-\delta)} = \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix},$$

the fundamental matrix $Y(t)$ of $\dot{y} = (A + D(t))y$, such that $Y(0) = I$ will be given by

$$\begin{cases} Y(t) = e^{At} & \text{for } 0 \leq t < T-\delta, \\ Y(t) = e^{R(t-(T-\delta))}e^{A(T-\delta)} = e^{R(t-T)}e^{R\delta}e^{A(T-\delta)}, & \text{for } T-\delta \leq t < T. \end{cases}$$

Then the monodromy matrix will be

$$Y(T) = e^{R\delta}e^{A(T-\delta)} = \begin{pmatrix} \cos\left(\dfrac{\pi\delta}{2\delta}\right) & \sin\left(\dfrac{\pi\delta}{2\delta}\right) \\ -\sin\left(\dfrac{\pi\delta}{2\delta}\right) & \cos\left(\dfrac{\pi\delta}{2\delta}\right) \end{pmatrix} \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix} = \begin{pmatrix} 0 & e^{-\beta(T-\delta)} \\ -e^{\alpha(T-\delta)} & 0 \end{pmatrix}.$$

Now we can find the eigenvalues of the monodromy $Y(T)$ and they will be the characteristic multipliers of (7.3)

$$Y(T) - \lambda I = \begin{pmatrix} -\lambda & e^{-\beta(T-\delta)} \\ -e^{\alpha(T-\delta)} & -\lambda \end{pmatrix}.$$

The characteristic polynomial is given by $p(\lambda) \doteq \lambda^2 + e^{(\alpha-\beta)(T-\delta)}$. Since $\beta > \alpha$ this implies that

$$|\lambda| = \sqrt{e^{(\alpha-\beta)(T-\delta)}} < 1.$$

Therefore $\dot{y} = (A + D(t))y$ is uniformly asymptotically stable.

8. **Stabilizing unstable linear ODE in infinite dimensions.** There is a classical example in Operator Theory due to S. Kakutani of a bounded operator in an infinite-dimensional Hilbert space whose spectrum shrinks drastically from a disk to a single point under an arbitrarily small bounded perturbation. The example can be found in [13] (p. 282) and [6] (p. 248) and it is also described in [21], where the present authors recently used it to build an example of the possibility of nonlinear stabilization of an unstable linear map under Fréchet differentiability hypotheses. It is also briefly described below. The purpose of the present section is, by means of two examples, to use the ideas of Kakutani's example to show this drastic stabilization in linear ordinary differential equations in infinite dimensional Hilbert spaces, of the form

$$\dot{x}(t) = Ax(t) + B(t)x(t), \tag{8.1}$$

when the system $\dot{x}(t) = Ax(t)$ is unstable and the perturbation $B(t)$ is small in some senses. Roughly speaking, we could say that the examples of this section show that while stability is a robust feature, instability does not need to be so.

Let us describe briefly the example of Kakutani with the notations and choices of [21]. In a real separable Hilbert space H with a Hilbert orthonormal basis $(e_n)_{n\geq 1}$ a weighted shift operator $W \in \mathcal{L}(H)$ is a bounded linear operator defined by the relations $We_i = \alpha_i e_{i+1}$ for a bounded sequence of real numbers $(\alpha_n)_{n\geq 1}$. One readily sees that

$$\|W\| = \sup\{|\alpha_n|\} \quad \text{and} \quad \|W^k\| = \sup\{|\alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1}|\}. \tag{8.2}$$

We choose first the sequence $\varepsilon_m = M/K^{m-1}$ for some $M > 0$ and some $K > 1$, and define a weighted shift W_ε by $\alpha_n = \varepsilon_m$ if $n = 2^{(m-1)}(2\ell + 1)$, where ℓ is a non-negative integer. This sophisticated way of distributing the numbers ε_m into a sequence α_n makes a number ε_m to appear for the first time in the α_n sequence at the position $n = 2^{(m-1)}$ and from that position onwards to appear periodically, infinitely many times, with a period of 2^m.

Then, one also defines the weighted shifts L_m by a sequence of weights α_n that are all of them equal to zero, except at the positions $n = 2^{(m-1)}(2\ell + 1)$, where ℓ is a non-negative integer, where $\alpha_n = \varepsilon_m$. With this choice, the operator $W_\varepsilon - L_m$ is also a weighted shift, and it has zeroes along its sequence of weights, distributed each 2^m places, and starting at the $2^{(m-1)}$ position. This means, according to 8.2, that $W_\varepsilon - L_m$ is nilpotent of index 2^m, $(W_\varepsilon - L_m)^{2^m} = 0$. Consequently, its spectral radius $\rho(W_\varepsilon - L_m) = 0$. One can also obtain, after some work, that $\rho(W_\varepsilon) = M/K$ and that the spectrum $\sigma(W_\varepsilon)$ is the whole disk of radius M/K centered at zero. Concerning the norms, by using (8.2) one gets that $\|W_\varepsilon\| = M$ and $\|-L_m\| = \varepsilon_m$.

In this way, Kakutani's example shows the existence of a bounded linear operator W_ε with positive spectral radius that is approximated, in the operator norm, by a sequence $W_\varepsilon - L_m$ of operators whose spectrum reduces to the single point 0.

Our first example of translation of these ideas to (8.1) is very simple. Let us choose a number R and the previous numbers M and K in such a way that $0 < R - M/K < R < 1 < R + M/K$ and with these choices define the new operator $T = RI + W_\varepsilon$, where I is the identity operator. The spectrum of T is a disk of radius M/K centered at the point R. This spectrum intersects the exterior of the unit circle and lies entirely in the half-plane $Re\, z \geq R - M/K > 0$. Because of this last property, the operator $A \doteq \log(T)$ can be defined, and by the Spectral Mapping Theorem

$$\|e^{tA}\| \geq \rho(e^{tA}) = e^{t\log(R+M/K)}, \tag{8.3}$$

which is unstable since $R + M/K > 1$.

We construct now the sequence of operators $S_m = R\,I + W_\varepsilon - L_m$. All of these operators have their spectra reduced to the single point $z = R$, and these operators converge in the operator norm to $T = R\,I + W_\varepsilon$, which spectrum is the disk of radius M/K centered at $z = R$. If we take now $A_m = \log(R\,I + W_\varepsilon - L_m)$, we again have that the sequence A_m tends to $A = \log(T)$ as $m \to \infty$ in the operator norm, by the continuity of the logarithm. Also, by the properties of the exponential, perhaps by using adapted norms, for all $\delta > 0$ and all m, there exists a number $D_{m,\delta}$ such that

$$\|e^{tA_m}\| \leq D_{m,\delta}\; e^{t(\log(R)+\delta)}, \tag{8.4}$$

which implies stability since $\log(R) < 0$, and δ can be chosen small enough.

In this way we have perturbed an autonomous unstable system $\dot{x}(t) = Ax(t)$ to a new autonomous system $\dot{x}(t) = Ax(t) + (A_m - A)x(t)$, with a perturbation that can be taken as small as we wish in the operator norm, and the new system is asymptotically stable.

This example deserves to be commented in relation of Theorem 4 of [10] (p. 2704). According to that theorem, if an equation $\dot{x}(t) = A(t)x(t)$ exhibits an exponential dichotomy with nontrivial stable and an unstable part (which in particular means that it is unstable), then a new system $\dot{x}(t) = A(t)x(t) + B(t)x(t)$ will exhibit a similar dichotomy (which means that it is also unstable) if $\sup\{\|B(t)\|; t \in \mathbb{R}\}$ is sufficiently small, and if some compactness conditions are met, that are automatically satisfied in our case since B does not depend on t. This robustness of the instability is broken in our example, since the spectrum of A is a connected set that has points both in $Re\, z < 0$ and in $Re\, z > 0$, but it is not possible to divide it into two spectral sets by the vertical line $Re\, z = 0$. This is something very typical from infinite dimensional functional analysis, that cannot be expected in finite dimensions.

Our second example, also based on Kakutani's construction, starts with the same system $\dot{x}(t) = Ax(t)$ as above, with $A = \log(R\,I + W_\varepsilon)$ and W_ε, with the relations $0 < R - M/K < R < 1 < R + M/K$, whose instability is expressed by the inequality (8.3) above. We want to add to it now a time-dependent perturbation $B(t)$, depending continuously on $t \geq 0$ such that $\sup\{\|B(t)\|; t \in [0, \infty)\}$ can be taken as small as we wish, but with the novelty that $\lim_{t\to\infty} \|B(t)\| = 0$. Despite this, we want to obtain a system $\dot{x}(t) = Ax(t) + B(t)x(t)$ that will be stable.

Let us name B_m the operators $A_m - A$ considered above. Let us say again that $\|B_m\| \to 0$ as $m \to \infty$ and that the spectra $\sigma(A + B_m) = \sigma(A_m) = \{\log R\}$. Let us fix now one value of $\delta > 0$ in (8.4) such that if we define $\omega = -\log(R) - \delta$ we still have $\omega > 0$. For example, $\delta = -\frac{1}{2}\log R$. If we write D_m for $D_{m,\delta}$ in (8.4) we will have $\|e^{tA_m}\| \leq D_m e^{-\omega t}$. We do not expect the sequence D_m to be bounded as $m \to \infty$. Let us choose an index $m_0 \geq 1$ and define

$$B(t) = \begin{cases} B_{m_0+k}, & \text{for } t_k \leq t \leq t_{k+1} - 1, \\ (t_{k+1} - t)B_{m_0+k} + (t - t_{k+1} + 1)B_{m_0+k+1} & \text{for } t_{k+1} - 1 \leq t \leq t_{k+1}, \end{cases} \tag{8.5}$$

for an increasing sequence t_k with $t_0 = 0$ and $t_k + 1 < t_{k+1}$, to be defined later. It is clear that $B(t)$ is a continuous function from $[0, \infty)$ to $\mathcal{L}(H)$. Since $\|B_m\| \to 0$ it is clear that

$$E_{m_0} \doteq \sup\{\|B_m\|; m \geq m_0\} \to 0 \text{ as } m_0 \to \infty.$$

Therefore, $\|B(t)\| \leq E_{m_0}$ for all $t \geq 0$, and this can be made as small as we like by choosing m_0 sufficiently large.

In order to define the sequence $(t_k)_{k\geq 0}$ let us now bound the solutions of

$$\begin{cases} \dot{x}(t) = Ax(t) + B(t)x(t), \\ x(0) = x_0. \end{cases} \tag{8.6}$$

For t between t_k and $t_{k+1}-1$ we will have $A+B(t) = A_{m_0+k}$ and, because of (8.4),

$$\|x(t)\| \leq \|x(t_k)\| D_{m_0+k} e^{-\omega(t-t_k)}.$$

To fix ideas, let us start with $k = 0$. For $0 = t_0 \leq t \leq t_1 - 1$ we can write $\|x(t)\| \leq D_{m_0} e^{-\omega t}\|x(0)\|$. Then, for $t_1 - 1 \leq t \leq t_1$ we can broadly bound as

$$\|x(t)\| \leq e^{(t-t_1+1)(\|A\|+E_{m_0})}\|x(t_1-1)\| \leq e^{(\|A\|+E_{m_0})}\|x(t_1-1)\|,$$

and, putting the two parts together

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega t}\|x(0)\|, \tag{8.7}$$

which obviously implies the weaker bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t}\|x(0)\|, \tag{8.8}$$

both for $0 \leq t \leq t_1$. Then, we continue with $t_1 \leq t \leq t_2 - 1$, and for this range of t we have $A + B(t) = A_{m_0+1}$ and

$$\|x(t)\| \leq D_{m_0+1} e^{-\omega(t-t_1)}\|x(t_1)\|,$$

and, as before,

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)}\|x(t_1)\|,$$

now for the whole $t_1 \leq t \leq t_2$. Putting this together with (8.7) we get, again for $t_1 \leq t \leq t_2$,

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega t_1}\|x(0)\|,$$

that we can write again as

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t_1} e^{-\frac{1}{2}\omega t_1}\|x(0)\|,$$

and at this point we see that we can choose t_1 large enough in such a way that

$$e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\frac{1}{2}\omega t_1} \leq 1.$$

With this choice we get

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_1)} e^{-\frac{1}{2}\omega t_1}\|x(0)\|, \tag{8.9}$$

for $t_1 \leq t \leq t_2$, which will be needed in the next interval, and also deduce, together with (8.8) the weaker but more global bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t}\|x(0)\|, \tag{8.10}$$

now for all t such that $0 \leq t \leq t_2$.

Now we proceed inductively. Suppose that along the interval $t_{k-1} \leq t \leq t_k$, where t_k is still to be chosen, we have obtained, as in (8.9), the bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_{k-1})} e^{-\frac{1}{2}\omega t_{k-1}}\|x(0)\|, \tag{8.11}$$

for $t_{k-1} \leq t \leq t_k$, and the weaker inequality

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t}\|x(0)\|, \tag{8.12}$$

for $0 \leq t \leq t_k$. Then we analyze for $t_k \leq t \leq t_{k+1}$ and obtain that

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0+k} e^{-w(t-t_k)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t_k-t_{k-1})} e^{-\frac{1}{2}\omega t_{k-1}}\|x(0)\|.$$

Then we choose t_k in such a way that

$$e^{(\|A\|+E_{m_0})} D_{m_0+k} e^{-\frac{1}{2}\omega(t_k - t_{k-1})} \leq 1,$$

and obtain

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_k)} e^{-\frac{1}{2}\omega t_k} \|x(0)\|, \tag{8.13}$$

for $t_k \leq t \leq t_{k+1}$, and the weaker inequality

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{8.14}$$

for $0 \leq t \leq t_{k+1}$.

With these choices of the t_k one can make $k \to \infty$ and obtain the final bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{8.15}$$

for all $t \geq 0$, that proves the exponential asymptotic stability of the solutions of (8.6).

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Robustness of Exponential Dichotomy in a Class of Generalised Almost Periodic Linear Differential Equations in Infinite Dimensional Banach Spaces

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Abstract

In this paper we study the robustness of the exponential dichotomy in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Some applications are obtained to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the stability is robust. These results extend to infinite dimensions some results given in Coppel (Dichotomies in stability theory. Lecture notes in mathematics, Springer, Berlin, 1970). Based in Rodrigues (Invariância para sistemas de equações diferenciais com retardamento e aplicações, Tese de Mestrado, Universidade de São Paulo, São Carlos, 1970) and in Kloeden and Rodrigues (Nonlinear Anal 74:2695–2719, 2011), Rodrigues et al. (Stability problems in non autonomous linear differential equations in infinite dimensions. arXiv:1906.04642, 2019) we use the class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable to model these oscillating perturbations. We also present an infinite dimensional example of the previous results.

Keywords Exponential dichotomy · Nonautonomous ordinary differential equations · Generalized almost period functions

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1 Introduction

The main objective of this paper is to study the robustness of the exponential dichotomy in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Also, we provide some applications to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the dichotomy is robust. In particular, our results extend some results given in Coppel [2] to infinite dimensions. Based in Rodrigues [6] and in Kloeden and Rodrigues [5], Rodrigues et al. [7] we use the class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable to model these oscillating perturbations. We also present an infinite dimensional example to illustrate the previous abstract results.

Let X be a Banach space and $A(t),\ B(t)$ be bounded operators defined in X, such that $\|A(t)\|,\ \|B(t)\|$ are bounded for every $t \in \mathbb{R}$. We consider the following systems:

$$\dot{x} = A(t)x \tag{1}$$
$$\dot{x} = A(t)x + B(t)x. \tag{2}$$

In Sect. 3 we show that if system (1) possesses an exponential dichotomy in $\mathbb{R}$ and $B(t)$ is integrably small, then system (2) has an exponential dichotomy in $\mathbb{R}$. Then if we suppose that $B(t)$ belongs to the class of generalized almost periodic functions and we consider the systems

$$\dot{x} = A(t)x \tag{3}$$
$$\dot{x} = A(t)x + B(\omega t)x, \tag{4}$$

if system (3) has an exponential dichotomy in $\mathbb{R}$ and ω is sufficiently large, then system (4) has also an exponential dichotomy in $\mathbb{R}$. We observe that if $B(t)$ is periodic then $B(\omega t)$ will have small period if ω is large.

In [7], page 17, there is a two dimensional example such that (1) has a non-trivial exponential dichotomy (and therefore it is not asymptotically stable), $B(t)$ is periodic with very large period and very small mean value and (2) is asymptotically stable.

In Sect. 4 we present an infinite dimensional example where $A(t) = A$ is constant, (3) admits an exponential dichotomy, $B(t)$ belongs to the class of generalized almost periodic functions and (4) has an exponential dichotomy with sufficiently large ω.

In Sect. 5 we consider a case where the linear part is constant, unbounded, generates a $\mathcal{C}_0$-semigroup and the perturbation $B(t)$ is small in some sense, and in Theorem 3 we present the necessary results for this case. In Example 1 we present an application of our abstract results to the heat equation.

2 Integral Inequalities

In the next lemma we prove a new integral inequality that will be very useful to show our main results.

Lemma 1 *Let s be a fixed number in $\mathbb{R}$. Let $u(t) \geq 0$ be a real continuous and bounded function for $t \geq s$, such that*

$$u(t) \leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^t e^{-\mu(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_s^t e^{-\alpha(t-\tau)}u(\tau)d\tau$$
$$+\mathcal{M}\int_t^\infty e^{\gamma(t-\tau)}u(\tau)d\tau, \tag{5}$$

where $K,\ \mathcal{N},\ \mathcal{L},\ \mathcal{M},\ \mu,\ \alpha,\ \gamma$ *are positive numbers, with* $\mu < \alpha$. *Let* $\beta \doteq \frac{\mathcal{N}}{\mu} + \frac{\mathcal{L}}{\alpha} + \frac{\mathcal{M}}{\gamma} < 1$. *Then*

$$u(t) \leq \frac{K}{1-\beta}e^{-\left(\alpha - \frac{\mathcal{N}+\mathcal{L}}{1-\beta}\right)(t-s)}, \quad t \geq s.$$

Also, if $u(t) \geq 0$ *is continuous and bounded for* $t \leq s$, *and*

$$u(t) \leq Ke^{\alpha(t-s)} + \mathcal{N}\int_t^s e^{\mu(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_t^s e^{\alpha(t-\tau)}u(\tau)d\tau$$
$$+\mathcal{M}\int_{-\infty}^t e^{-\gamma(t-\tau)}u(\tau)d\tau, \tag{6}$$

where $K,\ \mathcal{N},\ \mathcal{L},\ \mathcal{M},\ \mu,\ \alpha,\ \gamma$ *are positive numbers, with* $\mu < \alpha$, *then*

$$u(t) \leq \frac{K}{1-\beta}e^{\left(\alpha - \frac{\mathcal{N}+\mathcal{L}}{1-\beta}\right)(t-s)}, \quad t \leq s.$$

Proof We will first prove that $u(t) \to 0$ as $t \to \infty$. Suppose, by contradiction, that this is not true. Let $\delta \doteq \limsup_{t\to\infty} u(t)$. Then $\delta > 0$.

Let $0 < \nu < \theta < 1$. Then there exists $t_1 > s$ such that $u(t) \leq \frac{\delta}{\theta}$ for $t \geq t_1$.

Therefore for $t \geq t_1$,

$$\begin{aligned}
u(t) &\leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau + \mathcal{N}\int_{t_1}^t e^{-\mu(t-\tau)}u(\tau)d\tau \\
&\quad +\mathcal{L}\int_s^{t_1} e^{-\alpha(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_{t_1}^t e^{-\alpha(t-\tau)}u(\tau)d\tau \\
&\quad +\mathcal{M}\int_t^\infty e^{\gamma(t-\tau)}u(\tau)d\tau \\
&\leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau + \frac{\mathcal{N}\delta}{\theta}\int_{t_1}^t e^{-\mu(t-\tau)}d\tau \\
&\quad +\mathcal{L}\int_s^{t_1} e^{-\alpha(t-\tau)}u(\tau)d\tau + \frac{\mathcal{L}\delta}{\theta}\int_{t_1}^t e^{-\alpha(t-\tau)}d\tau \\
&\quad +\frac{\mathcal{M}\delta}{\theta}\int_t^\infty e^{\gamma(t-\tau)}u(\tau)d\tau \\
&\leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau + \frac{\mathcal{N}\delta}{\mu\theta} \\
&\quad +\mathcal{L}\int_s^{t_1} e^{-\alpha(t-\tau)}u(\tau)d\tau + \frac{\mathcal{L}\delta}{\alpha\theta} + \frac{\mathcal{M}\delta}{\gamma\theta} \\
&= Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{\mathcal{N}}{\mu}+\frac{\mathcal{L}}{\alpha}+\frac{\mathcal{M}}{\gamma}\right]\frac{\delta}{\theta}\\
&= Ke^{-\alpha(t-s)}+(\mathcal{N}+\mathcal{L})\int_s^{t_1} e^{-\mu(t-\tau)}u(\tau)d\tau+\beta\frac{\delta}{\theta}.
\end{aligned}
$$

Then $\delta = \limsup_{t\to\infty} u(t) \le \frac{\beta\delta}{\theta} < \delta$, which is a contradiction. Therefore $u(t) \to 0$ as $t \to \infty$.

Now for $t \ge s$ let $v(t) \doteq \sup_{\tau\ge t} u(\tau)$. We can see that $v(t)$ is a decreasing function for $t \ge s$.

Since $u(t) \to 0$ as $t \to \infty$, given $t \in [s,\infty)$ there exists $t_1 \ge t$ such that $v(t) = v(\tau) = u(t_1)$ for $t \le \tau \le t_1$ and $v(\tau) < v(t_1)$ if $\tau > t_1$. Indeed, let us prove this statement. Let $\bar{t}$ such that $u(\bar{t}) < v(t)$. Let $t_1 = \max\{\tau \in [t,\bar{t}]\}$ such that $v(\tau) = u(t_1)$. Then for $\tau \in [t,t_1]$ $v(\tau) = v(t) = u(t_1)$ and $v(\tau) < v(t_1)$ if $\tau > t_1$.

Then

$$
\begin{aligned}
v(t) = u(t_1) &\le Ke^{-\alpha(t_1-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t_1-\tau)}v(\tau)d\tau\\
&\quad +\mathcal{L}\int_s^{t_1} e^{-\alpha(t_1-\tau)}v(\tau)d\tau+\mathcal{M}\int_{t_1}^{\infty} e^{\gamma(t_1-\tau)}v(\tau)d\tau\\
&\le Ke^{-\alpha(t_1-s)}+\mathcal{N}\int_s^{t} e^{-\mu(t_1-\tau)}v(\tau)d\tau+\mathcal{N}\int_t^{t_1} e^{-\mu(t_1-\tau)}v(\tau)d\tau\\
&\quad +\mathcal{L}\int_s^{t} e^{-\alpha(t_1-\tau)}v(\tau)d\tau+\mathcal{L}\int_t^{t_1} e^{-\alpha(t_1-\tau)}v(\tau)d\tau\\
&\quad +\mathcal{M}\int_t^{\infty} e^{\gamma(t_1-\tau)}v(\tau)d\tau\\
&\le Ke^{-\alpha(t-s)}+\mathcal{N}\int_s^{t} e^{-\mu(t_1-\tau)}v(\tau)d\tau+\mathcal{N}v(t)\int_t^{t_1} e^{-\mu(t_1-\tau)}d\tau\\
&\quad +\mathcal{L}\int_s^{t} e^{-\alpha(t_1-\tau)}v(\tau)d\tau+\mathcal{L}v(t)\int_t^{t_1} e^{-\alpha(t_1-\tau)}d\tau\\
&\quad +\mathcal{M}v(t)\int_t^{\infty} e^{\gamma(t_1-\tau)}d\tau\\
&= Ke^{-\alpha(t-s)}+\left[\frac{\mathcal{N}}{\mu}+\frac{\mathcal{L}}{\alpha}+\frac{\mathcal{M}}{\gamma}\right]v(t)+\mathcal{N}\int_s^{t} e^{-\mu(t-\tau)}v(\tau)d\tau\\
&\quad +\mathcal{L}\int_s^{t} e^{-\alpha(t-\tau)}v(\tau)d\tau\\
&= Ke^{-\alpha(t-s)}+\beta v(t)+[\mathcal{N}+\mathcal{L}]\int_s^{t} e^{-\alpha(t-\tau)}v(\tau)d\tau.
\end{aligned}
$$

Therefore $(1-\beta)v(t) \le Ke^{-\alpha(t-s)} + [\mathcal{N}+\mathcal{L}]\int_s^t e^{-\alpha(t-\tau)}v(\tau)d\tau$ and

$$
\begin{aligned}
v(t) &\le \frac{K}{1-\beta}e^{-\alpha(t-s)}+\frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^{t} e^{-\alpha(t-\tau)}v(\tau)d\tau,\\
e^{\alpha(t-s)}v(t) &\le \frac{K}{1-\beta}+\frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^{t} e^{\alpha(t-s)}e^{-\alpha(t-\tau)}v(\tau)d\tau\\
&= \frac{K}{1-\beta}+\frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^{t} e^{\alpha(\tau-s)}v(\tau)d\tau
\end{aligned}
$$

Thanks to Gronwall's inequality,

$$e^{\alpha(t-s)}v(t) \leq \frac{K}{1-\beta}e^{\frac{\mathcal{N}+\mathcal{L}}{1-\beta}(t-s)}, \quad \text{and so} \quad u(t) \leq v(t) \leq \frac{K}{1-\beta}e^{-\left(\alpha-\frac{\mathcal{N}+\mathcal{L}}{1-\beta}\right)(t-s)}, \quad t \geq s.$$

The proof of the second integral inequality is similar. □

3 Robustness of Exponential Dichotomy in $\mathbb{R}$

Based on [4] we define the concept of exponential dichotomies. Suppose the evolution operators $T(t,s) \in L(X),\ t \geq s$, for $\dot{x} = A(t)x$ are defined in $\mathbb{R}$ (see [7] for a detailed description of the concepts used in this paper).

Definition 1 Equation $\dot{x} = A(t)x$ is said to have an exponential dichotomy in $\mathbb{R}$, with exponent $\beta > 0$ and bound M if there exist projections $P(t),\ t \in \mathbb{R}$ such that

1. $T(t,s)(I-P(s)) = (I-P(t))T(t,s),\ t \geq s,\ t,s \in \mathbb{R}$.
2. the restriction $T(t,s)|_{\mathcal{R}(I-P(s))},\ t \geq s$, is an isomorphism of $\mathcal{R}(I-P(s))$ onto $\mathcal{R}(I-P(t))$, and we define $T(s,t)$ as the inverse from $\mathcal{R}(I-P(t))$ to $\mathcal{R}(I-P(s))$.
3.

$$\begin{aligned} \|T(t,s)P(s)\| &\leq Me^{-\beta(t-s)} \quad for \quad t \geq s \ in\ \mathbb{R} \\ \|T(t,s)(I-P(s))\| &\leq Me^{-\beta(s-t)} \quad for \quad s \geq t \ in\ \mathbb{R} \end{aligned} \tag{7}$$

Remark 1 In [4], $P(t)$ projects X onto the unstable manifold, differing from the usual convention. In this paper, we chose to follow the usual convention, thus $P(t)$ will project X onto the stable manifold.

Suppose now that $t \in \mathbb{R} \to A(t) \in L(X)$ is continuous and that equation $\dot{x} = A(t)x$ has an exponential dichotomy in $\mathbb{R}$. Then, there is no solution $x(t)$ defined and bounded in $\mathbb{R}$. Let X_1 be the subspace of X of initial conditions on $t = 0$ of the solutions that are bounded for $t \geq 0$ and X_2 be the subspace of X of initial conditions on $t = 0$ of the solutions that are bounded for $t \leq 0$. Then, we have $X = X_1 \oplus X_2$ and $P_1,\ P_2$ the projections from X onto X_1 and X_2 respectively. Then we can take $P(t) = X(t)P_1X^{-1}(t)$, where $X(t)$ is the operator solution of the equation such that $X(0) = I$.

Theorem 1 *Let $A,\ B : \mathbb{R} \to L(X)$ be continuous functions such that there exists $M > 0$ and $\|A(t)\| \leq M$ and $\|B(t)\| \leq M$ for every $t \in \mathbb{R}$. Consider the equations:*

$$\dot{x} = A(t)x \tag{8}$$

$$\dot{y} = A(t)y + B(t)y \tag{9}$$

Let $T(t,s) = X(t)X^{-1}(s)$ be the evolution operator of (8) and $S(t,s) = Y(t)Y^{-1}(s)$ the evolution operator of (9). We suppose that system (8) admits an exponential dichotomy in $\mathbb{R}$, i.e., there exist projections $P(s),\ s \in \mathbb{R}$, constants $K > 1$, $\alpha > 0$, such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s \\ \|T(t,s)(I-P(s))\| &\leq Ke^{\alpha(t-s)}, \quad t \leq s \end{aligned} \tag{10}$$

Assume that there exist $\delta,\ h > 0$ such that $\| \int_{t_1}^{t_2} B(t)dt\| \leq \delta$ provided that $|t_2 - t_1| \leq h$, $t_1, t_2 \in \mathbb{R}$.

Then, there exist projections $Q(s)$, $s \in \mathbb{R}$ *and constants* $\widetilde{K}$ *and* $\widetilde{\alpha} > 0$, *such that we have* $S(t,s)Q(s) = Q(t)S(t,s)$ *and*

$$\begin{aligned} \|S(t,s)Q(s)\| &\leq \widetilde{K} e^{-\widetilde{\alpha}(t-s)}, \quad t \geq s \\ \|S(t,s)(I - Q(s))\| &\leq \widetilde{K} e^{\widetilde{\alpha}(t-s)}, \quad t \leq s \end{aligned} \tag{11}$$

where, $\widetilde{K} = \frac{K\left(1+\delta \frac{K}{1-\left(K+\frac{6KM}{\alpha}\right)\delta}\right)}{1-\beta}$, $\beta = \frac{6KM\delta}{\alpha} < 1$, *and* $\widetilde{\alpha} = \alpha - \frac{6KM\delta}{1-\beta}$.

Proof We first prove that there exists a projection $Q(s)$ such that $S(t,s)Q(s)$ is bounded for $t \geq s$ for $t,\ s \in \mathbb{R}$.

From the variation of constants formula it follows that

$$S(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)d\tau.$$

Since we look for $Q(s)$ as a perturbation of $P(s)$, we will show that the following implicit equation

$$S(t,s)Q(s) = T(t,s)P(s) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)Q(s)d\tau.$$

has a solution $S(t,s)Q(s)$ bounded for $t \geq s$. Let $Y(t,s) \doteq S(t,s)Q(s)$.

Then we should prove that the equation

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)B(\tau)Y(\tau,s)d\tau$$

has a solution $Y(t,s) \in L(X)$ bounded for $t \geqq s$ and $t,\ s \in \mathbb{R}$, and then $Q(s) = S(t,s)^{-1}Y(t,s)$:

$$\begin{aligned} Y(t,s) &= T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &\quad + \int_s^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &= T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &\quad + \int_s^\infty T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &\quad + \int_\infty^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &= T(t,s)\left[P(s) + \int_s^\infty T(s,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau\right] \\ &\quad + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &\quad + \int_\infty^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau. \end{aligned}$$

Since

$$\int_s^\infty T(s,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau = (I - P(s))\int_s^\infty T(s,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau$$

and $T(t,s)(I-P(s))\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau$ is bounded for $t \geq s$ this implies that this term must be equal zero.

Therefore we must solve the equation,

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau + \int_\infty^t T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau.$$

We will first estimate $Y(t,s)$ in an arbitrary interval of length h. To this end, we consider the strip $H_h \doteq \{(s,t) \in \mathbb{R}^2 : s \leq t \leq s+h\}$. For $(t,s) \in H_h$ consider the integral equation:

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \tag{12}$$

We prove the existence of a solution $Y(t,s)$ of this equation using the Banach Fixed Point Theorem.

Now we consider the space $\mathcal{Y}_h \doteq BC(H_h, X)$ of the bounded continuous functions Y from H_h to X with the norm $|Y| \doteq \sup_{(s,t)\in H_h} |Y(t,s)|$. This is a Banach space. For $Y \in \mathcal{Y}_h$ we define the operator $\mathcal{T}$ as

$$(\mathcal{T}Y)(t,s) \doteq T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau. \tag{13}$$

We first prove that $\mathcal{T}(\mathcal{Y}_h) \subset \mathcal{Y}_h$. The continuity is trivial. Let us prove the boundedness. Let $Y \in \mathcal{Y}_h$. For $(s,t) \in H_h$

$$\begin{aligned}
|(\mathcal{T}Y)(t,s)| &\leq |T(t,s)P(s)| + \left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\
&\quad + \left|\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| \\
&\leq Ke^{-\alpha(t-s)} + \int_s^t Ke^{-\alpha(t-\tau)}M|Y|\,d\tau \\
&\quad + \int_t^\infty Ke^{-\alpha(\tau-t)}M|Y|\,d\tau \\
&\leq K + \frac{2KM|Y|}{\alpha}
\end{aligned}$$

From (13) it follows that

$$(\mathcal{T}Y)(t,s) = T(t,s)P(s) + (\mathcal{T}_1Y)(t,s),$$

where

$$(\mathcal{T}_1Y)(t,s) = \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau. \tag{14}$$

Since $(\mathcal{T}_1 Y)(t,s)$ is linear, to prove that $(\mathcal{T} Y)(t,s)$ is a contraction it is sufficient to prove that $(\mathcal{T}_1 Y)(t,s)$ is a contraction.

Let us analyse the first integral of (14). Let $(s,t) \in H_h$ such that $s \le t \le s+h$. Consider the integral:

$$\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau.$$

We let $C_t(\tau) \doteq \int_t^\tau B(u)du$. In order to use the smallness of the integral $C_t(\tau) \doteq \int_t^\tau B(u)du$, we will perform an integration by parts taking the derivative of three terms:

$$\begin{aligned}
&\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)] \\
&\quad = -T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s) + T(t,\tau)P(\tau)B(\tau)Y(\tau,s) \\
&\qquad + T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).
\end{aligned}$$

Therefore

$$\begin{aligned}
T(t,\tau)P(\tau)B(\tau)Y(\tau,s) = &\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)] \\
&+ T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s) \\
&- T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).
\end{aligned}$$

Integrating,

$$\begin{aligned}
\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau &= \int_s^t \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\
&\quad + \int_s^t T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\
&= -T(t,s)P(s))C_t(s)Y(s,s) + \int_s^t T(t,\tau) \\
&\quad P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\
&\le T(t,s)P(s))C_t(s)Y(s,s)| + \left|\int_s^t T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau\right| \\
&\quad + \left|\int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau\right| \\
&\le K\delta e^{-\alpha(t-s)}|Y(s,s)| + 3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau.
\end{aligned}$$

We conclude that for $(s, t) \in H_h$, that is for $s \le t \le s + h$,

$$\begin{aligned}&\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\ &\quad \le K\delta e^{-\alpha(t-s)}|Y(s,s)| + 3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau. \qquad (15)\end{aligned}$$

Now if $s < t$ let $n \in \mathbb{N}$ such that $s + nh \le t \le s + (n+1)h$.

$$\begin{aligned}\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau &= \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &\quad + \int_{s+nh}^{t} T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &= \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\ &\quad + \int_{s+nh}^{t} T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\ &\quad -\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\ &\quad - \int_{s+nh}^{t} T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\ &\quad +\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &\quad + \int_{s+nh}^{t} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau.\end{aligned}$$

But

$$\begin{aligned}&\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &\quad + \int_{s+nh}^{t} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &= \int_{s}^{s+h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &\quad + \int_{s+h}^{s+2h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau + \cdots \\ &\quad + \int_{s+(n-2)h}^{s+(n-1)h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &\quad + \int_{s+(n-1)h}^{t} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\ &= [T(t,s+h)P(s+h)C_t(s+h)Y(s+h,s) - T(t,s)P(s)C_t(s)Y(s,s)] \\ &\quad +[T(t,s+2h)P(s+2h)C_t(s+2h)Y(s+2h,s)\end{aligned}$$

$$
\begin{aligned}
&-T(t,s+h)P(s+h)C_t(s+h)Y(s+h,s)]+\cdots\\
&+T(t,s+(n-1)h)P(s+(n-1)h)C_t(s+(n-1)h)Y(s+(n-1)h,s)\\
&-T(t,s+(n-2)h)P(s+(n-2)h)C_t(s+(n-2)h)Y(s+(n-2)h,s)\\
&-T(t,s+(n-1)h)P(s+(n-1)h)C_t(s+(n-1)h)Y(s+(n-1)h,s)\\
=&-T(t,s)P(s)C(s)Y(s,s).
\end{aligned}
$$

Therefore if $s < t$ and $n \in \mathbb{N}$ such that $s + nh \le t \le s + (n+1)h$, we have

$$
\begin{aligned}
\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau = &-T(t,s)P(s)C(s)Y(s,s)\\
&+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)A(\tau)C(\tau)Y(\tau,s)d\tau\\
&+\int_{s+nh}^{t} T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau\\
&+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)[A(\tau)\\
&\quad+B(\tau)]C_t(\tau)Y(\tau,s)d\tau\\
&+\int_{s+nh}^{t} T(t,\tau)P(\tau)[A(\tau)+B(\tau)]C_t(\tau)Y(\tau,s)d\tau.
\end{aligned}
$$

Estimating,

$$
\begin{aligned}
\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \le &\ |T(t,s)P(s)C_t(s)Y(s,s)|\\
&+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau|\\
&+\int_{s+nh}^{t} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)|d\tau\\
&+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} |T(t,\tau)P(\tau)[A(\tau)\\
&\quad+B(\tau)]C_t(\tau)Y(\tau,s)d\tau|\\
&+\int_{s+nh}^{t} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)|d\tau\\
\le &\ \delta K e^{-\alpha(t-s)}|Y(s,s)|\\
&+KM\delta\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau\\
&+KM\delta\int_{s+nh}^{t} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau\\
&+2KM\delta\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+2KM\delta \sum_{j=0}^{n-1} \int_{s+nh}^{t} e^{-\alpha(t-\tau)} |Y(\tau, s)| d\tau \\
&= 4KM\delta \int_{s}^{t} e^{-\alpha(t-\tau)} |Y(\tau, s)| d\tau.
\end{aligned}
$$

Therefore, if $s < t$, let $n \in \mathbb{N}$ such that $s + nh \leq t \leq s + (n+1)h$, we have

$$
\begin{aligned}
\left| \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \right| &\leq \delta K e^{-\alpha(t-s)} |Y(s,s)| \\
&\quad + 3KM\delta \int_s^t e^{-\alpha(t-\tau)} |Y(\tau,s)| d\tau. \qquad (16)
\end{aligned}
$$

Consider now $\int_t^\infty T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau$.
As before, we pick $s \leq t \leq s + h$, and we estimate the integral

$$
\int_s^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau.
$$

Let us denote $C_t(\tau) = \int_t^\tau B(u)du$.
Taking derivatives,

$$
\begin{aligned}
\frac{d}{d\tau}[T(t,\tau)(I - P(\tau))C_t(\tau)Y(\tau,s)] &= -T(t,\tau)(I - P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\quad + T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s) \\
&\quad + T(t,\tau)(I - P(\tau))C_t(\tau)(A(\tau) + B(\tau))Y(\tau,s).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s) &= \frac{d}{d\tau}[T(t,\tau)(I - P(\tau))C_t(\tau)Y(\tau,s)] \\
&\quad + T(t,\tau)(I - P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\quad - T(t,\tau)(I - P(\tau))C_t(\tau)(A(\tau) + B(\tau))Y(\tau,s).
\end{aligned}
$$

Integrating,

$$
\begin{aligned}
\int_s^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau &= \int_s^t \frac{d}{d\tau}[T(t,\tau)(I - P(\tau))C_t(\tau)Y(\tau,s)]d\tau \\
&\quad + \int_s^t T(t,\tau)(I - P(\tau))A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)(I - P(\tau))C_t(\tau)(A(\tau) + B(\tau))Y(\tau,s)d\tau \\
&= T(t,s)(I - P(s))C_t(s)Y(s,s) \\
&\quad + \int_s^t T(t,\tau)(I - P(\tau))A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)(I - P(\tau))C_t(\tau)(A(\tau) + B(\tau))Y(\tau,s)d\tau.
\end{aligned}
$$

We have

$$
\begin{aligned}
\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau &= \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&= \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} \frac{d}{d\tau}[T(t,\tau)(I-P(\tau))C_t(\tau)Y(\tau,s)]d\tau \\
&\quad + \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} [T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\quad - T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)]d\tau
\end{aligned}
$$

Since the first term is zero we have

$$
\begin{aligned}
\left|\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| &\le \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} Ke^{\alpha(t-\tau)}3M\delta|Y(\tau,s)|d\tau \\
&= 3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\le \frac{3KM\delta}{\alpha}|Y|.
\end{aligned} \tag{17}
$$

Therefore

$$
|(\mathcal{T}_1 Y)(t,s)| \le \delta\left[\frac{K}{1-e^{-\alpha h}} + \frac{3KM}{\alpha} + \frac{3KM}{\alpha}\right]|Y|,
$$

and we conclude that, if δ is sufficiently small, $\mathcal{T}_1$ is contraction and so $\mathcal{T}$ is a contraction. The Banach Fixed Point Theorem ensures the existence of a unique fixed point $Y(t,s)$.

From the above inequality also follows that for $s \le t$

$$
\left|\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| \le 3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau \tag{18}
$$

Therefore

$$
\begin{aligned}
Y(t,s) &= T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&\quad - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau,
\end{aligned} \tag{19}
$$

$$
\text{and } Y(s,s) = P(s) + \int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau.
$$

From (19) it follows that

$$
\begin{aligned}
Y(t,s)Y(s,s) &= T(t,s)P(s)Y(s,s) \\
&\quad + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)Y(s,s)d\tau \\
&\quad - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)Y(s,s)d\tau.
\end{aligned}
$$

But

$$
\begin{aligned}
P(s)Y(s,s) &= P(s) - P(s)\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&= P(s) - \int_s^\infty T(s,\tau)P(\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&= P(s).
\end{aligned}
$$

Then $Y(t,s)Y(s,s)$ is also a solution of (19) and so $Y(t,s)Y(s,s) = Y(t,s)$. This implies that $Y(s,s)Y(s,s) = Y(s,s)$ and so $Q(s) \doteq Y(s,s)$ is a projection. In particular,

$$
P(s)Q(s) = P(s).
$$

Also from (19), it follows that $Y(t,s)P(s)$ is a solution, and then $Y(t,s)P(s) = Y(t,s)$, which implies that $Y(s,s)P(s) = Y(s,s)$ and so $Q(s)P(s) = Q(s)$.

From (19) it follows that

$$
\begin{aligned}
Y(t,s)Q(s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)Q(s)d\tau \\
&- \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)Q(s)d\tau,
\end{aligned}
$$

and from (16) and (17) it follows that, for $s \le t$

$$
\begin{aligned}
|Y(t,s)Q(s)| \le K(1+\delta|Y(s,s)|)e^{-\alpha(t-s)} &+ 3KM\delta\int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau \\
&+3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau. \qquad (20)
\end{aligned}
$$

Thus we must estimate $|Y(s,s)|$. From (19), using the estimates (15) and 17) we obtain for $s \le t \le s+h$

$$
\begin{aligned}
|Y(t,s)| \le Ke^{-\alpha(t-s)} &+ \delta Ke^{-\alpha(t-s)}|Y(s,s)| \\
&+3KM\delta\int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&+3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau.
\end{aligned}
$$

and

$$
|Y| \le K + \delta K|Y| + \frac{3KM\delta}{\alpha}|Y| + \frac{3KM\delta}{\alpha}|Y| = K + \delta K|Y| + \frac{6KM\delta}{\alpha}|Y|.
$$

Then

$$
|Y| \le \frac{K}{1 - K\delta\left(1+\frac{6M}{\alpha}\right)}
$$

In particular

$$
|Y(s,s)| = |Q(s)| \le \frac{K}{1-\left(K+\frac{6KM}{\alpha}\right)\delta}
$$

and thus we have a bound for $|Q(s)|$.

$$
\begin{aligned}
|Y(t,s)Q(s)| \leq K &\left(1+\delta\frac{K}{1-\left(K+\frac{6KM}{\alpha}\right)\delta}\right)e^{-\alpha(t-s)} \\
&+3KM\delta\int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau \\
&+3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau
\end{aligned}
$$

If we let $S(t,s)Q(s)=Y(t,s)Q(s)$, from Lemma 1, we obtain

$$
|S(t,s)Q(s)| \leq \frac{K\left(1+\delta\frac{K}{1-\left(K+\frac{6KM}{\alpha}\right)\delta}\right)}{1-\beta}e^{-\left(\alpha-\frac{6KM\delta}{1-\beta}\right)(t-s)}, \quad t\geq s \tag{21}
$$

and $\beta=\frac{6KM\delta}{\alpha}$.

From the variation of constants formula it follows that

$$
S(t,s)=T(t,s)+\int_s^t T(t,\tau)B(\tau)S(\tau,s)d\tau.
$$

Since we are looking for a projection $W(s)$ as a perturbation of $I-P(s)$, we will show that the following implicit equation

$$
S(t,s)W(s)=T(t,s)(I-P(s))+\int_s^t T(t,\tau)B(\tau)S(\tau,s)W(s)d\tau.
$$

has a solution $S(t,s)W(s)$ bounded for $t\leq s$. Let $Z(t,s)\doteq S(t,s)W(s)$, then

$$
Z(t,s)=T(t,s)(I-P(s))+\int_s^t T(t,\tau)B(\tau)Z(\tau,s)d\tau,
$$

and if $Z(t,s)$ is bounded for $t\leq s$,

$$
\begin{aligned}
Z(t,s) &= T(t,s)(I-P(s)) \\
&\quad+\int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad+\int_s^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \\
Z(t,s) &= T(t,s)(I-P(s)) \\
&\quad+\int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad+\int_s^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&\quad+\int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&\quad+\int_t^{-\infty} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&= T(t,s)(I-P(s))+\int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau
\end{aligned}
$$

$$+\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau$$
$$-\int_{-\infty}^{s} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

But

$$\int_{-\infty}^{s} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau = T(t,s)P(s)\int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

Since this term should be bounded for $t \le s$, then it must be equal to 0. Therefore,

$$Z(t,s) = T(t,s)(I-P(s)) + \int_{s}^{t} T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau$$
$$+\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

Now we proceed as in (12). Let $H \doteq \{(t,s) \in \mathbb{R}^2 : \ t \le s\}$. For $(t,s) \in H$ we consider the integral equation:

$$Z(t,s) = T(t,s)(I-P(s)) + \int_{s}^{t} T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau$$
$$+\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \tag{22}$$

We now prove the existence of a solution $Z(t,s)$ for this equation by using the Banach Fixed Point Theorem. To this end, we consider the space $\mathcal{Z} \doteq BC(H,X)$ of the bounded continuous functions Z from H to X with the norm $|Z| \doteq \sup_{t\le s} |Z(t,s)|$. This is a Banach space. For $Z \in \mathcal{Z}$ we define the operator $\mathcal{T}_1$ as

$$(T_1 Z)(t,s) = T(t,s)(I-P(s)) + \int_{s}^{t} T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau$$
$$+\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

One can prove that $\mathcal{T}_1 Z \subset Z)$ and that $\mathcal{T}_1$ is a contraction. Then the integral equation (22) has a unique solution $Z(t,s)$ in $\mathcal{Z}$.

Since $Z(t,s)Z(s,s)$ is also a solution of that equation, this implies that $Z(s,s)Z(s,s) = Z(s,s)$ and so $Z(s,s)$ is a projection and

$$Z(s,s) = I - P(s) + \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

Now

$$\begin{aligned}(I-Q(s))Z(s,s) &= (I-Q(s))(I-P(s)) \\ &\quad + \int_{-\infty}^{s} (I-Q(s))T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\ &= I - Q(s) + \int_{-\infty}^{s} T(s,\tau)(I-Q(\tau))P(\tau)B(\tau)Z(\tau,s)d\tau\end{aligned}$$

$$
\begin{aligned}
&= (I - Q(s)) + \int_{-\infty}^{s} T(s,\tau)(P(\tau) - P(\tau)B(\tau)Z(\tau,s)d\tau \\
&= I - Q(s).
\end{aligned}
$$

Therefore $Z(s,s) = I - Q(s)$. From (22) it follows that

$$
\begin{aligned}
S(t,s)(I - Q(s)) = T(t,s)(I - P(s)) &+ \int_{s}^{t} T(t,\tau)(I - P(\tau))B(\tau)Z(\tau,s)d\tau \\
&+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \qquad (23)
\end{aligned}
$$

If we proceed as in the estimate of $S(t,s)Q(s)$ in 21, and use (1), we can prove that

$$
|S(t,s)(I - Q(s))| \le \frac{K\left(1 + \delta \frac{K}{1 - \left(K + \frac{6KM}{\alpha}\right)\delta}\right)}{1 - \beta} e^{\left(\alpha - \frac{6KM\delta}{1-\beta}\right)(s-t)}, \quad s \le t \qquad (24)
$$

and $\beta = \frac{6KM\delta}{\alpha}$. □

Following the ideas of [7] pages 9 and 10 and from Coppel [2] page 8 we obtain

Corollary 1 *Let $A, B : \mathbb{R} \to L(X)$ be continuous functions such that $\|A(t)\| \le M$ and $\|B(t)\| \le M$ for every $t \in \mathbb{R}$, where M is a positive constant. Suppose that $B(t)$ is a generalized almost periodic function ($\mathcal{GAP}$) with mean value zero. Consider the equations*

$$
\dot{x} = A(t)x \qquad (25)
$$

$$
\dot{x} = A(t)x + B(\omega t)x \qquad (26)
$$

Let $T(t,s)$ be the evolution operator of (25) and $S_\omega(t,s)$ be the evolution operator of (26). Suppose that there exist projections $P(s)$, $s \in \mathbb{R}$ such that $|T(t,s)P(s)| \le Ke^{-\alpha(t-s)}$ for $t \ge s$ and $|T(t,s)(I - P(s))| \le Ke^{\alpha(t-s)}$ for $t \le s$, $t, s \in \mathbb{R}$, where $\alpha > 0$ and $K > 1$. Then there exist projections $Q_\omega(s)$, $s \in \mathbb{R}$, $\widetilde{K} > K$, $\widetilde{\beta} < \alpha$ and $\omega_0 > 0$ such that for $\omega > \omega_0$

$$
\begin{aligned}
|S_\omega(t,s)Q_\omega(s)| &\le \widetilde{K}e^{-\widetilde{\beta}(t-s)}, \quad t \ge s, \\
|S_\omega(t,s)(I - Q_\omega(s))| &\le \widetilde{K}e^{\widetilde{\beta}(t-s)}, \quad t \le s,
\end{aligned}
$$

where $S_\omega(t,s)$ indicates the evolution operator of (26).

Consider now $A \in \mathcal{GAP}$. Then we have $A(t) = A_0 + B(t)$, where $A_0 = \mathcal{M}(A)$ and $\mathcal{M}(B) = 0$, where $\mathcal{M}$ denotes the mean value. We suppose that $|A_0| \le M$ and $|B(t)| \le M$ for every $t \in \mathbb{R}$. Consider the equations:

$$
\dot{x} = A_0 x \qquad (27)
$$

$$
\dot{x} = A_0 x + B(\omega t)x. \qquad (28)
$$

Let $T(t) \doteq e^{A_0 t}$ be the group generated by (27) and $S_\omega(t,s)$ be the evolution operator of (28). The next corollary follows from Corollary 1.

Corollary 2 *Assume that system (8) admits an exponential dichotomy in $\mathbb{R}$, i.e., there exist projections P, constants $K > 1$, $\alpha > 0$, such that*

$$
\|T(t)P\| \le Ke^{-\alpha(t-s)}, \quad t \ge s \qquad (29)
$$

$$
\|T(t)(I - P)\| \le Ke^{\alpha(t-s)}, \quad t \le s \qquad (30)
$$

Then there exist projections $Q_\omega(s),\ s \in \mathbb{R}$ *and constants* $\widetilde{K} > K$, $0 < \widetilde{\alpha} < \alpha$ *and* $\omega_0 > 0$ *such that for every* $\omega > \omega_0$ *we have*

$$S_\omega(t,s)Q_\omega(s) \leq \widetilde{K}e^{-\widetilde{\alpha}(t-s)}, \quad \forall t \geq s,$$
$$S_\omega(t,s)(I - Q_\omega(s)) \leq \widetilde{K}e^{\widetilde{\alpha}(t-s)}, \quad \forall t \leq s.$$

□

4 Example of Exponential Dichotomy

If we proceed as in the infinite dimensional example in [7], we can construct bounded operators $A_1,\ A_2$ from ℓ_2 to ℓ_2, such that $|e^{A_1 t}| \leq e^{\frac{-a}{2}t}$ for $t \geq 0$ and $|e^{A_2 t}| \leq e^{\frac{a}{2}t}$ for $t \leq 0$.

If we proceed as in [7], defining $L_{(a,\nu)} \doteq \begin{pmatrix} a & 0 \\ 0 & \nu J + aI \end{pmatrix}$, $L_1 \doteq L_{(1/2,1/4)}$, $L_2 \doteq L_{(3/2,1/4)}$, $A_1 \doteq log(L_1)$ and $A_2 \doteq log(L_2)$, where

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} \tag{31}$$

Now we consider the bounded linear operator

$$A \doteq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{32}$$

and projections

$$P_1 \doteq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 \doteq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and we have

$$|e^{At}P_1| \leq e^{\frac{-a}{2}t}, \quad t \geq 0, \quad |e^{At}P_2| \leq e^{\frac{a}{2}t}, \quad t \leq 0. \tag{33}$$

Let $B(t) \doteq \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$ be a generalized almost periodic function $\mathcal{GAP}$ with mean value zero, $|A| \leq M,\ \sup_{t\in\mathbb{R}} |B(t)| \leq M$.

Consider the equations:

$$\dot{x} = Ax \tag{34}$$
$$\dot{y} = Ay + B(\omega t)y \tag{35}$$

From the above assumptions and the last previous result it follows the next one (Figs. 1, 2).

Corollary 3 *Let* $S_\omega(t,s)$ *be the evolution operator of* (35). *Then there exist* $\omega_0 > 0$, *constants* $\widetilde{K} \geq 1,\ \widetilde{\alpha} \leq \alpha$, *projections* $Q(s),\ s \in \mathbb{R}$ *such that for* $\omega \geq \omega_0$

$$|S(t,s)Q(s)| \leq \widetilde{K}e^{-\widetilde{\alpha}(t-s)}, \quad t \geq s,$$
$$|S(t,s)(I - Q(s))| \leq \widetilde{K}e^{\widetilde{\alpha}(t-s)}, \quad t \leq s.$$

□

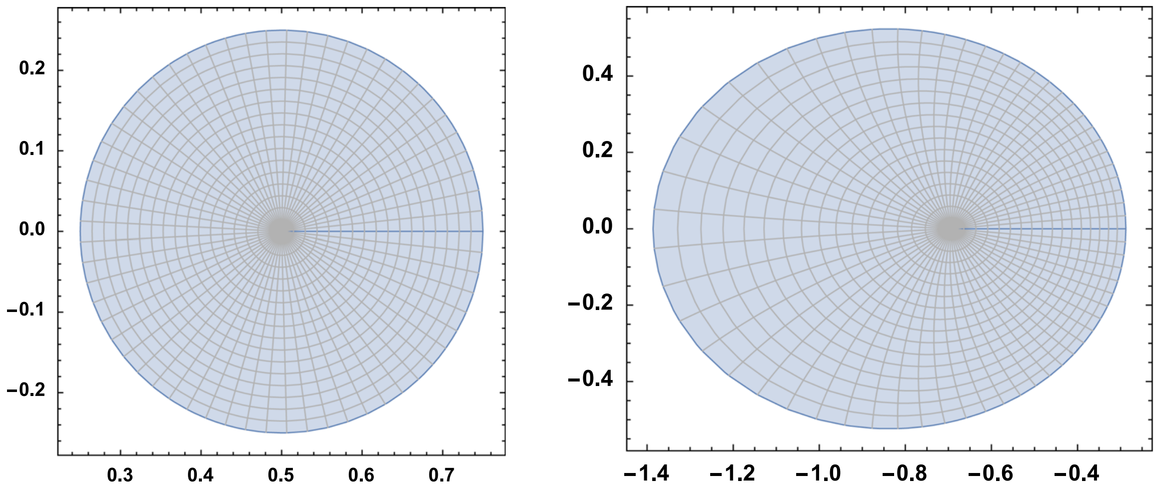

Fig. 1 Left: The spectrum of L_1 given by $\sigma(L_1) = B_{1/4}(1/2)$. Right: The spectrum of A_1 given by $\sigma(A_1) = \log(\sigma(L_1))$

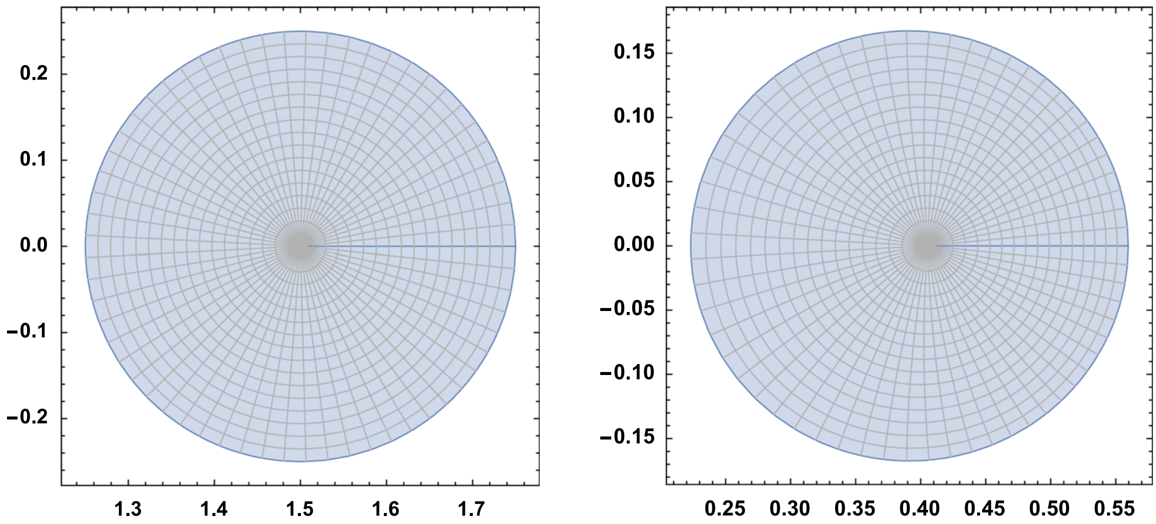

Fig. 2 Left: The spectrum of L_2 given by $\sigma(L_2) = B_{1/4}(3/2)$. Right: The spectrum of A_2 given by $\sigma(A_2) = \log(\sigma(L_2))$

5 A Case Where the Infinitesimal Generator is Unbounded

In this section we consider the equations

$$\begin{aligned}\dot{x} &= Ax\\ \dot{y} &= Ay + B(t)y,\end{aligned}$$

where we assume that $\mathcal{D}$ is dense in X and $A : \mathcal{D} \to X$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $T(t)$. We also assume that there exist a projection $P : X \to X$ and constants $K \geq 1$, $\alpha \in \mathbb{R}$, such that the following exponential dichotomy is satisfied:

$$\begin{aligned}\|T(t)P\| &\leq Ke^{-\alpha t}, \quad t \geq 0\\ \|T(t)(I-P)\| &\leq Ke^{\alpha t}, \quad t \leq 0\end{aligned} \tag{36}$$

Let us now recall an important result from Henry [4, page 30].

Theorem 2 *Suppose A is a closed operator in a Banach space X and assume that the spectrum of A can be decomposed as*

$$\sigma(A) = \sigma^+ \cup \sigma^1 \cup \sigma^2,\ \sigma^1 \cap \sigma^2 = \emptyset,$$

$\sigma^+ \subset \{\lambda \in \mathbb{C} : Re(\lambda) \geq \alpha > 0\}$ *is a bounded spectral set,* $\sigma^1 \cup \sigma^2 \subset \{\lambda \in \mathbb{C} : Re(\lambda) \leq -\alpha\}$, σ_1 *is a bounded spectral set of A, and* $\sigma_2 = \sigma(A) - \sigma_1$ *is closed and unbounded and so* $\sigma_2 \cup \{-\infty\}$ *is another spectral set.*

Let $I - P$, P_1*and* P_2 *be, respectively, the projections associated with these three spectral sets:* σ^+, σ^1, σ^2, *and* $X_+ \doteq (I-P)X$ *and* $X_j = P_j(X)$, $j = 1, 2$. *Then* $X_- \doteq X_1 \oplus X_2$ *and* X_j *are invariant under A, and if* A_j *is the restriction of A to* X_j, *for* $j = 1, 2$ *and* A_+ *is the restriction of A to* X_+ *then* $A_+ : X_+ \to X_+$ *is bounded* $\sigma(A_+) = \sigma^+$ *and*

$$A_1 : X_1 \to X_1 \text{ is bounded}, \sigma(A_1) = \sigma_1,\ \ \mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.$$

Furthemore, $P = P_1 + P_2$.

Now we will analyse some smallness conditions on the perturbation $B(t)$, such that equation (39) also admits an exponential dichotomy. The case when $B(t)$ is uniformly small is studied in Kloeden–Rodrigues [5] without leaving the continuous case. Similar results are obtained by Carvalho et al. [1], but they first find the result for the discrete case.

Similar results to the next theorem are treated by Carvalho et al. [1] and Dalekii-Krein [3] but they use the stronger assumption that $\int_\tau^t |B(t)|$ is small, with the norm inside the integral, and in the first one, they prove via a discretization method. Similar results are obtained by Henry [4, Theorem 7.6.11, page 238], where he also considers first the discrete case, and requires that $B(t)$ is uniformly small and integrally small.

Our next result is an extension of a classical result of Coppel [2] to the infinite dimensional case, and A being an unbounded operator.

Theorem 3 *Let h and* δ *be positive real numbers.*

Suppose that $A : \mathcal{D}(A) \subset X \to X$ *is the generator of a* C_0*-semigroup* $T(t)$, $t \geq 0$, $B(t) \in L(X)$ *and* $|B(t)| \leq M$ *for every* $t \in \mathbb{R}$. *Assume that for every* $t \in \mathbb{R}$ *we have that* $R(B(t)) \subset D(A)$, $AB(t)$ *is bounded and* $BA(t)$ *can be extended to a bounded operator. Also assume that* $B(t)$ *is integrally small, that is,*

$$\left| \int_t^u B(\tau) d\tau \right| \leq \delta \quad \text{whenever} \quad |t - u| \leq h.$$

Suppose we can decompose $\sigma(A) \doteq \sigma^+ \cup \sigma_1 \cup \sigma_2$, *as in Theorem 2, and define the respective projections* $I - P$, P_1 *and* P_2, *with* $P = P_1 + P_2$.

Suppose the equation

$$\dot{x} = Ax \tag{37}$$

admits an exponential dichotomy, or more specifically,

$$\begin{aligned} |T(t)(I-P)| &\leq Ke^{\alpha t}, \\ |T(t)P_1| &\leq Ke^{-\alpha t}, \\ |T(t)P_2| &\leq Ke^{-\mu t}, \end{aligned}$$

where $K > 0$ *and* $\mu > \alpha > 0$.

Assume that δ *is sufficiently small in such a way that* $\delta < \frac{\alpha}{6KM}$. *We also assume that* $|P_2 B(t)| < M\delta$, *for every* $t \in \mathbb{R}$ *(See Example 1 below).*

In analogy with the bounded case, if $C_t(u) \doteq \int_t^u B(\tau)d\tau$, we suppose that for $t \le u \le t+h$

$$\begin{array}{ll} |P_1 C_t(u)B| \le M\delta, & |(I-P)C_t(u)B| \le M\delta, \\ |P_1 AC_t(u)| \le M\delta, & |(I-P)AC_t(u)| \le M\delta, \\ |P_1 C_t(u)A| \le M\delta, & |(I-P)C_t(u)A| \le M\delta. \end{array} \tag{38}$$

If the above assumptions are satisfied, then the perturbed equation

$$\dot{y} = Ay + B(t)y \tag{39}$$

also admits an exponential dichotomy, that is,

$$|S(t,s)Q(s)| \le 2Ke^{-(\alpha-4KM\delta)(t-s)}, \quad t \ge s$$
$$|S(t,s)(I-Q(s))| \le 2Ke^{(\alpha-4KM\delta)(t-s)}, \quad t \le s.$$

Proof We will partially follow the steps of Theorem 1. Let us consider

$$\begin{aligned} S(t,s)Q(s) &= T(t-s)P + \int_s^t T(t-\tau)PB(\tau)S(\tau,s)Q(s)d\tau \\ &\quad + \int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau. \\ S(t,s)Q(s) &= T(t-s)P + \int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau \\ &\quad + \int_s^t T(t-\tau)P_1B(\tau)S(\tau,s)Q(s)d\tau \\ &\quad + \int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau. \\ |S(t,s)Q(s)| &\le |T(t-s)P| + \left|\int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau\right| \\ &\quad + \left|\int_s^t T(t-\tau)P_1B(\tau)S(\tau,s)Q(s)d\tau\right| \\ &\quad + \left|\int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau\right|. \end{aligned}$$

To estimate the two last integrals we proceed as in the proof of Theorem 1 using the fact that $B(t)$ is integrably small and in the first integral we use the estimate $|T(t)P_2| \le Ke^{-\mu t}$.

$$\left|\int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau\right| \le \int_s^t Ke^{-\mu(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau.$$

Therefore we obtain

$$\begin{aligned} |S(t,s)Q(s)| &\le Ke^{-\alpha(t-s)} + \int_s^t Ke^{-\mu(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau \\ &\quad + \int_s^t Ke^{-\alpha(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau \\ &\quad + \int_t^\infty Ke^{\alpha(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau. \end{aligned}$$

Now we use Lemma 1 with

$$\mathcal{N} = KM\delta,\ \mathcal{L} = KM\delta,\ \mathcal{M} = KM\delta,$$
$$\beta \doteq KM\delta\left(\frac{1}{\mu} + \frac{2}{\alpha}\right) < KM\delta\left(\frac{1}{\alpha} + \frac{2}{\alpha}\right) \le \frac{3KM\delta}{\alpha} < \frac{1}{2}.$$

Then we have that $\beta < \frac{1}{2}$ if $\delta < \frac{\alpha}{6KM}$, $-\beta > -\frac{1}{2}$ and so $1 - \beta > \frac{1}{2}$.

$$|S(t,s)Q(s)| \le 2Ke^{-\left(\alpha - \frac{2KM\delta}{1-KM\delta\left(\frac{1}{\mu}+\frac{2}{\alpha}\right)}\right)(t-s)},\ t \ge s$$

But

$$\frac{2KM\delta}{1 - KM\delta\left(\frac{1}{\mu} + \frac{2}{\alpha}\right)} \le \frac{2KM\delta}{1/2} \le 4KM\delta,$$
$$\left(\alpha - \frac{2KM\delta}{1 - KM\delta\left(\frac{1}{\mu} + \frac{2}{\alpha}\right)}\right) \ge \alpha - 4KM\delta,$$
$$-\left(\alpha - \frac{2KM\delta}{1 - KM\delta\left(\frac{1}{\mu} + \frac{2}{\alpha}\right)}\right) \le -(\alpha - 4KM\delta),$$

therefore,

$$|S(t,s)Q(s)| \le 2Ke^{-(\alpha - 4KM\delta)(t-s)}, \quad t \le s, \quad \text{if } \delta < \frac{\alpha}{6KM}.$$

For $t \le s$

$$\begin{aligned} S(t,s)(I - Q(s)) &= T(t-s)(I-P) \\ &\quad + \int_t^s T(t-\tau)P_2B(\tau)S(\tau,s)(I-Q(s))d\tau \\ &\quad + \int_t^s T(t-\tau)P_1B(\tau)S(\tau,s)(I-Q(s))d\tau \\ &\quad + \int_{-\infty}^t T(t-\tau)(I-P)B(\tau)S(\tau,s)(I-Q(s))d\tau. \\ |S(t,s)(I-Q(s))| &\le |T(t-s)(I-P)| \\ &\quad + \int_t^s |T(t-\tau)P_2B(\tau)S(\tau,s)(I-Q(s))|d\tau \\ &\quad + \int_t^s |T(t-\tau)P_1B(\tau)S(\tau,s)(I-Q(s)|)d\tau \\ &\quad + \int_{-\infty}^t |T(t-\tau)(I-P)B(\tau)S(\tau,s)(I-Q(s))|d\tau \\ &\le Ke^{\alpha(t-s)} + \int_t^s Ke^{\mu(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \\ &\quad + \int_t^s Ke^{\alpha(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \\ &\quad + \int_{-\infty}^t Ke^{-\alpha(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \end{aligned}$$

$$\begin{aligned}&\leq Ke^{\alpha(t-s)} + KM\delta\int_t^s e^{\mu(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau\\&+KM\delta\int_t^s e^{\alpha(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau\\&+KM\delta\int_{-\infty}^t e^{-\alpha(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau.\end{aligned}$$

Now we use Lemma 1 with

$$\begin{aligned}&\mathcal{N} = KM\delta, \quad \mathcal{L} = KM\delta, \quad \mathcal{M} = KM\delta,\\&\beta \doteq KM\delta\left(\frac{1}{\mu}+\frac{2}{\alpha}\right) < KM\delta\left(\frac{1}{\alpha}+\frac{2}{\alpha}\right) \leq \frac{3KM\delta}{\alpha} < \frac{1}{2}.\end{aligned}$$

We obtain

$$|S(t,s)(I-Q(s))| \leq K\frac{1}{1-KM\delta\left(\frac{1}{\mu}+\frac{2}{\alpha}\right)}e^{\left(\alpha-\frac{2KM\delta}{1-KM\delta\left(\frac{1}{\mu}+\frac{2}{\alpha}\right)}\right)(t-s)}, \quad t\leq s.$$

Therefore,

$$|S(t,s)(I-Q(s))| \leq 2Ke^{(\alpha-4KM\delta)(t-s)}, \quad t\leq s, \quad \text{if } \delta < \frac{\alpha}{6KM}.$$

□

Remark 2 The method used, the unbounded operator A and its domain impose restrictions on the class of perturbations $B(t)$ that can be used. For example if $D(t) \in L(X)$ is continuous and bounded for $t \in \mathbb{R}$, is integrally small and 0 belongs to the resolvent set of A, we could define $B(t) \doteq A^{-1}D(t)A^{-1}$ and then the above assumptions including (38) could be satisfied.

Remark 3 Another case is when $B(t) \in L(X)$ is continuous and bounded for $t \in \mathbb{R}$, is integrally small, commutes with A and $B(t)A$ can be considered as a bounded operator. In this case $B(t)$ acts as a smooth operator. This will be observed in some applications to the heat equation below.

Example 1 Application to the Heat Equation

In this part we use some results of Henry [4], page 119.

Let $X = L^2(0,\pi)$, $Au = -\frac{d^2u}{dx^2}$. Let $\mathcal{D}(A) = H_0^1(0,\pi)\cap H^2(0,\pi)$. If $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$ and $u = \sum_{n=1}^{\infty}\phi_n(\phi_n,u)$, then define $\|u\| = [\sum_{n=1}^{\infty}|(\phi_n,u)|^2]^{1/2}$.

$$\begin{aligned}&Au = \sum_{n=1}^{\infty}(-n^2)\phi_n(\phi_n,u), \quad e^{At}u = \sum_{n=1}^{\infty}e^{-n^2t}\phi_n(\phi_n,u),\\&\sigma(A) = \{-n^2,\ n=1,2,3,\ldots\}.\end{aligned}$$

Consider the equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u, \quad 0<x<\pi; \quad u=0 \quad \text{at} \quad x=0,\pi \tag{40}$$

This equation defines a local dynamical system in $X^{1/2} = H_0^1(0,\pi)$ and $\sigma(A-\lambda I) = \{\lambda - n^2 : n=1,2,3\ldots\}$.

Let $T(t)$ be the semigroup generated by $A - \lambda I$. If $\lambda > 1$ we can decompose the space $H_0^1(0,\pi) = E_- \oplus E_+$, with projections P_λ and $(I - P_\lambda)$, respectively on E_+ and E_- and we will have an exponential dichotomy:

$$\begin{cases} \|T(t)P_\lambda\| \le Ke^{-\alpha t}, & t \ge 0, \\ \|T(t)(I - P_\lambda)\| \le Ke^{\alpha t}, & t \le 0. \end{cases}$$

The subspace E_+ is generated by the eigenfunctions $\phi_n(x)$, such that $\lambda - n^2 > 0$.

For $n \in \mathbb{N}$, let $b_n(t)$ be real continuous functions in $t \in \mathbb{R}$.

Consider now the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u + B(t)u, \quad 0 < x < \pi; \quad u = 0 \quad \text{at} \quad x = 0,\ \pi, \tag{41}$$

where $B(t)u \doteq \sum_{n=1}^{\infty} b_n(t)\phi_n(\phi_n, u)$.

In order to simplify the calculations and to verify the assumptions (38) of Theorem 3, we will assume that $M = 1$ and $|b_n(t)| \le \frac{1}{n^2 2^{(n+1)/2}}$, $\forall\, t \in \mathbb{R},\ n \ge 1$. We will also assume that for $\delta > 0$, sufficiently small and $h > 0$ sufficiently large that $|\int_t^u b_n(\tau)d\tau| \le \frac{1}{n^2 2^{(n+1)/2}}\delta$ for $|t - u| \le h$.

In this case we consider

$$\begin{aligned} \sigma_\lambda^+ &= \{\lambda - n^2 > 0,\ n = 1, 2, \ldots, N_\lambda\}, \\ \sigma_\lambda^- &= \{\lambda - n^2 < 0,\ n = N_\lambda + 1, N_\lambda + 2, \ldots, M_\lambda\}, \\ \sigma_\lambda^{-\infty} &= \{\lambda - n^2,\ n = M_\lambda + 1, \ldots, \infty\},\ M_\lambda \ge N_\lambda + 1 \end{aligned}$$

Consider the projections: $(I - P_\lambda)u \doteq \sum_{n=1}^{N_\lambda} \phi_n(\phi_n, u)$ associated to σ_λ^+ and $P_\lambda u \doteq \sum_{n=N_\lambda+1}^{\infty} \phi_n(\phi_n, u)$ associated to $\sigma_\lambda^- \cup \sigma_\lambda^{-\infty} = \{n : \lambda - n^2 < 0\}$. Let $P_1 = P_1(\lambda)$ be the projection associated to σ_λ^- and $P_2 = P_2(\lambda)$ associated to $\sigma_\lambda^{-\infty}$, be given respectively by $P_1 u = \sum_{n=N_\lambda+1}^{M_\lambda} \phi_n(\phi_n, u)$, $P_2 u = \sum_{n=M_\lambda+1}^{\infty} \phi_n(\phi_n, u)$.

$$\begin{aligned} P_2 e^{At}u &= \sum_{n=M_\lambda+1}^{\infty} e^{-n^2 t}\phi_n(\phi_n, u), \\ \|P_2 e^{At}u\| &= \left\| \sum_{n=M_\lambda+1}^{\infty} e^{-n^2 t}\phi_n(\phi_n, u) \right\| = \left(\sum_{n=M_\lambda+1}^{\infty} \left| e^{-n^2 t}\phi_n(\phi_n, u) \right|^2 \right)^{\frac{1}{2}} \\ &= e^{-(M_\lambda+1)^2 t} \left(\sum_{n=M_\lambda+1}^{\infty} \left| e^{-(n^2 - (M_\lambda+1)^2)t}\phi_n(\phi_n, u) \right|^2 \right)^{\frac{1}{2}} \\ &\le e^{-(M_\lambda+1)^2 t} \left(\sum_{n=M_\lambda+1}^{\infty} |\phi_n(\phi_n, u)|^2 \right)^{\frac{1}{2}} = e^{-(M_\lambda+1)^2 t}\|u\|. \end{aligned}$$

Taking $\mu \doteq (M_\lambda + 1)^2$, we obtain $\|P_2 e^{At}u\| \le e^{-(M_\lambda+1)^2 t}\|u\| = e^{-\mu t}\|u\|$, for $t \ge 0$.

Next we prove that P_2 commutes with $B(t)$.

$$P_2 B(t)u = \sum_{n=M_\lambda+1}^{\infty} \phi_n(\phi_n, B(t)u) = \sum_{n=M_\lambda+1}^{\infty} \phi_n \left(\phi_n, \sum_{k=1}^{\infty} b_k(t)\phi_k(\phi_k, u) \right)$$

$$
\begin{aligned}
&= \sum_{n=M_\lambda+1}^{\infty} b_n(t)\phi_n(\phi_n, u) \\
B(t)P_2 u &= \sum_{n=1}^{\infty} b_n(t)\phi_n(\phi_n, Pu) = \sum_{n=1}^{\infty} b_n(t)\phi_n\left(\phi_n, \sum_{k=M_\lambda+1}^{\infty} \phi_n(\phi_k, u)\right) \\
&= \sum_{n=M_\lambda+1}^{\infty} b_n(t)\phi_n(\phi_n, u). \\
\|P_2 B(t)u\| &\le \left[\sum_{n=M_\lambda+1}^{\infty} |b_n(t)|^2\right]^{1/2} \left[\sum_{n=M_\lambda+1}^{\infty} |(\phi_n, u)|^2\right]^{1/2} \\
&\le \left[\sum_{n=M_\lambda+1}^{\infty} \left(\frac{\delta}{2^{(n+1)/2}}\right)^2\right]^{1/2} \|u\| \le \delta \left[\sum_{n=0}^{\infty} \left(\frac{1}{2^{(n+1)/2}}\right)^2\right]^{1/2} \|u\| \\
&= \delta\|u\|
\end{aligned}
$$

Therefore $B(t)$ commutes with P_λ and with $I - P_\lambda$ and also with P_1 and P_2, $|P_2 B(t)| \to 0$, as $M_\lambda \to \infty$, uniformly with respect to $t \in \mathbb{R}$. With a similar calculation we can prove $(I - P_\lambda)B(t)v = \sum_{n=1}^{N} b_n(t)\phi_n(\phi_n, v)$.

Also if $u \in \mathcal{D}(A)$ we have

$$
\begin{aligned}
AP_\lambda u &= \sum_{n=1}^{\infty}(-n^2)\phi_n(\phi_n, P_\lambda u) = \sum_{n=1}^{\infty}(-n^2)\phi_n\left(\phi_n, \sum_{k=N_\lambda+1}^{\infty} \phi_k(\phi_k, u)\right) \\
&= \sum_{n=N_\lambda+1}^{\infty} (-n^2)\phi_n(\phi_n, u) \\
P_\lambda A u &= \sum_{n=N_\lambda+1}^{\infty} \phi_n(\phi_n, Au) = \sum_{n=N_\lambda+1}^{\infty} \phi_n\left(\phi_n, \sum_{k=1}^{\infty}(-k^2)\phi_k(\phi_k, u)\right) \\
&= \sum_{n=N_\lambda+1}^{\infty} (-n^2)\phi_n(\phi_n, u).
\end{aligned}
$$

Therefore $AP_\lambda = P_\lambda A$ and so they commute and they are both bounded operators.

In order to use Theorem 3 we consider $C_t(u) \doteq \int_t^u B(\tau)d\tau$ and it is easy to see that $P_\lambda C_t(v) = C_t(u)P_\lambda$ and $P_\lambda C_t(v) = \sum_{n=N_\lambda+1}^{\infty} \int_t^u b_n(\tau)d\tau\phi_n(\phi_n, v)$. Hence, it can be seen that the conditions of Theorem 3 are satisfied.

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CHAPTER 3

RIGOROUS COMPUTATION USING HAAR WAVELETS

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A framework for rigorous computational methods using Haar wavelets for differential equations

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Abstract

This work presents a framework for a-posteriori error-estimating algorithms for differential equations which combines the radii polynomial approach with Haar wavelets. By using Haar wavelets, we obtain recursive structures for the matrix representations of the differential operators and quadratic nonlinearities, which can be exploited for the radii polynomial method in order to get error estimates in the L^2 sense. This allows the method to be applicable when the system or solution is not continuous, which is a limitation of other radii-polynomial-based methods. Numerical examples show how the method is implemented in practice.

Keywords: Rigorous computation, Computer-assisted proofs, Haar wavelets, Nonlinear dynamical systems.

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1 Introduction

Rigorous computation is an area under active development since the 1980s [18]. With an steady increase of computing power, numerical methods became viable tools for analyzing differential equations and gaining insight on structures such as invariant objects. However, standard numerical methods provide only approximations; the results are non-rigorous and cannot be used in formal proofs. They can only be used to gain insights on the true structures of the system. Moreover, some structures, such as bifurcations, may still be hidden even when using very accurate numerical methods.

Rigorous computational methods try to fill these gaps, providing mathematically valid estimates and bounds for truncation and rounding errors, and rigorously proving the existence of such hidden structures. Over the years, a number of such methods were developed, such as rigorous integration [29, 8, 17], Conley index methods [4, 14], self-consistent bounds [30, 28] and discretization methods [19, 9]. A more thorough review can be seen in [20] and references therein.

Of particular interest to us are the *radii polynomials* approach [5, 7, 27, 13, 11, 6, 2, 22, 26]. These methods recast the problem of investigating the existence of structures as finding solutions to functional equations, usually expanding the solutions in terms of a basis. Then, usual numerical methods are employed to find an approximate solution to these equations. Finally, using fixed point theorems, we can guarantee the existence of a true, rigorous solution of the functional equations within certain bounds of the numerical solution. The hypotheses of

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the fixed point theorems are in turn proven to be satisfied with the aid of the so-called radii polynomials.

In this work we present a new radii polynomial method employing Haar wavelets. While many other bases were already employed by this approach, such as Taylor series [22], Fourier series [27] and Chebyshev polynomials [13], to the best of our knowledge, no attempt has been made to combine wavelet methods and rigorous computations. We also believe that this method can be can be a framework to build other methods upon, such as rigorous continuation methods and methods for partial differential equations.

Wavelets are functions that form an orthonormal basis for the L^2 function space. While wavelet theory was only relatively recently formalized, their special properties — such as time and frequency localization — made them widely applicable in many fields, such as signal processing and compression algorithms. This poses an interesting case, because most of the aforementioned radii polynomial methods were proposed to work with smooth functions, while our wavelet-based radii polynomial method works in more general settings.

The radii polynomial method presented in this paper is based on the ideas of a numerical method for solving differential equations, the *Haar wavelet method*, which was first proposed in [3]. It is assumed that the highest-order derivative is expressed in terms of wavelets, and the solutions are given by the integral of the series, essentially rewriting the differential equation in its integral form. This allows the method to work with only Haar wavelets, the simplest wavelet available, and leads to a matrix representation of the integral operator. The simplicity of the Haar wavelet allows this matrix to be easily and recursively calculated. The original Haar wavelet method was further analyzed, developed and applied in several publications [12, 21, 1, 24, 16, 15].

In our method, we find the functional equations for the radii polynomial method using the same expansions of the Haar wavelet method. This essentially transforms the differential equation into an integral one, and allow us to use a radii polynomial theorem similar to [25]. Also, by using Haar wavelets, the integral operator and nonlinearities can be represented using infinite but recursive matrices, allowing us to make the estimates needed in the radii polynomial approach.

The work is organized in the following way. In Section 2 we introduce the Haar wavelet and its integral, and review some of their properties. In Section 3 we introduce the radii polynomial method and prove the theorems that guarantee the existence of a true solution, provided that certain estimates are satisfied. In Section 4 we study quadratic nonlinearities in order to prove estimates needed in the method. In Section 5 we present some examples illustrating the applications of the proposed method.

2 The Haar wavelets and their integral

Here we introduce the Haar wavelet and its integral, which are one of the pillars of the proposed methods. Consider the space $L^2([0,1])$ of the square-integrable functions $f : [0,1] \to \mathbb{R}$ with respect to the usual Lebesgue measure. The Haar wavelets are a family of functions $\{\phi, \psi_{j,k}\} \subset L^2([0,1])$ defined, for $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$, by

$$\phi(t) := \begin{cases} 1 & , \quad 0 \le t < 1 \\ 0 & , \quad \text{otherwise} \end{cases} \quad , \qquad \psi_{j,k}(t) := \begin{cases} 2^{j/2} & , \quad \frac{k}{2^j} \le t < \frac{k+0.5}{2^j} \\ -2^{j/2} & , \quad \frac{k+0.5}{2^j} \le t < \frac{k+1}{2^j} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{1}$$

The Haar wavelets form an orthonormal basis for $L^2([0,1])$; the proof is in many standard texts in wavelet theory, see e.g. [10]. Hence any function $y(t) \in L^2([0,1])$ can be expanded into a unique Haar wavelet series

$$y(t) = c_1\phi(t) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t),$$

where $c_1 = \int_0^1 y(t)\phi(t)\,dt$, $c_{j,k} = \int_0^1 y(t)\psi_{j,k}(t)\,dt$, and the sum converges in $L^2([0,1])$. If we make $i = 2^j + k + 1$, then the sequence $(c_i)_{i=1}^{\infty} \in \ell^2(\mathbb{R})$.

Notation. We can change between the "one-index" and "two-indices" notations, depending on which is more convenient in each case. One can be converted to the other by making, for all $i > 2$,

$$i = 2^j + k + 1 \iff \begin{array}{l} j = \lfloor \log_2 i \rfloor \\ k = i - 2^j - 1 \end{array}$$

where $\lfloor\cdot\rfloor$ is the floor function. The index $i = 1$ is reserved for the scaling function ϕ.

Conversely, any $(c_i)_{i=1}^{\infty} \in \ell^2(\mathbb{R})$ defines a unique $f(t) \in L^2([0,1])$ by making

$$f(t) = \sum_{i=1}^{\infty} c_i\psi_i(t) = c_1\phi(t) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t).$$

This means that we can define an invertible operator $\mathcal{H} : L^2([0,1]) \to \ell^2(\mathbb{R})$ defined element-wise as

$$(\mathcal{H}f)_i := \int_0^1 f(t)\psi_i(t)\,dt.$$

The inverse Haar transform $\mathcal{H}^{-1}$ is given by

$$(\mathcal{H}^{-1}\mathbf{c})(t) := \mathbf{h}^T(t)\mathbf{c} = \sum_{i=1}^{\infty} c_i\psi_i(t) \quad , \quad \mathbf{h}(t) := (\psi_1(t), \psi_2(t), ...)$$

It is worth noting that the Haar transform is an isometry, due to the fact that the Haar wavelet system is an orthonormal basis of $L^2([0,1])$.

Another interesting and useful property of the Haar wavelets is what we call the "nesting property":

Proposition 2.1 (Nesting property). *Let $\psi_{j,k}$ and $\psi_{m,n}$ be two Haar wavelets such that $j < m$. If supp $\psi_{j,k} \cap$ supp $\psi_{m,n} \neq \emptyset$, then either supp $\psi_{m,n} \subseteq [\frac{k}{2^j}, \frac{k+0.5}{2^j})$ or supp $\psi_{m,n} \subseteq [\frac{k+0.5}{2^j}, \frac{k+1}{2^j})$.*

The proof is simple and will be omitted; it consists in comparing the supports of the wavelets $\phi_{j,k}$, which are dyadic intervals of length 2^{-j}, for different j. Intuitively, it means that, if the supports of two wavelets at different resolutions overlap, then the support of the "finer" wavelet (that is, the higher-resolution one) is entirely nested within either the positive or the negative part of the "coarser" wavelet.

For this work, we are also interested in the integral of the Haar wavelets, and how the integral relates to the wavelet themselves. The integrals of the Haar wavelet family in the interval $[0,1]$ are the triangular functions given by

$$w_1(t) = t \qquad , \qquad w_{j,k}(t) = \begin{cases} 2^{j/2}\left(t - \frac{k}{2^j}\right) & , \quad \frac{k}{2^j} \le t \le \frac{k+0.5}{2^j} \\ 2^{j/2}\left(\frac{k+1}{2^j} - t\right) & , \quad \frac{k+0.5}{2^j} \le t \le \frac{k+1}{2^j} \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2}$$

We also extend the one-index notation to the Haar wavelet integrals w_i.

The Haar wavelet integrals are continuous functions in $[0, 1]$; thus they are square-integrable in that interval, and can be expanded in Haar wavelet series themselves:

$$w_i(t) = \sum_{l=1}^{\infty} P_{i,l}\psi_l(t) = P\mathbf{h}^T(t) \ , \ P_{i,l} := \int_0^1 \psi_l(t)w_i(t)\,dt. \tag{3}$$

While P is expressed by an infinite matrix, there is a recursive formula to compute it. We must first define the Haar matrix of our wavelet system:

Definition 2.1. *For a given resolution J, the **Haar matrix** H_M of order $M = 2^{J+1}$ is given element-wise by*

$$(H_M)_{p,q} := \psi_p(t_q) \tag{4}$$

where $t_q = \dfrac{q - 0.5}{M}$, for $q = 1, \ldots, M$.

*The **discrete Haar transform matrix** HT_M of order $M = 2^{J+1}$ is defined by*

$$HT_M := \frac{1}{\sqrt{M}} H_M \tag{5}$$

A particularly important fact is that HT_M is unitary for all J, and hence H_M is invertible and $(H_M)^{-1} = \frac{1}{M}H_M^T$.

Theorem 2.1. *The infinite matrix P can be recursively calculated as*

$$P_1 = \frac{1}{2} \qquad , \qquad P_{2m} = \begin{bmatrix} P_m & -\dfrac{1}{4\sqrt{m^3}}H_m \\ \dfrac{1}{4\sqrt{m^3}}H_m^T & 0_{m\times m} \end{bmatrix} \tag{6}$$

for $m = 2^j$ and $j = 0, 1, 2, \ldots$.

The proof for this formula is in [3], with some modifications to account for the fact that we are using normalized wavelets.

Let us prove that $P^T\mathbf{c} \in \ell^2(\mathbb{R})$ for all $\mathbf{c} \in \ell^2(\mathbb{R})$. We first need to define some projections. Given a resolution level J, let $M = 2^{J+1}$ as before, and define the projection Π_M as

$$\begin{aligned} \Pi_M : \ell^2(\mathbb{R}) &\to \mathbb{R}^M \\ \mathbf{c} &\mapsto (c_1, ..., c_M). \end{aligned} \tag{7}$$

We identify the vector $(c_1, ..., c_M) \in \mathbb{R}^M$ with its infinite-dimensional counterpart $(c_1, ..., c_M, 0, 0, ...) \in \ell^2(\mathbb{R})$. We also define the projection $\Pi_\infty \in B(\ell^2(\mathbb{R}))$ as

$$\Pi_\infty\mathbf{c} := (I - \Pi_M)\mathbf{c}.$$

Notation. Given $\mathbf{c} \in \ell^2(\mathbb{R})$, we sometimes divide it into blocks of length 2^n, $n = 0, 1, 2, ...$ as

$$\mathbf{c} = (\mathbf{c}_0^*, \mathbf{c}_1^*, \mathbf{c}_2^*, \mathbf{c}_4^*, ...)^T \ , \text{ where } \mathbf{c}_0^* = c_1 \ , \ \mathbf{c}_{2^n}^* = (c_{2^{n-1}+1}, c_{2^{n-1}+2}, ..., c_{2^n})^T . \tag{8}$$

Also, given a matrix A, we denote $A_{m_1:m_2,n_1:n_2}$ the submatrix of A given by

$$A_{m_1:m_2,n_1:n_2} = \begin{bmatrix} A_{m_1,n_1} & \cdots & A_{m_1,n_2} \\ \vdots & & \vdots \\ A_{m_2,n_1} & \cdots & A_{m_2,n_2} \end{bmatrix}$$

If we wish to take all rows or all columns of A, we denote $A_{*,n_1:n_2}$ and $A_{m_1:m_2,*}$, respectively. Lastly, to reduce notation clutter, we denote $\mathbf{c}_M = \Pi_M\mathbf{c}$ and $\mathbf{c}_\infty = \Pi_\infty\mathbf{c}$.

Proposition 2.2. *For $\mathbf{c} \in \ell^2(\mathbb{R})$, $M = 2^{J+1}$, $J = 0, 1, 2, ...$ and the projections as defined before,*

$$\left\|\Pi_M P^T \mathbf{c}\right\|_{\ell^2} \leq \frac{1}{\sqrt{3}} \left(4 - \frac{1}{2^{2J+2}}\right)^{\frac{1}{2}} \|\mathbf{c}\|_{\ell^2} \qquad , \qquad \left\|\Pi_\infty P^T \mathbf{c}\right\|_{\ell^2} \leq \frac{1}{\sqrt{3}} \frac{\|\mathbf{c}\|_{\ell^2}^2}{2^{J+1}}$$

In particular, letting $J \to \infty$, $P^T \mathbf{c} \in \ell^2(\mathbb{R})$ and $P^T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ is a bounded linear operator.

Proof. From the structure of P in Theorem 2.1 and the block representation of $\mathbf{c}$, we have

$$\left(P^T \mathbf{c}\right)_{2^j}^* = -\frac{1}{2^{\frac{3j}{2}+2}} H_{2^j}^T (\Pi_{2^j} \mathbf{c}) + \sum_{q=j+1}^{\infty} \frac{1}{2^{\frac{3q}{2}+2}} H_{2^q} \mathbf{c}_{2^q}^* \tag{9}$$

Recalling that $H_M = \sqrt{M} HT_M$ and HT_M is a unitary matrix, we can bound the above term by

$$\begin{aligned} \left\|\left(P^T \mathbf{c}\right)_{2^j}^*\right\|_{\ell^2} &\leq \frac{1}{2^{j+2}} \left\|HT_{2^j}^T (\Pi_{2^j} \mathbf{c})\right\|_{\ell^2} + \sum_{q=j+1}^{\infty} \frac{1}{2^{q+2}} \|HT_{2^q} \mathbf{c}_{2^q}^*\|_{\ell^2} \\ &\leq \frac{\|\mathbf{c}\|_{\ell^2}}{2^{j+2}} + \sum_{q=j+1}^{\infty} \frac{1}{2^{q+2}} \|\mathbf{c}_{2^q}^*\|_{\ell^2} \leq \frac{\|\mathbf{c}\|_{\ell^2}}{2^{j+1}} \end{aligned} \tag{10}$$

The only term left is $\left(P^T \bar{\mathbf{c}}\right)_1$. We can bound it with

$$\left|\left(P^T \bar{\mathbf{c}}\right)_1\right| = \left|\frac{1}{2} c_1 - \sum_{q=0}^{\infty} \frac{1}{2^{\frac{3q}{2}+2}} H_{2^q} \mathbf{c}_{2^q}^*\right| \leq \frac{1}{2} |c_1| + \sum_{q=0}^{\infty} \frac{1}{2^{q+2}} \|HT_{2^q} \mathbf{c}_{2^q}^*\|_{\ell^2} \leq \|\mathbf{c}\|_{\ell^2} \tag{11}$$

Thus, the norm of $\Pi_M P^T \mathbf{c}$ can be estimated by

$$\|\Pi_M P^T \mathbf{c}\|_{\ell^2}^2 \leq \left|\left(P^T \mathbf{c}\right)_1\right|^2 + \sum_{j=0}^{J} \left\|\left(P^T \mathbf{c}\right)_{2^j}^*\right\|_{\ell^2}^2 = \left(4 - \frac{1}{2^{2J+2}}\right) \frac{\|\mathbf{c}\|_{\ell^2}^2}{3}$$

Analogously, the norm of $\Pi_\infty P^T \mathbf{c}$ is bounded by

$$\|\Pi_\infty P^T \mathbf{c}\|_{\ell^2}^2 \leq \sum_{j=J+1}^{\infty} \left\|\left(P^T \bar{\mathbf{c}}\right)_{2^j}^*\right\|_{\ell^2}^2 \leq \sum_{j=J+1}^{\infty} \frac{\|\mathbf{c}\|_{\ell^2}^2}{2^{2j+2}} = \frac{1}{3} \frac{\|\mathbf{c}\|_{\ell^2}^2}{2^{2J+2}}$$

□

3 The radii polynomial approach

In this section, we introduce the radii polynomial [5, 7, 27, 13, 11, 6] approach for rigorous computation. Consider an initial value problem

$$\begin{cases} \dot{u} = f(u, t) \\ u(0) = u_0 \end{cases} \tag{12}$$

and suppose we find a numerical, approximate solution $\bar{u}(t)$. Our aim is to prove the existence of a true solution $\tilde{u}(t)$ in some neighborhood of $\bar{u}(t)$. This is done using the radii polynomial method.

For our work, suppose that $\dot{u}(t) \in L^2([0, 1])$. Then, we can write $\dot{u}$ and u using the Haar wavelet and its integral as

$$\dot{u}(t) = \sum_{i=1}^{\infty} c_i \psi_i(t) \quad , \quad u(t) = \sum_{i=1}^{\infty} c_i w_i(t) + u_0 \tag{13}$$

Substituting back into the differential equation (12) and taking the Haar transform,

$$F(\mathbf{c}) := \mathbf{c} - \mathcal{H}\left(f\left(t, \sum_{i=1}^{\infty} c_i w_i(t) + u_0\right)\right) = 0 \tag{14}$$

Thus, we have a map $F : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ such that finding a solution of the differential equation implies finding a zero of F. Conversely, due to the uniqueness of the wavelet series that represents $\dot{u}$ — and consequently of the series that represents $u(t)$ — finding a zero of F is equivalent to finding a solution to (12).

Now we recast the problem of finding the zeros of F to finding a fixed point of a map $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ near the numerical solution $\bar{u}$. This is done by showing that T is a contraction near $\bar{u}$. First, define the operator $A : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ as

$$A\mathbf{x} = A_M \Pi_M \mathbf{x} + \Pi_\infty \mathbf{x},$$

where $A_M : \mathbb{R}^M \to \mathbb{R}^M$ is a finite-dimensional, computational approximation for the inverse $D(\Pi_M F(\bar{\mathbf{x}}))^{-1}$, with $\bar{\mathbf{x}}$ the solution numerically obtained. Then we can define the fixed-point map $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ by

$$T(\mathbf{x}) := \mathbf{x} - AF(\mathbf{x}) = (\Pi_M - A_M \Pi_M F)(\mathbf{x}) + \Pi_\infty(\mathbf{x} - F(\mathbf{x})) \tag{15}$$

Its derivative, which is used for the radii polynomial method, is given by

$$DT(\mathbf{x}) = \Pi_M - A_M \Pi_M DF(\mathbf{x}) + \Pi_\infty(I - DF(\mathbf{x})) \tag{16}$$

Remark 3.1. Since many of the matrices in this work are block matrices, one can compute their inverse as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

provided that the matrices A and $D - CA^{-1}B$ are invertible. Calculating the inverse with this formula can be faster than directly inverting the full matrix $D(\Pi_M F(\bar{\mathbf{x}}))$.

Notation. In order to help visualize the operators and reduce clutter in notation, we employ the following "block matrix" notation for an operator $C : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ whenever it is convenient:

$$C\mathbf{x} = \begin{bmatrix} C_M & C_{M,\infty} \\ C_{\infty,M} & C_\infty \end{bmatrix} \begin{pmatrix} \Pi_M \mathbf{x} \\ \Pi_\infty \mathbf{x} \end{pmatrix}. \tag{17}$$

We refer to this as the **finite-infinite decomposition**.

Intuitively, the first term of (15) is a Newton-like map for the finite terms we computed numerically. In the second term, we hope the "tail" of F will contract to zero by itself — which is what happens with quadratic nonlinearities. This is all motivated by the fact that, as M increases, the new elements of the matrix P_M become smaller.

We now formally prove that a fixed point of T corresponds to a zero of F:

Proposition 3.1. *Suppose the map T as defined above in (15) is a contraction in some closed neighborhood of $\ell^2(\mathbb{R})$. Then T has a unique fixed point $\tilde{\mathbf{c}}$ in that neighborhood. Moreover, $\tilde{\mathbf{c}}$ is a fixed point of T if and only if it is a zero of F as defined earlier in (14).*

Proof. Since T is a contraction in a closed neighborhood of $\ell^2(\mathbb{R})$, the Banach Fixed Point Theorem guarantees that it has a fixed point $\tilde{\mathbf{c}}$ in the same neighborhood. Also, if $\tilde{\mathbf{c}}$ is a zero of F, then a straightforward calculation shows that it is a fixed point of T.

It remains to prove that the fixed point $\tilde{\mathbf{c}}$ is a zero of F. By (15)

$$T(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}} = 0 = A_M \Pi_M F(\tilde{\mathbf{c}}) + \Pi_\infty F(\tilde{\mathbf{c}}).$$

Since $A_M \Pi_M F(\tilde{\mathbf{c}}) \in \Pi_M(\ell^2(\mathbb{R}))$, we have

$$\Pi_\infty F(\tilde{\mathbf{c}}) = 0 \quad , \quad A_M \Pi_M F(\tilde{\mathbf{c}}) = 0$$

and since A_M is invertible, then $\Pi_M F(\tilde{\mathbf{c}}) = 0$ as well. Thus $F(\tilde{\mathbf{c}}) = 0$. □

To prove that T is actually a contraction near of our numerical solution $\bar{\mathbf{c}}$, we use the radii polynomials. First, we define the closed neighborhood

$$\overline{B_\omega(\bar{\mathbf{c}}, r)} = \left\{ \mathbf{y} \in \ell^2(\mathbb{R}) : \|\Pi_M(\mathbf{y} - \bar{\mathbf{c}})\| \le \omega r \text{ and } \|\Pi_\infty(\mathbf{y} - \bar{\mathbf{c}})\| \le (1-\omega) r \right\} \tag{18}$$

in which T will be a contraction. $\omega \in (0,1)$ is a "trade-off parameter": we can loosen the radius in the infinite part, at the cost of tightening the radius in the finite part, and vice-versa. For the next calculations, we assume ω is fixed, though in practice it is chosen later.

Next, we need bounds Y_M and Y_∞, and polynomials $Z_M(r)$ and $Z_\infty(r)$ such that

$$\|\Pi_M(T(\bar{\mathbf{c}}) - \bar{\mathbf{c}})\|_{\ell^2} \le Y_M \tag{19}$$
$$\|\Pi_\infty(T(\bar{\mathbf{c}}) - \bar{\mathbf{c}})\|_{\ell^2} \le Y_\infty \tag{20}$$
$$\sup_{x_1, x_2 \in \overline{B(r)}} \|\Pi_M(DT(\bar{\mathbf{c}} + x_1)x_2)\| \le Z_M(r) r \tag{21}$$
$$\sup_{x_1, x_2 \in \overline{B(r)}} \|\Pi_\infty(DT(\bar{\mathbf{c}} + x_1)x_2)\| \le Z_\infty(r) r \tag{22}$$

Then, we can define the radii polynomials as

$$p_M(r) := Z_M(r) r - \omega r + Y_M \quad , \quad p_\infty(r) \quad := Z_\infty(r) r - (1-\omega) r + Y_\infty \tag{23}$$

Theorem 3.1. *Consider the radii polynomials as defined in (23). If there exists an $r_0 > 0$ such that $p_M(r_0) < 0$ and $p_\infty(r_0) < 0$, then there exists a unique $\tilde{\mathbf{c}} \in B_\omega(\bar{\mathbf{c}}, r_0)$ such that $T(\tilde{\mathbf{c}}) = \tilde{\mathbf{c}}$.*

Proof. Due to the Banach Fixed Point Theorem, we only need to prove that $T\left(\overline{B_\omega(\bar{\mathbf{c}}, r_0)}\right) \subseteq \overline{B_\omega(\bar{\mathbf{c}}, r_0)}$ and that T is a contraction when restricted to $\overline{B_\omega(\bar{\mathbf{c}}, r_0)}$.

We first prove that T restricted to $\overline{B_\omega(\bar{\mathbf{c}}, r_0)}$ is a contraction. If $y_1, y_2 \in \overline{B_\omega(\bar{\mathbf{c}}, r_0)}$, then by the Mean Value Theorem

$$\|T(y_1) - T(y_2)\|_{\ell^2} \le \sup_{x \in \overline{B_\omega(\bar{\mathbf{c}}, r_0)}} \|DT(x)\| \|y_1 - y_2\|_{\ell^2} = \sup_{x_1 \in \overline{B(r_0)}} \|DT(\bar{\mathbf{c}} + x_1)\| \|y_1 - y_2\|_{\ell^2}$$

Hence, we must show that $\|DT(\bar{\mathbf{c}} + x_1)\| < 1$ for $x_1 \in \overline{B(r_0)}$. Observe that

$$\begin{aligned}
\sup_{x_1 \in \overline{B(r_0)}} \|DT(\bar{\mathbf{c}} + x_1)\| &= \frac{1}{r_0} \sup_{x_1, x_2 \in \overline{B(r_0)}} \|DT(\bar{\mathbf{c}} + x_1)x_2\|_{\ell^2} \\
&\le \frac{1}{r_0}\left(\sup_{x_1, x_2 \in \overline{B(r_0)}} \|\Pi_M DT(\bar{\mathbf{c}} + x_1)x_2\|_{\ell^2} + \sup_{x_1, x_2 \in \overline{B(r_0)}} \|\Pi_\infty DT(\bar{\mathbf{c}} + x_1)x_2\|_{\ell^2} \right) \\
&\le Z_M(r_0) + Z_\infty(r_0)
\end{aligned}$$

So we have

$$\|T(y_1) - T(y_2)\|_{\ell^2} \le (Z_M(r_0) + Z_\infty(r_0)) \|y_1 - y_2\|_{\ell^2}$$

But since $p_M(r_0) < 0$ and $p_\infty(r_0) < 0$,

$$\begin{aligned} Z_M(r_0)r_0 + Z_\infty(r_0)r_0 - r_0 &\leq (Z_M(r_0)r_0 - \omega r_0 + Y_M) + (Z_\infty(r_0)r_0 - (1-\omega)r_0 + Y_\infty) \\ &= p_M(r_0) + p_\infty(r_0) < 0 \end{aligned}$$

Thus $Z_M(r_0) + Z_\infty(r_0) < 1$, and T restricted to $\overline{B_\omega(\bar{\mathbf{c}}, r_0)}$ is a contraction.

Now we must prove that $T\left(\overline{B_\omega(\bar{\mathbf{c}}, r_0)}\right) \subseteq \overline{B_\omega(\bar{\mathbf{c}}, r_0)}$. If $y \in \overline{B_\omega(\bar{\mathbf{c}}, r_0)}$, then

$$\begin{aligned} \|\Pi_M(T(y) - \bar{\mathbf{c}})\|_{\ell^2} &\leq \|\Pi_M(T(y) - T(\bar{\mathbf{c}}))\|_{\ell^2} + \|\Pi_M(T(\bar{\mathbf{c}}) - \bar{\mathbf{c}})\|_{\ell^2} \\ &\leq Z_M(r_0)\|y - \bar{\mathbf{c}}\|_{\ell^2} + Y_M \leq Z_M(r_0)r_0 + Y_M < \omega r_0 \end{aligned}$$

and similarly for $\Pi_\infty(T(y) - \bar{\mathbf{c}})$

$$\begin{aligned} \|\Pi_\infty(T(y) - \bar{\mathbf{c}})\|_{\ell^2} &\leq \|\Pi_\infty(T(y) - T(\bar{\mathbf{c}}))\|_{\ell^2} + \|\Pi_\infty(T(\bar{\mathbf{c}}) - \bar{\mathbf{c}})\|_{\ell^2} \\ &\leq Z_\infty(r_0)\|y - \bar{\mathbf{c}}\|_{\ell^2} + Y_\infty \leq Z_\infty(r_0)r_0 + Y_\infty < (1-\omega)r_0 \end{aligned}$$

and hence $T(y) \in \overline{B_\omega(\bar{\mathbf{c}}, r_0)}$. □

Thus, if the radii polynomial method is successful in finding an r_0, then the solution $\bar{\mathbf{c}}$ found by the numerical method is "close" to the wavelet coefficients of true solution $\tilde{\mathbf{c}}$ in the $\ell^2(\mathbb{R})$ sense, that is, $\|\bar{\mathbf{c}} - \tilde{\mathbf{c}}\|_{\ell^2} \leq r_0$. Or equivalently, the numerical approximation $\bar{u}(t) = \bar{\mathbf{c}}^T\mathbf{h}(t)$ is "close" to the true solution $\tilde{u}(t)$ in the $L^2([0,1])$ sense, that is, $\|\bar{u} - \tilde{u}\|_{L^2} \leq r_0$.

4 Nonlinear terms

In this section we study quadratic nonlinearities in more depth. This may seem restrictive, but there are many interesting systems involving those, such as the Lorenz system. Furthermore, we believe that estimates for higher nonlinearities can be computed with similar techniques.

Consider two functions u and v such that $\dot{u}, \dot{v} \in L^2([0,1])$. Their expansions into Haar wavelet integrals as in (13) are

$$u(t) = u_0 + \sum_{i=1}^{\infty} c_i w_i(t) = u_0 + \mathbf{w}^T(t)\mathbf{c}\ ,\ \ v(t) = v_0 + \sum_{i=1}^{\infty} d_i w_i(t) = v_0 + \mathbf{w}^T(t)\mathbf{d}$$

Thus, considering the product $u(t)v(t)$, we have

$$u(t)v(t) = u_0v_0 + u_0\mathbf{w}^T(t)\mathbf{d} + v_0\mathbf{w}^T(t)\mathbf{c} + (W(t))(\mathbf{c}, \mathbf{d})$$

where, for $t \in [0,1]$, $(W(t))(\mathbf{c}, \mathbf{d}) := \mathbf{c}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{d}$. A crucial observation is that, for any given $t \in [0,1]$, *$W(t)$ is a symmetric bilinear form.*

The next theorem shows that $W(t)(\mathbf{c}, \mathbf{d}) \in L^2([0,1])$ for any pair $\mathbf{c}, \mathbf{d} \in \ell^2(\mathbb{R})$, which allows us to calculate its Haar transform and use the radii polynomial methods developed in Section 3. However, its proof is lengthy and will be left to A for clarity.

Theorem 4.1. *The bilinear form $(W(\mathbf{c}, \mathbf{d}))(t) := \mathbf{c}^T\mathbf{w}(t)\mathbf{w}(\mathrm{t})\mathbf{d}$ is bounded in $L^2([0,1])$ for all $\mathbf{c}, \mathbf{d} \in \ell^2(\mathbb{R})$, that is, there exists $C > 0$ such that*

$$\|W(\mathbf{c}, \mathbf{d})\|_{L^2} \leq C\, \|\mathbf{c}\|_{\ell^2}\, \|\mathbf{d}\|_{\ell^2} \ \text{ for all } \mathbf{c}, \mathbf{d} \in \ell^2(\mathbb{R}).$$

Additionally, making $\mathbf{a} := P^T\mathbf{c}$ and $\mathbf{b} := P^T\mathbf{d}$, we can write

$$\mathbf{c}^T\mathbf{w}(t)\mathbf{w}(\mathrm{t})\mathbf{d} = \mathbf{a}^T\Omega(t)\mathbf{b} + \mathbf{a}^T\Omega^T(t)\mathbf{b} + \mathbf{a}^T\Theta(t)\mathbf{b} \tag{24}$$

where the operator $\Omega(t)$ can be recursively defined as

$$\Omega_1(t) = 0 \quad , \quad \Omega_{2m}(t) = \begin{bmatrix} \Omega_m(t) & \Upsilon_m(t) \\ 0_{m\times m} & 0_{m\times m} \end{bmatrix} \tag{25}$$

with $\Upsilon_m(t)$ being a $m \times m$ matrix defined element-wise as

$$(\Upsilon_m)_{i,l}(t) = (H_m)_{i,l}\psi_{m+l}(t),$$

and the operator $\Theta(t)$ can be represented as an infinite diagonal matrix given by

$$\Theta(t) = \begin{bmatrix} \psi_1^2(t) & & & \\ & \psi_2^2(t) & & \\ & & \psi_3^2(t) & \\ & & & \ddots \end{bmatrix} \tag{26}$$

with zeros omitted for clarity. With the terms defined as above,

$$\mathcal{H}(\mathbf{c}^T\mathbf{w}(t)\mathbf{w}(\mathrm{t})\mathbf{d}) = \mathcal{H}(\mathbf{a}^T\Omega(t)\mathbf{b}) + \mathcal{H}(\mathbf{a}^T\Omega(t)\mathbf{b}) + \mathcal{H}(\mathbf{a}^T\Omega(t)\mathbf{b})$$

and

$$\begin{aligned} \mathcal{H}(\mathbf{a}^T\Omega(t)\mathbf{b}) &= (\tilde{\Omega}^T\mathbf{a}) \odot \mathbf{b} \\ \mathcal{H}(\mathbf{a}^T\Omega^T(t)\mathbf{b}) &= (\tilde{\Omega}^T\mathbf{b}) \odot \mathbf{a} \\ \mathcal{H}(\mathbf{a}^T\Theta(t)\mathbf{b}) &= \Gamma^T(\mathbf{a} \odot \mathbf{b}) \end{aligned} \tag{27}$$

where $\tilde{\Omega}^T$ and $\tilde{\Gamma}^T$ are recursively defined as

$$\left.\begin{aligned} \tilde{\Omega}_1 = 0 \quad , \quad \tilde{\Omega}_{2m} &= \begin{bmatrix} \tilde{\Omega}_m & H_m \\ 0_m & 0_m \end{bmatrix} \\ \Gamma_1 = 1 \quad , \quad \Gamma_{2m} &= \begin{bmatrix} \Gamma_m & 0_m \\ H_m^T & 0_m \end{bmatrix} \end{aligned}\right. \quad , \text{ for } m = 2^j \text{ and } j = 0, 1, 2, 3, \ldots$$

We now present some estimates required for the radii polynomial method. The full proof for those estimates are lengthy and left B. The main strategy consists in employing both the recursive block structures of the matrices from Theorems 2.1 and 4.1 and the finite-infinite decomposition from (17). These estimates provide tighter bounds which increase the likelihood of finding an r_0 which satisfies Theorem 3.1. We believe that similar estimates may be applied for higher-degree polynomial nonlinearities.

Proposition 4.1. *Given $\bar{\mathbf{c}}, \bar{\mathbf{d}} \in \mathbb{R}^M$, $M = 2^{J+1}$ for some $J \geq 0$, and $\bar{\mathbf{a}} = P^T\bar{\mathbf{c}}$, $\bar{\mathbf{b}} = P^T\bar{\mathbf{d}}$, the following estimates are valid:*

i) $\Pi_M P^T\bar{\mathbf{c}} = P_M^T\bar{\mathbf{c}}$

ii) $\Pi_M\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)\bar{\mathbf{b}}) = (\tilde{\Omega}_M^T P_M^T\bar{\mathbf{c}}) \odot (P_M^T\bar{\mathbf{d}})$

iii) $\Pi_M\mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)\bar{\mathbf{b}}) = \Gamma_M^T(\bar{\mathbf{a}}_M \odot \bar{\mathbf{b}}_M) + \Gamma_{\infty,M}^T(\bar{\mathbf{a}}_\infty \odot \bar{\mathbf{b}}_\infty)$*; moreover,*

$$\left\|\Gamma_\infty^T(\bar{\mathbf{a}}_\infty \odot \bar{\mathbf{b}}_\infty)\right\|_{\ell^2} \leq \frac{\sqrt{2}\,\|\bar{\mathbf{c}}\|_{\ell^2}\|\bar{\mathbf{d}}\|_{\ell^2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+3}}.$$

Proposition 4.2. *Given $\bar{\mathbf{c}}, \bar{\mathbf{d}} \in \mathbb{R}^M$, $M = 2^{J+1}$ for some $J \geq 0$, and $\bar{\mathbf{a}} = P^T\bar{\mathbf{c}}$, $\bar{\mathbf{b}} = P^T\bar{\mathbf{d}}$, the following estimates are valid:*

i) $\left\|\Pi_\infty P^T \bar{\mathbf{c}}\right\|_{\ell^2} \leq \dfrac{1}{\sqrt{3}} \dfrac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{J+2}}$

ii) $\left\|\Pi_\infty \mathcal{H}(\bar{\mathbf{a}}^T \Omega(t) \bar{\mathbf{b}})\right\|_{\ell^2} \leq \dfrac{1}{\sqrt{3}} \dfrac{\|\bar{\mathbf{c}}\|_{\ell^2} \left\|\bar{\mathbf{d}}\right\|_{\ell^2}}{2^{2J+4}}$

iii) $\left\|\Pi_\infty \mathcal{H}(\bar{\mathbf{a}}^T \Theta(t) \bar{\mathbf{b}})\right\|_{\ell^2} \leq \dfrac{1}{21\sqrt{7}} \dfrac{\|\bar{\mathbf{c}}\|_{\ell^2} \left\|\bar{\mathbf{d}}\right\|_{\ell^2}}{2^{3J+6}}$.

Proposition 4.3. *Given* $\mathbf{x}, \mathbf{y} \in \ell^2(\mathbb{R})$ *and* $\bar{\mathbf{c}} \in \mathbb{R}^M$,

i) $\left\|(P^T_{\infty,M}\, \mathbf{y}_\infty)\right\|_{\ell^2} \leq \dfrac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}}$

ii) $\left\|\left(\tilde{\Omega}^T P^T\right)_{M,\infty} \mathbf{y}_\infty\right\|_{\ell^2} \leq \left(1+\sqrt{2}\right) \dfrac{\|\mathbf{y}\|_{\ell^2}}{2^{\frac{J+3}{2}}}$

iii) $\left\|\Gamma^T_{\infty,M}(\bar{\mathbf{a}}_\infty \odot \Pi_\infty P^T \mathbf{y})\right\|_{\ell^2} \leq \dfrac{\sqrt{2}}{4-\sqrt{2}} \dfrac{\|\bar{\mathbf{c}}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}$

iv) $\left\|A_M \Pi_M \mathcal{H}(\mathbf{x}^T P\, \Omega(t) P^T \mathbf{y})\right\|_{\ell^2} \leq K_1 \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}$

v) $\left\|A_M \Pi_M \mathcal{H}(\mathbf{x}^T P \Theta(t) P^T \mathbf{y})\right\|_{\ell^2} \leq K_2 \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}$

where

$$K_1 = \left\|A_M \operatorname{diag}\left(\|(P^T_M)_{i,*}\|_{\ell^2}\right)\, \tilde{\Omega}^T_M P^T_M\right\| + \frac{\|A_M \tilde{\Omega}^T_M P^T_M\|}{2^{J+2}} + \frac{(1+\sqrt{2})\|A_M P^T_M\|}{2^{\frac{J+3}{2}}} + \frac{(1+\sqrt{2})\|A_M\|}{2^{\frac{3J+7}{2}}}$$

$$K_2 = \left\|A_M \Gamma^T_M \operatorname{diag}(\|(P^T_M)_{i,*}\|^2_{\ell}) P^T_M\right\| + \frac{\left\|A_M \Gamma^T_M P^T_M\right\|}{2^{J+1}} + \frac{\left\|A_M \Gamma^T_M\right\|}{2^{2J+4}} + \frac{\sqrt{2}\|A_M\|}{(4-\sqrt{2})2^{\frac{3J}{2}+4}}$$

Proposition 4.4. *Given* $\mathbf{x}, \mathbf{y} \in \ell^2(\mathbb{R})$ *and* $\bar{\mathbf{c}} \in \mathbb{R}^M$, *the following estimates are valid:*

i) $\left\|\Pi_\infty \mathcal{H}(\bar{\mathbf{c}}^T \mathbf{w}(t) \mathbf{w}^T(t) \mathbf{y})\right\|_{\ell^2} \leq D_1 \dfrac{\|\bar{\mathbf{c}}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}}} \quad , \quad D_1 = \dfrac{1}{8\sqrt{7}} \left(3 + \dfrac{\sqrt{2}}{4} + \dfrac{4}{4-\sqrt{2}}\right);$

ii) $\left\|\Pi_\infty \mathcal{H}(\mathbf{x}^T \mathbf{w}(t) \mathbf{w}^T(t) \mathbf{y})\right\|_{\ell^2} \leq D_2 \dfrac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+2}} \quad , \quad D_2 = \dfrac{8+6\sqrt{2}}{\sqrt{7}\,(4-\sqrt{2})}$.

5 Examples

In this section we illustrate the implementation of our method by means of three examples: the logistic equation, the logistic equation with a discontinuous forcing term, and the Lorenz system. The method was implemented in MATLAB R2021b using the INTLAB package for interval arithmetic [23]. The files for these examples are available at https://github.com/gknakassima/RigComp-HaarWavelet.

5.1 Logistic equation

As a first simple example we look at the logistic equation

$$\dot{u} = \lambda u(1-u) \quad , \quad u(0) = u_0 \tag{28}$$

since it has a polynomial nonlinearity and its analytical solution is given by $u(t) = \frac{u_0 e^{\lambda t}}{1-u_0+u_0 e^{\lambda t}}$. Using the expansions in (13), we have the functional equation equivalent to (28):

$$\mathbf{c}^T \mathbf{h}(t) - \lambda \left(\mathbf{c}^T \mathbf{w}(t) + u_0 - u_0^2 - 2u_0 \mathbf{c}^T \mathbf{w}(t) - (W(t))(\mathbf{c}, \mathbf{c})\right) = 0. \tag{29}$$

5.1.1 Obtaining a numerical approximation

In order to obtain a finite-dimensional approximation $\bar{\mathbf{c}}$ of the solution, we first consider a truncated version of our matrix equation. Given a resolution level $J > 0$ and making $M = 2^{J+1}$, we apply the projection Π_M to all sequences of (29), obtaining

$$\mathbf{c}_M^T \mathbf{h}_M(t) - \lambda \left(\mathbf{c}_M^T \mathbf{w}_M(t) + u_0 - u_0^2 - 2u_0 \mathbf{c}_M^T \mathbf{w}_M(t) - (W_M(t))(\mathbf{c}_M, \mathbf{c}_M)\right) = 0$$

where, for $\mathbf{a}_M, \mathbf{b}_M \in \mathbb{R}^M$, $(W_M(t))(\mathbf{a}_M, \mathbf{b}_M) := \mathbf{a}_M^T \mathbf{w}_M(t) \mathbf{w}_M^T(t) \mathbf{b}_M$. Since the equation holds for all $t \in [0, 1]$, we sample it at the times $t_l = \frac{l-0.5}{2^J}$ for $l = 1, ..., M$. Recalling that

$$H_M = [\mathbf{h}_M(t_1), \mathbf{h}_M(t_2), ..., \mathbf{h}_M(t_l), ..., \mathbf{h}_M(t_{2^{J+1}})],$$

we can organize the time samples in matrix form as

$$\mathbf{c}_M^T H_M + \lambda(2u_0 - 1)\mathbf{c}_M^T P_M H_M + \lambda(u_0^2 - u_0)\mathbf{e}^T + \lambda(\mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M))^T = 0$$

with $\mathbf{e}$ and $\mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M)$ being $M \times 1$ vectors given by

$$\mathbf{e} := (1, 1, ..., 1)^T \quad , \quad \left(\mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M)\right)_i := (W_M(t_i))(\mathbf{c}_M, \mathbf{c}_M).$$

By transposing this system and multiplying by $(H_M^T)^{-1} = \frac{1}{M} H_M$ we finally arrive at the equation to solve numerically:

$$\mathbf{c}_M + \lambda(2u_0 - 1)P_M^T \mathbf{c}_M + \frac{\lambda(u_0^2 - u_0)}{M} H_M \mathbf{e} + \frac{\lambda}{M} H_M \mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M) = 0. \tag{30}$$

As this is a nonlinear equation, we use Newton's method. Define $F_M : \mathbb{R}^M \to \mathbb{R}^M$ as

$$F_M(\mathbf{c}_M) := \mathbf{c}_M + \lambda(2u_0 - 1)P_M^T \mathbf{c}_M + \frac{\lambda(u_0^2 - u_0)}{M} H_M \mathbf{e} + \frac{\lambda}{M} H_M \mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M) \tag{31}$$

Thus, we apply Newton's method by iteratively calculating

$$\mathbf{c}_M^{p+1} = \mathbf{c}_M^p - A_M(\mathbf{c}_M^p) F_M(\mathbf{c}_M^p) \tag{32}$$

where $A_M(\mathbf{c}_M^p)$ is a numerical approximation for $\left(DF_M(\mathbf{c}_M^p)\right)^{-1}$ and $\mathbf{c}_M^p$ is the result from the p-th iteration.

5.1.2 Estimates for the radii polynomials

Here, we provide the bounds for Theorem 3.1. The maps T and DT for the fixed point theorem are given by (15) and (16), respectively. For this example, the map $F : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ is given as

$$F(\mathbf{x}) := \lambda(u_0^2 - u_0)e_1 + \mathbf{x} + \lambda(2u_0 - 1)P^T \mathbf{x} + \lambda \mathcal{H}\left(\mathbf{x}^T \mathbf{w}(t) \mathbf{w}^T(t) \mathbf{x}\right). \tag{33}$$

and its derivative $DF(\mathbf{x})$ applied to $\mathbf{y} \in \ell^2(\mathbb{R})$ is given by

$$(DF(\mathbf{x}))\mathbf{y} = \mathbf{y} + \lambda(2u_0 - 1)P^T \mathbf{y} + 2\lambda \mathcal{H}(\mathbf{x}^T \mathbf{w}(t) \mathbf{w}^T(t) \mathbf{y}). \tag{34}$$

where the last equality comes from the symmetry of the bilinear form.

Before proceeding, it is worth outlining the general strategy for the estimates. We separate the operator matrices according to the finite-infinite decomposition in (17). Then, all the finite-dimensional parts are collected together and left for the computer to calculate, while we use the analytic estimates from Section 4 for the infinite parts.

- Y_M: Using the decomposition of $\lambda \mathcal{H}\left(\bar{\mathbf{c}}^T \mathbf{w}(t) \mathbf{w}^T(t) \bar{\mathbf{c}}\right)$ and Proposition 4.1,

$$\begin{aligned}\Pi_M(T(\bar{\mathbf{c}})-\bar{\mathbf{c}}) &= -A_M\Pi_M\mathcal{H}F(\bar{\mathbf{c}})\\ &= -A_M\Big[\lambda(u_0^2-u_0)\mathbf{e}+\bar{\mathbf{c}}+\lambda(2u_0-1)P_M^T\bar{\mathbf{c}}+2\lambda\big((\tilde{\Omega}_M^T P_M^T\bar{\mathbf{c}})\odot(P_M^T\bar{\mathbf{c}})\big)\\ &\qquad +\lambda\Gamma_M^T(\bar{\mathbf{a}}_M\odot\bar{\mathbf{a}}_M)\Big]-\lambda A_M\Pi_M\Gamma_{\infty,M}^T(\bar{\mathbf{a}}_\infty\odot\bar{\mathbf{a}}_\infty),\end{aligned}$$

with $\bar{\mathbf{a}}=P^T\bar{\mathbf{c}}$. Note that the term in the brackets can be computationally evaluated. Using the bound from Proposition 4.1 (iii) for the last term, we can define Y_M as

$$\begin{aligned}Y_M &:= \left\|A_M\Big[\lambda(u_0^2-u_0)\mathbf{e}+\bar{\mathbf{c}}+\lambda(2u_0-1)P_M^T\bar{\mathbf{c}}+2\lambda\big((\tilde{\Omega}_M^T P_M^T\bar{\mathbf{c}})\odot(P_M^T\bar{\mathbf{c}})\big)+\lambda\Gamma_M^T(\bar{\mathbf{a}}_M\odot\bar{\mathbf{a}}_M)\Big]\right\|_{\ell^2}\\ &\quad+\frac{|\lambda|\|A_M\|\sqrt{2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+3}}\|\bar{\mathbf{c}}\|_{\ell^2}^2\end{aligned} \tag{35}$$

- Y_∞: Observe that

$$\Pi_\infty(T(\bar{\mathbf{c}})-\bar{\mathbf{c}})=\Pi_\infty\left(\lambda(2u_0-1)P^T\bar{\mathbf{c}}+\lambda\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)\bar{\mathbf{a}}+\bar{\mathbf{a}}^T\Omega^T(t)\bar{\mathbf{a}}+\bar{\mathbf{a}}^T\Theta(t)\bar{\mathbf{a}}\right).$$

With the estimates from Proposition 4.2, we can make Y_∞ as

$$Y_\infty := \frac{|\lambda(2u_0-1)|}{\sqrt{3}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{J+2}}+\frac{|\lambda|}{\sqrt{3}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}^2}{2^{2J+3}}+\frac{|\lambda|}{21\sqrt{7}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}^2}{2^{3J+6}}. \tag{36}$$

- Z_M: Using the finite-infinite decomposition and the fact that $\bar{\mathbf{c}}\in\mathbb{R}^M$,

$$\begin{aligned}\Pi_M(DT(\bar{\mathbf{c}}+\mathbf{x}))\mathbf{y} &= (I_M-A_MB_1)\,\mathbf{y}_M-A_MB_2P_{\infty,M}^T\mathbf{y}_\infty-2\lambda A_M\mathrm{diag}(P_M^T\bar{\mathbf{c}})\big(\tilde{\Omega}^TP^T\big)_{M,\infty}\mathbf{y}_\infty\\ &\quad-2\lambda A_M\Gamma_{\infty,M}^T\Pi_\infty\left(P^T\bar{\mathbf{c}}\odot P^T\mathbf{y}\right)-2\lambda A_M\Pi_M\mathcal{H}(\mathbf{x}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y})\end{aligned}$$

where

$$\begin{aligned}B_1 &:= I_M+\lambda(2u_0-1)P_M^T+2\lambda\mathrm{diag}\left(\bar{\mathbf{c}}^TP_M\tilde{\Omega}_M\right)P_M^T\\ &\qquad+2\lambda\mathrm{diag}(P_M^T\bar{\mathbf{c}})\tilde{\Omega}_M^TP_M^T+2\lambda\Gamma_M^T\mathrm{diag}\left(P_M^T\bar{\mathbf{c}}\right)P_M^T\\ B_2 &:= \lambda(2u_0-1)I_M+2\lambda\mathrm{diag}\left(\bar{\mathbf{c}}^TP_M\tilde{\Omega}_M\right)+2\lambda\mathrm{diag}(P_M^T\bar{\mathbf{c}})+2\lambda\Gamma_M^T\mathrm{diag}\left(P_M^T\bar{\mathbf{c}}\right)\end{aligned}$$

and I_M is the $M\times M$ identity matrix. While the expression of B_1 and B_2 seem complicated, all terms are finite-dimensional and hence their norms can be calculated computationally. Thus, using the estimates from Proposition 4.3 with the expression for DT to bound the terms which cannot be easily estimated computationally,

$$\|\Pi_M(DT(\bar{\mathbf{c}}+\mathbf{x}))\mathbf{y}\|_{\ell^2}\le(C_1+C_2\|\mathbf{x}\|_{\ell^2})\|\mathbf{y}\|_{\ell^2}$$

where

$$\begin{aligned}C_1 &:= \|I_M-A_MB_1\|+\frac{\|A_MB_2\|}{2^{J+2}}+\frac{(1+\sqrt{2})\left\|\lambda A_M\mathrm{diag}(P_M^T\bar{\mathbf{c}})\right\|}{2^{\frac{J+1}{2}}}+\frac{\sqrt{2}\,\|\lambda A_M\|\,\|\bar{\mathbf{c}}\|_{\ell^2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+2}}\\ C_2 &:= 2|\lambda|\,(2K_1+K_2)\end{aligned}$$

and K_1 and K_2 are as in Proposition 4.3. Hence, we can make $Z_M(r)$ as

$$Z_M(r):=C_1+C_2r \tag{37}$$

- Z_∞: We have that

$$\Pi_\infty(DT(\bar{\mathbf{c}}+\mathbf{x}))\mathbf{y} = \lambda(2u_0-1)\Pi_\infty P^T\mathbf{y} + 2\lambda\Pi_\infty\mathcal{H}(\bar{\mathbf{c}}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y}) + 2\lambda\Pi_\infty\mathcal{H}(\mathbf{x}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y})$$

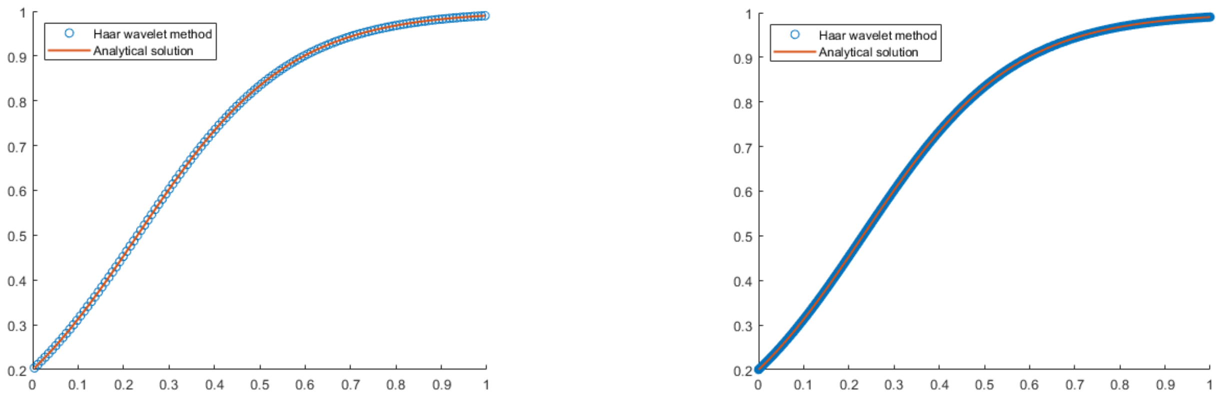
All terms are infinite-dimensional and need to be analitically estimated. Using the estimates from Propositions 4.4 and 2.2, we can make $Z_\infty(r)$ as

$$Z_\infty(r) := \left(\frac{|\lambda(2u_0-1)|}{2^{J+1}\sqrt{3}} + \frac{|\lambda|D_1\,\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{\frac{3J}{2}}}\right) + \frac{|\lambda|D_2}{2^{\frac{3J}{2}+2}}r \tag{38}$$

with D_1 and D_2 as in Proposition 4.4.

5.1.3 Results

Figure 1 shows the numerical solutions for $J=6$ and $J=10$ compared to the true solution, using $\lambda = 6$ and $u_0 = 0.2$ for both cases. Visually, the numerical solutions agrees with the true one.



(a) $J=6$ (b) $J=10$

Figure 1: Numerical and true solutions for the logistic equation.

Figure 2 shows the radius r_0 obtained as J increases for different ω. It is clear that, as J increases, the radius r_0 decreases; this is due to more terms being calculated more accurately, instead of only being bounded by analytical estimates. Also, smaller values of ω yield tighter radii; however, if ω is too small the method will not work, as there will not be a true solution within $\overline{B_\omega(\bar{\mathbf{c}}, r)}$. Figure 3a shows the radii obtained with ω optimized up to two significant digits.

Lastly, Figure 3b shows the radii and computation time. As J increases, the computation time is expected to increase; however, Figure 3b shows that after a certain point it increases more rapidly than r_0 decreases. This is expected as the size of the matrices quadruples for every increase of J, and so one must carefully balance the needed precision with computing time.

5.2 Logistic equation with a discontinuous forcing term

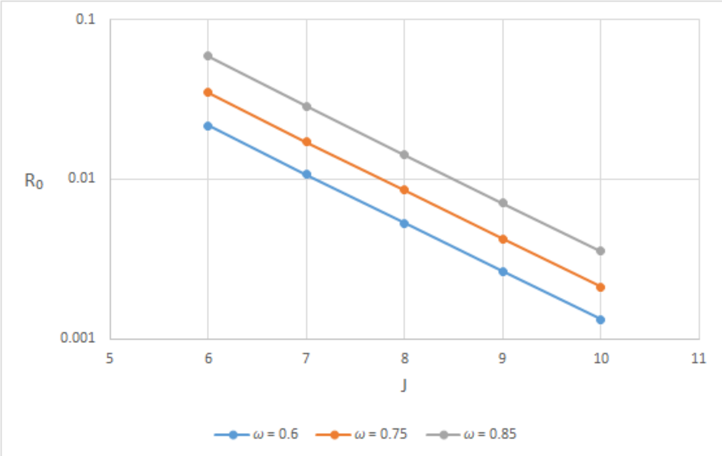
The next example is again the logistic equation, but with a discontinuous forcing term

$$\begin{aligned} \dot{u} &= \lambda u(1-u) + g(t) \\ u(0) &= u_0 \end{aligned} \quad , \quad g(t) = \begin{cases} 1 & \text{, if } t \le \frac{1}{2} \\ 0 & \text{, if } t > \frac{1}{2} \end{cases} \tag{39}$$

While this is a Ricatti equation which can be explicitly solved, we can see from the equation itself that the solution should not be smooth, since g is discontinuous. Nonetheless, we can find a verification radius in the L^2 sense.

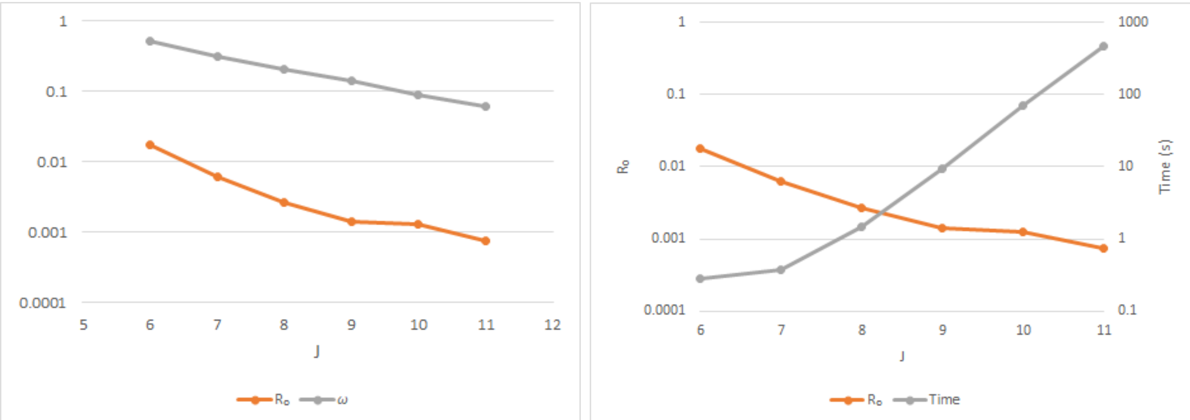
The functional equation for this case is similar to (28); making $\mathbf{g} = \mathcal{H}(g)$,

$$\mathbf{c}^T\mathbf{h}(t) - \lambda\left(\mathbf{c}^T\mathbf{w}(t) + u_0 - u_0^2 - 2u_0\mathbf{c}^T\mathbf{w}(t) - (W(t))(\mathbf{c},\mathbf{c})\right) - \mathbf{g}^T\mathbf{h}(t) = 0. \tag{40}$$



J	$r_0\ (\omega = 0.6)$	$r_0\ (\omega = 0.75)$	$r_0\ (\omega = 0.85)$
6	2.1677704×10^{-2}	3.4976922×10^{-2}	5.9222878×10^{-2}
7	1.0690405×10^{-2}	1.7163569×10^{-2}	2.8785441×10^{-2}
8	5.3120948×10^{-3}	8.5120420×10^{-3}	1.4224769×10^{-2}
9	2.6483940×10^{-3}	4.2402808×10^{-3}	7.0756199×10^{-3}
10	1.3223867×10^{-3}	2.1164777×10^{-3}	3.5294192×10^{-3}

Figure 2: Radius r_0 obtained for different values of ω



(a) (b)

J	ω	r_0	Time
6	0.51	1.7651807×10^{-2}	0.283
7	0.31	6.1826325×10^{-3}	0.377
8	0.21	2.6863885×10^{-3}	1.499
9	0.14	1.4138969×10^{-3}	9.447
10	0.089	1.2789917×10^{-3}	70.958
11	0.060	7.4420035×10^{-4}	474.220

Figure 3: (a) Radius r_0 obtained with more optimized ω for each resolution level J. (b) Comparison between r_0 and time elapsed.

5.2.1 Numerical approximation

Applying the same method as in the previous example, we get the equation to obtain the numerical, finite-dimensional approximation $\bar{\mathbf{c}}$, which is similar to before:

$$\mathbf{c}_M + \lambda(2u_0 - 1)P_M^T \mathbf{c}_M + \frac{\lambda(u_0^2 - u_0)}{M} H_M \mathbf{e} + \frac{\lambda}{M} H_M \mathbf{W}_M(\mathbf{c}_M, \mathbf{c}_M) - \frac{1}{M} H_M \mathbf{g}_M = 0 \tag{41}$$

where $\mathbf{g}_M := (g(t_1), \ldots, g(t_M))^T$. Again, since this is a nonlinear equation, we will use Newton's method.

5.2.2 Estimates for the radii polynomials

In order to apply the radii polynomial method, we use the following functional equation $F : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$:

$$F(\mathbf{x}) := \lambda(u_0^2 - u_0)e_1 + \mathbf{x} + \lambda(2u_0 - 1)P^T\mathbf{x} + \lambda\mathcal{H}\left(\mathbf{x}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{x}\right) - \mathcal{H}(g).$$

Observe that $g = \frac{1}{2}(\psi_1 + \psi_2)$, and thus $\mathcal{H}(g) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, ...\right)^T =: \mathbf{g}$; in particular, $\mathbf{g} \in \mathbb{R}^M$. Hence, the map used is given by

$$F(\mathbf{x}) = \lambda(u_0^2 - u_0)e_1 + \mathbf{x} + \lambda(2u_0 - 1)P^T\mathbf{x} + \lambda\mathcal{H}\left(\mathbf{x}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{x}\right) - \mathbf{g}. \tag{42}$$

The maps T and $DT(\mathbf{x})$ for the radii polynomials are the same as (15) and (16), respectively. Actually, since $\mathbf{g}$ does not depend on $\mathbf{x}$, the derivative DF is the same as in the non-forced logistic equation from (34).

Using the same methods as before, we have the following bounds for the radii polynomial method:

$$Y_M := C_0 + \frac{|\lambda|\|A_M\|\sqrt{2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+3}}\|\bar{\mathbf{c}}\|_{\ell^2}^2 \tag{43}$$

$$Y_\infty := \frac{|\lambda(2u_0-1)|}{\sqrt{3}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{J+2}} + \frac{|\lambda|}{\sqrt{3}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}^2}{2^{2J+3}} + \frac{|\lambda|}{21\sqrt{7}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}^2}{2^{3J+6}} \tag{44}$$

$$Z_M(r) := C_1 + C_2 r \tag{45}$$

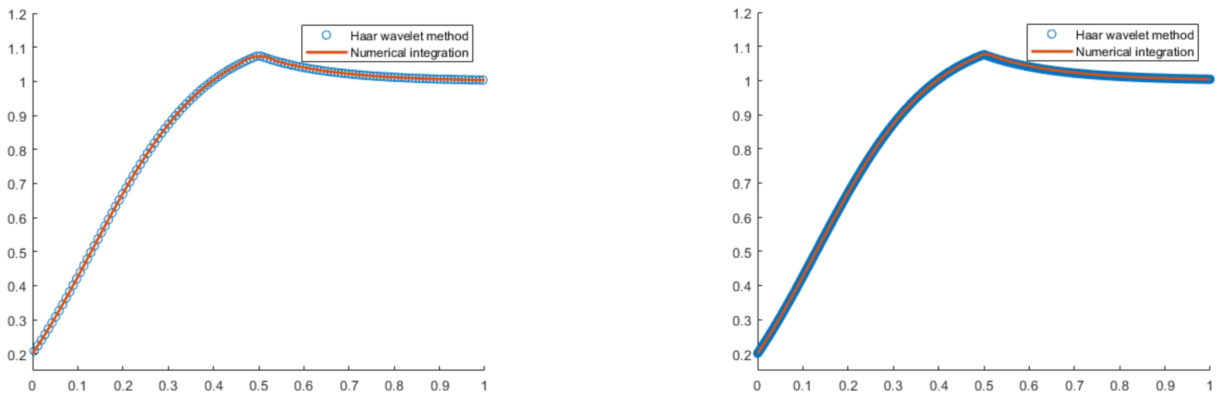
$$Z_\infty(r) := \left(\frac{|\lambda(2u_0-1)|}{2^{J+1}\sqrt{3}} + \frac{|\lambda|D_1\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{\frac{3J}{2}}}\right) + \frac{|\lambda|D_2}{2^{\frac{3J}{2}+2}}r \tag{46}$$

where C_1, C_2, D_1 and D_2 are as in the estimates for the non-forced logistic equation, and

$$C_0 = \Big\| A_M\Big[\lambda(u_0^2 - u_0)\mathbf{e} + \bar{\mathbf{c}} + \lambda(2u_0-1)P_M^T\bar{\mathbf{c}} + 2\lambda\Big((\tilde{\Omega}_M^T P_M^T\bar{\mathbf{c}}) \odot (P_M^T\bar{\mathbf{c}})\Big) + \lambda\Gamma_M^T(\bar{\mathbf{a}}_M \odot \bar{\mathbf{a}}_M) - \mathbf{g}\Big]\Big\|_{\ell^2}.$$

5.2.3 Results

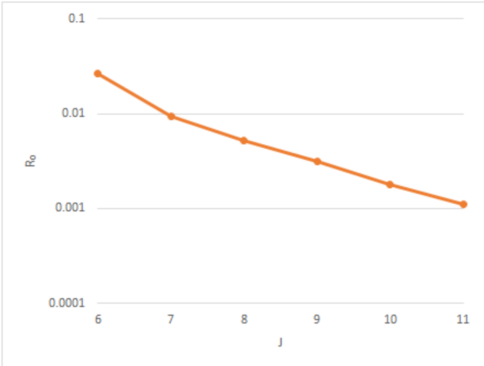
Figure 4 shows the results using the Haar wavelet method compared to numerical integration, using $\lambda = 6$ and $u_0 = 0.2$. For the numerical integration, we used the same amount of points as the Haar wavelet method, that is, 2^{J+1} points. It can be seen that the numerical integration tends to smooth the graph at $t = 0.5$, while our method preserves the original shape.



(a) $J = 6$ (b) $J = 10$

Figure 4: Numerical and true solutions for the logistic equation with forcing term

Figure 5 shows the verification radius as J increases. It is worth noting that the solution is not smooth; nonetheless, our method returned verification radii similar to the non-forced logistic equation.



J	ω	r_0
6	0.53	2.6161420×10^{-2}
7	0.31	9.2508029×10^{-3}
8	0.20	5.1495598×10^{-3}
9	0.13	3.1382945×10^{-3}
10	0.086	1.7710909×10^{-3}
11	0.057	1.1107730×10^{-3}

Figure 5: Verification radius for the forced logistic equation

5.3 Lorenz system

The Lorenz system is given by

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{47}$$

where σ, ρ and β are positive parameters, usually taken as $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$. This is a well-studied system, and with these parameters the system exhibits chaotic behavior with a strange attractor.

Since this is a system of equations, we must first some of our definitions in order to apply our method. First, we define the spaces $X_s := \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})$ and $X_f := L^2([0,1]) \times L^2([0,1]) \times L^2([0,1])$, and endow them with the norms

$$\begin{aligned} \|(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)\|_{X_s} &:= \max\{\|\mathbf{c}_1\|_{\ell^2}, \|\mathbf{c}_2\|_{\ell^2}, \|\mathbf{c}_3\|_{\ell^2}\} \\ \|(f_1, f_2, f_3)\|_{X_f} &:= \max\{\|f_1\|_{L^2}, \|f_2\|_{L^2}, \|f_3\|_{L^2}\} \end{aligned} \tag{48}$$

With these norms, X_s and X_f are still Banach spaces; though they are no longer Hilbert spaces, the methods of Section 3 are still applicable. Also, an operator $A : X_s \to X_s$ can be expressed using block matrix notation as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}$$

where $A_{i,j} : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ for $i, j = 1, 2, 3$. If those are bounded, A is bounded and

$$\|A\|_{B(X_s)} = \max_{1 \le i \le 3} \sum_{j=1}^{3} \|A_{i,j}\|_{B(\ell^2)}. \tag{49}$$

Similar notation will be used when $A_{i,j} \in B(\mathbb{R}^M)$. Lastly, we make a small abuse of notation and extend the notation for the operators in $\ell^2(\mathbb{R})$ such as the projections Π_M and Π_∞ to X_s by applying them element-wise:

$$\begin{aligned} \Pi_M(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3) &:= (\Pi_M \mathbf{c}_1, \Pi_M \mathbf{c}_2, \Pi_M \mathbf{c}_3) \\ \Pi_\infty(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3) &:= (\Pi_\infty \mathbf{c}_1, \Pi_\infty \mathbf{c}_2, \Pi_\infty \mathbf{c}_3). \end{aligned}$$

Applying the same methods used to obtain (29) to each equation in (47), we obtain the functional equation

$$
\begin{aligned}
&\mathbf{c}_x^T\mathbf{h}(t) - \sigma\mathbf{c}_y^T\mathbf{w}(t) + \sigma\mathbf{c}_x^T\mathbf{w}(t) - \sigma(y_0 - x_0) = 0\\
&\mathbf{c}_y^T\mathbf{h}(t) + x_0\mathbf{c}_z^T\mathbf{w}(t) - (\rho - z_0)\mathbf{c}_x^T\mathbf{w}(t) + \mathbf{c}_y^T\mathbf{w}(t) - (x_0(\rho - z_0) - y_0) + W(t)(\mathbf{c}_x, \mathbf{c}_z) = 0\\
&\mathbf{c}_z^T\mathbf{h}(t) - y_0\mathbf{c}_x^T\mathbf{w}(t) - x_0\mathbf{c}_y^T\mathbf{w}(t) + \beta\mathbf{c}_z^T\mathbf{w}(t) - (x_0y_0 - \beta z_0) - W(t)(\mathbf{c}_x, \mathbf{c}_y) = 0
\end{aligned}
\tag{50}
$$

where $\mathbf{c}_x = \mathcal{H}(\dot{x})$, $\mathbf{c}_y = \mathcal{H}(\dot{y})$ and $\mathbf{c}_z = \mathcal{H}(\dot{z})$.

5.3.1 Numerical approximation

Using the same techniques used to obtain (30) and (41) to each equation in (50), we obtain the system to be solved numerically with Newton's method:

$$
\begin{aligned}
&\bar{\mathbf{c}}_x - \frac{\sigma}{M}P_M^T\bar{\mathbf{c}}_y + \frac{\sigma}{M}P_M^T\bar{\mathbf{c}}_x - \frac{\sigma}{M}(y_0 - x_0)H_M\mathbf{e} = 0\\
&\bar{\mathbf{c}}_y + \frac{1}{M}H_M\mathbf{W}_M(\bar{\mathbf{c}}_x, \bar{\mathbf{c}}_z) + x_0P_M^T\bar{\mathbf{c}}_z - (\rho - z_0)P_M^T\bar{\mathbf{c}}_x + P_M^T\bar{\mathbf{c}}_y - \frac{1}{M}(x_0(\rho - z_0) - y_0)H_M\mathbf{e} = 0\\
&\bar{\mathbf{c}}_z - \frac{1}{M}H_M\mathbf{W}_M(\bar{\mathbf{c}}_x, \bar{\mathbf{c}}_y) - y_0P_M^T\bar{\mathbf{c}}_x - x_0P_M^T\bar{\mathbf{c}}_y + \beta P_M^T\bar{\mathbf{c}}_z - \frac{1}{M}(x_0y_0 - \beta z_0)H_M\mathbf{e} = 0
\end{aligned}
\tag{51}
$$

Remark 5.1. One interesting remark in [12] is that one can use results from lower resolutions as initial guesses for the Newton's method for higher resolution levels, instead of using a high resolution level right from the start. For the Lorenz system, this can reduce convergence problems and overall calculation time.

5.3.2 Estimates for the radii polynomials

For the Lorenz system, the maps T and DT are as in (15) and (16) respectively, with $\mathbf{c} = (\mathbf{c}_x, \mathbf{c}_y, \mathbf{c}_z) \in X_s$. The map $F : X_s \to X_s$ is given by

$$
F(\mathbf{c}) := \begin{pmatrix}
\mathbf{c}_x - \sigma P^T\mathbf{c}_y + \sigma P^T\mathbf{c}_x - \sigma(y_0 - x_0)\mathbf{e}_1\\
\mathbf{c}_y + \mathcal{H}\left(\mathbf{c}_x^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{c}_z\right) + x_0P^T\mathbf{c}_z - (\rho - z_0)P^T\mathbf{c}_x + P^T\mathbf{c}_y - (x_0(\rho - z_0) - y_0)\mathbf{e}_1\\
\mathbf{c}_z - \mathcal{H}\left(\mathbf{c}_x^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{c}_y\right) - y_0P^T\mathbf{c}_x - x_0P^T\mathbf{c}_y + \beta P^T\mathbf{c}_z - (x_0y_0 - \beta z_0)\mathbf{e}_1
\end{pmatrix}
$$

and for $\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) \in X_s$,

$$
DF(\mathbf{c})\mathbf{v} := \begin{pmatrix}
\mathbf{v}_x - \sigma P^T\mathbf{v}_y + \sigma P^T\mathbf{v}_x\\
\mathbf{v}_y + \mathcal{H}\left(\mathbf{c}_x^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{v}_z\right) + \mathcal{H}\left(\mathbf{c}_z^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{v}_x\right)\\
\qquad + x_0P^T\mathbf{v}_z - (\rho - z_0)P^T\mathbf{v}_x + P^T\mathbf{v}_y\\
\mathbf{c}_z - \mathcal{H}\left(\mathbf{c}_x^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{v}_y\right) - \mathcal{H}\left(\mathbf{c}_y^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{v}_x\right)\\
\qquad - y_0P^T\mathbf{v}_x - x_0P^T\mathbf{v}_y + \beta P^T\mathbf{v}_z
\end{pmatrix}.
$$

For the Lorenz system, we have the following estimates for the radii polynomial method. Using the norms defined in (48) and (49) and applying the same techniques and techniques as before to each equation, we find the following bounds:

$$
\begin{aligned}
Y_M &= \|A_MF_M(\bar{\mathbf{c}})\|_{X_s} + \frac{\|A_M\|_{B(X_s)}\|\bar{\mathbf{c}}_x\|_{\ell^2}}{(4 - \sqrt{2})2^{\frac{3J+5}{2}}}\max\left\{\|\bar{\mathbf{c}}_y\|_{\ell^2}, \|\bar{\mathbf{c}}_z\|_{\ell^2}\right\}\\
Y_\infty &= \max\{Y_\infty^1, Y_\infty^2, Y_\infty^3\}\\
Z_M(r) &= \alpha_1 + \alpha_2 r\\
Z_\infty(r) &= \gamma_1 + \gamma_2 r
\end{aligned}
\tag{52}
$$

where the constants above are given by

$$
\begin{aligned}
Y_\infty^1 &:= \frac{|\sigma|}{\sqrt{3}} \frac{\|\bar{\mathbf{c}}_x - \bar{\mathbf{c}}_y\|_{\ell^2}}{2^{J+2}} \\
Y_\infty^2 &:= \frac{\|\bar{\mathbf{c}}_x\|_{\ell^2}\|\bar{\mathbf{c}}_z\|_{\ell^2}}{\sqrt{3}\ 2^{2J+5}} + \frac{\|\bar{\mathbf{c}}_x\|_{\ell^2}\|\bar{\mathbf{c}}_z\|_{\ell^2}}{(21\sqrt{7})\ 2^{3J+6}} + \frac{\|x_0\bar{\mathbf{c}}_z - (\rho - z_0)\bar{\mathbf{c}}_x + \bar{\mathbf{c}}_y\|_{\ell^2}}{\sqrt{3}\ 2^{J+2}} \\
Y_\infty^3 &:= \frac{\|\bar{\mathbf{c}}_x\|_{\ell^2}\|\bar{\mathbf{c}}_y\|_{\ell^2}}{2^{2J+5}\ \sqrt{3}} + \frac{\|\bar{\mathbf{c}}_x\|_{\ell^2}\|\bar{\mathbf{c}}_y\|_{\ell^2}}{(21\sqrt{7})\ 2^{3J+6}} + \frac{\|\beta\bar{\mathbf{c}}_z - y_0\bar{\mathbf{c}}_x - x_0\bar{\mathbf{c}}_y\|_{\ell^2}}{\sqrt{3}\ 2^{J+2}}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_1 &:= \|I_M - A_M B_1\|_{B(X_s)} + \frac{\|A_M B_2\|_{B(X_s)}}{2^{J+2}} \\
&\quad + \frac{1+\sqrt{2}}{2^{\frac{J+3}{2}}} \|A_M B_2\|_{B(X_s)} + \frac{\sqrt{2}}{4-\sqrt{2}} \left\| A_M^\dagger \begin{pmatrix} 0 \\ \|\bar{\mathbf{c}}_x\|_{\ell^2} + \|\bar{\mathbf{c}}_z\|_{\ell^2} \\ \|\bar{\mathbf{c}}_x\|_{\ell^2} + \|\bar{\mathbf{c}}_y\|_{\ell^2} \end{pmatrix} \right\|_{X_s} \\
\alpha_2 &:= \|4C + 2D\|_{B(X_s)}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1 &:= \max\left\{ \frac{|\sigma|}{\sqrt{3}\ 2^{J+1}},\ \frac{D_1\big(\|\bar{\mathbf{c}}_z\|_{\ell^2} + \|\bar{\mathbf{c}}_x\|_{\ell^2}\big)}{2^{\frac{3J}{2}}} + \frac{1 + |\rho - z_0| + |x_0|}{\sqrt{3}\ 2^{J+2}},\right. \\
&\qquad\qquad \left. \frac{D_1\big(\|\bar{\mathbf{c}}_z\|_{\ell^2} + \|\bar{\mathbf{c}}_x\|_{\ell^2}\big)}{2^{\frac{3J}{2}}} + \frac{\beta + |x_0| + |y_0|}{\sqrt{3}\ 2^{J+2}} \right\} \\
\gamma_2 &:= \frac{D_2}{2^{\frac{3J}{2}+1}}.
\end{aligned}
$$

and the auxiliary quantities to calculate the constants are given by

$$
A_M^\dagger := \begin{bmatrix} \|A_{M_{x,x}}\|_{B(\ell^2)} & \|A_{M_{x,y}}\|_{B(\ell^2)} & \|A_{M_{x,z}}\|_{B(\ell^2)} \\ \|A_{M_{y,x}}\|_{B(\ell^2)} & \|A_{M_{y,y}}\|_{B(\ell^2)} & \|A_{M_{y,z}}\|_{B(\ell^2)} \\ \|A_{M_{z,x}}\|_{B(\ell^2)} & \|A_{M_{z,y}}\|_{B(\ell^2)} & \|A_{M_{z,z}}\|_{B(\ell^2)} \end{bmatrix}
$$

$$
B_1 := \begin{bmatrix} I_M + \sigma P_M^T & -\sigma P_M^T & 0 \\ (B_M(\bar{\mathbf{c}}_z) - (\rho - z_0) I_M) P_M^T & I_M + P_M^T & (B_M(\bar{\mathbf{c}}_x) + x_0 I_M) P_M^T \\ (-B_M(\bar{\mathbf{c}}_y) - y_0 I_M) P_M^T & -(B_M(\bar{\mathbf{c}}_x) + x_0 I_M) P_M^T & I_M + P_M^T \end{bmatrix}
$$

$$
B_M(\bar{\mathbf{c}}_i) := \mathrm{diag}(\tilde{\Omega}_M^T P_M^T \bar{\mathbf{c}}_i) + \Gamma_M^T \mathrm{diag}(P_M^T \bar{\mathbf{c}}_i)
$$

$$
B_2 := \begin{bmatrix} \sigma I_M & -\sigma I_M & 0 \\ B_M(\bar{\mathbf{c}}_z) & I_M & B_M(\bar{\mathbf{c}}_x) \\ -B_M(\bar{\mathbf{c}}_y) & -B_M(\bar{\mathbf{c}}_x) & I_M \end{bmatrix}
$$

$$
C = \begin{bmatrix} 0 & C_{x,y} & C_{x,z} \\ 0 & C_{y,y} & C_{y,z} \\ 0 & C_{z,y} & C_{z,z} \end{bmatrix} \quad , \quad D = \begin{bmatrix} 0 & D_{x,y} & D_{x,z} \\ 0 & D_{y,y} & D_{y,z} \\ 0 & D_{z,y} & D_{z,z} \end{bmatrix}
$$

and for $i, j \in \{x, y, z\}$

$$
\begin{aligned}
C_{i,j} &:= \|A_{M_{i,j}}\, \mathrm{diag}\left(\|(P_M^T)_{i,*}\|_{\ell^2}\right) \tilde{\Omega}_M^T P_M^T\| + \frac{\left\|A_{M_{i,j}} \tilde{\Omega}_M^T P_M^T\right\|}{2^{J+2}} \\
&\qquad + \frac{(1+\sqrt{2})\left\|A_{M_{i,j}} P_M^T\right\|}{2^{\frac{J+3}{2}}} + \frac{(1+\sqrt{2})\left\|A_{M_{i,j}}\right\|}{2^{\frac{3J+7}{2}}} \\
D_{i,j} &:= \left\|A_{M_{i,j}} \Gamma_{M_{i,j}}^T \mathrm{diag}(\|(P_M^T)_{i,*}\|_\ell^2) P_M^T\right\| + \frac{\left\|A_{M_{i,j}} \Gamma_{M_{i,j}}^T P_M^T\right\|}{2^{J+1}} \\
&\qquad + \frac{\left\|A_{M_{i,j}} \Gamma_{M_{i,j}}^T\right\|}{2^{2J+4}} + \frac{\sqrt{2}}{(4-\sqrt{2})} \frac{\|A_{M_{i,j}}\|}{2^{\frac{3J}{2}+4}}
\end{aligned}
$$

and D_1 and D_2 are as in Proposition 4.4.

5.3.3 Results

Figure 6 shows the approximation obtained with the Haar wavelet method using $J = 10$ and numerical integration (fewer points from the Haar wavelet method are displayed for clarity); for the latter, we used $M^2 = 2^{2J+2}$ points in order to obtain a precise result. Visually, there seems to be good agreement between both results.

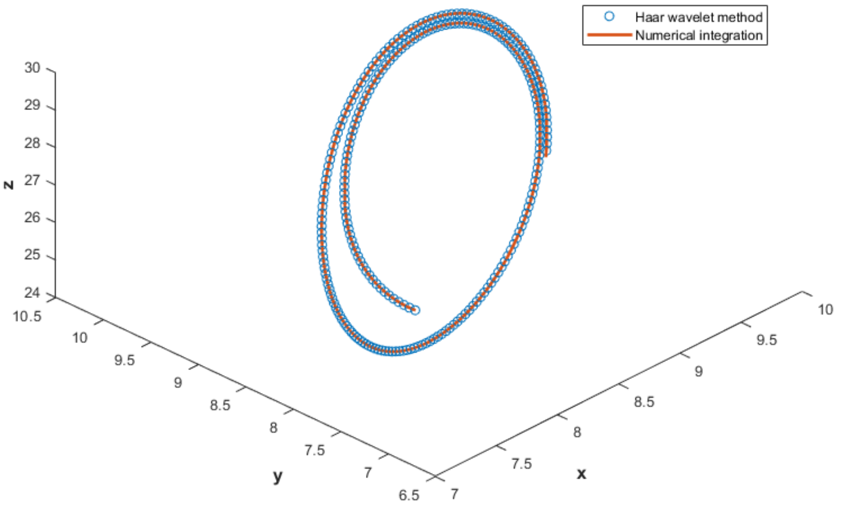


Figure 6: Results for the Lorenz system

For the same resolution level, the numerical result was rigorously verified by the radii polynomial method using the estimates described above, obtaining an $r_0 = 3.9868504 \times 10^{-2}$ for $\omega = 0.45$ in which the true solution lies in X_s.

6 Conclusions and future work

We developed a radii polynomial method using the Haar wavelet approach for differential equations, and illustrated the method by applying it to three differential equations. One advantage of our method over previous methods based on the radii polynomials approach is that, due to the use of the Haar wavelets, our method does not require the solutions to be smooth.

In the future we plan to develop the estimates for higher-order derivatives. While a higher-order differential equation can be transformed into a system of first-order equations, this increases the size of the matrices. Thus it might be interesting to use operators that directly represent higher-order derivatives. While some of those have already been used for usual numerical methods, we need to compute the estimates needed for our radii polynomial method.

Furthermore, we only presented the estimates needed for the radii polynomials for quadratic nonlinearities, since the main goal of this paper is to present the general method and illustrate how to compute the estimates and apply the method. In the future we plan to extend these estimates to include higher-order polynomial nonlinearities, as this would greatly expand the applicability of our method.

Lastly, we believe this method can be a basis to build other methods using similar techniques, such as continuation methods.

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Appendices

A Proof of Theorem 4.1

Here we present the full proof of Theorem 4.1. While lengthy, this proof not only validates the theorem that allows us to deal with quadratic nonlinearities, but also hints at how to find bounds for the radii polynomials.

Let us outline the general strategy. We first make heuristical calculations to find an expression for the Haar transforms; then, by proving that they are indeed in $\ell^2(\mathbb{R})$, the uniqueness of the Haar wavelet series justifies the calculations.

Let $\mathbf{c}, \mathbf{d} \in \ell^2(\mathbb{R})$, and assume that $\mathbf{c}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{d} \in L^2([0,1])$. We have

$$\mathbf{c}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{d} = \mathbf{a}^T\mathbf{h}(t)\mathbf{h}^T(t)\mathbf{b}$$

where we denote $\mathbf{a} = P^T\mathbf{c}$ and $\mathbf{b} = P^T\mathbf{d}$ for simplicity. Let us study the matrix $\mathbf{h}(t)\mathbf{h}^T(t)$ in more depth. First, by definition, we have $\psi_{j,k}(t)\psi_{j,q}(t) \neq 0$ if and only if $k = q$. For $\psi_{j,k}(t)\psi_{p,q}(t)$ when $j \neq p$, without loss of generality, consider $j < p$ for a fixed p. Because of the nesting property, $\psi_{j,k}$ is constant (possibly zero) in $\operatorname{supp}\psi_{p,q}$, so we can assume $\psi_{j,k}(t) = \psi_{j,k}\left(\frac{q+0.5}{2^p}\right)$ for $t \in \operatorname{supp}\psi_{p,q}$. Thus, we can write $\psi_{j,k}(t)\psi_{p,q}(t)$ as

$$\psi_{j,k}(t)\psi_{p,q}(t) = \psi_{j,k}\left(\frac{q+0.5}{2^p}\right)\psi_{p,q}(t) = (H_{m/2})_{i,l}\psi_l(t).$$

Hence, after adjusting indices and taking into account the symmetry of the matrix, we have proved the following:

Lemma .1. *The product $\mathbf{h}(t)\mathbf{h}^T(t)$ can be recursively calculated for $m = 2^j$, $j = 0, 1, 2, \ldots$ as*

$$\mathbf{h}_1(t)\mathbf{h}_1^T(t) = \psi_1^2(t) = \phi^2(t) \quad , \quad \mathbf{h}_{2m}(t)\mathbf{h}_{2m}^T(t) = \begin{bmatrix} \mathbf{h}_m(t)\mathbf{h}_m^T(t) & \Upsilon_m(t) \\ \Upsilon_m^T(t) & \Delta_m(t) \end{bmatrix}$$

where, for $i, l = 1, ..., 2^m$, Γ_m and Δ_m are $m \times m$ matrices defined element-wise as

$$(\Upsilon_m)_{i,l}(t) = (H_m)_{i,l}\psi_{m+l}(t) \quad , \quad (\Delta_m)_{i,l}(t) = \begin{cases} \psi_i^2(t) & , \ \textit{if } i = l \\ 0 & , \ \textit{otherwise.} \end{cases}$$

This justifies the decomposition of $\mathbf{h}(t)\mathbf{h}^T(t)$ as in Theorem 4.1, that is, $\mathbf{h}(t)\mathbf{h}^T(t) = \Omega(t) + \Omega^T(t) + \Theta(t)$. We treat each term separately. First, for $\mathbf{a}^T\Omega(t)\mathbf{b}$, dividing $\mathbf{b}$ in 2^j vector blocks, the product $\Omega(t)\mathbf{b}$ is given element-wise by

$$\begin{aligned}(\Omega(t)\mathbf{b})_i &= \sum_{p=j}^{\infty}(\Upsilon_{2^p})_{i,*}\mathbf{b}_{2^p}^* = \sum_{p=j}^{\infty}\sum_{q=1}^{2^p}(H_{2^p})_{i,q}\,\psi_{2^p+q}(t)\,b_{2^p+q} \\ &= \sum_{p=j}^{\infty}(H_{2^p})_{i,*}\,(\mathbf{b}_{2^p}^* \odot \mathbf{h}_{2^p}^*(t)) =: (\tilde{\Omega}(\mathbf{b} \odot \mathbf{h}(t)))_i\end{aligned}$$

where $\odot$ denotes the Hadamard (i.e. element-wise) product, and the matrix $\tilde{\Omega}$ can be recursively constructed as

$$\tilde{\Omega}_1 = 0 \quad , \quad \tilde{\Omega}_{2m} = \begin{bmatrix} \tilde{\Omega}_m & H_m \\ 0_m & 0_m \end{bmatrix} \quad , \text{ for } m = 2^j \text{ and } j = 0, 1, 2, \ldots$$

Thus, the product $\mathbf{a}^T\Omega(t)\mathbf{b}$ is given by

$$\mathbf{a}^T\Omega(t)\mathbf{b} = (\mathbf{c}^T P)\tilde{\Omega}\left(\mathbf{b}\odot\mathbf{h}(t)\right) = \sum_{i=1}^{\infty}\sum_{l=1}^{\infty} c_i (P\tilde{\Omega})_{i,l} b_l \psi_l(t).$$

In particular, if we assume $\mathbf{a}^T\Omega(t)\mathbf{b} \in L^2([0,1])$, then

$$\left(\mathcal{H}\left(\mathbf{a}^T\Omega(t)\mathbf{b}\right)\right)_l = b_l \sum_{i=1}^{\infty} c_i (P\tilde{\Omega})_{i,l} = (\mathbf{c}^T P\tilde{\Omega})_l b_l. \tag{53}$$

or written in another way, $\mathcal{H}\left(\mathbf{a}^T\Omega(t)\mathbf{b}\right) = \left(\tilde{\Omega}^T P^T\mathbf{c}\right)\odot\mathbf{b}$.

Also, since the output of $\mathbf{a}^T\Omega^T(t)\mathbf{b}$ is a real number, then $\mathbf{a}^T\Omega^T(t)\mathbf{b} = \left(\mathbf{a}^T\Omega^T(t)\mathbf{b}\right)^T = \mathbf{b}^T\Omega(t)\mathbf{a}$, and thus $\mathcal{H}\left(\mathbf{a}^T\Omega^T(t)\mathbf{b}\right) = \left(\tilde{\Omega}^T P^T\mathbf{d}\right)\odot\mathbf{a}$ as well.

For the term $\mathcal{H}(\mathbf{a}^T\Theta(t)\mathbf{b})$, we have

$$\mathbf{a}^T\Theta(t)\mathbf{b} = \sum_{i=1}^{\infty} a_i b_i \psi_i^2(t) \quad , \quad \psi_i^2(t) = \begin{cases} 2^j & , \text{ if } \frac{k}{2^j} \le t \le \frac{k+1}{2^j}; \\ 0 & , \text{ otherwise.} \end{cases}$$

Then $\psi_i^2 \in L^2([0,1])$, and we can write it as a Haar wavelet series:

$$\psi_i^2(t) = \sum_{l=1}^{\infty} \gamma_{i,l}\psi_l(t) = \Gamma\mathbf{h}(t) \quad , \quad \gamma_{i,l} := \int_0^1 \psi_i^2(t)\psi_l(t)\,dt.$$

Similar to P, the matrix Γ also has a recursive structure:

Lemma .2. *The matrix Γ can be recursively calculated by*

$$\Gamma_1 = 1 \quad , \quad \Gamma_{2m} = \begin{bmatrix} \Gamma_m & 0_m \\ H_m^T & 0_m \end{bmatrix} \quad , \textit{ for } m = 2^j \textit{ and } j = 0,1,2,3,... \tag{54}$$

Proof. First, calculating $\Gamma_1 = \gamma_{11}$ is straightforward. By definition, for $i,l = 1,2,...,m$, we have

$$(\Gamma_{2m})_{i,l} = \int_0^1 \psi_i^2(t)\psi_l(t)\,dt = (\Gamma_m)_{i,l}$$

Suppose now that $m+1 \le l \le 2m$, and denote using the two-index notation $\psi_l = \psi_{p,q}$ and $\psi_i = \psi_{j,k}$. For $2 \le i \le 2m$, due to the nesting property, $\psi_{j,k}$ is constant (possibly zero) in $\operatorname{supp}\psi_{p,q}$. However, even if $\psi_{j,k}$ is non-zero in $\operatorname{supp}\psi_{p,q}$,

$$(\Gamma_{2m})_{i,l} = \int_0^1 \psi_i^2(t)\psi_l(t)\,dt = 2^j \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \psi_{p,q}(t)\,dt = 2^j \int_{\frac{q}{2^p}}^{\frac{q+1}{2^p}} \psi_{p,q}(t)\,dt = 0.$$

Similar reasoning applies for $i = 1$ (in which case $\psi_1(t) = \phi(t) \equiv 1$ in $[0,1]$).

Suppose now that $m+1 \le i \le 2m$ and $l \le m$. For $l = 1$ a straightforward calculation show that $(\Gamma_{2m})_{i,l} = 1 = (H_m)_{1,i}$. For $l > 1$, since $p < j$, we can apply the same reasoning as in the proof of Lemma .1, yielding

$$\gamma_{i,l} = 2^j \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \psi_{p,q}(t)\,dt = 2^j \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \psi_{p,q}\left(\frac{k+0.5}{2^j}\right)dt = \psi_l(t_i) = (H_m)_{l,i}.$$

□

Hence,

$$\mathbf{a}^T\Theta(t)\mathbf{b} = \sum_{i=1}^{\infty} a_i b_i \psi_i^2(t) = \sum_{i=1}^{\infty} a_i b_i \sum_{l=1}^{\infty} \gamma_{i,l}\psi_l(t) = (\mathbf{a}\odot\mathbf{b})^T\Gamma\,\mathbf{h}(t)$$

Thus, if $\Gamma^T(\mathbf{a}\odot\mathbf{b}) \in \ell^2(\mathbb{R})$, then $\mathcal{H}(\mathbf{a}^T\Theta(t)\mathbf{b}) = \Gamma^T(\mathbf{a}\odot\mathbf{b})$.

Now we must prove that our tentative Haar transforms are indeed elements of $\ell^2(\mathbb{R})$; Theorem 4.1 follows then from the uniqueness of the Haar series. Before that, we prove a few lemmas:

Lemma .3. *The matrix $P\tilde{\Omega}$ is recursively given by*

$$P_{2m}\tilde{\Omega}_{2m} = \begin{bmatrix} P_m\tilde{\Omega}_m & P_m H_m \\ \frac{1}{4\sqrt{m^3}} H_m^T\tilde{\Omega}_m & \frac{1}{4\sqrt{m}} I_m \end{bmatrix}. \tag{55}$$

This is proven by multiplying the recursive formulas for P and $\tilde{\Omega}$.

Lemma .4. *The matrix $H_m^T\tilde{\Omega}_m$ is given element-wise as*

$$(H_m^T\tilde{\Omega}_m)_{i,l} = \begin{cases} 0 & \text{, if } l = 1; \\ \psi_l^2(t_i) & \text{, otherwise.} \end{cases} \tag{56}$$

Proof. First, observe that, since the first column of the matrix $\tilde{\Omega}_m$ is zero, then the first column of $H_m^T\tilde{\Omega}_m$ is also zero, proving the case $l = 1$.

For $l \geq 2$, fix an element $(H_m^T\tilde{\Omega}_m)_{i,l}$ and make $\psi_i = \psi_{j,k}$ and $\psi_l = \psi_{p,q}$ using the two-index notation. Due to the structure of H_m^T and $\tilde{\Omega}_m$,

$$(H_m^T\tilde{\Omega}_m)_{i,l} = \mathbf{h}_{2^p}^T(t_i)\mathbf{h}_{2^p}(t_{q+1}) = 1 + \sum_{r=0}^{p-1}\sum_{s=0}^{2^r-1} \psi_{r,s}(t_i)\psi_{r,s}(t_{q+1}).$$

where $t_i = 2^{-j}(k-0.5)$ and $t_{q+1} = 2^{-p}(q+0.5)$. Since t_i and t_{q+1} are fixed, for each r there is at most a single wavelet ψ_{r,s_r} whose support contains both t_i and t_{q+1}, because the intervals where wavelets at the same resolution level are non-zero do not overlap. Thus,

$$(H_m^T\tilde{\Omega}_m)_{i,l} = 1 + \sum_{r=0}^{p-1} \psi_{r,s_r}(t_i)\psi_{r,s_r}(t_{q+1})$$

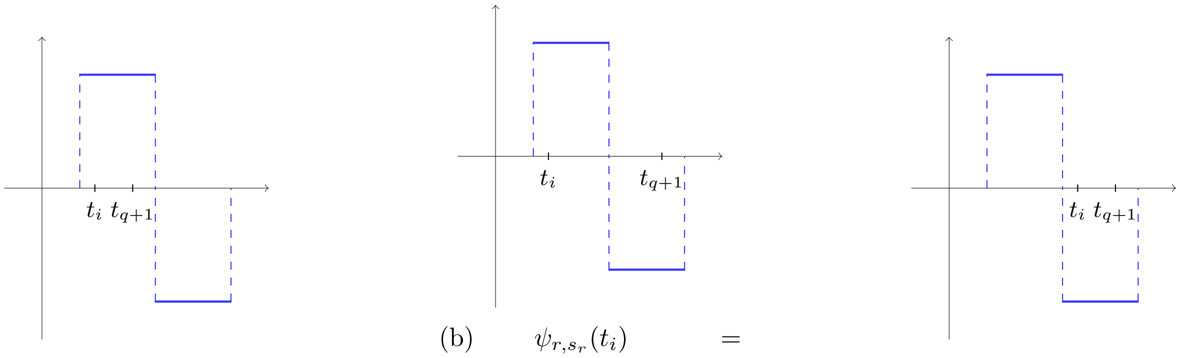
If neither $\psi_{r,s_r}(t_i)$ nor $\psi_{r,s_r}(t_{q+1})$ are zero, only two cases may occur: either $\psi_{r,s_r}(t_i) = \psi_{r,s_r}(t_{q+1})$ or $\psi_{r,s_r}(t_i) = -\psi_{r,s_r}(t_{q+1})$. The possible situations are depicted in Figure 7, supposing without loss of generality that $t_i \leq t_{q+1}$.

Let us study what happens when we change the resolution level r:

a) Suppose that $\psi_{r,s_r}(t_i) = \psi_{r,s_r}(t_{q+1})$ (situations (a) or (c) in Figure 7) for every $r = 0, ..., p-1$. Then $t_i = t_{q+1}$, since for $r = p-1$ we only sample ψ_{r,s_r} at the times $t_1 = \frac{s_r+0.25}{2^r}$ and $t_2 = \frac{s_r+0.75}{2^r}$, and we have $\psi_{r,s_r}(t_1) = -\psi_{r,s_r}(t_2)$. Thus

$$(H_m^T\tilde{\Omega}_m)_{i,l} = 1 + \sum_{r=0}^{p-1} \psi_{r,s_r}^2(t_i) = 1 + \sum_{r=0}^{p-1} 2^r = 2^p = \psi_l^2(t_i)$$

b) Suppose that $\psi_{r,s_r}(t_i) = -\psi_{r,s_r}(t_{q+1})$ (situation (b) in Figure 7) happens for some r for some $r \leq p-1$. Then $t_i \in [\frac{s}{2^r}, \frac{s+0.5}{2^r}]$ and $t_{q+1} \in [\frac{s+0.5}{2^r}, \frac{s+1}{2^r}]$. Due to the nesting property of the



(a) $\psi_{r,s_r}(t_i) = \psi_{r,s_r}(t_{q+1})$ (b) $\psi_{r,s_r}(t_i) = -\psi_{r,s_r}(t_{q+1})$ (c) $\psi_{r,s_r}(t_i) = \psi_{r,s_r}(t_{q+1})$

Figure 7: All possible situations for the product $\psi_{r,s_r}(t_i)\psi_{r,s_r}(t_{q+1})$

Haar wavelets, no finer wavelet has both t_i and t_{q+1} in its support, and for all coarser resolutions $\rho < r$ we have $\psi_{\rho,s_\rho}(t_i) = \psi_{\rho,s_\rho}(t_{q+1})$. Hence

$$(H_m^T\tilde{\Omega}_m)_{i,l} = 1 + \sum_{\rho=0}^{r-1} \psi^2_{\rho,s_\rho}(t_i) - \psi^2_{r,s_r}(t_i) = 1 + \sum_{\rho=0}^{r-1} 2^\rho - 2^r = 0$$

Lastly, since $p > r$ and $t_q \in \text{supp }\psi_{p,q}$, then $t_i \notin \text{supp}\psi_{p,q}$. Thus $\psi_l^2(t_i) = \psi^2_{p,q}(t_i) = 0 = (H_m^T\tilde{\Omega}_m)_{i,l}$.

$\square$

Now we finally prove that $(\tilde{\Omega}^T P^T \mathbf{c}) \odot \mathbf{b}$ and $\Gamma^T(\mathbf{a} \odot \mathbf{b})$ are indeed in $\ell^2(\mathbb{R})$.

Proposition .1. *The sequences $(\tilde{\Omega}^T P^T \mathbf{c}) \odot \mathbf{b}$ and $\Gamma^T(\mathbf{a} \odot \mathbf{b})$ are in $\ell^2(\mathbb{R})$ and satisfy, for some $C_1, C_2 > 0$,*

$$\left\|(\tilde{\Omega}^T P^T \mathbf{c}) \odot \mathbf{b}\right\|_{\ell^2} \leq C_1 \|\mathbf{c}\|_{\ell^2}\|\mathbf{d}\|_{\ell^2}$$
$$\left\|\Gamma^T(\mathbf{a} \odot \mathbf{b})\right\|_{\ell^2(\mathbb{R})} \leq C_2 \|\mathbf{c}\|_{\ell^2}\|\mathbf{d}\|_{\ell^2}$$

Proof. For $(\tilde{\Omega}^T P^T \mathbf{c}) \odot \mathbf{b}$, if we divide $\mathbf{c}$ as in (8) and using the block structure from Lemma .3, each block of $(\tilde{\Omega}^T P^T \mathbf{c})$ is given by

$$(\tilde{\Omega}^T P^T \mathbf{c})_1 = \sum_{r=0}^{\infty} \frac{1}{2^{\frac{3r}{2}+2}} (\mathbf{c}^*_{2^r})^T H^T_{2^r}\tilde{\Omega}_{2^r}$$
$$(\tilde{\Omega}^T P^T \mathbf{c})^*_{2^j} = (c_1, ..., c_{2^j})P_{2^j}H_{2^j} + \frac{1}{2^{\frac{j}{2}+2}}(\mathbf{c}^*_{2^j})^T + \sum_{r=j+1}^{\infty} \frac{1}{2^{\frac{3r}{2}+2}} (\mathbf{c}^*_{2^r})^T H^T_{2^r}\tilde{\Omega}_{2^r}$$

We bound each term in the right-hand side:

- $\|(c_1, ..., c_{2^j})P_{2^j}H_{2^j}\|_{\ell^2} \leq \left\|H^T_{2^j}\right\|\left\|P^T_{2^j}(c_1, ..., c_{2^j})\right\|_{\ell^2} \leq \frac{\|\mathbf{c}\|_{\ell^2}}{2^{\frac{j}{2}+1}}$
- $\left\|\frac{1}{2^{\frac{j}{2}+2}}(\mathbf{c}^*_{2^j})^T\right\|_{\ell^2} \leq \frac{\|\mathbf{c}\|_{\ell^2}}{2^{\frac{j}{2}+2}}$
- From Lemma .4, $\max H^T_{2^j}\Omega^T_{2^j} = 2^j$; thus

$$\left\|\sum_{r=j+1}^{\infty} \frac{1}{2^{\frac{3r}{2}+2}} (\mathbf{c}^*_{2^r})^T H^T_{2^r}\Omega^T_{2^r}\right\|_{\ell^2} \leq \sum_{r=j+1}^{\infty} \frac{1}{2^{\frac{3r}{2}+2}} \|\mathbf{c}\|_{\ell^2} \left\|H^T_{2^r}\Omega^T_{2^r}\right\|_{\ell^2} \leq \frac{1+\sqrt{2}}{2^{\frac{j}{2}+2}}\|\mathbf{c}\|_{\ell^2}$$

Hence,

$$\|(\tilde{\Omega}^T P^T \mathbf{c})^*_{2^j}\|_{\ell^2} \le \left(1+\frac{\sqrt{2}}{8}\right)\frac{\|\mathbf{c}\|_{\ell^2}}{2^{\frac{j}{2}}} \quad , \quad |(\tilde{\Omega}^T P^T \mathbf{c})_1| \le \frac{1+\sqrt{2}}{4}\|\mathbf{c}\|_{\ell^2} \tag{57}$$

Also, for $i = 2^j, 2^j+1, ..., 2^{2j}-1$, $\left|(P^T\mathbf{d})_i\right| \le \left\|(P^T\mathbf{d})^*_{2^j}\right\|_{\ell^2} \le \frac{\|\mathbf{d}\|_{\ell^2}}{2^{j+1}}$, and thus

$$\left\|\left((\tilde{\Omega}^T P^T \mathbf{c})\odot\mathbf{b}\right)^*_{2^j}\right\|_{\ell^2} \le \left\|(\tilde{\Omega}^T P^T \mathbf{c})^*_{2^j}\right\|_{\ell^2} \max_{2^j\le i\le 2^{2j}-1}\left|(P^T\mathbf{d})_i\right| \le \left(1+\frac{\sqrt{2}}{8}\right)\frac{\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}}{2^{\frac{3j}{2}+1}}$$

Therefore, we can bound $(\tilde{\Omega}^T P^T \mathbf{c})\odot\mathbf{b}$ by

$$\begin{aligned}\left\|(\tilde{\Omega}^T P^T \mathbf{c})\odot\mathbf{b}\right\|^2_{\ell^2} &= \left|\left((\tilde{\Omega}^T P^T \mathbf{c})\odot\mathbf{b}\right)_1\right|^2 + \sum_{j=0}^{\infty}\left\|\left((\tilde{\Omega}^T P^T \mathbf{c})\odot\mathbf{b}\right)^*_{2^j}\right\|^2_{\ell^2} \\ &\le \frac{1}{112}(169+79\sqrt{2})\,\|\mathbf{c}\|^2_{\ell^2}\,\|\mathbf{d}\|^2_{\ell^2} = C_1^2\,\|\mathbf{c}\|^2_{\ell^2}\,\|\mathbf{d}\|^2_{\ell^2}\,.\end{aligned}$$

For $\Gamma^T(\mathbf{a}\odot\mathbf{b})$, from (54) we have

$$\begin{aligned}\left|\left(\Gamma^T(\mathbf{a}\odot\mathbf{b})\right)_1\right| &= \left|a_1b_1 + \sum_{r=0}^{\infty} H_{2^r}\left(\mathbf{a}^*_{2^r}\odot\mathbf{b}^*_{2^r}\right)\right| \le |a_1b_1| + \sum_{r=0}^{\infty}\|H_{2^r}\|\,\|\mathbf{a}^*_{2^r}\|_{\ell^2}\max|\mathbf{b}^*_{2^r}| \\ &\le \|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2} + \sum_{r=0}^{\infty}\frac{\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}}{2^{\frac{3r}{2}+2}} \le \frac{18+\sqrt{2}}{14}\,\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}\end{aligned}$$

$$\begin{aligned}\left\|\left(\Gamma^T(\mathbf{a}\odot\mathbf{b})\right)^*_{2^j}\right\|_{\ell^2} &= \left\|\sum_{r=j+1}^{\infty} H_{2^r}\left(\mathbf{a}^*_{2^r}\odot\mathbf{b}^*_{2^r}\right)\right\|_{\ell^2} \le \sum_{r=j+1}^{\infty}\|H_{2^r}\|\,\|\mathbf{a}^*_{2^r}\|_{\ell^2}\max|\mathbf{b}^*_{2^r}| \\ &\le \sum_{r=j+1}^{\infty}\frac{\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}}{2^{\frac{3r}{2}+2}} = \frac{1+2\sqrt{2}}{4}\,\frac{\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}}{2^{\frac{3r}{2}}}\end{aligned}$$

Finally, we bound the sequence $\Gamma^T(\mathbf{a}\odot\mathbf{b})$ with

$$\begin{aligned}\|\Gamma^T(\mathbf{a}\odot\mathbf{b})\|^2_{\ell^2} &= \left|\left(\Gamma^T(\mathbf{a}\odot\mathbf{b})\right)_1\right|^2 + \sum_{j=0}^{\infty}\left\|\left(\Gamma^T(\mathbf{a}\odot\mathbf{b})\right)^*_{2^j}\right\|^2_{\ell^2} \\ &\le \frac{1}{49}(113+23\sqrt{2})\,\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2} = C_2^2\,\|\mathbf{c}\|_{\ell^2}\,\|\mathbf{d}\|_{\ell^2}\end{aligned}$$

□

B Proofs of quadratic estimates from Section 4

Here we prove the quadratic estimates from Propositions 4.1—4.4. As stated in the paper, the main strategy is to employ both the recursive block structures of the matrices from Theorems 2.1 and 4.1 and the finite-infinite decomposition from (17). For clarity, Figure 8 shows how they overlap for the operator P; the other matrices follow a similar pattern. We also draw insights from A to bound the sums that appear in the proof. We believe that similar procedures may be applied for higher-degree polynomial nonlinearities.

B.1 Proof of Proposition 4.1

i) For $i \le M$, since $\bar{\mathbf{c}} \in \mathbb{R}^M$, the i-th element of $\Pi_M P^T\bar{\mathbf{c}}$ is given by

$$\left(P^T\bar{\mathbf{c}}\right)_i = \sum_{l=1}^{M}\left(P^T\right)_{i,l} c_l = \left(P_M^T\bar{\mathbf{c}}\right)_i\,.$$

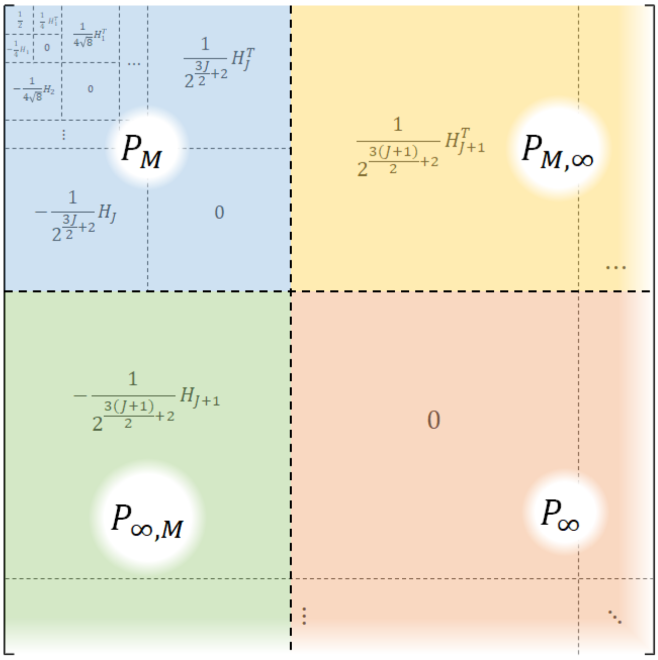


Figure 8: Overlay of the block structure and finite-infinite decomposition for P

ii) For $i \leq M$,

$$\big(\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)\bar{\mathbf{b}})\big)_i = (\bar{\mathbf{c}}^T P\tilde{\Omega})_i \bar{\mathbf{b}}_i = \sum_{l=1}^{M} c_l (P\tilde{\Omega})_{l,i}\bar{\mathbf{b}}_i = (\bar{\mathbf{c}}^T P_M \tilde{\Omega}_M)_i \bar{\mathbf{b}}_i.$$

with the last equality due to Lemma .3. Thus, from item *(i)*

$$\Pi_M \mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)\bar{\mathbf{b}}) = (\bar{\mathbf{c}}^T P_M\tilde{\Omega}_M)^T \odot \Pi_M \bar{\mathbf{b}} = (\tilde{\Omega}_M^T P_M^T \bar{\mathbf{c}}) \odot (P_M^T \bar{\mathbf{d}}).$$

iii) Using the finite-infinite decomposition, we can separate $\Pi_M\mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)\bar{\mathbf{b}})$ as

$$\Pi_M\mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)\bar{\mathbf{b}}) = \Gamma_M^T(\bar{\mathbf{a}}_M \odot \bar{\mathbf{b}}_M) + \Gamma_{\infty,M}^T(\bar{\mathbf{a}}_\infty \odot \bar{\mathbf{b}}_\infty)$$

We now need to bound the infinite sum in the second term. Adapting the expression of $\bar{\mathbf{b}} = P^T\bar{\mathbf{d}}$ in the proof of Proposition 2.2 for $2^j + l > M$ and $\bar{\mathbf{d}} \in \mathbb{R}^M$,

$$\bar{b}_{2^j+l} = -\frac{1}{2^{\frac{3j}{2}+2}} \sum_{q=1}^{M} (H_{2^j}^T)_{l,q}\, \bar{c}_q,$$

and thus, using the structure of Γ, (58) and the fact that $H_{2^j}H_{2^j}^T = 2^j I_{2^j}$,

$$\begin{aligned}
\left|\big(\Gamma_\infty^T(\Pi_\infty\bar{\mathbf{a}} \odot \Pi_\infty\bar{\mathbf{b}})\big)_i\right| &= \left|\sum_{j=J+1}^{\infty}\sum_{l=1}^{2^j} (H_{2^j})_{i,l}\, \bar{a}_{2^j+l}\bar{b}_{2^j+l}\right| \\
&\leq \sum_{j=J+1}^{\infty}\left|\sum_{l=1}^{2^j} -(H_{2^j})_{i,l}\left(\frac{1}{2^{\frac{3j}{2}+2}}\sum_{q=1}^{M}(H_{2^j}^T)_{l,q}\, c_q\right)\bar{b}_{2^j+l}\right| \\
&\leq \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{3j}{2}+2}}\left|\left(\sum_{q=1}^{M}\sum_{l=1}^{2^j}(H_{2^j})_{i,l}\,(H_{2^j}^T)_{l,q}\, c_q\right)\right|\frac{\|\bar{\mathbf{d}}\|_{\ell^2}}{2^{j+1}} = \frac{\sqrt{2}\,\|\bar{\mathbf{d}}\|_{\ell^2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+3}}|c_i|
\end{aligned}$$

and thus

$$\|\Gamma_\infty^T(\Pi_\infty\bar{\mathbf{a}} \odot \Pi_\infty\bar{\mathbf{b}})\|_{\ell^2} \leq \frac{\sqrt{2}\,\|\bar{\mathbf{c}}\|_{\ell^2}\|\bar{\mathbf{d}}\|_{\ell^2}}{(4-\sqrt{2})\,2^{\frac{3J}{2}+3}}.$$

B.2 Proof of Proposition 4.2

i) Using the finite-infinite decomposition, the recursive block structure of P and the fact that $\bar{\mathbf{c}} \in \mathbb{R}^M$, we have for $j > J$

$$\left\| \left(P^T \bar{\mathbf{c}}\right)^*_{2^j} \right\|_{\ell^2} = \left\| -\frac{1}{2^{\frac{3j}{2}+2}} H^T_{2^j} \bar{\mathbf{c}} \right\|_{\ell^2} \leq \frac{1}{2^{\frac{3j}{2}+2}} \left\| H^T_{2^j} \right\| \|\bar{\mathbf{c}}\|_{\ell^2} = \frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{j+2}} \tag{58}$$

Estimating as we did in Theorem .1,

$$\left\| \Pi_\infty \left(P^T \bar{\mathbf{c}}\right) \right\|_{\ell^2} = \sqrt{\sum_{j=J+1}^{\infty} \|(P^T \bar{\mathbf{c}})^*_{2^j}\|^2_{\ell^2}} \leq \frac{1}{\sqrt{3}} \frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{J+2}}.$$

ii) For $j > J$, dividing $\Pi_\infty \mathcal{H}(\bar{\mathbf{a}}^T \Omega(t) \bar{\mathbf{b}})$ in blocks,

$$\begin{aligned}
\left\| \left(\mathcal{H}(\bar{\mathbf{a}}^T \Omega(t) \bar{\mathbf{b}})\right)^*_{2^j} \right\|_{\ell^2} &= \left\| -\frac{1}{2^{\frac{3j}{2}+2}} \left(H^T_{2^j} P^T_{2^j} \bar{\mathbf{c}}\right) \odot \left(H^T_{2^j} \bar{\mathbf{d}}\right) \right\|_{\ell^2} \leq \frac{1}{2^{\frac{3j}{2}+2}} \left(\max_{1 \leq k \leq 2^j} \left| \left(H^T_{2^j} \bar{\mathbf{c}}\right)_k \right| \right) \left\| H^T_{2^j} P^T_{2^j} \bar{\mathbf{d}} \right\|_{\ell^2} \\
&\leq \frac{1}{2^{\frac{3j}{2}+2}} \left\| H^T_{2^j} \bar{\mathbf{c}} \right\|_{\ell^2} \left\| H^T_{2^j} P^T_{2^j} \bar{\mathbf{d}} \right\|_{\ell^2} \leq \frac{\|\bar{\mathbf{c}}\|_{\ell^2} \left\|\bar{\mathbf{d}}\right\|_{\ell^2}}{2^{j+4}}
\end{aligned}$$

and thus

$$\left\| \Pi_\infty \left(\mathcal{H}(\bar{\mathbf{a}}^T \Omega(t) \bar{\mathbf{b}})\right) \right\|_{\ell^2} = \sqrt{\sum_{j=J+1}^{\infty} \left\| \left(\mathcal{H}(\bar{\mathbf{a}}^T \Omega(t) \bar{\mathbf{b}})\right)^*_{2^j} \right\|^2_{\ell^2}} \leq \frac{1}{\sqrt{3}} \frac{\|\bar{\mathbf{c}}\|_{\ell^2} \left\|\bar{\mathbf{d}}\right\|_{\ell^2}}{2^{2J+4}}.$$

iii) To estimate $\Pi_\infty \mathcal{H}(\bar{\mathbf{a}}^T \Theta(t) \bar{\mathbf{a}})$, with the block matrix structure from (54) for $i > M$,

$$\begin{aligned}
\left| \left(\mathcal{H}(\bar{\mathbf{a}}^T \Theta(t) \bar{\mathbf{b}})\right)_i \right| = \left| \Gamma^T_{i,*} (\bar{\mathbf{a}} \odot \bar{\mathbf{b}}) \right| &= \left| \sum_{p=j+1}^{\infty} \frac{1}{2^{3p+4}} \sum_{q=1}^{2^p} (H_{2^p})_{i,q} \left((H^T_{2^p})_{q,*} \bar{\mathbf{c}}\right) \left((H^T_{2^p})_{q,*} \bar{\mathbf{d}}\right) \right| \\
&\leq \sum_{p=j+1}^{\infty} \frac{1}{2^{3p+4}} \max_{1 \leq q \leq 2^p} \left| \left((H^T_{2^p})_{q,*} \bar{\mathbf{d}}\right) \right| \sum_{q=1}^{2^p} \left| (H_{2^p})_{i,q} \left((H^T_{2^p})_{q,*} \bar{\mathbf{c}}\right) \right| \\
&\leq \sum_{p=j+1}^{\infty} \left| (I_{2^p})_{i,*} \bar{\mathbf{c}} \right| \frac{\|\bar{\mathbf{d}}\|_{\ell^2}}{2^{3p+6}} = \frac{|c_i|}{2^{3j+6}} \frac{\|\bar{\mathbf{d}}\|_{\ell^2}}{7}
\end{aligned}$$

and hence

$$\left\| \Pi_\infty \mathcal{H}(\bar{\mathbf{a}}^T \Theta(t) \bar{\mathbf{b}}) \right\|_{\ell^2} \leq \frac{1}{21\sqrt{7}} \frac{\|\bar{\mathbf{c}}\|_{\ell^2} \|\bar{\mathbf{d}}\|_{\ell^2}}{2^{3J+6}}.$$

B.3 Proof of Proposition 4.3

The following Lemma helps estimating terms of the type $A_M(B_M \bar{\mathbf{x}} \odot C_M \bar{\mathbf{y}})$ for arbitrary $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbb{R}^M$, which would be tricky otherwise.

Lemma .5. *Given $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbb{R}^M$ and $M \times M$ matrices A_M, B_M, C_M, then*

$$\|A_M(B_M \bar{\mathbf{x}} \odot C_M \bar{\mathbf{y}})\|_{\ell^2} \leq \|A_M \operatorname{diag}(\|(C_M)_{l,*}\|_{\ell^2}) B_M\| \, \|\bar{\mathbf{x}}\|_{\ell^2} \|\bar{\mathbf{y}}\|_{\ell^2}, \tag{59}$$

where

$$\operatorname{diag}(\bar{\mathbf{x}}) := \begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix}, \quad (\|(C_M)_{l,*}\|_{\ell^2}) = \begin{pmatrix} \|(C_M)_{1,*}\|_{\ell^2} \\ \|(C_M)_{2,*}\|_{\ell^2} \\ \vdots \\ \|(C_M)_{M,*}\|_{\ell^2} \end{pmatrix}.$$

Proof. For each element of $A_M(B_M\bar{\mathbf{x}}\odot C_M\bar{\mathbf{y}})$

$$\begin{aligned}|(A_M(B_M\bar{\mathbf{x}}\odot C_M\bar{\mathbf{y}}))_i| &= \left|\sum_{l=1}^{M}(A_M)_{i,l}(B_M\bar{\mathbf{x}})_l(C_M\bar{\mathbf{y}})_l\right| \\ &\leq \left|\sum_{l=1}^{M}(A_M)_{i,l}(B_M\bar{\mathbf{x}})_l\|(C_M)_{l,*}\|_{\ell^2}\|\bar{\mathbf{y}}\|_{\ell^2}\right| = |(A_M)_{i,*}\mathrm{diag}\left(\|(C_M)_{l,*}\|_{\ell^2}\right)B_M\bar{\mathbf{x}}|\,\|\bar{\mathbf{y}}\|_{\ell^2}.\end{aligned}$$

Taking the ℓ^2 norm, the result follows. □

i) Applying the finite-infinite decomposition to P and observing its block structure as in Figure 8, we have that

$$\begin{aligned}\left\|(P^T_{\infty,M}\,\mathbf{y}_\infty)\right\|_{\ell^2} &= \left\|\sum_{q=J+1}^{\infty}\frac{1}{2^{\frac{3q}{2}+2}}\left(H_{2^q}\right)_{1:M,*}\mathbf{y}^*_{2^q}\right\|_{\ell^2} \leq \sum_{q=J+1}^{\infty}\frac{1}{2^{\frac{3q}{2}+2}}\left\|\left(H_{2^q}\right)_{1:M,*}\right\|\,\|\mathbf{y}^*_{2^q}\|_{\ell^2} \\ &\leq \sum_{q=J+1}^{\infty}\frac{1}{2^{\frac{3q}{2}+2}}\|H_{2^q}\|\,\|\mathbf{y}^*_{2^q}\|_{\ell^2} \leq \frac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}}. \end{aligned} \tag{60}$$

ii) Fix $j>J$ and make $\mathbf{z}_{2^j} := H_{2^j}\mathbf{y}^*_{2^j}$. From Lemma .3,

$$\begin{aligned}\left(\tilde{\Omega}^TP^T\right)_{M,\infty}\mathbf{y}_\infty &= \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{3j}{2}+2}}\left(\tilde{\Omega}^T_{2^j}H_{2^j}\right)_{1:M,*}\mathbf{y}^*_{2^j} \\ &= \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{3j}{2}+2}}\left(\tilde{\Omega}^T_{2^j}\right)_{1:M,*}H_{2^j}\mathbf{y}^*_{2^j} = \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{3j}{2}+2}}\left(\tilde{\Omega}^T_{2^j}\right)_{1:M,*}\mathbf{z}_{2^j}\end{aligned}$$

Observe that, for $p=0,1,...,J$,

$$\left(\tilde{\Omega}^T_{2^j}\mathbf{z}_{2^j}\right)_1 = 0 \quad , \quad \left\|\left(\tilde{\Omega}^T_{2^j}\mathbf{z}_{2^j}\right)^*_{2^p}\right\|_{\ell^2} = \left\|H^T_{2^p}(z_1,...,z_{2^p})^T\right\|_{\ell^2} \leq 2^{\frac{p}{2}}\,\|\mathbf{z}_{2^j}\|_{\ell^2}$$

and therefore, for each $j>J$,

$$\left\|\tilde{\Omega}^T_{2^j}\mathbf{z}_{2^j}\right\|_{\ell^2} = \sqrt{\sum_{p=0}^{j}\left\|\left(\tilde{\Omega}^T_{2^j}\mathbf{z}_{2^j}\right)^*_{2^p}\right\|^2_{\ell^2}} \leq 2^{\frac{j+1}{2}}\,\|\mathbf{z}_{2^j}\|_{\ell^2} \leq 2^{j+\frac{1}{2}}\left\|\mathbf{y}^*_{2^j}\right\|_{\ell^2}.$$

Hence, we can bound the norm of $\left(\tilde{\Omega}^TP^T\right)_{M,\infty}\mathbf{y}_\infty$ with

$$\begin{aligned}\left\|\left(\tilde{\Omega}^TP^T\right)_{M,\infty}\mathbf{y}_\infty\right\|_{\ell^2} &\leq \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{3j}{2}+2}}\left\|\left(\tilde{\Omega}^T_{2^j}\right)_{1:M,*}\mathbf{z}_{2^j}\right\|_{\ell^2} \\ &\leq \sum_{j=J+1}^{\infty}\frac{1}{2^{\frac{j+3}{2}}}\left\|\mathbf{y}^*_{2^j}\right\|_{\ell^2} \leq \left(1+\sqrt{2}\right)\frac{\|\mathbf{y}\|_{\ell^2}}{2^{\frac{J+3}{2}}}\end{aligned}$$

iii) From the block structure of (54), and using (10) and (58),

$$\begin{aligned}\left\|\Gamma^T_{\infty,M}(\bar{\mathbf{a}}_\infty\odot\Pi_\infty P^T\mathbf{y})\right\|_{\ell^2} &= \left\|\sum_{q=J+1}^{\infty}\left(H_{2^q}\right)_{1:M,*}\left(\bar{\mathbf{a}}^*_{2^q}\odot(P^T\mathbf{y})^*_{2^q}\right)\right\|_{\ell^2} \\ &\leq \sum_{q=J+1}^{\infty}\|(H_{2^q})\|\left(\max_{1\leq r\leq 2^q}|(\bar{\mathbf{a}}^*_{2^q})_r|\right)\left\|(P^T\mathbf{y})^*_{2^q}\right\|_{\ell^2} \\ &\leq \sum_{q=J+1}^{\infty}2^{\frac{q}{2}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{q+2}}\frac{\|\mathbf{y}\|_{\ell^2}}{2^{q+1}} = \frac{\sqrt{2}}{4-\sqrt{2}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\,\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}\end{aligned}$$

iv) Applying the finite-infinite decomposition to $A_M \Pi_M \mathcal{H}(\mathbf{x}^T P\,\Omega(t) P^T \mathbf{y})$, we have

$$\Pi_M \mathcal{H}(\mathbf{x}^T P\,\Omega(t) P^T \mathbf{y}) = \left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)^T \odot \left(P_M^T \mathbf{y}_M\right) + \left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)^T \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right)$$
$$+ \left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \odot \left(P_M^T \mathbf{y}_M\right) + \left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right).$$

Using (59), we can bound the first term with Lemma .5:

$$\left\| A_M \left(\left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)^T \odot \left(P_M^T \mathbf{y}_M\right) \right) \right\|_{\ell^2} \leq \left\| A_M \operatorname{diag}\left(\|(P_M^T)_{i,*}\|_{\ell^2} \right) \tilde{\Omega}_M^T P_M^T \right\| \; \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}.$$

For the second term, using the bound from (60),

$$\left| \left(A_M \left(\left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)^T \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right)_i \right| = \left| \sum_{l=1}^{M} (A_M)_{i,l} \left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)_l \left(P_{\infty,M}^T \mathbf{y}_\infty\right)_l \right|$$
$$\leq \left| \sum_{l=1}^{M} (A_M)_{i,l} \left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)_l \right| \frac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}} = \left| \left(A_M \tilde{\Omega}_M^T P_M^T \mathbf{x}_M \right)_i \right| \frac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}}$$

and thus

$$\left\| A_M \left(\left(\mathbf{x}_M^T P_M \tilde{\Omega}_M\right)^T \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right\|_{\ell^2} \leq \frac{\left\| A_M \tilde{\Omega}_M^T P_M^T \right\|}{2^{J+2}} \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}.$$

For the third term, with a similar method to the previous term and the estimate for $\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}$,

$$\left| \left(A_M \left(\left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \odot \left(P_M^T \mathbf{y}_M\right) \right) \right)_i \right| = \left| \sum_{q=1}^{M} (A_M)_{i,q} \left(P_M^T \mathbf{y}_M\right)_q \left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)_q \right|$$
$$\leq \left(\max_{1 \leq q \leq M} \left| \left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)_q \right| \right) \left| \sum_{q=1}^{M} (A_M)_{i,q} \left(P_M^T \mathbf{y}_M\right)_q \right|$$
$$\leq \left\| \mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty} \right\|_{\ell^2} \left| \left(A_M P_M^T \mathbf{y}_M\right)_i \right| \leq \left(1 + \sqrt{2}\right) \frac{\|\mathbf{x}\|_{\ell^2}}{2^{\frac{J+3}{2}}} \left| \left(A_M P_M^T \mathbf{y}_M\right)_i \right|$$

and thus

$$\left\| A_M \left(\left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \odot \left(P_M^T \mathbf{y}_M\right) \right) \right\|_{\ell^2} \leq \left(1 + \sqrt{2}\right) \frac{\|\mathbf{x}\|_{\ell^2}}{2^{\frac{J+3}{2}}} \left\| A_M P_M^T \mathbf{y}_M \right\|_{\ell^2}$$
$$\leq \left(1 + \sqrt{2}\right) \left\| A_M P_M^T \right\| \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{J+3}{2}}}$$

For the fourth term, with the estimates for $\mathbf{x}_\infty^T \left((P\,\tilde{\Omega})_{M,\infty}\right)$ and $P_{\infty,M}^T \mathbf{y}_\infty$, we obtain

$$\left\| A_M \left(\left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right\|_{\ell^2} = \left\| A_M \operatorname{diag}\left(P_{\infty,M}^T \mathbf{y}_\infty\right) \left(\mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty}\right)^T \right\|_{\ell^2}$$
$$\leq \|A_M\| \left\| P_{\infty,M}^T \mathbf{y}_\infty \right\|_{\ell^2} \left\| \mathbf{x}_\infty^T (P\,\tilde{\Omega})_{M,\infty} \right\|_{\ell^2} \leq \|A_M\| \left(1 + \sqrt{2}\right) \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J+7}{2}}}$$

Finally, adding the four bounds, we obtain the desired bound.

v) Applying the finite-infinite decomposition to both P^T and Γ^T, we have

$$
\begin{aligned}
A_M \Pi_M \mathcal{H}(\mathbf{x}^T P \Theta(t) P^T \mathbf{y}) = A_M \Gamma_M^T \Big[& \left(P_M^T \mathbf{x}_M\right) \odot \left(P_M^T \mathbf{y}_M\right) + \left(P_M^T \mathbf{x}_M\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \\
& + \left(P_{\infty,M}^T \mathbf{x}_\infty\right) \odot \left(P_M^T \mathbf{y}_M\right) + \left(P_{\infty,M}^T \mathbf{x}_\infty\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \Big] \\
& + A_M \Gamma_{\infty,M}^T \Pi_\infty \left(P^T \mathbf{x} \odot P^T \mathbf{y}\right).
\end{aligned}
$$

We estimate it term by term again. For the first one, using Lemma .5,

$$
\left\| A_M \Gamma_M^T \left(\left(P_M^T \mathbf{x}_M\right) \odot \left(P_M^T \mathbf{y}_M\right) \right) \right\|_{\ell^2} \leq \left\| A_M \Gamma_M^T \mathrm{diag}(\|(P_M^T)_{i,*}\|_{\ell^2}) P_M^T \right\| \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}.
$$

For the second term, we can use (60) to bound it element-wise by

$$
\begin{aligned}
\left| \left(A_M \Gamma_M^T \left(\left(P_M^T \mathbf{x}_M\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right)_i \right| &= \left| \sum_{l=1}^{M} (A_M \Gamma_M^T)_{i,l} \left(P_M^T \mathbf{x}_M\right)_l \left(P_{\infty,M}^T \mathbf{y}_\infty\right)_l \right| \\
&\leq \left| \sum_{l=1}^{M} (A_M \Gamma_M^T)_{i,l} \left(P_M^T \mathbf{x}_M\right)_l \right| \frac{1}{2^{J+2}} \|\mathbf{y}\|_{\ell^2} = \frac{\left|(A_M \Gamma_M^T P_M^T \mathbf{x}_M)_i\right|}{2^{J+2}} \|\mathbf{y}\|_{\ell^2}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\left\| A_M \Gamma_M^T \left(\left(P_M^T \mathbf{x}_M\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right\|_{\ell^2} &= \left(\sum_{i=1}^{M} \left| \left(A_M \Gamma_M^T \left(\left(P_M^T \mathbf{x}_M\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right)_i \right|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{M} \left| (A_M \Gamma_M^T P_M^T \mathbf{x}_M)_i \right|^2 \right)^{\frac{1}{2}} \frac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}} = \left\| A_M \Gamma_M^T P_M^T \mathbf{x}_M \right\|_{\ell^2} \frac{\|\mathbf{y}\|_{\ell^2}}{2^{J+2}} \\
&\leq \frac{\left\| A_M \Gamma_M^T P_M^T \right\|}{2^{J+2}} \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}
\end{aligned}
$$

The same procedure applied to the third term yields the same bound as above. For the fourth term, using previous estimates,

$$
\begin{aligned}
\left\| A_M \Gamma_M^T \left(\left(P_{\infty,M}^T \mathbf{x}_\infty\right) \odot \left(P_{\infty,M}^T \mathbf{y}_\infty\right) \right) \right\|_{\ell^2} &\leq \left\| A_M \Gamma_M^T \right\| \left\| P_{\infty,M}^T \mathbf{x}_\infty \right\|_{\ell^2} \left\| P_{\infty,M}^T \mathbf{y}_\infty \right\|_{\ell^2} \\
&\leq \left\| A_M \Gamma_M^T \right\| \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{2J+4}}.
\end{aligned}
$$

For the fifth term, using the block structure of Γ^T in (54) and previous estimates,

$$
\begin{aligned}
\left\| A_M \Gamma_{\infty,M}^T \Pi_\infty \left(P^T \mathbf{x} \odot P^T \mathbf{y}\right) \right\|_{\ell^2} &= \left\| A_M \sum_{p=J+1}^{\infty} (H_{2^p})_{1:M,*} \left(\mathrm{diag}\left(P^T \mathbf{x}\right)_{2^p}^* \right) \left(P^T \mathbf{y}\right)_{2^p}^* \right\|_{\ell^2} \\
&\leq \|A_M\| \sum_{p=J+1}^{\infty} \|H_{2^p}\| \left\| \left(P^T \mathbf{x}\right)_{2^p}^* \right\|_{\ell^2} \left\| \left(P^T \mathbf{y}\right)_{2^p}^* \right\|_{\ell^2} \\
&\leq \|A_M\| \sum_{p=J+1}^{\infty} 2^{\frac{p}{2}} \frac{\|\mathbf{x}\|_{\ell^2}}{2^{p+2}} \frac{\|\mathbf{y}\|_{\ell^2}}{2^{p+2}} = \frac{\sqrt{2}\,\|A_M\|}{4-\sqrt{2}} \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+4}}
\end{aligned}
$$

Hence, adding the estimates, the result follows.

B.4 Proof of Proposition 4.4

i) To estimate $\mathcal{H}(\bar{\mathbf{c}}^T\mathbf{w}(t)\mathbf{w}^T\mathbf{y})$, we again decompose in three parts and estimate each one separately:

$$\mathcal{H}(\bar{\mathbf{c}}^T\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y}) = \mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y}) + \mathcal{H}(\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y}) + \mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)P^T\mathbf{y}).$$

- $\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})$: Since $\bar{\mathbf{c}} \in \mathbb{R}^M$, $\left(\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})\right)_i = (\bar{\mathbf{c}}^T P_{2^j}H_{2^j})_i(P^T\mathbf{y})_i$. Therefore, applying the block decomposition to $\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})$ and some previously calculated estimates from (10) and (58),

$$\begin{aligned}\left\|\left(\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})\right)^*_{2^j}\right\|_{\ell^2} &= \left\|\left(H^T_{2^j}P^T_{2^j}\bar{\mathbf{c}}\right)^*_{2^j} \odot \left(P^T\mathbf{y}\right)^*_{2^j}\right\|_{\ell^2} \leq \left\|\operatorname{diag}\left((P^T\mathbf{y})^*_{2^j}\right)\right\| \left\|\left(H^T_{2^j}P^T_{2^j}\bar{\mathbf{c}}\right)^*_{2^j}\right\|_{\ell^2}\\ &= \max_{2^j+1\leq i\leq 2^{j+1}} \left|\left(P^T\mathbf{y}\right)_i\right| \left\|\left(H^T_{2^j}P^T_{2^j}\bar{\mathbf{c}}\right)^*_{2^j}\right\|_{\ell^2} \leq \frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3j}{2}+3}}\end{aligned}$$

and thus

$$\left\|\Pi_\infty\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})\right\| = \sqrt{\sum_{j=J+1}^{\infty}\left\|\left(\mathcal{H}(\bar{\mathbf{a}}^T\Omega(t)P^T\mathbf{y})\right)^*_{2^j}\right\|^2} \leq \frac{1}{\sqrt{7}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}.$$

- $\mathcal{H}(\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y})$: Since $\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y} = (\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y})^T = (P^T\mathbf{y})^T\Omega(t)\bar{\mathbf{a}}$, then using the estimate from (57),

$$\begin{aligned}\left\|\left(\mathcal{H}(\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y})\right)^*_{2^j}\right\|_{\ell^2} &= \left\|\operatorname{diag}\left((P^T\bar{\mathbf{c}})^*_{2^j}\right)\left(\tilde{\Omega}^TP^T\mathbf{y}\right)^*_{2^j}\right\|_{\ell^2}\\ &\leq \max_{2^j+1\leq i\leq 2^{j+1}}\left|(P^T\bar{\mathbf{c}})_i\right|\left(1+\frac{\sqrt{2}}{8}\right)\frac{\|\mathbf{y}\|_{\ell^2}}{2^{\frac{j}{2}}} \leq \left(1+\frac{\sqrt{2}}{8}\right)\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3j}{2}+2}},\end{aligned}$$

and the bound for $\Pi_\infty\mathcal{H}(\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y})$ becomes

$$\left\|\Pi_\infty\mathcal{H}(\bar{\mathbf{a}}^T\Omega^T(t)P^T\mathbf{y})\right\|_{\ell^2} \leq \frac{1}{\sqrt{7}}\left(1+\frac{\sqrt{2}}{8}\right)\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+2}}.$$

- $\mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)P^T\mathbf{y})$: Using the block structure of Γ^T from (54) and the estimates (58) and (10), we have

$$\begin{aligned}\left\|\left(\Gamma^T\left(\bar{\mathbf{a}}\odot P^T\mathbf{y}\right)\right)^*_{2^j}\right\|_{\ell^2} &= \left\|\sum_{p=j+1}^{\infty}(H_{2^p})_{2^j+1:2^{j+1},*}\left(\bar{\mathbf{a}}^*_{2^p}\odot(P^T\mathbf{y})^*_{2^p}\right)\right\|_{\ell^2}\\ &\leq \sum_{p=j+1}^{\infty}\left\|(H_{2^p})_{2^j+1:2^{j+1},*}\right\|\left\|\operatorname{diag}(\bar{\mathbf{a}}^*_{2^p})\right\|\left\|(P^T\mathbf{y})^*_{2^p}\right\|_{\ell^2}\\ &\leq \sum_{p=j+1}^{\infty}\left\|H_{2^p}\right\|\max_{2^p+1\leq i\leq 2^{p+1}}\left|(P^T\bar{\mathbf{c}})_i\right|\left\|(P^T\mathbf{y})^*_{2^p}\right\|_{\ell^2}\\ &\leq \sum_{p=j+1}^{\infty}2^{\frac{p}{2}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}}{2^{p+2}}\frac{\|\mathbf{y}\|_{\ell^2}}{2^{p+1}} = \frac{\sqrt{2}}{4-\sqrt{2}}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3j}{2}+3}},\end{aligned}$$

and thus

$$\left\|\Pi_\infty\mathcal{H}(\bar{\mathbf{a}}^T\Theta(t)P^T\mathbf{y})\right\|_{\ell^2} \leq \frac{\sqrt{2}}{\sqrt{7}(4-\sqrt{2})}\frac{\|\bar{\mathbf{c}}\|_{\ell^2}\|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}.$$

The result follows from summing the estimates.

ii) We proceed in the same way as we did for $\Pi_\infty \mathcal{H}(\bar{\mathbf{c}}^T \mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y})$, that is, separate it in three parts

$$\mathcal{H}(\mathbf{x}^T \mathbf{w}(t)\mathbf{w}^T(t)\mathbf{y}) = \mathcal{H}(\mathbf{x}^T \Omega(t) P^T \mathbf{y}) + \mathcal{H}(\mathbf{x}^T \Omega^T(t) P^T \mathbf{y}) + \mathcal{H}(\mathbf{x}^T \Theta(t) P^T \mathbf{y})$$

and estimate each one.

- $\mathcal{H}(\mathbf{x}^T \Omega(t) P^T \mathbf{y})$: Applying the block decomposition to the expression of $\mathcal{H}(\mathbf{x}^T \Omega(t) P^T \mathbf{y})$ and the estimates from (10) and (58),

$$\begin{aligned}\left\| \left(\mathcal{H}(\mathbf{x}^T \Omega(t) P^T \mathbf{y})\right)^*_{2^j} \right\|_{\ell^2} &= \left\| (\tilde{\Omega}^T P^T \mathbf{x})^*_{2^j} \odot (P^T \mathbf{y})^*_{2^j} \right\|_{\ell^2} \leq \left\| \operatorname{diag}\left(P^T \mathbf{y}\right)^*_{2^j} \right\| \left\| (\tilde{\Omega}^T P^T \mathbf{x})^*_{2^j} \right\|_{\ell^2} \\ &\leq \frac{\|\mathbf{y}\|_{\ell^2}}{2^{j+1}} \left(3 + 2\sqrt{2}\right) \frac{\|\mathbf{x}\|_{\ell^2}}{2^{\frac{j}{2}+2}} = \frac{(3+2\sqrt{2})}{2^{\frac{3j}{2}+3}} \|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}\end{aligned}$$

and then

$$\left\| \Pi_\infty \mathcal{H}(\mathbf{x}^T \Omega(t) P^T \mathbf{y}) \right\|_{\ell^2} \leq \frac{(3+2\sqrt{2})}{\sqrt{7}} \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}$$

- $\mathcal{H}(\mathbf{x}^T \Omega^T(t) P^T \mathbf{y})$: Since $\mathbf{x}^T \Omega^T(t) P^T \mathbf{y} = (\mathbf{x}^T \Omega^T(t) P^T \mathbf{y})^T = \mathbf{y}^T \Omega^T(t) P^T \mathbf{x}$, repeating the same process for the previous item, we also have

$$\left\| \Pi_\infty \mathcal{H}(\mathbf{x}^T \Omega^T(t) P^T \mathbf{y}) \right\|_{\ell^2} \leq \frac{(3+2\sqrt{2})}{\sqrt{7}} \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+3}}$$

- $\mathcal{H}(\mathbf{x}^T \Theta(t) P^T \mathbf{y})$: Using the block decomposition and the block structure of Γ,

$$\begin{aligned}\left\| \left(\Gamma^T \left(P^T \mathbf{x} \odot P^T \mathbf{y}\right)\right)^*_{2^j} \right\|_{\ell^2} &= \left\| \sum_{p=j+1}^{\infty} (H_{2^p})_{2^j+1:2^{j+1},*} \left((P^T \mathbf{x})^*_{2^p} \odot (P^T \mathbf{y})^*_{2^p}\right) \right\|_{\ell^2} \\ &\leq \sum_{p=j+1}^{\infty} \|(H_{2^p})\| \left\| \operatorname{diag}((P^T \mathbf{x})^*_{2^p}) \right\| \left\| (P^T \mathbf{y})^*_{2^p} \right\|_{\ell^2} \\ &\leq \sum_{p=j+1}^{\infty} 2^{\frac{p}{2}} \frac{\|\mathbf{x}\|_{\ell^2}}{2^{p+1}} \frac{\|\mathbf{y}\|_{\ell^2}}{2^{p+1}} = \frac{\sqrt{2}}{4-\sqrt{2}} \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3j}{2}+2}}\end{aligned}$$

and thus

$$\left\| \Pi_\infty \mathcal{H}(\mathbf{x}^T \Theta(t) P^T \mathbf{y}) \right\|_{\ell^2} \leq \frac{\sqrt{2}}{\sqrt{7}\left(4-\sqrt{2}\right)} \frac{\|\mathbf{x}\|_{\ell^2} \|\mathbf{y}\|_{\ell^2}}{2^{\frac{3J}{2}+2}}$$

Putting together the estimates, we have the desired result.

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CHAPTER

4

GENERALIZED COMBINATORIAL MARCHING HYPERCUBES METHOD

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A Generalized Combinatorial Marching Hypercubes Algorithm

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Abstract

We present a Generalized Combinatorial Marching Hypercubes (GCMH) algorithm to compute a cell complex approximation of a manifold of any dimension and co-dimension, that is, manifolds of dimension $n - k$ embedded into an n-dimensional space. The algorithm uses combinatorial and topological methods to avoid the use of expensive lookup tables and hence be efficient in higher dimensions. We illustrate the effectiveness of our algorithm in higher dimensions and compare its performance with a similar algorithm based on a simplicial decomposition of the domain.

Keywords: Manifold approximation, High-dimensional manifolds, Marching Hypercubes, Combinatorial skeleton

1 Introduction

Methods to compute approximations of two- and three-dimensional manifolds (surfaces and volumes) are widely available and used in computer graphics and applied to animation, digital games, molecular modeling, object detection, object tracking, medical imaging, object reconstruction, etc. (see [1–7]).

Higher-dimensional manifolds are less prevalent, but they are also used in applications such as superstring theory [8], time-dependent 3D modeling and visualization [9, 10] and mathematical visualization [11]. Methods to compute such manifolds are also available; however, most of these methods are highly inefficient in high dimensions. An efficient algorithm, called Combinatorial Marching Hypercubes (CMH), to compute high dimensional manifolds of dimension $n-1$ embedded into a n-dimensional space was presented in [12].

In this paper, we present a generalization of the method presented in [12]. More precisely, we present the Generalized Combinatorial Marching Hypercubes (GCMH) algorithm to compute and represent implicitly defined $(n-k)$-dimensional manifolds as a cell complex embedded into a grid of n-dimensional hypercubes. The main advantage over the previous method is the ability to compute manifolds of dimension $n-k$, for any $k < n$, while the method in [12] is strictly limited to compute manifolds of dimension $n-1$. Our implementation of the algorithm can be found at https://github.com/gknakassima/GCMH.

An implicitly defined $(n-k)$-dimensional manifold $\mathcal{M}$ is the set of points where a function $F\colon \mathbb{R}^n \to \mathbb{R}^k$ takes on a given value c (see Section 2 for a precise definition). Figure 1 shows an example of the output of our algorithm to represent a Bagel Klein Bottle defined by $F(x,y,z,s,t) = (0,0,0)$, where $F\colon \mathbb{R}^5 \to \mathbb{R}^3$ is given by

$$F(x,y,z,s,t) = \begin{pmatrix} \cos(s)(3+\cos(s/2)\sin(t) - \sin(s/2)\sin(2t)) - x \\ \sin(s)(3+\cos(s/2)\sin(t) - \sin(s/2)\sin(2t)) - y \\ \sin(s/2)\sin(t) + \cos(s/2)\sin(2t) - z \end{pmatrix}.$$

Note that the Bagel Klein Bottle, as defined above, is a non-orientable two-dimensional manifold embedded in $\mathbb{R}^5$.

1.1 Literature review

One of the most renowned algorithms to approximate 2-dimensional manifolds embedded into a three-dimensional space is the Marching Cubes algorithm [13] by Lorensen and Cline. Improvements to this algorithm to deal with ambiguities and improve topological correctness have been presented in [14–22]. A survey of Marching Cubes type algorithms can be found in [23], and estimates on its complexity can be found in [24, 25]. Algorithms based on a grid of tetrahedra, rather than a cubical grid, called Marching Tetrahedra are presented in [26–29].

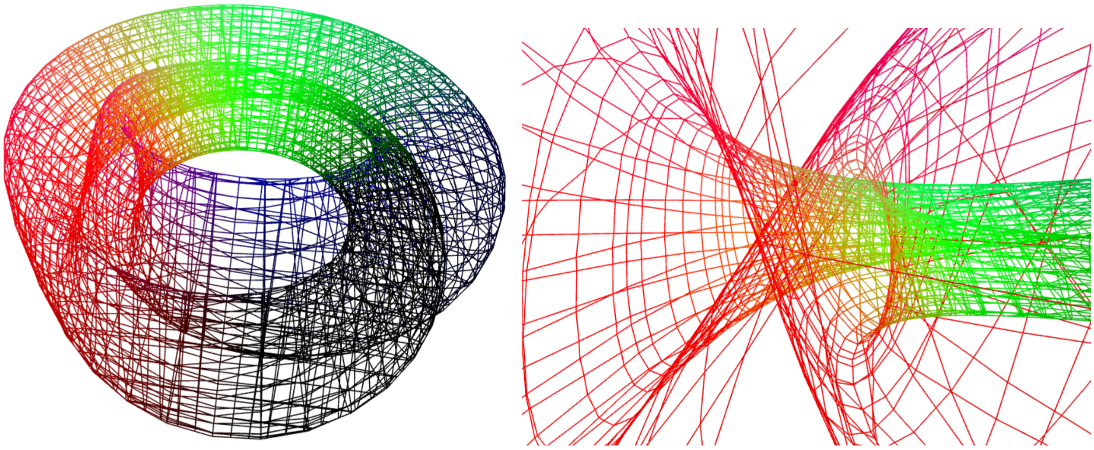

Fig. 1 Left: The Bagel Klein Bottle manifold in $\mathbb{R}^5$. Right: A zoom in of the same manifold showing its non-orientability.

Generalizations of Marching Cubes to compute manifolds of dimension $n-1$ embedded into an n-dimensional space are presented in [30–32] for $n = 4$, and in [33, 34] for $n \geq 4$. Generalizations of the Marching Tetrahedra algorithm to higher dimensions are presented in [35–37] and are usually called Marching Simplex algortihms. A continuation algorithm to create simplicial approximations of manifolds in arbitrary dimensions is presented in [38].

However, those generalized methods often employ data structures that are very memory intensive and, for this reason, are not efficient in high dimensions. For example, typical generalizations of Marching Cubes to higher dimensions use look-up tables, which grows with 2^{2^n}; thus those algorithms can be very memory-intensive even for relatively mild values of n. In [12] a memory-efficient algorithm, called Combinatorial Marching Hypercubes (CMH), was presented, to compute approximations to manifolds of dimension $n-1$ embedded into an n-dimensional space for arbitrary values of n. See [12] for more details of the relevant literature and for examples in dimensions $n > 4$.

1.2 Contributions and paper organization

In this paper, we present the Generalized Combinatorial Marching Hypercubes (GCMH) algorithm, a generalization of CMH to compute $(n-k)$-dimensional manifolds embedded into an n-dimensional space.

This paper is organized as follows. In Section 2 we present the background material and mathematical definitions used in the paper. In Section 3 we present the details of the GCMH algorithm. In Section 4 we compare the results of our method with the Combinatorial Marching Simplex algorithm from [12]. In Section 5 we present a variation of the GCMH called the Generalized Combinatorial Continuation Hypercube (GCCH) algorithm. Finally, in Section 6 we present results and comparison with other methods, and in Section 7 we present some concluding remarks.

2 Background

In this section we present the definition of an *implicitly defined manifold* and other objects used in the paper. These definitions are from [12].

Definition 1 Let $F\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. We say that a point $c \in \mathbb{R}$ is a *regular value* of F if $\nabla F(x) \neq 0$ for all $x \in F^{-1}(c)$. If $c \in \mathbb{R}$ is not a regular value of F we say that it is a *critical value* of F.

Definition 2 A set $\mathcal{M} \subset \mathbb{R}^n$ is called an *implicitly defined* $(n-1)$*-dimensional manifold* if there exists a differentiable function $F\colon \mathbb{R}^n \to \mathbb{R}$ and a regular value c of F such that

$$\mathcal{M} = F^{-1}(c) = \{x \in \mathbb{R}^n \mid F(x) = c\}.$$

The *tangent space* at $p \in \mathcal{M}$ is $T\mathcal{M}_p = \ker(\nabla F(p))$, where ker denotes the null space of a linear map [39].

We can assume without any loss of generality that $c = 0$, that is, that the manifold is given by

$$\mathcal{M} = F^{-1}(0) = \{x \in \mathbb{R}^n \mid F(x) = 0\}.$$

Definition 3 (Transversality) Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds of dimensions m and n, respectively, in $\mathbb{R}^k$ with $max\{m, n\} \leq k$. Given $p \in \mathcal{M} \cap \mathcal{N}$ we say that $\mathcal{M}$ and $\mathcal{N}$ are *transverse at* p if $T\mathcal{M}_p \oplus T\mathcal{N}_p = \mathbb{R}^k$, where $T\mathcal{M}_p$ and $T\mathcal{N}_p$ are the tangent spaces to $\mathcal{M}$ and $\mathcal{N}$, respectively, at p and $\oplus$ denotes the direct sum of two vector spaces [39]. We say that $\mathcal{M}$ and $\mathcal{N}$ are *transverse* if they are transverse at every point $p \in \mathcal{M} \cap \mathcal{N}$ or if $\mathcal{M} \cap \mathcal{N} = \emptyset$.

Definition 4 (Simplex) The *simplex* generated by the points $v_0, \ldots, v_m \in \mathbb{R}^n$ is the set

$$\sigma = \left\{ v \in \mathbb{R}^n \mid v = \sum_{i=0}^{m} \lambda_i v_i, \text{ with } \lambda_i \geq 0 \text{ and } \sum_{i=0}^{m} \lambda_i = 1 \right\},$$

and it is denoted by $\sigma = [v_0, \ldots, v_m]$. The *dimension* of σ is defined as $\dim(\sigma) = \dim(\mathrm{span}\{v_1 - v_0, \ldots, v_m - v_0\})$. For clarity, a simplex of dimension k is referred to as a k-simplex.

A simplex of dimension 0 is also referred to as a vertex and a simplex of dimension a 1 is also referred to as an edge. Notice that an edge may be represented by multiple collinear vertices.

Definition 5 (Simplex Face) Let $\sigma = [u_0, \ldots, u_{k_1}]$ be a k-simplex and $\tau = [v_0, \ldots, v_{k_2}]$ be an m-simplex with $\{v_0, \ldots, v_{k_2}\} \subseteq \{u_0, \ldots, u_{k_1}\}$. Then τ is a *m-face* (or simply a *face*) of σ if $\tau = \sigma$ or if the following conditions are satisfied:

1. $m < k$;

2. If η is an m-simplex with $\tau \subseteq \eta$, then $\eta = \tau$.

If τ is a face of σ with $\tau \neq \sigma$ we say that τ is a *proper face* of σ.

Definition 6 (Hypercube) The set $\mathcal{I}^n = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ with $a_i < b_i \in \mathbb{R}$ is called an *n-dimensional hypercube* (or simply a hypercube). The $(n-1)$-*faces* of $\mathcal{I}^n$ are the sets

$$\mathcal{I}_i^{n-1} = \{x \in \mathcal{I}^n \mid x_i = a_i\} \text{ and } \mathcal{I}_{n+i}^{n-1} = \{x \in \mathcal{I}^n \mid x_i = b_i\}$$

for each $i = 1, \ldots, n$. The faces of dimension less than $n-1$ can be obtained by the intersection of two or more faces of higher dimension, that is

- $\mathcal{I}_{i,j}^{n-2} = \mathcal{I}_i^{n-1} \cap \mathcal{I}_j^{n-1}$, for $j \neq n-i$ (since opposite faces have an empty intersection);
- $\mathcal{I}_{i,j,k}^{n-3} = \mathcal{I}_i^{n-1} \cap \mathcal{I}_j^{n-1} \cap \mathcal{I}_k^{n-1}$, for $j \neq n-i$, $k \neq n-i$, $k \neq n-j$.

The pattern above continues until the edges ($\mathcal{I}^1$) of the hypercube. Each edge is the intersection of $n-1$ faces of dimension $n-2$.

Definition 7 (Adjacency) Two simplices σ_1 and σ_2 are said to be *adjacent* if $\sigma_1 \cap \sigma_2$ is a common face to both σ_1 and σ_2.

Similarly, two hypercubes $\mathcal{I}$ and $\mathcal{I}'$ are said to be *adjacent* if $\mathcal{I} \cap \mathcal{I}'$ is a common face to both $\mathcal{I}$ and $\mathcal{I}'$.

Definition 8 (Incidence) A simplex τ is said to be *incident* to a simplex σ if τ is a proper face of σ.

Similarly, a hypercube $\mathcal{I}$ is said to be *incident* to a hypercube $\mathcal{J}$ if $\mathcal{I}$ is a face of $\mathcal{J}$.

Our algorithm produces a collection of cells, as described in [12], to represent the computed manifold approximation.

Definition 9 (Cells) We define a *cell* as follows: A cell of dimension 0 or 1 is a convex affine cell of dimension 0 or 1, respectively. A cell of dimension k is defined as a list of its vertices and a list of its lower dimensional faces represented as cells.

A cell of dimension k is referred to as a *k-cell* or simply a *cell*. We also refer to 0-cells and 1-cells as *vertices* and *edges*, respectively.

3 The Generalized Combinatorial Marching Hypercubes Algorithm

This section describes the Generalized Combinatorial Marching Hypercubes algorithm (GCMH). It computes an approximation of an $(n-k)$-dimensional implicit manifold $\mathcal{M} = F^{-1}(0)$, with $F\colon \mathbb{R}^n \to \mathbb{R}^k$. The main difference from the CMH algorithm in [12] is the application of a simplex decomposition process to both find the vertices of the manifold and solve possible ambiguities, as described in the following subsections.

3.1 Algorithm Description

Here, we present the main steps of the GCMH algorithm.

Input

The input of GCMH is:

- A C^1 function $F \colon \mathbb{R}^n \to \mathbb{R}^k$.
- A domain hypercube $D = \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$.
- A grid size tuple $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n$.

We subdivide each interval $[a_i, b_i]$ uniformly into k_i intervals; this way we divide D into a uniform grid of smaller hypercubes. We shall refer to it as the *domain grid*. Each $(n-k)$-cell will be incident to two grid hypercubes or one grid hypercube and the boundary.

Output

The output of GCMH is an approximation $\mathcal{M}_A$ for $\mathcal{M} = F^{-1}(0)$, represented by:

- The coordinates of the vertices of $\mathcal{M}_A$.
- A set of $(n-k)$-cells and their r-faces for $0 \leq r \leq n-k-1$.

Each $(n-k)$-cell in $\mathcal{M}_A$ will be a subset of a hypercube of the domain grid, and the algorithm does not create pairs of adjacent $(n-k)$-cells within the same hypercube. Thus, two adjacent $(n-k)$-cells in $\mathcal{M}_A$ necessarily belong to two different adjacent hypercubes.

Given a hypercube $\mathcal{H}$ of the domain grid we denote by $\mathcal{M}_{\mathcal{H}}$ the cells of $\mathcal{M}_A$ contained in $\mathcal{H}$, that is, $\mathcal{M}_{\mathcal{H}} = \mathcal{M}_A \cap \mathcal{H}$.

The main steps of the GCMH algorithm are:

1) Add a perturbation to each vertex v_i in the domain grid by adding a random n-dimensional vector ϵ_i to v_i.
2) For each hypercube $\mathcal{H}$ in the domain grid do:
   2.1) Compute the vertices and edges of $\mathcal{M}_{\mathcal{H}}$ and their incidence and adjacency relations. This computation results into a graph $G_{\mathcal{H}}$.
   2.2) For each connected component of $G_{\mathcal{H}}$, create an $(n-k)$-cell of $\mathcal{M}_{\mathcal{H}}$ and compute its r-faces using combinatorial techniques for $2 \leq r \leq n-k-1$.

Perturbations to each vertex v_i are added in order to displace any eventual vertices of $\mathcal{H}$ that are directly on the manifold, which could lead to errors. This step is further explained in [12].

The steps (2.1) and (2.2) of the GCMH algorithm are detailed in Sections 3.1.1 and 3.1.2, respectively.

3.1.1 Vertices and Edges Computation

Vertices of the approximation $\mathcal{M}_A$ are computed using the Coxeter-Freudenthal-Kuhn (CFK) simplex decomposition algorithm, described in [40–42], applied to each k-face. As stated before, this is the main difference between the previous CMH method, and allows the GCMH method to be applied to manifolds of any co-dimension $k < n$. It is worth noting that, while this calculation is carried out with a simplex decomposition algorithm, information about the simplices is not needed for the combinatorial skeleton, which greatly reduces processing time after the computation of vertices and edges.

For a given hypercube $\mathcal{H}$ of the domain grid, we break each of its k-faces f_k into $k!$ simplices of dimension k, using the CFK triangulation. Then, a vertex of $\mathcal{M}_\mathcal{H} = \mathcal{M}_A \cap \mathcal{H}$ may be created in each of those k-simplices by computing the linear interpolation of the vertices v_i incident to the given k-simplex, weighted by $F(v_i)$; that is, the intersection point v is given by $v = \sum_{i=0}^{k} \lambda_i v_i$, where λ_i, $i = 1, ..., k$, satisfy

$$\sum_{i=0}^{k} \lambda_i F(v_i) = 0, \quad \sum_{i=0}^{k} \lambda_i = 1.$$

If all $\lambda_i \geq 0$, a new vertex of $\mathcal{M}_\mathcal{H}$ will be created in that k-simplex; otherwise, no vertex is created.

To compute the edges of $\mathcal{M}_\mathcal{H}$, the algorithm loops through each $(k+1)$-face f_{k+1} connecting the manifold vertices created on the k-faces incident to f_{k+1}. For each f_{k+1}, three cases may occur:

- No vertices are found: The algorithm does not create any vertex or edge on this face.
- Two vertices are found: The algorithm simply connects both with an edge of $\mathcal{M}_\mathcal{H}$.
- More than two vertices are found: In this case there is more than one way of connecting the vertices with edges, leading to an ambiguity.

These ambiguities are solved using the CFK triangulation applied to the $(k+1)$-face. An example of the finished process can be seen in Figure 2 for a 2-face, where the blue dashed lines represent the edges added to $\mathcal{M}_\mathcal{H}$. A fully worked example is illustrated in Figure 3 with a 3-face, which is a 3-dimensional cube; thus, in this case we have $k = 2$. For clarity, we will denote faces and simplices using the cube vertices, which are numbered 0 to 7, in ascending order; for example, the top 2-face will be denoted face 4567.

Suppose that the algorithm found the manifold vertices a — d, marked with blue dots, as in Figure 3 (a). Vertices a and b are in face 0246, vertex c is in face 2367 and vertex d is in face 1357. In order to solve the ambiguity, the cube is divided into $(k+1)$-simplices in Figure 3 (b). For all k-simplices incident to each $(k+1)$-simplex we calculate the manifold vertices using the linear interpolation described above. In this example, suppose the vertices found are the red dots in Figure 3 (c), which are in the 2-simplices 027, 047 and 057. This will also recalculate the vertices a — d and show they are in the 2-simplices 046, 026, 237 and 157, respectively. Those 2-simplices are in shades of grey in Figure 3 (c).

After all k-simplices are analyzed, we pick a manifold vertex in a k-face; for example, vertex a. The algorithm then identifies which manifold vertex found in the previous step connects with vertex a, and connects them. This is done by checking incidence relations between the k-simplices and $(k+1)$-simplices. Then in the same manner it identifies and connects a third manifold vertex that connects to the second one and so on, until it finds one that is in a k-face of the cube. In our example, the algorithm identified that vertex a is in the 2-simplex 046, which is incident to the 3-simplex 0467. The second manifold vertex that a is connected to is the one in the same 3-simplex, which is the red vertex in the 2-simplex 047. This 2-simplex is also incident to the 3-simplex 0457, and thus this second vertex is connected to the red vertex in the 2-simplex 057. Finally, this vertex is connected to the blue vertex d. The result of this process is illustrated in Figure Figure 3 (d).

Once the algorithm identifies a second vertex in a 2-face (in our example, two blue vertices), it connects them with a single manifold edge (denoted by a blue dashed line in Figure Figure 3 (e) between vertices a and d), stores it, and pick a new manifold vertex in a k-face to start again. Lastly, once all manifold vertices in k-faces are exhausted, the algorithm only keeps the vertices and the manifold edges stored; all internal vertices and connections are discarded. The end result can be seen in Figure 3 (e).

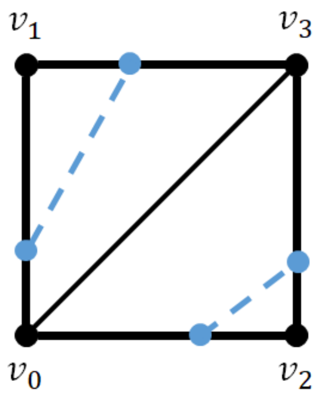

Fig. 2 Disambiguation for a 2-face using the CFK triangulation

3.1.2 Combinatorial Skeleton

Here we summarize the main definitions and ideas for the construction of the *Combinatorial Skeleton*, which is a structure used to compute higher-dimensional faces of the approximation $\mathcal{M}_A$. Further details can be found in [12].

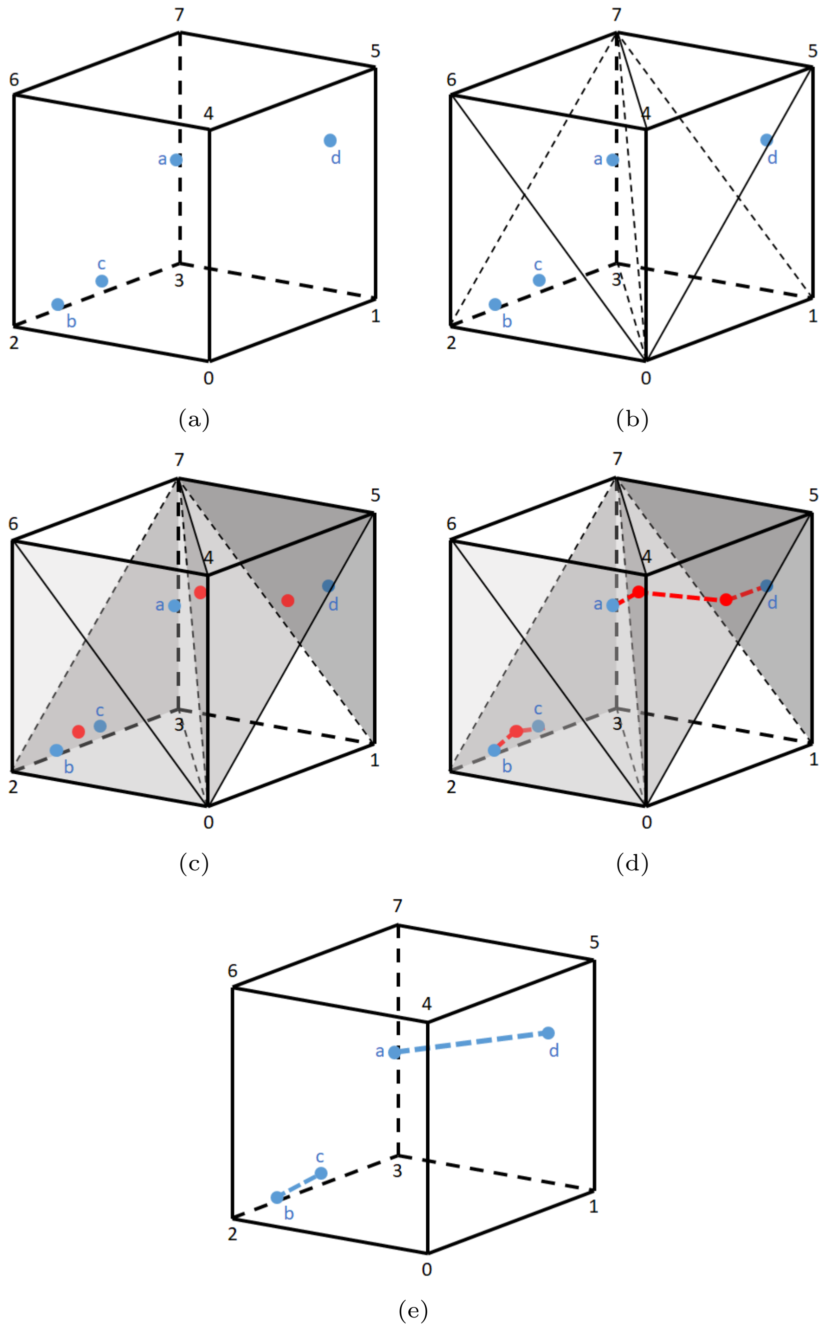

Fig. 3 Disambiguation process in a 3-face using the CFK triangulation. (a) Manifold vertices computed on the 2-faces. (b) 3-face divided into 3-simplices. (c) Internal manifold vertices found on the shaded faces of the 3-simplices. (d) Internal manifold vertices connected along the 3-simplices. (e) Final manifold edges created and stored in $\mathcal{M}_{\mathcal{H}}$.

Definition 10 (Face labeling) We represent the faces $\mathcal{I}_i^{n-1}$ and $\mathcal{I}_{n+i}^{n-1}$ of the n-dimensional hypercube $\mathcal{I}^n$ by the integers i and $n+i$, that is, we label $\ell(\mathcal{I}_i^{n-1}) = i$ and $\ell(\mathcal{I}_{n+i}^{n-1}) = n+i$.

The faces of dimensions lower than $n-1$ are labeled accordingly as $\ell(\mathcal{I}_{i,j}^{n-2}) = \{i,j\}$, $\ell(\mathcal{I}_{i,j,k}^{n-3}) = \{i,j,k\}$, etc. The pattern continues until the edges $\mathcal{I}^1$ of $\mathcal{I}^n$.

Due to the construction of the r-faces and the properties of hypercubes, one can prove the following [12]:

- Each $(n-k)$-cell $\mathcal{C}$ in $\mathcal{M}_{\mathcal{H}}$ is indeed a cell as defined in Definition 9;
- We can represent the incidence relations of $\mathcal{C}$ using the labels of $\mathcal{H}$ by labeling each r-face of $\mathcal{C}$ with the labels of the corresponding $(r+1)$-face of $\mathcal{H}$.

These facts are used to build the combinatorial skeleton. Let $\mathcal{L}_r$ be the set of labels of the r-faces of $\mathcal{C}$, for $r = 0, \ldots, n-2$, and set $\mathcal{L} = \bigcup_{r=0}^{n-2} \mathcal{L}_r$. We also define V as the set of manifold vertices of $\mathcal{M}_{\mathcal{H}}$.

The combinatorial skeleton is built as follows: The vertices and edges of $\mathcal{M}_{\mathcal{H}}$ obtained from Section 3.1.1 form a graph G, with possibly more than one connected component. For each component G' of G, we apply the following algorithm:

function CombinatorialSkeleton(n, V, $\mathcal{C}$, $\mathcal{H}$, $\mathcal{L}$)
  for all vertices $v \in V$ of G' on a k-face f of $\mathcal{H}$ **do**
    add v to $\mathcal{C}$
    $\ell(v) \leftarrow \ell(e)$
    add $\ell(v)$ to $\mathcal{L}_0$
  end for
  for r $\leftarrow 2$ to $n-2$ **do**
    for all $(r+1)$-faces $f_{\mathcal{H}}$ of $\mathcal{H}$ **do**
      if $\exists \ell(x) \in \mathcal{L}_{r-1}$ such that $\ell(f_{\mathcal{H}}) \subset \ell(x)$ **then**
        create a new r-face f_C
        add f_C to $\mathcal{C}$
        $\ell(f_C) \leftarrow \ell(f_{\mathcal{H}})$
        add $\ell(f_C)$ to $\mathcal{L}_r$
        for all $(r-1)$-faces f'_C of $\mathcal{C}$ **do**
          if $\ell(f_C) \subset \ell(f'_C)$ **then**
            Add f'_C as a face of f_C in $\mathcal{C}$
          end if
        end for
      end if
    end for
  end for
  return $\mathcal{C}$
end function

3.2 Consistency

This section addresses the consistency of the manifold approximation $\mathcal{M}_A$ generated by the GCMH algorithm, as described below.

Definition 11 (Consistency) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two adjacent hypercubes of the domain grid that share a common $(n-1)$-face f and let $\mathcal{M}_f = \mathcal{M}_{\mathcal{H}_1} \cap \mathcal{M}_{\mathcal{H}_2}$. We say that $\mathcal{M}_f$ is *consistent* if $\mathcal{M}_{\mathcal{H}_1} \cap f = \mathcal{M}_{\mathcal{H}_2} \cap f$. In this case we also say that each cell of $\mathcal{M}_f$ is consistent.

In other words, the approximation is consistent if it generates the same vertices, edges, and higher-dimensional cells on shared $(n-1)$-faces of adjacent hypercubes. In order for the algorithm to be correct, it is sufficient that the approximation is consistent for all pairs of adjacent hypercubes in the domain grid.

Since the construction of the combinatorial skeleton is the same as in [12], the proofs of the consistency results can mostly be adapted from there with minor changes. The only proof which needs a more significant modification is vertex consistency because the vertex generation uses a different process. Thus, for higher-dimensional consistency, we will only state those results and refer to [12] for details of their proofs.

Lemma 1 (Vertex consistency) *The vertices generated by the GCMH algorithm on* $\mathcal{M}_{\mathcal{H}_1} \cap f$ *and* $\mathcal{M}_{\mathcal{H}_2} \cap f$ *are the same.*

Proof Since the vertices of the $(n-1)$-face f are common to $\mathcal{H}_1$ and $\mathcal{H}_2$, the values of the function F at the vertices of f are the same on $\mathcal{H}_1$ and $\mathcal{H}_2$, that is, when f is viewed as a face of $\mathcal{H}_1$ and as a face of $\mathcal{H}_2$. Moreover, the consistency of the CFK triangulation ensures that the simplices which $\mathcal{M}_{\mathcal{H}_1} \cap f$ and $\mathcal{M}_{\mathcal{H}_2} \cap f$ are decomposed in having the same vertices. Therefore the vertices of $\mathcal{M}_f$, which are obtained by linear interpolation on the values of F at the vertices of the simplices in f, are the same on $\mathcal{M}_{\mathcal{H}_1}$ and $\mathcal{M}_{\mathcal{H}_2}$. $\square$

Lemma 2 (Edge consistency) *The edges generated by the GCMH algorithm on* $\mathcal{M}_{\mathcal{H}_1} \cap f$ *and* $\mathcal{M}_{\mathcal{H}_2} \cap f$ *are the same.*

Lemma 3 (Higher dimensional consistency) *The* k*-cells generated by the GCMH algorithm on* $\mathcal{M}_{\mathcal{H}_1} \cap f$ *and* $\mathcal{M}_{\mathcal{H}_2} \cap f$ *are the same for* $2 \leq r \leq n-2$.

The lemmas above prove the following theorem:

Theorem 3.1 (GCMH algorithm consistency) *The set of cells* $\mathcal{M}_f = \mathcal{M}_{\mathcal{H}_1} \cap \mathcal{M}_{\mathcal{H}_2}$ *generated by the GCMH algorithm is consistent for any pair* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *of adjacent hypercubes of the domain grid.*

3.3 Combinatorial methods for grid generation

In order to represent hypercubes and their vertices and generate grid elements, we use combinatorial techniques based on the enumeration of discrete Cartesian products. Those methods are presented in [12] with almost no modifications; we, therefore, refer to [12] for details on these combinatorial methods.

4 Comparison to Combinatorial Marching Simplex

In order to study the performance of the GCMH algorithm, we implemented the Combinatorial Marching Simplex (CMS) method, also described in [12]. This method is a triangulation method that also uses a combinatorial structure to compute an approximation $\mathcal{M}_A$ of a manifold implicitly defined by $\mathcal{M} = F^{-1}(0)$, with $F\colon \mathbb{R}^n \to \mathbb{R}^k$. This is the main reason for choosing CMS for a performance comparison, since the main advantage of GCMH over the CMH algorithm in [12] is the ability to compute manifolds of co-dimension higher than 1, and thus the comparison with CMH would not be very meaningful. Furthermore, since CMS also uses a combinatorial structure similar to GCMH, we were able to compare the performance of our method with a similar algorithm.

The CMS algorithm is broadly described as follows:

1. Subdivide a regular and finite grid of hypercubes of a compact subspace of $\mathbb{R}^n$ into simplices using the CFK triangulation. The resulting subdivision is the domain grid.
2. For each simplex $\mathcal{S}$ of each hypercube $\mathcal{H}$ in the domain do:
   (a) Compute the vertices of $\mathcal{M}_\mathcal{S} = \mathcal{S} \cap \mathcal{M}_A$ that are on the edges of $\mathcal{S}$. This is done with linear interpolation of the vertices of each face of the simplex, similar to the computation of vertices in the GCMH algorithm (Section 3.1.1).
   (b) Define the edges of $\mathcal{M}_\mathcal{S}$ that are on the 2-faces of $\mathcal{S}$. Contrary to GCMH there is an unique way to this.
   (c) Build the Combinatorial Skeleton to get the k-faces of $\mathcal{M}_\mathcal{S}$ for $k > 2$. The details of this step are analogous to GCMH.

It is worth noting that it is expected that CMS will require more memory and processing time in comparison to GCMH. This is because CMS breaks each hypercube into n-simplices, analyzes each k-face of every n-simplex, then stores the data from every manifold vertex found in an n-simplex in order to build the approximation $\mathcal{M}_A$. The CFK triangulation generates $n!$ simplices for every n-dimensional hypercube and then each $\binom{n}{k}$ k-face of every simplex needs to be analyzed; hence, the CMS algorithm will need to analyze $\binom{n}{k}n!$ k-simplices for every hypercube. On the other hand, GMHC only uses the data from the vertices found in the k-simplices in the k-faces of each hypercube. Since there are $2^{n-k}\binom{n}{k}$ k-faces in a hypercube and each k-face is decomposed into $k!$ k-simplices, this means there are $2^{n-k}\binom{n}{k}k!$ k-simplices for GCMH to analyze for every hypercube. This leads to CMS having a substantial increase not only in the number of simplex faces to search for manifold vertices, but also in the processing time of the edges and higher-dimensional cells of $\mathcal{M}_A$ in comparison to GCMH, as the number of vertices will also increase. This difference is illustrated in Figure 4 (it is worth noting that we necessarily must have $n \geq k+1$). In all cases, we can expect that GCMH will need to process substantially less k-simplices than CMS. This point will be made clearer in Section 6.

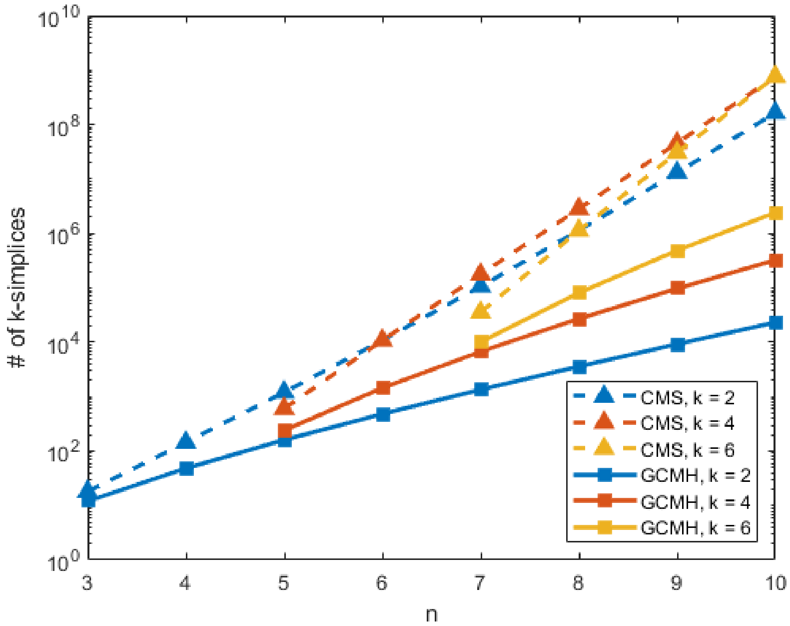

Fig. 4 Estimates on number of k-simplices analyzed by CMS and GCMH per hypercube.

5 The Generalized Combinatorial Continuation Hypercube extension

In a similar way to [12], here we also propose an extension to the GCMH algorithm called the Generalized Combinatorial Continuation Hypercube (GCCH). The aim is to improve the time efficiency of the GCMH algorithm by checking for vertices only in the hypercubes adjacent to the ones that are transversal to the manifold and therefore have a chance to be transversal themselves. This extension to a continuation method was mainly based on the continuation method in the work of Allgower and Georg [43].

5.1 Starting Point

Given a user-input starting point x_0, we first find a starting hypercube that contains it. Since the hypercubes are defined by intervals along each dimension, the starting hypercube is found by comparing the coordinates of x_0 with those intervals.

5.2 List of Hypercubes

After finding the initial hypercube $\mathcal{H}_0$ containing the initial point x_0, we put all of its adjacent hypercubes into a list L_{tbp} of hypercubes to be processed, and we process $\mathcal{H}_0$ using the GCMH algorithm. We then create a list L_p of already processed hypercubes, and add $\mathcal{H}_0$ to it. After that, the algorithm proceeds as follows:

While L_{tbp} is not empty do:

1. Remove an hypercube $\mathcal{H}_i$ from L_{tbp}.
2. Find all hypercubes adjacent to $\mathcal{H}_i$ that are not in L_p and are transverse to $\mathcal{M}$; add them to L_{tbp}.
3. Process $\mathcal{H}_i$ like in the GCMH algorithm.
4. Add $\mathcal{H}_i$ to L_p.

When $\mathcal{M}$ has more than one connected component, some components might not be computed by the steps above. In order to reach those components, the user can execute the algorithm again with another starting point on a different connected component.

5.3 Estimate on complexity

Tha main difference between GCMH and GCCH is their strategy for traversing the hypercube grid. The GCMH algorithm traverses the entire grid sequentially; therefore, supposing the the domain is divided in m segments for every dimension, the GCMH algorithm is expected to have $O(m^n)$ complexity.

On the other hand, the GCCH algorithm restricts the analysis to hypercubes adjacent to the ones already verified to be transversal to $\mathcal{M}$, and thus the ones most likely to have manifold vertices. Since the list of hypercubes grow along $\mathcal{M}$ and it has dimension $n-k$, it is expected that the GCCH algorithm will have $O(m^{n-k})$ complexity for smooth manifolds, which is a significant gain over GCMH.

6 Results

This section describes some experimental results to show the empirical effectiveness and efficiency of the GCMH algorithm and the GCCH extension.

We used a computer with a 2.3 GHz 8-Core Intel Core i9 Processor and 64 GB 2667 MHz DDR4 memory, and the software Matlab 2021b for all the computations.

In order to show the efficiency of the GCMH and GCCH we ran an experiment based on the complex cosine function: $\mathcal{M} = F^{-1}(0)$, where $F : \mathbb{C}^2 \to \mathbb{C}^1$ is given by $F(z, w) = z - \cos(w)$. For the purpose of our algorithm, F is treated as a function from $\mathbb{R}^4$ to $\mathbb{R}^2$; this was done by separating the real and imaginary parts and treating them as real numbers. We compared the results using GCMH, GCCH, CMS, and Combinatorial Continuation Simplex (CCS). The experiment shows that (see discussion in Section 6.1):

- The processing time of GCMH is significantly lower than the processing time of CMS. The same applies to GCCH when compared to CCS.
- The continuation method significantly decreases the processing time of GCCH and CCS.

div	CMS	GCMH	CCS	GCCH
5	3.83	2.16	0.69	0.54
10	55.54	34.20	2.25	1.54
20	2048.63	589.38	8.34	5.54
40	12362.45	7927.68	38.30	22.35
80			236.79	87.22
160			2837.64	396.12
320			41355.25	1912.98

Table 1 Processing time, in seconds, of CMS, GCMH, CCS, GCCH to approximate the complex cosine function $z = \cos(w)$ for each number *div* of divisions per dimension.

div	CCS	GCCH
5	1162	148
10	4378	612
20	17258	2112
40	68354	8336

Table 2 Output size (number of $(n-1)$-cells) of the approximation of $z = \cos(w)$ using CCS and GCCH.

- Although the output size of GCCH is smaller than the output size of CCS, the output approximation is smoother when using GCCH.

A second experiment was made to test the efficiency of GCMH and GCCH in higher dimensions. For that purpose we used the 3-torus (or 3D torus) embedded in $\mathbb{R}^6$, given by $\mathcal{M} = S^1 \times S^1 \times S^1$. More specifically, $F(x) = 0$ where $F\colon \mathbb{R}^6 \to \mathbb{R}^3$ is given by

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} x_1^2 + x_2^2 - \frac{1}{4} \\ x_3^2 + x_4^2 - 1 \\ x_5^2 + x_6^2 - 4 \end{pmatrix}.$$

6.1 Comparing CMS, GCMH, CCS and GCCH

In order to show the advantages of the GCMH algorithm over the CMS and the advantages of the continuation method for both GCCH and CCS, we measured the processing time needed to approximate the complex cosine function. For each run the number of hypercubes of the domain grid is set by a number *div* of divisions per dimension, that is, the total number of hypercubes is div^4.

The results can be seen on Table 1 and Figure 5. As expected, the continuation method significantly improves both algorithms. Also, the processing time using the GCMH algorithm is significantly lower than the CMS.

We also measured the output size, as the number of $(n-1)$-cells, of GCCH and CCS. Note that the continuation method does not change the output of the approximation when compared to GCMH and CMS. The results can be seen on Table 2. As expected, the output of GCCH is smaller.

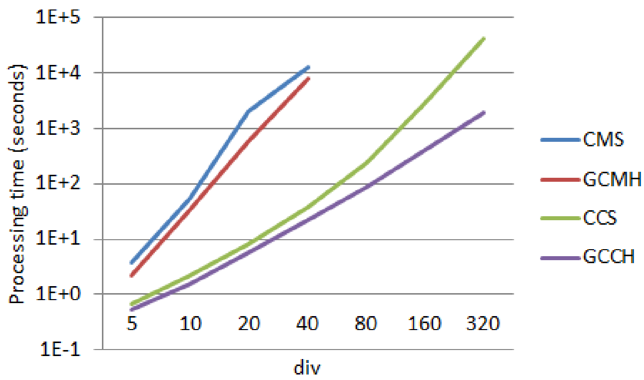

Fig. 5 Processing time, in logarithmic , of CMS, GCMH, CCS and GCMH to approximate the complex cosine function $z = \cos(w)$ as a function of the number *div* of subdivisions.

div	GCMH	GCCH	*div*	GCMH	GCCH
2	93.61	93.07	5	22686.60	2946.30
3	1055.67	734.47	6	67749.18	7126.96
4	5910.59	846.46	7	172271.80	13545.33

Table 3 Processing time, in seconds, of GCMH and GCCH to approximate the 3-torus for each number *div* of divisions per dimension.

Figures 6(a) and 6(b) show projections of the approximation of the complex cosine function using GCCH and CCS, respectively. Figures 6(c) and 6(d) show a zoom in of a portion of both projections. Although the output using GCCH is less dense, it is smoother. We indicate the smoothness of the results in Figures 6(c) and 6(d) by a series of red segments. The consecutive red segments represent "curve breaks"; a higher density of curve breaks indicates better smoothness.

6.2 Results in higher dimension

In order to show the effectiveness of GCMH and GCCH in higher dimensions, we measured the processing time needed to approximate the 3-torus embedded in $\mathbb{R}^6$ (see Figure 8). For each computation, the number of hypercubes of the domain grid was set by a number *div* of divisions per dimension, that is, the total number of hypercubes is given by div^6. The results can be seen on Table 3 and Figure 7.

7 Conclusion

This paper proposed a generalization of the Combinatorial Marching Hypercubes algorithm [12] that extends the renowned Marching Cubes algorithm to approximate manifolds of any dimension. In [12] the manifold was required to have co-dimension 1 while in the generalization proposed in this paper this requirement was eliminated.

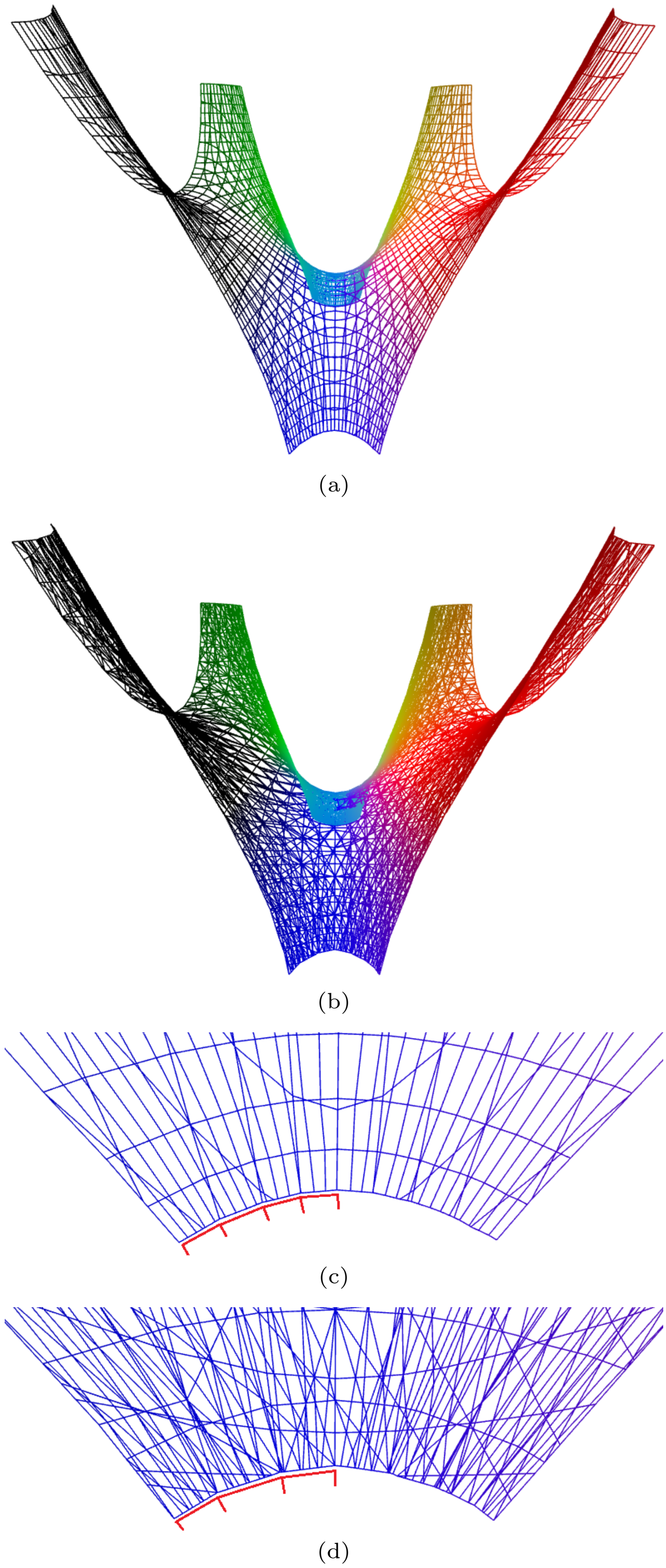

Fig. 6 Projections of the complex cosine function $z = cos(w)$ using (a) GCCH with $div = 40$ and (b) CCS with $div = 20$. In (c) and (d) we show a zoom in of a portion of the manifolds in (a) and (b), respectively.

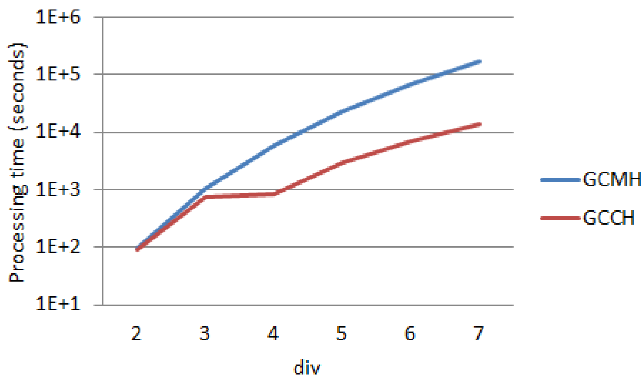

Fig. 7 Processing time of GCMH and GCMH to approximate the 3-torus, in logarithmic scale.

The proposed implementation did not rely on lookup tables or other expensive memory structure that grows exponentially with dimension. Our algorithm used combinatorial and topological techniques to build the manifold approximation by a cell complex. Each hypercube can be processed independently, making the proposed algorithm highly parallelizable.

We showed the effectiveness of our method in higher dimensions by approximating classical manifolds from the literature and we also showed that the algorithm outperformed a similar algorithm based on a simplicial decomposition of the domain grid.

As future work, we plan a parallel implementation, as the algorithm is highly parallelizable. We also plan to study the accuracy of the manifolds created, and apply it to practical problems.

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Statements and Declarations

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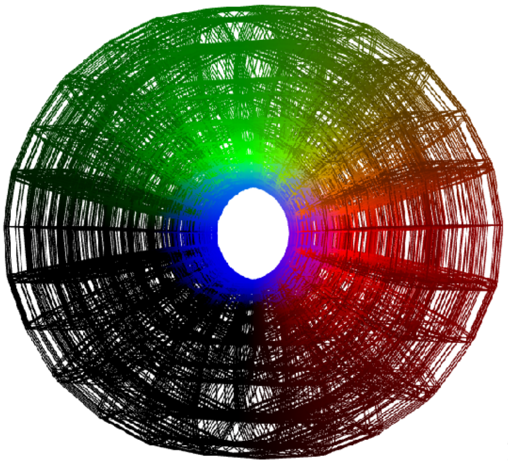
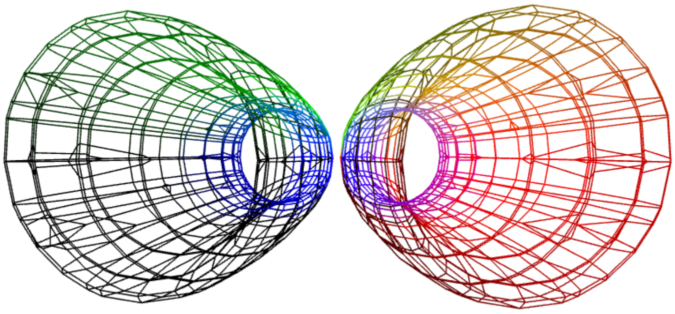
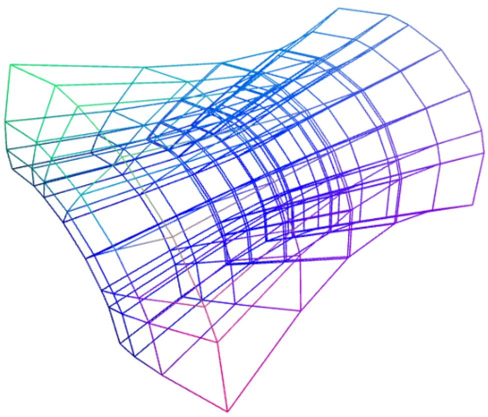


(a)



(b)



(c)

Fig. 8 Projections of the approximations computed by the GCCH algorithm of the 3-torus $\mathcal{M} = S^1 \times S^1 \times S^1$ embedded in $\mathbb{R}^6$. (a) A projection of $\mathcal{M}$ with $div = 8$. (b) A projection of $\mathcal{M} \cap \{x_6 = 0\}$ with $div = 8$. (c) A projection of $\mathcal{M} \cap \mathbb{R}^6_+$ with $div = 5$.

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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CHAPTER 5

CONCLUSION

In this work we showed works on three different aspects of differential equations:

- The first two works are concerned with the robustness of asymptotical stability and exponential dichotomy of differential equations in Banach spaces. Those works extend previous results in finite dimensions to infinite dimensions. We show that if the equation is in the class of Generalized Almost Periodic ($\mathscr{GAP}$) functions (which include important functions, such as periodic ones), then asymptotic stability and exponential dichotomy are robust under integrally small perturbations. The amplitude of those oscillations need not be small, as long as they are fast enough.

  We also show examples in infinite dimensions, including one that shows that asymptotic stability is not robust; we are able to stabilize a system with a forcing term that oscillates fast enough. This may have applications on control systems.

- The second object of interest is a rigorous computational method using wavelets. This method combines the radii polynomial approach for validated numerics with the Haar wavelet method for differential equations. This results in a rigorous computational method which can deal with less smooth problems, which is a limitation of previous methods. We believe further work using other types of nonlinearities, higher-order derivatives and other wavelet types will mitigate its weaknesses.

- The last topic is the Generalized Combinatorial Marching Hypercube (GCMH) algorithm. This is an algorithm which builds upon the classic Marching Cubes algorithm for generating isosurfaces. Our method is able to work in any dimension for both the domain and co-domain, and uses a combinatorial technique to avoid using the large lookup tables that other Marching Hypercubes methods would require. It also performs considerably faster than simplex-based algorithms, especially as dimensions increase.

We also study an extension called the Generalized Combinatorial Continuation Hypercube (GCCH) which combines the method with a continuation algorithm. This extension can be faster but has some drawbacks.

This method can still be improved via adaptative algorithms and parallelization. We are also working on combining the GCMH and GCCH algorithms with validated numerical methods, in order to both implement adaptive refinement and obtaining an efficient rigorous algorithm for multi-parameter differential equations and dynamical systems.

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